A novel equivalence relation in relativity

Armin Nikkhah Shirazi*
Department of Physics, University of Michigan, 450 Church Street, Ann Arbor MI 48109
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This paper proves a proposition which says that existence in a spacetime which meets four commonly satisfied principles is an equivalence relation by absolute dimensionality, and briefly discusses three of its implications: (1) the equivalence relation opens the serious possibility for things to exist in a physical sense without existing in spacetime, (2) it establishes a novel classical geometric correspondence between proper time intervals and Euclidean distances in a spacetime one dimension higher, and (3) it may serve as a theoretical tool for checking the internal consistency of higher-dimensional models of reality.

INTRODUCTION

This paper presents a novel equivalence relation which connects the so far merely philosophical concept of existence directly to physics, specifically relativity. What will here be called the *ontic equivalence relation* says that existence in a spacetime is an equivalence relation by the number of length dimensions characterizing an object. It will be proved for Minkowski spacetime, but the principles from which it is proved are essentially satisfied by any reasonable spacetime. It has definite implications for fundamental physics, of which three will be discussed here: first, it opens in a very general manner the serious possibility for things to exist in a physical sense without existing in spacetime; second, it establishes a classical correspondence between proper time intervals and Euclidean distances in a one-dimension higher spacetime; third, it may be used as a theoretical tool to check the internal consistency of models of reality with extra dimensions.

The organization is as follows: the next section presents a reinterpretation of special relativistic length contraction and time dilation which is meant to draw attention to four obvious yet so far unappreciated spacetime principles, discussed in the following section. The ontic equivalence relation is then proved from these four principles, and the final section briefly discusses the three implications.

LORENTZ TRANSFORMATIONS REINTERPRETED

When a body is Lorentz contracted, it can also intuitively be interpreted to take on a weaker three-dimensional character because while in that case both its volume and its surface area decrease, volume decreases faster. When the focus is on this implication of length contraction, the term *dimensional abatement* will be used. To make this mathematically precise, we define the *relative dimensionality* $dim_{rel}(a/b)$ between compact objects a and b, which in three spatial dimensions is given by

$$dim_{rel}(a/b) = \frac{\frac{\int dV_a}{\int dA_a}}{\frac{\int dV_b}{\int dA_b}} \tag{1}$$

Where a is the contracted comparison object, $\int dV_a$ its volume, $\int dA_a$ its surface area, b is the uncontracted reference object, $\int dV_b$ its volume, and $\int dA_b$ its surface area. This equation defines a dimensionless quantity in the interval $[0, 1]$ which can be interpreted as a quantitative measure of the weakness of the three-dimensional character (or, in terms of its inverse, the strength of the two-dimensional character) of a relative to b. If $0 < dim_{rel}(a/b) < 1$, then a has *absolute dimensionality* 3 but has a weaker three-dimensional character than b. We will refer to this as *dimensional diminution.* If $dim_{rel}(a/b) = 0$ then a has absolute dimensionality 2, and we will refer to this as *dimensional reduction.* Dimensional abatement can then be considered an umbrella term which refers either to dimensional diminution or dimensional reduction.

To prove that Lorentz contraction implies dimensional abatement for any arbitrary three-dimensional compact shape, consider an arbitrarily shaped three-dimensional body to be made up of infinitesimal cubical volume elements. Then, it is trivial to show that when the object is Lorentz contracted, each of its volume elements becomes dimensionally abated. Since this is true of every volume element, it must be true of the object itself.

Let us now consider time dilation. The key idea behind reinterpreting special relativistic time dilation is to reframe time as the duration of existence of something in spacetime between two spacetime events. Proper time is reinterpreted as the duration of existence in spacetime between two spacetime events of an observed object, and coordinate time is reinterpreted as the duration of existence in spacetime between the same two spacetime events of the observer (or a class of objects at rest with respect to the observer). When time dilation is interpreted this way, we will refer to it as *ontochronic abatement.*

The reinterpretation of time in terms of duration of existence opens the possibility to connect existence to physics. To do it formally, we postulate the following criterion for existence in spacetime: *A physical object exists in Minkowski spacetime if and only if it is characterized by a timelike spacetime interval.*

As a plausibility argument, one can argue that it is natural to express existence in spacetime in terms of a finite duration of existence. Also, this criterion connects existence to Lorentz invariance. However, notice that Lorentz invariance is merely necessary, not sufficient, as objects characterized by null intervals fail to satisfy the existence criterion. The case of null intervals will be addressed in the discussion of the implications of the ontic equivalence relation.

Given the existence criterion, we can then define a function $\exists_S : \mathfrak{O} \to \{0,1\}$ which maps the collection of all physical objects in the domain of physics $\mathfrak{O}$ to the numbers 1 and 0, depending on whether an object satisfies the existence criterion or not. If it does, its *spacetime ontic value* is 1, and it is 0 otherwise. The collection of objects which satisfy the criterion is then the set of spacetime objects $S \subset \mathfrak{O}$.

A quantitative measure of relative duration of existence in spacetime is defined by taking the ratio of the proper time interval between two spacetime intervals of the observed object, τ_a to the proper time interval of a reference object τ_b between the same two spacetime events:

$$ont_{\mathrm{rel}}(a/b) = \frac{\int d\tau_a}{\int d\tau_b} \tag{2}$$

We will call this *relative ontochronicity.* When the reference object is the observer, $\tau_b = t$, the coordinate time. As with relative dimensionality, this ratio is a dimensionless number in the interval $[0,1]$. It can be interpreted as a factor which indicates how much the duration of existence of an observed object in spacetime between two spacetime events is decreased relative to that of the observer. Notice that $ont_{\mathrm{rel}}(a/b)$ is similar to, but distinct from, the inverse Lorentz factor $\gamma^{-1} = \frac{d\tau}{dt}$. If $0 < ont_{\mathrm{rel}}(a/b) < 1$, then this means that an object satisfies the existence criterion but its duration of existence in spacetime between the two spacetime events has diminished relative to that of the observer, and so this will be called *ontochronic diminution.* If $ont_{\mathrm{rel}}(a/b) = 0$, then this implies that the object fails the existence criterion, so this will be called *ontic reduction.* Ontochronic abatement can then be considered an umbrella term which encompasses ontochronic diminution and ontic reduction.

The proof that special relativistic time dilation implies ontochronic abatement follows trivially from re-interpreting the proper time of an object as its observed duration of existence in spacetime, and coordinate time as the duration of existence in spacetime of the observer, between two given spacetime events.

FOUR SPACETIME PRINCIPLES

The reinterpretation of length contraction and time dilation, along with the conceptual structure constructed to articulate it, draws attention to four spacetime principles for which we were unable to find previous discussion in the physics literature [1]. They consist of two invariance and two symmetry principles, and the absence of their explicit discussion in the literature likely reflects the fact that under the standard interpretation of the Lorentz transformations, their physical significance is too obscure:

- **Invariance of Absolute Dimensionality:** *The absolute dimensionality of any compact body is invariant under spacetime coordinate transformations.*
- **Homodimensionality of Space:** *The dimensionality of every (maximally dimensional) space-like hypersurface of Minkowski spacetime is everywhere the same.*
- **Invariance of Spacetime Ontic Value:** *The spacetime ontic value of any compact body is invariant under spacetime coordinate transformations.*
- **Homodimensionality of Time:** *The dimensionality of every timelike hypersurface of Minkowski spacetime is everywhere the same.*

The Lorentz transformations obey these principles. It is not possible to violate the invariance of absolute dimensionality or of spacetime ontic value by means of a Lorentz transformation, the former because length contraction can never lead to dimensional reduction for speeds $v < c$, while less than complete length contraction can never be attained for $v = c$, and the latter because proper time is Lorentz invariant.

The Lorentz transformations also obey the homodimensionality principles because while spacetime coordinates may change according to the transformations, the number of components and their decomposition into n spatial components and one time component does not change.

PROOF OF THE ONTIC EQUIVALENCE RELATION

It is useful to observe the following:

- **The two invariance principles together couple absolute dimensionality to spacetime ontic value.** This is analogous to how the Lorentz transformations couple length contraction to time dilation, or, when reinterpreted, dimensional to ontochronic diminution. The two invariance principles together extend this coupling to that between dimensional and ontic reduction.

- **The two homodimensionality principles together ensure that the coupling of absolute dimensionality to spacetime ontic value holds globally.** In the absence of the homodimensionality principles, the coupling of the two invariance principles may only hold locally because in regions in which the dimensionality of maximally dimensional spacelike or timelike hypersurfaces is different inside than outside of those regions, the absolute dimensionality could decouple from spacetime ontic value. The two homodimensionality principles require that Minkowski spacetime everywhere decomposes into $n+1$ spacetime dimensions and thereby ensure that the coupling of absolute dimensionality and ontic value holds everywhere.

For convenience, a spacetime in which both homodimensionality principles hold will be defined to be *isodimensional.* It is now very easy to prove the following proposition from the four principles: *Physical existence in Minkowski spacetime is an equivalence relation by absolute dimensionality.*
Proof: An equivalence relation is determined by the properties of reflexivity, symmetry and transitivity. Consider an n-dimensional compact object A subject to the above principles. By the the coupling of ontic value to absolute dimensionality, it must exist in an $n+1$-dimensional Minkowski spacetime region. By the isodimensionality of Minkowski spacetime, this region is, in fact, all of $n+1$-dimensional spacetime. In particular, A exists in the $n+1$-dimensional Minkowski spacetime in which it exists. This proves reflexivity. Now consider an m-dimensional compact object B. By the same argument as given for reflexivity, it must exist in an $m+1$-dimensional spacetime. Suppose A exists in the same spacetime as B. This requires that $n+1=m+1$, and, consequently, that $n=m$. But that means B has the same absolute dimensionality as A, and therefore exists in the same spacetime as A. This proves symmetry. Finally, consider an l-dimensional compact object C. By the same argument as given for reflexivity, it must exist in an $l+1$-dimensional spacetime. Now suppose that B exists in the same spacetime as C, and that A exists in the same spacetime as B. This requires $m+1=l+1$ and $n+1=m+1$, respectively, from which it follows that $n=m=l$, so A has the same absolute dimensionality as C and therefore exists in the same spacetime as C. This proves transitivity. ■

THREE IMPLICATIONS OF THE ONTIC EQUIVALENCE RELATION

The ontic equivalence relation induces a partition on the collection of all physical objects in the domain of physics into equivalence classes based on an object's absolute dimensionality, such that each equivalence class of n-dimensional classes can only exist in an $n+1$-dimensional Minkowski spacetime. This has a number of implications, but here we will focus on just three.

- **The ontic equivalence opens the serious possibility for things to exist in a physical sense without existing in spacetime.** As lightspeed objects are dimensionally reduced, they belong to a different *ontic equivalence class* than spacetime objects. This is a consequence of the more general implication of the ontic equivalence relation that in any given $n+1$-dimensional spacetime, the possibility is open for things to exist without existing in that spacetime, namely when their absolute dimensionality is other than n. On the other hand, it is probably fair to say that under the current scientific paradigm existence in a physical sense is equated with existence in spacetime (setting speculative frameworks aside).
  The mathematics of special relativity gives hints that equating physical existence with existence in spacetime is incorrect: the fact that the duration of existence in spacetime of a lightspeed object is always zero between any two spacetime events, and the fact that it is impossible to obtain a four-volume, i.e. a spacetime region, by integrating a three-volume in the lightlike direction already suggest that lightspeed objects do not exist in spacetime. Indeed, if we already accepted this, then likely our first argument to justify it would be that no spacetime observer can transform to a lightspeed rest frame! But standard special relativity cannot tell us directly that lightspeed objects do not exist in spacetime because it does not contain any concept of existence. Embedding it via the existence criterion makes this explicit and gives meaning to features of special relativity which could previously only be regarded as meaningless curiosities. Sometimes, paradigm changes happen precisely when seemingly meaningless curiosities in a model of reality are recognized to carry novel significance, and arguably, the birth of special relativity itself was due to such a development [2].

- **The ontic equivalence relation establishes a novel classical geometric correspondence.** Because of dimensional reduction, lightspeed objects must by the ontic equivalence relation exist in a $2+1$ dimensional spacetime. The mathematics of special relativity gives hints of this: As an object approaches the speed of light, its direction of motion and its time direction are both observed to approach the lightlike direction. In the limit of c, they both become lightlike [3]. But that means that in a lightspeed frame, our spacetime is a vector space with a linearly dependent set of vectors, which in turn implies that in a lightspeed frame, $3+1$ spacetime has too many dimensions. Also, special relativity is clear that in such a frame there are only two independent spacelike directions.

The existence of lightspeed objects in $2+1$-dimensional spacetime can be interpreted in terms of a classical and a quantum picture.

The classical picture is that as a lightspeed object traverses a null geodesic in spacetime, it defines a rest frame on a null-plane, so that what is in $2+1$ dimensions the passage of proper time corresponds in $3+1$ dimensions to motion in space. Since the rest frame in a null-plane has to be represented in $3+1$ dimensions necessarily in an atemporal manner (due to the connection between existence in a spacetime and timelike proper time), it suggests a novel interpretation of the fact that the stabilizer subgroup of the Poincaré group ('Wigner's little group') for massless objects is isomorphic to $E(2)$, the group of isometries in the Euclidean plane [4].

As the correspondence involves analogous quantities associated with spaces of different dimensionality, we will for clarity use left subscripts to indicate the total number of dimensions of the space with which the quantity is associated. Let ${}_3d : \mathbb{E}^3 \rightarrow [0,\infty)$ be the distance function for three-dimensional Euclidean space, and ${}_3\tau* : \mathbb{M}^3 \rightarrow [0,\infty)$ be the timelike plus zero (indicated by adding *) interval function [5] for three-dimensional Minkowski spacetime, then

$$\mathfrak{C} : {}_3\tau* \rightarrow {}_3d \tag{3}$$

will be called the *classical* $\tau * -d$ *duality.* It reflects the fact that, unlike certain other relations such as set membership, existence in a spacetime is not inherited by a one-dimension higher embedding spacetime, an immediate consequence of the ontic equivalence relation.

In [6], it was implicitly shown that classical electrodynamics obeys the classical $\tau * -d$ duality, as the magnetic force field of an infinite line current can be reinterpreted as the line integral of a two-dimensional Coulomb force field such that a spatial distance covered by the line integral corresponds to the worldline of a Coulomb source in a two-dimensional leaf of a foliation of space normal to the direction of the current (i.e. really a worldline in $2+1$ dimensions). This reconceptualization permits the recognition of geometric relationships between Maxwell's equations not evident in the standard formulation.

Obviously, ${}_3\tau*$ is distinct from ${}_4\tau*$, the timelike plus zero interval characterizing objects in $3+1$ spacetime. It would be interesting to investigate whether positing that the classical $\tau * -d$ duality also holds for ${}_4\tau*$, at least at galactic or cosmological scales, suggests novel theoretical approaches to elucidating the nature of dark matter and/or dark energy [7].

The quantum picture effectively dissociates the 2+1 spacetime from definite trajectories in $3+1$ spacetime. In order to make this dissociation explicit, but also in order to promote 2+1 dimensional spacetime with a Lorentzian signature to an equal footing to our own repository of existence, we propose to give it its own name and call it *areatime.* An investigation fleshing out the connection between quantum theory and the concept of areatime is in preparation but will be considered outside the scope of this article.

- **The ontic equivalence relation may be used as a tool for identifying hidden inconsistencies in spacetime models with extra dimensions.** Although the ontic equivalence relation was derived for Minkowski spacetime, it is easy to see that it holds much more generally because the four spacetime principles from which it was derived are essential to all candidate models of spacetime one would take seriously. For instance, the very definition of a manifold M equipped with a metric g essentially[8] presupposes isodimensionality. Similarly, we would likely consider a violation of the invariance principles to be unphysical. The generality of the principles therefore allows the ontic equivalence relation to be used as a novel consistency check on models of reality which posit extra dimensions, even if the spacetimes involved are not Minkowski.

  Any model in which both the four spacetime principles hold and m-dimensional objects are supposed to exist in $n+1$-dimensional spacetime, where $m \neq n$, is a candidate for being inconsistent. For instance, models obeying the four principles which assume that fields, especially fields of particles with a timelike proper time, can propagate into additional dimensions are likely inconsistent. However, if in such models our spacetime is merely effectively four-dimensional but actually has the same dimensionality as the embedding space (the so-called "thick brane solutions"[9]), then the inconsistency may be avoided. Similarly, if the objects in a string theory model, which requires $9+1$ dimensions [10], are considered only effectively three-dimensional but are actually nine-dimensional, then this inconsistency may also be averted.

  Again, whether there actually is an inconsistency or not depends on the details of each model, but the ontic equivalence relation gives a new theoretical tool for researchers who investigate them.

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* armin@umich.edu

[1] Invariance of dimension is a theorem in topology, however. See L.E.J. Brouwer, Math. Ann. **70**, 161 (1911).

[2] A. Einstein, Ann. Phys., **17**, 891 (1905) famously opens with the observation in classical electrodynamics of "asymmetries that do not attach to the phenomena" i.e. a recognition that the fact that observers in relative motions observe different fields for the same phenomena may not be just a meaningless curiosity of Maxwell's theory. See also A. Einstein and L. Infeld *The evolution of physics* (Simon and Schuster, New York, 1954), p. 179.

[3] N.M.J. Woodhouse *Special Relativity* (Springer-Verlag, London, 2003), p.67.

[4] E.P. Wigner, Ann. Math. **40**, 149 (1939).

[5] Zero is included in the co-domain in order to account for the zero interval, i.e. the interval of the additive identity vector, not to be confused with null intervals.

[6] A. Nikkhah Shirazi, (unpublished), http://hdl.handle.net/2027.42/147435.

[7] J.E. Horvath, Cosm. Hist., **5**, 287 (2009).

[8] 'Essentially' because the homodimensionality principles may not hold within singular regions. However, this problem can be easily addressed for manifolds in which the four principles hold everywhere except at a countable number of point-like singularities by relaxing the constraint of isodimensionality to *isodimensionality almost everywhere* to exclude the applicability of the equivalence relation to that zero-measure collection of points.

[9] V. Dzhunushaliev, V. Folomeev, M. Minamitsuji, Rep. Prog. Phys.**73**:066901 (2010).

[10] J. Polchinksi, *String theory* (Cambridge University Press, Cambridge, 1998), Vol.1, p. 6.