Appendix

Transformation of SSP to SSP0: From (16), we have

$$y_i \geq y_{i+1} \geq 0,$$

for $0 \leq i \leq n$ in the given sequence. Then,

$$C_i - C_{i-1} \quad = \quad -\frac{\ln(y_i) - \ln(y_{i-1})}{R_i}, \qquad \text{for } 1 \leq i \leq n+1.$$

Observe that the completion time $C_i = \sum_{j=1}^{i}(C_j - C_{j-1})$. Then, we transform the precedence constraints for the C_i's into constraints in the $\ln(y_i)$ terms. To see this, we first rewrite precedence Constraints (1) as

$$C_k - C_i = \sum_{j=i+1}^{k}(C_j - C_{j-1}) \geq D_k, \qquad \text{for } 0 \leq i \leq n, k \in S_i,$$

and then,

$$\sum_{j=i+1}^{k}(C_j - C_{j-1}) = -\sum_{j=i+1}^{k}\frac{\ln(y_j) - \ln(y_{j-1})}{R_j} \geq D_k, \quad \text{for } 0 \leq i \leq n, k \in S_i,$$

$$\text{or equivalently} \qquad \sum_{j=i+1}^{k}\frac{\ln(y_j) - \ln(y_{j-1})}{R_j} \quad \leq \quad -D_k, \quad \text{for } 0 \leq i \leq n, k \in S_i.$$

Using the decision variables $y_0, \ldots, y_{n+1}$, Constraint (5) can be rewritten as

$$C_{n+1} = \sum_{j=1}^{n+1}(C_j - C_{j-1}) = -\sum_{j=1}^{n+1}\frac{\ln(y_j) - \ln(y_{j-1})}{R_j} \leq \Delta. \quad \square$$

Approximation of Constraints (17) and (18): We next linearize Constraints (17) and (18) approximately, to find lower and upper bounds on the optimal value of problem SSP0. It is straightforward to verify that, after a natural exponential transformation, the left-hand-side of Constraints (17) becomes

$$\prod_{j=i+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}}.$$

We next show how to approximate this expression using linear terms.

For $0 \leq m < k \leq n+1$, we define

$$B_1(m,k) \quad = \quad \left[\max\left\{\sum_{j=m+1}^{k-1}\frac{R_{m+1}}{R_{j+1}}\left(\frac{y_{j+1}}{y_m} - \frac{y_j}{y_m}\right) + \frac{y_{m+1}}{y_m}, 0\right\}\right]^{\frac{1}{R_{m+1}}},$$

$$B_2(m,k) \quad = \quad \left[\sum_{j=m+1}^{k-1}\frac{R_{m+1}}{R_{j+1}}\left(\frac{y_m}{y_{j+1}} - \frac{y_m}{y_j}\right) + \frac{y_m}{y_{m+1}}\right]^{-\frac{1}{R_{m+1}}},$$

$$B_3(m,k) = \left[\sum_{j=m+1}^{k-1} \frac{R_k}{R_j}\left(\frac{y_{j-1}}{y_k} - \frac{y_j}{y_k}\right) + \frac{y_{k-1}}{y_k}\right]^{-\frac{1}{R_k}},$$

$$B_4(m,k) = \left[\sum_{j=m+1}^{k-1} \frac{R_k}{R_j}\left(\frac{y_k}{y_{j-1}} - \frac{y_k}{y_j}\right) + \frac{y_k}{y_{k-1}}\right]^{\frac{1}{R_k}}.$$

Given these definitions of $B_i(m,k)$ for $i = 1,2,3,4$, we have the following bounding result.

Lemma 1 *Let $0 \le R_{i+1} \le R_i$ and $0 \le y_{i+1} \le y_i$ for $0 \le i \le n$. For $0 \le m < k \le n+1$, we have*

$$\begin{aligned}\left(\frac{y_m}{y_k}\right)^{-\frac{1}{R_k}} &\le \max\{B_1(m,k), B_3(m,k)\} \\ &\le \prod_{j=m+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \\ &\le \min\{B_2(m,k), B_4(m,k)\} \le \left(\frac{y_k}{y_m}\right)^{\frac{1}{R_{m+1}}}. \end{aligned} \tag{39}$$

Proof: First, we show that

$$B_1(m,k) \le \prod_{j=m+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \le B_2(m,k).$$

The proof is by induction. When $m+1 = k$, it is clear that the inequality holds as an equality. We assume that the inequality holds when $m+1 = t+1$ where $1 \le t \le k-1$. That is, $B_1(t,k) \le \prod_{j=t+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \le B_2(t,k)$. Then, we need to show that it holds when $m+1 = t$. Note that $\frac{R_t}{R_{t+1}} \ge 1$ and for $x \ge 0$ and $a \ge 1$, we have $x^a \ge a(x-1)+1$. If $\sum_{j=t+1}^{k-1} \frac{R_{t+1}}{R_{j+1}}\left(\frac{y_{j+1}}{y_t} - \frac{y_j}{y_t}\right) + \frac{y_{t+1}}{y_t} \ge 0$, then

$$\begin{aligned} B_1(t,k)\left(\frac{y_t}{y_{t-1}}\right)^{\frac{1}{R_t}} &= \left[\sum_{j=t+1}^{k-1} \frac{R_{t+1}}{R_{j+1}}\left(\frac{y_{j+1}}{y_t} - \frac{y_j}{y_t}\right) + \frac{y_{t+1}}{y_t}\right]^{\frac{R_t}{R_{t+1}}\frac{1}{R_t}} \left(\frac{y_t}{y_{t-1}}\right)^{\frac{1}{R_t}} \\ &\ge \left\{\frac{R_t}{R_{t+1}}\left[\sum_{j=t+1}^{k-1} \frac{R_{t+1}}{R_{j+1}}\left(\frac{y_{j+1}}{y_t} - \frac{y_j}{y_t}\right) + \frac{y_{t+1}}{y_t} - 1\right] + 1\right\}^{\frac{1}{R_t}} \left(\frac{y_t}{y_{t-1}}\right)^{\frac{1}{R_t}} \qquad (40) \\ &= \left[\sum_{j=t}^{k-1} \frac{R_t}{R_{j+1}}\left(\frac{y_{j+1}}{y_{t-1}} - \frac{y_j}{y_{t-1}}\right) + \frac{y_t}{y_{t-1}}\right]^{\frac{1}{R_t}} = B_1(t-1,k), \end{aligned}$$

where inequality (40) holds from the facts that $R_t \ge R_{t+1}$, $x^a \ge a(x-1)+1$ for $a \ge 1$, and $x \ge 0$. Then, from the induction hypothesis, $\prod_{j=t}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \ge B_1(t,k)\left(\frac{y_t}{y_{t-1}}\right)^{\frac{1}{R_t}} \ge B_1(t-1,k)$. Alternatively, if $\sum_{j=t+1}^{k-1} \frac{R_{t+1}}{R_{j+1}}\left(\frac{y_{j+1}}{y_t} - \frac{y_j}{y_t}\right) + \frac{y_{t+1}}{y_t} < 0$ so that $B_1(t,k) = 0$, then

$$\sum_{j=t}^{k-1} \frac{R_t}{R_{j+1}}\left(\frac{y_{j+1}}{y_{t-1}} - \frac{y_j}{y_{t-1}}\right) + \frac{y_t}{y_{t-1}} = \left\{\frac{R_t}{R_{t+1}}\left[\sum_{j=t+1}^{k-1} \frac{R_{t+1}}{R_{j+1}}\left(\frac{y_{j+1}}{y_t} - \frac{y_j}{y_t}\right) + \frac{y_{t+1}}{y_t} - 1\right] + 1\right\}\left(\frac{y_t}{y_{t-1}}\right)$$

$$
\begin{aligned}
&= \left\{\frac{R_t}{R_{t+1}}\left[\sum_{j=t+1}^{k-1}\frac{R_{t+1}}{R_{j+1}}\left(\frac{y_{j+1}}{y_t}-\frac{y_j}{y_t}\right)+\frac{y_{t+1}}{y_t}\right]-\frac{R_t}{R_{t+1}}+1\right\}\left(\frac{y_t}{y_{t-1}}\right)\\
&\le 0.
\end{aligned}
$$

Consequently, we have $B_1(t-1,k)=0$. Again, $\prod_{j=t}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \ge B_1(t,k)\left(\frac{y_t}{y_{t-1}}\right)^{\frac{1}{R_t}} \ge B_1(t-1,k)$. Thus, the induction step for $B_1(m,k)$ is proved.

For $B_2(m,k)$, we have

$$
\begin{aligned}
B_2(t,k)\left(\frac{y_t}{y_{t-1}}\right)^{\frac{1}{R_t}} &= \left[\sum_{j=t+1}^{k-1}\frac{R_{t+1}}{R_{j+1}}\left(\frac{y_t}{y_{j+1}}-\frac{y_t}{y_j}\right)+\frac{y_t}{y_{t+1}}\right]^{\frac{R_t}{R_{t+1}}\left(-\frac{1}{R_t}\right)}\left(\frac{y_{t-1}}{y_t}\right)^{-\frac{1}{R_t}}\\
&\le \left\{\frac{R_t}{R_{t+1}}\left[\sum_{j=t+1}^{k-1}\frac{R_{t+1}}{R_{j+1}}\left(\frac{y_t}{y_{j+1}}-\frac{y_t}{y_j}\right)+\frac{y_t}{y_{t+1}}-1\right]+1\right\}^{-\frac{1}{R_t}}\left(\frac{y_{t-1}}{y_t}\right)^{-\frac{1}{R_t}} \qquad (41)\\
&= \left[\sum_{j=t}^{k-1}\frac{R_t}{R_{j+1}}\left(\frac{y_{t-1}}{y_{j+1}}-\frac{y_{t-1}}{y_j}\right)+\frac{y_{t-1}}{y_t}\right]^{-\frac{1}{R_t}} = B_2(t-1,k),
\end{aligned}
$$

where inequality (41) holds from the facts that $R_t \ge R_{t+1}$, $x^a \ge a(x-1)+1$ for $a \ge 1$ and $x \ge 0$, and x^{-z} is decreasing in x for $z>0$ and $x \ge 0$. We can verify that $\sum_{j=t+1}^{k-1}\frac{R_{t+1}}{R_{j+1}}\left(\frac{y_t}{y_{j+1}}-\frac{y_t}{y_j}\right)+\frac{y_t}{y_{t+1}} \ge 1 > 0$ since $y_j \le y_i$ for $j>i$, which guarantees the requirements on x^a and x^{-z}. The remainder of the induction step proof follows that for $B_1(m,k)$.

Next, we show that

$$
B_3(m,k) \le \prod_{j=m+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \le B_4(m,k),
$$

again by induction. When $k=m+1$, it is clear that the inequality holds as an equality. We assume that the inequality holds when $k=t$ where $t \ge m+1$. That is, $B_3(m,t) \le \prod_{j=t+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \le B_4(m,t)$. Then, we need to show it holds when $k=t+1$.

For $B_3(m,k)$, we have

$$
\begin{aligned}
B_3(m,t)\left(\frac{y_{t+1}}{y_t}\right)^{\frac{1}{R_{t+1}}} &= \left[\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_{j-1}}{y_t}-\frac{y_j}{y_t}\right)+\frac{y_{t-1}}{y_t}\right]^{\frac{R_{t+1}}{R_t}\left(-\frac{1}{R_{t+1}}\right)}\left(\frac{y_t}{y_{t+1}}\right)^{-\frac{1}{R_{t+1}}}\\
&\ge \left\{\frac{R_{t+1}}{R_t}\left[\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_{j-1}}{y_t}-\frac{y_j}{y_t}\right)+\frac{y_{t-1}}{y_t}-1\right]+1\right\}^{-\frac{1}{R_{t+1}}}\left(\frac{y_t}{y_{t+1}}\right)^{-\frac{1}{R_{t+1}}} \qquad (42)\\
&= \left[\sum_{j=m+1}^{t}\frac{R_{t+1}}{R_j}\left(\frac{y_{j-1}}{y_{t+1}}-\frac{y_j}{y_{t+1}}\right)+\frac{y_t}{y_{t+1}}\right]^{-\frac{1}{R_{t+1}}} = B_3(m,t+1),
\end{aligned}
$$

where inequality (42) holds from the fact $R_t \ge R_{t+1}$, $x^a \le a(x-1)+1$ for $a \le 1$ and $x \ge 0$, and x^{-z} is decreasing in x for $z>0$ and $x \ge 0$. We can verify that $\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_{j-1}}{y_t}-\frac{y_j}{y_t}\right)+\frac{y_{t-1}}{y_t} > 0$

since $y_j \le y_i$ for any $j > i$, which guarantees the requirements on x^a and x^{-z}. The remainder of the proof follows that for $B_1(m,k)$ and $B_2(m,k)$.

For $B_4(m,k)$, we have

$$
\begin{aligned}
B_4(m,t)\left(\frac{y_{t+1}}{y_t}\right)^{\frac{1}{R_{t+1}}} &= \left[\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_t}{y_{j-1}}-\frac{y_t}{y_j}\right)+\frac{y_t}{y_{t-1}}\right]^{\frac{R_{t+1}}{R_t}\left(\frac{1}{R_{t+1}}\right)}\left(\frac{y_{t+1}}{y_t}\right)^{\frac{1}{R_{t+1}}} \\
&\le \left\{\frac{R_{t+1}}{R_t}\left[\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_t}{y_{j-1}}-\frac{y_t}{y_j}\right)+\frac{y_t}{y_{t-1}}-1\right]+1\right\}^{\frac{1}{R_{t+1}}}\left(\frac{y_{t+1}}{y_t}\right)^{\frac{1}{R_{t+1}}} \qquad (43) \\
&= \left[\sum_{j=m+1}^{t}\frac{R_{t+1}}{R_j}\left(\frac{y_{t+1}}{y_{j-1}}-\frac{y_{t+1}}{y_j}\right)+\frac{y_{t+1}}{y_t}\right]^{\frac{1}{R_{t+1}}} = B_4(m,t+1),
\end{aligned}
$$

where inequality (43) follows from the facts that $R_t \ge R_{t+1}$, $x^a \le a(x-1)+1$ for $a \le 1$ and $x \ge 0$, and x^z is increasing in x for $z > 0$ and $x \ge 0$. We can verify

$$
\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_t}{y_{j-1}}-\frac{y_t}{y_j}\right)+\frac{y_t}{y_{t-1}} = \sum_{j=m+1}^{t-1}\frac{y_t}{y_j}\left(\frac{R_t}{R_{j+1}}-\frac{R_t}{R_j}\right)+\frac{R_t}{R_{m+1}}\frac{y_t}{y_m} > 0
$$

since $R_j \le R_i$ for any $j > i$, which guarantees the requirements on x^a and x^z.

Next, we show that $\left(\frac{y_m}{y_k}\right)^{-\frac{1}{R_k}} \le B_3(m,k)$, i.e., $\sum_{j=m+1}^{k-1}\frac{R_k}{R_j}\left(\frac{y_{j-1}}{y_k}-\frac{y_j}{y_k}\right)+\frac{y_{k-1}}{y_k} \le \frac{y_m}{y_k}$. This result holds if $k = m+1$, since $y_j \le y_i$ for any $j > i$. Now, suppose it holds for $k = t$. For $k = t+1$,

$$
\begin{aligned}
\sum_{j=m+1}^{t}\frac{R_{t+1}}{R_j}\left(\frac{y_{j-1}}{y_{t+1}}-\frac{y_j}{y_{t+1}}\right)+\frac{y_t-y_m}{y_{t+1}} &= \frac{R_{t+1}}{R_t}\sum_{j=m+1}^{t}\frac{R_t}{R_j}\left(\frac{y_{j-1}}{y_{t+1}}-\frac{y_j}{y_{t+1}}\right)+\frac{y_t-y_m}{y_{t+1}} \\
&= \frac{R_{t+1}}{R_t}\left[\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_{j-1}}{y_{t+1}}-\frac{y_j}{y_{t+1}}\right)+\left(\frac{y_{t-1}}{y_{t+1}}-\frac{y_t}{y_{t+1}}\right)\right]+\frac{y_t-y_m}{y_{t+1}} \\
&= \frac{R_{t+1}}{R_t}\left[\frac{y_t}{y_{t+1}}\left(\sum_{j=m+1}^{t-1}\frac{R_t}{R_j}\left(\frac{y_{j-1}}{y_t}-\frac{y_j}{y_t}\right)+\frac{y_{t-1}}{y_t}\right)-\frac{y_t}{y_{t+1}}\right]+\frac{y_t-y_m}{y_{t+1}} \\
&\le \frac{R_{t+1}}{R_t}\left[\frac{y_t}{y_{t+1}}\frac{y_m}{y_t}-\frac{y_t}{y_{t+1}}\right]+\frac{y_t-y_m}{y_{t+1}} \\
&= \frac{y_m-y_t}{y_{t+1}}\left(\frac{R_{t+1}}{R_t}-1\right) \le 0,
\end{aligned}
$$

where the first inequality follows from the induction hypothesis. Therefore, $\sum_{j=m+1}^{t}\frac{R_{t+1}}{R_j}\left(\frac{y_{j-1}}{y_{t+1}}-\frac{y_j}{y_{t+1}}\right)+\frac{y_t}{y_{t+1}} \le \frac{y_m}{y_{t+1}}$, and the proof is complete.

Finally, we show that $B_2(m,k) \le \left(\frac{y_k}{y_m}\right)^{\frac{1}{R_{m+1}}} = \left(\frac{y_m}{y_k}\right)^{-\frac{1}{R_{m+1}}}$. Hence, we need to show $\sum_{j=m+1}^{k-1}\frac{R_{m+1}}{R_{j+1}}\left(\frac{y_m}{y_{j+1}}-\frac{y_m}{y_j}\right)+\frac{y_m}{y_{m+1}} \ge \frac{y_m}{y_k}$. If $k = m+1$, since $y_j \le y_i$ for any $j > i$, this inequality holds. For $k > m+1$, since $R_{m+1} \ge R_j$ for $j = m+2, \cdots, k-1$ and $y_j \le y_i$ for $j > i$, we have

$$
\sum_{j=m+1}^{k-1}\frac{R_{m+1}}{R_{j+1}}\left(\frac{y_m}{y_{j+1}}-\frac{y_m}{y_j}\right)+\frac{y_m}{y_{m+1}} \ge \sum_{j=m+1}^{k-1}\left(\frac{y_m}{y_{j+1}}-\frac{y_m}{y_j}\right)+\frac{y_m}{y_{m+1}} = \frac{y_m}{y_k}. \quad \square
$$

One issue with $B_1(m,k)$ is that when the first term within the bracket is negative, the applicable bound $B_1(m,k)$ is 0. Therefore, we use the term $B_3(m,k)$ as a lower bound on the original nonlinear term. Also, our preliminary computational studies show that in most cases $B_4(m,k)$ is smaller than $B_2(m,k)$. Hence, we use $B_4(m,k)$ as an upper bound on the original nonlinear term. Using bounds $B_3(m,k)$ and $B_4(m,k)$, we can approximate Constraints (17) as

$$\sum_{j=m+1}^{k-1} \frac{R_k}{R_j}\left(\frac{y_{j-1}}{y_k} - \frac{y_j}{y_k}\right) + \frac{y_{k-1}}{y_k} \geq \exp(R_k D_k), \quad \text{for } 0 \leq m \leq n, k \in S_m, \quad \text{and} \tag{44}$$

$$\sum_{j=m+1}^{k-1} \frac{R_k}{R_j}\left(\frac{y_k}{y_{j-1}} - \frac{y_k}{y_j}\right) + \frac{y_k}{y_{k-1}} \leq \exp(-R_k D_k), \quad \text{for } 0 \leq m \leq n, k \in S_m. \tag{45}$$

We observe that Constraints (44) can be linearized as Constraints (21):

$$\sum_{j=m+1}^{k-1} \frac{R_k}{R_j}(y_{j-1} - y_j) + y_{k-1} - y_k \exp(R_k D_k) \geq 0, \quad \text{for } 0 \leq m \leq n, k \in S_m.$$

However, Constraint (45) is still not linear, since the left-hand-side contains multiple terms with different nonconstant denominators. We need to approximate the left-hand-side of Constraint (45) further, in order to linearize it. Doing so requires the following result.

Lemma 2 *Let $a_i \geq 0, x_i \geq 0$ for $i = 1, 2, \ldots, n$. Let $x_{\min} = \min\{x_1, x_2, \ldots, x_n\}$ and $x_{\max} = \max\{x_1, x_2, \ldots, x_n\}$. We then have*

$$\begin{aligned} \max\left\{\frac{2\sum_{i=1}^n a_i}{x_{\min}} - \frac{\sum_{i=1}^n a_i x_i}{x_{\min}^2}, \frac{2\sum_{i=1}^n a_i}{x_{\max}} - \frac{\sum_{i=1}^n a_i x_i}{x_{\max}^2}\right\} &\leq \sum_{i=1}^n \frac{a_i}{x_i} \\ &\leq \left(\frac{1}{x_{\min}} + \frac{1}{x_{\max}}\right)\sum_{i=1}^n a_i - \frac{\sum_{i=1}^n a_i x_i}{x_{\min} x_{\max}}. \end{aligned}$$

Proof: Note that

$$\begin{aligned} \sum_{i=1}^n a_i x_i - x_{\max}^2 \sum_{i=1}^n \frac{a_i}{x_i} &= \sum_{i=1}^n a_i \left(\frac{x_i^2 - x_{\max}^2}{x_i}\right) \\ &= \sum_{i=1}^n a_i \left(1 - \frac{x_{\max}}{x_i}\right)(x_i + x_{\max}) \\ &\leq (x_{\min} + x_{\max})\left(\sum_{i=1}^n a_i - x_{\max}\sum_{i=1}^n \frac{a_i}{x_i}\right) \\ &= (x_{\min} + x_{\max})\sum_{i=1}^n a_i - x_{\min}x_{\max}\sum_{i=1}^n \frac{a_i}{x_i} - x_{\max}^2 \sum_{i=1}^n \frac{a_i}{x_i}. \end{aligned}$$

As a result,

$$\begin{aligned} x_{\min}x_{\max}\sum_{i=1}^n \frac{a_i}{x_i} &\leq (x_{\min} + x_{\max})\sum_{i=1}^n a_i - \sum_{i=1}^n a_i x_i \\ \Rightarrow \sum_{i=1}^n \frac{a_i}{x_i} &\leq \left(\frac{1}{x_{\min}} + \frac{1}{x_{\max}}\right)\sum_{i=1}^n a_i - \frac{1}{x_{\min}}\frac{1}{x_{\max}}\sum_{i=1}^n a_i x_i. \end{aligned}$$

On the other hand,

$$x_i - \frac{x_{\max}^2}{x_i} \quad = \quad x_i + \frac{x_{\max}^2}{x_i} - 2\frac{x_{\max}^2}{x_i} \geq 2x_{\max} - 2\frac{x_{\max}^2}{x_i}.$$

Consequently,

$$\begin{aligned} \sum_{i=1}^{n} a_i x_i - x_{\max}^2 \cdot \sum_{i=1}^{n} \frac{a_i}{x_i} &= \sum_{i=1}^{n} a_i \left(x_i - \frac{x_{\max}^2}{x_i} \right) \\ &\geq 2x_{\max} \sum_{i=1}^{n} a_i \left(1 - \frac{x_{\max}}{x_i} \right) \\ &= 2x_{\max} \sum_{i=1}^{n} a_i - 2x_{\max}^2 \sum_{i=1}^{n} \frac{a_i}{x_i}. \end{aligned}$$

As a result, we have

$$\sum_{i=1}^{n} \frac{a_i}{x_i} \quad \geq \quad \frac{2}{x_{\max}} \sum_{i=1}^{n} a_i - \frac{1}{x_{\max}^2} \sum_{i=1}^{n} a_i x_i.$$

Similarly, we have

$$x_i - \frac{x_{\min}^2}{x_i} \quad = \quad x_i + \frac{x_{\min}^2}{x_i} - 2\frac{x_{\min}^2}{x_i} \geq 2x_{\min} - 2\frac{x_{\min}^2}{x_i}.$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n} a_i x_i - x_{\min}^2 \sum_{i=1}^{n} \frac{a_i}{x_i} &= \sum_{i=1}^{n} a_i \left(x_i - \frac{x_{\min}^2}{x_i} \right) \\ &\geq 2x_{\min} \left(\sum_{i=1}^{n} a_i - x_{\min} \sum_{i=1}^{n} \frac{a_i}{x_i} \right) \\ &= 2x_{\min} \sum_{i=1}^{n} a_i - 2x_{\min}^2 \sum_{i=1}^{n} \frac{a_i}{x_i}. \end{aligned}$$

Finally, we have

$$\sum_{i=1}^{n} \frac{a_i}{x_i} \quad \geq \quad \frac{2}{x_{\min}} \sum_{i=1}^{n} a_i - \frac{1}{x_{\min}^2} \sum_{i=1}^{n} a_i x_i. \quad \square$$

The bounds on $\sum_{i=1}^{n} \frac{a_i}{x_i}$ defined in Lemma 2 are close to each other when the difference between $x_{\min}$ and $x_{\max}$ is small, as now shown.

Remark 5 *If* $|x_{\min} - x_{\max}| \to 0$*, then*

$$\left| \max \left\{ \frac{2\sum_{i=1}^{n} a_i}{x_{\min}} - \frac{\sum_{i=1}^{n} a_i x_i}{x_{\min}^2}, \frac{2\sum_{i=1}^{n} a_i}{x_{\max}} - \frac{\sum_{i=1}^{n} a_i x_i}{x_{\max}^2} \right\} - \left(\frac{1}{x_{\min}} + \frac{1}{x_{\max}} \right) \sum_{i=1}^{n} a_i + \frac{\sum_{i=1}^{n} a_i x_i}{x_{\min} x_{\max}} \right| \to 0.$$

We now apply Lemma 2 to relax the left-hand-side of (45), in order to linearize the constraint. We rewrite term $B_4(m,k)$ as follows:

$$B_4(m,k) \quad = \quad \left[\sum_{j=m+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j} \right) \frac{y_k}{y_j} + \frac{R_k}{R_{m+1}} \frac{y_k}{y_m} \right]^{\frac{1}{R_k}}.$$

Then, we have the following result.

Lemma 3 *We have*

$$\begin{aligned} B_4(m,k)^{R_k} &= \sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_k}{y_j}+\frac{R_k}{R_{m+1}}\frac{y_k}{y_m} \\ &\le 1+\frac{y_k}{y_m}-\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_m}+\frac{R_k}{R_{m+1}}\right]. \end{aligned}$$

Proof: Let $a_{m+1}=\frac{R_k}{R_{m+1}}$, $x_{m+1}=\frac{y_m}{y_k}$, and $a_i=\frac{R_k}{R_{i+1}}-\frac{R_k}{R_i}\ge 0$ and $x_i=\frac{y_i}{y_k}$ for $i=m+2,\cdots,k-1$. Note that $\sum_{i=m+1}^{k} a_i=1$, $x_{\min}=\frac{y_{k-1}}{y_k}$ and $x_{\max}=\frac{y_m}{y_k}$. Then, we have

$$\begin{aligned} \sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_k}{y_j}+\frac{R_k}{R_{m+1}}\frac{y_k}{y_m} &\le \frac{y_k}{y_m}+\frac{y_k}{y_{k-1}}-\frac{y_k}{y_m}\frac{y_k}{y_{k-1}}\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_k}+\frac{R_k}{R_{m+1}}\frac{y_m}{y_k}\right] \\ &\le \frac{y_k}{y_m}+1-\frac{y_k}{y_m}\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_k}+\frac{R_k}{R_{m+1}}\frac{y_m}{y_k}\right] \\ &= 1+\frac{y_k}{y_m}-\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_m}+\frac{R_k}{R_{m+1}}\right], \end{aligned}$$

where the first inequality follows from the second part of Lemma 2, and the second inequality follows from $\frac{y_k}{y_{k-1}}\le 1$ and

$$\begin{aligned} 1-\frac{y_k}{y_m}\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_k}+\frac{R_k}{R_{m+1}}\frac{y_m}{y_k}\right] &= 1-\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_m}+\frac{R_k}{R_{m+1}}\right] \\ &\ge 1-\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)+\frac{R_k}{R_{m+1}}\right] \\ &= 0. \quad \square \end{aligned}$$

Following Lemma 3, we can rewrite inequality (45) as

$$1+\frac{y_k}{y_m}-\left[\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_m}+\frac{R_k}{R_{m+1}}\right] \le \exp(-R_kD_k), \quad \text{for } 0\le m\le n, k\in S_m,$$

which is linear and can be further rewritten as Constraint (23):

$$\left[1-\frac{R_k}{R_{m+1}}-\exp(-R_kD_k)\right]y_m-\sum_{j=m+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)y_j+y_k\le 0, \quad \text{for } 0\le m\le n, k\in S_m.$$

This completes the approximation of Constraints (17) and (18). □

Proof of Theorem 1: To prove the theorem, we show that the polytope of problem SSP0 is contained in the polytope of problem SSP1. Note that Constraints (19) and (20) are the same in problems SSP0 and SSP1. Thus, we only need to show that Constraints (17) and (18) of problem SSP0 imply the corresponding constraints of problem SSP1, i.e., (21) and (22), respectively.

From Lemma 1, we have

$$B_3(i,k)=\left[\sum_{j=i+1}^{k-1}\frac{R_k}{R_j}\left(\frac{y_{j-1}}{y_k}-\frac{y_j}{y_k}\right)+\frac{y_{k-1}}{y_k}\right]^{-\frac{1}{R_k}} \le \prod_{j=i+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \le \exp(-D_k),$$

where the last inequality follows from Constraints (17) of problem SSP0. Observe that

$$\left[\sum_{j=i+1}^{k-1} \frac{R_k}{R_j}\left(\frac{y_{j-1}}{y_k} - \frac{y_j}{y_k}\right) + \frac{y_{k-1}}{y_k}\right]^{-\frac{1}{R_k}} \leq \exp(-D_k)$$

$$\Leftrightarrow \sum_{j=i+1}^{k-1} \frac{R_k}{R_j}\left(\frac{y_{j-1}}{y_k} - \frac{y_j}{y_k}\right) + \frac{y_{k-1}}{y_k} \geq \exp(R_k D_k)$$

$$\Leftrightarrow \sum_{j=i+1}^{k-1} \frac{R_k}{R_j}(y_{j-1} - y_j) + y_{k-1} - y_k \exp(R_k D_k) \geq 0,$$

which are Constraints (21) of problem SSP1. Hence, whenever Constraints (17) of problem SSP0 hold, Constraints (21) of problem SSP1 hold.

From Lemmas 1 and 3, we have

$$\begin{aligned} \exp(-\Delta) &\leq \prod_{j=1}^{n+1}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \leq B_4(0, n+1) \\ &\leq \left\{1 + \frac{y_{n+1}}{y_0} - \left[\sum_{j=1}^{n}\left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j}\right)\frac{y_j}{y_0} + \frac{R_{n+1}}{R_1}\right]\right\}^{\frac{1}{R_{n+1}}}, \end{aligned}$$

where the first inequality follows from deadline Constraint (18) of problem SSP0. Observe that

$$\begin{aligned} \left\{1 + \frac{y_{n+1}}{y_0} - \left[\sum_{j=1}^{n}\left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j}\right)\frac{y_j}{y_0} + \frac{R_{n+1}}{R_1}\right]\right\}^{\frac{1}{R_{n+1}}} &\geq \exp(-\Delta) \\ \Leftrightarrow 1 + \frac{y_{n+1}}{y_0} - \left[\sum_{j=1}^{n}\left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j}\right)\frac{y_j}{y_0} + \frac{R_{n+1}}{R_1}\right] &\geq \exp(-R_{n+1}\Delta) \\ \Leftrightarrow \left[1 - \frac{R_{n+1}}{R_1} - \exp(-R_{n+1}\Delta)\right] y_0 - \sum_{j=1}^{n}\left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j}\right) y_j + y_{n+1} &\geq 0, \end{aligned}$$

which are Constraints (22) of problem SSP1. Thus, whenever Constraints (18) of problem SSP0 hold, Constraints (22) of problem SSP1 hold. Therefore, we conclude that the feasible set of the original problem SSP0 is contained in the feasible set of problem SSP1, and hence problem SSP1 provides an upper bound on the optimal value of the original problem. □

Proof of Theorem 2: To prove the theorem, we show that the polytope of problem SSP2 is contained in the polytope of problem SSP0. Note that Constraints (19) and (20) are the same for problems SSP0 and SSP2. Thus, we only need to show that Constraints (23) and (24) of problem SSP2 imply the corresponding constraints of problem SSP0, i.e., (17) and (18), respectively.

From Lemmas 1 and 3, for $0 \leq i \leq n$ and $k \in S_i$, we have

$$\prod_{j=i+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \leq \left\{1 + \frac{y_k}{y_i} - \left[\sum_{j=i+1}^{k-1}\left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j}\right)\frac{y_j}{y_i} + \frac{R_k}{R_{i+1}}\right]\right\}^{\frac{1}{R_k}}.$$

Consequently, if

$$\left\{1+\frac{y_k}{y_i}-\left[\sum_{j=i+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_i}+\frac{R_k}{R_{i+1}}\right]\right\}^{\frac{1}{R_k}} \quad \leq \quad \exp(-D_k)$$

hold, then

$$\prod_{j=i+1}^{k}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \quad \leq \quad \exp(-D_k),$$

which are Constraints (17) of problem SSP0, hold. Note that

$$\begin{aligned}
\left\{1+\frac{y_k}{y_i}-\left[\sum_{j=i+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_i}+\frac{R_k}{R_{i+1}}\right]\right\}^{\frac{1}{R_k}} &\leq \exp(-D_k)\\
\Leftrightarrow 1+\frac{y_k}{y_i}-\left[\sum_{j=i+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)\frac{y_j}{y_i}+\frac{R_k}{R_{i+1}}\right] &\leq \exp(-R_kD_k)\\
\Leftrightarrow \left[1-\frac{R_k}{R_{i+1}}-\exp(-R_kD_k)\right]y_i-\sum_{j=i+1}^{k-1}\left(\frac{R_k}{R_{j+1}}-\frac{R_k}{R_j}\right)y_j+y_k &\leq 0,
\end{aligned}$$

which are Constraints (23) of problem SSP2. Hence, whenever Constraints (23) hold, Constraints (17) of problem SSP0 also hold.

From Lemma 1, we have

$$\prod_{j=1}^{n+1}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \geq B_3(0,n+1)=\left[\sum_{j=1}^{n}\frac{R_{n+1}}{R_j}\left(\frac{y_{j-1}}{y_{n+1}}-\frac{y_j}{y_{n+1}}\right)+\frac{y_n}{y_{n+1}}\right]^{-\frac{1}{R_{n+1}}}.$$

Consequently, if

$$\left[\sum_{j=1}^{n}\frac{R_{n+1}}{R_j}\left(\frac{y_{j-1}}{y_{n+1}}-\frac{y_j}{y_{n+1}}\right)+\frac{y_n}{y_{n+1}}\right]^{-\frac{1}{R_{n+1}}} \geq \exp(-\Delta)$$

holds, then

$$\prod_{j=1}^{n+1}\left(\frac{y_j}{y_{j-1}}\right)^{\frac{1}{R_j}} \geq \exp(-\Delta)$$

holds, which is Constraint (18) of problem SSP0. Observe that

$$\begin{aligned}
\left[\sum_{j=1}^{n}\frac{R_{n+1}}{R_j}\left(\frac{y_{j-1}}{y_{n+1}}-\frac{y_j}{y_{n+1}}\right)+\frac{y_n}{y_{n+1}}\right]^{-\frac{1}{R_{n+1}}} &\geq \exp(-\Delta)\\
\Leftrightarrow \sum_{j=1}^{n}\frac{R_{n+1}}{R_j}\left(\frac{y_{j-1}}{y_{n+1}}-\frac{y_j}{y_{n+1}}\right)+\frac{y_n}{y_{n+1}} &\leq \exp(R_{n+1}\Delta)\\
\Leftrightarrow \sum_{j=1}^{n}\frac{R_{n+1}}{R_j}(y_{j-1}-y_j)+y_n-y_{n+1}\exp(R_{n+1}\Delta) &\leq 0,
\end{aligned}$$

which is Constraint (24) of problem SSP2. Hence, whenever Constraint (24) holds, Constraint (18) of problem SSP0 also holds. Therefore, we conclude that the feasible set of problem SSP2 is contained in the feasible set of problem SSP0, and hence problem SSP2 provides a lower bound on the optimal value of the original problem SSP0. □

Proof of Theorem 3: First, from the one-to-one correspondence of C_i and y_i, for $i = 0, 1, \cdots, n+1$, we can change the conditions $C_0 = 0 \leq C_1 \leq C_2 \leq \cdots \leq C_l$ and $C_l \leq C_{l+1}, C_{l+2}, \ldots, C_n \leq C_{n+1}$ into $y_0 = 1 \geq y_1 \geq y_2 \geq \cdots \geq y_l$ and $y_l \geq y_{l+1}, y_{l+2}, \ldots, y_{n+1}$.

Note that if $l = n+1$, i.e., the partially given sequence includes all the tasks, then problem PSSP1 is just problem SSP1. Now, we consider the case $l < n+1$. Observe that (26) is exactly (21) in problem SSP1. For notational convenience, we denote any tasks finished between i and $k \in S_i$ by $i+1, i+2, \cdots, k-1$. Let $i \in \sigma'$ and $k \in \sigma''$. We now show that Constraints (21) with the appropriate Constraints (19) of problem SSP1 imply Constraints (27) of problem PSSP1. To do so, we need the following inequality:

$$\begin{aligned}
\sum_{j=l+1}^{k-1} \frac{R_k}{R_j}(y_{j-1} - y_j) + y_{k-1} - y_l &= \sum_{j=l+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j} \right) y_j + \left(\frac{R_k}{R_{l+1}} - 1 \right) y_l \\
&\leq y_l \left[\sum_{j=l+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j} \right) + \left(\frac{R_k}{R_{l+1}} - 1 \right) \right] \\
&= 0, \qquad (46)
\end{aligned}$$

where the inequality follows from $y_j \leq y_l$, for $j = l+1, \cdots, k-1$ in Constraints (19) of problem SSP1.

Then, from problem SSP1, by substituting Constraints (19) for $y_{i+1} - y_i \leq 0$ and $R_{i+1} \leq R_i$ into Constraints (21), the left-hand-side of Constraints (21) becomes

$$\begin{aligned}
& \sum_{j=i+1}^{k-1} \frac{R_k}{R_j}(y_{j-1} - y_j) + y_{k-1} - y_k \exp(R_k D_k) \\
= & \sum_{j=i+1}^{l} \frac{R_k}{R_j}(y_{j-1} - y_j) + \sum_{j=l+1}^{k-1} \frac{R_k}{R_j}(y_{j-1} - y_j) + y_{k-1} - y_k \exp(R_k D_k) \\
\leq & \sum_{j=i+1}^{l} \frac{R_k}{R_j}(y_{j-1} - y_j) + y_l - y_k \exp(R_{k,\min} D_k),
\end{aligned}$$

where the inequality holds since $R_{k,\min} \leq R_k$ from Inequality (25), and from (46).

Then, since $R_{k,\max} \geq R_k \geq R_{k,\min}$ from Inequality (25), we have

$$\begin{aligned}
& \sum_{j=i+1}^{k-1} \frac{R_k}{R_j}(y_{j-1} - y_j) + y_{k-1} - y_k \exp(R_k D_k) \\
\leq & \sum_{j=i+1}^{l} \frac{R_{k,\max}}{R_j}(y_{j-1} - y_j) + y_l - y_k \exp(R_{k,\min} D_k),
\end{aligned}$$

which is the left-hand-side of Constraints (27) of problem PSSP1. Hence, Constraints (21) with the appropriate Constraints (19) of problem SSP1 imply Constraints (27) of problem PSSP1.

We next show that Constraints (21) with the appropriate Constraints (19) of problem SSP1 imply Constraints (28) of problem PSSP1. Let $i \in \sigma'$ and $k \in \sigma''$ for $k \in S_i$. From Constraints (21) of problem SSP1, we have

$$\begin{aligned} & \sum_{j=i+1}^{k-1} \frac{R_k}{R_j} (y_{j-1} - y_j) + y_{k-1} - y_k \exp(R_k D_k) \\ = \; & y_i \frac{R_k}{R_{i+1}} + \sum_{j=i+1}^{k-1} y_j \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j} \right) - y_k \exp(R_k D_k) \\ \leq \; & y_i \left[\frac{R_k}{R_{i+1}} + \sum_{j=i+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j} \right) \right] - y_k \exp(R_k D_k) \\ \leq \; & y_i - y_k \exp(R_{k,\min} D_k), \end{aligned} \tag{47}$$

which is the left-hand-side of Constraints (28) of problem PSSP1. Therefore, Constraints (21) with the appropriate Constraints (19) of problem SSP1 imply Constraints (28) of problem PSSP1.

Finally, we show Constraints (22) with the appropriate Constraints (19) of problem SSP1 imply Constraints (29) of problem PSSP1. From Constraint (22) of problem SSP1, together with $y_{i+1} \leq y_i$ from Constraints (19),

$$\begin{aligned} & \left[1 - \frac{R_{n+1}}{R_1} - \exp(-R_{n+1}\Delta) \right] y_0 - \sum_{j=1}^{n} \left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j} \right) y_j + y_{n+1} \\ = \; & \left[1 - \frac{R_{n+1}}{R_1} - \exp(-R_{n+1}\Delta) \right] y_0 - \sum_{j=1}^{l} \left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j} \right) y_j - \sum_{j=l+1}^{n} \left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j} \right) y_j + y_{n+1} \\ \leq \; & \left[1 - \frac{R_{n+1}}{R_1} - \exp(-R_{n+1}\Delta) \right] y_0 - \sum_{j=1}^{l} \left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j} \right) y_j - \sum_{j=i}^{n} \left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j} \right) y_j + y_{n+1} \\ = \; & \left[1 - \frac{R_{n+1}}{R_1} - \exp(-R_{n+1}\Delta) \right] y_0 - \sum_{j=1}^{l} \left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j} \right) y_j + \frac{R_{n+1}}{R_i} y_i - \sum_{j=i+1}^{n} \frac{R_{n+1}}{R_{j+1}} (y_j - y_{j+1}) \\ \leq \; & \left[1 - \frac{R_{n+1}}{R_1} - \exp(-R_{n+1}\Delta) \right] y_0 - \sum_{j=1}^{l} \left(\frac{R_{n+1}}{R_{j+1}} - \frac{R_{n+1}}{R_j} \right) y_j + \frac{R_{n+1}}{R_i} y_i, \end{aligned}$$

where the first inequality follows from $i \geq l+1$. Note that for $i \in \sigma''$, we have $R_{i,\min} \leq R_i$. Therefore, Constraint (22) of problem SSP1 implies Constraints (29) of problem PSSP1.

As a result, the polytope of problem SSP1 is contained in the polytope of problem PSSP1, and thus the maximum value of problem PSSP1 is an upper bound on the maximum ENPV of the project scheduling problem. □

Proof of Theorem 4: Observe that for $i, k \in \sigma'$ where $k \in S_i$, since the partial sequence between i and k is given, Constraints (33) of problem PSSP2 are the same as Constraints (23) of problem

SSP2. For Constraint (34), since $l \le k-1$, from Constraints (19) and (23),

$$\begin{aligned} & \left[1 - \frac{R_k}{R_{i+1}} - \exp(-R_k D_k)\right] y_i - \sum_{j=i+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j}\right) y_j + y_k \\ \le\ & \left[1 - \frac{R_k}{R_{i+1}} - \exp(-R_k D_k)\right] y_i - \sum_{j=i+1}^{l} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j}\right) y_j + y_k. \end{aligned}$$

For a function $h(x) = \frac{x}{R_{i+1}} + \exp(-xD_k)$ for $x \ge 0$, we have $h'(x) = \frac{1}{R_{i+1}} - D_k \exp(-xD_k)$. Then, whenever $D_k R_{i+1} \le 1$, we have $h'(x) \ge 0$. Now, $D_k R_{i+1} \le 1$ is guaranteed by condition (32). As a result, $1 - \frac{R_k}{R_{i+1}} - \exp(-R_k D_k) \le 1 - \frac{R_{k,\min}}{R_{i+1}} - \exp(-R_{k,\min} D_k)$. Therefore,

$$\begin{aligned} & \left[1 - \frac{R_k}{R_{i+1}} - \exp(-R_k D_k)\right] y_i - \sum_{j=i+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j}\right) y_j + y_k \\ \le\ & \left[1 - \frac{R_{k,\min}}{R_{i+1}} - \exp(-R_{k,\min} D_k)\right] y_i - \sum_{j=i+1}^{l} \left(\frac{R_{k,\min}}{R_{j+1}} - \frac{R_{k,\min}}{R_j}\right) y_j + y_k. \end{aligned}$$

Then, Constraints (34) of problem PSSP2 imply Constraints (23) and the appropriate Constraints (19) of problem SSP2.

For Constraints (35) of problem PSSP2, the left-hand-side of Constraints (23) is

$$\begin{aligned} & \left[1 - \frac{R_k}{R_{i+1}} - \exp(-R_k D_k)\right] y_i - \sum_{j=i+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j}\right) y_j + y_k \\ \le\ & \left[1 - \frac{R_k}{R_{i+1}} - \exp(-R_k D_k)\right] y_i + y_k \\ \le\ & \left[1 - \frac{R_{k,\min}}{R_{i+1}} - \exp(-R_{k,\min} D_k)\right] y_i + y_k \\ \le\ & \left[1 - \frac{R_{k,\min}}{R_{i,\max} - r_i} - \exp(-R_{k,\min} D_k)\right] y_i + y_k, \end{aligned}$$

where the second inequality follows from the fact that $h(x) = \frac{x}{R_{i+1}} + \exp(-xD_k)$ is increasing in x for $x \ge 0$, and the third inequality follows from the fact that $R_{i,\max} - r_i \ge R_{i+1}$, as a consequence of $R_{i,\max} \ge R_i$ from Inequality (25). Then, Constraints (35) of problem PSSP2 imply Constraints (23) and the appropriate constraints of (19) of problem SSP2.

We now show that Constraint (36) of problem PSSP2 implies Constraint (24) of problem SSP2, by applying (47) in the proof of Theorem 3 to the tasks in σ''. From (47), we have

$$\sum_{j=i+1}^{k-1} \frac{R_k}{R_j} (y_{j-1} - y_j) + y_{k-1} \ \le\ y_i \left[\frac{R_k}{R_{i+1}} + \sum_{j=i+1}^{k-1} \left(\frac{R_k}{R_{j+1}} - \frac{R_k}{R_j}\right)\right] = y_i.$$

As a result, we have

$$\sum_{j=1}^{l} \frac{R_{n+1}}{R_j} (y_{j-1} - y_j) + y_l - y_{n+1} \exp(R_{n+1}\Delta)$$

$$\geq \sum_{j=1}^{l} \frac{R_{n+1}}{R_j} (y_{j-1} - y_j) + \sum_{j=l+1}^{n} \frac{R_{n+1}}{R_j} (y_{j-1} - y_j) + y_n - y_{n+1} \exp(R_{n+1}\Delta)$$
$$= \sum_{j=1}^{n} \frac{R_{n+1}}{R_j} (y_{j-1} - y_j) + y_n - y_{n+1} \exp(R_{n+1}\Delta).$$

Thus, when Constraint (36) of problem PSSP2 holds, i.e., $\sum_{j=1}^{l} \frac{R_{n+1}}{R_j} (y_{j-1} - y_j) + y_l - y_{n+1} \exp(R_{n+1}\Delta) \leq 0$, Constraint (24) of problem SSP2 also holds, i.e., $\sum_{j=1}^{n} \frac{R_{n+1}}{R_j} (y_{j-1} - y_j) + y_n - y_{n+1} \exp(R_{n+1}\Delta) \leq 0$. Hence, Constraint (36) of problem PSSP2 implies Constraint (24) and the appropriate Constraints (19) of problem SSP2.

As a result, the polytope of problem PSSP2 is contained in the polytope of problem SSP2. Hence, the optimal value of problem PSSP2 is a lower bound on the maximum ENPV of the project scheduling problem. □

Proof of Theorem 5. Let $C_i(\sigma') = C_i^*(\sigma)$ for any $i \in V \setminus \{l, m\}$, $C_l(\sigma') = C_m^*(\sigma)$ and $C_m(\sigma') = C_l^*(\sigma)$. Note that $\mathbf{C}(\sigma')$ defines a feasible schedule, since task l has no successors that are not shared with task m, and task m has no predecessors that are not shared with task l, $C_l^*(\sigma) \leq C_m^*(\sigma)$ as implied by $l < m$, and $D_l \geq D_m$. Note that the ENPV value of a task i in σ under completion time $\mathbf{C}^*(\sigma)$ is

$$\text{ENPV}_i^*(\sigma) = F_i \exp\left(-\sum_{j=1}^{i} R_j(C_j^*(\sigma) - C_{j-1}^*(\sigma))\right).$$

Then, ENPV$_i(\sigma')$ =ENPV$_i^*(\sigma)$ for $0 \leq i < l$. Since $r_l = r_m$, the total risk profile does not change due to the interchange of tasks. Then, for $l < i < m$, we have

$$\begin{aligned}
\text{ENPV}_i^*(\sigma) &= F_i \exp\Big(-\sum_{j=1}^{l-1} R_j(C_j^*(\sigma) - C_{j-1}^*(\sigma)) - R_l(C_l^*(\sigma) - C_{l-1}^*(\sigma)) \\
&\quad - R_{l+1}(C_{l+1}^*(\sigma) - C_l^*(\sigma)) - \sum_{j=l+2}^{i} R_j(C_j^*(\sigma) - C_{j-1}^*(\sigma))\Big) \\
&= F_i \exp\Big(-\sum_{j=1}^{l-1} R_j(C_j(\sigma') - C_{j-1}(\sigma')) - R_l(C_m(\sigma') - C_{l-1}(\sigma')) \\
&\quad - R_{l+1}(C_{l+1}(\sigma') - C_m(\sigma')) - \sum_{j=l+2}^{i} R_j(C_j(\sigma') - C_{j-1}(\sigma'))\Big) = \text{ENPV}_i(\sigma').
\end{aligned}$$

Similarly, the conclusion holds for $m < i \leq n+1$. For ease of exposition, we let δ_i denote the discount coefficient of cash flow F_i, for $i = l, m$. Again, since the risk profile does not change from the setting of $\mathbf{C}(\sigma')$, we have ENPV$_l^*(\sigma)$+ENPV$_m^*(\sigma)$ $= F_l\delta_l + F_m\delta_m$ and ENPV$_l(\sigma')$+ENPV$_m(\sigma')$ $= F_m\delta_l + F_l\delta_m$. Since $F_l\delta_l + F_m\delta_m - F_m\delta_l - F_l\delta_m = (F_l - F_m)(\delta_l - \delta_m) \leq 0$, we have ENPV$_l^*(\sigma)$+ENPV$_m^*(\sigma)$ $\leq$ENPV$_l(\sigma')$+ENPV$_m(\sigma')$. □

Definition of the Kendall tau Rank Correlation Coefficient

Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a set of n different evaluations. A pair of evaluations (x_i, y_i) and (x_j, y_j)

are *concordant* if $x_i > x_j$ and $y_i > y_j$, or $x_i < x_j$ and $y_i < y_j$; and are *discordant* if $x_i > x_j$ and $y_i < y_j$, or $x_i < x_j$ and $y_i > y_j$. Then, the Kendall coefficient τ, where $-1 \leq \tau \leq 1$, is defined as

$$\tau = \frac{2(\text{number of concordant pairs} - \text{number of discordant pairs})}{n(n-1)}. \quad \square$$

Cash Flow Generation in the Computational Study

In our formulations in Section 4, we assume that the cash flow of the end-of-project dummy task is 0. However, with a minor adjustment to those formulations, we can allow the cash flow of the end-of-project dummy task to be nonzero. For the results in Table 2 with two positive cash flows, the cash flows are generated as follows. First, we generate $n_c - 2$ negative cash flows from the continuous uniform distribution $U[-1.0, 0.0]$, and let C_N denote their sum. Second, a positive cash flow in the amount of $-C_N$ is assigned both to the end-of-project dummy task and to a mid-project task with average depth of $\frac{2}{3}((n_c - 2)I + 2) = (n_c + 2)/3$. Note that, including the end-of-project dummy task, the project depth is $(n_c - 2)I + 2$, and $I = 0.5$. If $(n_c + 2)/3$ is integer, then the positive cash flow is assigned to a task with depth of $(n_c + 2)/3$; whereas, if $(n_c + 2)/3$ is not integer, then the positive cash flow is assigned to a task with depth either $\lfloor (n_c + 2)/3 \rfloor$ or $\lceil (n_c + 2)/3 \rceil$, with an average depth of $(n_c + 2)/3$.

For the results in Table 3 with $n_p \in \{1, 2, 3\}$, we first generate $n_c - n_p$ negative cash flows from the continuous uniform distribution $U[-1.0, 0.0]$, and find their sum C_N. We index the tasks topologically with the last task n_c as a dummy task that completes the project. Then, we assign a positive cash flow in the amount of $-2C_N$ to task n_c, if $n_p = 1$; of $-C_N$ to tasks $n_c/2$ and n_c, if $n_p = 2$; and of $-2C_N/3$ to tasks $(n_c - 1)/3$, $2(n_c - 1)/3$ and n_c, if $n_p = 3$. Recall that $n_c = 16$, and hence all the task indices are integer. Finally, the negative cash flows are assigned to the other $n_c - n_p$ tasks. $\square$