THE ASYMPTOTIC FORMULA IN WARING'S PROBLEM

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§1. *Introduction.* Let s, k and n be positive integers and define $r_{s,k}(n)$ to be the number of solutions of the diophantine equation

$$n = x_1^k + x_2^k + \cdots + x_s^k \tag{1.1}$$

in positive integers x_i. In 1922, using their circle method, Hardy and Littlewood [2] established the asymptotic formula

$$r_{s,k}(n) = \mathfrak{S}_{s,k}(n) \frac{\Gamma(1+(1/k))^s}{\Gamma(s/k)} n^{(s/k)-1}(1+o(1)) \tag{1.2}$$

whenever $s \geqslant (k-2)2^{k-1}+5$. Here $\mathfrak{S}_{s,k}(n)$, the *singular series*, relates the local solubility of (1.1). For each k we define $\tilde{G}(k)$ to be the smallest value of s_0 such that for all $s \geqslant s_0$ we have (1.2), the *asymptotic formula in Waring's problem.* The main result of this memoir is the following theorem which improves upon bounds of previous authors when $k \leqslant 9$.

THEOREM 1. *Suppose that* $k \geqslant 6$. *Then* $\tilde{G}(k) \leqslant \frac{7}{8}2^k$.

Hardy and Littlewood were able to prove that $\tilde{G}(k) \leqslant (k-2)2^{k-1}+5$ by using Weyl's inequality, first discussed by Weyl [8] in 1914. In 1938, Hua [4] demonstrated that (1.2) holds provided $s \geqslant 2^k+1$, thus proving that $\tilde{G}(k) \leqslant 2^k+1$, and improving upon the bound of Hardy and Littlewood when $k \geqslant 4$. Hua's proof complemented Weyl's inequality with a new ingredient, Hua's inequality. For small values of k ($k \leqslant 11$), no progress was made on $\tilde{G}(k)$ over the next 48 years. For large values of k, frequent progress has been made—initiated by Vinogradov's pioneering work. Vinogradov, Hua, and others refined bounds for $\tilde{G}(k)$ leading to the bound $\tilde{G}(k) < (4+o(1))k^2 \log k$. Although it is quite probable that $\tilde{G}(k) \ll k$, and, perhaps, even that $\tilde{G}(k) = k+1$ (even though (1.1) may be insoluble), this is far beyond the reach of known methods. In 1986, Vaughan [6], [7] used important new ideas to establish (1.2) when $s = 2^k$ and $k \geqslant 3$. Using techniques unrelated to those of Vaughan, Heath-Brown [3] in 1988 proved that $\tilde{G}(k) \leqslant \frac{7}{8}2^k + 1$ when $k \geqslant 6$. His striking methods encompass both an improvement in Weyl's inequality and an associated improvement in Hua's inequality.

THEOREM (Heath-Brown [3], Theorem 1). *Let* $k \geqslant 6$ *and*

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}$$

with $(a, q) = 1$. *Set* $f(\alpha) = \sum_{x=1}^{P} e(\alpha x^k)$. *Then, for any* $\varepsilon > 0$,

$$f(\alpha) \ll P^{1+\varepsilon}(Pq^{-1} + P^{-2} + qP^{1-k})^{\frac{4}{3}2^{-k}}. \tag{1.3}$$

(Throughout this treatise we set $f(\alpha)$, the standard generating function for kth powers, to be as defined in the last theorem, with $P = [n^{1/k}]$.) This result yields its strongest bound when $P^3 \ll q \ll P^{k-3}$, and, for these q, $f(\alpha) \ll P^{1-\frac{8}{3}2^{-k}+\varepsilon}$ (*cf.* (6.2) and (6.4)). This may be compared to the bound $f(\alpha) \ll P^{1-2^{1-k}+\varepsilon}$ that is deduced from Weyl's inequality when $P \ll q \ll P^{k-1}$; many of the technical difficulties that we encounter can be traced to the disparity in the aforementioned ranges on q. More precisely, Heath-Brown's result improves upon the bound that arises from Weyl's inequality whenever $P^{\delta} \ll q \ll P^{k-\delta}$ with $\delta > \frac{5}{2}$.

THEOREM (Heath-Brown [3], Theorem 2). *Let* $k \geqslant 6$. *Then, for any* $\varepsilon > 0$,

$$\int_0^1 |f(\alpha)|^{\frac{7}{8}2^k} d\alpha \ll P^{\frac{7}{8}2^k - k + \varepsilon}. \tag{1.4}$$

To establish his bound on $\tilde{G}(k)$, Heath-Brown combined (1.4) with Weyl's inequality.

Extending ideas of Heath-Brown [3] and Vaughan [7] (the latter relying heavily upon the opera of Hooley), we establish Theorem 1 as a direct consequence of the following result.*

THEOREM 2. *Suppose that s, k and n are positive integers with* $k \geqslant 6$ *and* $s = \frac{7}{8}2^k$ *and that* $\chi(k)$ *is* 1 *if k is even and* 0 *otherwise. Then*

$$r_{s,k}(n) = \mathfrak{S}_{s,k}(n) \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{(s/k)-1} + O(n^{(s/k)-1}(\log n)^{\varepsilon + \eta(k)})$$

where

$$\eta(k) = 2(k-3)^{3/2} - \tfrac{1}{2}k^2 - \tfrac{3}{2}k + 13 + \chi(k)(d(k)-1).$$

(A formal definition of $\mathfrak{S}_{s,k}(n)$ is given in Section 5.) It is crucial that $\eta(k)$ is negative as $\mathfrak{S}_{s,k}(n) \gg 1$. Indeed: $\eta(6) = -0{\cdot}607\ldots$, $\eta(7) = -6$, $\eta(8) = -5{\cdot}639\ldots$, $\eta(9) = -11{\cdot}606\ldots$. The value of $\eta(k)$ may be reduced, but at the cost of rather cumbersome complications. We intend to pursue such methods in a later paper. The proof of Theorem 2 involves a delicate treatment of the

* Recently, Greaves has proved an estimate that allows us to take $\chi(k) = 0$ for all k in this result. See G. Greaves, On the representation of a number as a sum of two fourth powers II, (Russian) *Mat. Zametki*, 55 (1994), 44–58.

Hardy–Littlewood method that is motivated by the proof of (1.4) of Heath-Brown [3]. The essential difficulty in the proof is that we cannot appeal to a classical dissection into major and minor arcs. The major arcs would be too wide to accommodate the error terms in the standard analysis of the auxiliary functions that are employed to obtain (1.2) and, moreover, we need the "full" strength of (1.3) which demands that $q \gg P^3$. In classical treatments (see, for example, [5], Chapter 4), the major arcs "cover" those α which can be well-approximated with small values of q (*e.g.* $q \ll P$). It may be of some interest to note that in the arc dissection to follow the main contribution to the error term in Theorem 2 does not arise on the minor arcs (as formulated), but on *pruned* sections of the (non-classical) major arcs.

We would be remiss to not note that Wooley [9] has recently demonstrated that $\tilde{G}(k) < (2+o(1))k^2 \log k$. In particular, he has proved that $\tilde{G}(10) \leqslant 750$. His substantial improvements supersede the bounds of Heath-Brown when $k \geqslant 10$.

The author would like to thank Professors H. L. Montgomery, R. C. Vaughan, and T. D. Wooley for their support, encouragement, and many helpful discussions. This manuscript comprises a part of the author's Ph.D. thesis at the University of Michigan.

§2. *Notation.* As usual, ε denotes a sufficiently small positive number, $\ll$ and $\gg$ denote Vinogradov's well-known notation (where implicit constants are functions of, at most, ε and k unless otherwise specified), and $e(x) = \exp(2\pi i x)$. Further, we assume that $s = \frac{7}{8}2^k$, that k is a fixed positive integer that is at least 6, and that n is large (in terms of ε and k). We adopt the convention that the value of ε may change from one occurrence to the next. It will be convenient to define $\chi(k)$ to be the characteristic function on the even integers. As usual, $d_r(n)$ denotes the number of representations of n as the product of r ordered positive integers and $d_2(n) = d(n)$. Also, $[w]$ designates the greatest integer not exceeding w and $\|w\|$ the distance from w to the nearest integer.

§3. *Preliminary lemmata.* We initiate the proof of Theorem 2 in the next section. Presently, we give two germane results of potentially independent interest. To obtain strong estimates for divisor sums of restricted type we provide the following lemma.

LEMMA 1. *Let r, t, and A be fixed, $r \geqslant 2$, $t \geqslant 1$, $A \geqslant 3$, and put*

$$c(m) = \#\{(a_1, a_2, \ldots, a_r) : m = a_1 a_2 \ldots a_r,\ 1 \leqslant a_i \leqslant A\}.$$

Then

$$\sum_{m \geqslant 1} c(m)^t \ll A^r (\log A)^{r^t - (r-1)t - 1 + \varepsilon}.$$

Proof. Let τ be fixed and set $M=A^r(\log A)^{-\tau}$. It is clear that if $m>A^r$ we have $c(m)=0$. Set

$$H_1=\sum_{m\leqslant M} c(m)^t, \qquad H_2=\sum_{m>M} c(m)^t.$$

We treat H_1 by utilizing the basic inequality $c(m)\leqslant d_r(m)$. Employing well-known bounds for moments of the divisor function,

$$H_1\leqslant \sum_{m\leqslant M} d_r(m)^t \ll M(\log M)^{r^t-1}.$$

Our estimate for H_1 is acceptable if τ is taken to be sufficiently large. If $m>M$, we have $a_i>A(\log A)^{-\tau}$ for all i. For $j\geqslant 1$ we set

$$I(j)=(A(\log A)^{-\tau}e^{j-1}, A(\log A)^{-\tau}e^j].$$

Since

$$c(m)\leqslant \sum_{\substack{a_1a_2\dots a_{r-1}|m\\ 1\leqslant a_i\leqslant A}} 1,$$

we see that, if $m>M$,

$$c(m)\leqslant \sum_{j_1,j_2,\dots,j_{r-1}} \sum_{\substack{a_1a_2\dots a_{r-1}|m\\ a_i\in I(j_i)}} 1.$$

As each j_i takes on $O(\log\log A)$ values, it follows that

$$c(m)\ll(\log\log A)^{r-1} \max_{u_1,u_2,\dots,u_{r-1}} \sum_{\substack{a_1a_2\dots a_{r-1}|m\\ u_i<a_i\leqslant eu_i}} 1=(\log\log A)^{r-1}\Delta_r(m)$$

where $\Delta_r(m)$ is Hooley's (extended) divisor function. Setting $y=1$ in Theorem 3 of Hall and Tenenbaum [1] we obtain

$$\sum_{N\leqslant x} \Delta_r(N)^t \ll_{r,t} x(\log x)^{r^t-(r-1)t-1+\varepsilon}$$

(and the exponent of $\log x$ here is sharp). The desired result follows immediately.

The second lemma of this section, a mean value estimation, is quite extraordinary—in part because it can be established by elementary means.

LEMMA 2 (Vaughan, unpublished). *Let S denote the number of solutions of the simultaneous equations*

$$\begin{aligned} x_1^3+x_2^3+x_3^3&=y_1^3+y_2^3+y_3^3\\ x_1+x_2+x_3&=y_1+y_2+y_3 \end{aligned} \tag{3.1}$$

in positive integers x_i, y_j with $x_i\leqslant P$, $y_j\leqslant P$. Then

$$S=6P^3+O(P^{\frac{7}{3}+\varepsilon}).$$

Proof. If $x_i=y_j$ for some i and j, it follows easily that y_1, y_2, y_3 is a permutation of x_1, x_2, x_3. (The x_i are distinct for all but $O(P^2)$ triples x_1, x_2, x_3.) It therefore remains to show that the number of non-trivial solutions of

(3.1) is $O(P^{\frac{7}{3}+\varepsilon})$. On factoring $(x_1+x_2+x_3)^3-x_1^3-x_2^3-x_3^3$, it follows that system (3.1) is equivalent to the pair of simultaneous equations

$$(x_2+x_3)(x_3+x_1)(x_1+x_2)=(y_2+y_3)(y_3+y_1)(y_1+y_2)$$
$$x_1+x_2+x_3=y_1+y_2+y_3.$$

Upon making the substitutions

$$X_1=x_2+x_3, \qquad X_2=x_3+x_1, \qquad X_3=x_1+x_2,$$

and

$$Y_1=y_2+y_3, \qquad Y_2=y_3+y_1, \qquad Y_3=y_1+y_2,$$

we see that it suffices to show that the number of non-trivial solutions of

$$X_1X_2X_3=Y_1Y_2Y_3, \qquad X_1+X_2+X_3=Y_1+Y_2+Y_3$$

with $1\leqslant X_i\leqslant 2P$, $1\leqslant Y_j\leqslant 2P$, is $O(P^{\frac{7}{3}+\varepsilon})$. (*If* $X_i=Y_j$ for some i and some j then Y_1, Y_2, Y_3 is a permutation of X_1, X_2, X_3 and, hence, that $x_i=y_j$ for some i and j.) Writing

$$d_1=(Y_1,X_1), \qquad d_2=\left(\frac{Y_1}{d_1},X_2\right), \qquad Y_1=d_1d_2d_3,$$

$$e_1=\left(Y_2,\frac{X_1}{d_1}\right), \qquad e_2=\left(\frac{Y_2}{e_1},\frac{X_2}{d_2}\right), \qquad Y_2=e_1e_2e_3,$$

$$f_1=\left(Y_3,\frac{X_1}{d_1e_1}\right), \qquad f_2=\left(\frac{Y_3}{f_1},\frac{X_2}{d_2e_2}\right), \qquad Y_3=f_1f_2f_3,$$

we find that

$$\frac{X_1X_2}{d_1e_1f_1d_2e_2f_2}X_3=d_3e_3f_3$$

and that

$$\left(\frac{X_1X_2}{d_1e_1f_1d_2e_2f_2},\, d_3e_3f_3\right)=1.$$

Thus $X_1=d_1e_1f_1$, $X_2=d_2e_2f_2$, and $X_3=d_3e_3f_3$. We need, therefore, to bound the number of non-diagonal solutions of

$$d_1e_1f_1+d_2e_2f_2+d_3e_3f_3=d_1d_2d_3+e_1e_2e_3+f_1f_2f_3 \tag{3.2}$$

with the six terms in (3.2) each at most $2P$. It is sufficient to consider the solutions for which $f_3\leqslant f_2\leqslant f_1$, since every solution is of this form after a suitable rearrangement of indices. We therefore suppose that $f_3\ll P^{1/3}$. Multiplying (3.2) through by f_3 and rearranging terms we obtain

$$(f_3d_3-e_1e_2)(f_3e_3-d_1d_2)=(f_3f_1-d_2e_2)(f_3f_2-d_1e_1). \tag{3.3}$$

Since we are considering only non-diagonal solutions to (3.2), each of the four factors in (3.3) is non-zero. By standard estimates for the divisor function, given d_1, d_2, d_3, e_1, e_2, e_3 and f_3, the number of choices for f_1 and f_2 is thus at most

$$2d(|(f_3d_3-e_1e_2)(f_3e_3-d_1d_2)|)\ll P^{\varepsilon}.$$

The number of solutions of (3.2) is therefore at most

$$\sum_{d_1d_2d_3\leqslant 2P}\ \sum_{e_1e_2e_3\leqslant 2P}\ \sum_{f_3\ll P^{1/3}} P^{\varepsilon}\ll P^{\frac{1}{3}+\varepsilon}\Big(\sum_{d_1d_2d_3\leqslant 2P} 1\Big)^2\ll P^{\frac{7}{3}+\varepsilon}.$$

This completes the proof of the lemma.

§4. *Preparatory treatments.* One of the innovations of Heath-Brown in [3] was to iterate Weyl's shift operator only $k-3$ times to obtain cubic exponential sums. We begin the proof of Theorem 2 by refining an argument of Heath-Brown [3] that is based upon a symmetric difference operator. (Alternatively, one may use Weyl's forward differencing technique followed by a suitable linear transformation.) We first set

$$J(x)=\left[\frac{[P^3x]}{P^3},\frac{[P^3x]+1}{P^3}\right],\qquad T(x)=\max\left|\sum_{n\in I} e(yn^3+zn)\right| \tag{4.1}$$

where the maximum is taken over subintervals I of $[1, P]$, z in $[0, 1]$ and y in $J(x)$. Though we refrain from reproducing the details of the method of Heath-Brown [3] here, we begin with the bound

$$f(\alpha)^{2^{k-3}}\ll P^{2^{k-3}-k+2}\sum_{|h_i|<P/2} T\left(\alpha\frac{k!}{6}2^{k-3}h_1h_2\dots h_{k-3}\right)$$

(obtained by a differencing argument). (The maximum over y is a spurious condition as $y=x$. However, we shall find it convenient to take the max after realizing that $x\in J(x)$.) We now make two definitions:

$$c^*(h)=\{(h_1,h_2,\dots,h_{k-3}):h=h_1h_2\dots h_{k-3},\ |h_i|<P/2\},$$
$$c(h)=\{(h_1,h_2,\dots,h_{k-3}):h=h_1h_2\dots h_{k-3},\ 0\leqslant h_i<P/2\}.$$

It follows that

$$f(\alpha)^{2^{k-3}}\ll P^{2^{k-3}-k+2}\sum_{h=0}^{(P/2)^{k-3}} c^*(h)T(\alpha\kappa 2^{k-3}h)$$

where $\kappa=k!/6$. Since $c(0)\ll P^{k-4}$ and $T(0)=P$, we obtain, as $c^*(h)\ll c(h)$,

$$f(\alpha)^{2^{k-3}}\ll P^{2^{k-3}-1}+P^{2^{k-3}-k+2}\sum_{h=1}^{(P/2)^{k-3}} c(h)T(\alpha\kappa 2^{k-3}h). \tag{4.2}$$

It is at this point in our discourse that we consider two separate lines of attack—*via* two different applications of Hölder's inequality. From (4.2) we conclude that

$$f(\alpha)^{2^{k-3}} \ll P^{2^{k-3}-1} + P^{2^{k-3}-k+2}\left(\sum_{h=1}^{\kappa P^{k-3}} T(\alpha h)^3\right)^{1/3}\left(\sum_{h=1}^{P^{k-3}} c(h)^{3/2}\right)^{2/3}$$

and that

$$f(\alpha)^{2^{k-3}} \ll P^{2^{k-3}-1} + P^{2^{k-3}-k+2}\left(\sum_{h=1}^{\kappa P^{k-3}} T(\alpha h)^6\right)^{1/6}\left(\sum_{h=1}^{P^{k-3}} c(h)^{6/5}\right)^{5/6}.$$

We now appeal to Lemma 1 to obtain the respective bounds

$$f(\alpha)^{2^{k-3}} \ll P^{2^{k-3}-1} + P^{2^{k-3}-\frac{k}{3}}(\log P)^{\frac{2}{3}(k-3)^{3/2}-k+\frac{10}{3}+\varepsilon}\left(\sum_{h=1}^{\kappa P^{k-3}} T(\alpha h)^3\right)^{1/3} \tag{4.3}$$

and

$$f(\alpha)^{2^{k-3}} \ll P^{2^{k-3}-1} + P^{2^{k-3}-\frac{k}{6}-\frac{1}{2}}(\log P)^{\frac{5}{6}(k-3)^{6/5}-k+\frac{19}{6}+\varepsilon}\left(\sum_{h=1}^{\kappa P^{k-3}} T(\alpha h)^6\right)^{1/6}. \tag{4.4}$$

In Section 6 we use (4.4) to treat the integral over the minor arcs. The integral over the pruned arcs is more difficult to evaluate and we shall resort to additional devices to complement (4.3) in later sections of this lucubration. Before we proceed to invent a suitable arc demarcation, we provide a result which we shall use to extract key information concerning the "average" size of $T(x)^6$.

LEMMA 3. *Using the notation of* (4.1),

$$\sum_{m=1}^{P^3} T(mP^{-3})^6 \ll P^7.$$

Proof. From the definition in (4.1), $T(x)$ attains a maximum for some I, z and y. For $x = mP^{-3}$ let such an extremal triple be denoted by I_m, z_m and y_m. Then

$$T(mP^{-3}) = \left|\sum_{n\in I_m} e(y_m n^3 + z_m n)\right|$$
$$= \left|\int_0^1 \sum_{n=1}^{P} e(y_m n^3 + (z_m+\gamma)n) \sum_{\tau\in I_m} e(-\tau\gamma)\,d\gamma\right|.$$

Suppose γ is not an integer. Since I_m is a subinterval of $[1, P]$,

$$\left|\sum_{\tau\in I_m} e(-\tau\gamma)\right| \leqslant \frac{2}{|1-e(-\gamma)|} = \frac{1}{|\sin(-\pi\gamma)|} \ll \frac{1}{\|\gamma\|}.$$

If γ is near an integer, we supplement this with a trivial bound whence

$$T(mP^{-3}) \ll \int_0^1 \left|\sum_{n=1}^{P} e(y_m n^3 + (z_m+\gamma)n)\right| \min\left(P, \frac{1}{\|\gamma\|}\right) d\gamma.$$

By Hölder's inequality,

$$T(mP^{-3})^6 \ll \left(\int_0^1 \left|\sum_{n=1}^{P} e(y_m n^3 + (z_m+\gamma)n)\right|^6 d\gamma\right)\left(\int_0^1 \min\left(P, \frac{1}{\|\gamma\|}\right)^{6/5} \delta\gamma\right)^5.$$

Since

$$\int_0^1 \min\left(P, \frac{1}{\|\gamma\|}\right)^{6/5} d\gamma \ll P^{1/5},$$

employing the change of variable $z_m+\gamma=\mu$ we conclude that

$$\sum_{m=1}^{P^3} T(mP^{-3})^6 \ll P \sum_{m=1}^{P^3} \int_0^1 \left|\sum_{n=1}^{P} e(y_m n^3+\mu n)\right|^6 d\mu. \tag{4.5}$$

We define $R(u, v)$ to be the number of solutions of the simultaneous equations

$$n_1^3+n_2^3+n_3^3=v,$$

$$n_1+n_2+n_3=u,$$

with $1 \leqslant n_i \leqslant P$. Since

$$\left(\sum_{n=1}^{P} e(y_m n^3+\mu n)\right)^3 = \sum_u \sum_v R(u, v) e(y_m v+\mu u),$$

from (4.5) we discern that

$$\sum_{m=1}^{P^3} T(mP^{-3})^6 \ll P \sum_{m=1}^{P^3} \int_0^1 \left|\sum_u \left(\sum_v R(u, v) e(y_m v)\right) e(\mu u)\right|^2 d\mu.$$

By Parseval's indentity, the integral on the right-hand side is equal to

$$\sum_u \left|\sum_v R(u, v) e(y_m v)\right|^2$$

and therefore

$$\sum_{m=1}^{P^3} T(mP^{-3})^6 \ll P \sum_u \sum_{m=1}^{P^3} \left|\sum_v R(u, v) e(y_m v)\right|^2. \tag{4.6}$$

By assumption, each of the points y_m lies in the interval $[m/P^3, (m+1)/P^3]$ so that the set of y_m under consideration can be partitioned into two sets, by parity on m, in each of which they are spaced at least P^{-3} apart. Hence, by the large sieve inequality,

$$\sum_{m=1}^{P^3} \left|\sum_v R(u, v) e(y_m v)\right|^2 \ll P^3 \sum_v R(u, v)^2.$$

Thus, from (4.6),

$$\sum_{m=1}^{P^3} T(mP^{-3})^6 \ll P^4 \sum_u \sum_v R(u, v)^2.$$

Since the double sum is the number of solutions of (3.1) as counted by S, the desired result follows from Lemma 2.

This proof of Lemma 3 is due to R. C. Vaughan. From Lemma 3 we see that $T(\alpha h)$ is rarely larger than $P^{2/3}$. We shall exploit this phenomenon further (*cf.* (8.2)).

§5. *The essential demarcation.* By a theorem of Dirichlet, for any real α we can find integers a and q with $(a, q)=1$ and

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{qP^{k-3}} \tag{5.1}$$

where $1 \leqslant q \leqslant P^{k-3}$. We tacitly assume that (5.1) holds throughout this manuscript. It will be convenient to work on the unit interval

$$\mathscr{U} = \left[\frac{P}{2kn}, 1+\frac{P}{2kn}\right]$$

rather than [0, 1]. We partition $\mathscr{U}$ into three sections. We first set

$$\mathscr{M}(a, q) = \left\{\alpha : \left|\alpha - \frac{a}{q}\right| \leqslant \frac{P}{2kqn}\right\}, \qquad \mathscr{M} = \bigcup_{a,q}{}^{*} \mathscr{M}(a, q)$$

where the * signifies that the union is taken over all $1 \leqslant a \leqslant q \leqslant P$ with $(a, q) = 1$. It follows that $\mathscr{M} \subset \mathscr{U}$. Additionally, set

$$\mathfrak{m} = \left\{\alpha \in \mathscr{U} : \left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{qP^{k-3}} \text{ with } (a, q)=1 \text{ implies } \frac{P^3}{4} \leqslant q \leqslant P^{k-3}\right\}$$

and

$$\mathscr{N} = \mathscr{U} \backslash (\mathscr{M} \cup \mathfrak{m}),$$

pruned sections of the major arcs $\mathscr{M}$. (The intervals $\mathscr{M}(a, q)$ are clearly pairwise disjoint.) We have, by classical methods (see [**5**], Theorem 4.4),

$$\int_{\mathscr{M}} f(\alpha)^s e(-n\alpha)d\alpha = \mathfrak{S}_{s,k}(n)\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{(s/k)-1} + O(n^{(s/k)-1-\delta}) \tag{5.2}$$

for some explicit $\delta > 0$. As usual, the singular series $\mathfrak{S}_{s,k}(n)$ is given by

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (S(q, a)q^{-1})^s e\left(-\frac{an}{q}\right)$$

with

$$S(q,a)=\sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right).$$

It is well-known that $1 \gg \mathfrak{S}_{s,k}(n) \gg 1$ (see [5], Theorem 4.6). A salient feature of our major arcs is the inequality

$$\frac{P}{2kqn}<\frac{1}{qP^{k-3}},$$

so we may think of $\mathcal{M}$ as being embedded in the set of $\alpha\in\mathcal{U}$ that satisfy (5.1) with $1\leqslant q\leqslant P$. As noted in Section 1, we cannot take $\mathcal{M}$ to be the set of all $\alpha\in\mathcal{U}$ that satisfy (5.1) with $1\leqslant q\leqslant P$ for that would corrupt the treatment of the auxiliary functions that are (classically) used to approximate $f(\alpha)$ on the major arcs (see [5], Chapter 4). Pruning is our recourse. Since

$$r_{s,k}(n)=\int_0^1 f(\alpha)^s e(-n\alpha)\,d\alpha,$$

it follows that

$$\begin{aligned} r_{s,k}(n) &= \int_{\mathcal{M}} f(\alpha)^s e(-n\alpha)\,d\alpha + \int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\,d\alpha + \int_{\mathcal{N}} f(\alpha)^s e(-n\alpha)\,d\alpha \\ &= \int_{\mathcal{M}} f(\alpha)^s e(-n\alpha)\,d\alpha + O\left(\int_{\mathfrak{m}} |f(\alpha)|^s\,d\alpha + \int_{\mathcal{N}} |f(\alpha)|^s\,d\alpha\right). \end{aligned} \tag{5.3}$$

§6. *The treatment of the minor arcs.* We first obtain a refined form of Heath-Brown's analogue of Weyl's inequality (1.3). We use the following lemma.

LEMMA 4 (Heath-Brown [3], Lemma 6). *Let $|\alpha-(a/q)|=\bar{z}$ with $(a,q)=1$ and $\bar{z}\leqslant 1/q^2$. Then, for $H\geqslant 1$, $\Delta>0$, and any real μ,*

$$\#\{h\colon 1\leqslant h\leqslant H,\ \|\alpha h-\mu\|\leqslant\Delta\}\leqslant 8(1+(q\bar{z})^{-1}\Delta)(1+q\bar{z}H)$$

and

$$\#\{h\colon 1\leqslant h\leqslant H,\ \|\alpha h-\mu\|\leqslant\Delta\}\leqslant 4(1+q\Delta)(1+q^{-1}H).$$

We deduce from Lemma 4 that if $\bar{z}\leqslant 1/q^2$ and $(a,q)=1$,

$$\#\{h\colon 1\leqslant h\leqslant H,\ \|\alpha h-\mu\|\leqslant\Delta\}\ll 1+q\bar{z}H+(q\bar{z})^{-1}\Delta+H\Delta$$

and that

$$\#\{h\colon 1\leqslant h\leqslant H,\ \|\alpha h-\mu\|\leqslant\Delta\}\ll 1+q^{-1}H+q\Delta+H\Delta.$$

It follows that

$$\#\{h: 1 \leqslant h \leqslant H, \|\alpha h - \mu\| \leqslant \Delta\} \ll 1 + H\Delta + q\Delta Z + q^{-1}HZ^{-1}$$

where $Z = \max(1, (H\bar{z}/\Delta))$. Thus, for $Z = \max(1, P^k\bar{z})$,

$$\#\{h: h \leqslant \kappa P^{k-3}, \|\alpha h - mP^{-3}\| \leqslant P^{-3}\} \ll qZP^{-3} + P^{k-6} + q^{-1}Z^{-1}P^{k-3}. \tag{6.1}$$

Since

$$\sum_{h=1}^{\kappa P^{k-3}} T(\alpha h)^6 \ll \max_m \#\{h \leqslant \kappa P^{k-3}: \|\alpha h - mP^{-3}\| \leqslant P^{-3}\} \sum_{m=1}^{P^3} T(mP^{-3})^6,$$

from Lemma 3 we conclude that

$$\sum_{h=1}^{\kappa P^{k-3}} T(\alpha h)^6 \ll P^7(qZP^{-3} + P^{k-6} + q^{-1}Z^{-1}P^{k-3}).$$

Therefore, from (4.4),

$$f(\alpha) \ll P(PZ^{-1}q^{-1} + P^{-2} + qZP^{1-k})^{\frac{4}{3}2^{-k}}(\log P)^{\mathscr{W}(k)+\varepsilon}$$

where

$$\mathscr{W}(k) = \tfrac{4}{3}2^{-k}(5(k-3)^{6/5} - 6k + 19).$$

It follows from (only) using the second bound in Lemma 4 that whenever $|\alpha - (a/q)| \leqslant 1/q^2$ with $(a, q) = 1$,

$$f(\alpha) \ll P(Pq^{-1} + P^{-2} + qP^{1-k})^{\frac{4}{3}2^{-k}}(\log P)^{\mathscr{W}(k)+\varepsilon}. \tag{6.2}$$

To treat the minor arcs we first note that $s = \frac{3}{8}2^k + 2^{k-1}$ so

$$\int_{\mathfrak{m}} |f(\alpha)|^s d\alpha \ll \sup_{\alpha \in \mathfrak{m}} |f(\alpha)|^{\frac{3}{8}2^k} \int_{\mathscr{U}} |f(\alpha)|^{2^{k-1}} d\alpha. \tag{6.3}$$

From the definition of $\mathfrak{m}$ we deduce from (6.2) that for $\alpha \in \mathfrak{m}$

$$f(\alpha) \ll P^{1-\frac{8}{3}2^{-k}}(\log P)^{\mathscr{W}(k)+\varepsilon}. \tag{6.4}$$

To overcome the positive power of the logarithm in (6.4) we appeal to Theorem B of Vaughan [7] with subsequent remarks to obtain, with $l = k-1$,

$$\int_0^1 |f(\alpha)|^{2^{k-1}} d\alpha \ll P^{2^{k-1}-(k-1)}(\log P)^{\varepsilon - \frac{1}{2}(k-1)(k-2) + \chi(k)(d(k)-1)}. \tag{6.5}$$

Combining this with (6.3) and (6.4) we conclude that

$$\int_{\mathfrak{m}} |f(\alpha)|^s d\alpha \ll P^{s-k}(\log P)^{\Psi(k)+\varepsilon} \tag{6.6}$$

where

$$\Psi(k) = \tfrac{5}{2}(k-3)^{6/5} - \tfrac{1}{2}k^2 - \tfrac{3}{2}k + \tfrac{17}{2} + \chi(k)(d(k)-1).$$

For $k \geqslant 6$, $\Psi(k)$ is negative: $\Psi(6) = -6{\cdot}157\ldots$, $\Psi(7) = -13{\cdot}304\ldots$, $\Psi(8) = -15{\cdot}253\ldots$, and $\Psi(9) = -24{\cdot}035\ldots$. It suffices to treat the integral over the pruned arcs to establish Theorem 2.

§7. *A partition of the pruned arcs.* We construct two subsets of $\mathcal{N}$ corresponding to rational approximations to α as in (5.1) with q large and q small. To this end we set

$$I(q, z) = \bigcup{}^{*} \left\{ \alpha \in \mathcal{U} : z - P^{-k} \leqslant \left| \alpha - \frac{a}{q} \right| \leqslant 2z \right\}$$

where the * indicates that the union is taken over all a with $1 \leqslant a \leqslant q$ and $(a, q) = 1$. Clearly $I(q, z)$ and $I(q, 2z)$ are not disjoint. Let $i_0 = i_0(q)$ be the smallest integer i with

$$2^i \left(P^{-k} + \frac{P}{2kqn} \right) \geqslant \frac{1}{2qP^{k-3}}.$$

Define, for $0 \leqslant i < i_0$,

$$Z_i(q) = 2^i \left(P^{-k} + \frac{P}{2kqn} \right)$$

and let

$$Z_{i_0}(q) = \frac{1}{2qP^{k-3}}.$$

If we set

$$\mathcal{N}_1 = \bigcup_{1 \leqslant q \leqslant P} \; \bigcup_{0 \leqslant i \leqslant i_0} I(q, Z_i(q)),$$

then $\mathcal{N}_1$ includes all $\alpha \in \mathcal{U}$ that satisfy (5.1) with $1 \leqslant q \leqslant P$ that do not lie in $\mathcal{M}$. We define $j_0 = j_0(q)$ to be the smallest integer j such that

$$2^j P^{-k} \geqslant \frac{1}{2qP^{k-3}},$$

and subsequently let, if $0 \leqslant j < j_0$,

$$Z_j'(q) = 2^j P^{-k}$$

and set

$$Z_{j_0}'(q) = \frac{1}{2qP^{k-3}}.$$

We next define

$$\mathcal{N}_2 = \bigcup_{P < q < \frac{1}{4}P^3} \; \bigcup_{0 \leqslant j \leqslant j_0} I(q, Z_j'(q)).$$

It follows that $\mathcal{N}_2$ includes all $\alpha \in \mathcal{U}$ that satisfy (5.1) with q in the intermediate range $P < q < \frac{1}{4}P^3$. Thus

$$\int_{\mathcal{N}} |f(\alpha)|^s d\alpha \ll \sum_{1 \leqslant q \leqslant P} \sum_{0 \leqslant i \leqslant i_0} \int_{I(q, Z_i(q))} |f(\alpha)|^s d\alpha$$

$$+ \sum_{P < q < \frac{1}{4}P^3} \sum_{0 \leqslant j \leqslant j_0} \int_{I(q, Z_j'(q))} |f(\alpha)|^s d\alpha.$$

For each of the two sums on q we break up the range of q into dyadic blocks. For convenience we refer to the first sum above by subscript 1 and the second sum with, naturally, subscript 2. There are $O(\log P)$ such blocks for each sum. It follows that there exist Q_1 and Q_2 such that

$$\int_{\mathcal{N}} |f(\alpha)|^s d\alpha \ll \log P \sum_{Q_1 < q \leqslant 2Q_1} \sum_{0 \leqslant i \leqslant i_0} \int_{I(q, Z_i(q))} |f(\alpha)|^s d\alpha$$

$$+ \log P \sum_{Q_2 < q \leqslant 2Q_2} \sum_{0 < j \leqslant j_0} \int_{I(q, Z_j'(q))} |f(\alpha)|^s d\alpha. \qquad (7.1)$$

We next make each range for $Z_i(q)$ and $Z_j'(q)$ independent of q by annealing our structure. Let i_1 be the smallest integer i with

$$2^i\left(P^{-k} + \frac{P}{4kQ_1 n}\right) \geqslant \frac{1}{2Q_1 P^{k-3}}$$

and define, for $0 \leqslant i < i_1$,

$$\bar{Z}_i(q) = 2^i\left(P^{-k} + \frac{P}{4kQ_1 n}\right)$$

and set

$$\bar{Z}_{i_1}(q) = \frac{1}{2Q_1 P^{k-3}}.$$

For the sum on j we define j_1 to be the smallest integer j that satisfies

$$2^j P^{-k} \geqslant \frac{1}{2Q_2 P^{k-3}}$$

and let

$$\bar{Z}_j'(q) = 2^j P^{-k}$$

if $0 \leqslant j < j_1$ and set

$$\bar{Z}_{j_1}'(q) = \frac{1}{2Q_2 P^{k-3}}.$$

(Clearly $\bar{Z}'_{j_1}(q) > Z'_{j_0}(q)$.) This newly generated cover of $|\alpha-(a/q)|$ for $Q_v<q\leqslant 2Q_v$ $(v=1,2)$ is larger than the former given implicitly in (7.1). We rewrite (7.1), inverting the orders of summation, to find that

$$\int_{\mathcal{N}} |f(\alpha)|^s d\alpha \ll \log P \sum_{0\leqslant i\leqslant i_1} \sum_{Q_1<q\leqslant 2Q_1} \int_{I(q,\bar{Z}_i(q))} |f(\alpha)|^s d\alpha$$
$$+\log P \sum_{0\leqslant j\leqslant j_1} \sum_{Q_2<q\leqslant 2Q_2} \int_{I(q,\bar{Z}'_j(q))} |f(\alpha)|^s d\alpha.$$

Since i_1 and j_1 are $O(\log P)$ we conclude that there exist Z_1 and Z_2 such that

$$\int_{\mathcal{N}} |f(\alpha)|^s d\alpha \ll (\log P)^2 \max_{v} \sum_{Q_v<q\leqslant 2Q_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha \tag{7.2}$$

where

$$Q_1 \ll P, \qquad P \ll Q_2 \ll P^3, \qquad \frac{P^{1-k}}{Q_1} \ll Z_1 \ll \frac{P^{3-k}}{Q_1}, \qquad P^{-k} \ll Z_2 \ll \frac{P^{3-k}}{Q_2}. \tag{7.3}$$

In the next section we employ the fact that the $I(q, Z_v)$ are pairwise disjoint (for v fixed). To establish this assertion suppose α is in both $I(q_1, Z_v)$ and $I(q_2, Z_v)$ where $q_1 \neq q_2$. Then

$$\frac{1}{q_1q_2} \leqslant \left|\frac{a_1}{q_1}-\frac{a_2}{q_2}\right| \leqslant \left|\alpha-\frac{a_1}{q_1}\right| + \left|\alpha-\frac{a_2}{q_2}\right| \leqslant 4Z_v \leqslant \frac{2}{Q_vP^{k-3}}$$

from which it follows that

$$P^{k-3}Q_v \leqslant 2q_1q_2.$$

Since $q_1 \leqslant 2Q_v$ we obtain a contradiction as $q_2 < \frac{1}{4}P^3$ on $\mathcal{N}$.

§8. *The small values case.* Following (4.3) and (7.2), we need to bound

$$\sum_{h=1}^{\kappa P^{k-3}} T(\alpha h)^3 \tag{8.1}$$

for those $\alpha \in I(q, Z_v)$. We cannot, unfortunately, evaluate the integral over $\mathcal{N}$ in (5.3) in a direct manner. Instead, we consider a bifurcation contingent upon the size of the quantity in (8.1). Suppose that this function attains its maximum value for $\alpha \in I(q, Z_v)$ at α_q. Let Υ_v denote those values of q with $Q_v<q\leqslant 2Q_v$ for which

$$\sum_{h=1}^{\kappa P^{k-3}} T(\alpha_q h)^3 \geqslant P^{k-1}(\log P)^2(\log\log P). \tag{8.2}$$

For any $\alpha \in I(q, Z_v)$ where $q \notin \Upsilon_v$ (where we are henceforth assuming that $Q_v < q \leqslant 2Q_v$),

$$\sum_{h=1}^{\kappa P^{k-3}} T(\alpha_q h)^3 \ll P^{k-1}(\log P)^{2+\varepsilon}$$

so that, from (4.3),

$$f(\alpha)^{2^{k-3}} \ll P^{2^{k-3}-\frac{1}{3}}(\log P)^{\frac{2}{3}(k-3)^{3/2}-k+4+\varepsilon}. \tag{8.3}$$

We first note the trivial bound (after (7.2))

$$\int_{\mathcal{N}} |f(\alpha)|^s d\alpha \ll (\log P)^2 \max_v \left(\sum_{q \in \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha + \sum_{q \notin \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha \right). \tag{8.4}$$

We now deal with the second sum in (8.4) by using (8.3) and (6.5). Set

$$\mathscr{A}(k) = 2(k-3)^{3/2} - 3k + 14.$$

Since it was demonstrated in the previous section that the $I(q, Z_v)$ are pairwise disjoint,

$$\begin{aligned}(\log P)^2 \sum_{q \notin \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha &\ll P^{\frac{3}{8}2^k-1}(\log P)^{\mathscr{A}(k)+\varepsilon} \sum_{q \notin \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^{2^{k-1}} d\alpha \\ &\ll P^{\frac{3}{8}2^k-1}(\log P)^{\mathscr{A}(k)+\varepsilon} \int_{\mathcal{U}} |f(\alpha)|^{2^{k-1}} d\alpha \\ &\ll P^{s-k}(\log P)^{\eta(k)+\varepsilon}\end{aligned} \tag{8.5}$$

where

$$\eta(k) = 2(k-3)^{3/2} - \tfrac{1}{2}k^2 - \tfrac{3}{2}k + 13 + \chi(k)(d(k)-1). \tag{8.6}$$

A direct calculation shows that $0 > \eta(k) > \Psi(k)$ when $k \geqslant 6$. (Explicit values of $\eta(k)$ are given in Section 1.)

To wit, from (5.2), (5.3), (6.6), (8.4) and (8.5),

$$\begin{aligned} r_{s,k}(n) = \mathfrak{S}_{s,k}(n) \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{(s/k)-1} + O(n^{(s/k)-1}(\log n)^{\varepsilon+\eta(k)}) \\ + O\left((\log P)^2 \max_v \sum_{q \in \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha \right). \end{aligned} \tag{8.7}$$

§9. *A large values sieve.* The goal of this section is to show that there cannot be "many" q with $q \in \Upsilon_v$. From (4.3) we obtain

$$\sum_{q \in \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha \ll \sum_{q \in \Upsilon_v} V(q) \int_{I(q,Z_v)} |f(\alpha)|^{\frac{1}{8}2^k} d\alpha \tag{9.1}$$

where

$$V(q) = P^{\frac{6}{8}2^k - 6} + P^{\frac{6}{8}2^k - 2k + \varepsilon}\left(\sum_{h=1}^{\kappa P^{k-3}} T(\alpha_q h)^3\right)^2.$$

Define

$$S_v = \sum_{q \in \Upsilon_v} \left(\sum_{h=1}^{\kappa P^{k-3}} T(\alpha_q h)^3\right)^2$$

and, as D and T run over (appropriate) powers of 2, let

$$\Xi(D, T, q) = \{h \in [1, \kappa P^{k-3}] : D < (h, q) \leqslant 2D,\ T < T(\alpha_q h) \leqslant 2T\}.$$

(Trivially, $T(\alpha_q h) \geqslant 1$.) Thus, for some D_v and T_v,

$$\sum_{h=1}^{\kappa P^{k-3}} T(\alpha_q h)^3 \ll (\log P)^2 T_v^3 \# \Xi(D_v, T_v, q). \tag{9.2}$$

In the other direction (8.2) holds whence

$$\# \Xi(D_v, T_v, q) \gg P^{k-1} T_v^{-3} \log \log P. \tag{9.3}$$

Observing that

$$\begin{aligned} \# \Xi(D_v, T_v, q) &\leqslant \# \{h \leqslant \kappa P^{k-3} : (h, q) > D_v\} \\ &\ll \sum_{\substack{d \mid q \\ d > D_v}} \# \{h \leqslant \kappa P^{k-3} : d \mid h\} \\ &\ll d(q) P^{k-3} D_v^{-1} \\ &\ll P^{k-3+\varepsilon} D_v^{-1} \end{aligned}$$

(a bound that can be worse than trivial), we conclude from (9.2) that

$$S_v \ll P^{k-3+\varepsilon} T_v^6 D_v^{-1} \sum_{q \in \Upsilon_v(D_v, T_v)} \# \Xi(D_v, T_v, q) \tag{9.4}$$

for a suitable subset $\Upsilon_v(D_v, T_v)$ of Υ_v. We next consider

$$S_v' = \sum_{q \in \Upsilon_v(D_v, T_v)} \# \Xi(D_v, T_v, q).$$

Since the treatment of S_v' that we need is (essentially) the same as that given by Heath-Brown [3], Section 3, we suppress the details here (the argument relies upon (6.1)). To strengthen our rhetoric, we remark upon an improvement arising from the precision of Lemma 3:

$$S_v' \ll P^{4+\varepsilon} D_v T_v^{-6} (Q_v Z_v)^{-1} + \# \Upsilon_v(D_v, T_v) P^{k-\frac{9}{2}} \left(\sum_{m=1}^{P^3} \frac{T(mP^{-3})^6}{T_v^6}\right)^{1/2}.$$

From (9.3),

$$S_v' \gg \# \Upsilon_v(D_v, T_v) P^{k-1} T_v^{-3} \log \log P$$

hence, we deduce after Lemma 3 that

$$S_v' \ll P^{4+\varepsilon} D_v T_v^{-6} (Q_v Z_v)^{-1}.$$

Thus, from (9.4),

$$S_v \ll P^{k+1+\varepsilon}(Q_v Z_v)^{-1}. \tag{9.5}$$

§10. *The conclusion of the proof of Theorem* 2. We first observe that if $\bar{k} = 2[\frac{1}{2}(k+1)]$ then $\frac{1}{8}2^k > \bar{k} \geqslant k$ for all $k \geqslant 6$. Additionally,

$$\int_{I(q,Z_v)} |f(\alpha)|^{\frac{1}{8}2^k} d\alpha \ll \sup_{\alpha \in I(q,Z_v)} |f(\alpha)|^{\frac{1}{8}2^k - \bar{k}} \int_{I(q,Z_v)} |f(\alpha)|^{\bar{k}} d\alpha$$

$$\ll \sup_{\alpha \in I(q,Z_v)} |f(\alpha)|^{\frac{1}{8}2^k - \bar{k}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-2Z_v}^{2Z_v} \left| f\left(\frac{a}{q} + \beta\right) \right|^{\bar{k}} d\beta. \tag{10.1}$$

We now give a very useful mean value result.

LEMMA 5 (Health-Brown [3], Lemma 8). *Let h be a fixed positive integer such that* $2h \geqslant k$. *Then*

$$\sum_{a=1}^{q} \int_{-2z}^{2z} \left| f\left(\frac{a}{q} + \beta\right) \right|^{2h} d\beta \ll qzP^{2h-1+\varepsilon} + P^{2h-k+\varepsilon}.$$

Lemma 5 may be refined so the P^{ε} terms are quantified as slowly growing functions of q—but, as we shall soon see, this is an unnecessary complication.

In order to treat the supremum in (10.1), we provide one last lemma.

LEMMA 6. *Suppose that* $|\alpha - (a/q)| \leqslant 2/(qP^{k-3})$ *where* $(a,q)=1$ *and* $1 \leqslant q < \frac{1}{4}P^3$. *Suppose, further, there is a positive constant A such that whenever* $1 \leqslant q \leqslant P$ *we have*

$$\left| \alpha - \frac{a}{q} \right| > \frac{A}{qP^{k-1}}.$$

Then

$$f(\alpha) \ll_A P^{1-2^{1-k}+\varepsilon}.$$

Proof. Clearly $|\alpha - (a/q)| \leqslant 1/q^2$. For $q > P$, the desired result is thus a direct consequence of Weyl's inequality (see [5], Lemma 2.4). We suppose, therefore, that $1 \leqslant q \leqslant P$. By Dirichlet's theorem, we can find integers r and s with $(r,s)=1$ and $1 \leqslant s \leqslant P^{k-1}/A$ satisfy

$$\left| \alpha - \frac{r}{s} \right| \leqslant \frac{A}{sP^{k-1}}.$$

Since a/q does not satisfy this inequality, $a/q \neq r/s$ hence

$$\frac{1}{sq} \leqslant \left| \frac{r}{s} - \frac{a}{q} \right| \leqslant \left| \alpha - \frac{r}{s} \right| + \left| \alpha - \frac{a}{q} \right| \leqslant \frac{A}{sP^{k-1}} + \frac{2}{qP^{k-3}}.$$

It follows that $s \gg P^{k-3}$. Thus

$$\left|\alpha - \frac{r}{s}\right| \leqslant \frac{1}{s^2}$$

where $(r, s) = 1$ and $P^{k-3} \ll s \ll_A P^{k-1}$. The desired result follows from Weyl's inequality.

We now address the supremum in (10.1)—if $\alpha \in I(q, Z_v)$,

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{Q_v P^{k-3}} \leqslant \frac{2}{q P^{k-3}}$$

with $(a, q) = 1$. If $q \leqslant P$ (*i.e.*, $v = 1$),

$$Z_1 \gg \frac{1}{q P^{k-1}}$$

by construction. From (10.1) and Lemmata 5 and 6 we infer that

$$\int_{I(q,Z_v)} |f(\alpha)|^{\frac{1}{8}2^k} d\alpha \ll P^{(1-2^{1-k})(\frac{1}{8}2^k - \bar{k}) + \varepsilon}(Q_v Z_v P^{\bar{k}-1} + P^{\bar{k}-k}).$$

(We have dropped the restriction $(a, q) = 1$ to majorize the right-hand side of (10.1).) Combining this with (9.1) we find that

$$\sum_{q \in \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha \ll (P^{-6} Q_v + P^{-2k} S_v) P^{s - \frac{1}{4} + 2^{1-k}\bar{k} + \varepsilon}(Q_v Z_v P^{-1} + P^{-k}).$$

On the first factor on the right we use the trivial bound $Q_v \ll P^3$. Then, from (9.5),

$$\sum_{q \in \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha \ll P^{s-k-\frac{1}{4}+2^{1-k}\bar{k}+\varepsilon}(Q_v Z_v P^{k-4} + 1 + P^{1-k}(Q_v Z_v)^{-1}). \qquad (10.2)$$

Since (7.3) provides that

$$P^{1-k} \ll Q_v Z_v \ll P^{3-k},$$

the second factor on the right-hand side of (10.2) is bounded. We conclude that

$$\sum_{q \in \Upsilon_v} \int_{I(q,Z_v)} |f(\alpha)|^s d\alpha \ll P^{s-k-\delta+\varepsilon}$$

where $\delta = \frac{1}{4} - 2^{1-k}\bar{k}$. From (8.7) we consequently obtain

$$r_{s,k}(n) = \mathfrak{S}_{s,k}(n) \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} n^{(s/k)-1}$$
$$+ O(n^{(s/k)-1}(\log n)^{\varepsilon + \eta(k)}) + O(n^{(s/k)-1+\varepsilon-\delta/k}) \qquad (10.3)$$

where $\eta(k)$ is given in (8.6). A simple check verified that $\delta > 0$ whence the latter error term in (10.3) is subsumed by the former—and, *mirabile dictu*, Theorem 2 is proved.

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11P05: *NUMBER THEORY; Additive number theory, partitions; Waring's problem.*

Received on the 6th of July, 1993.