Gromov-Witten Theory and Type II Extremal Transitions

by

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ABSTRACT

Extremal transitions are a contract-deform surgery, which play a significant role in spacetime topology change in string theory. From a physical point of view, we expect that mirror symmetry might be preserved under extremal transitions. Li-Ruan initiated the study of the change of Gromov-Witten theory under conifold transitions twenty years ago and some generalizations to other Type I transitions have been made over the years, but no work on Type II extremal transitions has been known before. In this thesis, we propose a very general conjectural framework which relates two quantum D-modules under a primitive extremal transition. We verify this framework for a number of examples of Type II extremal transitions. In the case of cubic extremal transitions, we find an interesting connection between the quantum D-module and the genus zero FJRW theory of the cubic singularity.

CHAPTER I

Introduction

The interplay between mathematics and physics has proven extremely fruitful over the past few decades. String theory, a physical model of the universe proposed by physicists in hopes of reconciling quantum physics with general relativity, has stimulated plenty of new developments in mathematics. One of the most intriguing impetuses comes from the study of physical dualities. Interesting dualities often arise naturally in physics where different models may describe the same physical theories. On the mathematical side, however, such a duality can give rise to a highly nontrivial correspondence between two rather different mathematical theories.

One prominent example is *mirror symmetry*, a physical duality between two forms of string theory. Mirror symmetry conjectures that there exists a pair of Calabi-Yau 3-folds $(X, \hat{X})$ such that the symplectic geometry of X is equivalent to the complex geometry of $\hat{X}$. On the symplectic side of X, we can define *Gromov-Witten invariants*, which can be viewed as the virtual number of holomorphic curves of a fixed genus and degree in X. If mirror symmetry holds for $(X, \hat{X})$, then we expect the Gromov-Witten invariants of X can be computed from the period integrals on $\hat{X}$, which purely depend on the complex geometry of $\hat{X}$. The idea of mirror symmetry has led to many important breakthroughs, including the solution to a long-standing problem in enumerative geometry: counting rational curves on a general quintic 3-fold (see e.g. [4, 17, 28]).

This thesis focuses on understanding how Gromov-Witten invariants change under *extremal transitions*. Extremal transitions are a topological surgery, which serves as the basic building blocks for topological changes of Calabi-Yau 3-folds. If one can establish a mirror symmetry statement under extremal transitions, then mirror symmetry conjectures may be proved for a huge family of Calabi-Yau 3-folds from known examples. Our current approach is through *quantum D-modules*, which encode the genus zero Gromov-Witten invariants. In this work, we will propose a conjectural framework that relates two quantum D-modules under an extremal transition, and we will verify this framework for a number of examples involving Calabi-Yau 3-folds. In the case of cubic extremal transitions, we will also show an interesting connection between the quantum D-modules and the *Fan-Jarvis-Ruan-Witten (FJRW) theory* of the cubic singularity.

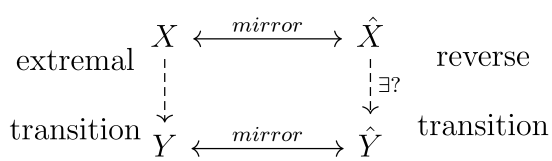
1.1 Reid's Fantasy

Classification of Calabi-Yau 3-folds presents an important problem in algebraic geometry. A construction by Batyrev using toric geometry [2] revealed more than 470 million families of Calabi-Yau 3-folds, with more than 30,000 distinct Hodge diamonds, which made the classification problem nearly impossible. It is even unknown whether the number of topologically distinct Calabi-Yau 3-folds is finite or not.

A popular classifying scheme from the 1990s is to classify Calabi-Yau 3-folds up to surgeries. There are two types of surgeries involved in this picture: flops and extremal transitions. Flops are a birational surgery and usually arise as different resolutions of a singular variety. It is known that any two Calabi-Yau 3-folds are birationally equivalent if and only if they are connected by a sequence of flops. On the other hand, for two birationally non-equivalent classes of Calabi-Yau 3-folds, Miles Reid speculated that they might be connected through a sequence of extremal transitions (known as *Reid's fantasy* [32]). One strong support for Reid's Fantasy is that there are 7555 Calabi-Yau hypersurfaces in weighted $\mathbb{P}^4$ that are indeed connected through extremal transitions [5].

1.2 Gromov-Witten theory under surgeries

Two decades ago, Li-Ruan [27] initiated a program to study the change of Gromov-Witten theory under flops and conifold transitions. Their motivation is to extend mirror symmetry to a larger class of Calabi-Yau 3-folds. The strategy can be summarized as follows: one starts with a mirror pair $(X, \hat{X})$, and assumes that another Calabi-Yau 3-fold Y is obtained by applying a flop/extremal transition on X. Then conjecturally there should exist a reverse flop/transition which allows one to obtain the mirror partner $\hat{Y}$ for Y from the mirror partner $\hat{X}$ for X.



Assuming Reid's Fantasy holds, if one can prove a mirror symmetry statement for flops and extremal transitions, then mirror symmetry for all Calabi-Yau 3-folds would follow directly from one single mirror pair $(X, \hat{X})$. Hence the very first step would be to understand how Gromov-Witten invariants change under flops as well as under extremal transitions.

In their pioneering work [27], Li-Ruan show that a flop between two Calabi-Yau 3-

folds induces a natural equivalence between their genus zero Gromov-Witten theories. More precisely, there is an isomorphism between their quantum cohomology rings, up to analytic continuation over the extended Kähler moduli space. Here quantum cohomology rings can be viewed as encoding the genus zero Gromov-Witten invariants. Later, Lee-Lin-Wang [25] generalized the invariance of quantum cohomology to higher dimensional flops in various settings. A general expectation is that two Gromov-Witten theories related by a flop are essentially equivalent via an appropriate analytic continuation of the quantum variables.

In sharp contrast to flops, the change of Gromov-Witten theory under an extremal transition is much subtler and more complicated. If X and Y are related by an extremal transition, a general expectation is that the Gromov-Witten theory of Y can be realized as a sub-theory of the Gromov-Witten theory of X. However, it is not clear what statement one can prove for extremal transitions.

1.3 Extremal Transitions

Let X and Y be two smooth varieties over $\mathbb{C}$. The process of going from X to Y is called an *extremal transition* if Y is obtained by applying a birational contraction $\pi : X \to \overline{Y}$ followed by a complex smoothing $\overline{Y} \rightsquigarrow Y$.

$$\begin{array}{ccc} X & \dashrightarrow & \\ \downarrow{\scriptstyle \pi} & & \searrow \\ \overline{Y} & \rightsquigarrow & Y \end{array}$$

An extremal transition is called *primitive* if the birational contraction cannot be factored further in the algebraic category. According to Wilson's classification [33], we have the following three types of primitive extremal transitions:

- Type I (small transition): π contracts a union of curves

- Type II: π contracts a divisor to a point
- Type III: π contracts a divisor to a curve

In the case of Type II transitions between 3-folds, when the exceptional locus of $\pi : X \to \overline{Y}$ is a del Pezzo surface of degree k, we will say this a *degree-k extremal transition.*

Li-Ruan [27] showed that a conifold transition (or more generally, any Type I extremal transition) between two Calabi-Yau 3-folds induces a natural homomorphism between their quantum cohomology rings. There are many further generalizations (see e.g. [21, 23, 26]) for Type I extremal transitions using different formulations.

1.4 The main results

This thesis work is devoted to studying Gromov-Witten theory under Type II extremal transitions since no such work[1] had been published in the prior twenty years and the most difficult problem is that it is unclear what kind of statement one can prove. we will formulate a conjectural framework using the language of quantum D-modules and verify this framework for the local model and Calabi-Yau examples of Type II extremal transitions in both degree-3(cubic) and degree-4 cases. In the cubic case, we will show an interesting connection between the quantum D-modules and the Fan-Jarvis-Ruan-Witten (FJRW) theory of the cubic singularity. This thesis work is primarily based on [30, 31].

The notion of the quantum D-module of X, denoted by $\mathcal{H}(X)$, is a cyclic D-model generated by the generating function of Gromov-Witten invariants of X with only one insertion. Due to a reconstruction theorem in Gromov-Witten theory (see e.g. [24]), the quantum D-module can be viewed as encoding the genus zero Gromov-

[1] Lee-Lin-Wang initiated the study on A-model and B-model transitions of k-fold singularities. However, their results have not been published and we were only aware of their results after the completion of the current work.

Witten theory of X. When X is a complete intersection in toric variety, we usually consider the ambient part quantum D-module, which is still denoted by $\mathcal{H}(X)$ if there is no risk of confusion. Detailed explanations can be found in Chapter II.

I propose the following framework in [30] that relates the quantum D-modules under a primitive extremal transition.

Conjecture I.1. *Suppose two smooth projective varieties X and Y are related by a primitive extremal transition. One may perform analytic continuation of $\mathcal{H}(X)$ over the extended Kähler moduli space of X to obtain a D-module $\bar{\mathcal{H}}(X)$. There is a divisor E and a submodule $\bar{\mathcal{H}}^E(X) \subseteq \bar{\mathcal{H}}(X)$ with maximum trivial E-monodromy such that*

$$\bar{\mathcal{H}}^E(X)|_E \simeq \mathcal{H}(Y),$$

where $\bar{\mathcal{H}}^E(X)|_E$ is the restriction to E.

The above conjecture is based on several observations: first, the quantum D-modules of X and Y usually involve different quantum variables. To compare their quantum D-modules, one must find a way to relate their quantum variables. This is usually done by analytic continuation over the extended Kähler moduli, and in this way, we obtain a D-module $\bar{\mathcal{H}}(X)$. Moreover, a discrepancy exists between the rank of $\bar{\mathcal{H}}(X)$ and $\mathcal{H}(Y)$, which leads us to identify $\mathcal{H}(Y)$ as a submodule of $\bar{\mathcal{H}}(X)$ after restriction to a transition divisor E in the extended Kähler moduli space. There is also the issue of monodromy involved when we want to make sense of the restriction of a quantum D-module. For this reason, we need to require this submodule admit only trivial monodromy around the transition divisor in question. Finally, it turns out that in many cases, this submodule should have maximum trivial E-monodromy.

The above conjectural framework is verified for the following examples.

Theorem I.2. *Conjecture I.1 holds for the following cases:*

1. $X := \mathbb{P}(K_S \oplus \mathcal{O})$, *and* Y *is a cubic 3-fold in* $\mathbb{P}^4$, *where* S *is a cubic surface;*
2. *Degree-3 Type II extremal transitions between two Calabi-Yau 3-folds* $\{X_0, Y_0\}$*;*
3. $X' := \mathbb{P}(K_E \oplus \mathcal{O})$, *and* Y' *is a (2,2) complete intersection in* $\mathbb{P}^5$, *where* E *is a del-Pezzo surface of degree 4;*
4. *Degree-4 Type II extremal transitions between two Calabi-Yau 3-folds* $\{X_1, Y_1\}$*;*
5. *Degree-4 Type II extremal transitions between two Calabi-Yau 3-folds* $\{X_2, Y_2\}$*.*

The descriptions of these examples are as follows. More detailed explanations and the proof of Theorem I.2 can be found in Chapter III and Chapter V, respectively.

1. The local model of degree-3 extremal transition $\{X, Y\}$: Let S be a cubic surface inside a Calabi-Yau 3-fold V and assume the curves on S generate an extremal ray in the Mori cone of V. One may contract S to a point with cubic singularity whose local equation is given by

$$x^3 + y^3 + z^3 + w^3 = 0,$$

Deforming this equation, we obtain a smoothing

$$x^3 + y^3 + z^3 + w^3 = t \quad (t \neq 0).$$

Consider a tubular neighborhood around E, which is identified with its normal bundle $N_{E/X}$. Since the ambient variety is a Calabi-Yau 3-fold, we also have $N_{E/X} \simeq K_E$. Compactifying both sides, we take $X := \mathbb{P}(K_S \oplus \mathcal{O})$, and Y is a cubic 3-fold in $\mathbb{P}^4$. This is the local model of degree-3 Type II extremal transitions.

2. A Calabi-Yau example of cubic extremal transitions $\{X_0, Y_0\}$ (appeared in [30]) is obtained in the following way: Let Y_0 be a quintic hypersurface in $\mathbb{P}^4$ which

can be deformed into a singular quintic $\bar{Y}_0$ with a unique triple point singularity, then let X_0 be the blow-up of $\bar{Y}_0$ at the unique singular point, which is a crepant resolution. The passage from X_0 to Y_0 is an example of degree-3 Type II extremal transition between two Calabi-Yau 3-folds.

3. The local model of degree-4 Type II extremal transitions $\{X', Y'\}$: Let E be the degree-4 del Pezzo surface inside a Calabi-Yau 3-fold X, which gets contracted to a point under π. Consider a tubular neighborhood around E, which is identified with its normal bundle $N_{E/X}$. Since the ambient variety is a Calabi-Yau 3-fold, we also have $N_{E/X} \simeq K_E$. Under the birational morphism π, the divisor is contracted to a point with singularity locally given by a complete intersection of two equations with quadratic leading terms. Smoothing out the singularity locally yields a (2,2) complete intersection in $\mathbb{P}^5$. We compactify $N_{E/X} \simeq K_E$ and take $X' := \mathbb{P}(K_E \oplus \mathcal{O})$, while Y' is a (2,2) complete intersection in $\mathbb{P}^5$. The transition from X' to Y' is the *local model* of Type II extremal transitions in degree 4.

4. A Calabi-Yau example of degree-4 Type II extremal transitions $\{X_1, Y_1\}$: In the first example, Y_1 is a (2,4) complete intersection in $\mathbb{P}^5$, which can be deformed into $\bar{Y}_1$ with a unique singularity, and the Taylor expansions of the two polynomials at the singular point begin with a general quadratic term. Then we take X_1 to be the blow-up of $\bar{Y}_1$ at the unique singular point. Thus the passage from X_1 to Y_1 is a Calabi-Yau example.

5. The second Calabi-Yau example $\{X_2, Y_2\}$ is obtained in a similar fashion. We take Y_2 to be a $(3, 3)$ complete intersection in $\mathbb{P}^5$ which can be also deformed into $\bar{Y}_1$ with a unique singularity, and the Taylor expansions of the two polynomials at the singular point begin with a general quadratic term. Then we take X_2 to

be the blow-up of $\bar{Y}_2$ at the unique singular point. The passage from X_2 to Y_2 is another Calabi-Yau example.

The second major result in this work is to show that, in the case of cubic extremal transitions $\{X, Y\}$, there is an interesting connection between the quantum D-module of X and the genus zero FJRW theory of the cubic singularity (represented as *FJRW D-module*). We note that Lee-Lin-Wang [23] also observed that the monodromy non-invariant part of the quantum D-module $\bar{\mathcal{H}}(X)$ should be related to the deformation of the singularity of $\bar{Y}$. Thus we confirm Lee-Lin-Wang's observation in the case of cubic extremal transitions.

In the early days of mirror symmetry, physicists observed that the defining equation of a Calabi- Yau 3-fold in weighted projective space naturally leads to a Landau-Ginzburg (LG) model of the singularity. A Gromov-Witten-type theory for singularities was later worked out by Fan, Jarvis, and Ruan [12, 13, 14], based on Wittens proposal on the LG A-model, and now this theory is referred to as Fan-Jarvis-Ruan-Witten (FJRW) theory. Inspired by physics, a correspondence between Calabi-Yau geometry (considered from the point of view of Gromov-Witten theory) and the Landau-Ginzburg model for singularity (considered from the point of view of FJRW theory) has been established (known as *LG/CY correspondence*) and led to many important consequences (see e.g. [6, 7, 8]). Later an analogous *LG/(Fano/General Type) correpondence* was studied in Acosta's thesis work [1]. The genus zero FJRW invariants are encoded in FJRW I-function, which is a formal hypergeometric series valued in FJRW state space. In Fano case, Acosta introduces a notion of *regularized FJRW I-function*, which determines the ordinary FJRW I-function by Laplace transformation and asymptotic expansion. The D-module attached to the regularized FJRW I-function is defined as the *FJRW D-module* in this work, where more detailed explanations can be found in Chapter IV.

The connection between the quantum D-module of X and the FJRW D-module of the cubic singularity is the following.

Theorem I.3. *Let $\bar{\mathcal{H}}(X)$ be the D-module obtained by analytic continuation considered in the example (1) or (2) in Theorem I.2, then there is another divisor F in the extended Kähler moduli and a submodule $\bar{\mathcal{H}}^F(X) \subseteq \bar{\mathcal{H}}(X)$ which has maximal trivial F-monodromy such that*

$$\bar{\mathcal{H}}^F(X)/\bar{\mathcal{H}}^E(X)|_F \simeq \mathcal{L}(W),$$

where $\mathcal{L}(W)$ is the FJRW D-module of the polynomial $W = x^3 + y^3 + z^3 + u^3$.

Theorem I.3 is stated and proved in Chapter IV. Combining Theorem I.2 and Theorem I.3, we can conclude that in the case of cubic extremal transitions, the genus zero Gromov-Witten theory of X recovers not only the genus zero Gromov-Witten theory of Y but also the genus zero FJRW theory of the cubic singularity. This can be schematically summarized as follows:

$$\text{GW theory } (X) \Longrightarrow \begin{cases} \text{GW theory } (Y) \\ \qquad + \\ \text{FJRW theory (cubic singularity)} \end{cases} \qquad (\text{for } g = 0).$$

1.5 Outline

The thesis is organized as follows: In Chapter II, we will review the basics of Gromov-Witten theory and quantum D-modules. We will establish the necessary notations and terminologies used throughout this work. In Chapter III, we will explain the local model and the Calabi-Yau example of cubic extremal transitions and prove the part (1) and (2) of Theorem I.2. In Chapter IV, we will give a brief survey of FJRW theory and prove Theorem I.3. In Chapter V, we will describe the local model and Calabi-Yau examples of degree 4 Type II extremal transitions. We will prove the part (3), (4) and (5) of Theorem I.2.

CHAPTER II

Background in Gromov-Witten theory

In this chapter, we will begin reviewing the basics of Gromov-Witten theory. General references on this subject are [10, 15, 20].

2.1 Gromov-Witten invariants and quantum D-module

Let P be a smooth projective variety, $H^*(P)$ be its cohomology ring in even degrees, with coefficients in $\mathbb{Q}$ unless specified otherwise. Let $\overline{\mathcal{M}}_{0,n}(P,\beta)$ be the moduli space of genus zero stable maps $f:(C,x_1,\cdots,x_n)\to P$ from a rational nodal curve C with n markings and $f_*[C]=\beta\in NE(P)$, the Mori cone of effective curves. Given $\gamma_1,\cdots,\gamma_n\in H^*(P)$, $a_1,\cdots,a_n\in\mathbb{N}$, the descendent Gromov-Witten invariant is defined as

$$\langle\tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\rangle_\beta := \int_{[\overline{\mathcal{M}}_{0,n}(P,\beta)]^{vir}} \psi_1^{a_1}\mathrm{ev}_1^*\gamma_1\cdots\psi_n^{a_n}\mathrm{ev}_n^*\gamma_n,$$

where

- $\mathrm{ev}_i:\overline{\mathcal{M}}_{0,n}(P,\beta)\to P$ is the i-th evaluation map.
- $\psi_i=c_1(\mathbb{L}_i)$, where $\mathbb{L}_i$ is the line bundle over $\overline{\mathcal{M}}_{0,n}(P,\beta)$ whose fiber over a moduli point $[(C,x_1,\cdots,x_n,f)]$ is the cotangent line $T^*_{x_i}C$ at the i-th marked point.

When $a_1 = \cdots = a_n = 0$, we call this primary Gromov-Witten invariant, and write it as

$$\langle \gamma_1 \cdots \gamma_n \rangle_\beta := \int_{[\overline{\mathcal{M}}_{0,n}(P,\beta)]^{vir}} \mathrm{ev}_1^* \gamma_1 \cdots \mathrm{ev}_n^* \gamma_n.$$

To introduce the notion of quantum D-module of $\mathcal{H}(P)$, we choose a basis $\{T_i\}_{i=0}^m$ for $H^*(P)$ such that $T_0 = \mathbf{1}$ and $\{T_1, \cdots, T_r\}$ is a nef basis for $H^2(P)$. Let $\{T^i\}$ be the dual basis for $\{T_i\}$. Choose a generic cohomology class $T = \sum t_i T_i$ with coordinate $t = (t_i)_{i=0}^m$, then the genus zero Gromov-Witten potential is defined as

$$\Phi(T) := \sum_{n=0}^{\infty} \sum_{\beta \in NE(P)} \frac{Q^\beta}{n!} \langle \underbrace{T, T, \cdots, T}_{n\text{-tupe}} \rangle_\beta.$$

The (big) quantum product is defined as

$$T_i \star_t T_j := \sum_{k=0}^{m} \Phi_{ijk} T^k,$$

where the structure coefficients are $\Phi_{ijk} = \partial_{t_i} \partial_{t_j} \partial_{t_k} \Phi$. This product is associative due to WDVV equation.

In Givental's approach to the mirror theorem [16], the following generating function is introduced, which is usually called the *big J-function*

$$J_{\mathrm{big}}(t, z^{-1}) := 1 + \frac{T}{z} + \sum_{n=0}^{\infty} \sum_{\beta \in NE(P)} \frac{Q^\beta}{n!} \sum_{k=0}^{m} \left\langle \frac{T_k}{z(z - \psi_1)}, \underbrace{T, T, \cdots, T}_{n\text{-tupe}} \right\rangle_\beta T^k.$$

It is usually convenient to consider the small J-function, which is obtained by restricting the big J-function to $t_{r+1} = \cdots = t_m = 0$. Let $q_i = e^{t_i}$ for $i = 1, \cdots, r$. These variables are usually called *small parameters* and may be viewed as local coordinates of the Kähler moduli space. In this setting, the Novikov varible Q may be eliminated by setting $Q \equiv 1$. By divisor equation, the small J-function takes the following form

$$J_{\mathrm{sm}}(t_0, q, z^{-1}) := e^{t_0/z} q^{T/z} \left(1 + \sum_{\beta \neq 0} q^\beta \sum_{k=0}^{m} \left\langle \frac{T_k}{z(z - \psi_1)} \right\rangle_\beta T^k \right), \tag{2.1}$$

where

$$q^{T/z} := \prod_{i=1}^{r} q_i^{T_i/z}, \qquad q_i^{T_i/z} := \exp\left(\frac{T_i}{z}\log q_i\right), \qquad q^{\beta} := q_1^{T_1\cdot\beta}\cdots q_r^{T_r\cdot\beta}.$$

Let $\mathbb{D}_q$ be the ring generated by the log differential operator $z\delta_{q_i} := zq_i\partial_{q_i}$ over $\mathbb{C}[q_i, q_i^{-1}, z]$, where we view z as a formal parameter. Let $J(q, z^{-1}) := J_{\mathrm{sm}}(0, q, z^{-1})$. We make the following definition

Definition II.1. The (small) *quantum D-module* $\mathcal{H}(P)$ is the cylic $\mathbb{D}_q$-module generated by $J(q, z^{-1})$.

Remark II.2. It is well known that the cyclic D-module generated by $J_{\mathrm{big}}(t, z^{-1})$ can be identified with the Dubrovin connection associated to the big quantum product. Our definition above coincides with the Dubrovin connection restricted to the small parameter space $H^2(P)$.

2.2 Quantum Lefschetz theorem and ambient quantum D-module

Having defined quantum D-module for a general smooth projective variety P, now we turn into the case where Z is a smooth complete intersection inside P. Let L_i be convex line bundles over P and $\mathcal{V} = \oplus L_i$, and assume $Z = \sigma^{-1}(0)$ for a section $\sigma \in H^0(\oplus\mathcal{V})$. The quantum Lefschetz theorem gives a way to compute $QH^*(Z)$ from $QH^*(P)$.

In Coates-Givental's work [9], a twisted version of J-function is introduced for the pair $(P, \mathcal{V})$ as follows

$$J^{\mathcal{V}}_{\mathrm{big}}(t, z^{-1}) := 1 + \frac{T}{z} + \sum_{n=0}^{\infty} \frac{Q^d}{n!}(\mathrm{ev}_{n+1})_* \left(\frac{\mathcal{V}'_{0,n+1,d}}{z(z-\psi_{n+1})} \prod_{i=1}^{n} \mathrm{ev}_i^* T \right),$$

where $\mathcal{V}'_{0,n+1,d}$ is the kernel of the map $R^0\pi_*\mathrm{ev}_{n+1}^*\mathcal{V} \to \mathrm{ev}_{n+1}^*\mathcal{V}$ and $\pi : \overline{\mathcal{M}}_{0,n+1}(P,\beta) \to \overline{\mathcal{M}}_{0,n}(P,\beta)$ forgets the last marking. Let $J^P_{\mathrm{big}}(t, z^{-1})$ be the big J-function for P, we write

$$J^P_{\rm big}(t,z^{-1})=\sum_\beta Q^\beta J_\beta(t,z).$$

Then we define the function $I^{\mathcal V}_{\rm big}(t,z)$ as the following

$$I^{\mathcal V}_{\rm big}(t,z,z^{-1}):=\sum_\beta Q^\beta J_\beta(t,z)\prod_i\prod_{k=1}^{L_i\cdot\beta}(c_1(\mathcal L)+kz).$$

When $c_1(Z)\geq 0$, the mirror theorem shows that $J^P_{\rm big}(\tau(t),z^{-1})=I^P_{\rm big}(t,z^{-1})$ for the mirror transformation $t\mapsto\tau(t)$. However, without the restriction $c_1(Z)\geq 0$, the function $I^{\mathcal V}_{\rm big}(t,z,z^{-1})$ may have positive powers of z. In this situation, a fundamental result in Coates-Givental's work [9] is the following.

Theorem II.3. *$J^{\mathcal V}_{\rm big}(\tau,z)$ is recovered from $I^{\mathcal V}_{\rm big}(t,z,z^{-1})$ via the Birkhoff factorization procedure followed by a (generalized) mirror transformation $t\mapsto\tau(t)$.*

Let $J^Z_{\rm big}(t,z^{-1})$ be the big J-function for Z. The full genus zero Gromov-Witten theory of Z is hard to determine due to the existence of primitive cohomology classes, which lie in the kernel of ι_*. However, we have the following relation

$$\mathbf{e}(\mathcal V)J^{\mathcal V}_{\rm big}(t,z^{-1})=\iota_*J^Z_{\rm big}(\iota^*T,z^{-1}).$$

As before, we consider the small version of the big J(resp. I)-functions. By setting $t_{r+1}=\cdots=t_m=0$, $Q\equiv 1$ and $q:=(q_i)=(e^{t_i})$ for $i=1,\cdots,r$, we obtain the (multivalued) small J-function $J^{\mathcal V}_{\rm sm}(t_0,q,z^{-1})$ (resp. small I-function $I^{\mathcal V}_{\rm sm}(t_0,q,z,z^{-1})$). Now we make the following definition for the ambient part quantum D-module of Z.

Definition II.4. The *ambient part quantum D-module* of Z is the cyclic $\mathbb D_q$-module generated by $\mathbf{e}(\mathcal V)J^{\mathcal V}_{\rm sm}(0,q,z^{-1})$, which is still denoted by $\mathcal H(Z)$ when there is no risk of confusion.

Remark II.5. It follows directly from Theorem II.3 that the ambient part quantum D-module of Z may be identified with the cyclic $\mathbb D$-module generated by $\mathbf{e}(\mathcal V)I^{\mathcal V}_{\rm sm}(0,q,z,z^{-1})$.

It is often easy to write down the explicit expression of $\mathbf{e}(\mathcal{V})I^{\mathcal{V}}(0, q, z, z^{-1})$ provided that P is a toric projective variety or certain GIT quotients. The left ideal in $\mathbb{D}_q$ that annihilates $I^{\mathcal{V}}_{\mathrm{sm}}(0, q, z, z^{-1})$ is called *Picard-Fuchs ideal.*

Remark II.6. Suppose $H^*(P)$ is generated by divisors, then by a reconstruction theorem [24], we can recover the genus zero Gromov-Witten potential of P from the small J-function. In the case above where Z is a complete intersection, this reconstruction procedure yields all the genus zero Gromov-Witten invariants in which the insertions are pulled back from the ambient space P. This allows us to use the ambient part quantum D-module $\mathcal{H}(Z)$ to represent the genus zero Gromov-Witten theory of Z.

Remark II.7. One may also adopt the viewpoint of integrable connection to define the ambient part quantum D-module. A closely related definition for toric nef complete intersections is given by Mann-Mignon [29].

2.3 Conifold transition and Li-Ruan's theorem

In this section, we will reformulate Li-Ruan's result on conifold transitions [27, Corollary B.1]. Let P be a smooth Calabi-Yau 3-fold, then the genus zero Gromov-Witten theory of P boils down to the following numbers

$$N^P_\beta := \deg([\overline{\mathcal{M}}_{0,0}(X, \beta)]^{vir}) \in \mathbb{Q} \qquad \forall \beta \in NE(X),$$

as the virtual dimension of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ is zero.

Let X and Y be smooth Calabi-Yau 3-folds such that Y is obtained by contracting finitely many $O_{\mathbb{P}^1}(-1)+O_{\mathbb{P}^1}(-1)$-rational curves on X followed by a smoothing. This surgery is called a *conifold transition.* The following theorem is proved in [27].

Theorem II.8 (Li-Ruan)**.** *The conifold transition from X to Y induces the following morphisms:*

$$\varphi_* : H_2(X) \to H_2(Y), \quad \varphi^* : H^*(Y) \to H^*(X)$$

such that

1. φ_* *is surjective;*
2. $\varphi^* : H^2(Y) \to H^2(X)$ *is dual to* φ_* *and the dual map to* $\varphi_* : H^4(Y) \to H^4(X)$ *gives a right inverse to* φ_**;*
3. *For every* $0 \neq \beta' \in H_2(Y)$*, the set* $\Lambda = \{\beta \in NE(X) : \varphi_*(\beta) = \beta'\}$ *is finite;*
4. *For every* $0 \neq \beta' \in H_2(Y)$*, the following relation holds*

$$\sum_{\beta\in\Lambda} N^X_\beta = N^Y_{\beta'}.$$

We will study the implication of this theorem on quantum D-modules. For a Calabi-Yau 3-fold P, the small J-function takes a simpler form, which is proved in [10, Section 10.3.2].

Lemma II.9. *Setting* $t_0 = 0$*, the* J*-function of* P *defined in Equation* (2.1) *is*

$$J^P(q,z) = q^{T/z}\left(1 + z^{-2}\sum_{\beta\neq 0} q^\beta N^P_\beta[\beta] - 2z^{-3}\sum_{\beta\neq 0} q^\beta N^P_\beta[pt]\right),$$

where $[\cdots]$ *means the Poincare dual.*

This lemma is useful way in comparing the small J-functions of X and Y. The topological change of the cornifold transition has been studied in [21]. It is shown that there is an exact sequence:

$$\oplus_{i=1}^k \mathbb{C}[E_i] \longrightarrow H_2(X) \xrightarrow{\varphi_*} H_2(Y) \longrightarrow 0,$$

where φ_* is the morphism in Theorem II.8, and $[E_i]$ are the exceptional curve classes in the contraction of X. According the multiple cover formula, we have

$$N^X_{nE_i} = \frac{1}{n^3}, \quad i = 1, 2, \cdots, k.$$

Following [21], we choose a basis $b_1, \cdots, b_{r+m}$ of $H_2(X)$ in such a way that $b_1, \cdots, b_{r+m}$ span a cone containing the Mori cone of X, and the last m elements

$b_{r+1}, \cdots, b_{r+m}$ span a cone containing the classes $[E_1], \cdots, [E_k]$, where m is the dimension of the cone spanned by the classes $[E_1], \cdots, [E_k]$. In general we have $m \le k$, since the curve classes $[E_i]$ have a nontrivial linear relation. Once such a basis is chosen, we get a natural basis $\varphi_* b_1, \cdots, \varphi_* b_r$ for $H_2(Y)$. So we may use $q_1, \cdots, q_{r+m}$ as the small parameters for X which are dual to $b_1, \cdots, b_{r+m}$, and $\tilde{q}_1, \cdots, \tilde{q}_r$ for Y dual to $\varphi_* b_1, \cdots, \varphi_* b_r$.

Lemma II.10. *The small J-function of X can be written as a sum*

$$J^X(q, z^{-1}) = J_1^X(q, z^{-1}) + J_2^X(q, z^{-1}), \tag{2.2}$$

such that

$$\lim_{\substack{q_i \to 1 \\ r+1 \le i \le r+m}} L \circ J_1^X(q, z^{-1}) = J^Y(\tilde{q}, z^{-1}), \tag{2.3}$$

where $L : H^*(X) \to H^*(Y)$ is the dual morphism to φ_* given in Theorem II.8, and $\tilde{q}_i \mapsto q_i$ for $i = 1, 2, \cdots, r$.

Proof. We define the following series:

$$J_1^X(q, z^{-1}) := \prod_{i=1}^{b+m} q_i^{[b_i]/z} \left(1 + z^{-2} \sum_{\varphi_*(\beta) \neq 0} q^\beta N_\beta^X [\beta] - 2z^{-3} \sum_{\varphi_*(\beta) \neq 0} q^\beta N_\beta^X [pt]_X \right),$$

$$\begin{aligned} J_2^X(q, z) &= \prod_{i=1}^{b+m} q_i^{[b_i]/z} \left(z^{-2} \sum_{\substack{\varphi_*(\beta)=0 \\ \beta \neq 0}} q^\beta N_\beta^X [\beta] - 2z^{-3} \sum_{\substack{\varphi_*(\beta)=0 \\ \beta \neq 0}} q^\beta N_\beta^P [pt]_X \right) \\ &= \prod_{i=1}^{b+m} q_i^{[b_i]/z} \left(z^{-2} \sum_{i=1}^{k} \sum_{n=1}^{\infty} q^{nE_i} N_{nE_i}^X [nE_i] - 2z^{-3} \sum_{i=1}^{k} \sum_{n=1}^{\infty} q^{nE_i} N_{nE_i}^X [pt]_X \right) \\ &= \prod_{i=1}^{b+m} q_i^{[b_i]/z} \left(z^{-2} \sum_{i=1}^{k} \sum_{n=1}^{\infty} \frac{q^{nE_i}}{n^2} [E_i] - 2z^{-3} \sum_{i=1}^{k} \sum_{n=1}^{\infty} \frac{q^{nE_i}}{n^3} [pt]_X \right). \end{aligned}$$

Obviously, we have

$$J^X(q, z^{-1}) = J_1^X(q, z^{-1}) + J_2^X(q, z^{-1}).$$

The small J-function of Y is the following

$$J_1^X(\tilde{q}, z^{-1}) := \prod_{i=1}^{b} \tilde{q}_i^{[b_i]/z} \left(1 + z^{-2} \sum_{\beta' \neq 0} \tilde{q}^{\beta'} N_{\beta'}^{Y} [\beta'] - 2z^{-3} \sum_{\beta' \neq 0} \tilde{q}^{\beta'} N_{\beta'}^{Y} [pt]_Y \right).$$

Since L is dual to φ^*, by Theorem II.8(1-2) we have that

$$L[\beta] = [\varphi_*(\beta)], \quad L[pt]_X = [pt]_Y.$$

Comibing this with Theorem II.8 gives the desired result. □

Let E be the locus $\{q_{r+1} = \cdots = q_{r+m} = 1\}$ in $\mathbb{C}^{r+m}$. We call E the *transition locus*. Consider a small punctured neighborhood U of the point $\mathbf{q} = (q_1 = \cdots = q_r = 0, q_{r+1} = \cdots = q_{r+m} = 1)$ with each $q_i = 0$(or 1) deleted. Fixing $q_1, \cdots, q_r$, we choose a path to analytically continue $J^X(q, z^{-1})$ to a point in U, so that we obtain a function $\bar{J}^X(q, z^{-1})$. As in the decomposition (2.2), we have

$$\bar{J}^X(q, z^{-1}) = \bar{J}_1^X(q, z^{-1}) + \bar{J}_2^X(q, z^{-1}).$$

Lemma II.11. *$\bar{J}_1^X(q, z^{-1})$ has trivial E-monodromy, i.e. $\bar{J}_1^X(q, z^{-1})$ remains unchanged under analytical continuation of $\bar{J}_1^X(q, z^{-1})$ along any loop circulating around E (with fixed $q_1, \cdots, q_r$).*

Proof. This follows from the fact that the set $\Lambda = \{\beta \in NE(X) : \varphi_*(\beta) = \beta'\}$ is finite (in Theorem II.8). For fixed $q_1, \cdots, q_r$, both terms $\sum_{\varphi_*(\beta) \neq 0} q^{\beta} N_{\beta}^{P} [\beta]$ and $\sum_{\varphi_*(\beta) \neq 0} q^{\beta} N_{\beta}^{P} [pt]$ are polynomials in $q_{r+1}, \cdots, q_{r+m}$, thus admitting trivial E-monodromy after performing the analytic continuation.

□

Remark II.12. We note that $\bar{J}_2^X(q, z^{-1})$ has nontrivial monodromy. This follows from the fact that the polylogarithm function

$$\mathrm{Li}_s(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^s}, \qquad s = 2, 3,$$

is analytic in $|q| < 1$ and branched at $q = 1$ with the monodromy operator M_1 around $q = 1$ given by

$$M_1(\mathrm{Li}_s(q)) = \mathrm{Li}_s(q) + \frac{2\pi i}{\Gamma(s)} \log^{s-1}(q).$$

This has been computed in [3].

Now we are in a position to reformulate Li-Ruan's result (Theorem II.8) as follows

Theorem II.13. *For conifold transition of Calabi-Yau 3-folds X to Y, one may analytically continue the quantum D-module $\mathcal{H}(X)$ to obtain a D-module $\bar{\mathcal{H}}(X)$ over U, let E be the transition locus $q_{r+1} = \cdots = q_{r+m} = 1$, then there is a submodule $\bar{\mathcal{H}}^E(X) \subseteq \bar{\mathcal{H}}(X)$ which has maximal trivial E-monodromy such that*

$$\bar{\mathcal{H}}^E(X)|_E \simeq \mathcal{H}(Y),$$

Here $\bar{\mathcal{H}}^E(X)|_E$ is the restriction of $\bar{\mathcal{H}}^E(X)$ to the transition locus E.

Proof. By definition, $\mathcal{H}(X)$ is identified with the $\mathbb{D}_q$-module generated by $J^X(q, z^{-1})$. After performing the analytic continuation, we see that $\bar{J}_1^X(q, z^{-1})$ has trivial monodromy around E. Let $\bar{\mathcal{H}}^E(X)$ be the sub-local system attached to $\bar{J}_1^X(q, z^{-1})$. After performing the analytic continuation, by Lemma II.11 and Remark II.12, we see that $\bar{\mathcal{H}}^E(X)$ is the submodule of $\bar{\mathcal{H}}(X)$ with maximum trivial E-monodromy. Restriction this to E amounts to taking the limit $q_i \to 1$ $(i = r+1, \cdots, r+m)$. It follows from Lemma II.10 that $\bar{\mathcal{H}}^E(X)|_E \simeq \mathcal{H}(Y)$. □

CHAPTER III

Cubic Extremal Transitions

In this chapter, we will study the example of cubic extremal transitions. Let V be a Calabi-Yau 3-fold that contains a smooth cubic surface S. Assume that the rational curves on S generate an extremal ray of the Mori cone of V, then we can birationally contract S to a point to obtain a singular Calabi-Yau 3-fold $\bar{V}$ with local singularity given by

$$x^3 + y^3 + z^3 + u^3 = 0.$$

To smooth out the singularity, we consider the following deformation of the above equation

$$x^3 + y^3 + z^3 + u^3 = t \quad (t \neq 0).$$

This is a local surgery. According to Gross' work [18], this surgery can be done globally. In other words, there is a Calabi-Yau 3-fold $\widetilde{V}$ obtained by the smoothing. The process of going from V to $\tilde{V}$ is called a *cubic extremal transition.*

3.1 The local model

If we only consider a small neighborhood where this surgery takes place, we obtain the *local model.* Viewing S as a smooth cubic surface in $\mathbb{P}^3$, we close up a neighborhood of S and then define

$$X := \mathbb{P}(N_{S/V} \oplus O) = \mathbb{P}(K_S \oplus O).$$

The second equality follows from $K_V \simeq \mathcal{O}_V$ and the adjunction formula.

Smoothing out the cubic singularity gives a cubic 3-fold in $\mathbb{P}^4$, i.e.

$$Y = \{x^3 + y^3 + z^3 + u^3 = tv^3\} \subseteq \mathbb{P}^4.$$

We note first that both X and Y can be naturally embedded into smooth projective toric varieties. Indeed, for X, by adjunction formula

$$K_S = K_{\mathbb{P}^3} \otimes O_{\mathbb{P}^3}(S)|_S = O_{\mathbb{P}^3}(-1)|_S,$$

so we have the following diagram

$$\begin{array}{ccc} X & \xhookrightarrow{i_X} & \mathbb{P}(O_{\mathbb{P}^3}(-1) \oplus O_{\mathbb{P}^3}) \\ \downarrow & & \downarrow \pi \\ S & \hookrightarrow & \mathbb{P}^3 \end{array}$$

and X is the vanishing locus of a section of $\pi^* O_{\mathbb{P}^3}(3)$. Moreover, Y is a hypersurface defined by a section of $O_{\mathbb{P}^4}(3)$

$$\begin{array}{ccc} & & O_{\mathbb{P}^4}(3) \\ & & \downarrow \\ Y & \xhookrightarrow{i_Y} & \mathbb{P}^4 \end{array}$$

Let $\mathcal{H}(X)$ (resp. $\mathcal{H}(Y)$) be the ambient part quantum D-module of X (resp. Y). To give an explicit description of the quantum D-modules, we introduce the following notations:

Notation III.1. $\mathcal{O}_{\widetilde{X}}(1)$ is the anti-tautological line bundle over $\widetilde{X} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1))$.

- $h := c_1(\pi^* \mathcal{O}_{\mathbb{P}^3}(1))$, $\xi := c_1(\mathcal{O}_{\widetilde{X}}(1))$.

- $p := c_1(\mathcal{O}_{\mathbb{P}^4}(1))$.

By toric geometry, we know that $H^*(\widetilde{X})$ is generated by h, ξ, so we use q_1 and q_2 as small parameters for X_0, which correspond to h and ξ, respectively. On the other hand, the cohomology of $\widetilde{Y}_0 := \mathbb{P}^4$ is generated by p, and $\dim H^2(Y) = \dim H^2(\widetilde{Y}) = 1$, so we use y as the small parameter for Y, which corresponds to p. We also introduce the following differential operators:

$$z\delta_{q_1} := zq_1\frac{\partial}{\partial q_1}, \qquad z\delta_{q_2} := zq_2\frac{\partial}{\partial q_2}, \qquad z\delta_y = zy\frac{\partial}{\partial y}.$$

The small I-function for X is the following

$$I^X(q_1,q_2) := (3h)q_1^{h/z}q_2^{\xi/z}\sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1}q_2^{d_2}\frac{\prod_{m=-\infty}^{0}(\xi-h+mz)\prod_{m=1}^{3d_1}(3h+mz)}{\prod_{m=1}^{d_1}(h+mz)^4\prod_{m=1}^{d_2}(\xi+mz)\prod_{m=-\infty}^{d_2-d_1}(\xi-h+mz)},$$

subject to the relation $h^4 = h\xi - \xi^2 = 0$.

On the other hand, the small I-function for Y is the following

$$I^Y(y) = (3p)y^{p/z}\sum_{d\in\mathbb{N}}\frac{\prod_{m=1}^{3d}(3p+mz)}{\prod_{m=1}^{d}(p+mz)^5},$$

subject to the relation $p^5 = 0$.

According to Remark II.5, the ambient quantum D-module $\mathcal{H}(X)$ for X may be identified with the cyclic $\mathbb{D}_q$-module generated by $I^X(q_1,q_2)$, and $\mathcal{H}(Y)$ may be identified with the cyclic $\mathbb{D}_y$-module generated by $I^Y(y)$. Our goal is to study the relation between $\mathcal{H}(X)$ and $\mathcal{H}(Y)$.

Lemma III.2. *The Picard-Fuchs ideal associated to $I^X(q_1,q_2)$ is generated by $\triangle_1$ and $\triangle_2$, where*

$$\triangle_1 = (z\delta_{q_1})^3 - 3q_1\left(3z\delta_{q_1}+z\right)\left(3z\delta_{q_1}+2z\right)\left(z\delta_{q_2}-z\delta_{q_1}\right),$$

$$\triangle_2 = (z\delta_{q_2})\,(z\delta_{q_2} - z\delta_{q_1}) - q_2.$$

In other words, the components of I^X *gives a full basis of solutions to* $\triangle_1 I = \triangle_2 I = 0$ *at any point near the origin in* $(\mathbb{C}^*)^2$*.*

Proof. By a ratio test, we find that $I^X(q_1, q_2)$ is holomorphic when $(q_1, q_2) \in (\mathbb{C}^*)^2$ is sufficiently close to the origin. It is straightforward to check that $\triangle_1$, $\triangle_2$ annihilates $I^X(q_1, q_2)$: if we write $I^X(q_1, q_2)$ in the following way

$$I^X(q_1, q_2) = (3h) q_1^{h/z} q_2^{\xi/z} \sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1} q_2^{d_2} A_{d_1,d_2},$$

then the differential equation $\triangle_1 I = 0$ (resp. $\triangle_2 I = 0$) follows from the recursion relation between A_{d_1,d_2} and A_{d_1-1,d_2} (resp. A_{d_1,d_2-1}), and the cohomology relation $h^4 = \xi(\xi - h) = 0$ is equivalent to the fact that $A_{d_1,d_2} = 0$ for $d_1 < 0$ or $d_2 < 0$. Moreover, we see that the 6 components of $I^X(q_1, q_2)$ are linearly independent due to their initial terms, but the differential equation system has at most 6-dimensional solution space by a holonomic rank computation. So we obtain a full basis to the differential equation system, and the lemma is proved. □

We note that I^X involves two small parameters q_1 and q_2, whereas I^Y involves only a single small parameter y. To compare them, we introduce the following auxiliary function in two variables x and y.

$$\bar{I}^Y(x, y) := (3p) y^{p/z} \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} x^i y^j \frac{\prod_{m=-\infty}^{0} (p + mz)^4 \prod_{m=-\infty}^{3j-3i} (3p + mz)}{\prod_{m=-\infty}^{j-i} (p + mz)^4 \prod_{m=1}^{j} (p + mz) \prod_{m=1}^{i} (mz) \prod_{m=-\infty}^{0} (3p + mz)},$$

subject to the relation $p^5 = 0$. This may be viewed as an extension of $I^Y(y)$ in the following sense

Lemma III.3. $\bar{I}^Y(x, y)$ *has trivial monodromy around* $x = 0$ *and we have*

$$\lim_{x\to 0} \bar{I}^Y(x, y) = I^Y(y).$$

Proof. Since the initial term $y^{p/z}$ does not involve x, the monodromy around $x = 0$ is trivial. The second part is straightforward to check. □

In a similar fashion, we obtain the partial differential equation satisfied by $\bar{I^Y}(x,y)$ by studying the recursion relation in their coefficients of $x^i y^j$. However, the natural partial differential equation system attached to $\bar{I^Y}(x,y)$ has 6-dimensional solution space, but the components of $\bar{I^Y}(x,y)$ give only 4 linearly independent solutions. We introduce another two hypergeometric series as follows.

$$I_5(x,y) = x^{\frac{1}{3}} \sum_{i \geqslant j \geqslant 0} x^i y^j \frac{(-1)^{i-j}\Gamma(\frac{1}{3}+i-j)^4}{\Gamma(\frac{4}{3}+i)\Gamma(3i-3j+1)\Gamma(1+j)z^{2j}},$$

$$I_6(x,y) = x^{\frac{2}{3}} \sum_{i \geqslant j \geqslant 0} x^i y^j \frac{(-1)^{i-j}\Gamma(\frac{2}{3}+i-j)^4}{\Gamma(\frac{5}{3}+i)\Gamma(3i-3j+2)\Gamma(1+j)z^{2j}}.$$

Lemma III.4. *The function $\bar{I}^Y(x,y)$ is analytic at any point near the origin in $(\mathbb{C}^*)^2$, and can be annihilated by the following differential operators:*

$$\triangle_1' := x\left(z\delta_y - z\delta_x\right)^3 - 3\left(3(z\delta_y - z\delta_x) + z\right)\left(3(z\delta_y - z\delta_x) + 2z\right)\left(z\delta_x\right),$$

$$\triangle_2' := \left(z\delta_y\right)\left(z\delta_x\right) - xy.$$

The components of $\bar{I}^Y(q_1,q_2)$, together with I_5 and I_6, give a full basis of solutions to the differential equation system $\triangle_1' I = \triangle_2' I = 0$ at any point near the origin in $(\mathbb{C}^*)^2$.

Proof. The proof is parallel to that of Lemma 3.2, so we omit it. □

We notice that the differential equation systems $\{\triangle_1 I = \triangle_2 I = 0\}$ and $\{\triangle_1' I = \triangle_2' I = 0\}$ both have 6 dimensional solution spaces. Under an appropriate change of variables, these two systems are indeed equivalent. Our key lemma is the following:

Lemma III.5. *The change of variables $x \mapsto q_1^{-1}$ and $y \mapsto q_1 q_2$ induces an equivalence between the differential equation systems $\{\triangle_1 I = \triangle_2 I = 0\}$ and $\{\triangle_1' I = \triangle_2' I = 0\}$.*

Proof. The change of variables $x \mapsto q_1^{-1}$ and $t \mapsto q_1 q_2$ yields the following relation of the differential operators:

$$z\delta_{q_1} = z\delta_y - z\delta_x, \qquad z\delta_{q_2} = z\delta_y.$$

It follows directly that $\triangle_2 I = 0$ is converted to $\triangle_2' I = 0$ and vice versa. For $\triangle_1 I = 0$ and $\triangle_1' I = 0$, one just has to notice that since $x \neq 0$, $\triangle_1' I = 0$ is equivalent to

$$\left[(\delta_y - \delta_x)^3 - 3x^{-1}\left(3(\delta_y - \delta_x) + z\right)\left(3(\delta_y - \delta_x) + 2z\right)(\delta_x)\right] I = 0,$$

which is converted to $\triangle_1 I = 0$ term by term under the relation among the differential operators. □

From Lemma III.5, we see that the function $I^X(q_1, q_2)$ and $\bar{I}^Y(x, y)$ satisfies the same system of differential equations under an appropriate change of variables. Since they are both holomorphic on certain domains, it is expected that $\bar{I}^Y(x, y)$ may be obtained by analytic continuation of $I^X(q_1, q_2)$ followed by a linear transformation $L : H^*(\widetilde{X}) \to H^*(\widetilde{Y})$. This subsection is devoted to working out this analytic continuation using Mellin-Barnes method. For a similar computation, we refer the reader to [6].

We will frequently use the following identity.

Lemma III.6. *For any $a \in \mathbb{Z}$, we have*

$$\frac{\prod_{m=-\infty}^{a}(u + mz)}{\prod_{m=-\infty}^{0}(u + mz)} = \frac{z^a \Gamma\left(1 + \frac{u}{z} + a\right)}{\Gamma\left(1 + \frac{u}{z}\right)}.$$

By this lemma, we can rewrite $I^X(q_1,q_2)$ and $\bar{I}^Y(x,y)$ in the following way.

$$I^X(q_1,q_2) = 3q_1^{\frac{h}{z}}q_2^{\frac{\xi}{z}} \cdot \frac{h\Gamma(1+\frac{\xi-h}{z})\Gamma(1+\frac{h}{z})^4\Gamma(1+\frac{\xi}{z})}{\Gamma(1+\frac{3h}{z})} \cdot$$

(3.1)
$$\sum_{\substack{d_1\geqslant 0\\ d_2\geqslant 0}} q_1^{d_1}q_2^{d_2}\frac{\Gamma(1+\frac{3h}{z}+3d_1)}{\Gamma(1+\frac{h}{z}+d_1)^4\Gamma(1+\frac{\xi}{z}+d_2)\Gamma(1+\frac{\xi-h}{z}+(d_2-d_1))z^{2d_2}},$$

subject to the relation $h^4 = h\xi - \xi^2 = 0$.

(3.2)
$$\bar{I}^Y(x,y) = \frac{(3p)y^{\frac{p}{z}}\Gamma(1+\frac{p}{z})^5}{\Gamma(1+\frac{3p}{z})}\sum_{\substack{i\geqslant 0\\ j\geqslant 0}} x^i y^j \frac{\Gamma(1+\frac{3p}{z}+3j-3i)}{\Gamma(1+\frac{p}{z}+j-i)^4\Gamma(1+\frac{p}{z}+j)\Gamma(1+i)z^{2j}},$$

subject to relation $p^5 = 0$.

Theorem III.7. *One may analytically continue the function $I^X(q_1,q_2)$ to obtain a holomorphic function $\bar{I}^X(x,y)$ near the origin in $(\mathbb{C}^*)^2$, where the change of variables is given by $x \mapsto q_1^{-1}$ and $y \mapsto q_1q_2$. There exists a degree-preserving linear transformation $L : H^*(\tilde{X}) \to H^*(\tilde{Y})$ such that $I^Y(y)$ is recovered by*

$$I^Y(y) = \lim_{x\to 0} L\circ \bar{I}^X(x,y).$$

Proof. By Lemma III.3, we need to show there exists a degree-preserving linear transformation $L : H^*(\tilde{X}) \to H^*(\tilde{Y})$ such that $\bar{I}^Y(x,y) = L\circ \bar{I}^X(q_1,q_2)$ under the change of variables $x \mapsto q_1^{-1}$ and $y \mapsto q_1q_2$. To begin with, for every $d_2 \in \mathbb{N}$, we define the following function

$$\begin{aligned}\varphi_{d_2}(s) :&= (-1)^{d_2}\frac{\sin(\frac{\xi-h}{z})\pi}{\sin(\frac{3h}{z})\pi}\cdot\frac{\sin(-\frac{3h}{z}-3s)\pi}{\sin(s-d_2-\frac{\xi-h}{z})\pi}\\ &= (-1)^{d_2}\frac{\sin(\frac{\xi-h}{z})\pi}{\sin(\frac{3h}{z})\pi}\cdot\frac{\pi/\sin(s-d_2-\frac{\xi-h}{z})\pi}{\pi/\sin(-\frac{3h}{z}-3s)\pi}\\ &= (-1)^{d_2}\frac{\sin(\frac{\xi-h}{z})\pi}{\sin(\frac{3h}{z})\pi}\cdot\frac{\Gamma(1+\frac{\xi-h}{z}+d_2-s)\Gamma(s-d_2-\frac{\xi-h}{z})}{\Gamma(1+\frac{3h}{z}+3s)\Gamma(-\frac{3h}{z}-3s)}.\end{aligned}$$

It is clear that $\varphi_{d_2}(s)$ is periodic with period 1, and takes value 1 at any integer $s \in \mathbb{Z}$. Using (3.1), the function $I^X(q_1, q_2)$ may be further written as:

$$\begin{aligned}
I^X(q_1,q_2) &= 3q_1^{h/z}q_2^{\xi/z}\frac{h\Gamma(1+\frac{\xi-h}{z})\Gamma(1+\frac{h}{z})^4\Gamma(1+\frac{\xi}{z})}{\Gamma(1+\frac{3h}{z})}\cdot \\
&\quad \sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1}q_2^{d_2}\frac{\Gamma(1+\frac{3h}{z}+3d_1)\varphi_{d_2}(d_1)}{\Gamma(1+\frac{h}{z}+d_1)^4\Gamma(1+\frac{\xi}{z}+d_2)\Gamma(1+\frac{\xi-h}{z}+(d_2-d_1))z^{2d_2}}, \\
&= 3q_1^{h/z}q_2^{\xi/z}\frac{h\sin(\pi\frac{\xi-h}{z})\Gamma(1+\frac{\xi-h}{z})\Gamma(1+\frac{h}{z})^4\Gamma(1+\frac{\xi}{z})}{\sin(\pi\frac{3h}{z})\Gamma(1+\frac{3h}{z})}\cdot \\
&\quad \sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1}q_2^{d_2}\frac{(-1)^{d_2}\Gamma(d_1-d_2-\frac{\xi-h}{z})}{\Gamma(1+\frac{h}{z}+d_1)^4\Gamma(1+\frac{\xi}{z}+d_2)\Gamma(-\frac{3h}{z}-3d_1)z^{2d_2}}.
\end{aligned} \tag{3.3}$$

Then define a sequence of functions $g_{d_2}(s, q_1)$ for each $d_2 \in \mathbb{N}$ as follows:

$$g_{d_2}(s,q_1) := \frac{\Gamma(s-d_2-\frac{\xi-h}{z})q_1^s}{(e^{2\pi\sqrt{-1}s}-1)\Gamma(1+\frac{h}{z}+s)^4\Gamma(-\frac{3h}{z}-3s)}.$$

It is clear that $g_{d_2}(s, q_1)$ is a meromorphic function in s with simple poles at every integer, as well as $s = d_2 + \frac{\xi-h}{z} - l$ for $l \in \mathbb{N}$.

We claim that the function $I^X(q_1, q_2)$ may be represented as the following integral:

$$\begin{aligned}
I^X(q_1,q_2) &= 3q_1^{h/z}q_2^{\xi/z}\frac{h\sin(\pi\frac{\xi-h}{z})\Gamma(1+\frac{\xi-h}{z})\Gamma(1+\frac{h}{z})^4\Gamma(1+\frac{\xi}{z})}{\sin(\pi\frac{3h}{z})\Gamma(1+\frac{3h}{z})} \\
&\quad \cdot\sum_{d_2\in\mathbb{N}}\frac{(-q_2)^{d_2}}{\Gamma(1+\frac{\xi}{z}+d_2)z^{2d_2}}\int_{C^+} g_{d_2}(s,q_1)ds,
\end{aligned} \tag{3.4}$$

where for a fixed d_2, the contour C^+ goes along the imaginary axis and closes to the right in such a way that only the simple poles at nonnegative integers are enclosed inside C^+.

Indeed, according to the residue theorem, for each $d_2 \in \mathbb{N}$ we have

$$\begin{aligned}
\int_{C^+} g_{d_2}(s,q_1)ds &= 2\pi\sqrt{-1}\sum_{d_1\in\mathbb{N}}\operatorname*{Res}_{s=d_1} g_{d_2}(s,q_1) \\
&= \sum_{d_1\in\mathbb{N}}\frac{\Gamma(d_1-d_2-\frac{\xi-h}{z})q_1^{d_1}}{\Gamma(1+\frac{h}{z}+d_1)^4\Gamma(-\frac{3h}{z}-3d_1)}.
\end{aligned} \tag{3.5}$$

Substuiting (3.5) into (3.4) for each $d_2 \in \mathbb{N}$, we obtain (3.3), hence the the claim follows.

To perform the analytic continuation, we notice that for $|q_1|$ sufficiently large, we may close up the imaginary axis to the left in such a way that all the remaining poles are enclosed in this contour, denoted by C^-. Using the residue theorem again, we obtain

$$\begin{aligned}
\int_{C^-} g_{d_2}(s,q_1)ds &= 2\pi\sqrt{-1}\sum_{l\in\mathbb{N}}\left(\operatorname*{Res}_{s=-l-1} g_{d_2}(s,q_1) + \operatorname*{Res}_{s=d_2+\frac{\xi-h}{z}-l} g_{d_2}(s,q_1)\right)\\
&= 2\pi\sqrt{-1}\sum_{l\in\mathbb{N}}\frac{\Gamma(-l-1-d_2-\frac{\xi-h}{z})q_1^{-l-1}}{\Gamma(\frac{h}{z}-l)^4\Gamma(-\frac{3h}{z}+3l+3)}+\\
&\quad 2\pi\sqrt{-1}\sum_{l\in\mathbb{N}}\frac{(-1)^l q_1^{d_2-l+\frac{\xi-h}{z}}}{(e^{2\pi\sqrt{-1}\frac{\xi-h}{z}}-1)\Gamma(1+l)\Gamma(1+\frac{\xi}{z}+d_2-l)^4\Gamma(3l-3d_2-\frac{3\xi}{z})}. \qquad (3.6)
\end{aligned}$$

Replacing C^+ by C^- in (3.4), and substuiting (3.6) into (3.4) for each $d_2 \in \mathbb{N}$, we obtain the following analytic continuation of $I^X(q_1,q_2)$:

$$\bar{I}^X(q_1,q_2) = 3(q_1q_2)^{\xi/z}\sum_{(l,d_2)\in\mathbb{N}^2} q_1^{d_2-l}(q_2)^{d_2}A_{l,d_2} + h^4 f(q_1,q_2,\log q_1,\log q_2,h,\xi), \qquad (3.7)$$

for some $f(q_1,q_2,\log q_1,\log q_2,h,\xi) \in \mathbb{C}[[q_1,q_2,\log q_1,\log q_2,h,\xi]]$ and

$$\begin{aligned}
A_{l,d_2} &= \frac{(-1)^{l+d_2}(2\pi\sqrt{-1})h\sin\pi(\frac{\xi-h}{z})}{\sin\pi(\frac{3h}{z})\Gamma(1+\frac{3h}{z})(e^{2\pi\sqrt{-1}\frac{\xi-h}{z}}-1)}\\
&\quad\cdot\frac{\Gamma(1+\frac{\xi-h}{z})\Gamma(1+\frac{h}{z})^4\Gamma(1+\frac{\xi}{z})}{\Gamma(1+\frac{\xi}{z}+d_2)\Gamma(1+l)\Gamma(1+\frac{\xi}{z}+d_2-l)^4\Gamma(3l-3d_2-\frac{3\xi}{z})z^{2d_2}}\\
&= \frac{(2\sqrt{-1})h\sin(\frac{3\xi}{z}\pi)\sin(\frac{\xi-h}{z}\pi)\Gamma(1+\frac{\xi-h}{z})\Gamma(1+\frac{h}{z})^4\Gamma(1+\frac{\xi}{z})}{\sin(\frac{3h}{z}\pi)(e^{2\pi\sqrt{-1}\frac{\xi-h}{z}}-1)\Gamma(1+\frac{3h}{z})}\\
&\quad\cdot\frac{\Gamma(1+3\frac{\xi}{z}+3d_2-3l)}{\Gamma(1+\frac{\xi}{z}+d_2)\Gamma(1+l)\Gamma(1+\frac{\xi}{z}+d_2-l)^4z^{2d_2}}.
\end{aligned}$$

Replacing q_1, q_2 by x, y using the change of variables, and repeatedly applying the relation $h^4 = h\xi - \xi^2 = 0$, (3.7) eventually reduces to

(3.8)

$$\bar{I}^X(x,y) = \frac{(3\xi)y^{\frac{\xi}{z}}\Gamma(1+\frac{\xi}{z})^5}{\Gamma(1+\frac{3\xi}{z})}\sum_{(i,j)\in\mathbb{N}^2} x^iy^j\frac{\Gamma(1+3\frac{\xi}{z}+3j-3i)}{\Gamma(1+\frac{\xi}{z}+j)\Gamma(1+i)\Gamma(1+\frac{\xi}{z}+j-i)^4z^{2j}}.$$

Comparing (3.8) with (3.2), it is clear that the following linear transformation does the job:

$$L : \xi \mapsto p, \qquad \xi^2 \mapsto p^2, \qquad \xi^3 \mapsto p^3, \qquad \xi^4 \mapsto p^4,$$

hence we are done. □

Now we are in a position to state and prove Theorem I.2(1) for the local model.

Theorem III.8 (=Theorem I.2(1))**.** *One may perform analytic continuation of $\mathcal{H}(X)$ to obtain a D-module $\bar{\mathcal{H}}(X)$, then there is a divisor E in the extended Kähler moduli space and a submodule $\bar{\mathcal{H}}^E(X) \subseteq \bar{\mathcal{H}}(X)$ with maximum trivial E-monodromy such that*

$$\bar{\mathcal{H}}^E(X)|_E \simeq \mathcal{H}(Y),$$

where $\bar{\mathcal{H}}^E(X)|_E$ is the restriction to E.

Proof. We may identify the ambient part quantum D-module $\mathcal{H}(X)$ with the local system attached to $I^X(x, y)$ (Remark II.5). By Theorem III.7, we see that $I^X(x, y)$ can be analytically continued to $\bar{I}^Y(x, y)$ up to a linear transformation. Let $\bar{\mathcal{H}}(X)$ be the D-module that correpsonds to the system $\triangle_1' I = \triangle_2' I = 0$.

Next, we claim the components of $\bar{I}^Y(x, y)$ give a sub-solution space which has maximum trivial x-monodromy. Indeed, their x-monodromy is trivial by Lemma 3.4, and maximal because the remaining two solutions I_5 and I_6 have non-trivial x-monodromy due to their initial terms $x^{1/3}$ and $x^{2/3}$. Hence our claim follows.

Consider the sub-local system $\bar{\mathcal{H}}^E(X)$ attached to the components of $\bar{I}^Y(x, y)$, since their x-monodromy is trivial, we may consider the natural restriction of $\bar{I}^Y(x, y)$ to $x = 0$. By Lemma III.4, we recover the I-function $I^Y(y)$ for Y through this process, so the local system $\mathcal{H}(Y)$ is obtained by restricting the local system of $\bar{\mathcal{H}}^E(X)$ to

$x = 0$, which is viewed as a divisor E in the extended Kähler moduli. Therefore the theorem is proved. $\square$

3.2 A Calabi-Yau Example

In this section, we will study one particular example of cubic extremal transitions for Calabi-Yau 3-folds (which appears in [11]), and we will show that Conjecture I.1 holds in this case.

Let Y_0 be a quintic hypersurface in $\mathbb{P}^4$ defined by the following homogenous equation:

$$x_0^2(tx_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3) + x_0 f(x_1, x_2, x_3, x_4) + g(x_1, x_2, x_3, x_4) = 0, \quad (t \neq 0)$$

where f and g are generic homogenous polynomials of degree 4 and 5, respectively. Y_0 is generically a smooth Calabi-Yau 3-folds, which can be deformed into the following singular quintic $\overline{Y}_0$ with singularity only at $[1:0:0:0:0] \in \mathbb{P}^4$:

$$\overline{Y}_0 : \quad x_0^2(x_1^3 + x_2^3 + x_3^3 + x_4^3) + x_0 f(x_1, x_2, x_3, x_4) + g(x_1, x_2, x_3, x_4) = 0.$$

Let X_0 be the Calabi-Yau 3-folds obtained by blowing up $\overline{Y}_0$ at the triple point $[1:0:0:0:0] \in \mathbb{P}^4$. Clearly X_0 is a smooth Calabi-Yau 3-fold and the exceptional divisor over $[1:0:0:0:0] \in \mathbb{P}^4$ is a cubic surface. The passage from X_0 to Y_0 is a global example of cubic extremal transitions.

On the same line of reasoing, X_0 is a hypersurface inside $\widetilde{X_0} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1))$. We will still adopt the same notations in III.1, where

- $\pi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)) \to \mathbb{P}^3$ is the projection map, and $\mathcal{O}_{\widetilde{X_0}}(1)$ is the anti-tautological line bundle over $\widetilde{X_0} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1))$.
- $h := c_1(\pi^* \mathcal{O}_{\mathbb{P}^3}(1))$, $\xi := c_1(\mathcal{O}_{\widetilde{X_0}}(1))$. Let q_1, q_2 be the small parameters for X_0, correpsonding to h and ξ.

- $p := c_1(\mathcal{O}_{\mathbb{P}^4}(1))$. Let y be the small parameter for Y_0, corresponding to p.

Then X_0 represents the divisor class $3h + 2\xi$ in $\widetilde{X_0}$, whereas Y_0 represents the divisor class $5p$ in $\widetilde{Y_0} = \mathbb{P}^4$. The I-functions of X_0 and Y_0 are the following:

$$I^{X_0}(q_1, q_2) := (3h+2\xi)q_1^{h/z}q_2^{\xi/z} \sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1}q_2^{d_2} \frac{\prod_{m=-\infty}^{0}(\xi - h + mz)\prod_{m=1}^{3d_1+2d_2}(3h+2\xi+mz)}{\prod_{m=1}^{d_1}(h+mz)^4 \prod_{m=1}^{d_2}(\xi+mz)\prod_{m=-\infty}^{d_2-d_1}(\xi-h+mz)},$$

subject to the relation $h^4 = h\xi - \xi^2 = 0$.

$$I^{Y_0}(y) = (5p)y^{p/z}\sum_{d\in\mathbb{N}} \frac{\prod_{m=1}^{5d}(5p+mz)}{\prod_{m=1}^{d}(p+mz)^5},$$

subject to the relation $p^5 = 0$.

Following Section 3.1, we may also construct a hypergeometric series $I^{Y_0}(x, y)$ in x and y such that

$$\lim_{x\to 0} \bar{I}^{Y_0}(x, y) = I^{Y_0}(y),$$

where $\bar{I}^{Y_0}(x, y)$ is given by:

$$\bar{I}^{Y_0}(x, y) := (5p)e^{(p\log y)/z}\sum_{\substack{i\geqslant 0\\ j\geqslant 0}} x^i y^j \frac{\prod_{m=-\infty}^{0}(p+mz)^4 \prod_{m=-\infty}^{5j-3i}(5p+mz)}{\prod_{m=-\infty}^{j-i}(p+mz)^4 \prod_{m=1}^{j}(p+mz)\prod_{m=1}^{i}(mz)\prod_{m=-\infty}^{0}(5p+mz)},$$

subject to the relation $p^5 = 0$.

With the explicit formulas of $I^{X_0}(q_1, q_2)$, we have the following lemmas. (The differential operators below are also computed in [29].)

Lemma III.9. *The components of $I^{X_0}(q_1, q_2)$ give a full basis of solutions to the system $\triangle_1 I = \triangle_2 I = P_0 I = 0$ near the origin in $\mathbb{C}^*$, where*

$$\triangle_1 = (z\delta_{q_1})^4 - q_1(3z\delta_{q_1} + 2z\delta_{q_2} + z)(3z\delta_{q_1} + 2z\delta_{q_2} + 2z)(3z\delta_{q_1} + 2z\delta_{q_2} + 3z)(z\delta_{q_2} - z\delta_{q_1}),$$

$$\triangle_2 = (z\delta_{q_2})(z\delta_{q_2} - z\delta_{q_1}) - q_2(3z\delta_{q_1} + 2z\delta_{q_2} + z)(3z\delta_{q_1} + 2z\delta_{q_2} + 2z),$$

$$P_0 = -5(z\delta_{q_1})^3 + 2(z\delta_{q_1})^2(z\delta_{q_2}) + 15q_1(z\delta_{q_2} - z\delta_{q_1})(3z\delta_{q_1} + 2z\delta_{q_2} + z)(3z\delta_{q_1} + 2z\delta_{q_2} + 2z)$$
$$- 4q_2(z\delta_{q_2})^2(3z\delta_{q_1} + 2z\delta_{q_2} + z).$$

Proof. By a computation in [29], the differential system $\triangle_1 I = \triangle_2 I = P_0 I = 0$ has a six-dimensional solution space. It is straightforward to check that I^{X_0} are annihilated by $\triangle_1, \triangle_2$ and P_0. Due to the cohomological relation $h^4 = h\xi - \xi^2 = 0$, the components of I^{X_0} would give 6 dimensional linearly-independent solutions. Hence they must form a basis. □

In a similar fashion, we introduce the following two hypergeometric series:

$$I_5(x,y) = x^{\frac{1}{3}} \sum_{i \geqslant j \geqslant 0} x^i y^j \frac{(-1)^{i-j}\Gamma(\frac{1}{3}+i-j)^4}{\Gamma(\frac{4}{3}+i)\Gamma(3i-5j+1)\Gamma(1+j)},$$

$$I_6(x,y) = x^{\frac{2}{3}} \sum_{i \geqslant j \geqslant 0} x^i y^j \frac{(-1)^{i-j}\Gamma(\frac{2}{3}+i-j)^4}{\Gamma(\frac{5}{3}+i)\Gamma(3i-5j+2)\Gamma(1+j)}.$$

Lemma III.10. *The components of $\bar{I}^{Y_0}(q_1,q_2)$, together with I_5, I_6, give a full basis to the system $\triangle_1' I = \triangle_2' I = P_0' I = 0$, where*

$$\triangle_1' = x(z\delta_y - z\delta_x)^4 - (5z\delta_y - 3z\delta_x + 3z)(5z\delta_y - 3z\delta_x + 2z)(5z\delta_y - 3z\delta_x + z + z)(z\delta_x),$$

$$\triangle_2' = (z\delta_x)(z\delta_y) - xy(5\delta_y - 3\delta_x + z)(5\delta_y - 3\delta_x + 2z),$$

$$P_0' = -5(z\delta_y - z\delta_x)^3 + 2(z\delta_y - z\delta_x)^2(z\delta_y) + 15x^{-1}(z\delta_x)(5z\delta_y - 3z\delta_x + z)(5z\delta_y - 3z\delta_x + 2z)$$
$$- 4xy(z\delta_y - z\delta_x)^2(5z\delta_y - 3z\delta_x + z).$$

Proof. This is parallel to Lemma III.9, one can check this directly. □

The differential equation systems $\{\triangle_1 I = \triangle_2 I = P_0 I = 0\}$ and $\{\triangle_1' I = \triangle_2' I = P_0' I = 0\}$ both have 6-dimensional solution spaces. We can relate them using an appropriate change of variables as follows

Lemma III.11. *The change of variables $x \mapsto q_1^{-1}$ and $y \mapsto q_1 q_2$ induces an equivalence between the differential equation systems $\{\triangle_1 I = \triangle_2 I = P_0 I = 0\}$ and $\{\triangle'_1 I = \triangle'_2 I = P'_0 I = 0\}$.*

Proof. Notice that under the change of variables $x \mapsto q_1^{-1}$ and $y \mapsto q_1 q_2$, the differential operators have the following relations:

$$\delta_{q_1} = \delta_y - \delta_x, \quad \delta_{q_2} = \delta_y.$$

It is straightfoward to check that this change of variables converts $\triangle_1 I = \triangle_2 I = P_0 I = 0$ directly into $\triangle'_1 I = \triangle'_2 I = P_0 I = 0$, provided that $x \neq 0$ Hence we are done. □

Since $I^{X_0}(q_1, q_2)$ and $\bar{I}^{Y_0}(x, y)$ satisfy the the same differential equation up to a change of variables, we would expect that $I^{X_0}(q_1, q_2)$ can be analytic continued to $\bar{I}^{Y_0}(x, y)$ up to a linear transformation. Indeed, we may still work out this analytic continuation explicitly using Mellin-Barnes method, which is completely parallel to Section 3.1.

First, we rewrite $I^{X_0}(q_1, q_2)$ and $\bar{I}^{Y_0}(x, y)$ using Lemma III.6 as follows.

$$(3.9) \quad \begin{aligned} I^{X_0}(q_1, q_2) = (3h + 2\xi) q_1^{\frac{h}{z}} q_2^{\frac{\xi}{z}} \cdot \frac{\Gamma(1 + \frac{\xi - h}{z})\Gamma(1 + \frac{h}{z})^4 \Gamma(1 + \frac{\xi}{z})}{\Gamma(1 + \frac{3h+2\xi}{z})} \cdot \\ \sum_{\substack{d_1 \geqslant 0 \\ d_2 \geqslant 0}} q_1^{d_1} q_2^{d_2} \frac{\Gamma(1 + \frac{3h+2\xi}{z} + 3d_1 + 2d_2)}{\Gamma(1 + \frac{h}{z} + d_1)^4 \Gamma(1 + \frac{\xi}{z} + d_2)\Gamma(1 + \frac{\xi - h}{z} + (d_2 - d_1))}, \end{aligned}$$

subject to the relation $h^4 = h\xi - \xi^2 = 0$.

$$(3.10) \quad \bar{I}^{Y_0}(x, y) = \frac{(5p) y^{\frac{p}{z}} \Gamma(1 + \frac{p}{z})^5}{\Gamma(1 + \frac{5p}{z})} \sum_{\substack{i \geqslant 0 \\ j \geqslant 0}} x^i y^j \frac{\Gamma(1 + \frac{5p}{z} + 5j - 3i)}{\Gamma(1 + \frac{p}{z} + j - i)^4 \Gamma(1 + \frac{p}{z} + j)\Gamma(1 + i)},$$

subject to relation $p^5 = 0$.

For every $d_2 \in \mathbb{N}$, we define the following function

$$\begin{aligned}
\varphi_{d_2}(s) :&= (-1)^{d_2} \frac{\sin(\frac{\xi-h}{z})\pi}{\sin(\frac{3h+2\xi}{z})\pi} \cdot \frac{\sin(-\frac{3h+2\xi}{z} - 3s - 2d_2)\pi}{\sin(s - d_2 - \frac{\xi-h}{z})\pi} \\
&= (-1)^{d_2} \frac{\sin(\frac{\xi-h}{z})\pi}{\sin(\frac{3h+2\xi}{z})\pi} \cdot \frac{\pi/\sin(s - d_2 - \frac{\xi-h}{z})\pi}{\pi/\sin(-\frac{3h+2\xi}{z} - 3s - 2d_2)\pi} \\
&= (-1)^{d_2} \frac{\sin(\frac{\xi-h}{z})\pi}{\sin(\frac{3h+2\xi}{z})\pi} \cdot \frac{\Gamma(1 + \frac{\xi-h}{z} + d_2 - s)\Gamma(s - d_2 - \frac{\xi-h}{z})}{\Gamma(1 + \frac{3h}{z} + 3s + 2d_2)\Gamma(-\frac{3h+2\xi}{z} - 3s - 2d_2)}.
\end{aligned}$$

By definition, $\varphi_{d_2}(s)$ is periodic with period 1, and takes value 1 at every integer $s \in \mathbb{Z}$. It follows that $I^{X_0}(x, y)$ can be further written as

$$\begin{aligned}
I^{X_0}(q_1, q_2) &= (3h + 2\xi) q_1^{h/z} q_2^{\xi/z} \frac{\Gamma(1 + \frac{\xi-h}{z})\Gamma(1 + \frac{h}{z})^4 \Gamma(1 + \frac{\xi}{z})}{\Gamma(1 + \frac{3h+2\xi}{z})} \cdot \\
&\quad \sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1} q_2^{d_2} \frac{\Gamma(1 + \frac{3h+2\xi}{z} + 3d_1 + 2d_2)\varphi_{d_2}(d_1)}{\Gamma(1 + \frac{h}{z} + d_1)^4 \Gamma(1 + \frac{\xi}{z} + d_2)\Gamma(1 + \frac{\xi-h}{z} + (d_2 - d_1))}, \\
&= (3h + 2\xi) q_1^{h/z} q_2^{\xi/z} \frac{\sin(\pi\frac{\xi-h}{z})\Gamma(1 + \frac{\xi-h}{z})\Gamma(1 + \frac{h}{z})^4 \Gamma(1 + \frac{\xi}{z})}{\sin(\pi\frac{3h+2\xi}{z})\Gamma(1 + \frac{3h+2\xi}{z})} \cdot
\end{aligned}$$

$$\sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1} q_2^{d_2} \frac{(-1)^{d_2}\Gamma(d_1 - d_2 - \frac{\xi-h}{z})}{\Gamma(1 + \frac{h}{z} + d_1)^4 \Gamma(1 + \frac{\xi}{z} + d_2)\Gamma(-\frac{3h+2\xi}{z} - 3d_1 - 2d_2)}. \tag{3.11}$$

Similarly, we define a sequence of functions $g_{d_2}(s, q_1)$ for each $d_2 \in \mathbb{N}$ as follows.

$$g_{d_2}(s, q_1) := \frac{\Gamma(s - d_2 - \frac{\xi-h}{z}) q_1^s}{(e^{2\pi\sqrt{-1}s} - 1)\Gamma(1 + \frac{h}{z} + s)^4 \Gamma(-\frac{3h+2\xi}{z} - 3s - 2d_2)}.$$

We see that $g_{d_2}(s, q_1)$ is a meromorphic function in s with simple poles at every integer, as well as $s = d_2 + \frac{\xi-h}{z} - l$ for $l \in \mathbb{N}$. We claim that $I^{X_0}(x, y)$ admits the following integral representation.

$$I^{X_0}(q_1, q_2) = (3h + 2\xi) q_1^{h/z} q_2^{\xi/z} \frac{\sin(\pi\frac{\xi-h}{z})\Gamma(1 + \frac{\xi-h}{z})\Gamma(1 + \frac{h}{z})^4 \Gamma(1 + \frac{\xi}{z})}{\sin(\pi\frac{3h+2\xi}{z})\Gamma(1 + \frac{3h+2\xi}{z})}$$

$$\cdot \sum_{d_2\in\mathbb{N}} \frac{(-q_2)^{d_2}}{\Gamma(1 + \frac{\xi}{z} + d_2)} \int_{C^+} g_{d_2}(s, q_1) ds, \tag{3.12}$$

where for a fixed d_2, the contour C^+ goes along the imaginary axis and closes to the right in such a way that only the simple poles at nonnegative integers are enclosed inside C^+.

Again, this follows from the residue computation. For each $d_2 \in \mathbb{N}$ we have

$$\int_{C^+} g_{d_2}(s,q_1)ds = 2\pi\sqrt{-1}\sum_{d_1\in\mathbb{N}} \operatorname*{Res}_{s=d_1} g_{d_2}(s,q_1)$$

$$= \sum_{d_1\in\mathbb{N}} \frac{\Gamma(d_1-d_2-\frac{\xi-h}{z})q_1^{d_1}}{\Gamma(1+\frac{h}{z}+d_1)^4\Gamma(-\frac{3h+2\xi}{z}-3d_1-2d_2)}. \tag{3.13}$$

Substuiting (3.13) into (3.12) gives (3.11). Hence our claim is proved.

To perform the analytic continuation, we notice that for $|q_1|$ sufficiently large, we may close up the imaginary axis to the left in such a way that all the remaining poles are enclosed in this contour, denoted by C^-. Using the residue theorem again, we obtain

$$\int_{C^-} g_{d_2}(s,q_1)ds = 2\pi\sqrt{-1}\sum_{l\in\mathbb{N}}\left(\operatorname*{Res}_{s=-l-1} g_{d_2}(s,q_1) + \operatorname*{Res}_{s=d_2+\frac{\xi-h}{z}-l} g_{d_2}(s,q_1)\right)$$

$$= 2\pi\sqrt{-1}\sum_{l\in\mathbb{N}} \frac{\Gamma(-l-1-d_2-\frac{\xi-h}{z})q_1^{-l-1}}{\Gamma(\frac{h}{z}-l)^4\Gamma(-\frac{3h+2\xi}{z}+3l-2d_2+3)}+$$

$$2\pi\sqrt{-1}\sum_{l\in\mathbb{N}} \frac{(-1)^l q_1^{d_2-l+\frac{\xi-h}{z}}}{(e^{2\pi\sqrt{-1}\frac{\xi-h}{z}}-1)\Gamma(1+l)\Gamma(1+\frac{\xi}{z}+d_2-l)^4\Gamma(3l-5d_2-\frac{5\xi}{z})}. \tag{3.14}$$

Replacing C^+ by C^- in (3.12), and substuiting (3.14) into (3.12) for each $d_2 \in \mathbb{N}$, we obtain the following analytic continuation of $I^{X_0}(q_1,q_2)$.

(3.15)

$$\bar{I}^{X_0}(q_1,q_2) = (3h+2\xi)(q_1q_2)^{\xi/z}\sum_{(l,d_2)\in\mathbb{N}^2} q_1^{-l}(q_1q_2)^{d_2}A_{l,d_2} + h^4 f(q_1,q_2,\log q_1,\log q_2,h,\xi),$$

for some $f(q_1,q_2,\log q_1,\log q_2,h,\xi) \in \mathbb{C}[[q_1,q_2,\log q_1,\log q_2,h,\xi]]$ and

$$A_{l,d_2} = \frac{(2\sqrt{-1})h\sin(\frac{5\xi}{z}\pi)\sin(\frac{\xi-h}{z}\pi)\Gamma(1+\frac{\xi-h}{z})\Gamma(1+\frac{h}{z})^4\Gamma(1+\frac{\xi}{z})}{\sin(\frac{3h+2\xi}{z}\pi)(e^{2\pi\sqrt{-1}\frac{\xi-h}{z}}-1)\Gamma(1+\frac{3h+2\xi}{z})}$$

$$\cdot\frac{\Gamma(1+5\frac{\xi}{z}+5d_2-3l)}{\Gamma(1+\frac{\xi}{z}+d_2)\Gamma(1+l)\Gamma(1+\frac{\xi}{z}+d_2-l)^4}.$$

Replacing q_1, q_2 by x, y using the change of variables, and repeatedly applying the relation $h^4 = h\xi - \xi^2 = 0$, (3.15) eventually reduces to

$$\bar{I}^{X_0}(x,y) = \frac{(5\xi)y^{\frac{\xi}{z}}\Gamma(1+\frac{\xi}{z})^5}{\Gamma(1+\frac{3\xi}{z})}\sum_{(i,j)\in\mathbb{N}^2} x^i y^j \frac{\Gamma(1+5\frac{\xi}{z}+5j-3i)}{\Gamma(1+\frac{\xi}{z}+j)\Gamma(1+i)\Gamma(1+\frac{\xi}{z}+j-i)^4}.$$

Thus using the linear transformation $L : H^*(\widetilde{X}_0) \to H^*(\widetilde{Y}_0)$ given by

$$L : \xi \mapsto p, \qquad \xi^2 \mapsto p^2, \qquad \xi^3 \mapsto p^3, \qquad \xi^4 \mapsto p^4,$$

we have

$$\bar{I}^{Y_0}(x, y) = L \circ \bar{I}^{X_0}(x, y),$$

and

$$I^{Y_0}(y) = \lim_{x \to 0} \bar{I}^{Y_0}(x, y) = \lim_{x \to 0} L \circ \bar{I}^{X_0}(x, y).$$

Following a similar argument as in Theorem III.8, now we may conclude that Theorem I.2(2) holds, namely

Theorem III.12 (=Theorem I.2(2))**.** *For cubic extremal transition between two Calabi-Yau 3-folds X_0 to Y_0 given above, one may perform analytic continuation of $\mathcal{H}(X_0)$ to obtain a D-module $\bar{\mathcal{H}}(X_0)$, then there is a divisor E in the extended Kähler moduli space and a submodule $\bar{\mathcal{H}}^E(X_0) \subseteq \bar{\mathcal{H}}(X_0)$ with maximum trivial E-monodromy such that*

$$\bar{\mathcal{H}}^E(X_0)|_E \simeq \mathcal{H}(Y_0).$$

CHAPTER IV

Connection to FJRW theory

In this chapter, we will begin by reviewing Fan-Jarvis-Ruan-Witten (FJRW) theory and LG/Fano correspondence. We will introduce our key notion – FJRW D-module, which encodes the genus zero FJRW theory. Following the notations in Chapter III, we will prove Theorem I.3, which gives a relationship between the quantum D-module of X and the FJRW D-module of the cubic singularity.

4.1 FJRW theory and LG/Fano correspondence

Fan-Jarvis-Ruan-Witten(FJRW) theory is a quantum singuarity theory, worked out by Fan, Jarvis and Ruan [12, 13, 14], based on Witten's propsal on the Landau-Ginzburg A-model. FJRW theory turns out to be a kind of cohomological field theory, and in many ways very similar to Gromov-Witten theory. Here we will give a very short review of FJRW theory, and we refer the reader to [22] for a detailed survery of this theory.

One may associate FJRW theory to a pair (W, G), where W is a non-degenerate quasihomogenous polynomial of degree d in $x_1, \cdots, x_n$, where x_i has weight w_i. Here W has an isolated singularity at the origin, and G is a subgroup of $(\mathbb{C}^*)^n$ that fixes

W and contains $\langle J_W \rangle$, where

$$J_W := \left(\exp \frac{2\pi\sqrt{-1}w_1}{d}, \cdots, \exp \frac{2\pi\sqrt{-1}w_n}{d} \right).$$

Given W, a canonical choice of G would simply be $\langle J_W \rangle$.

The FJRW theory of (W, G) has the following ingredients:

- A state space $\mathcal{H}$ which is defined as the relative Chen-Ruan cohomology:

$$\mathcal{H} := \bigoplus_{g \in G} \mathcal{H}_g = \bigoplus_{g \in G} H^*_{CR}([\mathrm{Fix}(g)/G], W_g^{+\infty}),$$

  where $[\mathrm{Fix}(g)/G]$ is the quotient stack obtained by the G-action on the fixed locus $\mathrm{Fix}(g)$ under $g \in G$, and $W_g^\infty := \mathfrak{Re}^{-1}(M, +\infty) \cap \mathrm{Fix}(g)$ for $M >> 0$. The state space $\mathcal{H}$ can be decomposed into the direction sum of narrow sectors ($\mathrm{Fix}(g) = \mathbf{0}$) and broad sectors ($\mathrm{Fix}(g) \neq \mathbf{0}$).

- A moduli stack $\mathcal{W}_{g,n}^{W,G}$ which parametrizes (the equivalence classes of) genus g, n-pointed $(x_1, \cdots, x_n)$ stable orbicurve C equipped with obifold line bundles $\mathcal{L}_i$ such that for every monoimals W_j involved in W we have

$$W_j(L_1, \cdots, L_n) \simeq \omega_{C,\log} := \omega_C(x_1 + \cdots + x_n).$$

- A virtual cycle $[\mathcal{W}_{g,n}^{W,G}]^{vir}$ on $\mathcal{W}_{g,n}^{W,G}$ and FJRW invariants defined by integrating the narrow insertions and ψ classes against the virtual cycle.

Analogous to Gromov-Witten theory, one can define the J-functions and the I-function for FJRW theory of (W, G). The J-function is a generating series of genus zero FJRW invariants, while the I-function arises as period integrals at the Gepner point. *FJRW mirror theorem* basically says that the J-function of FJRW theory can be identified with the I-function of FJRW theory after an explicit change of variables. Since I-functions are usually explicit hypergeometric series, one can extract genus zero FJRW invariants from the identification $J = I$ in FJRW mirror theorem. We note that FRJW mirror theorem has been proved in many different settings (see e.g. [6, 19]).

The celebrated LG/CY correspondence was initially proposed by physicists, who observed that the defining equation of a Calabi-Yau 3-fold as a hypersurface in weighted projective space will naturally lead to a Landau-Ginzburg model of the singularity. This correpsondence has been formulated mathematically under the framework of Gromov-Witten theory and FJRW theory (see e.g. [6, 7]). More precisely, for the FJRW theory of (W, G), when $\sum w_i = d$, we call (W, G) satisfies *Calabi-Yau* condition, in which case one can define a Calabi-Yau hypersurface $[X_W/\widetilde{G}]$ in weighted projective space $\mathbb{P}(w_1, \cdots, w_n)$, where X_W is defined by $\{W = 0\}$ and $\widetilde{G} = G/\langle J_W \rangle$. The LG/CY correspondence predicts that the FJRW theory of (W, G) should be equivalent in all genera to the Gromov-Witten theory of $[X_W/\widetilde{G}]$ up to certain analytic continuation and a symplectic transformation bewteen their state spaces.

While the LG/CY correspondence has been verified in many different settings, Acosta [1] generized this correspondence to the case where W does not satisfy the Calabi-Yau condition. We will give a brief account of Acosta's work in the Fano case, which is known as LG/Fano correspondence.

When $\sum w_i < d$, (W, G) would corresponds to a Fano hypersurface $[X_W/\widetilde{G}]$, and then the FJRW I-function of (W, G) would have zero radius of convergence, which makes it only a formal power series. Acosta introduced the notion of *regularized FJRW I-function* $I^{reg}_{FJRW}(\tau)$, which is obtained by term-by-term modification of the original FJRW I-function $I_{FJRW}(t)$. The regularized I-function has the property that the radius of convergence is positive. Furthermore, if we define

$$\mathbb{I}_{FJRW}(u) := u\int_0^\infty e^{-u\tau} I_{FJRW}^{reg}(\tau)d\tau,$$

then by Watson's lemma, $\mathbb{I}_{FJRW}(u)$ recovers the original FJRW I-function $I_{FJRW}(q)$ via asymptotic expansion. Let $H_{CR}^{\mathrm{amb}}([X_W/\widetilde{G}])$, $\mathcal{H}_{(W,G)}^{\mathrm{nar}}$ be the ambient part cohomology of $[X_W/\widetilde{G}]$ and the narrow part of the FJRW state space of (W,G). In [1], Acosta shows the following LG/Fano correspondence:

Theorem IV.1. *There is a linear transformation* $L : H_{CR}^{\mathrm{amb}}([X_W/\widetilde{G}]) \to \mathcal{H}_{(W,G)}^{\mathrm{nar}}$ *such that*

$$L \cdot I_{GW}(q) = \mathbb{I}_{FJRW}(u)$$

after an explicit change of variable $u \mapsto u(q)$*. Furthermore,* $\mathbb{I}_{FJRW}(u)$ *recovers the original FJRW* I*-function* $I_{FJRW}(t)$ *via asymptotic expansion.*

The key ingredient in this theorem is the regularized FJRW I-function $I_{FJRW}^{reg}(\tau)$, which may be viewed as encoding the genus zero FJRW theory of (W,G). We may associate a D-module to this function, as stated in the following definition.

Definition IV.2. The *FJRW* D*-module* of $(W,\langle J_W\rangle)$ is the cyclic D-module generated by the components of the regularized FJRW I-function $I_{FJRW}^{reg}(\tau)$, denoted by $\mathcal{L}(W)$.

4.2 Proof of Theorem I.3

Let $W = x^3+y^3+z^3+w^3$, which is precisely the local equation for the singularity in cubic extremal transition. We consider the FJRW I-function associated to the pair (W,G), where $G = \langle J_W\rangle$ is generated by

$$J_W := \left(\exp\left(\frac{2\sqrt{-1}\pi}{3}\right), \exp\left(\frac{2\sqrt{-1}\pi}{3}\right), \exp\left(\frac{2\sqrt{-1}\pi}{3}\right), \exp\left(\frac{2\sqrt{-1}\pi}{3}\right)\right).$$

The (small) FJRW I-function of (W,G) is a formal power series given by:

$$I_{FJRW}(t) := -\sum_{l=0}^{\infty} t^{3l+1}\frac{(-1)^l\Gamma(l+\frac{1}{3})^4}{(3l)!\Gamma(\frac{1}{3})^4}\phi_0 + \sum_{l=0}^{\infty} t^{3l+2}\frac{(-1)^l\Gamma(l+\frac{2}{3})^4}{(3l)!\Gamma(\frac{2}{3})^4}\phi_1, \tag{4.1}$$

where ϕ_0 and ϕ_1 are generators of the narrow sector of the state space of (W,G).

The regularized FJRW I-function is defined in [1] as follows:

$$I^{reg}_{FJRW}(\tau) := \sum_{l=0}^{\infty} \frac{\tau^{l+\frac{1}{3}}(-1)^{3l+1}\Gamma(l+\frac{1}{3})^4}{(3l)!\Gamma(l+\frac{4}{3})\Gamma(\frac{1}{3})^4}\phi_0 + \sum_{l=0}^{\infty} \frac{\tau^{l+\frac{2}{3}}(-1)^{3l+2}\Gamma(l+\frac{2}{3})^4}{(3l+1)!\Gamma(l+\frac{5}{3})\Gamma(\frac{2}{3})^4}\phi_1,$$

Remark IV.3. The components of the regularized FJRW I-function of (W,G) are both analytic near $\tau = 0$. It is shown in [1] that this $I^{reg}_{FJRW}(\tau)$ may be used to recover the ordinary FJRW I-function $I_{FJRW}(t, z = 1)$, through the method of asymptotic expansion. So $I^{reg}_{FJRW}(\tau)$ can be viewed as encoding the genus zero data of the (narrow part) FJRW theory of (W,G).

In Chapter III Section 1 (or Section 2), we defined the hypergeometric series $I_5(x,y)$ and $I_6(x,y)$ as the "extra" solutions coming from the analytic continuation of the ambient part quantum D-module of X. Interestingly, these functions are directly related to the regularized FJRW theory of (W,G) in the following way:

Proposition IV.4. *The function $I_5(x,y)$ (resp. $I_6(x,y)$) is analytic in y near the origin, and also it has trivial monodromy around $y = 0$. It recovers, up to scalar multiple, the coefficient of ϕ_0 (resp. ϕ_1) in $I^{reg}_{FJRW}(\tau)$ by setting $x = \tau$, $z = 1$ and $y \to 0$.*

Proof. It is analytic by a ratio test, and monodromy around $y = 0$ is trivial because it is an ordinary power series in y for every fixed value of x. As for the last part, it is straightforward to check. □

As a direct corollary, we restated Theorem I.3 and prove it below:

Theorem IV.5 (=Theorem I.3)**.** *Let $\bar{\mathcal{H}}(X)$ be the D-module obtained by analytic continuation considered in Theorem III.8 (or Theorem III.12), then there is another divisor F in the extended Kähler moduli and a submodule $\bar{\mathcal{H}}^F(X) \subseteq \bar{\mathcal{H}}(X)$ which has maximal trivial F-monodromy such that*

$$\bar{\mathcal{H}}^F(X)/\bar{\mathcal{H}}^E(X)|_F \simeq \mathcal{L}(W),$$

where $\mathcal{L}(W)$ represents the FJRW D-module of the pair (W, G), in which $W = x^3 + y^3 + z^3 + u^3$ and $G = \langle J_W \rangle$.

Proof. Using the notations from Chapter III Section 1 (or Section 2), we let I_1, I_2, I_3, I_4 be the components of $\bar{I}^Y(x, y)$, where I_1 represents the constant term in $\bar{I}^Y(x, y)$. We notice that I_1 is analytic in an open neighborhood of the origin, and I_5 and I_6 have trivial monodromy around $y = 0$. Thus to single out the components I_5, I_6, we should take the maximal components that have trivial y-monodromy, namely I_1, I_5, I_6, and then modulo the maximal components with trivial $x-$mondromy, namely I_1, I_2, I_3, I_4. In terms of D-modules, we just need to take the restriction $\bar{\mathcal{H}}^F(X)/\bar{\mathcal{H}}^E(X)|_F$, in which the divisor F corresponds to $y = 0$ in the extended Kähler moduli. Hence the theorem is proved. □

CHAPTER V

Type II deg-4 extremal transitions

Let us first introduce the setup of the Type II extremal transition in degree 4: We have a pair of Calabi-Yau 3-folds (X, Y) related in the following diagram

$$\begin{array}{ccc} X & \cdots\cdots & \\ \downarrow{\scriptstyle \pi} & & \searrow \\ \overline{Y} & \rightsquigarrow & Y \end{array}$$

in which $\pi : X \to \overline{Y}$ is a birational contraction of a divisor $E \hookrightarrow X$ to a point $p \in \overline{Y}$, and going from $\overline{Y}$ to Y is given by smoothing out the singularity at p. We further require that E be a del Pezzo surface of degree 4 in $\mathbb{P}^4$, and the curve classes on E generate an extremal ray in the Mori cone of X. In this case, the singularity aournd p is a complete intersection of two equations with quadratic leading terms. We call the process of going form X to Y *a Type II extremal transition in degree 4* (degree-4 transition for short). We are going to consider the following local model and global examples.

5.1 The local model

In this section, we begin to study the change of quantum D-modules associted to the local model of degree-4 transition. Let E be the degree-4 del Pezzo surface inside a Calabi-Yau 3-fold X, which gets contracted to a point under π. Consider a tubular neighborhood around E, which is identified with its normal bundle $N_{E/X}$. Since the ambient variety is a Calabi-Yau 3-fold, we also have $N_{E/X} \simeq K_E$. Under the birational morphism $\pi : X \to \overline{Y}$, the divisor is contracted to a point with singularity given by a complete intersection of two equations with quadratic leading terms. Smoothing out the singularity locally yields a (2,2) complete intersection in $\mathbb{P}^5$. To study the Gromov-Witten theory of this local picutre, we usually compactify $N_{E/X} \simeq K_E$ and take $X' := \mathbb{P}(K_E \oplus \mathcal{O})$, while $Y' :=$ $(2,2)$ complete intersection in $\mathbb{P}^5$. The transition from X' to Y' is the *local model* of the Type II extremal transition in degree 4. Let $i : E \hookrightarrow \mathbb{P}^4$ be the embedding. By adjuction formula, we have

$$K_E = i^*(K_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(2) \otimes \mathcal{O}_{\mathbb{P}^4}(2)) = i^*\mathcal{O}_{\mathbb{P}^4}(-1).$$

Thus X' is embedded into $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})$ as a complete intersection. Let $\pi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}) \to \mathbb{P}^4$ be the natural projection, then X' can be viewed as the zero locus of a section of $\pi^*(\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(2))$. On the other hand, Y' is the zero locus of a section of $\mathcal{O}_{\mathbb{P}^5}(2) \oplus \mathcal{O}_{\mathbb{P}^5}(2)$. We will adopt the following notations in this section.

- $h := c_1(\pi^*\mathcal{O}_{\mathbb{P}^4}(1))$, corresponding to small parameter q_1
- Let $\mathcal{O}_{\mathbb{P}}(1)$ be the anti-tautological bundle over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})$, and $\xi := c_1(\mathcal{O}_{\mathbb{P}}(1))$, corresponding to small parameter q_2.
- $p := c_1(\mathcal{O}_{\mathbb{P}^5}(1))$, corresponding to small parameter y.

According to the combinatorical data of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})$ and $\mathbb{P}^5$, it is straightforward to write down the twisted I-functions for X' and Y' as follows.

$$I^{X'}(q_1, q_2) := (2h)(2h)q_1^{h/z}q_2^{\xi/z} \sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1}q_2^{d_2} \frac{\prod_{m=-\infty}^{0}(\xi - h + mz)\prod_{m=1}^{2d_1}(2h+mz)^2}{\prod_{m=1}^{d_1}(h+mz)^5\prod_{m=1}^{d_2}(\xi+mz)\prod_{m=-\infty}^{d_2-d_1}(\xi-h+mz)},$$

subject to the relation $h^5 = \xi(\xi - h) = 0$,

$$I^{Y'}(y) := (2p)(2p)y^{p/z} \sum_{j \in \mathbb{N}} \frac{\prod_{m=1}^{2j} (2p + mz)^2}{\prod_{m=1}^{j} (p + mz)^6},$$

subject to the relation $p^6 = 0$.

To compare these two functions, it is helpful to introduce the following auxcillary hypergeometric series:

$$\bar{I}^{Y'}(x, y) := (2p)(2p)y^{p/z} \sum_{(i,j) \in \mathbb{N}^2} x^i y^j \frac{\prod_{m=-\infty}^{0} (p + mz)^5 \prod_{m=-\infty}^{2j-2i} (2p + mz)^2}{\prod_{m=-\infty}^{j-i} (p + mz)^5 \prod_{m=1}^{j} (p + mz) \prod_{m=1}^{i} (mz) \prod_{m=-\infty}^{0} (2p + mz)^2}.$$

We note that $\bar{I}^{Y'}(x, y)$ involves two variables x, y. It's easy to check that $\bar{I}^{Y'}$ is a holomorphic function on a small domain minus the origin. We also observe that $\bar{I}^{Y'}$ has trivial monodromy around $x = 0$, thus it makes sense to take the limit $x \to 0$, we obtain

$$\lim_{x \to 0} \bar{I}^{Y'}(x, y) = I^{Y'}(y),$$

which precisely recovers the twisted I-function for Y'.

Lemma V.1. *The components of $I^{X'}(q_1, q_2)$ comprise a basis of solutions to the differential equation system $\{\triangle_1 I = \triangle_2 I = 0\}$ at any point around the origin in $(\mathbb{C}^*)^2$, where*

$$\triangle_1 := (z\delta_{q_1})^3 - 4q_1(2z\delta_{q_1} + z)^2,$$

$$\triangle_2 := z\delta_{q_2}(z\delta_{q_2} - z\delta_{q_1}) - q_2.$$

Proof. First we write

$$I^{X'}(q_1, q_2) = (2h)(2h)q_1^{h/z} q_2^{\xi/z} \sum_{(d_1, d_2) \in \mathbb{N}^2} q_1^{d_1} q_2^{d_2} A_{d_1, d_2}.$$

These two differential operators are obtained precisely by the recursion reltaions between A_{d_1,d_2} and A_{d_1+1,d_2} (or A_{d_1,d_2+1}), and the cohomology relation $h^5 = \xi(\xi - h) = 0$ amounts to the fact that $A_{d_1,d_2} = 0$ unless $(d_1, d_2) \in \mathbb{N}^2$. The components of $I^{X'}(q_1, q_2)$ give rise to 6 linearly-independent solutions to the differential equation system. On the other hand, this differential equation system can have at most 6-dimensional solution space due to a holonomic rank computation. Hence the lemma follows. $\square$

Insipred by the work of Lee-Lin-Wang[26], we apply the following change of variable to the above differential equation system

$$q_1 \mapsto x^{-1}, \quad q_2 \mapsto xy.$$

Then we have the relation

$$\delta_{q_1} = \delta_y - \delta_x, \quad \delta_{q_2} = \delta_y.$$

Let $\triangle_1'$, $\triangle_2'$ denote the differential operators obtained by applying the above change of variable, then we have the following lemma

Lemma V.2. *The components of* $\bar{I}^{Y'}(x, y)$ *comprise 4 linearly independent solutions to the differential equation system* $\{\triangle_1' I = \triangle_2' I = 0\}$ *at any point around the origin in* $(\mathbb{C}^*)^2$, *where*

$$\triangle_1' := x(z\delta_y - z\delta_x)^3 - 4(2(z\delta_y - z\delta_x) + z)^2],$$

$$\triangle_2' := (z\delta_y)(z\delta_x) - xy.$$

Proof. This is straightfoward to check. $\square$

To find the extra solutions to the differetial equation system $\{\triangle_1' I = \triangle_2' I = 0\}$, we define $\bar{I}^{Y'}_{ext}(x,y)$ in the following way,

$$\bar{I}^{Y'}_{ext}(x,y) = x^{\frac{1}{2}+u} \sum_{(i,j)\in\mathbb{N}^2} x^i y^j C_{i,j}, \tag{5.1}$$

where $\{C_{i,j}\}$ satisfies the following recursion relations for $(i,j) \in \mathbb{Z}^2$:

$$C_{i-1,j}(j-i+\frac{1}{2}-u)^3 z = 16C_{i,j}(j-i-u)^2, \tag{5.2}$$

$$C_{i-1,j-1} = C_{i,j}(zj)(zi+\frac{1}{2}+u). \tag{5.3}$$

Lemma V.3. *Let $I_5, I_6 \in \mathbb{C}[[x,y,\log x]][x^{\frac{1}{2}}]$ be the components of $\bar{I}^{Y'}_{ext}(x,y)$ in the following sense*

$$\pi : \mathbb{C}[[x,y,u,\log x]][x^{\frac{1}{2}}] \longrightarrow \mathbb{C}[[x,y,u,\log x]][x^{\frac{1}{2}}]/(u^2),$$
$$x^{\frac{1}{2}} e^{u\log x} \sum_{(i,j)\in\mathbb{N}^2} x^i y^j C_{i,j} \longmapsto I_5 + I_6 u,$$

where π is the obvious projection map, and $C_{i,j}$ are defined by the recursion relations (5.2) and (5.3) with initial condition $C_{0,0} = 1$. Then I_5 and I_6, together with the components of $\bar{I}^{Y'}(x,y)$, comprise a basis of solutions to the differential equation system $\{\triangle_1' I = \triangle_2' I = 0\}$ at any point around the origin in $(\mathbb{C}^*)^2$.

Proof. Given the solution form (5.1), it is straightfoward to check that the recursion relation (5.2) and (5.3) correspond precisely to the differential operator $\triangle_1'$ and $\triangle_2'$, respectively. Choosing initial condition $C_{0,0} = 1$, it follows that $C_{i,j}$ are uniquely determined for all $(i,j) \in \mathbb{N}^2$. If we require $u^2 = 0$, we see that $C_{i,j} = 0$ if $i < 0$ or $j < 0$. Thus I_5 and I_6 are solutions to the differential equation system $\{\triangle_1' I = \triangle_2' I = 0\}$. It is clear that I_5, I_6, as well as the components of $\bar{I}^{Y'}(x,y)$, are all linearly-independent because of their initial terms. On the other hand, the differential equation system $\{\triangle_1' I = \triangle_2' I = 0\}$ should have 6-dimensional solution space, hence the lemma is proved. □

Now we are ready to prove the main theorem in this section.

Theorem V.4 (=Theorem I.2(3)). *The Conjecture I.1 holds for the local model $\{X', Y'\}$, namely: one may perform analytic continuation of $\mathcal{H}(X')$ over the extended Kähler moduli to obtain a D-module $\bar{\mathcal{H}}(X')$, then there exists a divisor E and a submodule $\bar{\mathcal{H}}^E(X') \subseteq \bar{H}(X')$ with maximum trivial monodromy around E, such that*

$$\bar{\mathcal{H}}^E(X')|_E \simeq \mathcal{H}(Y),$$

where $\bar{\mathcal{H}}^E(X')$ is the restriction to E.

Proof. We begin by identifying the ambient part quantum D-module $\mathcal{H}(X')$ and $\mathcal{H}(Y')$ with the cyclic D-modules generated by $I^{X'}(q_1, q_2)$ and $I^{Y'}(y)$, respectively. The change of variable $x \mapsto q_1^{-1}$ and $y \mapsto q_1 q_2$ give rise to the anayltic continuation $\mathcal{H}(X') \rightsquigarrow \bar{\mathcal{H}}(X')$.

By Lemma V.1 and Lemma V.2, we may consider the submodule of $\bar{\mathcal{H}}(X')$ corresponding to the sub D-module generated by the components of $\bar{I}^{Y'}(x, y)$. It has trivial monodromy around $x = 0$ as the initial term of $\bar{I}^{Y'}(x, y)$ does not involve x. This trivial monodromy is also maximal because by Lemma III.3, the remaining two solutions I_5, I_6 have non-trivial monodromy around $x = 0$. Let E denote the transition divisor $x = 0$, and $\bar{\mathcal{H}}^E(X')$ denote this submodule.

Since $I^{Y'}(y)$ is recovered by $I^{Y'}(y) = \lim_{x \to 0} \bar{I}^{Y'}(x, y)$, we see that $\mathcal{H}(Y')$ is isomophic to the restriction of $\bar{\mathcal{H}}^E(X')$ to E. Hence our theorem is proved. □

5.2 A Calabi-Yau example of type (2,4)

In this section, we study our first global example of Type II transition in degree 4. Let Y_1 be a (2,4) complete intersection in $\mathbb{P}^5$ defined by the following equations:

$$f(x_1, x_2, x_3, x_4, x_5) + t x_0^2 = 0,$$

$$x_0^2(g(x_1,x_2,x_3,x_4,x_5)+tx_0^2)+x_0g_1(x_1,x_2,x_3,x_4,x_5)+g_2(x_1,x_2,x_3,x_4,x_5)=0,$$

where f and g are quadratic homogenous polynomial which form a complete intersection, g_1 is a generic cubic homogenous polynomial and g_2 is a generic quartic homogenous polynomial.

By deforming the above equations to $t=0$, we obtain a singular variety $\bar{Y}_1 \subseteq \mathbb{P}^5$ defined by

$$f(x_1,x_2,x_3,x_4,x_5)=0,$$

$$x_0^2g(x_1,x_2,x_3,x_4,x_5)+x_0g_1(x_1,x_2,x_3,x_4,x_5)+g_2(x_1,x_2,x_3,x_4,x_5)=0.$$

Choosing g_1 and g_2 appropriately, we may assume $\overline{Y_1}$ has a unique singularity at $[1:0:0:0:0:0]$ arised from the (2,2) complete intersection (f,g).

Now we take X_1 to be the blow up of $\bar{Y}_1$ at the point $[1:0:0:0:0:0]$. The expectional divisor is given by $\{f=g=0\}$ in $\mathbb{P}^4$, which is a del Pezzo surface in degree 4. Clearly, both X_1 and Y_1 are Calabi-Yau 3-folds. The transition from X_1 to Y_1 is our primary example in this section. In our case here, Y_1 is the zero locus of a section of $\mathcal{O}_{\mathbb{P}^5}(2)\oplus\mathcal{O}_{\mathbb{P}^5}(4)$, whereas X_1 is natrually embeded into $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1)\oplus\mathcal{O}_{\mathbb{P}^4})$ as a complete intersection.

We adopt the following notations throughout this section.

- $h := c_1(\pi^*\mathcal{O}_{\mathbb{P}^4}(1))$, corresponding to small parameter q_1
- Let $\mathcal{O}_{\mathbb{P}}(1)$ be the anti-tautological bundle over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1)\oplus\mathcal{O}_{\mathbb{P}^4})$, and $\xi := c_1(\mathcal{O}_{\mathbb{P}}(1))$, corresponding to small parameter q_2.
- $p := c_1(\mathcal{O}_{\mathbb{P}^5}(1))$, corresponding to small parameter y.

Then X_1 is the zero locus of a section of the vector bundle $\mathcal{O}(2h)\oplus\mathcal{O}(2h+2\xi)$ over

$\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1)\oplus\mathcal{O}_{\mathbb{P}^4})$, whose I-function is the following:

$$I^{X_1}(q_1,q_2):=(2h)(2h+2\xi)q_1^{h/z}q_2^{\xi/z}$$
$$\cdot\sum_{(d_1,d_2)\in\mathbb{N}^2}q_1^{d_1}q_2^{d_2}\frac{\prod_{m=-\infty}^{0}(\xi-h+mz)\prod_{m=1}^{2d_1}(2h+mz)\prod_{m=1}^{2d_1+2d_2}(2h+2\xi+mz)}{\prod_{m=1}^{d_1}(h+mz)^5\prod_{m=1}^{d_2}(\xi+mz)\prod_{m=-\infty}^{d_2-d_1}(\xi-h+mz)},$$

subject to the relation $h^5=\xi(\xi-h)=0$.

On the other hand, the I-function for Y_1 is the following:

$$I^{Y_1}(y):=(2p)(4p)y^{p/z}\sum_{j\in\mathbb{N}}\frac{\prod_{m=1}^{4j}(4p+mz)\prod_{m=1}^{2j}(2p+mz)}{\prod_{m=1}^{j}(p+mz)^6},$$

subject to the relation $p^6=0$.

Similar to the local model, we introduce the following auxcillary hypergeometric series $\bar{I}^{Y_1}$ in two variables x and y.

$$\bar{I}^{Y_1}(x,y):=(2p)(4p)y^{p/z}\sum_{(i,j)\in\mathbb{N}^2}x^iy^j\frac{\prod_{m=-\infty}^{0}(p+mz)^5\prod_{m=-\infty}^{2j-2i}(2p+mz)\prod_{m=-\infty}^{4j-2i}(2p+mz)}{\prod_{m=-\infty}^{j-i}(p+mz)^5\prod_{m=1}^{j}(p+mz)\prod_{m=1}^{i}(mz)\prod_{m=-\infty}^{0}(2p+mz)^2}.$$

It's easy to check that $\bar{I}^{Y_1}$ is a holomorphic function on a small domain minus the origin, and it has trivial monodromy around $x=0$. Taking the limit $x\to 0$, we obtain

$$\lim_{x\to 0}\bar{I}^{Y_1}(x,y)=I^{Y_1}(y),$$

which precisely recovers the I-function for Y_1.

To study the relation between the ambient part quantum D-modules of X_1 and Y_1, we want to find a way to relate $I^{X_1}(x,y)$ and $I^{Y_1}(y)$. As I^{Y_1} is recovered from $\bar{I}^{Y_1}$, it is tempting to study the relation between I^{X_1} and I^{Y_1} as they both involve 2 parameters.

We consider the Picard-Fuchs equations that annihilate $I^{X'}$, which usually originates from a GKZ system attached to the toric data. We have the following lemma:

Lemma V.5. *The components of $I^{X_1}(q_1, q_2)$ comprise a basis of solutions to the differential equation system $\{\triangle_1 I = \triangle_2 I = \mathcal{L} I = 0\}$ at any point around the origin in $(\mathbb{C}^*)^2$, where*

$$\triangle_1 := (\delta_{q_1})^5 - 4(\delta_{q_1})(\delta_{q_1} + \delta_{q_2})(2\delta_{q_1} - 1)(\delta_{q_2} - \delta_{q_1} + 1)(2\delta_{q_1} + 2\delta_{q_2} - 1)q_1,$$

$$\triangle_2 := \delta_{q_2}(\delta_{q_2} - \delta_{q_1}) - 2(\delta_{q_1} + \delta_{q_2})(2\delta_{q_1} + 2\delta_{q_2} - 1)q_2,$$

$$\mathcal{L} := (2\delta_{q_1}^3 - 2\delta_{q_1}^2\delta_{q_2} + \delta_{q_1}\delta_{q_2}^2) - 8(2\delta_{q_1} - 1)(\delta_{q_2} - \delta_{q_1} + 1)(2\delta_{q_2} + 2\delta_{q_1} - 1)q_1 - 2\delta_{q_1}\delta_{q_2}(2\delta_{q_1} + 2\delta_{q_2} - 1)q_2.$$

Moreover, these differential operators are related in the following factorization

$$2\triangle_1 + \delta_{q_1}^2\delta_{q_2}\triangle_2 = (\delta_{q_1} + \delta_{q_2})\delta_{q_1}\mathcal{L}. \tag{5.4}$$

Proof. Indeed, the GKZ system attached to the toric data yields the generators $\triangle_1$ and $\triangle_2$. By the factorization (5.4), we obtain a differential operator $\mathcal{L}$ of order 3, thus the system $\{\triangle_1 I = \triangle_2 I = \mathcal{L} I = 0\}$ can have at most 6-dimensional solution space. It is direct to check that the components of I^{X_1} give 6 linearly independent solution to this differential equation system, hence they must form a basis. $\square$

Similar to Section 3, we apply the following change of variables

$$q_1 \mapsto x^{-1}, \quad q_2 \mapsto xy,$$

which yields the following relations between differential operators

$$\delta_{q_1} = \delta_y - \delta_x, \quad \delta_{q_2} = \delta_y.$$

Let $\triangle_1'$, $\triangle_2'$, $\mathcal{L}'$ denote the differential operators obtained by applying the above change of variables to $\triangle_1, \triangle_2, \mathcal{L}$, respectively. Then we have

Lemma V.6. *The components of* $\bar{I}^{Y_1}(x,y)$ *comprise 4 linearly independent solutions to the differential equation system* $\{\triangle_1' I = \triangle_2' I = \mathcal{L}' I = 0\}$ *at any point around the origin in* $(\mathbb{C}^*)^2$*, where*

$$\triangle_1' := (\delta_y - \delta_x)^5 - 4(\delta_y - \delta_x)(2\delta_y - \delta_x)(2\delta_y - 2\delta_x - 1)(\delta_x + 1)(4\delta_y - 2\delta_x - 1)x^{-1},$$

$$\triangle_2' := \delta_y \delta_x - 2(2\delta_y - \delta_x)(4\delta_y - 2\delta_x - 1)xy,$$

$$\mathcal{L}' := (2(\delta_y - \delta_x)^3 - 2(\delta_y - \delta_x)^2\delta_y + (\delta_y - \delta_x)\delta_y^2) - 8(2\delta_y - 2\delta_x - 1)(\delta_x + 1)(4\delta_y - 2\delta_x - 1)q_1$$
$$- 2(\delta_y - \delta_x)\delta_y(4\delta_y - 2\delta_x - 1)xy.$$

Proof. This can be checked directly. □

To find the extra solutions to the above differetial equation system $\{\triangle_1' I = \triangle_2' I = 0\}$, we introduce $\bar{I}^{Y_1}_{ext}(x,y)$ in the following way,

$$\bar{I}^{Y_1}_{ext}(x,y) = x^{\frac{1}{2}+u} \sum_{(i,j)\in\mathbb{N}^2} x^i y^j C_{i,j}, \tag{5.5}$$

where $\{C_{i,j}\}$ satisfies the following recursion relations for $(i,j) \in \mathbb{Z}^2$:

(5.6)
$$C_{i,j}(j-i-\frac{1}{2}-u)^4 = C_{i+1,j}(2j-u-i-\frac{1}{2})(2j-2u-2i-2)(u+i+\frac{3}{2})(4j-2u-2i-2),$$

$$(4j-2u-2i-1)(4j-2i-2u-2)C_{i-1,j-1} = C_{i,j}(j)(i+\frac{1}{2}+u). \tag{5.7}$$

Lemma V.7. *Let* $I_5, I_6 \in \mathbb{C}[[x,y,\log x]][x^{\frac{1}{2}}]$ *be the components of* $\bar{I}^{Y_1}_{ext}(x,y)$ *in the following sense*

$$\pi : \mathbb{C}[[x,y,u,\log x]][x^{\frac{1}{2}}] \longrightarrow \mathbb{C}[[x,y,u,\log x]][x^{\frac{1}{2}}]/(u^2),$$
$$x^{\frac{1}{2}} e^{u\log x} \sum_{(i,j)\in\mathbb{N}^2} x^i y^j C_{i,j} \longmapsto I_5 + I_6 u,$$

where π is the obvious projection map, and $C_{i,j}$ are defined recursively by (5.6) and (5.7) with initial condition $C_{0,0} = 1$. Then I_5 and I_6, together with the components of $\bar{I}^{Y_1}(x, y)$, comprise a basis of solutions to the differential equation system $\{\triangle_1' I = \triangle_2' I = \mathcal{L}' I = 0\}$ at any point around the origin in $(\mathbb{C}^*)^2$.

Proof. Given the solution form (5.5), it is straightfoward to check that the recursion relation (5.6) and (5.7) are compatible with $\triangle_1'$, $\triangle_2'$ and $\mathcal{L}'$. If we require $u^2 = 0$, we see that $C_{i,j} = 0$ if $i < 0$ or $j < 0$. Given initial condition $C_{0,0} = 1$, it is clear that $C_{i,j}$ are uniquely determined for all $(i, j) \in \mathbb{N}^2$. Thus I_5 and I_6 are solutions to the differential equation system $\{\triangle_1' I = \triangle_2' I = \mathcal{L}' I = 0\}$. We also note that I_5, I_6, together with the components of $\bar{I}^{Y_1}(x, y)$, are linearly-independent because of their initial terms. On the other hand, the differential equation system $\{\triangle_1' I = \triangle_2' I = \mathcal{L}' I = 0\}$ should have at most 6-dimensional solution space, hence the lemma follows. □

We are now in a position to prove the main theorem in this section.

Theorem V.8 (=Theorem I.2(4)). *The Conjecture I.1 holds for the $\{X_1, Y_1\}$, namely: one may perform analytic continuation of $\mathcal{H}(X_1)$ over the extended Kähler moduli to obtain a D-module $\bar{\mathcal{H}}(X_1)$, then there exists a divisor E and a submodule $\bar{\mathcal{H}}^E(X_1) \subseteq \bar{H}(X_1)$ with maximum trivial monodromy around E, such that*

$$\bar{\mathcal{H}}^E(X_1)|_E \simeq \mathcal{H}(Y_1),$$

where $\bar{\mathcal{H}}^E(X_1)$ is the restriction to E.

Proof. Following the argument in Chapter III, we first identify the ambient part quantum D-module $\mathcal{H}(X_1)$ and $\mathcal{H}(Y_1)$ with the cyclic D-modules generated by $I^{X_1}(q_1, q_2)$ and $I^{Y_1}(y)$, respectively. The change of variable $x \mapsto q_1^{-1}$ and $y \mapsto q_1 q_2$ give rise to the anayltic continuation $\mathcal{H}(X_1) \rightsquigarrow \bar{\mathcal{H}}(X_1)$.

By Lemma V.5 and Lemma V.6, we consider the submodule of $\bar{\mathcal{H}}(X_1)$ corresponding to the sub D-module attached to the components of $\bar{I}^{Y_1}(x,y)$. It has trivial monodromy around $x=0$ as the initial term of $\bar{I}^{Y_1}(x,y)$ does not involve x. This trivial monodromy is also maximal because by Lemma V.7, it is clear that the remaining two solutions I_5, I_6 have non-trivial monodromy around $x=0$. Let E denote the transition divisor corresponding to $x=0$, and $\bar{\mathcal{H}}^E(X_1)$ denote this submodule.

As $I^{Y_1}(y)$ is recovered by $I^{Y_1}(y) = \lim_{x\to 0} \bar{I}^{Y_1}(x,y)$, we obtain immediately that $\mathcal{H}(Y_1)$ is isomophic to the restriction of $\bar{\mathcal{H}}^E(X_1)$ to E. Hence the theorem is proved.

□

5.3 A Calabi-Yau example of type (3,3)

In this section, our goal is to verify Conjecuture 1.1 for another global example of degree-4 transition. Let Y_2 be a $(3,3)$-complete intersection in $\mathbb{P}^5$ defined by the following two equations:

$$x_0(f(x_1,x_2,x_3,x_4,x_5)+tx_0^2)+f_1(x_1,x_2,x_3,x_4,x_5)=0,$$

$$x_0(g(x_1,x_2,x_3,x_4,x_5)+tx_0^2)+g_1(x_1,x_2,x_3,x_4,x_5)=0,$$

where f and g are quadratic homogenous polynomials that form a complete interesection, while f_1 and g_1 are generic homogenous polynomial in degree 3. Deforming the above equations by letting $t=0$, we obtain a singular 3-fold $\bar{Y}_2 \subseteq \mathbb{P}^5$, whose defining equations are:

$$x_0 f(x_1,x_2,x_3,x_4,x_5)+f_1(x_1,x_2,x_3,x_4,x_5)=0,$$

$$x_0 g(x_1,x_2,x_3,x_4,x_5)+g_1(x_1,x_2,x_3,x_4,x_5)=0.$$

As f_1 and g_1 are choosen to be generic, we may assume that $\bar{Y}_2$ has a unique singularity at $[1:0:0:0:0:0]$, which is a (2,2) complete intersection singularity. If we blow up Y_0 at this point, we obtain a smooth Calabi-Yau 3-fold X_2, where the exceptional divisor is a (2,2)-complete intersection in $\mathbb{P}^4$, namely, a del Pezzo surface in degree 4.

Going backwards, we see that Y_2 is obtained by birationally contracting a degree-4 del Pezzo surface in X_2 and followed by smoothing out the singularity. Thus $\{X_2, Y_2\}$ is another global example of degree-4 transitions, where both sides are Calabi-Yau 3-folds. In this case, Y_2 is the zero locus of a section of $\mathcal{O}_{\mathbb{P}^5}(3) \oplus \mathcal{O}_{\mathbb{P}^5}(3)$, whereas X_2 natrually sits inside $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})$ as a complete intersection.

As before, we shall adopt the following notations throughout this section.

- $h := c_1(\pi^* \mathcal{O}_{\mathbb{P}^4}(1))$, corresponding to small parameter q_1
- Let $\mathcal{O}_{\mathbb{P}}(1)$ be the anti-tautological bundle over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})$, and $\xi := c_1(\mathcal{O}_{\mathbb{P}}(1))$, corresponding to small parameter q_2.
- $p := c_1(\mathcal{O}_{\mathbb{P}^5}(1))$, corresponding to small parameter y.

Then X_2 is the zero locus of a section of the vector bundle $\mathcal{O}(2h+\xi) \oplus \mathcal{O}(2h+\xi)$ over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})$, whose I-function is the following:

$$I^{X_2}(q_1, q_2) := (2h+\xi)^2 q_1^{h/z} q_2^{\xi/z} \cdot \sum_{(d_1,d_2)\in\mathbb{N}^2} q_1^{d_1} q_2^{d_2} \frac{\prod_{m=-\infty}^{0} (\xi - h + mz) \prod_{m=1}^{2d_1} (2h+mz) \prod_{m=1}^{2d_1+d_2} (2h+\xi+mz)^2}{\prod_{m=1}^{d_1} (h+mz)^5 \prod_{m=1}^{d_2} (\xi+mz) \prod_{m=-\infty}^{d_2-d_1} (\xi - h + mz)},$$

subject to the relation $h^5 = \xi(\xi - h) = 0$.

On the other hand, the I-function for Y_2 is the following:

$$I^{Y_2}(y) := (3p)^2 y^{p/z} \sum_{j\in\mathbb{N}} \frac{\prod_{m=1}^{3j} (3p+mz)^2}{\prod_{m=1}^{j} (p+mz)^6},$$

subject to the relation $p^6 = 0$.

To compare I^{X_2} and I^{Y_2}, we introduce the following auxcillary hypergeometric series $\bar{I}^{Y_2}$ in two variables x and y.

$$\bar{I}^{Y_2}(x,y) := (3p)^2 y^{p/z} \sum_{(i,j)\in\mathbb{N}^2} x^i y^j \frac{\prod_{m=-\infty}^{0}(p+mz)^5 \prod_{m=-\infty}^{3j-2i}(3p+mz)^2}{\prod_{m=-\infty}^{j-i}(p+mz)^5 \prod_{m=1}^{j}(p+mz) \prod_{m=1}^{i}(mz) \prod_{m=-\infty}^{0}(2p+mz)^2}.$$

It is clear that $\bar{I}^{Y_2}$ is a holomorphic function on a small domain minus the origin, and it has trivial monodromy around $x=0$. Taking the limit $x\to 0$, we obtain

$$\lim_{x\to 0}\bar{I}^{Y_2}(x,y) = I^{Y_2}(y),$$

which still recovers the I-function for Y_2.

To compare I^{X_2} and I^{Y_2}, we turn our attention to the relation betwen I^{X_2} and $\bar{I}^{Y_2}$, as $\bar{I}^{Y_2}$ involves two variables x and y, and recovers I^{Y_2} naturally. We consider the Picard-Fuchs equations that annihilates I^{X_2}, which usually arises from a GKZ system attached to the toric data.

Lemma V.9. *The components of $I^{X_2}(q_1,q_2)$ comprise a basis of solutions to the differential equation system $\{\triangle_1 I = \triangle_2 I = \mathcal{L}I = 0\}$ at any point around the origin in $(\mathbb{C}^*)^2$, where*

$$\triangle_1 := (\delta_{q_1})^5 - q_1(\delta_{q_2}-\delta_{q_1})(2\delta_{q_1}+\delta_{q_2}+1)^2(2\delta_{q_1}+\delta_{q_2}+2)^2,$$

$$\triangle_2 := \delta_{q_2}(\delta_{q_2}-\delta_{q_1}) - q_2(2\delta_{q_2}+\delta_{q_1}+1)^2,$$

$$\mathcal{L} := 9\delta_{q_1}^3 - 5\delta_{q_2}^3 - 36(\delta_{q_2}-\delta_{q_1}+1)(2\delta_{q_1}+\delta_{q_2}-1)^2 q_1 + (36\delta_{q_1}^3+45\delta_{q_1}^2\delta_{q_2}+25\delta_{q_1}\delta_{q_2}^2+5\delta_{q_2}^3)q_2.$$

Moreover, these differential operators are related in the following factorization

$$36\triangle_1 - (36\delta_{q_1}^3+45\delta_{q_1}^2\delta_{q_2}+25\delta_{q_1}\delta_{q_2}^2+5\delta_{q_2}^3)\triangle_2 = (2\delta_{q_1}+\delta_{q_2})^2\mathcal{L}. \tag{5.8}$$

Proof. The GKZ system attached to the toric data gives rise to the generators $\triangle_1$ and $\triangle_2$. By the factorization (5.8), we obtain a differential operator $\mathcal{L}$ of order 3, thus the system $\{\triangle_1 I = \triangle_2 I = \mathcal{L}I = 0\}$ can have at most 6-dimensional solution space. It is straightforward to check that the components of I^{X_2} comprise 6 linearly independent solution to this differential equation system, hence they must form a basis. □

On the other hand, by making the change of variable $x \mapsto q_1^{-1}$ and $y \mapsto q_1 q_2$, the resulting differential operators turn out to be the annihilators of $\bar{I}^{Y_2}$. We have the following lemma.

Lemma V.10. *The components of $\bar{I}^{Y_2}(x,y)$ comprise 4 linearly independent solutions to the differential equation system $\{\triangle_1' I = \triangle_2' I = \mathcal{L}' I = 0\}$ at any point around the origin in $(\mathbb{C}^*)^2$, where*

$$\triangle_1' := x(\delta_y - \delta_x)^5 - \delta_x(3\delta_y - 2\delta_x + 1)^2(3\delta_y - 2\delta_x + 2)^2,$$

$$\triangle_2' := \delta_y\delta_x - xy(3\delta_y - 2\delta_x + 1)^2,$$

$$\begin{aligned}\mathcal{L}' := {} & 9(\delta_y - \delta_x)^3 - 5\delta_y^3 - 36(\delta_x + 1)(3\delta_y - 2\delta_x - 1)^2 x^{-1} + (36(\delta_y - \delta_x)^3 + 45(\delta_y - \delta_x)^2\delta_y \\ & + 25(\delta_y - \delta_x)\delta_y^2 + 5\delta_y^3)xy.\end{aligned}$$

Proof. This is easy to check. □

To find the extra solutions to the above differential equation system, we adopt the same method used in previous sections, namely, define $\bar{I}^{Y_2}_{ext}(x,y)$ in the following way.

$$\bar{I}^{Y_2}_{ext}(x,y) = x^{\frac{1}{2}+u} \sum_{(i,j)\in\mathbb{N}^2} x^i y^j C_{i,j}, \tag{5.9}$$

where $\{C_{i,j}\}$ satisfies the following recursion relations for $(i,j) \in \mathbb{Z}^2$:

$$C_{i-1,j}(j - u + \frac{1}{2})^5 = C_{i,j}(i + u + \frac{1}{2})(3j - 2i - 2u)(3j - 2i - 2u + 1)^2, \tag{5.10}$$

$$(3j - 2i - 2u - 1)^2 C_{i-1,j-1} = C_{i,j}(j)(i + \frac{1}{2} + u). \tag{5.11}$$

Lemma V.11. *Let $I_5, I_6 \in \mathbb{C}[[x, y, \log x]][x^{\frac{1}{2}}]$ be the components of $\bar{I}^{Y_2}_{ext}(x, y)$ in the following sense*

$$\pi : \mathbb{C}[[x, y, u, \log x]][x^{\frac{1}{2}}] \longrightarrow \mathbb{C}[[x, y, u, \log x]][x^{\frac{1}{2}}]/(u^2),$$
$$x^{\frac{1}{2}} e^{u \log x} \sum_{(i,j)\in\mathbb{N}^2} x^i y^j C_{i,j} \longmapsto I_5 + I_6 u,$$

where π is the obvious projection map, and $C_{i,j}$ are defined recursively by (5.10) and (5.11) with initial condition $C_{0,0} = 1$. Then I_5 and I_6, together with the components of $\bar{I}^{Y_2}(x, y)$, comprise a basis of solutions to the differential equation system $\{\triangle'_1 I = \triangle'_2 I = \mathcal{L}' I = 0\}$ at any point around the origin in $(\mathbb{C}^*)^2$.

Proof. As $\bar{I}^{Y_1}_{ext}(x, y)$ is of form (5.9), it is easy to check that the recursion relation (5.10) and (5.11) are compatible with $\triangle'_1$, $\triangle'_2$ and $\mathcal{L}'$. If we require $u^2 = 0$, we see that $C_{i,j} = 0$ if $i < 0$ or $j < 0$. Moreover, the initial condition $C_{0,0} = 1$ allows us to determine $C_{i,j}$ uniquely for all $(i, j) \in \mathbb{N}^2$. Thus I_5 and I_6 are solutions to the differential equation system $\{\triangle'_1 I = \triangle'_2 I = \mathcal{L}' I = 0\}$. We also note that I_5, I_6, together with the components of $\bar{I}^{Y_2}(x, y)$, are linearly-independent because of their initial terms. On the other hand, the differential equation system $\{\triangle'_1 I = \triangle'_2 I = \mathcal{L}' I = 0\}$ should have at most 6-dimensional solution space, hence the lemma follows. □

Now we arrive at the proof of the main theorem in this section.

Theorem V.12 (=Theorem I.2(5)). *The Conjecture I.1 holds for the $\{X_2, Y_2\}$, namely: one may perform analytic continuation of $\mathcal{H}(X_2)$ over the extended Kähler moduli to obtain a D-module $\bar{\mathcal{H}}(X_2)$, then there exists a divisor E and a submodule*

$\bar{\mathcal{H}}^E(X_2) \subseteq \bar{H}(X_2)$ *with maximum trivial monodromy around* E*, such that*

$$\bar{\mathcal{H}}^E(X_2)|_E \simeq \mathcal{H}(Y_2),$$

where $\bar{\mathcal{H}}^E(X_2)$ *is the restriction to* E*.*

Proof. Following the same line of arguments in previous sections, we identify the ambient part quantum D-module $\mathcal{H}(X_2)$ and $\mathcal{H}(Y_2)$ with the cyclic D-modules generated by $I^{X_2}(q_1, q_2)$ and $I^{Y_2}(y)$, respectively. The change of variable $x \mapsto q_1^{-1}$ and $y \mapsto q_1 q_2$ give rise to the anayltic continuation $\mathcal{H}(X_2) \rightsquigarrow \bar{\mathcal{H}}(X_2)$.

By Lemma V.9 and Lemma V.10, we may consider the submodule of $\bar{\mathcal{H}}(X_2)$ corresponding to the sub D-module attached to the components of $\bar{I}^{Y_2}(x, y)$. It has trivial monodromy around $x = 0$ as the initial term of $\bar{I}^{Y_2}(x, y)$ does not involve x. This trivial monodromy is also maximal because by Lemma 5.3, the remaining two solutions I_5, I_6 have non-trivial monodromy around $x = 0$. Let E denote the transition divisor $x = 0$, and $\bar{\mathcal{H}}^E(X_1)$ denote this submodule.

Since $I^{Y_2}(y)$ is recovered by $I^{Y_2}(y) = \lim_{x\to 0} \bar{I}^{Y_2}(x, y)$, we conclude that $\mathcal{H}(Y_2)$ is recovered as the restriction of $\bar{\mathcal{H}}^E(X_2)$ to E. Hence the theorem is proved. $\square$

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