THE VALUES OF A TRIGONOMETRICAL POLYNOMIAL AT WELL SPACED POINTS

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1. A study of the recent papers of Roth† and Bombieri‡ on the large sieve has led us to the following simple result on the sum of the squares of the absolute values of a trigonometric polynomial at a finite set of points.

THEOREM 1. *Let $a_{-N}, \ldots, a_N$ be any complex numbers and let*

$$S(x) = \sum_{n=-N}^{N} a_n e(nx), \tag{1}$$

where $e(\theta) = e^{2\pi i\theta}$. Let $x_1, x_2, \ldots, x_R$ ($R \geqslant 2$) be any real numbers, and define§

$$\delta = \min_{j \neq k} \| x_j - x_k \|. \tag{2}$$

Then

$$\sum_{r=1}^{R} | S(x_r)|^2 \leqslant 2{\cdot}2 \max(\delta^{-1}, 2N) \sum_{n=-N}^{N} |a_n|^2. \tag{3}$$

The numerical constant 2·2 arises from the use of a particular auxiliary function and could be improved by using other functions or a combination of functions¶. It would be of interest to know if there is any simple and best possible inequality which includes (3).

We prove Theorem 1 (very simply) in §2, and in §3 we deduce slightly sharper versions of Bombieri's Theorems 2 and 3. Theorem 3 was the basis for Bombieri's work on the average of the error term in the prime number theorem for arithmetic progressions.

If $N\delta$ is small, it is possible to prove a result which is stronger than (3), apart from the numerical constant. This result (Theorem 4 of §4) attaches greater weight to those points x_r which are well separated from their neighbours. Let

$$\delta_r = \min_{j \neq r} \| x_j - x_r \| ; \tag{4}$$

then the result is that

$$\sum_{r=1}^{R} \min(1, N\delta_r) | S(x_r)|^2 \leqslant 6 \cdot 1N \sum_{-N}^{N} | a_n |^2. \tag{5}$$

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† *Mathematika*, 12 (1965), 1–9.

‡ *Mathematika*, 12 (1965), 201–225.

§ We denote by $\|\theta\|$ the distance from θ to the nearest integer.

¶ If δN is sufficiently small, the factor on the right of (3) can be improved to $(1+\epsilon)\,\delta^{-1}$ by replacing $\psi(x)$ in (8) by the function which is 1 for $\|x\| < \eta$ and 0 otherwise. We are indebted to Dr. H. Stark for the remark that the constant 2·2 cannot be replaced by a number less than $2 - 8\pi^{-2}$. This is shown by the following example (taking N to be large): $a_{-N} = \ldots = a_N = 1$; $R = 2N$ and $x_1, \ldots, x_R$ are the numbers 0, $\frac{1}{2}$, $\pm \frac{2n+1}{2(2N+1)}$ ($n = 1, \ldots, N-1$), so that $\delta^{-1} = 2N+1$.

The justification for our describing this as a stronger result lies in the fact that $\min(1, N\delta_r) \geqslant N\delta$ if $N\delta \leqslant 1$. As corollaries to Theorem 4 we have modified forms of Theorems 2 and 3.

Finally, in §5 we point out a modified form of (3) containing $|S(x_r)|^k$, where $k \geqslant 2$.

2. *Proof of Theorem* 1. For $0 < \eta \leqslant \frac{1}{2}\delta$, let†

$$\psi(x) = \sum_{-\infty}^{\infty} b_n e(nx)$$

be any real function of integrable square which vanishes for $\|x\| > \eta$. Let

$$T(x) = \sum_{-N}^{N} b_n^{-1} a_n e(nx).$$

Then, by the usual " convolution " formula,

$$S(x) = \int_0^1 \psi(y)\, T(x-y)\, dy = \int_{-\eta}^{\eta} \psi(y)\, T(x-y)\, dy,$$

whence

$$|S(x)|^2 \leqslant \left(\int_{-\eta}^{\eta} \psi^2(y)\, dy\right)\left(\int_{-\eta}^{\eta} |T(x-y)|^2\, dy\right).$$

On replacing x by x_r, and summing over r, we obtain

$$\sum_{r=1}^{R} |S(x_r)|^2 \leqslant \left(\int_{-\eta}^{\eta} \psi^2(y)\, dy\right)\int_0^1 |T(z)|^2\, dz, \tag{6}$$

since the intervals $(x_r - \eta, x_r + \eta)$ do not overlap. Since

$$\int_{-\eta}^{\eta} \psi^2(y)\, dy = \int_0^1 \psi^2(y)\, dy = \sum_{-\infty}^{\infty} b_n^2,$$

we deduce that

$$\sum_{r=1}^{R} |S(x_r)|^2 \leqslant \left(\sum_{-\infty}^{\infty} b_n^2\right)\left(\sum_{-N}^{N} b_n^{-2} |a_n|^2\right). \tag{7}$$

We now take

$$\psi(x) = \sum_{-\infty}^{\infty} \left(\frac{\sin n\pi\eta}{n\pi\eta}\right)^2 e(nx) = \begin{cases} \eta^{-1}(1 - \eta^{-1}\|x\|) & \text{if } \|x\| \leqslant \eta, \\ 0 & \text{if } \|x\| \geqslant \eta. \end{cases} \tag{8}$$

For this function,

$$\sum_{-\infty}^{\infty} b_n^2 = \int_0^1 \psi^2(x)\, dx = \tfrac{2}{3}\eta^{-1},$$

and for $|n| \leqslant N$ we have

$$b_n^{-2} \leqslant b_N^{-2} = \left(\frac{N\pi\eta}{\sin N\pi\eta}\right)^4$$

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† Functions of this type played an important part in Roth's argument.

provided $N\eta \leqslant \frac{1}{2}$. Thus

$$\sum_{r=1}^{R} |S(x_r)|^2 \leqslant \tfrac{2}{3}\pi N \frac{(N\pi\eta)^3}{(\sin N\pi\eta)^4} \sum_{-N}^{N} |a_n|^2.$$

The function of η on the right is least when $N\pi\eta = \theta_0$, where $\theta_0 = 0{\cdot}845\ldots$, and its value then is $4{\cdot}34 \ldots N$. If the value of η so determined satisfies $\eta \leqslant \frac{1}{2}\delta$, we use it and get an upper bound $4{\cdot}34 \ldots N$. If not, we take $\eta = \frac{1}{2}\delta$, and get an upper bound

$$\tfrac{2}{3}\eta^{-1}\left(\frac{N\pi\eta}{\sin N\pi\eta}\right)^4 \leqslant \tfrac{2}{3}(2\delta^{-1})\left(\frac{\theta_0}{\sin\theta_0}\right)^4 < 2{\cdot}2\delta^{-1}.$$

In either case the condition $N\eta \leqslant \frac{1}{2}$ is amply satisfied.

3. THEOREM 2. *Let the a_n be any complex numbers, and let*

$$S(x) = \sum_{n=Y+1}^{Y+U} a_n e(nx).$$

Then

$$\sum_{q \leqslant X} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |S(a/q)|^2 \leqslant 2{\cdot}2 \max(U, X^2) \sum_{Y+1}^{Y+U} |a_n|^2.$$

Proof. When the numbers x_r are the rational numbers a/q with $q \leqslant X$, we have $\delta \geqslant X^{-2}$. The result follows from Theorem 1 on taking $N = \frac{1}{2}U$ or $\frac{1}{2}(U-1)$, and putting $n = n' + Y + 1 + N$, so that n' goes from $-N$ to N or $N-1$.

THEOREM 3. *For any character χ to the modulus q, let*

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m)\, e(m/q),$$

and let the a_n be any complex numbers. Then

$$\sum_{q \leqslant X} \frac{1}{q} \sum_{\chi} |\tau(\chi)|^2 \left| \sum_{Y+1}^{Y+U} \chi(n)\, a_n \right|^2 \leqslant 2{\cdot}2 \max(U+X, X^2) \sum_{Y+1}^{Y+U} d(n) |a_n|^2.$$

Proof. When $(n, q) = 1$ we have

$$\tau(\bar{\chi})\chi(n) = \sum_{\substack{m=1 \\ (m,q)=1}}^{q} \bar{\chi}(m)\, e(mn/q),$$

whence

$$\tau(\bar{\chi}) \sum_{Y+1}^{Y+U} \chi(n)\, a_n = \sum_{\substack{m=1 \\ (m,q)=1}}^{q} \bar{\chi}(m) \sum_{\substack{n=Y+1 \\ (n,q)=1}}^{Y+U} a_n\, e(mn/q).$$

On multiplying this by its complex conjugate and summing over χ, we obtain

$$\sum_{\chi} |\tau(\chi)|^2 \left| \sum_{Y+1}^{Y+U} \chi(n)\, a_n \right|^2 = \phi(q) \sum_{\substack{m=1 \\ (m,q)=1}}^{q} |S_q(m/q)|^2,$$

where

$$S_q(x) = \sum_{\substack{n=Y+1 \\ (n,q)=1}}^{Y+U} a_n e(nx).$$

We have

$$S_q(x) = \sum_{d \mid q} \mu(d) \sum_{\substack{n=Y+1 \\ d \mid n}}^{Y+U} a_n e(nx),$$

whence, by Cauchy's inequality,

$$|S_q(x)|^2 \leqslant \left(\sum_{d \mid q} \mu^2(d)/d\right)\left(\sum_{d \mid q} d \left|\sum_{\substack{n=Y+1 \\ d \mid n}}^{Y+U} a_n e(nx)\right|^2\right).$$

Since

$$\sum_{d \mid q} \mu^2(d)/d = \prod_{p \mid q}\left(1+\frac{1}{p}\right) \leqslant q/\phi(q),$$

we obtain

$$\phi(q) \sum_{\substack{m=1 \\ (m,q)=1}}^{q} |S_q(m/q)|^2 \leqslant q \sum_{d \mid q} d \sum_{\substack{m=1 \\ (m,q)=1}}^{q} \left|\sum_{\substack{n=Y+1 \\ d \mid n}}^{Y+U} a_n e(nm/q)\right|^2.$$

Thus the sum under consideration in the theorem does not exceed

$$\sum_{d \leqslant X} d \sum_{\substack{q \leqslant X \\ d \mid q}} \sum_{\substack{m=1 \\ (m,q)=1}}^{q} \left|\sum_{\substack{n=Y+1 \\ d \mid n}}^{Y+U} a_n e(nm/q)\right|^2.$$

For each d we apply Theorem 1 to the sum over q and m in the same way as in the proof of Theorem 2. We take the numbers x_r of Theorem 1 to be the rationals a/q with $q \leqslant X$ and $d \mid q$, and this allows us to take $\delta \geqslant dX^{-2}$. We also put $n = dn'$, so that the range for n' is

$$U' \leqslant 1 + U/d.$$

We therefore get, for each d,

$$\sum_{\substack{q \leqslant X \\ d \mid q}} \sum_{\substack{m=1 \\ (m,q)=1}}^{q} \left|\sum_{\substack{n=Y+1 \\ d \mid n}}^{Y+U} a_n e(nm/q)\right|^2 \leqslant 2\cdot 2 \max(X^2 d^{-1}, 1 + Ud^{-1}) \sum_{\substack{n=Y+1 \\ d \mid n}}^{Y+U} |a_n|^2$$

$$\leqslant (2\cdot 2d^{-1}) \max(X^2, X+U) \sum_{\substack{n=Y+1 \\ d \mid n}}^{Y+U} |a_n|^2.$$

Multiplying by d and summing over d we get the result.

4. THEOREM 4. *With the hypotheses of Theorem* 1, *let*

$$\delta_r = \min_{j \neq r} \| x_j - x_r \|.$$

Then

$$\sum_{r=1}^{R} \min(1, N\delta_r) |S(x_r)|^2 \leqslant 6\cdot 1 N \sum_{-N}^{N} |a_n|^2.$$

Proof. We argue as in the proof of Theorem 1, but without supposing that $\eta \leqslant \frac{1}{2}\delta$. The modified form of (6) is

$$\sum_{r=1}^{R} \min(1, N\delta_r)|\,S(x_r)|^2 \leqslant \left(\int_{-\eta}^{\eta} \psi^2(y)\,dy\right)\int_0^1 F(z)|\,T(z)|^2\,dz,$$

where

$$F(z) = \sum_{z-\eta<x_r<z+\eta} \min(1, N\delta_r). \tag{9}$$

If the last sum contains no point x_r, or only one point x_r, we have $F(z) \leqslant 1$. Now suppose that the points occurring are $x_{r_1}, \ldots, x_{r_s}$, in increasing order. Since the intervals

$$(x_{r_1}, x_{r_1}+\delta_{r_1}),\ (x_{r_2}, x_{r_2}+\delta_{r_2}),\ \ldots,\ (x_{r_{s-1}}, x_{r_{s-1}}+\delta_{r_{s-1}})$$

are contained in $(z-\eta, z+\eta)$ and do not overlap, we have

$$\delta_{r_1}+\delta_{r_2}+\ldots+\delta_{r_{s-1}} \leqslant 2\eta.$$

Hence

$$F(z) \leqslant 1 + 2N\eta.$$

It follows that

$$\sum_{r=1}^{R} \min(1, N\delta_r)|\,S(x_r)|^2 \leqslant (1+2N\eta)\left(\sum_{-\infty}^{\infty} b_n{}^2\right)\left(\sum_{-N}^{N} b_n{}^{-2}|\,a_n|^2\right). \tag{10}$$

With the same function as before for $\psi(x)$, the right-hand side is

$$\leqslant (1+2N\eta)\,\tfrac{2}{3}\eta^{-1}\left(\frac{N\pi\eta}{\sin N\pi\eta}\right)^4 \sum_{-N}^{N} |\,a_n|^2,$$

provided $N\eta \leqslant \frac{1}{2}$. This (apart from the sum) is

$$\tfrac{2}{3}N\left(2+\frac{\pi}{\theta}\right)\left(\frac{\theta}{\sin\theta}\right)^4,$$

where $\theta = N\pi\eta$; and on taking $\theta = 0{\cdot}75$ we obtain an amount less than $6\cdot 1N$.

COROLLARY 1. *With the notation of Theorem* 2,

$$\sum_{q\leqslant X} \min\left(1, \frac{U}{2Xq}\right) \sum_{\substack{a=1\\(a,q)=1}}^{q} |\,S(a/q)|^2 \leqslant 3\cdot 1\;U \sum_{Y+1}^{Y+U} |\,a_n|^2.$$

This follows as before on taking $N = \frac{1}{2}U$ or $\frac{1}{2}(U-1)$ and on noting that the distance from a/q to the nearest rational number a'/q' with $q' \leqslant X$ is at least $1/(Xq)$.

COROLLARY 2. *With the notation of Theorem* 3,

$$\sum_{q\leqslant X} \frac{1}{q}\min\left(1, \frac{U}{2Xq}\right) \sum_{\chi} |\,\tau(\chi)|^2 \left|\sum_{Y+1}^{Y+U} \chi(n)\,a_n\right|^2 \leqslant 3\cdot 1(U+X) \sum_{Y+1}^{Y+U} d(n)|\,a_n|^2.$$

5. Suppose that $1/k+1/l=1$ and that $1<l\leqslant 2$. We can replace the inequality preceding (6) by

$$|S(x)|^k \leqslant \left(\int_{-\eta}^{\eta} |\psi(y)|^l\,dy\right)^{k/l} \left(\int_{-\eta}^{\eta} |T(x-y)|^k\,dy\right).$$

This gives

$$\sum_{r=1}^{R} |S(x_r)|^k \leqslant \left(\int_0^1 |\psi(y)|^l\,dy\right)^{k/l} \int_0^1 |T(z)|^k\,dz$$

By the Young–Hausdorff theorem†,

$$\left(\int_0^1 |T(z)|^k\,dz\right)^{1/k} \leqslant \left(\sum_{-N}^{N} b_n^{-l} |a_n|^l\right)^{1/l}.$$

If we use the same function ψ as before, then

$$\int_0^1 \psi^l(y)\,dy = \frac{2}{l+1}\,\eta^{-l+1},$$

and the conclusion is that

$$\sum_{r=1}^{R} |S(x_r)|^k \leqslant \left(\frac{2}{l+1}\,\eta^{-l+1}\right)^{k/l} \left(\frac{N\pi\eta}{\sin N\pi\eta}\right)^{2k} \left(\sum_{-N}^{N} |a_n|^l\right)^{k/l},$$

subject to $\eta\leqslant\frac{1}{2}\delta$ and $N\eta\leqslant\frac{1}{2}$.

Putting $N\pi\eta=\theta$, we see that the last expression is

$$\leqslant \frac{N\pi}{\theta}\left(\frac{\theta}{\sin\theta}\right)^{2k} \left(\sum_{-N}^{N} |a_n|^l\right)^{k/l}.$$

Choosing θ as in the proof of Theorem 1, but now as a function of k, we obtain

$$\sum_{r=1}^{R} |S(x_r)|^k \leqslant A\,k^{1/2} \max(\delta^{-1}, N) \left(\sum_{-N}^{N} |a_n|^l\right)^{k/l},$$

where A is an absolute constant.

The analogous generalization of Theorem 4 is

$$\sum_{r=1}^{R} \min(1, N\delta_r)\,|S(x_r)|^k \leqslant A\,k^{1/2}\,N \left(\sum_{-N}^{N} |a_n|^l\right)^{k/l}.$$

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† Zygmund, *Trigonometric Series* (2nd ed.), II, 101.