Cluster Algebras of Open Richardson Varieties

by

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

ABSTRACT

This thesis shows that the coordinate ring of the open Richardson variety $\mathcal{R}^{u,w}$ in type A has the structure of an upper cluster algebra. We begin by deriving inequalities related to northwest rank conditions on a general collection of columns of a matrix in B_-vB_+. In the special case where $\mathbf{w}$ is the unipeak expression for w, the chamber minors in Marsh and Rietsch's generalized chamber ansatz for the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ factor into products of minors $\Delta_{\hat{C}_j}^{\hat{R}_j}$, which turn out to be flag-invariant regular functions. We show that each minor in this change of basis is either globally nonvanishing on $\mathcal{R}^{u,w}$ or vanishes on precisely one boundary divisor. Using augmenting paths in a weighted, oriented bridge diagram, we show that there is an upper cluster algebra structure on $\mathcal{R}^{u,w}$ with initial cluster given by the minors $\Delta_{\hat{C}_j}^{\hat{R}_j}$ and exchange relations induced by the chamber ansatz quiver defined by Berenstein, Fomin and Zelevinsky.

CHAPTER I

Introduction

The purpose of this thesis is to show that for every open Richardson variety $\mathcal{R}^{u,w}$ in type A, the coordinate ring $\mathbb{C}[\mathcal{R}^{u,w}]$ has the structure of an upper cluster algebra.

Fomin and Zelevinsky introduced cluster algebras after their classification of minimal total positivity tests for double Bruhat cells revealed a beautiful, fractal-like pattern formed by connecting pairs of total positivity tests which differ by exchanging a single element.[10] Pairs of elements that are exchanged satisfy binomial relations of the form $XX' = \mathcal{M}_1 + \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are products of the elements common to both positivity tests.

Total positivity was first studied in the context of matrices whose minors are all nonnegative real numbers, motivated by applications in data interpolation[20] and modeling mechanical vibrations.[15] More recent applications of total positivity include the study of Markov structures in statistics[8] and Ising models of ferromagnetism in physics.[25]

Gasca and Peña proved that in order to test a matrix for total positivity, it suffices to check the collection of n^2 minors of the form $\Delta^{[1,h]}_{[c+1,c+h]}$ or $\Delta^{[r+1,r+h]}_{[1,h]}$, where the row indices and column indices are given by intervals of consecutive integers and each minor contains either the first row or the first column.[16]

On the torus where the minors Δ_k in this positivity test are nonvanishing, determinantal identities of the form $XY = AD + BC$ give expansions for the remaining matrix minors as Laurent polynomials in the variables Δ_k with positive coefficients.

Fomin and Zelevinsky characterized minimal total positivity tests for totally nonnegative invertible matrices stratified by the double Bruhat cell decomposition in [9]. In 2000, they introduced *cluster algebras* to formalize the study of total positivity in algebraic varieties.[10] A cluster algebra is generated by the variables appearing in a *seed pattern*. A seed Σ consists of an n-element *cluster* (generalizing a minimal total positivity test), together with a quiver or skew-symmetrizable matrix encoding subtraction-free binomial exchange relations. Seed mutation is an involutive operation giving a new seed whose cluster differs from the initial cluster by a exchanging a single element according to the exchange relations; seeds Σ and Σ' are called *mutation-equivalent* if there is a sequence of mutation operations taking Σ to Σ'. Fomin and Zelevinsky proved that if Σ and Σ' are mutation-equivalent seeds, then the cluster variables of Σ' are Laurent polynomials in the cluster variables of the seed Σ. They conjectured that the coefficients in this expansion are positive; this was later proved by Lee and Schiffler for cluster algebras with skew-symmetric exchange relations in [24], and by Gross, Hacking, Keel and Kontsevich for skew-symmetrizable cluster algebras in [18].

Applications of cluster algebras include the study of scattering amplitudes (see [1]).

In [3], Berenstein, Fomin and Zelevinsky introduced the *upper cluster algebra* $\overline{\mathcal{A}}(\Sigma)$ generated by a seed Σ, defined as of the intersection of the Laurent rings in the cluster variables of all seeds mutation equivalent to Σ, and containing the cluster algebra $\mathcal{A}(\Sigma)$ as a subalgebra. They proved that coordinate rings of double Bruhat cells have an upper cluster algebra structure. It was later shown by Goodearl and Yakimov that the upper cluster algebra of a double Bruhat cell coincides with the cluster algebra in [17].

Our primary goal is to show that there is an upper cluster algebra structure on open Richardson varieties $\mathcal{R}^{u,w}$ in type A, where the cluster variables in the initial seed come from a minimal total positivity test for $\mathcal{R}^{u,w}$. It is an open question whether the cluster algebra generated by the associated seed pattern is always equal to the upper cluster algebra.

Lusztig defined total positivity for flags in various Lie types in [26] and conjectured a decomposition of the totally nonnegative part of the flag variety into algebraic cells. In [29], Rietsch proved this conjecture and showed that each cell in the decomposition is contained in an open Richardson variety $\mathcal{R}^{u,w}$ indexed by a pair of Weyl group elements u and w.

Given a matrix $g \in G = SL_n(\mathbb{C})$, the spans of the first i columns of g determine a *flag* $F = F_1 \subset F_2 \subset \cdots \subset F_n$ where $\dim(F_i) = i$. Matrices g_1 and g_2 determine the same flag if and only if $g_2 = g_1 b$ where b is an element of the upper triangular subgroup $B_+ \subset SL_n(\mathbb{C})$.

Let $\Delta_C^R(M)$ denote the minor of a matrix M with row set R and column set C. Right multiplication by elements of B_+ preserves ratios of left-justified minors: if $g_2 = g_1 b$, there are nonzero scalars $\lambda_1, \cdots, \lambda_{n-1}$ so that $\Delta_{[1,h]}^R(g_2) = \lambda_h \Delta_{[1,h]}^R(g_1)$. In particular, for each h, the row sets R for which the minor $\Delta_{[1,h]}^R$ is nonzero is independent of the flag representative g. For each flag F, there are uniquely determined permutations $u, w \in \mathfrak{S}_n$ so that $u([1,h])$ is the minimal row set with $\Delta_{[1,h]}^R \neq$

0 and $w([1,h])$ is the maximal row set with $\Delta^R_{[1,h]} \neq 0$. Flags with the same northwest rank conditions as F are the opposite Schubert cell $B_- \dot{u} B_+/B_+$ and flags with the same southwest rank conditions are the Schubert cell $B_+ \dot{w} B_+/B_+$. The open Richardson variety $\mathcal{R}^{u,w}$ is the intersection $B_- \dot{u} B_+/B_+ \cap B_+ \dot{w} B_+/B_+$. When $u \leqslant w$ in the Bruhat order, $\mathcal{R}^{u,w}$ is a variety of dimension $\ell(w) - \ell(u)$; otherwise, $\mathcal{R}^{u,w}$ is empty.

Let F be a flag in the Schubert cell $B_+ \dot{w} B_+/B_+$ and let $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$ be any reduced word for w. We write $w_{(i)} = s_{h_1} \cdots s_{h_i}$ for the product of the first i factors, where $w_{(0)} = 1$. There is a unique sequence of flags $F^0, F^1, \cdots, F^\ell = F$ satisfying the following conditions: for each $0 \leqslant i \leqslant \ell$, the flag F^i is in the Schubert cell $B_+ \dot{w}_{(i)} B_+/B_+$, and the subspaces F^i_j and F^{i-1}_j are equal for all $j \neq h_i$.[27]

Deodhar[7] showed that for each open Richardson variety $\mathcal{R}^{u,w}$ and each reduced word $\mathbf{w}$, the sequences of permutations $\mathbf{v}^j = v_{(i)}$ which occur as northwest rank conditions for a sequence $F^0, F^1, \cdots, F^\ell$ with $F^\ell \in \mathcal{R}^{u,w}$ are the *distinguished subexpressions* of $\mathbf{w}$ with product u (see Definition II.16). Distinguished subexpressions for u give a partition of $\mathcal{R}^{u,w}$ into a dense open torus $(\mathbb{C}^*)^{\ell(w)-\ell(u)}$ and a union of lower dimensional subvarieties $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$.

Marsh and Rietsch showed in [27] that algebraic cell $\mathcal{R}^{u,w}_{>0}$ given by the totally positive points of $\mathcal{R}^{u,w}$ is contained in the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ for any choice of reduced expression $\mathbf{w}$. They define a minimal total positivity test by regular functions Δ_k giving coordinates on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, and provide parametrizations for each subvariety in Deodhar's decomposition.

For each reduced word $\mathbf{w}$ for w and each distinguished subexpression $\mathbf{v}$, they define a sequence of coset representatives $g_{(0)}, g_{(1)}, \cdots, g_{(\ell)}$ for the flags $F^0, F^1, \cdots, F^\ell \in \mathcal{D}^{\mathbf{v},\mathbf{w}}$ using a combination of 1-parameter subgroups and signed transpositions corresponding to crossings in a wiring diagram, which they partition into index sets $J^\circ_{\mathbf{v}}$, $J^+_{\mathbf{v}}$ and $J^-_{\mathbf{v}^j}$ (see Definition II.18).

The inverse maps Marsh and Rietsch define from $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ to $(\mathbb{C}^*)^{|J^\circ_{\mathbf{v}}|} \times (\mathbb{C})^{|J^-_{\mathbf{v}}|}$ use chamber ansatz formulas which closely parallel Berenstein, Fomin and Zelevinsky's chamber ansatz formulas for double Bruhat cells. An important distinction is that chamber minors for the open torus in $\mathcal{R}^{u,w}$ need not be irreducible.

Example I.1. Let $\mathbf{w} = s_3s_2s_1s_4s_3s_2s_3s_4$ and let $u = s_3$. Then the minors Δ^{123}_{124} and Δ^{1234}_{1245} are both chamber minors, but the identity $\Delta^{123}_{124} = \Delta^{12}_{24}\Delta^{1234}_{1245}$ holds everywhere on $\mathcal{R}^{u,w}$.

Karpman proved that for Deodhar strata $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ which project onto positroid cells, Marsh and Rietsch's wiring diagram can be converted into a plabic graph by first constructing a *bridge diagram* and then deleting strand endpoints to obtain a *bridge graph*. We will interpret the partial products $g_{(i)}$ in Marsh and Rietsch's parametrization of $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ as weight matrices for Karpman's bridge diagrams, viewed as non-planar weighted networks, in order to express minors as sums of flow weights via Lindström's lemma. After reorienting along certain distinguished flows, the ratios of minors may be computed in terms of weights of simple *augmenting paths*. This technique is used by Postnikov in the case of planar networks mapping to the Grassmannian via the boundary measurement map, and is also closely related to Ford and Fulkerson's classic algorithm for finding a maximal flow in a network with edge capacities.[1]

Following Berenstein, Fomin and Zelevinsky's construction of the upper cluster algebra structure on double Bruhat cells,[3] we will show that there are regu-

[1] In the network flow sense studied by Ford and Fulkerson, paths in a flow need not be vertex-disjoint and the value of the flow through a given edge need not be integer-valued. A flow in the sense of Lindström, Gessel and Viennot requires that the flow through each vertex and each edge is either 0 or 1, so that a finite network has a finite number of possible flows; rather than the value of a flow, the relevant data is the weight of the flow, given by the product of the edge weights.

lar functions $(X_1, \cdots, X_{\ell(w)-\ell(u)})$ which are pairwise coprime and satisfy the conditions that for each variable X_i, either X_i is identically nonvanishing on $\mathcal{R}^{u,w}$, or there is a function X_i' which is regular on $\mathcal{R}^{u,w}$ so that X_i' is relatively prime to $(X_1, \cdots, X_{\ell(w)-\ell(u)})$ and the functions X_i and X_i' satisfy the exchange relation $X_iX_i' = \mathcal{M}_+ + \mathcal{M}_-$ for coprime monomials $\mathcal{M}_+$ and $\mathcal{M}_-$ in the variables $\{X_j : j \neq i\}$. We will show that the mutated cluster $(X_1, \cdots, X_{i-1}, X_i', \cdots, X_{\ell(w)-\ell(u)})$ gives coordinates for a new torus.

There are two major ways that our work extends previous work. We provide a factorization of the chamber minors into products of irreducible minors, allowing us to define a cluster structure that holds generally, where previous results have placed restrictions on either the word w or the subword u. This requires us to compute minors in terms of path weights in non-planar graphs.

Similar work has been done by Leclerc[23], using cluster-tilting objects to construct a cluster algebra $\mathcal{A}$ contained in $\mathbb{C}[\mathcal{R}^{u,w}]$. Leclerc showed that $\mathcal{A}$ is equal to $\mathbb{C}[\mathcal{R}^{u,w}]$ in the special case where w has a reduced word of the form $\mathbf{w} = \mathbf{vu}$, where $\mathbf{u}$ is a reduced word for u. Serhiyenko, Sherman-Bennett and Williams used Leclerc's result to show that the coordinate ring of an open Schubert variety in the Grassmannian is a cluster algebra in [30]. Galashin and Lam[14] generalized this result, proving that if the open Richardson variety $\mathcal{R}^{u,w}$ projects onto a positroid cell in the Grassmannian, then $\mathbb{C}[\mathcal{R}^{u,w}]$ has a cluster algebra structure. This holds when w is a Grassmannian permutation, which in particular implies that w has a unique commutation class of reduced words.

We give a further generalization by describing an initial seed Σ for a specific commutation class of reduced words $\mathbf{w}$ with positive subexpression $\mathbf{u}_+$, where $w \in \mathfrak{S}_n$ is any permutation. A word $\mathbf{w}$ is unipeak if its wiring diagram satisfies the condition that each strand first rises monotonically to a maximum height and then falls monotonically. Every permutation $w \in \mathfrak{S}_n$ has a unique commutation class of unipeak words (see [22] on canonical sequences). In the case where the permutation w is not fully commutative, the chamber minors Δ_k from Marsh and Rietsch's minimal total positivity test need not be irreducible in $\mathbb{C}[\mathcal{R}^{u,w}]$. In Chapter V, we describe a construction for factoring each chamber minor into a product of irreducible minors. An irreducible minor X_j is obtained from Marsh and Rietsch's minor Δ_j by removing strands and row indices along a broken-line "path" traveling left to right through the wiring diagram. We note that there are close parallels between this construction and the technique of deleting redundant portions of strands, used by Galashin and Lam in [14] and defined by Karpman in [21]. The strand portions removed in our algorithm are in general not redundant in Karpman's sense, and in particular a strand α may be removed when solving for the minor X_j but retained in solving for some minor X_k corresponding to a crossing k to the right of j. Another subtle distinction is that we remove strands based on crossings to the right of a chamber rather than to the left. That is, a chamber minor which is irreducible for matrices $z \in B_+$ with $z\dot{w}_{(i)} \in \mathcal{R}^{u_{(i)},w_{(i)}}$ may factor if we require that $z\dot{w}_{(k)} \in \mathcal{R}^{u_{(k)},w_{(k)}}$ for some $k > i$.

Chapter II gives an introduction to the symmetric group, the Schubert and opposite Schubert cell decompositions of the flag variety, and the open Richardson varieties $\mathcal{R}^{u,w}$ under Deodhar's decomposition.

Chapter III gives background on wiring diagrams with applications to Marsh and Rietsch's Chamber Ansatz and describes key characteristics of unipeak wiring diagrams which will enable us first to find regular functions which are pairwise coprime, and later to compute entries in a distinguished flag representative and verify Laurentness in the mutated cluster.

In Chapter IV, we develop combinatorics for showing when a minor on a collection of columns which is not left justified vanishes due to northwest rank conditions. We show that chambers labeled by minors which vanish on a given divisor form a simply connected region in the planar projection of the wiring diagram and are bounded by a connected cycle in the non-planar wiring diagram.

In Chapter V, we give a chamber weighting on the unipeak wiring diagram for the open Deodhar stratum, so that each minor is a product of irreducible minors $\Delta^{\widehat{R}^j}_{\widehat{C}^j}$ indexed by *nearly positive sequences* $\mathbf{v}^j$. In particular, the distinguished subexpressions giving hypersurfaces in Deodhar's decomposition are a subset of the $\mathbf{v}^j$. We show that if $\mathbf{v}^j$ is a distinguished subexpression, then the minor $\Delta^{\widehat{R}^j}_{\widehat{C}^j}$ vanishes identically on $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$, and that for any $i \neq j$ the minor $\Delta^{\widehat{R}^i}_{\widehat{C}^i}$ is generically nonvanishing where $\Delta^{\widehat{R}^j}_{\widehat{C}^j} = 0$.

In Chapter VI, we lay out the background for computing minors of a weight matrix for a weighted directed network using augmenting flows in a reoriented graph and describe Karpman's bridge diagram construction for identifying Marsh and Rietsch's wiring diagrams with weighted directed networks.

In Chapter VII, we recall Berenstein, Fomin and Zelevinsky's Starfish Lemma for giving the coordinate ring of a normal variety the structure of an upper cluster algebra. We define a quiver with vertices indexed by the variables X_j and arrows induced by arrows in the chamber ansatz quiver. We use augmenting flows in an acyclic weighted network to first show that the exchange relations send coordinate functions on the torus to Laurent polynomials in the new cluster, and later to show that each exchange variable X_i' can be computed in terms of the *special chamber minor* defined by Marsh and Rietsch in the parametrization of the boundary divisor

$\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$.

CHAPTER II

Background and Notation

2.1 The symmetric group $\mathfrak{S}_n$

Definition II.1. An *interval* $[a, b]$ in the integers is given by the consecutive integers i with $a \leqslant i \leqslant b$.

Definition II.2. The *symmetric group* $\mathfrak{S}_n$ is the group of permutations on n letters, where a permutation is a bijection from $[1, n] \to [1, n]$. We write vw for the composition $v \circ w$ given by $v \circ w(i) = v(w(i))$.

Definition II.3. The *one-line notation* for a permutation $w \in \mathfrak{S}_n$ is the sequence of letters $w(1)w(2) \cdots w(n)$.

The group $\mathfrak{S}_n$ is generated by the *elementary transpositions* $\{s_i | 1 \leqslant i \leqslant n-1\}$, where s_i swaps the letters i and $i+1$ and fixes $j \notin \{i, i+1\}$, with the following defining relations.

- $s_i^2 = 1$.
- $s_i s_j = s_j s_i$ when $|i - j| > 1$.
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Definition II.4. An *inversion* of $w \in \mathfrak{S}_n$ is an ordered pair (i, j) with $i < j$ and $w(i) > w(j)$. We write $\mathrm{inv}\,(w)$ for the set of inversions of w.

Note that $\operatorname{inv}(s_i) = \{(i, i+1)\}$.

Definition II.5. The *length* of w is $\ell(w) = \#\operatorname{inv}(w)$.

Note that $\ell(w) = 1$ if and only if $w = s_i$ for some i.

For any $w \in \mathfrak{S}_n$ and any elementary transposition s_i, either $\ell(ws_i) = \ell(w) + 1$ or $\ell(ws_i) = \ell(w) - 1$.[4]

Definition II.6. A *word* for w is a sequence $s_{i_1}, \cdots, s_{i_k}$ with product $s_{i_1} \cdots s_{i_k} = w$. A word for w is *reduced* if $k = \ell(w)$.

Definition II.7. A *subword* of $s_{i_1} \cdots s_{i_k}$ is an ordered subsequence $s_{i_{j_1}} \cdots s_{i_{j_m}}$.

Definition II.8. The *permutation matrix* corresponding to $w \in \mathfrak{S}_n$ is the $n \times n$ matrix $\mathcal{P}(w)$ with entries $\mathcal{P}(w)_{w(j),j} = 1$ and $\mathcal{P}(w)_{ij} = 0$ if $i \neq w(j)$. This gives an isomorphism between $\mathfrak{S}_n$ and the group of $n \times n$ matrices with exactly one 1 in each row and each column and other entries 0.

Definition II.9. Let M be an $n \times n$ invertible matrix and denote the submatrix of M with rows I and columns J by M^I_J. We define the northwest and southwest rank matrices of M as follows:

${}^{\mathrm{NW}}\mathrm{Rank}_{ij}(M) = \mathrm{Rank}\left(M^{[1,i]}_{[1,j]}\right)$

${}_{\mathrm{SW}}\mathrm{Rank}_{ij}(M) = \mathrm{Rank}\left(M^{[i,n]}_{[1,j]}\right)$

Definition II.10. Let M be an $n \times n$ matrix. Let R and C be subsets of $[1, n]$ with $|R| = |C|$. We denote the minor of the submatrix M^R_C by Δ^R_C.

In the case of a permutation matrix $\mathcal{P}(w)$, the rank of any submatrix is its number of nonzero entries, so that the entry ${}^{\mathrm{NW}}\mathrm{Rank}_{ij}(\mathcal{P}(w))$ is the number of 1s in the northwest block $\mathcal{P}(w)^{[1,i]}_{[1,j]}$, and the entry ${}_{\mathrm{SW}}\mathrm{Rank}_{ij}(\mathcal{P}(w))$ is the number of 1s in the southwest block $\mathcal{P}(w)^{[i,n]}_{[1,j]}$.

$$\mathcal{P}(s_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad {}^{\text{NW}}\text{Rank}\,(\mathcal{P}(s_2)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad {}_{\text{SW}}\text{Rank}\,(\mathcal{P}(s_2)) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{P}(s_2 s_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad {}^{\text{NW}}\text{Rank}\,(\mathcal{P}(s_2 s_1)) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad {}_{\text{SW}}\text{Rank}\,(\mathcal{P}(s_2 s_1)) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Figure 2.1: Permutation matrices and rank matrices for the words $u = s_2$ and $w = s_2 s_1$

We say that $u \leqslant w$ in the *Bruhat order* if it meets one of the following criteria. (See Corollary 2.2.3 and Theorem 2.1.5 [4].)

Proposition II.11. *Let u and w be permutations in $\mathfrak{S}_n$. The following are equivalent:*

1. *Every reduced word $s_{i_1} \cdots s_{i_k}$ for w has a subword that is a reduced word for u.*
2. *Some reduced word $s_{i_1} \cdots s_{i_k}$ for w has a subword that is a reduced word for u.*
3. *For every i and j in $[1, n]$, $\#\{a \in [1, i] : u(a) \geqslant j\} \leqslant \#\{a \in [1, i] : w(a) \geqslant j\}$.*
4. *For every i and j in $[1, n]$, ${}_{\text{SW}}\mathit{Rank}_{ij}\,(\mathcal{P}(u)) \leqslant {}^{\text{NW}}\mathit{Rank}_{ij}\,(\mathcal{P}(w))$.*
5. *For every i and j in $[1, n]$, ${}^{\text{NW}}\mathit{Rank}_{ij}\,(\mathcal{P}(u)) \geqslant {}_{\text{SW}}\mathit{Rank}_{ij}\,(\mathcal{P}(w))$.*

In Figure 2.1, we note that for the words $u = s_2$ and $w = s_2 s_1$, the matrices ${}^{\text{NW}}\text{Rank}\,(\mathcal{P}(u))$ and ${}^{\text{NW}}\text{Rank}\,(\mathcal{P}(w))$ agree in columns 2 and 3, and so do the matrices ${}_{\text{SW}}\text{Rank}\,(\mathcal{P}(u))$ and ${}_{\text{SW}}\text{Rank}\,(\mathcal{P}(w))$, while the entries in the first columns satisfy inequalities ${}_{\text{SW}}\text{Rank}_{i1}\,(\mathcal{P}(u)) \leqslant {}_{\text{SW}}\text{Rank}_{i1}\,(\mathcal{P}(w))$ and ${}^{\text{NW}}\text{Rank}_{i1}\,(\mathcal{P}(u)) \geqslant {}^{\text{NW}}\text{Rank}_{i1}\,(\mathcal{P}(w))$. The rank matrices corresponding to pairs of permutations u and w satisfy the same type of inequalities whenever u can be expressed as a subword of a reduced word for w.

2.2 Flags

Let G denote $SL_n(\mathbb{C})$, the group of $n \times n$ matrices with entries in $\mathbb{C}$ and determinant 1. Let B_+ and B_- denote the Borel subgroups of upper and lower triangular matrices. The type A *flag variety* $\mathcal{F}\ell_n = G/B_+$ is the variety of sequences of subspaces $F = F^1 \subset F^2 \subset \cdots \subset \cdots F^n = \mathbb{C}^n$ satisfying $\dim F^i = i$. Given a matrix $g \in G$, the successive spans of the first i columns determine a flag. Matrices g_1 and g_2 determine the same flag if there is an upper triangular matrix $b \in B_+$ so that $g_1 b = g_2$. We identify a flag F with its coset of matrix representatives gB_+. Since ratios of maximal minors on a given collection of vectors are preserved by elementary column operations, the following data are independent of the choice of g.

1. The condition that $\Delta^R_{[1,h]} \neq 0$.
2. The value of the ratio $\frac{\Delta^{R_1}_{[1,h]}(g)}{\Delta^{R_2}_{[1,h]}(g)}$, where R_2 is chosen so that the denominator is nonvanishing.

In particular, the matrices ${}^{\mathrm{NW}}\mathrm{Rank}\,(g)$ and ${}_{\mathrm{SW}}\mathrm{Rank}\,(g)$ are well-defined functions of gB_+. For each flag gB_+, there are unique permutations u and w so that ${}^{\mathrm{NW}}\mathrm{Rank}\,(g)$ $= {}^{\mathrm{NW}}\mathrm{Rank}\,(\mathcal{P}(u))$ and ${}_{\mathrm{SW}}\mathrm{Rank}\,(g) = {}_{\mathrm{SW}}\mathrm{Rank}\,(\mathcal{P}(w))$. In the general linear group $GL_n(\mathbb{C})$ ${}^{\mathrm{NW}}\mathrm{Rank}\,(g) = {}^{\mathrm{NW}}\mathrm{Rank}\,(\mathcal{P}(u))$ if and only if $g = L\mathcal{P}(u)U$ for some lower triangular L and upper triangular U, and ${}_{\mathrm{SW}}\mathrm{Rank}\,(g) = {}_{\mathrm{SW}}\mathrm{Rank}\,(\mathcal{P}(w))$ if and only if $g = U_1\mathcal{P}(w)U_2$ for some upper triangular matrices U_1 and U_2 (see Fulton[13]). Since the determinant of a permutation matrix $\mathcal{P}(w)$ is given by $(-1)^{\#\ell(w)} = \pm 1$, the matrices $\mathcal{P}(u)$ and $\mathcal{P}(w)$ need not be elements of G, but there are analogous conditions ${}^{\mathrm{NW}}\mathrm{Rank}\,(g) = {}^{\mathrm{NW}}\mathrm{Rank}\,(g')$ if and only if $B_-gB_+ = B_-g'B_+$ and ${}_{\mathrm{SW}}\mathrm{Rank}\,(g) =$ ${}_{\mathrm{SW}}\mathrm{Rank}\,(g')$ if and only if $B_+gB_+ = B_+g'B_+$.

2.2.1 1-parameter subgroups and signed transposition matrices

Following Marsh and Rietsch,[27] we write $y_h(t)$ and $x_h(t)$ for the 1-parameter subgroups of G whose elements look like the identity matrix with the $(h, h+1)$

diagonal block replaced by the blocks $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$.

There is a set-theoretic inclusion of $\mathfrak{S}_n$ into G induced by sending each elementary transposition s_h to the matrix $\dot{s}_h = y_h(1)x_h(-1)y_h(1)$, giving the block $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ at the $(h, h+1)$ position. The matrices $\dot{s}_h$ have relations $\dot{s}_h\dot{s}_k = \dot{s}_k\dot{s}_h$ for $|h-k| \geqslant 2$ and $\dot{s}_h\dot{s}_{h+1}\dot{s}_h = \dot{s}_{h+1}\dot{s}_h\dot{s}_{h+1}$; we note that $\dot{s}_h^{-1} = y_h(-1)x_h(1)y_h(-1)$, with $(h, h+1)$ block so that $\dot{s}_h$ is not an involution. For each $w \in \mathfrak{S}_n$, let $s_{h_1} \cdots s_{h_\ell}$ be a reduced word for w and let $\dot{w}$ be the product $\dot{s}_{h_1} \cdots \dot{s}_{h_\ell}$. From Deodhar, the matrix $\dot{w}$ is independent of the choice of reduced word.[7]

Remark II.12. By construction, the matrices $\dot{w}$ have the property that the minor on rows $w([1, h])$ and columns $[1, h]$ has determinant 1, since a column is multiplied by -1 exactly when it introduces an involution.

2.3 Schubert cells, opposite Schubert cells and open Richardson varieties

The flag variety G/B_+ has two natural decompositions into *Schubert cells* $B_+\dot{w}B_+/B_+$ consisting of flags with southwest rank matrix ${}_{\mathrm{SW}}\mathrm{Rank}\,(\mathcal{P}(w))$ and *opposite Schubert cells* $B_-\dot{u}B_+/B_+$ corresponding to flags with northwest rank matrix ${}^{\mathrm{NW}}\mathrm{Rank}\,(\mathcal{P}(u))$. Following Fulton,[13] each flag gB_+ in the Schubert cell $B_+\dot{w}B_+/B_+$ has a unique coset representative of the form $\hat{z}\dot{w}$, where $\hat{z}$ is upper triangular with diagonal entries 1 and $\hat{z}\dot{w}_{w(i),j} = 0$ whenever $j > i$. The entries indexed by permutation positions give $\ell\,(w)$ independent coordinates on $B_+\dot{w}B_+/B_+$.

Remark II.13. It follows immediately that if $\lambda_1, \cdots, \lambda_n$ are scalars satisfying $\prod_{i=1}^{n}\lambda_i = 1$, there is a unique $z \in B_+$ so that the flag $z\dot{w}B_+ = gB_+$, the diagonal entry $z_{ii} = \lambda_i$ and $z\dot{w}_{w(i),j} = 0$ for $j > i$.

Definition II.14. The *open Richardson variety* $\mathcal{R}^{u,w}$ is the intersection $B_+\dot{w}B_+/B_+ \cap B_-\dot{u}B_+/B_+$.

From Marsh and Rietsch,[27] whenever $u \leqslant w$ in the Bruhat order, $\mathcal{R}^{u,w}$ is nonempty and has dimension $\ell(w) - \ell(u)$, while if $u \nleqslant w$, then $\mathcal{R}^{u,w}$ is empty. We recall that if $u \nleqslant w$, then there exists some index h so that $u([1,h]) \nleqslant w([1,h])$, so that $\Delta_{[1,h]}^{u([1,h])}$ vanishes identically on $B_+\dot{w}B_+/B_+$, while $\Delta_{[1,h]}^{u([1,h])}$ is everywhere nonvanishing on $B_-\dot{u}B_+/B_+$.

2.4 Flag reductions and Deodhar's decomposition

Given any reduced word $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$, Deodhar[7] showed that the open Richardson variety $\mathcal{R}^{u,w}$ breaks down into a disjoint union of simpler strata. The largest piece is a dense open torus, and the remaining strata are of lower dimension and isomorphic to products of tori with affine spaces.

Notation II.15. Let $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$ be a reduced expression for w. We write $w_{(i)}$ for the partial product $s_{h_1} \cdots s_{h_i}$.

Let $F = F_1 \subset \cdots \subset F_n$ be a flag in $\mathcal{R}^{u,w}$ and let $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$ be a reduced word for w. There is a unique flag $\tilde{F}$ in $B_+\dot{w}_{(\ell-1)}B_+/B_+$ so that the subspaces $\tilde{F}_h = F_h$ agree for $h \neq h_\ell$. We say that $\tilde{F}$ is the projection of F to the Schubert cell $B_+\dot{w}_{(\ell-1)}B_+/B_+$. If z is an upper triangular matrix with $z\dot{w}$ is a coset representative for the flag F, then $z\dot{w}_{(\ell-1)}$ is a coset representative for $\tilde{F}$. (See [27].)

Inductively, if $gB_+ \in \mathcal{R}^{u,w}$, then there is a unique sequence of flags $F^\ell = gB_+$, $F^{\ell-1}, \cdots, F^0 = B_+$ obtained by projecting the flag F^i in $B_+\dot{w}_{(i)}B_+/B_+$ to the Schubert cell $B_+\dot{w}_{(i-1)}B_+/B_+$. Deodhar considered the sequence of permutations $u_{(k)}$ so that the flag F^k is in the opposite Schubert cell $B_-\dot{u}_{(k)}B_+/B_+$, and gave the following characterization.

Definition II.16. Let $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$ be a reduced expression for w. A sequence of permutations $\mathbf{v} = v_{(0)}, \cdots, v_{(\ell)}$ is a *distinguished subexpression* of $\mathbf{w}$ if $v_{(0)} = 1$ and for each index i with $1 \leqslant i \leqslant \ell$, the following conditions hold.

1. Either $v_{(i)} = v_{(i-1)}$ or $v_{(i)} = v_{(i-1)}s_{h_i}$.
2. Whenever $v_{(i-1)}s_{h_i} < v_{(i-1)}$ in the Bruhat order, $v_{(i)} = v_{(i-1)}s_{h_i}$.

We say that $\mathbf{v}$ is a distinguished subexpression for u if $\mathbf{v}$ is a distinguished subexpression satisfying $v_{(\ell)} = u$.

Notation II.17. [27] We use the notation $\mathbf{v} \prec \mathbf{w}$ to mean that $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$.

Definition II.18. [27] Let w and $u \in \mathfrak{S}_n$ with $u < w$. Fix a reduced expression $\mathbf{w}$ for w, and let $\mathbf{v} \prec \mathbf{w}$ be a distinguished subexpression for u in $\mathbf{w}$. We define $J^{\circ}_{\mathbf{v}} = \{j : v_{(j)} = v_{(j-1)}\}$, $J^{+}_{\mathbf{v}} = \{j : v_{(j)} > v_{(j-1)}\}$ and $J^{-}_{\mathbf{v}} = \{j : v_{(j)} < v_{(j-1)}\}$.

Notation II.19. [27] Given a reduced expression $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$, we encode a distinguished subexpression $\mathbf{v}$ by writing $\underline{s}_{h_i}$ to indicate that $v_{(i)} = v_{(i-1)}s_{h_i}$.

Theorem II.20. *[7] Fix a reduced word* $\mathbf{w}$ *for* w *and let* $\mathbf{v} = v_{(0)}, \cdots, v_{(\ell)}$ *be a sequence of permutations. The locus* $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ *given by* $\{z\dot{w}B_+ \in \mathcal{R}^{u,w} : z\dot{w}_{(i)}B_+ \in \mathcal{R}^{v_{(i)},w_{(i)}}$ *for all* $0 \leqslant i \leqslant \ell\}$ *is non-empty if and only if* $\mathbf{v}$ *is a distinguished subexpression of* $\mathbf{w}$*. For each distinguished subexpression* $\mathbf{v}$*,* $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ *is isomorphic to* $(\mathbb{C}^*)^{|J^{\circ}_{\mathbf{v}}|} \times \mathbb{C}^{|J^{-}_{\mathbf{v}}|}$*.*

Definition II.21. Let $\mathbf{v}$ be a distinguished subexpression of $\mathbf{w}$ with product u. $\mathbf{v}$ is called a *positive subexpression* if for each $1 \leqslant i \leqslant \ell$, we have $v_{(i-1)} \leqslant v_{(i)}$.

Marsh and Rietsch proved the following lemma, which implies that the decomposition of $\mathcal{R}^{u,w}$ corresponding to a reduced word $\mathbf{w}$ has a unique maximal torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}} \cong (\mathbb{C}^*)^{\ell(w)-\ell(u)}$.

Lemma II.22 (Lemma 3.2[27]). *Let* $u \leqslant w$ *in the Bruhat order and fix a reduced word* $\mathbf{w}$ *for* w*. There is a unique positive subexpression of* $\mathbf{w}$ *with product* u*, obtained by setting* $u_{(\ell)} = u$ *and taking* $u_{(i-1)} = u_{(i)}s_{h_i}$ *whenever* $u_{(i)}s_{h_i} < u_{(i)}$*.*

One of our initial goals is to show that there are regular functions $X_1, \cdots, X_{\ell(w)-\ell(u)}$ giving a parametrization of $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ which satisfy the coprimeness condition that for each $i \neq j$, the locus $\{X_i = 0\} \cap \{X_j = 0\}$ has complex codimension $\geqslant 2$. In particular, to describe the regular functions on $\mathcal{R}^{u,w}$, it suffices to check regularity on the torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ and on each hypersurface $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$, where $\mathbf{v}^j$ is a distinguished subexpression of $\mathbf{w}$ which gets shorter at a unique index.

Definition II.23. Let $\mathbf{u}_+ = u_{(0)}, \cdots, u_{(\ell)}$ be the positive subexpression for u in $\mathbf{w}$. Let j be an index with $u_{(j-1)} = u_{(j)}$. We define the *nearly positive sequence* $\mathbf{v}^j$ by setting $v^j_{(k)} = u_{(k)}$ for all $k \geqslant j$, $v^j_{(j-1)} = u_{(j)} s_{h_j}$ and $v^j_{(i-1)} = v^j_{(i)} s_{h_j}$ whenever $v^j_{(i)} s_{h_i} < v^j_{(i)}$. $\mathbf{v}^j$ is a *nearly positive subexpression* for u if $v^j_{(0)} = 1$.

Corollary II.24. *Let $\mathbf{v}$ be a distinguished subexpression for u in $\mathbf{w}$. The Deodhar stratum $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ is a hypersurface if and only if $\mathbf{v}$ is the nearly positive subexpression $\mathbf{v}^j$ for some index j with $u_{(j)} = u_{(j-1)}$ and $u_{(0)} = 1$.*

Proof. We note that if $\mathbf{v}$ is a nearly positive subexpression $\mathbf{v}^j$, then by construction it has $\ell(u) + 2$ factors and hence gives a hypersurface by Theorem II.20. The distinguished subexpression $\mathbf{v}$ gives a hypersurface if and only if there is a unique index where $v_{(j-1)} > v_{(j)}$. Since $\mathbf{v}$ and $\mathbf{u}_+$ are both distinguished subexpressions for u, we have $v_{(\ell)} = u = u_{(\ell)}$. For $k > j$, both sequences decrease whenever possible, giving $v_{(j)} = u_{(j)}$, and hence $v_{(j-1)} = u_{(j)} s_{h_j} > u_{(j)}$, so that $u_{(j-1)} = u_{(j)}$ by positivity of $\mathbf{u}_+$. By hypothesis, $\mathbf{v}^j$ has $\ell(u) + 2$ factors, so that $v_{(0)}, \cdots, v_{(j-1)}$ must be the unique positive subexpression for $v_{(j-1)} = u_{(j-1)} s_{h_j}$ inside $w_{(0)}, \cdots, w_{(j-1)}$. So $\mathbf{v} = \mathbf{v}^j$, and hence $\mathbf{v}^j$ must be a distinguished subexpression. □

In Example II.25, we compute the nearly positive sequences for the expression $\mathbf{w} = s_4 s_3 s_2 s_1 \underline{s}_2 s_3 s_4$. In general, if $i \in J^\circ_{\mathbf{u}_+}$ is the index of the first factor of s_h in $\mathbf{w}$, the sequence $\mathbf{v}^i$ is not a nearly positive subexpression. In this example, we will see that this is not an equivalence: the sequence $\mathbf{v}^6$ fails to be a nearly positive subexpression. This means that the cluster variable X_6 (as described in Chapter V) is frozen.

Example II.25. Let $\mathbf{w} = s_4 s_3 s_2 s_1 \underline{s_2} s_3 s_4$, with positive subexpression $\mathbf{u}_+$ given by the underlined factor $\underline{s_2}$. In sequence notation, $\mathbf{u}_+ = (u_{(0)}, \cdots, u_{(7)}) = (1, 1, 1, 1, 1, s_2, s_2, s_2)$. The nearly positive sequences of $\mathbf{w}$ are given in the following table. We note that each $\mathbf{v}^i$ agrees with $\mathbf{u}_+$ for indices $i, i+1, \cdots, \ell = 7$.

Index $i \in J^\circ_{\mathbf{u}_+}$	$v^i_{(i-1)} := u_{(i-1)} s_{h_i}$	$\mathbf{v}^i = (v^i_{(0)}, v^i_{(1)}, \cdots, v^i_{(7)})$
1	s_4	$(s_4, 1, 1, 1, 1, s_2, s_2, s_2)$
2	s_3	$(s_3, s_3, 1, 1, 1, s_2, s_2, s_2)$
3	s_2	$(s_2, s_2, s_2, 1, 1, s_2, s_2, s_2)$
4	s_1	$(s_1, s_1, s_1, s_1, 1, s_2, s_2, s_2)$
6	$s_2 s_3$	$(s_2 s_3, s_2 s_3, s_2 s_3, s_2 s_3, s_2 s_3, s_2, s_2)$
7	$s_2 s_4$	$(1, s_4, s_4, s_4, s_2 s_4, s_2 s_4, s_2 s_4, s_2)$

The sequence $\mathbf{v}^7$ is a nearly positive subexpression of $\mathbf{w}$, since it satisfies the condition $v^7_{(0)} = 1$. We do not define a nearly positive sequence $\mathbf{v}^5$ since the index 5 is in $J^+_{\mathbf{u}_+}$, corresponding to the underlined factor $\underline{s_2}$.

2.5 Cluster algebras from quivers

Cluster algebras are an algebraic structure introduced by Fomin and Zelevinsky, motivated by the study of total positivity. In [9], they characterized minimal total nonnegativity criteria for matrices in the double Bruhat cell $G^{u,v}$. Using determinantal identities, they showed that if the minors indexed by the chambers of a double wiring diagram for the pair of permutations (u, v) are nonzero, then every minor of the matrix can be expanded as a subtraction-free Laurent polynomial in the chamber minors. Further, they showed that one set of total positivity criteria can be transformed to another by a sequence of *exchanges*, where one function X is exchanged for another function X' obeying a subtraction-free binomial exchange relation $XX' = M_1 + M_2$.

In [10], Fomin and Zelevinsky defined cluster algebras to formalize the study of the rings generated by *clusters* related by a similar subtraction-free binomial exchange recurrence.

We will restrict our attention to cluster algebras from quivers, as described by Williams in [34].

Definition II.26 (Ice quivers)**.** An *ice quiver* is a directed multigraph $Q = (V, E)$ so that whenever there is an arrow e from v_1 to v_2, there is no arrow from v_2 to v_1, together with the following data and constraints.

1. Each vertex $v \in V$ is designated either *mutable* or *frozen*.
2. For each arrow $e \in E$, at least one endpoint of e is mutable.
3. For each mutable vertex v_k, there is a *mutation rule* μ_k sending Q to a new quiver $Q' = (V, \mu_k(E))$. The arrows $\mu_k(E)$ are obtained from E as follows.
   - For any length 2 path $v_i \xrightarrow{e_1} v_k \xrightarrow{e_2} v_j$ in Q with at least one of the endpoints v_i and v_j mutable, an arrow $v_i \to v_j$ is added to E'.
   - Arrows starting or ending at v_j are reversed.
   - Cycles of length 2 are eliminated by removing pairs of arrows $v_i \to v_j$ and $v_j \to v_i$.

The vertices of the ice quiver are labeled by a *cluster* $\mathbf{X}$ of algebraically independent indeterminates, given the same designation of frozen or mutable as the corresponding vertices. When the quiver is mutated at a mutable vertex v_k, the variable X_k is exchanged for a variable X'_k, where the product $X_k X'_k$ is a binomial in the other cluster variables encoded by the arrows of the quiver. The elements of the original cluster $\mathbf{X}$ and the mutated cluster $\mu_k(\mathbf{X})$ generate the same field extension over $\mathbb{C}$.

Definition II.27. Given an ice quiver Q with mutable vertices $v_1, \cdots, v_N$ and frozen vertices $v_{N+1}, \cdots, v_{N+M}$, a *cluster* is a tuple of variables indexed by V with X_i designated mutable if v_i is mutable and frozen if v_i is frozen. The pair $\Sigma = (\mathbf{X}, Q)$ is called a *seed*. The *seed mutation* $\mu_k(\Sigma)$ is the mutation of Q at vertex v_k together with the *cluster mutation* $\mu_k(\mathbf{X}) = \mathbf{X} \backslash \{X_k\} \cup \{X'_k\}$, where X_k and X'_k satisfy the exchange relation $X_k X'_k = \prod_{v_k \xrightarrow{e} v_j} X_j + \prod_{v_j \xrightarrow{e} v_k} X_j$.

For any mutable vertex v_k, the mutation $\mu_k(\Sigma)$ is an involution. In general, the mutations μ_i and μ_k do not commute under composition.

Definition II.28 (Mutation equivalence)**.** Two seeds Σ and Σ' are *mutation-equivalent* if there is a sequence of mutations transforming Σ to Σ'. In this case we write $\Sigma \sim \Sigma'$.

Definition II.29 (The cluster algebra generated by an initial seed)**.** Let $\Sigma = (\mathbf{X}, Q)$ be a seed. The *cluster algebra* generated by Σ is the $\mathbb{C}$-algebra $\mathcal{A}\Sigma$ generated by all cluster variables appearing in seeds mutation-equivalent to Σ.

Fomin and Zelevinsky proved that whenever $\Sigma = (\mathbf{X}, Q)$ and $\Sigma' = (\mathbf{X}', Q')$ are mutation-equivalent seeds, every cluster variable X'_k in the cluster $\mathbf{X}'$ can be written as a Laurent polynomial in the cluster variables $\mathbf{X}$. That is, the cluster algebra $\mathcal{A}(\Sigma)$ is contained in the ring of Laurent polynomials in the variables of any given cluster $\mathbf{X}$. Given an initial seed Σ, Berenstein, Fomin and Zelevinsky defined the upper bound algebra $\mathcal{U}(\Sigma)$ as the intersection of the Laurent rings in the cluster variables belonging to seeds within one mutation from Σ and the upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$ as the ring of functions that are Laurent in the cluster variables of every seed mutation equivalent to Σ.

Definition II.30 (Upper bounds and upper cluster algebras)**.** Let $\Sigma = (\mathbf{X}, Q)$ be a seed. The *upper bound* for Σ is the algebra given by $\mathcal{U}(\Sigma) = \mathbb{C}[\mathbf{X}^{\pm}] \cap \bigcap_{X_k \text{ mutable}} \mathbb{C}[\mu_k(\mathbf{X})^{\pm}]$. The *upper cluster algebra* generated by Σ is given by $\overline{\mathcal{A}}\Sigma = \bigcap_{\substack{\Sigma' = (\mathbf{X}', Q') \\ \Sigma' \sim \Sigma}} \mathbb{C}[\mathbf{X}^{\pm}]$.

By Theorem 1.5 in [3], the upper bound algebra $\mathcal{U}(\Sigma)$ is equal to the upper cluster algebra in the case where exchange binomials in every seed are pairwise coprime.

We will find it computationally convenient to describe the exchange relations for a cluster variable in the initial seed in terms of the ratio of the two monomials on the right hand side; that is, the $\hat{y}$-variables defined by Fomin and Zelevinsky in [11].

Definition II.31 ($\hat{y}$-variables)**.** Let v_i be a mutable vertex. The *$\hat{y}$-variable* for the cluster variable X_k is $\hat{y}_k = \dfrac{\prod_{v_k \xrightarrow{e} v_j} X_j}{\prod_{v_j \xrightarrow{e} v_k} X_j}$.

CHAPTER III

Wiring Diagrams

3.1 Wiring diagrams

Wiring diagrams are a combinatorial object for keeping track of where a left-justified interval $[1, h]$ is sent by partial products of the first i terms of a word $s_{h_1} \cdots s_{h_\ell}([1, h])$, which applies in our context to keeping track of which minors of an upper triangular matrix z are determined by ratios of left-justified minors of the flag $z\dot{w}_{(i)}B_+$.

We begin with a general discussion of the wiring diagram associated to a word $\mathbf{w}$.[2] [3]

Let $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$ be a word for w (not necessarily reduced). The *wiring diagram* associated to $\mathbf{w}$ is given by the following procedure: We begin with n horizontal line segments or *strands*, with endpoints labeled from bottom to top as $\lambda_1, \cdots, \lambda_n$ on the left and $\rho_1, \cdots, \rho_n$ on the right. The *height* of a strand α is the number of strands weakly below it.

For each index i, we insert a *crossing* at height h_i corresponding to the transposition s_{h_i} by extending the diagram at the right, so that the strands with endpoints labeled ρ_{h_i} and ρ_{h_i+1} cross transversely, switching heights.

Notation III.1. We refer to the strand that travels up at crossing i as the *rising*

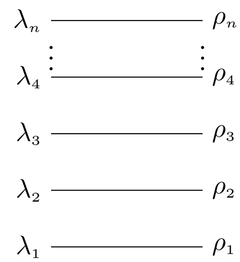

Figure 3.1: Wiring diagram for the empty word $\mathbf{w} = 1$

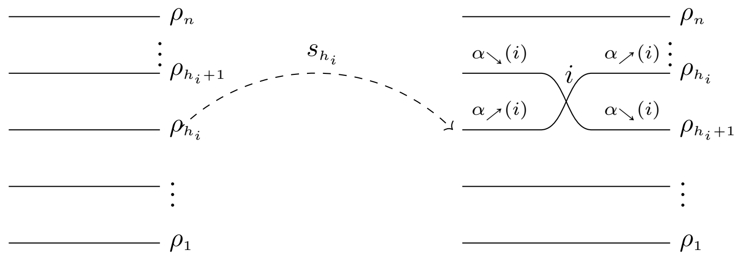

Figure 3.2: Adding a crossing at height h_i

strand, denoted by $\alpha_{\nearrow}(i)$; the strand that travels down is the *falling strand*, denoted by $\alpha_{\searrow}(i)$.

We will also need refer to the set of crossings with a particular rising or falling strand.

Notation III.2. For each strand α, we define the *ascending indices* of α to be the set

$$J_{\nearrow}(\alpha) = \{j : \alpha_{\nearrow}(j) = \alpha\}.$$

We define the *descending indices* of α to be

$$J_{\searrow}(\alpha) = \{j : \alpha_{\searrow}(j) = \alpha\}.$$

Notation III.3. We write $\lambda(\mathbf{w})(\alpha) = h$ if the strand α has left endpoint λ_h. We write $\rho^i(\mathbf{w})(\alpha) = h$ if α has right endpoint ρ_h after crossing i is added (or equivalently, if α is at height h immediately to the right of crossing i).

The wiring diagram for $\mathbf{w} = s_{h_1} \cdots s_{h_\ell}$ is *reduced* if $\mathbf{w}$ is a reduced word, which is equivalent to the condition that each pair of strands α and β cross at most once.

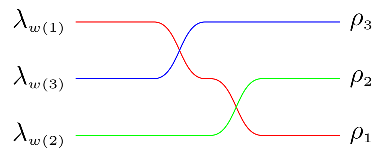

Figure 3.3: Wiring diagram for the reduced word $\mathbf{w} = s_2s_1$

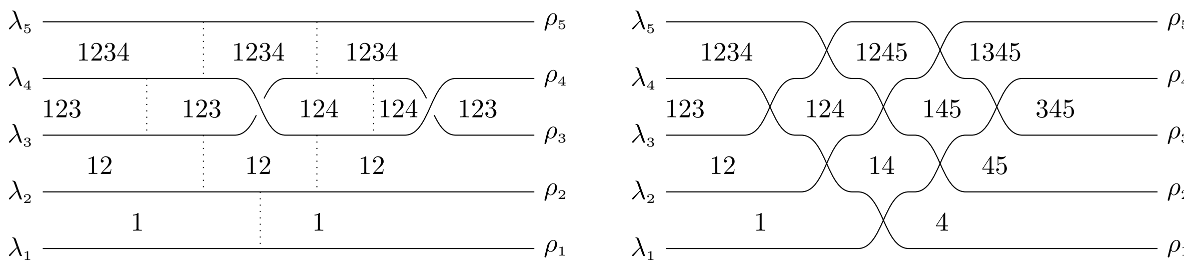

Figure 3.4: Upper and lower arrangements for $s_3s_2s_1s_4\underline{s}_3s_2s_4\underline{s}_3$

Definition III.4. A *left-to-right path* in a wiring diagram is a sequence of strands $\alpha_i, \cdots, \alpha_m$ indexed by consecutive crossings $i \leqslant j \leqslant m$, such that for each $j \geqslant i+1$, either $\alpha_j = \alpha_{j-1}$ or α_j and α_{j-1} cross at index j. That is, $\alpha_i, \cdots, \alpha_m$ is a left-to-right path if there is some $h \in [1, n]$ and a subexpression $\mathbf{v}$ of $\mathbf{w}$ so for each $i \leqslant j \leqslant m$ we have $\left(v_{(j)}\right)^{-1}(h) = \rho^j(\mathbf{w})(\alpha_j)$. A left-to-right path is *connected* if for each crossing j where $\alpha_j \neq \alpha_{j-1}$, we have $j \in J^\circ$.

In general, a left-to-right path may refer to a sequence of edges in the non-planar graph which projects onto a path.

Definition III.5 (Upper and Lower Arrangements)**.** Let $\mathbf{w}$ be a reduced expression for w with distinguished subexpression $\mathbf{v}$. The *upper arrangement* is the wiring diagram with crossings corresponding to the factors of $\mathbf{v}$. The *lower arrangement* is the wiring diagram corresponding to $\mathbf{w}$.

Definition III.6. Let r be an index with $1 \leqslant r \leqslant n$. The *geodesic path* γ_r is the left-to-right path in the wiring diagram for $\mathbf{w}$ corresponding to the strand in the upper arrangement with left endpoint λ_r and right endpoint $\rho_{u^{-1}(r)}$.

Notation III.7. We write $\lambda^i(\mathbf{u}_+)(\alpha) = r$ if the right endpoint of the strand α at index i is at height h where $u_{(i)}(h) = r$ (or equivalently, if the right endpoint of α at index h is on the geodesic path γ_r). We write $\rho^i(\mathbf{u}_+)(r) = h$ if the geodesic path with left endpoint λ_r has right endpoint ρ_h at index i.

Definition III.8. We say that the geodesic path γ_r *crosses below* the strand $\alpha_{\nearrow}(i)$ if either of the following holds:

1. $i \in J^+_{\mathbf{u}_+}$ and γ_r follows $\alpha_{\searrow}(i)$ immediately to the left and right of crossing i.
2. $i \in J^\circ_{\mathbf{u}_+}$ and γ_r follows $\alpha_{\nearrow}(i)$ to the left of i and $\alpha_{\searrow}(i)$ to the right of i.

3.2 Chambers in reduced wiring diagrams

Consider an embedding of the wiring diagram in a rectangle so that the endpoints of the strands are on the boundary of the rectangle.

A *closed chamber* at height h in a wiring diagram is the bounded region between consecutive crossings at height h. A *left boundary chamber* at height h is the region between the strands with left endpoints λ_h and λ_{h+1} and the left boundary of the rectangle; a *right boundary chamber* at height h is enclosed by the strands with endpoints ρ_h and ρ_{h+1} and the right boundary of the rectangle. The wiring diagram in Figure 3.5 has a single closed chamber, C. Chamber D is both a left boundary chamber and a right boundary chamber.

We will use the Korean letter ㅊ ("chieut") to represent a chamber, and we write Strands(ㅊ) for the set of strands below ㅊ. We denote the four chambers surrounding the crossing at index i by ㅊ$_{\leftarrow}(i)$, ㅊ$_{\uparrow}(i)$, ㅊ$_{\downarrow}(i)$ and ㅊ$_{\rightarrow}(i)$, as shown in Figure 3.6.[1] The pairs of chambers {ㅊ$_{\leftarrow}(i)$, ㅊ$_{\rightarrow}(i)$} and {ㅊ$_{\uparrow}(i)$, ㅊ$_{\downarrow}(i)$} are

[1] These can be pronounced as "the west chamber of i" for ㅊ$_{\leftarrow}(i)$, etc.

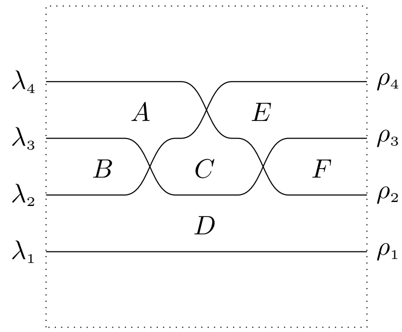

Figure 3.5: Chambers in a wiring diagram

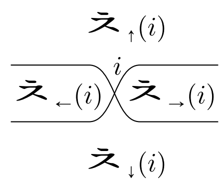

Figure 3.6: Relative indexing for the chambers west, north, south and east of a crossing i

opposing; we refer to the other pairs of chambers surrounding ㅊ as *adjacent*.

3.3 Unipeak wiring diagrams

In this section, we describe a distinguished class of reduced wiring diagrams that will enable us to define an initial seed for each open Richardson variety $\mathcal{R}^{u,w}$. Unipeak diagrams are the wiring diagrams associated to the *canonical sequences* studied by Kassel, Lascoux and Reutenauer[22] in the context of factorizations of the Schubert form representative of a flag in $B_+\dot{v}B_+$.

Definition III.9. A wiring diagram is *unipeak* if for each strand α there is an index j so that the sequence $\rho^i(\mathbf{w})(\alpha)$ is weakly increasing for all $i \leqslant j$ and weakly decreasing for all $i \geqslant j$.

Proposition III.10. *Every unipeak wiring diagram is reduced.*

Proof. Suppose to the contrary that there is a pair of strands that cross twice. Then there are indices $i < j$ so that $\alpha_{\nearrow}(i) = \alpha_{\searrow}(j)$ and $\alpha_{\searrow}(i) = \alpha_{\nearrow}(j)$, which contradicts the hypothesis that no strand travels up after traveling down. $\square$

Proposition III.11. *A reduced wiring diagram is unipeak if and only if for every crossing j, there is no strand β which crosses below $\alpha_{\nearrow}(j)$ to the left of i and crosses above $\alpha_{\searrow}(j)$ to the right of j.*

Proof. It is immediate from the definition that if a diagram is unipeak, no such strand β exists since a strand in a unipeak diagram cannot travel up after traveling down. Conversely, if a reduced diagram fails to be unipeak, then choose some strand β so that there are indices $i < k$ with $\alpha_{\searrow}(i) = \beta = \alpha_{\nearrow}(k)$. Since the wiring diagram is reduced, the strand $\alpha_{\nearrow}(i)$ stays above β to the right of i and the strand $\alpha_{\searrow}(k)$ is above β to the left of k. So there must be some index j with $i < j < k$ with $\alpha_{\nearrow}(j) = \alpha_{\nearrow}(i)$ and $\alpha_{\searrow}(j) = \alpha_{\searrow}(i)$. $\square$

In [22], Kassel, Lascoux and Reutenauer showed that every permutation $w \in \mathfrak{S}_n$ has a unique commutation class of unipeak wiring diagrams, which they characterized by the equivalent description from Proposition III.11. They define the canonical sequence of a permutation w as follows. For each index h with $1 \leqslant h \leqslant n$, let $m_h = h + \#\{h' : (h, j) \in \operatorname{inv}(w)\}$, and let C_h be the cycle $s_{m_h - 1} \cdots s_h$, where the product is empty if $m_h = h$. The canonical sequence for w is the expression $\mathbf{w} = C_1 \cdots C_n$. Its wiring diagram is unipeak: for the strand α ending at target vertex ρ_h, the crossings where α travels down are precisely the crossings corresponding to the cycle C_h.

Proposition III.12. *Let i and k be crossings in a unipeak wiring diagram with $i < k$. Then the following hold:*

1. *If $\alpha_{\nearrow}(i) = \alpha_{\nearrow}(k)$, then*

$$\mathit{Strands}(\text{ㅊ}_{\leftarrow}(k)) = \mathit{Strands}(\text{ㅊ}_{\leftarrow}(i)) \cup \{\alpha_{\searrow}(j) : i \leqslant j < k, \alpha_{\nearrow}(j) = \alpha_{\nearrow}(i).$$

2. *If* $\alpha_{\nearrow}(i) = \alpha_{\nearrow}(k)$, *then for all* $j \geqslant i$, $\rho^{j}(\mathbf{w})\left(\alpha_{\searrow}(i)\right) < \rho(\mathbf{w})\left(\alpha_{\searrow}(k)\right)$.
3. *If* $\alpha_{\searrow}(i) = \alpha_{\searrow}(k)$, *then*

$$\mathit{Strands}(\text{え}_{\leftarrow}(k)) = \mathit{Strands}(\text{え}_{\leftarrow}(i)) \backslash \{\alpha_{\nearrow}(j) : i \leqslant j < k, \alpha_{\searrow}(j) = \alpha_{\searrow}(i)\}.$$

4. *If* $\alpha_{\searrow}(i) = \alpha_{\searrow}(k)$, *then* $\lambda(\mathbf{w})\left(\alpha_{\nearrow}(i)\right) > \lambda(\mathbf{w})\left(\alpha_{\nearrow}(k)\right)$.

Proof. We can think of the strands in a chamber of the form $\text{え}_{\leftarrow}(k)$ either as the collection of strands with targets weakly below $\alpha_{\nearrow}(k)$ at index $k-1$ or as the collection of strands with targets strictly below $\alpha_{\searrow}(k)$ at index $k-1$.

In general, if $\mathbf{w}$ is reduced, and $\alpha_{\nearrow}(i) = \alpha_{\nearrow}(k)$, then

$$\begin{aligned}\mathrm{Strands}(\text{え}_{\leftarrow}(k)) = \mathrm{Strands}(\text{え}_{\leftarrow}(i)) \setminus \{\alpha_{\nearrow}(j) : i < j < k, \alpha_{\searrow}(j) = \alpha_{\nearrow}(i)\} \\ \cup \{\alpha_{\searrow}(j) : i \leqslant j < k, \alpha_{\nearrow}(j) = \alpha_{\nearrow}(i)\}.\end{aligned}$$

If $\alpha_{\nearrow}(k) = \alpha_{\nearrow}(i)$, then the strand $\alpha_{\nearrow}(i)$ is traveling strictly up between indices i and k, so that the set $\{\alpha_{\nearrow}(j) : i < j < k, \alpha_{\searrow}(j) = \alpha_{\nearrow}(i)\}$ is empty. The strand $\alpha_{\searrow}(i)$ is traveling strictly down for indices greater than i, so we must have $\rho^{i}(\mathbf{w})\left(\alpha_{\searrow}(i)\right) < \rho^{k}(\mathbf{w})\left(\alpha_{\searrow}(k)\right)$.

Similarly, if $\alpha_{\searrow}(i) = \alpha_{\searrow}(k)$, then

$$\begin{aligned}\mathrm{Strands}(\text{え}_{\leftarrow}(k)) = \mathrm{Strands}(\text{え}_{\leftarrow}(i)) \setminus \{\alpha_{\nearrow}(j) : i \leqslant j < k, \alpha_{\searrow}(j) = \alpha_{\searrow}(i)\} \\ \cup \{\alpha_{\searrow}(j) : i < j < k, \alpha_{\nearrow}(j) = \alpha_{\searrow}(i)\}.\end{aligned}$$

In a unipeak diagram, the set $\{\alpha_{\searrow}(j) : i < j < k, \alpha_{\nearrow}(j) = \alpha_{\searrow}(i)\}$ is empty, since $\alpha_{\searrow}(i)$ cannot cross up over any other strand at an index $j > i$.

Since $\alpha_{\searrow}(i) = \alpha_{\searrow}(k)$, then $\alpha_{\nearrow}(i)$ and $\alpha_{\nearrow}(k)$ must each be traveling up in the region to the left of $\alpha_{\searrow}(i)$. In particular, $\alpha_{\nearrow}(i)$ and $\alpha_{\nearrow}(k)$ do not cross each other before crossing $\alpha_{\searrow}(j)$. Since $\alpha_{\searrow}(i)$ crosses down over $\alpha_{\nearrow}(i)$ first, $\alpha_{\nearrow}(i)$ must be above $\alpha_{\nearrow}(k)$ to the left of $\alpha_{\searrow}(i)$. $\square$

Proposition III.13. *[31] Let i be a crossing in a unipeak wiring diagram. Then the following hold:*

1. $\{\alpha : \lambda(\mathbf{w})\,(\alpha) \leqslant \lambda(\mathbf{w})\,(\alpha_{\nearrow}(i))\,\} \subseteq$ *Strands*$(\text{え}_{\leftarrow}(i))$.
2. *For each $j \geqslant i-1$, the set $\{\beta : \rho^j(\mathbf{w})\,(\beta) < \rho^j(\mathbf{w})\,(\alpha_{\searrow}(i))\,\} \subseteq$ Strands$(\text{え}_{\leftarrow}(i))$.*

Corollary III.14. *For each $j \geqslant i-1$, the set $\{\alpha : \lambda(\mathbf{w})\,(\alpha) \leqslant \lambda(\mathbf{w})\,(\alpha_{\nearrow}(i))\,\} \cup \{\beta : \rho^j(\mathbf{w})\,(\beta) < \rho^j(\mathbf{w})\,(\alpha_{\searrow}(i))\,\} =$ Strands$(\text{え}_{\leftarrow}(i))$, where in general the union is not disjoint.*

Proposition III.15. *Let $\pi = \alpha_i, \cdots, \alpha_m$ be a left-to-right path in a unipeak wiring diagram. If $j < k$ are indices so that $\alpha_j = \alpha_{\nearrow}(j)$ and $\alpha_k = \alpha_{\nearrow}(k)$, then $\lambda(\mathbf{w})\,(\alpha_j) \geqslant \lambda(\mathbf{w})\,(\alpha_k)$, with equality if and only if $\alpha_{j'} = \alpha_j$ for all $j \leqslant j' \leqslant k$.*

Proof. We claim that the sequence $\{\lambda(\mathbf{w})\,(\alpha_{\nearrow}(j)) : \pi$ travels through crossing $j\}$ is weakly decreasing. Let j and k be consecutive crossings along π. If $\alpha_j = \alpha_{\nearrow}(j)$, then either $\alpha_{\nearrow}(k) = \alpha_{\nearrow}(j)$ or $\alpha_{\nearrow}(k)$ crosses above $\alpha_{\nearrow}(j)$ so that $\lambda(\mathbf{w})\,(\alpha_{\nearrow}(k)) < \lambda(\mathbf{w})\,(\alpha_{\nearrow}(j))$. Otherwise, $\alpha_{\nearrow}(j)$ and $\alpha_{\nearrow}(k)$ are consecutive strands crossing above α_j, and hence $\lambda(\mathbf{w})\,(\alpha_{\nearrow}(k)) > \lambda(\mathbf{w})\,(\alpha_{\searrow}(k))$. $\square$

Corollary III.16. *Let $\mathbf{w}$ be a unipeak word with positive subexpression $\mathbf{u}_+$. Let γ_r be a geodesic path and suppose that there are crossings $i < k$ so that γ_r follows $\alpha_{\nearrow}(i)$ immediately to the left of i and $\alpha_{\nearrow}(k)$ immediately to the left of k. If $\alpha_{\nearrow}(i)$ has left endpoint λ_a and $\alpha_{\nearrow}(k)$ has left endpoint λ_b, then $a \geqslant b$. In particular, $\alpha_{\nearrow}(k) \in$ Strands$(\text{え}_{\leftarrow}(i))$.*

Corollary III.17. *Let $\mathbf{w}$ be a unipeak expression with positive subexpression $\mathbf{u}_+$. If $j < k$ are indices so that $\lambda^j(\mathbf{u}_+)\,(\alpha_{\nearrow}(j)) = \lambda^k(\mathbf{u}_+)\,(\alpha_{\nearrow}(k))$, then $\lambda(\mathbf{w})\,(\alpha_{\nearrow}(j)) \geqslant \lambda(\mathbf{w})\,(\alpha_{\nearrow}(k))$, with equality if and only if $\lambda^{j'}(\mathbf{u}_+)\,(\alpha_{\nearrow}(j)) = \lambda^j(\mathbf{u}_+)\,(\alpha_{\nearrow}(j))$ for all $j \leqslant j' \leqslant k$.*

Corollary III.18. *If $\alpha_{\nearrow}(i) = \alpha_{\nearrow}(j)$ and $\lambda^j(\mathbf{u}_+)\,(\alpha_{\nearrow}(i)) = r$, then for all crossings $k \geqslant j$ with $\lambda^k(\mathbf{u}_+)\,(\alpha_{\nearrow}(k)) = r$, we have $\alpha_{\nearrow}(k) \in \text{え}_{\leftarrow}(i)$.*

Proposition III.19. *Let* $\mathbf{w}$ *correspond to a unipeak diagram for* w *and let* $\mathbf{u}_+$ *be a positive subexpression of* $\mathbf{w}$. *Let* α *be a strand in the wiring diagram and let* k *be an index so that* $\rho^i(\mathbf{w})(\alpha)$ *is weakly increasing for* $i \leqslant k$ *and weakly decreasing for* $i \geqslant k$. *Then* $\lambda^i(\mathbf{u}_+)(\alpha)$ *is weakly increasing for* $i \leqslant k$ *and weakly decreasing for* $i \geqslant k$.

Proof. Since $\mathbf{u}_+$ is a positive subexpression, for any crossing i, we have $\lambda^{i-1}(\mathbf{u}_+)(\alpha_{\nearrow}(i)) < \lambda^{i-1}(\mathbf{u}_+)(\alpha_{\searrow}(i))$. If $i \in J^{\circ}_{\mathbf{u}_+}$, $\lambda^i(\mathbf{u}_+)(\alpha_{\nearrow}(i)) = \lambda^{i-1}(\mathbf{u}_+)(\alpha_{\searrow}(i))$ and $\lambda^i(\mathbf{u}_+)(\alpha_{\searrow}(i)) = \lambda^{i-1}(\mathbf{u}_+)(\alpha_{\nearrow}(i))$. If $i \in J^{+}_{\mathbf{u}_+}$, then $\lambda^i(\mathbf{u}_+)(\alpha_{\nearrow}(i)) = \lambda^{i-1}(\mathbf{u}_+)(\alpha_{\nearrow}(i))$ and $\lambda^i(\mathbf{u}_+)(\alpha_{\searrow}(i)) = \lambda^{i-1}(\mathbf{u}_+)(\alpha_{\searrow}(i))$. □

Lemma III.20 (Forbidden crossings)**.** *Fix a unipeak diagram corresponding to* $\mathbf{u}_+$ *and* $\mathbf{w}$. *Let* α *and* β *be two strands and suppose that for some* i *we have* $\rho^i(\mathbf{w})(\alpha) < \rho^i(\mathbf{w})(\beta)$ *and* $\lambda^i(\mathbf{u}_+)(\alpha) > \lambda^i(\mathbf{u}_+)(\beta)$. *Then* $\rho^k(\mathbf{w})(\alpha) < \rho^k(\mathbf{w})(\beta)$ *for all* $k \geqslant i$. *(That is,* α *and* β *do not cross at any index* $k \geqslant i$.*)*

Proof. Assume for contradiction that there is some $k > i$ so that $\rho^{k-1}(\mathbf{w})(\alpha) < \rho^{k-1}(\mathbf{w})(\beta)$ and $\rho^k(\mathbf{w})(\alpha) > \rho^k(\mathbf{w})(\beta)$). Let $r_\alpha = \lambda^i(\mathbf{u}_+)(\alpha)$ and let $r_\beta = \lambda^i(\mathbf{u}_+)(\beta)$. Write $w_{(k)} = w_{(k-1)}s_{h_k}$. Since α must cross up at index k, $\rho^j(\mathbf{w})(\alpha)$ is weakly increasing for $j \leqslant k$ and so $\lambda^{k-1}(\mathbf{u}_+)(\alpha) \geqslant r_\alpha$. If $\lambda^{k-1}(\mathbf{u}_+)(\beta) \leqslant r_\beta < r_\alpha$, then $u_{(k-1)}s_{h_k} < u_{(k-1)}$ which contradicts positivity of $\mathbf{u}_+$. So α and β cannot cross at k. If $\lambda^{k-1}(\mathbf{u}_+)(\beta) > r_\beta$, then there is some $j \in J^{\circ}_{\mathbf{u}_+}$ with $i < j < k$, $\lambda^{j-1}(\mathbf{u}_+)(\beta) = r_\beta$ and $\lambda^j(\mathbf{u}_+)(\beta) > r_\beta$. So $\rho^j(\mathbf{w})(\beta) > \rho^j(\mathbf{w})(\alpha_{\searrow}(j)) > \rho(\mathbf{w})(\alpha_j)$ with $\alpha_{\searrow}(j)$ traveling down. Hence α must cross above $\alpha_{\searrow}(j)$ at some index j' between j and k. But $\lambda^{j'}(\mathbf{u}_+)(\alpha_{\searrow}(j)) \leqslant \lambda^j(\mathbf{u}_+)(\alpha_{\searrow}(j)) = r_\beta$, contradicting positivity of $\mathbf{u}_+$. So α and β do not cross at any index $k > i$. □

Corollary III.21. *Let* $i \in J^{+}_{\mathbf{u}_+}$ *and write* $r^+ = u_{(i)}(h_i)$ *and* $r_- = u_{(i)}(h_i + 1)$. *Let* α *and* β *be strands so that* γ_{r^+} *follows* α *at some index* $k > i$ *and* γ_{r_-} *follows* β *at some index*

$k' > i$. Then the right endpoint of α is above the right endpoint of β.

3.4 Chamber ansatz formulas

In this section, we describe the Chamber Ansatz formulas of Marsh and Rietsch, which generalize earlier work of Berenstein and Zelevinsky.

From Deodhar's theorem, given a reduced expression $\mathbf{w}$ with distinguished subexpression $\mathbf{v}$, the Deodhar stratum $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ is isomorphic to the product of a torus $(\mathbb{C}^*)^{|J^\circ_{\mathbf{v}}|}$ and an affine space $(\mathbb{C})^{|J^-_{\mathbf{v}}|}$, where we recall that $J^\circ_{\mathbf{v}}$ is the set of indices i with $v_{(i)} = v_{(i-1)}$ and $J^-_{\mathbf{v}}$ is the set of indices i with $v_{(i)} < v_{(i-1)}$. Marsh and Rietsch give the following explicit parametrization of $\mathcal{D}^{\mathbf{v},\mathbf{w}}$.

Formula III.22 (Parametrizing the Deodhar stratum $\mathcal{D}^{\mathbf{v},\mathbf{w}}$[27]). Let $\mathbf{w}$ be a reduced expression for w and let $\mathbf{v} \prec \mathbf{w}$. There is a bijective map from $(\mathbb{C}^*)^{|J^\circ_{\mathbf{v}}|} \times (\mathbb{C})^{|J^-_{\mathbf{v}}|}$ to $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ given by $(t_i)_{i\in J^\circ_{\mathbf{v}}} \times (m_i)_{i \in J^-_{\mathbf{v}}} \mapsto gB_+$ where g is the coset representative $g = \prod_{i=1}^{\ell} g_i$, with

$$g_i = \begin{cases} y_i(t_i) & i \in J^\circ_{\mathbf{v}} \\ \dot{s}_i & i \in J^+_{\mathbf{v}} \\ x_i(m_i)\dot{s}_i^{-1} & i \in J^-_{\mathbf{v}} \end{cases}$$

To construct the inverse map $\mathcal{D}^{\mathbf{v},\mathbf{w}} \to (\mathbb{C}^*)^{|J^\circ_{\mathbf{v}}|} \times (\mathbb{C})^{|J^-_{\mathbf{v}}|}$, Marsh and Rietsch define *chamber ansatz* formulas, where a flag $gB_+ \in \mathcal{D}^{\mathbf{v},\mathbf{w}}$ is sent to a point described by ratios of minors of an upper triangular matrix z satisfying $z\dot{w}B_+ = gB_+$.

To each chamber え in the wiring diagram for $\mathbf{w}$ and $\mathbf{v}$, Marsh and Rietsch associate a minor of an upper triangular matrix z such that flag determined by $z\dot{w}$ is an element of $\mathcal{D}^{\mathbf{v},\mathbf{w}}$. Rows are determined by the strands in the upper arrangement below the region containing え, while columns are determined by the strands under ㅊ in the lower arrangement. For indices i with $i \in J^+_{\mathbf{v}}$ or $i \in J^-_{\mathbf{v}}$, so that there is a crossing in the upper arrangement at index i, Marsh and Rietsch also consider the *special minor* with columns taken from the chamber to the right of i and rows taken from the region to the left of i in the upper arrangement.

Notation III.23. We denote the minor of a matrix M with row set R and column set C by $\Delta^R_C(M)$. Except where otherwise specified, row indices and column indices are taken in increasing order. When no matrix is specified, Δ^R_C refers to a minor of a fixed upper triangular matrix z.

Definition III.24 (Chamber minors [27])**.** Label each region R in the upper arrangement by the left endpoints of the strands γ under R and label each chamber ㅊ in the lower arrangement with the left endpoints of the strands α under ㅊ. Fix an upper triangular matrix z with $z\dot{w}B_+ \in \mathcal{D}^{\mathbf{v},\mathbf{w}}$. The *standard chamber minor* for a chamber $\text{ㅊ} = \text{ㅊ}_{\rightarrow}(i)$ is the minor $\Delta^{v_{(i)}([1,h_i])}_{w_{(i)}([1,h_i])} = \Delta^{v_{(i)}([1,h_i])}_{w_{(i)}([1,h_i])}(z)$. If $i \in J^+_{\mathbf{v}}$ or $i \in J^-_{\mathbf{v}}$, the *special chamber minor* for ㅊ is the minor $\Delta^{v_{(i-1)}([1,h_i])}_{w_{(i)}([1,h_i])} = \Delta^{v_{(i-1)}([1,h_i])}_{w_{(i)}([1,h_i])}(z)$.

Marsh and Rietsch observed that the chamber minors obey two relations, implied by an identity proved by Dodgson for minors of a general matrix. We will use the following form of Dodgson's identity, as restated by Curtis, Ingerman and Morrow in [6].

Formula III.25 (Dodgson's Identity). Let M be an $n \times n$-matrix and let R and C be subsets of $[1, n]$ with $|R| = |C|$. Let $a, b \in R$ and let $\alpha, \beta \in C$ with $a < b$ and $\alpha < \beta$. Then

$$\Delta^R_C \Delta^{R\setminus\{a,b\}}_{C\setminus\{\alpha,\beta\}} = \Delta^{R\setminus\{a\}}_{C\setminus\{\alpha\}} \Delta^{R\setminus\{b\}}_{C\setminus\{\beta\}} - \Delta^{R\setminus\{a\}}_{C\setminus\{\beta\}} \Delta^{R\setminus\{b\}}_{C\setminus\{\alpha\}}.$$

For each $i \in J^+_{\mathbf{v}}$, the special minor $\Delta^{v_{(i-1)}([1,h_i])}_{w_{(i)}([1,h_i])}$ vanishes, so that Dodgson's identity gives the relation

$$\Delta^{v_{(i-1)}([1,h_i+1])}_{w_{(i-1)}([1,h_i+1])}\Delta^{v_{(i-1)}([1,h_i-1])}_{w_{(i-1)}([1,h_i-1])} = \Delta^{v_{(i-1)}([1,h_i])}_{w_{(i-1)}([1,h_i])}\Delta^{v_{(i)}([1,h_i])}_{w_{(i)}([1,h_i])}$$

for the standard chamber minors indexed by the chambers ㅊ$_{\leftarrow}(i)$, ㅊ$_{\uparrow}(i)$, ㅊ$_{\downarrow}(i)$ and ㅊ$_{\rightarrow}(i)$ surrounding crossing i. Similarly, for each $i \in J^{-}_{\mathbf{v}}$, the minor $\Delta^{v_{(i)}([1,h_i])}_{w_{(i-1)}([1,h_i])}$ vanishes, so that Dodgson's identity gives the relation

$$\Delta^{v_{(i-1)}([1,h_i+1])}_{w_{(i-1)}([1,h_i+1])}\Delta^{v_{(i-1)}([1,h_i-1])}_{w_{(i-1)}([1,h_i-1])} = -\Delta^{v_{(i-1)}([1,h_i])}_{w_{(i-1)}([1,h_i])}\Delta^{v_{(i)}([1,h_i])}_{w_{(i)}([1,h_i])}$$

for the standard chamber minors indexed by the chambers ㅊ$_{\leftarrow}(i)$, ㅊ$_{\uparrow}(i)$, ㅊ$_{\downarrow}(i)$ and ㅊ$_{\rightarrow}(i)$ surrounding crossing i. The negative sign comes from the condition that $v_{(i-1)}([1,h_i]) > v_{(i)}([1,h_i])$ while $w_{(i-1)}([1,h_i]) < w_{(i)}([1,h_i])$.

Definition III.26. A *chamber weighting* is a function Q(ㅊ) assigning a monomial weight to each chamber ㅊ in the wiring diagram, satisfying the following conditions.

1. If $i \in J^{+}_{\mathbf{u}_+}$, then Q(ㅊ$_{\leftarrow}(i)$)Q(ㅊ$_{\rightarrow}(i)$) = Q(ㅊ$_{\uparrow}(i)$)Q(ㅊ$_{\downarrow}(i)$).
2. If $i \in J^{-}_{\mathbf{u}_+}$, then Q(ㅊ$_{\leftarrow}(i)$)Q(ㅊ$_{\rightarrow}(i)$) = −Q(ㅊ$_{\uparrow}(i)$)Q(ㅊ$_{\downarrow}(i)$).

Formula III.27 (Chamber Ansatz [27]). Let $\mathbf{w}$ be a reduced expression for w and let $\mathbf{v} \prec \mathbf{w}$. Let z be an upper triangular matrix so that $z\dot{w}B_+ \in \mathcal{D}^{\mathbf{v},\mathbf{w}}$. The inverse map is given by

$$z\dot{w}B_+ \mapsto \left(\frac{\Delta^{\lambda^{i-1}(\mathbf{v})([1,h_i+1])}_{\lambda^{i-1}(\mathbf{w})([1,h_i+1])}\Delta^{\lambda^{i-1}(\mathbf{v})([1,h_i-1])}_{\lambda^{i-1}(\mathbf{w})([1,h_i-1])}}{\Delta^{\lambda^{i-1}(\mathbf{v})([1,h_i])}_{\lambda^{i-1}(\mathbf{w})([1,h_i])}\Delta^{\lambda^{i}(\mathbf{v})([1,h_i])}_{\lambda^{i}(\mathbf{w})([1,h_i])}}\right)_{i\in J^{\circ}_{\mathbf{v}}}$$

$$\times\left(-\frac{\Delta^{\lambda^{i-1}(\mathbf{v})([1,h_i])}_{\lambda^{i-1}(\mathbf{w})([1,h_i])}\Delta^{\lambda^{i}(\mathbf{v})([1,h_i])}_{\lambda^{i}(\mathbf{w})([1,h_i])}}{\Delta^{\lambda^{i-1}(\mathbf{v})([1,h_i+1])}_{\lambda^{i-1}(\mathbf{w})([1,h_i+1])}\Delta^{\lambda^{i-1}(\mathbf{v})([1,h_i-1])}_{\lambda^{i-1}(\mathbf{w})([1,h_i-1])}} - \Delta^{v_{(i-1)}([1,h_i])}_{s_{h_i}([1,h_i])}g_{(i-1)}\right)_{i\in J^{-}_{\mathbf{v}}}$$

We note that if the permutation $w \in \mathfrak{S}_n$ is not the order-reversing permutation w_0, there are multiple choices of $z \in B_+$ which have the same chamber minors

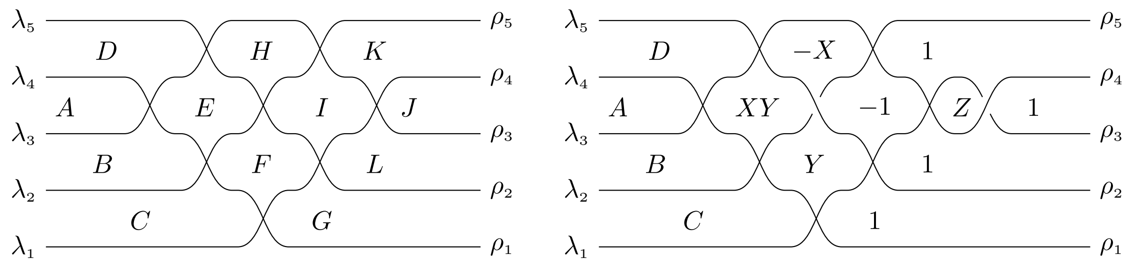

Figure 3.7: Chamber weightings for the wiring diagram for the word $\mathbf{w} = s_3s_2s_1s_4s_3s_2s_4s_3$ as a double wiring diagram for the double Bruhat cell $G^{w,1}$ and as a decorated wiring diagram for the Deodhar stratum $\mathcal{D}^{\mathbf{v},\mathbf{w}}$, where $\mathbf{v}$ is given by the underlined factors in $s_3s_2s_1s_4\underline{s_3}s_2s_4\underline{s_3}$

(see [27]). We will work with a canonical choice of z, satisfying the condition that $z\dot{w}$ is the renormalization of Fulton's canonical form for flags in the Schubert cell $B_+\dot{w}B_+/B_+$ with minimal nonvanishing minors equal to 1.

Notation III.28. We write $\Upsilon^{u,w}$ for the unique coset representative for a flag $F \in \mathcal{R}^{u,w}$ satisfying the following conditions.

1. $\Upsilon^{u,w}_{w(h),j} = 0$ whenever $h < j$.
2. $\Upsilon^{u,w}_{i,j} = 0$ whenever $i > w(j)$.
3. $\Delta^{u([1,j])}_{[1,j]}(\Upsilon^{u,w}) = 1$.

Our canonical choice of z is the matrix $\Upsilon^{u,w}\dot{w}^{-1}$.

CHAPTER IV

Pivots

Marsh and Rietsch's parametrization of the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ uses ratios of chamber minors which obey certain binomial equations. In Chapter V, we will give solutions to those equations in terms of parameters X_j indexed by $J^\circ_{\mathbf{u}_+}$, which will be the cluster variables for an initial seed. Recall that in Definition II.23, we defined nearly positive sequences $\mathbf{v}^j$ indexed by $J^\circ_{\mathbf{u}_+}$, and in Corollary II.24, we showed that the boundary divisors in Deodhar's decomposition are of the form $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$, where $\mathbf{v}^j$ satisfies the stronger condition of being a nearly positive subexpression. The key feature of our change of coordinates is that for each j, either the parameter X_j vanishes on precisely one boundary divisor, indexed by $\mathbf{v}^j$, or X_j is nonzero everywhere on $\mathcal{R}^{u,w}$ and $\mathbf{v}^j$ does not index a boundary divisor.We will give a chamber weighting for the Deodhar torus so that for each divisor $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$, the parameter X_j divides the weighting of the chamber minors which vanish on $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$.

This requires understanding the relationship between the nonreduced distinguished subexpression $\mathbf{v}^j \prec \mathbf{w}$ and the positive subexpression $\mathbf{u}_+ \prec \mathbf{w}$. The goal of this chapter is to describe the rank conditions on the boundary divisor $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$ in terms of rank conditions on collections of columns, which do not necessarily correspond to chamber minors. We will first need to address the rank conditions for a general collection of columns of a matrix M so that MB_+ is in the opposite Schubert cell $B_-\dot{v}B_+/B_+$.

Notation IV.1. Let R and S be sets of size k for some k. We say that $R \leqslant S$ if R has elements $r_1 < r_2 < \cdots < r_k$ and S has elements $s_1 < s_2 < \cdots < s_k$ with $r_i \leqslant s_i$ for each index i from 1 to k.

Let $w \in \mathfrak{S}_n$ and let $u \leqslant w$. Let $F = gB_+$ be a flag in $\mathcal{R}^{u,w}$ and let $z \in B_+$ be an upper triangular matrix so that $z\dot{w}B_+ = F$. Fix a reduced expression $\mathbf{w}$ for w. By Deodhar's theorem, there is a unique distinguished subexpression $\mathbf{v}^j$ of $\mathbf{w}$ so that $v_{(\ell)} = u$ and for each $0 \leqslant i \leqslant \ell$ and $1 \leqslant h \leqslant n$, $\Delta^{v_{(i)}([1,h])}_{w_{(i)}([1,h])}(z) \neq 0$ and $\Delta^{R}_{w_{(i)}([1,h])}(z)$ $= 0$ for any $R \subseteq [1,n]$ with $|R| = h$ and $v_{(i)}([1,h]) \not\leqslant R$.

That is, for each $0 \leqslant i \leqslant \ell$, the flag $z\dot{w}_{(i)}B_+$ is an element of $\mathcal{R}^{v_{(i)},w_{(i)}}$.

Given a flag $F \in \mathcal{R}^{u,w}$, there is a canonical matrix $\Upsilon^{u,w}$ so that $\Upsilon^{u,w}B_+ = F$ as elements of G/B_+, $\Upsilon^{u,w}\dot{w}^{-1}$ is upper triangular, and $\Delta^{u([1,h])}_{[1,h]}(\Upsilon^{u,w}) = 1$ for each $1 \leqslant h \leqslant n$.

If $j \in J^{\circ}_{\mathbf{u}_+}$ is an index so that the nearly positive sequence $\mathbf{v}^j$ satisfies $v^j_{(0)} = 1$, then $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$ is nonempty and has codimension 1 in $\mathcal{R}^{u,w}$. Since $\mathbf{v}^j$ is defined recursively from the formulas $v^k_{(j)} = u_{(k)}$ for $k \geqslant j$ and $v^{j-1}_{(j)} = u_{(j-1)}s_{h_j}$ where $u_{(j-1)} < u_{(j-1)}s_{h_j}$, it's clear that if $\text{ㅊ}_{\leftarrow}(j)$ is the chamber to the west of j so that $\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(j)) =$ $[1,h_j]$, the minor $\Delta^{u_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\leftarrow}(j))}(\Upsilon^{u,w}\dot{w}^{-1}) = \Delta^{u_{(j-1)}([1,h_j])}_{[1,h_j]}\left(\Upsilon^{u,w}\dot{w}^{-1}\dot{w}_{(j-1)}\right)$ vanishes on $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$. However, the minor $\Delta^{u_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\leftarrow}(j))}(\Upsilon^{u,w}\dot{w}^{-1})$ can also vanish on a boundary divisor $\mathcal{D}^{\mathbf{v}^{\mathbf{k}},\mathbf{w}}$ for some $k \neq j$.

We will show that for each nearly positive subexpression $\mathbf{v}^j$, there is a minor $\Delta^{\widehat{R}^j}_{\lambda(\mathbf{w})(\widehat{C}^j)}$ so that $\Delta^{\widehat{R}^j}_{\lambda(\mathbf{w})(\widehat{C}^j)}$ vanishes on $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$ and $\Delta^{\widehat{R}^j}_{\lambda(\mathbf{w})(\widehat{C}^j)}$ is generically nonvanishing on $\mathcal{D}^{\mathbf{v}^{\mathbf{k}},\mathbf{w}}$ for any $k \neq j$. In general, $\widehat{C}^j$ need not satisfy $\rho^k(\mathbf{w})\left(\widehat{C}^j\right) = [1,h]$ for some k.

Remark IV.2. In this chapter, we will look at minors Δ_C^R as functions of $n \times n$ matrices M, rather than functions defined for a flag $z\dot{w}B_+$ in terms of minors of upper triangular matrices.

Definition IV.3. Let M be a matrix and suppose that M_C^R and $M_{C'}^{R'}$ are square submatrices of size m. If $R' \leqslant R$ and $C' \leqslant C$, we say that $M_{C'}^{R'}$ is *northwest* of M_C^R and M_C^R is *southeast* of $M_{C'}^{R'}$.

Proposition IV.4. *Let R and C be subsets of $[1, n]$ of size m. Suppose that there exist sets $R' \leqslant R$ and $C' \leqslant C$ so that $\mathcal{P}(v)_{C'}^{R'}$ is an $m \times m$ permutation matrix. Then the minor Δ_C^R is generically nonvanishing on $B_-\dot{v}B_+$. Otherwise, Δ_C^R vanishes identically on $B_-\dot{v}B_+$.*

The following example is an illustration of the property that when a column set C is not an interval of the form $[1, h]$, the minor Δ_C^R can vanish identically on $B_-\dot{v}B_+$ even when the smallest northwest block containing rows R and columns C has full rank.

Example IV.5. Let $n = 4$ and let $v = s_1s_2s_1$. Then v has permutation matrix and northwest rank matrix given by

$$\mathcal{P}(v) = \begin{pmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix} \text{ and } {}^{\text{NW}}\text{Rank}\,(v) = \begin{pmatrix} 0&0&1&1 \\ 0&1&2&2 \\ 1&2&3&3 \\ 1&2&3&4 \end{pmatrix}.$$

Consider the minor Δ_{14}^{12}. Every northwest block containing the rows $\{1, 2\}$ and columns $\{1, 4\}$ has rank at least 2. However, for all $M \in B_-\dot{v}B_+$ the minor Δ_{14}^{12} vanishes, since the submatrix M_1^{12} has rank 0. We note that there is no permutation submatrix of $\mathcal{P}(v)$ northwest of the submatrix $\mathcal{P}(v)_{14}^{12}$. In contrast,the matrix $\mathcal{P}(v)$ has permutation submatrices $\mathcal{P}(v)_{13}^{13}$and $\mathcal{P}(v)_{23}^{12}$ which are in positions northwest of Δ_{14}^{13} and Δ_{24}^{12}, respectively. Although for $M = \mathcal{P}(v)$ the minors Δ_{14}^{13} and Δ_{24}^{12} are both zero, if

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then $\Delta^{13}_{14}(M) = \Delta^{12}_{24}(M) = -1$. So the minors Δ^{13}_{14} and Δ^{12}_{24} are generically non-vanishing on $B_-\dot{v}B_+$.

Definition IV.6. Let $v \in \mathfrak{S}_n$ and let C be a subset of $[1, n]$. We define the *pivots* of C with respect to v to be the minimal set $R \subseteq [1, n]$ with $|R| = |C|$ so that $\mathcal{P}(v)$ has a $|C| \times |C|$ permutation submatrix $\mathcal{P}(v)^{R'}_{C'}$ northwest of $\mathcal{P}(v)^{R}_{C}$. Write $R = \mathrm{Pivots}_C(v)$.

Since the condition that there is a 1 in position (r, c) in the permutation matrix $\mathcal{P}(v)$ is equivalent to each of the statements $r = v(c)$ and $c = v^{-1}(r)$, the definition may be restated as follows.

Proposition IV.7 (Minimality criterion)**.** *Let $v \in \mathfrak{S}_n$ and let C be a subset of $[1, n]$. Then $\mathit{Pivots}_C(v) = \min\limits_{C' \leqslant C} v(C') = \min\limits_{\substack{R \subset [1,n] \\ v^{-1}(R) \leqslant C}} R$.*

Corollary IV.8 (Partial order reversing rule)**.** *Let $v \in \mathfrak{S}_n$ and let C and D be subsets of $[1, n]$ with $|C| = |D|$ and $C \leqslant D$. Then $\mathit{Pivots}_C(v) \geqslant \mathit{Pivots}_D(v)$ and $v^{-1}(\mathit{Pivots}_c(v)) \leqslant v^{-1}(\mathit{Pivots}_D(v))$.*

Corollary IV.9 (Sandwich rule)**.** *Suppose that $C \leqslant D$. The following are equivalent:*

1. *$\mathit{Pivots}_C(v) = \mathit{Pivots}_D(v)$.*
2. *$v^{-1}(\mathit{Pivots}_D(v)) \leqslant C$.*
3. *If $v^{-1}(\mathit{Pivots}_C(v)) \leqslant S \leqslant D$, then $v(S) \leqslant \mathit{Pivots}_C(v)$.*

Proposition IV.10. *The set $\mathit{Pivots}_C(v)$ gives the minimal pivots of the column set C when generic rightward column operations are applied to the permutation matrix for v. We have $\mathit{Pivots}_C(v) \leqslant v(C)$, with equality in the special case where $C = [1, h]$ for some h.*

Example IV.11. Let $n = 3$ and let $v = s_1$. Then $\mathrm{Pivots}_{\{1,3\}}(v) = \{1, 2\}$, since $v^{-1}(\{1, 2\}) = \{1, 2\} \leqslant \{1, 3\}$.

We can apply rightward column operations to the permutation matrix for s_1 to put columns 1 and 3 in column-echelon form with pivots in rows 1 and 2.

$$\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix} \xrightarrow{C_2+C_3\to C_3} \begin{pmatrix} \boxed{0}&1&\boxed{1}\\ \boxed{1}&0&\boxed{0}\\ \boxed{0}&0&\boxed{1} \end{pmatrix}$$

4.1 Pivot inequalities for subsets

Proposition IV.12. *Let $v \in \mathfrak{S}_n$ and let C and D be subsets of $[1, n]$ with $C \subseteq D$. Then Pivots$_C(v) \subseteq$ Pivots$_D(v)$.*

Proof. From a linear algebra perspective, if M is a generic element of $B_-\dot{v}B_+$, then Pivots$_S(v)$ is the set of minimal pivots of the submatrix $M_S^{[1,n]}$ under downward row operations. If $C \subseteq D$, then $M_C^{[1,n]}$ is a submatrix of $M_D^{[1,n]}$, so the minimal pivots of $M_C^{[1,n]}$ are a subset of the minimal pivots of $M_D^{[1,n]}$. □

Definition IV.13. Let $v \in \mathfrak{S}_n$ and let C be a subset of $[1, n]$. Write $C = c_1 < \cdots < c_m$. The *ordered pivots* of C with respect to v are given by $r_1, \cdots, r_m$, where r_{i+1} is the unique element of Pivots$_{c_1,\cdots,c_{i+1}}(v) \setminus$ Pivots$_{c_1,\cdots,c_i}(v)$.

Proposition IV.14 (Partition property)**.** *Suppose that $a, b \in$ Pivots$_C(v)$ with $a < b$. Then there is some $C' \subset C$ with $a \in$ Pivots$_C(v)$ and $b \notin$ Pivots$_C(v)$.*

Proof. Write $C = c_1 < \cdots < c_m$ and let $r_1, \cdots, r_m$ be the ordered pivots of C. Take $C' = \{c_i : r_i < b\}$. Then Pivots$_{C'}(v) = \{r_i : r_i < b\}$, which contains a and does not contain b. □

In the next few propositions, we derive several inequalities relating pivots of sets $C \subseteq D$.

Proposition IV.15 (Bubble sort property[1])**.** *Let $v \in \mathfrak{S}_n$ and let $C = c_1 < \cdots < c_\ell$ and*

[1]Bubble sort is a sorting algorithm in which larger elements in an array gradually "bubble" to the right via a series of swaps, notorious for being neither efficient nor intuitive.

$D = d_1 < \cdots < d_m$ be subsets of $[1, n]$ with $C \subseteq D$. Write $r_1, \cdots, r_\ell$ for the ordered pivots of C and write $t_1, \cdots, t_m$ for the ordered pivots of D. If $c_i = d_j$, then $r_i \leqslant t_j$.

Proof. Since r_i and t_j depend only on $c_1 < \cdots < c_i$ and $d_1 < \cdots < d_j$, we may assume $i = \ell$ and $j = m$. By induction on $|D \backslash C|$, we may assume that $C = D \backslash \{d_j\}$ for some j, so that $c_i = d_i$ for $i < j$ and $c_i = d_{i+1}$ for $i \geqslant j$.

For each $i \geqslant j$, write $\hat{t}_i$ for the unique element of $\text{Pivots}_{d_1, \cdots, d_i}(v) \setminus \text{Pivots}_{c_1, \cdots, c_{i-1}}(v)$. Since $\text{Pivots}_{c_1, \cdots, c_{i-1}}(v) \subset \text{Pivots}_{c_1, \cdots, c_i}(v) \subset \text{Pivots}_{d_1, \cdots, d_{i+1}}(v) = \text{Pivots}_{c_1, \cdots, c_{i-1}}(v) \cup \{\hat{t}_i, t_{i+1}\}$. By minimality of ordered pivots, $r_i = \min\{\hat{t}_i, t_{i+1}\}$, so that in particular $r_\ell \leqslant t_{\ell+1}$. □

Corollary IV.16. *Let $v \in \mathfrak{S}_n$ and let $C = c_1, \cdots, c_m$ be a subset of $[1, n]$. Let b be the unique element of $\text{Pivots}_C(v) \setminus \text{Pivots}_{C \backslash \{c_j\}}(v)$. Then $b = \max\{r_k : k \geqslant j\}$.*

Corollary IV.17. *Let $v \in \mathfrak{S}_n$ and let C and D be subsets of $[1, n]$ with $C \subseteq D$. Let α be an element of C, and let $\text{Pivots}_D(v) \setminus \text{Pivots}_{D \backslash \{\alpha\}}(v) = \{b\}$. Then $\text{Pivots}_C(v) \leqslant \text{Pivots}_{C \backslash \{\alpha\}}(v) \cup \{b\}$.*

Proof. It's equivalent to show that if a is the unique element of $\text{Pivots}_C(v) \setminus \text{Pivots}_{C \backslash \alpha}(v)$, then $a \leqslant b$. Write $r_1, \cdots, r_\ell$ for the ordered pivots of C and write $t_1, \cdots, t_m$ for the ordered pivots of D. We have $a = \max\{r_j : c_j \geqslant \alpha\}$ and $b = \max\{t_k : d_k \geqslant \alpha\}$. Given j and k with $c_j = d_k$ we have $r_j \leqslant t_k$; since $\{c_j \geqslant \alpha\} \subseteq \{d_k \geqslant \alpha\}$, it follows that $\max\{r_j : c_j \geqslant \alpha\} \leqslant \max\{t_k : d_k \geqslant \alpha\}$. □

4.2 Elementary transpositions and pivots

Let s_h be an elementary transposition. Then s_h acts on subsets of $[1, n]$ by exchanging the letters h and $h+1$ and on elements of $\mathfrak{S}_n$ by right multiplication. In this section, we consider how pivots can change when a transposition s_h acts on the index C, the permutation v, or both simultaneously.

Lemma IV.18. *Let $v < vs_h$ and let $C = c_1 < \cdots < c_m$ be any collection of indices. Then the following hold:*

1. *$Pivots_C\,(v) = Pivots_{s_h(C)}\,(v)$.*
2. *$Pivots_C\,(v) \leqslant Pivots_C\,(vs_h)$, with equality if $h \notin C$ or $h+1 \in C$.*[2] *If $Pivots_C\,(v) \neq Pivots_C\,(vs_h)$, then they differ by exactly one element.*

Proof. We will use the criterion $\text{Pivots}_C\,(v) = \min_{C' \leqslant C} v(C')$. Write C' for the set $v^{-1}(\text{Pivots}_C\,(v)\,)$. Consider the action of s_h on the sets C and C'.

Since s_h is the transposition that swaps h and $h+1$, s_h acts nontrivially on a set $S \subseteq [1, n]$ if and only if S contains exactly one of the indices h and $h+1$; we have $s_h(S) > S$ if $h \in S$ and $h+1 \notin S$, $s_h(S) < S$ if $h \notin S$ and $h+1 \in S$.

First, we show that if $h+1 \in C'$, then $h \in C'$, so that $C' \leqslant s_h(C')$.

Suppose to the contrary that $s_h(C') = C' \backslash \{h+1\} \cup \{h\}$. By transitivity, $s_h\,(C') < C' \leqslant C$. Since $v < vs_h$ and $v(h) < v(h+1)$ are equivalent conditions, this implies that $v\,(s_h\,(C'))\, < v\,(C')$, contradicting the minimality of $v(C')$.

We claim that $C' \leqslant s_h\,(C)$.

If $C \leqslant s_h\,(C)\,$, then $C' \leqslant s_h\,(C)\,$ by transitivity.

Suppose that $s_h(C) < C$. Then $h+1 \in C$ and $h \notin C$. Let i be the index of $h+1$ in C, so that $\#\{c \in C : c < h\} = i-1$. Since $C' \leqslant C$, we must have $\#\{c' \in C' : c' < h\} \geqslant i-1$ and $\#\{c' \in C' : c' \leqslant h+1\} \geqslant i$. Combining this with the condition that if $h+1 \in C'$, then $h \in C'$, we obtain that C' has at least i elements c' with $c' \leqslant h$. So $C' \leqslant s_h\,(C)$.

Next, we apply these two inequalities to deduce the statements in the lemma.

[2]This condition is sufficient but not necessary; in particular if there is at least one $h' < h$ with $v(h') < v(h)$, then $\text{Pivots}_C\,(v) = \text{Pivots}_C\,(vs_h)$.

1. Since $C' \leqslant s_h(C)$, it suffices to show that if $S \leqslant s_h(C)$ then $v(S) \leqslant v(C')$. Suppose $S \leqslant s_h(C)$. If $S \leqslant C$, then $v(C') \leqslant v(S)$ by choice of C'. Otherwise, we must have $h+1 \in S$ and $h+1 \notin C$, so that $s_h(S) < S$ and $s_h(S) \leqslant C$. Since $v(h) < v(h+1)$, this gives $v(C') \leqslant v(s_h(S)) < v(S)$. By Proposition IV.7, $v(C') = \text{Pivots}_{s_h(C)}(v)$.

2. Let $S \subseteq [1,n]$ be a set with $S \leqslant C$. Since s_h is an involution, we have $vs_h(s_h(S)) = v(S)$. If $s_h(S) \leqslant C$, then Proposition IV.7 implies that $vs_h(S) = v(s_h(S)) = \geqslant v(C')$. If $s_h(S) \nleqslant C$, then we must have $s_h(S) = S\backslash\{h\} \cup \{h+1\}$ so that in particular $vs_h(S) = v(S\backslash\{h\} \cup \{h+1\}) > v(S) \geqslant v(C')$. Since for any $S \leqslant C$ we have $vs_h(S) \geqslant v(C')$, $\min_{S \leqslant C} vs_h(S) \geqslant v(C') = \text{Pivots}_C(v)$. Suppose that $h \notin C$ or $h+1 \in C$. We claim that $s_h(C') \leqslant C$. If $s_h(C') = C'$, this is clear. Otherwise, we must have $h \in C'$ and $h+1 \notin C'$ with $s_h(C') = C'\backslash\{h\} \cup \{h+1\}$. If $h \notin C$, then for any $c \in C$ with $h \leqslant c$ we have $h+1 \leqslant c$, so that $s_h(C') \leqslant C$. If h and $h+1$ are both elements of C, write i for the index of $h+1$ in C. Since there are at least i elements c' in C' satisfying $c' \leqslant h+1$ and $h+1 \notin C'$, we must have $\#\{c' \in C' : c' < h\} \geqslant i-1$. So $\#\{c' : s_h(c') \leqslant h\} \geqslant i-1$ and $\#\{c' : s_h(c') \leqslant h+1\} \geqslant i$, and hence $s_h(C') \leqslant C$.

$\square$

Corollary IV.19. *Let v and w be elements of $\mathfrak{S}_n$ and let $s_{h_1} \cdots s_{h_\ell}$ be a reduced word for w. Suppose that for each $1 \leqslant i \leqslant \ell$ we have $v < vs_{h_i}$. Then the action of w on column indices fixes pivots: for each $C \subseteq [1,n]$, Pivots$_{w^{-1}(C)}(v) =$ Pivots$_C(v)$.*

Example IV.20. Let $n = 3$ and let v and w be the permutations $v = s_1$ and $w = s_2s_1$, so that v has permutation matrix
$\mathcal{P}(v) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Consider the set $C = \{1\}$. Then $\text{Pivots}_C\,(v) = \{v(1)\} = \{2\}$, while $\text{Pivots}_{w^{-1}(C)}\,(v)$ $= \text{Pivots}_{\{2\}}\,(v) = \{1\}$. This does not contradict our corollary since s_2 is a factor of w with $vs_2 = s_2 s_2 = 1 < v$.

We remark that $\ell\left(vs_{h_1}\cdots s_{h_i}\right) = \ell(v) + i$ for each $1 \leqslant i \leqslant \ell$ is not a sufficient condition for the action of w on column indices to preserve pivots.

Corollary IV.21. *$\text{Pivots}_C\,(v) \leqslant \text{Pivots}_{s_h(C)}\,(vs_h)$, with equality if $h+1 \notin C$ or $h \in C$. If $\text{Pivots}_C\,(v) \neq \text{Pivots}_{s_h(C)}\,(vs_h)$, then they differ by exactly one element.*

Proof. We have $\text{Pivots}_C\,(v) = \text{Pivots}_{s_h(C)}\,(v) \leqslant \text{Pivots}_{s_h(C)}\,(vs_h)$. Since the transposition s_h swaps h and $h+1$, the index $h \in C$ if and only if $h+1 \in s_h(C)$, while the index $h+1 \in C$ if and only if $h \in s_h(C)$, so that equality holds if $h+1 \notin C$ or $h \in C$. $\square$

Proposition IV.22 (Ordered pivot criterion)**.** *Let $v < vs_h$ and let $C = c_1 < \cdots < c_m$ be a subset of $[1,n]$ such that $h \in C$ and $h+1 \notin C$, and let i be the index of h. Write $R = r_1, \cdots, r_m$ for the ordered pivots of C. The following are equivalent:*

1. *$\text{Pivots}_C\,(vs_h) \neq \text{Pivots}_C\,(v)$*
2. *$(vs_h)^{-1}(R) \nleqslant C$.*
3. *$r_i = v(h)$ and $v^{-1}(r_j) > h+1$ for all $j > i$.*
4. *$v(h+1) \notin R$ and $|R \cap v([1,h])| = i$.*
5. *$\text{Pivots}_{c_1,\cdots,c_i}\,(vs_h) \nsubseteq \text{Pivots}_C\,(v)$.*

Proof. Since $v < vs_h$, $\text{Pivots}_C\,(vs_h) \geqslant \text{Pivots}_C\,(v)$, with strict inequality if and only if $(vs_h)^{-1}(R) \nleqslant C$. Given a set $S = s_1 < \cdots < s_m$, we have $S \nleqslant C$ if and only for some index j we have $s_j > c_j$, which is equivalent to the condition $\#\{s \in S : s \leqslant c_j\} < j$.

For any $c \notin \{h, h+1\}$ and any r, we have $(vs_h)^{-1}(r) \leqslant c$ if and only if $v^{-1}(r) \leqslant c$; hence, if $c_j \neq h$, then $\#\{r \in R : (vs_h)^{-1}(r) \leqslant c_j\} = \#\{r \in R : v^{-1}(r_j) \leqslant c_j\} \geqslant j$.

Since $(vs_h)^{-1}(r) \leqslant h$ if and only if either $v^{-1}(r) < h$ or $r = v(h+1)$, we have $R \cap vs_h([1,h]) = R \cap v([1,h-1]) \cup (R \cap \{v(h+1)\}) = (R \cap v([1,h]))\backslash\{v(h)\} \cup (R \cap \{v(h+1)\})$. Since $(vs_h)^{-1}(r_j) < h$ for $j < i$, the set $|R \cap vs_h([1,h])| < i$ if and only if $|R \cap v([1,h-1])| = i-1$ and $v(h+1) \notin R$. Since $\text{Pivots}_{c_1,\cdots,c_i}(vs_h)$ is a set of size i contained in $v([1,h])$, this implies that $\text{Pivots}_{c_1,\cdots,c_i}(vs_h) \nsubseteq R$. □

Corollary IV.23. *Suppose that $\text{Pivots}_C(vs_h) \neq \text{Pivots}_C(v)$ and $\text{Pivots}_{C\alpha}(vs_h) = \text{Pivots}_{C\alpha}(v)$. Then $\alpha > h$.*

Proof. Write $C = c_1 < \cdots < c_m$ and write $R = r_1, \cdots, r_m$ for the ordered pivots of C. Write $\text{Pivots}_{C\alpha}(v) = Ra$. Then we must have $\text{Pivots}_C(vs_h) = R\backslash\{r_-\} \cup \{a\}$ for some $r_- < a$. Write $h = c_i$. Then $r_i = v(h)$ and $v^{-1}(r_j) > h+1$ for all $j > i$.

Suppose that $\alpha < h$. The $\text{Pivots}_{(C\alpha)\cap[1,h]}(R)$ is a set of size $i+1$ contained in $Ra \cap ([1,h])$. Since $|R \cap v([1,h])| = i$, we must have $v^{-1}(a) < h$, so that in particular Ra does not contain $v(h+1)$ and $|Ra \cap v([1,h])| = i+1$, which is the index of h in $C\alpha$. Hence $\text{Pivots}_{C\alpha}(vs_h) \neq \text{Pivots}_{C\alpha}(v)$, a contradiction. □

Definition IV.24. Let $v < vs_h$. We define the *jump set* of v and s_h to be

$$\mathcal{J}(v, s_h) = \{C \subset [1,n] : \text{Pivots}_C(vs_h) > \text{Pivots}_C(v)\}.$$

Lemma IV.25. *Let $v \in \mathfrak{S}_n$ with $v < vs_h$. Let C be a subset of $[1,n]$ and suppose that $\alpha < \beta$ are elements of $[1,n] \setminus C$ with $\text{Pivots}_{C\alpha}(v) \neq \text{Pivots}_{C\beta}(v)$. Then*

$$|\mathcal{J}(v, s_h) \cap \{C, C\alpha\beta\}| = |\mathcal{J}(v, s_h) \cap \{C\alpha, C\beta\}|.$$

Proof. Write J for the set $\mathcal{J}(v, s_h) \cap \{C, C\alpha, C\beta, C\alpha\beta\}$.

We will show that if P is one of the sets $\{C, C\alpha\beta\}$ and $\{C\alpha, C\beta\}$, then $P \cap J = \varnothing$

implies that $J = \varnothing$ and $P \subseteq J$ implies that $J = \{C, C\alpha, C\beta, C\alpha\beta\}$.

Let $R = \text{Pivots}_C(v)$. Since $\text{Pivots}_{C\alpha}(v) \neq \text{Pivots}_{C\beta}(v)$, we must have $\text{Pivots}_{C\alpha}(v) = Ra$ and $\text{Pivots}_{C\beta}(v) = Rb$ for some $a, b \notin R$ with $a \neq b$. By the order-reversing property, $C\alpha < C\beta$ implies that $Ra \geqslant Rb$, and so $a > b$. Since $C\alpha$ and $C\beta$ are subsets of $C\alpha\beta$, $\text{Pivots}_{C\alpha\beta}(v)$ must contain Ra and Rb, so by cardinality $\text{Pivots}_{C\alpha\beta}(v) = Rab$.

Suppose that $\text{Pivots}_C(vs_h) = R$ and $\text{Pivots}_{C\alpha\beta}(vs_h) = Rab$. By the containment property, $R \subset \text{Pivots}_{C\alpha}(vs_h) \subset Rab$. Since $Ra > Rb$ and $\text{Pivots}_{C\alpha}(vs_h) \geqslant \text{Pivots}_{C\alpha}(vs_h) = Ra$, we must have $\text{Pivots}_{C\alpha}(vs_h) = Ra$. Since $(vs_h)^{-1}(R) \leqslant C$ and $(vs_h)^{-1}(Rab) \leqslant C\alpha\beta$, we must have $(vs_h)^{-1}(Rb) \leqslant \max\{C\alpha, C\beta\} = C\beta$. So $\text{Pivots}_{vs_h}(C\beta) \leqslant Rb = \text{Pivots}_v(C\beta) \leqslant \text{Pivots}_{vs_h}(C\beta)$.

Suppose that $\text{Pivots}_{C\alpha}(vs_h) = Ra$ and $\text{Pivots}_{C\beta}(vs_h) = Rb$. By the containment property, $\text{Pivots}_C(vs_h) \subseteq Ra \cap Rb = R$ and $Rab \subseteq \text{Pivots}_{C\alpha\beta}(vs_h)$, so by cardinality it follows that $\text{Pivots}_C(vs_h) = R$ and $\text{Pivots}_{C\alpha\beta}(vs_h) = Rab$.

Suppose now that J is nonempty and contains one of the pairs $\{C, C\alpha\beta\}$ or $\{C\alpha, C\beta\}$. By the ordered pivot criterion, this implies that both elements of the pair contain h but not $h+1$ and have pivots containing $v(h)$ but not $v(h+1)$. In particular, $h \in C$ and $h+1 \notin C\alpha\beta$, while $v(h) \in Rab$ and $v(h+1) \notin Rab$. Let i be the index of h in C. Since α, $\beta \notin C$, neither of them is equal to h. By hypothesis, $\alpha < \beta$. So there are three possible orderings of the indices $\{h, \alpha, \beta\}$, distinguished by whether the index h is added at the beginning, middle or end.

If $h < \alpha < \beta$, then for any $D \in \{C, C\alpha, C\beta, C\alpha\beta\}$, the smallest i elements of D are $c_1, \cdots, c_i$ with $c_i = h$, so that $\text{Pivots}_{c_1,\cdots,c_i}(vs_h) \subseteq \text{Pivots}_D(vs_h)$. By the ordered pivot criterion, if $\{C, C\alpha\beta\} \subset J$, then $\text{Pivots}_{c_1,\cdots,c_i}(vs_h) \nsubseteq R \cup Rab = Rab$ and if $\{C\alpha, C\beta\} \subset J$ then $\text{Pivots}_{c_1,\cdots,c_i}(vs_h) \nsubseteq Ra \cup Rb = Rab$. In particular,

$\text{Pivots}_{c_1,\cdots,c_i}(vs_h) \nsubseteq \text{Pivots}_D(v) \subseteq Rab$, and hence $\text{Pivots}_D(vs_h) \neq \text{Pivots}_D(v)$.

If $\alpha < h < \beta$, then h has index i in C and $C\beta$ and has index $i+1$ in $C\alpha$ and $C\alpha\beta$. By the ordered pivot condition, C and $C\alpha\beta$ are in J if and only if $|R \cap v([1,h])| = i$ and $|Rab \cap v([1,h])| = i+1$, while $C\beta, C\alpha$ are in J if and only if $|Rb \cap v([1,h])| = i$ and $|Ra \cap v([1,h])| = i+1$, which are equivalent conditions.

If $\alpha < \beta < h$, then h has index i in C, index $i+1$ in $C\alpha$ and $C\beta$ and index $i+2$ in $C\alpha\beta$. So C and $C\alpha\beta$ are in J if and only if $|R \cap v([1,h])| = i$ and $|Rab \cap v([1,h])| = i+2$, while $C\alpha$ and $C\beta$ are in J if and only if $|Ra \cap v([1,h])| = i+1$ and $|Rb \cap v([1,h])| = i+1$; these conditions are equivalent. □

4.3 Pivots and unipeak wiring diagrams

Proposition IV.26. *Let* $\mathbf{w}$ *correspond to a unipeak diagram for* w *and let* $\mathbf{u}_+$ *be a positive subexpression of* $\mathbf{w}$*. Let* C *be a collection of strands and let* α *be a strand with* $\rho^j(\mathbf{w})(\alpha)$ *increasing for* $j \leqslant k$*. Let* $i \leqslant k$ *be some index and write* $R = \mathit{Pivots}_{\rho^i(\mathbf{w})(C)}(u_{(i)})$*. If* $\lambda^i(\mathbf{u}_+)(\alpha) \notin R$*, then* $\lambda^j(\mathbf{u}_+)(\alpha) \notin R$ *for any* $i \leqslant j \leqslant k$*.*

Proof. Write $r_\alpha = \lambda^i(\mathbf{u}_+)(\alpha)$. By Proposition IV.7, if $r \in R$ with $\rho^i(\mathbf{w})(\alpha) < u_{(i)}^{-1}(r)$, then $r < r_\alpha$. Since $\lambda^j(\mathbf{u}_+)(\alpha)$ is weakly increasing for $j \leqslant k$ and $\mathbf{u}_+$ is a positive subexpression, α cannot cross above any strand β with $\lambda^j(\mathbf{u}_+)(\beta) \in R$ before reaching its peak. □

Proposition IV.27. *Let* C *be a collection of strands and let* β *be a strand so that* $\rho^j(\mathbf{w})(\beta)$ *is weakly decreasing for* $j \geqslant i$*. Write* $R = \mathit{Pivots}_C(u_{(i)})$*. If* $\lambda^i(\mathbf{u}_+)(\beta) \in R$*, then* $\lambda^j(\mathbf{u}_+)(\beta) \in R$ *for any* $j \geqslant i$*.*

Proof. Write $r_\beta = \lambda^i(\mathbf{u}_+)(\beta)$. By positivity of $\mathbf{u}_+$, if β crosses below a strand α at an index $j+1 > i$, then $\lambda^j(\mathbf{u}_+)(\alpha) < \lambda^j(\mathbf{u}_+)(\beta) \leqslant \lambda^i(\mathbf{u}_+)(\beta)$. Let $r = \lambda^j(\mathbf{u}_+)(\alpha)$. Since $\mathbf{u}_+$ is a positive word, we must have $u_{(i)}^{-1}(r) < \rho^i(\mathbf{w})(\beta)$. By Proposition IV.7, since $r_\beta \in \text{Pivots}_C\left(u_{(i)}\right)$ and $r < r_\beta$ with $u_{(i)}^{-1}(r) < u_{(i)}^{-1}(r_\beta)$, we must have $r \in \text{Pivots}_C\left(u_{(i)}\right)$. □

Corollary IV.28 (Pivot stabilization criterion)**.** *Let C be a collection of strands and let $R = Pivots_C\left(u_{(i)}\right)$. Suppose that C satisfies the following closure property: If $j \geqslant i$ and $\lambda^j(\mathbf{u}_+)(\alpha) \in R$, then either $\alpha \in C$ or α is traveling down. Then $Pivots_C\left(u_{(j)}\right) = R$ for all $j \geqslant i$.*

Definition IV.29. Fix a reduced expression $\mathbf{w}$ for w with positive subexpression $\mathbf{u}_+$. Let $j \in J^\circ_{\mathbf{u}_+}$. The *jump chambers* of j are given by

$$\mathtt{JC}(j) = \{\, ㅊ \text{ a chamber in the wiring diagram } : \rho^{j-1}(\mathbf{w})\,(\,ㅊ\,) \in \mathcal{J}\left(u_{(j-1)}, s_{h_j}\right)\}.$$

Definition IV.30. The *closure* of $\mathtt{JC}(j)$, denoted $\overline{\mathtt{JC}(j)}$, is the union of the chambers $\mathtt{JC}(j)$ and the strand segments surrounding each of them.

A strand segment e between two crossings is adjacent to exactly two chambers, one above e and one below e.

Definition IV.31. We say that a strand α or geodesic path intersects $\overline{\mathtt{JC}(j)}$ if it contains a strand segment e which is adjacent to a chamber in $\mathtt{JC}(j)$.

Definition IV.32. The strand segment e belongs to the *boundary* $\partial\mathtt{JC}(j)$ if exactly one of the adjacent chambers is in $\mathtt{JC}(j)$. The *upper boundary* of $\mathtt{JC}(j)$, denoted $\partial_\uparrow\mathtt{JC}(j)$, is the union of all strand segments e so that the chamber below e is in $\mathtt{JC}(j)$ and the chamber above e is not in $\mathtt{JC}(j)$. The *lower boundary* $\partial_\downarrow\mathtt{JC}(j)$ is the union of strand segments e so that the chamber below e is not in $\mathtt{JC}(j)$ and the chamber above e is in $\mathtt{JC}(j)$. A geodesic path γ_r is an upper boundary geodesic if it contains a strand segment $e \in \partial_\uparrow\mathtt{JC}(j)$; γ_r is a lower boundary geodesic if it contains a strand segment $e \in \partial_\downarrow\mathtt{JC}(j)$.

4.4 The geometry of the region $\mathtt{JC}(j)$ in a unipeak wiring diagram

Our goal for this section is to describe the region in a unipeak wiring diagram consisting of the chambers $\mathtt{JC}(j)$ and the strand segments bounding or intersecting them. We will show that whenever the nearly positive sequence $\mathbf{v}^j$ is a distinguished subexpression, there is a simple closed cycle $\partial\mathtt{JC}(j)$ so that a chamber ㅊ belongs to $\mathtt{JC}(j)$ if and only if it is in the interior of $\partial\mathtt{JC}(j)$. The region $\mathtt{JC}(j)$ is bounded on the right by the descending strand $\alpha_{\searrow}(j)$ and the ascending strand $\alpha_{\nearrow}(j)$. Each strand segment on $\partial\mathtt{JC}(j)$ can be classified as either belonging to the upper boundary $\partial_{\uparrow}\mathtt{JC}(j)$ or belonging to the lower boundary $\partial_{\downarrow}\mathtt{JC}(j)$, where upper and lower boundary segments follow disjoint collections of strands. We will show that for each chamber ㅊ in $\mathtt{JC}(j)$, the set of pivots of $\rho^{j-1}(\mathbf{w})$ (ㅊ) with respect to the permutations $u_{(j-1)}$ and $v^j_{(j-1)}$ differ by indices r_- and r^+ so that the geodesic path γ_{r_-} follows a portion of $\partial_{\downarrow}\mathtt{JC}(j)$, travels under ㅊ in $\mathtt{JC}(j)$ and eventually crosses above the strand $\alpha_{\searrow}(j)$, while the geodesic path γ_{r^+} follows a portion of $\partial_{\uparrow}\mathtt{JC}(j)$, travels over ㅊ, and eventually crosses below $\alpha_{\nearrow}(j)$.

We begin by showing that every chamber in $\mathtt{JC}(j)$ is bounded from the right, and that the condition that every chamber in $\mathtt{JC}(j)$ is bounded from the left is equivalent to the condition that the nearly positive sequence $\mathbf{v}^j$ is a distinguished subexpression of $\mathbf{w}$ (so that $\mathbf{v}^j$ indexes a Deodhar hypersurface). Intuitively, chambers which are open on the right correspond to left-justified minors of the flag, which must have the upper rank conditions corresponding to $B_-\dot{u}B_+/B_+$, while chambers which are open on the left correspond to left-justified minors of an invertible upper triangular matrix, and hence the principal minors must be nonvanishing.

Proposition IV.33. *Fix a reduced expression $\mathbf{w}$ for w with positive subexpression $\mathbf{u}_+$. Let*

$j \in J^{\circ}_{\mathbf{u}_+}$. Then the following hold:

1. *If ㅊ $\in \mathtt{JC}(j)$, then ㅊ $=$ ㅊ$_{\leftarrow}(i)$ for some $i \leqslant j$.*
2. *The nearly positive subsequence $\mathbf{v}^j$ is a distinguished subexpression of $\mathbf{w}$ if and only if every ㅊ $\in \mathtt{JC}(j)$ is bounded.*

Proof. Suppose that ㅊ is a chamber at height h which is not given by ㅊ$_{\leftarrow}(i)$ for any $i \leqslant j$. Then there is some index $k \geqslant j$ so that $\rho^k(\mathbf{w})$ (ㅊ) $= [1, h]$. Since $\rho(\mathbf{w})\,(\alpha_{\nearrow}(j)) > \rho(\mathbf{w})\,(\alpha_{\searrow}(j))$ for all $k \geqslant j$, if $\alpha_{\nearrow}(j) \in$ ㅊ then $\alpha_{\searrow}(j) \in$ ㅊ. So if $h_j \in \rho^{j-1}(\mathbf{w})$ (ㅊ) then $h_j + 1 \in \rho^{j-1}(\mathbf{w})$ (ㅊ) and hence $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}s_{h_j}\right) = \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}\right)$. So every ㅊ $\in \mathtt{JC}(j)$ must be given by ㅊ$_{\leftarrow}(i)$ for some $i \leqslant j$.

It follows that any open chamber ㅊ $\in \mathtt{JC}(j)$ must be on the left boundary of the wiring diagram. If ㅊ is an open chamber at height h, then $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}\right) = [1, h]$ and $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}s_{h_j}\right) = v^0_{(j)}([1, h])$. The nearly positive subsequence $\mathbf{v}^j$ is a distinguished subexpression if and only if $v^0_{(j)} = 1$, which is equivalent to the condition $v^0_{(j)}([1, h]) = [1, h]$ for every $1 \leqslant h \leqslant n$. □

We next observe that the region $\mathtt{JC}(j)$ is weakly above the ascending strand $\alpha_{\nearrow}(j)$ and below the descending strand $\alpha_{\searrow}(j)$. In the wiring diagram for the partial word $w_{(j-1)}$, this corresponds to the strands below a chamber connecting to right endpoints indexed by a set C that includes h_j and excludes $h_j + 1$.

Proposition IV.34. *Let ㅊ be a chamber satisfying ㅊ $\in \mathtt{JC}(j)$ and let i be a crossing so that ㅊ is one of the four chambers surrounding i. Then the chamber ㅊ and the crossing i are weakly above $\alpha_{\nearrow}(j)$ and below $\alpha_{\searrow}(j)$. If $\mathbf{w}$ is unipeak, this implies that either $\alpha_{\searrow}(i) = \alpha_{\searrow}(j)$ or $\rho^{j-1}(\mathbf{w})\,(\alpha_{\searrow}(i)) < h_j$.*

Proof. We must have $h_j \in \rho^{j-1}(\mathbf{w})$ (ㅊ) and $h_j + 1 \notin \rho^{j-1}(\mathbf{w})$ (ㅊ), which means

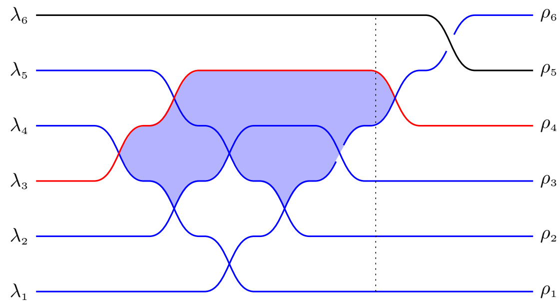

Figure 4.1: Strands in $C^{\uparrow}(8)$ are shown in red and strands in $C^{\downarrow}(8)$ are shown in blue. Immediately to the left of crossing 8, strands in $C^{\uparrow}(8)$ are at heights $\geqslant h_8 + 1$ and strands in $C^{\downarrow}(8)$ are at heights $\leqslant h_8$. The strand beginning at source λ_6 does not intersect $\overline{\texttt{JC}(8)}$.

that the $\alpha_{\nearrow}(j)$ is below ㅊ and the strand $\alpha_{\searrow}(j)$ is above ㅊ. By Corollary IV.21, if $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right) \neq \text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right)$, then $h_j \in \rho^{j-1}(\mathbf{w})(ㅊ)$ and $h_j + 1 \notin \rho^{j-1}(\mathbf{w})(ㅊ)$. We have $\rho^{j-1}(\mathbf{w})\left(\alpha_{\nearrow}(j)\right) = h_j$ and $\rho^{j-1}(\mathbf{w})\left(\alpha_{\searrow}(j)\right) = h_j + 1$. A strand which is strictly between $\alpha_{\nearrow}(j)$ and $\alpha_{\searrow}(j)$ at some index $i < j$ must either cross above $\alpha_{\searrow}(j)$ or cross below $\alpha_{\nearrow}(j)$ to the left of j; a strand which begins descending before index j must cross below $\alpha_{\nearrow}(j)$. $\square$

We can partition the set of strands that intersect $\overline{\texttt{JC}(j)}$ into sets $C^{\uparrow}(j)$ and $C^{\downarrow}(j)$ based on their right endpoints in the wiring diagram for $w_{(j-1)}$, and similarly we partition the geodesic paths that intersect $\overline{\texttt{JC}(j)}$ into sets $R^{\uparrow}(j)$ and $R^{\downarrow}(j)$.

Definition IV.35. We define $C^{\uparrow}(j)$ to be the set of strands which intersect $\overline{\texttt{JC}(j)}$ and have left endpoints in $w_{(j-1)}([h_j + 1, n])$, and we define $C^{\downarrow}(j)$ to be the set of strands intersecting $\overline{\texttt{JC}(j)}$ with left endpoints in $w_{(j-1)}([1, h_j])$. [3]

(See Figure 4.1.)

Definition IV.36. Let $R^{\downarrow}(j)$ denote the set of indices $r \in v^j_{(j-1)}([1, h_j])$ so that γ_r intersects $\overline{\texttt{JC}(j)}$ and let $R^{\uparrow}(j)$ denote the set of indices $r \in v^j_{(j-1)}([h_j + 1, n])$ so that γ_r intersects $\overline{\texttt{JC}(j)}$.

[3] That is, immediately to the left of crossing j, the strands $C^{\uparrow}(j)$ are at heights in the interval $[h_j + 1, n]$ and the strands $C^{\downarrow}(j)$ are at heights in the interval $[1, h_j]$.

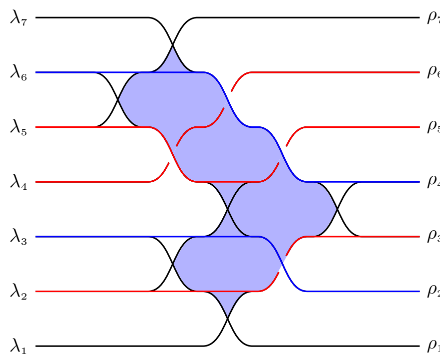

Figure 4.2: Geodesic paths in $R^{\uparrow}(10)$ are shown in red and geodesics in $R^{\downarrow}(10)$ are shown in blue. The geodesic beginning at λ_4 is not a boundary geodesic.

(See Figure 4.4.)

A key property of $\mathtt{JC}(j)$ is the following statement about chambers surrounding a crossing $i \in J^{+}_{\mathbf{u}_+}$, which follows immediately from Lemma IV.25.

Corollary IV.37. *Let $i \in J^{+}_{\mathbf{u}_+}$, and suppose that some chamber ㅊ $\in \{$ㅊ$_{\leftarrow}(i)$, ㅊ$_{\uparrow}(i)$, ㅊ$_{\downarrow}(i)$, ㅊ$_{\rightarrow}(i)\}$ is in $\mathtt{JC}(j)$. Then there is a chamber ㅊ$'$ intersecting ㅊ along a strand segment at i so that ㅊ$'$ is in $\mathtt{JC}(j)$. If a pair of opposite chambers $\{$ㅊ$_{\leftarrow}(i)$, ㅊ$_{\rightarrow}(i)\}$ or $\{$ㅊ$_{\uparrow}(i)$, ㅊ$_{\downarrow}(i)\}$ is in $\mathtt{JC}(j)$, then so is the other pair.*

We will need to refer to the previous crossings in $J^{\circ}_{\mathbf{u}_+}$ which share the same ascending or descending strand as the crossing j. We will eventually show that these crossings (where defined) are on the boundary $\partial\mathtt{JC}(j)$.

Notation IV.38. We denote the index $u_{(j-1)}(h_j) = v^{j}_{(j-1)}(h_j + 1)$ by r_* and we denote the index $u_{(j-1)}(h_j + 1) = v^{j}_{(j-1)}(h_j + 1)$ by r^*.

Notation IV.39. Let $j^{-}_{\nearrow} = \max\{i < j : i \in J_{\nearrow}(\alpha_{\nearrow}(j)) \cap J^{\circ}_{\mathbf{u}_+}\}$, and let $j^{-}_{\searrow} = \max\{i < j : i \in J_{\searrow}(\alpha_{\searrow}(j)) \cap J^{\circ}_{\mathbf{u}_+}\}$, where we take the maximum of an empty set to be 0.

We will see that if $v^{j}_{(0)} = 1$, then $j^{-}_{\nearrow} > 0$ while $j^{-}_{\searrow}$ may equal 0. Note that there may be crossings $i \in J^{+}_{\mathbf{u}_+}$ where $i \in J_{\nearrow}(\alpha_{\nearrow}(j))$ and $j^{-}_{\nearrow} < i < j$ or $i \in J_{\searrow}(\alpha_{\searrow}(j))$

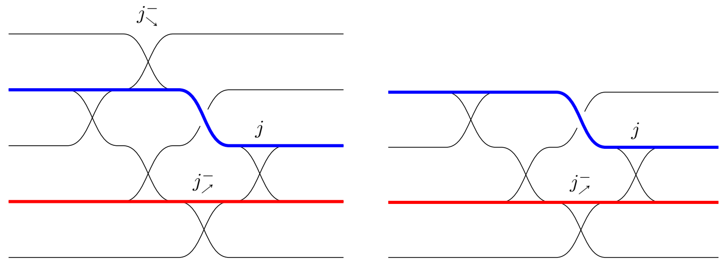

Figure 4.3: At left, the strand $\alpha_{\searrow}(j)$ joins γ_{r^*} by traveling down at the index $j^-_{\searrow}$. At right, the strand $\alpha_{\searrow}(j)$ joins γ_{r^*} by traveling up.

and $j^-_{\searrow} < i < j$. That is, between $j^-_{\nearrow}$ and j the strand $\alpha_{\nearrow}(j)$ follows the geodesic path γ_{r_*}; for indices in $J_{\searrow}(\alpha_{\searrow}(j))$ between $j^-_{\searrow}$ and j, the strand $\alpha_{\searrow}(j)$ follows the geodesic path γ_{r^*}.

In the following proposition, we describe the part of $\partial\mathtt{JC}(j)$ on the strand $\alpha_{\nearrow}(j)$ and the descending portion of the strand $\alpha_{\searrow}(j)$. We show that this coincides with the segment where $\alpha_{\nearrow}(j)$ is traveling up along the geodesic path γ_{r_*} and the segment where $\alpha_{\searrow}(j)$ is traveling down along the geodesic path γ_{r^*}, together with the crossing j.

Proposition IV.40. *Let i be an index with $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$. If $\alpha_{\nearrow}(i) = \alpha_{\nearrow}(j)$, then $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\nearrow}(i)) = r_*$. If $\alpha_{\searrow}(i) = \alpha_{\searrow}(j)$, then $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\searrow}(i)) = r^*$. Conversely, if $i \in J_{\nearrow}(\alpha_{\nearrow}(j))$ and $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\nearrow}(j)) = r_*$ or $i \in J_{\searrow}(\alpha_{\searrow}(j))$ and $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\searrow}(j)) = r^*$, then $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$.*

Proof. By Lemma III.20, if $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\nearrow}(i)) \neq u_{(j-1)}\,(h_j)$, then $\rho^{i-1}(\mathbf{u}_+)\,(u_{(j-1)}) > h_i$ so that $u_{(j-1)}(h_j) \notin \lambda^{i-1}(\mathbf{u}_+)\,(\,\text{ㅊ}_{\leftarrow}(i)) = \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$. So $\rho^{j-1}(\mathbf{w})\,(\,\text{ㅊ}_{\leftarrow}(i))$ $\notin \mathcal{J}\left(u_{(j-1)}, s_{h_j}\right)$.

Similarly, if $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\searrow}(i)) \neq u_{(j-1)}(h_j+1)$, then $\rho^{i-1}(\mathbf{u}_+)\,(u_{(j-1)}\,(h_j+1)) \leqslant h_i$ and so $u_{(j-1)}(h_j+1) \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$. So $\rho^{j-1}(\mathbf{w})\,(\,\text{ㅊ}_{\leftarrow}(i)) \notin \mathcal{J}\left(u_{(j-1)}, s_{h_j}\right)$.

For the converse, we note that if $i \in J^+_{\mathbf{u}_+}$ is an index with $i < j$ such that

$\rho^{j-1}(\mathbf{w})\,(\alpha_{\nearrow}(i)) = h_j$ and $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\nearrow}(i)) = u_{(j-1)}(h_j)$, then $\alpha_{\nearrow}(i) = \alpha_{\nearrow}(j)$ is traveling up before index j with $\lambda(\mathbf{u}_+)\,(\alpha_{\nearrow}(i)) = u_{(j-1)}(h_j)$ between i and j, so that in particular for all $\alpha \in \text{ㅊ}_{\leftarrow}(i)$ we have $\rho^{j-1}(\mathbf{w})\,(\alpha) \leqslant h_j$ and for all $r \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$ $= u_{(i-1)}([1, h_i])$, we have $\rho^{j-1}(\mathbf{u}_+)\,(r) \leqslant h_j$. Since we have $h_j \in \rho^{j-1}(\mathbf{w})\,(\text{ㅊ}_{\leftarrow}(i))$, $h_j + 1 \notin \rho^{j-1}(\mathbf{w})\,(\text{ㅊ}_{\leftarrow}(i))$, $u_{(j-1)}(h_j) \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$ and $u_{(j-1)}(h_j + 1) \notin$ $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$, we have $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)}) < \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,\left(u_{(j-1)}s_{h_j}\right)$.

If $i \in J^+_{\mathbf{u}_+}$ is an index with $i < j$ so that $\rho^{j-1}(\mathbf{w})\,(\alpha_{\searrow}(i)) = h_j + 1$ and $\lambda^{i-1}(\mathbf{u}_+)\,(\alpha_{\nearrow}(i))$ $= u_{(j-1)}(h_j + 1)$, then for all indices between i and j, $\alpha_{\searrow}(i) = \alpha_{\searrow}(j)$ is traveling down with $\lambda(\mathbf{u}_+)\,(\alpha_{\searrow}(i)) = u_{(j-1)}(h_j)$. In particular, if k is an intermediate crossing along $\alpha_{\searrow}(j)$, then $k \in J^+_{\mathbf{u}_+}$ with $\alpha_{\nearrow}(k) \in \text{ㅊ}_{\leftarrow}(i)$ and $\lambda^k(\mathbf{u}_+)\,(\alpha_{\nearrow}(k)) \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$ $= u_{(i-1)}([1, h_i])$ so that $\#\{c \in \rho^{j-1}(\mathbf{w})\,(\text{ㅊ}_{\leftarrow}(i)) : c \leqslant h_j\} = \#\{r \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)}) :$ $\rho^{j-1}(\mathbf{u}_+)\,(r) \leqslant h_j\}$. Since we have $h_j \in \rho^{j-1}(\mathbf{w})\,(\text{ㅊ}_{\leftarrow}(i))$, $h_j + 1 \notin \rho^{j-1}(\mathbf{w})\,(\text{ㅊ}_{\leftarrow}(i))$, $u_{(j-1)}(h_j) \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$ and $u_{(j-1)}(h_j + 1) \notin \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$, we have $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)}) < \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,\left(u_{(j-1)}s_{h_j}\right)$. $\square$

In the following two propositions, we show that any chamber in $\mathtt{JC}(j)$ except the chamber $\text{ㅊ}_{\leftarrow}(j)$ is connected on the right to another chamber in $\mathtt{JC}(j)$; this goes back to the condition that if two strands cross before index j, then the descending strand has a lower right endpoint at index $j-1$.

Proposition IV.41. *Let i be an index in $J^\circ_{\mathbf{u}_+}$ with $i < j$. Suppose that $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$. Then the chamber $\text{ㅊ}_{\rightarrow}(i)$ is in $\mathtt{JC}(j)$.*

Proof. Since $i \in J^\circ_{\mathbf{u}_+}$, we may write $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\rightarrow}(i))}\,(u_{(j-1)}) = \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)})$ $= R$. Since $\mathbf{w}$ is a reduced word and $j > i$, we have $\rho^{j-1}(\mathbf{w})\,(\text{ㅊ}_{\rightarrow}(i)) < \rho^{j-1}(\mathbf{w})\,(\text{ㅊ}_{\leftarrow}(i))$ and so $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,\left(u_{(j-1)}s_{h_j}\right) \geqslant \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,\left(u_{(j-1)}s_{h_j}\right) > R$. $\square$

For unipeak wiring diagrams, if ㅊ is a chamber in $\mathtt{JC}(j)$ and the strand α

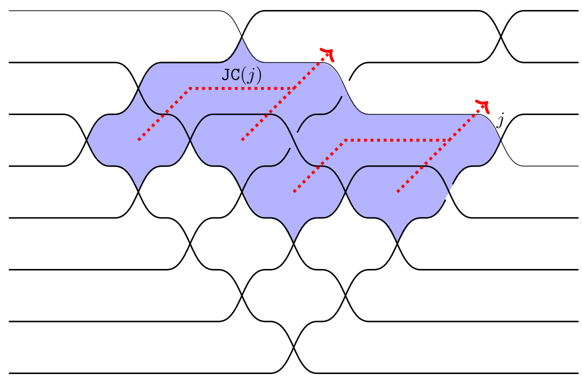

Figure 4.4: Paths traveling northeast between chambers in $\texttt{JC}(j)$ eventually reach the descending strand $\alpha_{\searrow}(j)$.

bounds ㅊ from the northeast, then either α is $\alpha_{\searrow}(j)$ or the chamber $\text{ㅊ}'$ northeast of ㅊ is also in $\texttt{JC}(j)$. Inductively, there is a sequence of chambers in $\texttt{JC}(j)$ beginning with ㅊ and traveling northeast until hitting $\alpha_{\searrow}(j)$ (see Figure 4.4). (This is false in general for a diagram that is not unipeak.)

Proposition IV.42. *Let i be a crossing of any type. Then the following hold.*

1. *If the chamber $\text{ㅊ}_{\downarrow}(i) \in \texttt{JC}(j)$ and the strand $\alpha_{\searrow}(i) \neq \alpha_{\searrow}(j)$, then $\text{ㅊ}_{\rightarrow}(i) \in \texttt{JC}(j)$.*
2. *If the chamber $\text{ㅊ}_{\leftarrow}(i) \in \texttt{JC}(j)$ and the strand $\alpha_{\searrow}(i) \neq \alpha_{\searrow}(j)$, then $\text{ㅊ}_{\uparrow}(i) \in \texttt{JC}(j)$.*

Proof. Since $h_j \in \rho^{j-1}(\mathbf{w})\left(\text{ㅊ}_{\downarrow}(i)\right)$ and $h_j + 1 \notin \rho^{j-1}(\mathbf{w})\left(\text{ㅊ}_{\downarrow}(i)\right)$, the crossing i is weakly between the strands $\alpha_{\nearrow}(j)$ and $\alpha_{\searrow}(j)$. If $\alpha_{\searrow}(i) \neq \alpha_{\searrow}(j)$, then it must cross below $\alpha_{\nearrow}(j)$ to the left of j by the unipeak property. So $\rho^{j-1}(\mathbf{w})\left(\alpha_{\searrow}(j)\right) < h_k$, and hence by Corollary IV.23, if $C \in \mathcal{J}\left(u_{(j-1)}, s_{h_j}\right)$, then so is $C\rho^{j-1}(\mathbf{w})\left(\alpha_{\searrow}(j)\right)$. Apply this to the pair with $C = \rho^{j-1}(\mathbf{w})\left(\text{ㅊ}_{\downarrow}(i)\right)$ and $C\rho^{j-1}(\mathbf{w})\left(\alpha_{\searrow}(j)\right) = \rho^{j-1}(\mathbf{w})\left(\text{ㅊ}_{\rightarrow}(i)\right)$ and to the pair with $C = \rho^{j-1}(\mathbf{w})\left(\text{ㅊ}_{\leftarrow}(i)\right)$ and $C\rho^{j-1}(\mathbf{w})\left(\alpha_{\searrow}(j)\right) = \rho^{j-1}(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(i)\right)$.

$\square$

Corollary IV.43. *Suppose that i is a crossing where a segment of $\partial_{\uparrow}\texttt{JC}(j)$ meets a segment of $\partial_{\downarrow}\texttt{JC}(j)$. Then $i \in J^{\circ}_{\mathbf{u}_+}$ and one of the following cases holds.*

1. *The chambers* $\text{ㅊ}_{\leftarrow}(i), \text{ㅊ}_{\uparrow}(i)$ *and* $\text{ㅊ}_{\downarrow}(i)$ *are not in* $\mathtt{JC}(j)$ *and* $\text{ㅊ}_{\rightarrow}(i) \in \mathtt{JC}(j)$.

2. *The chamber* $\text{ㅊ}_{\leftarrow}(i)$ *is not in* $\mathtt{JC}(j)$ *and the chambers* $\text{ㅊ}_{\uparrow}(i)$, $\text{ㅊ}_{\downarrow}(i)$ *and* $\text{ㅊ}_{\rightarrow}(i)$ $\in \mathtt{JC}(j)$.

3. *The crossing* i *is equal to* j.

Corollary IV.44. *Suppose that* $\alpha \in C^{\uparrow}(j)$ *intersects* $\overline{\mathtt{JC}(j)}$ *only on* $\partial_{\uparrow} \mathtt{JC}(j)$. *Then either* $\alpha = \alpha_{\searrow}(j)$, *or* $\alpha = \alpha_{\nearrow}(j_{\searrow}^{-})$.

In the following proposition, we show that if $\text{ㅊ} = \text{ㅊ}_{\leftarrow}(i)$ is a chamber in $\mathtt{JC}(j)$ and r^+ and r_- are the indices of the pivots added to and removed from $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}\right)$ to obtain $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}')}\left(v^j_{(j-1)}\right)$, then every chamber between the geodesic paths γ_{r^+} and γ_{r_-} immediately to the left of the crossing i is in $\mathtt{JC}(j)$.

Proposition IV.45. *Let* $\text{ㅊ} \in \mathtt{JC}(j)$ *and write* r^+ *and* r_- *for the indices so that* $\mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(v^j_{(j-1)}\right) = \mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}\right) \setminus \{r_-\} \cup \{r^+\}$. *Write* $\text{ㅊ} = \text{ㅊ}_{\leftarrow}(i)$ *for some* $i < j$. *At index* $i-1$, *the paths* γ_{r^+} *and* γ_{r_-} *are in* $\overline{\mathtt{JC}(j)}$, *with* γ_{r^+} *above* ㅊ *and* γ_{r_-} *below* ㅊ.

Proof. A geodesic path γ_r is below ㅊ if $r \in \mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})(\text{ㅊ}')}\left(u_{(k-1)}\right)$ for $k \geqslant i$ and above ㅊ if and only if $r \notin \mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})(\text{ㅊ}')}\left(u_{(k-1)}\right)$ for $k \geqslant i$.

Let $\text{ㅊ}'$ be any chamber above ㅊ, so that $\mathrm{Strands}(\text{ㅊ}) \subseteq \mathrm{Strands}(\text{ㅊ}')$. Then $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(v^j_{(j-1)}\right) \subseteq \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}')}\left(v^j_{(j-1)}\right)$. Since $r^+ \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}')}\left(v^j_{(j-1)}\right)$, we must have $r^+ \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}')}\left(v^j_{(j-1)}\right)$. So if $\text{ㅊ}' \notin \mathtt{JC}(j)$, then γ_{r^+} is below $\text{ㅊ}'$. Hence γ_{r^+} is in $\overline{\mathtt{JC}(j)}$ at index $i-1$.

Now suppose that $\text{ㅊ}'$ is a chamber below ㅊ, so that $\mathrm{Strands}(\text{ㅊ}') \subseteq \mathrm{Strands}(\text{ㅊ})$. Then $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}')}\left(v^j_{(j-1)}\right) \subseteq \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(v^j_{(j-1)}\right)$. In particular, if $\text{ㅊ}' \notin \mathtt{JC}(j)$, then $r_- \notin \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}')}\left(u_{(j-1)}\right)$, and so γ_{r_-} is above $\text{ㅊ}'$. So γ_{r_-} is in $\overline{\mathtt{JC}(j)}$ at index $i-1$. $\square$

In the next few propositions, we will show that γ_{r^+} being an upper boundary geodesic is equivalent to the existence of some chamber ㅊ in $\texttt{JC}(j)$ so that r^+ is the "jump" pivot, and γ_{r_-} being a lower boundary geodesic is equivalent to the existence of a chamber ㅊ $\in \texttt{JC}(j)$ so that r_- is the pivot removed when the pivots of ㅊ jump.

Proposition IV.46. *If γ_{r^+} is an upper boundary geodesic for $\texttt{JC}(j)$, then there is some chamber ㅊ $\in \texttt{JC}(j)$ so that $r^+ \notin \mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right)$ and $r^+ \in \mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right)$.*

If γ_{r_-} is a lower boundary geodesic for $\texttt{JC}(j)$,then there is some chamber ㅊ $\in \texttt{JC}(j)$ so that $r_- \in \mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right)$ and $r_- \notin \mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right)$.

Proof. We first note that γ_{r^+} is an upper boundary geodesic if and only if there are chambers ㅊ $\in \texttt{JC}(j)$ and ㅊ$' \notin \texttt{JC}(j)$ and a strand α so that Strands(ㅊ$'$) $=$ Strands(ㅊ) $\cup \{\alpha\}$ and $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right) = R$, $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ')}\left(u_{(j-1)}\right) = R \cup \{r^+\}$. We have $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right) \subseteq \text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ')}\left(v^j_{(j-1)}\right) = R \cup \{r^+\}$ since ㅊ$' \notin \texttt{JC}(j)$. Since ㅊ $\in \texttt{JC}(j)$, $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right) \neq R$, so by cardinality $r^+ \in \text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right)$.

Similarly, γ_{r_-} is a lower boundary geodesic if and only if there are chambers ㅊ $\in \texttt{JC}(j)$ and ㅊ$' \notin \texttt{JC}(j)$ and a strand α so that Strands(ㅊ) $=$ Strands(ㅊ$'$) $\cup \{\alpha\}$ and $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ')}\left(u_{(j-1)}\right) = R$, $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right) = R \cup \{r_-\}$. Since ㅊ$' \notin \texttt{JC}(j)$, $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ')}\left(v^j_{(j-1)}\right) = R$, so that $R \subseteq \text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right)$. Since ㅊ $\in \texttt{JC}(j)$, $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right) \neq R \cup \{r_-\}$, so by cardinality $r_- \notin \text{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right)$. □

Corollary IV.47. *If γ_r is an upper boundary geodesic, then $r \in R^{\downarrow}(j)$, and if γ_r is a lower boundary geodesic, then $r \in R^{\uparrow}(j)$.*

Corollary IV.48. *Suppose that $r \in R^{\downarrow}(j)$ and γ_r is a geodesic path that goes through $\texttt{JC}(j)$*

but is not an upper boundary geodesic. Then there is an index $r^+ < r$ so that γ_{r^+} is an upper boundary geodesic and γ_r crosses below γ_{r^+} on $\partial_\uparrow \mathtt{JC}(j)$.

Suppose that $r \in R^\uparrow(j)$ and γ_r is a geodesic path that goes through $\mathtt{JC}(j)$ but is not a lower boundary geodesic. Then there is an index $r_- > r$ so that γ_{r_-} is a lower boundary geodesic and γ_r crosses above γ_{r_-} on $\partial_\downarrow \mathtt{JC}(j)$.

Proof. If γ_r goes through $\mathtt{JC}(j)$ but is not a boundary geodesic, then it first enters the interior of $\mathtt{JC}(j)$ by either crossing below an upper boundary geodesic γ_{r^+} with $r^+ < r$ along some segment of $\partial_\uparrow \mathtt{JC}(j)$ or by crossing below a lower boundary geodesic γ_{r_-} with $r_- > r$ along some segment of $\partial_\downarrow \mathtt{JC}(j)$. Since crossings on $\partial \mathtt{JC}(j)$ are to the left of crossing j, if γ_r crosses below γ_{r^+} along $\partial_\uparrow \mathtt{JC}(j)$, then since $\gamma_{r^+} \in R^\downarrow(j)$, we must have $\gamma_r \in R^\downarrow(j)$. Similarly, if γ_r crosses above γ_{r_-} along $\partial_\downarrow \mathtt{JC}(j)$, then since $\gamma_{r_-} \in R^\uparrow(j)$, γ_r is also in $R^\uparrow(j)$. $\square$

Proposition IV.49. *Suppose that ㅊ $\in \mathtt{JC}(j)$ and write*

$$\mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right) = \mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right) \setminus \{r_-\} \cup \{r^+\}.$$

Then γ_{r^+} is an upper boundary geodesic and γ_{r_-} is a lower boundary geodesic.

Proof. Write ㅊ $=$ ㅊ$_\leftarrow(i)$. By the proof of the previous proposition, γ_{r^+} and γ_{r_-} are in $\overline{\mathtt{JC}(j)}$ at index $i-1$. By the previous corollary, if γ_{r^+} is not an upper boundary geodesic, then there is some $r' < r^+$ so that $r' \in R^\downarrow(j)$ and $r' \notin \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right)$. So by minimality $r^+ \notin \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right)$. If γ_{r_-} is not a lower boundary geodesic, then there is some $r' > r_-$ so that $r' \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right)$ and $r' \in R^\uparrow(j)$. By the ordered pivot criterion, the unique element of $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(v^j_{(j-1)}\right) \setminus \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(ㅊ)}\left(u_{(j-1)}\right)$ is at least r', a contradiction. $\square$

The following proposition shows that if $i \in J^\circ_{\mathbf{u}_+}$ is any interior crossing of $\mathtt{JC}(j)$ in a unipeak diagram, then both the ascending strand and the descending strand belong to the lower partition $C^{\downarrow}(j)$. Note that this implies that if $\alpha \in C^{\uparrow}(j)$, then while α is traveling through the interior of $\mathtt{JC}(j)$, every crossing along α is in $J^{+}_{\mathbf{u}_+}$, so that α follows a single geodesic path.

Proposition IV.50. *Suppose that $i \in J^{\circ}_{\mathbf{u}_+}$ and $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$. Then $\alpha_{\nearrow}(j) \in C^{\downarrow}(j)$.*[4]

Proof. This holds by construction if $i = j$, so we may assume that $i < j$. By the previous proposition, since $i \in J^{\circ}_{\mathbf{u}_+}$ we must have $\alpha_{\nearrow}(i) \neq \alpha_{\nearrow}(j)$ and $\alpha_{\searrow}(i) \neq \alpha_{\searrow}(j)$, so that the crossing i is strictly below $\alpha_{\searrow}(j)$ and strictly above $\alpha_{\nearrow}(j)$. Suppose for contradiction that $\alpha_{\nearrow}(i) \in C^{\uparrow}(j)$. Since $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$, by the ordered pivot criterion we must have that $\alpha_{\nearrow}(j) \in \text{Strands}(\text{ㅊ}_{\leftarrow}(i))$, $\alpha_{\searrow}(j) \notin \text{Strands}(\text{ㅊ}_{\leftarrow}(i))$, $u_{(j-1)}(h_j) \in \text{Pivots}_{[1,h_i]}(u_{(i-1)})$ and that at index $j-1$, the number of strands in $\text{Strands}(\text{ㅊ}_{\leftarrow}(i))$ with heights at most h_j is equal to the number of geodesics γ_r where $r \in \text{Pivots}_{[1,h_i]}(u_{(i-1)})$ at heights at most h_j.

Suppose that immediately to the left of crossing i the strand β is below $\alpha_{\nearrow}(j)$ and on the geodesic path γ_r. Since $\alpha_{\nearrow}(j)$ is in $\text{Strands}(\text{ㅊ}_{\leftarrow}(i))$ with $i < j$ and the diagram is unipeak, the strand β is in both $\text{Strands}(\text{ㅊ}_{\leftarrow}(i))$ and $\text{Strands}(\text{ㅊ}_{\leftarrow}(j))$ and γ_r is below both $\text{ㅊ}_{\leftarrow}(i)$ and $\text{ㅊ}_{\leftarrow}(j)$, so that r is in both $\text{Pivots}_{\rho^{i-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}(u_{(i-1)})$ and $\text{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(j))}(u_{(j-1)})$. It therefore suffices to consider the number of strands and geodesic paths which cross below $\alpha_{\nearrow}(j)$ between indices i and j.

Suppose that $r \in \text{Pivots}_{\rho^{i-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}(u_{(i-1)})$ and the geodesic path γ_r does not cross below $\alpha_{\nearrow}(j)$ to the left of crossing j. Since the crossing i is below $\alpha_{\searrow}(j)$ and above $\alpha_{\nearrow}(j)$, γ_r must cross above $\alpha_{\searrow}(j)$ at some index k to the left of crossing j. It follows that $\alpha_{\nearrow}(k)$ is a strand in $\text{Strands}(\text{ㅊ}_{\leftarrow}(i))$ which does not cross below $\alpha_{\nearrow}(i)$ to the left of crossing j.

Since $\alpha_{\nearrow}(i) \in C^{\uparrow}(j)$, we must have that $\alpha_{\nearrow}(i)$ crosses above $\alpha_{\searrow}(j)$ to the left of

[4]This is false if the diagram is not unipeak.

j. Because $i \in J^{\circ}_{\mathbf{u}_+}$, $\lambda^i(\mathbf{u}_+)\,(\alpha_{\nearrow}(i)) \notin \mathrm{Pivots}_{\rho^{i-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(i-1)})$ and hence by Proposition IV.26, the crossing where $\alpha_{\nearrow}(i)$ crosses above $\alpha_{\searrow}(j)$ is on a geodesic path γ_r with $r \notin \mathrm{Pivots}_{\rho^{i-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(i-1)})$.

In particular, there are strictly more geodesic paths γ_r below $\alpha_{\nearrow}(j)$ at index $j-1$ such that $r \in \mathrm{Pivots}_{\rho^{i-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(i-1)})$ than there are strands β below $\alpha_{\nearrow}(j)$ at index $j-1$, a contradiction. $\square$

In the next proposition, we show that if a crossing i is on the left boundary of $\mathtt{JC}(j)$, then the ascending strand $\alpha_{\nearrow}(i)$ is in the upper partition $C^{\uparrow}(j)$ and the descending strand $\alpha_{\searrow}(j)$ is in the lower partition $C^{\downarrow}(j)$.

Proposition IV.51. *Suppose that for some i we have $\text{ㅊ}_{\leftarrow}(i) \notin \mathtt{JC}(j)$ and $\text{ㅊ}_{\rightarrow}(i) \in \mathtt{JC}(j)$. Then $\alpha_{\nearrow}(i) \in C^{\uparrow}(j)$ and $\alpha_{\searrow}(i) \in C^{\downarrow}(j)$.*

Proof. Since $\text{ㅊ}_{\leftarrow}(i) \notin \mathtt{JC}(j)$ and $\text{ㅊ}_{\rightarrow}(i) \in \mathtt{JC}(j)$, the crossing i is between the strands $\alpha_{\searrow}(j)$ and $\alpha_{\nearrow}(j)$ so that the strands $\alpha_{\nearrow}(i)$ and $\alpha_{\searrow}(i)$ must each be in one of the sets $C^{\uparrow}(j)$ and $C^{\downarrow}(j)$ depending on whether they cross above $\alpha_{\searrow}(j)$ or below $\alpha_{\nearrow}(j)$ to the right of i. Since the diagram is unipeak, $\alpha_{\searrow}(i)$ travels strictly down to the right of i, and hence $\alpha_{\searrow}(i) \in C^{\downarrow}(j)$. We claim that if $\alpha_{\nearrow}(i) \in C^{\downarrow}(j)$ and $\text{ㅊ}_{\rightarrow}(i) \in \mathtt{JC}(j)$, then $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$. If $i \in J^{\circ}_{\mathbf{u}_+}$, then $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\,(u_{(j-1)}) = \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\rightarrow}(i))}\,(u_{(j-1)})$ and the sets $\mathrm{Strands}(\text{ㅊ}_{\leftarrow}(i))$ and $\mathrm{Strands}(\text{ㅊ}_{\rightarrow}(i))$ have the same number of strands below height h_j at index $j-1$, so that by the ordered pivot criterion $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$ if and only if $\text{ㅊ}_{\rightarrow}(i) \in \mathtt{JC}(j)$. So if $i \in J^{\circ}_{\mathbf{u}_+}$ with $\text{ㅊ}_{\leftarrow}(i) \notin \mathtt{JC}(j)$ and $\text{ㅊ}_{\rightarrow}(i) \in \mathtt{JC}(j)$, then $\alpha_{\nearrow}(i) \in C^{\uparrow}(j)$. Suppose that $i \in J^{+}_{\mathbf{u}_+}$. If $\alpha_{\nearrow}(i)$ is an upper boundary strand, then there is some index $i' \in J^{\circ}_{\mathbf{u}_+}$ with $i' < i$ and $\alpha_{\nearrow}(i') = \alpha_{\nearrow}(i)$ so that $\text{ㅊ}_{\leftarrow}(i') \in \mathtt{JC}(j)$ and $\text{ㅊ}_{\rightarrow}(i') \notin \mathtt{JC}(j)$. So $\alpha_{\nearrow}(i) \in C^{\uparrow}(j)$. Otherwise, the strand $\alpha_{\searrow}(i)$ is a lower boundary strand. Write $r = u_{(i)}(h_{i+1})$ and $r_- = u_{(i)}(h_i)$, where $r < r_-$. Then γ_{r_-} is a

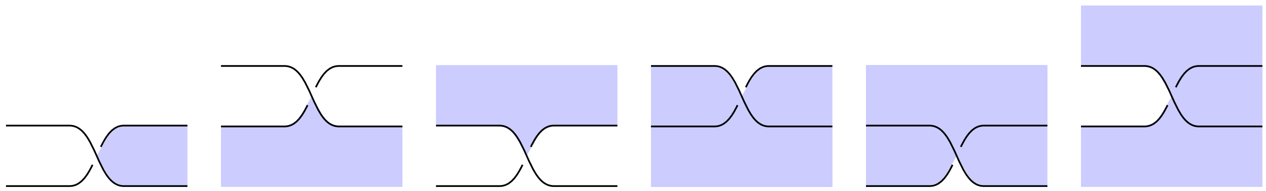

Figure 4.5: Since $\mathtt{JC}(j)$ is simply connected, any discontinuity of the boundary $\partial\mathtt{JC}(j)$ would correspond to switching strands at a crossing $i \in J^+_{\mathbf{u}_+}$, so that $\mathtt{JC}(j)$ would contain an odd number of the chambers surrounding i.

lower boundary geodesic, so that $r_- \in R^{\uparrow}(j)$. If $\alpha_{\nearrow}(i) \notin C^{\uparrow}(j)$, then $\alpha_{\nearrow}(i)$ must cross below $\alpha_{\nearrow}(j)$, and hence must cross below the geodesic path γ_{r_-} to the right of the crossing i. This violates Lemma III.20, so that we must have $\alpha_{\nearrow}(i) \in C^{\uparrow}(j)$. $\square$

Corollary IV.52. *Suppose that* $\text{ㅊ}_{\leftarrow}(i) \notin \mathtt{JC}(j)$ *and* $\text{ㅊ}_{\rightarrow}(i) \in \mathtt{JC}(j)$. *Write r for* $u_{(i)}(h_i + 1)$, *so that the geodesic path γ_r follows $\alpha_{\nearrow}(i)$ at index i. Let k be the index where $\alpha_{\nearrow}(i)$ crosses above $\alpha_{\searrow}(j)$, where we have $i < k < j$. Then γ_r follows $\alpha_{\nearrow}(i)$ between the indices i and k.*

Proof. The strand $\alpha_{\nearrow}(i)$ is in $C^{\uparrow}(j)$ and hence travels strictly up between indices i and k. If i' is the index of some crossing with $\alpha_{\nearrow}(i') = \alpha_{\nearrow}(i)$ and $\text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(j)$, then $i \in J^+_{\mathbf{u}_+}$, so that the geodesic path γ_r follows $\alpha_{\nearrow}(i)$ until it first crosses above $\partial_{\uparrow}\mathtt{JC}(j)$. Every connected component of $\partial_{\uparrow}\mathtt{JC}(j)$ which is not on $\alpha_{\searrow}(j)$ is a portion of a strand in $C^{\uparrow}(j)$ which travels strictly up before crossing above $\alpha_{\searrow}(j)$, so $\alpha_{\nearrow}(i)$ cannot cross above the upper boundary before crossing $\alpha_{\searrow}(j)$. $\square$

Proposition IV.53. *The region $\mathtt{JC}(j)$ is a simply connected region bounded by the closed cycle $\partial\mathtt{JC}(j)$.*

Proof. By Proposition IV.42, given any chamber ㅊ in $\mathtt{JC}(j)$, there is a path in $\mathtt{JC}(j)$ traveling northeast until reaching $\alpha_{\searrow}(j)$. Traveling southeast along $\alpha_{\searrow}(j)$ gives a path from ㅊ to $\text{ㅊ}_{\leftarrow}(j)$. So $\mathtt{JC}(j)$ is path-connected. The region $\mathtt{JC}(j)$ cannot have any holes; otherwise, there must be some crossing i bounding a hole from the left, so that ㅊ$_{\downarrow}(i) \in \mathtt{JC}(j)$ and ㅊ$_{\rightarrow}(i) \notin \mathtt{JC}(j)$. By Corollary IV.43, crossings where $\partial_{\uparrow}\mathtt{JC}(j)$ meets $\partial_{\downarrow}\mathtt{JC}(j)$ are in $J^{\circ}_{\mathbf{u}_+}$ (so that the strands meet at a point). At any crossing i where an upper boundary component or lower boundary component switches from following one strand to another, an odd number of the surrounding chambers are in $\mathtt{JC}(j)$, so that $i \in J^{\circ}_{\mathbf{u}_+}$. It follows that the boundary $\partial\mathtt{JC}(j)$ is a connected cycle. □

CHAPTER V

Regularity of Cluster Variables

In the previous chapter, we described the collection of chambers $\mathtt{JC}(j)$ where $j \in J^{\circ}_{\mathbf{u}_+}$. In this chapter, we introduce a chamber weighting for a unipeak expression $\mathbf{w}$ with positive subexpression $\mathbf{u}_+$, where the weight of each chamber is a product of algebraically independent indeterminates X_j where j ranges over $J^{\circ}_{\mathbf{u}_+}$. We will use pivot combinatorics to show that each of these variables X_j is given by $\Delta^{\widehat{R}^j}_{\lambda(\mathbf{w})(\widehat{C}^j)}$ for some collection of strands $\widehat{C}^j$ and rows $\widehat{R}^j = \mathrm{Pivots}_{\rho(\mathbf{w})(\widehat{C}^j)}(u)$ where $\Delta^{\widehat{R}^j}_{\lambda(\mathbf{w})(\widehat{C}^j)}$ vanishes identically on the Deodhar stratum $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$ corresponding to the nearly positive subexpression $\mathbf{v}^j$.

5.1 Nearly positive sequences and chamber weightings

Proposition V.1. *Let* $\mathbf{w}$ *be a unipeak expression with positive subexpression* $\mathbf{u}_+$*. Then there is a valid chamber weighting given by* $Q(\text{ㅊ}) = \prod_{j:\, \text{ㅊ} \in \mathit{JC}(j)} X_j$.

Proof. Since $\mathbf{u}_+$ is a positive subexpression, the set $J^{-}_{\mathbf{u}_+}$ is empty, so it suffices to show that for each $i \in J^{+}_{\mathbf{u}_+}$, we have $\mathrm{Q}(\text{ㅊ}_{\uparrow}(i))\mathrm{Q}(\text{ㅊ}_{\downarrow}(i)) = \mathrm{Q}(\text{ㅊ}_{\leftarrow}(i))\mathrm{Q}(\text{ㅊ}_{\rightarrow}(i))$. We must show that for each $j \in J^{\circ}_{\mathbf{u}_+}$, the monomials $\mathrm{Q}(\text{ㅊ}_{\uparrow}(i))\mathrm{Q}(\text{ㅊ}_{\downarrow}(i))$ and $\mathrm{Q}(\text{ㅊ}_{\leftarrow}(i))\mathrm{Q}(\text{ㅊ}_{\rightarrow}(i))$ have the same degree in the variable X_j. If $j < i$, then for any chamber ㅊ incident to i, we have $\text{ㅊ} \notin \mathtt{JC}(j)$ and so X_j does not divide

$\mathrm{Q}(\text{ㅊ})$. We note that there is no variable X_i since $i \in J^+_{\mathbf{u}_+}$. Suppose that $j > i$. By Lemma IV.28, for each chamber ㅊ incident to i, $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}(u_{(j-1)}) = \mathrm{Pivots}_{\rho^{i}(\mathbf{w})(\text{ㅊ})}(u_{(i)})$, so we have $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\leftarrow(i)})}(R) = u_{(i-1)}([1, h_i]) \neq u_{(i)}([1, h_i])$ $= \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ}_{\rightarrow(i)})}(R)$. By Corollary IV.37,

$$|\{\text{ㅊ}_{\uparrow}(i), \text{ㅊ}_{\downarrow}(i)\} \cap \mathtt{JC}(j)| = |\{\text{ㅊ}_{\leftarrow}(i), \text{ㅊ}_{\rightarrow}(i)\} \cap \mathtt{JC}(j)|.$$

So the products $\mathrm{Q}(\text{ㅊ}_{\uparrow}(i))\mathrm{Q}(\text{ㅊ}_{\downarrow}(i))$ and $\mathrm{Q}(\text{ㅊ}_{\leftarrow}(i))\mathrm{Q}(\text{ㅊ}_{\rightarrow}(i))$ have the same degree in X_j.

Iterating over all $\{j : j \in J^{\circ}_{\mathbf{u}_+}, j > i\}$ we have $\mathrm{Q}(\text{ㅊ}_{\uparrow}(i))\mathrm{Q}(\text{ㅊ}_{\downarrow}(i)) = \mathrm{Q}(\text{ㅊ}_{\leftarrow}(i))\mathrm{Q}(\text{ㅊ}_{\rightarrow}(i))$. $\square$

Corollary V.2. *If $\lambda(\mathbf{w})(\text{ㅊ}) = [1, h]$ for some h, then for each factor X_j dividing $Q(\text{ㅊ})$, the nearly positive subsequence $\mathbf{v}^j$ has $v^0_{(j)} \neq 1$. If $\rho^{\ell}(\mathbf{w})(\text{ㅊ}) = [1, h]$, then $Q(\text{ㅊ}) = 1$.*

We write $\mathbf{X}$ for the tuple $(X_j : j \in J^{\circ}_{\mathbf{u}_+})$. Given $i \in J^{\circ}_{\mathbf{u}_+}$, we write $t_i(\mathbf{X})$ for the ratio $\frac{\mathrm{Q}(\text{ㅊ}_{\uparrow(i)})\mathrm{Q}(\text{ㅊ}_{\downarrow(i)})}{\mathrm{Q}(\text{ㅊ}_{\leftarrow(i)})\mathrm{Q}(\text{ㅊ}_{\rightarrow(i)})}$ from Marsh and Rietsch's chamber ansatz formula.

We will now show that assignments of nonzero scalar values to the X_j is in bijection with right-normalized chamber weightings for the Deodhar torus, and therefore with Marsh and Rietsch's coordinates t_j. This will imply that the X_j are well-defined regular functions on the torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, given by ratios of chamber minors. We will eventually show that the functions X_j can be extended to regular functions on $\mathcal{R}^{u,w}$.

Proposition V.3. *Let $\mathbf{w}$ be a unipeak word with positive subexpression $\mathbf{u}_+$. The change of coordinates $\left(X_j : j \in J^{\circ}_{\mathbf{u}_+}\right) \to \left(t_j : j \in J^{\circ}_{\mathbf{u}_+}\right)$ is invertible.*

Proof. From the proof of Proposition 8.1 in [27], $\left(\Delta^{\lambda(\mathbf{u}_+)(\text{ㅊ}_{\leftarrow(j)})}_{\lambda(\mathbf{w})(\text{ㅊ}_{\leftarrow(j)})} : j \in J^{\circ}_{\mathbf{u}_+}\right) \to \left(t_j : j \in J^{\circ}_{\mathbf{u}_+}\right)$ is an isomorphism. It therefore suffices to show that the map $\left(X_j : j \in J^{\circ}_{\mathbf{u}_+}\right) \to$

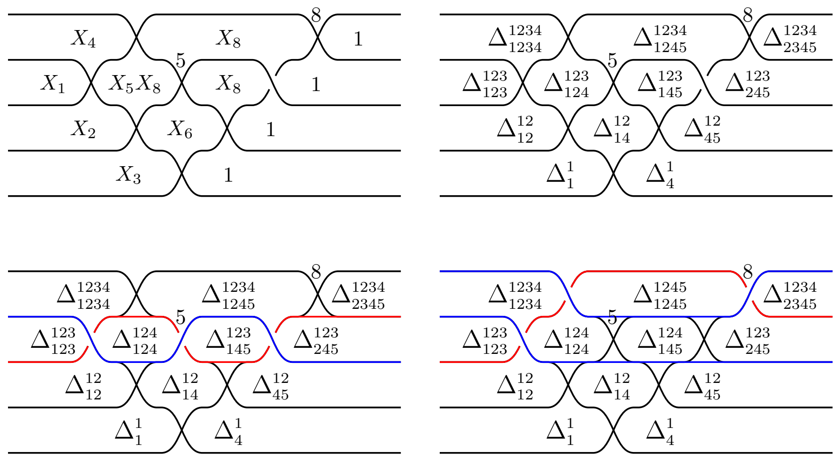

Figure 5.1: The first row shows the weighting $\mathrm{Q}(\text{え}) = \prod_{\text{え} \in \mathtt{JC}(k)} X_k$ together with the chamber labeling for the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$. The second row shows the chamber labelings for the boundary divisors $\mathcal{D}^{\mathbf{v}^{5},\mathbf{w}}$ and $\mathcal{D}^{\mathbf{v}^{8},\mathbf{w}}$.

$\left(\Delta^{\lambda(\mathbf{u}_+)(\text{え}_{\leftarrow}(i))}_{\lambda(\mathbf{w})(\text{え}_{\leftarrow}(i))} : i \in J^{\circ}_{\mathbf{u}_+}\right)$ is invertible. For each $i \in J^{\circ}_{\mathbf{u}_+}$, we have $\Delta^{\lambda(\mathbf{u}_+)(\text{え}_{\leftarrow}(i))}_{\lambda(\mathbf{w})(\text{え}_{\leftarrow}(i))} = \mathrm{Q}(\text{え}_{\leftarrow}(i)) = \prod_{j:\, \text{え}_{\leftarrow}(i) \in \mathtt{JC}(j)} X_j$. We have that $\text{え}_{\leftarrow}(i) \in \mathtt{JC}(i)$ and for each j with $\text{え}_{\leftarrow}(i) \in \mathtt{JC}(j)$, $j \geqslant i$. So the matrix giving the change of coordinates is a lower unitriangular matrix with rows and columns indexed by $J^{\circ}_{\mathbf{u}_+}$ and entry $(j,i) = 1$ if $\text{え}_{\leftarrow}(i) \in \mathtt{JC}(j)$. Row j is the bit vector representing the set $\{i \in J^{\circ}_{\mathbf{u}_+} : \text{え}_{\leftarrow}(i) \in \mathtt{JC}(j)\}$ and column i is the exponent vector for $\mathrm{Q}(\text{え}_{\leftarrow}(i))$. $\square$

One of our primary goals for this chapter is to give an algorithm for writing the parameter X_i as an irreducible minor in the case where the weighting on the chamber $\text{え}_{\leftarrow}(i)$ is given by a nontrivial product $\mathrm{Q}(\text{え}_{\leftarrow}(i)) = X_i \prod_{\substack{k>i \\ \text{え}_{\leftarrow}(i) \in \mathtt{JC}(k)}} X_k$. We first consider the following example, where $\mathrm{Q}(\text{え}_{\leftarrow}(i))$ has the form X_iX_k and the parameter X_i can be recovered by deleting a single row and column from the minor $\Delta^{u_{(i-1)}([1,h_i])}_{\lambda(\mathbf{w})(\text{え}_{\leftarrow}(i))}$.

Example V.4. Let w be the unipeak expression $s_3s_2s_1s_4s_3s_2\underline{s_3}s_4$ from our running example. In Figure 5.2, the chamber to the left of crossing 5 is given a weight of

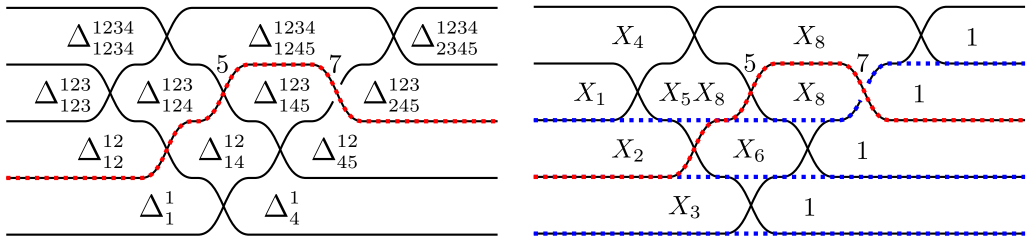

Figure 5.2: At crossing 7, the strand $\alpha_{\nearrow}(7)$ with left endpoint λ_1 and the geodesic path γ_3 cross above the strand $\alpha_{\nearrow}(5)$. The minor Δ^{123}_{124} and the minor Δ^{12}_{24} obtained by deleting row 3 and column 1 satisfy the same scalar relation $\Delta^{R\setminus\{3\}}_{C\setminus\{1\}} = \frac{\Delta^R_C}{X_8}$ as the pairs of chamber minors surrounding crossing 7.

X_5X_8, where X_5 does not appear in any other chamber weights. On the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, X_5 is a regular function with formula given by $X_5 = \frac{\Delta^{123}_{124}}{\Delta^{1234}_{1245}}$. This formula does not extend to the boundary divisor $\mathcal{D}^{\mathbf{v^8},\mathbf{w}}$, where the function $X_8 = \Delta^{1234}_{1245}$ vanishes. However, we may extend X_5 to a regular function on $\mathcal{R}^{u,w}$ by showing that there is a non-chamber minor which is equal to X_5 on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$.

We note that at index 7, the strand $\alpha_{\nearrow}(5)$ crosses down over the strand $\alpha_{\nearrow}(7)$. The minors in the four chambers surrounding crossing 7 are related by a common scalar, with $\frac{\Delta^{124}_{245}}{\Delta^{1234}_{1245}} = \frac{\Delta^{12}_{45}}{\Delta^{123}_{145}} = \frac{1}{X_8}$. We will show that this property that removing column 1 and row 3 corresponds to dividing by X_8 also applies to the minor Δ^{123}_{124} $= X_5X_8$, so that $\Delta^{12}_{24} = X_5$.

For the specific upper triangular matrix $z = \Upsilon^{u,w}\dot{w}^{-1}$, we have

$$\Delta^{12}_{24} = \det\begin{pmatrix} \frac{X_2+X_3X_5}{X_6} & 1 \\ \frac{X_2}{X_3} & \frac{X_6}{X_3} \end{pmatrix} = \frac{X_2}{X_3} + X_5 - \frac{X_2}{X_3} = X_5.$$

While the formula $X_5 = \frac{\Delta^{123}_{245}}{\Delta^{1234}_{1245}}$ expressing X_5 as a regular function on the Deodhar torus is not defined on the boundary divisor $\mathcal{D}^{\mathbf{v^8},\mathbf{w}}$, X_5 can be extended to a regular function on $\mathcal{R}^{u,w}$ by setting $X_5 = \Delta^{12}_{24}$.

5.2 The left-to-right path π^i

In this chapter, we prove that for each index $i \in J^{\circ}_{\mathbf{u}_+}$ in a unipeak wiring diagram, the variable X_i is a regular function on $\mathcal{R}^{u,w}$ by expressing X_i as a minor of the upper triangular matrix $\Upsilon^{u,w}\dot{w}^{-1}$. Given i, there is a collection of rows $\widehat{R}^i \subseteq \lambda^{i-1}(\mathbf{u}_+)\left(\text{ㅊ}_{\leftarrow}(i)\right)$ and a collection of strands $\widehat{C}^i \subseteq \mathrm{Strands}(\text{ㅊ}_{\leftarrow}(i))$ so that $\Delta^{\widehat{R}^i}_{\lambda(\mathbf{w})\left(\widehat{C}^i\right)} = X_i$.

We begin with an outline of our proof strategy.

Consider the collection of strands $\widehat{C}^i_i := \mathrm{Strands}(\text{ㅊ}_{\leftarrow}(i))$ and the collection of row indices $\widehat{R}^i_i := u_{(i-1)}([1,h])$, so that $\widehat{R}^i_i = \mathrm{Pivots}_{\rho^{i-1}(\mathbf{w})\left(\widehat{C}^i_i\right)}(u_{(i-1)}) = \mathrm{Pivots}_{\rho(\mathbf{w})\left(\widehat{C}^i_i\right)}(u)$. We note that $\Delta^{\widehat{R}^i_i}_{\lambda(\mathbf{w})\left(\widehat{C}^i_i\right)} = \mathrm{Q}(\text{ㅊ}_{\leftarrow}(i)) = X_i \prod_{\substack{k>i \\ \text{ㅊ}_{\leftarrow}(i) \in \mathtt{JC}(k)}} X_k = X_i \prod_{\substack{k>i \\ \rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_i\right) \in \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)}} X_k.$

Following Example V.4, we will iteratively define collections of strands $\widehat{C}^i_j$ and row indices $\widehat{R}^i_j$ by deleting one strand and one row index at a time. We will use the following criteria for performing a deletion.

1. The crossing j is in $J^+_{\mathbf{u}_+}$ and the strands $\alpha_{\nearrow}(j)$ and $\alpha_{\searrow}(j)$ are in $\widehat{C}^i_{j-1}$.
2. The set $\widehat{C}^i_{j-1}$ is contained in $\mathrm{Strands}(\text{ㅊ}_{\uparrow}(j))$.
3. The row index $r = \lambda^{j-1}(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$ is in $\widehat{R}^i_{j-1}$.

In this case, we will delete the strand $\alpha_{\nearrow}(j)$ from the set $\widehat{C}^i_{j-1}$ and the row index $r = \lambda^{j}(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$ from $\widehat{R}^i_{j-1}$ to obtain the sets $\widehat{C}^i_j$ and $\widehat{R}^i_j$. We will show that the minors $\Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)}$ and $\Delta^{\widehat{R}^i_{j}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j}\right)}$ are related by the identity $\dfrac{\Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)}}{\Delta^{\widehat{R}^i_{j}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j}\right)}} = \dfrac{\Delta^{\lambda(\mathbf{u}_+)(j-1)([1,h_j+1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}}{\Delta^{\lambda(\mathbf{u}_+)(j)([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}} = \dfrac{\mathrm{Q}(\text{ㅊ}_{\uparrow}(j))}{\mathrm{Q}(\text{ㅊ}_{\rightarrow}(j))}.$

We will also show by induction that for each $j \geqslant i$, the minor $\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)}$ is equal to X_i times the correction factor $\displaystyle\prod_{\substack{k>j \\ \rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right)\in\mathcal{J}\left(u_{(k-1)},s_{h_k}\right)}} X_k$. In particular, for $j = \ell$, the minor $\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)}$ will be equal to X_i.

Since strands and row indices are deleted only at crossings j where the strand $\alpha_{\searrow}(j)$ is in $\widehat{C}^i_{j-1}$ and the set $\widehat{C}^i_{j-1}$is contained in the set Strands$(\text{ㅊ}_{\uparrow}(j))$, we need only consider the crossings where the uppermost strand in the collection $\widehat{C}^i_{j-1}$ travels down. We will therefore consider the left-to-right path π^i consisting of the uppermost strand segment in the collection $\widehat{C}^i_k$ for each $k \geqslant i$; the crossings $j \in J^+_{\mathbf{u}_+}$ where the strand $\alpha_{\nearrow}(j)$ and row index $r = \lambda^j(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$ are deleted will be precisely those crossings where π^i crosses down over a geodesic path γ_r where $r \in \widehat{R}^i_{j-1}$. We emphasize that π^i is not a connected left-to-right path; while it switches from one strand to another only at crossings, consecutive strand segments need not meet at a common vertex.

Definition V.5. Let $i \in J^\circ_{\mathbf{u}_+}$. We define π^i as follows: π^i follows $\alpha_{\nearrow}(i)$ until it begins to travel down (necessarily at some index $j > i$). For crossings $k > i$, π^i travels up whenever possible. If the strand π^i is following crosses down at some crossing k, π^i continues to follow the strand $\alpha_{\searrow}(k)$ if $\lambda^k(\mathbf{u}_+)\left(\alpha_{\nearrow}(k)\right)$ is in $\text{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}(u)$, and otherwise switches strands.

That is, π^i travels greedily up, subject to the constraint that it follows only geodesic paths γ_r where $r \notin \text{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}(u)$. It is immediate that π^i can travel down only at indices $j \in J^+_{\mathbf{u}_+}$. By Proposition III.16, since π^i never switches from an ascending strand to a descending strand at a crossing, if π^i follows a segment of a strand α, then $\alpha \in \text{ㅊ}_{\leftarrow}(i)$.

Definition V.6. We define the sets $\widehat{C}^i_k$ and $\widehat{R}^i_k$ as follows. Let $\widehat{C}^i_i = \text{Strands}(\text{ㅊ}_{\leftarrow}(i))$

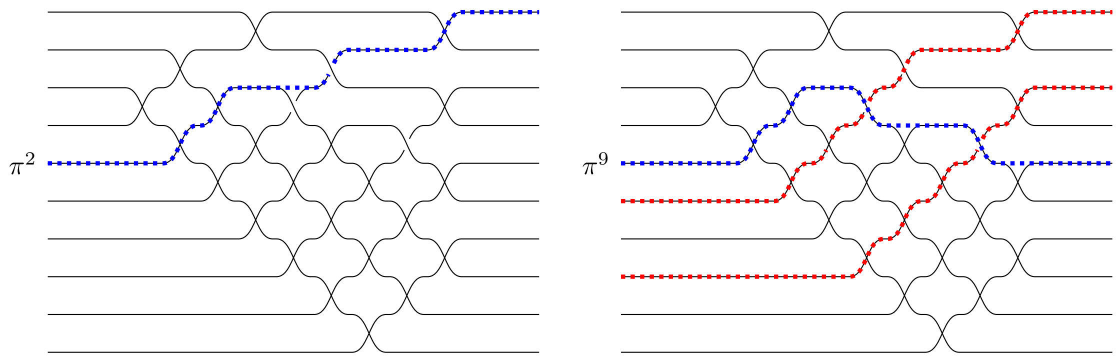

Figure 5.3: The crossings 2 and 9 have the same rising strand $\alpha_{\nearrow}(2) = \alpha_{\nearrow}(9)$, but give rise to different paths π^2 and π^9.

and let $\widehat{R}^i_i = \mathrm{Pivots}_{\widehat{C}^i_i}(u) = u_{(i-1)}([1, h_i])$. For each $k \geqslant i$, there are two cases.

1. If π^i follows the strand $\alpha_{\searrow}(k)$ immediately to the right of k, we take $\widehat{C}^i_k = \widehat{C}^i_{k-1} \setminus \{\alpha_{\nearrow}(k)\}$ and $\widehat{R}^i_k = \widehat{R}^i_{k-1} \setminus \{\lambda^k(\mathbf{u}_+)(\alpha_{\nearrow}(k))\}$.
2. Otherwise, we take $\widehat{C}^i_k = \widehat{C}^i_{k-1}$ and $\widehat{R}^i_k = \widehat{R}^i_{k-1}$.

Example V.7. Let $\mathbf{w} = s_7 s_6 s_5 s_4 s_3 s_2 s_1 s_8 \underline{s}_7 s_6 s_5 s_4 s_3 s_2 s_7 s_6 s_5 s_4 s_3 \underline{s}_6 s_5 s_9 \underline{s}_8 s_7 s_9$. Figure 5.3 shows the paths π^2 and π^9 corresponding to crossings with the same rising strand. Since π^2 never travels down, we have $\widehat{R}^2 = \lambda^1(\mathbf{u}_+)(\text{ㅊ}_{\leftarrow}(2)) = \{1, 2, 3, 4, 5, 6\}$ and $\widehat{C}^2 = \lambda^1(\mathbf{w})(\text{ㅊ}_{\leftarrow}(2)) = \{1, 2, 3, 4, 5, 6\}$, so that $X_2 = \Delta^{123456}_{123456}$. Since π^9 travels down at crossings 15 and 20, $\widehat{R}^9 = \lambda^8(\mathbf{u}_+)(\text{ㅊ}_{\leftarrow}(9)) \setminus \{\lambda^{15}(\mathbf{u}_+)(\alpha_{\nearrow}(15)), \lambda^{20}(\mathbf{u}_+)(\alpha_{\nearrow}(20))\} = \{1, 2, 3, 4, 5, 6, 7\} \setminus \{7, 6\}$ and $\widehat{C}^9 = \lambda^8(\mathbf{w})(\text{ㅊ}_{\leftarrow}(9)) \setminus \{\lambda^{15}(\mathbf{w})(\alpha_{\nearrow}(15)), \lambda^{20}(\mathbf{w})(\alpha_{\nearrow}(20))\} = \{1, 2, 3, 4, 5, 6, 8\} \setminus \{5, 3\}$. This gives $X_9 = \Delta^{12345}_{12468}$.

We note that the operation taking a chamber $\text{ㅊ}_{\leftarrow}(i)$ to a minor $\Delta^{\widehat{R}^i}_{\widehat{C}^i}$ does not preserve containments of row and column indices or even relative sizes of minors: in this case, the strands below $\text{ㅊ}_{\leftarrow}(2)$ are a subset of the strands below $\text{ㅊ}_{\leftarrow}(9)$, but X_9 is given by a 5×5-minor and X_2 is given by a 6×6-minor.

5.3 Deletion of strands and geodesics crossing above the path π^i

Our main goal for this section is to show by induction on $j > i$ that the minor $\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})(\widehat{C}^i_j)}$ is equal to $X_i \prod_{\substack{k>j \\ \rho^{k-1}(\mathbf{w})(\widehat{C}^i_j)\in\mathcal{J}(u_{(k-1)},s_{h_k})}} X_k$. In order to determine for which $k > j$ we have $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right) \in \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$, we will first need to show that the set $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right)$ has pivots $\mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})(\widehat{C}^i_j)}\left(u_{(k-1)}\right) = \widehat{R}^i_j$.

Proposition V.8. *Fix an index $k \geqslant i$, and let h be the height of π^i at index k. Then the following hold.*

1. *Suppose that $\alpha \in$ Strands($ㅊ_{\leftarrow}(i)$). Then $\rho^k(\mathbf{w})(\alpha) \leqslant h$ if and only if $\alpha \in \widehat{C}^i_k$.*
2. *Suppose that $r \in$ Pivots$_{\rho(\mathbf{w})(ㅊ_{\leftarrow}(i))}(u)$. Then $r \in u_{(k)}([1, h])$ if and only if $r \in \widehat{R}^i_k$.*

Proof. Suppose that a strand β crosses above π^i at a crossing $k \in J^+_{\mathbf{u}_+}$. We claim that β stays above π^i for all $k' > k$. To see this, note that π^i "partitions" the diagram in the following way: Write $r^* = \lambda^j(\mathbf{u}_+)(\alpha_{\nearrow}(i))$. By Lemma III.20, if $r \in$ Pivots$_{\rho(\mathbf{w})(ㅊ_{\leftarrow}(i))}(u)$ and $r > r^*$, since $\lambda^j(\mathbf{u}_+)(\alpha_{\searrow}(j)) = r^*$, for all $k \geqslant j$ we have $u^{-1}_{(k)}(r) < \rho^k(\mathbf{w})(\alpha_{\searrow}(j)) < \rho^k(\mathbf{w})(\pi^i)$. So if $k > i$ is the index of a crossing (β,α) along π^i where $u_{(k)}(\beta) \in$ Pivots$_{\rho(\mathbf{w})(ㅊ_{\leftarrow}(i))}(u)$ so that β "escapes" from π^i, then $\lambda^k(\mathbf{u}_+)(\beta)$ $< r^*$.

For any $k > i$, we claim that $\lambda(\mathbf{u}_+)(k)(\pi^i_k) \geqslant r^*$.

We note that if $\rho^j(\mathbf{w})(\alpha) < \rho^j(\mathbf{w})(\alpha_{\nearrow}(i))$, then $\lambda^j(\mathbf{u}_+)(\alpha) < r^*$ implies that $\lambda^j(\mathbf{u}_+)(\alpha) \in$ Pivots$_{\rho(\mathbf{w})(ㅊ_{\leftarrow}(i))}(u)$. Suppose that for some $k \geqslant j$ we have that $\lambda^{k-1}(\mathbf{u}_+)(\pi^i)$ $\geqslant r^*$ and if $\rho^{k-1}(\mathbf{w})(\alpha) < \rho^{k-1}(\mathbf{w})(\pi^i)$, then $\lambda^{k-1}(\mathbf{u}_+)(\alpha) < r^*$ implies that $\lambda^{k-1}(\mathbf{u}_+)(\alpha)$ $\in$ Pivots$_{\rho(\mathbf{w})(ㅊ_{\leftarrow}(i))}(u)$. If the crossing k is not on π^i or if $k \in J^{\circ}_{\mathbf{u}_+}$ and π^i switches strands at k, then $\lambda^{k-1}(\mathbf{u}_+)(\pi^i) = \lambda^k(\mathbf{u}_+)(\pi^i)$ and $u_{(k)}([1, \rho^k(\mathbf{w})(\pi^i)]) = u_{(k)}([1, \rho^{k-1}(\mathbf{w})(\pi^i)])$, so that the statement also holds for k.

If $k \in J^+_{\mathbf{u}_+}$ and π^i switches strands at k, then $\rho^{k-1}(\mathbf{w})\,(\alpha_{\nearrow}(k)) < \rho^{k-1}(\mathbf{w})\,(\pi^i)$ and $\lambda^{k-1}(\mathbf{u}_+)\,(\alpha_{\nearrow}(k)) \notin \mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\leftarrow}(i))}\,(u)$, so that $\lambda^k(\mathbf{u}_+)\,(\pi^i) = \lambda^{k-1}(\mathbf{u}_+)\,(\alpha_{\nearrow}(k)) \geqslant r^*$ and $u_{(k)}([1, \rho^k(\mathbf{w})\,(\pi^i)\,]) = u_{(k-1)}([1, \rho^{k-1}(\mathbf{w})\,(\pi^i)\,])$.

If π^i travels up at k, then $\lambda^{k-1}(\mathbf{u}_+)\,(\alpha_{\searrow}(k)) > \lambda^{k-1}(\mathbf{u}_+)\,(\pi^i) \geqslant r^*$ so that $u_{(k)}([1, \rho^k(\mathbf{w})\,(\pi^i)\,])$ $= u_{(k)}([1, \rho^{k-1}(\mathbf{w})\,(\pi^i)\,]) \cup \{\lambda^{k-1}(\mathbf{u}_+)\,(\alpha_{\searrow}(k))\,\}$ and $\lambda^k(\mathbf{u}_+)\,(\pi^i) \geqslant \lambda^{k-1}(\mathbf{u}_+)\,(\pi^i)$.

If $k \in J^+_{\mathbf{u}_+}$ is a crossing where π^i travels down, then $\lambda^k(\mathbf{u}_+)\,(\pi^i) = \lambda^{k-1}(\mathbf{u}_+)\,(\pi^i)$ and $u_{(k)}([1, \rho^k(\mathbf{w})\,(\pi^i)\,]) = u_{(k-1)}([1, \rho^{k-1}(\mathbf{w})\,(\pi^i)\,])$.

So $\lambda^k(\mathbf{u}_+)\,(\pi^i) \geqslant r^*$ and if $\rho^k(\mathbf{w})\,(\alpha) < \rho^k(\mathbf{w})\,(\pi^i)$, then $\lambda^k(\mathbf{u}_+)\,(\alpha) = r^*$ implies that $\lambda^k(\mathbf{u}_+)\,(\alpha) \in \mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\leftarrow}(i))}\,(u)$.

By Lemma III.20, it follows that if a strand α crosses above the path π^i at a crossing k and $r' = \lambda^k(\mathbf{u}_+)\,(\alpha)$, then $r' < r^*$ and for all $k' > k$, we have $\rho^{k'}(\mathbf{w})\,(\alpha) \geqslant u^{-1}_{(k-1)}(r') > \rho^{k'}(\mathbf{w})\,(\pi^i)$. $\square$

In the next proposition, we remark that for any subset S of the strands below height h at some index j, the pivots of S with respect to $u_{(k)}$ is contained in $u_{(j)}([1, h])$.

Proposition V.9. *Let $\mathbf{u}_+$ be a positive subexpression of a reduced expression $\mathbf{w}$. Let S be a collection of strands with $\rho^j(\mathbf{w})\,(S) \subseteq [1, h]$ for some index j and height h. Let $r \notin u_{(j)}([1, h])$. Then $r \notin$ Pivots$_{\rho(\mathbf{w})(S)}\,(u)$.*

Proof. Let T be the collection of strands with $\rho^j(\mathbf{w})\,(T) = [1, h]$. We have $\mathrm{Pivots}_{\rho(\mathbf{w})(S)}\,(u)$ $\subseteq \mathrm{Pivots}_{\rho(\mathbf{w})(T)}\,(u) = u_{(j)}([1, h]) \not\ni r$. $\square$

Corollary V.10. *We have Pivots$_{\rho(\mathbf{w})\left(\widehat{C}^i_k\right)}\,(u) = \widehat{R}^i_k$.*

Suppose now that for some index $j \geqslant i$, the collection of strands $\widehat{C}^i_{j-1}$ satisfies the condition $\Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)} = X_i \prod_{\substack{k > j-1 \\ \rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right) \in \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)}} X_k$. In order to prove the induction step, we must check that several conditions are satisfied. If $j \in J^{+}_{\mathbf{u}_+}$ is an index where we obtain the collections $\widehat{C}^i_j$ and $\widehat{R}^i_j$ by deleting the strand $\alpha_{\nearrow}(j)$ from $\widehat{C}^i_{j-1}$ and the row index $r = \lambda^j(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$ from $\widehat{R}^i_{j-1}$, we must show the following.

1. The minors $\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)}$ and $\Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)}$ satisfy the equation

$$\frac{\Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)}}{\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)}} = \frac{\Delta^{\lambda(\mathbf{u}_+)(j-1)([1,h_j+1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}}{\Delta^{\lambda(\mathbf{u}_+)(j)([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}}.$$

   (See Proposition V.11.)

2. For any $k > j$, the condition that $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_{j-1}\right) \in \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$ and $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right)$ $\notin \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$ is equivalent to the condition that $\text{ㅊ}_{\uparrow}(j) \in \mathtt{JC}(k)$ and $\text{ㅊ}_{\rightarrow}(j)$ $\notin \mathtt{JC}(k)$.

3. For any $k > j$, the condition that $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_{j-1}\right) \notin \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$ and $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right)$ $\in \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$ is equivalent to the condition that $\text{ㅊ}_{\uparrow}(j) \notin \mathtt{JC}(k)$ and $\text{ㅊ}_{\rightarrow}(j)$ $\in \mathtt{JC}(k)$.

We will prove this in more generality for a collection of strands $S \subset \text{Strands}(\text{ㅊ}_{\rightarrow}(j))$ so that the pivots of the sets $\rho(\mathbf{w})(S)$ and $\rho(\mathbf{w})(S \cup \{\alpha_{\nearrow}(j)\})$ differ by the row index $r = \lambda^j(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$. On the other hand, if $j \in J^{\circ}_{\mathbf{u}_+}$, then no strand or row index deletions are performed, so that $\widehat{C}^i_j = \widehat{C}^i_{j-1}$ and $\widehat{R}^i_j = \widehat{R}^i_{j-1}$; we must therefore show that $\rho^{j-1}(\mathbf{w})\left(\widehat{C}^i_j\right) \notin \mathcal{J}\left(u_{(j-1)}, s_{h_j}\right)$.

Fix $j \in J^{+}_{\mathbf{u}_+}$ and let r be the row index $\lambda^j(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$. Let S be a subset of $\text{Strands}(\text{ㅊ}_{\rightarrow}(j))$, and write $\text{Pivots}_{\rho^j(\mathbf{w})(S)}\left(u_{(j)}\right)$. The condition that $S \subseteq \text{Strands}(\text{ㅊ}_{\rightarrow}(j))$ implies that $r \notin \text{Pivots}_{\rho^j(\mathbf{w})(S)}\left(u_{(j)}\right)$. The following proposition shows that if the set $\rho^j(\mathbf{w})\left(S \cup \alpha_{\nearrow}(j)\right)$ has pivots $R \cup \{r\}$, then the minors $\Delta^{R}_{\lambda(\mathbf{w})(S)}$ and $\Delta^{R \cup \{\lambda^j(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)\}}_{\lambda(\mathbf{w})\left(S \cup \alpha_{\nearrow}(j)\right)}$ are related by the same scalar factor as the minors $\Delta^{u_{(j)}([1,h_j+1])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}$ and $\Delta^{u_{(j)}([1,h_j])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}$.

Proposition V.11. *Fix $j \in J^+_{\mathbf{u}_+}$ and let r be the row index $u_{(j)}(h_j) = \lambda^j(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$. Let S be a subset of Strands$(\text{ㅊ}_{\rightarrow}(j))$, and let R be the set Pivots$_{\rho^j(\mathbf{w})(S)}\left(u_{(j)}\right)$. The condition that $S \subseteq$ Strands$(\text{ㅊ}_{\rightarrow}(j))$ implies that $r \notin$ Pivots$_{\rho^j(\mathbf{w})(S)}\left(u_{(j)}\right)$. Suppose that Pivots$_{\rho^j(\mathbf{w})\left(S \cup \alpha_{\nearrow}(j)\right)}\left(u_{(j)}\right) = R \cup \{r\}$. Then the minors $\Delta^R_{\lambda(\mathbf{w})(S)}$ and $\Delta^{R\cup\{r\}}_{\lambda(\mathbf{w})\left(S\cup\alpha_{\nearrow}(j)\right)}$ satisfy the relation $\dfrac{\Delta^{R\cup\{r\}}_{\lambda(\mathbf{w})\left(S\cup\alpha_{\nearrow}(j)\right)}}{\Delta^R_{\lambda(\mathbf{w})(S)}} = \dfrac{\Delta^{u_{(j)}([1,h_j])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}}{\Delta^{u_{(j)}([1,h_j+1])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}}$ on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$.*

Proof. Let $m = |\text{Strands}(\text{ㅊ}_{\rightarrow}(j))\backslash S|$ and apply induction on m. Let $S_m = S \subset S_{m-1} \subset \cdots \subset S_0 = \text{Strands}(\text{ㅊ}_{\rightarrow}(j))$ be a nested sequence of sets so that for each d the set S_d is obtained from S_{d-1} by adding a single strand. Write R_d for the set Pivots$_{\rho^j(\mathbf{w})(S_d)}\left(u_{(j)}\right)$. Since $S_d \subseteq \text{Strands}(\text{ㅊ}_{\rightarrow}(j))$ the index r is not in R_d; since $S \cup \{\alpha_{\nearrow}(j)\} \subseteq S_d \cup \{\alpha_{\nearrow}(j)\}$, we must have Pivots$_{\rho^j(\mathbf{w})\left(S_d\cup\{\alpha_{\nearrow}(j)\}\right)}\left(u_{(j)}\right) = R_d \cup \{r\}$.

For the base case, we take $d = 0$. Since $S_0 = \text{Strands}(\text{ㅊ}_{\rightarrow}(j))$ and $S_0 \cup \alpha_{\nearrow}(j) = \text{Strands}(\text{ㅊ}_{\uparrow}(j))$, the statement $\Delta^{R_d}_{\lambda(\mathbf{w})(S_d)} = \dfrac{\Delta^{\lambda(\mathbf{u}_+)(\text{ㅊ}_{\rightarrow}(j))}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}}{\Delta^{\lambda(\mathbf{u}_+)\left(\text{ㅊ}_{\uparrow}(j)\right)}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}} \Delta^{R_d\cup\{r\}}_{\lambda(\mathbf{w})\left(S_d\cup\{\alpha_{\nearrow}(j)\}\right)}$ holds trivially for $d = 0$. For the induction step, suppose that for some $d < m$ we have $\Delta^{R_d}_{\lambda(\mathbf{w})(S_d)} = \dfrac{\Delta^{\lambda(\mathbf{u}_+)(\text{ㅊ}_{\rightarrow}(j))}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}}{\Delta^{\lambda(\mathbf{u}_+)\left(\text{ㅊ}_{\uparrow}(j)\right)}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}} \Delta^{R_d\cup r}_{S_d\cup\{\alpha_{\nearrow}(j)\}}$. Let β be the unique strand in $S_d \setminus S_{d+1}$ and let a be the unique row index in $R_d \setminus R_{d+1}$. By Corollary IV.16, we note that a must be larger than r. By Dodgson's identity,

$$\Delta^{R_{d+1}\cup\{r\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\alpha_{\nearrow}(j)\})} \Delta^{R_{d+1}\cup\{a\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\beta\})} - \Delta^{R_{d+1}\cup\{r\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\beta\})} \Delta^{R_{d+1}\cup\{a\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\alpha_{\nearrow}(j)\})} = \Delta^{R_{d+1}}_{\lambda(\mathbf{w})(S_{d+1})} \Delta^{R_{d+1}\cup\{r,a\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\alpha_{\nearrow}(j),\beta\})}.$$

$$\Delta^{R_{d+1}\cup\{r\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\alpha_{\nearrow}(j)\})} \Delta^{R_d}_{\lambda(\mathbf{w})(S_d)} - \Delta^{R_{d+1}\cup\{r\}}_{\lambda(\mathbf{w})(S_d)} \Delta^{R_{d+1}}_{\lambda(\mathbf{w})(S_{d+1})} = \Delta^{R_d}_{\lambda(\mathbf{w})(S_d)} \Delta^{R_{d+1}\cup\{r\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\alpha_{\nearrow}(j)\})}$$

Since $a > r$, we have that $R_{d+1} \cup \{r\} \not\leq$ Pivots$_{\rho^j(\mathbf{w})(S_d)}\left(u_{(j)}\right) = R_{d+1} \cup \{a\}$. Since a minor of the form $\Delta^R_{\lambda(\mathbf{w})(S)}$ vanishes on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ if there is any index j where $R \not\leq$

$\mathrm{Pivots}_{\rho^j(\mathbf{w})(S)}\left(u_{(j)}\right)$, we must have $\Delta^{R_{d+1}\cup\{r\}}_{\lambda(\mathbf{w})(S_d)} = 0$. So

$$\Delta^{R_{d+1}}_{\lambda(\mathbf{w})(S_{d+1})} = \frac{\Delta^{R_d}_{\lambda(\mathbf{w})(S_d)}}{\Delta^{R_d\cup\{r\}}_{\lambda(\mathbf{w})\left(S_d\cup\{\alpha_{\nearrow}(j)\}\right)}}\Delta^{R_{d+1}\cup\{r\}}_{\lambda(\mathbf{w})(S_{d+1}\cup\{\alpha_{\nearrow}(j)\})},$$

which is equal to $\dfrac{\Delta^{u_{(j)}([1,h_j+1])}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(j))}}{\Delta^{u_{(j)}([1,h_j])}_{\lambda(\mathbf{w})\left(\text{え}_{\uparrow}(j)\right)}}\Delta^{R_{d+1}\cup\{r\}}_{S_{d+1}\cup\{\alpha_{\nearrow}(j)\}}$ by the induction hypothesis. $\square$

Combining the previous results,

$$\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)} = \Delta^{\lambda(\mathbf{u}_+)(\text{え}_{\leftarrow}(i))}_{\lambda(\mathbf{w})(\text{え}_{\leftarrow}(i))}\prod_{\substack{j'\leqslant j\\ \widehat{C}^i_{j'}\neq\widehat{C}^i_{j'-1}}}\frac{\Delta_{\text{え}_{\rightarrow}(j')}}{\Delta_{\text{え}_{\uparrow}(j')}}$$

$$= X_i\left(\prod_{\substack{k>i\\ \text{え}_{\leftarrow}(i)\in\mathtt{JC}(k)}}X_k\right)\left(\prod_{\substack{j'\leqslant j\\ \widehat{C}^i_{j'}\neq\widehat{C}^i_{j'-1}}}\frac{\Delta_{\text{え}_{\rightarrow}(j')}}{\Delta_{\text{え}_{\uparrow}(j')}}\right).$$

All j' appearing in the second product satisfy $j' > i$, so the chamber weightings of $\text{え}_{\uparrow}(j')$ and $\text{え}_{\rightarrow}(j')$ are monomials in $\{X_k : k > i\}$. In particular,

$$\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)} = X_i\mathcal{M}$$

where $\mathcal{M}$ is a Laurent monomial in $\{X_k : k > i\}$.

We will show that

$$\mathcal{M} = \prod_{\substack{k>i\\ \rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right)\in\mathcal{J}\left(u_{(k-1)},s_{h_k}\right)}}X_k$$

$$= \prod_{\substack{k>j\\ \rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right)\in\mathcal{J}\left(u_{(k-1)},s_{h_k}\right)}}X_k.$$

Suppose that $j \in J^+_{\mathbf{u}_+}$ is an index with $j > i$ and $S \subseteq \text{え}_{\rightarrow}(j)$ with $\mathrm{Pivots}_S\left(\rho^j(\mathbf{w})\left(u_{(j)}\right)\right) = R$ and $\mathrm{Pivots}_{\rho^j(\mathbf{w})\left(S\cup\{\alpha_{\nearrow}(j)\}\right)}\left(u_{(j)}\right) = R\cup\{r\}$ where $r = \lambda^j(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right)$. The following lemma shows that if $\Delta^{R\cup\{r\}}_{\lambda(\mathbf{w})\left(S\cup\{\alpha_{\nearrow}(j)\}\right)} = X_i\prod_{\substack{k>i\\ \rho^k(\mathbf{w})\left(S\cup\{\alpha_{\nearrow}(j)\}\right)\in\mathcal{J}\left(u_{(k-1)},s_{h_k}\right)}}X_k$, then

$\Delta^{R}_{\lambda(\mathbf{w})(S)}$ satisfies the weaker condition $\Delta^{R}_{\lambda(\mathbf{w})(S)} = X_i\mathcal{M}$, where $\mathcal{M}$ is a squarefree monomial in $\{X_k : k > i\}$.

Lemma V.12. *Let $j \in J^{+}_{\mathbf{u}_+}$ and let $r_- = \lambda^{j-1}(\mathbf{u}_+)\left(\alpha_{\nearrow}(j)\right) < \lambda^{j-1}(\mathbf{u}_+)\left(\alpha_{\searrow}(j)\right) = r^+$. Then the following hold:*

1. *If $\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v^{k}_{(k-1)}\right) \neq \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(u_{(k-1)}\right)$ and $\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^{k}_{(k-1)}\right)$ $= \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right)$, then for any $S \subseteq \text{ㅊ}_{\uparrow}(j)$ with $r_- \in \mathrm{Pivots}_S\left(u_{(k-1)}\right)$, we have $\mathrm{Pivots}_S\left(v^{k}_{(k-1)}\right) \neq \mathrm{Pivots}_S\left(u_{(k-1)}\right)$.*

2. *Suppose that for some $k > j$ we have $\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v^{k}_{(k-1)}\right) = \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(u_{(k-1)}\right)$ and $\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^{k}_{(k-1)}\right) \neq \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right)$. Then $u_{(k-1)}(\rho^{k-1}(\mathbf{w})\left(\alpha_{\searrow}(k)\right))$ $= r_-$, and for any $S \subseteq \text{ㅊ}_{\uparrow}(j)$ with $r_- \in \mathrm{Pivots}_S\left(u_{(k-1)}\right)$ we have $\mathrm{Pivots}_S\left(v^{k}_{(k-1)}\right) =$ $\mathrm{Pivots}_S\left(u_{(k-1)}\right)$.*

Proof. 1. We have

$$\begin{aligned}\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v^{k}_{(k-1)}\right) &\supseteq \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^{k}_{(k-1)}\right)\\ = \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right) &= \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(u_{(k-1)}\right)\setminus\{r_-\}.\end{aligned}$$

Since $\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v^{k}_{(k-1)}\right) \neq \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(u_{(k-1)}\right)$, we must have $r_- \notin$ $\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v^{k}_{(k-1)}\right)$. So if $r_- \in \mathrm{Pivots}_S\left(u_{(k-1)}\right)$ with $S \subseteq \text{ㅊ}_{\uparrow}(j)$, then $\mathrm{Pivots}_S\left(v^{k}_{(k-1)}\right)$ $\subseteq \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v^{k}_{(k-1)}\right) \not\ni r$, and hence $\mathrm{Pivots}_S\left(v^{k}_{(k-1)}\right) \neq \mathrm{Pivots}_S\left(u_{(k-1)}\right)$.

2. We have

$$\begin{aligned}\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^{k}_{(k-1)}\right) &\subseteq \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v^{k}_{(k-1)}\right)\\ = \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(u_{(k-1)}\right) &= \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right) \cup \{r\},\end{aligned}$$

so if $\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^{k}_{(k-1)}\right) \neq \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right)$, then $\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^{k}_{(k-1)}\right)$ $= \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right) \setminus \{a\} \cup \{r_-\}$ for some $a < r_-$.

So either $\rho^k(\mathbf{w})\,(\alpha_\searrow(k))\,u_{(k-1)} = r_-$ or $\rho^k(\mathbf{w})\,(\alpha_\searrow(k))\,u_{(k-1)} > r_- \notin \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_\uparrow(j)\right)}\,(u_{(k-1)})$.

Suppose that $u_{(k-1)}(\rho^k(\mathbf{w})\,(\alpha_\searrow(k))\,) > r_-$. Writing m for the index of $\rho^{k-1}(\mathbf{w})\,(\alpha_\nearrow(k))$ in $\rho^{k-1}(\mathbf{w})\,(\text{ㅊ}_\rightarrow(j))$ and $r_1, \cdots, r_{|\text{ㅊ}_\rightarrow(j)|}$ for $\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_\rightarrow(j))}\left(v^k_{(k-1)}\right)$, we must have $r_m = r$ and

$$\#\{\alpha \in \text{ㅊ}_\rightarrow(j) : \rho^{k-1}(\mathbf{w})\,(\alpha)\ < h_k\} = \#\{r \in \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_\rightarrow(j))}\left(u_{(k-1)}\right)\ : u^{-1}_{(k-1)}\,(r)\ \leqslant h_k\}.$$

But $r_m = r_-$ implies that $\rho^{k-1}(\mathbf{w})\,(\alpha_\nearrow(j))\ \leqslant u^{-1}_{(k-1)}(r_-) < \rho^{k-1}(\mathbf{w})\,(\alpha_\nearrow(k))$, so

$$\#\{\alpha \in \text{ㅊ}_\uparrow(j) : \rho^{k-1}(\mathbf{w})\,(\alpha)\ < h_k\} = \#\{\alpha \in \text{ㅊ}_\rightarrow(j) : \rho^{k-1}(\mathbf{w})\,(\alpha)\ < h_k\} + 1$$

$$= \#\{r \in \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_\rightarrow(j))}\left(u_{(k-1)}\right)\ \cup \{r_-\} : u^{-1}_{(k-1)}\,(r)\ \leqslant h_k\}$$

$$= \#\{r \in \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_\uparrow(j)\right)}\left(u_{(k-1)}\right)\ : u^{-1}_{(k-1)}\,(r)\ \leqslant h_k\}.$$

So $\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_\rightarrow(j))}\left(v^k_{(k-1)}\right) \neq \mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_\uparrow(j)\right)}\left(u_{(k-1)}\right)$, a contradiction.

Suppose that $r_- \in \mathrm{Pivots}_S\,(u_{(k-1)})$. The condition $u_{(k-1)}(\rho^k(\mathbf{w})\,(\alpha_\searrow(k))\,) \in \mathrm{Pivots}_S\,(u_{(k-1)})$ implies that $\mathrm{Pivots}_S\left(v^k_{(k-1)}\right) = \mathrm{Pivots}_S\,(u_{(k-1)})$.

□

It follows immediately from the the second part of this lemma that if for some $j \in J^+_{\mathbf{u}_+}$ and $k \in J^\circ_{\mathbf{u}_+}$ we have $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_{j-1}\right) \notin \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$ but $\rho^{k-1}(\mathbf{w})\left(\widehat{C}^i_j\right) \in \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$, then the crossing k is above π^i_k so that in particular $\widehat{C}^k_i \notin \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$.

Corollary V.13. *Let $j \in J^+_{\mathbf{u}_+}$ be a crossing where π^i travels down, so that $\widehat{C}^i_j = \widehat{C}^i_{j-1} \setminus \{\alpha_\nearrow(j)\}$ and $\widehat{R}^i_j = \widehat{R}^i_{j-1} \setminus \{r\}$ where $r = \lambda^{j-1}(\mathbf{u}_+)\,(\alpha_\nearrow(j))$. If k is an index so that $\mathrm{Pivots}_{\widehat{C}^i_{j-1}}\left(v^k_{(k-1)}\right) = \mathrm{Pivots}_{\widehat{C}^i_{j-1}}\,(u_{(k-1)})$ but $\mathrm{Pivots}_{\widehat{C}^i_j}\left(v^k_{(k-1)}\right) \neq \mathrm{Pivots}_{\widehat{C}^i_j}\,(u_{(k-1)})$, then the strand $\alpha_\nearrow(k)$ is in $\widehat{C}^i_j$, the row index $\lambda^{k-1}(\mathbf{u}_+)\,(\alpha_\nearrow(k))$ is in $\widehat{R}^i_j$, and at index $k-1$ the strand $\alpha_\nearrow(k)$ is strictly above π^i. In particular, the strand $\alpha_\nearrow(k)$ is not in the collection $\widehat{C}^i_{k-1}$ and the row index $\lambda^{k-1}(\mathbf{u}_+)\,(\alpha_\nearrow(k))$ is not in $\widehat{R}^i_{k-1}$.*

Lemma V.14. *Let $j \in J^{+}_{\mathbf{u}_+}$, and let $r_- = \lambda^{\alpha_{\nearrow}(j)}(\mathbf{u}_+)\,(j-1)$. Suppose that for some $k > j$ one of the following conditions holds.*

1. *Neither of the chambers $\text{ㅊ}_{\uparrow}(j)$ and $\text{ㅊ}_{\rightarrow}(j)$ is in $\mathtt{JC}(j)$.*
2. *Both of the chambers $\text{ㅊ}_{\uparrow}(j)$ and $\text{ㅊ}_{\rightarrow}(j)$ are in $\mathtt{JC}(j)$.*

Let S be a subset of Strands$(\text{ㅊ}_{\rightarrow}(j))$ satisfying the pivot stabilization criterion from Corollary IV.28, and write $R = \mathit{Pivots}_S\,(u)$. Suppose that $\mathit{Pivots}_{\rho(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\,(u) = R \cup \{r_-\}$. Then $\rho^{k-1}(\mathbf{w})\,(S)$ is in $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$ if and only if $\rho^{k-1}(\mathbf{w})\,(S \cup \{\alpha_{\nearrow}(j)\})$ is in $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$.

Proof. Suppose that neither of the chambers $\text{ㅊ}_{\uparrow}(j)$ and $\text{ㅊ}_{\rightarrow}(j)$ is in $\mathtt{JC}(j)$, so that r_- is the unique element of $\text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(v_{(k-1)}\right) \setminus \text{Pivots}_{\rho^{k-1}(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^k_{(k-1)}\right)$.

We first consider the case where $\rho^{k-1}(\mathbf{w})\,(S)$ is not in the set $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$, so that $\text{Pivots}_{\rho^{k-1}(\mathbf{w})(S)}\left(v^k_{(k-1)}\right) = R$. By Corollary IV.17, the set $\text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(v^k_{(k-1)}\right)$ is weakly dominated by the set $R \cup \{r_-\} = \text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(u_{(k-1)}\right)$. On the other hand, since $u_{(k-1)} < v^k_{(k-1)}$ in the Bruhat order, we must have $\text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(u_{(k-1)}\right) \leqslant \text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(v^k_{(k-1)}\right)$. So equality holds, and hence $\rho^{k-1}(\mathbf{w})\,(S \cup \{\alpha_{\nearrow}(j)\})$ is not in $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$.

Conversely, if $\rho^{k-1}(\mathbf{w})\,(S)$ is in the set $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$, then $\text{Pivots}_{\rho^{k-1}(\mathbf{w})(S)}\left(v^k_{(k-1)}\right)$ is of the form $R \backslash \{r_a\} \cup \{r^b\}$. Since S is a subset of Strands$(\text{ㅊ}_{\rightarrow}(j))$, the index r^b must be an element of $\text{Pivots}_{\rho^{k-1}(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(v^k_{(k-1)}\right)$, which does not contain the index r_-. So $\text{Pivots}_{\rho^{k-1}(\mathbf{w})(S)}\left(v^k_{(k-1)}\right)$ contains an index r^b which is not in the set $R \cup \{r_-\} = \text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(u_{(k-1)}\right)$, and hence $\text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(v^k_{(k-1)}\right)$ also contains the index r^b. So $\text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(u_{(k-1)}\right) \neq \text{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S \cup \{\alpha_{\nearrow}(j)\}\right)}\left(v^k_{(k-1)}\right)$, and thus $\rho^{k-1}(\mathbf{w})\,(S \cup \alpha_{\nearrow}(j))$ is in $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$.

Suppose now that both of the chambers $\text{ㅊ}_{\uparrow}(j)$ and $\text{ㅊ}_{\rightarrow}(j)$ are in $\mathtt{JC}(j)$. We claim that $u^{-1}_{(k)}(r_-) = \left(v^k_{(k-1)}\right)^{-1}(r_-)$; that is, r_- is not one of the indices $u_{(k-1)}(h_k)$ and

$u_{(k-1)}(h_k+1)$. By Corollary IV.21, the index $u_{(k-1)}(h_k+1)$ is not in $\mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(u_{(k-1)}\right)$ and the index $u_{(k-1)}(h_k)$ is in $\mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right)$. Since r_- is in $\mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\left(u_{(k-1)}\right)$, we must have $r_- \neq u_{(k-1)}(h_k+1)$; since r_- is not in $\mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}\left(u_{(k-1)}\right)$, we must have $r_- \neq u_{(k-1)}(h_k)$.

Since $u_{(k-1)}^{-1}(R) \leqslant \rho^{k-1}(\mathbf{w})(S)$ and $u_{(k-1)}^{-1}(R \cup \{r_-\}) \leqslant \rho^{k-1}(\mathbf{w})(S \cup \{\alpha_{\nearrow}(j)\})$ and $u_{(k-1)}^{-1}(r_-) = \left(v_{(k-1)}^k\right)^{-1}(r_-)$, we have $\left(v_{(k-1)}^k\right)^{-1}(R) \leqslant \rho^{k-1}(\mathbf{w})(S)$ if and only if $\left(v_{(k-1)}^k\right)^{-1}(R \cup \{r_-\}) \leqslant \rho^{k-1}(\mathbf{w})(S \cup \{\alpha_{\nearrow}(j)\})$. By Proposition IV.22, this implies that $\mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})(S)}\left(v_{(k-1)}^k\right) = R$ if and only if $\mathrm{Pivots}_{\rho^{k-1}(\mathbf{w})\left(S\cup\{\alpha_{\nearrow}(j)\}\right)}\left(v_{(k-1)}^k\right) = R \cup \{r_-\}$. Taking the contrapositive of this statement, we have that $\rho^{k-1}(\mathbf{w})(S)$ is in the set $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$ if and only if $\rho^{k-1}(\mathbf{w})(S \cup \alpha_{\nearrow}(j))$ is in the set $\mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)$. □

5.4 The variable X_i is regular on $\mathcal{R}^{u,w}$ and vanishes on the Deodhar divisor $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$

Lemma V.15. *Let $i \in J_{\mathbf{u}_+}^{\circ}$ and let $j \geqslant i$. Then we have $\mathrm{Pivots}_{\rho^{j'-1}(\mathbf{w})\left(\widehat{C}_j^i\right)}\left(v_{(j'-1)}^{j'}\right) = \mathrm{Pivots}_{\rho^{j'-1}(\mathbf{w})\left(\widehat{C}_j^i\right)}\left(u_{(j'-1)}\right)$ for all $j' \leqslant j$ with $j' \neq i$, and*

$$\Delta^{\widehat{R}_j^i}_{\lambda(\mathbf{w})\left(\widehat{C}_j^i\right)} = X_i \prod_{\substack{k>i \\ \rho^{k-1}(\mathbf{w})\left(\widehat{C}_j^i\right)\in\mathcal{J}\left(u_{(k-1)},s_{h_k}\right)}} X_k.$$

Proof. Apply induction on j. For $j = i$, the set of strands $\widehat{C}_j^i$ is given by $\mathrm{Strands}(\text{ㅊ}_{\leftarrow}(i))$ and the row indices are $\widehat{R}_j^i = \mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\leftarrow}(i))}\left(u_{(i-1)}\right)$, so that statement follows trivially from our choice of chamber weighting Q(ㅊ).

Suppose that for some $j > i$, we have $\mathrm{Pivots}_{\widehat{C}_{j-1}^i}\left(v_{(j'-1)}^{j'}\right) = \mathrm{Pivots}_{\widehat{C}_{j-1}^i}\left(u_{(j'-1)}\right)$ for all $j' \in J_{\mathbf{u}_+}^{\circ}$ with $i < j' < j-1$ and $\Delta^{\widehat{R}_{j-1}^i}_{\lambda(\mathbf{w})\left(\widehat{C}_{j-1}^i\right)} = X_i \prod_{\substack{k>i \\ \rho^{k-1}(\mathbf{w})\left(\widehat{C}_{j-1}^i\right)\in\mathcal{J}\left(u_{(k-1)},s_{h_k}\right)}} X_k.$

Suppose that $\widehat{C}_j^i = \widehat{C}_{j-1}^i$. If $j \in J_{\mathbf{u}_+}^{+}$, the statement follows trivially.

If $j \in J_{\mathbf{u}_+}^{\circ}$, it suffices to show that $\mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})\left(\widehat{C}_{j-1}^i\right)}\left(v_{(j-1)}^j\right) = \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})\left(\widehat{C}_{j-1}^i\right)}\left(u_{(j-1)}\right)$.

Suppose otherwise. Then $\lambda^{j-1}(\mathbf{u}_+)(\alpha_{\nearrow}(j)) \in \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})\left(\widehat{C}_{j-1}^i\right)}\left(u_{(j-1)}\right)$,

$\lambda^{j-1}(\mathbf{u}_+)\,(\alpha_{\searrow}(j)) \notin \mathrm{Pivots}_{\rho^{j-1}(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)}\,(u_{(j-1)})$, and

$$\#\{\alpha \in \widehat{C}^i_{j-1} : \rho^{j-1}(\mathbf{w})\,(\alpha) \leqslant h_j + 1\} = \#\{r \in \widehat{R}^i_{j-1} : u^{-1}_{(j-1)}\,(r) \leqslant h_j + 1\}.$$

Suppose that $\widehat{C}^i_j = \widehat{C}^i_{j-1} \setminus \alpha_{\nearrow}(j)$. Then $\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)} = \Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)} \dfrac{\Delta^{\mathrm{Pivots}\,\text{え}_{\to(j)}(\dot{u})}_{\text{え}_{\to(j)}}}{\Delta^{\mathrm{Pivots}\,\text{え}_{\uparrow(j)}(\dot{u})}_{\text{え}_{\uparrow(j)}}}$ so that X_k divides $\Delta^{\widehat{R}^i_j}_{\lambda(\mathbf{w})\left(\widehat{C}^i_j\right)}$ if and only if either X_k divides $\Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)}$ and divides both or neither of $\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\to(j)})}(u)}_{\text{え}_{\to(j)}}$ and $\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{え}_{\uparrow(j)}\right)}(u)}_{\text{え}_{\uparrow(j)}}$ or X_k divides $\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\to(j)})}(u)}_{\text{え}_{\to(j)}}$ and does not divide $\Delta^{\widehat{R}^i_{j-1}}_{\lambda(\mathbf{w})\left(\widehat{C}^i_{j-1}\right)}$ or $\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{え}_{\uparrow(j)}\right)}(u)}_{\text{え}_{\uparrow(j)}}$. For $k \neq i$, these are precisely the conditions for when $\mathrm{Pivots}_{\widehat{C}^i_j}\left(v^k_{(k-1)}\right) \neq \mathrm{Pivots}_{\widehat{C}^i_j}\,(u_{(k-1)})$ ($= \widehat{R}^i_j$). $\square$

Lemma V.16. $\Delta^{\widehat{R}^i_\ell}_{\lambda(\mathbf{w})\left(\widehat{C}^i_\ell\right)} = X_i$.

Proof. $\Delta^{\widehat{R}^i_\ell}_{\lambda(\mathbf{w})\left(\widehat{C}^i_\ell\right)} = X_i \prod_{\substack{k>\ell \\ \rho(\mathbf{w})\left(\widehat{C}^i_\ell\right) \in \mathcal{J}\left(u_{(k-1)}, s_{h_k}\right)}} X_k$, where the product is empty since there are no crossings k with $k > \ell$. $\square$

Lemma V.17. *Let $i \in J^\circ_{\mathbf{u}_+}$ be an index with $v^0_{(i)} = 1$. Then $\Delta^{\widehat{R}^i_\ell}_{\lambda(\mathbf{w})\left(\widehat{C}^i_\ell\right)}$ vanishes identically on the Deodhar boundary divisor $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$.*

Proof. The relation

$$\Delta^{\widehat{R}^i_\ell}_{\widehat{C}^i_\ell} = \Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\leftarrow(i)})}(u)}_{\text{え}_{\leftarrow(i)}} \prod_{\substack{k>i \\ \pi^i_k \neq \pi^i_{k-1}}} \frac{\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\to(k)})}(u)}_{\lambda(\mathbf{w})(\text{え}_{\to(k)})}}{\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{え}_{\uparrow(k)}\right)}(u)}_{\lambda(\mathbf{w})\left(\text{え}_{\uparrow(k)}\right)}}$$

holds on the dense open subset $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ and hence holds wherever the minors $\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{え}_{\uparrow(k)}\right)}(u)}_{\text{え}_{\uparrow(k)}}$ are all nonzero. But these are chamber minors to the right of j in the wiring diagram for $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$ and hence are nonzero on $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$. On $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$, $\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\leftarrow(i)})}(u)}_{\text{え}_{\leftarrow(i)}}$ $= 0$ and hence

$$\Delta^{\widehat{R}^i_\ell}_{\widehat{C}^i_\ell}\,(\Upsilon^w_u) = \Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\leftarrow(i)})}(u)}_{\text{え}_{\leftarrow(i)}} \prod_{\substack{k>i \\ \pi^i_k \neq \pi^i_{k-1}}} \frac{\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{え}_{\to(k)})}(u)}_{\text{え}_{\to(k)}}}{\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{え}_{\uparrow(k)}\right)}(u)}_{\text{え}_{\uparrow(k)}}}$$

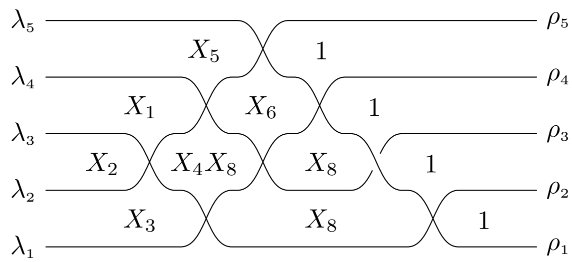

Figure 5.4: Chamber weights for the non-unipeak expression $\mathbf{w} = s_2s_1s_3s_2s_4s_3\underline{s}_2s_1$

$$= 0 \cdot \prod_{\substack{k>i \\ \pi^i_k \neq \pi^i_{k-1}}} \frac{\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})(\text{ㅊ}_{\to(k)})}(u)}_{\text{ㅊ}_{\to(k)}}}{\Delta^{\mathrm{Pivots}_{\rho(\mathbf{w})\left(\text{ㅊ}_{\uparrow(k)}\right)}(u)}_{\text{ㅊ}_{\uparrow(k)}}}$$

$= 0.$ $\square$

Corollary V.18. *Let $i \in J^\circ_{\mathbf{u}_+}$ with $v^i_{(0)} = 1$ and suppose that $\mathit{Pivots}_{\lambda(\mathbf{w})(\text{ㅊ})}\left(v^i_{(i-1)}\right) \neq \mathit{Pivots}_{\lambda(\mathbf{w})(\text{ㅊ})}\left(u_{(i-1)}\right)$. Then X_i does not divide $\Delta^{\mathit{Pivots}_{\lambda(\mathbf{w})(\text{ㅊ})}\left(v^i_{(i-1)}\right)}_{\lambda(\mathbf{w})(\text{ㅊ})}$.*

Proof. By Deodhar's theorem, the minor $\Delta^{\mathrm{Pivots}_{\lambda(\mathbf{w})(\text{ㅊ})}\left(v^i_{(i-1)}\right)}_{\lambda(\mathbf{w})(\text{ㅊ})}$ is nonzero everywhere on $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$. We note that if X_i divided $\Delta^{\mathrm{Pivots}_{\lambda(\mathbf{w})(\text{ㅊ})}\left(v^i_{(i-1)}\right)}_{\lambda(\mathbf{w})(\text{ㅊ})}$ in our parametrization of $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, then $\Delta^{\widehat{R}^i}_{\widehat{C}^i}$ would divide $\Delta^{\mathrm{Pivots}_{\lambda(\mathbf{w})(\text{ㅊ})}\left(v^i_{(i-1)}\right)}_{\lambda(\mathbf{w})(\text{ㅊ})}$ on $\mathcal{R}^{u,w}$ and hence we would have $\Delta^{\mathrm{Pivots}_{\lambda(\mathbf{w})(\text{ㅊ})}\left(v^i_{(i-1)}\right)}_{\lambda(\mathbf{w})(\text{ㅊ})} = 0$ on $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$. $\square$

Remark V.19. The formula we describe for obtaining X_j by removing strands and pivots along the path π_j is specific to unipeak wiring diagrams. In Figure 5.4, we show the chamber weighting $\mathbf{w} = s_2s_1s_3s_2s_4s_3\underline{s}_2s_1$, which has a "univalley diagram" corresponding to the vertical reflection of our running example. The upper triangular matrix $\Upsilon^{u,w}\dot{w}^{-1}$ corresponding to the weighting Q(ㅊ) on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ is given by

$$\Upsilon^{u,w}\dot{w}^{-1} = \begin{pmatrix} X_2 & 0 & X_8 & -\frac{X_2X_3+X_1X_6}{X_3X_4} & 1 \\ 0 & \frac{X_1}{X_2} & \frac{X_4X_8}{X_2} & -\frac{X_1X_6}{X_2X_3} & \frac{X_4}{X_2} \\ 0 & 0 & \frac{X_3}{X_1} & 0 & \frac{X_1+X_3}{X_1X_8} \\ 0 & 0 & 0 & \frac{X_5}{X_3} & \frac{X_3+X_4X_5}{X_3X_6} \\ 0 & 0 & 0 & 0 & \frac{1}{X_5} \end{pmatrix}.$$

Although the variable X_4 cannot be expressed as a minor obtained by removing row and column indices from $\mathrm{Q}(ㅊ_{\leftarrow}(4)) = \Delta_{13}^{12} = X_4X_8$, it is given by the minor Δ_{15}^{12} obtained by replacing the column index $\lambda(\mathbf{w})\,(\alpha_{\nearrow}(8)) = 3$ with the column index $\lambda(\mathbf{w})\,(\alpha_{\searrow}(8)) = 5$ while keeping the same row set. We have verified that for unipeak diagrams, the variable X_j can be expressed as a minor with rows $\lambda(\mathbf{u}_+)\,(ㅊ_{\leftarrow}(j))$ and columns obtained by iteratively "uncrossing" strands of the form $\alpha_{\nearrow}(k)$ and $\alpha_{\searrow}(k)$ to rescale the minor by the ratio $\frac{\mathrm{Q}(ㅊ_{\rightarrow(k)})}{\mathrm{Q}(ㅊ_{\leftarrow(k)})}$, and we believe this construction applies to any reduced wiring diagram with positive subexpression. In general, this procedure requires uncrossing strands along multiple paths in the diagram, and removed strands may be reintroduced later.

CHAPTER VI

Flows in Oriented Bridge Diagrams

In order to describe the structure of the coordinate ring $\mathbb{C}[\mathcal{R}^{u,w}]$, we will need to compute some additional minors of the matrices $g_{(i)}$ given by Marsh and Rietsch's parametrization of the Deodhar strata $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$. In the first part of this chapter, we recall a lemma originally due to Lindström for computing minors of the weight matrix of a weighted, directed graph in terms of *flows*. We will need the version proved by Fomin and Zelevinsky in [9] for non-planar graphs.

We will use Karpman's *bridge diagram* construction to convert the wiring diagram for the pair $(u_{(k)}, w_{(k)})$ into a weighted, directed network $\mathcal{G}^k$ with weight matrix $g_{(k)}$. After reversing the edges of $\mathcal{G}_k$ along a flow, we can expand the relevant minors of $g_{(k)}$ as weights of simple *augmenting paths* between boundary vertices λ_a and λ_b.

6.1 Weighted networks, flows and augmenting paths

Weighted networks are used extensively in the study of total positivity. In the case of a planar, acyclic weighted network, every minor of the weight matrix M can be expressed as a subtraction-free polynomial in the edge weights rather than an alternating sum in the entries M_{ij}. Fomin and Zelevinsky gave several classes of weighted networks parametrizing the totally nonnegative matrices in [9].

We adopt their hypothesis that a weighted network has no directed cycles. In the more general case, Talaska [33] and Postnikov [28] gave formulas for expressing minors of a weight matrix in terms of rational functions in the edge weights.

Definition VI.1 (Weighted networks and weight matrices)**.** A *weighted network* $G = (V, E)$ is a directed acyclic graph with source vertices $\lambda_1, \cdots, \lambda_n$ and target vertices $\rho_1, \cdots, \rho_n$, equipped with a weighting function ω on the edges E. The weight of a path π is the product of edge weights $\omega(\pi) = \prod_{e \in \pi} \omega(e)$. The *weight matrix* M is the matrix with entries given by summing the weights of all paths from source λ_i to target ρ_j, so that $M_{ij} = \sum_{\lambda_i \xrightarrow{\pi} \rho_j} \omega(\pi)$.

When the entries M_{ij} are sums of path weights for paths $\lambda_i \to \rho_j$, the minors $\Delta^I_J(M)$ can be expressed in terms of weights of *flows* from sources indexed by I to targets indexed by J.

Definition VI.2 (Flows)**.** Let I and J be collections of vertices in a directed acyclic graph with $|I| = |J|$, and fix orderings of I and J. A *flow* $\mathcal{F} : I \to J$ is a collection of pairwise vertex-disjoint paths $\pi_1, \cdots, \pi_{|I|}$ so that every vertex in I is the start of some π_i and and every vertex in J is the end of some π_i. We write $e \in \mathcal{F}$ if e is an edge in one of the paths π_i. The *weight* of a flow is $\omega(\mathcal{F}) = \prod_{e \in \mathcal{F}} \omega(e)$, and the *sign* sgn$(\mathcal{F})$ of a flow is the sign of the permutation determined by the reordering of the endpoints.

We refer to Fomin and Zelevinsky's proof of Lindström's lemma.

Formula VI.3 (Lindström's lemma[9])*.* Let $\mathcal{G}$ be a weighted network with weight matrix M, and let R and C be subsets of $[1, n]$ with $|R| = |C|$. The minor $\Delta^R_C(M)$ is given by

$$\Delta^R_C(M) = \sum_{\mathcal{F}: R \to C} \mathrm{sgn}(\mathcal{F}) \omega(\mathcal{F}).$$

We are interested in computing ratios of minors of the form $\Delta^R_{[1,h_k]}(g_{(k)})$, which correspond to ratios of minors on the column set indexed by the chamber え$_{\rightarrow}(k)$. Although every $\Delta^R_{[1,h_k]}(g_{(k)})$ can be computed using flows in the oriented bridge diagram $\mathcal{G}_k$ which we will discuss in Section 6.2, it suffices for our purposes to compute minors where the row set R has the form $S\backslash\{s\} \cup \{r\}$, where S is one of the sets $u_{(k)}([1, h_k])$ or $w_{(k)}([1, h_k])$.

To simplify our computations, we will use the standard technique of reversing edges along a $\mathcal{F} : S \rightarrow T$ in a graph G and identifying *augmenting paths* $\pi : r \rightarrow s$ in the resulting graph $\widehat{G}$ with flows $\mathcal{F}' : S\backslash\{s\} \cup \{r\} \rightarrow T$ in the original graph G.

Definition VI.4 (Augmenting paths)**.** Let G be a directed acyclic graph and let $\mathcal{F} : S \rightarrow T$ be a flow. Let $\widehat{G}$ be the graph obtained from G by reversing edges along $\mathcal{F}$. For vertices $s \in S$ and $r \notin S \cup T$, an *augmenting path* from r to s is a directed path $\pi : r \rightarrow s$ in $\widehat{G}$. Let $\mathcal{F}^\pi := \{e : e \in \mathcal{F}, e \notin \pi\} \cup \{e : e \in \pi, e \notin \mathcal{F}\}$. Then $\mathcal{F}^\pi$ is a flow from $S\backslash\{s\} \cup \{r\}$ to T in G, denoted the *augmentation of $\mathcal{F}$ along π*.

Augmenting paths are used in Ford and Fulkerson's algorithm for finding a maximal flow in a directed network with edge capacities[12] and Hopcroft and Karp's maximal bipartite matching algorithm[19]. In the context of total positivity for Grassmannians, Postnikov showed that for a perfectly oriented planar network, the boundary measurement map is preserved when we reverse all edges along a flow and invert their weights.

More generally, if both the directed graph G and the graph $\widehat{G}$ obtained by reversing edges along a flow $\mathcal{F} : S \rightarrow T$ are acyclic, then there is a bijection between augmenting paths $\pi : r \rightarrow s$ in $\widehat{G}$ and flows $\mathcal{F} : S\backslash\{s\} \cup \{r\} \rightarrow T$ in G. A sufficient condition is that G is acyclic and there is a unique flow $\mathcal{F} : S \rightarrow T$ in G. For completeness, we include the following proof, outlined by Speyer in an email.

Proposition VI.5. *[32] Let G be a directed acyclic graph, and let S and T be collections of vertices such that there is a unique flow $\mathcal{F}$ from S to T. Let $\widehat{G}$ be the graph obtained from G by reversing all edges in $\mathcal{F}$. Then $\widehat{G}$ is acyclic, and for each $s \in S$ and $r \notin S \cup T$, there is a bijection between paths $\{\pi : r \to s\}$ in $\widehat{G}$ and flows $\{\mathcal{F}' : S\backslash\{s\} \cup \{r\}\}$ given by $\pi \leftrightarrow \mathcal{F}^{\pi}$.*

Proof. Without loss of generality, we may assume that the sets S and T are disjoint; if v is a vertex in $S \cap T$, the path from v to v in $\mathcal{F}$ has no edges by acyclicity of G.

Suppose that C is a directed cycle in $\widehat{G}$. Write C_1 for the edges of C which have the same orientation in G and $\widehat{G}$ and write C_2 for the edges of C which have opposite orientations. Since G is acyclic, both C_1 and C_2 must be nonempty: otherwise, either $C = C_1$ is a cycle in G, or $C = C_2$ so that the reverse of C is a cycle in G. Hence, there is a flow $\mathcal{F}'$ from S to T obtained from $\mathcal{F}$ by using the edges C_1 instead of the edges C_2, contradicting uniqueness of $\mathcal{F}$. So $\widehat{G}$ is acyclic.

Fix $s \in S$ and $r \notin S \cup T$. If π is a path from r to s in $\widehat{G}$, then $\mathcal{F}^{\pi}$ is a flow from $S\backslash\{s\} \cup \{r\}$ in G. Conversely, if $\mathcal{F}' : S\backslash\{s\} \cup \{r\} \to T$ is a flow in G, consider the set of edges E in the symmetric difference between $\mathcal{F}$ and $\mathcal{F}'$. We claim that in $\widehat{G}$, the edges of E form an oriented path from r to s. It is immediate from the definition that a flow in G uses exactly one inbound edge and one outbound edge for each non-source, non-target vertex it contains. For an acyclic graph, a flow with disjoint source and target sets uses one outbound edge for each source and one inbound edge for each target. It follows that in $\widehat{G}$, for each vertex v contained in both $\mathcal{F}$ and $\mathcal{F}'$, E contains an edge of $\mathcal{F}'$ directed toward v if and only if it contains an edge of $\mathcal{F}$ directed away from v, and vice versa. Begin at the vertex r and follow a path π that alternates between edges of $\mathcal{F}'\backslash\mathcal{F}$ and edges of $\mathcal{F}\backslash\mathcal{F}'$ whenever possible. By the previous statement, π must eventually either revisit a vertex or arrive at the vertex s. Since s is a source of $\mathcal{F}$ in G and $s \notin \mathcal{F}'$, the vertex s has no outbound edges in E. Because $\widehat{G}$ is acyclic, π must reach s and terminate.

We claim that π uses all the edges of E. Otherwise, choose an edge e in E which was not used by π and form a maximal alternating path π' using edges of E. We claim that π' uses no edges of π. Otherwise, consider the first edge $e = (v_1, v_2)$ where π and π' agree. The edge e cannot be the first edge of π' by hypothesis, and it cannot be the first edge of π since there are no inbound edges to the vertex r in $\widehat{G}$. So π and π' arrive at v_1 from different inbound edges, and hence one uses an inbound edge from $\mathcal{F}$ and the other uses an inbound edge from $\mathcal{F}'$. But both π and π' are alternating, so they cannot both leave v_1 using e. Hence π' does not share any edges with π, and so it cannot reach s. It follows that π' eventually self-intersects, contradicting acyclicity of $\widehat{G}$. $\square$

Under the hypotheses of Proposition VI.5, we will define the weight of an augmenting path π to be the ratio of the *signed weights* of $\mathcal{F}^\pi$ and $\mathcal{F}$.

Definition VI.6 (Weight of an augmenting path)**.** Let G be a directed acyclic graph, and let S and T be collections of vertices such that there is a unique flow $\mathcal{F}$ from S to T. Let $\widehat{G}$ be the graph obtained from G by reversing all edges in $\mathcal{F}$. Let $\pi : s \to t$ be an augmenting path. We define the weight of π to be the ratio $\frac{\operatorname{sgn}(\mathcal{F}^\pi)\omega(\mathcal{F}^\pi)}{\operatorname{sgn}(\mathcal{F})\omega(\mathcal{F})}$.

6.2 Converting a wiring diagram to a weighted network

Fix a reduced expression w with positive subexpression $\mathbf{u}_+$. Karpman gave the following construction for converting the wiring diagram for $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ to a weighted, directed acyclic network, or *oriented bridge diagram*.[21]

1. For each $i \in J^\circ_{\mathbf{u}_+}$, replace the crossing i with a vertical *bridge* of weight t_i directed down from level $h_i + 1$ to h_i, as in Figure 6.1.

Figure 6.1: Blocks in the wiring diagram and oriented bridge diagram corresponding to $i \in J^{\circ}_{\mathbf{u}_+}$.

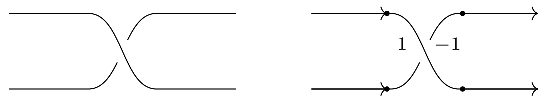

Figure 6.2: Blocks in the wiring diagram and oriented bridge diagram corresponding to $i \in J^{+}_{\mathbf{u}_+}$.

2. For each $i \in J^{+}_{\mathbf{u}_+}$, add vertices on either side of the crossing i, giving a non-planar block with horizontal and inclined edges oriented left-to-right between heights h_i and h_{i+1}, as in Figure 6.2. Assign weight -1 to the upward-inclined edge and weight 1 to the downward-inclined edge.

3. Strand segments between crossings are horizontal edges of weight 1, directed from left to right.

We note that a block containing a bridge has weight matrix $g_i = y_{h_i}(t_i)$ and a block containing inclined edges has weight matrix $\dot{s}_{h_i}$. Following Fomin and Zelevinsky's discussion of weighted networks[9], concatenating these blocks from left to right corresponds to multiplying their weight matrices from left to right.

For each index k with $0 \leqslant k \leqslant \ell$, let $\mathcal{G}_k$ denote the oriented bridge diagram for the first k indices of $\mathbf{w}$ and $\mathbf{u}_+$. It follows that $\mathcal{G}_k$ has the weight matrix $g_{(k)} = g_1 g_2 \cdots g_k$ from Marsh and Rietsch's parametrization.

Since Karpman's bridge diagrams are non-planar in general, the path permutation on the endpoints of a flow need not be the identity, so that some flows $\mathcal{F}$ may have $\operatorname{sgn}(\mathcal{F}) = -1$. However, when the targets of $\mathcal{F}$ are indexed by a left-justified interval $[1, h]$, the inversions of the path permutation of $\mathcal{F}$ are in bijection with upward-inclined edges used by $\mathcal{F}$. Since upward-inclined edges have weight -1, the term $\operatorname{sgn}(\mathcal{F})\omega(\mathcal{F})$ has coefficient 1 as a monomial in the t_i. This is consistent with the result of Marsh and Rietsch that every left-justified minor of $g_{(k)}$ is a subtraction-free polynomial in the variables $\{t_i : i \leqslant k\}$.[27]

By Lemma 7.4 in [27], for each $\mathcal{G}_k$, the unique flow $\mathcal{F}_{u_{(k)}}$ from the source vertices $\lambda_{u_{(k)}(1)}, \cdots, \lambda_{u_{(k)}(h_k)}$ to targets $\rho_1, \cdots, \rho_{h_k}$ has weight $\Delta^{u_{(k)}([1,h_k])}_{[1,h_k]}\big(g_{(k)}\big) = 1$, and the unique flow $\mathcal{F}_{w_{(k)}}$ from source vertices $\lambda_{w_{(k)}(1)}, \cdots, \lambda_{w_{(k)}(h_k)}$ to targets $\rho_1, \cdots, \rho_{h_k}$ has weight $\Delta^{w_{(k)}([1,h_k])}_{[1,h_k]}\big(g_{(k)}\big)$, where these minors are related to minors of the upper triangular matrix z by the equation

$$\frac{1}{\Delta^{w_{(i)}([1,h_k])}_{[1,h_k]}\big(g_{(k)}\big)} = \frac{\Delta^{u_{(k)}([1,h_k])}_{[1,h_k]}\big(g_{(k)}\big)}{\Delta^{w_{(k)}([1,h_k])}_{[1,h_k]}\big(g_{(k)}\big)} = \frac{\Delta^{\lambda(\mathbf{u}_+)(\text{え}_{\rightarrow}(k))}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(k))}}{\Delta^{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(k))}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(k))}}.$$

Definition VI.7 (The graphs $\widehat{\mathcal{G}}^{\mathbf{u}_+}_k$ and $\widehat{\mathcal{G}}^{\mathbf{w}}_k$)**.** Fix k with $0 \leqslant k \leqslant \ell$. We denote the graph obtained from $\mathcal{G}_k$ by reversing the edges of $\mathcal{F}_{u_{(k)}}$ by $\widehat{\mathcal{G}}^{\mathbf{u}_+}_k$ and we denote the graph obtained from $\mathcal{G}_k$ by reversing the edges of $\mathcal{F}_{w_{(k)}}$ by $\widehat{\mathcal{G}}^{\mathbf{w}}_k$.

We note that an augmenting path in one of the graphs $\widehat{\mathcal{G}}^{\mathbf{u}_+}_k$ or $\widehat{\mathcal{G}}^{\mathbf{w}}_k$ is a nonempty directed path $\pi : \lambda_a \to \lambda_b$ beginning and ending on the left edge of the rectangle bounding the planar projection of $\mathcal{G}_k$.

Proposition VI.8. *The weight of an augmenting path π in $\widehat{\mathcal{G}}^{\mathbf{w}}_k$ is given by the ratio* $\dfrac{\prod_{\substack{i \in J^\circ_{\mathbf{u}_+} \cap \pi \\ i \text{ is oriented down}}} t_i}{\prod_{\substack{i \in J^\circ_{\mathbf{u}_+} \cap \pi \\ i \text{ is oriented up}}} t_i}$. *The weight of an augmenting path π in $\widehat{\mathcal{G}}^{\mathbf{u}_+}_k$ is given by* $\omega(\pi) = \prod_{i \in \pi \cap J^\circ_{\mathbf{u}_+}} t_i$.

Proof. For each flow $\mathcal{F}$ in $\{\mathcal{F}_{w_{(k)}}, \mathcal{F}_{u_{(k)}}\}$, we have that $\operatorname{sgn}(\mathcal{F}_\pi)\omega(\mathcal{F}_\pi) = \prod_{i \in \mathcal{F}_\pi \cap J^\circ_{\mathbf{u}_+}} t_i$ and $\operatorname{sgn}(\mathcal{F})\omega(\mathcal{F}) = \prod_{i \in \mathcal{F} \cap J^\circ_{\mathbf{u}_+}} t_i$. The augmenting path π travels down bridge i if and only if $\mathcal{F}_\pi$ uses i and $\mathcal{F}$ does not use i, while π travels up bridge i if and only if $\mathcal{F}_\pi$ does not use i and $\mathcal{F}$ uses i. In $\widehat{\mathcal{G}}^{\mathbf{w}}_k$, the bridge i is oriented up if $i \in \mathcal{F}_{w_{(k)}}$ and oriented down otherwise. In $\widehat{\mathcal{G}}^{\mathbf{u}_+}_k$, all bridges are oriented down since $\mathcal{F}_{u_{(k)}}$ uses no bridges. $\square$

6.3 Graph theoretic properties of $\overline{\texttt{JC}(j)}$ in an oriented bridge diagram

In this section, we will use our results from Section 4.4 to give a graph theoretic description of the region $\overline{\texttt{JC}(j)}$ in the oriented bridge diagram $\mathcal{G}_k$ for $k \geqslant j$.

Proposition VI.9. *For $k \geqslant j$, $\partial\texttt{JC}(j)$ is a simple closed cycle in the underlying undirected graph of $\mathcal{G}_k$.*

Proof. By Proposition IV.53, the planar projection of $\overline{\texttt{JC}(j)}$ is a simply connected region, and the boundary $\partial\texttt{JC}(j)$ is a cycle which switches strands only at crossings $i \in J^\circ_{\mathbf{u}_+}$. Replacing a crossing $i \in J^\circ_{\mathbf{u}_+}$ with a bridge takes a connected cycle to a connected cycle. We note that the crossing j is the rightmost crossing on $\partial\texttt{JC}(j)$. $\square$

Definition VI.10 (Interior vertices of $\texttt{JC}(j)$)**.** Let v be a vertex in $\mathcal{G}_k$. We say that v is an interior vertex of $\texttt{JC}(j)$ if either v is a vertex on some bridge i where all three chambers incident to i are in $\texttt{JC}(j)$ or v is an endpoint of an inclined edge where both chambers incident to v are in $\texttt{JC}(j)$.

Definition VI.11. We say that a vertex v is in $\overline{\texttt{JC}(j)}$ if either v is on $\partial\texttt{JC}(j)$ or v is an interior vertex of $\texttt{JC}(j)$. We say that an edge e is an *interior edge* of $\texttt{JC}(j)$ if both endpoints are in $\overline{\texttt{JC}(j)}$ and at least one endpoint is an interior vertex.

We say that an inclined edge e *enters* $\texttt{JC}(j)$ or *escapes from* $\texttt{JC}(j)$ if exactly one of its endpoints is an interior vertex of $\texttt{JC}(j)$.

Definition VI.12 (X_j-degrees)**.** Let G be a weighted directed graph and let $e \in G$ be a directed edge with weight $\omega(e)$. Write $\omega(e) = X_j^d M(X)$, where $M(X)$ is a Laurent monomial in the variables $\{X_i : i \neq j\}$. We say that the edge e has X_j-degree d and write $\deg_{X_j}(e) = d$. Similarly, we define the X_j-degree of a path or flow as the

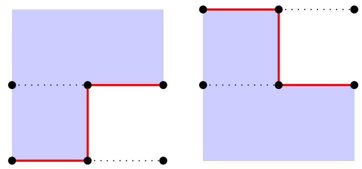

Figure 6.3: Forbidden boundary segments for reduced wiring diagrams.

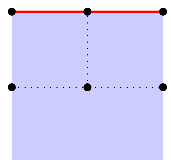

Figure 6.4: Forbidden boundary segments for unipeak wiring diagrams.

power of X_j dividing its weight, or equivalently as the sum of the X_j-degrees of its edges.

It is immediate from the definition that if the edge e is not a bridge or if e is vertex-disjoint from $\partial\mathtt{JC}(j)$, then e has degree 0 in any orientation.

Definition VI.13. An edge e is *$\partial\mathtt{JC}(j)$-incident* if at least one of its vertices is on the cycle $\partial\mathtt{JC}(j)$.

Definition VI.14. Fix $j \in J^\circ_{\mathbf{u}_+}$ with $v^j_{(0)} = 1$. Let $i \leqslant j$ be an index with $i \in J^\circ_{\mathbf{u}_+}$ and suppose that $\mathrm{Pivots}_{\text{辶}\leftarrow(i)}\left(v^j_{(j-1)}\right) = \mathrm{Pivots}_{\text{辶}\leftarrow(i)}\left(u_{(j-1)}\right)$ and $\mathrm{Pivots}_{\text{辶}\rightarrow(i)}\left(v^j_{(j-1)}\right)$ $\neq \mathrm{Pivots}_{\text{辶}\rightarrow(i)}\left(u_{(j-1)}\right)$. Let $r^+ = \lambda^{i-1}(\mathbf{u}_+)\left(\alpha_\searrow(i)\right)$ and let $r_- = \lambda^{i}(\mathbf{u}_+)\left(\alpha_\searrow(i)\right)$. We say that i is a *convex left corner* of $\mathtt{JC}(j)$ if $r^+ \in v^j_{(j-1)}([1, h_j])$ and $r_- \in v^j_{(j-1)}([h_j+1, n])$. We say that i is a *concave left corner* if $r^+ \in v^j_{(j-1)}([h_j+1, n])$ and $r_- \in v^j_{(j-1)}([1, h_j])$.

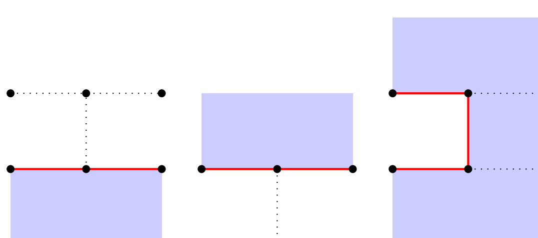

Figure 6.5: Bridges with X_j-degree $+1$ when oriented down and X_j-degree -1 when oriented up.

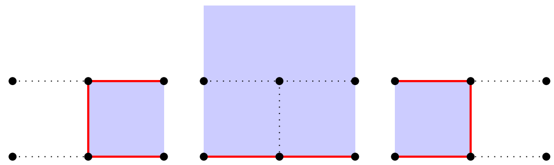

Figure 6.6: Bridges with X_j-degree -1 when oriented down and X_j-degree $+1$ when oriented up.

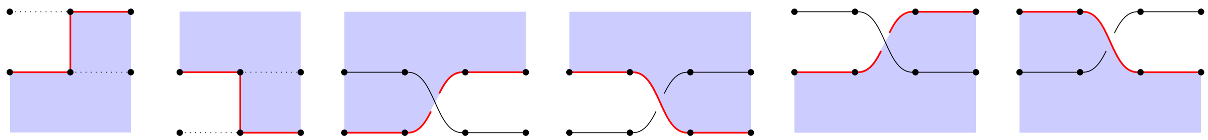

Figure 6.7: Boundary-incident edges of X_j-degree 0.

Proposition VI.15. *Let i be a $\partial\mathtt{JC}(j)$-incident bridge. If i is an upper or lower boundary edge, then $deg_{X_j}(i) = 0$.*

Otherwise, the degree of i in X_j is as follows.[1]

1. *If i is directed toward $\partial_\uparrow\mathtt{JC}(j)$ or away from $\partial_\downarrow\mathtt{JC}(j)$, then $deg_{X_j}(i) = 1$.*
2. *If i is directed away from $\partial_\uparrow\mathtt{JC}(j)$ or toward $\partial_\downarrow\mathtt{JC}(j)$, then $deg_{X_j}(i) = -1$.*

Proof. If i is a $\partial\mathtt{JC}(j)$-incident bridge which is not an upper or lower boundary edge, then either i satisfies description 1 when oriented down and description 2 when oriented up or vice versa. Since the weight of a bridge i is t_i if i is directed down and $\frac{1}{t_i}$ if i is directed up, it suffices to verify the claim for downward oriented bridges. For a unipeak wiring diagram, the possible types of bridges with nonzero weight in X_j are as follows:

[1]Mnemonically, going to the upper boundary makes the power of X_j go up; going to the lower boundary makes the power of X_j lower.

	ㅊ$_{\leftarrow}(i)$	ㅊ$_{\uparrow}(i)$	ㅊ$_{\downarrow}(i)$	ㅊ$_{\rightarrow}(i)$	$\deg_{X_j}(t_i)$
$\deg_{X_j}$(Q(ㅊ))	1	0	0	0	-1
	0	1	0	0	1
	0	0	1	0	1
	1	1	1	0	1
	1	1	0	1	-1
	0	0	0	1	-1

Case 1 occurs only for $i = j$; for unipeak expressions, case 3 occurs only for $i = j^{-}_{\searrow}$. □

Corollary VI.16. *Let i be a bridge so that at least one vertex of i is in $\overline{\mathtt{JC}(j)}$. Let γ_r and $\gamma_{r'}$ be the geodesic paths containing the endpoints of i. Then the degree of i is as follows.*

1. *If $r, r' \in R^{\downarrow}(j)$ or $r, r' \in R^{\uparrow}(j)$, then deg$_{X_j}(i) = 0$.*

2. *If $r \in R^{\downarrow}(j)$ and $r' \notin R^{\downarrow}(j)$, then the bridge i has degree 1 when directed toward γ_r and degree -1 otherwise.*

3. *If $r \in R^{\uparrow}(j)$ and $r' \notin R^{\uparrow}(j)$, then the bridge i has degree -1 when directed toward γ_r and degree 1 otherwise.*

Proof. Every lower boundary component follows a union of geodesic paths in $R^{\uparrow}(j)$, every upper boundary component follows a union of geodesic paths in $R^{\downarrow}(j)$, and every bridge between geodesic paths $\gamma_r \in R^{\uparrow}(j)$ and $\gamma_{r'} \in R^{\uparrow}(j)$ is incident to $\partial\mathtt{JC}(j)$. If both $r, r' \in R^{\uparrow}(j)$, then the bridge i must be on $\partial_{\downarrow}\mathtt{JC}(j)$ and hence has degree 0 in X_j for both orientations. If both $r, r' \in R^{\downarrow}(j)$, then either i is on $\partial_{\uparrow}\mathtt{JC}(j)$ or i is an interior bridge, so that i has degree 0 in X_j. □

Corollary VI.17. *Let $\widehat{\mathcal{G}_k}$ be any weighted, directed graph obtained from $\mathcal{G}_k$ by reversing some of the edges and inverting their weights. If $\pi = v_0, \cdots, v_{m+1}$ is a directed path in $\mathcal{G}_k$ so that $v_1, \cdots, v_m$ is contained in $\overline{\mathtt{JC}(j)}$, and if $e_{i_1}, \cdots, e_{i_m} = (v_{i_1}, v_{i_1+1}), \cdots, (v_{i_m}, v_{i_m+1})$ is the subsequence of edges with $deg_{X_j}(e) \neq 0$, then $deg_{X_j}\left(e_{i_{d+1}}\right) = -deg_{X_j}(e_{i_d})$ so that $deg_{X_j}(\pi) \in \{-1, 0, 1\}$.*

Proof. Each geodesic path γ_r intersecting $\overline{\mathtt{JC}(j)}$ satisfies exactly one of $r \in R^{\uparrow}(j)$ and $r \in R^{\downarrow}(j)$, and the path π switches from one geodesic path to another exactly when it crosses a bridge. Since π is a path, it cannot go from $R^{\uparrow}(j)$ to $R^{\downarrow}(j)$ twice without going from $R^{\downarrow}(j)$ to $R^{\uparrow}(j)$ in between, and vice versa. □

Corollary VI.18. *Let $\widehat{\mathcal{G}_k}$ be any weighted, directed graph obtained from $\mathcal{G}_k$ by reversing some of the edges and inverting their weights. Suppose that $\pi = v_0, \cdots, v_{m+1}$ is a directed path in $\mathcal{G}_k$ so that $v_1, \cdots, v_m$ is contained in $\overline{\mathtt{JC}(j)}$. Let d be an index with $1 \leqslant d \leqslant m+1$, and let v_d be on the geodesic path γ_r. Then the X_j-degree of the first d edges of π is given by*

$$deg_{v_d}(v_0) = \begin{cases} \epsilon & r \in R^{\uparrow}(j) \\ \epsilon + 1 & r \in R^{\downarrow}(j) \end{cases}$$

where $\epsilon \in \{-1, 0\}$.

CHAPTER VII

Regularity of Mutated Variables

In this chapter, we will describe the exchange relations corresponding to the cluster given by the regular functions X_j described in Chapter V. We will verify that the initial cluster variables together with the mutated variables X'_j satisfy the conditions of Berenstein, Fomin and Zelevinsky's Starfish Lemma, giving the coordinate ring $\mathbb{C}[\mathcal{R}^{u,w}]$ of the open Richardson variety the structure of an upper cluster algebra.

Using our geometric description of the region $\mathtt{JC}(j)$ from Chapter IV, we will consider the exchange relations obtained by multiplying the $\hat{y}$-variables corresponding to Berenstein, Fomin and Zelevinsky's chamber ansatz quiver. We will show that for each mutable variable X_j, the term $\widehat{Y}_j = \frac{\mathcal{M}_+}{\mathcal{M}_-}$ is the unsigned path weight of a simple cycle around the boundary $\partial\mathtt{JC}(j)$. By analyzing possible paths between boundary vertices under certain perfect orientations of oriented bridge diagrams, we will show that on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, the coordinates of the Schubert form $\Upsilon^{u,w}$ are Laurent polynomials in the variables $\{X_i : i \neq j\} \cup \{X'_j\}$. This will allow us to show that the mutated cluster variables X'_j are globally regular functions on $\mathcal{R}^{u,w}$ and to verify coprimeness conditions for pairs of distinct variables X_i and X_j in the initial cluster as well as pairs of variables X_i and X'_j.

7.1 Starfish Lemma

In Theorem 2.10 and Lemma 2.12 from [3], Berenstein, Fomin and Zelevinsky prove that the coordinate ring $\mathbb{C}[G^{u,v}]$ of the double Bruhat cell $G^{u,v} = B_+uB_+ \cap B_-vB_-$ has the structure of the upper cluster algebra, where any reduced double wiring diagram for u, v determines an initial seed with cluster variables given by the chamber minors and exchange relations given by a quiver structure they defined. Their proof uses the property that $G^{u,v}$ is a normal algebraic variety, so that the coordinate ring $\mathbb{C}[G^{u,v}]$ contains all functions f with singular locus of complex codimension $\geqslant 2$. Berenstein, Fomin and Zelevinsky begin with an initial seed Σ and consider the "starfish" of seeds $\Sigma' = \mu_i(\Sigma)$ obtained from Σ by a single cluster mutation. They show that the initial cluster variables and the mutated variables are globally regular functions, with frozen variables everywhere nonvanishing, and that for the initial seed Σ and each of the neighboring seeds $\Sigma' = \mu_i(\Sigma)$, the locus where all cluster variables are nonzero is isomorphic to a torus $(\mathbb{C}^*)^{N+M}$. For distinct mutable variables X_i and X_j in the initial cluster, they show that the functions X_i and X_j are *coprime*—that is, the hypersurfaces $\{X_i = 0\}$ and $\{X_j = 0\}$ intersect in codimension $\geqslant 2$. Similarly, they show that the mutated variables $X'_i = \mu_i(X_i)$ and the initial cluster variables X_j are pairwise coprime. It follows that $\mathbb{C}[G^{u,v}]$ consists of all functions which are regular on all the tori determined by the seeds Σ and Σ', so that $\mathbb{C}[G^{u,v}]$ is the *upper bound* algebra $\mathcal{U}(\Sigma)$ given by the intersection of the Laurent rings $\mathbb{C}[\mathbf{X}^{\pm}]$ and $\mathbb{C}[\mu_i(\mathbf{X})^{\pm}]$ for all mutable X_i. We will use the following restatement of the Starfish Lemma.

Lemma VII.1. *[3] Let $\mathcal{V}$ be a normal algebraic variety with coordinate ring $\mathbb{C}[\mathcal{V}]$. Sup-*

pose that there are regular functions $X_1, \cdots, X_N, X_{N+1}, \cdots, X_{N+M}$ *and an ice quiver with mutable vertices* $v_1, \cdots v_N$ *and frozen vertices* $v_{N+1}, \cdots, v_{N+M}$ *so that the following conditions hold.*

1. *The map* $\mathcal{V} \to \mathbb{C}^{N+M}$ *given by* $g \mapsto (X_1(g), \cdots, X_{N+M}(g))$ *restricts to a biregular isomorphism* $\{X_1, \cdots, X_{N+M} \neq 0\} \to (\mathbb{C}^*)^{N+M}$.
2. *The functions* $X_{N+1}, \cdots, X_{N+M}$ *are nonvanishing on* $\mathcal{V}$.
3. *If* $1 \leqslant i, j \leqslant N$ *with* $i \neq j$*, then the locus* $X_i = X_j = 0$ *has complex codimension* $\geqslant 2$.
4. *For each* $1 \leqslant i \leqslant N$*, there is a regular function* X_i' *so that* $X_i X_i' = \mathcal{M}_+^i + \mathcal{M}_-^i$.
5. *The map* $\mathcal{V} \to \mathbb{C}^{N+M}$ *given by* $g \mapsto (X_1(g), \cdots, X_{i-1}(g), X_i'(g), X_{i+1}(g), \cdots, X_{N+M}(g))$ *restricts to a biregular isomorphism* $\{X_1, \cdots, X_{i-1}, X_i', X_{i+1}, \cdots, X_{N+M} \neq 0\} \to (\mathbb{C}^*)^{N+M}$.
6. *For any* i *and* j *with* $1 \leqslant i, j \leqslant N$*, the locus* $X_i' = X_j = 0$ *has complex codimension* $\geqslant 2$.

Then $\mathbb{C}[\mathcal{V}]$ *is isomorphic to the upper cluster algebra* $\overline{\mathcal{A}}(X_1, \cdots, X_{N+M})$.

Brion gave a proof that the open Richardson variety $\mathcal{R}^{u,w}$ is a normal algebraic variety in [5]. In order to show that the coordinate ring $\mathbb{C}[\mathcal{R}^{u,w}]$ is isomorphic to an upper cluster algebra, it therefore suffices to give an initial cluster and quiver and verify the conditions of Berenstein, Fomin and Zelevinsky's Starfish Lemma. We have shown that for each $w \in \mathfrak{S}_n$ and each $u < w$ in the Bruhat order, the wiring diagram for the unipeak expression $\mathbf{w}$ for w with positive subexpression $\mathbf{u}_+$ for u gives $\dim \mathcal{R}^{u,w} = \ell(w) - \ell(u)$ regular functions X_i indexed by the crossings $J^\circ_{\mathbf{u}_+}$ so that the evaluation map gives a birational isomorphism from the locus $\{X_i \neq 0 : i \in J^\circ_{\mathbf{u}_+}\}$ to $(\mathbb{C}^*)^{|J^\circ_{\mathbf{u}_+}|}$. We have also shown that the function X_i is nowhere vanishing on $\mathcal{R}^{u,w}$ if and only if the nearly positive expression $\mathbf{v}^i$ fails to be a distinguished subexpression of $\mathbf{w}$, so that X_i divides the chamber weighting Q(ㅊ) for some open chamber ㅊ at the left boundary of the wiring diagram.

We will now describe a quiver structure inherited from Berenstein, Fomin and Zelevinsky's Chamber Ansatz quiver. We will verify that for each $j \in J^{\circ}_{\mathbf{u}_+}$ such that the nearly positive sequence $\mathbf{v}^j$ is a distinguished subexpression of $\mathbf{w}$, there is a regular function $X'_j \in \mathbb{C}[\mathcal{R}^{u,w}]$ so that $X_j X'_j = \mathcal{M}_+ + \mathcal{M}_-$, where $\mathcal{M}_+$ and $\mathcal{M}_-$ are the monomials defined by the outbound and inbound arrows incident to the vertex corresponding to X_j in the quiver. Specifically, the function X'_j is proportional to a binomial in the expansion of the *special chamber minor* $\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}$ appearing in Marsh and Rietsch's Chamber Ansatz for the Deodhar hypersurface $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$. As an intermediate step, we will show that for distinct indices i and j, the variables X_i and X_j are coprime. At the end of the chapter, we will show that the variables X_i and X'_j are also coprime.

Berenstein, Fomin and Zelevinsky defined the following ice quiver structure on the chambers of a wiring diagram in [3].

1. In each chamber ㅊ in the wiring diagram, draw a vertex $v_{\text{ㅊ}}$, designated mutable if ㅊ is a closed chamber and frozen if ㅊ is an open chamber.

2. For each crossing i, draw a left-to-right horizontal arrow from $v_{\text{ㅊ}_{\leftarrow}(i)}$ to $v_{\text{ㅊ}_{\rightarrow}(i)}$, and draw a right-to-left inclined arrow between each pair of chambers that meet along a strand segment.

3. Erase 2-cycles and arrows between frozen vertices.

Definition VII.2 (Richardson quiver)**.** The *induced quiver on* $\mathcal{R}^{u,w}$ is as follows.

1. Draw one vertex v_j for each $\{X_j : j \in J^{\circ}_{\mathbf{u}_+}\}$, with X_j designated mutable if the

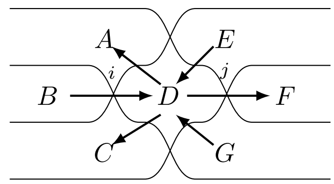

Figure 7.1: The $\hat{y}$-variable for the chamber D is $\frac{ACF}{BEG}$, which is equal to the ratio $\frac{t_i}{t_j}$.

nearly positive sequence $\mathbf{v}^j$ is a distinguished subexpression of $\mathbf{w}$ and frozen otherwise. (Equivalently, X_j is frozen if and only if there is an open chamber ㅊ so that X_j divides Q(ㅊ).)

2. For each pair of distinct vertices v_i and v_j, draw one arrow from v_i to v_j for each arrow $e : v_{ㅊ} \to v_{ㅊ'}$ in the chamber ansatz quiver so that X_i divides Q(ㅊ) and X_j divides Q(ㅊ $'$).

3. Erase loops, 2-cycles and arrows between frozen vertices.

We note that the $\hat{y}$-variables for the chamber ansatz quiver are given by the following formula.

Formula VII.3. Let ㅊ be a chamber in a wiring diagram bounded on the left and right by crossings $i < j$ where $h_i = h_j = h$ and for all $i < i' < j$ we have $h_{i'} \neq h$. The $\hat{y}$-variable for ㅊ with respect to the chamber ansatz quiver is given by $\widehat{y}\,(ㅊ) = \frac{\Delta_{ㅊ_\uparrow(i)}\Delta_{ㅊ_\downarrow(i)}\Delta_{ㅊ_\rightarrow(j)}}{\Delta_{ㅊ_\leftarrow(i)}\Delta_{ㅊ_\uparrow(j)}\Delta_{ㅊ_\downarrow(j)}} = \frac{t_i}{t_j}$ (where if $k \in J^+_{\mathbf{u}_+}$ we define $t_k = 1$).

This is easiest to see by example. In Figure 7.1, the ratio $\frac{t_i}{t_j}$ for the crossings i and j to the left of D is given by $\frac{t_i}{t_j} = t_i t_j^{-1} = \frac{AC}{BD}\frac{DF}{EG} = \frac{ACF}{BEG} = \frac{\prod_{D \to X} X}{\prod_{X \to D} X}$, which is the $\hat{y}$-variable for the chamber D. Note that $\widehat{y}\,(ㅊ)$ has a natural interpretation as the path weight of a counterclockwise cycle around the boundary of the chamber ㅊ, where traveling up at a crossing corresponds to inverting the edge weight.

The outbound and inbound arrows for the vertex labeled by X_j are computed by taking the net number of outbound and inbound arrows for the chambers in $\mathtt{JC}(j)$, which corresponds to multiplying the $\hat{y}$-variables from the chamber ansatz quiver.

Formula VII.4. Let $j \in J^{\circ}_{\mathbf{u}_+}$ be an index with $v^j_{(0)} = 1$. The $\hat{y}$-variable for the variable X_j is given by $\widehat{Y}_j = \prod_{\text{え} \in \mathtt{JC}(j)} \widehat{y}(\text{え})$.

Since the region $\overline{\mathtt{JC}(j)}$ is simply connected, the product of the path weights of the counterclockwise cycles around each chamber in $\mathtt{JC}(j)$ is the weight of the counterclockwise cycle around $\partial\mathtt{JC}(j)$.

7.2 The upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$ is the upper bound $\mathcal{U}(\Sigma)$ for the initial seed

At the end of this chapter we will verify that $\mathbb{C}[\mathcal{R}^{u,w}]$ is equal to the upper bound algebra $\mathcal{U}(\Sigma)$ for the initial seed $\Sigma = (\mathbf{X}, Q)$. In order to conclude that $\mathbb{C}[\mathcal{R}^{u,w}]$ is the upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$, we will need to show that the algebras $\overline{\mathcal{A}}(\Sigma) \subseteq \mathcal{U}(\Sigma)$ coincide.In [3], Berenstein, Fomin and Zelevinsky give a criterion for equality of the upper bound and the upper cluster algebra in terms of the rank of a matrix $\tilde{B}(Q)$ describing the exchange relations of a quiver Q.

Definition VII.5. Let Q be a quiver. The signed adjacency matrix of Q is the matrix

$$A(Q) = (a_{ij}) = \#(v_i \to v_j) - \#(v_j \to v_i),$$

where the rows and columns are indexed by the vertices of Q and the entry a_{ij} is the net number of arrows from v_i to v_j. The $\tilde{B}$-matrix of Q is the submatrix $\tilde{B}(Q)$ consisting of the columns of $A(Q)$ indexed by mutable variables.

By Corollary 1.9 in [3], if the matrix $\tilde{B}(Q)$ has full rank, then for any seed Σ the upper bound $\mathcal{U}(\Sigma)$ is equal to the upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$. Combining this with their result that the chamber ansatz quiver Q_0 satisfies the condition $\tilde{B}(Q_0)$ is full rank, we show that the Richardson quiver Q also has a full rank $\tilde{B}$-matrix.

The proof of the following proposition is due to Speyer.

Proposition VII.6. *Let Q be the Richardson quiver. The matrix $\tilde{B}(Q)$ has full rank.*

Proof. Write $J = J^{\circ}_{\mathbf{u}_+} \cup J^{+}_{\mathbf{u}_+}$ for the index set of the crossings in the wiring diagram of $\mathbf{w}$, and write $J^{interior}$ for the set $\{j \in J : \exists j^- < j, h_{j^-} = h_j\}$. That is, $J^{interior}$ is the index set for the mutable vertices in the chamber ansatz quiver. Let $J^{\circ,interior}_{\mathbf{u}_+}$ be the set $\{j \in J^{\circ}_{\mathbf{u}_+} : \exists j^- < j, h_{j^-} = h_j\}$. Let L be the lattice $\mathbb{Z}^J$ and let $L^{interior}$ be the sublattice $\mathbb{Z}^{J^{interior}}$. We may think of a point $v \in L$ as the exponent vector of a Laurent monomial in some weighting on the chambers of the wiring diagram, where an index $j \in J$ gives the exponent for the weight of the chamber $\text{ㅊ}_{\leftarrow}(j)$. Let $M \subset L$ be the lattice $\mathbb{Z}^{J^{\circ}_{\mathbf{u}_+}}$, and let $M^{interior}$ be the sublattice of M with index set $J^{\circ,interior}_{\mathbf{u}_+}$. Let Q_0 be the chamber ansatz quiver. The matrix $A(Q_0)$ gives a skew form $\langle\, , \,\rangle$ on L, and the matrix $\tilde{B}(Q_0)$ gives a map from $L^{interior}$ to the dual lattice $\check{L}$ corresponding to restricting the form $\langle\, , \,\rangle$. Let Q be the Richardson quiver. For each $i \in J^{+}_{\mathbf{u}_+}$, let $i_{\uparrow}$, $i_{\downarrow}$ and $i_{\rightarrow}$ denote the crossings to the right of the chambers $\text{ㅊ}_{\uparrow}(i)$, $\text{ㅊ}_{\downarrow}(i)$ and $\text{ㅊ}_{\rightarrow}(i)$. Let $\varphi_i \in \check{L}$ be the weight vector that looks like $(1, -1, -1, 1)_{i, i_{\uparrow}, i_{\downarrow}, i_{\rightarrow}}$. The dual lattice $\check{M}$ is the quotient of $\check{L}$ by $M^{\perp} := \mathrm{Span}(\varphi_i)_{i \in J^{+}_{\mathbf{u}_+}}$, which corresponds to the condition that chambers surrounding a crossing i in $J^{+}_{\mathbf{u}_+}$ must satisfy Dodgson's identity.

Write $B^{interior}(Q)$ for the submatrix of $A(Q)$ with columns indexed by $J^{\circ,interior}_{\mathbf{u}_+}$, so that $\tilde{B}(Q)$ is a submatrix of $B^{interior}(Q)$. The following diagram commutes.

$$\begin{tikzcd}
& & M^\perp \arrow[d, hook] \\
L^{interior} \arrow[r, hook] \arrow[rr, bend left, "\tilde{B}(Q_0)"] & L \arrow[r, "A(Q_0)"] & \breve{L} \arrow[d, two heads] \\
M^{interior} \arrow[u, hook] \arrow[r, hook] \arrow[rr, bend right, "B^{interior}(Q)"'] & M \arrow[u, hook] \arrow[r, "A(Q)"] & \breve{M}
\end{tikzcd}$$

We note that in order to show that $\tilde{B}(Q)$ is full rank, it suffices to show that $B^{interior}(Q)$ is full rank.

Consider the map $B^{interior}(Q) : M^{interior} \to \breve{M}$ and suppose that $\vec{v} \in M^{interior}$ is an element of the kernel. Viewing $\vec{v}$ as an element of the larger lattice L and identifying crossings i with the chambers ㅊ$_{\leftarrow}(i)$, choose i to be minimal so that $\vec{v}$ is supported on ㅊ$_{\rightarrow}(i)$. Then i must be in $J^{\circ}_{\mathbf{u}_+}$, since if i were in $J^{+}_{\mathbf{u}_+}$ then the condition that $\varphi_i(\vec{v}) = 0$ for $\vec{v} \in M$ would imply that $\vec{v}$ be supported on at least one of ㅊ$_{\uparrow}(i)$ and ㅊ$_{\downarrow}(i)$, contradicting the minimality of i.

Write j for the index so that ㅊ$_{\rightarrow}(i) =$ ㅊ$_{\leftarrow}(j)$. Let $N \neq 0$ be the value of $\vec{v}_j$. By hypothesis, $B^{interior}(Q)\vec{v} = 0$. So there must be an element $\sum_{k \in J^+_{\mathbf{u}_+}} \lambda_k \varphi_k \in M^\perp$ so that $\tilde{B}(Q_0)\vec{v} + \sum_{k \in J^+_{\mathbf{u}_+}} \lambda_k \varphi_k = 0$. Now the condition ㅊ$_{\rightarrow}(i) =$ ㅊ$_{\leftarrow}(j)$ implies that $\tilde{B}(Q_0)$ has entry $b_{ij} = 1$, so $\tilde{B}(Q_0)\vec{v}_i = N$. By minimality of i, $\tilde{B}(Q_0)\vec{v}_{i'} = 0$ for all $i' < i$. In particular, this holds for all i' so that ㅊ$_{\leftarrow}(i')$ is below the strand $\alpha_{\searrow}(i)$ and above the strand $\alpha_{\nearrow}(i)$. So the sum $\sum_{\substack{\text{ㅊ}_{\leftarrow}(i') \text{ below } \alpha_{\searrow}(i) \\ \text{and above } \alpha_{\nearrow}(i)}} (\tilde{B}(Q_0)\vec{v})_{i'} = N$. Since the crossing i is in $J^{\circ}_{\mathbf{u}_+}$, for any crossing k in $J^{+}_{\mathbf{u}_+}$ the components of the weight vector φ_k indexed by chambers above the strand $\alpha_{\nearrow}(i)$ and below the strand $\alpha_{\searrow}(i)$ must sum to zero.

So any weight vector of the form $\omega = \tilde{B}(Q_0)\vec{v} + \sum_{k \in J^+_{\mathbf{u}_+}} \lambda_k \varphi_k$ satisfies the condition $\sum_{i'} \omega_{i'} = N$, where i' ranges over the crossings that are between $\alpha_{\searrow}(i)$ and $\alpha_{\nearrow}(i)$. In particular, ω cannot be the zero weight vector, since summing over a subset of the components gives a nonzero sum N.

This is a contradiction, so the matrix $B^{interior}(Q)$ is full rank, and hence $\tilde{B}(Q)$ is full rank.

$\square$

7.3 Coordinate functions of $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ are Laurent in the mutated variables

In this section, we will use augmenting paths in the family of graphs $\widehat{\mathcal{G}}_k^{\mathbf{w}}$ to show that whenever the variable X_j vanishes on a Deodhar hypersurface $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$, the restriction of any Schubert coordinate to $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ is Laurent in the variables $\{X_i : i \neq j\} \cup \{X_j'\}$. It will be convenient to work with the *left-normalized Schubert coordinates* described by Fulton in [13], corresponding to entries in the upper unitriangular matrix $\hat{z}$ with $z = \hat{z}\ \mathrm{diag}\,(z_{ii})$.

By the following result of Kassel, Lascoux and Reutenauer, for any unipeak expression $\mathbf{w}$, the entries of $\hat{z}$ satisfy a stabilization property.

Theorem VII.7 (Left-normalized Schubert coordinates in unipeak diagrams, Kassel, Lascoux and Reutenauer[22])**.** *Let $\mathbf{w}$ be a unipeak expression for w. Let F be a flag in $B_+\dot{w}B_+/B_+$ and let $\hat{z}$ be the upper unitriangular matrix so that the matrix $\hat{z}\dot{w}$ has zeros to the right of each permutation position and $\hat{z}\dot{w}B_+ = F$. Then the matrix $\hat{z}\dot{w} = \prod_{i=1}^{\ell} x_{h_i}(f_i)\dot{s}_{h_i}$. Write $\hat{z}^j$ for the upper unitriangular matrix $\left(\prod_{i=1}^{j} x_{h_i}(f_i)\dot{s}_{h_i}\right)\dot{w}_{(j)}^{-1}$. Write λ_a for the left endpoint of $\alpha_{\nearrow}(k)$ and write λ_b for the left endpoint of $\alpha_{\searrow}(k)$. Then the entry $\hat{z}_{ab}^j$ is given by*

$$\hat{z}_{ab}^j = \begin{cases} 0 & \text{if } j < k \\ \hat{z}_{ab} & \text{if } j \geqslant k. \end{cases}$$

By Proposition VI.5, the ratio of minors $\frac{\Delta^{w_{(k)}([1,h_k])\setminus\{b\}\cup\{a\}}_{[1,h_k]}(g_{(k)})}{\Delta^{w_{(k)}([1,h_k])}_{[1,h_k]}(g_{(k)})} = \pm\, \hat{z}_{ab}$ is given by

$$\frac{\Delta^{w_{(k)}([1,h_k])\setminus\{b\}\cup\{a\}}_{[1,h_k]}(g_{(k)})}{\Delta^{w_{(k)}([1,h_k])}_{[1,h_k]}(g_{(k)})} = \sum_{\mathcal{F}:\lambda_a\to\lambda_b} \omega(\mathcal{F}),$$

where the sum is over augmenting paths from λ_a to λ_b in the directed graph $\widehat{\mathcal{G}}^{\mathbf{w}}_k$. We note that $z_{ab} = \hat{z}_{ab} z_{bb}$ where z_{bb} is a ratio of frozen variables $\{X_i : v^i_{(0)} \neq 1\}$, so that for any mutable variable X_j, the X_j-degree of an augmenting path from λ_a to λ_b in $\widehat{\mathcal{G}}^{\mathbf{w}}_k$ is the same as the X_j-degree in the corresponding term in the expansion of z_{ab}.

Our main goal for this section is to give a characterization of augmenting paths of negative degree in X_j in the graph $\widehat{\mathcal{G}}^{\mathbf{w}}_k$.

By Proposition VI.17, if $\pi = v_0, \cdots, v_{m+1}$ is a directed path in $\widehat{\mathcal{G}}^{\mathbf{w}}_k$ so that the vertices $v_1, \cdots, v_m$ are contained in $\overline{\mathtt{JC}(j)}$, then the X_j-degrees of the edges of π satisfy a weak alternation property so that $\deg_{X_j}(\pi)$ is either $-1, 0$ or 1.

The following elementary proposition shows that we can decompose any augmenting path $\mathcal{F}$ into segments which either satisfy this property or are vertex-disjoint from $\overline{\mathtt{JC}(j)}$, and therefore have degree 0 in X_j.

Proposition VII.8 (Partitioning an augmenting path $\mathcal{F}$). *Let $\mathcal{F} : \lambda_a \to \lambda_b$ be an augmenting path in $\widehat{\mathcal{G}}^{\mathbf{w}}_k$. Then $\mathcal{F}$ has a unique decomposition into a sequence of consecutive directed paths $\pi^1, \cdots, \pi^c$ satisfying the following conditions.*

1. *Either π^d is vertex-disjoint from $\overline{\mathtt{JC}(j)}$ or the vertices of π^d can be enumerated as $\pi^d = v_0, \cdots, v_{m+1}$, where the vertices $v_0, v_{m+1} \notin \overline{\mathtt{JC}(j)}$ and $v_1, \cdots, v_m \in \overline{\mathtt{JC}(j)}$, with $m > 0$.*
2. *If π^d is vertex-disjoint from $\overline{\mathtt{JC}(j)}$, then π^{d+1} intersects $\overline{\mathtt{JC}(j)}$.*

Proof. Every vertex of $\widehat{\mathcal{G}}_k^{\mathbf{w}}$ either belongs to $\overline{\mathtt{JC}(j)}$ or to the complement of $\overline{\mathtt{JC}(j)}$, and the vertices λ_a and λ_b are not in $\overline{\mathtt{JC}(j)}$. Split up the path $\mathcal{F}$ at each vertex $v_i \notin \overline{\mathtt{JC}(j)}$ so that one of the vertices v_{i-1} or $v_{i+1} \in \overline{\mathtt{JC}(j)}$. □

We will now introduce some notation for the paths intersecting $\overline{\mathtt{JC}(j)}$ in this decomposition of an augmenting path.

Definition VII.9 ($\overline{\mathtt{JC}(j)}$-path components)**.** Let $\mathcal{F} : \lambda_a \to \lambda_b$ be an augmenting path in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$. We say that a path π contained in $\mathcal{F}$ is a
$\overline{\mathit{JC}(j)}$*-path component* of $\mathcal{F}$ if $\pi = v_0, \cdots, v_{m+1}$ where $m > 0$, the vertices $v_0, v_{m+1} \notin \overline{\mathtt{JC}(j)}$ and all vertices $v_1, \cdots, v_m \in \overline{\mathtt{JC}(j)}$.

Definition VII.10 ($\partial\mathtt{JC}(j)$-path components)**.** Let $\pi = v_0, \cdots, v_{m+1}$ be a
$\overline{\mathtt{JC}(j)}$-path component of an augmenting path. If $v_0, \cdots, v_m \subset \partial\mathtt{JC}(j)$, we say that π is a ∂*JC*(j)*-path component* and designate π as *clockwise* or *counterclockwise* depending on the orientation of the boundary arc $v_0, \cdots, v_m$.

We will also refer to a path π as being a $\overline{\mathtt{JC}(j)}$-path component or a
$\partial\mathtt{JC}(j)$-path component, leaving the augmenting path or paths containing π implicit. This should always be understood to mean that there is at least one directed path from λ_a to v_0 and at least one directed path from v_{m+1} to λ_b. The following proposition shows that we can reduce the task of characterizing augmenting paths $\mathcal{F}$ with $\deg_{X_j}(\mathcal{F}) < 0$ to the more local task of describing $\overline{\mathtt{JC}(j)}$-path components of X_j-degree -1.

Proposition VII.11. *Let $\mathcal{F} : \lambda_a \to \lambda_b$ be an augmenting path in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$, and suppose that deg$_{X_j}(\mathcal{F})$ is negative. Then $\mathcal{F}$ has a $\overline{JC(j)}$-path component $\pi = v_0, \cdots, v_{m+1}$ so that deg$_{X_j}(\pi) = -1$.*

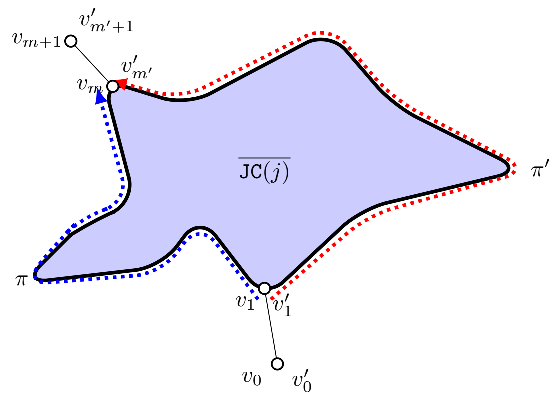

Figure 7.2: Flipping a negative weight $\overline{\mathtt{JC}(j)}$-path component

Proof. Let $\pi^1, \cdots, \pi^c$ be the decomposition of $\mathcal{F}$ from Proposition VII.8. Then $\deg_{X_j}(\mathcal{F}) = \sum_{i=1}^{c} \deg_{X_j}(\pi^i)$. If π^i is vertex-disjoint from $\overline{\mathtt{JC}(j)}$, then all edges $e \in \pi^i$ have degree 0 in X_j and so $\deg_{X_j}(\pi^i) = 0$.

If π^i is a $\overline{\mathtt{JC}(j)}$-path component of $\mathcal{F}$, then by Corollary VI.17, the edges of nonzero X_j degree are of alternating signs, so that $\deg_{X_j}(\pi^i) \in \{-1, 0, 1\}$. In particular, if $\sum_{i=1}^{c} \deg_{X_j}(\pi^i) < 0$, then at least one $\overline{\mathtt{JC}(j)}$-path component π^i must have $\deg_{X_j}(\pi^i) = -1$. $\square$

We wish to show that whenever $\mathcal{F}$ is an augmenting path from λ_a to λ_b with $\deg_{X_j}(\mathcal{F}) < 0$, there exists an augmenting path $\mathcal{F}'$ from λ_a to λ_b so that $\omega(\mathcal{F}) + \omega(\mathcal{F}') = M\frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j}$ where M is a Laurent monomial in $\mathbb{C}[X_i^{\pm} : i \neq j]$. [1]

The overall strategy we use is as follows. Given an augmenting path $\mathcal{F} : \lambda_a \to \lambda_b$ in the graph $\widehat{\mathcal{G}}_k^{\mathbf{w}}$, let π be a $\overline{\mathtt{JC}(j)}$-path component of $\mathcal{F}$ as in Proposition VII.8 and suppose that $\deg_{X_j}(\pi) = -1$. We will show that knowing the path π completely determines the orientation of $\overline{\mathtt{JC}(j)}$ in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$. In particular, we will prove the following.

1. The path $\pi = v_0, \cdots, v_{m+1}$ is a $\partial\mathtt{JC}(j)$-path component.

2. The vertices v_1 and v_m partition $\partial\mathtt{JC}(j)$ into a clockwise arc and a counterclock-

[1] It is not obvious a priori that every augmenting path with negative degree in X_j has degree -1 in X_j; it's false in general for the subset of the open torus inside $\mathcal{R}^{1,w}$ parametrized by the specialized chamber weighting $\mathbf{Q}(ネ)$.

wise arc.

3. If $\tilde{\pi}$ is any $\overline{\mathtt{JC}(j)}$-path component in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$ with $\deg_{X_j}(\tilde{\pi}) = -1$, then $\tilde{\pi}$ is either π or the path π' which travels from v_0 to v_{m+1} along the complementary arc of $\partial\mathtt{JC}(j)$.

Combining this result with the acyclicity of the orientation, we will conclude that if $\mathcal{F}$ is an augmenting path from λ_a to λ_b, then $\deg_{X_j}(\mathcal{F}) \geqslant -1$, and when equality holds, there is an augmenting path $\mathcal{F}'$ which differs from $\mathcal{F}$ by an oriented cycle around $\partial\mathtt{JC}(j)$, so that $\omega(\mathcal{F}) + \omega(\mathcal{F}')$ is divisible by $\frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j}$.

We begin with some observations about the oriented graph $\widehat{\mathcal{G}}_k^{\mathbf{w}}$.

Paths in the flow $\mathcal{F}_{w_{(k)}}$ in the oriented bridge diagram $\mathcal{G}_k$ join source vertices $\lambda_{w_{(k)}(h)}$ where $1 \leqslant h \leqslant h_k$ to target vertices $\rho_{h'}$ where $1 \leqslant h' \leqslant k$; a path switches from one strand to another if it reaches a bridge i where both $\alpha_{\nearrow}(i)$ and $\alpha_{\searrow}(i)$ have targets in the interval $[1, h_k]$. In particular, a bridge i is an edge of $\mathcal{F}_{w_{(k)}}$ (and hence directed up in the graph $\widehat{\mathcal{G}}_k^{\mathbf{w}}$) if and only if $\alpha_{\nearrow}(i)$ ends at a target in the interval $[h_k + 1, n]$ and $\alpha_{\searrow}(i)$ ends at a target in the interval $[1, h_k]$.

Proposition VII.12. *Let α be a strand directed left-to-right in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$. Then the following hold for indices $i, i' \leqslant k$.*

1. *Suppose that i and i' are indices in $J_{\nearrow}(\alpha)$ with $i < i'$. If $\alpha_{\searrow}(i')$ is oriented right-to-left, then so is $\alpha_{\searrow}(i)$.*

2. *If $i \in J_{\searrow}(\alpha)$, then $\alpha_{\nearrow}(i)$ is oriented left-to-right.*

Proof. If β is a strand with right endpoint ρ_h, then β is oriented right-to-left if $h \leqslant h_k$ and left-to-right if $h \geqslant h_k + 1$.

1. The strands $\alpha_{\searrow}(i)$ and $\alpha_{\searrow}(i')$ cannot cross each other after crossing below α, so the right endpoint of $\alpha_{\searrow}(i)$ is lower than the right endpoint of $\alpha_{\searrow}(i')$.

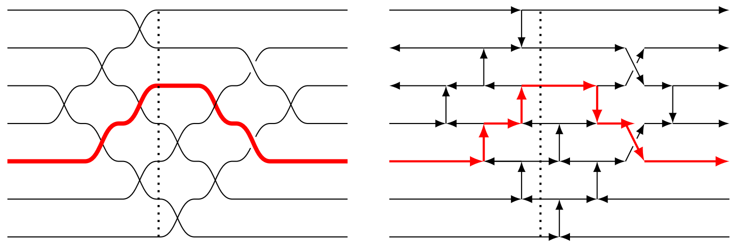

Figure 7.3: A unipeak wiring diagram and the orientation of its bridge diagram obtained by reversing edges along the flow from sources $\{\lambda_{w(1)}, \lambda_{w(2)}\}$ to targets $\{\rho_1, \rho_2\}$. Every edge incident to or above the red path to the right of the dotted line is oriented left-to-right.

2. If $i \in J_{\searrow}(\alpha)$, the strand $\alpha_{\nearrow}(i)$ crosses above α and hence has a higher right endpoint.

□

Corollary VII.13. *Let α be a strand directed left-to-right in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$. Let i be a crossing along α such that both strands are directed left-to-right. Then the portion of $\widehat{\mathcal{G}}_k^{\mathbf{w}}$ above the strand $\alpha_{\searrow}(i)$ to the right of the crossing i is oriented left-to-right.*

Corollary VII.14. *Let $\mathcal{F}$ be an augmenting path in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$. Whenever the path $\mathcal{F}$ switches directions, it must travel up a bridge.*

Proof. Horizontal and inclined edges on a single strand α are either all left-to-right or all right-to-left, so that $\mathcal{F}$ must switch from one strand to another in order to change directions. Strands in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$ meet at a vertex exactly when they have a bridge in common; if $\alpha_{\nearrow}(i)$ and $\alpha_{\searrow}(i)$ have opposite orientations, then the bridge i is directed up. □

We will now describe several types of directed edges that are never used by any augmenting path π in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$, since augmenting the flow $\mathcal{F}_{w(k)}$ along $\pi : \lambda_a \to \lambda_b$ must give a left-justified flow with targets $[1, h_k]$ in the oriented bridge diagram $\mathcal{G}_k$. In particular, a directed path through $\overline{\texttt{JC}(j)}$ containing any of these *forbidden edges* fails to be a $\overline{\texttt{JC}(j)}$-path component.

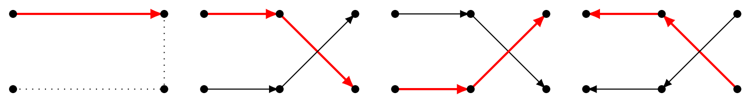

Figure 7.4: Directed edges which are never used by an augmenting path in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$.

Proposition VII.15 (Forbidden edges). *Let $\pi : \lambda_a \to \lambda_b$ be an augmenting path in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$. If e is any of the following types of directed edges, then e is not in π. (See Figure 7.4.)*

1. *The edge $e = (s,t)$ is a left-to-right horizontal edge at height h where either t is a sink vertex or there is an edge (t,t') where t' is a vertex at level $h-1$.*
2. *The edge $e = (t,t')$ is a bridge directed down from height h to height $h-1$, where the horizontal edge (s,t) is directed left-to-right.*
3. *The edge $e = (s,t)$ is an inclined edge directed southeast.*
4. *The edge $e = (s,t)$ is an inclined edge oriented northeast and the corresponding edge (s',t') is oriented southeast.*
5. *The edge $e = (s,t)$ is an inclined edge oriented northwest and the corresponding edge (s',t') is oriented northeast.*

Proof. For the first three cases, every edge reachable after following e is directed left-to-right, so if $\mathcal{F}$ uses one of these edges then $\mathcal{F}$ cannot end at the vertex λ_b. In the last two cases, there is a crossing $i \in J^+_{\mathbf{u}_+}$ where the flow $\mathcal{F}^\pi$ obtained by augmenting $\mathcal{F}_{w_{(k)}}$ along the path π uses an upward inclined edge without using the corresponding downward inclined edge. This does not occur for any flow through the oriented bridge diagram $\mathcal{G}_k$ which has left-justified endpoints $\rho_1, \cdots, \rho_{h_k}$. $\square$

Proposition VII.16. *, Let $i = (v_-, v_+)$ be a bridge directed upward from γ_{r_-} to γ_{r^+} in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$. Then there are connected left-to-right paths π^- and π^+ in the wiring diagram for $w_{(k)}$ so that π^- travels from v_- to a target vertex ρ_{h_-} where $h_- \leqslant h_k$, π^+ travels from v^+ to a*

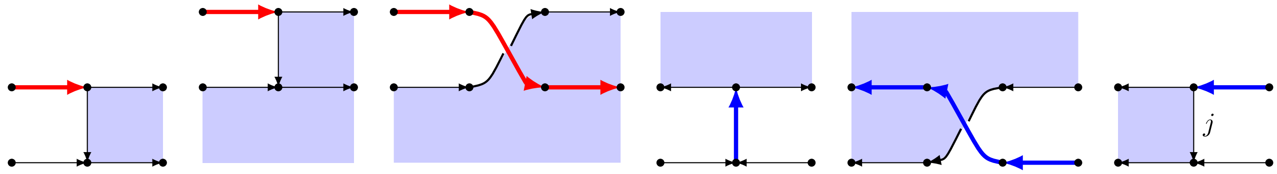

Figure 7.5: Types of directed edges arriving at $\overline{\texttt{JC}(j)}$ that either follow a geodesic γ_r where $r \in R^{\downarrow}(j)$ or follow a bridge toward a geodesic γ_r where $r \in R^{\uparrow}(j)$. In the first three diagrams on the left, the red edges arriving at $\overline{\texttt{JC}(j)}$ are forbidden.

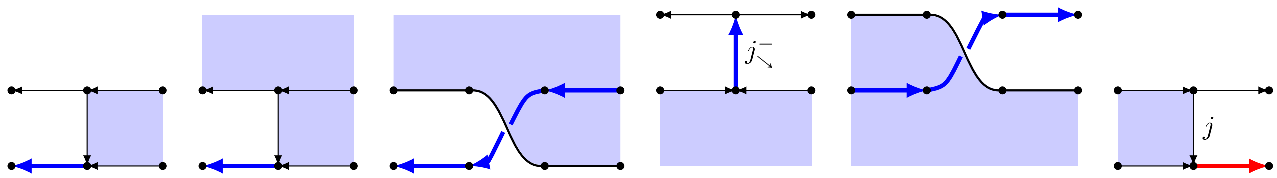

Figure 7.6: Types of directed edges leaving $\overline{\texttt{JC}(j)}$ that either follow a geodesic γ_r where $r \in R^{\uparrow}(j)$ or follow a bridge away from a geodesic γ_r where $r \in R^{\downarrow}(j)$. In the diagram on the far right, the red edge leaving $\overline{\texttt{JC}(j)}$ on γ_{r_*} is forbidden.

target vertex ρ_{h^+} with $h^+ \geqslant h_k + 1$, and π^- travels only down at bridges while π^+ travels only up at bridges.

Proof. Since the bridge i is oriented up in $\widehat{\mathcal{G}}_k^{\mathbf{w}}$, the strand $\alpha_{\nearrow}(i)$ has right endpoint in the interval $[h_k + 1, n]$ and the strand $\alpha_{\searrow}(i)$ has left endpoint in the interval $[1, h_k]$. Take the path π^- to follow $\alpha_{\searrow}(i)$ to the right of i and let π^+ follow $\alpha_{\nearrow}(i)$ initially and travel up whenever possible and down only at inclined edges. The strand $\alpha_{\nearrow}(i)$ has right endpoint in the interval $[1, h_k]$, and traveling maximally up in a reduced wiring diagram weakly increases the right endpoint. $\square$

In order for the $\overline{\texttt{JC}(j)}$-path component π to have degree -1 in X_j, there are two requirements that must be satisfied. First, for the indices where the vertex $v_d \in \overline{\texttt{JC}(j)}$, if v_d is on the geodesic path γ_r, the value of $\deg_{X_j}(v_0, \cdots, v_d)$ must be -1 when $r \in R^{\uparrow}(j)$ and 0 when $r \in R^{\downarrow}(j)$. This is equivalent to the condition that the edge (v_0, v_1) is either a horizontal or inclined edge on a geodesic path γ_r where $r \in R^{\downarrow}(j)$ or a bridge directed toward a geodesic path γ_r where $r \in R^{\uparrow}(j)$ (see Figure 7.5). In the notation of Proposition VI.18, we will say that π has $\epsilon = -1$. Second,

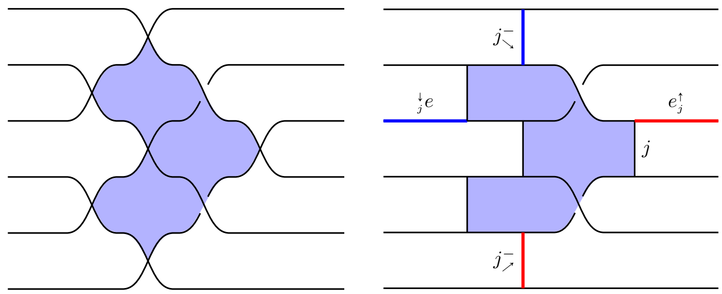

Figure 7.7: Augmenting paths with negative X_j-degree arrive at $\overline{\mathtt{JC}(j)}$ from one of the red edges $j^-_\nearrow$ or $e^\uparrow_j$ and leave $\overline{\mathtt{JC}(j)}$ using one of the blue edges $j^-_\nwarrow$ or ${}^\downarrow_j e$.

if the vertex v_m is on γ_r where $r \in R^\uparrow(j)$, then (v_m, v_{m+1}) must be a horizontal or inclined edge on the geodesic γ_r, and if $r \in R^\downarrow(j)$, then (v_m, v_{m+1}) must be a bridge directed away from γ_r (see Figure 7.6). In this case, we say that π has X_j-degree ϵ.

Definition VII.17. We define the *upper left corner* of $\overline{\mathtt{JC}(j)}$ to be the minimal index i so that え$_\rightarrow(i) \in \mathtt{JC}(j)$ and

$$\alpha_\nearrow(i) = \begin{cases} \alpha_\searrow(j) & \text{if } j^-_\nwarrow = 0 \\ \alpha_\nearrow(j^-_\nwarrow) & \text{if } j^-_\nwarrow \neq 0 \end{cases}.$$

Note that this index i is necessarily in $J^\circ_{\mathbf{u}_+}$.

Notation VII.18. Let i be the upper left corner of $\overline{\mathtt{JC}(j)}$. We denote the horizontal edge on $\alpha_\nearrow(i)$ to the left of i by ${}^\downarrow_j e$. We denote the horizontal edge on $\alpha_\nearrow(j)$ to the right of j by $e^\uparrow_j$.

Proposition VII.19. *Suppose that i is a crossing incident to $\overline{\mathtt{JC}(j)}$. If $\alpha_\nearrow(i) \neq \alpha_\searrow(j)$, then the left endpoint of $\alpha_\nearrow(i)$ is below the left endpoint of $\alpha_\searrow(j)$.*

Proof. The region $\overline{\mathtt{JC}(j)}$ is below the strand $\alpha_\searrow(j)$, so that if a strand β starts above $\alpha_\searrow(j)$ then it must cross below $\alpha_\searrow(j)$ at some point weakly to the left of the crossing where it first intersects $\overline{\mathtt{JC}(j)}$. Since $\mathbf{w}$ is unipeak, if $\beta = \alpha_{\nearrow}(i)$ for some i incident to $\overline{\mathtt{JC}(j)}$, then it does not travel down before index i. $\square$

Corollary VII.20. *Suppose that i is a crossing on $\partial_{\downarrow}\mathtt{JC}(j)$ with $\alpha_{\nearrow}(i) \neq \alpha_{\nearrow}(j)$ and $\alpha_{\nearrow}(i) \in C^{\downarrow}(j)$. Then the strand $\alpha_{\nearrow}(i)$ is below $\alpha_{\searrow}(j)$ for all indices $0 \leqslant k \leqslant \ell$.*

Corollary VII.21. *Suppose that $\alpha \neq \alpha_{\nearrow}(j)$ is a strand with $\alpha = \alpha_{\nearrow}(i)$ for some crossing i on $\partial_{\downarrow}\mathtt{JC}(j)$. Let β be a strand in $C^{\uparrow}(j)$. If β starts below α, then β crosses above α at some index between i and j. Otherwise, β stays above α for all $0 \leqslant k \leqslant \ell$.*

Proposition VII.22. *Let $\pi = v_0, \cdots, v_{m+1}$ be a $\overline{\mathtt{JC}(j)}$-path component, and suppose that $\epsilon = -1$. Suppose that the edge (v_0, v_1) is not an edge of $\alpha_{\nearrow}(i)$. Then $deg_{X_j}(\pi) = 0$.*

Proof. The edge (v_0, v_1) must be either a bridge with upper vertex on $\partial_{\downarrow}\mathtt{JC}(j)$ or an inclined edge crossing $\partial_{\downarrow}\mathtt{JC}(j)$ traveling northwest. In particular, the edge (v_0, v_1) is an edge of a crossing i where the strand $\alpha_{\nearrow}(i)$ is a lower boundary strand for $\overline{\mathtt{JC}(j)}$ with right endpoint in the interval $[h_k + 1, n]$, and the strand $\alpha_{\searrow}(i)$ has right endpoint in the interval $[1, h_k]$. Suppose that the crossing i is a bridge. Since (v_0, v_1) is not on the strand $\alpha_{\nearrow}(j)$, we have $\alpha_{\nearrow}(i) \neq \alpha_{\nearrow}(j)$. Suppose that $\beta \in C^{\uparrow}(j)$. We claim that either β is oriented left-to-right, or there is no directed path from the edge (v_0, v_1) to β in $\overline{\mathtt{JC}(j)}$. By Corollary VII.21, if β crosses a portion of $\partial_{\downarrow}\mathtt{JC}(j)$ above and to the left of the crossing i, then β must be oriented left-to-right.

Suppose that the edge (v_0, v_1) is an inclined edge corresponding to the crossing $i \in J^{+}_{\mathbf{u}_+}$. If $\alpha_{\nearrow}(i) \neq \alpha_{\nearrow}(j)$, the same argument shows that every strand $\beta \in C^{\uparrow}(j)$ is either directed left-to-right or cannot be reached by a $\overline{\mathtt{JC}(j)}$-path component beginning with the edge (v_0, v_1).

Suppose that $\alpha_{\nearrow}(i) = \alpha_{\nearrow}(j)$. We note that every strand in $C^{\uparrow}(j)$ starts above $\alpha_{\nearrow}(j)$. By definition of $C^{\uparrow}(j)$, if $\beta \in C^{\uparrow}(j)$ then β does not cross below $\alpha_{\nearrow}(j)$ to the left of j. By Lemma III.20, if $\beta \notin \{\alpha_{\searrow}(j), \alpha_{\nearrow}(j^{-}_{\searrow})\}$, then β stays above $\alpha_{\nearrow}(j)$. By Proposition VII.16, there is some connected left-to-right path in the wiring diagram traveling up and to the right from the vertex v_0; by Corollary III.21, this path cannot cross the strand $\alpha_{\searrow}(j)$ and hence cannot cross $\alpha_{\nearrow}(j^{-}_{\searrow})$. Hence every strand in $C^{\uparrow}(j)$ is oriented left-to-right, and so the $\overline{\texttt{JC}(j)}$-path component π has X_j-degree $\epsilon + 1 = 0$. $\square$

Proposition VII.23. *Suppose that every strand in the set $C^{\uparrow}(j) \setminus \{\alpha_{\nearrow}(j)\} \cup \{\alpha_{\searrow}(j)\}$ is oriented right-to-left and every strand in the set $C^{\downarrow}(j) \setminus \{\alpha_{\searrow}(j), \alpha_{\nearrow}(j^{-}_{\searrow})\}$ is oriented left-to-right. Then the following hold.*

1. *Exactly one of the edges $j^{-}_{\nearrow}$ and $e^{\uparrow}_{j}$ is oriented toward $\partial\,\texttt{JC}(j)$.*
2. *Exactly one of the edges $j^{-}_{\searrow}$ and ${}^{\downarrow}_{j}e$ is oriented toward $\partial\,\texttt{JC}(j)$.*
3. *Writing v_1 and v_m for the endpoints of these edges which are on $\partial\,\texttt{JC}(j)$, the vertices v_1 and v_m partition $\partial\,\texttt{JC}(j)$ into a clockwise arc and a counterclockwise arc.*
4. *No path from v_1 to v_m that stays inside $\overline{\texttt{JC}(j)}$ goes through the interior of $\texttt{JC}(j)$.*

Proof. 1. Since $\alpha_{\nearrow}(j^{-}_{\nearrow}) = \alpha_{\nearrow}(j)$ and $\alpha_{\searrow}(j^{-}_{\nearrow}) \in C^{\downarrow}(j)$, the bridge $j^{-}_{\nearrow}$ is directed up and toward $\partial_{\downarrow}\texttt{JC}(j)$ if and only if $\alpha_{\nearrow}(j)$ is oriented left-to-right, while $e^{\uparrow}_{j}$ is oriented toward $\partial\texttt{JC}(j)$ if and only if $\alpha_{\nearrow}(j)$ is oriented right-to-left.

2. Suppose that $j^{-}_{\searrow} \neq 0$ so that it indexes a bridge. Since the strand $\alpha_{\searrow}(j^{-}_{\searrow}) = \alpha_{\searrow}(j)$ is oriented right-to-left, the bridge $j^{-}_{\searrow}$ is directed up and away from $\partial_{\uparrow}\texttt{JC}(j)$ if and only if the strand $\alpha_{\nearrow}(j^{-}_{\searrow})$ is oriented left-to-right (so that ${}^{\downarrow}_{j}e$ is oriented toward $\partial\texttt{JC}(j)$). If $j^{-}_{\searrow} = 0$, then there is no bridge $j^{-}_{\searrow}$, while ${}^{\downarrow}_{j}e$ is an edge of $\alpha_{\searrow}(j)$ and hence oriented right-to-left and away from $\partial\texttt{JC}(j)$.

3. By Proposition IV.51, if $i \neq j$ is a bridge on $\partial\texttt{JC}(j)$, then $\alpha_{\nearrow}(i) \in C^{\uparrow}(j)$ and $\alpha_{\searrow}(j) \in C^{\downarrow}(j)$; if i is a $\partial\texttt{JC}(j)$-incident bridge that is not on $\partial\texttt{JC}(j)$, then either

$i = j^-_\searrow$ (and hence both $\alpha_\nearrow(i)$ and $\alpha_\searrow(i)$ are in $C^\uparrow(j)$) or both $\alpha_\nearrow(i)$ and $\alpha_\searrow(i)$ are in $C^\downarrow(j)$. Since a bridge i is oriented up if and only if $\alpha_\nearrow(i)$ is left-to-right and $\alpha_\searrow(i)$ is right-to-left, we have that the portion of $\partial\mathtt{JC}(j)$ not on the strands $\alpha_\nearrow(j)$, $\alpha_\searrow(j)$ or $\alpha_\nearrow(j^-_\searrow)$ is oriented clockwise. The boundary portion $\partial_\uparrow\mathtt{JC}(j) \cap \alpha_\searrow(j)$ is oriented counterclockwise, and $\partial_\uparrow\mathtt{JC}(j) \cap \alpha_\nearrow(j^-_\searrow)$ is counterclockwise if $\alpha_\nearrow(j^-_\searrow)$ is right-to-left, clockwise if $\alpha_\nearrow(j^-_\searrow)$ is left-to-right, while $\partial_\downarrow\mathtt{JC}(j) \cap \alpha_\nearrow(j)$ is counterclockwise if $\alpha_\nearrow(j)$ is left-to-right, clockwise if $\alpha_\nearrow(j)$ is right-to-left.

4. Suppose that e is an edge with one vertex on $\partial\mathtt{JC}(j)$ and one edge in the interior of $\mathtt{JC}(j)$. If e is a bridge i, then the strands $\alpha_\nearrow(i)$ and $\alpha_\searrow(i)$ are strands in $C^\downarrow(i) \setminus \{\alpha_\nearrow(j)\}$, so that they are both directed right-to-left and the bridge i is directed down. Since the boundary vertex is necessarily on $\partial_\downarrow\mathtt{JC}(j)$, the edge e is directed toward $\partial\mathtt{JC}(j)$. Similarly, if e is a horizontal edge on a strand $\alpha_\searrow(i)$ where i is a boundary bridge, then the strand $\alpha_\searrow(i) \in C^\downarrow(i)$ so that e is directed right-to-left toward $\partial\mathtt{JC}(j)$. If e is a horizontal edge on a strand $\alpha_\nearrow(i)$ where i is a boundary bridge, then there are no bridges on the strand $\alpha_\nearrow(i)$ to the right of i and in $\overline{\mathtt{JC}(j)}$.

$\square$

Proposition VII.24. *Let $\pi = v_0, \cdots, v_{m+1}$ be a $\overline{\mathtt{JC}(j)}$-path component and suppose that $deg_{X_j}(\pi) = -1$. Then the following hold.*

1. *The edge (v_0, v_1) is on the strand $\alpha_\nearrow(j)$ so that either (v_0, v_1) is the bridge $j^-_\nearrow$, directed up, or (v_0, v_1) is the horizontal edge at height $h_j + 1$ immediately to the right of j.*

2. *The edge (v_m, v_{m+1}) is either the bridge $j^-_\searrow$, directed up, or the horizontal edge below and to the left of the upper left corner of $\overline{\mathtt{JC}(j)}$.*

3. *The path π is a $\partial\,\mathtt{JC}(j)$-path component, and there is a $\partial\,\mathtt{JC}(j)$-path component $\pi' = v'_0, \cdots, v'_{m'+1}$ with $(v'_0, v'_1) = (v_0, v_1)$ and $(v'_{m'}, v'_{m'+1}) = (v_m, v_{m+1})$ which follows the complementary arc of $\partial\,\mathtt{JC}(j)$.*

Proof. 1. Since $\deg_{X_j}(\pi) = -1$, the path π must have $\epsilon = -1$, and hence either the first edge (v_0, v_1) is on the strand $\alpha_{\nearrow}(j)$ or it is a bridge directed to $\partial_{\downarrow}\mathtt{JC}(j)$ or inclined edge traveling northwest over $\partial_{\downarrow}\mathtt{JC}(j)$. By Proposition VII.22, (v_0, v_1) must be on $\alpha_{\nearrow}(j)$.

2. Since π must have X_j-degree ϵ, either the edge (v_m, v_{m+1}) is on a right-to-left strand $\alpha \in C^{\uparrow}(j)$, or (v_m, v_{m+1}) crosses $\partial_{\uparrow}\mathtt{JC}(j)$ by traveling northeast over the strand $\alpha_{\searrow}(j)$ at some crossing $i \in J^{+}_{\mathbf{u}_+}$. We claim that if $\alpha \notin \{\alpha_{\searrow}(j), \alpha_{\nearrow}(j^{-}_{\searrow})\}$, then α is oriented left-to-right in $\widehat{\mathcal{G}}^{\mathbf{w}}_k$.

   If (v_0, v_1) is the bridge $j^{-}_{\nearrow}$, then $\alpha_{\nearrow}(j)$ is directed left-to-right. If(v_0, v_1) is the horizontal edge e to the right of j, then by Proposition VII.16, there is a connected left-to-right path in the wiring diagram traveling from e to a target vertex ρ_h where $h \geqslant h_k{+}1$. By Corollary III.21, if $\alpha \in C^{\uparrow}(j)$ and $\alpha \notin \{\alpha_{\searrow}(j), \alpha_{\nearrow}(j^{-}_{\searrow})\}$, then the strand α must end above height $h_k + 1$ and so α is oriented left-to-right. So (v_m, v_{m+1}) is either an edge on one of the strands $\{\alpha_{\searrow}(j), \alpha_{\nearrow}(j^{-}_{\searrow})\}$ or an inclined edge crossing $\alpha_{\searrow}(j)$ traveling northeast at a crossing $i \in J^{+}_{\mathbf{u}_+}$. By Corollary VII.14, since every augmenting path containing π must eventually turn left, there must be an upward oriented bridge on the strand $\alpha_{\nearrow}(i)$ to the right of $\alpha_{\searrow}(j)$, so that there is a connected left-to-right path in the wiring diagram from the strand segment corresponding to the edge (v_m, v_{m+1}) to a target vertex ρ_h with $h \leqslant h_k$. But this path must cross below the left-to-right path from $\alpha_{\searrow}(j)$ ending in the interval $[h_k + 1, n]$, contradicting Corollary III.21. Hence, the edge (v_m, v_{m+1}) must be on one of the strands $\{\alpha_{\searrow}(j), \alpha_{\nearrow}(j^{-}_{\searrow})\}$.

3. We note that $k \geqslant j$, since the strand $\alpha_{\nearrow}(j)$ or some strand crossing above it to the right of j ends at height $\geqslant h_k + 1$ while the strand $\alpha_{\searrow}(j)$ and possibly the strand $\alpha_{\nearrow}(j^-_{\searrow})$ ends at height $\leqslant h_k$. The claim follows from Proposition VII.23.

□

Corollary VII.25.

$$\sum_{\mathcal{F}:\lambda_a \to \lambda_b} \omega(\mathcal{F}) = P_1 X'_j + P_2,$$

where $P_1 \in \mathbb{C}[X_i^{\pm} : i \neq j]$ *and* $P_2 \in \mathbb{C}[X_i^{\pm} : i \neq j][X_j]$.

Proof. If every augmenting path from λ_a to λ_b has nonnegative degree in X_j, then this is clear. Otherwise, fix some $\overline{\mathtt{JC}(j)}$-path component $\pi = v_0, \cdots, v_{m+1}$ of degree -1 in X_j, and let π' be the $\overline{\mathtt{JC}(j)}$-path component following the complementary arc of $\partial \mathtt{JC}(j)$. By Proposition VII.24, every $\overline{\mathtt{JC}(j)}$-path component of degree -1 in X_j is either π or π' and hence has starting vertex v_0 and ending vertex v_{m+1}. Since $\widehat{\mathcal{G}}_k^{\mathbf{w}}$ is acyclic, an augmenting path $\mathcal{F}$ can contain at most one $\overline{\mathtt{JC}(j)}$-path component of degree -1, and hence $\deg_{X_j}(\mathcal{F}) \geqslant -1$. So if $\mathcal{F} : \lambda_a \to \lambda_b$ is an augmenting path with $\deg_{X_j}(\mathcal{F}) = -1$, then there is an augmenting path $\mathcal{F}' \neq \mathcal{F}$ obtained from $\mathcal{F}$ by replacing π with π' or vice versa, and we have $\omega(\mathcal{F}) + \omega(\mathcal{F}') = M\frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j}$ where M is some Laurent monomial in the variables $\{X_i : i \neq j\}$. Pairing augmenting paths of degree -1 in X_j gives $\sum_{\mathcal{F}:\lambda_a \to \lambda_b} \omega(\mathcal{F}) = P_1 X'_j + P_2$ for some $P_1 \in \mathbb{C}[X_i^{\pm} : i \neq j]$ and $P_2 \in \mathbb{C}[X_i^{\pm} : i \neq j][X_j]$. □

Lemma VII.26. *Let* $j \in J^{\circ}_{\mathbf{u}_+}$ *be an index with* $v^j_{(0)} = 1$. *On the Deodhar torus* $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, *the entries of the Schubert form* $\Upsilon^{u,w}$ *are Laurent polynomials in the variables* $\{X_i : i \neq j\} \cup \{X'_j\}$.

Proof. On $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, the entries of $\Upsilon^{u,w}$ are Laurent in the variables $\{X_i : i \in J^{\circ}_{\mathbf{u}_+}\}$ and $X'_j = \frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j}$. By Theorem VII.7, it suffices to show that if k is any index with

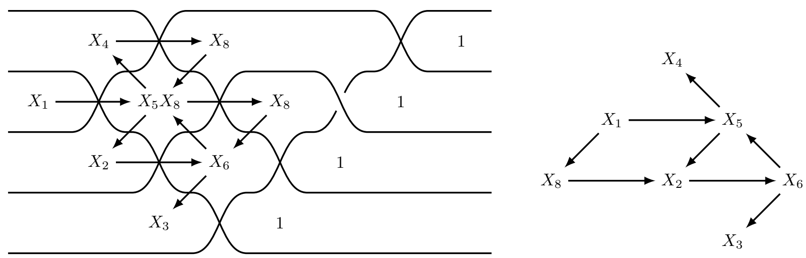

Figure 7.8: The chamber ansatz quiver and the induced Richardson quiver

$1 \leqslant k \leqslant \ell$, then writing $a = w_{(k)}(h_k)$ and $b = w_{(k)}(h_k + 1)$, the ratio of minors $\dfrac{\Delta^{w_{(k)}([1,h_k])\setminus\{b\}\cup\{a\}}_{[1,h_k]}(g_{(k)})}{\Delta^{w_{(k)}([1,h_k])}_{[1,h_k]}(g_{(k)})} = \sum\limits_{\substack{\mathcal{F}:\lambda_a\to\lambda_b \\ \text{in } \widehat{\mathcal{G}}_k^{\mathbf{w}}}} \omega(\mathcal{F})$ is Laurent in $\{X_i : i \neq j\} \cup \{X'_j\}$. Apply Corollary VII.25 for each $1 \leqslant k \leqslant \ell$. $\square$

7.4 Mutated variables as terms in the special chamber minors

In the previous section, we showed that if $j \in J^\circ_{\mathbf{u}_+}$ is an index so that the function $X_j = \Delta^{\widehat{R}_j}_{\widehat{C}_j} \in \mathcal{R}^{u,w}$ vanishes on a Deodhar hypersurface $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$, the Schubert coordinate functions are Laurent in the variables $\{X_i : i \neq j\} \cup \{X'_j = \frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j}\}$ on the torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$. It remains to show that X'_j extends to a regular function on $\mathcal{R}^{u,w}$. That is, we have shown that if there are terms in the expansion of the Schubert coordinates on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ which have negative degree in X_j, then they are unique up to multiplication by Laurent polynomials in the variables $\{X_i : i \neq j\}$, but we have not yet shown that such terms exist. We will need to show that there is a globally regular function Δ so that on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, the coefficient of X'_j in the expansion of Δ is a nonzero Laurent *monomial* in the $\{X_i : i \neq j\}$.

Example VII.27. Let $\mathbf{w} = s_3s_2s_1s_4s_3s_2\underline{s_3}s_4$. Figure 7.8 shows the chamber ansatz quiver and induced Richardson quiver. The chamber weights are minors of the upper triangular matrix

$$z = \Upsilon^{u,w}\dot{w}^{-1} = \begin{pmatrix} X_3 & \frac{X_2+X_3X_5}{X_6} & -\frac{(X_1+X_2)X_3}{X_2X_8} & 1 & -\frac{X_2X_4+X_3X_5X_4+X_1X_6}{X_4X_5X_6} \\ 0 & \frac{X_2}{X_3} & 0 & \frac{X_6}{X_3} & -\frac{X_2X_4+X_1X_6}{X_3X_4X_5} \\ 0 & 0 & \frac{X_1}{X_2} & \frac{X_5X_8}{X_2} & -\frac{X_1X_8}{X_2X_4} \\ 0 & 0 & 0 & \frac{X_4}{X_1} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{X_4} \end{pmatrix}.$$

The exchange relations for the mutable variables are $X_5X_5' = X_2X_4+X_1X_6$, $X_6X_6' = X_2 + X_3X_5$, and $X_8X_8' = X_1 + X_2$. In the following table, we compute the special minor $\Delta^{v_{(j)}^{j-1}([1,h_j])}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(j))}$ for the Deodhar boundary stratum $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$ for each mutable variable X_j.

X_j	X_j'	$\Delta^{v_{(j)}^{j-1}([1,h_j])}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(j))}$
X_5	$\frac{X_2X_4+X_1X_6}{X_5}$	$\Delta^{124}_{145} = \frac{X_2X_4+X_1X_6}{X_1X_5}$
X_6	$\frac{X_2+X_3X_5}{X_6}$	$\Delta^{13}_{45} = \frac{(X_2+X_3X_5)X_8}{X_2X_6}$
X_8	$\frac{X_1+X_2}{X_8}$	$\Delta^{1245}_{2345} = \frac{X_1+X_2}{X_1X_8}$

In this particular case, each of the minors $\Delta^{v_{(j)}^{j-1}([1,h_j])}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(j))}$ has the form MX_j', where M is a ratio of variables $\{X_k : k \neq j\}$. In general, $\Delta^{v_{(j)}^{j-1}([1,h_j])}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(j))} = MX_j' + P$ for some $P \in \mathbb{C}[X_k^{\pm} : k \neq j][X_j]$.

In this section, we will use augmenting paths in the graph $\widehat{\mathcal{G}}_j^{\mathbf{u}_+}$ to show that the function $X_j' \in \mathbb{C}[\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}]$ has an alternate formula which is regular on a dense open subset of $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$. We will show that on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, a multiple of $X_j' = \frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j}$ appears in the expansion of the minor $\Delta^{v_{(j-1)}^{j}[1,h_j]}_{\lambda(\mathbf{w})(\text{え}_{\rightarrow}(j))}$, which is a *special chamber minor* in Marsh and Rietsch's Chamber Ansatz for the Deodhar hypersurface $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$. At the end of the chapter, we will use specialized chamber weightings to show the following, thus verifying the last unchecked conditions for the Starfish Lemma.

1. If X_i and X_j are distinct mutable variables in the initial cluster, then the locus $\{X_i = X_j = 0\}$ has complex codimension $\geqslant 2$.

2. If X_j is a mutable variable, then the function X'_j is globally regular on $\mathcal{R}^{u,w}$.

3. If X_i and X_j are mutable variables in the initial cluster, then the locus $\{X_i = X'_j = 0\}$ has complex codimension $\geqslant 2$.

We begin with a roadmap of our proof strategy.

By Proposition VI.5, since $v^j_{(j-1)}([1,h_j]) = u_{(j)}([1,h_j])\backslash\{r_*\} \cup \{r^*\}$ and $\text{ㅊ}_{\rightarrow}(j)$ is the chamber above the strands with right endpoints $[1,h_j]$ in the oriented bridge graph $\mathcal{G}_j$, the special chamber minor $\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}$ is given by

$$\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))} = \mathrm{Q}(\text{ㅊ}_{\rightarrow}(j)) \sum_{\mathcal{F}:\lambda_{r^*}\rightarrow\lambda_{r_*}} \omega(\mathcal{F}),$$

where the sum is over augmenting paths in the directed graph $\widehat{\mathcal{G}}^{\mathbf{u}_+}_j$. Since the chamber $\text{ㅊ}_{\rightarrow}(j)$ is not in $\mathtt{JC}(j)$, $\mathrm{Q}(\text{ㅊ}_{\rightarrow}(j))$ has degree 0 in X_j so that the path weight $\omega(\mathcal{F})$ and the corresponding term $\mathrm{Q}(\text{ㅊ}_{\rightarrow}(j))\omega(\mathcal{F})$ in the expansion of $\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))}$ have the same degree in X_j.

We will show that there are distinguished augmenting paths $\mathcal{F}_L$ and $\mathcal{F}_R$ which have degree -1 in X_j and differ by a cycle around $\partial\mathtt{JC}(j)$, with $\mathcal{F}_L$ following the left portion of $\partial\mathtt{JC}(j)$ and $\mathcal{F}_R$ following the right portion of $\partial\mathtt{JC}(j)$. For any augmenting path $\mathcal{F} \neq \mathcal{F}_R$, we will show that $\mathcal{F}$ stays weakly to the left of the $\mathcal{F}_L$ and that the degree difference $\deg_{X_j}(\mathcal{F}) - \deg_{X_j}(\mathcal{F}_L)$ is the number of times that $\mathcal{F}$ leaves $\mathcal{F}_L$. Since the vertices λ_{r^*} and λ_{r_*} have valence 1, every augmenting path from λ_{r^*} to λ_{r_*} has the same first and last edges. Hence, if $\mathcal{F}$ is an augmenting path other than $\mathcal{F}_L$ and $\mathcal{F}_R$ then $\deg_{X_j}(\mathcal{F})$ is nonnegative.

This shows that the remainder term $\displaystyle\sum_{\substack{\mathcal{F}:\lambda_{r^*}\rightarrow\lambda_{r_*}\\ \mathcal{F}\neq\mathcal{F}_L,\mathcal{F}_R}} \omega(\mathcal{F})$ is polynomial in X_j.

There are several important consequences of the fact that the graph $\widehat{\mathcal{G}}_j^{\mathbf{u}_+}$ is obtained from $\mathcal{G}_j$ by reversing edges along the flow $\mathcal{F}_j$, which follows the geodesic paths γ_r where $r \in u_{(j-1)}([1, h_j])$, and in particular uses no bridges of $\mathcal{G}_j$.

Like paths in the oriented bridge diagram $\mathcal{G}_j$, directed paths π in $\mathcal{G}_j$ always go from higher-indexed geodesic paths to lower-indexed geodesic paths.

Proposition VII.28. *Let π be a directed path in $\widehat{\mathcal{G}}_j^{\mathbf{u}_+}$. Let $e_1, \cdots, e_m$ be the subsequence of non-bridge edges of π and write γ_{r_d} for the geodesic path containing the edge e_d. Then the sequence $r_1, \cdots, r_m$ is weakly decreasing.*

Proof. The path π travels from one geodesic path to another exactly when it follows a bridge i. Since $\mathbf{u}_+$ is a positive subexpression of $\mathbf{w}$, whenever i is a bridge from γ_r to $\gamma_{r'}$with γ_r above $\gamma_{r'}$, we have $r > r'$. All bridges in $\widehat{\mathcal{G}}_j^{\mathbf{u}_+}$ are oriented down, so that whenever π uses a bridge i the index of the geodesic path it follows decreases. $\square$

Proposition VII.29. *Let v be a vertex in $\widehat{\mathcal{G}}_k^{\mathbf{u}_+}$ for some k, and suppose that v is not a vertex of the form λ_h or $\rho_{h'}$. If v is the upper vertex of a bridge, it has one inbound edge and two outbound edges, while if v is the lower vertex of a bridge, it has two inbound edges and one outbound edge. Otherwise, v has one inbound edge and one outbound edge.*

Proof. Every vertex not of the form λ_h or $\rho_{h'}$ is incident to two edges on the same geodesic path and possibly one bridge. Edges on a single geodesic path are either oriented left-to-right or right-to-left, while all bridges are oriented down. $\square$

Proposition VII.30. *Let $\mathcal{F} : \lambda_{r^*} \to \lambda_{r_*}$ be an augmenting path in $\widehat{\mathcal{G}}_j^{\mathbf{u}_+}$. Then $\mathcal{F} \cap \overline{\mathtt{JC}(j)} \subseteq \partial\,\mathtt{JC}(j)$.*

Proof. Assume for contradiction that the augmenting path $\mathcal{F}$ crosses $\partial\mathtt{JC}(j)$, and let e be the first edge of $\mathcal{F}$ containing an interior vertex of $\mathtt{JC}(j)$. Then e cannot be a bridge, since all bridges with one endpoint on $\partial\texttt{JC}(j)$ and one endpoint in the interior of $\texttt{JC}(j)$ are oriented down toward $\partial_\downarrow\texttt{JC}(j)$.

Similarly, all inclined edges crossing the upper right boundary are oriented from the interior of $\texttt{JC}(j)$ to the outside, and inclined edges crossing the lower right boundary are inaccessible from the left.

So e must be a left-to-right edge crossing the left boundary.

□

Proposition VII.31. *Consider the augmenting paths from λ_{r^*} to λ_{r_*} in $\widehat{\mathcal{G}}_j^{\mathbf{u}_+}$. There are two distinguished augmenting paths $\mathcal{F}_R$ and $\mathcal{F}_L$, so that the following hold.*

1. *$\mathcal{F}_R$ travels clockwise around the right boundary of $\texttt{JC}(j)$ and $\mathcal{F}_L$ travels counterclockwise around the left boundary of $\texttt{JC}(j)$.*
2. *$\omega(\mathcal{F}_R) = t_j$ and $\omega(\mathcal{F}_L) = \widehat{Y}_j t_j$.*
3. *If $\mathcal{F} : \lambda_{r^*} \to \lambda_{r_*}$ is an augmenting path other than $\mathcal{F}_R$ and $\mathcal{F}_L$, then $\mathcal{F}$ stays weakly to the left of $\mathcal{F}_L$.*

Proof. Since $r^* = u_{(j)}(h_j + 1)$ and $r_* = u_{(j)}(h_j)$, the geodesic path γ_{r^*} is directed left-to-right and the geodesic path γ_{r_*} is directed right-to-left, and the bridge j is directed down from to the geodesic path γ_{r_*}. Take $\mathcal{F}_R$ to be the path that follows γ_{r^*} until reaching bridge j and then follows γ_{r_*} back to λ_{r_*}. The weight of $\mathcal{F}_R$ is $\omega(i) = t_i$.

By Proposition IV.47, if γ_r is an upper boundary geodesic of $\overline{\texttt{JC}(j)}$, then $r \in v^j_{(j-1)}([1, h_j])$, while if γ_r is a lower boundary geodesic, then $r \in v^j_{(j-1)}([h_j + 1, n])$. Since $v^j_{(j-1)} = u_{(j-1)} s_{h_j}$, upper boundary geodesics other than γ_{r^*} are oriented right-to-left while lower boundary geodesics other than γ_{r_*} are oriented left-to-right. Since we also have that bridges are oriented down, it follows that the left boundary of $\overline{\mathtt{JC}(j)}$ is oriented counterclockwise, so that there is an augmenting path $\mathcal{F}_L$ which differs from $\mathcal{F}_R$ by traveling counterclockwise around $\overline{\mathtt{JC}(j)}$ rather than clockwise. So the ratio $\frac{\omega(\mathcal{F}_L)}{\omega(\mathcal{F}_R)}$ is the weight of a counterclockwise cycle around $\overline{\mathtt{JC}(j)}$ and hence equal to $\widehat{Y}_j$, and so $\omega(\mathcal{F}_L) = \widehat{Y}_j t_j$.

Suppose that $\mathcal{F} : \lambda_{r^*} \to \lambda_{r_*}$ is an augmenting path with $\mathcal{F} \neq \mathcal{F}_R$. By Proposition VII.30, $\mathcal{F} \cap \overline{\mathtt{JC}(j)} \subseteq \partial\mathtt{JC}(j)$; since an augmenting path that uses an edge of the right boundary must agree with $\mathcal{F}_R$, the edges of $\mathcal{F} \cap \overline{\mathtt{JC}(j)}$ must consist of left boundary edges. So $\mathcal{F}$ stays weakly to the left of the left boundary of $\mathtt{JC}(j)$, hence weakly to the left of $\mathcal{F}_L$. □

Proposition VII.32. *Let $\mathcal{F} : \lambda_{r^*} \to \lambda_{r_*}$ be an augmenting path that stays weakly to the left of $\mathcal{F}_L$. If e_1 and e_2 are consecutive edges of $\mathcal{F}$ so that $e_1 \notin \mathcal{F}_L$ and $e_2 \in \mathcal{F}_L$, then one of the following holds.*

1. *The edges e_1 and e_2 are horizontal edges on a lower boundary geodesic γ_r for some $r_* < r < r^*$, with $e_2 \in \partial_\downarrow \mathtt{JC}(j)$.*
2. *The edge e_2 is a horizontal edge on γ_{r_*} and e_1 is a bridge, with e_2 vertex-disjoint from $\overline{\mathtt{JC}(j)}$. The edge e_1 is either vertex disjoint from $\overline{\mathtt{JC}(j)}$ or has X_j-degree 1.*

Proof. Let e_1' be the edge of $\mathcal{F}_L$ preceding e_2, where by hypothesis $e_1' \neq e_1$. Since the edges e_1, e_1' and e_2 intersect at a trivalent vertex v, one of them must be a bridge and the others horizontal edges on the same geodesic path. Since the paths $\mathcal{F}$ and $\mathcal{F}'$ converge, by Proposition VII.29 the bridge must be one of the edges e_1 or e_1'. So the edge e_2 is horizontal. By Proposition VII.28, $\mathcal{F}$ cannot leave and rejoin γ_{r^*}, so $e_2 \notin \gamma_{r^*}$. Since $e_2 \in \mathcal{F}_L$, it can either be on $\partial\mathtt{JC}(j)$ or on γ_{r_*}. By Corollary IV.47, every lower boundary geodesic γ_r satisfies $r \in R^\uparrow(j)$ and every upper boundary geodesic γ_r satisfies $r \in R^\downarrow(j)$. So every lower boundary geodesic except γ_{r_*} is oriented left-to-right and every upper boundary geodesic except γ_{r^*} is oriented right-to-left. The path $\mathcal{F}$ stays weakly to the left of $\mathcal{F}_L$, so if e_1 is a horizontal edge it must be directed left-to-right, and hence it must be an edge on a lower boundary geodesic γ_r where $r_* < r < r^*$. Since $r \neq r_*$, the edge e_2 must be on $\partial\mathtt{JC}(j)$. If the edge e_1 is a bridge, it must either be directed toward $\partial\mathtt{JC}(j)$ or toward γ_{r_*}. Every bridge incident to $\partial_\uparrow\mathtt{JC}(j)$ and weakly below γ_{r^*} is on $\partial\mathtt{JC}(j)$. If i is a bridge with lower endpoint on $\partial_\downarrow\mathtt{JC}(j)$, then either i is on $\partial\mathtt{JC}(j)$ or the upper endpoint of i is an interior vertex of $\mathtt{JC}(j)$. Since $\mathcal{F}$ cannot go through the interior of $\mathtt{JC}(j)$, if the edge e_1 is a bridge then e_2 must be an edge on γ_{r_*}. Since $\mathcal{F}_L$ leaves $\partial\mathtt{JC}(j)$ as soon as it reaches γ_{r_*}, the condition that $\mathcal{F}$ stays weakly to the left of $\mathcal{F}_L$ implies that the edge e_2 is vertex-disjoint from $\overline{\mathtt{JC}(j)}$. We note that if e_1 is a bridge with upper endpoint on $\partial\mathtt{JC}(j)$ and lower endpoint on γ_{r_*} but not on $\partial\mathtt{JC}(j)$, then e_1 is directed away from $\partial_\downarrow\mathtt{JC}(j)$ and hence has X_j-degree 1. $\square$

Proposition VII.33. *Let $\mathcal{F} : \lambda_{r^*} \to \lambda_{r_*}$ be an augmenting path that stays weakly to the left of $\mathcal{F}_L$. If e_1 and e_2 are consecutive edges of $\mathcal{F}$ so that $e_1 \in \partial\mathtt{JC}(j)$ and $e_2 \notin \mathcal{F}_L$, then one of the following holds.*

1. *The edge e_1 is a horizontal edge on γ_{r^*} and e_2 is a bridge, with e_1 and e_2 vertex-disjoint from $\overline{\mathtt{JC}(j)}$.*
2. *The edges e_1 and e_2 are horizontal edges on an upper boundary geodesic γ_r for some $r_* < r < r^*$, with $e_1 \in \partial_\uparrow\mathtt{JC}(j)$.*
3. *The edge e_1 is a horizontal edge on a lower boundary geodesic γ_r for some $r_* < r < r^*$ and e_2 is a bridge with lower endpoint outside $\overline{\mathtt{JC}(j)}$.*

Proof. Write e_2' for the edge following e_1 in $\mathcal{F}_L$. The edges e_1, e_2 and e_2' must intersect at a trivalent vertex v with one inbound edge and two outbound edges, so that the edge e_1 is a horizontal edge on some geodesic path γ_r and one of the edges e_2, e_2' is the following edge on γ_r while the other is a bridge with upper vertex on γ_r. By Proposition VII.28, the edge e_1 cannot be on γ_{r_*} since an augmenting path that leaves γ_{r_*} cannot rejoin it. Since $\mathcal{F}$ stays weakly to the left of $\mathcal{F}_L$, the edges e_2 and e_2' must both be weakly to the left of $\overline{\mathtt{JC}(j)}$. The leftmost bridge from γ_{r^*} to a vertex in $\overline{\mathtt{JC}(j)}$ is on the path $\mathcal{F}_L$, so if e_2 is a bridge leaving γ_{r^*} it must be vertex disjoint from $\overline{\mathtt{JC}(j)}$. If the edge e_1 is not on γ_{r^*}, then $e_1 \in \partial\mathtt{JC}(j)$. The geodesic path γ_r containing e_1 must satisfy $r_* < r < r^*$. So if γ_r is an upper boundary geodesic then it is oriented right-to-left, while if γ_r is a lower boundary geodesic then it is oriented left-to-right. We note that the leftmost edge leaving the vertex v is the horizontal edge to the left of v if γ_r is oriented right-to-left and the bridge if γ_r is oriented left-to-right. In the latter case, the lower endpoint of the bridge e_2 must be outside $\overline{\mathtt{JC}(j)}$ because the upper vertex is on $\partial\mathtt{JC}(j)$ and the region $\overline{\mathtt{JC}(j)}$ is simply connected. $\square$

Corollary VII.34. *Let $\mathcal{F} = v_1, \cdots, v_{|\mathcal{F}|}$ be an augmenting path from λ_{r^*} to λ_{r_*}. Suppose that for some m and m' with $m < m'$ the path $\mathcal{F}$ leaves $\mathcal{F}_L$ via the edge (v_m, v_{m+1}) and rejoins $\mathcal{F}_L$ via the edge $(v_{m'}, v_{m'+1})$, so that $v_m \in \mathcal{F}_L$ and $v_{m+1} \notin \mathcal{F}_L$ while $v_{m'} \notin \mathcal{F}_L$ and $v_{m'+1} \in \mathcal{F}_L$. Let γ_r be the geodesic path containing v_m and let $\gamma_{r'}$ be the geodesic path containing $v_{m'+1}$. Then $r' < r$.*

Proof. From Proposition VII.33, either the edge (v_m, v_{m+1}) is a bridge oriented down, so that the index of the geodesic path followed by $\mathcal{F}$ decreases, or $\mathcal{F}$ travels left at a horizontal edge while $\mathcal{F}_L$ travels down a bridge to a lower-indexed geodesic. Since the indices of the geodesics followed by $\mathcal{F}$ and $\mathcal{F}_L$ respectively are weakly decreasing, we must have $r' < r$. $\square$

In the following proposition, we combine the previous results to show that if $\mathcal{F} : \lambda_{r^*} \to \lambda_{r_*}$ is an augmenting path in $\widehat{\mathcal{G}}_j^{\mathbf{u}+}$ that stays weakly to the left of $\mathcal{F}_L$, then enumerating the edges of $\mathcal{F}$ as $\mathcal{F} = e_1, \cdots, e_{|\mathcal{F}|}$, we have $\deg_{X_j}(\mathcal{F}) = -1 + \#\{d : e_{d-1} \in \mathcal{F}_L, e_d \notin \mathcal{F}_L\}$, so that the weight of every augmenting path other than $\mathcal{F}_L$ and $\mathcal{F}_R$ is polynomial in X_j.

Proposition VII.35. *Let $\mathcal{F} : \lambda_{r^*} :\to \lambda_{r_*}$ be an augmenting path in $\widehat{\mathcal{G}}_j^{\mathbf{u}+}$. Suppose that $\mathcal{F}$ is not one of the distinguished augmenting paths $\mathcal{F}_L$ and $\mathcal{F}_R$. Then $\deg_{X_j}(\mathcal{F})$ is nonnegative.*

Proof. Write the edges of $\mathcal{F}$ as $e_1, \cdots, e_{|\mathcal{F}|}$ and let d be the minimal index so that $e_d \notin \mathcal{F}_L$. We claim that the path $e_1, \cdots, e_d$ has degree 0 in X_j. The edges $e_1, \cdots, e_{d-1}$ are the first $d-1$ edges of $\mathcal{F}_L$. Consider the cases from Proposition VII.32. If the edge e_{d-1} is an edge of γ_{r^*} and e_2 is a bridge that is vertex-disjoint from $\overline{\mathtt{JC}(j)}$, then all of the edges $e_1, \cdots, e_d$ are vertex-disjoint from $\overline{\mathtt{JC}(j)}$ and hence have degree 0 in X_j. If e_{d-1} and e_d are horizontal edges on an upper boundary geodesic γ_r with $r < r^*$, then the path $e_1, \cdots, e_{d-1}$ has degree 0 in X_j. If e_{d-1} is a horizontal edge on a lower boundary geodesic γ_r and e_d is a bridge directed away from γ_r, then the path $e_1, \cdots, e_{d-1}$ has degree -1 and the bridge e_d has degree 1 so that $e_1, \cdots, e_d$ has degree 0.

Suppose now that m is an index so that $e_{m-1} \notin \mathcal{F}_L$ while $e_m \in \mathcal{F}_L$ and the path $e_1, \cdots, e_{m-1}$ has nonnegative degree in X_j. By Proposition VII.32, the edge e_m is a horizontal edge on a lower boundary geodesic γ_r (where r may equal r_*), so that the path of $\mathcal{F}_L$ ending with e_m has degree -1 and $\deg_{X_j}(e_1, \cdots, e_m) = \deg_{X_j}(e_1, \cdots, e_{m-1})$. In particular, if $\mathcal{F}$ continues to follow $\mathcal{F}_L$ for indices $m \leqslant i \leqslant m' - 1$, then $\deg_{X_j}(e_1, \cdots, e_{m'}) \geqslant \deg_{X_j}(e_1, \cdots, e_{m-1})$, with strict inequality if $e_{m'} \notin \mathcal{F}_L$. Inductively, if $\mathcal{F}$ is an augmenting path other than $\mathcal{F}_L$ and $\mathcal{F}_R$, then $\deg_{X_j}(\mathcal{F}) \geqslant 0$. $\square$

Proposition VII.36. *Let* ㅊ $=$ ㅊ$_{\to}(i)$ *be a chamber in* $\mathtt{JC}(j)$*. Let* $R = \mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}\right)$ *and write* $\mathit{Pivots}_{\rho^{j-1}(\mathbf{w})(\text{ㅊ})}\left(u_{(j-1)}\right) = R\backslash\{r_-\} \cup \{r^+\}$*. Then the minor* $\Delta^{R\backslash\{r_-\}\cup\{r^+\}}_{\lambda(\mathbf{w})(\text{ㅊ})}$ *has a term of degree* 0 *in* X_j*. Further, if* $r_a \in R$ *and* $r^b \notin R$ *are indices with* $r_- \leqslant r_a$ *and* $r^b \leqslant r^+$ *and at least one of the inequalities is strict, then every term of* $\Delta^{R\backslash\{r_a\}\cup\{r^b\}}_{\lambda(\mathbf{w})(\text{ㅊ})}$ *has positive degree in* X_j*.*

Proof. Consider the graph $\widehat{\mathcal{G}}_i^{\mathbf{u}_+}$. We have that $\Delta^{R\backslash\{r_a\}\cup\{r^b\}}_{\lambda(\mathbf{w})(\text{ㅊ})} = \mathrm{Q}(\text{ㅊ}) \sum_{\mathcal{F}:\lambda_{r^b}\to\lambda_{r_a}} \omega(\mathcal{F})$, where the sum is over augmenting paths in $\widehat{\mathcal{G}}_i^{\mathbf{u}_+}$. By hypothesis, $\mathrm{Q}(\text{ㅊ})$ has X_j-degree 1. By Marsh and Rietsch, if $\mathcal{F}$ is an augmenting path from λ_{r^b} to λ_{r_a}, then the path weight $\omega\mathcal{F}$ is nonnegative as a Laurent monomial in the X_k, so there is no cancellation of terms. So X_j divides the minor $\Delta^{R\backslash\{r_a\}\cup\{r^b\}}_{\lambda(\mathbf{w})(\text{ㅊ})}$ if and only if every augmenting path $\mathcal{F}$ from λ_{r^b} to λ_{r_a} satisfies $\deg_{X_j}(\mathcal{F}) \geqslant 0$.

We claim that there is at least one augmenting path from λ_{r^+} to λ_{r_-} with X_j-degree -1. In particular, we claim that there is an augmenting path $\mathcal{F}$ that follows γ_{r^+} until reaching $\overline{\mathtt{JC}(j)}$ and then travels clockwise along $\partial\mathtt{JC}(j)$ until joining γ_{r_-}. By Proposition VI.18, such a path $\mathcal{F}$ would have $\deg_{X_j}(\mathcal{F}) = -1$.

Let v^+ be the leftmost vertex on γ_{r^+} that is on $\partial\mathtt{JC}(j)$, and let v_- be the leftmost vertex of γ_{r_-} on $\partial\mathtt{JC}(j)$. We must show that between the vertices v^+ and v_-, every portion of $\partial_{\uparrow}\mathtt{JC}(j)$ is oriented right-to-left and every portion of $\partial_{\downarrow}\mathtt{JC}(j)$ is oriented left-to-right. The indices of the boundary geodesics along $\partial\mathtt{JC}(j)$ are weakly decreasing between γ_{r^*} and γ_{r_*}, so that if γ_r intersects $\partial\mathtt{JC}(j)$ between v^+ and v_- then $r_- < r < r^+$. Since r^+ must be the minimal index in $R^{\downarrow}(j)$ with γ_{r^+} above ㅊ, if γ_r is an upper boundary geodesic intersecting $\partial\mathtt{JC}(j)$ between v^+ and v_- then γ_r is below ㅊ, hence oriented right-to-left. Since r_- must be the maximal index in $R^{\uparrow}(j)$ with γ_{r_-} below ㅊ, if γ_r is a lower boundary geodesic intersecting $\partial\mathtt{JC}(j)$ between v^+ and v_-, γ_r is above ㅊ and hence oriented left to right. So $\partial\mathtt{JC}(j)$ is oriented clockwise between v^+ and v_- and hence there is an augmenting path $\mathcal{F}$ from λ_{r^+} to λ_{r_-} of X_j-degree -1.

Now suppose that r^b and r_a are indices with $r_a \in R$ and $r^b \notin R$ so that the inequalities $r_- \leqslant r_a$ and $r^b \leqslant r^+$ hold. Suppose that $\mathcal{F} : \lambda_{r^b} \to \lambda_{r_a}$ is an augmenting path in $\widehat{\mathcal{G}}_i^{\mathbf{u}_+}$. By positivity of $\mathbf{u}_+$, the path $\mathcal{F}$ cannot cross above γ_{r^+} and cannot cross below γ_{r_-}. Suppose that $\mathcal{F}$ has negative degree in X_j. Then there must be some segment $\pi = v_0, \cdots, v_{m+1}$ of $\mathcal{F}$ so that the vertices v_0 and v_{m+1} are not in $\overline{\mathtt{JC}(j)}$, all vertices $v_1, \cdots, v_m \in \overline{\mathtt{JC}(j)}$ and $\deg_{X_j}(\pi) = -1$. Since all bridges are oriented down in $\widehat{\mathcal{G}}_i^{\mathbf{u}_+}$, the edge (v_0, v_1) must be an edge on a geodesic path γ_r where $r \in R^{\downarrow}(j)$ and the edge (v_m, v_{m+1}) must be an edge on a geodesic path γ_r where $r \in R^{\uparrow}(j)$. In particular, the path π must meet or cross $\partial\mathtt{JC}(j)$ somewhere between the vertices v^+ and v_-. Since the vertices v_0 and v_{m+1} must be to the left of $\partial\mathtt{JC}(j)$, the edge (v_0, v_1) must be left-to-right and the edge (v_m, v_{m+1}) must be right-to-left. This implies that (v_0, v_1) is on γ_{r^+} and (v_m, v_{m+1}) is on γ_{r_-}. Since the indices of the geodesic paths used by $\mathcal{F}$ are weakly decreasing, we must have $r^b \geqslant r^+$ and $r_a \leqslant r_-$. Hence if either of the inequalities is strict, then every term in the expansion of $\Delta^{R\setminus\{r_a\}\cup\{r^b\}}_{\lambda(\mathbf{w})(\text{え})}$ has positive degree in X_j. $\square$

7.5 Regularity and coprimeness conditions

We are now ready to show that the locus where distinct variables X_i and X_j vanish has codimension $\geqslant 2$ in $\mathcal{R}^{u,w}$. We will explicitly construct a path in the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ where $X_j = -\epsilon$ for a parameter $\epsilon \in \mathbb{C}^*$ and the variables $\{X_i : i \neq j\}$ are constants with values ± 1 and verify that the limit of this path is a flag in the divisor $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$.

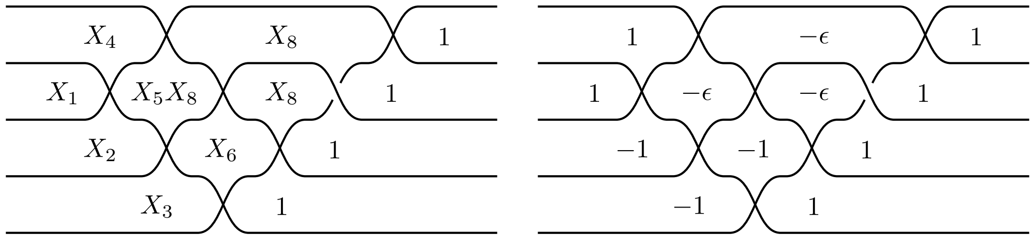

Figure 7.9: In the chamber weighting at right, the variables $(X_1, X_2, X_3, X_4, X_5, X_6, X_8)$ indexed by crossings in $J^\circ_{\mathbf{u}_+}$ specialize to $(1, -1, -1, 1, 1, -1, -\epsilon)$. Letting ϵ approach zero gives a path through the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ with limit in the boundary divisor $\mathcal{D}^{\mathbf{v^8},\mathbf{w}}$.

Lemma VII.37. *Let i and j be distinct indices in $J^\circ_{\mathbf{u}_+}$ with $v^j_{(0)} = 1$. Then the minor $\Delta^{\widehat{R}^i}_{\widehat{C}^i}$ does not vanish identically on the Deodhar stratum $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$.*

Proof. We will show that there is an element $F \in \mathcal{R}^{u,w}$ so that $\Delta^{\widehat{R}^j}_{\widehat{C}^j} = 0$ and for every $i \neq j$ we have $\Delta^{\widehat{R}^i}_{\widehat{C}^i} = \pm 1$. The Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ is an open subvariety of $\mathcal{R}^{u,w}$, so every assignment of nonzero values to the variables X_i gives an element of $\mathcal{R}^{u,w}$. We will construct a 1-parameter family of flags in $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ where we set $X_j = -\epsilon$ and fix the values of the other X_i as ± 1 in such a way that taking the limit as $\epsilon \to 0$ gives an element of $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$.

Specialize the chamber weighting as follows. If ㅊ is a chamber above $\alpha_{\nearrow}(j)$ and below $\alpha_{\searrow}(j)$, then the weighting of ㅊ is $-\epsilon^{\deg_{X_j}(\mathbf{Q}(\text{ㅊ}))}$. Otherwise, the weighting of ㅊ is 1. (See Figure 7.9.) For each $i \neq j$, writing $d = \deg_{X_j}(t_i)$ for the standard chamber weighting, the parameter t_i specializes to ϵ^d. The parameter t_j specializes to $-\frac{1}{\epsilon}$.

We have $X_j = \Delta^{\widehat{R}^j}_{\widehat{C}^j} = -\epsilon$, since

$$\Delta^{\widehat{R}^j}_{\widehat{C}^j} = \mathrm{Q}(\text{ㅊ}_{\leftarrow}(j)) \prod_{k>j:\widehat{C}^k_i \neq \widehat{C}^{k-1}_i} \frac{\mathrm{Q}(\text{ㅊ}_{\rightarrow}(k))}{\mathrm{Q}(\text{ㅊ}_{\uparrow}(k))} = -\epsilon \prod_{\substack{k>j \\ \widehat{C}^k_i \neq \widehat{C}^{k-1}_i}} 1.$$

Suppose that $i \in J^\circ_{\mathbf{u}_+}$ is an index with $i \neq j$. We have $\Delta^{\widehat{R}^i}_{\widehat{C}^i} = \mathrm{Q}(\text{ㅊ}_{\leftarrow}(i)) \prod_{\substack{k>i \\ \widehat{C}^k_i \neq \widehat{C}^{k-1}_i}} \frac{\mathrm{Q}(\text{ㅊ}_{\rightarrow}(k))}{\mathrm{Q}(\text{ㅊ}_{\uparrow}(k))}$.

Under this specialization, all factors in the product are either $\pm\epsilon, \pm\frac{1}{\epsilon}$ or ± 1. Since

$\Delta^{\widehat{R}^i}_{\widehat{C}^i}$ has degree 0 in X_j on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, it has degree 0 in ϵ under the given specialization. So $\Delta^{\widehat{R}^i}_{\widehat{C}^i} = \pm 1$.

We note that the distinguished augmenting paths $\mathcal{F}_L$ and $\mathcal{F}_R$ in $\widehat{\mathcal{G}}^{\mathbf{u}_+}_j$ have weight $\frac{1}{\epsilon}$ and $-\frac{1}{\epsilon}$, so that $\widehat{Y}_j = \frac{\mathcal{M}_+}{\mathcal{M}_-} = -1$ and so $X'_j = \frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j} = 0$.

By Proposition VII.25, every entry of $\Upsilon^{u,w}$ can be written in the form $P_1 X'_j + P_2$ where P_1 is Laurent in the variables $\{X_i : i \neq j\}$ and $P_2 \in \mathbb{C}[X_i^{\pm} : i \neq j][X_j]$.

Since $X'_j = 0$ under this specialization, the entries of $\Upsilon^{u,w}$ are polynomial in X_j, so that evaluating at $\epsilon = 0$ gives an element of $\mathcal{R}^{u,w}$. We have $\Delta^{\widehat{R}^j}_{\widehat{C}^j} = -\epsilon = 0$ and for each $i \neq j$, $\Delta^{\widehat{R}^i}_{\widehat{C}^i} = \pm 1 \neq 0$.

If ㅊ is a chamber which is not in $\mathtt{JC}(j)$, so ㅊ has the same row labels $\lambda(\mathbf{v}^j)(\text{ㅊ})$ $= \lambda(\mathbf{u}_+)(\text{ㅊ})$ in the upper arrangements for $\mathbf{v}^j$ and $\mathbf{u}_+$, then $\Delta^{\lambda(\mathbf{u}_+)(\text{ㅊ})}_{\lambda(\mathbf{w})(\text{ㅊ})}$ is identically ± 1 for all ϵ. If ㅊ $\in \mathtt{JC}(j)$, by Proposition VII.36 the minor $\Delta^{\lambda(\mathbf{v}^j)(\text{ㅊ})}_{\lambda(\mathbf{w})(\text{ㅊ})}$ has a term of degree zero in X_j on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, so under this specialization we have $\Delta^{\lambda(\mathbf{v}^j)(\text{ㅊ})}_{\lambda(\mathbf{w})(\text{ㅊ})}$ $= \epsilon P + m$ where P is polynomial in ϵ and m is an integer. The $\Delta^{\lambda(\mathbf{v}^j)(\text{ㅊ})}_{\lambda(\mathbf{w})(\text{ㅊ})}$ can be expanded as path weights of augmenting paths which all have the same sign as monomials in the parameters $\{t_i : i < j\}$. Under our specialization, t_j is the only parameter which is negative as a Laurent monomial in ϵ, so there can be no cancellation of terms. It follows that the integer m is nonzero.

So in the limit as $\epsilon \to 0$, the minor $\Delta^{\lambda(\mathbf{v}^j)(\text{ㅊ})}_{\lambda(\mathbf{w})(\text{ㅊ})}$ approaches a nonzero integer m.

On the other hand, if R is a collection of row indices with $\lambda(\mathbf{u}_+)(\text{ㅊ}) < R <$ $\lambda(\mathbf{v}^j)(\text{ㅊ})$, then by Proposition VII.36, $\Delta^{R}_{\lambda(\mathbf{w})(\text{ㅊ})}$ specializes to ϵP where P is a polynomial in ϵ, and hence in the limit as $\epsilon \to 0$ the minor $\Delta^{R}_{\lambda(\mathbf{w})(\text{ㅊ})}$ approaches 0. Hence, this specialized chamber weighting gives a path through $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ so that the limit as $\epsilon \to 0$ is a flag in the Deodhar divisor $\mathcal{D}^{\mathbf{v}^j,\mathbf{w}}$. $\square$

By upper triangularity in the parameters t_i, we may change the sign of a parameter t_i with $i > j$ to obtain a different path to the Deodhar divisor $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$.

Corollary VII.38. *Let $\{t_{i,\epsilon} : i \in J^\circ_{\mathbf{u}_+}\}$ be the specialization of parameters from the proof of Lemma VII.37. Suppose that $\{t'_{i,\epsilon} : i \in J^\circ_{\mathbf{u}_+}\}$ satisfies $t'_{i,\epsilon} = t_{i,\epsilon}$ for all $i \leqslant j$ and $t'_{i,\epsilon_j} = \pm 1$ for $i > j$. Then the entries of $\Upsilon^{u,w}$ under this specialization are polynomial in ϵ and taking the limit as $\epsilon \to 0$ gives an element of $\mathcal{R}^{u,w}$ with $X_j = 0$ and all other $X_i = \pm 1$.*

Proof. The $\hat{y}$-variable $\widehat{Y}_j$ depends only on values t_i where $i \leqslant j$, and the variables X_k are ratios of the t_i with coefficient 1. □

Lemma VII.39. *Let $j \in J^\circ_{\mathbf{u}_+}$ be an index with $v^j_{(0)} = 1$. Then X'_j is a regular function on $\mathcal{R}^{u,w}$.*

Proof. It suffices to show that the singular locus of X'_j has codimension at least 2. On $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$, we have $X'_j = \frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j} = M\Delta^{v^j_{(j-1)}([1,h_j])}_{\text{え}_{\to}(j)} - P$ where M is a Laurent monomial in $\{X_i : i \neq j\}$ and P is polynomial in X_j and Laurent in $\{X_i : i \neq j\}$. Since all the X_i are regular functions on $\mathcal{R}^{u,w}$, $\frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j}$ is regular on the locus where $X_j \neq 0$. $M\Delta^{v^j_{(j-1)}([1,h_j])}_{\text{え}_{\to}(j)} - P$ is regular on $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$ except where some X_i appearing in the denominator of M or P vanishes. Since $i \neq j$ implies that X_i does not vanish identically on $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$, the locus of $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$ where $M\Delta^{v^j_{(j-1)}([1,h_j])}_{\text{え}_{\to}(j)} - P$ is singular has codimension 2 in $\mathcal{R}^{u,w}$.

□

Lemma VII.40. *Let X_i be any variable in the initial cluster. Then the locus where $X_i = 0$ and $X'_j = 0$ has complex codimension $\geqslant 2$.*

Proof. It's straightforward to show that the variable X_i is generically nonvanishing when $X'_j = 0$. In particular, all the variables X_i are nonzero on the Deodhar torus

$\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$; in the proof of Lemma VII.37, we gave a family of flags in $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ satisfying $X'_j = 0$.

To show that X'_j is generically nonzero when $X_i = 0$, we will need to consider several cases.

First, we consider the case where $i = j$. By Lemma V.17, X_j vanishes identically on the the Deodhar stratum $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$. We will show that the mutated variable X'_j is generically nonzero on $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$. Write X'_j in the form $M\Delta^{v^j_{(j-1)}([1,h_j])}_{\text{ㅊ}_{\rightarrow}(j)} - P$, where M is a Laurent monomial in $\{X_i : i \neq j\}$ and P is polynomial in X_j and Laurent in $\{X_i : i \neq j\}$, so that this formula is generically defined on $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$. By Marsh and Rietsch's Chamber Ansatz formula in [27], there is a birational isomorphism from $(\mathbb{C}^*)^{\ell(w)-\ell(u)-2} \times \mathbb{C}$ to $\mathcal{D}^{\mathbf{v}^{\mathbf{j}},\mathbf{w}}$ so that the inverse map gives

$$m_j = \frac{\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\leftarrow}(j))}}{\Delta^{v^j_{(j-1)}([1,h_j+1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\Delta^{v^j_{(j-1)}([1,h_j-1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\downarrow}(j)\right)}}\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))} - \Delta^{v^j_{(j-1)}}_{s_{h_j}([1,h_j])}\left(g_{(j-1)}\right),$$

where the minors on the chambers $\text{ㅊ}_{\uparrow}(j)$, $\text{ㅊ}_{\leftarrow}(j)$ and $\text{ㅊ}_{\downarrow}(j)$ are nonvanishing on $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ and determined by the parameters $\{t_i : i < j\}$ and the term $\Delta^{v^j_{(j-1)}}_{s_{h_j}([1,h_j])}\left(g_{(j-1)}\right)$ is also determined by the $\{t_i : i < j\}$.

Rearranging terms, we have

$$\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\rightarrow}(j))} = \frac{\Delta^{v^j_{(j-1)}([1,h_j+1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\Delta^{v^j_{(j-1)}([1,h_j-1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\downarrow}(j)\right)}}{\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\leftarrow}(j))}}\left(m_j + \Delta^{v^j_{(j-1)}}_{s_{h_j}([1,h_j])}\left(g_{(j-1)}\right)\right).$$

Substituting into the formula for X'_j, we have

$$\begin{aligned} X'_j &= M\frac{\Delta^{v^j_{(j-1)}([1,h_j+1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\uparrow}(j)\right)}\Delta^{v^j_{(j-1)}([1,h_j-1])}_{\lambda(\mathbf{w})\left(\text{ㅊ}_{\downarrow}(j)\right)}}{\Delta^{v^j_{(j-1)}([1,h_j])}_{\lambda(\mathbf{w})(\text{ㅊ}_{\leftarrow}(j))}}\left(m_j + \Delta^{v^j_{(j-1)}}_{s_{h_j}([1,h_j])}\left(g_{(j-1)}\right)\right) + P \\ &= M'\left(m_j + \Delta^{v^j_{(j-1)}}_{s_{h_j}([1,h_j])}\left(g_{(j-1)}\right)\right) + P \end{aligned}$$

$$= M'm_j + R$$

where M' and R are determined by $\{t_i : i < j\}$ and M' is generically nonvanishing on $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$. Since the parameter m_j takes values in $\mathbb{C}$, the function $X'_j = M'm_j + R$ is generically nonvanishing on $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$.

Now suppose that X_i is a mutable variable with $i \neq j$. The exchange monomials $\mathcal{M}_+$ and $\mathcal{M}_-$ were defined as the numerator and denominator of a Laurent monomial in reduced form, so they do not have a common factor. Suppose that X_i divides one of the exchange monomials—without loss of generality, $\mathcal{M}_+$. Then $\mathcal{M}_+$ vanishes on the locus $\{X_i = 0\}$, while there is a dense open subset of $\{X_i = 0\}$ where all the variables $\{X_k : k \neq i\}$ are nonzero, so that in particular X_j and $\mathcal{M}_-$ are nonzero. On this locus, we have $X'_j = \frac{\mathcal{M}_+ + \mathcal{M}_-}{X_j} = \frac{0+\mathcal{M}_-}{X_j} = \frac{\mathcal{M}_-}{X_j} \neq 0$.

Suppose now that $i \neq j$ and X_i does not divide $\mathcal{M}_+$ or $\mathcal{M}_-$, so that $\deg_{X_i}\left(\widehat{Y}_j\right) = 0$. We will show that there is a flag in $\mathcal{D}^{\mathbf{v^i},\mathbf{w}}$ where $X'_j = \pm 2$. Consider the specialized chamber weighting corresponding to setting $X_i = -\epsilon$ and the other $X_k = \pm 1$, from the proof of Lemma VII.37. Then the parameter t_i specializes to $\frac{1}{\epsilon}$, with other t_k specializing to ϵ^d where $\deg_{X_i}(t_k) = d$. Since $\widehat{Y}_j$ is a ratio of parameters t_k and by hypothesis $\deg_{X_i}\left(\widehat{Y}_j\right) = 0$, under this specialization we must have

$$\widehat{Y}_j = \begin{cases} 1 & \text{if the bridge } i \text{ is not an edge of } \partial\mathtt{JC}(j) \\ -1 & \text{if the bridge } i \text{ is an edge of } \partial\mathtt{JC}(j) \end{cases}.$$

The condition that $X_k = \pm 1$ if $k \neq i$ implies that the monomials $\mathcal{M}_+$ and $\mathcal{M}_-$ have value ± 1, so that

$$X'_j = \begin{cases} \pm 2 & \text{if } \widehat{Y}_j = 1 \\ 0 & \text{if } \widehat{Y}_j = -1 \end{cases}.$$

So if the bridge i is not an edge of $\partial\texttt{JC}(j)$, then X_j' specializes to ± 2, and letting $\epsilon \to 0$ gives a flag in $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$ with $X_j' = \pm 2 \neq 0$.

On the other hand, if i is an edge of $\partial\texttt{JC}(j)$, then since j is the rightmost crossing of $\partial\texttt{JC}(j)$ we must have $j > i$ so that t_j specializes to 1. By Corollary VII.38, changing the specialization to set $t_j = -1$ gives another family of flags in $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ so that $X_k = \pm 1$ for $k \neq i$ and taking the limit as $\epsilon \to 0$ gives an element of $\mathcal{D}^{\mathbf{v}^{\mathbf{i}},\mathbf{w}}$. Since changing the value of t_j from 1 to -1 changes the sign of $\widehat{Y}_j$, this revised specialization has $\widehat{Y}_j = \pm 1$, so that $X_j' = \pm 2$. $\square$

We have shown that the variables X_j and the mutated variables X_j' are globally regular functions on $\mathcal{R}^{u,w}$, and that the complement of the locus where all cluster variables are nonzero for at least one cluster in the "starfish" about the initial cluster has codimension $\geqslant 2$ in $\mathcal{R}^{u,w}$. Since $\overline{\mathcal{A}}(\mathbf{X})$ is the ring of functions that are Laurent in every cluster mutation equivalent to $\mathbf{X}$, given any $f \in \overline{\mathcal{A}}(\mathbf{X})$, we can verify that f is globally regular on $\mathcal{R}^{u,w}$ by expressing it as a Laurent polynomial in each of the clusters $\mathbf{X}, \mu_j(\mathbf{X})$ to show that it is regular on the locus where the variables from the given cluster are nonzero. In the following lemma, we verify that the coordinate ring of $\mathcal{R}^{u,w}$ is contained in the upper cluster algebra determined by the initial seed $(\mathbf{X}, Q)$ where Q is the quiver corresponding to the variables $\widehat{Y}_j$. The proof technique was outlined by Speyer in an email.

Lemma VII.41. *[32] Let $\mathbf{X} = \{X_j : j \in J^\circ_{\mathbf{u}_+}\}$ be the initial cluster for a unipeak expression* $\mathbf{w}$ *with exchange relations given by $\{\widehat{Y}_j : j \in J^\circ_{\mathbf{u}_+}, \mathbf{v}^0[j] = 1\}$. Then $\mathbb{C}[\mathcal{R}^{u,w}] \subseteq \overline{\mathcal{A}}(\mathbf{X})$.*

Proof. By [3], the upper cluster algebra $\overline{\mathcal{A}}(\mathbf{X}) = \bigcap_{\mathbf{X}' \sim \mathbf{X}} \mathbb{C}[(\mathbf{X}')^{\pm}]$ determined by the initial seed $\Sigma = (\mathbf{X}, Q)$ is equal to the upper bound algebra $\mathcal{U}(\Sigma) = \bigcap_{\substack{X_j \text{ mutable} \\ \mathbf{X}' = \mu_j(\mathbf{X})}} \mathbb{C}[(\mathbf{X}')^{\pm}]$.

So we must show that for each cluster $\mathbf{X}' \in \{\mathbf{X}, \mu_j(\mathbf{X}) : X_j \text{ mutable }\}$, the Laurent ring $\mathbb{C}[(\mathbf{X}')^{\pm}]$ is the ring of regular functions on the locus where all cluster variables in $\mathbf{X}'$ are nonvanishing.

For $\mathbf{X}' = \mathbf{X}$, this is clear. Marsh and Rietsch showed in [27] that the coordinate ring of $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ is isomorphic to the Laurent ring $\mathbb{C}[t_i^{\pm} : i \in J^{\circ}_{\mathbf{u}_+}]$. Applying the chamber ansatz formulas to our weighting Q(ㅊ) shows that the t_i are Laurent monomials in the X_j, while Lemma V.16 shows that the X_j are Laurent monomials in the t_i. Suppose that $f \in \mathbb{C}[\mathcal{R}^{u,w}]$ is any function. Then the restriction of f to the Deodhar torus $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ is Laurent in the cluster variables $\mathbf{X}$. We claim that f is Laurent in the cluster variable $\mu_j(\mathbf{X})$. It suffices to show that there is a generating set for $\mathbb{C}[\mathcal{R}^{u,w}]$ so that all generators restrict to Laurent polynomials in the variables $\mu_j(\mathbf{X})$.

By [13] and [22], the coordinate ring of the Schubert cell $B_+\dot{w}B_+/B_+$ is generated by the entries $\hat{z}_{ij}$, where $\hat{z}$ is the upper unitriangular matrix so that $\hat{z}\dot{w}$ has zeros to the right of permutation positions.Let Y be the intersection of the Schubert cell $B_+\dot{w}B_+/B_+$ with the opposite Schubert variety $\overline{B_-\dot{u}B_+/B_+} = \{gB_+ : {}^{\mathrm{NW}}\mathrm{Rank}\,(g)_{ij} \leqslant {}^{\mathrm{NW}}\mathrm{Rank}\,(u)_{ij}\}$. Since Y is a closed subvariety of $B_+\dot{w}B_+/B_+$, the coordinate ring $\mathbb{C}[Y]$ is a quotient of $\mathbb{C}[B_+\dot{w}B_+/B_+]$, and hence generated by the $\hat{z}_{ij}$. The open Richardson variety is the locus of Y where all left-justified minors $\Delta^{u([1,h])}_{[1,h]}(g)$ are nonzero for flag representatives g with $gB_+ \in \mathcal{R}^{u,w}$, so that $\mathbb{C}[\mathcal{R}^{u,w}]$ $= \mathbb{C}[Y][(\Delta^{\lambda(\mathbf{u}_+)([1,h])}_{\lambda(\mathbf{w})([1,h])}\hat{z})^{-1}]$ so that in particular $\mathbb{C}[\mathcal{R}^{u,w}]$ is generated by the entries z_{ij} and the reciprocals of minors $(\Delta^{\lambda(\mathbf{u}_+)([1,h])}_{\lambda(\mathbf{w})([1,h])}\hat{z})^{-1}$. The matrix $\hat{z}$ differs from the matrix $z = \Upsilon^{u,w}\dot{w}^{-1}$ by the gauge transformation $\hat{z} = z\mathrm{diag}\left(z_{ii}^{-1}\right)$ where the z_{ii} are ratios of frozen variables X_i. It follows that $\mathbb{C}[\mathcal{R}^{u,w}]$ is generated by the entries z_{ij} and the inverses of the frozen variables X_i.

By Proposition VII.26, the entries z_{ij} are Laurent polynomials in the cluster variables $\mu_j(\mathbf{X})$ for all mutable X_j, so that in particular there is a set of ring generators for $\mathbb{C}[\mathcal{R}^{u,w}]$ contained in the intersection of Laurent rings $\bigcap_{\substack{X_j \text{ mutable} \\ \mathbf{X}'=\mu_j(\mathbf{X})}} \mathbb{C}[(\mathbf{X}')^{\pm}] = \overline{\mathcal{A}}(\mathbf{X})$. $\square$

CHAPTER VIII

Conclusions

In this thesis, we constructed an upper cluster algebra structure on the coordinate ring of the open Richardson variety $\mathcal{R}^{u,w}$ in type A. Points in the open Richardson variety are flags, so that regular functions must be expressible in terms of ratios of minors on a left-justified collection of minors. Since $\mathcal{R}^{u,w}$ is contained in the Schubert cell $B_+\dot{w}B_+/B_+$, if we choose an ordered basis for a flag F of the form $z\dot{w}$ where z is an upper triangular matrix, ratios of minors on columns indexed by chambers in a wiring diagram for a reduced expression for w are independent of z. Each choice of reduced expression $\mathbf{w}$ gives a different decomposition of $\mathcal{R}^{u,w}$ into disjoint Deodhar strata indexed by the distinguished subexpressions of the expression $\mathbf{w}$ which have product u. In particular, there is a unique distinguished subexpression $\mathbf{u}_+ \prec \mathbf{w}$ that is a reduced expression for u. Marsh and Rietsch defined a parametrization of each Deodhar stratum $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ with inverse given by a generalized chamber ansatz. Each chamber is labeled by a minor Δ_C^R, where the column set C is indexed by the left endpoints of the strands below the chamber and the row set R is minimal so that the minor is nonvanishing on $\mathcal{D}^{\mathbf{v},\mathbf{w}}$.

If a distinguished subexpression $\mathbf{v} \prec \mathbf{w}$ is non-reduced, then there is at least one chamber え which has additional rank conditions on $\mathcal{D}^{\mathbf{v},\mathbf{w}}$ than on the torus

$\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$. In particular, if i is an index where the partial product $v_{(i)}$ becomes shorter, then the chamber to the left of the i^{th} crossing receives different row labels. In general, if a chamber is labeled by the minor Δ_C^R for the Deodhar torus, the minor Δ_C^R may vanish on multiple boundary divisors.

We showed that given a unipeak expression $\mathbf{w}$, the chamber minors for $\mathcal{D}^{\mathbf{u}_+,\mathbf{w}}$ can be factored into products of minors $\Delta_{\widehat{C}_j}^{\widehat{R}_j}$ indexed by *nearly positive sequences* $\mathbf{v}^j$, which generalize the distinguished subexpressions giving boundary divisors. Although the minors $\Delta_{\widehat{C}_j}^{\widehat{R}_j}$ are not always chamber minors, we showed that they are well-defined regular functions on $\mathcal{R}^{u,w}$ using determinantal identities. Using augmenting paths in the oriented bridge diagrams defined by Karpman, we verified that when the vanishing locus of a minor $\Delta_{\widehat{C}_j}^{\widehat{R}_j}$ is nonempty, it contains a single boundary divisor $\mathcal{D}^{\mathbf{v^j},\mathbf{w}}$. We showed that defining an initial seed with cluster variables $X_j = \Delta_{\widehat{C}_j}^{\widehat{R}_j}$ and quiver obtained from Berenstein, Fomin and Zelevinsky's chamber ansatz quiver gives an upper cluster algebra structure to the coordinate ring $\mathbb{C}[\mathcal{R}^{u,w}]$.

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