MEASURE FUNCTION PROPERTIES OF THE ASYMMETRIC CAUCHY PROCESS

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§1. *Preliminaries.* A Cauchy process in d-dimensional Euclidean space, R^d, is a stochastic process, $X_t(\omega)$, with stationary independent increments and with a continuous transition density, $p(t, y-x)$ defined by

$$\int_{R^d} \exp\,[i(z, y-x)]\, p(t, y-x)\, dy = \exp\,[-t\psi(z)], \tag{1}$$

and

$$\psi(z) = |z| \int_{S^d} w(z, \theta)\, m(d\theta), \tag{2}$$

where m, the isotropic measure, is a probability measure on S^d, the unit sphere in R^d, such that when $d > 1$ the support of m is not contained in any $d-1$ dimensional subspace. In (2) w is given by

$$w(z, \theta) = |(\hat{z}, \theta)| + \frac{2i}{\pi}(\hat{z}, \theta) \log |(z, \theta)|,$$

where $\hat{z} = z/|z|$. It follows that for each $t > 0$ and y we have $p(t, y) > 0$ and that for each $t > 0$ $p(t, y)$ is a bounded and continuous function of y. $X_t(\omega)$ can be considered as being a standard Markov process (for a full description of the definition of such a process see Chapter 1 of **[1]**) and in particular we can assume that the sample functions of $X_t(\omega)$ are right continuous and have left limits. We can also assume that $X_t(\omega)$ enjoys the strong Markov property. We write P_x and E_x for probabilities and expectations conditional on $X_0(\omega) = x$, and we write P for P_0.

If $d = 1$ the unit sphere consists of just two points and $\psi(z)$ is then given by

$$\psi(z) = |z|[1 + ih\, \mathrm{sgn}\,(z) \log |z|], \tag{3}$$

where $h = 2\beta/\pi$, $\beta = p - q, q = 1 - p$ and p is the mass put at $\{+1\}$ by the isotropic measure. It turns out that the sample function properties of $X_t(\omega)$ are largely determined by whether, or not, $h = 0$, that is whether, or not, the distribution of $X_t(\omega) - X_0(\omega)$ is symmetric. For example, if $h = 0, X_t(\omega)$ is neighbourhood recurrent; whilst, if $h \neq 0$, $X_t(\omega)$ is transient (see **[5]**). In **[7]** S. Orey showed that if $h = 0$ then single points are polar for $X_t(\omega)$. We recall that this means that if

$$T_y(\omega) = \inf\,[t > 0 : X_t(\omega) = y]$$

then $P_x(T_y < \infty) = 0$ for all x. On the other hand he also showed that if $h = 2/\pi$ then $P_x(T_y < \infty) > 0$ for all pairs (x, y). More recently in **[8]** S. C. Port and C. Stone established the following surprising result.

Proposition 1. *Suppose $h \neq 0$. Then, for any y and all x, $P_x(T_y < \infty) > 0$ and $P_x(T_x = 0) = 1$, that is x is regular for $\{x\}$.*

Let $R(\omega)$, $G(\omega)$ and $Z^x(\omega)$ be defined by $R(\omega) = [x : x = X_t(\omega), 0 \leqslant t \leqslant 1]$,

$$G(\omega) = [(t, X_t(\omega)) : 0 \leqslant t \leqslant 1], \quad Z^x(\omega) = [t : x = X_t(\omega) : 0 \leqslant t \leqslant 1],$$

and
$$Z(\omega) = Z^0(\omega).$$

Then $R(\omega)$, $G(\omega)$ and $Z(\omega)$ are called the range, graph and zero set of $X_t(\omega)$.

In **[10]** S. J. Taylor showed that, for the symmetric process, the exact measure function of the range, by which is meant a measure function ϕ such that $\phi - mR(\omega)$ is finite and positive with probability one, is $\phi(h) = h \log (1/h) \log \log \log (1/h)$. In particular this implies that the range of the symmetric Cauchy process has zero Lebesgue measure with probability one. In **[9]** the authors showed that the exact measure function for the graph is $\phi(h) = h$. By Orey's result the zero set for the symmetric process consists of at most one point so there is no problem.

Our object in this paper is to obtain Hausdorff measure results for the range, graph and zero set in the case where $X_t(\omega)$ is asymmetric. The problem of obtaining the exact measure functions of these sets is made difficult by the fact that we do not have sufficiently accurate estimates for the distributions we need, nor can we fall back on the scaling property which frequently simplifies matters. We thus content ourselves with establishing the three theorems below.

Henceforth we let $X_t(\omega)$ be an asymmetric Cauchy process in the line. We suppose that $h > 0$ (for otherwise we could just consider the process $-X_t(\omega)$), and we let Λ denote Lebesgue measure.

THEOREM 1. *For any $x \in R$ we have*

$$P_x\{\Lambda[R(\omega)] > 0\} = 1.$$

Since $R(\omega) \subset R$, this implies that $\phi(h) = h$ is the exact measure function for the range of $X_t(\omega)$.

The result by Port and Stone ensures that the zero set for the asymmetric process is non-empty; however, it is very small. In fact a result by R. M. Blumenthal and R. K. Getoor (p. 64 of **[4]**) ensures that the zero set has zero Hausdorff dimension. We obtain the following result which is a kind of " logarithmic dimension " result for the zero set.

THEOREM 2. *Let $\phi(h) = (\log (1/h))^{-\alpha}$. Then*

(i) *if* $0 < \alpha < 1$ $\quad \phi - mZ(\omega) = \infty$, *P almost surely*;

(ii) *if* $\alpha = 1$ $\quad \phi - mZ(\omega) < \infty$, *P almost surely*.

THEOREM 3. *Let $\psi(h) = h(\log (1/h))^{-\alpha}$. Then*

(i) *if* $0 < \alpha < 1$ $\quad \psi - mG(\omega) = \infty$, *P almost surely*;

(ii) *if* $\alpha = 1$ $\quad \psi - mG(\omega) < \infty$, *P almost surely*.

In §2 we prove Theorem 1 whilst in §3 we reduce the truth of Theorems 2 and 3 to the truth of two simpler propositions, which we prove in §4 and §5 respectively.

§2. The idea of the proof is very simple but we must take care with the measurability difficulties. We first establish some lemmas.

LEMMA 1.1. *$P_x(T_y < 1) > 0$ for all pairs (x, y).*

Proof. Let $g(x) = \int_0^\infty p(t, x)\,dt$, then (see [8]) there is a positive constant c such that

$$P_x(T_y < \infty) = cg(y - x).$$

Now

$$P_x(T_y < 1) \geqslant P_x(T_y < \infty) - P_x(X_t = y \text{ for some } t \geqslant 1)$$

$$= c\int_0^\infty p(t, y - x)\,dt - c\int_R dz p(1, z - x)\int_0^\infty p(s, y - z)\,ds,$$

by the Markov property,

$$= c\int_0^1 p(t, y - x)\,dt > 0,$$

and the lemma is proved.

LEMMA 1.2. *Suppose that ϕ is a measure function. Then*

(i) $P_x(\phi - mR(\omega) > 0) > 0$ *implies* $P_x(\phi - mR(\omega) > 0) = 1$

and $P_x(\phi - mR(\omega) < \infty) > 0$ *implies* $P_x(\phi - mR(\omega) < \infty) = 1$;

(ii) $P_x(\phi - mG(\omega) > 0) > 0$ *implies* $P_x(\phi - mG(\omega) > 0) = 1$

and $P_x(\phi - mG(\omega) < \infty) > 0$ *implies* $P_x(\phi - mG(\omega) < \infty) = 1$.

Proof. The first implication is proved in [2; Theorem 9.1] for the case where $\phi(h) = h^\alpha$. The proof does not depend on the particular choice of ϕ and the argument involved also serves to prove the rest of the lemma.

LEMMA 1.3. *Let $\mathscr{B}$ be the class of Borel subsets of R and let $I(y, \omega)$ be the indicator function of $\overline{R(\omega)}$. Then $I(y, \omega)$ is measurable with respect to the σ-field $\mathscr{B} \times \mathscr{F}$ (see* [1] *for the definition of $\mathscr{F}$).*

Proof. It is sufficient to show that $I^{-1}(0) \in \mathscr{B} \times \mathscr{F}$. Now

$$I^{-1}(0) = [(y, \omega) : y \notin \overline{R(\omega)}].$$

Let B_n be an enumeration of those open intervals with rational endpoints, then

$$I^{-1}(0) = \bigcup_{n=1}^{\infty} B_n \times [\omega : \overline{R(\omega)} \subset B_n^c],$$

so that it is sufficient to show that

$$[\omega : \overline{R(\omega)} \subset B_n^c] \in \mathscr{F}.$$

Since $X_t(\omega)$ is right continuous with left limits we have

$$\overline{R(\omega)} = [x : x = X_{t-}(\omega) \text{ or } X_t(\omega) \quad 0 \leqslant t \leqslant 1],$$

and hence

$$[\omega : \overline{R(\omega)} \subset B_n^c] = \bigcap_{\substack{0 \leqslant r \leqslant 1 \\ r \text{ rational}}} [\omega : X_r(\omega) \in B_n^c].$$

Each set in the intersection on the right is in $\mathscr{F}$ and so the lemma is proved.

Proof of Theorem 1. Since $\overline{R(\omega)}\setminus R(\omega)$ is at most countable we have

$$\Lambda[\overline{R(\omega)}] = \Lambda[\overline{R(\omega)}] = \int_R I(y, \omega)\, dy,$$

and

$$E_x \Lambda[R(\omega)] = \int P_x(d\omega) \int_R I(y, \omega)\, dy.$$

Fubini's Theorem and Lemma 1.3 show that this is

$$\int_R dy \int I(y, \omega)\, P_x(d\omega).$$

Now

$$\begin{aligned}\int I(y, \omega)\, P_x(d\omega) &= P_x[\overline{R(\omega)} \cap \{y\} \neq \phi] \\ &\geqslant P_x[X_t(\omega) = y \text{ for some } t, \quad 0 \leqslant t \leqslant 1] \\ &> 0,\end{aligned}$$

by Lemma 1.1. Hence $E_x \Lambda\ [R(\omega)]$ is positive and $P_x\{\Lambda[R(\omega)] > 0\} > 0$. Theorem 1 now follows from Lemma 1.2.

§3. Let ϕ be a measure function with $\phi(2h) = O(1)\phi(h)$, E be a plane set and $\psi(h) = h\phi(h)$. If A is a measurable subset of the x axis and $E_x = [y : (x, y) \in E]$ is such that

$$\phi - mE_x > p \quad \text{for all} \quad x \in A,$$

we have $\psi - m(E) \geqslant kp\Lambda(A)$ for some absolute constant k, greater than zero. This is proved by Marstrand in [**6**] for the case where $\phi(h) = h^\alpha$ and the proof extends to cover this case.

Now let $\pi = [(-x, \omega) : \phi - mZ^x(\omega) = \infty]$. Then $\pi \in \mathscr{B} \times \mathscr{F}$, see [**3**; p. 314]. Let $\pi_x = [\omega : \phi - mZ^x(\omega) = \infty]$ and $\pi_\omega = [x : \phi - mZ^x(\omega) = \infty]$. Now by the right continuity of the paths we have

$$X_{T_x(\omega)}(\omega) = x, \quad P \text{ almost surely on } (T_x < \infty),$$

so if ϕ is a measure function with $P[\phi - mZ(\omega) = \infty] > 0$ then the strong Markov property and the independent increment property imply that

$$P[\omega : \phi - mZ^x(\omega) = \infty,\ T_x(\omega) < 1] > 0 \text{ for each } x.$$

Thus $P[\pi_x] > 0$ for each x and hence $\Lambda \times P(\pi) > 0$ and $P[\Lambda(\pi_\omega) > 0] > 0$. Let $\Omega = [\omega : \Lambda(\pi_\omega) > 0]$. If $\omega \in \Omega$ we can apply the result in the last paragraph to the set $G(\omega)$ to show that if ϕ also satisfies $\phi(2h) = O(1)\phi(h)$ and if $\psi(h) = h\phi(h)$ then

$$\psi - mG(\omega) = \infty.$$

Thus $P[\psi - mG(\omega) = \infty] > 0$ and hence, by Lemma 1.2, $P[\psi - mG(\omega) = \infty] = 1$. It is clear that our problem reduces to proving the following two propositions.

PROPOSITION 2. *If* $0 < \alpha < 1$ *and* $\phi(h) = (\log(1/h))^{-\alpha}$ *then*

$$P[\phi - mZ(\omega) = \infty] = 1.$$

PROPOSITION 3. *If* $\psi(h) = h\,(\log(1/h))^{-1}$ *then* $P[\psi - mG(\omega) < \infty] = 1$.

§4. Since $X_t(\omega)$ has 0 regular for $\{0\}$ it follows that there exists a local time at zero in the sense of Blumenthal and Getoor **[4]**. Their arguments show that the measure function properties of the zero set of $X_t(\omega)$ are the same as those of the range of the subordinator, $T_t(\omega)$, whose subordinator exponent, $g(\lambda)$, is given by

$$[g(\lambda)+\gamma]^{-1} = \int_0^\infty \exp(-\lambda t)\, p(t,0)\, dt,$$

and

$$\gamma^{-1} = \int_0^\infty p(t,0)\, dt.$$

Let $F_t(x)$ be the distribution of $T_t(\omega) - T_0(\omega)$. We shall need a reliable estimate for the behaviour of $F_t(x)$ as x and t tend to zero. As a first step we prove

LEMMA 2.1.

$$g(\lambda) \sim \frac{\pi h^2}{2p} \log \lambda \quad \textit{as} \quad \lambda \to +\infty.$$

Proof. From (1) we obtain

$$p(t,0) = \frac{p(1, -h\log t)}{t},$$

and, as noted in **[8]**, we also have $x^2 p(1,x)$ bounded and $p(1,x) \sim 2p/x^2$ as $x \to +\infty$. Let

$$T = \frac{\log \lambda}{\lambda}, \quad S = \frac{1}{\lambda \log \lambda},$$

and consider

$$\frac{1}{g(\lambda)+\gamma} = \int_0^\infty \exp(-\lambda t) \frac{p(1, -h\log t)\, dt}{t}$$

$$= \int_T^\infty + \int_S^T + \int_0^S = I_1 + I_2 + I_3, \text{ say.} \tag{4}$$

Now

$$I_1 \leqslant \exp(-\lambda T) \int_T^\infty \frac{p(1, -h\log t)\, dt}{t},$$

and, putting $y = -h \log t$,

$$I_1 \leqslant \frac{\exp(-\lambda T)}{h} \int_{-\infty}^\infty p(1,y)\, dy = \frac{1}{\lambda h}. \tag{5}$$

Now

$$I_2 = O(1) \int_S^T \frac{dt}{t(\log t)^2} = O(1) \left[\frac{1}{\log S} - \frac{1}{\log T}\right],$$

and

$$I_2 = o(1) \frac{1}{\log \lambda}. \tag{6}$$

Now, as $\lambda \to \infty$,

$$I_3 \sim \frac{2p}{\pi h^2} \int_0^S \frac{dt}{t(\log t)^2} = \frac{2p}{\pi h^2} \frac{1}{\log \lambda + \log \log \lambda}. \tag{7}$$

Equations (4)–(7) combine to prove the lemma.

LEMMA 2.2. *There exist positive constants A and k such that, whenever* $0 < 2x < t < 1$,

$$F_t(x) \leqslant Ax^{kt}.$$

Proof. Since $g(\lambda)$ and $\log \lambda$ are continuous and bounded away from zero on $\lambda \geqslant 2$, Lemma 2.1 shows that there is a positive constant k such that $g(\lambda) \geqslant k \log \lambda$ if $\lambda \geqslant 2$.

Now

$$\begin{aligned} \exp(-\lambda x) F_t(x) &\leqslant \int_0^x \exp(-\lambda x) F_t(dx) \\ &\leqslant \int_0^\infty \exp(-\lambda x) F_t(dx) \\ &= \exp[-tg(\lambda)]. \end{aligned}$$

Therefore

$$\begin{aligned} F_t(x) &\leqslant \exp[\lambda x - tg(\lambda)] \\ &\leqslant \exp[\lambda x - tk \log \lambda], \quad \text{if} \quad \lambda \geqslant 2. \end{aligned}$$

Now put $\lambda = t/x$. Then, if $0 < 2x < t < 1$, we have $\lambda > 2$, and so

$$\begin{aligned} F_t(x) &\leqslant \exp[t - tk \log t + kt \log x] \\ &= \exp[t - tk \log t]\, x^{kt}. \end{aligned}$$

The first term in the product is bounded for $0 \leqslant t \leqslant 1$ so the lemma is proved.

LEMMA 2.3. *Let* $h(t) = \exp[-t^{-1/\alpha}]$ *and* $\phi(h) = (\log(1/h))^{-\alpha}$ *where* $0 < \alpha < 1$. *Define* $P(a, \omega) = \inf(S : T_S(\omega) > a)$. *Then*

(i) $\liminf_{t \to 0} \dfrac{T_t(\omega)}{h(t)} \geqslant 1$ *P almost surely,*

and

(ii) $\limsup_{a \to 0} \dfrac{P(a, \omega)}{\phi(a)} \leqslant 1$ *P almost surely.*

Proof. Let $r_q = 2^{-q}$ for each integer q, and let A_q be the event

$$T_{r_{q+1}}(\omega) < h(r_q).$$

Then, by Lemma 2.2, $\sum_{q=1}^{\infty} P(A_q) < \infty$, so that by the Borel Cantelli Lemma $P(\limsup A_q) = 0$ and hence

$$\liminf_{q \to \infty} \frac{T_{r_{q+1}}(\omega)}{h(r_q)} \geqslant 1 \quad P \text{ almost surely.}$$

Since T_t and $h(t)$ are monotonic increasing $t \in [r_{q+1}\, r_q)$ implies that

$$\frac{T_t(\omega)}{h(t)} \geqslant \frac{T_{r_{q+1}}(\omega)}{h(r_q)},$$

and hence

$$\liminf_{t\to 0} \frac{T_t(\omega)}{h(t)} \geqslant \liminf_{q\to\infty} \frac{T_{r_{q+1}}(\omega)}{h(r_q)} \geqslant 1,$$

P almost surely. The lemma now follows from the observations that $h[\phi(t)] = t$ and $[\omega : P(a, \omega) > r] = [\omega : T_r(\omega) < a]$.

LEMMA 2.4. *Suppose that F is a measure defined on real Borel sets and that E is a Borel set such that for every $x \in E$ and some measure function ψ we have*

$$\limsup_{h\to 0} \frac{F[x, x+h]}{\psi(h)} \leqslant k < \infty.$$

Then $k\psi - m(E) \geqslant F(E)$.

Proof. This is Lemma 4 of **[11]**.

Proof of Proposition 2. Define

$$P(t, a, \omega) = \inf [S : T_{t+S}(\omega) - T_t(\omega) > a].$$

Let $\phi(h) = (\log(1/h))^{-\alpha}$, $0 < \alpha < 1$, and let

$$\pi = \left[(t, \omega) \in [0, 1] \times \Omega : \limsup_{a\to 0} \frac{P(t, a, \omega)}{\phi(a)} \leqslant 1\right].$$

Then $\pi \in \mathscr{B} \times \mathscr{F}$. Let

$$\pi_t = \left[\omega : \limsup_{a\to 0} P(t, a, \omega)/\phi(a) \leqslant 1\right],$$

be the t-section of π and let

$$\pi_\omega = \left[t : 0 \leqslant t \leqslant 1, \limsup_{a\to 0} P(t, a, \omega)/\phi(a) \leqslant 1\right]$$

be the ω-section of π. Then, since T_t has stationary independent increments, $P(t, a, \omega)$ and $P(0, a, \omega)$ have the same distribution and so, by Lemma 2.3, $P(\pi_t) = 1$ for all t, $0 \leqslant t \leqslant 1$. Fubini's Theorem now applies to show that for P almost all ω, $\pi_\omega \in \mathscr{B}$ and $\Lambda(\pi_\omega) = 1$. Now take one of these ω. Then, if

$$A_\omega = [x : x = T_t(\omega) \text{ for some } t \in \pi_\omega],$$

A_ω is a Borel set. Consider the measure F_ω, defined on the Borel subsets B, of R, by $F_\omega(B) = \Lambda[t : T_t(\omega) \in B]$. Then $F_\omega(A_\omega) = \Lambda(\pi_\omega) = 1$ and

$$F_\omega[T_t(\omega), T_t(\omega) + h] = P(t, h, \omega),$$

if $x \in A_\omega$ then $x = T_t(\omega)$ for some $t \in \pi_\omega$ and hence

$$\limsup_{h\to 0} \frac{F_\omega[x, x+h]}{\phi(h)} \leqslant 1.$$

Now by Lemma 2.4 we have $\phi - mA_\omega \geqslant F_\omega(A_\omega) = 1$. Since this is true for any α with $0 < \alpha < 1$ and since $A_\omega \subset T([0, 1], \omega)$ we have

$$\phi - mT([0, 1], \omega) = \infty \quad P \text{ almost surely,}$$

and hence

$$\phi - mZ(\omega) = \infty \quad P \text{ almost surely.}$$

Thus Proposition 2 is proved.

§5. We start by proving a simple lemma.

LEMMA 3.1. *Let $\{X_i\}$ be a sequence of mutually independent random variables and let $\{Y_i\}$ also be mutually independent random variables. Suppose that for each x and i we have*

$$P(X_i < x) \leqslant P(Y_i < x).$$

Then for any n and x we have

$$P\left(\sum_{i=1}^{n} X_i < x\right) \leqslant P\left(\sum_{i=1}^{n} Y_i < x\right).$$

Proof. It suffices to consider the case where $n = 2$. Let $G_i(x) = P(Y_i < x)$ and $F_i(x) = P(X_i < x)$. Then

$$\begin{aligned} P(X_1 + X_2 < x) &= \int_R F_2(x-y)\, F_1(dy) \\ &\leqslant \int_R G_2(x-y)\, F_1(dy) \\ &= \int_R F_1(x-y)\, G_2(dy), \text{ integrating by parts,} \\ &\leqslant \int_R G_1(x-y)\, G_2(dy) \\ &= P(Y_1 + Y_2 < x), \end{aligned}$$

and the lemma is proved.

Proof of Propositon 3. It follows from (1) that, for each $r > 0$,

$rX_t(\omega)$ and $X_{rt}(\omega) + rth \log r$ have the same P distribution. (8)

Now take ε_0 such that $0 < \varepsilon < \varepsilon_0$ implies that $1 + h \log(1/\varepsilon) < \log(1/\varepsilon)$ (note $0 < h \leqslant 2/\pi$). For any ε we define a sequence $\{\sigma_k^{\varepsilon}\}$ of stopping times as follows

$$\sigma_0^{\varepsilon} \equiv 0,$$

$$\tau_k^{\varepsilon} = \inf\,[t \geqslant \sigma_{k-1}^{\varepsilon} : |X_t - X_{\sigma^{\varepsilon}_{k-1}}| > \varepsilon \log(1/\varepsilon)],$$

$$\sigma_k^{\varepsilon} = \min\,[\tau_k^{\varepsilon}, \sigma_{k-1}^{\varepsilon} + \varepsilon \log(1/\varepsilon)].$$

The strong Markov property now applies to show

$\{\sigma_k^{\varepsilon} - \sigma_{k-1}^{\varepsilon}\}$ are independent identically distributed random variables. (9)

Take c and ε with $0 < c \leqslant \varepsilon < \varepsilon_0$, then

$$P(\sigma_1{}^{\varepsilon} < c) = P\left(\sup_{t<c} |X_t| > \varepsilon \log(1/\varepsilon)\right)$$
$$= P\left(\sup_{t<c} (1/\varepsilon)|X_t| > \log(1/\varepsilon)\right)$$
$$= P\left\{\sup_{t<c} |X_{t\varepsilon^{-1}} + (t/\varepsilon) h \log(1/\varepsilon)| > \log(1/\varepsilon)\right\}, \text{ by (8)},$$
$$= P\left\{\sup_{S<(c/\varepsilon)} |X_S + Sh \log(1/\varepsilon)| > \log(1/\varepsilon)\right\}.$$

Now $S \leqslant 1$ implies

$$(\omega : |X_S + Sh \log(1/\varepsilon)| > \log(1/\varepsilon)) \subset (\omega : |X_S| + Sh \log(1/\varepsilon) > \log(1/\varepsilon))$$
$$\subset (\omega : |X_S| > 1).$$

Thus we have

$$P(\sigma_1{}^{\varepsilon} < c) \leqslant P\left(\sup_{S<(c/\varepsilon)} |X_S| > 1\right) = P(\tau_1{}^1 < (c/\varepsilon))$$
$$= P(\sigma_1{}^1 < (c/\varepsilon))$$

and so, by (9),

$$P(\sigma_k{}^{\varepsilon} - \sigma^{\varepsilon}_{k-1} < c) \leqslant P(\sigma_k{}^1 - \sigma^1_{k-1} < (c/\varepsilon)).$$

Let $H_k = \min[(\sigma_k{}^1 - \sigma^1_{k-1}), 1]$ then the H_k are independent and identically distributed and for each x, k and $\varepsilon < \varepsilon_0$ we have

$$P(\sigma_k{}^{\varepsilon} - \sigma^{\varepsilon}{}_{k-1} < \varepsilon x) \leqslant P(H_k < x)$$

(for $x \leqslant 1$ put $x = (c/\varepsilon)$, for $x > 1$ use the definition of H_k).

Let $m = EH_1$ be the expectation of H_1, so that $m > 0$, and define

$$S_\varepsilon = \min[k : \sigma_k{}^{\varepsilon} \geqslant \tfrac{1}{2}m].$$

Now

$$P(\varepsilon S_\varepsilon > 1) = P\left[\sum_{i=1}^{[1/\varepsilon]} (\sigma_i{}^{\varepsilon} - \sigma^{\varepsilon}_{i-1}) < \tfrac{1}{2}m\right],$$

and so, by Lemma 3.1,

$$P(\varepsilon S_\varepsilon > 1) \leqslant P\left[\varepsilon \sum_{i=1}^{[1/\varepsilon]} H_i < \tfrac{1}{2}m\right]. \tag{10}$$

By the definition of m and the weak law of large numbers the right-hand side of (10) tends to zero as ε tends to zero. So we can find a sequence, ε_n, decreasing to zero such that

$$P(\varepsilon_n S_{\varepsilon_n} > 1) < 2^{-n}.$$

By the Borel Cantelli Lemma we now have

$$\limsup_{n\to\infty} \varepsilon_n S_{\varepsilon_n} \leqslant 1, \tag{11}$$

P almost surely.

Let $G'(\omega) = [(t, X_t(\omega)) : 0 \leqslant t \leqslant \tfrac{1}{2}m]$ be the graph of $X_t(\omega)$ up to time $\tfrac{1}{2}m$. We now cover G' by the sets A_k where

$$A_k = [\sigma^{\varepsilon}_{k-1}, \sigma_k{}^{\varepsilon}] \times X[\sigma^{\varepsilon}_{k-1}, \sigma_k{}^{\varepsilon}].$$

Each of these has diameter less than $3\varepsilon \log(1/\varepsilon)$, and S_ε of these cover G'. If $\psi(h) = h(\log(1/h))^{-1}$ we have

$$\psi(3\varepsilon \log(1/\varepsilon)) \sim 3\varepsilon \quad \text{as} \quad \varepsilon \to 0. \tag{12}$$

Since

$$\psi - mG' \leqslant \liminf_{\varepsilon \to 0} S_\varepsilon \psi(3\varepsilon \log(1/\varepsilon)),$$

(11) and (12) combine to show that $\psi - mG'(\omega) \leqslant 3$ P almost surely.

By the independent increment property we now have

$$\psi - mG(\omega) \leqslant 12/m,$$

P almost surely, which proves Proposition 3.

Remarks. 1. If $X_t(\omega)$ is an asymmetric Cauchy Process in R^d then its projection onto some linear subspace will be an asymmetric Cauchy Process on that line. The methods of §4 would apply to give a lower bound for the measure of the graph. We could then use the d-dimensional equivalent of (8) to obtain an upper bound. We can thus show that the conclusions of Theorem 3 are valid in higher dimensions.

2. We could improve our arguments in §4 to show that, if

$$\phi(h) = \frac{\log\log\log(1/h)}{\log(1/h)},$$

then $\phi - mZ(\omega) > 0$, P almost surely, and since this is the best result that the methods of that section would give, it seems likely that this is the exact measure function of the zero set. We would also expect that the exact measure function for the graph (and the range in dimensions $\geqslant 2$) is

$$\frac{h \log\log\log(1/h)}{\log(1/h)}.$$

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