ENUMERATION OF SELF-CONVERSE DIGRAPHS

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How many digraphs are isomorphic with their own converses? Our object is to derive a formula for the counting polynomial $d_p'(x)$ which has as the coefficient of x^q, the number of " self-converse " digraphs with p points and q lines. Such a digraph D has the property that its converse digraph D' (obtained from D by reversing the orientation of all lines) is isomorphic to D. The derivation uses the classical enumeration theorem of Pólya [**9**] as applied to a restriction of the power group [**6**] wherein the permutations act only on 1–1 functions.

1. *Self-converse digraphs.* A *directed graph* D (or more briefly a *digraph*) consists of a finite set V of *points* $v_1, v_2, \ldots, v_p$ together with a prescribed collection of ordered pairs of distinct points of V; see [**5**]. Each such ordered pair (u, v) is called a *directed line* and is usually denoted by uv. The point u is *adjacent to* v and v is *adjacent from* u. The converse D' of D is the digraph with the same set of points as D and in which u is adjacent to v if and only if v is adjacent to u in D. A digraph and its converse are shown in Fig. 1.

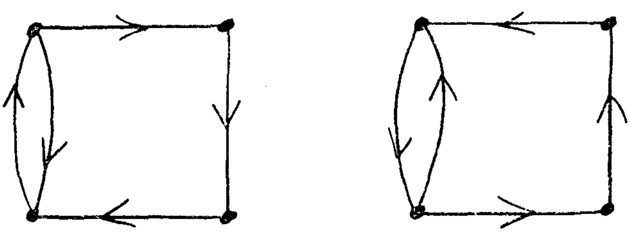

Fig. 1.

If D and its converse D' are isomorphic, written $D \cong D'$, then D is called *self-converse.* All of the self-converse digraphs with three points are shown in Fig. 2.

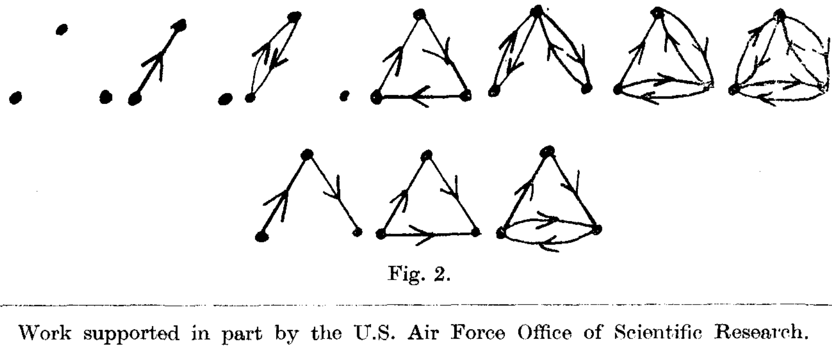

Fig. 2.

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Thus the counting polynomial which enumerates these self-converse digraphs is

$$d_3'(x) = 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6.$$

The complement $\bar{D}$ of D has the same set of points as D and in it u is adjacent to v if and only if u is not adjacent to v in D. The next result, which appears in [**3**], is simple but useful.

THEOREM 1. $(\bar{D})' = (\bar{D'})$, *i.e., the converse and the complement of a digraph commute.*

An immediate consequence of Theorem 1 accounts for the symmetry of the coefficients of $d_p'(x)$.

COROLLARY 1a. *A digraph is self-converse if and only if its complement is.*

2. *Restriction of the power group.* Let A be a permutation group of order $|A|$ acting on the set X of d objects. For each permutation α in A, let $j_k(\alpha)$ be the number of cycles of length k in the disjoint cycle decomposition of α.

The *cycle index* $Z(A)$ of A is the polynomial in the variables $a_1, a_2, \ldots, a_d$ defined by

$$Z(A) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{d} a_k^{j_k(\alpha)} \tag{1}$$

For any polynomial $h(x)$ in the variable x, we denote by $Z\big(A, h(x)\big)$ the polynomial obtained from $Z(A)$ on replacing each a_k by $h(x^k)$.

We also find the next formula useful (see [**7**]):

$$j_1(\alpha^k) = \sum_{s|k} s j_s(\alpha). \tag{2}$$

Let B be another permutation group acting on the set Y of e objects. Then as defined in [**6**], the power group B^A acts on Y^X, the functions from X into Y. For each pair of permutations α in A and β in B there is a unique permutation, written $(\alpha;\ \beta)$, in B^A such that for each function f in Y^X and all x in X,

$$(\alpha;\ \beta)\, f(x) = \beta f(\alpha x). \tag{3}$$

Suppose $d \leqslant e$ and denote by B^{*A} the permutation group obtained from B^A by restricting its permutations to the 1–1 functions in Y^X. Let m be the degree of B^{*A}, so that $m = e(e-1)\ldots(e-d+1)$.

It is easy to show that the order, $|B^{*A}|$, of B^{*A} is $|B|\,|A|$ unless A and B are both S_2. In this case $S_2{}^{*S_2}$ is also S_2.

Let the cycle index of B^{*A} be the polynomial in the variables $c_1, c_2, \ldots, c_m$ given by

$$Z(B^{*A}) = \frac{1}{|B^{*A}|} \sum_{(\alpha;\ \beta) \in B^{*A}} \prod_{k=1}^{m} c_k^{j_k(\alpha;\ \beta)}.$$

The formulas which give the numbers $j_k(\alpha;\ \beta)$ in terms of $j_k(\alpha)$ and $j_k(\beta)$ are

$$j_1(\alpha;\ \beta) = \prod_{k=1}^{d} \left(k^{j_k(\alpha)} \prod_{s=0}^{j_k(\alpha)-1} \left(j_k(\beta) - s \right) \right), \tag{4}$$

where by convention, the product over s is 1 if $j_k(\alpha) = 0$; and for $k > 1$.

$$j_k(\alpha;\ \beta) = \frac{1}{k} \sum_{s|k} \mu\left(\frac{k}{s}\right) j_1(\alpha^s;\ \beta^s), \tag{5}$$

and μ denotes the familiar Möbius function; see Rota [**11**] for a modern treatment.

Now we justify formula (4). Suppose f is a 1–1 function in Y^X which is fixed by the permutation $(\alpha;\ \beta)$. Let z_k be any cycle of length k in the disjoint cycle decomposition of α. Since f is fixed by $(\alpha;\ \beta)$, f must map the elements permuted by z_k onto the elements permuted by some cycle z_s in the disjoint cycle decomposition of β. Since f is 1–1, we must have $k = s$, and hence $j_k(\alpha) \leqslant j_k(\beta)$ for each k. Also there are exactly k ways in which f can map the elements of z_k onto the k elements permuted by the cycle of β. The elements permuted by another cycle of length k in α must be mapped by f onto the elements of one of the remaining $j_k(\beta) - 1$ cycles of the same length in β, again in one of k ways. Thus the contribution to $j_1(\alpha;\ \beta)$ of the cycles of length k is

$$k^{j_k(\alpha)}\left(j_k(\beta)\right)\left(j_k(\beta) - 1\right)\ldots\left(j_k(\beta) - j_k(\alpha) + 1\right).$$

Formula (4) follows immediately; it is implicit in de Bruijn [**1**].

Since it is easily verified from the definition of $(\alpha;\ \beta)$ that $(\alpha^k;\ \beta^k) = (\alpha;\ \beta)^k$, (2) may be used to express $j_1(\alpha^k;\ \beta^k)$ in terms of the numbers $j_s(\alpha;\ \beta)$ with $s|k$, and on applying the Möbius inversion formula [**11**], we obtain (5). The expressions $j_1(\alpha^s;\ \beta^s)$ occurring in (5) can be evaluated with the aid of (4).

3. *Enumeration of digraphs.* Since we will use the counting polynomial $d_p(x)$ which enumerates digraphs, we give a brief explanation of the formula derived in [**2**] for $d_p(x)$.

For convenience, let $X = \{1, 2, \ldots, p\}$. The set of ordered pairs (i, j) of distinct elements of X is denoted by $X^{[2]}$. Let the symmetric group of degree p, denoted by S_p, act on X. The *reduced ordered pair group* $S_p{}^{[2]}$, defined in [**4**], acts on $X^{[2]}$, and each of its permutations is induced by a permutation in S_p. That is, for each permutation α in S_p, if α' is the induced permutation in $S_p{}^{[2]}$, then for all (i, j), $\alpha'(i, j) = (\alpha i, \alpha j)$.

An application of Pólya's theorem gives the next theorem, which was presented in [**2**], together with an explicit formula for $Z(S_p{}^{[2]})$.

THEOREM 2. *The counting polynomial $d_p(x)$ which enumerates digraphs on p points is*

$$d_p(x) = Z(S_p^{[2]}, 1+x). \tag{6}$$

4. *Enumeration of digraphs up to conversion.* Two digraphs D_1 and D_2 with the same set of points are *equivalent up to conversion* if either $D_1 \cong D_2$ or $D_1' \cong D_2$. Our objective here is to find a formula for $c_p(x)$, the counting polynomial which enumerates digraphs with p points up to conversion. To do this, we must find, as in the case for digraphs, the appropriate permutation group to which Pólya's theorem may be applied.

Let S_2 act on $\{1, 2\}$ and consider the power group $S_p^{S_2}$ acting on $X^{\{1,2\}}$, the functions from $\{1, 2\}$ into X. Observe the natural correspondence between the elements of $X^{[2]}$ and the 1–1 functions in $X^{\{1,2\}}$. Each ordered pair (i, j) in $X^{[2]}$ corresponds to the function in $X^{\{1,2\}}$ which sends 1 to i and 2 to j. Thus we may consider the restricted power group $S_p^{*S_2}$ as acting on the elements of $X^{[2]}$. More specifically the permutations of $S_p^{*S_2}$ consist of ordered pairs $(\alpha; \beta)$ of permutations α in S_2 and β in S_p so that for any (i, j) in $X^{[2]}$,

$$(\alpha;\ \beta)(i, j) = \begin{cases} (\beta i, \beta j) & \text{if } \alpha = (1)(2) \\ (\beta j, \beta i) & \text{if } \alpha = (12). \end{cases} \tag{7}$$

Now let E_2 be the identity group acting on the set $Y = \{0, 1\}$. Consider the power group E_2^T with $T = S_p^{*S_2}$ acting on $Y^{X^{[2]}}$, the functions from $X^{[2]}$ into Y. Each function f in $Y^{X^{[2]}}$ represents a digraph whose points are the elements of $X = \{1, 2, \ldots, p\}$, in which i is adjacent to j whenever $f(i, j) = 1$. Thus the elements 0 and 1 of Y indicate the absence or presence of directed lines.

Let f_1 and f_2 be two functions in $Y^{X^{[2]}}$, and let their digraphs be D_1 and D_2 respectively. Then $D_1 \cong D_2'$ or $D_1 \cong D_2$ if and only if there is a permutation γ in E_2^T with $T = S_p^{*S_2}$ such that $\gamma f_1 = f_2$. This follows from the fact that for $\gamma = (\alpha; \beta)$, the digraph of γf_1 is isomorphic to D_1 or D_1' according as α is (1) (2) or (12).

Thus equivalence of digraphs up to conversion corresponds to equivalence of functions in $Y^{X^{[2]}}$ determined by the power group E_2^T with $T = S_p^{*S_2}$.

Now applying Pólya's theorem, we obtain the desired result.

THEOREM 3. *The counting polynomial $c_p(x)$ which enumerates digraphs up to conversion is*

$$c_p(x) = Z(S_p^{*S_2}, 1+x). \tag{8}$$

Formulas (1), (4) and (5) can be used to express the cycle index of any restricted power group B^{*A}. But in the special case $A = S_2$ and $B = S_p$, a more explicit formula can be given.

For each permutation α in S_p, the partition of α is denoted by $(j) = (j_1, j_2, \ldots, j_p)$, where j_k is the number of disjoint cycles of length k in α. Then the contribution to $Z(S_p{}^{*S_2})$ of $\big((12);\ \alpha\big)$ is

$$I(\alpha) = \prod_{k=1}^{p} a_{m(2,k)}^{d(2,k)\,k\binom{j_k}{2}} \prod_{1 \leq r < s \leq p} a_{m(2,\,m(r,s))}^{d(2,\,m(r,s))\,d(r,s)\,j_r\,j_s}$$

$$\prod_{k \text{ odd}} a_{2k}^{j_k(k-1)/2} \prod_{k \text{ even}} a_k^{(k-2)\,j_k}\, a_{k/2}^{\eta(k)\,2j_k}\, a_k^{(1-\eta(k))\,j_k}, \tag{9}$$

where $\eta(k) = 1$ if $k/2$ is an odd integer and 0 otherwise, and $d(r, s)$ and $m(r, s)$ are the g.c.d. and l.c.m. respectively.

Hence the cycle index of $S_p{}^{*S_2}$ can be expressed as

$$Z(S_p{}^{*S_2}) = \frac{1}{2(p!)}\left(p!\, Z(S_p{}^{[2]}) + \sum_{\alpha \in S_p} I(\alpha)\right). \tag{10}$$

5. *Enumeration of self-converse digraphs.* Now we make the simple observation for self-converse digraphs which corresponds to that made by Read [**10**] for self-complementary graphs. Namely, the polynomial $2c_p(x)$ counts each digraph twice if it is self-converse and once if not. Hence the polynomial $2c_p(x) - d_p(x)$ counts each self-converse digraph just once. Thus we have

$$d_p'(x) = 2c_p(x) - d_p(x). \tag{11}$$

This together with formulas (6) and (8) gives the next result.

THEOREM 4. *The counting polynomial $d_p'(x)$ for self-converse digraphs is*

$$d_p'(x) = 2Z(S_p{}^{*S_2}, 1+x) - Z(S_p{}^{[2]}, 1+x). \tag{12}$$

To use formula (12) for $d_p'(x)$ let $F(S_p{}^{*S_2}) = \frac{1}{p!} \sum_{\alpha \in S_p} I(\alpha)$. By $F(S_p{}^{*S_2}, 1+x)$ we mean the polynomial obtained by replacing each variable a_k in $F(S_p{}^{*S_2})$ by $1 + x^k$. Combining Theorem 4 and formula (10) for $Z(S_p{}^{*S_2})$ we obtain

$$d_p'(x) = F(S_p{}^{*S_2}, 1+x). \tag{13}$$

To illustrate, we develop the polynomial $d_3'(x)$ for the self-converse digraphs on three points, shown in Fig. 2. The cycle index of the symmetric group S_3 is

$$Z(S_3) = \tfrac{1}{6}(a_1^3 + 3a_1 a_2^2 + 2a_3).$$

From this and formula (9) for $I(\alpha)$, we have

$$F(S_3{}^{*S_2}) = \tfrac{1}{6}(a_2^3 + 3a_1^2 a_2^2 + 2a_6).$$

Formula (13) gives

$$d_3'(x) = \tfrac{1}{6}\Big((1+x^2)^3 + 3(1+x)^2(1+x^2)^2 + 2(1+x^6)\Big)$$
$$= 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6.$$

Similarly formula (13) gives for $p = 4$:

$$d_4'(x) = \tfrac{1}{24}\Big((1+x^2)^6 + 6(1+x)^2(1+x^2)^5 + 8(1+x^6)^2 + 3(1+x)^4(1+x^2)^4 + 6(1+x^4)^3\Big)$$
$$= 1 + x + 3x^2 + 5x^3 + 9x^4 + 10x^5 + 12x^6 + 10x^7 + 9x^8 + 5x^9 + 3x^{10} + x^{11} + x^{12}.$$

These coefficients may be checked by examining the diagrams of the four point digraphs in [8]. In Fig. 3 we show the five self-converse digraphs with four points and three lines.

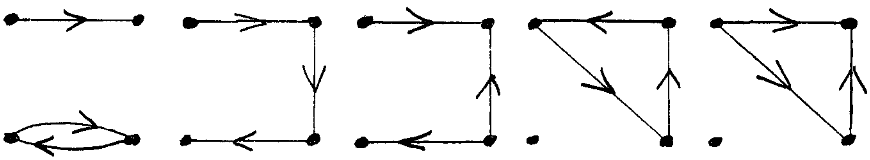

Fig. 3.

6. *Self-converse relations.* A slight modification of formula (12) results in the polynomial $r_p'(x)$ that enumerates self-converse digraphs in which loops are permitted. Digraphs with loops are, of course, just relations. It is easy to see how the power group $S_p^{S_2}$ can be used to count such digraphs up to conversion. The *ordered pair group* S_p^2 acts on all ordered pairs as induced by the symmetric group S_p. As shown in [2], the polynomial $r_p(x)$ which counts relations is

$$r_p(x) = Z(S_p^2, 1+x). \tag{14}$$

Then $r_p'(x)$ is given by

$$r_p'(x) = 2Z(S_p^{S_2}, 1+x) - Z(S_p^2, 1+x). \tag{15}$$

To use equation (15), for each permutation α in S_p we let

$$J(\alpha) = I(\alpha) \prod_{k=1}^{p} a_k^{j_k}. \tag{16}$$

Then the cycle index of the power group $S_p^{S_2}$ can be expressed by

$$Z(S_p^{S_2}) = \frac{1}{2(p!)}\Big(p!\,Z(S_p^2) + \sum_{\alpha \in S_p} J(\alpha)\Big) \tag{17}$$

Now let $G(S_p^{S_2}) = \dfrac{1}{p!} \sum_{\alpha \in S} J(\alpha)$. Then the formula for $r_p'(x)$ can be written

$$r_p'(x) = G(S_p^{S_2}, 1+x). \tag{18}$$

7. *Self-converse digraphs with p points.* Let d_p' be the total number of self-converse digraphs with p points. Then, referring to (12), we see that $d_p' = d_p'(1)$. In order to express a formula for d_p' in relatively manageable form, we introduce the following notation. For each α in S_p, let

$$\epsilon(\alpha) = \sum_{k=1}^{p} \left[d(2, k)\left\{ \frac{k-1}{2} j_k + k\binom{j_k}{2} \right\} + \eta(k) j_k \right] + \sum_{1 \leq r < s \leq p} d\big(2, m(r, s)\big)\, d(r, s) j_r j_s. \quad (19)$$

Since the replacement in (13) and (9) of each a_k in $F(S_p{}^{*S_2})$ by 2 gives $d_p'(1)$, we have

$$d_p' = \frac{1}{p!} \sum_{\alpha \in S_p} 2^{\epsilon(\alpha)}. \quad (20)$$

A similar formula is easily obtained for the total number r_p' of self-converse relations with p points.

To compute these numbers, we use the fact that the number of permutations in S_p with partition (j) is $p!\Big/\left(\prod_{k=1}^{p} k^{j_k} j_k!\right)$. Here are the totals for $p = 1$ to 6.

p	1	2	3	4	5	6
d_p'	1	3	10	70	709	47,960
r_p'	2	8	44	436	7176	484,256

8. *Unsolved problem.* How many self-converse oriented graphs (directed graphs with no symmetric pairs of lines) are there with a given number of points and lines?

References

1. N. G. de Bruijn, "Pólya's theory of counting", Chapter 5 in *Applied Combinatorial Mathematics*, E. F. Beckenbach (ed.) (New York, 1964), pp. 144–184.
2. F. Harary, "The number of linear, directed, rooted and connected graphs", *Trans. American Math. Soc.*, 78 (1955), 445–463.
3. ———, "Structural duality", *Behavioral Science*, 2 (1957), 255–265.
4. ———, *A seminar on graph theory* (New York, 1967), to appear.
5. F. Harary, R. Norman, D. Cartwright, *Structural models: an introduction to the theory of directed graphs* (New York, 1965).
6. F. Harary and E. M. Palmer, "The power group of two permutation groups", *Proc. Nat. Acad. Sci. U.S.A.*, 54 (1965), 680–682.
7. ———, ———, "The power group enumeration theorem", *J. Combinatorial theory*, 1 (1966), 157–173.
8. ———, ———, "The groups of the small digraphs", *J. Indian Stat. Assoc.*, to appear.
9. G. Pólya, "Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen", *Acta Math.*, 68 (1937), 145–254.
10. R. C. Read, "On the number of self-complementary graphs and digraphs", *Journal London Math. Soc.*, 38 (1963), 99–104.
11. G. C. Rota, "On the foundations of combinatorial theory, I. Theory of Möbius functions", *Z. Wahrscheinlichkeitstheorie*, 2 (1964), 340–368.

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