Optimal Reinsurance and Investment in Danger-Zone and Safe-Region

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Dear Editor,

Optimal Control Applications and Methods provides a forum for papers on the full range of optimal control and related control design methods. The aim is to encourage new developments in optimal control theory and design methodologies that may lead to advances in real control applications. In the following, we give two reasons to illustrate that our paper is absolutely relevant to this journal. **Firstly**, in this paper, we take reinsurance and investment as the control variables and investigate two optimal reinsurance-investment problems, where the criteria are minimizing the probability of drawdown and minimizing the expected time to reach a given capital level. The optimal reinsurance-investment strategy and the corresponding value functions are derived by using the stochastic dynamic programming and solving the corresponding boundary-value problems. Note that short-selling is prohibited in our risk model and the reinsurance proportion is constrained into the interval [0, 1]. Thus, the topics in the paper are relevant to dynamic optimization techniques, constrained control and dynamic programming. **Secondly**, risk models taking reinsurance into consideration have received a great deal of attention in the literature because reinsurance is an integral component of risk management practices for insurance companies. Indeed, reinsurance is one of the major vehicles for an insurer to transfer risk exposure and thereby allow the insurer to increase underwriting capacity, stabilize profits, and provide protection against a catastrophic loss. Subject to controlling reinsurance with or without controlling investment, optimization problems under various objective functions have become a popular research topic in the actuarial literature. Thus, the risk model in our paper is some kind of interesting optimal control applications and design studies. Therefore, we deem that the optimization issues we deal with and the corresponding technique we apply are particularly relevant to this journal.

Compared to the literature existed, there are four main differences and contributions in this paper. Firstly, we point out that when the surplus is relatively low, the insurer prefers to pay more attention to reducing the risk; but when the surplus becomes relatively high, the insurer may be more interested in reaching a goal as quickly as possible. Thus, it is meaningful to consider the objectives of survival and growth in two complementary regions, and our optimal results for both aspects of the problems will therefore complement the results in Han et al. [15, 16]. Furthermore, it is more reasonable for the insurer to bear drawdown than ruin where the wealth drops below a fixed level. In particular, minimizing the probability of drawdown is a general criterion which includes minimizing the probability of ruin as a special case when $\alpha = 0$. Moreover, since the domain and boundary conditions for drawdown problem are very different from ruin, the minimum drawdown probability depends not only on the value of surplus u but also on the maximum value m, which makes the optimization problem much more complicated and practical than the case of $\alpha = 0$. Secondly, we assume that the insurer takes both investment and reinsurance into consideration and the price process of risky asset is correlated to the claim process. Short-selling is prohibited and the reinsurance proportion is constrained into $[0, 1]$. These all present a challenge when finding the explicit optimal risk control policies and solving the value functions in closed-form. Besides, several detail comparisons are provided to study the impact of some important parameters on the optimal strategies and we illustrate the observations from the perspective of finance. Thirdly, we investigate the behavior of the surplus process and find a rather surprising result that in the danger-zone, the optimally controlled surplus never reaches the safe level before drawdown. Further, when minimizing the expected time to reach the goal in the safe-region, the optimal strategies make the low boundary inaccessible from above and the insurer will stay in the safe-region forever, almost surely. Fourthly, to the best of our knowledge, only Luo et al. [22] and Liang and Bai [19] studied the objective of minimizing expected time to reach a given capital level before ruin for risk models with cheap proportional reinsurance.

By contrast, we limit the surplus into the safe-region, under the same criterion, we find the optimal policies for the risk model with non-cheap reinsurance, which makes the optimization problem more practical.

Keywords: Stochastic optimal control; Hamilton-Jacobi-Bellman equation; Proportional reinsurance; Investment; Diffusion approximation model