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ON POINTWISE CONVERGENCE OF SCHRÖDINGER MEANS

EVANGELOS DIMOU AND ANDREAS SEEGER

Abstract. For functions in the Sobolev space H^s and decreasing sequences $t_n \to 0$ we examine convergence almost everywhere of the generalized Schrödinger means on the real line, given by

$$S^a f(x, t_n) = \exp(it_n(-\partial_{xx})^{a/2}) f(x);$$

here $a > 0$, $a \neq 1$. For decreasing convex sequences we obtain a simple characterization of convergence a.e. for all functions in H^s when $0 < s < \min\{a/4, 1/4\}$ and $a \neq 1$. We prove sharp quantitative local and global estimates for the associated maximal functions. We also obtain sharp results for the case $a = 1$.

§1. *Introduction.* For Schwartz functions f defined on the real line consider the initial value problem

$$i\partial_t u(x, t) + (-\partial_{xx})^{a/2} u(x, t) = 0, \quad u(x, 0) = f(x);$$

so that for $a = 2$ we recover the Schrödinger equation. The solutions are given by

$$S^a f(x, t) = \int_{\mathbb{R}} e^{i(x\xi + t|\xi|^a)} \widehat{f}(\xi) \, \frac{d\xi}{2\pi},$$

and, for fixed time, the solution operator extends to all $f \in H^s$, where H^s is the Sobolev space of all distributions f with $\|f\|_{H_s} := \big(\int (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi\big)^{1/2} < \infty$.

One refers to the operators $f \mapsto S^a f(\cdot, t)$ as generalized Schrödinger means. For Schwartz functions f it is clear that $\lim_{t\to 0} S^a f(x, t) = f(x)$ and that the convergence is uniform in x. One is interested in almost everywhere (a.e.) convergence for functions in H^s for suitable $s > 0$. Following the fundamental result by Carleson [**2**], many authors have considered this question. It was shown in [**2**, **15**] that

$$\lim_{t\to 0} S^a f(x, t) = f(x) \ \text{ a.e.}, \quad f \in H^{1/4},$$

when $a > 1$ and this result fails for some $f \in H^s$, if $s < 1/4$ [**5**, **15**]. Related results are in [**4**, **10**, **14**, **20**]. If $0 < a < 1$, pointwise convergence for $f \in H^s$ holds when $s > a/4$ and may fail for $f \in H^s$ when $s < a/4$; see [**21**]. We remark that the problem in higher dimensions is much harder and not considered here. For the Schrödinger equation in higher dimensions a complete solution up to endpoints has been recently found in [**7**, **8**] and relies on sophisticated methods from Fourier restriction theory. We refer to these papers for more references and a historical prospective.

In this paper, we consider, in one spatial dimension, the question of the solution converging to the initial data when the limit is taken over a decreasing sequence $\{t_n\}_{n=1}^{\infty}$, converging to zero. Here we always use the term "decreasing" as synonymous with "nonincreasing." Given such a sequence we seek to find the precise range of s such that $\lim_{n\to\infty} S^a f(x, t_n) = f(x)$ a.e. holds for every $f \in H^s$. This is partially motivated by the work [**3**] on approach regions for

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pointwise convergence for solutions of the Schrödinger equation, and also by the work [13] on the pointwise convergence of spherical means of L^p functions (although the mathematical issues and expected outcomes for the latter problem are different).

For the class of convex decreasing sequences and any $s \in (0, \min\{a/4, 1/4\})$ we obtain a complete characterization of when pointwise convergence holds for all $f \in H^s$. This characterization involves the Lorentz space $\ell^{r,\infty}(\mathbb{N})$. By definition, for $0 < r < \infty$,

$$\{t_n\} \in \ell^{r,\infty} \quad \Longleftrightarrow \quad \sup_{b>0} b^r \#\{n \in \mathbb{N} : |t_n| > b\} < \infty.$$

Note that $\ell^{r_1,\infty}(\mathbb{N}) \subset \ell^{r_2}(\mathbb{N}) \subset \ell^{r_2,\infty}(\mathbb{N}) \subset \ell^{\infty}(\mathbb{N})$ if $r_1 < r_2 < \infty$ and all inclusions are strict. A model example is given by $t_n = n^{-\gamma}$ which belongs to $\ell^{r,\infty}$ if and only if $r \geqslant 1/\gamma$. Another example is $\{n^{-\gamma} \log n\}$ which belongs to $\ell^{r,\infty}$ if and only if $r > 1/\gamma$.

THEOREM 1.1. *Let $a > 0$, $a \neq 1$, and assume $0 < s < \min\{a/4, 1/4\}$. Let $\{t_n\}_{n=1}^{\infty}$ be a decreasing sequence with $\lim_{n\to\infty} t_n = 0$ and assume that $t_n - t_{n+1}$ is also decreasing. Then the following four statements are equivalent.*

(a) *The sequence $\{t_n\}$ belongs to $\ell^{r(s),\infty}(\mathbb{N})$, where $r(s) = \frac{2s}{a-4s}$.*

(b) *There is a constant C_1 such that for all $f \in H^s$ and for all sets B of diameter at most 1 we have*

$$\Big\| \sup_{n\in\mathbb{N}} |S^a f(x, t_n)| \Big\|_{L^2(B)} \leqslant C_1 \|f\|_{H^s}.$$

(c) *There is a constant C_2 such that for all $f \in H^s$, for all sets B of diameter at most 1, and for all $\alpha > 0$,*

$$\operatorname{meas}\left(\left\{x \in B : \sup_{n\in\mathbb{N}} |S^a f(x, t_n)| > \alpha\right\}\right) \leqslant C_2\, \alpha^{-2} \|f\|_{H^s}^2.$$

(d) *For every $f \in H^s$ we have*

$$\lim_{n\to\infty} S^a f(x, t_n) = f(x) \quad a.e.$$

Here and in what follows we write $\operatorname{meas}(A)$ for the Lebesgue measure of $A \subset \mathbb{R}$. The equivalence of (b) and (c) seems nontrivial, and we do not have a direct proof for it, without going through condition (a). In Theorem 1.1 the convexity assumption can be dropped for the sufficiency, i.e. statements (b), (c), (d) hold whenever t_n is decreasing and belongs to $\ell^{\frac{2s}{a-4s},\infty}(\mathbb{N})$; see Proposition 2.3.

Regarding the maximal function inequalities we also have a global version:

THEOREM 1.2. *Let $a > 0$, $a \neq 1$, and assume $0 < s < a/4$. Let $\{t_n\}_{n=1}^{\infty}$ be a decreasing sequence with $\lim_{n\to\infty} t_n = 0$, and assume that $t_n - t_{n+1}$ is also decreasing. Then the following statements (a), (b), (c) are equivalent.*

(a) *The sequence $\{t_n\}$ belongs to $\ell^{\frac{2s}{a-4s},\infty}(\mathbb{N})$.*

(b) *There is a constant C_1 such that for all $f \in H^s$ we have*

$$\Big\| \sup_{n\in\mathbb{N}} |S^a f(x, t_n)| \Big\|_{L^2(\mathbb{R})} \leqslant C_1 \|f\|_{H^s}.$$

(c) *There is a constant C_2 such that for all $f \in H^s$ and all $\alpha > 0$,*

$$\operatorname{meas}\left(\left\{x \in \mathbb{R} : \sup_{n\in\mathbb{N}} |S^a f(x, t_n)| > \alpha\right\}\right) \leqslant C_2\, \alpha^{-2} \|f\|_{H^s}^2.$$

We contrast the above results with the exceptional case $a = 1$ which covers solutions of the wave equation. Now the critical $r(s) = \frac{2s}{a-4s}$ in Theorem 1.1 has to be replaced with the smaller $\frac{2s}{1-2s}$, for all $s < 1/2$. Note that S^1 corresponds to a family of translation operators, when acting on functions with spectrum in $[0, \infty)$ or $(-\infty, 0]$. The analysis is somewhat similar to the one for spherical means in [13]; see also [12]. For $a = 1$ we have

THEOREM 1.3. *Let $0 < s < 1/2$ and let $\{t_n\}_{n=1}^{\infty}$ be a decreasing sequence with $\lim_{n\to\infty} t_n = 0$ such that $t_n - t_{n+1}$ is also decreasing. Then the following four statements are equivalent.*

(a) *The sequence $\{t_n\}$ belongs to $\ell^{\rho(s),\infty}(\mathbb{N})$, where $\rho(s) = \frac{2s}{1-2s}$.*

(b) *There is a constant C_1 such that for all $f \in H^s$ we have*

$$\Big\| \sup_{n\in\mathbb{N}} \big| S^1 f(x, t_n) \big| \Big\|_{L^2(\mathbb{R})} \leqslant C_1 \|f\|_{H^s}.$$

(c) *There is a constant C_2 such that for all $f \in H^s$, for all sets B of diameter at most 1, and for all $\alpha > 0$,*

$$\operatorname{meas}\Big(\Big\{ x \in B : \sup_{n\in\mathbb{N}} \big| S^1 f(x, t_n) \big| > \alpha \Big\} \Big) \leqslant C_2 \, \alpha^{-2} \|f\|_{H^s}^2.$$

(d) *For every $f \in H^s$ we have*

$$\lim_{n\to\infty} S^1 f(x, t_n) = f(x) \quad a.e.$$

The convexity condition is satisfied for the model case $t_n = n^{-\gamma}$ with $\gamma > 0$ and thus Theorems 1.1 and 1.3, and the known results for $s = 1/4$, when $a > 1$, yield.

COROLLARY 1.4. *Let $0 < \gamma < \infty$.*

(i) *If $a > 1$, then $\lim_{n\to\infty} S^a f(x, n^{-\gamma}) = f(x)$ a.e. holds for every $f \in H^s(\mathbb{R})$ if and only if $s \geqslant \min\{\frac{a}{2\gamma+4}, \frac{1}{4}\}$.*

(ii) *If $0 < a < 1$, then $\lim_{n\to\infty} S^a f(x, n^{-\gamma}) = f(x)$ a.e. holds for every $f \in H^s(\mathbb{R})$ if and only if $s \geqslant \frac{a}{2\gamma+4}$.*

(iii) *If $a = 1$, then $\lim_{n\to\infty} S^1 f(x, n^{-\gamma}) = f(x)$ a.e. holds for every $f \in H^s(\mathbb{R})$ if and only if $s \geqslant \frac{1}{2\gamma+2}$.*

This answer for the sequence $\{n^{-\gamma}\}$ reveals a perhaps surprising phenomenon for the case $a > 1$, namely that there is a gain over the general pointwise convergence result when $\gamma > 2(a-1)$, but not when $0 < \gamma \leqslant 2(a-1)$. When $0 < a \leqslant 1$, we have for all $\gamma \in (0, \infty)$ a gain over the general convergence result. The same remarks apply to the local $H^s \to L^2(B)$ maximal inequality. In contrast we get for the global maximal operator and $a \neq 1$:

COROLLARY 1.5. *Let $0 < \gamma < \infty$ and $a \in (0, \infty) \setminus \{1\}$. Then the global maximal function inequality*

$$\Big\| \sup_{n} \big| S^a f(x, n^{-\gamma}) \big| \Big\|_{L^2(\mathbb{R})} \leqslant C \|f\|_{H^s}$$

holds for some $C > 0$ and all $f \in H^s$ if and only if $s \geqslant \frac{a}{2\gamma+4}$.

Remarks. (i) Our results for the special case $\{n^{-\gamma}\}$, $a \neq 1$ as stated in Corollaries 1.4 and 1.5 were already incorporated in the 2016 thesis [**6**] of the first author. Moreover sufficiency in Theorem 1.1, merely for decreasing sequences but under the more restrictive assumption $\{t_n\} \in \ell^r$ for $r < \frac{2s}{a-4s}$, follows already from [**6**, Proposition 1.6].

(ii) The problem of convergence of Schrödinger means $S^a(f, t_n)$ for a decreasing sequence $\{t_n\}$ was independently considered in recent papers by Sjölin [**17**] and by Sjölin and Strömberg [**18**]. Their conditions are more restrictive, but apply in all dimensions. In [**17**] it is proved for $a > 1$ that the condition $\{t_n\} \in \ell^{2s/a}$ is sufficient for pointwise convergence. This is improved in [**18**] where for $s \leqslant 1/2$, $a > 2s$, the condition $\{t_n\} \in \ell^r$ for $r < \frac{2s}{a-2s}$ is shown to be sufficient for pointwise convergence. Proposition 2.3 yields an improvement of these results and Theorem 1.1 gives the optimal result for decreasing convex sequences.

(iii) For $a \neq 1$ there are natural analogous open questions of necessary and sufficient conditions in higher dimensions, given the recent groundbreaking results for the full local Schrödinger maximal operator in [**7**, **8**] which are sharp up to endpoints.

(iv) For $0 < a < 1$ there is still the open problem whether $S^a f(x, t) \to f(x)$ a.e. holds for all $f \in H^{a/4}(\mathbb{R})$. Likewise there is the problem of a global bound for the maximal function if $s = a/4$, and $a > 1$. One can show using a variant of the arguments in [**16**, **21**] that a.e. convergence holds in the Besov space $B^{a/4}_{2,1}(\mathbb{R})$ which is properly contained in $B^{a/4}_{2,2} \equiv H^{a/4}$; see Proposition 2.4. For the case $a = 1$ we have pointwise convergence in $B^{1/2}_{2,1}(\mathbb{R})$, but pointwise convergence fails for some functions in $H^{1/2}(\mathbb{R})$; see Proposition 4.3.

This paper. In § 2 we show for decreasing sequences that the $\ell^{r(s),\infty}$ condition is sufficient for pointwise convergence and the appropriate boundedness properties of the maximal operators. The necessity for decreasing convex sequences (converging to 0) is proved in § 3. The case $a = 1$ is separately considered in § 4. We include a short Appendix regarding the relevant application of Stein–Nikishin theory.

§2. *Upper bounds for maximal functions.* In the present section we prove maximal function results which imply the positive results of the theorems stated in the introduction. We already know the local estimate

$$\Big\| \sup_{t\in[0,1]} |S^a f(\cdot, t)| \Big\|_{L^2(B)} \leqslant C\|f\|_{H^{1/4}}, \tag{2.1}$$

which was established by Kenig and Ruiz [**11**] when $a = 2$ and Sjölin [**15**] for general $a > 1$. In view of (2.1) it now suffices to give the proof of the $L^2(\mathbb{R})$ bound in part (b) of Theorem 1.2, under the assumption of $\{t_n\} \in \ell^{\frac{2s}{a-4s}}$, whenever $s < a/4$.

Throughout this section we assume that $\{t_n\}$ is decreasing but we drop the convexity assumption in the introduction. Without loss of generality (dropping a finite number of terms in the sequence) we can assume that $t_n \in (0, 1)$ for all $n \in \mathbb{N}$. We first restrict our attention to the frequency-localized operator

$$S^a_\lambda f(x, t) = \int_{\mathbb{R}} e^{i(x\xi + t|\xi|^a)} \widehat{f}(\xi)\chi(\xi/\lambda)\,\frac{d\xi}{2\pi},$$

where $\chi \in C^\infty$ is a real-valued, smooth function, supported in $\{1/2 \leqslant |\xi| \leqslant 1\}$. The following result is a variant of the inequality given in [**11**].

PROPOSITION 2.1. *If $J \subseteq [0,1]$ is an interval and $0 < a \neq 1$, then*

$$\Big\| \sup_{t\in J} \big|S^a_\lambda f(\cdot,t)\big| \Big\|_{L^2(\mathbb{R})} \leqslant C(1+|J|^{1/4}\lambda^{a/4})\|f\|_2.$$

Proof. We use the Kolmogorov–Seliverstov–Plessner method, by linearizing the maximal operator: let $x \mapsto t(x)$ be a measurable function, with values in J. It will then suffice to prove

$$\Big(\int_{\mathbb{R}} |S^a_\lambda(x,t(x))|^2\,dx\Big)^{1/2} \leqslant C(1+|J|^{1/4}\lambda^{a/4})\|f\|_2,$$

where the constant C is independent of $t(\cdot)$ and f. Note that

$$S^a_\lambda f(x,t(x)) = \int e^{i(x\xi+t(x)|\xi|^a)}\widehat{f}(\xi)\chi(\xi/\lambda)\,\tfrac{d\xi}{2\pi} = \lambda\, T^a_\lambda[\widehat{f}(\lambda\cdot)](x),$$

where

$$T^a_\lambda g(x) = \int e^{i(\lambda x\xi+\lambda^a t(x)|\xi|^a)}\chi(\xi)g(\xi)\,\tfrac{d\xi}{2\pi}.$$

Since $\|\widehat{f}(\lambda\cdot)\|_2 = c\lambda^{-1/2}\|f\|_2$ we need to show that

$$\|T^a_\lambda\|_{L^2\to L^2} \lesssim \lambda^{(a-2)/4}|J|^{1/4}+\lambda^{-1/2},$$

which in turn follows from

$$\|T^a_\lambda(T^a_\lambda)^*\|_{L^2\to L^2} \lesssim \lambda^{(a-2)/2}|J|^{1/2}+\lambda^{-1}. \tag{2.2}$$

The kernel of $T^a_\lambda(T^a_\lambda)^*$ is

$$K^a_\lambda(x,y) = \int e^{i[\lambda(x-y)\xi+\lambda^a(t(x)-t(y))|\xi|^a]}\chi^2(\xi)\,\tfrac{d\xi}{2\pi},$$

and the derivative of the phase $\Phi^a_\lambda(\xi) = \lambda(x-y)\xi+\lambda^a(t(x)-t(y))|\xi|^a$ is equal to

$$(\Phi^a_\lambda)'(\xi) = \lambda(x-y)+a\lambda^a(t(x)-t(y))\,(\operatorname{sign}\xi)\,|\xi|^{a-1}.$$

Therefore, if $|x-y| \gg \lambda^{a-1}|t(x)-t(y)|$, we have that $|(\Phi^a_\lambda)'(\xi)| \gtrsim \lambda|x-y|$ and integration by parts gives

$$|K^a_\lambda(x,y)| \lesssim_N (\lambda|x-y|)^{-N}.$$

In the case where $|x-y| \lesssim \lambda^{a-1}|t(x)-t(y)|$ we use van der Corput's lemma. The second derivative of the phase is $(\Phi^a_\lambda)''(\xi) = c_a\lambda^a(t(x)-t(y))|\xi|^{a-2}$, hence

$$|K^a_\lambda(x,y)| \lesssim \lambda^{-a/2}|t(x)-t(y)|^{-1/2} \lesssim (\lambda|x-y|)^{-1/2}.$$

Thus $\int_{\mathbb{R}}|K^a_\lambda(x,y)|\,dy$ can be estimated by

$$\int\limits_{\substack{|x-y|\lesssim\\ \lambda^{a-1}|t(x)-t(y)|}} \lambda^{-1/2}|x-y|^{-1/2}dy + \int\limits_{\substack{|x-y|\gg\\ \lambda^{a-1}|t(x)-t(y)|}} (1+\lambda|x-y|)^{-N}dy$$

$$\leqslant \int_{|x-y|\lesssim\lambda^{a-1}|J|}\lambda^{-1/2}|x-y|^{-1/2}dy + \int_{\mathbb{R}}(1+\lambda|x-y|)^{-N}dy \lesssim \lambda^{(a-2)/2}|J|^{1/2}+\lambda^{-1}.$$

Therefore $\sup_{x\in\mathbb{R}}\int|K^a_\lambda(x,y)|\,dy \lesssim \lambda^{(a-2)/2}|J|^{1/2}+\lambda^{-1}$ and by symmetry we get the same bound for $\sup_{y\in\mathbb{R}}\int|K^a_\lambda(x,y)|\,dx$. Hence Schur's test gives the required bound (2.2). □

We now use Proposition 2.1 to prove a sharp result for the frequency-localized operators S^a_λ.

LEMMA 2.2. *Let $0 < a \neq 1$, $0 < r < \infty$ and let $\{t_n\}$ be a sequence in $[0,1]$ which belongs to $\ell^{r,\infty}$. Then for $\lambda > 1$*

$$\Big\| \sup_n \big| S^a_\lambda f(\cdot, t_n) \big| \Big\|_{L^2(\mathbb{R})} \leqslant C \lambda^{\frac{ar}{2+4r}} \|f\|_{L^2(\mathbb{R})}.$$

Proof. We start by writing

$$\Big\| \sup_n \big| S^a_\lambda f(\cdot, t_n) \big| \Big\|_2 \leqslant \Big\| \sup_{n:\, t_n \leqslant b} \big| S^a_\lambda f(\cdot, t_n) \big| \Big\|_2 + \Big\| \sup_{n:\, t_n > b} \big| S^a_\lambda f(\cdot, t_n) \big| \Big\|_2 .$$

By Proposition 2.1 we can bound the first term by $b^{1/4} \lambda^{a/4} \|f\|_2$. On the other hand, using Plancherel's theorem and our assumption, we get

$$\Big\| \sup_{n:\, t_n > b} \big| S^a_\lambda f(\cdot, t_n) \big| \Big\|_{L^2(\mathbb{R})} \leqslant \Bigg(\sum_{n:\, t_n > b} \|S^a_\lambda f(\cdot, t_n)\|_2^2 \Bigg)^{1/2} \leqslant \#(\{n : t_n > b\})^{1/2} \|f\|_2 \lesssim b^{-r/2} \|f\|_2.$$

We therefore have

$$\Big\| \sup_n |S^a_\lambda f(\cdot, t_n)| \Big\|_2 \lesssim (b^{1/4} \lambda^{a/4} + b^{-r/2}) \|f\|_2$$

and choosing b such that $b^{1/4} \lambda^{a/4} = b^{-r/2}$, namely $b = \lambda^{-\frac{a}{1+2r}}$, finishes the proof. □

We wish to apply the Lemma 2.2 for $\lambda = 2^k$, $k > 1$. A more refined argument is needed to combine the dyadic scales.

PROPOSITION 2.3. *Let $0 < a \neq 1$, and assume that $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$ is decreasing. Then*

$$\Big\| \sup_n |S^a f(\cdot, t_n)| \Big\|_{L^2(\mathbb{R})} \leqslant C \|f\|_{H^s}, \quad s = \frac{ar}{2+4r}. \tag{2.3}$$

Moreover $S^a f(x, t_n) \to f(x)$ a.e. whenever $f \in H^\sigma$ for $\sigma \geqslant \min\{\frac14, \frac{ar}{2+4r}\}$.

Proof. Define projection operators P_k by

$$\widehat{P_0 f}(\xi) = \mathbb{1}_{[-1/2,1/2]}(\xi) \widehat{f}(\xi)$$

$$\widehat{P_k f}(\xi) = (\mathbb{1}_{[2^{k-1},2^k]} + \mathbb{1}_{[-2^k,-2^{k-1}]}) \widehat{f}(\xi), \quad k \geqslant 1$$

Clearly $P_k P_k = P_k$ and $\sum_{k \geqslant 0} P_k f = f$.

Next, for each integer $l \geqslant 0$ we set

$$\mathfrak{N}_l = \big\{ n \in \mathbb{N} : 2^{-(l+1)\frac{a}{1+2r}} < t_n \leqslant 2^{-l\frac{a}{1+2r}} \big\}.$$

By the assumption that $\{t_n\} \in \ell^{r,\infty}$ there is $C > 0$ so that

$$\#(\mathfrak{N}_l) \leqslant C 2^{l\frac{ar}{1+2r}} = C 2^{2ls}. \tag{2.4}$$

We can then write

$$\sup_n |S^a f(x,t_n)| \leqslant \mathcal{E}_1(x) + \mathcal{E}_2(x) + \mathcal{E}_3(x)$$

where

$$\mathcal{E}_1(x) = \sup_l \sup_{n\in\mathfrak{N}_l} \Big| \sum_{k<\frac{l}{1+2r}} S^a P_k f(x,t_n) \Big|$$

$$\mathcal{E}_2(x) = \sup_l \sup_{n\in\mathfrak{N}_l} \Big| \sum_{\frac{l}{1+2r}\leqslant k<l} S^a P_k f(x,t_n) \Big|$$

$$\mathcal{E}_3(x) = \sup_l \sup_{n\in\mathfrak{N}_l} \Big| \sum_{k\geqslant l} S^a P_k f(x,t_n) \Big|$$

We first give the estimate for $\|\mathcal{E}_3\|_2$. We make the change of variable $k = l + m$ and get

$$\mathcal{E}_3(x) \leqslant \sum_{m\geqslant 0} \Big(\sum_{l\geqslant 0} \sup_{n\in\mathfrak{N}_l} |S^a P_{l+m} f(x,t_n)|^2 \Big)^{1/2}.$$

From this,

$$\begin{aligned}\|\mathcal{E}_3\|_2 &\leqslant \sum_{m\geqslant 0} \Big(\sum_{l\geqslant 0} \Big\| \sup_{n\in\mathfrak{N}_l} |S^a P_{l+m} f(\cdot,t_n)| \Big\|_2^2 \Big)^{1/2} \\ &\leqslant \sum_{m\geqslant 0} \Big(\sum_{l\geqslant 0} \sum_{n\in\mathfrak{N}_l} \|S^a P_{l+m} f(\cdot,t_n)\|_2^2 \Big)^{1/2} \\ &\leqslant \sum_{m\geqslant 0} \Big(\sum_{l\geqslant 0} \#(\mathfrak{N}_l) \|P_{l+m} f\|_2^2 \Big)^{1/2}\end{aligned}$$

and using (2.4) this is further estimated by

$$\sum_{m\geqslant 0} \Big(\sum_{l\geqslant 0} 2^{2sl} \|P_{l+m} f\|_2^2 \Big)^{1/2} = \sum_{m\geqslant 0} 2^{-ms} \Big(\sum_{l\geqslant 0} 2^{2s(l+m)} \|P_{l+m} f\|_2^2 \Big)^{1/2} \lesssim \|f\|_{H^s}.$$

In order to deal with the first and second terms we use that by definition of $\mathfrak{N}_l$ the t_n with $n \in \mathfrak{N}_l$ lie in the interval

$$J_l = [0, 2^{-l\frac{a}{1+2r}}].$$

For the term $\mathcal{E}_2$ we make the change of variables $k = l - j$ and estimate

$$\mathcal{E}_2(x) \leqslant \sup_l \sup_{n\in\mathfrak{N}_l} \Big| \sum_{0<j\leqslant\frac{2r}{1+2r}l} S^a P_{l-j} f(x,t_n) \Big|$$

$$\leqslant \left(\sum_{l \geqslant 0} \left(\sum_{0<j\leqslant \frac{2r}{1+2r} l} \sup_{n \in \mathfrak{N}_l} |S^a P_{l-j} f(x,t_n)| \right)^2 \right)^{1/2}$$

$$\leqslant \sum_{j>0} \left(\sum_{l \geqslant j \frac{1+2r}{2r}} \sup_{n \in \mathfrak{N}_l} |S^a P_{l-j} f(x,t_n)|^2 \right)^{1/2}.$$

We can now use Proposition 2.1, with $J = J_l$ amd $\lambda = 2^k = 2^{l-j}$. Note that $l \geqslant j \frac{1+2r}{r}$ implies that $|J_l|^{\frac{1}{4}} 2^{\frac{a}{4}(l-j)} = 2^{-l \frac{a}{4(1+2r)}} 2^{\frac{a}{4}(l-j)} \geqslant 1$. Using $P_k P_k = P_k$ we then get

$$\|\mathcal{E}_2\|_2 \leqslant \sum_{j \geqslant 0} \left(\sum_{l \geqslant j \frac{1+2r}{r}} \left\| \sup_{n \in \mathfrak{N}_l} |S^a P_{l-j} f(\cdot, t_n)| \right\|^2_{L^2(\mathbb{R})} \right)^{1/2}$$

$$\lesssim \sum_{j \geqslant 0} \left(\sum_{l \geqslant j \frac{1+2r}{2r}} \left[(1 + 2^{\frac{a}{4}(l-j)}\, 2^{-l \frac{a}{4(1+2r)}}) \|P_{l-j} f\|_2 \right]^2 \right)^{1/2}$$

$$\lesssim \sum_{j \geqslant 0} \left(\sum_{l \geqslant j \frac{1+2r}{2r}} \left[2^{(l-j)\frac{a}{4}(1-\frac{1}{1+2r})}\, 2^{-j \frac{a}{4(1+2r)}} \|P_{l-j} f\|_2 \right]^2 \right)^{1/2}$$

$$= \sum_{j \geqslant 0} 2^{-j \frac{a}{4(1+2r)}} \left(\sum_{l \geqslant j \frac{1+2r}{2r}} \left[2^{s(l-j)} \|P_{l-j} f\|_2 \right]^2 \right)^{1/2} \lesssim \|f\|_{H^s}.$$

Finally we consider the term $\mathcal{E}_1$ and estimate

$$\|\mathcal{E}_1\|_2 \leqslant \left\| \sup_l \sup_{n \in \mathfrak{N}_l} \sum_{k < \frac{l}{1+2r}} |S^a P_k f(x,t_n)| \right\|_2$$

$$\leqslant \sum_{k \geqslant 0} \left\| \sup_{l > k(1+2r)} \sup_{n \in \mathfrak{N}_l} |S^a P_k f(\cdot, t_n)| \right\|_2$$

The t_n with $n \in \cup_{l > k(1+2r)} \mathfrak{N}_l$ lie in an interval of length $O(2^{-ka})$, namely in $J_{l(k)}$ with $l(k) = \lfloor k(1+2r) \rfloor$. We use again Proposition 2.1, with $J = J_{l(k)}$ and $\lambda = 2^k$, and observe that now $2^{ka/4} |J_{l(k)}|^{1/4} \lesssim 1$. Thus we can bound for each $k > 0$

$$\left\| \sup_{l > k(1+2r)} \sup_{n \in \mathfrak{N}_l} |S^a P_k f(\cdot, t_n)| \right\|_2 \lesssim (1 + 2^{ka/4} |J_{l(k)}|^{1/4}) \|P_k f\|_2 \lesssim \|P_k f\|_2.$$

We sum in k and deduce that

$$\|\mathcal{E}_1\|_2 \lesssim \sum_{k \geqslant 0} \|P_k f\|_2 \leqslant C(s) \|f\|_{H^s}, \quad s > 0.$$

We combine the estimates for $\|\mathcal{E}_i\|$, $i = 1, 2, 3$ to finish the proof of the maximal inequality (2.3). Since $\lim_{t \to 0} S^a f(x,t) = f(x)$ for all $x \in \mathbb{R}$ whenever f is a Schwartz function, and since Schwartz functions are dense in H^σ the stated pointwise convergence result follows from (2.1), (2.3) by a standard argument (see e.g. [**15**] or [**18**]). $\square$

Finally we mention an endpoint result involving the Besov space $B^s_{2,1}(\mathbb{R})$ when $s = a/4$. We do not know whether $B^{a/4}_{2,1}$ can be replaced with $H^{a/4}$ in the following proposition.

PROPOSITION 2.4. *Let $a > 0$, $a \neq 1$. Then, for all $f \in B^{a/4}_{2,1}(\mathbb{R})$,*

$$\Big\| \sup_{t\in[0,1]} |S^a f(\cdot, t)| \Big\|_{L^2(\mathbb{R})} \leqslant C\|f\|_{B^{a/4}_{2,1}}.$$

Proof. Write $f = \sum_{k\geqslant 0} S^a P_k P_k f$ as in the proof of Proposition 2.3. By Proposition 2.1 we have

$$\Big\| \sup_{t\in[0,1]} S^a f \Big\|_2 \leqslant \sum_{k\geqslant 0} \Big\| \sup_{t\in[0,1]} |S^a P_k P_k f| \Big\|_2 \lesssim \sum_{k\geqslant 0} 2^{ka/4} \|P_k f\|_2$$

and using Plancherel's theorem and the definition of Besov spaces via dyadic frequency decompositions we see that the last expression is dominated by $C\|f\|_{B^{a/4}_{2,1}}$. $\square$

§3. *Necessary conditions.* In order to prove necessity in Theorem 1.1 we use arguments from Nikishin–Stein theory. We include the standard argument for the proof of the following proposition in Appendix.

PROPOSITION 3.1. *Assume that for every $f \in H^s$, the limit $\lim_{n\to\infty} S^a f(x, t_n)$ exists for almost every $x \in \mathbb{R}$. Then for any compact set $K \subset \mathbb{R}$, there is a constant C_K, such that for all $\alpha > 0$,*

$$\text{meas}\bigg(\Big\{x \in K : \sup_n |S^a f(x, t_n)| > \alpha\Big\}\bigg) \leqslant C_K \bigg(\frac{\|f\|_{H^s}}{\alpha}\bigg)^2.$$

We also need the following elementary lemma.

LEMMA 3.2. *Let $\{t_n\}$ be a sequence of positive numbers in $[0,1]$, let $0 < r < \infty$ and assume that $\sup_{b>0} b^r \#(\{n : b < t_n \leqslant 2b\}) \leqslant A$. Then $\{t_n\} \in \ell^{r,\infty}$.*

Proof. For every $\beta > 0$,

$$\beta^r \#(\{n : t_n > \beta\}) = \beta^r \sum_{k\geqslant 0} \#(\{n : 2^k\beta < t_n \leqslant 2^{k+1}\beta\}) \leqslant \sum_{k\geqslant 0} A 2^{-kr} \lesssim A. \qquad \square$$

We now turn to the proof of the necessity of the $\ell^{r,\infty}$-condition in Theorems 1.1 and 1.2.

PROPOSITION 3.3. *Assume that $\{t_n\}$ is a decreasing sequence such that $t_n - t_{n+1}$ is also decreasing and $\lim_{n\to\infty} t_n = 0$. For $0 < s < a/4$, let*

$$r(s) = \frac{2s}{a - 4s}.$$

(i) *If $s < \min\{a/4, 1/4\}$ and if*

$$\text{meas}\bigg(\Big\{x \in [0,1] : \sup_n |S^a f(x, t_n)| > 1/2\Big\}\bigg) \leqslant C_\circ \|f\|^2_{H_s} \tag{3.1}$$

holds for all $f \in H^s$, then $\{t_n\} \in \ell^{r(s),\infty}$.

(ii) *If $s < a/4$ and if the global weak type inequality*

$$\operatorname{meas}\left(\left\{x \in \mathbb{R} : \sup_n |S^a f(x,t_n)| > 1/2\right\}\right) \leqslant C_\circ \|f\|_{H_s}^2 \tag{3.2}$$

holds for all $f \in H^s$, then $\{t_n\} \in \ell^{r(s),\infty}$.

Proof. We argue by contradiction and assume that $\{t_n\} \notin \ell^{r(s),\infty}$ while (3.1) holds if $s < \min\{a/4, 1/4\}$ or (3.2) holds in the case $a > 1$ and $1/4 \leqslant s < a/4$. By Lemma 3.2, this means

$$\sup_{0<b<1/2} b^{r(s)} \#(\{n : b < t_n \leqslant 2b\}) = \infty.$$

Hence there exists an increasing sequence $\{R_j\}$ with $\lim_{j\to\infty} R_j = \infty$ and a sequence of positive numbers b_j with $\lim_{j\to\infty} b_j = 0$ so that

$$\#(\{n : b_j < t_n \leqslant 2b_j\}) \geqslant R_j b_j^{-r(s)}. \tag{3.3}$$

We take another sequence

$$M_j \leqslant R_j \text{ with } \lim_{j\to\infty} M_j = \infty$$

such that in the case where $s < 1/4$

$$a M_j^{\frac{2(a-1)}{a}} b_j^{\frac{1-4s}{a-4s}} \leqslant 1. \tag{3.4}$$

In the case $1/4 \leqslant s < a/4$ we simply take $M_j = R_j$.

We now show that

$$t_n - t_{n+1} \leqslant 2 M_j^{-1} b_j^{\frac{a-2s}{a-4s}}, \quad \text{if } t_n \leqslant b_j. \tag{3.5}$$

Indeed since $n \mapsto t_n - t_{n+1}$ is decreasing we get, for $t_n \leqslant b_j$,

$$t_n - t_{n+1} \leqslant \min\{t_m - t_{m+1} : t_m > b_j\} \leqslant \frac{2b_j}{\#(\{n : b_j < t_n \leqslant 2b_j\})} \leqslant \frac{2b_j}{R_j b_j^{-r(s)}} \leqslant \frac{2b_j}{M_j b_j^{-r(s)}},$$

by (3.3), and (3.5) follows since $r(s) + 1 = \frac{a-2s}{a-4s}$.

For our construction of a counterexample we rely on the idea originally proposed by Dahlberg and Kenig [**5**]. We introduce a family of Schwartz functions which is used to test (3.1). Choose a C^∞ function g with compact support in $[-1/2, 1/2]$ such that $g(\xi) \geqslant 0$ and $\int g(\xi)\, d\xi = 1$ and consider a family of functions $f_{\lambda,\rho}$, with large λ and $\rho \ll \lambda$, defined via the Fourier transform by

$$\widehat{f_{\lambda,\rho}}(\eta) = \rho^{-1} g((\eta + \lambda)/\rho).$$

Thus $\widehat{f_{\lambda,\rho}}$ is supported in an interval of length $\rho \ll \lambda$ contained in $[-2\lambda, -\lambda/2]$. The assumption $\rho \ll \lambda$ clearly implies

$$\|f_{\lambda,\rho}\|_{H^s} \lesssim \lambda^s \rho^{-1/2}. \tag{3.6}$$

We now examine the action of S^a on $f_{\lambda,\rho}$. We have

$$|S^a f_{\lambda,\rho}(x,t_n)| = \left| \int e^{i(x\eta + t_n|\eta|^a)} \rho^{-1} g((\eta+\lambda)/\rho)\, \frac{d\eta}{2\pi} \right| = \left| \int e^{i\Phi_{\lambda,\rho}(\xi; x, t_n)} g(\xi)\, \frac{d\xi}{2\pi} \right|,$$

where

$$\Phi_{\lambda,\rho}(\xi; x, t_n) = x(\rho\xi - \lambda) + t_n(\lambda - \rho\xi)^a.$$

We shall use, for x in a suitable interval $I_j \subset I$, and for suitable choices of λ_j, ρ_j and $n(x, j)$, the estimate

$$|S^a f_{\lambda_j,\rho_j}(x, t_{n(x,j)})| \geqslant \int g(\xi)\, d\xi - \int |e^{i\Phi_{\lambda_j,\rho_j}(\xi;x,t_{n(x,j)})} - 1|\, g(\xi)\, \tfrac{d\xi}{2\pi}$$
$$\geqslant 1 - \max_{|\xi|\leqslant 1/2} \left| e^{i\Phi_{\lambda_j,\rho_j}(\xi;x,t_{n(x,j)})} - 1 \right| \tag{3.7}$$

and we will have to show that the subtracted term is small for our choices of x, $n(x, j)$ and (λ_j, ρ_j).

By a standard Taylor expansion, we see that

$$(1 - \rho\xi/\lambda)^a = 1 - a\rho\xi/\lambda + \tfrac{a(a-1)}{2}(\rho\xi/\lambda)^2 + E_3(\rho\xi/\lambda)$$

where $E_3(t) = -\frac{1}{2}a(a-1)(a-2)(\int_0^1 (1-st)^{a-3}(1-s)^2 ds)\, t^3$. Hence

$$\Phi_{\lambda,\rho}(\xi; x, t_n) = (x - a\lambda^{a-1}t_n)\rho\xi + \tfrac{a(a-1)}{2}\rho^2\lambda^{a-2}t_n\xi^2 + \lambda^a t_n E_3(\rho\xi/\lambda) + \lambda^a t_n - \lambda x. \tag{3.8}$$

Since terms that are independent of ξ do not affect the absolute value of our integral, we only need to show an upper bound of the first three terms. We consider t_n with $t_n \leqslant b_j/2$ and let ε be such that $\varepsilon < 10^{-1}(a+2)^{-1}$. We chose $(\lambda, \rho) = (\lambda_j, \rho_j)$ as

$$\lambda_j = M_j^{2/a} b_j^{-\frac{1}{a-4s}}, \qquad \rho_j = \varepsilon b_j^{-1/2}\lambda_j^{1-a/2} = \varepsilon M_j^{\frac{2-a}{a}} b_j^{-\frac{1-2s}{a-4s}} \tag{3.9}$$

and we consider these choices for large j when $b_j \ll 1$ and $M_j \gg 1$. We then get

$$\rho_j/\lambda_j = \varepsilon M_j^{-1} b_j^{\frac{2s}{a-4s}} \leqslant \varepsilon;$$

hence for $|\xi| \leqslant 1/2$

$$|\tfrac{a(a-1)}{2}\rho_j^2\lambda_j^{a-2}t_n\xi^2| \leqslant \tfrac{(a+1)^2}{2}\rho_j^2\lambda_j^{a-2}b_j\xi^2$$
$$\leqslant (a+1)^2\varepsilon^2 M_j^{\frac{4-2a}{a}} b_j^{-\frac{2-4s}{a-4s}} M_j^{\frac{2(a-2)}{a}} b_j^{-\frac{a-2}{a-4s}} b_j = (a+1)^2\varepsilon^2 \tag{3.10a}$$

and similarly

$$|\lambda_j^a t_n E_3(\rho_j\xi/\lambda)| \leqslant (a+2)^3\lambda_j^a b_j \left(\tfrac{\rho_j}{2\lambda_j}\right)^3 \leqslant (a+2)^3\varepsilon\lambda_j^{a-2}\rho_j^2 b_j \leqslant (a+2)^3\varepsilon^3. \tag{3.10b}$$

Next we consider x in the interval

$$I_j := [0, a\lambda_j^{a-1}b_j/2].$$

Note that in the case $s < 1/4$,

$$a\lambda_j^{a-1}b_j/2 = \frac{a}{2} M_j^{\frac{2(a-1)}{a}} b_j^{\frac{1-4s}{a-4s}} \leqslant 1/2,$$

by (3.4) and hence $I_j \subset [0, 1/2]$ in this case. If $a > 1$ and $1/4 \leqslant s < a/4$, no restriction on I_j is required (as we are trying to disprove the global inequality (3.2) in this case). Each $x \in I_j$ is contained in an interval $(a\lambda_j^{a-1}t_{n+1}, a\lambda_j^{a-1}t_n]$ for a unique n, which we label $n(x, j)$. By (3.5) we have that

$$0 \leqslant t_{n(x,j)} - t_{n(x,j)+1} \leqslant 2M_j^{-1} b_j^{\frac{a-2s}{a-4s}}.$$

Hence

$$|(x - a\lambda_j^{a-1} t_{n(x,j)})\rho_j \xi| \leqslant a\lambda_j^{a-1}\rho_j(t_{n(x,j)} - t_{n(x,j)+1})$$
$$\leqslant aM_j^{\frac{2(a-1)}{a}} b_j^{-\frac{a-1}{a-4s}} \varepsilon M_j^{\frac{2-a}{a}} b_j^{-\frac{1-2s}{a-4s}} 2M_j^{-1} b_j^{\frac{a-2s}{a-4s}} = 2a\varepsilon. \quad (3.10c)$$

As $\varepsilon \leqslant 10^{-1}(a+2)^{-1}$ we obtain from (3.10a), (3.10b) and (3.10c)

$$\max_{|\xi|\leqslant 1/2} |e^{i\Phi_{\lambda_j,\rho_j}(\xi; x, t_{n(x,j)})} - 1| \leqslant 1/2$$

and thus from (3.7)

$$\sup_n |S^a f_{\lambda_j,\rho_j}(x, t_n)| \geqslant |S^a f_{\lambda_j,\rho_j}(x, t_{n(x,j)})| \geqslant \tfrac{1}{2}, \text{ for } x \in I_j = [0, a\lambda_j^{a-1} b_j/2], \quad (3.11)$$

and, as noted before, $I_j \subset [0, 1]$ if $s < 1/4$. The assumption of (3.1) (in the case $s < \min\{a/4, 1/4\}$) or the assumption of (3.2), both yield

$$\operatorname{meas}(I_j) \leqslant 4C_\circ \|f_{\lambda_j,\rho_j}\|_{H^s}^2 = \widetilde{C}\lambda_j^{2s}\rho_j^{-1}. \quad (3.12)$$

This leads to

$$aM_j^{\frac{2(a-1)}{a}} b_j^{\frac{1-4s}{a-4s}} \leqslant \widetilde{C} M_j^{\frac{4s}{a}} b_j^{-\frac{2s}{a-4s}} \varepsilon^{-1} M_j^{\frac{a-2}{a}} b_j^{\frac{1-2s}{a-4s}}$$

and hence to

$$a\varepsilon\widetilde{C}^{-1} \leqslant M_j^{-\frac{a-4s}{a}}.$$

Since $\lim_{j\to\infty} M_j = \infty$ the right hand side converges to 0 as $j \to \infty$ and we obtain a contradiction. This means that if $\{t_n\} \notin \ell^{\frac{2s}{a-4s},\infty}$ then (3.1) (and therefore (3.2)) cannot hold with $s < \min\{a/4, 1/4\}$ and (3.2) cannot hold with $1/4 \leqslant s < a/4$. Thus both parts of the proposition are proved. □

We are now able to combine previous results to give a proof of the theorems in the introduction.

Proof of Theorem 1.1. The implications (a) ⇒ (b) and (a) ⇒ (d) follow from Proposition 2.3. The implication (b) ⇒ (c) is immediate by Tshebyshev's inequality. The implication (c) ⇒ (a) follows from part (i) of Proposition 3.3. Finally, the implication (d) ⇒ (c) follows from Proposition 3.1. □

Proof of Theorem 1.2. The implication (a) ⇒ (b) follows from Proposition 2.3. The implication (b) ⇒ (c) is again immediate by Tshebyshev's inequality. The implication (c) ⇒ (a) follows from part (ii) of Proposition 3.3. □

§4. *The case $a = 1$.* We now give the sketch of the proof of Theorem 1.3. We start with an analog to Lemma 2.2, for the frequency-localized operator S^1_λ.

LEMMA 4.1. (i) *Let $b > \lambda^{-1}$. Then*

$$\Big\| \sup_{0\leqslant t\leqslant b} \big|S^1_\lambda f(\cdot, t)\big| \Big\|_2 \lesssim (\lambda b)^{1/2} \|f\|_2.$$

(ii) *Let* $0 < r < \infty$ *and let* $\{t_n\}$ *be a sequence in* $[0,1]$ *which belongs to* $\ell^{r,\infty}$. *Then for* $\lambda > 1$

$$\Big\| \sup_n \big|S^1_\lambda f(\cdot, t_n)\big| \Big\|_{L^2(\mathbb{R})} \leqslant C\lambda^{\frac{r}{2+2r}} \|f\|_{L^2(\mathbb{R})}.$$

Proof. We use the elementary inequality

$$\Big\| \sup_{c\leqslant t\leqslant c+\lambda^{-1}} \big|S^1_\lambda f(x,t)\big| \Big\|_2 \lesssim \|f\|_2 \tag{4.1}$$

which just follows from L^2 estimates for $S^1_\lambda f(\cdot, t)$ and $\partial_t S^1_\lambda f(\cdot, t)$. Now

$$\Big\| \sup_{0<t\leqslant b} \big|S^1_\lambda f(\cdot, t)\big| \Big\|_{L^2(\mathbb{R})} \leqslant \Bigg(\sum_{\substack{m\geqslant 0:\\ 0\leqslant m\lambda^{-1}\leqslant b}} \Big\| \sup_{m\lambda^{-1}\leqslant t\leqslant (m+1)\lambda^{-1}} \big|S^1_\lambda f(\cdot, t)\big| \Big\|_2^2 \Bigg)^{1/2}$$

which is bounded by a constant times $(\lambda b)^{1/2}\|f\|_2$.

To prove part (ii) we write as in the proof of Lemma 2.2, for $b > \lambda^{-1}$ to be determined,

$$\Big\| \sup_n |S^1_\lambda f(\cdot, t_n)| \Big\|_2 \leqslant \Big\| \sup_{n:\, t_n\leqslant b} \big|S^1_\lambda f(\cdot, t_n)\big| \Big\|_2 + \Big\| \sup_{n:\, t_n>b} \big|S^1_\lambda f(\cdot, t_n)\big| \Big\|_2 .$$

For the first term we have

$$\Big\| \sup_{n:\, t_n\leqslant b} \big|S^1_\lambda f(\cdot, t_n)\big| \Big\|_{L^2(\mathbb{R})} \lesssim (\lambda b)^{1/2}\|f\|_2,$$

by part (i). For the second term we may estimate as in Lemma 2.2

$$\Big\| \sup_{n:\, t_n>b} \big|S^1_\lambda f(\cdot, t_n)\big| \Big\|_{L^2(\mathbb{R})} \lesssim b^{-r/2}\|f\|_2.$$

Choosing b such that $(\lambda b)^{1/2} = b^{-r}$ yields the claimed result. □

PROPOSITION 4.2. *Let* $0 < r < \infty$ *and assume that* $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$ *is decreasing. Then*

$$\Big\| \sup_n \big|S^1 f(\cdot, t_n)\big| \Big\|_{L^2(\mathbb{R})} \leqslant C\|f\|_{H^s}, \quad s = \frac{r}{2+2r}. \tag{4.2}$$

Proof. We set for $l \geqslant 0$ we set

$$\mathfrak{N}(l) = \Big\{ n \in \mathbb{N} : 2^{-(l+1)\frac{1}{1+r}} < t_n \leqslant 2^{-l\frac{1}{1+r}} \Big\}.$$

By the assumption that $\{t_n\} \in \ell^{r,\infty}$ there is $C > 0$ so that

$$\#(\mathfrak{N}(l)) \leqslant C2^{l\frac{r}{1+r}} = C2^{2ls}. \tag{4.3}$$

Arguing as in the proof of Proposition 4.2 we can estimate $\| \sup_n |S^a f(\cdot, t_n)| \|_2 \leqslant \|\mathcal{E}_1\|_2 + \|\mathcal{E}_2\|_2$, where

$$\mathcal{E}_1(x) = \sum_{j \geqslant 0} \left(\sum_{l \geqslant j} \sup_{n \in \mathfrak{N}(l)} |S^1 P_{l-j} f(x, t_n)|^2 \right)^{1/2}$$

$$\mathcal{E}_2(x) = \sum_{m \geqslant 0} \left(\sum_{l \geqslant 0} \sup_{n \in \mathfrak{N}(l)} |S^1 P_{l+m} f(x, t_n)|^2 \right)^{1/2}.$$

Again as in the proof of Proposition 2.3

$$\|\mathcal{E}_2\|_2 \leqslant \sum_{m \geqslant 0} \left(\sum_{l \geqslant 0} \#(\mathfrak{N}(l)) \|P_{l+m} f\|_2^2 \right)^{1/2} \lesssim \sum_{m \geqslant 0} \left(\sum_{l \geqslant 0} 2^{2sl} \|P_{l+m} f\|_2^2 \right)^{1/2}$$

$$= \sum_{m \geqslant 0} 2^{-ms} \left(\sum_{l \geqslant 0} 2^{2s(l+m)} \|P_{l+m} f\|_2^2 \right)^{1/2} \lesssim \|f\|_{H^s}.$$

In order to deal with the first sum, we use that $\mathfrak{N}(l) \subset [0, b_l]$ with $b_l = 2^{-l/(1+r)}$. Hence by Lemma 4.1

$$\|\mathcal{E}_1\|_2 \leqslant \sum_{j \geqslant 0} \left(\sum_{l \geqslant j} \left\| \sup_{n \in \mathfrak{N}(l)} \left| S^1 P_{l-j} f(\cdot, t_n) \right| \right\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \lesssim \sum_{j \geqslant 0} \left(\sum_{l \geqslant j} \left[2^{\frac{l-j}{2}} 2^{-l \frac{1}{2+2r}} \|P_{l-j} f\|_2 \right]^2 \right)^{1/2}$$

$$\lesssim \sum_{j \geqslant 0} 2^{-j \frac{1}{2+2r}} \left(\sum_{l \geqslant j} \left[2^{(l-j) \frac{r}{2+2r}} \|P_{l-j} f\|_2 \right]^2 \right)^{1/2} \lesssim \|f\|_{H^s}$$

with $s = \frac{r}{2r+2}$. □

For completeness we state the case $s = 1/2$, $a = 1$ analog of Proposition 2.4 which is sharp in this case.

PROPOSITION 4.3. *For all $f \in B^{1/2}_{2,1}(\mathbb{R})$,*

$$\left\| \sup_{t \in [0,1]} \left| S^1 f(\cdot, t) \right| \right\|_{L^2(\mathbb{R})} \leqslant C \|f\|_{B^{1/2}_{2,1}}.$$

The space $B^{1/2}_{2,1}$ cannot be replaced by $B^{1/2}_{2,\nu}$ for any $\nu > 1$.

Proof. The first part is immediate from Lemma 4.1. For the second part one recalls that there are unbounded functions in $B^{1/2}_{2,\nu}$ whose Fourier transform is supported in $(-\infty, 0]$; cf. [**1**]. For such functions $S^1 f(x, t) = f(x - t)$ and thus, for $\nu > 1$ one can easily find $f \in B^{1/2}_{2,\nu}$ such that $\sup_{t \in [0,1]} |S^1 f(x, t)| = \infty$ on a set $A \subset [0, 1]$ with $\text{meas}(A) > 0$. □

PROPOSITION 4.4. *Assume that* $\{t_n\}$ *is a decreasing sequence such that* $t_n - t_{n+1}$ *is also decreasing and* $\lim_{n\to\infty} t_n = 0$. *For* $s < 1/2$ *let*

$$\rho(s) = \frac{2s}{1-2s}.$$

Then the validity of the inequality

$$\text{meas}\left(\left\{x \in [0,1] : \sup_n \left|S^1 f(x,t_n)\right| > 1/2\right\}\right) \leqslant C_\circ \|f\|_{H_s}^2. \tag{4.4}$$

for all $f \in H^s$, *implies that* $\{t_n\} \in \ell^{\rho(s),\infty}$.

Proof. Assume that $\{t_n\} \notin \ell^{\rho(s),\infty}$. Arguing as in the proof of Proposition 3.3 we find an increasing sequence M_j with $\lim M_j = \infty$ and a sequence of positive numbers b_j with $\lim_{j\to\infty} b_j = 0$ so that

$$\#(\{n : b_j < t_n \leqslant 2b_j\}) \geqslant M_j b_j^{-\rho(s)}.$$

As in the previous proof we also have

$$t_n - t_{n+1} \leqslant 2M_j^{-1} b_j^{\rho(s)+1}, \quad \text{if } t_n \leqslant b_j. \tag{4.5}$$

Let $g \in C^\infty$ be nonnegative, supported in $(-1/2, 1/2)$, such that $\int g(\xi)\, d\xi = 1$. Define f_λ, for large λ, by

$$\widehat{f_\lambda}(\xi) = 10\lambda^{-1} g(10\lambda^{-1}(\xi + \lambda)).$$

Then $\|f_\lambda\|_{H^s} \leqslant C_\circ \lambda^{s-1/2}$. We write

$$|S^1 f_\lambda(x,t)| = \left|\int e^{i10^{-1}\lambda(x-t)\xi} g(\xi) \tfrac{d\xi}{2\pi}\right| \geqslant 1 - \left|\int (e^{i10^{-1}\lambda(x-t)\xi} - 1) g(\xi)\, \tfrac{d\xi}{2\pi}\right|$$

and see that

$$|S^1 f_\lambda(x,t)| \geqslant 1/2, \quad \text{if } |x-t| \leqslant \lambda^{-1}. \tag{4.6}$$

We now set $\lambda_j = M_j b_j^{-\frac{1}{1-2s}}$. Note that $\rho(s) + 1 = (1-2s)^{-1}$, and thus we have $t_n - t_{n+1} \leqslant M_j \lambda_j^{-1}$ for $t_n \leqslant b_j$, by (4.5). Hence by (4.6) we see that

$$\sup_n |S^1 f_{\lambda_j}(x)| \geqslant 1/2, \quad \text{for } 0 < x < b_j/2.$$

Therefore the asserted weak type inequality implies

$$b_j/2 \leqslant 4\|f_{\lambda_j}\|_{H^s}^2 \leqslant 4C_\circ \lambda_j^{2s-1} = 4C_\circ M_j^{2s-1} b_j$$

and thus $8C_\circ M_j^{2s-1}$ is bounded below as $j \to \infty$. This yields a contradiction as we have $\lim_{j\to 0} M_j^{2s-1} = 0$ for $s < 1/2$. □

Proof of Theorem 1.3. The implication (a) ⇒ (b) follows from Proposition 4.2. The implication (b) ⇒ (c) follows from Tshebyshev's inequality. The implication (c) ⇒ (a) follows from Proposition 4.4. The implication (c) ⇒ (d) follows by a standard argument using the weak type inequality and the density of Schwartz functions. The implication (d) ⇒ (c) follows from Proposition 3.1. □

Appendix. Proof of Proposition 3.1. We need to use a theorem by Nikishin, whose proof can be found, e.g. in [**9**, Chapter VI, Corollary 2.7]; see also [**19**]. Nikishin's theorem asserts that if $M : L^2(Y, \mu) \to L^0(\mathbb{R}^d, |\cdot|)$ is a continuous sublinear operator (with (Y, μ) an arbitrary measure space), then there exists a measurable function w with $w(x) > 0$ a.e. such that

$$\int_{\{x:\,|Mf(x)|>\alpha\}} w(x)\,dx \leqslant \alpha^{-2}\|f\|^2_{L^2(\mu)}.$$

To prove Proposition 3.1, let $M^a f(x) = \sup_n |S^a f(x, t_n)|$ and consider $T^a_n g(x) = (2\pi)^{-1}\int e^{i(x\xi + t_n|\xi|^a)} g(\xi)\,d\xi$, so that

$$T^a_n \widehat{f}(x) = S^a f(x, t_n).$$

Then T^a_n acts on functions in the weighted L^2 space $L^2(\mu_s)$, where $d\mu_s(\xi) = (1 + |\xi|^2)^s d\xi$. Define the corresponding maximal operator, $\widetilde{M}^a g = \sup_n |T^a_n g|$.

Now assuming that $\lim_n S^a f(x, t_n)$ exists a.e. for every $f \in H^s$, we see that $\widetilde{M}^a g(x) < \infty$ a.e. for every $g \in L^2(\mu_s)$. Then by [**9**, Proposition 1.4, p. 529], this implies that the sublinear operator $\widetilde{M}^a : L^2(\mu_s) \to L^0(|\cdot|)$ is continuous. By the abovementioned Nikishin's theorem,

$$\int_{\{x:\,|\widetilde{M}^a g(x)|>\alpha\}} w(x)\,dx \leqslant \alpha^{-2}\|g\|^2_{L^2(\mu_s)}$$

for some weight w with $w(x) > 0$ a.e. As we can replace w with $\min\{w, 1\}$ we may further assume that w is bounded.

Next, for $f \in H^s$, $\widetilde{M}^a \widehat{f} = M^a f$ and $\|\widehat{f}\|_{L^2(\mu_s)} = \|f\|_{H^s}$, so

$$\int_{\{x:\,|M^a f(x)|>\alpha\}} w(x)\,dx \leqslant \alpha^{-2}\|f\|^2_{H^s}.$$

After the change of variables $x \to x + y$ and using the translation invariance of M^a, we can replace the integrand $w(x)$ by $w(x - y)$ for any y. Arguing as in [**9**, Chapter VI] multiply both sides of the resulting inequality by $h(y)$, where h is a strictly positive continuous function with $\int h = 1$, and then integrate in y, to arrive at

$$\int_{\{x:\,|M^a f(x)|>\alpha\}} h * w(x)\,dx \leqslant \alpha^{-2}\|f\|^2_{H^s}.$$

Since $h * w$ is continuous it attains a minimum over any compact set. We therefore conclude that

$$\operatorname{meas}(\{x \in K : |M^a f(x)| > \alpha\}) \leqslant C_K \alpha^{-2}\|f\|^2_{H^s}$$

must hold true for every compact set K, as desired.

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Evangelos Dimou,
Department of Mathematics,
University of Michigan,
530 Church Street,
Ann Arbor, MI 48109-1043,
U.S.A.

Current address:
Department of Mathematics,
University of Virginia,
P. O. Box 400137,
Charlottesville, VA 22904,
U.S.A.
Email: ed8bg@virginia.edu

Andreas Seeger,
Department of Mathematics,
University of Wisconsin-Madison,
480 Lincoln Drive,
Madison, WI 53706,
U.S.A.
Email: seeger@math.wisc.edu

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