Filtration Theorems and Bounding Generators of Symbolic Multi-powers

by

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To my family and friends.

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PREFACE

This dissertation is submitted for the degree of Doctor of Philosophy at the University of Michigan. The research described herein was conducted under the supervision of Professor Melvin Hochster in the Department of Mathematics, University of Michigan, between September 2016 and May 2019.

This work is to the best of my knowledge original, except where acknowledgements and references are made to previous work. Neither this, nor any substantially similar dissertation has been or is being submitted for any other degree, diploma or other qualification at any other university.

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ABSTRACT

We prove a very powerful generalization of the theorem on generic freeness that gives countable ascending filtrations, by prime cyclic A-modules A/P, of finitely generated algebras R over a Noetherian ring A and of finitely generated R-modules such that the number of primes P that occur is finite. Moreover, we can control, in a sense that we can make precise, the number of factors of the form A/P that occur.

In the graded case, the number of occurrences of A/P up to a given degree is eventually polynomial. The degree is at most the number of generators of R over A. By multi-powers of a finite sequence of ideals we mean an intersection of powers of the ideals with exponents varying. Symbolic multi-powers are defined analogously using symbolic powers instead of powers. We use our filtration theorems to give new results bounding the number of generators of the multi-powers of a sequence of ideals and of the symbolic multi-powers as well under various conditions. This includes the case of ordinary symbolic powers of one ideal.

Furthermore, we give new results bounding, by polynomials in the exponents, the number of generators of multiple Tor when each input module is the quotient of R by a power of an ideal. The ideals and exponents vary. The bound is given by a polynomial in the exponents. There are similar results for Ext when both of the input modules are quotients of R by a power of an ideal. Typically, the two ideals used are different, and the bound is a polynomial in two exponents.

CHAPTER I

Introduction

It appears to be very difficult to give a bound on the number of generators of a symbolic power or of an intersection of powers. In this thesis, we will introduce a powerful tool to give a bound for some particular cases.

We prove a very powerful generalization of the theorem on generic freeness that gives countable ascending filtrations, by prime cyclic A-modules $A/\mathfrak{p}$, of finitely generated algebras R over a Noetherian ring A and of finitely generated R-modules such that the number of primes $\mathfrak{p}$ that occur is finite. Moreover, we can control, in a sense that we can make precise, the number of factors of the form $A/\mathfrak{p}$ that occur.

When A is a domain, the theorem on generic freeness follows at once: one simply localizes at one element of $A - \{0\}$ in all of the finitely many nonzero primes of A that occur in the filtration.

In the graded case, the number of occurrences of $A/\mathfrak{p}$ up to a given degree is eventually polynomial. The degree is at most the number of generators of R over A. Therefore, in a sense, the results have generalized the standard theory of Hilbert functions for standard graded algebras over a field. We use these theorems to give new results bounding the number of generators of the multi-powers of ideals, *i.e.*, $I_1^{n_1} \cap \cdots \cap I_k^{n_k}$, and of symbolic multi-powers, $I_1^{(n_1)} \cap \cdots \cap I_k^{(n_k)}$, under various conditions. This includes the case of ordinary symbolic powers $I^{(n)}$.

Furthermore, we give new results bounding, by polynomials in the n_i, the number of generators of $\mathrm{Tor}_h^R(\frac{R}{I_1^{n_1}}, \cdots, \frac{R}{I_k^{n_k}})$ and of other functors, *e.g.*, $\mathrm{Ext}_R^h(\frac{R}{I_1^{n_1}}, \frac{R}{I_2^{n_2}})$, some of which are needed to prove the results mentioned above.

In the paper [13], Craig Huneke and Ilya Smirnov prove related results on prime filtrations of R/I^n.

Enescu and Yao define the notion of *Frobenius complexity* [7], and it follows from the results of [15] that for a complete local normal domain R of positive prime characteristic p, the Frobenius complexity is finite if and only if there is a polynomial in d that bounds the number of generators of $I^{(d)}$ for a suitably chosen ideal I of R. This question remains open. This gives further motivation for studying the problems considered here.

Although we do not study the containment problem for symbolic powers here, we do want to point out that there is considerable recent literature on the existence of constants c such that $P^{(cn)}$ is contained in P^n for all n. P need not be prime, although that case is of great importance, and there are results giving a single choice of c for all ideals (e.g., in regular rings and certain isolated singularities) as well as results that place an extra hypothesis on R/P. Containment results may be found in [2], [8], [3], [4], [5], [9], [12], [14], and [16].

1.1 Outline and main results

In Chapter II, we provide background material necessary in understanding the thesis work. First, we establish some notations to be used throughout the thesis in Section 2.1. Next, we give a review of the basic standard facts in the commutative algebra in the rest of this Chapter.

In Chapter III, we discuss the notion of ω^r-filtrations, defined just below. Next, we prove several useful properties of ω^r-filtrations. Also, we prove the existence of ω^r-filtrations with an important property that we will describe in the following theorem.

Definition 3.1.1. Let M be a R-module. We define recursively the notion of an ω^r-filtration of M. If $r=1$, an ω-filtration of M is just an ascending sequence of submodules denoted by the following.

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

where M_i is a submodule of M and $\bigcup_{i=0}^{\infty} M_i = M$. Recursively, if we have already defined an ω^{r-1}-filtration of an arbitrary R-module for $r \geqslant 2$, an ω^r-filtration of M is an ascending sequence of submodules denoted by $M_0, M_1, M_2, \cdots$ such that $\bigcup_{i=0}^{\infty} M_i = M$, and each M_i/M_{i-1} has an ω^{r-1}-filtration.

Theorem 3.1.15. Let A be a Noetherian commutative ring. Let R be an A-algebra with r generators and M be a finitely generated R-module. Then M has an ω^r-filtration in which all the factors are prime cyclic A-modules. Furthermore, only finitely many distinct factors will occur.

In Section 3.2, we give an explicit construction of ω^r-filtrations. For several particular cases, we calculate the factors of these ω^r-filtrations.

Proposition 3.2.2. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. Then R has an ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ where $R_{i_1,i_2,\cdots,i_r}$ is defined as the following.

$$\begin{aligned} R_{i_1,i_2,\cdots,i_r} = &\sum_{i_1'=0}^{i_1-1} A[\theta_1,\theta_2,\cdots,\theta_{r-1}]\theta_r^{i_1'} && (1.1)\\ &+ \sum_{i_2'=0}^{i_2-1} A[\theta_1,\theta_2,\cdots,\theta_{r-2}]\theta_{r-1}^{i_2'}\theta_r^{i_1} && (1.2)\\ &+ \sum_{i_3'=0}^{i_3-1} A[\theta_1,\theta_2,\cdots,\theta_{r-3}]\theta_{r-2}^{i_3'}\theta_{r-1}^{i_2}\theta_r^{i_1} && (1.3)\\ &+ \cdots && (1.4)\\ &+ \sum_{i_r'=0}^{i_r-1} A\theta_1^{i_r'}\theta_2^{i_{r-1}}\cdots\theta_r^{i_1} && (1.5) \end{aligned}$$

$$= \sum_{j=1}^{r} \sum_{i'_j=0}^{i_j-1} A[\theta_1, \theta_2, \cdots, \theta_{r-j}]\theta_{r-j+1}^{i'_j} \prod_{k=1}^{j-1} \theta_{r+1-k}^{i_k} \tag{1.6}$$

Note that if the upper index is less than the lower index of the sum, we define the sum to be zero. All the factors are cyclic A-modules. These cyclic A-modules may be replaced, by filtration, by prime cyclic A-modules, *i.e.*, modules of the form $A/\mathfrak{p}$ with $\mathfrak{p}$ prime. Only finitely many distinct $\mathfrak{p}$ occur.

Last but not least, we introduce the definition of rectangularly and triangularly normal ω^r-filtrations. Futhermore, we construct rectangularly and triangularly normal ω^r-filtrations in several particular cases.

Definition 3.3.1. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$.

For $l_1, l_2, \cdots, l_r \in \mathbb{N}$, we define $R_{[l_1,l_2,\cdots,l_r]}$ as the following.

$$R_{[l_1,l_2,\cdots,l_r]} = \sum_{i'_1 \leqslant l_1, i'_2 \leqslant l_2, \cdots, i'_r \leqslant l_r} A\theta_1^{i'_2}\theta_1^{i'_2} \cdots \theta_r^{i'_r} \tag{1.7}$$

We call $R_{[l_1,l_2,\cdots,l_r]}$ a *rectangular* submodule of R.

For $d \in \mathbb{N}$, we define $R_{\{d\}}$ as the following.

$$R_{\{d\}} = \sum_{i'_1+i'_2+\cdots+i'_r \leqslant d} A\theta_1^{i'_2}\theta_1^{i'_2} \cdots \theta_r^{i'_r} \tag{1.8}$$

Similarly, $R_{\{d\}}$ is called a *triangular* submodule of R.

The notions of rectangular and triangular submodules depend on the choice of generators θ_i.

An ω^r-filtration of R is said to be rectangularly normal (respectively, triangularly normal) if all the inherited ω^r-filtrations on rectangular (respectively, triangular) submodules produce only finitely many factors.

Proposition 3.3.9. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. If $\theta_1, \theta_2, \cdots, \theta_h$ are indeterminates and $\theta_{h+1}, \theta_{h+2}, \cdots, \theta_r$ are integral over A, then the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly and triangularly normal.

We cannot prove that ω^r-filtrations are rectangularly or triangularly normal in general. However, in Chapter IV, we derive an ω-filtration in the graded case from the ω^r-filtration. By using a suitable asscending ω-filtration of R or M, we may reduce to studying the graded case. By this method, we bypass all the difficulties that appear in Chapter III.

Proposition 4.1.5. Let A be a Noetherian ring and R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. If R is a standard $\mathbb{N}$-graded ring, say $R = R_0 \oplus R_1 \oplus R_2 \cdots$ where $R_h = \sum\limits_{i_1+\cdots+i_r=h} A\theta_1^{i_1}\cdots\theta_r^{i_r}$ for any $h \geqslant 0$, there exists an ω-filtration of R in which the factors are prime cyclic A-modules and only finitely many distinct factors occur. Furthermore, the length of the inherited finite filtration of R_h is at most Ch^{r-1}, where $h \geqslant 0$ and C is a constant. For $h \gg 0$ and any factor in the filtration of R_h, the number of copies of this factor is a polynomial of degree at most $r-1$.

Theorem 4.1.10. Let A be a Noetherian ring and R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. There exists an ω-filtration of R in which the factors are prime cyclic A-modules and only finitely many distinct factors occur. All the $R_{\{h\}}$ occur in the filtration. Furthermore, the length of the inherited finite filtration on $R_{\{h\}}/R_{\{h-1\}}$ is at most Ch^{r-1}, where $h \geqslant 0$. Notice that $R_{\{-1\}} = 0$. For all $h \gg 0$, the number of copies of $A/\mathfrak{p}$ occurring as a factor in the filtration of $R_{\{h\}}$ agrees with a polynomial in h of degree at most r.

Theorem 4.1.12. Let A be a Noetherian ring, R be a finitely generated A-algebra, and M be a finitely generated R-module. There exists an ω-filtration of M in which factors are prime cyclic A-modules and only finitely many distinct factors will occur. For all $h \gg 0$, the number of copies of $A/\mathfrak{p}$ occurring as a factor in the filtration of $M_{\{h\}}$ agrees with a polynomial in h of degree at most r.

In the very last section, we use these ω-filtrations to give a bound on the number of generators of an intersection of powers of two ideals or the ordinary symbolic powers $I^{(n)}$ under particular restrictions that we will describe later.

Theorem 4.2.7. Let R be a ring that is semi-local or finitely generated over a field. Let $I_1, \cdots, I_k$ be k ideals of R with $d_1, \cdots, d_k$ generators, respectively. Suppose also that $\dim(\frac{R}{I_1+\cdots+I_k}) \leqslant 1$. For $n_1, \cdots, n_k \gg 0$, we have $\mu(\mathrm{Tor}_h^R(\frac{R}{I_1^{n_1}}, \cdots, \frac{R}{I_k^{n_k}})) = \mathcal{O}(n_1^{d_1} \cdots n_k^{d_k})$. If $k = 2$, the corresponding fact also holds for Ext_R^h, hence, for Hom_R.

Theorem 4.2.10. Let R be a ring that is semi-local or finitely generated over a field, let I, J be two ideals of R with d and d' generators, respectively. Suppose also that $\dim(\frac{R}{I+J}) \leqslant 1$. For $m, n \gg 0$, there is a polynomial upper bound on the number of generators of $I^m \cap J^n$. Specifically, we have $\mu(I^m \cap J^n) = \mathcal{O}(m^d n^{d'})$.

Discussion 1.1.1. Let R be a Noetherian ring and $\mathfrak{p}$ be a prime ideal of R. We have $\mathrm{gr}_{\mathfrak{p}}(R) = R/\mathfrak{p} \oplus \mathfrak{p}/\mathfrak{p}^2 \oplus \mathfrak{p}^2/\mathfrak{p}^3 \oplus \cdots$. The ideal $J = \ker(\mathrm{gr}_{\mathfrak{p}}(R) \to (R-\mathfrak{p})^{-1}\,\mathrm{gr}_{\mathfrak{p}}(R))$ is a finitely generated ideal of $\mathrm{gr}_{\mathfrak{p}}(R)$. Thus, there exists $a \in R - \mathfrak{p}$ such that it kills this kernel. Let I_0 be the set of all elements of R that kill J. Thus, I_0 is an ideal of R such that $\mathfrak{p} \subsetneq I_0$ and for any $a \in I_0 - \mathfrak{p}$ we have that $\mathfrak{p}^{(n)} = \mathfrak{p}^n : a^n$, by Proposition 2.7.30.

Theorem 4.2.8. Let $R, \mathfrak{p}, I_0$ be the same as in Discussion 1.1.1. Assume that R is semi-local or finitely generated over a field. Let $h = \mathrm{height}(I_0 \frac{R}{\mathfrak{p}})$. Suppose that $\dim(R/\mathfrak{p}) = \dim(\frac{R}{I_0+\mathfrak{p}}) + h$ and $\dim(\frac{R}{I_0+\mathfrak{p}}) \leqslant 1$. For $n \gg 0$, we have $\mu(\mathfrak{p}^{(n)}) = \mathcal{O}(n^{d+h})$.

In Chapter V, we give a formula to calculate the number of generators of the symbolic multi-powers of the intersection of prime monomial ideals, *i.e.*, the intersection of powers of these prime monomial ideals.

Definition 5.1.1. Suppose $k \geqslant 1$. Let N_k^n be the number of non-negative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = n$ if $n \geqslant 0$. This is the number of monomials with k variables of degree n. It will be convenient to make the convention that $N_k^n = 1$ if $n < 0$. We make the corresponding convention for powers of ideals, *i.e.*, $I^n = I^0 = R$ if $n \leqslant 0$.

For the rest of this introduction, we are working in the polynomial ring $R = K[x_1, \cdots, x_N]$.

Definition 5.1.4. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_c$ be prime monomial ideals. Actually $\mathfrak{p}_i$ is the ideal generated by a subset A_i of the variables where $1 \leqslant i \leqslant c$. Also, σ_c denotes a binary string whose entries are 0 or 1, containing at least one 1, with c characters. Since c is fixed, we replace σ_c by σ for simplicity. Σ is the set of all σ. The j-th character of σ is denoted by $\sigma(j)$ where $1 \leqslant j \leqslant c$. Let A_i' be the set of variables not in A_i. Denote $A_\sigma = (\bigcap\limits_{\sigma(i)=1} A_i) \cap (\bigcap\limits_{\sigma(j)=0} A_j')$. We denote the cardinality of A_σ by m_σ.

Definition 5.1.7. A degree restriction is a function d from Σ to the nonnegative integer whose value on σ is denoted by d_σ. Let $s_1, \cdots, s_c$ be nonnegative integers, and let $\Delta(s_1, \cdots, s_c)$ be the set of all degree restrictions such that for all i, $1 \leqslant i \leqslant c$, and for all $\sigma \in \Sigma$, we have that $\sum\limits_{\sigma \in \Sigma, \sigma(i)=1} d_\sigma \geqslant s_i$ and for every σ, either $d_\sigma = 0$ or there exists i such that $\sigma(i) = 1$ and $\sum\limits_{\sigma \in \Sigma, \sigma(i)=1} d_\sigma = s_i$.

Theorem 5.1.8. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_c$ be prime monomial ideals. Then we have the following equation

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2} \cap \cdots \cap \mathfrak{p}_c^{s_c}) = \sum_{d \in \Delta(s_1, \cdots, s_c)} \prod_{\sigma \in \Sigma} N_{m_\sigma}^{d_\sigma}$$

In the second section, we give a polynomial upper bound on the number of generators of the intersection of the powers of two prime monomial ideals.

Theorem 5.2.8. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two prime monomial ideals. m_{01}, m_{10}, m_{11} are defined in Definition 5.1.4 above. For fixed m_{01}, m_{10}, m_{11} and $s_2 \gg s_1 \gg 0$, $\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2})$ is a polynomial function of s_1 and $s_2 - s_1$. Denote $a = m_{01} - 1$, $b = m_{10} - 1$, and $c = m_{11} - 1$. For $0 \leqslant v \leqslant a$, denote $\Phi_{a,b,c}(v) = \binom{a}{v} \sum\limits_{u=0}^{c} \frac{(-1)^u}{a+b+u-v+1} \binom{c}{u}$. We have $\Phi_{a,b,c}(v) > 0$ and

$$\begin{aligned}\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) \sim & \frac{1}{a!b!c!} \sum_{v=0}^{a} s_1^{a+b+c-v+1} (s_2 - s_1)^v \Phi_{a,b,c}(v) \\ & + \frac{1}{a!c!} \sum_{u=0}^{c} \frac{(-1)^{c-u} \binom{c}{u}}{a+c-u+1} s_2^u (s_2 - s_1)^{a+c-u+1}\end{aligned}$$

CHAPTER II

Conventions and technical background

In this chapter, we will establish some notations to be used throughout the thesis, as well as a review of the basic standard facts in commutative algebra. It may not contain new materials.

2.1 Conventions and basic notations

When we say a ring, we always require this ring to be a commutative associate ring with an identity 1.

When we say a local (semi-local) ring, we always require this ring to be a Noetherian ring with a unique (respectively, with finitely many) maximal ideal.

$\mathrm{Ass}(M)$ is the set of associated primes of M. See definition 2.5.17.

$\mathrm{Ann}_R(S)$ is the annihilator of S over R. See definition 2.5.16.

$\mathrm{bigheight}(I)$ is the big height of I. See definition 2.10.4.

$\mathfrak{an}(I)$ is the analytic spread of I. See definition 2.10.1.

$\dim(R)$ means the Krull dimension of R. See definition 2.5.22.

$\operatorname{depth}(M)$ means the depth of M. See definition 2.4.4.

$\operatorname{Frac}(R)$ is the fraction field of R. See definition 2.5.30.

$\operatorname{gr}_I(R)$ is the associated graded ring of R with respect to the ideal I. See definition 2.5.11.

$\operatorname{height}(I)$ is the height of I. See definition 2.5.23.

I is usually an ideal of a ring R.

I^e the extension of ideal I. See definition 2.7.8.

I^c the contraction of ideal I. See definition 2.7.8.

$I^{(\underline{s})}$ is the symbolic multi-powers of I. See definition 2.7.29.

$I^{(n)}$ is the nth-symbolic power of I. See definition 2.7.10.

K means a field.

$l(M)$ denotes the length of M. See definition 2.6.1.

M is usually a module over a ring R.

$\mathfrak{m}$ is a maximal ideal.

$\{M_{i_1,i_2,\cdots,i_{r-1},i_r}\}_{(i_1,i_2,\cdots,i_{r-1},i_r)\in\omega^r}$ denotes an ω^r-filtration of M. See definition 3.1.5.

$\mu(I)$ denotes the minimal number of generators of I. See definition 2.7.17.

$\mathbb{N}$ is the set of all non-negative numbers.

N_k^n is the number of non-negative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = n$. See definition 5.1.1.

$\mathfrak{p}$ is a prime ideal.

$\mathfrak{Q}$ is a primary ideal. See definition 2.7.3.

R is usually a ring.

$\mathbb{R}$ is the set of all real numbers.

R_a means the localization of R at $\{a^k\}_{k\geqslant 0}$. See definition 2.3.2.

$R_\mathfrak{p}$ means the localization of R at $R - \mathfrak{p}$. See definition 2.3.2.

$R_{[l_1,l_2,\cdots,l_r]}$ is a rectangular submodule of R. See definition 3.3.1.

$R_{\{d\}}$ is a triangular submodule of R. See definition 3.3.1.

$S^{-1}R$ means the localization of R at S. See definition 2.3.2.

$\mathcal{S}(I)$ is the symbolic Rees algebra of I. See definition 2.7.15.

$\operatorname{Rad}(I)$ is a radical of an ideal I. See definition 2.7.1.

$R[x]$ is a polynomial ring over R.

$(R, \mathfrak{m}, K)$ is a local ring. See definition 2.5.14.

σ_i is a projective map. See definition 3.3.5.

$sl(I)$ is the symbolic analytic spread of I. See definition 2.7.26.

$\operatorname{supp}(\mathfrak{p})$ is the support of $\mathfrak{p}$. See definition 2.7.28.

trdeg is transcendence degree. See definition 2.5.26.

ω^r is the set $\mathbb{N}^r$ identified with an ordinal number. See definition 3.1.4.

$f(x) = \mathcal{O}(g(x))$ means that f is dominated by g asymptotically. See definition 2.7.21.

$f(x) \sim g(x)$ means that f is equal to g asymptotically. See definition 2.7.25.

$\twoheadrightarrow$ means a surjective map.

$\hookrightarrow$ means an injective map.

$\cong$ means an isomorphism.

2.2 Integral and module-finite extensions

Definition 2.2.1. Let R be a commutative ring with a unit element and S be an R-algebra with structural homomorphism $f : R \to S$. We call that $s \in S$ is *integral over* R if there exists $d \in \mathbb{N}_+$ and $r_0, r_1, \cdots, r_{d-1} \in R$ such that we have

$$s^d = r_{d-1}d^{d-1} + \cdots + r_1 s + r_0 \tag{2.1}$$

We say that S is *integral over* R if s is integral over R for any $s \in S$.

Proposition 2.2.2. Let S be a ring, R a subring of S. The following are equivalent:

1. $s \in$ S is integral over R.
2. $R[s]$ is a finitely generated R-module.
3. $R[s]$ is contained in a subring S' of S such that S' is a finitely generated R-module.

Proof. See proposition 5.1 in the chapter 5 of the book [1]. □

Definition 2.2.3. If $R \subseteq S$ and S is integral over R, then S is said to be an *integral extension* of R. S is said to be module-finite over R if S is finitely generated as an R-module. If $R \subseteq S$ and S is module-finite over R, then S is said to be a *module-finite extension* of R.

Theorem 2.2.4. Let S be module-finite over the ring R. Then every element of S is integral over R.

Proof. See corollary 4.5 in the chapter 4 of the book [6]. □

Proposition 2.2.5. Let $R \to S \to T$ be ring homomorphisms such that S is module-finite over R with generators $s_1, s_2, \cdots, s_m$ and T is module-finite over S with generators $t_1, t_2, \cdots, t_n$. Then the composition of $R \to T$ is module-finite with mn generators $s_i t_j$ where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Proof. For any $t \in T$, there exists $a_j \in S$ such that $t = \sum_{j=1}^{n} a_j t_j$, and each a_j can be written as $\sum_{i=1}^{m} b_{ij} s_i$ for some $b_{ij} \in R$. Thus, we have the following equation.

$$t = \sum_{j=1}^{n} a_j t_j = \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ij} s_i t_j \tag{2.2}$$

This actually implies the proposition. $\square$

Corollary 2.2.6. The elements of S integral over R form a subring of S.

Proof. We can replace R by its image in S and assume $R \subseteq S$. For any two $s, t \in S$ which are integral over R. $R[s,t] = R[s][t]$ is integral is module-finite over $R[s]$ since t is integral over R and $R[s]$ is module-finite R. According to the previous proposition, we know that $R[s,t]$ is module-finite over R. According to theorem 2.2.4, we know $s \pm t$ and st are integral over R since they are in $R[s,t]$. $\square$

Theorem 2.2.7. Let S be an R-algebra. Then S is module-finite over R if and only if S is finitely generated as an R-algebra and integral over R. For S to be module-finite over R, it suffices that if S is generated over R by finitely many elements, each of which is integral over R.

Proof. According to theorem 2.2.4, we know that module-finite extensions are integral, and it is clear that they are finitely generated as R-algebras.

Without loss of generality, we suppose that $R \subset S$ and $S = R[s_1, s_2, \cdots, s_n]$. According to proposition 2.2.2, $R[s_1]$ is module-finite over R since s_1 is integral over R. Suppose that $R[s_1, \cdots, s_k]$ is module-finite over R where $1 \leqslant k < n$. We know $R[s_1, \cdots, s_k, s_{k+1}]$ is integral over $R[s_1, \cdots, s_k]$, thus $R[s_1, \cdots, s_k, s_{k+1}]$ is module-finite over $R[s_1, \cdots, s_k]$ which implies $R[s_1, \cdots, s_k, s_{k+1}]$ is integral over R according to proposition 2.2.5. By induction, we know S is module-finite over R. □

Definition 2.2.8. A union of a family of sets, subgroups, submodules, subrings or subalgebras is called a *directed union* if any two of them are contained in a third.

Corollary 2.2.9. *S is integral over R if and only if it is a directed union of module-finite extensions of R.*

Proof. If S is a directed union of module-finite extensions of R, then for any $s \in S$, s will be in one of the module-finite extensions and therefore s is integral over R. This implies that S is integral over R.

As we all know, S is the directed union of its finitely generated R-subalgebras, each of which will be module-finite over R. □

2.3 Normal rings and the Noether normalization theorem

Definition 2.3.1. The set of elements of $S \supseteq R$ that is integral over R is a ring according to corollary 2.2.6. This ring is said to be the *integral closure* of R in S. A domain R is called *normal* if every element of the faction field of R that is integral over R is in R.

Definition 2.3.2. A non-empty subset S of R that is closed under multiplication is called a *multiplicative system* of R. The *localization* of R at S is denoted by $S^{-1}R$. It is constructed by enlarging R to have inverses for the elements of S while changing R as little as possible in any other way.

Remark 2.3.3. For a prime ideal $\mathfrak{p}$, $R - \mathfrak{p}$ is a multiplicative system of R. In fact, for any $a, b \in R - \mathfrak{p}$, if $ab \notin R - \mathfrak{p}$, then $ab \in \mathfrak{p}$ implies either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ which is a contradiction to $a, b \in R - \mathfrak{p}$. The localization of R at $R - \mathfrak{p}$ is denoted by $R_{\mathfrak{p}}$.

Remark 2.3.4. For any nonzero $a \in R$, $S = \{a^k\}_{k\geqslant 1}$ is a multiplicative system. We denote $R_a = S^{-1}R$.

Proposition 2.3.5. Let R be a domain. The following are equivalent:

1. R is a normal domain.

2. For every prime ideal $\mathfrak{p}$, the local ring $R_{\mathfrak{p}}$ is a normal domain.

3. For every maximal ideal $\mathfrak{m}$, the ring $R_{\mathfrak{m}}$ is a normal domain.

Proof. See the proof of lemma 10.36.10 in chapter 10 of [20]. $\square$

Definition 2.3.6. A ring R is called *normal* if for every prime $\mathfrak{p} \subseteq R$ the localization $R_{\mathfrak{p}}$ is normal domain.

Lemma 2.3.7. A localization of a normal ring is a normal ring. Particularly, a localization of a normal domain is a normal domain.

Proof. See the proof of lemma 10.36.7 in chapter 10 of [20]. $\square$

Lemma 2.3.8. Let R be a normal ring. Then $R[x]$ is a normal ring where x is indeterminate. Particularly, if R is a normal domain, then $R[x]$ is a normal domain.

Proof. See the proof of lemma 10.36.8 in chapter 10 of [20]. $\square$

Definition 2.3.9. Let R be an A-algebra and $z_1, z_2, \cdots, z_d \in R$. We shall say that the elements $z_1, z_2, \cdots, z_d$ are *algebraically independent* over A if the unique A-algebra homomorphism from the polynomial ring $A[x_1, \cdots, x_d] \to R$ that sends x_i to z_i for $1 \leqslant i \leqslant n$ is injective.

Theorem 2.3.10. Let K be a field and let R be any finitely generated K-algebra. Then there are algebraically independent elements $z_1, z_2, \cdots, z_d$ in R such that R is module-finite over its subring $K[z_1, \ldots, z_d]$, which is isomorphic to a polynomial ring (d may be zero). That is, every finitely generated K-algebra is isomorphic with a module-finite extension of polynomial ring.

Proof. See the proof of theorem 13.3 in chapter 13 of the book [6]. □

Remark 2.3.11. Let D be an integral domain and let R be any finitely generated D-algebra extensions of D. Then there is a nonzero element $c \in D$ and elements $z_1, z_2, \cdots, z_d$ in R_c algebraically independent over D_c such that R_c is module-finite over its subring $D_c[z_1, z_2, \cdots, z_d]$, which is isomorphic to a polynomial ring (d may be zero) over D_c.

Lemma 2.3.12. Let A be a Noetherian integral domain. R is a finitely generated A-algebra. And, R is a domain. Then, there exists nonzero $a \in A$, elements $u_1, \cdots, u_m \in R$ algebraically independent over A_a, and elements $v_1, \cdots, v_n \in R$ integral over $A[u_1, u_2, \cdots, u_m]$ such that $R_a = A_a[u_1, u_2, \cdots, u_m][v_1, v_2, \cdots, v_n]$. It is easy to see that $m = \text{trdeg}_A(R)$. Furthermore, We denote

$$B = A[u_1, u_2, \cdots, u_m][v_1, v_2, \cdots, v_n] \tag{2.3}$$

There exists $f_1, f_2, \cdots, f_r \in B$ such that $R = B[f_1/a, f_2/a, \cdots, f_r/a]$.

Proof. According to the remark of theorem 2.3.10, there exists nonzero $a \in A$ and elements $u_1, u_2, \cdots, u_m$ in R_a algebraically independent over A_a such that R_a is module-finite over its subring $A_a[u_1, u_2, \cdots, u_m]$. Thus, there exists $v_1, v_2, \cdots, v_n \in R_a$ integral over $A_a[u_1, u_2, \cdots, u_m]$ such that $R_a = A_a[u_1, u_2, \cdots, u_m][v_1, v_2, \cdots, v_n]$.

We can require that $v_1, v_2, \cdots, v_n$ are the module-basis of R_a over $A_a[u_1, u_2, \cdots, u_m]$. Since if there is a linear relation between $v_1, v_2, \cdots, v_n$, we say $r_1v_1 + r_2v_2 + \cdots + r_nv_n = 0$, and we assume r_1 is nonzero without loss of generality, then we can replace a by ar_1, then $R_a = A_a[u_1, u_2, \cdots, u_m][v_2, \cdots, v_n]$. Thus, without loss of generality, we say $v_1, v_2, \cdots, v_n$ are module-basis.

We can also require $u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n$ are elements in R. In fact, for any $u_i, v_j \in R_a$, there exists $u_i', v_j' \in R$ such that $u_i = u_i'/a^{n_i}, v_i = v_i'/a^{n_j}$, then we have $R_a = A_a[u_1', u_2', \cdots, u_m'][v_1', v_2', \cdots, v_n']$. It is easy to see that $u_1', u_2', \cdots, u_m'$ are also algebraically independent over A_a and $v_1', v_2', \cdots, v_n'$ are module-basis. We may replace $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ by $u_1', u_2', \cdots, u_m'$ and $v_1', v_2', \cdots, v_n'$, respectively. Furthermore, we can require $v_1, v_2, \cdots, v_n$ are integral over $A[u_1, u_2, \cdots, u_m]$. For each v_i where $1 \leqslant i \leqslant n$, it is integral over $A_a[u_1, u_2, \cdots, u_m]$. There exists $r_0, r_1, \cdots, r_{s-1} \in A_a[u_1, u_2, \cdots, u_m]$ such that $v_i^s = r_0 + r_1 v_i + \cdots + r_{s-1} v_i^{s-1}$. As we all know, there exists N large enough such that each r_j has the form r'_j/a^N where $r'_j \in A[u_1, u_2, \cdots, u_m]$ for $1 \leqslant j \leqslant m$. Thus, we have

$$(a^N v_i)^s = a^{Ns} v_i^s = a^{N(s-1)} r'_0 + a^{N(s-2)} r'_1 (a^N v_i) + \cdots + r'_{s-1} (a^N v_i)^{s-1} \tag{2.4}$$

It is obvious that $a^N v_i$ is still the module-basis of module R_a over $A_a[u_1, u_2, \cdots, u_m]$. We may replace v_i by $a^N v_i$. Thus, each v_i is integral over $A[u_1, u_2, \cdots, u_m]$.

We denote $B = A[u_1, u_2, \cdots, u_m][v_1, v_2, \cdots, v_n]$. Then we have

$$B_a = A_a[u_1, u_2, \cdots, u_m][v_1, v_2, \cdots, v_n] = R_a \tag{2.5}$$

R is a finitely generated A-algebra, without loss of generality, we assume $R = A[\theta_1, \cdots, \theta_r]$. There exists $f_1, f_2, \cdots, f_r \in B$ and M large enough such that $\theta_i = f_i/a^M$ for $1 \leqslant i \leqslant r$. We may replace a by a^M. Then $R = B[f_1/a, \cdots, f_r/a]$. □

2.4 Depth and Cohen-Macaulay rings

Definition 2.4.1. Let R be a ring. Let M be an R-module. A sequence of elements $r_1, r_2, \cdots, r_n \in R$ is called an *M-regular* sequence if the following conditions hold:

1. r_i is a nonzero divisor on $M/(r_1, r_2, \cdots, r_{i-1})M$ for any $1 \leqslant i \leqslant n$, and

2. the module $M/(r_1, r_2, \cdots, r_n)M$ is not zero.

If I is an ideal of R and $r_1, r_2, \cdots, r_n \in I$ then we call $r_1, r_2, \cdots, r_n$ a *M-regular* sequence in I. If $M = R$, we simply call $r_1, r_2, \cdots, r_n$ a *regular* sequence.

Remark 2.4.2. The empty sequence is regular sequence on every nonzero module M.

Remark 2.4.3. If $r_1, r_2, \cdots, r_m, r_{m+1}, \cdots, r_n \in R$ is a regular sequence on M if and only if $r_1, r_2, \cdots, r_m$ is a regular sequence on M and $r_{m+1}, \cdots, r_n \in R$ is a regular sequence on $M/(r_1, r_2, \cdots, r_m)M$.

Definition 2.4.4. Let R be a ring, and $I \subseteq R$ an ideal. Let M be a finitely generated R-module. The *I-depth* of M, denoted by $\operatorname{depth}_I(M)$, is defined as follows:

1. if $IM \neq M$, then $\operatorname{depth}_I(M)$ is the supremum of the lengths of M-regular sequences in I,

2. if $IM = M$, we set $\operatorname{depth}_I(M) = \infty$.

If $(R, \mathfrak{m}, K)$ is local, we call $\operatorname{depth}_{\mathfrak{m}}(M)$ the *depth* of M which is denoted by $\operatorname{depth}(M)$.

Theorem 2.4.5. Let $(R, \mathfrak{m}, K)$ be a local ring. Then, the depth of R is at most the Krull dimension of R.

Definition 2.4.6. Let $(R, \mathfrak{m}, K)$ be a local ring. This ring is called *Cohen-Macaulay* if its depth is equal to its dimension.

Definition 2.4.7. Let R be a Noetherian ring. This ring is called *Cohen-Macaulay* if all of its localizations at maximal ideals (equivalently, at prime ideals) are Cohen-Macaulay.

2.5 Reductions of ideals in local rings

Definition 2.5.1. Let R be a Noetherian commutative ring with a unit element. We assume $\mathfrak{a}$ and $\mathfrak{b}$ are two proper ideals of R. We will call $\mathfrak{b}$ a *reduction* of $\mathfrak{a}$ if $\mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{b}\mathfrak{a}^r = \mathfrak{a}^{r+1}$ for at least one positive integer r.

Remark 2.5.2. It is easy to see that every ideal is a reduction of itself. Also, if $\mathfrak{b}\mathfrak{a}^r = \mathfrak{a}^{r+1}$ then $\mathfrak{b}\mathfrak{a}^n = \mathfrak{a}^{n+1}$ for all $n \geqslant r$ and $\mathfrak{b}^m\mathfrak{a}^r = \mathfrak{a}^{r+m}$ for all positive integers m.

Definition 2.5.3. A module M over a ring R is called *faithful* if for any $a \in R$, $a \neq 0$, then we have $aM \neq 0$.

Lemma 2.5.4. $a \in R$ is integral over $\mathfrak{a}$ if and only if $\mathfrak{a}$ is a reduction of $\mathfrak{a} + aR$.

Proof. $\mathfrak{a}$ is a reduction of $\mathfrak{a} + aR$ is equivalent to say that there exists a positive integer r such that $\mathfrak{a}(\mathfrak{a} + aR)^r = (\mathfrak{a} + aR)^{r+1}$. Also, we have

$$\mathfrak{a}(\mathfrak{a} + aR)^r = \mathfrak{a}(\mathfrak{a}^r + a\mathfrak{a}^{r-1} + \cdots + a^r\mathfrak{a}^{r-t} + \cdots + a^r R) \tag{2.6}$$

$$= \mathfrak{a}^{r+1} + a\mathfrak{a}^r + \cdots + a^r\mathfrak{a}^{r+1-t} + \cdots + a^r\mathfrak{a} \tag{2.7}$$

and

$$(\mathfrak{a} + aR)^{r+1} = \mathfrak{a}^{r+1} + a\mathfrak{a}^r + \cdots + a^r\mathfrak{a}^{r+1-t} + \cdots + a^r\mathfrak{a} + a^{r+1}R \tag{2.8}$$

Then $\mathfrak{a}(\mathfrak{a}+aR)^r = (\mathfrak{a}+aR)^{r+1}$ is equivalent to $a^{r+1} \in \mathfrak{a}^{r+1} + a\mathfrak{a}^r + \cdots + a^r\mathfrak{a}^{r+1-t} + \cdots + a^r\mathfrak{a}$, and this is precisely the condition for a to satisfy an equation of integral dependence on $\mathfrak{a}$ of degree $r + 1$. This actually proves the lemma. □

Lemma 2.5.5. If R is a domain, $I \subseteq J$ are two ideals of R and M is finitely generated faithful R-module such that $JM = IM$ then J is integral over I.

Proof. Let $u_1, \ldots, u_n$ be generators for M and μ be an element of J. Then for each j we can write $\mu u_j = \sum_{i=1}^{n} v_{ij}u_i$ where $v_{ij} \in I$. Let $\mathbb{I}$ denote the size n identity matrix, let B denote the size b matrix (v_{ij}). Let U be an n column vector whose entries are the u_i. Then, in matrix notation, we have $\mu U = BU$. It follows that $(\mu\mathbb{I} - B)U = 0$. Let C be the transpose of the cofactor matrix of $\mu\mathbb{I} - B$. Then $C(\mu\mathbb{I} - B) = D\mathbb{I}$, where $D = \det(\mu\mathbb{I} - B)$. So we have $DU = 0$, which means that D kills all the generators of M and M is faithful, it follows that $D = 0$. Now, we proved the lemma. □

Lemma 2.5.6. If J is integral over I in R is equivalent to $J(R/\mathfrak{p})$ is integral over $I(R/\mathfrak{p})$ for any minimal prime $\mathfrak{p}$ of R.

Proposition 2.5.7. Let R be a Noetherian commutative ring. $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ iff $\mathfrak{a}$ is integral over $\mathfrak{b}$.

Proof. Since R is a Noetherian ring and $\mathfrak{b} \subseteq \mathfrak{a}$, $\mathfrak{a}$ is finitely generated over $\mathfrak{b}$. First, we assume that $\mathfrak{a}$ is integral over $\mathfrak{b}$, we can prove it by the induction on the number of elements needed to generate $\mathfrak{a}$ over $\mathfrak{b}$. We say $\mathfrak{a}_1 = r_1R+r_2R+\cdots+r_nR+\mathfrak{b}$, and suppose that $\mathfrak{a}_1^{k+1} = \mathfrak{b}\mathfrak{a}_1^k$, we try to prove that $\mathfrak{a}^{m+1} = \mathfrak{b}\mathfrak{a}^m$ for some positive integer m where $\mathfrak{a} = r_1R+\cdots+r_nR+r_{n+1}R+\mathfrak{b}$. The previous lemma tells us that there exists some positive integer l such that $\mathfrak{a}^{l+1} = \mathfrak{a}_1\mathfrak{a}^l$. Then $\mathfrak{a}^{l+1}\mathfrak{a}_1^k = \mathfrak{a}_1^{k+1}\mathfrak{a}^l = \mathfrak{b}\mathfrak{a}_1^k\mathfrak{a}^l$, it follows that $\mathfrak{a}^{kl+1}\mathfrak{a}_1^k = \mathfrak{b}\mathfrak{a}^{kl}\mathfrak{a}_1^k$. Also we have $\mathfrak{a}^{(l+1)k} = \mathfrak{a}_1^k\mathfrak{a}^{kl}$. So $\mathfrak{a}^{kl+1}\mathfrak{a}_1^k = \mathfrak{a}(\mathfrak{a}^{(l+1)k}) = \mathfrak{b}\mathfrak{a}^{kl}\mathfrak{a}_1^k = \mathfrak{b}\mathfrak{a}^{(l+1)k}$. We can just choose $m = (l+1)k$.

Next, we assume that $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ which means $\mathfrak{a}^{m+1} = \mathfrak{b}\mathfrak{a}^m$ for some positive integer m. This equation still holds if we consider the images of $\mathfrak{a}$, $\mathfrak{b}$ modulo a minimal prime of R, and so if suffices to consider the case where R is a domain. We can also assume that $I \neq 0$. Otherwise, if $I = (0)$ the result is immediate. Thus, J^n is a faithful R-module. According to the previous lemma, we actually proved the proposition. $\square$

Remark 2.5.8. This proposition actually gives us another equivalent definition of reduction.

Definition 2.5.9. A *homogenous ideal* I in a graded ring $S = \bigoplus S_i$ is an ideal generated by a set of homogenous elements.

Definition 2.5.10. S is called a *standard graded* R-algebra if S is finitely generated over R and $\mathbb{N}$-graded with $S_0 = R$ and 1-forms S_1 of S generate S as an R-algebra. If S is a standard graded K-algebra, where K is a field, then S has a unique homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{n=1}^{\infty} S_n$.

Definition 2.5.11. Generally, if R is a commutative ring and I is an ideal of R, then the *associated graded ring*, denoted by $\mathrm{gr}_I(R)$, of R with respect to the ideal I is the $\mathbb{N}$-graded ring

$$R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots \tag{2.9}$$

so that the k-th graded piece is I^k/I^{k+1}. The multiplication is such that if $u \in I^j$ represents an element $\bar{u} \in I^j/I^{j+1}$ and $v \in I^k$ represents an element $\bar{v} \in I^k/I^{k+1}$, then uv represents the product $\bar{u}\bar{v} \in I^{j+k}/I^{j+k+1}$. Actually, this ring is generated by its forms of degree 1: moreover, given a set of generators of I as an ideal, the images of these elements in I/I^2 generate $\mathrm{gr}_I(R)$ as an R/I-algebra. It follows that $\mathrm{gr}_I(R)$ is Noetherian if R is Noetherian.

Theorem 2.5.12 (Nakayama's lemma, homogeneous form)**.** Let R be an $\mathbb{N}$-graded ring and let M be any $\mathbb{Z}$-graded module such that M has only finitely many nonzero negative components. Let I be the ideal of R generated by elements of positive degree. If $M = IM$, then $M = 0$. Hence, if N is a graded submodule such that $M = N + IM$, then $N = M$, and a homogeneous set of generators for M/IM generates M.

Lemma 2.5.13. Let $S \to T$ be a degree preserving K-algebra homomorphism of standard graded K-algebras. Let $\mathfrak{m} \subseteq S$ and $\mathfrak{n} \subseteq T$ be the homogeneous maximal ideals. Then T is a finitely generated S-module if and only if the image of S_1 in T_1 generates an $\mathfrak{n}$-primary ideal where S_1 and T_1 is degree 1 component of S and T, respectively.

Proof. By the homogeneous form of Nakayama's lemma, T is finitely generated as a module over S if and only if $T/\mathfrak{m}T$ is a finite-dimensional K-vector space, and this will hold if and only if and homogeneous components $[T/\mathfrak{m}T]_s$ are 0 for all large enough positive integer s, which holds if and only if $\mathfrak{n}^s \subseteq \mathfrak{m}T$ for all $s \gg 0$. $\square$

Definition 2.5.14. A local ring is a ring R that contains a single maximal ideal. We denote the local ring by $(R, \mathfrak{m}, K)$ where $\mathfrak{m}$ is the maximal ideal and $K = R/\mathfrak{m}$ is a field.

Proposition 2.5.15. Let $(R, \mathfrak{m}, K)$ be a local ring. If $I \subseteq J \subseteq \mathfrak{m}$ are ideals, then J is integral over I if and only if the image of I in $J/\mathfrak{m}J$ generates $\mathfrak{n}$-primary ideal in $T = K \otimes_R \mathrm{gr}_J(R)$, where $\mathfrak{n}$ is the homogeneous maximal ideal in T.

Proof. Note that J is integral over I if and only if $R[Jt]$ is integral over $R[It]$, and this is equivalent to the assertion that $R[Jt]$ is module-finite over $R[Jt]$, since $R[Jt]$ is finitely generated as an R-algebra, hence, as an $R[It]$-algebra.

If this holds, we have $K \otimes_R R[Jt]$ is finitely generated module over $K \otimes_R R[It]$, and, since the image of I generates the maximal ideal $\mathfrak{M}$ in $S = K \otimes_R \mathrm{gr}_I(R) \cong K \otimes_R R[It]$, the preceding lemma implies that the latter statement in proposition is true if and only if the image of I in $J/\mathfrak{m}J = [K \otimes_R \operatorname{gr}_J(R)]_1$ generates an $\mathfrak{n}$-primary ideal in $T = K \otimes_R \operatorname{gr}_J(R)$.

We will prove the proposition completely if we can show that when T is module-finite over S, then $R[Jt]$ is module-finite over $R[It]$. Let $j_1 \in J^{d_1}, \ldots, j_h \in J^{d_h}$ be elements whose images in $J^{d_1}/\mathfrak{m}J^{d_1}, \ldots, J^{d_n}/\mathfrak{m}J^{d_h}$, respectively, generate T as an S-module. We claim that $j_{d_1}t^{d_1}, \ldots, j_{d_h}t^{d_h}$ generate $R[Jt]$ over $R[It]$. To prove the claim, note that these elements generate T over S implies that for every N,

$$J^N = \sum_{1 \leqslant i \leqslant h \text{ such that } d_i \leqslant N} I^{N-d_i} j_{d_i} + \mathfrak{m}J^N \tag{2.10}$$

For each fixed N, we apply Nakayama's lemma to conclude that

$$J^N = \sum_{1 \leqslant i \leqslant h \text{ such that } d_i \leqslant N} I^{N-d_i} j_{d_i} \tag{2.11}$$

then, for all N, we have

$$J^N t^N = \sum_{1 \leqslant i \leqslant h \text{ such that } d_i \leqslant N} I^{N-d_i} t^{N-d_i} j_{d_i} t^{d_i} \tag{2.12}$$

It implies that $j_{d_1}t^{d_1}, \ldots, j_{d_h}t^{d_h}$ generate $R[Jt]$ over $R[It]$. □

Definition 2.5.16. Let R be a ring, and let M be a module over R. For a nonempty subset S of M. The annihilator of S, denoted by $\operatorname{Ann}_R(S)$, is the set of all elements $r \in R$ such that , for all $s \in S$, $rs = 0$.

Definition 2.5.17. Let R be a Noetherian ring. A prime ideal $\mathfrak{p}$ of R is called an *associated prime* of the R-module M if there is an element $m \in M$ whose annihilator is $\mathfrak{p}$. Or, equivalently, there is an injection $R/\mathfrak{p} \hookrightarrow M$.

Remark 2.5.18. The set of associated primes of M is denoted by $\operatorname{Ass}(M)$. If I is an ideal, we denote $\operatorname{Ass}(I)$ as the associated primes of I as an ideal and it should be the same as $\operatorname{Ass}(R/I)$.

Actually, $\mathrm{Ass}(R/I)$ is the same as the set of primes that occurs as radicals of primary ideas in an irredundant primary decomposition of I. That is the reason why we also call it an associated prime of I as an ideal. One important fact is that $\mathrm{Ass}(R/\mathfrak{p}) = \mathrm{Ass}(\mathfrak{p}) = \{\mathfrak{p}\}$ where $\mathfrak{p}$ is a prime ideal. And another useful fact is that $\mathrm{Ass}(M)$ is finite and non-empty if M is a nonzero Noetherian module.

Definition 2.5.19. A prime ideal $\mathfrak{p}$ is said to be a *minimal prime ideal* over an ideal I if it is minimal among all primes ideals containing I. A prime ideal is said to be a *minimal prime ideal* if it is a minimal prime ideal over the zero ideal.

Remark 2.5.20. If I is a prime ideal, then I is the only minimal prime over it.

Definition 2.5.21. The support of a module M over a ring R is the set of all prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}} \neq 0$. It is denoted by $\mathrm{Supp}(M)$.

Definition 2.5.22. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_d$ be a chain of prime ideals in a ring R. We call the integer d the *length* of the chain. The supremum of lengths of finite strictly ascending chains of prime ideals of R is called the *Krull dimension* of the ring R which is denoted by $\dim(R)$.

Definition 2.5.23. If $\mathfrak{p}$ is a prime ideal of R, by the *height* of $\mathfrak{p}$ we mean the supremum of lengths of finite strictly ascending chains of prime ideals contained in $\mathfrak{p}$. The height of any proper ideal I is the minimum of the heights of the prime ideals containing I which is denotes by $\mathrm{height}(I)$.

Remark 2.5.24. It should be clear that the dimensions of R is the same as the supremum of heights of all prime ideals, and that this will be the same as the supremum of heights of all maximal ideals.

Theorem 2.5.25. If $R \hookrightarrow S$ is an integral extension then $\dim(R) = \dim(S)$.

Proof. See the proof of lemma 11.26 in chapter 11 of the book [1]. $\square$

Definition 2.5.26. The *transcendence degree* of a field extension L/K is the largest cardinality of an algebraically independent subset of L over K. A subset S of L is a transcendence basis of L/K if it is algebraically independent over K and if furthermore L is an algebraic extension of the field $K(S)$ (the field obtained by adjoining the elements of S to K).

Remark 2.5.27. We can show that every field extension has a transcendence basis, and that all transcendence bases have the same cardinality. This cardinality is equal to the transcendence degree of the extension and is denoted by $\operatorname{trdeg}_K(L)$.

Definition 2.5.28. Let A be an integral domain. R is a finitely generated A-algebra and it is also an integral domain. The transcendence degree of R over A is the largest cardinality of an algebraically independent subset of R over A. It is the same as the transcendence degree of the field extension $\operatorname{Frac}(R)/\operatorname{Frac}(A)$.

Definition 2.5.29. An affine k-algebra is an integral domain that is also a finite-dimensional algebra over a field k.

Definition 2.5.30. R is an integral domain. Then we denote the fraction field of R by $\operatorname{Frac}(R)$.

Theorem 2.5.31. If R is an affine k-algebra, then $\dim(R) = \operatorname{trdeg}_k \operatorname{Frac}(R)$.

Theorem 2.5.32. If $\mathfrak{p}$ is a prime ideal of the affine k-algebra R, then $\operatorname{height}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$.

Theorem 2.5.33. Let M be an $\mathbb{N}$-graded module over an $\mathbb{N}$-graded Noetherian ring S. Then every associated prime of M is homogeneous. Hence, every minimal prime of the support of M is homogeneous and, in particular the associated(hence, the minimal) primes of S are homogeneous.

Proof. By definition, any associated prime $\mathfrak{p}$ of M is the annihilator of some elements u of M, and then every nonzero multiple of $u \neq 0$ can be thought of as a nonzero element of

$S/\mathfrak{p} \cong Su \subseteq M$, and so has annihilator $\mathfrak{p}$. If u_i is a nonzero homogeneous component of u of degree i, its annihilator J_i is a homogeneous ideal of S. If $J_h \neq J_i$ we can choose a form F in one and not the other, and then Fu is nonzero with fewer homogeneous components than u. Thus, the homogeneous ideals J_i are all equal to J and $J \subseteq \mathfrak{p}$. Suppose that $s \in \mathfrak{p} - J$ and subtract off all components of s that are in J, so that no nonzero component is in J. Let $s_a \notin J$ be the lowest degree component of s and u_b be the lowest degree component in u. Then $s_a u_b$ is the only term of degree $a + b$ occuring in $su = 0$, and so must be 0. But then $s_a \in \operatorname{Ann}_S u_b = J_b = J$, a contradiction. □

Corollary 2.5.34. Let S be a standard graded K-algebra of dimension d with homogeneous maximal ideal $\mathfrak{m}$, where K is an infinite field. Then there are forms $L_1, \ldots, L_d$ of degree 1 in S_1 such that $\mathfrak{m}$ is the radical of $(L_1, \ldots, L_d)S$.

Proof. The minimal primes of a graded algebra are homogeneous, and $\dim(S)$ is the same as $\dim(S/\mathfrak{p})$ for some minimal prime $\mathfrak{p}$ of R. Then $\mathfrak{p} \subset \mathfrak{m}$, and

$$\dim(S) = \dim(S/\mathfrak{p}) = \dim(S/\mathfrak{p})_{\mathfrak{m}} \leqslant \dim S_{\mathfrak{m}} \leqslant \dim(S) \tag{2.13}$$

so that $\dim(S) = \dim(S_{\mathfrak{m}}) = \operatorname{height} \mathfrak{m}$. If $\dim(S) = 0$, $\mathfrak{m}$ must be the unique minimal prime of S, and therefore it is nilpotent. Otherwise, S_1 can't be contained in the union of the minimal primes of S, or it will imply that it is contained in one of them, and S_1 generates $\mathfrak{m}$. Choose $L_1 \in S_1$ not in any minimal prime, and then $\dim(S/L_1) = d - 1$. We can prove the corollary by induction. If $L_1, \ldots, L_k$ have been chosen in S_1 such that $\dim(S/(L_1, \ldots, L_k)S) = d - k < d$, choose $L_{k+1} \in S_1$not in any minimal prime of $(L_1, \ldots, L_k)S$(if S_1 were contained in one of these, $\mathfrak{m}$ would be, and it would follow that height $\mathfrak{m} \leqslant k$, a contradiction). Thus, we have $L_1, \ldots, L_d$ such $\dim(S/(L_1, \ldots, L_d)S) = 0$, and then by the case where $d = 0$ we have that $\mathfrak{m}$ is nilpotent modulo $(L_1, \ldots, L_d)S$. □

Proposition 2.5.35. If $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then $\mathfrak{a}$ and $\mathfrak{b}$ have the same minimal prime ideals.[18]

Proof. We have $\mathfrak{b}\mathfrak{a}^r = \mathfrak{a}^{r+1}$ for some positive integer r by definition. It follows that $\mathfrak{a} \supseteq \mathfrak{b} \supseteq \mathfrak{a}^{r+1}$. Also if a prime ideal $\mathfrak{p} \supseteq \mathfrak{b}$, then $\mathfrak{p} \supseteq \mathfrak{a}^{r+1}$, which means that $\mathfrak{p} \supseteq \mathfrak{a}$. Now, we actually proved the proposition. □

Definition 2.5.36. A reduction $\mathfrak{b}$ of $\mathfrak{a}$ is called a *minimal reduction* of $\mathfrak{a}$ if there is no ideal strictly contained in $\mathfrak{b}$ which is a reduction of $\mathfrak{a}$.

Definition 2.5.37. An ideal is called a *basic ideal* if it doesn't have reduction other than itself.

Remark 2.5.38. An ideal which is a minimal reduction of a given ideal is a basic ideal. It follows from the following proposition.

Proposition 2.5.39. If $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ and $\mathfrak{c}$ is a reduction of $\mathfrak{b}$, then $\mathfrak{c}$ is a reduction of $\mathfrak{a}$.

Proof. It is easy to see that $\mathfrak{c} \subseteq \mathfrak{a}$. By definition, there exist positive integers r and s such that $\mathfrak{b}\mathfrak{a}^r = \mathfrak{a}^{r+1}$ and $\mathfrak{c}\mathfrak{b}^s = \mathfrak{b}^{s+1}$, then we have

$$\mathfrak{c}\mathfrak{a}^{r+s} = \mathfrak{c}\mathfrak{b}^s\mathfrak{a}^r = \mathfrak{b}^{s+1}\mathfrak{a}^r = \mathfrak{a}^{r+s+1} \tag{2.14}$$

□

Theorem 2.5.40. We fix a local ring $(R, \mathfrak{m}, K)$ such that K is infinite. If $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then there exists an ideal $\mathfrak{c}$ contained in $\mathfrak{b}$ which is a minimal reduction of $\mathfrak{a}$.

Proof. We need two lemmas first. As to the detailed proof, we can refer to the paper [18]. □

Lemma 2.5.41. If the ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are such that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 + \mathfrak{a}_1\mathfrak{m}$, then $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$.

Proof. From $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 + \mathfrak{a}_1\mathfrak{m}$, we have $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 + (\mathfrak{a}_2 + \mathfrak{a}_1\mathfrak{m})\mathfrak{m} \subseteq \mathfrak{a}_2 + \mathfrak{a}_1\mathfrak{m}^2$, it follows that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 + \mathfrak{a}_1\mathfrak{m}^k$ by induction where k is any positive integer. Thus,

$$\mathfrak{a}_1 \subseteq \bigcap_{k=1}^{\infty}(\mathfrak{a}_2 + \mathfrak{a}_1\mathfrak{m}^k) \subseteq \bigcap_{k=1}^{\infty}(\mathfrak{a}_2 + \mathfrak{m}^k) = \mathfrak{a}_2 \tag{2.15}$$

$\square$

Lemma 2.5.42. We have two ideals $\mathfrak{b} \subseteq \mathfrak{a}$, then $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ iff $\mathfrak{b}+\mathfrak{a}\mathfrak{m}$ is a reduction of $\mathfrak{a}$.

Proof. If $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then there exists r such that $\mathfrak{b}\mathfrak{a}^r = \mathfrak{a}^{r+1}$, it follows that $(\mathfrak{b}+\mathfrak{a}\mathfrak{m})\mathfrak{a}^r = \mathfrak{b}\mathfrak{a}^r + \mathfrak{m}\mathfrak{a}^{r+1} = \mathfrak{a}^{r+1}$, which means $\mathfrak{b}+\mathfrak{a}\mathfrak{m}$ is a reduction of $\mathfrak{a}$. Conversely, if $\mathfrak{b}+\mathfrak{a}\mathfrak{m}$ is a reduction of $\mathfrak{a}$, there exists r such that $(\mathfrak{b}+\mathfrak{a}\mathfrak{m})\mathfrak{a}^r = \mathfrak{a}^{r+1}$, we have $\mathfrak{b}\mathfrak{a}^r + \mathfrak{m}\mathfrak{a}^{r+1} = \mathfrak{a}^{r+1}$. According to the previous lemma, we have $\mathfrak{a}^{r+1} \subseteq \mathfrak{b}\mathfrak{a}^r$. And, it is easy to see, $\mathfrak{b}\mathfrak{a}^r \subseteq \mathfrak{a}^{r+1}$, then $\mathfrak{a}^{r+1} = \mathfrak{b}\mathfrak{a}^r$. $\square$

2.6 Finite filtrations

Definition 2.6.1. A *finite filtration* of an R-module M is a sequence $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ of submodules of n. The filtration is said to have *length* n. The modules M_{i+1}/M_i, $0 \leqslant i \leqslant n-1$ are called the *factors* of the filtration.

Definition 2.6.2. A nonzero module over a ring R is called *simple* if, equivalently, (1) it have no nonzero proper submodule or (2) it is isomorphic with $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

Definition 2.6.3. A module is said to have *finite length* if it has a filtration in which every factor is simple.

Remark 2.6.4. If M has finite length, the *length* $l(M)$ is defined to be the number of simple factors in any finite filtration such that all factors are simple or 0. It is well-defined because of Jordan-Hölder theorem.

Proposition 2.6.5. If we have a short exact sequence of modules,

$$0 \to M' \to M \to M'' \to 0 \tag{2.16}$$

then M has finite length iff both M' and M'' have finite length, and then $l(M) = l(M') + l(M'')$.

Proposition 2.6.6. Let M be a finitely generated R-module where R is Noetherian. Then

(a) If $u \neq 0$ is any element of M, one can choose $s \in R$ such that $\operatorname{Ann}_R su$ is a prime ideal $\mathfrak{p}$ of R, and $\mathfrak{p} \in \operatorname{Ass}(M)$. In particular, if $M \neq 0$, then $\operatorname{Ass}(M)$ is nonempty.

(b) If $ru = 0$ where $r \in R$ and $u \in M - \{0\}$, then one can choose $s \in R$ such that $\operatorname{Ann}_R(su) = \mathfrak{p}$. Note that $r \in \mathfrak{p}$. Consequently, the set of elements of R that are zerodivisors on M is the union of the set of associated primes of M.

(c) if $M \neq 0$, it has a finite filtration $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ in which all the factors M_i/M_{i-1} for $1 \leqslant i \leqslant n$ are prime cyclic modules, *i.e.*, have the form $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

Proof. (a) We consider a family of ideals $\{\operatorname{Ann}_R tu : t \in R \text{ and } tu \neq 0\}$ is nonempty since we may take $t = 1$. R is Noetherian, then R has ACC, thus it has a maximal element $\operatorname{Ann}_R su = \mathfrak{p}$. We claim that $\mathfrak{p}$ is prime. If $ab \in \mathfrak{p}$, then $absu = 0$. If $a \notin \mathfrak{p}$, we must have $b \in \mathfrak{p}$, otherwise $bsu \neq 0$, then $\operatorname{Ann}_R bsu$ containing $\mathfrak{p} + aR$ is strictly larger than $\mathfrak{p}$. This is a contradiction to the fact that $\mathfrak{p}$ is a maximal element in the family of ideals.

(b) From (a), we can choose $s \in R$ such that $\operatorname{Ann}_R su$ is a prime ideal $\mathfrak{p}$. Since $ru = 0$, we have $r \in \mathfrak{p}$. It implies that the set of elements of R that are zerodivisors on M is a subset of the union of the set of the set of associated primes of M. Furthermore, it is obvious that if $\mathfrak{p} = \operatorname{Ann}_R u$ with $u \in M$, then $u \neq 0$, and so every element of $\mathfrak{p}$ is a zerodivisor on M.

(c) Choose a sequence of elements $u_1, u_2, \cdots$ in M recursively as follows. Choose u_1 to be any element of M such than $\operatorname{Ann}_R u_1 = \mathfrak{p}_1$ is prime. If $u_1, u_2, \cdots, u_i$ have been chosen and $Ru_1 + Ru_2 + \cdots + Ru_i = M$, the sequence stops. If not, we can choose $u_{i+1} \in M$

such that its image $\overline{u_{i+1}}$ in $M/(Ru_1 + Ru_2 + \cdots + Ru_i)$ has annihilator $\mathfrak{p}_{i+1}$ that is prime. Let $M_i = Ru_1 + \cdots + Ru_i$. The sequence must stop, since the M_i are strictly increasing and M has ACC. By construction, the factors are prime cyclic modules.

□

2.7 Symbolic powers and symbolic multi-powers

Definition 2.7.1. Let I be an ideal of a ring R. The intersection of all the prime ideals of R that contain I is called the *radical* of I which is denoted as $\mathrm{Rad}(I)$. [1]

Remark 2.7.2. As we all know, $\mathrm{Rad}(I) = \{a \in R |$ there exists some $n \geqslant 1$ such that $a^n \in I\}$.

Definition 2.7.3. An ideal I in a ring R is called *primary* if whenever $ab \in I$ then either $a \in I$ or $b \in \mathrm{Rad}(I)$. If I is primary, $\mathrm{Rad}(I)$ is prime, say $\mathfrak{p}$. We can say that I is *primary* to $\mathfrak{p}$.

Remark 2.7.4. It's not true that I is primary simply because its radical is prime. See examples in chapter 4 of the book [1]. However, if $\mathrm{Rad}(I)$ is maximal, then I is primary. See the proposition 4.2 in chapter 4 of the book [1].

Definition 2.7.5. A *primary decomposition* of an ideal I is a representation of I as a finite intersection of some primary ideals *i.e.*, $I = \mathfrak{Q}_1 \cap \mathfrak{Q}_2 \cap \cdots \cap \mathfrak{Q}_c$ where $\mathfrak{Q}_i$ is primary for all $1 \leqslant i \leqslant c$. Furthermore, the decomposition is said to be *irredundant* if $\mathfrak{Q}_i$ are all distinct and we have $\mathfrak{Q}_i \not\supseteq \bigcap\limits_{j \neq i} \mathfrak{Q}_j$.

Theorem 2.7.6. Every proper ideal I of a Noetherian ring R has an irredundant primary decomposition.

Definition 2.7.7. If I, J are two ideals in a ring R, their *ideal quotient* is

$$(I : J) = \{r \in R | rJ \subseteq I\} \tag{2.17}$$

which is an ideal. Particularly, for any $a \in R$, we have $(I : a) = \{r \in R | ra \in I\}$.

Definition 2.7.8. Let $f : A \to B$ be a ring homomorphism. If I is an ideal in A, we define the extension I^e of I to be the ideal $Bf(I)$ generated by $f(I)$ in B. If J is an ideal of B, then $f^{-1}(J)$ is always an ideal of A, called the contraction J^c of J.

Theorem 2.7.9. Let R be an arbitrary commutative ring. $I \subseteq R$ is an ideal. If I has a primary decomposition, it has an irredundant one, say $I = \mathfrak{Q}_1 \cap \mathfrak{Q}_2 \cap \cdots \cap \mathfrak{Q}_c$. In this case the prime ideals $\mathfrak{p}_i = \mathrm{Rad}(\mathfrak{Q}_i)$ are distinct, by the definition of irredundant, and are uniquely determined. In fact, a prime $\mathfrak{p}$ occurs if and only if it has the form $\mathrm{Rad}(I : r)$ for some $r \in R$. Thus, the number of terms c is uniquely determined. The minimal elements among $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_c$, when intersected, give an irredundant primary decomposition of $\mathrm{Rad}(I)$, and are the same as the minimal primes of I. The primary ideal $\mathfrak{Q}$ in the decomposition corresponding to $\mathfrak{p}$, where $\mathfrak{p}$ is one of the minimal primes among $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_c\}$, is the contraction of $IR_{\mathfrak{p}}$ to R, and so is uniquely determined as well.

Proof. See the proof of theorem 4.5 in chapter 4 of the book [1]. □

Definition 2.7.10. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of dimension d and let I be an ideal of R. We can define the *nth-symbolic power* of I via the following formula[3].

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} (I^n R_{\mathfrak{p}} \cap R) \tag{2.18}$$

where $R_{\mathfrak{p}}$ is the localization of R to $\mathfrak{p}$ and the intersection runs through all of the minimal primes of I denoted by $\mathrm{Min}(I)$. To be clearer, $I^n R_{\mathfrak{p}} \cap R$ means the contraction of $I^n R_{\mathfrak{p}}$ to R.

Remark 2.7.11. Suppose we have $\mathrm{Min}(I) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_c\}$, the corresponding primary components of I are denoted by $\{\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_c\}$, it is easy to see that $I^{(n)} = \bigcap_{j=1}^{c} (\mathfrak{Q}_j^n R_{\mathfrak{p}_j} \cap R) = \bigcap_{j=1}^{c} \mathfrak{Q}_j^{(n)}$.

Remark 2.7.12. Once we have defined symbolic powers, we may wish they had some relations with the ordinary powers. They do not coincide with the ordinary powers in general, but they have deep relations. Actually, we can construct a counterexample. Let us first focus on symbolic powers of prime ideals. We say $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}^{(n)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$ by definition. In fact, $\mathfrak{p}^{(n)} = \{r \in R : \text{for some } w \in R - \mathfrak{p}, wr \in \mathfrak{p}^n\}$. We should have $\mathfrak{p}^n \subseteq \mathfrak{p}^{(n)}$ for all n, now we construct an example to explain why the converse fails.

Example 2.7.13 (F. S. Macaulay)**.** Let $R = K[x, y, z]$, the polynomial ring in three variables over a field. And map R by a K-algebra homomorphism onto $K[t^3, t^4, t^5] \subseteq K[t]$, where t is another variable, via $x \mapsto t^3$, $y \mapsto t^4$ and $z \mapsto t^5$. Of course, the kernel of the map denoted by $\mathfrak{p}$ is a prime ideal of R. And we can show that $\mathfrak{p} = (f, g, h)R$ where $f = xz - y^2$, $g = x^3 - yz$, and $h = x^3y - z^2$. If we consider $fh - g^2$ mod x, we can see it should be 0, which means that x divides $fh - g^2$. We say $xl = fh - g^2 \in \mathfrak{p}^2$ while $x \notin \mathfrak{p}$, by definition, we may have $l \in \mathfrak{p}^{(2)}$. And, l could not be in $\mathfrak{p}^2$. Actually, if we assign degrees to x, y, z so that x, y, z have degrees 3, 4, 5, respectively, then the generators $f^2, g^2, h^2, fg, gh, hf$ of $\mathfrak{p}^2$ all have degree 16 or more while l has degree 15.

Definition 2.7.14. The *Rees algebra* of an ideal I in a commutative ring R is defined to be

$$R[It] = \bigoplus_{n=0}^{\infty} I^n t^n \subseteq R[t] \tag{2.19}$$

Definition 2.7.15. Let I be an ideal, we can define the *symbolic Rees algebra* of I as $\mathcal{S}(I) := \bigoplus_{n \geqslant 0} I^{(n)} t^n$.

Remark 2.7.16. Generally speaking, $\mathcal{S}(I)$ is not Noetherian. There are many counterexamples. P. Roberts[19] found a counterexample based on the counterexamples of Nagata[17] to the 14th problem of Hilbert. However, for some interesting classes of ideals such as monomial ideals [11], $\mathcal{S}(I)$ is Noetherian.

Definition 2.7.17. Let I be an ideal. We denote by $\mu(I)$, the minimal number of generators of I, *i.e.*, the least element in the set $\{k \in \mathbb{N} |$ there exists $r_1, r_2, \cdots, r_k \in I$ such that $I = (r_1, r_2, \cdots, r_k)\}$. If the set is empty, we say $\mu(I) = \infty$.

Definition 2.7.18. Let R be a commutative ring and M be a R-module. We denote by $\mu(M)$, the minimal number of generators of M, *i.e.*, the least element in the set

$$\{k \in \mathbb{N} | \text{ there exists } r_1, r_2, \cdots, r_k \in M \text{ such that } M = r_1R + r_2R + \cdots + r_kR\} \tag{2.20}$$

If the set is empty, we say $\mu(M) = \infty$.

Proposition 2.7.19. Let M_1, M_2, M be R-modules. And we have the exact sequence defined by the following:

$$0 \to M_1 \to M \to M_2 \to 0 \tag{2.21}$$

Then we have $\mu(M) \leqslant \mu(M_1) + \mu(M_2)$.

Proposition 2.7.20. Let M be a R-module. Suppose M has a finite filtration denoted by the following:

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \tag{2.22}$$

Then, we have

$$\mu(M) \leqslant \sum_{i=1}^{n} \mu(\frac{M_i}{M_{i-1}}) \tag{2.23}$$

Definition 2.7.21. Let $f(x), g(x)$ be two functions of $x \in \mathbb{R}$. We say $f(x) = \mathcal{O}(g(x))$ if and only if there exists a positive real number M and a positive real number x_0 such that $|f(x)| \leqslant Mg(x)$ for all $x \geqslant x_0$.

Definition 2.7.22. For $s = (s_1, \cdots, s_n), s' = (s'_1, \cdots, s'_n) \in \mathbb{R}^n$, we call $s > s'$ if $s_i > s'_i$ for all $1 \leqslant i \leqslant n$, $s = s'$ if $s_i = s'_i$ for all $1 \leqslant i \leqslant n$, and $s < s'$ if $s_i < s'_i$ for all $1 \leqslant i \leqslant n$. Also, we call $s \to \infty$ if and only if $s_i \to \infty$ for all $1 \leqslant i \leqslant n$.

Definition 2.7.23. Let $f(s), g(s)$ be two functions of $s \in \mathbb{R}^n$. We say $f(s) = \mathcal{O}(g(s))$ if and only if there exists a positive real number M and $s' \in \mathbb{R}^n_+$ such that $|f(s)| \leqslant Mg(s)$ for all $s \geqslant s'$.

Remark 2.7.24. This a general version of definition 2.7.21.

Definition 2.7.25. Let $f(s), g(s)$ be two functions of $s \in \mathbb{R}^n$. We say $f(s) \sim g(s)$ if and only if for any $\epsilon > 0$, there exists $s' \in \mathbb{R}^n$ such that $|\frac{f(s)}{g(s)} - 1| < \epsilon$ for any $s \geqslant s'$, *i.e.*,

$$\lim_{s\to\infty} \frac{f(s)}{g(s)} = 1 \tag{2.24}$$

Definition 2.7.26. We denote by $sl(I)$, the *symbolic analytic spread* of I,

$$sl(I) = \min\{t|\mu(I^{(n)}) = \mathcal{O}(n^{t-1})\} \tag{2.25}$$

A priori, $sl(I)$ may be infinite. Actually, from the definition, $sl(I)$ gives control of the growth of least number of generators of $I^{(n)}$ as function of n.

Remark 2.7.27. It is a major open question whether $sl(I)$ is always finite.

Definition 2.7.28. As usual, let $R = k[x_1, \ldots, x_d]$ the polynomial ring over the field k and $\mathfrak{m} = (x_1, \ldots, x_d)$. For $(a_1, \ldots, a_d) \in \mathbb{N}^d$, let $\mathbf{x}^{\mathbf{a}}$ denote the monomial ideal of $x_1^{a_1} \cdots x_d^{a_d}$. For a monomial prime ideal $\mathfrak{p}$ we denote by $\operatorname{supp}(\mathfrak{p}) := \{i|x_i \in \mathfrak{p}\}$ the *support* of $\mathfrak{p}$.

Definition 2.7.29. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of dimension d and let I be an ideal of R. Suppose we have $\operatorname{Min}(I) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_c\}$, and the corresponding primary components of I are denoted by $\{\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_c\}$. We can define the *symbolic multi-powers* of I via the following formula

$$I^{(\underline{s})} = \bigcap_{j=1}^{c} \mathfrak{Q}_j^{(s_j)} \tag{2.26}$$

where $\underline{s}$ is $(s_1, \ldots, s_c) \in \mathbb{N}^c$.

Proposition 2.7.30. Let R be a Noetherian ring and $\mathfrak{p}$ be a prime ideal of R. We have $\operatorname{gr}_{\mathfrak{p}}(R) = R/\mathfrak{p} \oplus \mathfrak{p}/\mathfrak{p}^2 \oplus \mathfrak{p}^2/\mathfrak{p}^3 \oplus \cdots$. The ideal $\ker(\operatorname{gr}_{\mathfrak{p}}(R) \to (R-\mathfrak{p})^{-1}\operatorname{gr}_{\mathfrak{p}}(R))$ is a finitely generated ideal of $\operatorname{gr}_{\mathfrak{p}}(R)$. Thus, there exists $a \in R - \mathfrak{p}$ such that it kills this kernel. Let I_0 be the set of all such elements, *i.e.*, the set of elements that kill the $(R-\mathfrak{p})$-torsion in $\operatorname{gr}_{\mathfrak{p}}(R)$. Thus, I_0 is an ideal of R and for any $a \in I_0 - \mathfrak{p}$ we have that $\mathfrak{p}^{(n)} = \mathfrak{p}^n : a^n$. Generally, for any $a_1, a_2, \cdots, a_h \in I_0 - \mathfrak{p}$, we have $\mathfrak{p}^{(n)} = \mathfrak{p}^n : (a_1^h, \cdots, a_h^h)^n$.

Proof. It is easy to see that J is an ideal.

By definition, we have $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R = \{u \in R |$ there exists $c \in R - \mathfrak{p}$ such that $cu \in \mathfrak{p}^n\}$.

It is easy to see that $\mathfrak{p}^n : a^n \subseteq \mathfrak{p}^{(n)}$. For any $u \in \mathfrak{p}^n : a^n$, $ua^n \in \mathfrak{p}^n$, then $u \in \mathfrak{p}^{(n)}$ since $a \in R - \mathfrak{p}$ implies that $a^n \in R - \mathfrak{p}$.

Say $u \in \mathfrak{p}^{(n)} - \mathfrak{p}^n$, then there exists $c \in R - \mathfrak{p}$ such that $cu \in \mathfrak{p}^n$. There exists $0 \leqslant k \leqslant n-1$ such that $u \in \mathfrak{p}^k - \mathfrak{p}^{k+1}$ since $R \supseteq \mathfrak{p} \supseteq \mathfrak{p}^2 \supseteq \cdots$ and $u \notin \mathfrak{p}^n$. Consider $\overline{u} \in \mathfrak{p}^k/\mathfrak{p}^{k+1}$. It can be viewed as an element in $\mathrm{gr}_{\mathfrak{p}}(R)$. Since $cu \in \mathfrak{p}^n \subseteq \mathfrak{p}^{k+1}$, $\frac{\overline{u}}{1} = \frac{\overline{cu}}{c} = 0$. Furthermore, a kills the kernel, thus, we have $au \in \mathfrak{p}^{k+1}$. Now, there exists $l \geqslant k+1$ such that $au \in \mathfrak{p}^l - \mathfrak{p}^{l+1}$, by induction, we have $a^h u \in \mathfrak{p}^{k+h}$. Thus, $a^n u \in \mathfrak{p}^n$ which implies that $\mathfrak{p}^n : a^n \supseteq \mathfrak{p}^{(n)}$.

For any $u \in \mathfrak{p}^n : (a_1^h, \cdots, a_h^h)^n$, then $u(a_1^h, \cdots, a_h^h)^n \subseteq \mathfrak{p}^n$. Say $c = a_1^{hn} \in R - \mathfrak{p}$, according to previous argument, we have $a_1^n u \in \mathfrak{p}^n$. Thus, $\mathfrak{p}^n : (a_1^h, \cdots, a_h^h)^n \subseteq \mathfrak{p}^n : a_1^n$.

For any $u \in \mathfrak{p}^{(n)}$, we claim that $u(a_1^h, \cdots, a_h^h)^n \subseteq \mathfrak{p}^n$. In fact, for any element x in $(a_1^h, \cdots, a_h^h)^n$, there exists $1 \leqslant i \leqslant h$ and $b \in R$ such that $x = a_i^n b$. We know that $\mathfrak{p}^{(n)} = \mathfrak{p}^n : a_i^n$, thus, we have $ua_i^n \in \mathfrak{p}^n$ implies that $ux \in \mathfrak{p}^n$. □

Proposition 2.7.31. Let R be a Noetherian ring and $\mathfrak{p}_1, \cdots, \mathfrak{p}_k$ be prime ideals of R. For any $1 \leqslant i < j \leqslant k$, neither $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ nor $\mathfrak{p}_j \subseteq \mathfrak{p}_i$. Assume no element in $\mathrm{Ass}(R)$ strictly contains any of the $\mathfrak{p}_i$ (This is automatic if R is a domain). There exists $x \in R - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$ such that $\mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_k^{(n_k)} = (\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}) : x^n$ where $n = \max(n_1, \cdots, n_k)$.

Proof. Denote the set of elements that kill the $(R - \mathfrak{p}_i)$-torsion in $\mathrm{gr}_{\mathfrak{p}_i}(R)$ by J_i for any $1 \leqslant i \leqslant k$. Clearly, $\mathfrak{p}_i \subsetneq J_i$.

We claim that $J_i \nsubseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k \cup (\bigcup_{q \in \mathrm{Ass}(R)} q)$. Otherwise, there exists j such that $J_i \subseteq \mathfrak{p}_j$ or $q \in \mathrm{Ass}(R)$ such that $J_i \subseteq q$. Then we have $\mathfrak{p}_i \subsetneq J_i \subseteq \mathfrak{p}_j$ or $\mathfrak{p}_i \subsetneq J_i \subseteq q$ which is a contradiction.

Then there exists $x_i \in J_i - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k \cup (\bigcup_{q \in \mathrm{Ass}(R)} q)$ such that $\mathfrak{p}_i^{(n)} = \mathfrak{p}_i^n : x_i^n$ according to proposition 2.7.30.

Denote $x = x_1 \cdots x_k$. Since x_i is not a zero-divisor, x is not a zero-divisor. We claim that $\mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_k^{(n_k)} = (\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}) : x^n$ where $n = \max(n_1, \cdots, n_k)$.

Clearly, for any m, $\mathfrak{p}_i^{(m)} = \mathfrak{p}_i^m : x_i^m \subseteq \mathfrak{p}_i^m : x^m \subseteq \mathfrak{p}_i^{(m)}$ since $x \in R - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$. Then we have $\mathfrak{p}_i^{(n_i)} = \mathfrak{p}_i^{n_i} : x^{n_i} \subseteq \mathfrak{p}_i^{n_i} : x^n \subseteq \mathfrak{p}_i^{(n_i)}$.

In conclusion, we have that

$$\mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_k^{(n_k)} = (\mathfrak{p}_1^{n_1} : x^n) \cap \cdots \cap (\mathfrak{p}_k^{n_k} : x^n) \tag{2.27}$$

$$= (\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}) : x^n \tag{2.28}$$

□

Proposition 2.7.32. Let R be a Noetherian ring and $\mathfrak{p}_1, \cdots, \mathfrak{p}_k$ be prime ideals of R. For any $1 \leqslant i < j \leqslant k$, neither $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ nor $\mathfrak{p}_j \subseteq \mathfrak{p}_i$, *i.e.*, they are distinct and incomparable. Assume no $\mathfrak{p}_i$ is strictly contained in any associated prime $q \in \mathrm{Ass}(R)$. There exists $x \in R - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k \cup (\bigcup_{q \in \mathrm{Ass}(R)} q)$ such that $\mu(\mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_k^{(n_k)}) = \mu(\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k} \cap (x)^n)$ where $n = \max(n_1, \cdots, n_k)$.

Proof. According to Proposition 2.7.31, we have $\mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_k^{(n_k)} = (\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}) : x^n$.

Then we have

$$\mu(\mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_k^{(n_k)}) = \mu((\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}) : x^n) \tag{2.29}$$

$$= \mu(((\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}) : x^n) x^n) \tag{2.30}$$

$$= \mu(\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k} \cap (x)^n) \tag{2.31}$$

according to Lemma 2.7.34. □

Proposition 2.7.33. Let R be a Noetherian ring and $I = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_k$ where $\mathfrak{Q}_i$ is primary to $\mathfrak{p}_i$. The $\mathfrak{p}_i$ are mutually incomparable. Assume no $\mathfrak{p}_i$ is strictly contained in any associated prime $q \in \mathrm{Ass}(R)$. There exists a non zero-divisor $x \in R - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k \cup (\bigcup_{q \in \mathrm{Ass}(R)} q)$ such that $I^{(n)} = \bigcap \mathfrak{Q}_i^{(n)} = I^n : x^n$ for any n.

Proof. Let $W = R - \bigcup_i \mathfrak{p}_i$. In $W^{-1}R$, the $\mathfrak{p}_i W^{-1}R$ to maximal ideals and give all the maximal ideals. We have that

$$I^n W^{-1}R = (\mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_k)^n W^{-1}R \tag{2.32}$$

$$\supseteq \mathfrak{Q}_1^n \cdots \mathfrak{Q}_k^n W^{-1}R \tag{2.33}$$

$$= \mathfrak{Q}_1^n W^{-1}R \cap \cdots \cap \mathfrak{Q}_k^n W^{-1}R \tag{2.34}$$

We know that $I^{(n)} = I^n W^{-1}R \cap R = \mathfrak{Q}_1^{(n)} \cap \cdots \cap \mathfrak{Q}_k^{(n)}$.

I_0 kills all W-torsion in $\mathrm{gr}_I(R)$. We know that $I_0 \nsubseteq \bigcup \mathfrak{p}_i$. For any $x \in I_0 - \bigcup \mathfrak{p}_i$, we have $I^{(n)} = \bigcap \mathfrak{Q}_i^{(n)} = I^n : x^n$. □

Lemma 2.7.34. Let R be a commutative ring and I be an ideal of R. For any non zero-divisor $x \in R$, we have $(I : x)x = I \cap (x)$.

2.8 Subquotients

Definition 2.8.1. Let M be a R-module. A R-module M' is called a *subquotient* of M if there exists two submodules $M_1 \supseteq M_2$ of M such that M' has the form M_1/M_2.

Remark 2.8.2. Any submodule (and, hence any quotient) of M is a subquotient of M.

Lemma 2.8.3. A submodule M'' of a subquotient M' of M is a subquotient of M.

Proof. By definition, there exists $M_1 \supseteq M_2$ such that $M' = M_1/M_2$. Since M'' is a submodule of M', there exists a submodule M_1' of M_1 containing M_2 such that $M'' \cong M_1'/M_2$ implies that M'' is a subquotient of M. □

Remark 2.8.4. If A_1/A_2 is a subquotient of M, then $\frac{A_1 \cap N}{A_2 \cap N}$ is a subquotient of M. Actually, $\frac{A_1 \cap N}{A_2 \cap N}$ is a submodule of A_1/A_2.

Lemma 2.8.5. A subquotient M'' of a subquotient M' of M is a subquotient of M. Particularly, a quotient of a subquotient of M is a subquotient of M.

Proof. By definition, there exists $M_1 \supseteq M_2$ such that $M' = M_1/M_2$. Also, there exists M_1' and M_2' such that $M'' = M_1'/M_2'$. Since M_1' and M_2' are two submodules of $M' = M_1/M_2$, there exists two submodules M_1'' and M_2'' of M_1 containing M_2 such that $M_1' = M_1''/M_2$ and $M_2' = M_2''/M_2$. Thus, we have $M'' = M_1'/M_2' \cong M_1''/M_2''$ which implies that it is a subquotient of M. □

Lemma 2.8.6. If M has a finite flitration with factors M_j such that M_j is a subquotient of N_j, then the same is true for any subquotient of M.

Proof. For a submodule of M, the inherited filtration works: the factors are submodules of the original factors. For a quotient, the quotient filtration works: the factors are quotients of the factors in the original filtration. □

Remark 2.8.7. If a complex consists of modules with filtrations in which the factors are subquotients of certain modules $N_1, \ldots, N_h$, the same is true for homology. If the E_2 terms of the spectral sequence of a finite double complex have such a filtration, so does the homology of the total complex, since a finite associated graded module of that homology is obtained by repeatedly taking homology of the E_2 terms.

2.9 Bounds on the number of generators of submodules of one dimensional modules

Lemma 2.9.1. Given a finitely generated one-dimensional module M over a local ring R, there is a bound on the number of generators of all submodules (and, hence, of all subquotients) of M.

Proof. If N is a submodule of M, then $\hat{N} \subseteq \hat{M}$ has the same number of generators as N. We may replace R and M by their completions $\hat{R}$ and $\hat{M}$, respectively.

We may take a prime cyclic filtration of M. This induces a filtration of any submodule N of M whose factors are submodules of $R/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of R. Then, the bound on the number of generators of all submodules of $R/\mathfrak{p}$ give us a bound on the number of generators of all submodules of M since there are only finitely many distinct $\mathfrak{p}$. By construction, we have $M_i/M_{i-1} \cong R/\mathfrak{p}$ implies that $\dim(R/\mathfrak{p}) \leqslant \dim(M) \leqslant 1$.

1. if $\dim(R/\mathfrak{p}) = 0$, then $R/\mathfrak{p}$ is a field. It is easy to see the bound is 1.
2. if $\dim(R/\mathfrak{p}) = 1$, as we all know, $R/\mathfrak{p}$ is a complete local ring, then $R/\mathfrak{p}$ is module finite over V (structure theory) where V is a complete regular one-dimensional local domain *i.e.*, V is DVR implies that V is PID. Then, we have $R/\mathfrak{p} \cong V^r$ where r is the rank of $R/\mathfrak{p}$ over V since a finitely generated module over a PID is free if and only if it is torsion-free. All submodules of V^r are free over V of rank less than or equal to r and need at most r generators over V, hence, they need at most r generators over R.

□

Remark 2.9.2. One can generalize to the case where R is semi-local: after completion, R becomes a finite product of local rings, and the module becomes a product and so has a filtration in which each factor is a module over a local ring.

There is no bound on generators of ideals for a Noetherian ring that is not semi-local in general, although this is true for a ring that is finitely generated over a field. This reduces to the domain case (dimension one) and then the ring is module-finite over a polynomial ring in one variable and the same argument works.

2.10 Analytic spread of ideals

Definition 2.10.1. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring. For any ideal I in R, the *analytic spread* of I, which is denoted by $\mathfrak{an}(I)$, is defined to be the Krull dimension of $R[It]/\mathfrak{m}R[It] \cong R/\mathfrak{m} \oplus (I/\mathfrak{m}I)t \oplus (I^2/\mathfrak{m}I^2)t^2 \oplus \cdots \cong R/\mathfrak{m} \otimes_R R[It]$. [21]

Remark 2.10.2. Note that $(R/\mathfrak{m}) \otimes_R (R/I) \cong R/I$, it follows that

$$R/\mathfrak{m} \otimes_R R[It] \cong ((R/\mathfrak{m}) \otimes_R (R/I)) \otimes_R R[It] \tag{2.35}$$

$$\cong (R/\mathfrak{m}) \otimes_R ((R/I) \otimes_R R[It]) \tag{2.36}$$

$$\cong R/\mathfrak{m} \otimes_R \operatorname{gr}_I(R) \tag{2.37}$$

which means that $l(I)$ can be also defined to be the Krull dimension of $R/\mathfrak{m} \otimes_R \operatorname{gr}_I(R)$. In the rest of this section, the ring means a local ring.

Proposition 2.10.3. The analytic spread of a nilpotent ideal is 0.

Proof. First we assume that I is a nilpotent ideal which means that there exists a positive integer n such that $I^n = 0$. Then $R/\mathfrak{m} \otimes R[It]$ is a finite dimensional vector space over field $R/\mathfrak{m}$. The Krull dimension of finite dimensional vector space is 0. □

Definition 2.10.4. The *big height* of a proper ideal I of a Noetherian ring is defined to be the largest height of any minimal prime of I.

Remark 2.10.5. Notice that the height of a proper ideal I is the smallest height of any minimal prime of I. It is obvious that height less than big height. Also, the height of an $\mathfrak{m}$-primary ideal is the same as its big height.

Theorem 2.10.6. Let $(R, \mathfrak{m}, K)$ be local and $J \subseteq R$ an ideal. Then any reduction I of J has at least $\mathfrak{an}(J)$ generators. Moreover, if K is infinite, there is a reduction with $\mathfrak{an}(J)$ generators.

Proof. According to Proposition 2.5.15, the problem of giving $i_1, \dots, i_h \in J$ such that J is integral over $(i_1, \dots, i_h)R$ is equivalent to giving h elements of $J/\mathfrak{m}J$ that generate an $\mathfrak{M}$-primary ideal of $S = K \otimes_R \operatorname{gr}_J(R)$, where $\mathfrak{M}$ is the homogeneous maximal ideal of S. We have $h \geqslant \dim(S) = \mathfrak{an}(J)$. If K is infinite, the existence of suitable elements follows from the Corollary 2.5.34. □

Lemma 2.10.7. Let $(R, \mathfrak{m}, K)$ be a local ring and J be any proper ideal, then $\dim(R) = \dim(\operatorname{gr}_J R)$. It follows that $\dim(K \otimes_R \operatorname{gr}_J(R)) \leqslant \dim(R)$.

Proposition 2.10.8. For any proper ideal I of $(R, \mathfrak{m}, K)$, the analytic spread of I lies between the big height of I and $\dim(R)$. In particular, the analytic spread of an $\mathfrak{m}$-primary ideal is the dimension of the ring.

Proof. From the preceding lemma, we have $\mathfrak{an}(J) \leqslant \dim(R)$. Let $\mathfrak{p}$ be a minimal prime of J. We want to show that height $\mathfrak{p} \leqslant \mathfrak{an}(J)$. According to previous theorem, we have that J is integral over an ideal I with $\mathfrak{an}(J)$ generators. Then J is contained in the radical of I. In $R_\mathfrak{p}$ we have that $\mathfrak{p}$ is the radical of $JR_\mathfrak{p}$, since $\mathfrak{p}$ is a minimal prime of J, and so is contained in the radical of $IR_\mathfrak{p}$. Thus, height $\mathfrak{p} \leqslant \mathfrak{an}(J)$, as desired. □

2.11 Multi-Tors

We follow the discussion in [8] here.

Definition 2.11.1. Let R be a commutative ring. Given k modules $M_1, \cdots, M_k$ over ring R, define the R-module $\mathrm{Tor}_h^R(M_1, \ldots, M_k)$ by choosing a projective resolution $G_\bullet^{(i)}$ for each M_i, tensoring together all of these projective resolutions, with the modules M_i removed as usual, taking the total complex of this tensor product, and taking the homology of the total complex

Remark 2.11.2. One obtains a functor of several R-modules $M_1, \cdots, M_k$, covariant in each of the M_i.

Remark 2.11.3. For $k \geqslant 2$, this construction has many of the same properties as the usual 2-variables Tor. For example:

1. A short exact sequence in any of the variables yields a long exact sequence as usual when the other modules are held fixed.

2. $\mathrm{Ann}_R\, M_i$ also kills $\mathrm{Tor}_h^R(M_1, \cdots, M_k)$.

3. $\mathrm{Tor}_0^R(M_1, \cdots, M_k) = M_1 \otimes_R \cdots \otimes_R M_k$.

4. If all but one of the M_i is flat, then $\mathrm{Tor}_h^R(M_1, \cdots, M_n) = 0$ for all $h > 0$.

5. If the M_i are all finitely generated over a Noetherian ring R, so is $\mathrm{Tor}_h^R(M_1, \cdots, M_k)$.

6. Let R be a Noetherian ring. If $S_1, \cdots, S_k$ are finitely generated R-algebras and M_i is a finitely generated S_i-module, $1 \leqslant i \leqslant k$, then $\mathrm{Tor}_h^R(M_1, \cdots, M_k)$ is a finitely generated module over $S = S_1 \otimes_R \cdots \otimes_R S_k$.

Proof. We only prove the sixth property. Let T_i be a polynomial ring in finite many variables over R such that $S_i \cong T_i/J_i$. Let $G_\bullet^{(i)}$ be a free resolution of M_i over T_i such that modules in $G_\bullet^{(i)}$ are finitely generated T_i-modules. The $G_\bullet^{(i)}$ are also free over R since T_i is a polynomial ring over R. Let $G_\bullet$ be the total tensor product of the $G_\bullet^{(i)}$ over R. Then $\mathrm{Tor}_\bullet^R(M_1, \cdots, M_k) \cong H_\bullet(G_\bullet)$. The modules in $G_\bullet$ are finitely generated and free over $T = T_1 \otimes_R \cdots \otimes_R T_k$. Therefore $H_\bullet(G_\bullet)$ is finitely generated over T. For each i, we have a surjection $T_i \twoheadrightarrow S_i$ with kernel J_i. Since J_i kills M_i, it follows that $H_\bullet(G_\bullet)$ is a module over $T/(J_1T, J_2T, \cdots, J_kT) \cong S$. $\square$

CHAPTER III

Filtration theorems

In this Chapter, we discuss the notion of ω^r-filtrations. We also prove several useful properties of ω^r-filtrations and prove the existence of ω^r-filtration with an important property that we will describe later in this Chapter.

In Section 3.2, we give an explicit construction of ω^r-filtrations. For several particular cases, we calculate the factors of these ω^r-filtrations.

Last but not least, we introduce the definition of rectangularly and triangularly normal ω^r-filtrations. Futhermore, we construct rectangularly and triangularly normal ω^r-filtrations in several particular cases.

We will use the results that appears in this Chapter to prove main theorems in Chapter IV.

3.1 Introduction to ω^r-filtrations

In this section, we first discuss the notion of ω^r-filtrations. Then we prove several useful properties of ω^r-filtrations and prove the existence of ω^r-filtration with an important property that we will describe later in this section.

Definition 3.1.1. Let M be a R-module. We define recursively the notion of an ω^r-filtration of M. If $r = 1$, an ω-filtration of M is just an ascending sequence of submodules denoted by the following.

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \tag{3.1}$$

where M_i is a submodule of M and $\bigcup_{i=0}^{\infty} M_i = M$. Recursively, if we have already defined an ω^{r-1}-filtration of an arbitrary R-module for $r \geqslant 2$, an ω^r-filtration of M is an ascending sequence of submodules denoted by $M_0, M_1, M_2, \cdots$ such that $\bigcup_{i=0}^{\infty} M_i = M$, and each M_i/M_{i-1} has an ω^{r-1}-filtration.

Remark 3.1.2. We can also recursively define the factors of an ω^r-filtration of M. If $r = 1$, the factor has the form M_{i+1}/M_i where $i \geqslant 0$. For $r > 1$, the factor of the ω^r-filtration of M is actually the factor of the ω^{r-1}-filtration of M_{i+1}/M_i where $i \geqslant 0$.

Proposition 3.1.3. Let M be a R-module with a submodule M'. M/M' has an ω^r-filtration is equivalent to that there is an ω^r-filtration from M' to M.

Proof. This simply comes from the Noether correspondence between submodules of M/M' and submodules of M containing M' gotten by taking inverse images under $\phi : M \twoheadrightarrow M/M'$. We also need the following fact. $\bigcup_{i=0}^{\infty} M_i = M$ implies $\bigcup_{i=0}^{\infty} \phi(M_i) = M/M'$ where M_i is a submodule of M. $\bigcup_{i=0}^{\infty} \overline{M_i} = M/M'$ implies $\bigcup_{i=0}^{\infty} \phi^{-1}(M_i) = M$ where $\overline{M_i}$ is a submodule of M/M'. □

Definition 3.1.4. We can define a totally ordered set denoted by ω^r. $(i_1, i_2, \cdots, i_r) < (i'_1, i'_2, \cdots, i'_r)$ if for some k, $1 \leqslant k \leqslant r$, $i_t = i'_t$ for $t < k$ and $i_k < i'_k$.

Definition 3.1.5. According to Proposition 3.1.3, we have an alternative definition of an ω^r-filtration. An ω^r-filtration of M can be denoted by $\{M_{i_1,i_2,\cdots,i_{r-1},i_r}\}_{(i_1,i_2,\cdots,i_{r-1},i_r)\in\omega^r}$.

$$\begin{aligned} 0 = M_{0,0,\cdots,0,0} &\subseteq M_{0,0,\cdots,0,1} \subseteq M_{0,0,\cdots,0,2} \subseteq \cdots \\ &\subseteq M_{0,0,\cdots,1,0} \subseteq M_{0,0,\cdots,1,1} \subseteq M_{0,0,\cdots,1,2} \subseteq \cdots \\ &\subseteq M_{0,0,\cdots,2,0} \subseteq M_{0,0,\cdots,2,1} \subseteq M_{0,0,\cdots,2,2} \subseteq \cdots \\ &\cdots \end{aligned}$$

$$
\begin{aligned}
&\subseteq M_{1,0,\cdots,0,0} \\
&\cdots \\
&\subseteq M_{2,0,\cdots,0,0} \\
&\cdots
\end{aligned}
$$

where $\bigcup_{i_1=0}^{\infty} M_{i_1,0,\cdots,0} = M$. Factors have the form $M_{i_1,i_2,\cdots,i_{r-1},i_r+1}/M_{i_1,i_2,\cdots,i_{r-1},i_r}$ for any $(i_1, i_2, \cdots, i_r) \in \omega^r$.

Remark 3.1.6. Particularly, if $r = 2$, we have

$$
\begin{aligned}
0 = M_{0,0} &\subseteq M_{0,1} \subseteq M_{0,2} \subseteq \cdots \\
&\subseteq M_{1,0} \subseteq M_{1,1} \subseteq M_{1,2} \subseteq \cdots \\
&\subseteq M_{2,0} \subseteq M_{2,1} \subseteq M_{2,2} \subseteq \cdots \\
&\subseteq \cdots
\end{aligned}
$$

where $\bigcup_{i_1=0}^{\infty} M_{i_1,0} = M$.

Proposition 3.1.7. Let M be a R-module. Let M' and N are submodules of M. If there is an ω^r-filtration from M' to M, then there is an ω^r-filtration from $M' \cap N$ to $N = M \cap N$. Particularly, an ω^r-filtration of M deduces an ω^r-filtration of $N = M \cap N$. We also call this ω^r-filtration of N is inherited from the ω^r-filtration of M.

Proof. If $r = 1$, an ω-filtration from M' to M is denoted by the following.

$$M' = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \tag{3.2}$$

where M_i is a submodule of M containing M' and $\bigcup_{i=0}^{\infty} M_i = M$. We denote $N_i = M_i \cap N$. We have

$$M' \cap N = N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \tag{3.3}$$

and $\bigcup_{i=0}^{\infty} N_i = \bigcup_{i=0}^{\infty}(M_i \cap N) = M \cap N$. Recursively, suppose the proposition holds for any $1 \leqslant r \leqslant s$ where $s \geqslant 1$. An ω^{s+1}-filtration from M' to M is denoted by the following.

$$M' = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \tag{3.4}$$

where M_i is a submodule of M containing M' and $\bigcup_{i=0}^{\infty} M_i = M$. We denote $N_i = M_i \cap N$. We have an ascending sequence of submodules $M' \cap N = N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ with $\bigcup_{i=0}^{\infty} N_i = \bigcup_{i=0}^{\infty}(M_i \cap N) = M \cap N$. For each $i \geqslant 0$, by definition, there is an ω^s-filtration from M_i to M_{i+1} which implies there is an ω^s-filtration from $M_i \cap N = N_i$ to $M_{i+1} \cap N = N_{i+1}$. By definition, there is an ω^{s+1}-filtration from $M' \cap N$ to $M \cap N$. $\square$

Definition 3.1.8. Let M be a R-module, we say M has a general ω^r-filtration, if there is a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M \tag{3.5}$$

such that each M_i/M_{i-1} has an ω^r-filtration where $1 \leqslant i \leqslant n$.

Remark 3.1.9. It is obvious that an ω^r-filtration of M is also a general ω^r-filtration of M. And a general ω^r-filtration corresponds to an ω^{r+1}-filtration of M in which the ω^r-filtration submodules are eventually all the same. The factor of a general ω^r-filtration is defined similarly to the factor of an ω^r-filtration. Furthermore, Proposition 3.1.3 and Proposition 3.1.7 hold for general ω^r-filtrations.

Lemma 3.1.10. Let A be a Noetherian commutative ring, and B_1 be a finitely generated A-module. We have the following surjective morphism over A-modules.

$$B_1 \twoheadrightarrow B_2 \twoheadrightarrow B_3 \twoheadrightarrow B_4 \twoheadrightarrow \cdots \tag{3.6}$$

We claim that there exists $T \geqslant 1$ such that $B_T \cong B_{T+1} \cong B_{T+2} \cong \cdots$.

Proof. We have the following surjective map via composition.

$$\phi_i : B_1 \twoheadrightarrow B_i \tag{3.7}$$

where $i \geqslant 1$. For any $i < j$, we have $\ker \phi_i \subseteq \ker \phi_j$ since ϕ_j is a composition of $\phi_i : B_1 \twoheadrightarrow B_i$ and $f : B_i \twoheadrightarrow B_j$. Thus, we have

$$\ker \phi_1 \subseteq \ker \phi_2 \subseteq \ker \phi_3 \subseteq \ker \phi_4 \subseteq \cdots \tag{3.8}$$

Since B_1 has ACC, the ascending chain will be eventually stable, *i.e.*, there exists $T \geqslant 1$ such that $\ker \phi_T \cong \ker \phi_{T+1} \cong \ker \phi_{T+2} \cong \cdots$. According to isomorphism theorems, we have

$$\frac{B_1}{\ker \phi_i} \cong B_i \tag{3.9}$$

which means that there exists $T \geqslant 1$ such that $B_T \cong B_{T+1} \cong B_{T+2} \cong \cdots$. $\square$

Lemma 3.1.11. Let A be a Noetherian commutative ring, and R be an A-algebra generated by an element θ. $R = \overline{A}[\theta]$ where $\overline{A}$ is an image under the map $A[x] \twoheadrightarrow R$ with $x \to \theta$. Then R has an ω-filtration over A such that by prime cyclic A-modules involving only finitely many prime ideal of A.

Proof. We may replace A by its image in R and assume $R = A[\theta]$. Consider the following filtration

$$A \subseteq A + A\theta \subseteq A + A\theta + A\theta^2 \subseteq A + A\theta + A\theta^2 + A\theta^3 \subseteq \cdots \tag{3.10}$$

We claim that there are only finitely many distinct factors in this filtration. Let us denote the factor $\frac{A+A\theta+\cdots+A\theta^{i-1}+A\theta^i}{A+A\theta+\cdots+A\theta^{i-1}}$ by B_i where $i \geqslant 1$. Of course, B_i is an A-module. We have the following morphism:

$$\begin{aligned} B_i &\longrightarrow B_{i+1} \\ \overline{b} &\longrightarrow \overline{\overline{b\theta}} \end{aligned} \tag{3.11}$$

First, it is well-defined. Since for any element $a \in A + A\theta + \cdots + A\theta^{i-1}$, we have $a\theta \in A + A\theta + \cdots + A\theta^{i-1} + A\theta^{i}$. Second, the morphism is surjective. Actually, any element in B_{i+1} has the form $\overline{\overline{a\theta^{i+1}}}$ where $a \in A$, as we know $\overline{a\theta^{i}} \to \overline{\overline{a\theta^{i+1}}}$. Thus it is a surjective morphism. Then we have the following surjective morphism.

$$B_1 \twoheadrightarrow B_2 \twoheadrightarrow B_3 \twoheadrightarrow B_4 \twoheadrightarrow \cdots \tag{3.12}$$

B_1 is an A-module generated by one element. According to Lemma 3.1.10, there are only finitely many distinct B_i. Now we can prove the lemma. B_i can be viewed as an A-module with only one generator. According to Proposition 2.6.6, each B_i has a finite filtration in which all the factors are prime cyclic A-modules. Suppose we have the following filtration

$$0 = B_{i,0} \subseteq B_{i,1} \subseteq B_{i,2} \subseteq \cdots \subseteq B_{i,i_n} = B_i \tag{3.13}$$

then we can lift to the following filtration

$$A + A\theta + \cdots + A\theta^{i-1} = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{i_n} = A + A\theta + \cdots + A\theta^{i-1} + A\theta^{i} \tag{3.14}$$

Actually, we have $A + A\theta + \cdots + A\theta^{i-1} + A\theta^{i} \twoheadrightarrow B_i$, then C_j is the inverse image of $B_{i,j} \subseteq B_i$ where $0 \leqslant j \leqslant i_n$. According to isomorphism theorems, we have

$$C_j/C_{j-1} \cong B_{i,j}/B_{i,j-1} \tag{3.15}$$

Now, we actually construct a countable filtration with only finitely many distinct prime cyclic modules. □

Proposition 3.1.12. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. Then R has an ω^r-filtration in which all the factors are prime cyclic A-modules. Also, only finitely many distinct prime cyclic modules occur.

Proof. According to Lemma 3.1.11, we know the proposition holds when $r = 1$. Suppose the proposition holds when $r = j$. We claim that it still holds for $r = j + 1$. We denote $A[\theta_1, \theta_2, \cdots, \theta_j]$ by S_j, then $S_{j+1} = S_j[\theta_{j+1}]$ can be viewed as a S_j-algebra. According to Hilbert Theorem, S_j is a Noetherian ring. Thus, S_{j+1} has an ω-filtration in which all the factors have the form $S_j/\mathfrak{Q}_i$ where $\mathfrak{Q}_i$ is prime. As we know, only finitely many distinct factors occur. We have the following natural isomorphism

$$S_j/\mathfrak{Q}_i = \frac{A}{A \cap \mathfrak{Q}_i}[\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_j}] = \frac{A}{\mathfrak{p}_i}[\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_j}] \tag{3.16}$$

where $\mathfrak{p}_i = A \cap \mathfrak{Q}_i$ is prime. We can view $S_j/\mathfrak{Q}_i$ as a $A/\mathfrak{p}_i$-algebra. As we all know, $A/\mathfrak{p}_i$ is Noetherian since A is Noetherian. By assumption, $S_j/\mathfrak{Q}_i$ has an ω^j-filtration in which all the factors are prime cyclic $A/\mathfrak{p}_i$-modules which are also prime cyclic A-modules. Also, there are only finitely many distinct factors. $\square$

Remark 3.1.13. When A is a domain, the theorem on generic freeness follows at once: one simply localizes at one element of $A - \{0\}$ in all of the finitely many nonzero primes of A that occur in the filtration. See section 6.9 in [10].

Theorem 3.1.14. Let A be a Noetherian commutative ring. Let R be an A-algebra with r generators and M be a finitely generated R-module. Then M has a general ω^r-filtration in which all the factors are prime cyclic A-modules. Furthermore, only finitely many distinct factors occur.

Proof. According to Proposition 2.6.6, M has a finite filtration in which all the factors are prime cyclic R-modules.

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M \tag{3.17}$$

where $M_i/M_{i-1} \cong R/\mathfrak{Q}_i$ where $\mathfrak{Q}_i$ is prime. $R/\mathfrak{Q}_i$ is a finitely generated $A/\mathfrak{p}_i$-algebra with at most r generators where $\mathfrak{p}_i = \mathfrak{Q}_i \cap A$. According to Proposition 3.1.12, $R/\mathfrak{Q}_i$ has an ω^r-filtration in which all the factors are prime cyclic $A/\mathfrak{p}_i$-modules. These factors are also prime cyclic A-modules. Only finitely many distinct factors occur. □

Theorem 3.1.15. Let A be a Noetherian commutative ring. Let R be an A-algebra with r generators and M be a finitely generated R-module. Then M has an ω^r-filtration in which all the factors are prime cyclic A-modules. Furthermore, only finitely many distinct factors occur.

Proof. We assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$ and $M = R\alpha_1 + R\alpha_2 + \cdots + R\alpha_s$. Denote $A\alpha_1 + A\alpha_2 + \cdots + A\alpha_s$ by B.

If $r = 1$, we have an ascending sequence of submodules of an A-module $B[\theta]$ where $\theta = \theta_1$.

$$B \subseteq B + B\theta \subseteq B + B\theta + B\theta^2 \subseteq B + B\theta + B\theta^2 + B\theta^3 \subseteq \cdots \tag{3.18}$$

As in the proof of Lemma 3.1.11, we have the following well-defined surjective A-module morphisms.

$$B \twoheadrightarrow \frac{B + B\theta}{B} \twoheadrightarrow \frac{B + B\theta + B\theta^2}{B + B\theta} \twoheadrightarrow \frac{B + B\theta + B\theta^2 + B\theta^3}{B + B\theta + B\theta^2} \twoheadrightarrow \cdots \tag{3.19}$$

B is Noetherian as B is a finitely generated A-module. Then the sequence of factors will be eventually stable according to Lemma 3.1.10. Thus, there are finitely many distinct factors. Furthermore, we have the following natural isomorphism.

$$\frac{B + B\theta + \cdots + B\theta^{i-1} + B\theta^i}{B + B\theta + \cdots + B\theta^{i-1}} \cong A\overline{\theta^i} + A\overline{\theta^i\alpha_1} + A\overline{\theta^i\alpha_2} + \cdots + A\overline{\theta^i\alpha_s} \tag{3.20}$$

where $i \geqslant 1$. It is a finitely generated A-module. According to Proposition 2.6.6, each factor has a finite filtration in which all the factors are prime cyclic A-modules. Then only finitely many distinct factors occur. It is obvious that $M = B[\theta]$ when $r = 1$.

Suppose the theorem holds when $r = j$, we claim that it still holds for $r = j + 1$. Denote

$B[\theta_1, \theta_2, \cdots, \theta_j]$ by B', $A[\theta_1, \theta_2, \cdots, \theta_j]$ by A' and θ_{j+1} by θ. Then $B[\theta_1, \theta_2, \cdots, \theta_{j+1}] = B'[\theta]$. According to the base case, $B'[\theta]$ has an ω-filtration in which all the factors have the form $A'/\mathfrak{Q}_k$ where $\mathfrak{Q}_k$ is a prime ideal of A'. Also, only finitely many distinct factors occur. There is a natural isomorphism:

$$A'/\mathfrak{Q}_k \cong \frac{A}{\mathfrak{p}_k}[\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_j}] \tag{3.21}$$

where $\mathfrak{p}_k = \mathfrak{Q}_k \cap A$. According to Proposition 3.1.12, $A'/\mathfrak{Q}_k$ has an ω^j-filtration over A such that by prime cyclic A-modules involving only finitely many distinct prime ideal of A. The last part of the proof is to prove that $B[\theta_1, \theta_2, \cdots, \theta_r] = M$. We can easily check it is true. □

3.2 Explicit construction of ω^r-filtrations

In this section, we give an explicit construction of ω^r-filtrations. For several particular cases, we calculate the factors of these ω^r-filtrations.

Proposition 3.2.1. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. Consider the case $r = 2$ first. Then R has an ω^2-filtration $\{R_{i,j}\}_{(i,j)\in\omega^2}$ where $R_{i,j}$ is defined as the following.

$$R_{i,j} = \sum_{u=0}^{i-1} A[\theta_1]\theta_2^u + \sum_{v=0}^{j-1} A\theta_1^v\theta_2^i \tag{3.22}$$

Note that if the upper index is less than the lower index of the sum, we say the sum is zero. All the factors are cyclic A-modules, *i.e.*, $R_{i,j+1}/R_{i,j} \cong A/I$ where I is an ideal of A for any $i, j \geqslant 0$. Only finitely many distinct I occur.

Proof. First, we claim than $R_{i,j} \subseteq R_{i',j'}$ for any $(i,j) < (i',j') \in \omega^2$. If $i = i'$, $j < j'$, it is simply from the definition. If $i < i'$, we have $R_{i,0} \subseteq R_{i,j}$ and $R_{i',0} \subseteq R_{i',j'}$. Also, we have

$$\begin{aligned} R_{i,j} = \sum_{u=0}^{i-1} A[\theta_1]\theta_2^u + \sum_{v=0}^{j-1} A\theta_1^v\theta_2^i &\subseteq \sum_{u=0}^{i-1} A[\theta_1]\theta_2^u + \sum_{v=0}^{\infty} A\theta_1^v\theta_2^i \\ &\subseteq \sum_{u=0}^{i-1} A[\theta_1]\theta_2^u + \sum_{v=0}^{\infty} A\theta_1^v\theta_2^i \\ &\subseteq \sum_{u=0}^{i-1} A[\theta_1]\theta_2^u + A[\theta_1]\theta_2^i \\ &\subseteq \sum_{u=0}^{i} A[\theta_1]\theta_2^u = R_{i+1,0} \end{aligned}$$

Thus, $R_{i,j} \subseteq R_{i+1,0} \subseteq R_{i',0} \subseteq R_{i',j'}$. Furthermore, $R_{i,j+1} = R_{i,j} + A\theta_1^j\theta_2^i$ which means $R_{i,j+1}/R_{i,j}$ is a cyclic A-module.

We claim that distinct I only occur finitely many times. Consider the following surjective morphism.

$$\frac{R_{1,0}}{R_{0,0}} \twoheadrightarrow \frac{R_{2,0}}{R_{1,0}} \twoheadrightarrow \frac{R_{3,0}}{R_{2,0}} \twoheadrightarrow \frac{R_{4,0}}{R_{3,0}} \twoheadrightarrow \cdots \tag{3.23}$$

According to Lemma 3.1.10, there exists T such that we have

$$\frac{R_{T,0}}{R_{T-1,0}} \cong \frac{R_{T+1,0}}{R_{T,0}} \cong \frac{R_{T+2,0}}{R_{T+1,0}} \cong \frac{R_{T+3,0}}{R_{T+2,0}} \cong \cdots \tag{3.24}$$

For each fixed t, there exists S_t such that

$$\frac{R_{t,S_t+1}}{R_{t,S_t}} \cong \frac{R_{t,S_t+2}}{R_{t,S_t+1}} \cong \frac{R_{t,S_t+3}}{R_{t,S_t+2}} \cong \cdots \tag{3.25}$$

We can denote $\mathrm{Max}\{S_t\}_{t\leqslant T}$ by S. For any k, we have

$$\frac{R_{T+k,1}}{R_{T+k,0}} \twoheadrightarrow \frac{R_{T+k,2}}{R_{T+k,1}} \twoheadrightarrow \frac{R_{T+k,3}}{R_{T+k,2}} \twoheadrightarrow \frac{R_{T+k,4}}{R_{T+k,3}} \twoheadrightarrow \cdots \tag{3.26}$$

By construction, we also have the following injective morphism

$$\frac{R_{T+k,l}}{R_{T+k,0}} \hookrightarrow \frac{R_{T+k+1,0}}{R_{T+k,0}} \tag{3.27}$$

for any k and l. Consider the following isomorphism

$$\phi_{T+k} : \frac{R_{T+k+1,0}}{R_{T+k,0}} \xrightarrow{\overline{\theta_2}} \frac{R_{T+k+2,0}}{R_{T+k+1,0}} \tag{3.28}$$

restricted on $\frac{R_{T+k,l}}{R_{T+k,0}}$. By defintion, we have

$$\phi_{T+k}(\frac{R_{T+k,l}}{R_{T+k,0}}) = \frac{R_{T+k+1,l}}{R_{T+k+1,0}} \tag{3.29}$$

which means we have

$$\frac{R_{T+k,l}}{R_{T+k,0}} \cong \frac{R_{T+k+1,l}}{R_{T+k+1,0}} \tag{3.30}$$

where $l \in \mathbb{N}$. Thus, for any $l > 0$, we have

$$\frac{R_{T+k,l}}{R_{T+k,l-1}} \cong \frac{R_{T+k+1,l}}{R_{T+k+1,l-1}} \tag{3.31}$$

In conclusion, we have the commutative diagram 3.1.

Now, we actually prove that distinct I only occur finitely many times. $\square$

Theorem 3.2.2. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. Then R has an ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ where $R_{i_1,i_2,\cdots,i_r}$ is defined as the following.

$$\begin{aligned} R_{i_1,i_2,\cdots,i_r} = &\sum_{i'_1=0}^{i_1-1} A[\theta_1, \theta_2, \cdots, \theta_{r-1}]\theta_r^{i'_1} \\ &+ \sum_{i'_2=0}^{i_2-1} A[\theta_1, \theta_2, \cdots, \theta_{r-2}]\theta_{r-1}^{i'_2}\theta_r^{i_1} \end{aligned}$$

$$
\begin{tikzcd}
\frac{R_{0,1}}{R_{0,0}} \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{0,2}}{R_{0,1}} \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{0,3}}{R_{0,2}} \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{0,S}}{R_{0,S-1}} \arrow[r, "\sim"] \arrow[d, two heads] & \frac{R_{0,S+1}}{R_{0,S}} \arrow[r, "\sim"] \arrow[d, two heads] & \cdots \arrow[d, two heads] \\
\frac{R_{1,1}}{R_{1,0}} \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{1,2}}{R_{1,1}} \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{1,3}}{R_{1,2}} \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{1,S}}{R_{1,S-1}} \arrow[r, "\sim"] \arrow[d, two heads] & \frac{R_{1,S+1}}{R_{1,S}} \arrow[r, "\sim"] \arrow[d, two heads] & \cdots \arrow[d, two heads] \\
\frac{R_{2,1}}{R_{2,0}} \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{2,2}}{R_{2,1}} \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{2,3}}{R_{2,2}} \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & \frac{R_{2,S}}{R_{2,S-1}} \arrow[r, "\sim"] \arrow[d, two heads] & \frac{R_{2,S+1}}{R_{2,S}} \arrow[r, "\sim"] \arrow[d, two heads] & \cdots \arrow[d, two heads] \\
\cdots \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, "\sim"] \arrow[d, two heads] & \cdots \arrow[r, "\sim"] \arrow[d, two heads] & \cdots \arrow[d, two heads] \\
\frac{R_{T,1}}{R_{T,0}} \arrow[r, two heads] \arrow[d, "\wr"] & \frac{R_{T,2}}{R_{T,1}} \arrow[r, two heads] \arrow[d, "\wr"] & \frac{R_{T,3}}{R_{T,2}} \arrow[r, two heads] \arrow[d, "\wr"] & \cdots \arrow[r, two heads] \arrow[d, "\wr"] & \frac{R_{T,S}}{R_{T,S-1}} \arrow[r, "\sim"] \arrow[d, "\wr"] & \frac{R_{T,S+1}}{R_{T,S}} \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots \arrow[d, "\wr"] \\
\frac{R_{T+1,1}}{R_{T+1,0}} \arrow[r, two heads] \arrow[d, "\wr"] & \frac{R_{T+1,2}}{R_{T+1,1}} \arrow[r, two heads] \arrow[d, "\wr"] & \frac{R_{T+1,3}}{R_{T+1,2}} \arrow[r, two heads] \arrow[d, "\wr"] & \cdots \arrow[r, two heads] \arrow[d, "\wr"] & \frac{R_{T+1,S}}{R_{T+1,S-1}} \arrow[r, "\sim"] \arrow[d, "\wr"] & \frac{R_{T+1,S+1}}{R_{T+1,S}} \arrow[r, "\sim"] \arrow[d, "\wr"] & \cdots \arrow[d, "\wr"] \\
\cdots \arrow[r, two heads] & \cdots \arrow[r, two heads] & \cdots \arrow[r, two heads] & \cdots \arrow[r, two heads] & \cdots \arrow[r, "\sim"] & \cdots \arrow[r, "\sim"] & \cdots
\end{tikzcd}
$$

Figure 3.1: Commutative diagram of factors of an ω^2-filtration

$$
\begin{aligned}
&+\sum_{i_3'=0}^{i_3-1} A[\theta_1,\theta_2,\cdots,\theta_{r-3}]\theta_{r-2}^{i_3'}\theta_{r-1}^{i_2}\theta_r^{i_1}\\
&+\cdots\\
&+\sum_{i_r'=0}^{i_r-1} A\theta_1^{i_r'}\theta_2^{i_{r-1}}\cdots\theta_r^{i_1}\\
=&\sum_{j=1}^{r}\sum_{i_j'=0}^{i_j-1} A[\theta_1,\theta_2,\cdots,\theta_{r-j}]\theta_{r-j+1}^{i_j'}\prod_{k=1}^{j-1}\theta_{r+1-k}^{i_k}
\end{aligned}
$$

Note that if the upper index is less than the lower index of the sum, we define the sum to be zero. All the factors are cyclic A-modules. These cyclic A-modules may be replaced, by filtration, by prime cyclic A-modules, *i.e.*, modules of the form $A/\mathfrak{p}$ with $\mathfrak{p}$ prime. Only finitely many distinct $\mathfrak{p}$ occur.

Proof. We can prove the theorem by induction. Assume it holds for case $r-1$. $A[\theta_1,\theta_2,\cdots,\theta_r]$ is an $A[\theta_1]$-module. Thus, for any $(i_1,i_2,\cdots,i_r)\in\omega^r$, there are only finitely many distinct

$R_{i_1,i_2,\cdots,i_{r-1}+1,0}/R_{i_1,i_2,\cdots,i_{r-1},0}$. It is equivalent to say, for any $(i_1, i_2, \cdots, i_{r-2}) \in \omega^{r-2}$, there exists i_{r-1} so that we have

$$\frac{R_{i_1,i_2,\cdots,i_{r-1}+1,0}}{R_{i_1,i_2,\cdots,i_{r-1},0}} \cong \frac{R_{i_1,i_2,\cdots,i_{r-1}+k+1,0}}{R_{i_1,i_2,\cdots,i_{r-1}+k,0}} \tag{3.32}$$

for any k. The induced map

$$\frac{R_{i_1,i_2,\cdots,i_{r-1},l}}{R_{i_1,i_2,\cdots,i_{r-1},0}} \to \frac{R_{i_1,i_2,\cdots,i_{r-1}+k,l}}{R_{i_1,i_2,\cdots,i_{r-1}+k,0}} \tag{3.33}$$

is injective and surjective. It is similar to the proof of case $r = 2$. Thus, for any l, we have

$$\frac{R_{i_1,i_2,\cdots,i_{r-1},l}}{R_{i_1,i_2,\cdots,i_{r-1},0}} \cong \frac{R_{i_1,i_2,\cdots,i_{r-1}+k,l}}{R_{i_1,i_2,\cdots,i_{r-1}+k,0}} \tag{3.34}$$

which implies that

$$\frac{R_{i_1,i_2,\cdots,i_{r-1},l+1}}{R_{i_1,i_2,\cdots,i_{r-1},l}} \cong \frac{R_{i_1,i_2,\cdots,i_{r-1}+k,l+1}}{R_{i_1,i_2,\cdots,i_{r-1}+k,l}} \tag{3.35}$$

In conclusion, only finitely many distinct factors appear in $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$. □

Corollary 3.2.3. Let A be a commutative Noetherian ring, and $R = A[x_1, x_2, \cdots, x_r]$ where $x_1, x_2, \cdots, x_r$ are indeterminates. Then R has an ω^r-filtration in which factors are cyclic A-modules. Also, only one distinct factor occurs.

Proof. We define

$$R_{i_1,i_2,\cdots,i_r} = \sum_{j=1}^{r} \sum_{i'_j=0}^{i_j-1} A[x_1, x_2, \cdots, x_{r-j}] x_{r-j+1}^{i'_j} \prod_{k=1}^{j-1} x_{r+1-k}^{i_k} \tag{3.36}$$

Then we have

$$\frac{R_{i_1,i_2,\cdots,i_r+1}}{R_{i_1,i_2,\cdots,i_r}} = \frac{R_{i_1,i_2,\cdots,i_r} + A\prod_{k=1}^{r} x_{r+1-k}^{i_k}}{R_{i_1,i_2,\cdots,i_r}} \cong A \tag{3.37}$$

since $x_1, x_2, \cdots, x_r$ are indeterminates. □

Corollary 3.2.4. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. For θ_k, there exists $a_{k,0}, a_{k,1}, \cdots, a_{k,d_k-1} \in A$ such that

$$\theta_k^{d_k} = \sum_{l=0}^{d_k-1} a_{k,l}\theta_k^l \tag{3.38}$$

Then R has an ω^r-filtration in which factors are cyclic A-modules. The number of distinct nonzero factors is at most $d_1 d_2 \cdots d_r$.

Proof. We define $R_{i_1,i_2,\cdots,i_r}$ as the same in Proposition 3.2.1. Thus, we have

$$\frac{R_{i_1,i_2,\cdots,i_r+1}}{R_{i_1,i_2,\cdots,i_r}} = \frac{R_{i_1,i_2,\cdots,i_r} + A\prod_{k=1}^{r}\theta_{r+1-k}^{i_k}}{R_{i_1,i_2,\cdots,i_r}} \cong A\overline{\prod_{k=1}^{r}\theta_{r+1-k}^{i_k}} \tag{3.39}$$

If there exists l such that $i_l \geqslant d_l$, then we have

$$\overline{\prod_{k=1}^{r}\theta_{r+1-k}^{i_k}} = \overline{\prod_{k\neq l}\theta_{r+1-k}^{i_k}\theta_l^{i_l-d_l}\sum_{j=0}^{d_l-1}a_{l,j}\theta_l^j} = \overline{\sum_{j=0}^{d_l-1}\prod_{k\neq l}a_{l,j}\theta_{r+1-k}^{i_k}\theta_l^{i_l-d_l+j}} \tag{3.40}$$

For any $j \leqslant d_l - 1$, $\prod_{k\neq l} a_{l,j}\theta_{r+1-k}^{i_k}\theta_l^{i_l-d_l+j} \in R_{i_1,i_2,\cdots,i_r}$ since $i_l - d_l + j < i_l$. This actually proves that if there exists l such that $i_l \geqslant d_l$, then $\frac{R_{i_1,i_2,\cdots,i_r+1}}{R_{i_1,i_2,\cdots,i_r}} \cong 0$. Furthermore, the number of distinct factors is at most $d_1 d_2 \cdots d_r$. □

Remark 3.2.5. In this particular case, we can see that the ω^r-filtration reduces to a finite filtration.

Proposition 3.2.6. Let K be a commutative Noetherian ring, $A = K[x_1, x_2, \cdots, x_n]$ where x_i is indeterminate for $1 \leqslant i \leqslant n$, y_1 be another indeterminate, and $R = A[y_1, y_2]$ where $y_2 = \frac{y_1}{x_1}x_2$. Then the factors of the ω^2-filtration $\{R_{i,j}\}_{(i,j)\in\omega^2}$ are isomorphic to either A or A/Ax_1.

Proof. According to Proposition 3.2.1, we have

$$\frac{R_{i,j+1}}{R_{i,j}} = \frac{R_{i,j} + Ay_1^j y_2^i}{R_{i,j}} \tag{3.41}$$

When $i = 0$, we know that

$$\frac{R_{0,j+1}}{R_{0,j}} = \frac{\sum_{v=0}^{j} Ay_1^v}{\sum_{v=0}^{j-1} Ay_1^v} \cong A\overline{y_1^j} \cong A \tag{3.42}$$

since y_1 is indeterminate. When $i \geqslant 1$, we have

$$\frac{R_{i,j+1}}{R_{i,j}} = \frac{R_{i,j} + Ay_1^j y_2^i}{R_{i,j}} \cong A\overline{y_1^j y_2^i} = A\overline{\frac{y_1^{i+j} x_2^i}{x_1^i}} \tag{3.43}$$

Furthermore, we have the following natural surjective morphism.

$$A \twoheadrightarrow A\overline{\frac{y_1^{i+j} x_2^i}{x_1^i}} \tag{3.44}$$

$$\phi : a \rightarrow a\overline{\frac{y_1^{i+j} x_2^i}{x_1^i}} \tag{3.45}$$

We claim that $\ker \phi = Ax_1$.

First, for any $ax_1 \in Ax_1$, we have $\phi(ax_1) = a\overline{\frac{y_1^{i+j} x_2^i}{x_1^{i-1}}} = 0$ since $\frac{y_1^{i+j} x_2^i}{x_1^{i-1}} = x_2 y_1^{j+1} y_2^{i-1} \in R_{i,j}$ for $i \geqslant 1$.

Second, $a\overline{\frac{y_1^{i+j} x_2^i}{x_1^i}} = 0$ implies that $a\frac{y_1^{i+j} x_2^i}{x_1^i} \in R_{i,j}$. Since x_1, x_2, y_1 are indeterminates, a has to be the form bx_1 where $b \in A$. In fact, generators of $R_{i,j}$ have the form $y_1^t y_2^s$ where either any $t \in \mathbb{N}, s \leqslant i-1$ or $t \leqslant j-1, s = i$. By definition, we have

$$y_1^t y_2^s = \frac{y_1^{t+s} x_2^s}{x_1^s} \tag{3.46}$$

If $a \notin Ax_1$, $a\frac{y_1^{i+j} x_2^i}{x_1^i} = ay_1^j y_2^i$ must be generated by $y_1^t y_2^i$ where $t \leqslant j-1$. Contradiction. We actually prove the following isomorphism.

$$\frac{R_{i,j+1}}{R_{i,j}} = A\overline{\frac{y_1^{i+j}x_2^i}{x_1^i}} \cong A/\ker\phi = A/Ax_1 \tag{3.47}$$

Thus, factors of the ω^2-filtration $\{R_{i,j}\}_{(i,j)\in\omega^2}$ are isomorphic to either A or A/Ax_1. $\square$

Remark 3.2.7. Let $R = A[y_1, y_2, \cdots, y_r]$ where $2 \leqslant r \leqslant n$. Then the factors of the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ are isomorphic to either A or $A/(Ax_1 + Ax_2 + \cdots + Ax_k)$ for any $1 \leqslant k \leqslant r-1$.

Proof. According to Proposition 3.2.1, we have

$$\frac{R_{i_1,i_2,\cdots,i_r+1}}{R_{i_1,i_2,\cdots,i_r}} = \frac{R_{i_1,i_2,\cdots,i_r} + A\prod_{k=1}^{r} y_{r+1-k}^{i_k}}{R_{i_1,i_2,\cdots,i_r}} \cong A\overline{\prod_{k=1}^{r} y_{r+1-k}^{i_k}} \tag{3.48}$$

If $(i_1, i_2, \cdots, i_{r-1}) = 0$, we have

$$\frac{R_{0,0,\cdots,i_r+1}}{R_{0,0,\cdots,i_r}} = \frac{R_{0,0,\cdots,i_r} + Ay_1^{i_r}}{R_{0,0,\cdots,i_r}} \cong A\overline{y_1^{i_r}} \cong A \tag{3.49}$$

since y_1 is indeterminate and $y_1^{i_r} \notin R_{0,0,\cdots,i_r}$.

If $(i_1, i_2, \cdots, i_{r-1}) \neq 0$, there exists l such that $(i_1, \cdots, i_{l-1}) = 0$ and $i_l > 0$. We have

$$\frac{R_{i_1,i_2,\cdots,i_r+1}}{R_{i_1,i_2,\cdots,i_r}} \cong A\overline{\prod_{k=1}^{r} y_{r+1-k}^{i_k}} = A\overline{y_1^{i_r}(\frac{y_1}{x_1}x_2)^{i_{r-1}}\cdots(\frac{y_1}{x_1}x_r)^{i_1}} \tag{3.50}$$

$$= A\overline{\frac{y_1^{i_1+i_2+\cdots+i_r}x_2^{i_{r-1}}x_3^{i_{r-2}}\cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} \tag{3.51}$$

Similar to the case $r = 2$, we have the following natural surjective morphism.

$$A \twoheadrightarrow A\overline{\frac{y_1^{i_1+i_2+\cdots+i_r}x_2^{i_{r-1}}x_3^{i_{r-2}}\cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} \tag{3.52}$$

$$\phi : a \to a\overline{\frac{y_1^{i_1+i_2+\cdots+i_r}x_2^{i_{r-1}}x_3^{i_{r-2}}\cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} \tag{3.53}$$

We claim that $\ker\phi = Ax_1 + Ax_2 + \cdots + Ax_{r-l}$.

First, for $x_1 \in Ax_1$, we know that

$$\begin{aligned}
\phi(x_1) &= x_1 \overline{\frac{y_1^{i_1+i_2+\cdots+i_r} x_2^{i_{r-1}} x_3^{i_{r-2}} \cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} && (3.54)\\
&= \overline{\frac{y_1^{i_1+i_2+\cdots+i_r} x_2^{i_{r-1}} x_3^{i_{r-2}} \cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}-1}}} && (3.55)
\end{aligned}$$

since $i_{r-k+1} \geqslant 1$ where $k = r - l + 1$, then we have

$$\begin{aligned}
\phi(x_1) &= \overline{\frac{y_1^{i_1+i_2+\cdots+i_r} x_2^{i_{r-1}} x_3^{i_{r-2}} \cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}-1}}} && (3.56)\\
&= \overline{y_1^{i_r+1} y_2^{i_{r-1}} \cdots y_{k-1}^{i_{r-k+2}} y_k^{i_{r-k+1}-1} y_{k+1}^{i_{r-k}} \cdots y_r^{i_1}} && (3.57)\\
&= 0 && (3.58)
\end{aligned}$$

because we have $y_1^{i_r+1} y_2^{i_{r-1}} \cdots y_{k-1}^{i_{r-k+2}} y_k^{i_{r-k+1}-1} y_{k+1}^{i_{r-k}} \cdots y_r^{i_1} \in R_{i_1,i_2,\cdots,i_r}$. For $x_k \in Ax_k$ where $2 \leqslant k \leqslant r-l$, we have

$$\begin{aligned}
\phi(x_k) &= x_k \overline{\frac{y_1^{i_1+i_2+\cdots+i_r} x_2^{i_{r-1}} x_3^{i_{r-2}} \cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} && (3.59)\\
&= \overline{\frac{y_1^{i_1+i_2+\cdots+i_r} x_2^{i_{r-1}} \cdots x_{k-1}^{i_{r-k+2}} x_k^{i_{r-k+1}+1} x_{k+1}^{i_{r-k}} \cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} && (3.60)\\
&= x_l \overline{y_1^{i_r} y_2^{i_{r-1}} \cdots y_{k-1}^{i_{r-k+2}} y_k^{i_{r-k+1}+1} y_{k+1}^{i_{r-k}} \cdots y_{r-l}^{i_{l-1}} y_{r-l+1}^{i_l-1}} && (3.61)\\
&= 0 && (3.62)
\end{aligned}$$

since $y_1^{i_r} y_2^{i_{r-1}} \cdots y_{k-1}^{i_{r-k+2}} y_k^{i_{r-k+1}+1} y_{k+1}^{i_{r-k}} \cdots y_{r-l}^{i_{l-1}} y_{r-l+1}^{i_l-1} \in R_{i_1,\cdots,i_l-1,\cdots,i_{r-k+1}+1,\cdots} \subseteq R_{i_1,\cdots,i_r}$.

Second, for any $a \in \ker\phi$,

$$a \overline{\frac{y_1^{i_1+i_2+\cdots+i_r} x_2^{i_{r-1}} x_3^{i_{r-2}} \cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} = 0 \qquad (3.63)$$

implies that

$$a\frac{y_1^{i_1+i_2+\cdots+i_r}x_2^{i_{r-1}}x_3^{i_{r-2}}\cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}} \in R_{i_1,i_2,\cdots,i_r} \tag{3.64}$$

Since $x_1, x_2, \cdots, x_r, y_1$ are indeterminates, we know that generators of $R_{i_1,i_2,\cdots,i_r}$ have the form $y_1^{i'_r}y_2^{i'_{r-1}}\cdots y_r^{i'_1}$ where $(i'_1, i'_2, \cdots, i'_r) < (i_1, i_2, \cdots, i_r)$. Suppose $a \notin Ax_1+Ax_2+\cdots+Ax_{r-l}$, since y_1 is an indeterminate, we have $i'_1 + i'_2 + \cdots + i'_r = i_1 + \cdots + i_r$. Also, $(i'_1, i'_2, \cdots, i'_r) <$ $(i_1, i_2, \cdots, i_r)$ means there exists $v \geqslant l$ such that $i'_v < i_v$. Then there exists $u > v \geqslant l$ such that $i'_u > i_u$. Then the degree of x_{r+1-u} should be greater than i_u. As we know, $x_{r+1-u} \in Ax_1 + Ax_2 + \cdots + Ax_{r-l}$. Contradiction.

Thus, we have the following isomorphism.

$$\frac{R_{i_1,i_2,\cdots,i_r+1}}{R_{i_1,i_2,\cdots,i_r}} \cong A\overline{\frac{y_1^{i_1+i_2+\cdots+i_r}x_2^{i_{r-1}}x_3^{i_{r-2}}\cdots x_r^{i_1}}{x_1^{i_1+i_2+\cdots+i_{r-1}}}} \tag{3.65}$$

$$\cong A/\ker\phi \tag{3.66}$$

$$= A/(Ax_1 + Ax_2 + \cdots + Ax_{r-l}) \tag{3.67}$$

In conclusion, factors of the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ are isomorphic to either A or $A/(Ax_1 + Ax_2 + \cdots + Ax_k)$ for any $1 \leqslant k \leqslant r-1$. □

Proposition 3.2.8. If f, g forms a regular sequence of A, the factors of the ω-filtration of $A[f/g]$ have the form A/gA.

Proof. For any $k \in \mathbb{N}$, we have

$$\frac{R_{k+1}}{R_k} = \frac{A + A\frac{f}{g} + \cdots + A(\frac{f}{g})^k + A(\frac{f}{g})^{k+1}}{A + A\frac{f}{g} + \cdots + A(\frac{f}{g})^k} \tag{3.68}$$

$$\cong \frac{Ag^{k+1} + Afg^k + \cdots + Af^kg + Af^{k+1}}{Ag^{k+1} + Afg^k + \cdots + Af^kg} \tag{3.69}$$

$$\cong \frac{A}{I_k} \tag{3.70}$$

where $I_k = (g^{k+1}, fg^k, \cdots, f^kg) : f^{k+1}$.

First, we have $I_k \subseteq gA : f^{k+1} = gA$ since f, g forms a regular sequence of A.

Second, $I_k \supseteq gA$ since $gAf^{k+1} \subseteq (g^{k+1}, fg^k, \cdots, f^k g)$. Thus, we have $I_k = gA$.

In conclusion, the factors have the form A/gA. □

Example 3.2.9. Let K be a commutative Noetherian ring, $A = \frac{K[x,y,z]}{(x^3+y^3+z^3)}$, and $R = A[\theta_1, \theta_2]$ where $\theta_1 = \frac{y}{x}, \theta_2 = \frac{z}{x}$. We know x, y forms a regular sequence of A. According to Proposition 3.2.8, we have

$$\frac{R_{0,j+1}}{R_{0,j}} \cong \frac{A}{Ax} \tag{3.71}$$

For $i \geqslant 1$, we have that

$$\frac{R_{i,j+1}}{R_{i,j}} \cong \frac{R_{i,j} + A\theta_1^j \theta_2^i}{R_{i,j}} = \frac{R_{i,j} + A\frac{y^j z^i}{x^{i+j}}}{R_{i,j}} \tag{3.72}$$

If $i \geqslant 3$, then $z^i = -z^{i-3}(x^3 + y^3)$ which implies

$$\frac{y^j z^i}{x^{i+j}} = \frac{-y^j z^{i-3}(x^3 + y^3)}{x^{i+j}} = -(\theta_1^j \theta_2^{i-3} + \theta_1^{j+3}\theta_2^{i-3}) \in R_{i,j} \tag{3.73}$$

Thus, $\frac{R_{i,j+1}}{R_{i,j}} = 0$.

If $i = 2$, we claim that

$$\frac{R_{i,j+1}}{R_{i,j}} \cong \frac{R_{i,j} + A\frac{y^j z^i}{x^{i+j}}}{R_{i,j}} \cong A\overline{\frac{y^j z^i}{x^{i+j}}} \cong \frac{A}{Ax + Ay + Az} \tag{3.74}$$

As usual, consider the following surjective map.

$$A \twoheadrightarrow A\overline{\frac{y^j z^i}{x^{i+j}}} \tag{3.75}$$

$$\phi : a \to a\overline{\frac{y^j z^i}{x^{i+j}}} \tag{3.76}$$

First, we have $\ker \phi \supseteq Ax + Ay + Az$ since $x\frac{y^j z^2}{x^{2+j}} = \theta_1^j \theta_2 z \in R_{2,j}$, $y\frac{y^j z^2}{x^{2+j}} = \theta_1^{j+1}\theta_2 z \in R_{2,j}$, and $z\frac{y^j z^2}{x^{2+j}} = -\frac{y^j(x^3+y^3)}{x^{2+j}} \in R_{2,j}$.

Second, for any $a \in K$, $a\frac{y^j z^2}{x^{2+j}} \notin R_{2,j}$ since the degree of y is at most j, the degree of θ_1 is at most j, we have the degree of θ_2 is at least 2 because the denominator is x^{2+j}.

While the degree of θ_2 is 2, we have the degree of θ_1 is j. We know $\theta_1^j\theta_2^2 \notin R_{2,j}$. Thus, $\ker\phi = Ax + Ay + Az$ implies that

$$\frac{R_{i,j+1}}{R_{i,j}} \cong \frac{A}{Ax + Ay + Az} \tag{3.77}$$

If $i = 1$, we claim that

$$\frac{R_{i,j+1}}{R_{i,j}} \cong \frac{R_{i,j} + A\frac{y^j z^i}{x^{i+j}}}{R_{i,j}} \cong A\overline{\frac{y^j z^i}{x^{i+j}}} \cong \frac{A}{Ax + Ay + Az^2} \tag{3.78}$$

Similarly, we have the following surjective map.

$$A \twoheadrightarrow A\overline{\frac{y^j z^i}{x^{i+j}}} \tag{3.79}$$

$$\phi : a \to a\overline{\frac{y^j z^i}{x^{i+j}}} \tag{3.80}$$

We only need to prove that $\ker\phi = Ax + Ay + Az^2$.

First, since $x\frac{y^j z}{x^{1+j}} = \theta_1^j z \in R_{1,j}$, $y\frac{y^j z}{x^{1+j}} = \theta_1^{j+1} z \in R_{1,j}$, and $z^2\frac{y^j z^1}{x^{1+j}} = -\frac{y^j(x^3+y^3)}{x^{1+j}} \in R_{1,j}$, we have $\ker\phi \supseteq Ax + Ay + Az^2$.

Second, for any $a \in K$, $a\theta_1^j\theta_2 = a\frac{y^j z}{x^{1+j}} \notin R_{1,j}$, and $az\frac{y^j z}{x^{1+j}} = a\frac{y^j z^2}{x^{1+j}}$, the degree of y is at most j which means the degree of θ_1 is at most j, then the degree of θ_2 is at least one since the denominator is x^{j+1}. As we know, $\theta_1^j\theta_2 \notin R_{1,j}$ and $\theta_1^{j-1}\theta_2^2 \notin R_{1,j}$. This actually means $az\frac{y^j z}{x^{1+j}} \notin R_{1,j}$.

In conclusion, we know that the factors of ω^2-filtration have the form $A/Ax, A/(Ax + Ay + Az), A/(Ax + Ay + Az^2)$.

3.3 Rectangularly and triangularly normal ω^r-filtrations

In this section, we first introduce the definition of rectangularly and triangularly normal ω^r-filtrations. Then we construct rectangularly and triangularly normal ω^r-filtrations in several particular cases.

Definition 3.3.1. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$.

For $l_1, l_2, \cdots, l_r \in \mathbb{N}$, we define $R_{[l_1,l_2,\cdots,l_r]}$ as the following.

$$R_{[l_1,l_2,\cdots,l_r]} = \sum_{i'_1 \leqslant l_1, i'_2 \leqslant l_2, \cdots, i'_r \leqslant l_r} A\theta_1^{i'_2}\theta_1^{i'_2} \cdots \theta_r^{i'_r} \tag{3.81}$$

We call $R_{[l_1,l_2,\cdots,l_r]}$ a *rectangular* submodule of R.

For $d \in \mathbb{N}$, we define $R_{\{d\}}$ as the following.

$$R_{\{d\}} = \sum_{i'_1 + i'_2 + \cdots + i'_r \leqslant d} A\theta_1^{i'_2}\theta_1^{i'_2} \cdots \theta_r^{i'_r} \tag{3.82}$$

Similarly, $R_{\{d\}}$ is called a *triangular* submodule of R.

An ω^r-filtration of R is said to be rectangularly normal (respectively, triangularly normal) if all the inherited filtrations on rectangular (respectively, triangular) submodules produce only finitely many factors.

Lemma 3.3.2. Let A, R be the same as in Definition 3.3.1. For any finitely generated A-module N, the set $\{N \cap R_{i_1,i_2,\cdots,i_r} | (i_1, i_2, \cdots, i_r) \in \omega^r\}$ is a finite set which means the ω^r-filtration $N \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is actually a finite filtration. Thus, there are only finitely many distinct factors of $N \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$.

Proof. According to Proposition 3.1.7, we know $N \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$is actually an ω^r-filtration of $N \cap R$.

We claim that there are only finitely many distinct $N \cap R_{i_1,i_2,\cdots,i_r}$.

In fact, N is Noetherian. If there are infinitely many distinct $N \cap R_{i_1,i_2,\cdots,i_r}$, we can pick countably many distinct $N \cap R_{i_1,i_2,\cdots,i_r}$ such that they form a strictly increasing sequence which is a contradiction to the fact N has ACC. We have only finitely many distinct $N \cap R_{i_1,i_2,\cdots,i_r}$ implies only finitely many distinct $\frac{N \cap R_{i_1,i_2,\cdots,i_r+1}}{N \cap R_{i_1,i_2,\cdots,i_r}}$ will occur. $\square$

Corollary 3.3.3. Let A, R be the same as in Definition 3.3.1. For any fixed $d, l_1, \cdots, l_r$, there are only finitely many distinct factors of ω^r-filtration $R_{\{d\}} \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ and $R_{[l_1,l_2,\cdots,l_r]} \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$.

Proof. This comes from Lemma 3.3.2 directly since both $R_{\{d\}}$ and $R_{[l_1,l_2,\cdots,l_r]}$ are finitely generated A-modules. □

Proposition 3.3.4. Let A, R be the same as in Corollary 3.2.3. For any $l_1, l_2, \cdots, l_r \in \mathbb{N}$, we have the factors of ω^r-filtration $R_{[l_1,l_2,\cdots,l_r]} \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ are isomorphic to either 0 or A. It actually implies that $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly normal. Similarly, for any $d \in \mathbb{N}$, factors of ω^r-filtration $R_{\{d\}} \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ are isomorphic to either 0 or A which implies that $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is triangularly normal.

Proof. By definition, we have

$$\frac{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r}} = \frac{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r} + R_{[l_1,l_2,\cdots,l_r]} \cap A \prod_{k=1}^{r} x_{r+1-k}^{i_k}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r}} \tag{3.83}$$

Since $x_1, x_2, \cdots, x_r$ are indeterminates, if $\prod_{k=1}^{r} x_{r+1-k}^{i_k} \notin R_{[l_1,l_2,\cdots,l_r]}$, then

$$R_{[l_1,l_2,\cdots,l_r]} \cap A \prod_{k=1}^{r} x_{r+1-k}^{i_k} = 0 \tag{3.84}$$

which implies

$$\frac{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r}} = 0 \tag{3.85}$$

If $\prod_{k=1}^{r} x_{r+1-k}^{i_k} \in R_{[l_1,l_2,\cdots,l_r]}$, then we have

$$R_{[l_1,l_2,\cdots,l_r]} \cap A \prod_{k=1}^{r} x_{r+1-k}^{i_k} = A \prod_{k=1}^{r} x_{r+1-k}^{i_k} \tag{3.86}$$

which implies

$$\frac{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r}} \cong A \tag{3.87}$$

Similarly, we can prove $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is triangularly normal.

$$\frac{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r}} = \frac{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r} + R_{\{d\}} \cap A \prod_{k=1}^{r} x_{r+1-k}^{i_k}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r}} \tag{3.88}$$

Since $x_1, x_2, \cdots, x_r$ are indeterminates, if $\prod_{k=1}^{r} x_{r+1-k}^{i_k} \notin R_{\{d\}}$, then $R_{\{d\}} \cap A \prod_{k=1}^{r} x_{r+1-k}^{i_k} = 0$ implies that $\frac{R_{\{d\}}\cap R_{i_1,i_2,\cdots,i_r+1}}{R_{\{d\}}\cap R_{i_1,i_2,\cdots,i_r}} = 0$. If $\prod_{k=1}^{r} x_{r+1-k}^{i_k} \in R_{\{d\}}$, then $R_{\{d\}} \cap A \prod_{k=1}^{r} x_{r+1-k}^{i_k} = A \prod_{k=1}^{r} x_{r+1-k}^{i_k}$ implies that $\frac{R_{\{d\}}\cap R_{i_1,i_2,\cdots,i_r+1}}{R_{\{d\}}\cap R_{i_1,i_2,\cdots,i_r}} \cong A$. □

Definition 3.3.5. For any $x \in \mathbb{N}^r$, we denote the i-th coordinate of x by $\sigma_i(x)$ where $1 \leqslant i \leqslant r$.

Lemma 3.3.6. Let Δ_r be a subset of $\mathbb{N}^r$ containing infinitely many elements. Then, we can construct a sequence $\delta_1, \delta_2, \cdots, \delta_n, \cdots$ such that $\sigma_i(\delta_1), \sigma_i(\delta_2), \cdots, \sigma_i(\delta_n), \cdots$ is a non-decreasing sequence for any $1 \leqslant i \leqslant r$, and $\delta_n \in \Delta_r$ for any $n \in \mathbb{N}$. Also, $\delta_j \neq \delta_k$ for any $j \neq k$.

Proof. If $r = 1$, it is obvious.

If we prove the lemma for any $r \leqslant k - 1$, we consider the case k.

We denote $\Delta'_{k-1} = \{(i_1, i_2, \cdots, i_{k-1}) \in \mathbb{N}^{k-1} | (i_1, i_2, \cdots, i_{k-1}, i_k) \in \Delta_k\}$. Since Δ_k contains infinitely many elements, if Δ'_{k-1} is a finite set, there exists $(l_1, \cdots, l_{k-1})$ such that $\Delta_{k,k-1} = \{\delta \in \Delta_k | \sigma_i(\delta) = l_i \text{ where } 1 \leqslant i \leqslant k-1\}$ contains infinitely many elements. We can choose $\delta_1, \delta_2, \cdots \in \Delta_{k,k-1}$ such that $\sigma_k(\delta_1), \sigma_k(\delta_2), \cdots$ is strictly increasing. If Δ'_{k-1} is an infinite set, by induction, there exists a sequence $\delta'_1, \delta'_2, \cdots \in \Delta'_{k-1}$ such that $\sigma_i(\delta'_1), \sigma_i(\delta'_2), \cdots, \sigma_i(\delta'_n), \cdots$ is a non-decreasing sequence for any $1 \leqslant i \leqslant k-1$. Thus, we can choose $\delta_1, \delta_2, \cdots \in \Delta_k$ such that $\sigma_i(\delta_1), \sigma_i(\delta_2), \cdots, \sigma_i(\delta_n), \cdots$ is a non-decreasing sequence for any $1 \leqslant i \leqslant k-1$. For this particular sequence, if the set $\{\sigma_k(\delta_i) | i \in \mathbb{N}\}$ contains only finitely many elements, there exists $l \in \mathbb{N}$ such that $\{\delta_i | \sigma_k(\delta_i) = l, i \in \mathbb{N}\}$ contains infinitely many elements which gives us the desired sequence. If the set $\{\sigma_k(\delta_i) | i \in \mathbb{N}\}$ contains infinitely many elements, we can choose a strictly increasing sequence in $\{\sigma_k(\delta_i) | i \in \mathbb{N}\}$. Thus, this subsequence of $\delta_1, \delta_2, \cdots$ is the desired sequence. □

Proposition 3.3.7. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. If R is also a finite generated A-module, then the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly and triangularly normal.

Proof. We claim there are only finitely many distinct $R_{[l_1,l_2,\cdots,l_r]}$ where $l_1, l_2, \cdots, l_r \in \mathbb{N}$.

Suppose there are infinitely many distinct $R_{[l_1,l_2,\cdots,l_r]}$. We denote the set of distinct $R_{[l_1,l_2,\cdots,l_r]}$ by Δ. Furthermore, we define

$$\Delta_r = \{(l_1, l_2, \cdots, l_r) \in \mathbb{N}^r | R_{[l_1,l_2,\cdots,l_r]} \in \Delta\} \tag{3.89}$$

According to Lemma 3.3.6, there exists a sequence $\delta_1, \delta_2, \cdots, \delta_n, \cdots$ such that $\delta_n \in \Delta_r$ for any $n \in \mathbb{N}$, and $\sigma_i(\delta_1), \sigma_i(\delta_2), \cdots, \sigma_i(\delta_n), \cdots$ is a non-decreasing sequence for any $1 \leqslant i \leqslant r$. Thus, $R_{\delta_1}, R_{\delta_2}, \cdots$ is a strictly increasing sequence of submodules of R. Contradiction to the fact R has ACC. According to Corollary 3.3.3, we know the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly normal.

Since R has ACC, the increasing sequence $R_{\{d\}}$ will be eventually stable which means there are only finitely many distinct $R_{\{d\}}$. According to Corollary 3.3.3, we know the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is also triangularly normal. □

Remark 3.3.8. As we all know, $R = A[\theta_1, \theta_2, \cdots, \theta_r]$ is a finite generated A-module is equivalent to say θ_k is integral over A for any $1 \leqslant k \leqslant r$ *i.e.*, for θ_k, there exists $a_{k,0}, a_{k,1}, \cdots, a_{k,d_k-1} \in A$ such that

$$\theta_k^{d_k} = \sum_{l=0}^{d_k-1} a_{k,l}\theta_k^l \tag{3.90}$$

Then, there are at most $d_1 d_2 \cdots d_r$ distinct $R_{[l_1,l_2,\cdots,l_r]}$. For any $l_1, l_2, \cdots, l_r \in \mathbb{N}$, if there exists $1 \leqslant k \leqslant r$ such that $l_k \geqslant d_k$, then we have

$$R_{[l_1,l_2,\cdots,l_r]} = R_{[l_1,\cdots,l_{k-1},l_k-1,l_{k+1},\cdots,l_r]} = \cdots = R_{[l_1,\cdots,l_{k-1},d_k-1,l_{k+1},\cdots,l_r]} \tag{3.91}$$

since $\theta_k^{l_k} = \theta_k^{l_k-d_k} \sum_{l=0}^{d_k-1} a_{k,l}\theta_k^l$. The degree of θ_k will decrease one. We can keep going until the degree of θ_k is $d_k - 1$. Now, we actually prove that for any $l_1, l_2, \cdots, l_r$, there exists $l'_1, l'_2, \cdots, l'_r$ where $l'_k < d_k$ for all $1 \leqslant k \leqslant r$ such that $R_{[l_1,l_2,\cdots,l_r]} = R_{[l'_1,l'_2,\cdots,l'_r]}$. Thus, the number of distinct $R_{[l_1,l_2,\cdots,l_r]}$ is at most $d_1 d_2 \cdots d_r$. According to Corollary 3.3.3, there are finitely many distinct factors of $R_{[l_1,l_2,\cdots,l_r]} \cap \{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ for any fixed $l_1, l_2, \cdots, l_r$. Thus, for finitely distinct $(l_1, l_2, \cdots, l_r)$, there are only finitely many distinct factors. In conclusion, for any $(l_1, l_2, \cdots, l_r)$, the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly normal.

We also claim that there are only $d_1 + d_2 + \cdots + d_r - r + 1$ distinct $R_{\{d\}}$. If $d \geqslant d_1 + d_2 + \cdots + d_r - r + 1$, since $i'_1 + \cdots + i'_r = d$, there exists k such that $i'_k \geqslant d_k$, then $\theta_1^{i'_1} \cdots \theta_r^{i'_r} \in R_{\{d-1\}}$ since $\theta_k^{i'_k} = \theta_k^{i'_k-d_k} \sum\limits_{l=0}^{d_k-1} a_{k,l}\theta_k^l$. Then, we have

$$R_{\{d\}} = R_{\{d-1\}} = \cdots = R_{\{d_1+d_2+\cdots+d_r-r\}} \tag{3.92}$$

which implies that there are at most $d_1 + d_2 + \cdots + d_r - r + 1$ distinct $R_{\{d\}}$. According to Corollary 3.3.3, we know the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is also triangularly normal.

Proposition 3.3.9. Let A be a Noetherian commutative ring. Let R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. If $\theta_1, \theta_2, \cdots, \theta_h$ are indeterminates and $\theta_{h+1}, \theta_{h+2}, \cdots, \theta_r$ are integral over A, then the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly and triangularly normal.

Proof. We may replace $\theta_1, \theta_2, \cdots, \theta_h$ by $x_1, x_2, \cdots, x_h$. In order to prove the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is triangularly normal, it suffices to prove that there are only finitely many distinct factors.

$$\frac{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r}} = \frac{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r} + R_{\{d\}} \cap A\prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r}} \tag{3.93}$$

Since $x_1, x_2, \cdots, x_h$ are indeterminates, if $d < \sum_{k=1}^{h} i_{r+1-k}$, we have

$$R_{\{d\}} \cap A\prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}} = 0 \tag{3.94}$$

implies that

$$\frac{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r}} = 0 \tag{3.95}$$

If $d \geqslant \sum_{k=1}^{h} i_{r+1-k}$, say $d' = d \sum_{k=1}^{h} i_{r+1-k}$ and $R' = A[\theta_h, \cdots, \theta_r]$, then we have

$$R_{\{d\}} \cap A\prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}} \tag{3.96}$$

$$=\left(\sum_{\sum i'_k \leqslant d'} A \prod_{k=h+1}^{r} \theta_k^{i'_k} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}\right) \prod_{k=1}^{h} x_k^{i_{r+1-k}} \tag{3.97}$$

$$=\left(R'_{\{d'\}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}\right) \prod_{k=1}^{h} x_k^{i_{r+1-k}} \tag{3.98}$$

Also, by definition, we have

$$R_{i_1,i_2,\cdots,i_r} \cap A\prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}} \tag{3.99}$$

$$=\left(R_{i_1,i_2,\cdots,i_{r-h},0,\cdots,0} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}\right) \prod_{k=1}^{h} x_k^{i_{r+1-k}} \tag{3.100}$$

$$=\left(R'_{i_1,i_2,\cdots,i_{r-h}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}\right) \prod_{k=1}^{h} x_k^{i_{r+1-k}} \tag{3.101}$$

Thus, we have

$$R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r} \cap A\prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}} \tag{3.102}$$

$$=(R'_{\{d'\}} \cap R'_{i_1,i_2,\cdots,i_{r-h}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}) \prod_{k=1}^{h} x_k^{i_{r+1-k}} \tag{3.103}$$

As we all know,

$$\frac{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r}} = \frac{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r} + R_{\{d\}} \cap A \prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r}} \tag{3.104}$$

$$\cong \frac{R_{\{d\}} \cap A \prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}}{R_{\{d\}} \cap R_{i_1,i_2,\cdots,i_r} \cap A \prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}} \tag{3.105}$$

$$= \frac{(R'_{\{d'\}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}) \prod_{k=1}^{h} x_k^{i_{r+1-k}}}{(R'_{\{d'\}} \cap R'_{i_1,i_2,\cdots,i_{r-h}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}) \prod_{k=1}^{h} x_k^{i_{r+1-k}}} \tag{3.106}$$

$$\cong \frac{R'_{\{d'\}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}}{R'_{\{d'\}} \cap R'_{i_1,i_2,\cdots,i_{r-h}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}} \tag{3.107}$$

$$\cong \frac{R_{\{d'\}} \cap R'_{i_1,i_2,\cdots,i_{r-h}+1}}{R_{\{d'\}} \cap R'_{i_1,i_2,\cdots,i_{r-h}}} \tag{3.108}$$

According to Proposition 3.3.7, we know there are only finitely many distinct factors which implies that the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is triangularly normal. Similarly, we can prove this ω^r-filtration is rectangularly normal. In fact, since $x_1, x_2, \cdots, x_h$ are indeterminates, if there exists one $1 \leqslant k \leqslant h$ such that $l_k < i_{r+1-k}$, then we have

$$R_{[l_1,l_2,\cdots,l_r]} \cap A\prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}} = 0 \tag{3.109}$$

implies that

$$\frac{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r}} = 0 \tag{3.110}$$

Now, we assume for all $1 \leqslant k \leqslant h$, we have $l_k \geqslant i_{r+1-k}$. Denote $R' = A[\theta_h, \cdots, \theta_r]$. Then, we have

$$R_{[l_1,l_2,\cdots,l_r]} \cap A \prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}} = (R'_{[l_{h+1},l_{h+2},\cdots,l_r]} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}) \prod_{k=1}^{h} x_k^{i_{r+1-k}} \tag{3.111}$$

Thus,

$$\frac{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r+1}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r}} \tag{3.112}$$

$$= \frac{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r} + R_{[l_1,l_2,\cdots,l_r]} \cap A \prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r}} \tag{3.113}$$

$$\cong \frac{R_{[l_1,l_2,\cdots,l_r]} \cap A \prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}}{R_{[l_1,l_2,\cdots,l_r]} \cap R_{i_1,i_2,\cdots,i_r} \cap A \prod_{k=1}^{h} x_k^{i_{r+1-k}} \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}} \tag{3.114}$$

$$= \frac{(R'_{[l_{h+1},l_{h+2},\cdots,l_r]} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}) \prod_{k=1}^{h} x_k^{i_{r+1-k}}}{(R'_{[l_{h+1},l_{h+2},\cdots,l_r]} \cap R'_{i_1,i_2,\cdots,i_{r-h}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}) \prod_{k=1}^{h} x_k^{i_{r+1-k}}} \tag{3.115}$$

$$\cong \frac{R'_{[l_{h+1},l_{h+2},\cdots,l_r]} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}}{R'_{[l_{h+1},l_{h+2},\cdots,l_r]} \cap R'_{i_1,i_2,\cdots,i_{r-h}} \cap A \prod_{k=h+1}^{r} \theta_k^{i_{r+1-k}}} \tag{3.116}$$

$$\cong \frac{R'_{[l_{h+1},l_{h+2},\cdots,l_r]} \cap R'_{i_1,i_2,\cdots,i_{r-h}+1}}{R'_{[l_{h+1},l_{h+2},\cdots,l_r]} \cap R'_{i_1,i_2,\cdots,i_{r-h}}} \tag{3.117}$$

According to Proposition 3.3.7, we know there are only finitely many distinct factors which implies that the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly normal. □

Remark 3.3.10. Use the same method, if $R = A[\theta_1, \theta_2, \cdots, \theta_r]$ where θ_i is either an indeterminate or integral over A for all $1 \leqslant i \leqslant r$, then the ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is rectangularly and triangularly normal.

Lemma 3.3.11. Let K be a field, $A = K[x_1, x_2, \cdots, x_n]$ where x_i is indeterminate for $1 \leqslant i \leqslant n$, y_1 be another indeterminate, and $R = A[y_1, y_2]$ where $y_2 = \frac{y_1}{x_1}x_2$. For fixed $t, s \in \mathbb{N}$, we have

$$Ay_1^j y_2^i \cap R_{[t,s]} = \begin{cases} 0 & i+j > t+s \\ Ay_1^j y_2^i x_2^{j-t} & j > t, i+j \leqslant t+s \\ Ay_1^j y_2^s x_2^{i-s} & i > s, i+j \leqslant t+s \\ Ay_1^j y_2^i & j \leqslant t, i \leqslant s \end{cases} \tag{3.118}$$

Proof. For any $\alpha, \beta \in N$, if $\alpha + \beta \neq i + j$, we claim that $Ay_1^j y_2^i \cap Ay_1^\alpha y_2^\beta = 0$. Actually, we know that $y_1^j y_2^i = \frac{x_2^i}{x_1^i} y_1^{i+j}, y_1^\alpha y_2^\beta = \frac{x_2^\beta}{x_1^\beta} y_1^{\alpha+\beta}$, and $y_1 \notin A$ is indeterminate which implies $Ay_1^j y_2^i \cap Ay_1^\alpha y_2^\beta = 0$. If $\alpha + \beta = i + j$, we have

$$Ay_1^j y_2^i \cap Ay_1^\alpha y_2^\beta = Ay_1^{i+j} \frac{x_2^{\max(i,\beta)}}{x_1^{\min(i,\beta)}} \tag{3.119}$$

which implies

$$Ay_1^j y_2^i \cap R_{[t,s]} = \sum_{\alpha \leqslant t, \beta \leqslant s} Ay_1^j y_2^i \cap Ay_1^\alpha y_2^\beta = \sum_{\substack{\alpha+\beta=i+j \\ \alpha \leqslant t, \beta \leqslant s}} Ay_1^{i+j} \frac{x_2^{\max(i,\beta)}}{x_1^{\min(i,\beta)}} \tag{3.120}$$

If $t + s < i + j$, we have $Ay_1^j y_2^i \cap R_{[t,s]} = 0$. Otherwise,

1. if $t < j, s \geqslant i$, then $\alpha \leqslant t < j$ implies $\beta = i + j - \alpha > i$.

   Thus, we have

$$\begin{aligned} Ay_1^j y_2^i \cap R_{[t,s]} &= \sum_{\substack{\alpha+\beta=i+j \\ \alpha \leqslant t, \beta \leqslant s}} Ay_1^{i+j} \frac{x_2^{\max(i,\beta)}}{x_1^{\min(i,\beta)}} && (3.121) \\ &= \sum_{\alpha=0}^{t} Ay_1^{i+j} \frac{x_2^{\max(i,i+j-\alpha)}}{x_1^{\min(i,i+j-\alpha)}} && (3.122) \end{aligned}$$

$$= \sum_{\alpha=0}^{t} Ay_1^{i+j}\frac{x_2^{i+j-\alpha}}{x_1^i} \tag{3.123}$$

$$= Ay_1^{i+j}\frac{x_2^{i+j-t}}{x_1^i} \tag{3.124}$$

$$= Ay_1^j y_2^i x_2^{j-t} \tag{3.125}$$

2. if $t \geqslant j, s < i$, then $\beta \leqslant s < i$ implies $\alpha = i + j - \beta > j$.

   Thus, we have

$$Ay_1^j y_2^i \cap R_{[t,s]} = \sum_{\substack{\alpha+\beta=i+j \\ \alpha\leqslant t,\beta\leqslant s}} Ay_1^{i+j}\frac{x_2^{\max(i,\beta)}}{x_1^{\min(i,\beta)}} \tag{3.126}$$

$$= \sum_{\beta=0}^{s} Ay_1^{i+j}\frac{x_2^{\max(i,\beta)}}{x_1^{\min(i,\beta)}} \tag{3.127}$$

$$= \sum_{\beta=0}^{s} Ay_1^{i+j}\frac{x_2^{i}}{x_1^{\beta}} \tag{3.128}$$

$$= Ay_1^{i+j}\frac{x_2^{i}}{x_1^{s}} \tag{3.129}$$

$$= Ay_1^j y_2^s x_2^{i-s} \tag{3.130}$$

3. if $t \geqslant j, s \geqslant i$, we have $Ay_1^j y_2^i \cap R_{[t,s]} = Ay_1^j y_2^i$.

$\square$

Proposition 3.3.12. Let K, A, R be the same as in Lemma 3.3.11. Then the ω^2-filtration $\{R_{i,j}\}_{(i,j)\in\omega^2}$ is rectangularly and triangularly normal.

Proof. For fixed $t, s \in \mathbb{N}$, factors of the ω^2-filtration $R_{[t,s]} \cap \{R_{i,j}\}_{(i,j)\in\omega^2}$ have the following form

$$\frac{R_{i,j+1} \cap R_{[t,s]}}{R_{i,j} \cap R_{[t,s]}} \cong \frac{R_{i,j} \cap R_{[t,s]} + Ay_1^j y_2^i \cap R_{[t,s]}}{R_{i,j} \cap R_{[t,s]}} \tag{3.131}$$

If $i = 0$, we have

$$Ay_1^j \cap R_{[t,s]} = \begin{cases} 0 & j > t+s \\ Ay_1^j x_2^{j-t} & t < j \leqslant t+s \\ Ay_1^j & j \leqslant t \end{cases} \tag{3.132}$$

Since the degree of y_1 in $R_{0,j}$ is at most $j-1$ and y_1 is indeterminate, we have $\frac{R_{0,j+1} \cap R_{[t,s]}}{R_{0,j} \cap R_{[t,s]}} \cong A$ for $j \leqslant t+s$ and $\frac{R_{0,j+1} \cap R_{[t,s]}}{R_{0,j} \cap R_{[t,s]}} = 0$ for $j > t+s$.

Now, we consider the case $i \geqslant 1$.

According to Lemma 3.3.11, if $i + j > t + s$, then $\frac{R_{i,j+1} \cap R_{[t,s]}}{R_{i,j} \cap R_{[t,s]}} = 0$.

If $j \leqslant t, i \leqslant s$, we have $Ay_1^j y_2^i \cap R_{[t,s]} = Ay_1^j y_2^i$. Consider the following surjective map.

$$A \twoheadrightarrow A\overline{y_1^j y_2^i} \tag{3.133}$$

$$\phi : a \to a\overline{y_1^j y_2^i} \tag{3.134}$$

First, $x_1 y_1^j y_2^i = y_1^j y_2^{i-1} x_2 \in R_{i,j} \cap R_{[t,s]}$ implies that $Ax_1 \subseteq \ker \phi$. Second, if $ay_1^j y_2^i \in R_{i,j} \cap R_{[t,s]}$, then $ay_1^j y_2^i \in R_{i,j}$ implies $a \in Ax_1$ according to the proof of Proposition 3.2.6. Thus, $\ker \phi = Ax_1$. We have $\frac{R_{i,j+1} \cap R_{[t,s]}}{R_{i,j} \cap R_{[t,s]}} \cong A/Ax_1$.

If $j > t, i \leqslant t+s-j$, we have $Ay_1^j y_2^i \cap R_{[t,s]} = Ay_1^j y_2^i x_2^{j-t}$. Similarly, consider the following surjective map.

$$A \twoheadrightarrow A\overline{y_1^j y_2^i x_2^{j-t}} \tag{3.135}$$

$$\phi : a \to a\overline{y_1^j y_2^i x_2^{j-t}} \tag{3.136}$$

According to the proof of Proposition 3.2.6, $ay_1^j y_2^i \in R_{i,j}$ is equivalent to $a \in Ax_1$. Thus, $ay_1^j y_2^i x_2^{j-t} \in R_{i,j} \cap R_{[t,s]}$ implies that $ax_2^{j-t} \in Ax_1$, then $a \in Ax_1$. Furthermore, if $a \in Ax_1$, we say $a = bx_1$, then $ay_1^j y_2^i x_2^{j-t} = by_1^{j+1} y_2^{i-1} x_2^{j+1-t} \in Ay_1^{j+1} y_2^{i-1} x_2^{j+1-t} = Ay_1^{j+1} y_2^{i-1} \cap R_{[t,s]} \subseteq R_{i,j} \cap R_{[t,s]}$. It actually means that $\frac{R_{i,j+1} \cap R_{[t,s]}}{R_{i,j} \cap R_{[t,s]}} \cong A/Ax_1$.

If $i > s, j \leqslant t+s-i$, we have $Ay_1^j y_2^i \cap R_{[t,s]} = Ay_1^j y_2^s x_2^{i-s} \subseteq Ay_1^j y_2^s \subseteq R_{i,j}$. Thus, $Ay_1^j y_2^i \cap R_{[t,s]} \subseteq R_{i,j} \cap R_{[t,s]}$ implies $\frac{R_{i,j+1} \cap R_{[t,s]}}{R_{i,j} \cap R_{[t,s]}} = 0$.

In conclusion, for any $t, s \in \mathbb{N}$, factors of the ω^2-filtration $R_{[t,s]} \cap \{R_{i,j}\}_{(i,j) \in \omega^2}$ are isomorphic to $0, A$, or A/Ax_1. Thus, the ω^2-filtration $\{R_{i,j}\}_{(i,j) \in \omega^2}$ is rectangularly normal.

For fixed $d \in \mathbb{N}$, factors of the ω^2-filtration $R_{\{d\}} \cap \{R_{i,j}\}_{(i,j) \in \omega^2}$ have the following form

$$\frac{R_{i,j+1} \cap R_{\{d\}}}{R_{i,j} \cap R_{\{d\}}} \cong \frac{R_{i,j} \cap R_{\{d\}} + Ay_1^j y_2^i \cap R_{\{d\}}}{R_{i,j} \cap R_{\{d\}}} \tag{3.137}$$

If $j + i \leqslant d$, we have $Ay_1^j y_2^i \cap R_{\{d\}} = Ay_1^j y_2^i$. According to the proof of Proposition 3.2.6, $ay_1^j y_2^i \in R_{i,j}$ is equivalent to $a \in Ax_1$. If $ay_1^j y_2^i \in R_{i,j} \cap R_{\{d\}}$, then $ay_1^j y_2^i \in R_{i,j}$ implies that $a \in Ax_1$. If $a \in Ax_1$, then $ay_1^j y_2^i \in R_{i,j}$ and $Ay_1^j y_2^i \cap R_{\{d\}} = Ay_1^j y_2^i$ imply $ay_1^j y_2^i \in R_{i,j} \cap R_{\{d\}}$. Thus, $\frac{R_{i,j+1} \cap R_{\{d\}}}{R_{i,j} \cap R_{\{d\}}} \cong A/Ax_1$.

If $j + i > d$, we have $Ay_1^j y_2^i \cap R_{\{d\}} = 0$ since $Ay_1^j y_2^i = Ay_1^{j+i} \frac{x_2^i}{x_1^i}$ and $y_1 \notin A$ is indeterminate. Then $\frac{R_{i,j+1} \cap R_{\{d\}}}{R_{i,j} \cap R_{\{d\}}} = 0$.

In conclusion, for any $d \in \mathbb{N}$, factors of the ω^2-filtration $R_{\{d\}} \cap \{R_{i,j}\}_{(i,j) \in \omega^2}$ are isomorphic to $0, A$, or A/Ax_1. Thus, the ω^2-filtration $\{R_{i,j}\}_{(i,j) \in \omega^2}$ is triangularly normal. $\square$

CHAPTER IV

The main result on ω-filtrations and applications

We cannot prove that ω^r-filtrations are rectangularly or triangularly normal in general. However, in this Chapter, we derive an ω-filtration in the graded case from the ω^r-filtration. By using a suitable ascending ω-filtration of R or M, we may reduce to studying the graded case. By this method, we bypass all the difficulties that appear in Chapter III.

In the second section, we use these ω-filtrations to give a bound on the number of generators of an intersection of powers of two ideals or the ordinary symbolic powers $I^{(n)}$ under particular restrictions that we will describe later.

4.1 ω-filtrations of rings and modules

In this section, we derive an ω-filtration in the graded case from the ω^r-filtration. By using a suitable ascending ω-filtration of R or M, we may reduce to studying the graded case.

Definition 4.1.1. Let A be a commutative ring and R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. We define the *triangular* graded-ring of R over A as follows:

$$\mathrm{Tri}_A(R) = A \oplus \frac{R_{\{1\}}}{A} \oplus \frac{R_{\{2\}}}{R_{\{1\}}} \oplus \cdots \tag{4.1}$$

where $R_{\{d\}}$ is the triangular submodule of R for $d \in \mathbb{N}$. It is actually the associated graded ring of the filtration $\{R_{\{d\}}\}_{d \geqslant 0}$ and depends on both A and the choices of $\theta_1, \cdots, \theta_r$.

Discussion 4.1.2. When we form the graded ring $\mathrm{Tri}(R)$, the images of the θ_i in degree 1 generate, call them u_i. The ideal of relations on the u_i is homogeneous. One gets a relation in degree d if and only of one has a homogeneous polynomial of degree d in the θ_i that is equal to one of smaller degree. This means that if one maps $A[X_1, ..., X_n]$ onto R so that X_j maps to θ_j and the ideal of relations is J, the ideal of relations on the u_i is generated by all leading (top degree) forms of elements of J.

One can say a bit more from this point of view. Given a polynomial in F in J, one can homogenize it it by using an auxiliary variable, say z, and inserting a power of z into each term of F that is not of highest degree to bring it up to degree $d = \deg(F)$. Formally, this is the same as $z^d F(X_1/z, ..., X_r/z)$.

Consider the ideal generated by all these homogenized polynomials in $A[x_1, ..., x_r, z]$. They generated a homogeneous ideal J^h.

Let $S = A[x, z]/J^h$. Then if one kills $z - 1$ in S, one gets R,. If one kills z, one gets the associated graded ring $\mathrm{Tri}(R)$.

In many cases, this will show enable one to compare the dimensions of $\mathrm{Tri}(R)$ and R: often, they will both be $\dim(S) - 1$.

Remark 4.1.3. The ascending filtrations $R_{\{0\}} \subseteq R_{\{1\}} \subseteq R_{\{2\}} \subseteq R_{\{3\}} \subseteq \cdots$ are interesting. The following example shows that the associated graded ring $\mathrm{Tri}(R)$ depends heavily on the generators.

Suppose we start with $R = K[x]$ and use the generator $\theta_1 = x$. The associated graded is isomorphic with R.

Suppose we use $\theta_1 = x$ and $\theta_2 = x^2$. Then the n-th submodule consists of everything of degree at most $2n$, and the quotient is spanned by the images of x^{2n-1}, x^{2n}.

Suppose we denote the images of x, x^2 in the degree 1 piece by u, v. Then the degree n piece is spanned by v^n, uv^{n-1}. Note that $u^2 = 0$. This ring is isomorphic with $K[u, v]/(u^2)$.

It is not isomorphic to R.

Map $A[x_1, \cdots, x_r] \twoheadrightarrow R$. Let J be the kernel and J^h be generated by all homogeneous polynomial $F(x_1, \cdots, x_r, z) \in A[x_1, \cdot, x_r, z]$ such that $F(x_1, \cdots, x_r, 1) \in J$. Then z is not a zero-divisor on J^h from the definition and $z-1$ is not a zero-divisor because J^h is homogeneous. Let $S = A[x_1, \cdots, x_r, z]/J^h$. Then we have that $S/(z) \cong \mathrm{Tri}(R)$ and $S/(z-1) \cong R$. When $A = K$ and in many other situations, this implies $\dim(\mathrm{Tri}(R)) = \dim(R)$.

Proposition 4.1.4. $\mathrm{Tri}(R)$ is a standard $\mathbb{N}$-graded ring. If A is Noetherian, so is $\mathrm{Tri}(R)$.

Proof. Clearly, for any $i, j \geqslant 0$, we have $R_{\{i\}}R_{\{j\}} \subseteq R_{\{i+j\}}$. Notice that we denote $R_{\{0\}} = A$ and $R_{\{-1\}} = (0)$. Since $R_{\{i\}}R_{\{j-1\}}, R_{\{i-1\}}R_{\{j\}} \subseteq R_{\{i+j-1\}}$, there is a natural morphism as follows:

$$\frac{R_{\{i\}}}{R_{\{i-1\}}} \otimes_A \frac{R_{\{j\}}}{R_{\{j-1\}}} \to \frac{R_{\{i+j\}}}{R_{\{i+j-1\}}} \tag{4.2}$$

which means that $\mathrm{Tri}(R)$ is an $\mathbb{N}$-graded ring. Furthermore, $\mathrm{Tri}(R)$ is a finitely generated A-algebra with generators $\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_r} \in \frac{R_{\{1\}}}{A}$. This actually proves that $\mathrm{Tri}(R)$ is a standard $\mathbb{N}$-graded ring. Clearly, if A is Noetherian, so is $\mathrm{Tri}(R)$. $\square$

Proposition 4.1.5. Let A be a Noetherian ring and R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. If R is a standard $\mathbb{N}$-graded ring, say $R = R_0 \oplus R_1 \oplus R_2 \cdots$ where $R_h = \sum_{i_1+\cdots+i_r=h} A\theta_1^{i_1}\cdots\theta_r^{i_r}$ for any $h \geqslant 0$, there exists an ω-filtration of R in which the factors are prime cyclic A-modules and only finitely many distinct factors occur. Furthermore, the length of the inherited finite filtration of R_h is at most Ch^{r-1} where $h \geqslant 0$ and C is a constant. For $h \gg 0$ and any factor in the filtration of R_h, the number of copies of this factor is a polynomial of degree at most $r-1$.

Proof. According to Proposition 3.2.1, R has an ω^r-filtration $\{R_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ in which all the factors are cyclic A-modules and only finitely many distinct factors will occur.

Now we consider the inherited ω^r-filtration of R_h where $h \geqslant 0$. Since R_h is a finitely generated A-module , this inherited ω^r-filtration of R_h is actually a finite filtration according to Lemma 3.3.2. By definition,

$$R_{i_1,i_2,\cdots,i_r+1} = R_{i_1,i_2,\cdots,i_r} + A\prod_{k=1}^{r}\theta_{r+1-k}^{i_k} \tag{4.3}$$

We denote the piece of $R_{i_1,i_2,\cdots,i_r}$ of degree h by $R_{i_1,i_2,\cdots,i_r||h}$, then we have

$$\frac{R_{i_1,i_2,\cdots,i_r+1}}{R_{i_1,i_2,\cdots,i_r}} = \frac{R_{i_1,i_2,\cdots,i_r} + A\prod\limits_{k=1}^{r}\theta_{r+1-k}^{i_k}}{R_{i_1,i_2,\cdots,i_r}} \cong \frac{R_{i_1,i_2,\cdots,i_r+1||i_1+\cdots+i_r}}{R_{i_1,i_2,\cdots,i_r||i_1+\cdots+i_r}} \tag{4.4}$$

which means this factor is isomorphic to the factor of the finite filtration of $R_{i_1+\cdots+i_r}$. Thus, only finitely many distinct factors of the inherited finite filtrations of all R_h for all $h \geqslant 0$ may occur.

For simplicity, we denote the finite filtration of R_h by $0 = R_{h|0} \subseteq R_{h|1} \subseteq R_{h|2} \subseteq \cdots \subseteq R_{h|n_h} = R_h$. As we know all, the length n_h of this filtration is less than or equal to the number of monomials of r variables of degree h which is $\binom{h+r-1}{r-1}$. Then, we can construct the following filtration of R.

$$\begin{aligned} 0 =& R_{0|0} \subseteq R_{0|1} \subseteq \cdots \subseteq R_{0|n_0} \subseteq R_0 \oplus R_{1|0} \subseteq R_0 \oplus R_{1|1} \subseteq \cdots \subseteq R_0 \oplus R_{1|n_1} \\ &\subseteq \cdots \subseteq R_0 \oplus \cdots \oplus R_h \oplus R_{h+1|0} \subseteq \cdots \subseteq R_0 \oplus \cdots \oplus R_h \oplus R_{h+1|n_{h+1}} \subseteq \cdots \end{aligned}$$

We have

$$\bigcup_{h=0}^{\infty}\bigcup_{k=0}^{n_h}\bigoplus_{j=0}^{h-1} R_j \oplus R_{h|k} = \bigcup_{h=0}^{\infty}\bigoplus_{j=0}^{h} R_j = R \tag{4.5}$$

which means this is an ω-filtration in which the factors are cyclic A-modules and only finitely many distinct factors occur.

Since for each cyclic A-module, there is a finite filtration in which the factors are prime cyclic. Thus, there exists an ω-filtration of R in which the factors are prime cyclic A-modules and only finitely many distinct factors occur.

The above argument also works for cyclic R-modules. We denote $M = R\alpha$ and $D = A\alpha$. We may replace A and R by D and M, respectively.

We claim that for $h \gg 0$ and any factor in the filtration of M_h, the number of copies of this factor is a polynomial of degree at most $r-1$.

If $r = 1$, it is obviously true since $M_h = D\theta_1^h$.

If $r < s$, the claim holds. We denote $N = D[\theta_1, \cdots, \theta_{s-1}] = N_0 \oplus N_1 \oplus N_2 \oplus \cdots$ and $M = N[\theta_s] = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$. By induction hypothesis, M has a ω-filtration in which the factors are prime cyclic $A[\theta_1, \cdots, \theta_{s-1}]$-modules and only finitely many distinct factors occur. For $h \gg 0$ any factor in the filtration of M_h, the number of this factor is a constant. As we all know, $A[\theta_1, \cdots, \theta_{s-1}]/\mathfrak{Q} \cong \frac{A}{A\cap\mathfrak{Q}}[\overline{\theta_1}, \cdots, \overline{\theta_{s-1}}]$. By induction hypothesis, $\frac{A}{A\cap\mathfrak{Q}}[\overline{\theta_1}, \cdots, \overline{\theta_{s-1}}]$ has a ω^{s-1}-filtration in which the factors are prime cyclic $A[\theta_1, \cdots, \theta_{s-1}]$-modules and only finitely many distinct factors occur. For $h \gg 0$ any factor in the filtration of degree h piece of $\frac{A}{A\cap\mathfrak{Q}}[\overline{\theta_1}, \cdots, \overline{\theta_{s-1}}]$, the number of this factor is a polynomial of degree $s-1$. Furthermore, $M = D[\theta_1, \cdots, \theta_s] = M_1' \oplus M_2' \oplus M_3' \oplus \cdots$, M_h' is sum of the degree j piece of the factors in the filtration of M_{h-j}. Thus, the number of copies of the given factor is the sum of the number of copies of this factor in degree j piece of the factors in the filtration of M_{h-j}. It is a polynomial of degree at most $r-1$ since the sum of h polynomials is a polynomial and the degree increases at most 1. $\square$

Remark 4.1.6. This Proposition also works for a standard $\mathbb{N}$-graded module M over R.

Remark 4.1.7. If R (and M) are graded by $\mathbb{N}$ or $\mathbb{N}^h$, then one can form the filtration to be compatible with the given grading. This shows that one can localize at one nonzero element of A so that all components become A-free. (If one only knows the whole ring or the whole module is A-free, it is automatic that the graded or multi-graded pieces are projective, but one does not know they are actually free). In the argument, simply choose all generators of M and all of the θ_i to be homogeneous (or multi-homogeneous). The resulting filtration will then factor all homogeneous or multi-homogeneous components.

We give an immediate consequence as an example.

Corollary 4.1.8. Let R be a Noetherian ring and $M_1, \cdots, M_k$ be finitely generated R-modules. Let $I_1, \cdots, I_k$ be k ideals of R with $d_1, \cdots, d_k$ generators, respectively. Denote $A = \frac{R}{I_1+\cdots+I_k}$. Then there exists an ω-filtration of $\operatorname{Tor}_h(\operatorname{gr}_{I_1} M_1, \cdots, \operatorname{gr}_{I_k} M_k)$ in which the factors are prime cyclic A-modules and only finitely many distinct factors occur. For any $n \gg 0$ and any factor in the filtration of $\bigoplus_{s_1+\cdots+s_k=n} \operatorname{Tor}_h(\frac{I_1^{s_1}M_1}{I_1^{s_1+1}M_1}, \cdots, \frac{I_k^{s_k}M_k}{I_k^{s_k+1}M_k})$, the number of copies of this factor is a polynomial function in terms of n of degree at most $d_1+\cdots+d_k-1$. Then we have that $\mu(\bigoplus_{s_1+\cdots+s_k=n} \operatorname{Tor}_h(\frac{I_1^{s_1}M_1}{I_1^{s_1+1}M_1}, \cdots, \frac{I_k^{s_k}M_k}{I_k^{s_k+1}M_k})) = \mathcal{O}(n^{d_1+\cdots+d_k-1})$.

Proof. According to Remark 2.11.3, we know that $\operatorname{Tor}_h(\operatorname{gr}_{I_1} M_1, \cdots, \operatorname{gr}_{I_k} M_k)$ is a finitely generated module over $\operatorname{gr}_{I_1} R \otimes \cdots \otimes \operatorname{gr}_{I_k} R$. We have that

$$\operatorname{Tor}_h(\operatorname{gr}_{I_1} M_1, \cdots, \operatorname{gr}_{I_k} M_k) = \bigoplus_n (\bigoplus_{s_1+\cdots+s_k=n} \operatorname{Tor}_h(\frac{I_1^{s_1}M_1}{I_1^{s_1+1}M_1}, \cdots, \frac{I_k^{s_k}M_k}{I_k^{s_k+1}M_k})) \tag{4.6}$$

We also know that $\operatorname{gr}_{I_1} R \otimes \cdots \otimes \operatorname{gr}_{I_k} R$ is a finitely generated graded algebra over $A = \frac{R}{I_1+\cdots+I_k}$. According to Proposition 4.1.5, we are done. □

Remark 4.1.9. Let R be a Noetherian ring and M_1, M_2 be finitely generated R-modules. Let I_1, I_2 be two ideals of R with d_1, d_2 generators, respectively. Denote $A = \frac{R}{I_1+I_2}$. Then there exists an ω-filtration of $\operatorname{Ext}^h(\operatorname{gr}_{I_1} M_1, \operatorname{gr}_{I_2} M_2)$ in which the factors are prime cyclic A-modules and only finitely many distinct factors occur. For any $n \gg 0$ and any factor in the filtration of $\bigoplus_{s_1+s_2=n} \operatorname{Ext}^h(\frac{I_1^{s_1}M_1}{I_1^{s_1+1}M_1}, \frac{I_2^{s_2}M_2}{I_2^{s_2+1}M_2})$, the number of copies of this factor is a polynomial function in terms of n of degree at most d_1+d_2-1. Then we have that $\mu(\bigoplus_{s_1+s_2=n} \operatorname{Ext}^h(\frac{I_1^{s_1}M_1}{I_1^{s_1+1}M_1}, \frac{I_2^{s_2}M_2}{I_2^{s_2+1}M_2})) = \mathcal{O}(n^{d_1+d_2-1})$.

Theorem 4.1.10. Let A be a Noetherian ring and R be a finitely generated A-algebra. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$. There exists an ω-filtration of R in which the factors are prime cyclic A-modules and only finitely many distinct factors occur. All the $R_{\{h\}}$ occur in the filtration. Furthermore, the length of the inherited finite filtration from $R_{\{h-1\}}$ to $R_{\{h\}}$ is at most Ch^{r-1}, where $h \geqslant 0$. Notice that

$R_{\{-1\}} = 0$. For all $h \gg 0$, the number of copies of $A/\mathfrak{p}$ occurring as a factor in the filtration of $R_{\{h\}}$ agrees with a polynomial in h of degree at most r.

Proof. According to Proposition 4.1.4, $\mathrm{Tri}(R)$ is a standard $\mathbb{N}$-graded ring. It is a finitely generated A-algebra with generators $\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_r} \in \frac{R_{\{1\}}}{A}$. Denote $R' = \mathrm{Tri}(R)$. And we denote $R'_h = R_{\{h\}}/R_{\{h-1\}}$. Thus, we have $R' = R'_0 \oplus R'_1 \oplus R'_2 \cdots$. According to the previous proposition, we have the finite filtration of R'_h denoted by $0 = R'_{h|0} \subseteq R'_{h|1} \subseteq R'_{h|2} \subseteq \cdots \subseteq R'_{h|n_h} = R'_h$. Since $R'_h = R_{\{h\}}/R_{\{h-1\}}$, there is a finite filtration from $R_{\{h-1\}}$ to $R_{\{h\}}$ such that factors are the same as the factors of the finite filtration of R'_h. Furthermore, $R = \bigcup_{d=0}^{\infty} R_{\{d\}}$ implies that we actually construct an ω-filtration of R in which the factors are prime cyclic A-modules and only finitely many distinct factors occur.

For all $h \gg 0$, the number of copies of $A/\mathfrak{p}$ occurring as a factor in the filtration of $R_{\{h\}}/R_{\{h-1\}}$ agrees with a polynomial in h of degree at most $r-1$ according to the previous proposition. The sum of polynomials of degree $r-1$ is a polynomial of degree r. Thus, the number of copies of $A/\mathfrak{p}$ occurring as a factor in the filtration of $R_{\{h\}}$ agrees with a polynomial in h of degree at most r $\square$

Corollary 4.1.11. Let A be a Noetherian ring, R be a finitely generated A-algebra, and M be a finitely generated R-module. There exists a general ω-filtration in which the factors are prime cyclic A-modules and only finitely many distinct factors will occur.

Proof. According to Proposition 2.6.6, there exists $0 = M_0 \subseteq M_1 \subseteq \cdots M_n = M$ such that M_{h+1}/M_h are prime cyclic R-modules for $h \geqslant 0$. According to Theorem 4.1.10, M_{h+1}/M_h has an ω-filtration in which the factors are prime cyclic A-modules and only finitely many distinct factors will occur. This actually gives us a general ω-filtration in which the factors are prime cyclic A-modules and only finitely many distinct factors will occur. $\square$

Theorem 4.1.12. Let A be a Noetherian ring, R be a finitely generated A-algebra, and M be a finitely generated R-module. There exists an ω-filtration M in which the factors are prime cyclic A-modules and only finitely many distinct factors will occur. For all $h \gg 0$, the number of copies of $A/\mathfrak{p}$ occurring as a factor in the filtration of $M_{\{h\}}$ agrees with a polynomial in h of degree at most r.

Proof. We may replace A by its image in R and assume $R = A[\theta_1, \theta_2, \cdots, \theta_r]$ and $M = R\alpha_1 + R\alpha_2 + \cdots + R\alpha_s$. Denote $B = A\alpha_1 + A\alpha_2 + \cdots + A\alpha_s$. Then, $M = B[\theta_1, \theta_2, \cdots, \theta_r]$. We denote $M_{\{d\}} = \sum\limits_{i_1+\cdots+i_r \leqslant d} B\theta_1^{i_1} \cdots \theta_r^{i_r}$. $\mathrm{Tri}(M) = B \oplus M_{\{1\}}/B \oplus M_{\{2\}}/M_{\{1\}} \oplus \cdots$ is an $\mathbb{N}$-graded module over $\mathrm{Tri}(R)$. Similar to $R_{i_1,i_2,\cdots,i_r}$, we can define $M_{i_1,i_2,\cdots,i_r}$ as follows:

$$\begin{aligned}
M_{i_1,i_2,\cdots,i_r} &= \sum_{i_1'=0}^{i_1-1} B[\theta_1, \theta_2, \cdots, \theta_{r-1}]\theta_r^{i_1'} \\
&\quad + \sum_{i_2'=0}^{i_2-1} B[\theta_1, \theta_2, \cdots, \theta_{r-2}]\theta_{r-1}^{i_2'}\theta_r^{i_1} \\
&\quad + \sum_{i_3'=0}^{i_3-1} B[\theta_1, \theta_2, \cdots, \theta_{r-3}]\theta_{r-2}^{i_3'}\theta_{r-1}^{i_2}\theta_r^{i_1} \\
&\quad + \cdots \\
&\quad + \sum_{i_r'=0}^{i_r-1} B\theta_1^{i_r'}\theta_2^{i_{r-1}} \cdots \theta_r^{i_1} \\
&= \sum_{j=1}^{r} \sum_{i_j'=0}^{i_j-1} B[\theta_1, \theta_2, \cdots, \theta_{r-j}]\theta_{r-j+1}^{i_j'} \prod_{k=1}^{j-1} \theta_{r+1-k}^{i_k}
\end{aligned}$$

We can see that $\{M_{i_1,i_2,\cdots,i_r}\}_{(i_1,i_2,\cdots,i_r)\in\omega^r}$ is an ω^r-filtration of M in which all the factors are finitely generated A-modules and only finitely many distinct factors will occur.

As in the proof of Theorem 4.1.10, we know that there exists an ω-filtration of M in which the factors are finitely generated A-modules and only finitely many distinct factors will occur. For these factors, according to Proposition 2.6.6, since they are finitely generated A-module, there is a finite filtration in which all factors are prime cyclic A-modules. There exists C which is the uniform bound of length of each finite filtration. Thus, there exists an ω-filtration of M in which the factors are prime cyclic A-modules and only finitely many distinct factors may occur.

For all $h \gg 0$, the number of copies of $A/\mathfrak{p}$ occurring as a factor in the filtration of $M_{\{h\}}$

agrees with a polynomial in h of degree at most r. $\square$

Remark 4.1.13. It is easy to see that this ω-filtration is triangularly normal.

Remark 4.1.14. In the situation of the theorem above, suppose that $\mathfrak{p} \in \operatorname{Spec}(A)$ is minimal in the support, over A, of $\operatorname{Tri}(M)$. Then $A_0 = (A/\operatorname{Ann}_A(\operatorname{Tri}(M)))_{\mathfrak{p}}$ is an Artin local ring because $\mathfrak{p}$ is minimal. Then the number of copies of $A/\mathfrak{p}$ in the filtration of $M_{\{d\}}$ or of $[\operatorname{Tri}(M)]_{\leqslant d}$, *i.e.*, the pieces of $\operatorname{Tri}(M)$ with degree at most d, is the same as $\operatorname{length}_{A_0}([\operatorname{Tri}(M_{\mathfrak{p}})]_{\leqslant d})$ which may be thought of as a finitely generated graded module over a standard graded algebra over A_0. But this length agrees with a cumulative Hilbert function and for $d \gg 0$ agree with a polynomial of degree at most the Krull dimension of $\operatorname{Tri}(M)_{\mathfrak{p}}$.

Remark 4.1.15. The degree bound for the eventual behavior of the number of occurrences of a specific prime cyclic A-module for a finitely generated R-module M over a ring R can sometimes be improved, or utilized in a more general context, as follows.

(1) In the module case, one may replace the ring by its quotient by the annihilator of M, either before or after passing to the graded case.

(2) In the graded case, if R is integral (hence, module-finite) over an A-subalgebra R' generated by r' forms of degree 1, where $r' < r$, one may view M or R as a finite module over R', and improve the degree bound to r'.

(3) If R is not standard, but has generators of varying degrees, let L be the least common multiple of these degrees. The generators of R have powers of degree L, and these will generate an A-subalgebra of R, call it R', that may be thought of as standard once the degrees are divided by L. R is module-finite over R'. If $0 \leqslant \rho \leqslant L-1$, let ${}_{\rho}M$ denote the direct sum of the homogeneous components of M in degrees that are congruent to ρ mod L. Then M is the direct sum of these finitely many ${}_{\rho}M$, and every ${}_{\rho}M$ is a finitely generated module over R', to which our results already apply. This will yield L polynomials, F_{ρ}, $0 \leqslant \rho \leqslant L-1$ such that the number of occurrences of $A/\mathfrak{p}$ in degree d (we may write $d = \lfloor d/L \rfloor L + \rho$) is $F_{\rho}(\lfloor d/L \rfloor)$. This is entirely similar to the behaviour of Hilbert functions in non-standard $\mathbb{N}$-graded algebras over a field.

4.2 Upper bounds on the number of minimal generators

In this section, we first construct a finite filtration of R/I^n with properties that we will describe later in the section. Then we use these filtrations to give a bound on the number of generators of an intersection of powers of two ideals or the ordinary symbolic powers $I^{(n)}$ under particular restrictions that we will describe in this section.

Lemma 4.2.1. Let R be a Noetherian ring and I be an ideal of R with r generators. For any $n \in \mathbb{N}$, there exists a finite filtration of R/I^n in which the factors have the form $R/\mathfrak{Q}$ where $\mathfrak{Q}$ is a prime ideal of R containing I. There are only finitely many distinct $R/\mathfrak{Q}$ in all of the filtrations of R/I^n, and the number of copies of each is eventually polynomial in n of degree at most r.

Proof. For each R/I^n, we have a finite filtration defined as follows:

$$0 = \frac{I^n}{I^n} \subseteq \frac{I^{n-1}}{I^n} \subseteq \cdots \subseteq \frac{I^1}{I^n} \subseteq \frac{I^0}{I^n} = \frac{R}{I^n} \tag{4.7}$$

Each factor has the form $\frac{I^{h-1}}{I^n}/\frac{I^h}{I^n} \cong \frac{I^{h-1}}{I^h}$ where $1 \leqslant h \leqslant n$. The associated graded ring $\mathrm{gr}_I(R)$ is a standard $\mathbb{N}$-graded ring over $A = R/I$, according to Proposition 4.1.5, there is an ω-filtration of $\mathrm{gr}_I(R)$ in which all factors are prime cyclic $\frac{R}{I}$-modules and only finitely many distinct factors occur. The factors of inherited finite filtration of $\frac{I^{h-1}}{I^h}$ have the form $\frac{R}{I}/\mathfrak{Q}' \cong \frac{R}{I}/\frac{\mathfrak{Q}}{I} \cong \frac{R}{\mathfrak{Q}}$ where $\mathfrak{Q}'$ is a prime ideal of $\frac{R}{I}$ and $\mathfrak{Q}$ is the corresponding ideal of $\mathfrak{Q}'$ in R containing I. Thus, there is a finite filtration from $\frac{I^h}{I^n}$ to $\frac{I^{h-1}}{I^n}$. Then, there is a finite filtration of $\frac{R}{I^n}$ in which the factors have the form $R/\mathfrak{Q}$ where $\mathfrak{Q}$ is a prime ideal of R containing I. The number of copies of each $R/\mathfrak{Q}$ is eventually polynomial in n of degree at most r since the sum of polynomials of degree $r-1$ is a polynomial of degree r. $\square$

Remark 4.2.2. Craig Huneke and Ilya Smirnov prove that for all n , simultaneously, they can choose prime filtrations of R/I^n such that the set of primes appearing in these filtrations is finite in the paper [13].

Theorem 4.2.3. Let T be a functor of k variable R-modules (it may be covariant in some variables and contravariant in others) such that if all but the i th module are held fixed, producing a functor F of the module in the i th spot, and one has a short exact sequence $0 \to M' \to M \to M'' \to 0$, then the sequence $(*) \quad F(M') \to F(M) \to F(M'')$ (with the roles of M'' and M' interchanged in the contravariant case) is exact at the middle spot. Suppose we have modules $M_1, \cdots, M_k$ and M_i has a finite filtration $0 = M_{i|0} \subseteq M_{i|1} \subseteq \cdots \subseteq M_{i|n_i} = M_i$ with factors $M_{i,j} = M_{i|j}/M_{i|j-1}$, $1 \leqslant j \leqslant n_i$ of M_i, $1 \leqslant i \leqslant k$. Then $T(M_1, \cdots, M_k)$ has a finite filtration whose factors are subquotients of the $n_1 \cdots n_k$ modules $T(M_{1,j_1}, \cdots, M_{k,j_k})$.

Proof. We use induction on k and on n_k. Assume $k = 1$. We give the argument for the covariant case. The argument for the contravariant case is identical. If $n_1 = 1$ there is nothing to prove. If $n_1 > 1$, we have an exact sequence $T(M_{1|n_1-1}) \to T(M_1) \to T(M_{1,n_1})$. By the induction hypothesis and the Lemma 2.8.6, the first term has a filtration by subquotients of the modules $T(M_{1,j})$, $1 \leqslant j \leqslant n_1 - 1$, which induces such a filtration on the image N of the first map, while the quotient of the middle module by N is a submodule of $T(M_{1,n_1})$. If $k > 1$, and we hold $M_1, \cdots, M_{k-1}$ fixed, we get a filtration of $T(M_1, \cdots, M_{k-1}, M_k)$ by subquotients of the modules $T(M_1, \cdots, M_{k-1}, M_{k,j_k})$ using the case $k = 1$. The result is then immediate from the induction hypothesis and the Lemma 2.8.6. □

Corollary 4.2.4. Let R be a Noetherian ring. Let I and J be two ideals of R with d and d' generators, respectively. For $m, n \gg 0$, $\mathrm{Tor}_h(\frac{R}{I^m}, \frac{R}{J^n})$ has a finite filtration with length at most $Cm^d n^{d'}$ where C is a constant. The factors in this filtration are subquotients of $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}}, \frac{R}{\mathfrak{Q}'})$ where $\mathfrak{Q}$ and $\mathfrak{Q}'$ are two prime ideals of R containing I and J, respectively. Furthermore, there are only finitely many distinct $\mathfrak{Q}$ and $\mathfrak{Q}'$ occurring in two filtrations.

Proof. According to Lemma 4.2.1 and Theorem 4.2.3, we get the corollary directly. □

Corollary 4.2.5. Let R be a ring that is semi-local or finitely generated over a field. Let I, J be two ideals of R. We suppose that $\dim(\frac{R}{I+J}) \leqslant 1$. For any prime ideals $\mathfrak{Q}$ and $\mathfrak{Q}'$

containing I and J, respectively, there is a upper bound on the number of generators of any submodules (and, hence, of all subquotients) of $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}}, \frac{R}{\mathfrak{Q}'})$.

Proof. $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}}, \frac{R}{\mathfrak{Q}'})$ is a finitely generated $\frac{R}{\mathfrak{Q}+\mathfrak{Q}'}$-module. Since we have $\frac{R}{I+J} \twoheadrightarrow \frac{R}{\mathfrak{Q}+\mathfrak{Q}'}$,

$$\dim(\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}}, \frac{R}{\mathfrak{Q}'})) \leqslant \dim(\frac{R}{\mathfrak{Q}+\mathfrak{Q}'}) \leqslant \dim(\frac{R}{I+J}) \leqslant 1$$

According to Lemma 2.9.1, we actually prove the corollary. □

Theorem 4.2.6. Let R be a ring that is semi-local or finitely generated over a field. Let I, J be two ideals of R with d and d' generators, respectively. Suppose also that $\dim(\frac{R}{I+J}) \leqslant 1$. For $m, n \gg 0$, we have $\mu(\mathrm{Tor}_h(\frac{R}{I^m}, \frac{R}{J^n})) = \mathcal{O}(m^d n^{d'})$.

Proof. According to Corollary 4.2.4, $\mathrm{Tor}_h(\frac{R}{I^m}, \frac{R}{J^n})$ has a finite filtration with length $Cm^d n^{d'}$ where C is a constant. The factors in this filtration are subquotients of $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}}, \frac{R}{\mathfrak{Q}'})$ where $\mathfrak{Q}$ and $\mathfrak{Q}'$ are two prime ideals of R containing I and J, respectively.

According to Corollary 4.2.5, there is a bound on the number of generators of subquotients of $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}}, \frac{R}{\mathfrak{Q}'})$. Furthermore, there are only finitely many distinct $\mathfrak{Q}$ and $\mathfrak{Q}'$ occurring in two filtrations. Thus, there is a bound on the number of generators of factors of the finite filtration of $\mathrm{Tor}_h(\frac{R}{I^m}, \frac{R}{J^n})$.

According to Proposition 2.7.20, we know that $\mu(I^m \cap J^n) = \mathcal{O}(m^d n^{d'})$. □

Theorem 4.2.7. Let R be a ring that is semi-local or finitely generated over a field. Let $I_1, \cdots, I_k$ be k ideals of R with $d_1, \cdots, d_k$ generators, respectively. Suppose also that $\dim(\frac{R}{I_1+\cdots+I_k}) \leqslant 1$. For $n_1, \cdots, n_k \gg 0$, we have $\mu(\mathrm{Tor}_h^R(\frac{R}{I_1^{n_1}}, \cdots, \frac{R}{I_k^{n_k}})) = \mathcal{O}(n_1^{d_1} \cdots n_k^{d_k})$. If $k = 2$, the corresponding fact also holds for Ext_R^h, hence, for Hom_R.

Proof. According to Lemma 4.2.1 and Theorem 4.2.3, for $n_1, \cdots, n_k \gg 0$, $\mathrm{Tor}_h(\frac{R}{I_1^{n_1}}, \cdots, \frac{R}{I_k^{n_k}})$ has a finite filtration with length at most $Cn_1^{d_1} \cdots n_k^{d_k}$ where C is a constant. The factors in this filtration are subquotients of $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}_1}, \cdots, \frac{R}{\mathfrak{Q}_k})$ where $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_k$ are prime ideals of R

containing $I_1, \cdots, I_k$, respectively. Furthermore, there are only finitely many distinct prime ideals $\mathfrak{Q}_j$ occurring.

As we all know, $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}_1}, \cdots, \frac{R}{\mathfrak{Q}_k})$ is a finitely generated $\frac{R}{\mathfrak{Q}_1+\cdots+\mathfrak{Q}_k}$-module. Then, we have $\dim(\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}_1}, \cdots, \frac{R}{\mathfrak{Q}_k})) \leqslant 1$. According to Lemma 2.9.1, there is a bound on the number of generators of any submodules (and, hence, of all subquotients) of $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}_1}, \cdots, \frac{R}{\mathfrak{Q}_k})$. Since there are only finitely many distinct $\mathfrak{Q}_j$, there is a uniform upper bound on the number of generators of any submodules (and, hence, of all subquotients) of $\mathrm{Tor}_h(\frac{R}{\mathfrak{Q}_1}, \cdots, \frac{R}{\mathfrak{Q}_k})$.

According to Proposition 2.7.20, we know that $\mu(\mathrm{Tor}_h(\frac{R}{I_1^{n_1}}, \cdots, \frac{R}{I_k^{n_k}})) = \mathcal{O}(n_1^{d_1} \cdots n_k^{d_k})$. $\square$

Theorem 4.2.8. Let $R, \mathfrak{p}, I_0$ be the same as in Proposition 2.7.30. Assume that R is semi-local or finitely generated over a field. Let $h = \mathrm{height}(I_0 \frac{R}{\mathfrak{p}})$. Suppose that $\dim(R/\mathfrak{p}) = \dim(\frac{R}{I_0+\mathfrak{p}}) + h$ and $\dim(\frac{R}{I_0+\mathfrak{p}}) \leqslant 1$. For $n \gg 0$, we have $\mu(\mathfrak{p}^{(n)}) = \mathcal{O}(n^{d+h})$.

Proof. Denote $R/\mathfrak{p} = D$. Pick $a_1 \in I_0 - \bigcup$ minimal primes of D, then $\dim(D/(a_1)) \leqslant \dim(D) - 1$ and $\mathrm{height}(I_0/(a_1)) \geqslant h - 1$. By induction we can pick $a_1, \cdots, a_h \in I_0 - \mathfrak{p}$ such that $\dim(D/(a_1, \cdots, a_h)) \leqslant \dim(D) - h = \dim(\frac{R}{I_0+\mathfrak{p}}) \leqslant 1$.

Denote $J = (a_1^h, \cdots, a_h^h)$. According to Proposition 2.7.30, we have $\mathfrak{p}^{(n)} = \mathfrak{p}^n : J^n$. As we all know, $\mathrm{Hom}(\frac{R}{A}, \frac{R}{B}) \cong \frac{B:A}{B}$, then $\mathrm{Hom}(R/J^n, R/\mathfrak{p}^n) \cong \frac{\mathfrak{p}^n:J^n}{\mathfrak{p}^n} \cong \frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^n}$. We have that $\dim(\frac{R}{J+\mathfrak{p}}) = \dim(\frac{D}{(a_1,\cdots,a_h)}) \leqslant 1$. According to Theorem 4.2.7, we have $\mu(\frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^n}) = \mathcal{O}(n^{d+h})$.

According to Proposition 2.7.19, we know that $\mu(\mathfrak{p}^{(n)}) \leqslant \mu(\mathfrak{p}^n) + \mu(\frac{\mathfrak{p}^{(n)}}{\mathfrak{p}^n})$, thus $\mu(\mathfrak{p}^{(n)}) = \mathcal{O}(n^{d+h})$. $\square$

Lemma 4.2.9. Let R be a commutative ring and I, J be two ideals of R. Then we have the following inequality:

$$\mu(I^m \cap J^n) \leqslant \mu(I^m J^n) + \mu(\mathrm{Tor}_1(\frac{R}{I^m}, \frac{R}{J^n})) \tag{4.8}$$

where $n, m \in \mathbb{N}$.

Proof. We have the following short exact sequence:

$$0 \to I^m J^n \to I^m \cap J^n \to \frac{I^m \cap J^n}{I^m J^n} \to 0 \tag{4.9}$$

According to Proposition 2.7.19, we know that $\mu(I^m \cap J^n) \leqslant \mu(I^m J^n) + \mu(\frac{I^m \cap J^n}{I^m J^n})$. As we all know, $\mathrm{Tor}_1(\frac{R}{I^m}, \frac{R}{J^n}) \cong \frac{I^m \cap J^n}{I^m J^n}$ which implies the desired inequality. □

Theorem 4.2.10. Let R be a ring that is semi-local or finitely generated over a field. Let I, J be two ideals of R with d and d' generators, respectively. Suppose that $\dim(\frac{R}{I+J}) \leqslant 1$. For $m, n \gg 0$, there is a polynomial upper bound on the number of generators of $I^m \cap J^n$. Specifically, we have $\mu(I^m \cap J^n) = \mathcal{O}(m^d n^{d'})$.

Proof. According to Lemma 4.2.9, we have

$$\mu(I^m \cap J^n) \leqslant \mu(I^m J^n) + \mu(\mathrm{Tor}_1(\frac{R}{I^m}, \frac{R}{J^n})) \tag{4.10}$$

$\mu(I^m J^n) \leqslant \binom{m+d-1}{d-1}\binom{n+d'-1}{d'-1} = \mathcal{O}(m^{d-1} n^{d'-1})$.

According to Theorem 4.2.6, we know that $\mu(I^m \cap J^n) = \mathcal{O}(m^d n^{d'})$. □

Corollary 4.2.11. Let R be a Noetherian ring and $I = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_k$ with d generators where $\mathfrak{Q}_i$ is primary to $\mathfrak{p}_i$. The $\mathfrak{p}_i$ are mutually incomparable and $1 \leqslant \dim(R/\mathfrak{p}_i) \leqslant 2$. Assume no $\mathfrak{p}_i$ is strictly contained in any associated prime $q \in \mathrm{Ass}(R)$. For $n \gg 0$, $\mu(I^{(n)}) = \mathcal{O}(n^{d+1})$.

Proof. According to Proposition 2.7.33, there exists a non zero-divisor $x \in R - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k \cup (\bigcup_{q \in \mathrm{Ass}(R)} q)$ such that $I^{(n)} = I^n : x^n$ for any n.

Then we have

$$\mu(I^{(n)}) = \mu(I^n : x^n) = \mu((I^n : x^n)x^n) = \mu(I^n \cap (x^n)) = \mu(I^n \cap (x)^n) \tag{4.11}$$

according to Lemma 2.7.34.

According to Theorem 4.2.10, we have $\mu(I^{(n)}) = \mu(I^n \cap (x)^n) = \mathcal{O}(n^{d+1})$. □

Remark 4.2.12. Dutta's paper [4] gives a better polynomial bound with degree at most $d-2$. However, it requires that R is S_2. The proof uses the notion of analytic spread. See Section 2.10.

CHAPTER V

The number of generators of the symbolic multi-power of the intersection of prime monomial ideals

In this Chapter, we are working in the polynomial ring $R = K[x_1, \cdots, x_N]$. We give a formula to calculate the number of generators of the symbolic multi-power of the intersection of prime monomial ideals, *i.e.*, the intersection of powers of these prime monomial ideals.

In the second section, we give a polynomial upper bound on the number of generators of the intersection of the powers of two prime monomial ideals.

5.1 The number of generators of the symbolic multi-power of the intersection of prime monomial ideals

In this section, we first introduce some useful notations, and then we give a formula to calculate the number of generators of the intersection of powers of prime monomial ideals.

Definition 5.1.1. Suppose $k \geqslant 1$. Let N_k^n be the number of non-negative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = n$ if $n \geqslant 0$. This is the number of monomials with k variables of degree n. It will be convenient to make the convention that $N_k^n = 1$ if $n < 0$. We make the corresponding convention for powers of ideals, *i.e.*, $I^n = I^0 = R$ if $n \leqslant 0$.

Remark 5.1.2. It is well known that

$$N_k^n = \binom{n+k-1}{k-1} = \binom{n+k-1}{n} \tag{5.1}$$

if $n \geqslant 0$.

Example 5.1.3. Let $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_c$ where $\mathfrak{p}_i$ is a prime monomial ideal and the generators of $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are disjoint if $i \neq j$. Then

$$\begin{aligned} \mu(I^{(\underline{s})}) &= \mu(\bigcap_{j=1}^{c} \mathfrak{p}_j^{(s_j)}) = \mu(\bigcap_{j=1}^{c} \mathfrak{p}_j^{s_j}) \\ &= \mu(\mathfrak{p}_1^{s_1}\mathfrak{p}_2^{s_2}\dots\mathfrak{p}_c^{s_c}) \\ &= \mu(\mathfrak{p}_1^{s_1})\mu(\mathfrak{p}_2^{s_2})\dots\mu(\mathfrak{p}_c^{s_c}) \\ &= N_{\mu(\mathfrak{p}_1)}^{s_1} N_{\mu(\mathfrak{p}_2)}^{s_2} \dots N_{\mu(\mathfrak{p}_c)}^{s_c} \end{aligned} \tag{5.2}$$

If we assume $\mu(\mathfrak{p}_i) = m_i$, then

$$\mu(I^{(\underline{s})}) = \frac{(s_1+m_1-1)!}{(m_1-1)!(s_1)!}\frac{(s_2+m_2-1)!}{(m_2-1)!(s_2)!}\cdots\frac{(s_c+m_c-1)!}{(m_c-1)!(s_c)!} \tag{5.3}$$

As we know, for fixed m_i,

$$\frac{(s_i+m_i-1)!}{(m_i-1)!(s_i)!} = \mathcal{O}(s_i^{m_i-1}) \tag{5.4}$$

if $s_i \gg 0$. Furthermore, we have

$$\mu(I^{(\underline{s})}) = \mathcal{O}(s_1^{m_1-1}s_2^{m_2-1}\dots s_c^{m_c-1}) \tag{5.5}$$

Actually,

$$\lim_{s_i\to\infty} \frac{\frac{(s_i+m_i-1)!}{(m_i-1)!(s_i)!}}{s_i^{m_i-1}} = \frac{1}{(m_i-1)!} \tag{5.6}$$

Generally, we have

$$\lim_{\underline{s}\to\infty} \frac{\mu(I^{(\underline{s})})}{\prod_{i=1}^{i=c} s_i^{m_i-1}} = 1/\prod_{i=1}^{i=c}(m_i-1)! \tag{5.7}$$

Then, we can say

$$\mu(I^{(\underline{s})}) \sim \prod_{i=1}^{i=c} s_i^{m_i-1} / \prod_{i=1}^{i=c} (m_i - 1)! \tag{5.8}$$

Definition 5.1.4. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_c$ be prime monomial ideals. Actually $\mathfrak{p}_i$ is the ideal generated by a subset A_i of the variables where $1 \leqslant i \leqslant c$. Also, σ_c denotes a binary string whose entries are 0 or 1, containing at least one 1, with c characters. Since c is fixed, we replace σ_c by σ for simplicity. Σ is the set of all σ. The j-th character of σ is denoted by $\sigma(j)$ where $1 \leqslant j \leqslant c$. Let A_i' be the set of variables not in A_i. Denote $A_\sigma = (\bigcap_{\sigma(i)=1} A_i) \cap (\bigcap_{\sigma(j)=0} A_j')$. We denote the cardinality of A_σ by m_σ.

Remark 5.1.5. The set of variables is the disjoint union of all the A_σ.

Remark 5.1.6. If $c = 2$, we have the Venn diagram showing in the figure 5.1.

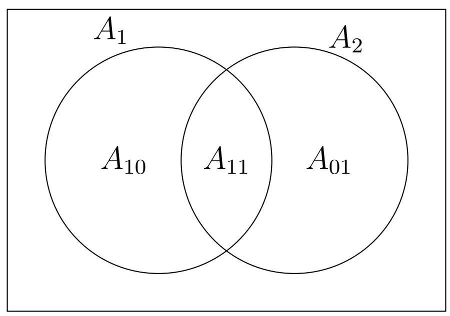

Figure 5.1: Venn diagram of two sets

Definition 5.1.7. A *degree restriction* is a function d from Σ to the nonnegative integer whose value on σ is denoted by d_σ. Let $s_1, \cdots, s_c$ be nonnegative integers, and let $\Delta(s_1, \cdots, s_c)$ be the set of all degree restrictions such that for all i, $1 \leqslant i \leqslant c$, and for all $\sigma \in \Sigma$, we have that $\sum_{\sigma\in\Sigma, \sigma(i)=1} d_\sigma \geqslant s_i$ and for every σ, either $d_\sigma = 0$ or there exists i such that $\sigma(i) = 1$ and $\sum_{\sigma\in\Sigma,\sigma(i)=1} d_\sigma = s_i$.

Theorem 5.1.8. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_c$ be prime monomial ideals. Then we have the following equation

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2} \cap \cdots \cap \mathfrak{p}_c^{s_c}) = \sum_{d\in\Delta(s_1,\cdots,s_c)} \prod_{\sigma\in\Sigma} N_{m_\sigma}^{d_\sigma}$$

Proof. We claim that any monomial $\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma}$ is a minimal generator of $\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2} \cap \cdots \cap \mathfrak{p}_c^{s_c} = I$ where d_σ satisfies degree restrictions and $\mathbf{x}_\sigma^{d_\sigma}$ is an element of the set:

$$\{x_{\sigma,1}^{d_{\sigma,1}} x_{\sigma,2}^{d_{\sigma,2}} \cdots x_{\sigma,m_\sigma}^{d_{\sigma,m_\sigma}} | d_{\sigma,i} \in \mathbb{N} \text{ and } x_{\sigma,i} \in A_\sigma \text{ where } 1 \leqslant i \leqslant m_\sigma, \sum_{i=1}^{m_\sigma} d_{\sigma,i} = d_\sigma\} \tag{5.9}$$

First, we prove $\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma} \in I$. Since $\sum_{\sigma\in\Sigma,\sigma(i)=1} d_\sigma \geqslant s_i$, we have

$$\prod_{\sigma\in\Sigma,\sigma(i)=1} \mathbf{x}_\sigma^{d_\sigma} \in \mathfrak{p}_i^{s_i} \tag{5.10}$$

where $1 \leqslant i \leqslant c$. Also, $\prod_{\sigma\in\Sigma,\sigma(i)=1} \mathbf{x}_\sigma^{d_\sigma} | \prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma}$ which implies $\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma} \in I$.

Second, if $\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d'_\sigma} | \prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma}$ and $\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d'_\sigma} \neq \prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma}$ where d_σ satisfies degree restrictions and d'_σ may not satisfy degree restrictions, then $\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d'_\sigma} \notin I$.

In fact, there exists σ_* such that $d'_{\sigma_*} < d_{\sigma_*}$. Since d_{σ_*} satisfies degree restrictions, at least on inequality in degree restrictions hold the equality. Without loss of generality, we have the following equation.

$$\sum_{\sigma\in\Sigma,\sigma(i)=1} d_\sigma = s_i \tag{5.11}$$

$\sigma_* \in \{\sigma \in \Sigma | \sigma(i) = 1\}$. Since $d'_{\sigma_*} < d_{\sigma_*}$, we have

$$\sum_{\sigma\in\Sigma,\sigma(i)=1} d'_\sigma < s_i \tag{5.12}$$

which means that

$$\prod_{\sigma\in\Sigma,\sigma(i)=1} \mathbf{x}_\sigma^{d'_\sigma} \notin \mathfrak{p}_i^{s_i} \tag{5.13}$$

Furthermore,

$$\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d'_\sigma} = \prod_{\sigma\in\Sigma,\sigma(i)=1} \mathbf{x}_\sigma^{d'_\sigma} \prod_{\sigma\in\Sigma,\sigma(i)=0} \mathbf{x}_\sigma^{d'_\sigma} \tag{5.14}$$

the variables appearing in $\prod\limits_{\sigma\in\Sigma,\sigma(i)=0} \mathbf{x}_\sigma^{d'_\sigma}$ are not in the ideal $\mathfrak{p}_i$. Thus,

$$\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d'_\sigma} \notin \mathfrak{p}_i^{s_i} \tag{5.15}$$

which implies $\prod\limits_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d'_\sigma} \notin I$.

From above argument, we can say

$$\prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma} \nmid \prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d'_\sigma} \tag{5.16}$$

if both d_σ and d'_σ satisfy degree restrictions and (d_σ) is not identical to (d'_σ).

We also claim that a minimal generator of I has the form $\prod\limits_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma}$ where d_σ satisfies degree restrictions. Actually, we have

$$\sum_{\sigma\in\Sigma,\sigma(i)=1} d_\sigma \geqslant s_i \tag{5.17}$$

for any i since

$$\prod_{\sigma\in\Sigma,\sigma(i)=1} \mathbf{x}_\sigma^{d_\sigma} \in \mathfrak{p}_i^{s_i} \tag{5.18}$$

It implies that $\sum\limits_{\sigma\in\Sigma,\sigma(i)=1} d_\sigma \geqslant s_i$ and $d_\sigma \geqslant 0$ for any i and σ. For any particular σ_*, if none of inequalities containing $d_{\sigma*}$ holds the equality, let $d'_{\sigma_*} = d_{\sigma*} - 1$. We define a new monomial $\mathbf{x}_{\sigma_*}^{d_{\sigma*}} \prod\limits_{\sigma\neq\sigma_*} \mathbf{x}_\sigma^{d_\sigma} \in I$. We have

$$\mathbf{x}_{\sigma_*}^{d_{\sigma*}} \prod_{\sigma\neq\sigma_*} \mathbf{x}_\sigma^{d_\sigma} \neq \prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma} \text{ and } \mathbf{x}_{\sigma_*}^{d_{\sigma*}} \prod_{\sigma\neq\sigma_*} \mathbf{x}_\sigma^{d_\sigma} \mid \prod_{\sigma\in\Sigma} \mathbf{x}_\sigma^{d_\sigma} \tag{5.19}$$

which means that at least one of inequalities containing σ_* hold the equality.

The number of elements in the set

$$\{x_{\sigma,1}^{d_{\sigma,1}} x_{\sigma,2}^{d_{\sigma,2}} \cdots x_{\sigma,m_\sigma}^{d_{\sigma,m_\sigma}} | d_{\sigma,i} \in \mathbb{N} \text{ and } x_{\sigma,i} \in A_\sigma \text{ where } 1 \leqslant i \leqslant m_\sigma, \sum_{i=1}^{m_\sigma} d_{\sigma,i} = d_\sigma\} \tag{5.20}$$

is $N_{m_\sigma}^{d_\sigma}$. Thus, above all, we have

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2} \cap \cdots \cap \mathfrak{p}_c^{s_c}) = \sum_{d \in \Delta(s_1, \cdots, s_c)} \prod_{\sigma \in \Sigma} N_{m_\sigma}^{d_\sigma} \tag{5.21}$$

□

Corollary 5.1.9. If $c = 2$, without loss of generality, assume that $s_1 \leqslant s_2$. Denote

$$\Delta'(d_{10}, d_{11}, d_{01}) = \{(d_{10}, d_{11}, d_{01}) \in \mathbb{N}^3 | d_{11} + d_{01} = s_2, d_{10} = \mathrm{Max}(0, s_1 - d_{11})\} \tag{5.22}$$

Then we have the following equation:

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) = \sum_{(d_{10}, d_{11}, d_{01}) \in \Delta'(d_{10}, d_{11}, d_{01})} N_{m_{10}}^{d_{10}} N_{m_{11}}^{d_{11}} N_{m_{01}}^{d_{01}} \tag{5.23}$$

Proof. By definition, we only need to prove $\Delta(d_{10}, d_{11}, d_{01})$ is the same as $\Delta'(d_{10}, d_{11}, d_{01})$.

First, for any $(d_{10}, d_{11}, d_{01}) \in \Delta(d_{10}, d_{11}, d_{01})$. If $d_{01} = 0$, we have $d_{11} \geqslant s_1 - d_{10}$ and $d_{11} \geqslant s_2$. $d_{11} = s_2$ since $s_2 \geqslant s_1$. Then $d_{10} = 0$ since one of $d_{10} \geqslant s_1 - d_{11}$ and $d_{10} \geqslant 0$ holds the equality. $(0, s_2, 0)$ is also a point in $\Delta'(d_{10}, d_{11}, d_{01})$. If $d_{01} > 0$, then $d_{11} + d_{01} = s_2$. $d_{10} \geqslant s_1 - d_{11}$ and $d_{10} \geqslant 0$ tell us $d_{10} = \mathrm{Max}(0, s_1 - d_{11})$.

Second, for any $(d_{10}, d_{11}, d_{01}) \in \Delta'(d_{10}, d_{11}, d_{01})$. $d_{10} = \mathrm{Max}(0, s_1 - d_{11})$ implies $d_{10} \geqslant s_1 - d_{11}$. $d_{11} + d_{10} = s_2$ implies $d_{11} + d_{10} \geqslant s_2$. According to $d_{11} + d_{10} = s_2$, both d_{11} and d_{10} hold at least one equality. $d_{10} = \mathrm{Max}(0, s_1 - d_{11})$ tells us d_{01} holds at least one equality. □

Remark 5.1.10. Actually, we have

$$\begin{aligned}
\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) &= \sum_{\Delta'(d_{10}, d_{11}, d_{01})} N_{m_{10}}^{d_{10}} N_{m_{11}}^{d_{11}} N_{m_{01}}^{d_{01}} && (5.24)\\
&= \sum_{0 \leqslant d_{11} \leqslant s_2} N_{m_{10}}^{\mathrm{Max}(0, s_1 - d_{11})} N_{m_{11}}^{d_{11}} N_{m_{01}}^{s_2 - d_{11}} && (5.25)\\
&= \sum_{d_{11}=0}^{s_1} N_{m_{10}}^{s_1 - d_{11}} N_{m_{11}}^{d_{11}} N_{m_{01}}^{s_2 - d_{11}} + \sum_{d_{11}=s_1+1}^{s_2} N_{m_{11}}^{d_{11}} N_{m_{01}}^{s_2 - d_{11}} && (5.26)
\end{aligned}$$

Particularly, if $m_{11} = 1$, we have

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) = \sum_{d_{11}=0}^{s_1} N_{m_{10}}^{s_1-d_{11}} N_{m_{01}}^{s_2-d_{11}} + \sum_{d_{11}=s_1+1}^{s_2} N_{m_{01}}^{s_2-d_{11}} \tag{5.27}$$

5.2 Two monomial prime ideals

In this section, we give a polynomial upper bound on the number of generators of the intersection of the powers of two prime monomial ideals. These prime monomial ideals are simply the ideals generated by a subset of the variables.

Lemma 5.2.1. For fixed $\alpha, \beta \in \mathbb{N}$, $\sum_{k=0}^{n} k^\beta (k+d)^\alpha$ is a polynomial function of n, d. If $n \gg 0, d \gg 0$, we have the following equation.

$$\sum_{k=0}^{n} k^\beta (k+d)^\alpha \sim \sum_{i=0}^{\alpha} \frac{\binom{\alpha}{i}}{\beta+i+1} n^{\beta+i+1} d^{\alpha-i} \tag{5.28}$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n} k^\beta (k+d)^\alpha &= \sum_{k=0}^{n} k^\beta \left(\sum_{i=0}^{\alpha} k^i d^{\alpha-i} \binom{\alpha}{i}\right) && (5.29)\\
&= \sum_{k=0}^{n} \sum_{i=0}^{\alpha} k^{\beta+i} d^{\alpha-i} \binom{\alpha}{i} && (5.30)\\
&= \sum_{k=0}^{n} d^{\alpha-i} \binom{\alpha}{i} \left(\sum_{i=0}^{\alpha} k^{\beta+i}\right) && (5.31)
\end{aligned}$$

As we all know, $\sum_{i=0}^{\alpha} k^{\beta+i}$ is a polynomial function of n with the leading term $n^{\beta+i+1}/(\beta+i+1)$ which means $\sum_{i=0}^{\alpha} k^{\beta+i} \sim n^{\beta+i+1}/(\beta+i+1)$. Thus, $\sum_{k=0}^{n} k^\beta (k+d)^\alpha$ is a polynomial function of n, d. And, we have

$$\sum_{k=0}^{n} k^\beta (k+d)^\alpha \sim \sum_{i=0}^{\alpha} \frac{\binom{\alpha}{i}}{\beta+i+1} n^{\beta+i+1} d^{\alpha-i} \tag{5.32}$$

□

Lemma 5.2.2. For fixed $a, b \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{k+b}{b} \binom{k+d+a}{a}$ is a polynomial function of n, d. If

$n \gg 0, d \gg 0$, we have the following equation.

$$\sum_{k=0}^{n} \binom{k+b}{b}\binom{k+d+a}{a} \sim \frac{1}{a!b!} \sum_{i=0}^{a} \frac{\binom{a}{i}}{b+i+1} n^{b+i+1} d^{a-i} \tag{5.33}$$

Proof.

$$\sum_{k=0}^{n} \binom{k+b}{b}\binom{k+d+a}{a} = \frac{1}{a!b!} \sum_{k=0}^{n} \prod_{i=1}^{b}(k+i) \prod_{j=1}^{a}(k+d+j) \tag{5.34}$$

$$\prod_{i=1}^{b}(k+i) \prod_{j=1}^{a}(k+d+j) = k^b(k+d)^a + \sum_{\substack{0\leqslant u\leqslant b \\ 0\leqslant v\leqslant a \\ (u,v)\neq(0,0)}} k^{b-u}(k+d)^{a-v}\phi_{u,v}(a,b) \tag{5.35}$$

where $\phi_{u,v}(a,b)$ is a function of a and b. $\phi_{u,v}(a,b)$ is fixed as both a, b are fixed and u, v are given. According to Lemma 5.2.1, we have

$$\sum_{k=0}^{n} k^b(k+d)^a \sim \sum_{i=0}^{a} \frac{\binom{a}{i}}{b+i+1} n^{b+i+1} d^{a-i} \tag{5.36}$$

$$\sum_{k=0}^{n} \sum_{\substack{0\leqslant u\leqslant b \\ 0\leqslant v\leqslant a \\ (u,v)\neq(0,0)}} k^{b-u}(k+d)^{a-v}\phi_{u,v}(a,b) \tag{5.37}$$

$$= \sum_{\substack{0\leqslant u\leqslant b \\ 0\leqslant v\leqslant a \\ (u,v)\neq(0,0)}} \phi_{u,v}(a,b) \sum_{k=0}^{n} k^{b-u}(k+d)^{a-v} \tag{5.38}$$

$$\sim \sum_{\substack{0\leqslant u\leqslant b \\ 0\leqslant v\leqslant a \\ (u,v)\neq(0,0)}} \phi_{u,v}(a,b) \sum_{i=0}^{a} \frac{\binom{a}{i}}{b+i+1} n^{b-u+i+1} d^{a-v-i} \tag{5.39}$$

Thus, $\sum\limits_{k=0}^{n} \binom{k+b}{b}\binom{k+d+a}{a}$ is a polynomial function of n, d for fixed a and b. Furthermore, we have

$$\sum_{k=0}^{n} \binom{k+b}{b}\binom{k+d+a}{a} \sim \frac{1}{a!b!}\sum_{i=0}^{a} \frac{\binom{a}{i}}{b+i+1} n^{b+i+1} d^{a-i} \tag{5.40}$$

since $n^{b-u+i+1}d^{a-v-i} | n^{b+i+1}d^{a-i}$ and $n^{b-u+i+1}d^{a-v-i} \neq n^{b+i+1}d^{a-i}$ for any $(u, v) \neq (0, 0)$. $\square$

Proposition 5.2.3. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two prime monomial ideals and $m_{11} = 1$. For fixed m_{10}, m_{01} and $s_2 \gg s_1 \gg 0$, we have that $\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2})$ is a polynomial function of s_1 and $s_2 - s_1$. Denote $a = m_{01} - 1$ and $b = m_{10} - 1$. We have

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) \sim \frac{1}{a!b!}\sum_{i=0}^{a} \frac{\binom{a}{i}}{b+i+1} s_1^{b+i+1}(s_2 - s_1)^{a-i} + \frac{(s_2 - s_1)^{(a+1)}}{(a+1)!} \tag{5.41}$$

Proof. According to the remark of Corollary 5.1.9, we have

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) = \sum_{d_{11}=0}^{s_1} N_{m_{10}}^{s_1-d_{11}} N_{m_{01}}^{s_2-d_{11}} + \sum_{d_{11}=s_1+1}^{s_2} N_{m_{01}}^{s_2-d_{11}} \tag{5.42}$$

$$= \sum_{t=0}^{s_1} \binom{s_1 - t + m_{10} - 1}{m_{10} - 1}\binom{s_2 - t + m_{01} - 1}{m_{01} - 1} + \sum_{t=s_1+1}^{s_2} \binom{s_2 - t + m_{01} - 1}{m_{01} - 1} \tag{5.43}$$

$$= \sum_{t=0}^{s_1} \binom{s_1 - t + b}{b}\binom{s_2 - t + a}{a} + \sum_{t=s_1+1}^{s_2} \binom{s_2 - t + a}{a} \tag{5.44}$$

where $a = m_{01} - 1$ and $b = m_{10} - 1$. For the first term, we have

$$\sum_{t=0}^{s_1} \binom{s_1 - t + b}{b}\binom{s_2 - t + a}{a} = \sum_{k=0}^{s_1} \binom{k+b}{b}\binom{k+d+a}{a} \tag{5.45}$$

where $d = s_2 - s_1$ and $k = s_1 - t$. According to Lemma 5.2.2, $\sum\limits_{t=0}^{s_1} \binom{s_1-t+b}{b}\binom{s_2-t+a}{a}$ is a polynomial function of s_1 and $d = s_2 - s_1$.

$$\sum_{t=0}^{s_1} \binom{s_1 - t + b}{b} \binom{s_2 - t + a}{a} \sim \frac{1}{a!b!} \sum_{i=0}^{a} \frac{\binom{a}{i}}{b+i+1} s_1^{b+i+1} (s_2 - s_1)^{a-i} \tag{5.46}$$

As to the second term, we have

$$\sum_{t=s_1+1}^{s_2} \binom{s_2 - t + a}{a} = \sum_{k=0}^{d-1} \binom{k+a}{a} \tag{5.47}$$

where $d = s_2 - s_1$ and $k = s_2 - t$. It is easy to check that $\sum\limits_{k=0}^{d-1} \binom{k+a}{a} = \binom{a+d}{a+1}$. a is fixed, we have

$$\sum_{k=0}^{d-1} \binom{k+a}{a} \sim \frac{(s_2 - s_1)^{(a+1)}}{(a+1)!} \tag{5.48}$$

In conclusion,

$$\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) \sim \frac{1}{a!b!} \sum_{i=0}^{a} \frac{\binom{a}{i}}{b+i+1} s_1^{b+i+1} (s_2 - s_1)^{a-i} + \frac{(s_2 - s_1)^{(a+1)}}{(a+1)!} \tag{5.49}$$

where $a = m_{01} - 1$ and $b = m_{10} - 1$. Both of a and b are fixed. $\square$

Lemma 5.2.4. For fixed $\alpha, \beta, \gamma \in \mathbb{N}$, $\sum\limits_{k=0}^{n} k^\beta (n-k)^\gamma (k+d)^\alpha$ is a polynomial function of n, d. Denote $\Phi_{\alpha,\beta,\gamma}(v) = \binom{\alpha}{v} \sum\limits_{u=0}^{\gamma} \frac{(-1)^u}{\alpha+\beta+u-v+1} \binom{\gamma}{u}$. We have $\Phi_{\alpha,\beta,\gamma}(v) > 0$. If $n \gg 0, d \gg 0$, we have the following equation.

$$\sum_{k=0}^{n} k^\beta (n-k)^\gamma (k+d)^\alpha \sim \sum_{v=0}^{\alpha} n^{\alpha+\beta+\gamma-v+1} d^v \Phi_{\alpha,\beta,\gamma}(v) \tag{5.50}$$

Proof.

$$\sum_{k=0}^{n} k^\beta (n-k)^\gamma (k+d)^\alpha = \sum_{k=0}^{n} k^\beta (\sum_{u=0}^{\gamma} (-k)^u n^{\gamma-u} \binom{\gamma}{u})(\sum_{v=0}^{\alpha} k^{\alpha-v} d^v \binom{\alpha}{v}) \tag{5.51}$$

$$= \sum_{u=0}^{\gamma} \sum_{v=0}^{\alpha} (-1)^u \binom{\gamma}{u} \binom{\alpha}{v} n^{\gamma-u} d^v \sum_{k=0}^{n} k^{\alpha+\beta+u-v} \tag{5.52}$$

$$\sim \sum_{u=0}^{\gamma}\sum_{v=0}^{\alpha}(-1)^u\binom{\gamma}{u}\binom{\alpha}{v}n^{\gamma-u}d^v\frac{n^{\alpha+\beta+u-v+1}}{\alpha+\beta+u-v+1} \tag{5.53}$$

since $\sum_{k=0}^{n} k^{\alpha+\beta+u-v}$ is a polynomial function of n and d with the leading term $\frac{n^{\alpha+\beta+u-v+1}}{\alpha+\beta+u-v+1}$.

$$\begin{aligned}
\sum_{k=0}^{n} k^{\beta}(n-k)^{\gamma}(k+d)^{\alpha} &\sim \sum_{u=0}^{\gamma}\sum_{v=0}^{\alpha}(-1)^u\binom{\gamma}{u}\binom{\alpha}{v}n^{\gamma-u}d^v\frac{n^{\alpha+\beta+u-v+1}}{\alpha+\beta+u-v+1} && (5.54)\\
&= \sum_{v=0}^{\alpha} n^{\alpha+\beta+\gamma-v+1}d^v\binom{\alpha}{v}\sum_{u=0}^{\gamma}\frac{(-1)^u}{\alpha+\beta+u-v+1}\binom{\gamma}{u} && (5.55)\\
&= \sum_{v=0}^{\alpha} n^{\alpha+\beta+\gamma-v+1}d^v\Phi_{\alpha,\beta,\gamma}(v) && (5.56)
\end{aligned}$$

where $\Phi_{\alpha,\beta,\gamma}(v) = \binom{\alpha}{v}\sum_{u=0}^{\gamma}\frac{(-1)^u}{\alpha+\beta+u-v+1}\binom{\gamma}{u}$. In conclusion, $\sum_{k=0}^{n} k^{\beta}(n-k)^{\gamma}(k+d)^{\alpha}$ is a polynomial function of n and d. Now, we prove that $\Phi_{\alpha,\beta,\gamma}(v) > 0$. Denote $\alpha+\beta-v+1$ by δ. $\delta \geqslant 1$ since $v \leqslant \alpha$. Thus, we have

$$\begin{aligned}
\binom{\alpha}{v}\sum_{u=0}^{\gamma}\frac{(-1)^u}{u+\delta}\binom{\gamma}{u} &= \binom{\alpha}{v}\sum_{u=0}^{\gamma}(-1)^u\binom{\gamma}{u}\int_0^1 x^{u+\delta-1}dx && (5.57)\\
&= \binom{\alpha}{v}\int_0^1\sum_{u=0}^{\gamma}(-1)^u\binom{\gamma}{u}x^{u+\delta-1}dx && (5.58)\\
&= \binom{\alpha}{v}\int_0^1 x^{\delta-1}\sum_{u=0}^{\gamma}(-1)^u\binom{\gamma}{u}x^{u}dx && (5.59)\\
&= \binom{\alpha}{v}\int_0^1 x^{\delta-1}(1-x)^{\gamma}dx && (5.60)\\
&> 0 && (5.61)
\end{aligned}$$

□

Lemma 5.2.5. For fixed $a,b,c \in \mathbb{N}$, $\sum_{k=0}^{n}\binom{k+b}{b}\binom{n-k+c}{c}\binom{k+d+a}{a}$ is a polynomial function of n,d. Denote $\Phi_{a,b,c}(v) = \binom{a}{v}\sum_{u=0}^{c}\frac{(-1)^u}{a+b+u-v+1}\binom{c}{u}$. We have $\Phi_{a,b,c}(v) > 0$. If $n \gg 0, d \gg 0$, we have the following equation.

$$\sum_{k=0}^{n}\binom{k+b}{b}\binom{n-k+c}{c}\binom{k+d+a}{a} \sim \frac{1}{a!b!c!}\sum_{v=0}^{a} n^{a+b+c-v+1}d^v\Phi_{a,b,c}(v) \tag{5.62}$$

Proof.

$$\sum_{k=0}^{n}\binom{k+b}{b}\binom{n-k+c}{c}\binom{k+d+a}{a} \tag{5.63}$$

$$=\frac{1}{a!b!c!}\sum_{k=0}^{n}\prod_{i=1}^{b}(k+i)\prod_{l=1}^{c}(n-k+j)\prod_{j=1}^{a}(k+d+j) \tag{5.64}$$

$$\prod_{i=1}^{b}(k+i)\prod_{l=1}^{c}(n-k+j)\prod_{j=1}^{a}(k+d+j) \tag{5.65}$$

$$=k^b(n-k)^c(k+d)^a + \sum_{\substack{0\leqslant i\leqslant b\\ 0\leqslant j\leqslant a\\ 0\leqslant l\leqslant c\\ (a,b,c)\neq(0,0,0)}} k^{b-i}(n-k)^{c-l}(k+d)^{a-j}\phi_{i,j,l}(a,b,c) \tag{5.66}$$

where $\phi_{i,j,l}(a,b,c)$ is a function of a,b,c. $\phi_{i,j,l}(a,b,c)$ is fixed as both a,b,c are fixed and i,j,l are given. According to Lemma 5.2.4, we have

$$\sum_{k=0}^{n} k^b(n-k)^c(k+d)^a \sim \sum_{v=0}^{a} n^{a+b+c-v+1}d^v\Phi_{a,b,c}(v) \tag{5.67}$$

$$\sum_{k=0}^{n}\sum_{\substack{0\leqslant i\leqslant b\\ 0\leqslant j\leqslant a\\ 0\leqslant l\leqslant c\\ (a,b,c)\neq(0,0,0)}} k^{b-i}(n-k)^{c-l}(k+d)^{a-j}\phi_{i,j,l}(a,b,c) \tag{5.68}$$

$$= \sum_{\substack{0\leqslant i\leqslant b\\ 0\leqslant j\leqslant a\\ 0\leqslant l\leqslant c\\ (a,b,c)\neq(0,0,0)}} \phi_{i,j,l}(a,b,c)\sum_{k=0}^{n} k^{b-i}(n-k)^{c-l}(k+d)^{a-j} \tag{5.69}$$

$$\sim \sum_{\substack{0\leqslant i\leqslant b\\0\leqslant j\leqslant a\\0\leqslant l\leqslant c\\(a,b,c)\neq(0,0,0)}} \phi_{i,j,l}(a,b,c)\sum_{v=0}^{a} n^{a+b+c-(i+j+l)-v+1}d^v\Phi_{a-i,b-j,c-l}(v) \tag{5.70}$$

Thus,

$$\sum_{k=0}^{n}\binom{k+b}{b}\binom{n-k+c}{c}\binom{k+d+a}{a} \tag{5.71}$$

$$\sim\frac{1}{a!b!c!}\sum_{v=0}^{a} n^{a+b+c-v+1}d^v\Phi_{a,b,c}(v) \tag{5.72}$$

since $i+j+l>1$ which means the order of $n^{a+b+c-(i+j+l)-v+1}d^v$ is strictly less than $n^{a+b+c-v+1}d^v$. From above all, $\sum\limits_{k=0}^{n}\binom{k+b}{b}\binom{n-k+c}{c}\binom{k+d+a}{a}$ is a polynomial function of n,d for fixed a,b,c. $\square$

Lemma 5.2.6. For fixed $\alpha,\gamma\in\mathbb{N}$, $\sum\limits_{k=0}^{d-1}(n+d-k)^\gamma k^\alpha$ is a polynomial function of n and d. If $n\gg 0, d\gg 0$, we have the following equation.

$$\sum_{k=0}^{d-1}(n-k)^\gamma k^\alpha \sim \sum_{u=0}^{\gamma}\frac{(-1)^{\gamma-u}\binom{\gamma}{u}}{\alpha+\gamma-u+1}(n+d)^u d^{\alpha+\gamma-u+1} \tag{5.73}$$

Proof.

$$\sum_{k=0}^{d-1}(n+d-k)^\gamma k^\alpha = \sum_{k=0}^{d-1}\sum_{u=0}^{\gamma}(-1)^{\gamma-u}(n+d)^u k^{\gamma-u}k^\alpha\binom{\gamma}{u} \tag{5.74}$$

$$= \sum_{u=0}^{\gamma}(-1)^{\gamma-u}\binom{\gamma}{u}(n+d)^u\sum_{k=0}^{d-1}k^{\alpha+\gamma-u} \tag{5.75}$$

$$\sim \sum_{u=0}^{\gamma}(-1)^{\gamma-u}\binom{\gamma}{u}(n+d)^u\frac{(d-1)^{\alpha+\gamma-u+1}}{\alpha+\gamma-u+1} \tag{5.76}$$

$$= \sum_{u=0}^{\gamma}\frac{(-1)^{\gamma-u}\binom{\gamma}{u}}{\alpha+\gamma-u+1}(n+d)^u d^{\alpha+\gamma-u+1} \tag{5.77}$$

$\square$

Lemma 5.2.7. For fixed $a,c\in\mathbb{N}$, $\sum\limits_{k=0}^{d-1}\binom{n+d-k+c}{c}\binom{k+a}{a}$ is a polynomial function of n and d. If

$n \gg 0, d \gg 0$, we have the following equation.

$$\sum_{k=0}^{d-1} \binom{n+d-k+c}{c} \binom{k+a}{a} \sim \frac{1}{a!c!} \sum_{u=0}^{c} \frac{(-1)^{c-u}\binom{c}{u}}{a+c-u+1} (n+d)^u d^{a+c-u+1} \tag{5.78}$$

Proof.

$$\sum_{k=0}^{d-1} \binom{n+d-k+c}{c} \binom{k+a}{a} = \frac{1}{a!c!} \sum_{k=0}^{d-1} \prod_{i=1}^{c} (n+d-k+i) \prod_{j=1}^{a} (k+j) \tag{5.79}$$

$$\prod_{i=1}^{c} (n+d-k+i) \prod_{j=1}^{a} (k+j) = (n+d-k)^c k^a + \sum_{\substack{0 \leqslant i \leqslant a \\ 0 \leqslant j \leqslant c \\ (i,j) \neq (0,0)}} (n+d-k)^{c-j} k^{a-i} \tag{5.80}$$

According to Lemma 5.2.6, we have

$$\sum_{k=0}^{d-1} \binom{n-k+c}{c} \binom{k+a}{a} \sim \frac{1}{a!c!} \sum_{u=0}^{c} \frac{(-1)^{c-u}\binom{c}{u}}{a+c-u+1} (n+d)^u d^{a+c-u+1} \tag{5.81}$$

And, $\sum\limits_{k=0}^{d-1} \binom{n+d-k+c}{c} \binom{k+a}{a}$ is a polynomial function of n, d. □

Theorem 5.2.8. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two prime monomial ideals. m_{01}, m_{10}, m_{11} are defined in Definition 5.1.4. For fixed m_{01}, m_{10}, m_{11} and $s_2 \gg s_1 \gg 0$, $\mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2})$ is a polynomial function of s_1 and $s_2 - s_1$. Denote $a = m_{01} - 1$, $b = m_{10} - 1$, and $c = m_{11} - 1$. For $0 \leqslant v \leqslant a$, denote $\Phi_{a,b,c}(v) = \binom{a}{v} \sum\limits_{u=0}^{c} \frac{(-1)^u}{a+b+u-v+1} \binom{c}{u}$. We have $\Phi_{a,b,c}(v) > 0$ and

$$\begin{aligned} \mu(\mathfrak{p}_1^{s_1} \cap \mathfrak{p}_2^{s_2}) \sim & \frac{1}{a!b!c!} \sum_{v=0}^{a} s_1^{a+b+c-v+1} (s_2 - s_1)^v \Phi_{a,b,c}(v) \\ & + \frac{1}{a!c!} \sum_{u=0}^{c} \frac{(-1)^{c-u}\binom{c}{u}}{a+c-u+1} s_2^u (s_2 - s_1)^{a+c-u+1} \end{aligned}$$

Proof. According to the remark of Corollary 5.1.9, we have

$$\begin{aligned}
\mu(\mathfrak{p}_1^{s_1}\cap\mathfrak{p}_2^{s_2}) &= \sum_{d_{11}=0}^{s_1} N_{m_{10}}^{s_1-d_{11}} N_{m_{11}}^{d_{11}} N_{m_{01}}^{s_2-d_{11}} + \sum_{d_{11}=s_1+1}^{s_2} N_{m_{11}}^{d_{11}} N_{m_{01}}^{s_2-d_{11}} \\
&= \sum_{t=0}^{s_1} \binom{s_1-t+m_{10}-1}{m_{10}-1}\binom{t+m_{11}-1}{m_{11}-1}\binom{s_2-t+m_{01}-1}{m_{01}-1} \\
&\quad + \sum_{t=s_1+1}^{s_2} \binom{t+m_{11}-1}{m_{11}-1}\binom{s_2-t+m_{01}-1}{m_{01}-1} \\
&= \sum_{t=0}^{s_1} \binom{s_1-t+b}{b}\binom{t+c}{c}\binom{s_2-t+a}{a} + \sum_{t=s_1+1}^{s_2}\binom{t+c}{c}\binom{s_2-t+a}{a}
\end{aligned}$$

where $a=m_{01}-1$, $b=m_{10}-1$ and $c=m_{11}-1$. For the first term, we have

$$\sum_{t=0}^{s_1}\binom{s_1-t+b}{b}\binom{t+c}{c}\binom{s_2-t+a}{a} = \sum_{k=0}^{s_1}\binom{k+b}{b}\binom{s_1-k+c}{c}\binom{k+d+a}{a} \tag{5.82}$$

where $d=s_2-s_1$ and $k=s_1-t$. According to Lemma 5.2.5, $\sum_{t=0}^{s_1}\binom{s_1-t+b}{b}\binom{t+c}{c}\binom{s_2-t+a}{a}$ is a polynomial function of s_1 and s_2-s_1. Also,

$$\sum_{t=0}^{s_1}\binom{s_1-t+b}{b}\binom{t+c}{c}\binom{s_2-t+a}{a} \sim \frac{1}{a!b!c!}\sum_{v=0}^{a} s_1^{a+b+c-v+1}(s_2-s_1)^v\Phi_{a,b,c}(v) \tag{5.83}$$

As to the second term, we have

$$\sum_{t=s_1+1}^{s_2}\binom{t+c}{c}\binom{s_2-t+a}{a} = \sum_{k=0}^{d-1}\binom{s_1+d-k+c}{c}\binom{k+a}{a} \tag{5.84}$$

where $d=s_2-s_1$ and $k=s_2-t$. According to Lemma 5.2.7, we have

$$\sum_{t=s_1+1}^{s_2}\binom{t+c}{c}\binom{s_2-t+a}{a} \sim \frac{1}{a!c!}\sum_{u=0}^{c}\frac{(-1)^{c-u}\binom{c}{u}}{a+c-u+1}s_2^u(s_2-s_1)^{a+c-u+1} \tag{5.85}$$

We actually prove the theorem. $\square$

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BIBLIOGRAPHY

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