ULAM FLOATING BODIES

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Abstract. We study a new construction of bodies from a given convex body in $\mathbb{R}^n$ which are isomorphic to (weighted) floating bodies. We establish several properties of this new construction, including its relation to p-affine surface areas. We show that these bodies are related to Ulam's long-standing floating body problem which asks whether Euclidean balls are the only bodies that can float, without turning, in any orientation.

1. Introduction

1.1. **Metronoids.** Let K be a convex body in $\mathbb{R}^n$ (i.e. a compact convex set with non-empty interior), and denote its Lebesgue volume by $|K|$. The purpose of this paper is to study a new family of convex bodies $\mathrm{M}_\delta(K)$ associated to K, where $0 < \delta < |K|$ is a parameter.

The construction of this family arises from the notion of metronoids which was recently introduced in [24] in order to study extensions of problems concerning the approximation of convex bodies by polytopes. Given a Borel measure μ on $\mathbb{R}^n$, the *metronoid* associated to μ is the convex set defined by

$$\mathrm{M}(\mu) = \bigcup_{\substack{0\le f\le 1,\\ \int_{\mathbb{R}^n} f\,\mathrm{d}\mu=1}} \left\{ \int_{\mathbb{R}^n} y f(y)\,\mathrm{d}\mu(y) \right\},$$

where the union is taken over all functions $0 \le f \le 1$ for which $\int_{\mathbb{R}^n} f\,\mathrm{d}\mu = 1$ and $\int_{\mathbb{R}^n} yf(y)\,\mathrm{d}\mu(y)$ exists. Note that for a discrete measure of the form $\sum_{i=1}^N \delta_{x_i}$, the corresponding metronoid is the convex hull of $x_1, \dots, x_N$. Hence $\mathrm{M}(\mu)$ can be thought of as a fractional extension of the convex hull.

1.2. **Ulam's floating body.** Our main object $\mathrm{M}_\delta(K)$ is the metronoid generated by the uniform measure on K with total mass $\delta^{-1}|K|$. Namely, let $\mathbb{1}_K$ be the characteristic function of K, and μ the measure whose density with respect to Lebesgue measure is $\delta^{-1}\mathbb{1}_K$. Then $\mathrm{M}_\delta(K) := \mathrm{M}(\mu)$. It turns out that $M_\delta(K)$ is intimately related to the following long-standing problem proposed by Ulam, see e.g., [5, 41, 15, 18]: Is a solid of uniform density which floats in water in every position a Euclidean ball? While counterexamples were found in $\mathbb{R}^2$ (convex and non-convex) and $\mathbb{R}^3$ (only non-convex), this problem remains open in arbitrary dimensions. For a full account of the progress made on this problem, see [58] and references therein.

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As we show in Section 2.2 below, along with a precise description of Ulam's problem, one can restate Ulam's problem in terms of $M_\delta(K)$ as follows: If $M_\delta(K)$ is a Euclidean ball, must K be a Euclidean ball as well? For that reason, we call $M_\delta(K)$ an *Ulam floating body*. As far as we know, this construction and its relation to Ulam's problem is not mentioned anywhere in the literature.

We also define weighted variations of $\mathrm{M}_\delta(K)$ where the weight is given by a positive continuous function $\phi : K \to \mathbb{R}$. Namely, we define

$$\mathrm{M}_\delta(K,\phi) := \mathrm{M}\left(\frac{\phi(x)}{\delta}\mathbb{1}_K(x)\,\mathrm{d}x\right).$$

To understand $\mathrm{M}_\delta(K)$ geometrically, recall that a convex body $K \subseteq \mathbb{R}^n$ is determined by its support function $h_K(\theta) = \max_{x\in K}\langle x,\, \theta\rangle$, where $\langle\cdot,\,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^n$. For every direction $\theta \in \mathbb{S}^{n-1}$, let $H(\delta,\theta)$ be the hyperplane orthogonal to θ that cuts a set of volume δ from K. That is

$$C_\delta(\theta) = K \cap \{x \,:\, \langle x,\, \theta\rangle \geq \langle y_\theta,\, \theta\rangle\}$$

has volume δ for any $y_\theta \in H(\delta,\theta)$. Then the barycenter of $C_\delta(\theta)$ is a point on the boundary of $\mathrm{M}_\delta(K)$. More precisely, by [24, Proposition 2.1], we have that for any direction θ,

$$h_{\mathrm{M}_\delta(K)}(\theta) = \frac{1}{\delta}\int_{C_\delta(\theta)} \langle x,\, \theta\rangle\,\mathrm{d}x.$$

As illustrated in Figure 1.1, the body $\mathrm{M}_\delta(K)$ is closely related to the convex floating body K_δ, introduced independently in [6] and [52]. Using the above notation, we have that

$$K_\delta = \bigcap_{\theta\in\mathbb{S}^{n-1}} \{x \,:\, \langle x,\, \theta\rangle \leq \langle y_\theta,\, \theta\rangle\},$$

which is a non-empty convex set for a sufficiently small $0 < \delta$. In fact, $\mathrm{M}_\delta(K)$ is isomorphic to K_δ in the sense that $K_{\frac{e-1}{e}\delta} \subseteq \mathrm{M}_\delta(K) \subseteq K_{\frac{\delta}{e}}$. We discuss this property in the more general case of weighted Ulam floating bodies in Section 2.3 below (also see Theorem 1.1).

The convex floating body is a natural variation of Dupin's floating body [16] from 1822. Dupin's floating body $K_{[\delta]}$ is defined as the body whose boundary is the set of points that are the barycenters of all the sections of K of the form $K \cap H(\delta,\theta)$, where $H(\delta,\theta)$ are the aforementioned hyperplanes that cut a set of volume δ from K. However, while K_δ coincides with $K_{[\delta]}$ whenever $K_{[\delta]}$ is convex (e.g., for centrally-symmetric K, see [43]), in the non-centrally symmetric case, Dupin's floating body need not be convex, as in the case of some triangles in $\mathbb{R}^2$ (see e.g., [31]). Restating the above, every point on the boundary of K_δ is the barycenter of $K \cap H(\delta,\theta)$ for some θ, but the converse holds only if Dupin's floating body is convex.

Note that our construction $M_\delta(K)$ corresponds nicely to both definitions, that of the floating body and that of the convex floating body in the sense that it enjoys being convex as well

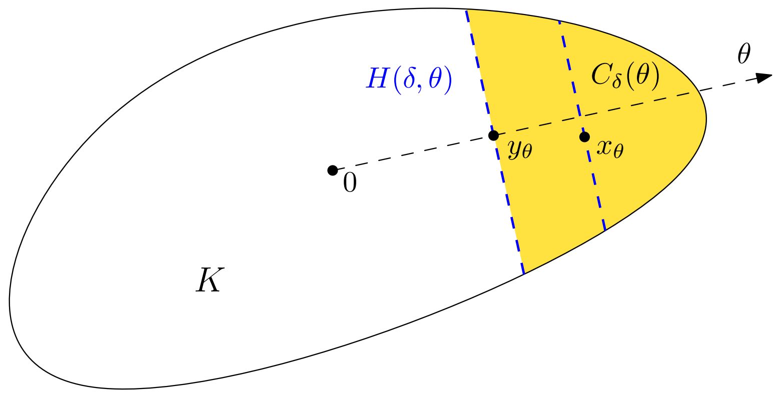

FIGURE 1.1. $H(\delta,\theta)$ is the hyperplane orthogonal to θ that cuts a set $C_\delta(\theta)$ of volume δ from a convex body K: $|C_\delta(\theta)| = |K \cap \{x : \langle x, \theta\rangle \geq \langle y_\theta, \theta\rangle\}| = \delta$. The point x_θ is the barycenter of $C_\delta(\theta)$. Then

$$K_\delta \subseteq K \cap \{x : \langle x, \theta\rangle \leq \langle y_\theta, \theta\rangle\}$$

while

$$\mathrm{M}_\delta(K) \subseteq K \cap \{x : \langle x, \theta\rangle \leq \langle x_\theta, \theta\rangle\}.$$

as having the property that a point is on the boundary of $\mathrm{M}_\delta(K)$ if and only if it is the barycenter of a set of volume δ that is cut off by a hyperplane.

1.3. **Main results.** We present three main theorems concerning Ulam's floating bodies. While the first result establishes an explicit relation between (weighted) floating bodies and (weighted) Ulam's floating bodies, the other two results are the analogous counterparts to the classical floating bodies.

1.3.1. ***Relation to floating bodies.*** Our first theorem shows that (weighted) Ulam's floating bodies are isomorphic, in a sense, to (weighted) floating bodies. Weighted floating bodies were introduced in [59] (also see [7, 9] for recent applications) as follows. Let $K \subseteq \mathbb{R}^n$ be a convex body, $0 < \delta$, and $\phi : K \to \mathbb{R}$ be integrable and such that $\phi > 0$ almost everywhere with respect to Lebesgue measure. For a hyperplane H in $\mathbb{R}^n$, let $H^\pm$ be the half-spaces separated by H. Then the weighted floating body $F_\delta(K, \phi)$ is defined as

$$F_\delta(K,\phi) = \bigcap \left\{ H^- : \int_{H^+\cap K} \phi(x)\,\mathrm{d}x \leq \delta \right\}.$$

Note that for $\phi \equiv 1$, we have that $F_\delta(K,\phi) = K_\delta$.

We prove the following.

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^n$, and let $\phi : K \to \mathbb{R}^+$ be an integrable log-concave function. Then for all $0 < \delta < |K|$, we have*

$$F_{\frac{e-1}{e}\delta}(K,\phi) \subseteq \mathrm{M}_\delta(K,\phi) \subseteq F_{\frac{\delta}{e}}(K,\phi).$$

In particular, for $\phi \equiv 1$ we have that

$$K_{\frac{e-1}{e}\delta} \subseteq \mathrm{M}_\delta(K, \phi) \subseteq K_{\frac{\delta}{e}}.$$

We remark that for $\phi \equiv 1$, Theorem 1.1 was proven in [24].

1.3.2. ***Smoothness of Ulam's floating bodies.*** Our second main result states that the boundary $\partial \mathrm{M}_\delta(K)$ of an Ulam floating body $\mathrm{M}_\delta(K)$ is always smoother than the boundary of K.

Theorem 1.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body, Suppose that $\partial K \in C^k$ for some $k \geq 0$. Then for any $0 < \delta < |K|$, we have that $\partial \mathrm{M}_\delta(K) \in C^{k+1}$.*

We remark that in the case of the convex floating body, an analogous result to Theorem 1.2 is known only in the centrally-symmetric case [43]. The main reason for this is that the proof in [43] relies on the above mentioned fact that in the centrally-symmetric case the convex floating convex body and Dupin's floating body coincide.

1.3.3. ***Affine Surface Area.*** The affine surface area was introduced by W. Blaschke [10] in 1923 for smooth convex bodies in Euclidean space of dimensions 2 and 3, and extended to $\mathbb{R}^n$ by K. Leichtweiss [29]. Given a convex body $K \subseteq \mathbb{R}^n$ with a sufficiently smooth boundary, let $\kappa_K(x)$ be the Gaussian curvature at $x \in \partial K$, and μ_K the surface area measure on ∂K. The affine surface area of K is defined by

$$as(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \, \mathrm{d}\mu_K.$$

Even though it proved to be much more difficult to extend the notion of affine surface area to general convex bodies than other notions, like surface area measures or curvature measures, successively such extensions were achieved, by e.g., K. Leichtweiss [29], E. Lutwak [35], who also proved the long conjectured upper semicontinuity of affine surface area [35] and by C. Schütt and E. Werner [52] who showed that the affine surface area arises as a limit of the volume difference of the convex body and its floating body. All these extensions coincide as was shown in [50, 30].

Affine surface area is among the most powerful tools in equiaffine differential geometry (see B. Andrews [2, 3], A. Stancu [55, 56], M. Ivaki [26], M. Ivaki and A. Stancu [27] and M. Ludwig and M. Reitzner [34]). It appears naturally as the Riemannian volume of a smooth convex hypersurface with respect to the affine metric (or Berwald-Blaschke metric), see e.g., the thorough monograph of K. Leichtweiss [31] or the book by K. Nomizu and T. Sasaki [45]. In particular the upper semicontinuity proved to be critical in the solution of the affine Plateau problem by N. S. Trudinger and X. J. Wang [57].

Applications of affine surface areas have been manifold. For instance, affine surface area appears in best and random approximation of convex bodies by polytopes, see K. Böröczky Jr. [12, 11], P. M. Gruber [21, 22, 23], M. Ludwig [33], M. Reitzner [47], C. Schütt [49, 51] and J. Grote and E. Werner, [20] and C. Schütt and E. Werner [53]. Furthermore, recent contributions indicate astonishing developments which open up new connections of affine surface area to, e.g., concentration of volume (e.g. [17, 37]), spherical and hyperbolic spaces [8, 9], geometric inequalities [40, 61] and information theory (e.g. [4, 14, 38, 39, 62, 46]).

The L_p-affine surface area is a generalization of the classical affine surface area and a central part in the L_p-Brunn-Minkowski theory. It was introduced by E. Lutwak [36] for $p > 1$ (see also D. Hug [25] and M. Meyer and E. Werner [44]) and extended for all $p \in [-\infty, \infty]$ in [54]. For $-\infty < p < \infty$, the L_p-affine surface area of a convex body $K \subseteq \mathbb{R}^n$ is given by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x), \tag{1.1}$$

where $N_K(x)$ is the outer normal of K at x. For $p = \pm\infty$, it is given by

$$as_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} d\mu_K(x). \tag{1.2}$$

As in the case of the classical affine surface area, several geometric extensions for the L_p-affine surface area have been proven. We refer to [54, 60] and references therein. These extensions all involve a construction of a special family of convex bodies $\{K_t\}_{t>0}$ which is related to a given convex body K, where the L_p-affine surface area can be written as a limit involving their volume difference.

We prove the following theorem which shows that this can also be achieved using weighted Ulam floating bodies.

Theorem 1.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $\phi : K \to (0, \infty)$ be a continuous function. Then*

$$\lim_{\delta \searrow 0} \frac{|K| - |\mathrm{M}_\delta(K, \phi)|}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \, \mathrm{d}\mu_K(x), \tag{1.3}$$

where $c_n = 2\frac{n+1}{n+3}\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$, and B_2^n is the Euclidean unit ball in $\mathbb{R}^n$.

For $-\infty \le p \le \infty$, $p \ne -n$, define the function $\phi_p : \partial K \to [0, \infty]$ by

$$\phi_p(x) = \frac{\langle x,\, N_K(x)\rangle^{\frac{n(n+1)(p-1)}{2(n+p)}}}{\kappa_K(x)^{\frac{n(p-1)}{2(n+p)}}}. \tag{1.4}$$

Note that $\phi_1(x) = 1$ for all $x \in \partial K$. If $\kappa_K(x) = 0$, which is the case, e.g., when $K = P$ is a polytope and x belongs to an $(n-1)$-dimensional facet of P, then

$$\phi_p(x) = \begin{cases} \infty & p > 1 \text{ or } p < -n \\ 0 & -n < p < 1. \end{cases}$$

If $\kappa_K(x) = \infty$, which is the case, e.g., when $K = P$ is a polytope and x is a vertex of P, then

$$\phi_p(x) = \begin{cases} 0 & p > 1 \text{ or } p < -n \\ \infty & -n < p < 1. \end{cases}$$

If K and p are such that ϕ_p is continuous on ∂K, we extend ϕ_p to a continuous function on K which we call again ϕ_p.

Applying Theorem 1.3 with ϕ_p yields the following extension of L_p-affine surface areas.

Corollary 1.4. *Let $K \subseteq \mathbb{R}^n$ be a convex body. If ϕ_p is continuous on K, then*

$$\lim_{\delta \searrow 0} \frac{|K| - |\mathrm{M}_\delta(K, \phi_p)|}{\delta^{\frac{2}{n+1}}} = c_n \, as_p(K).$$

In particular, for $p = 1$ we have

$$\lim_{\delta \searrow 0} \frac{|K| - |M_\delta(K)|}{\delta^{\frac{2}{n+1}}} = c_n \, as_1(K).$$

1.4. **Some additional notation.** Throughout the paper we denote by $B_2^n(u, \rho)$ the Euclidean ball with radius $\rho > 0$ centered at u. Let $\|\cdot\|$ denote the standard Euclidean norm on $\mathbb{R}^n$. For $u, v \in \mathbb{R}^n$, $[u, v]$ will denote the line segment between u and v. We denote the interior of a set $C \subseteq \mathbb{R}^n$ by $\mathrm{int}(C)$. In the sequel, we will always assume that our convex body K contains the origin in its interior. Finally, $c, c_0, c_1,$ etc. shall denote absolute constants that may change from line to line. Let O_n denote the orthogonal group of dimension n.

The paper is organized as follows. In Section 2 we discuss some properties of Ulam's floating bodies, and prove Theorems 1.1 and 1.2. Section 3 is devoted for the proof of Theorem 1.3.

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2. Properties of Ulam's floating bodies

2.1. **Basic properties.** For $\theta \in \mathbb{S}^{n-1}$ and $d \in \mathbb{R}$, we denote the hyperplane orthogonal to θ at distance d from the origin by $H(\theta, d) := \{x \in \mathbb{R}^n : \langle x, \theta \rangle = d\}$. We also denote the closed half-space $H^+(\theta, d) := \{x \in \mathbb{R}^n : \langle x, \theta \rangle \geq d\}$. Given a convex body $K \subseteq \mathbb{R}^n$ and a continuous function $\phi : K \to (0, \infty)$, the function

$$\mathbb{S}^{n-1} \times \mathbb{R} \longrightarrow \left[0, \int_K \phi(z) dz\right],$$

$$(\theta, d) \longrightarrow \delta(\theta, d) := \int_{K \cap H^+(\theta, d)} \phi(z) \, \mathrm{d}z$$

is continuous in the product metric, e,g., by using Lebesgue's dominated convergence theorem. Observe also that the function $(\theta, r) \to (\theta, \delta(\theta, r))$ is a bijection from

$$\left\{(\theta, r) \, : \, \theta \in \mathbb{S}^{n-1}, \, -h_K(-\theta) \leq r \leq h_K(\theta)\right\}$$

to $\mathbb{S}^{n-1} \times \left[0,\, \int_K \phi(x)\,\mathrm{d}x\right]$. We denote

$$(\theta,\,\delta) \to (\theta,\, d(\theta,\delta)) \tag{2.1}$$

as the inverse function of $(\theta,\, d) \to (\theta,\, \delta(\theta,\, d))$, which is also a continuous function. Abusing the notation we denote

$$H^+(\theta,\,\delta) := H^+(\theta,\, d(\theta,\,\delta)), \tag{2.2}$$

Let $h_{\mathrm{M}_\delta(K,\phi)}(\theta)$ be the support function of $\mathrm{M}_\delta(K,\phi)$. By definition of $\mathrm{M}_\delta(K,\phi)$,

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \max_{x\in \mathrm{M}_\delta(K,\phi)} \langle \theta,\, x\rangle = \sup_{0\le f\le 1, \int_K \frac{f(y)\phi(y)}{\delta}\,\mathrm{d}y=1} \int_K \langle y,\,\theta\rangle \frac{f(y)}{\delta}\phi(y)dy. \tag{2.3}$$

It follows from [24, Proposition 2.1] that the maximum in the above equation is attained for the function

$$f = \mathbb{1}_{K\cap H^+(\theta,\,\delta)}$$

and this maximal function is unique as $\phi(y)\mathbb{1}_K\,\mathrm{d}y$ is absolutely continuous with respect to Lebesgue measure. Thus we have the following proposition which is essentially a restatement of Proposition 2.1 of [24].

Proposition 2.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $\phi : K \to (0,\infty)$ be a continuous function. Let $\theta \in \mathbb{S}^{n-1}$ and $\delta \in \left(0, \int_K \phi(y)\,\mathrm{d}y\right)$. Then, the barycenter of $K \cap H^+(\theta,\,\delta)$ with respect to the weight function ϕ,*

$$x_{K,\,\phi}(\theta,\,\delta) := \frac{\int_{K\cap H^+(\theta,\,\delta)} y\phi(y)\,\mathrm{d}y}{\delta}$$

is the unique point in $\partial \mathrm{M}_\delta(K,\phi)$ with normal θ. In particular, $\mathrm{M}_\delta(K,\phi)$ is strictly convex. Moreover,

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \frac{\int_{K\cap H^+(\theta,\,\delta)} \langle\theta,\, y\rangle\phi(y)\,\mathrm{d}y}{\delta}.$$

Extending by limit, $h_{\mathrm{M}_\delta(K,\phi)}$ is a continuous function on $\mathbb{S}^{n-1} \times \left[0, \int_K \phi(y)\,\mathrm{d}y\right]$ and $h_{\mathrm{M}_0(K,\phi)}$ is the support function h_K of K.

We remark that we will use $x(\theta,\,\delta)$ in short for $x_{K,\,\phi}(\theta,\,\delta)$ whenever there is no ambiguity (which is actually everywhere, except for the proof of Theorem 1.2).

Proof. We only need to show that $h_{\mathrm{M}_\delta(K,\phi)}$ is continuous as a function of θ and δ. We put $g(\theta,\, d) = \int_{K\cap H^+(\theta,\, d)} \langle\theta,\, y\rangle\phi(y)\,\mathrm{d}y$. Then g is continuous in the product metric. By the above, the function $(\theta,\,\delta) \to (\theta,\, d(\theta,\delta))$ is continuous in the product metric. Now

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \frac{g(\theta,\, d(\theta,\,\delta))}{\delta},$$

and therefore it is continuous for $0 < \delta \le \int_K \phi(y)\,\mathrm{d}y$, $\theta \in \mathbb{S}^{n-1}$. Moreover, for all $\theta \in \mathbb{S}^{n-1}$ and for all $\delta \in (0, \int_K \phi(y)\,\mathrm{d}y]$,

$$d(\theta,\delta) \le h_{\mathrm{M}_\delta(K,\phi)}(\theta) \le h_K(\theta).$$

Note that for $\delta = 0$, $d(\theta, 0) = h_K(\theta)$. Let $\theta_0 \in \mathbb{S}^{n-1}$ be fixed. For $\varepsilon > 0$, there exists an open ball O containing $(\theta_0, 0) \in \mathbb{S}^{n-1} \times \left[0, \int_K \phi(y)\,\mathrm{d}y\right]$ such that for $(\theta_1, \delta_1) \in O$ we have $|h_K(\theta_0) - d(\theta_1, \delta_1)| < \varepsilon$. Thus, we conclude that $\left|h_K(\theta_0) - h_{\mathrm{M}_{\delta_1}(K,\phi)}(\theta_1)\right| < \varepsilon$ and hence $h_{\mathrm{M}_\delta(K,\phi)}(\theta)$ is continuous at $(\theta_0, 0)$ if we define $h_{\mathrm{M}_0(K,\phi)}(\theta_0) := h_K(\theta_0)$. □

2.2. **Ulam's floating body problem.** Let $K \subseteq \mathbb{R}^n$ be a body with a uniform density $0 < \rho < 1$. Suppose we put K in a liquid of uniform density 1, such that the surface of the liquid is orthogonal to the direction u. Let g be the barycenter of K, and b its center of buoyancy, that is the barycenter of the portion of K which is submerged in the liquid. We say that K floats in equilibrium in direction u if the barycenter of K is directly above its buoyancy center, namely $g - b$ is parallel to u.

A well-known fact in hydrostatics which was pointed out to us by Ning Zhang (see e.g., [19, Theorem 2]) states that if a body floats in liquid, then its barycenter, its center of buoyancy, and the barycenter of the portion of the body that is above the surface of the liquid, are all collinear. In terms of $M_\delta(K)$, this property translates to the following proposition:

Proposition 2.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body with* $\mathrm{bar}(K) = 0$ *and* $|K| = 1$. *Then,* $\mathrm{M}_{1-\delta}(K) = -\frac{\delta}{1-\delta}\mathrm{M}_\delta(K)$.

Remark 2.3. An immediate consequence of the above proposition is that for any convex body $K \subseteq \mathbb{R}^n$, $\mathrm{M}_{\frac{1}{2}}(K)$ is centrally-symmetric. Moreover, by Theorem 1.1 and Proposition 2.6 below, it follows that $\mathrm{M}_{\frac{1}{2}}(K)$ is isomorphic to B_2^n.

Proof. Recall that $h_{M_\delta(K)}(\theta) = \langle x(\theta,\delta), \theta\rangle$ where

$$x(\theta,\, \delta) := \frac{\int_{K\cap H^+(\theta,\,\delta)} y\,\mathrm{d}y}{\delta}$$

and $H^+(\theta,\,\delta)$ is the half space in direction θ such that $|K \cap H^+(\theta,\,\delta)| = \delta$. Observe that

$$0 = \mathrm{bar}(K) = \int_K x\,\mathrm{d}x = \int_{K\cap H^+(\theta,\,\delta)} x\,\mathrm{d}x + \int_{K\cap H^-(\theta,\,\delta)} x\,\mathrm{d}x,$$

which is equivalent to

$$0 = \delta x(\theta,\delta) + (1-\delta)x(-\theta, 1-\delta).$$

Therefore, $x(-\theta, 1-\delta) = -\frac{\delta}{1-\delta}x(\theta,\,\delta)$, which is equivalent to $\mathrm{M}_{1-\delta}(K) = -\frac{\delta}{1-\delta}\mathrm{M}_\delta(K)$. □

As mentioned in the introduction, Ulam's long-standing floating problem asks whether the only body of uniform density that floats in equilibrium in every orientation must be a Euclidean ball. A direct consequence of Proposition 2.2 is that Ulam's floating problem can be restated in terms of $M_\delta(K)$:

Corollary 2.4. *Ulam's floating problem is equivalent to the following problem: Suppose $M_\delta(K)$ is a Euclidean ball. Must K be a Euclidean ball?*

We remark that this new form of Ulam's problem remains open if one replaces $M_\delta(K)$ with the convex floating body K_δ. Another related open problem asks whether a convex body K is centrally-symmetric if and only if K_δ is symmetric. When replaced with $M_\delta(K)$, this problem seems also interesting. Note that Auerbach's counterexample in [5] to Ulam's problem in the plane, provides an example for a non-centrally-symmetric convex body $K \subseteq \mathbb{R}^2$ for which $M_\delta(K)$ is a Euclidean ball, thus answer both of the above problems in this case.

2.3. **Connection to floating bodies.** We begin with the proof of Theorem 1.1:

Proof of Theorem 1.1 . By Proposition 2.1 we have that

$$h_{\mathrm{M}_\delta(K,\phi)}(\theta) = \frac{1}{\delta}\int_{K\cap\{y\in\mathbb{R}^n\,:\,\langle y,\,\theta\rangle\geq d(\theta,\,\delta)\}} \langle x,\,\theta\rangle\phi(x)\,\mathrm{d}x \geq d(\theta,\,\delta) \geq h_{F_\delta(K,\,\phi)}(\theta).$$

Therefore, $F_\delta(K,\phi) \subseteq \mathrm{M}_\delta(K,\phi)$.

Fix $\delta > 0$ and $\theta \in \mathbb{S}^{n-1}$. For $\beta \in \mathbb{S}^{n-1}$, let $H_\beta^+ := \{y \in \mathbb{R}^n : \langle y,\,\beta\rangle \geq \langle x(\theta,\,\delta),\,\beta\rangle\}$. Consider the function $g_\beta(t) := \int_{\{y\,:\,\langle y,\,\beta\rangle=t\}} \mathbf{1}_{K\cap H^+(\theta,\,\delta)}(y)\phi(y)\,\mathrm{d}y$. Since ϕ is log-concave, it follows by Prékopa-Leindler's inequality that g_β is also log-concave. By [32, Lemma 5.4] (a generalization of Grünbaum's inequality), we have that

$$\frac{1}{e}\int g_\beta(t)\,\mathrm{d}t \leq \int_{t\geq\langle x(\theta,\,\delta),\,\beta\rangle} g_\beta(t)\,\mathrm{d}t \leq \left(1-\frac{1}{e}\right)\int g_\beta(t)\,\mathrm{d}t$$

or equivalently,

$$\frac{1}{e}\int_{K\cap H^+(\theta,\,\delta)} \phi(y)\,\mathrm{d}y \leq \int_{H_\beta^+\cap K\cap H^+(\theta,\,\delta)} \phi(y)\,\mathrm{d}y \leq \left(1-\frac{1}{e}\right)\int_{K\cap H^+(\theta,\,\delta)} \phi(y)\,\mathrm{d}y.$$

Taking $\beta = \theta$, we have $H_\theta^+ \cap K \cap H^+(\theta,\,\delta) = H_\theta^+ \cap K$. Since $\int_{H_\theta^+\cap K}\phi(y)\,\mathrm{d}y \leq \left(1-\frac{1}{e}\right)\delta$, we obtain

$$h_{F_{\left(1-\frac{1}{e}\right)\delta}(K,\,\phi)}(\theta) \leq d\left(\theta,\,\left(1-\frac{1}{e}\right)\delta\right) \leq \langle x(\theta,\,\delta),\,\theta\rangle = h_{\mathrm{M}_\delta(K,\phi)}(\theta),$$

and thus $F_{\left(1-\frac{1}{e}\right)\delta}(K,\,\phi) \subseteq \mathrm{M}_\delta(K,\phi)$. On the other hand (see Figure 2.1), for $\beta \in \mathbb{S}^{n-1}$ we have

$$\int_{H_\beta^+\cap K} \phi(y)\,\mathrm{d}y \geq \int_{H_\beta^+\cap K\cap H^+(\theta,\,\delta)} \phi(y)\,\mathrm{d}y \geq \frac{\delta}{e} = \int_{H^+\left(\beta,\,\frac{\delta}{e}\right)\cap K} \phi(y)\,\mathrm{d}y.$$

Hence, $d\left(\beta,\,\frac{\delta}{e}\right) \geq \langle x(\theta,\,\delta),\,\beta\rangle$. Therefore we have

$$x(\theta,\,\delta) \in \bigcap_{\beta\in\mathbb{S}^{n-1}} \left\{y\,:\,\langle y,\,\beta\rangle \leq d\left(\theta,\,\frac{\delta}{e}\right)\right\} = F_{\frac{\delta}{e}}(K,\,\phi).$$

Since $\mathrm{M}_\delta(K,\phi)$ and $F_{\frac{\delta}{e}}(K,\,\phi)$ are convex sets, we conclude that $\mathrm{M}_\delta(K,\phi) \subseteq F_{\frac{\delta}{e}}(K,\,\phi)$. $\square$

The L_p centroid bodies were introduced by Lutwak and Zhang [40] (using a different normalization) as follows: For a convex body K in $\mathbb{R}^n$ of volume 1 and $1 \leq p \leq \infty$, the L_p

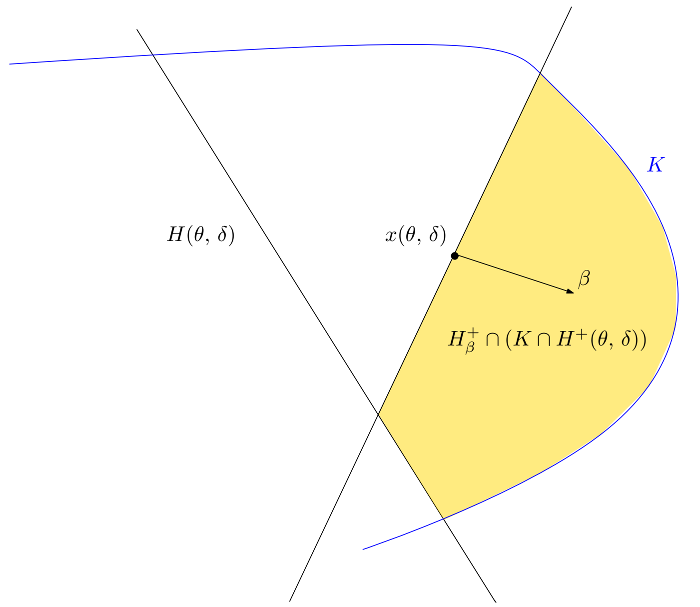

FIGURE 2.1.

centroid body $Z_p(K)$ is this convex body whose support function is given by:

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x,\theta\rangle|^p dx\right)^{1/p}. \tag{2.4}$$

It is known that the floating body K_δ is close to some L_p *centroid body* of K. More precisely, one has:

Theorem 2.5. *([46, Theorem 2.2]) Let K be a symmetric convex body of volume 1. For $\delta \in \left(0, \frac{1}{2}\right)$, we have*

$$c_1 Z_{\log\left(\frac{e}{2\delta}\right)}(K) \subseteq K_\delta \subseteq c_2 Z_{\log\left(\frac{e}{2\delta}\right)}(K),$$

where $c_1, c_2 > 0$ are universal constants.

We obtain a similar result for Ulam floating bodies:

Proposition 2.6. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1 . Then there is an absolute constant $c_1 > 0$ such that for all $\delta < \frac{1}{e}$*

$$c_1 Z_{\log\left(\frac{e}{2\delta}\right)}(K) \subseteq K_\delta \subseteq \mathrm{M}_\delta(K) \subseteq e Z_{\log\left(\frac{1}{\delta}\right)}(K).$$

Proof. The first inclusion holds by Theorem 2.5. The second one, $K_\delta \subseteq \mathrm{M}_\delta(K)$, follows from Theorem 1.1. By Hölder's inequality, we have for $p \in [1, \infty]$,

$$\begin{aligned}\int_{K\cap H^+(\theta,\,\delta)} \langle y,\,\theta\rangle\,\mathrm{d}y &\leq \left(\int_{K\cap H^+(\theta,\,\delta)} 1^q\,\mathrm{d}y\right)^{\frac{1}{q}} \left(\int_K |\langle \theta,\,y\rangle|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\\ &= \delta^{\frac{1}{q}} h_{Z_p(K)}(\theta),\end{aligned}$$

where q satisfies $\frac{1}{p}+\frac{1}{q}=1$. Dividing both sides by δ, we get

$$h_{\mathrm{M}_\delta(K)}(\theta,\,\delta)\leq\left(\frac{1}{\delta}\right)^{\frac{1}{p}}h_{Z_p(K)}(\theta).$$

Putting $p=\log\left(\frac{1}{\delta}\right)$ yields

$$h_{\mathrm{M}_\delta(K)}(\theta,\,\delta)\leq e\;h_{Z_{\log\left(\frac{1}{\delta}\right)}(K)}(\theta).$$

Therefore, we have that

$$\mathrm{M}_\delta(K)\subseteq e\;Z_{\log\left(\frac{1}{\delta}\right)}(K).$$

□

2.4. **Smoothness of Ulam floating bodies.** In this section we prove Theorem 1.2. To this end, let $\rho_v(\cdot)$ denote the radial function of K with center v. That is,

$$\rho_v(\theta)=\max\bigl\{r\in\mathbb{R}^+\,:\,v+r\theta\in K\bigr\}.$$

We will need the following fact, which can be found implicitly in e.g., [48].

Fact 2.7. *Let $K\subseteq\mathbb{R}^n$ be a convex body. Then, the following are equivalent:*

(1) K has C^k boundary;
(2) The function $(v,\,\theta)\to\rho_v(\theta)$ is C^k for every $v\in\mathrm{int}(K)$ and $\theta\in\mathbb{S}^{n-1}$;
(3) There exists $v\in\mathrm{int}(K)$ such that $\theta\to\rho_v(\theta)$ is C^k.

Proof of Theorem 1.2. For $a\in\mathbb{R}^n\setminus\{0\}$, let $H:=\{x\,:\,\langle x,\,a\rangle=1\}$, $\delta(a)=|K\cap\{\langle x,\,a\rangle\geq 1\}|$, and $U(a):=\int_{K\cap\{\langle x,\,a\rangle\geq 1\}}x\,\mathrm{d}x$. We would like to show that

$$\nabla\delta(a)=\frac{1}{\|a\|}\int_{K\cap H}x\,\mathrm{d}x \tag{2.5}$$

$$DU=\frac{1}{\|a\|}\left(\int_{K\cap\{\langle x,\,a\rangle=1\}}x_ix_j\,\mathrm{d}x\right)_{i,j\in[n]}, \tag{2.6}$$

where DU denotes the differential of U and $[n]=\{1,\cdots,n\}$. Equation (2.5) was proved in [42, Lemma 5]. Using the same ideas, we prove (2.6) as follows. Pick a direction θ so that θ is not parallel to a, and let $H_\varepsilon:=\{x\,:\,\langle x,\,a+\varepsilon\theta\rangle=1\}$. As illustrated in Figure 2.2, we also define:

$$\begin{aligned}K_-(\varepsilon)&=\mathrm{int}(K)\cap\{y\in\mathbb{R}^n\,:\,\langle y,\,a\rangle\geq 1,\,\langle y,\,a+\varepsilon\theta\rangle\leq 1\},\\ K_+(\varepsilon)&=\mathrm{int}(K)\cap\{y\in\mathbb{R}^n\,:\,\langle y,\,a\rangle\leq 1,\,\langle y,\,a+\varepsilon\theta\rangle\geq 1\}.\end{aligned}$$

Let U_j denote the jth coordinate of U. We have

$$U_j(a+\varepsilon\theta)-U_j(a)=\int_{K_+(\varepsilon)}\langle x,\,e_j\rangle\,\mathrm{d}x-\int_{K_-(\varepsilon)}\langle x,\,e_j\rangle\,\mathrm{d}x.$$

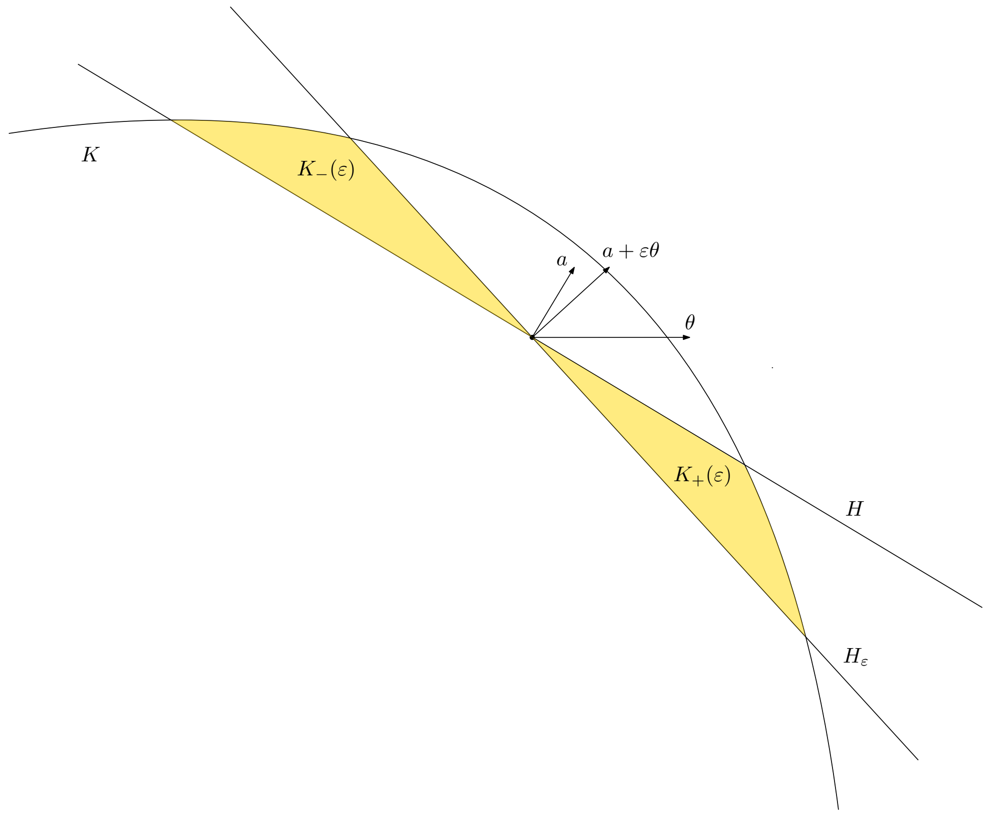

FIGURE 2.2.

From now on we choose $\varepsilon > 0$ small enough so that $\langle a, a+\varepsilon\theta\rangle > 0$. For $y \in \mathbb{R}^n$, we write y uniquely in the form $x + t\frac{a}{\|a\|}$, where $x = y + \frac{1-\langle y, a\rangle}{\langle a, a\rangle}a$ and $t = -\frac{1-\langle y, a\rangle}{\langle a, a\rangle}\|a\|$. Notice that $x \in H$. Then,

$$
\begin{aligned}
&\{y \in \mathbb{R}^n : \langle y, a\rangle \geq 1, \langle y, a+\varepsilon\theta\rangle \leq 1\} = \\
&\left\{x + ta : x \in H, t \in \mathbb{R}, \langle x + t\frac{a}{\|a\|}, a\rangle \geq 1, \langle x + t\frac{a}{\|a\|}, a+\varepsilon\theta\rangle \leq 1\right\} = \\
&\left\{x + ta : x \in H, 0 \leq t \leq \frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle}\right\} = \\
&\left\{x + ta : x \in H, \langle x, \theta\rangle \leq 0, 0 \leq t \leq \frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle}\right\}.
\end{aligned}
$$

Thus,

$$K_-(\varepsilon) = \left\{x + ta : x \in H, \langle x, \theta\rangle \leq 0, 0 \leq t \leq \frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle}\right\} \cap \mathrm{int}(K).$$

Let

$$O_-(\varepsilon) := \left\{x \in H : \langle x, \theta\rangle \leq 0, \left[x, x + \frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle}a\right] \cap \mathrm{int}(K) \neq \emptyset\right\}.$$

For $x \in H$ such that $\langle x, \theta\rangle \leq 0$, we have that

$$\frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle} = \frac{\varepsilon|\langle x, \theta\rangle|\|a\|}{\langle a, a+\varepsilon\theta\rangle} = \frac{|\langle x, \theta\rangle|\|a\|}{\langle a, a\rangle\varepsilon^{-1} + \langle a, \theta\rangle}$$

decrease to 0 as $\varepsilon \searrow 0$. Thus, $O(\varepsilon)$ shrinks to

$$\begin{aligned} O_-(0) &= \{x \in H \,:\, \langle x, \theta\rangle \leq 0, \, [x, x] \cap \operatorname{int}(K) \neq \emptyset\} \\ &= \{x \in H \cap \operatorname{int}(K) \,:\, \langle x, \theta\rangle \leq 0\}. \end{aligned}$$

For $x \in O_-(\varepsilon)$, let $0 \leq t_1(\varepsilon, x) \leq t_2(\varepsilon, x) \leq \frac{-\varepsilon\langle x, \theta\rangle}{\langle a, a+\varepsilon\theta\rangle}\|a\|$ be defined such that

$$\left\{x + ta \,:\, 0 \leq t \leq \frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle}\right\} \cap \operatorname{int}(K) = \{x + ta \,:\, t_1(\varepsilon, x) < t < t_2(\varepsilon, x)\}.$$

Then, by Fubini's theorem, we have

$$\begin{aligned} \int_{K_-(\varepsilon)} \langle y, e_j\rangle \, \mathrm{d}y &= \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon, x)}^{t_2(\varepsilon, x)} \langle x + t\frac{a}{\|a\|}, e_j\rangle \, \mathrm{d}t \, \mathrm{d}x \\ &= \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon, x)}^{t_2(\varepsilon, x)} \langle x, e_j\rangle \, \mathrm{d}t \, \mathrm{d}x + \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon, x)}^{t_2(\varepsilon, x)} \langle t\frac{a}{\|a\|}, e_j\rangle \, \mathrm{d}t \, \mathrm{d}x. \end{aligned}$$

We analyze each of the above terms, separately, as follows.

Firstly, we have that

$$\begin{aligned} \left|\int_{O_-(\varepsilon)} \int_{t_1(\varepsilon, x)}^{t_2(\varepsilon, x)} \langle t\frac{a}{\|a\|}, e_j\rangle \, \mathrm{d}t \, \mathrm{d}x\right| &\leq \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon, x)}^{t_2(\varepsilon, x)} t \, \mathrm{d}t \, \mathrm{d}x \\ &\leq \int_{O_-(\varepsilon)} \int_0^{\frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle}} t \, \mathrm{d}t \, \mathrm{d}x \\ &\leq \frac{1}{2}\frac{\varepsilon^2\|a\|^2}{\langle a, a+\varepsilon\theta\rangle^2} \int_{O_-(\varepsilon)} \langle x, \theta\rangle^2 \, \mathrm{d}x. \end{aligned}$$

Since $O_-(\varepsilon)$ is bounded and shrinks as ε decreases, we conclude that

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{O_-(\varepsilon)} \int_{t_1(\varepsilon, x)}^{t_2(\varepsilon, x)} \langle t\frac{a}{\|a\|}, e_j\rangle \, \mathrm{d}t \, \mathrm{d}x = 0.$$

Secondly, we have that

$$\frac{\int_{O_-(\varepsilon)} \int_{t_1(\varepsilon, x)}^{t_2(\varepsilon, x)} \langle x, e_j\rangle \, \mathrm{d}t \, \mathrm{d}x}{\varepsilon} = \int_H \frac{(t_2(x, \varepsilon) - t_1(x, \varepsilon))\langle x, e_j\rangle \mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon} \, \mathrm{d}x.$$

Fix $\varepsilon_0 > 0$. For $\varepsilon_0 > \varepsilon > 0$, we have that

$$\left|\frac{(t_2(x, \varepsilon) - t_1(x, \varepsilon))\langle x, e_j\rangle \mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon}\right| \leq \frac{|\langle x, \theta\rangle|\|a\|}{\langle a, a\rangle - \varepsilon_0|\langle a, \theta\rangle|} |\langle x, e_j\rangle| \mathbf{1}_{O_-(\varepsilon_0)},$$

where the function on the right hand side is integrable.

Suppose $x \notin O_-(0)$. Then, $\frac{(t_2(x,\varepsilon)-t_1(x,\varepsilon))\langle x, e_j\rangle \mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon} \to 0$ as $\varepsilon \searrow 0$ since $\mathbf{1}_{O_-(\varepsilon)}(x) = 0$ for small $\varepsilon > 0$. For $x \in O_-(0)$, we have $t_1(x) = 0$ and $t_2(x) = \frac{-\varepsilon\langle x, \theta\rangle\|a\|}{\langle a, a+\varepsilon\theta\rangle}$ for sufficiently small

ε. We conclude that, as $\varepsilon \searrow 0$,

$$\frac{(t_2(x,\,\varepsilon) - t_1(x,\,\varepsilon))\langle x,\, e_j\rangle \mathbf{1}_{O_-(\varepsilon)}(x)}{\varepsilon} \to \frac{-\langle x,\,\theta\rangle\langle x,\, e_j\rangle}{\|a\|}\mathbf{1}_{O_-(0)}(x).$$

By Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}&\lim_{\varepsilon\searrow 0} -\frac{\int_{K_-(\varepsilon)}\langle x,\, e_j\rangle\,\mathrm{d}x}{\varepsilon}\\ =&\lim_{\varepsilon\searrow 0} -\frac{\int_{O_-(\varepsilon)}\int_{t_1(\varepsilon,\,x)}^{t_2(\varepsilon,\,x)}\langle x,\, e_j\rangle\,\mathrm{d}t\,\mathrm{d}x}{\varepsilon}\\ =&\frac{1}{\|a\|}\int_{K\cap H\cap\{\langle x,\,\theta\rangle\le 0\}}\langle x,\,\theta\rangle\langle x,\, e_j\rangle\,\mathrm{d}x.\end{aligned}$$

Via the same argument, one also shows that

$$\lim_{\varepsilon\searrow 0}\frac{\int_{K_+(\varepsilon)}\langle x,\, e_j\rangle\,\mathrm{d}x}{\varepsilon} = \frac{1}{\|a\|}\int_{K\cap H\cap\{\langle x,\,\theta\rangle\ge 0\}}\langle x,\,\theta\rangle\langle x,\, e_j\rangle\,\mathrm{d}x.$$

Thus we conclude that

$$\lim_{\varepsilon\searrow 0}\frac{U_j(a+\varepsilon\theta) - U_j(a)}{\varepsilon} = \frac{1}{\|a\|}\int_{K\cap H}\langle x,\,\theta\rangle\langle x,\, e_j\rangle\,\mathrm{d}x.$$

This completes the proof of (2.6).

Next, we show that $DU(a)$ and $\nabla\delta(a)$ are C^k functions.

Pick $v \in \mathrm{int}(K)\cap H$. Let σ_a be the normalized Haar measure on $S(a) = \mathbb{S}^{n-1}\cap a^\perp$. Then

$$\begin{aligned}\int_{K\cap H} x\,\mathrm{d}x &= (n-1)\left|B_2^{n-1}\right|\int_{S(a)}\int_0^{\rho_v(\theta)} r^{n-2}(v+r\theta)\,\mathrm{d}r\,\mathrm{d}\sigma_a(\theta)\\ &= \left|B_2^{n-1}\right|\int_{S(a)}\left(\rho_v^{n-1}(\theta)v + \frac{n-1}{n}\rho_v^n(\theta)\theta\right)\mathrm{d}\sigma_a(\theta).\end{aligned}\tag{2.7}$$

Fix $a_0\in\mathbb{R}^n$ so that $\mathrm{int}(K)\cap\{\langle x,\, a_0\rangle = 1\}\neq\emptyset$ and let $v_0\in\mathrm{int}(K)\cap\{\langle x,\, a_0\rangle = 1\}$. By Fact 2.7, $(v,\,\theta)\to\rho_v(\theta)$ is C^k, and hence the function $F_{a_0}:\mathbb{R}^n\times O_n\to\mathbb{R}^n$ defined by

$$(v,\,T)\mapsto\left|B_2^{n-1}\right|\int_{S(a_0)}\left(\rho_v^{n-1}(T\theta)v + \frac{n-1}{n}\rho_v^n(T\theta)T\theta\right)\mathrm{d}\sigma_{a_0}(\theta)$$

is also C^k. We can find a smooth function $a\mapsto(v(a),\,T(a))$ in a neighborhood of a_0 so that $v(a)\in\mathrm{int}(K)\cap\{\langle x,\, a\rangle = 1\}$ and $T(a)S(a_0) = \mathbb{S}^{n-1}\cap a^\perp$. Indeed, for a close to a_0, we define the unique two-dimensional rotation $T(a)$ satisfying $T(a)\frac{a_0}{\|a_0\|} = \frac{a}{\|a\|}$ and $T(a)v = v$ for all $v\in\mathrm{span}(a,\,a_0)^\perp$. In particular, $a\mapsto T(a)$ is a smooth function around a_0. Also, $T(a)(S(a_0)) = S(a)$. Let $v(a)$ be the projection of v_0 onto $\{\langle x,\, a\rangle = 1\}$. In other words,

$$v(a) := v_0 - \langle v_0,\,\frac{a}{\|a\|}\rangle\frac{a}{\|a\|} + \frac{a}{\|a\|^2},$$

which is again smooth when $a\neq 0$. Also, $v(a_0) = v_0$, and $v(a)\in\mathrm{int}(K)$ if a is close to a_0.

Next, we express $\nabla\delta$ in terms of $v(a)$ and $T(a)$: By (2.7) we have

$$\begin{aligned}\nabla\delta(a) &= \int_{K\cap\{\langle x,\,a\rangle=1\}} x\,\mathrm{d}x\\ &= \frac{1}{\|a\|}\left|B_2^{n-1}\right|\int_{S(a)}\left(\rho_{v(a)}^{n-1}(\theta)v(a)+\frac{n-1}{n}\rho_{v(a)}^{n}(\theta)\theta\right)\mathrm{d}\sigma_a(\theta)\\ &= \frac{1}{\|a\|}\left|B_2^{n-1}\right|\int_{S(a_0)}\left(\rho_{v(a)}^{n-1}(T(a)\theta)v(a)+\frac{n-1}{n}\rho_{v(a)}^{n}(T(a)\theta)T(a)\theta\right)\mathrm{d}\sigma_{a_0}(\theta)\\ &= \frac{1}{\|a\|}F_{a_0}(v(a),\,T(a)).\end{aligned}$$

We conclude that $\nabla\delta(a)$ is C^k and thus $\delta(a)$ is C^{k+1}.

Recall that $\delta(\theta,\,d)=|K\cap\{\langle x,\,\theta\rangle\geq d\}|$. Consider the function from $\mathbb{S}^{n-1}\times\mathbb{R}$ to $\mathbb{S}^{n-1}\times\mathbb{R}$ defined by

$$(\theta,\,d)\mapsto\left(\theta,\,\delta\left(\frac{1}{d}\theta\right)\right)=(\theta,\,\delta(\theta,\,d)).$$

By the above, it is C^{k+1} whenever $\mathrm{int}(K)\cap\{\langle x,\,\theta\rangle=d\}\neq\emptyset$. Thus, its inverse function $(\theta,\,d(\theta,\,\delta))$ is also C^{k+1} for $(\theta,\,\delta)\in\mathbb{S}^{n-1}\times[0,\,|K|]$. Repeating the same argument as for $\nabla\delta(a)$ implies that $U(a)$ is also C^{k+1}.

Recall that if $d(\theta,\,\delta)>0$,

$$x_K(\theta,\,\delta)=\frac{1}{\delta}\int_{K\cap\{\langle x,\,\theta\rangle\geq d(\theta,\,\delta)\}}x\,\mathrm{d}x=\frac{1}{\delta}U\left(\frac{\theta}{d(\theta,\,\delta)}\right).$$

Therefore, for a fixed $0<\delta<|K|$, and θ such that $d(\theta,\,\delta)>0$, the function $\theta\mapsto\frac{x_K(\theta,\,\delta)}{\|x_K(\theta,\,\delta)\|}$ is C^{k+1}. Moreover, it is invertible since $\mathrm{M}_\delta(K)$ is strictly convex. Thus its inverse, denoted by $G_\delta:\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}$ is also C^{k+1}. Therefore, the radial function of $\mathrm{M}_\delta(K)$, which is given by $\rho(\theta)=\|x(G_\delta(\theta),\,\delta)\|$ is also C^{k+1}.

Finally, we need to show that $\theta\to x_K(\theta,\,\delta)$ is C^{k+1} whenever $d(\theta,\,\delta)\leq 0$. Indeed, we may choose some vector $v\in\mathbb{R}^n$ and consider $M_\delta(v+K)$. Then, $x_K(\theta,\,\delta)=x_{v+K}(\theta,\,\delta)-v$. Clearly, we can always choose v such that, for $v+K$, $d(\theta,\,\delta)>0$. Thus, following the same argument, we can show $x_{v+K}(\theta,\,\delta)$ is C^{k+1}. As a consequence, $x_K(\theta,\,\delta)$ is C^{k+1}. Therefore, we conclude that $\rho(\theta)$ is C^{k+1} on $\mathbb{S}^{n-1}$. By Fact (2.7), the boundary of $\mathrm{M}_\delta(K)$ is C^{k+1}. □

3. Relation to p-affine surface area

This section is devoted to the proof of Theorem 1.3.

3.1. Preliminary results.

For the proof of Theorem 1.3, we will need a few preliminary results.

First, we focus on $\mathrm{M}_\delta(\rho B_2^n,\phi)$, where ρB_2^n is the Euclidean ball centered at 0 and with radius ρ, and $\phi(x)$ is a constant function. By symmetry, we know that $\mathrm{M}_\delta(\rho B_2^n,\phi)$ is again a Euclidean ball with the same center. Let $\Delta(\rho,\,\delta)$ be the difference of the radius of ρB_2^n and

$\mathrm{M}_\delta(\rho B_2^n, \phi)$. If $\phi : \rho B_2^n \to (0, \infty)$, is a constant function, $\phi(x) = s$, for all $x \in \rho B_2^n$, then, we define $\Delta(\rho, \delta, s)$ to be the difference of radius of ρB_2^n and $\mathrm{M}_\delta(\rho B_2^n, s)$. One easily verifies that

$$\Delta(\rho, \delta, s) = \Delta\left(\rho, \frac{\delta}{s}\right). \tag{3.1}$$

Proposition 3.1. *$\lim_{\delta \searrow 0} \Delta(\rho, \delta)/\delta^{\frac{2}{n+1}} \rho^{\frac{n+1}{n-1}} = c_n$, where $c_n = \frac{1}{2} \frac{n+1}{n+3} \left(\frac{n+1}{|B_2^{n-1}|}\right)^{\frac{2}{n+1}}$.*

Proof. We denote $h(\rho, \delta)$ to be height of the cap of ρB_2^n which has volume δ. To be specific, $h(\rho, \delta)$ satisfies the equality

$$\delta = \left|B_2^{n-1}\right| \int_0^{h(\rho,\delta)} g^{n-1}(t)\,\mathrm{d}t,$$

where $g(t) = \left(\rho^2 - (\rho - t)^2\right)^{1/2}$. Moreover,

$$g(t) = \left(\rho^2 - (\rho - t)^2\right)^{1/2} = \rho\left(1 - (1 - t/\rho)^2\right)^{1/2} = \rho(2 - t/\rho)^{1/2}(t/\rho)^{1/2}.$$

We have

$$\delta = \left|B_2^{n-1}\right|\rho^{n-1} \int_0^{h(\rho,\,\delta)} (2 - t/\rho)^{\frac{n-1}{2}} (t/\rho)^{\frac{n-1}{2}}\,\mathrm{d}t.$$

Thus, we have the inequality

$$\begin{aligned} \left|B_2^{n-1}\right|\rho^{n-1}(2 - h(\rho,\,\delta)/\rho)^{\frac{n-1}{2}} \int_0^{h(\rho,\,\delta)} (t/\rho)^{\frac{n-1}{2}}\,\mathrm{d}t \le\ & \delta \\ \le\ & \left|B_2^{n-1}\right|\rho^{n-1} 2^{\frac{n-1}{2}} \int_0^{h(\rho,\,\delta)} (t/\rho)^{\frac{n-1}{2}}\,\mathrm{d}t. \end{aligned}$$

Since

$$\int_0^{h(\rho,\,\delta)} (t/\rho)^{\frac{n-1}{2}}\,\mathrm{d}t = \frac{2}{n+1} h(\rho,\,\delta)^{\frac{n+1}{2}} \rho^{-\frac{n-1}{2}},$$

we obtain

$$\frac{1}{2}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}} \le \frac{h(\rho,\,\delta)}{\delta^{\frac{2}{n+1}}} \le \frac{1}{2 - h(\rho,\,\delta)/\rho}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}}.$$

We conclude that

$$\lim_{\delta \searrow 0} \frac{h(\rho,\,\delta)}{\delta^{\frac{2}{n+1}}} = \frac{1}{2}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}}.$$

We have that

$$\Delta(\rho,\,\delta) = \frac{\left|B_2^{n-1}\right| \int_0^{h(\rho,\,\delta)} t g(t)^{n-1}\,\mathrm{d}t}{\left|B_2^{n-1}\right| \int_0^{h(\rho,\,\delta)} g(t)^{n-1}\,\mathrm{d}t}.$$

To compute the next limit, we apply twice L'Hospital's Rule,

$$\lim_{h\to 0}\frac{h}{\Delta}=\lim\frac{h\int_0^h h^{n-1}\,\mathrm{d}t}{\int_0^h tg^{n-1}\,\mathrm{d}t}\overset{L}{=}\lim\frac{\int_0^h g^{n-1}\,\mathrm{d}t+hg(h)^{n-1}}{hg(h)^{n-1}}=1+\lim\frac{\int_0^h g^{n-1}\,\mathrm{d}t}{hg(h)^{n-1}}$$

$$\overset{L}{=}1+\lim\frac{\rho^{n-1}\left(2-\frac{r}{\rho}\right)^{\frac{n-1}{2}}\left(\frac{r}{\rho}\right)^{\frac{n-1}{2}}}{\rho^n\left(\frac{1}{\rho}\frac{n+1}{2}\left(\frac{r}{\rho}\right)^{\frac{n-1}{2}}\left(2-\frac{r}{\rho}\right)^{\frac{n-1}{2}}-\frac{1}{\rho}\frac{n-1}{2}\left(\frac{r}{\rho}\right)^{\frac{n+1}{2}}\left(2-\frac{r}{\rho}\right)^{\frac{n-3}{2}}\right)}$$

$$=1+\lim\frac{\left(2-\frac{r}{\rho}\right)}{\frac{n+1}{2}\left(2-\frac{r}{\rho}\right)-\frac{n-1}{2}\left(\frac{r}{\rho}\right)}=1+\frac{2}{n+1}=\frac{n+3}{n+1}.$$

So,

$$\lim_{\delta\searrow 0}\frac{\Delta(\rho,\delta)}{\delta^{\frac{2}{n+1}}}=\lim_{\delta\searrow 0}\frac{h(\rho,\delta)}{\delta^{\frac{2}{n+1}}}\cdot\frac{\Delta(\rho,\delta)}{h(\rho,\delta)}=\frac{1}{2}\,\frac{n+1}{n+3}\left(\frac{n+1}{\left|B_2^{n-1}\right|}\right)^{\frac{2}{n+1}}\rho^{-\frac{n-1}{n+1}}.$$

□

We will also need the next lemma from [52]:

Lemma 3.2. *Let K and L be convex bodies in $\mathbb{R}^n$ such that $0\in int(L)$ and such that $L\subseteq K$. Then*

$$|K|-|L|=\frac{1}{n}\int_{\partial K}\langle x,N(x)\rangle\left(1-\left|\frac{\|x_L\|}{\|x\|}\right|^n\right)\mathrm{d}\mu_K(x),$$

where x_L is the unique point in the intersection $\partial L\cap[0,x]$.

For the next lemma we need a notion that was introduced in [52]. Let K be a convex body in $\mathbb{R}^n$ and let $x\in\partial K$ be such that $N_K(x)$ is unique. We put $r(x)$ to be the radius of the biggest Euclidean ball contained in K that touches K in x,

$$r(x)=\max\{\rho : B_2^n(x-\rho N_K(x),\rho)\subseteq K\}.$$

If $N_K(x)$ is not unique, $r(x)=0$. It was shown in [52, Lemma 5] that for any convex body K in $\mathbb{R}^n$ and any $0\le\alpha<1$,

$$\int_{\partial K} r(x)^{-\alpha}d\mu(x)<\infty. \tag{3.2}$$

Lemma 3.3. *Let K be a convex body in $\mathbb{R}^n$. Let $x\in\partial K$ and let $x_{M,\delta}=\partial(\mathrm{M}_\delta(K,\phi))\cap[0,x]$. Then*

$$\frac{\langle x,\,N_K(x)\rangle}{\delta^{\frac{2}{n+1}}}\left(1-\left|\frac{\|x_{M,\delta}\|}{\|x\|}\right|^n\right)\le c\;n\;r(x)^{-\frac{n-1}{n+1}},$$

where c is a constant independent of x and δ.

Proof. Let $x_{F,\delta}=\partial(F_\delta(K,\phi))\cap[0,x]$. By Theorem 1.1, we have that $F_\delta(K,\phi)\subseteq M_\delta(K,\phi)$ and hence $\|x_{\mathbb{F},\delta}\|\le\|x_{M,\delta}\|$. Therefore

$$\frac{\langle x,\,N_K(x)\rangle}{\delta^{\frac{2}{n+1}}}\left(1-\left|\frac{\|x_{M,\delta}\|}{\|x\|}\right|^n\right)\le\frac{\langle x,\,N_K(x)\rangle}{\delta^{\frac{2}{n+1}}}\left(1-\left|\frac{\|x_{F,\delta}\|}{\|x\|}\right|^n\right)$$

and it was shown in [52], Lemma 8, that the latter is smaller than or equal $c\, n\, r(x)^{-\frac{n-1}{n+1}}$. $\square$

The next lemma was proved in [52]. There, and in the proof of the main theorem, we need the indicatrix of Dupin (see, e.g., [53]). A theorem of Alexandrov [1] and Busemann and Feller [13] shows that the indicatrix of Dupin exists almost everywhere on ∂K and is an ellipsoid or an elliptic cylinder. We also use the notation $C(r,h)$ for the cap of a Euclidean ball with radius r and height h.

Lemma 3.4. [52] *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \partial K$ and $N_K(0) = -e_n = (0,\cdots,0,-1)$. Suppose the indicatrix of Dupin at 0 exists and is an $(n-1)$-dimensional sphere with radius $\sqrt{\rho}$. Let ξ be an interior point of K.*

(i) Let H be the hyperplane orthogonal to $N_K(0)$ and passing through z in $[0,\xi]$. We put $z_n = \langle z, e_n\rangle$. Then we have for $0 \le z_n \le \rho$,

$$\left|K \cap H^+\right| \le f(z_n)^{n-1} |C(\rho, z_n)|.$$

(ii) Let $d = dist\big(z, B_2^n(\rho\, e_n, \rho)^C\big)$. There is $\varepsilon_0 > 0$ such that we have for all $z \in [0,\xi]$ with $\|z\| \le \varepsilon_0$

$$d \le z_n \le d + \frac{2\, d^2}{\rho \langle \frac{\xi}{\|\xi\|}, N_K(0)\rangle^2}.$$

(iii) There is $\varepsilon_0 > 0$ and an absolute constant $c > 0$ such that for all $z \in [0,\xi]$ with $\|z\| \le \varepsilon_0$ and all hyperplanes H passing through z

$$\left|K \cap H^+\right| \ge f(\gamma)^{-n+1} |C(\rho, d(1 - c(f(\gamma)-1))|.$$

Here, $\gamma = 2\sqrt{2\, \rho\, d}$ and f is a monotone function on $\mathbb{R}^+$ such that $\lim_{t\to 0} f(t) = 1$.

The function f in Lemma 3.4 (iii) depends on K. It controls the error between the approximating ellipsoid and K at a boundary point of K.

Lemma 3.5. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Moreover, we assume that $0 \in \partial K$ and that $N_K(0) = -e_n$ is the unique outer normal to ∂K at 0. Let $\phi : K \to (0,\infty)$ be a continuous function. We set $H_t^+ = H^+(-e_n, -t) = \{y : \langle y, e_n\rangle < t\}$. Then, for each $t > 0$, there exists $r > 0$ such that for any $\delta > 0$,*

$$\mathrm{M}_\delta(K,\phi) \cap B_2^n(0,\, r) = \mathrm{M}_\delta\big(K \cap H_t^+, \phi\big) \cap B_2^n(0,\, r).$$

Proof. It is obvious that

$$\mathrm{M}_\delta\big(K \cap H_t^+, \phi\big) \cap B_2^n(0,\, r) \subseteq \mathrm{M}_\delta(K,\phi) \cap B_2^n(0,\, r).$$

Therefore, it is sufficient to show the other inclusion. Let $d \ge 0$. Observe that if $(\theta,\, d)$ is sufficiently close to $(-e_n,\, 0)$, then $H^+(\theta, -d) \cap K \subseteq H_t^+$, where $H^+(\theta,-d) = \{y : \langle y, -\theta\rangle < d\}$. As noted in (2.1), the function $d(\theta,\delta)$ is continuous in $(\theta,\, \delta)$. Therefore, there exists $\delta_0 > 0$ and $\varepsilon > 0$ such that

$$K \cap H^+(\theta,\, d(\theta,\, \delta)) \subseteq H_t^+, \tag{3.3}$$

for $\|\theta - (-e_n)\| < \varepsilon$ and $0 \leq \delta < \delta_0$. For each x in the interior of K, let $\delta(x)$ be the value such that $x \in \partial \mathrm{M}_{\delta(x)}(K, \phi)$ and $\theta(x)$ denote the unique outer normal at x of $\mathrm{M}_{\delta(x)}(K, \phi)$.

Claim : For any $\delta_0 > 0$ and $\varepsilon > 0$, there exists $r > 0$ such that $\delta(x) < \delta_0$ and $\|\theta(x) - (-e_n)\| < \varepsilon$, for $x \in \mathrm{int}(K) \cap B_2^n(0, r)$.

Indeed, note that $\mathrm{M}_{\delta_0}(K, \phi)$ is strictly contained in K. Thus, $0 \notin \mathrm{M}_{\delta_0}(K, \phi)$. Since $\mathrm{M}_{\delta_0}(K, \phi)$ is convex, there exists $r > 0$ so that $B_2^n(0, r) \cap \mathrm{M}_{\delta_0}(K, \phi) = \emptyset$. Then, $\delta(x) < \delta_0$ for $x \in \mathrm{int}(K) \cap B_2^n(0, r)$.

It remains to show that there exists $r > 0$ such that $\|\theta(x) - (-e_n)\| < \varepsilon$ for $\mathrm{int}(K) \cap B_2^n(0, r)$. Suppose it is false. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathrm{int}(K)$ such that $x_k \to 0$ and such that $\|\theta(x_k) - (-e_n)\| > \varepsilon$. By the compactness of $\mathbb{S}^{n-1}$, we may replace $(x_k)_{k\in\mathbb{N}}$ by a subsequence, again denoted by $(x_k)_{k\in\mathbb{N}}$, so that $\theta(x_k)$ converges to some $\theta_1 \neq -e_n$. Moreover, $\delta(x_k) \to 0$ since the first claim is true. Continuity of $h_{\mathrm{M}_{\delta(K,\phi)}}(\theta)$ implies that $h_{\mathrm{M}_{\delta(x_k)}(K,\phi)}(\theta(x_k)) \to h_K(\theta_1)$. As $-e_n$ is the unique outer normal to ∂K in 0, $h_K(\theta_1) > \langle 0, \theta_1 \rangle = 0$. Therefore, we obtain a contradiction, as $h_{\mathrm{M}_{\delta(x_k)}(K,\phi)}(\theta(x_k)) = \langle x_k, \theta(x_k) \rangle$, which converges to 0 as $x_k \to 0$. This completes the proof of the claim.

Hence, with the assumptions on δ_0 and ε, we conclude that there exists $r > 0$ such that for $x \in \mathrm{int}(K) \cap B_2^n(0, r)$,

$$K \cap H^+(\theta(x), d(\theta(x), \delta(x))) \subseteq H_t^+.$$

Let $x \in M_\delta(K, \phi) \cap B_2^n(0, r)$. Since $x \in \mathrm{int}(K) \cap B_2^n(0, r)$,

$$K \cap H^+(\theta(x), d(\theta(x), \delta(x))) \subseteq H_t^+,$$

and thus $x \in \mathrm{M}_{\delta(x)}\big(K \cap H_t^+, \phi\big)$. Moreover, notice that $\delta(x) \geq \delta$ and hence we have

$$\mathrm{M}_{\delta(x)}\big(K \cap H_t^+, \phi\big) \subseteq \mathrm{M}_\delta\big(K \cap H_t^+, \phi\big).$$

Hence, $x \in \mathrm{M}_\delta\big(K \cap H_t^+, \phi\big)$. Therefore, $\mathrm{M}_\delta(K, \phi) \cap B(0, r) \subseteq \mathrm{M}_\delta\big(K \cap H_t^+, \phi\big) \cap B(0, r)$. $\square$

3.2. **Proof of Theorem 1.3.** Recall that x_M is the unique point in $\partial(\mathrm{M}_\delta(K, \phi)) \cap [0, x]$. We will sometimes write in short x_M for $x_{M,\delta}$. By Lemmas 3.2 and 3.3, we have that

$$\lim_{\delta\to 0} \frac{|K| - |\mathrm{M}_\delta(K, \phi)|}{\delta^{\frac{2}{n+1}}} = \frac{1}{n} \int_{\partial K} \lim_{\delta\to 0} \delta^{-\frac{2}{n+1}} \langle x, N_K(x)\rangle \left(1 - \left|\frac{\|x_M\|}{\|x\|}\right|^n\right) \mathrm{d}\mu_K(x).$$

For $x \in \partial K$ fixed, the goal is to understand

$$\lim_{\delta\searrow 0} \frac{1}{n} \int_{\partial K} \delta^{-\frac{2}{n+1}} \langle x, N_K(x)\rangle \left(1 - \left|\frac{\|x_M\|}{\|x\|}\right|^n\right) \mathrm{d}\mu_K(x).$$

As x and x_M are collinear and as for all $0 \leq a \leq 1$,

$$1 - na \leq (1 - a)^n \leq 1 - na + \frac{n(n-1)}{2} a^2,$$

we get for δ sufficiently small that

$$\frac{\|x-x_M\|}{\|x\|}\left(1-\frac{n-1}{2}\frac{\|x-x_M\|}{\|x\|}\right)\le\frac{1}{n}\left(1-\left|\frac{\|x_M\|}{\|x\|}\right|^n\right)=$$

$$\frac{1}{n}\left[1-\left(1-\frac{\|x-x_M\|}{\|x\|}\right)^n\right]\le\frac{\|x-x_M\|}{\|x\|}. \tag{3.4}$$

(i) We assume first that the indicatrix of Dupin at $x\in\partial K$ is an ellipsoid. In fact, by a change of the coordinate system, we may also assume that $x=0$ and $N_K(0)=-e_n$. Let $\zeta\in\mathbb{R}^n$ be the origin in the previous coordinate system. Let $y_{M,\delta}:=\partial(\mathrm{M}_\delta(K,\phi))\cap[0,\zeta]$. Notice that $\|y_{M,\delta}\|=\|x-x_{M,\delta}\|$ and that $y_{M,\delta}\to 0$ as $\delta\searrow 0$. Thus

$$\lim_{\delta\searrow 0}\langle x,\,N_K(x)\rangle\frac{\|x-x_{M,\delta}\|}{\|x\|}=\lim_{\delta\searrow 0}\langle\zeta,\,e_n\rangle\frac{\|y_{M,\delta}\|}{\|\zeta\|}=\lim_{\delta\searrow 0}\langle y_{M,\,\delta},\,e_n\rangle. \tag{3.5}$$

There exists a volume preserving positive definite linear transform T such that $N_{TK}(0)=-e_n$ and such that the indicatrix of Dupin at 0 becomes a Euclidean ball with radius $\sqrt{\rho}$ (see, e.g., equation (5) in [53]). Moreover, ρ satisfies

$$\kappa_K(0)=\frac{1}{\rho^{n-1}}.$$

Let H^+ be the half space such that

$$\delta=\int_{K\cap H^+}\phi(y)\,\mathrm{d}y\qquad\text{and}\qquad y_{M,\,\delta}=\frac{\int_{K\cap H^+}y\phi(y)\,\mathrm{d}y}{\delta}.$$

As T is volume preserving, $\int_{TK\cap TH^+}\phi\big(T^{-1}y\big)\,\mathrm{d}y=\delta$, and thus

$$\begin{aligned}Ty_{M,\,\delta}\;&=\;\int_{K\cap H^+}Ty\phi(y)\,\mathrm{d}y/\delta=\int_{TK\cap TH^+}y\phi\big(T^{-1}y\big)\,\mathrm{d}y/\delta\\ &\in\;\partial\mathrm{M}_\delta\big(TK,\phi\circ T^{-1}\big).\end{aligned}$$

As a consequence we have

$$[0,\,T\zeta]\cap\partial\mathrm{M}_\delta\big(TK,\phi\circ T^{-1}\big)=Ty_{M,\,\delta},$$

$$\phi\big(T^{-1}0\big)=\phi(0),$$

and

$$\langle Ty_{M,\delta},\,e_n\rangle=\langle y_{M,\delta},\,Te_n\rangle=\langle y_{M,\delta},\,e_n\rangle.$$

Hence we have reduced the problem to the case when the indicatrix of Dupin at $0\in\partial K$ is a Euclidean sphere with radius $\sqrt{\rho}$ and $\kappa_K(0)=\frac{1}{\rho^{n-1}}$.
Moreover, ∂K can be approximated in 0 by a Euclidean ball $B_2^n(\rho e_n,\rho)$ of radius ρ and center ρe_n in the following sense (see, e.g., [54, Proof of Lemma 23]):
Let $\varepsilon>0$ be given. Let $B_2^n((1-\varepsilon)\rho e_n,(1-\varepsilon)\rho)$ be the Euclidean ball centered at $(1-\varepsilon)\rho e_n$ whose radius is $(1-\varepsilon)\rho$. Similarly, let $B_2^n((1+\varepsilon)\rho e_n,(1+\varepsilon)\rho)$ be the Euclidean ball centered

at $(1+\varepsilon)\rho$ with radius $(1+\varepsilon)\rho$. Then,

$$0 \in \partial[B_2^n(\rho e_n, \rho)], \quad 0 \in \partial[B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho)], \\ 0 \in \partial[B_2^n((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho)],$$

and

$$N_{B_2^n(\rho e_n, \rho)} = N_{B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho)} = N_{B_2^n((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho)} = -e_n$$

and (see, e.g., [54, Proof of Lemma 23]) there exists Δ_ε^0 such that for $0 < t < \Delta_\varepsilon^0$, the half-space $H_t^+ = \{y : \langle y, e_n\rangle \le t\}$ determined by the hyperplane orthogonal to e_n through the point te_n is such that

$$H_t^+ \cap B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho) \subseteq H_t^+ \cap K \\ \subseteq H_t^+ \cap B_2^n((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho). \tag{3.6}$$

By continuity of ϕ there exists $s > 0$ such that for all $y \in \mathrm{int}(B_2^n(0, s))$,

$$(1-\varepsilon)\phi(0) \le \phi(y) \le (1+\varepsilon)\phi(0). \tag{3.7}$$

We will apply Lemma 3.5 with $t = \Delta_\varepsilon^0$ simultaneously to K, $B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho)$ and $B_2^n((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho)$ with weights ϕ, $(1-\varepsilon)\phi(0)$, and $(1+\varepsilon)\phi(0)$ respectively.
Let $H_{\Delta_\varepsilon}^+ = \{y : \langle y, e_n\rangle \le \Delta_\varepsilon\}$. We choose $\Delta_\varepsilon \le \Delta_\varepsilon^0$ so small that

$$H_{\Delta_\varepsilon}^+ \cap B_2^n((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho) \subseteq B_2^n(0, \min\{s, r\}),$$

where r is given by Lemma 3.5. We denote

$$d_{M,\delta}^- = \mathrm{dist}(y_{M,\delta}, B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho)^c)$$

and

$$d_{M,\delta}^+ = \mathrm{dist}(y_{M,\delta}, B_2^n((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho)^c).$$

Boundedness of ϕ on $B_2^n(0, s)$ and (3.6) imply that for $\delta \ge 0$,

$$\mathrm{M}_\delta\big(B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho) \cap H_{\Delta_\varepsilon}^+, (1-\varepsilon)\phi(0)\big) \subseteq \mathrm{M}_\delta\big(K \cap H_{\Delta_\varepsilon}^+, \phi\big) \\ \subseteq \mathrm{M}_\delta\big(B_2^n((1+\varepsilon)\rho e_n, (1+\varepsilon)\rho) \cap H_{\Delta_\varepsilon}^+, (1+\varepsilon)\phi(0)\big).$$

By Lemma 3.5 and the choice of Δ_ε we have

$$\mathrm{M}_\delta(B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho), (1-\varepsilon)\phi(0)) \cap H_{\Delta_\varepsilon}^+ \subseteq \mathrm{M}_\delta(K, \phi) \cap H_{\Delta_\varepsilon}^+ \\ \subseteq \mathrm{M}_\delta(B_2^n((1+\varepsilon)\rho e_n, (1-\varepsilon)\rho), (1+\varepsilon)\phi(0)) \cap H_{\Delta_\varepsilon}^+.$$

Choose δ so small that $y_{M,\delta} \in H_{\Delta_\varepsilon}^+$. Then

$$y_{M,\delta} \notin \mathrm{int}\,(\mathrm{M}_\delta(B_2^n((1-\varepsilon)\rho e_n, (1-\varepsilon)\rho), (1-\varepsilon)\phi(0)))$$

and

$$y_{M,\delta} \in \mathrm{int}\,(\mathrm{M}_\delta(B_2^n((1-\varepsilon)\rho e_n, (1+\varepsilon)\rho), (1+\varepsilon)\phi(0))).$$

Thus, we conclude that

$$d^-_{M,\,\delta} \le \Delta((1-\varepsilon)\rho,\, (1-\varepsilon)\delta\phi(0)) \;\text{ and }\; d^+_{M,\,\delta} \ge \Delta((1+\varepsilon)\rho,\, (1+\varepsilon)\delta\phi(0)),$$

where $\Delta((1+\varepsilon)\rho,\, (1+\varepsilon)\delta\phi(0))$ and $\Delta((1-\varepsilon)\rho,\, (1-\varepsilon)\delta\phi(0))$ are the differences of the radii of $(1+\varepsilon)\rho B_2^n$ and $\mathrm{M}_\delta(\rho B_2^n, (1+\varepsilon)\phi(0))$, and of $(1-\varepsilon)\rho B_2^n$ and $\mathrm{M}_\delta(\rho B_2^n, (1-\varepsilon)\phi(0))$, respectively. Applying Lemma 3.4(ii) with $z = y_{M,\,\delta}$ and Proposition 3.1 for sufficiently small δ, yields

$$(1-\varepsilon)^{\frac{n+1}{n-1}+\frac{2}{n+1}} \le \frac{\langle y_{M,\,\delta},\, e_n\rangle}{c_n \delta^{\frac{2}{n+1}} \rho^{-\frac{n-1}{n+1}} \phi(0)^{\frac{2}{n+1}}} \le (1+\varepsilon)^{\frac{n+1}{n-1}+\frac{2}{n+1}}.$$

Since $\varepsilon > 0$ can be chosen arbitrary, we obtain, also using (3.5),

$$\lim_{\delta\to 0} \phi(x)^{\frac{2}{n+1}} \;\langle x,\, N_K(x)\rangle \frac{\|x - x_{M,\delta}\|}{\|x\|\delta^{\frac{2}{n+1}}} = c_n\; \rho(x)^{-\frac{n-1}{n+1}} = c_n\; \kappa_K(x)^{\frac{1}{n+1}}.$$

(ii) Now we assume that x is such that the indicatrix of Dupin at x is an elliptic cylinder. We will show that then

$$\lim_{\delta\to 0}\langle x,\, N_K(x)\rangle \frac{\|x - x_{M,\delta}\|}{\|x\|\delta^{\frac{2}{n+1}}} = 0.$$

We only need to show that $\lim_{\delta\to 0}\langle x,\, N_K(x)\rangle \frac{\|x - x_{M,\delta}\|}{\|x\|\delta^{\frac{2}{n+1}}} \le 0$.

We may assume that the first k axes of the elliptic cylinder have infinite lengths and the others not. Then, as above (see, e.g., [54, Proof of Lemma 23]) for all $\varepsilon > 0$ there is an approximating ellipsoid $\mathcal{E}$ and Δ_ε such that the hyperplane $H(N_K(x), x - \Delta_\varepsilon)N_K(x))$ orthogonal to $N_K(x)$ through the point $x - \Delta_\varepsilon N_K(x)$ is such that

$$H^+(N_K(x), x - \Delta_\varepsilon)N_K(x)) \;\cap\; \mathcal{E} \subseteq H^+(N_K(x), x - \Delta_\varepsilon)N_K(x)) \;\cap\; K$$

and such that the lengths of the k first principal axes of $\mathcal{E}$ are larger than $\frac{1}{\varepsilon}$. As noted above, there is a support hyperplane H_δ to $F_\delta(K,\phi)$ such that $x_{F,\delta} \in H_\delta$ and such that $\delta = \int_{K\cap H_\delta^+} \phi(y)dy$ [59]. Then

$$\delta \ge \min_{y\in K} \phi(y)|K \cap H_\delta^+| \ge \min_{y\in K} \phi(y)|\mathcal{E} \cap H_\delta^+|.$$

As above, we may assume that the approximating ellipsoid $\mathcal{E}$ is a Euclidean ball with radius $\rho = \rho(x)$ where $\rho \ge \frac{1}{\varepsilon}$. Then

$$\begin{aligned}
\langle x,\, N_K(x)\rangle \frac{\|x - x_{M,\delta}\|}{\|x\|\delta^{\frac{2}{n+1}}} \quad &\le \quad \langle x,\, N_K(x)\rangle \frac{\|x - x_{F,\delta}\|}{\|x\|\delta^{\frac{2}{n+1}}} \\
&\le \quad \frac{\langle \frac{x}{\|x\|}, N_K(x)\rangle \;\|x - x_{F,\delta}\|}{(\min_{y\in K}\phi(y))^{\frac{2}{n+1}} \left(|B_2^n(x - \rho N_K(x), \rho) \cap H_\delta^+|\right)^{\frac{2}{n+1}}} \\
&\le \quad \frac{\rho^{-\frac{n-1}{n+1}}}{c_n(\min_{y\in K}\phi(y))^{\frac{2}{n+1}}}.
\end{aligned}$$

The last inequality can be shown using similar methods as in the case (i). Or, one notices that we are precisely in the situation of Lemmas 7 and 10 of [52] where exactly this estimate is proved. As ρ is arbitrarily small, the proof is completed.

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