THE NUMBER OF PLANE TREES WITH A GIVEN PARTITION

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This note is a continuation of the articles [**6**] and [**2**]. In [**1**], trees with a given partition $\alpha = (a_1, a_2, \ldots)$, where a_i is the number of vertices (points) of valency (degree) i were enumerated. After the determination of the number of plane trees in [**2**], the number of planted plane trees with a given partition α was found explicitly in [**6**]. In the present note, the number of plane trees with a given partition is expressed as a function of the number of planted trees with a given partition. The method, which is not new, consists of an application of the enumeration techniques of Otter [**3**] and Pólya [**4**]; it was used in [**1**] and also by Riordan [**5**].

Our notation tends to follow that of [**1**] and [**2**]; the terminology [**2**] and [**6**]. Let $\bar{R}_{n,\alpha}$ be the number of planted plane trees with $n+1$ vertices and partition $\alpha = (a_1, a_2, \ldots)$ where a_1 is the number of monovalent vertices (end points) *other than the root* and for $i > 1$, a_i is the number of vertices of valency i. Of course α determines n since $n = \Sigma a_i$, but it is convenient to retain n. Let $t_1, t_2, \ldots$ be an infinite set of variables and write symbolically $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \ldots$. Then the generating function for planted plane trees is denoted by

$$\bar{R}(x, t) = \sum_{n,\alpha} \bar{R}_{n,\alpha} x^n t^\alpha. \tag{1}$$

In [**6**] it was shown that $\bar{R}_{n,\alpha} = 0$ unless the partition α satifies

$$\sum_1^\infty m a_m = 2\Sigma a_m - 1 = 2n - 1, \tag{2}$$

and that if (2) holds, then

$$\bar{R}_{n,\alpha} = \frac{(n-1)!}{\Pi a_m!}. \tag{3}$$

Thus $\bar{R}(x, t)$ is known.

Now let $R_{n,\alpha}$ be the number of rooted plane trees with n vertices and partition α (note that n is now the total number of vertices, including the root), and let their generating function be

$$R(x, t) = \sum_{n,\alpha} R_{n,\alpha} x^n t^\alpha. \tag{4}$$

Similarly let

$$r(x, t) = \sum_{n,\alpha} r_{n,\alpha} x^n t^\alpha \tag{5}$$

be the generating function for the numbers $r_{n,\alpha}$ which give the number of plane trees with n vertices and partition α.

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Two vertices (or edges) of a tree are *similar* if there is an automorphism of the tree sending one onto the other. Otter [**3**] has shown that the numbers p^* and q^* of similarity classes of vertices and edges respectively of any tree satisfy the equation

$$1 = p^* - (q^* - q_s), \tag{6}$$

where q_s is the number of symmetric edges, whose two vertices are similar to each other. It is easy to verify that the equation (6) also holds for plane trees. Summing (6) over all plane trees with a given partition α and with n vertices, one can show that

$$r_{n,\alpha} = R_{n,\alpha} - \tfrac{1}{2} \sum_{\substack{n_1+n_2=n \\ \alpha_1+\alpha_2=\alpha}} 2\bar{R}_{n_1,\alpha_1} \bar{R}_{n_2,\alpha_2} + \tfrac{1}{2} \begin{cases} 0 & (n \text{ or } \alpha \text{ odd}) \\ \bar{R}_{\frac{1}{2}n,\frac{1}{2}\alpha} & (n \text{ and } \alpha \text{ even.}) \end{cases} \tag{7}$$

The details of the manipulations (multiplying (7) by $x^n t^\alpha$ and summing over n and α) involved in proceeding from (7) to (8) are so entirely analogous to the methods and reasoning used in [**1**] and [**5**] that we omit them and simply state:

$$r(x, t) = R(x, t) - \tfrac{1}{2}[\bar{R}^2(x, t) - \bar{R}(x^2, t^2)]. \tag{8}$$

Note that while $\bar{R}(x, t)$ is known by (1), (2) and (3) and while equation (8) determines $r(x, t)$ in terms of $\bar{R}(x, t)$ and $R(x, t)$, it still remains to find a formula for the generating function $R(x, t)$ which enumerates rooted plane trees with a given partition.

Let C_m be the cyclic (permutation) group of degree m and order m. Then it is well known (see Pólya [**4**]) that its cycle index is

$$Z(C_m) = \frac{1}{m} \sum_{d|m} \phi(d)\, y_d^{m/d}, \tag{9}$$

where the letters y_d are indeterminates. In an analogous manner to the reasoning in [**2**], we find that $R(x, t)$ can be expressed in terms of $\bar{R}(x, t)$ by

$$R(x, t) = xt_1 \sum_{m=0}^{\infty} Z\Big(C_m, \bar{R}(x, t)\Big), \tag{10}$$

where as usual the substitution of $\bar{R}(x, t)$ into $Z(C_m)$ entails the replacement of each variable y_d by $\bar{R}(x^d, t^d)$.

It would be interesting to sum the equation (10) as Read [**7**] successfully did for a similar summation involving all the cycle indexes $Z(C_m)$, but we have not been able to accomplish this.

Note that the first few terms of these three generating functions are given by

$$r(x,t) = xt_1+x^2t_1^2+x^3t_1^2t_2+x^4(t_1^2t_2^2+t_1^3t_3)+x^5(t_1^2t_2^3+t_1^3t_2t_3+t_1^4t_4)+\dots$$

$$R(x,t) = xt_1+x^2t_1^2+2x^3t_1^2t_2 +x^4(2t_1^2t_2^2+2t_1^3t_3)+x^5(3t_1^2t_2^3+5t_1^3t_2t_3+2t_1^4t_4)+\dots$$

$$\bar{R}(x,t) = xt_1+x^2t_1t_2+x^3(t_1t_2^2+t_1^2t_3)+x^4(t_1t_2^3+3t_1^2t_2t_3+t_1^3t_4)+\dots .$$

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