ON THE DIMENSION OF A GRAPH

PAUL ERDÖS, FRANK HARARY *and* WILLIAM T. TUTTE

Our purpose in this note is to present a natural geometrical definition of the dimension of a graph and to explore some of its ramifications. In §1 we determine the dimension of some special graphs. We observe in §2 that several results in the literature are unified by the concept of the dimension of a graph, and state some related unsolved problems.

We define the *dimension* of a graph G, denoted dim G, as the minimum number n such that G can be embedded into Euclidean n-space E_n with every edge of G having length 1. The vertices of G are mapped onto distinct points of E_n, but there is no restriction on the crossing of edges.

1. *Some graphs and their dimensions.* Let K_n be the *complete graph* with n vertices in which every pair of vertices are adjacent (joined by an edge). The triangle K_3 and the tetrahedron K_4 are shown in Figure 1.

Fig. 1.

The dimension of K_3 is 2 since it may be drawn as a unit equilateral triangle. But clearly, $\dim K_4 = 3$ and in general $\dim K_n = n-1$.

By $K_n - x$ we mean the graph obtained from the complete graph K_n by deleting any one edge, x. For example $K_3 - x$ and $K_4 - x$ are shown in Figure 2.

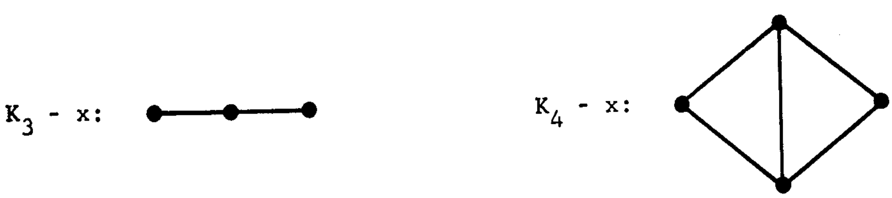

Fig. 2.

From this figure, we see at once that $\dim(K_3 - x) = 1$ and that $\dim(K_4 - x) = 2$ since it can be drawn as two equilateral triangles with the same base. By a similar construction it is easy to show that in general $\dim(K_n - x) = n-2$.

The *complete bicoloured graph* $K_{m,n}$ has m vertices of one colour, n of another colour, and two vertices are adjacent if and only if they have different colours. We shall see how to determine the dimension of $K_{m,n}$ for all positive integers m and n. In Figure 3 are shown three of these graphs, each of which we will see has a different dimension.

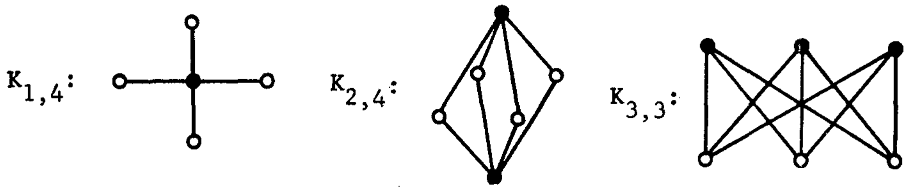

Fig. 3.

Which of the graphs $K_{m,n}$ have dimension 2? Since $K_{1,1}=K_2$, $\dim K_{1,1}=1$, and as shown in Figure 3, $\dim K_{1,4}=2$. Obviously, for every $n>1$, $\dim K_{1,n}=2$. There is also one other complete bicoloured graph with dimension 2, namely the rhombus $K_{2,2}$. Again from the figure, we see that $\dim K_{2,4}=3$ and in general that $\dim K_{2,n}=3$ when $n \geqslant 3$. Finally, it is easy to show that the dimension of every other graph $K_{m,n}$ not already mentioned in this paragraph is 4, including the famous 3 houses-3 utilities graph $K_{3,3}$. The proof is due to Lenz, as mentioned in a paper by Erdös [2], and proceeds as follows.

Let $\{u_i\}$ be the m vertices of the first colour and let $\{v_j\}$ be the n vertices of the second colour. We assign coordinates in E_4 to $u_i=(x_i, y_i, 0, 0)$ and $v_j=(0, 0, z_j, w_j)$ in such a way that $x_i^2+y_i^2=\frac{1}{2}$ and $z_j^2+w_j^2=\frac{1}{2}$. Then every distance $d(u_i, v_j)=1$, proving the assertion.

In the next two illustrations of the dimension of a graph we use the operations of the "join" and the "product" of two graphs G_1 and G_2. Let V_1 and V_2 be their respective vertex sets. The *join* G_1+G_2 of two disjoint graphs contains both of them and also has an edge joining each vertex of G_1 with each vertex of G_2. The *cartesian product* $G_1 \times G_2$ of G_1 and G_2 has $V_1 \times V_2$ as its set of vertices. Two vertices $u=(u_1, u_2)$ and $v=(v_1, v_2)$ are adjacent in $G_1 \times G_2$ if and only if $u_1=v_1$ and u_2v_2 is an edge of G_2 or $u_2=v_2$ and u_1v_1 is in G_1. Let P_n denote the polygon with n sides. By the *wheel* with n spokes is meant the graph P_n+K_1; see Figure 4,

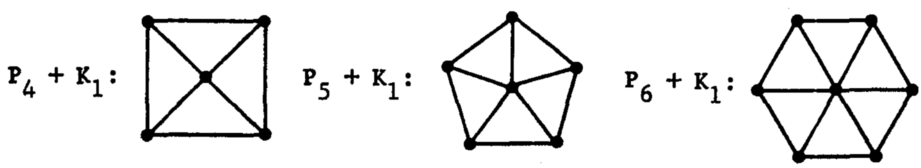

Fig. 4.

What is the dimension of a wheel? We already have one example since the smallest wheel $P_3+K_1=K_4$ has dimension 3. From Figure 4, we see that $\dim(P_4+K_1)=\dim(P_5+K_1)=3$ and that $\dim(P_6+K_1)=2$. By making expeditious use of the unit sphere, the reader can verify that for all $n > 6$, $\dim(P_n + K_1) = 3$. Thus we observe that the dimension of the n-spoked wheel is 3 except for "the odd number 6".

The *n-cube* Q_n is defined as the cartesian product of n copies of K_2; see Figure 5. Since $Q_1 = K_2$, $\dim Q_1 = 1$. Since $Q_2 = K_{2,2} = P_4$, $\dim Q_2 = 2$.

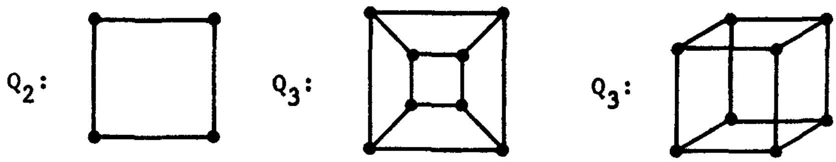

Fig. 5.

The 3-cube Q_3 is drawn twice in Figure 5. Its first appearance might suggest that its dimension is 3. But its second depiction (in which two pairs of edges intersect) shows that $\dim Q_3 = 2$. Similarly, for all $n > 1$, $\dim Q_n = 2$.

A modest generalization of this observation asserts that for any graph G, $\dim(G \times K_2)$ equals $\dim G$, if $\dim G \geqslant 2$, and equals $\dim G + 1$, if $\dim G = 0$ or 1.

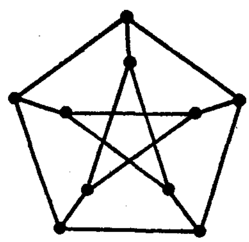

Fig. 6.

The well-known Petersen graph is shown in Figure 6. What is its dimension? It is easy to see (especially after seeing it) that the answer is 2; see Figure 7.

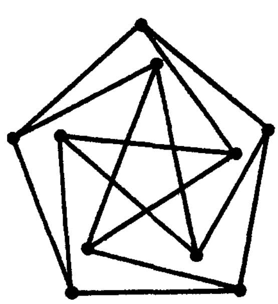

Fig. 7.

By the way, note that the dimension of any tree is at most 2. A *cactus* is a graph in which no edge is on more than one polygon. Since the definition of dim G allows edges to intersect, it is easily seen that the dimension of any cactus is at most 2.

In this section we have evaluated the dimension of a few special graphs. But for a given graph G, we know of no systematic method for determining the number dim G. Thus the calculation of the dimension of a given graph is at present in the nature of mathematical recreation.

2. *Some theorems on dimension.* In the theorems of this section we use the following concepts: the girth of a graph, the chromatic number of a graph, and the chromatic number of a Euclidean space. The *girth* of a graph G is the number of edges in its smallest polygon (if any). The *chromatic number* $\chi(G)$ of G is the least integer n such that the vertices of G can be coloured using n colours so that no two adjacent vertices have the same colour. The *chromatic number* $\chi(E_n)$ *of a Euclidean space* E_n is the smallest number of point sets into which E_n can be partitioned so that in no set does the distance 1 occur.

THEOREM 1. *For any graph* G, $\dim G \leqslant 2\chi(G)$.

The proof of this theorem is a simple generalization of the argument used in §1 to establish that $\dim K_{m,n} \leqslant 4$; see [**2**]. The next two theorems do not deal with the dimension of a graph, but will be used in later proofs.

THEOREM 2. (Erdös [**1**]). *There exists a graph with arbitrarily high girth and arbitrarily high chromatic number.*

THEOREM 3. (Erdös [**4**]). *If* G *is a graph with* n *vertices and girth greater than* $C \log n$, *for* C *sufficiently large, then* $\chi(G) \leqslant 3$.

COROLLARY. *Under the above hypothesis,* $\dim G \leqslant 6$.

It is possible that the above hypothesis implies $\dim G \leqslant 3$ or even $\dim G \leqslant 2$, but we could not decide this question.

THEOREM 4. (Erdös [**3**]). *Among all graphs with* n *vertices,* q *edges, and dimension* $2k$ *or* $2k+1$,

$$\lim_{n=\infty} \max \frac{q}{n^2} = \tfrac{1}{2}\left(1 - \frac{1}{k}\right)$$

The following question was posed by Erdös [**2**]: What is the maximum number of edges among all graphs of dimension d which have n vertices? The next theorem gives the answer for $d = 4$.

THEOREM 5. (Erdös, unpublished). *Among any* n *points of* E_4 *the distance* 1 *between pairs of points can occur at most* $n + [n^2/4]$ *times, and this number can be realized if* $n \equiv 0 \pmod 8$.

We now turn to some results concerning the chromatic number of a Euclidean space. The brothers Moser [**6**] called for a proof of the inequality $\chi(E_2) > 3$. Hadwiger [**5**] found the following inequalities.

THEOREM 6. $4 \leqslant \chi(E_2) \leqslant 7$.

COROLLARY. *If* $\dim G = 2$, *then* $\chi(G) \leqslant 7$.

Klee (unpublished) proved the next theorem.

THEOREM 7. *For every positive integer* n, $\chi(E_n)$ *is finite.*

This result has some consequences for the dimension of a graph, but they are not as sharp as Theorem 1.

COROLLARY 1. *If* $\dim G$ *is large, so is* $\chi(G)$.

COROLLARY 2. *There exist graphs with arbitrarily high dimension and girth.*

One might think that a graph of sufficiently high dimension must contain a complete subgraph K_n of specified order $n > 2$. That this is not necessarily so follows from the last corollary.

Unsolved problems.

I. Call a graph G *critical of dimension* n if $\dim G = n$ and for any proper subgraph H, $\dim H < n$. For example, K_{n+1} is critical of dimension n. Characterize the critical n-dimensional graphs, at least for $n = 3$ (this is trivial for $n = 2$).

II. Let G have n vertices and assume that every subgraph H with k vertices has dimension at most m. How large can $\dim G$ be? (For chromatic number instead of dimension, Erdös investigates this in [**4**].)

References

1. P. Erdös, " Graph theory and probability ", *Canad. J. Math.*, 11 (1959), 34–38.
2. P. Erdös, " On sets of distances of n points in Euclidean space ", *Publ. Math. Inst. Hung. Acad. Sci.*, 5 (1960), 165–169.
3. P. Erdös, " Some unsolved problems ", *Publ. Math. Inst. Hung. Acad. Sci.*, 6 (1961), 221–254, esp. p. 244.
4. P. Erdös, " On circuits and subgraphs of chromatic graphs ", *Mathematika*, 9 (1962), 170–175.
5. H. Hadwiger, " Ungelöste Probleme No. 40 ", *Elemente der Math.*, 16 (1961), 103–104.
6. L. Moser and W. Moser, " Solution to Problem 10 ", *Canad. Math. Bull.*, 4 (1961), 187–189.

Mathematical Institute, Budapest, Hungary.
University of Michigan, Ann Arbor, U.S.A.
University of Waterloo, Waterloo, Canada.

(*Received on the 7th of January,* 1965.)