ON THE ZEROS OF EPSTEIN'S ZETA FUNCTION

H. M. STARK

1. *Introduction.* Let $Q(x, y) = ax^2 + bxy + cy^2$ be a positive definite quadratic form with discriminant $d = b^2 - 4ac$. The Epstein zeta function associated with Q is given by

$$\zeta(s, Q) = \tfrac{1}{2} \sum_{x,y}{}' Q(x, y)^{-s} \quad (\sigma > 1), \tag{1}$$

where Σ' means the sum is over all pairs (x, y) of integers not both zero, and as usual, $s = \sigma + it$. Except for a first order pole at $s = 1$, $\zeta(s, Q)$ can be continued throughout the complex plane and has a functional equation similar to that of the Riemann zeta function. If we put $Q_1(x, y) = x^2 + \frac{b}{a} xy + \frac{c}{a} y^2$, then we see that the zeros of $\zeta(s, Q)$ and $\zeta(s, Q_1)$ are identical. The discriminant of Q_1, d/a^2, will be important in discussing the zeros of $\zeta(s, Q)$. As in **[1]**, we let

$$k = \frac{\sqrt{|d|}}{2a}. \tag{2}$$

Potter and Titchmarsh **[2]** have shown that $\zeta(s, Q)$ has an infinity of zeros on the line $\sigma = \frac{1}{2}$. However, the analogue of the Riemann hypothesis is not always true for $\zeta(s, Q)$ since Bateman and Grosswald **[1]** have shown that $\zeta(s, Q)$ has a real zero between $\frac{1}{2}$ and 1 if $k > 7{\cdot}0556$. In fact, in the case where a, b, c are integers, d is a fundamental discriminant, and the class number $h(d) > 1$, Davenport and Heilbronn **[3]** had previously shown that $\zeta(s, Q)$ has an infinity of zeros in the half plane $\sigma > 1$ arbitrarily close to the line $\sigma = 1$. We prove here two complements to these results.

THEOREM 1. *There exists a number K such that if $k > K$ then all the zeros of $\zeta(s, Q)$ in the region $-1 < \sigma < 2$, $-2k \leqslant t \leqslant 2k$ are simple zeros; with the exception of two real zeros between 0 and 1, all are on the line $\sigma = \frac{1}{2}$.*

THEOREM 2. *Let $N(T, Q)$ denote the number of zeros of $\zeta(s, Q)$ in the region $-1 < \sigma < 2$, $0 \leqslant t \leqslant T$. If $k > K$ and $0 < T \leqslant 2k$, then*

$$N(T, Q) = \frac{T}{\pi} \log\left(\frac{kT}{\pi e}\right) + O\{\log^{1/3}(T+3)[\log\log(T+3)]^{1/6}\}.$$

The constant implied by "O" is independent of k.

Thus, for large k, the infinity of zeros of $\zeta(s, Q)$ off the line $\sigma = \frac{1}{2}$ found in certain cases by Davenport and Heilbronn are far removed from the σ axis; in any event, the "first" $\frac{4}{\pi} k \log k + O(k)$ complex zeros of $\zeta(s, Q)$ are on the line $\sigma = \frac{1}{2}$.

2. *Notation and results from other sources.* We use the notation $f(t) \ll g(t)$ to mean that there is a positive constant c such that $f(t) \leqslant cg(t)$. All constants implied by the $\ll$ and O notations are absolute constants independent of t, σ, and sufficiently large k. We make the convention that every equation or statement involving k should be interpreted as holding for sufficiently large k only.

[MATHEMATIKA **14** (1967), 47–55]

We will use the following three well-known estimates of the gamma function: for $t \geqslant 2$ and $-1 \leqslant \sigma \leqslant 2$,

$$1 \ll \frac{|\Gamma(s)|}{t^{\sigma-1/2}\, e^{-\pi t/2}} \ll 1, \tag{3}$$

for $t \geqslant 2$ and $-1 \leqslant \sigma \leqslant 2$,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log t + \frac{i\pi}{2} + O\left(\frac{1}{t}\right), \tag{4}$$

for $t > 0$,

$$\arg \Gamma(\tfrac{1}{2} + it) = t \log t - t + O(1). \tag{5}$$

We will also use some refined estimates of $\zeta(s)$ coming from Vinogradoff's method [**6**, part V and p. 226; see also **4**, sections 3.10, 3.11]: there is a positive constant A such that in the region

$$\sigma \geqslant 1 - \frac{A}{\log^{2/3} t (\log\log t)^{1/3}}, \quad t \geqslant 3,$$

$\zeta(s)$ has no zeros and

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll \log^{2/3} t (\log\log t)^{1/3}, \tag{6}$$

$$\left|\frac{1}{\zeta(s)}\right| \ll \log^{2/3} t (\log\log t). \tag{7}$$

The functional equation [**4**] for $\zeta(s)$ is

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)} \zeta(1-s). \tag{8}$$

We see from [**1**] that we can write

$$a^s \zeta(s, Q) = \zeta(2s) + k^{1-2s} \zeta(2s-1) \frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)} + h(s), \tag{9}$$

where $h(s)$ is an entire function. If we put

$$\begin{aligned} b(s) &= \left(\frac{k}{\pi}\right)^s \Gamma(s), \\ \alpha(s) &= a^s \zeta(s, Q)\, b(s), \\ f(s) &= \zeta(2s)\, b(s), \\ g(s) &= h(s)\, b(s), \end{aligned} \tag{10}$$

then (9) can be written as

$$\alpha(s) = f(s) + f(1-s) + g(s). \tag{11}$$

As shown in [**1**], $g(s)$ is an entire function and

$$g(s) = g(1-s) \tag{12}$$

([1] works with the function $H(s) = k^{-\frac{1}{2}}g(s)$). As a result,

$$\alpha(s) = \alpha(1-s), \tag{13}$$

which is the functional equation for $\zeta(s, Q)$. Thus both $\alpha(s)$ and $g(s)$ are real valued on the line $\sigma = \frac{1}{2}$. Clearly $\log f(s)$ is well defined in the region $\sigma \geqslant \frac{1}{2}$, $s \neq \frac{1}{2}$. We will have occasion to use $\arg f(s) = \operatorname{Im} \log f(s)$; we use the branch of $\arg f(s)$ which is 0 for real $s > \frac{1}{2}$.

Various estimates of $g(s)$ and $h(s)$ are known. For s near 1, the estimate in [1] is the best that has yet appeared, but for s well removed from the σ-axis, the estimate of [5] is better: for $n \geqslant 1$ and $\sigma \geqslant \frac{1}{2}$,

$$|h(s)| < 2\pi \left(\frac{2n}{2n-1}\right)^2 \left(\frac{2|s| + 2n - 1}{\pi k\sqrt{2}}\right)^{2n}. \tag{14}$$

Only the special case $Q(x, y) = x^2 + xy + \frac{p+1}{4}y^2$ of discriminant $-p$ was considered in [5], but the proof is the same for $Q(x, y) = x^2 + \frac{b}{a}xy + \frac{c}{a}y^2$.

3. *Preliminary estimates and lemmas.* If in (14) we take

$$n = \left[\frac{\log(10k)}{2\log\left(\frac{4\cdot 4}{4\cdot 1}\right)}\right] + 1,$$

then we see that for $\sigma \geqslant \frac{1}{2}$ and $|s| < 2k + 3$,

$$|h(s)| < 10\left(\frac{4\cdot 1\, k}{4\cdot 4 k}\right)^{\log(10k)/\log\left(\frac{4\cdot 4}{4\cdot 1}\right)}$$

$$= \frac{1}{k}. \tag{15}$$

For $\sigma \geqslant \sigma_1 = 1 + 1/\log k$,

$$|\zeta(s)| \leqslant \sum_{n=1}^{\infty} n^{-\sigma_1}$$

$$\ll \log k. \tag{16}$$

From (3) and (8), we then get for $t \geqslant 2$,

$$\left|\zeta\left(-\frac{1}{\log k} + it\right)\right| \ll (\log k)\, t^{\frac{1}{2} + (1/\log k)}.$$

For $-1/\log k \leqslant \sigma \leqslant 1 + 1/\log k$,

$$|\zeta(\sigma + 2i)| \ll 1.$$

As a result, the standard Phragmén–Lindelöf theorem for a strip says that for $-1/\log k \leqslant \sigma \leqslant 1 + 1/\log k$ and $t \geqslant 2$,

$$|\zeta(s)| \ll (\log k)\, t^{\frac{1}{2}[1 + (1/\log k) - \sigma]}. \tag{17}$$

LEMMA 1. *If* $\frac{1}{2} + (\log k)^{-7/8} \leqslant \sigma \leqslant 2$, $2 \leqslant t \leqslant 2k+1$, *or if* $\sigma = 2$, $0 \leqslant t \leqslant 2$, *then* $|f(s)| > |f(1-s)|$ *and* $|f(s)+f(1-s)| > |g(s)|$.

Proof. From (7), if s is in the above region,

$$|f(s)| = |b(s)|.|\zeta(2s)| > 2|b(s)|(\log k)^{-7/8}. \tag{18}$$

For $\frac{1}{2} + (\log k)^{-7/8} \leqslant \sigma \leqslant 1 + \dfrac{1}{2\log k}$ and $2 \leqslant t \leqslant 2k+1$, we see from (3) and (17) that

$$\begin{aligned}\left|\frac{f(1-s)}{b(s)}\right| &= \left|\frac{(k/\pi)^{1-s}\Gamma(1-s)\zeta(2-2s)}{(k/\pi)^s\Gamma(s)}\right| \\ &\leqslant (k/\pi)^{1-2\sigma} t^{1-2\sigma} (\log k)(2t)^{\frac{1}{2}[1+(1/\log k)-(2-2\sigma)]} \\ &\leqslant (\log k)\, k^{1-2\sigma} \\ &\leqslant (\log k) \exp[-2(\log k)^{1/8}] \\ &< (\log k)^{-7/8}.\end{aligned} \tag{19}$$

For $1 + \dfrac{1}{2\log k} \leqslant \sigma \leqslant 2$ and $t \geqslant 2$, we see from (3), (9) and (16) that

$$\begin{aligned}\left|\frac{f(1-s)}{b(s)}\right| &= \left|k^{1-2s}\zeta(2s-1)\frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}\right| \\ &\leqslant k^{1-2\sigma}(\log k)\, t^{-\frac{1}{2}} \\ &< (\log k)^{-7/8}.\end{aligned} \tag{20}$$

The result of (20) clearly holds for $\sigma = 2$, $0 \leqslant t \leqslant 2$ also. Lemma 1 now follows from (15), (18), (19) and (20).

LEMMA 2. *If* $\frac{1}{2} - (\log k)^{-7/8} \leqslant \sigma \leqslant \frac{1}{2} + (\log k)^{-7/8}$ *and* $2k \leqslant t \leqslant 2k+1$, *then*

$$\operatorname{Re}\left\{\frac{f'(s)}{f(s)}\right\} > \log k \text{ and } \left|\operatorname{Im}\left\{\frac{f'(s)}{f(s)}\right\}\right| < \log k.$$

Proof.

$$\frac{f'(s)}{f(s)} = \log\left(\frac{k}{\pi}\right) + \frac{\Gamma'(s)}{\Gamma(s)} + 2\frac{\zeta'(2s)}{\zeta(2s)}.$$

Thus,

$$\begin{aligned}\operatorname{Re}\left\{\frac{f'(s)}{f(s)}\right\} &> \log\left(\frac{k}{\pi}\right) + \log t - (\log t)^{7/8} - O\left(\frac{1}{t}\right) \\ &> \log k,\end{aligned}$$

and,

$$\begin{aligned}\left|\operatorname{Im}\left\{\frac{f'(s)}{f(s)}\right\}\right| &< \frac{\pi}{2} + (\log t)^{7/8} + O\left(\frac{1}{t}\right) \\ &< \log k.\end{aligned}$$

LEMMA 3. *There exists a number* T_0 *such that* $2k < T_0 < 2k+1$ *and*

$$\arg f(\tfrac{1}{2} + iT_0) \equiv 0 \pmod{2\pi}.$$

Thus $f(\frac{1}{2} + iT_0)$ *and* $f(\frac{1}{2} - iT_0)$ *are positive real numbers.*

Proof. For $2k \leqslant t \leqslant 2k+1$,

$$\begin{aligned} \frac{d}{dt} \arg f(\tfrac{1}{2} + it) &= \operatorname{Re}\left\{\frac{f'(\frac{1}{2}+it)}{f(\frac{1}{2}+it)}\right\} \\ &> \log k \\ &> 2\pi. \end{aligned}$$

The lemma follows.

LEMMA 4. *For* $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + (\log k)^{-7/8}$ *and* $t = T_0$,

$$|f(s)| \geqslant |f(1-s)| \text{ and } |f(s) + f(1-s)| > |g(s)|$$

with equality in the first part if and only if $\sigma = \frac{1}{2}$.

Proof. For the interval I, given by $\frac{1}{2} - (\log k)^{-7/8} \leqslant \sigma \leqslant \frac{1}{2} + (\log k)^{-7/8}$, we have

$$\begin{aligned} \frac{d}{d\sigma} \log|f(\sigma + iT_0)| &= \operatorname{Re}\left(\frac{f'(\sigma+iT_0)}{f(\sigma+iT_0)}\right) \\ &> \log k. \end{aligned}$$

Thus if σ_1 and σ_2 are in I, $\sigma_1 < \sigma_2$, then

$$\left|\frac{f(\sigma_2 + iT_0)}{f(\sigma_1 + iT_0)}\right| > k^{\sigma_2 - \sigma_1}. \tag{21}$$

In particular, $|f(\sigma + iT_0)|$ is strictly increasing on I and this gives the first part of Lemma 4. In order to derive the second part of the lemma, it is convenient to introduce the number

$$\sigma_0 = \tfrac{1}{2} + \frac{\pi}{3\log k}. \tag{22}$$

Then σ_0 is in I, and (21) therefore gives

$$\begin{aligned} |f(\sigma_0 + iT_0)| &> k^{\pi/(3\log k)} |f(\tfrac{1}{2} + iT_0)| \\ &> 2|f(\tfrac{1}{2} + iT_0)|. \end{aligned} \tag{23}$$

If $\sigma_0 \leqslant \sigma \leqslant \frac{1}{2} + (\log k)^{-7/8}$, then

$$\begin{aligned} |f(\sigma + iT_0)| - |f(\tfrac{1}{2} + iT_0)| &\geqslant |f(\sigma_0 + iT_0)| - |f(\tfrac{1}{2} + iT_0)| \\ &> |f(\tfrac{1}{2} + iT_0)|, \end{aligned}$$

and hence

$$|f(\sigma + iT_0)| - |f(\tfrac{1}{2} + iT_0)| > \tfrac{1}{2}|f(\sigma + iT_0)|.$$

Therefore if $\sigma_0 \leqslant \sigma \leqslant \frac{1}{2} + (\log k)^{-7/8}$ and $t = T_0$, then

$$\begin{aligned} |f(s)+f(1-s)| &\geqslant |f(s)|-|f(1-s)| \\ &> |f(s)|-|f(\tfrac{1}{2}+iT_0)| \\ &> \tfrac{1}{2}|f(s)|. \end{aligned} \tag{24}$$

We now derive the equivalent of equation (24) with σ in the interval $\frac{1}{2} \leqslant \sigma \leqslant \sigma_0$. If $1-\sigma_0 \leqslant \sigma_1 < \sigma_2 \leqslant \sigma_0$ then, by Lemma 2,

$$\begin{aligned} |\arg f(\sigma_2+iT_0) - \arg f(\sigma_1+iT_0)| &= \left| \int_{\sigma_1}^{\sigma_2} \frac{d}{d\sigma} \arg f(\sigma+iT_0)\, d\sigma \right| \\ &= \left| \int_{\sigma_1}^{\sigma_2} \operatorname{Im} \left\{ \frac{f'(\sigma+iT_0)}{f(\sigma+iT_0)} \right\} d\sigma \right| \\ &< (\sigma_2-\sigma_1)\log k. \end{aligned}$$

Thus if $\frac{1}{2} \leqslant \sigma \leqslant \sigma_0$,

$$\begin{aligned} |\arg f(\sigma+iT_0) - \arg f(\tfrac{1}{2}+iT_0)| &< (\sigma_0-\tfrac{1}{2})\log k \\ &= \frac{\pi}{3}. \end{aligned}$$

Therefore, by the definition of T_0 in Lemma 3, if $\frac{1}{2} \leqslant \sigma \leqslant \sigma_0$ and $t = T_0$, then

$$\cos\{\arg f(s)\} > \tfrac{1}{2}. \tag{25}$$

In like manner, if $\frac{1}{2} \leqslant \sigma \leqslant \sigma_0$ and $t = T_0$,

$$\cos\{\arg f(1-s)\} > \tfrac{1}{2}. \tag{26}$$

Thus if $\frac{1}{2} \leqslant \sigma \leqslant \sigma_0$ and $t = T_0$ then

$$\begin{aligned} |f(s)+f(1-s)| &\geqslant |\operatorname{Re}\{f(s)+f(1-s)\}| \\ &= |f(s)|\cos\{\arg f(s)\} + |f(1-s)|\cos\{\arg f(1-s)\} \\ &> \tfrac{1}{2}|f(s)|. \end{aligned} \tag{27}$$

Combining (24) and (27), if $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2} + (\log k)^{-7/8}$ and $t = T_0$, then

$$\begin{aligned} |f(s)+f(1-s)| &> \tfrac{1}{2}|f(s)| \\ &= \tfrac{1}{2}|b(s)|\,.\,|\zeta(2s)| \\ &> \tfrac{1}{2}|b(s)|(\log T)^{-1} \\ &> |b(s)|\,.\,\frac{1}{k} \\ &> |g(s)|, \end{aligned}$$

which is the second assertion of Lemma 4.

LEMMA 5. *Let R be the interior of the rectangle with corners at $2 \pm iT_0$, $-1 \pm iT_0$. Then the number of zeros of $\alpha(s)$ (multiple zeros counted according to their multiplicity) in R is exactly*

$$2 + \frac{2}{\pi} \arg f(\tfrac{1}{2} + iT_0).$$

Proof. Let C_1 be the boundary of R. Since C_1 is symmetric about the lines $\sigma = \frac{1}{2}$ and $t = 0$, we see from (12) and Lemmas 1 and 4 that for s on C_1

$$|f(s) + f(1-s)| > |g(s)|.$$

Let N and P denote the number of zeros and poles of $\alpha(s)$ in R. Rouché's theorem then tells us that

$$N - P = \frac{1}{2\pi} \Delta_{C_1} \arg\{f(s) + f(1-s)\},$$

the change in argument being calculated once around C_1 in the positive direction. If we let C be the curve consisting of the two straight line segments from 2 to $2 + iT_0$ and then to $\frac{1}{2} + iT_0$, then due to symmetry

$$N - P = \frac{2}{\pi} \Delta_C \arg\{f(s) + f(1-s)\}, \tag{28}$$

the change in argument now being calculated on C. But now, $|f(1-s)| < |f(s)|$ on C except at $\frac{1}{2} + iT_0$, and

$$\arg\{f(2) + f(-1)\} \equiv \arg f(2) \equiv 0 \pmod{2\pi},$$

$$\arg\{f(\tfrac{1}{2} + iT_0) + f(\tfrac{1}{2} - iT_0)\} \equiv \arg f(\tfrac{1}{2} + iT_0) \equiv 0 \pmod{2\pi}.$$

Therefore,

$$N - P = \frac{2}{\pi} \Delta_C \arg f(s)$$

$$= \frac{2}{\pi} \arg f(\tfrac{1}{2} + iT_0).$$

Since $\alpha(s)$ has first order poles at 0 and 1 and is analytic elsewhere,

$$N = 2 + \frac{2}{\pi} \arg f(\tfrac{1}{2} + iT_0).$$

4. *Proof of Theorem* 1. The zeros of $\zeta(s, Q)$ in R are exactly the same as the zeros of $\alpha(s)$ in R. We now locate these zeros. As $\varepsilon \to 0$,

$$f(\tfrac{1}{2} + \varepsilon) = \sqrt{\frac{k}{\pi}}\, [1 + O(|\varepsilon| \log k)][\sqrt{\pi} + O(|\varepsilon|)]\left[\frac{1}{2\varepsilon} + O(1)\right]$$

$$= \frac{\sqrt{k}}{2\varepsilon} + O(\sqrt{k}.\log k).$$

Thus

$$\arg f(\tfrac{1}{2} + i0^+) = -\frac{\pi}{2}.$$

Let

$$n = \frac{1}{\pi} \arg f(\tfrac{1}{2} + iT_0), \tag{29}$$

so that n is an integer. Since $\arg f(\frac{1}{2} + it)$ is continuous for $t > 0$, there exist numbers $0 < t_0 < t_1 < \ldots < t_n = T_0$ such that

$$\arg f(\tfrac{1}{2} + it_j) = \pi j \quad (j = 0, 1, \ldots, n). \tag{30}$$

Thus $f(\frac{1}{2} + it_j)$ is real valued, positive if j is even, negative if j is odd. From (7) we see that for $\frac{3}{2} \leqslant t \leqslant T_0$,

$$|\zeta(1 + 2it)| > \frac{1}{k},$$

and this is clearly true for $0 < t < \frac{3}{2}$. Thus for $0 < t \leqslant T_0$,

$$\begin{aligned} |f(\tfrac{1}{2} + it)| &= |b(\tfrac{1}{2} + it)| \, . \, |\zeta(1 + 2it)| \\ &> |g(\tfrac{1}{2} + it)|. \end{aligned}$$

Hence, if we put

$$s_j = \tfrac{1}{2} + it_j \quad (j = 0, 1, \ldots, n), \tag{31}$$

then we see from

$$\begin{aligned} \alpha(s_j) &= f(s_j) + f(1 - s_j) + g(s_j) \\ &= 2f(s_j) + g(s_j) \end{aligned}$$

that $\alpha(s_j)$ and $f(s_j)$ have the same sign. Therefore the sequence of numbers $\alpha(s_0)$, $\alpha(s_1), \ldots, \alpha(s_n)$ alternates in sign. Thus there exist n distinct points $\frac{1}{2} + iv_j$ with

$$0 < t_{j-1} < v_j < t_j$$

and

$$\alpha(\tfrac{1}{2} + iv_j) = 0 \quad (j = 1, 2, \ldots, n).$$

The n points $\frac{1}{2} - iv_j$ are also zeros of $\alpha(s)$. Bateman and Grosswald **[1]** have shown that $\alpha(s)$ has a real zero between $\frac{1}{2}$ and 1 and hence also one between 0 and $\frac{1}{2}$. Thus we have found $2n + 2$ distinct points in R where $\alpha(s) = 0$. But Lemma 5 tells us that there are exactly $2n + 2$ zeros of $\alpha(s)$ in R (multiplicity included). This concludes the proof of Theorem 1.

5. *Proof of Theorem* 2. Now we see that v_j is unique; that is, there is exactly one zero of $\alpha(s)$ between s_{j-1} and s_j. As a result, if $t_{j-1} < t \leqslant t_j$ ($j = 0, 1, \ldots, n$ and we let $t_{-1} = 0$), then

$$|\arg f(\tfrac{1}{2} + it_j) - \arg f(\tfrac{1}{2} + it)| < 2\pi.$$

Therefore, if $0 < T \leqslant T_0$, then

$$N(T, Q) = \frac{1}{\pi} \arg f(\tfrac{1}{2} + iT) + O(1). \tag{32}$$

But,
$$\arg f(\tfrac{1}{2}+iT) = \arg\left(\frac{k}{\pi}\right)^{\frac{1}{2}+iT} + \arg\Gamma(\tfrac{1}{2}+iT) + \arg\zeta(1+2iT)$$
$$= T\log\left(\frac{k}{\pi}\right) + T\log T - T + \arg\zeta(1+2iT) + O(1). \quad (33)$$

It remains to estimate $\arg\zeta(1+2iT)$.

For convenience, when $T > 3$ put
$$\delta = \log^{-1/3} T(\log\log T)^{-1/6}.$$

Now, for $T > 3$, we have
$$|\arg\zeta(1+\delta+2iT)| \leqslant |\log\zeta(1+\delta+2iT)|$$
$$\leqslant \sum_p \sum_{m=1}^{\infty} \frac{1}{m} p^{-m(1+\delta)}$$
$$\ll 1/\delta,$$

and by (6) we also have
$$|\arg\zeta(1+\delta+2iT) - \arg\zeta(1+2iT)| \ll 1/\delta.$$

For $0 < T \leqslant 3$, we clearly have
$$|\arg\zeta(1+2iT)| \ll 1.$$

Thus for all $T > 0$,
$$\arg\zeta(1+2iT) = O\{\log^{1/3}(T+3)[\log\log(T+3)]^{1/6}\}. \quad (34)$$

Theorem 2 follows from (32), (33) and (34).

References

1. P. T. Bateman and E. Grosswald, " On Epstein's zeta function ", *Acta Arithmetica*, 9 (1964), 365–373.
2. H. S. A. Potter and E. C. Titchmarsh, " The zeros of Epstein's zeta-functions ", *Proc. London Math. Soc.* (2), 39 (1935), 372–384.
3. H. Davenport and H. Heilbronn, " On the zeros of certain Dirichlet series I, II ", *Journal London Math. Soc.*, 11 (1936), 181–185 and 307–312.
4. E. C. Titchmarsh, *The theory of the Riemann zeta-function* (Oxford, 1951).
5. H. M. Stark, " On complex quadratic fields with class number equal to one ", *Trans. American Math. Soc.*, 122 (1966), 112–119.
6. A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie* (Berlin, 1963).

The Department of Mathematics,
The University of Michigan,
Ann Arbor, Michigan, U.S.A.

(*Received on the 20th of July*, 1966.)