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Optimal reinsurance and investment in danger-zone and safe-region

Xia Han[1,2] | **Zhibin Liang**[1,2]

[1]School of Mathematical Sciences Institute of Finance Statistics, Nanjing Normal University, Jiangsu, China,

[2]Department of Mathematics, University of Michigan, Ann Arbor, Michigan,

Correspondence
Zhibin Liang, School of Mathematical Sciences Institute of Finance Statistics, Nanjing Normal University, Jiangsu 210023, China.
Email: liangzhibin111@hotmail.com

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Summary
This paper studies the optimal reinsurance-investment problems for an insurance company where the claim process follows a Brownian motion with drift. It turns out that there is a region where the probability of drawdown, namely, the probability that the value of the insurer's surplus process reaches some fixed fractional value of its maximum value to date is positive. Then in the complementary region, drawdown can be avoided with certainty. For this reason, we call the former region the "danger-zone" and "safe-region" for the latter. In the danger-zone, we consider the problem of minimizing the probability of drawdown; and in the safe-region, we turn our attention to the optimization problem of minimizing the expected time to reach a given capital level. Using the technique of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation, explicit expressions of the optimal reinsurance-investment strategies and the associated value functions are derived for the two optimization problems. Moreover, we provide several detailed comparisons to investigate the impact of some important parameters on the optimal strategies and illustrate the observation from behavior finance of view.

KEYWORDS
diffusion approximation model, Hamilton-Jacobi-Bellman equation, investment, proportional reinsurance, stochastic optimal control

1 | INTRODUCTION

In the past few decades, optimal investment and reinsurance problems for various risk models have attracted a great deal of attention in actuarial literature. This is due to the fact that the insurance company can reduce its risk exposure by purchasing reinsurance and increase its profit by investing its surplus into the risky and risky-free assets. The technique of stochastic control theory and the corresponding Hamilton- Jacobi-Bellman (HJB) equation are widely used to cope with these problems.

The most common criterion of the optimization is to minimize the probability of ruin or maximize the expected utility of terminal wealth. For example, Browne[1] used a diffusion risk model to describe the surplus of the insurance company. The optimal investment strategies are obtained not only for the criterion of maximizing the expected exponential utility of terminal wealth but also for the one of minimizing the probability of ruin. Zhang et al[2] minimized the probability of ruin by finding the optimal combination of quota-share and excess-of-loss reinsurance.

Liang and Yuen[3] adopted the variance premium principle to study the optimal proportional reinsurance problem for both the compound Poisson risk model and the diffusion approximation risk model under the criterion of maximizing the expected exponential utility. Liang and Young[4] computed the optimal investment and reinsurance strategy for an insurance company that wishes to minimize its probability of ruin when the risk process follows a compound Poisson process.

In this paper, we will consider other two important risk-measure criteria, namely, minimizing the probability of drawdown and minimizing the expected time to reach a given capital level. With drawdown, the decision-makers want to adopt strategies which minimize the probability that the value of the surplus process drops below some fixed proportion, say $\alpha \in [0, 1)$ of its maximum value to date. Note that when $\alpha = 0$, minimizing the probability of drawdown is equal to minimizing the probability of ruin. Angoshtari et al[5] and Han et al[6,7] minimized the probability of drawdown over an infinite-time horizon and showed that the strategy which minimizes the probability of ruin also minimizes the probability of drawdown. Besides, Angoshtari et al[8] and Chen et al[9] computed the optimal investment strategy to minimize the probability of lifetime drawdown for an individual investor. They found that the optimal strategy for a random (or finite) maturity setting is different from that of the corresponding ruin-minimization problem. In some other research involving drawdown, such as Grossman and Zhou,[10] Cvitanić and Karatzas,[11] and Elie and Touzi,[12] drawdown was used as a constraint associated with maximizing expected utility of consumption and terminal wealth. As for the criterion of minimizing the expected time to reach a goal, we can see the related works given in Heath et al,[13] Bayraktar and Young,[14] Frostig,[15] Luo et al,[16] and Liang and Bai.[17]

In the mathematical formulation, we suppose that the surplus process of the insurer is described by the diffusion model which is an approximation of the classical Cramér-Lundberg model. We assume that the insurer can purchase proportional reinsurance and invest its surplus in a financial market consisting of one risky asset and one risk-free asset. It turns out that the state space for wealth can be divided into two regions by a safe level, which we will call the "danger-zone" and the "safe-region." In the former region, drawdown is possible, and thus we aim to obtain the optimal strategy to minimize the probability of drawdown. In the latter region, the insurer will never face the possibility of drawdown, and thus we can concentrate purely on the growth aspects of the insurer and investigate the problem of minimizing the expected time to reach a given capital level. By the technique of stochastic dynamic programming, the explicit expressions for the optimal strategies and the corresponding value functions are derived for the two different optimization problems.

Compared to the existing literature, there are four main differences and contributions in this paper. Firstly, note that when the surplus is relatively low, the insurer prefers to pay more attention to reducing the risk; but when the surplus becomes relatively high, the insurer may be more interested in reaching a goal as quickly as possible. Thus, it is meaningful to consider the objectives of survival and growth in two complementary regions, and our optimal results for both aspects of the problems will therefore complement the results in Han et al.[6,7] Secondly, we assume that the insurer takes both investment and reinsurance into consideration and the price process of risky asset is correlated to the claim process. Short-selling is prohibited and the reinsurance proportion is constrained into $[0, 1]$. These issues all present a challenge when finding the explicit optimal risk control policies and solving the value functions in closed-form. Besides, several detailed comparisons are provided to study the impact of some important parameters on the optimal strategies and we illustrate the observations from the perspective of finance. Thirdly, we investigate the behavior of the surplus process and find a rather surprising result that in the danger-zone, the optimally controlled surplus never reaches the safe level before drawdown. Further, when minimizing the expected time to reach the goal in the safe-region, the optimal strategies make the low boundary inaccessible from above and the insurer will stay in the safe-region forever, almost surely. Fourthly, to the best of our knowledge, only Luo et al[16] and Liang and Bai[17] studied the objective of minimizing expected time to reach a given capital level before ruin for risk models with cheap proportional reinsurance. We would like to point out that, under the same criterion, we limit the surplus into the safe-region and find the optimal policies for the risk model with noncheap reinsurance, which makes the optimization problem more practical.

The rest of the paper is organized as follows. In Section 2, we describe the model and optimization problems. In Section 3, we derive explicit expressions for the optimal strategy and the corresponding minimum probability of drawdown. The optimization problem of minimizing the expected time to reach a given capital level is considered in Section 4. In Section 5, we present some numerical examples which show the impact of model parameters on the optimal results. We conclude the paper in Section 6.

2 | MODEL FORMULATION

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}, \mathbb{P})$ be a probability space containing all the objects defined in the following. We first introduce the classical Cramér-Lundberg risk model for the uncontrolled surplus process $X = \{X_t\}_{t\geq 0}$:

$$X_t = u + ct - \sum_{i=1}^{N_t} Y_i,$$

in which $X_0 = u \geq 0$ is the initial surplus and c is the premium rate. Moreover, $N = \{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, Y_i represents the size of the ith claim, and the claim sizes $Y_1, Y_2, \dots$ are independent and identically distributed, positive random variables, independent of N. Let Y be a generic random variable which has the same distribution as $Y_i (i \geq 1)$. Then, we assume that $F_Y(y)$ is the common cumulative distribution function of $Y_i (i \geq 1)$ with $F_Y(0) = 0$ and $0 < F_Y(y) \leq 1$ for $y > 0$. Assume that $\mathbb{E}(Y) < \infty$ and $\mathbb{E}(Y^2) < \infty$.

In this paper, the insurer is allowed to purchase proportional reinsurance to reduce its risk and q_t represents the proportion reinsured at time t. A retention strategy $q = \{q_t\}_{t\geq 0}$ is said to be *admissible* if it is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies $0 \leq q_t \leq 1$ for all $t \geq 0$. Let $U = \{U_t\}_{t\geq 0}$ denote the associated surplus process, that is, U_t is the surplus of the insurer at time t under the retention strategy q_t. Furthermore, we suppose that the surplus can be invested in a risk-free asset (bond or bank account) which earns a constant rate r and a risky asset (stock) whose price follows the Black-Scholes dynamics

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_{1t},$$

where $\mu > r$ and $\sigma > 0$ are constants, and $B_1 = \{B_{1t}\}_{t\geq 0}$ is a standard Brownian motion. Let π_t denote the amount invested in the risky asset at time $t \geq 0$, and then the rest of the surplus $(U_t - \pi_t)$ is invested in the risk-free asset. An investment strategy $\pi = \{\pi_t\}_{t\geq 0}$ is *admissible* if it is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and satisfies $\pi_t \geq 0$ (short-selling is prohibited) and $\int_0^t \pi_s^2 \, ds < \infty$ almost surely for all $t \geq 0$.

Denote the set of admissible strategies (q, π) by $\mathcal{D}$. Then, given any reinsurance-investment policy $\nu = \{q, \pi\} \in \mathcal{D}$, the surplus process has the following dynamics

$$dU_t = \left[rU_t + (\mu - r)\pi_t + c - \delta(q_t) \right] dt + \sigma \pi_t \, dB_{1t} - q_t \, d \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where $\delta(q_t)$ is the accumulated reinsurance premiums up to time t paid to the reinsurer.

We suppose that both the insurer and the reinsurer charge the premiums according to the expected-value principle, that is,

$$\begin{cases} c = (1+\theta)\lambda \mathbb{E}Y, \\ \delta(q_t) = (1+\eta)(1-q_t)\lambda \mathbb{E}Y, \end{cases}$$

where θ and η are the safety loadings of the insurer and the reinsurer. Without loss of generality, we assume that $\eta > \theta$, otherwise the problem becomes trivial. To derive the explicit expressions of the optimal results, we solve the optimization problems by approximating the jump process in Equation (1) with a diffusion, as in Bai et al,[18] Grandell,[19] and Liang and Yuen.[3] Specifically,

$$d \sum_{i=1}^{N_t} Y_i \approx \lambda \mathbb{E}Y \, dt - \sqrt{\lambda \mathbb{E}\left(Y^2\right)} \, dB_{2t},$$

in which $B_2 = \{B_{2t}\}_{t\geq 0}$ is a standard Brownian motion. Assume that the claim process is correlated to the price process of risky asset and we use $\rho (\rho \neq 0)$ to describe the correlation coefficient between B_1 and B_2, that is, $\mathbb{E}B_1 B_2 = \rho t$. For notational convenience, we denote $a = \lambda \mathbb{E}Y$ and $b = \sqrt{\lambda \mathbb{E}(Y^2)}$. Thus, the resulting process $\hat{U} = \{\hat{U}_t\}_{t\geq 0}$ evolves according to the dynamics

$$d\hat{U}_t = \left[r\hat{U}_t + (\mu - r)\pi_t + (\theta - \eta + \eta q_t)a \right] dt + \sigma \pi_t \, dB_{1t} + bq_t \, dB_{2t},$$

or equivalently,

$$d\hat{U}_t = \left[r\hat{U}_t + (\mu - r)\pi_t + (\theta - \eta + \eta q_t)a\right] dt + \sqrt{\sigma^2\pi_t^2 + 2\sigma b\rho q_t\pi_t + b^2q_t^2}d\, B_t, \tag{2}$$

in which $\hat{U}_0 = u$ and $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion.

Define the maximum surplus process $M = \{M_t\}_{t\geq 0}$ by

$$M_t = \max\left\{\sup_{0\leq s\leq t} \hat{U}_s,\ M_0\right\} \tag{3}$$

with $M_0 = m \geq u$. Note that the surplus process is allowed to have a financial past, as embodied by the term M_0 in Equation (3). Drawdown is the time when the value of the surplus process reaches $\alpha \in [0, 1)$ times its maximum value, that is, at the hitting time τ_α given by

$$\tau_\alpha = \inf\{t \geq 0 :\ \hat{U}_t \leq \alpha M_t\}.$$

If $\alpha = 0$, then drawdown is the same as ruin with the ruin level 0. In our paper, we shall consider the following two stochastic control problems:

Problem 1. Suppose that the insurer is interested in minimizing the probability of drawdown. The corresponding value function ϕ is defined by

$$\phi(u, m) = \inf_{v\in D} \mathbb{P}^{u,m}(\tau_\alpha < \infty) = \inf_{v\in D} \mathbb{E}^{u,m}\left(\mathbf{1}_{\{\tau_\alpha<\infty\}}\right),$$

in which $\mathbb{P}^{u,m}$ and $\mathbb{E}^{u,m}$ denote the probability and expectation, respectively, conditional on $\hat{U}_0 = u$ and $M_0 = m$.

Note that if the value of the surplus is greater than or equal to

$$u_s = \frac{a(\eta - \theta)}{r}, \tag{4}$$

then the insurer can buy full reinsurance and invest all the surplus in the risk-free asset to earn interest rate r, and the surplus of the insurer will never drop below its current value. For this reason, we call u_s the *safe level*.

Problem 2. Suppose that $\kappa > u_s$ with u_s defined by Equation (4). Let $\tau_\kappa = \inf\{t \geq 0 : \hat{U}_t \geq \kappa\}$ denote the first time when the surplus of the insurer reaches κ. Our goal is to minimize the expected time to reach the given capital level κ, that is,

$$\varphi(u) = \inf_{v\in D} \mathbb{E}^u(\tau_\kappa),$$

in which $\mathbb{E}^u$ denotes the expectation conditional on $\hat{U}_0 = u$.

3 | MINIMIZING THE PROBABILITY OF DRAWDOWN IN THE DANGER-ZONE

In this section, we investigate the optimal reinsurance-investment strategy to minimize the probability of drawdown (see Problem 2.1). From the discussion above, it follows that, if $u \leq \alpha m$, then $\phi(u, m) = 1$, and if $u \geq u_s$ and $u > \alpha m$, then $\phi(u, m) = 0$. It remains for us to determine the minimum probability of drawdown ϕ on the domain

$$\mathcal{O} = \left\{(u, m) \in (\mathbb{R}^+)^2 : \alpha m \leq u \leq \min(m, u_s)\right\}. \tag{5}$$

To that end, we first present a verification theorem in Section 3.1, which we use to find ϕ for the risk model in Equation (2). Combining with the verification theorem, the expressions of the optimal results for both the cases of $m \geq u_s$ and $m < u_s$ are derived explicitly in Section 3.2. Besides, we investigate the behavior of the process $\hat{U}_t$ and find that the optimal strategy will never achieve the safe level u_s with positive probability before drawdown in Section 3.3. Finally, we give several special cases of our risk model and show the impact of some important parameters on the optimal results in Section 3.4.

3.1 | Verification theorem

For a given admissible strategy v, we define the differential operator $\mathcal{A}^v$ on appropriately differentiable functions as follows

$$\mathcal{A}^v h(u,m) = \left[ru + (\mu - r)\pi_t + (\theta - \eta + q_t\eta)a\right] h_u + \frac{1}{2}\left(\sigma^2\pi_t^2 + 2\sigma b\rho q_t\pi_t + b^2 q_t^2\right) h_{uu}. \tag{6}$$

The verification theorem follows readily from the corresponding proof given in Han et al.[20] We omit the details here.

Theorem 1. (*Verification Theorem*) *Suppose $h : \mathcal{O} \to \mathbb{R}^+$ is a bounded, continuous function, which satisfies the following conditions:*

(i) $h(\cdot, m) \in C^2((\alpha m, \min(m, u_s)))$ is a nonincreasing, convex function with bounded first derivative,
(ii) $h(u, \cdot)$ is continuously differentiable, except possibly at u_s,
(iii) $h_m(m,m) \ge 0$ and $\frac{h_m(u,m)}{1-h(u,m)}$ decreases with respect to u if $m < u_s$,
(iv) $h(\alpha m, m) = 1$,
(v) $h(u_s, m) = 0$ if $m \ge u_s$,
(vi) $\mathcal{A}^v h \ge 0$ for all $v \in \mathcal{D}$

Then, $h \le \phi$ on $\mathcal{O}$.

Furthermore, suppose that the function h satisfies all the above conditions in such a way that conditions (iii) and (vi) hold with equality for some admissible strategy v^* defined in feedback form via $v_t^* = (q^*(\hat{U}_t^*), \pi^*(\hat{U}_t^*))$, in which we slightly abuse notation. Then, $h = \phi$ on $\mathcal{O}$, and v^* is the optimal reinsurance-investment strategy. Here, $\hat{U}_t^*$ denotes the optimally controlled process under the optimal policy v_t^*.*

Remark 1. Because Hamilton-Jacobi-Bellman equation that results from $\min_v \mathcal{A}^v\phi = 0$ is independent of αm and u_s, the optimal strategy also minimizes the probability of drawdown before reaching the upper level $\kappa_0 < u_s$. Taksar and Markussen[21] observed a similar phenomenon in their setting; see their Remark 2.1.

3.2 | Probability of drawdown

In this subsection, we use Theorem 1 to determine the minimum probability of drawdown ϕ. Recall from the definition of the domain $\mathcal{O}$ in Equation (5); if $\hat{U}_0 = u < u_s$, we have either $\hat{U}_t < u_s$ almost surely for all $t \ge 0$, or $\hat{U}_t = u_s$ for some $t > 0$. In the case of $m \ge u_s$, $M_t = m$ holds almost surely for all $t \ge 0$, that is, the maximum level of surplus does not increase, and then avoiding drawdown is equivalent to avoiding ruin with a (fixed) ruin level of αm. However, in the case of $m < u_s$, M_t can be larger than m, therefore, the drawdown level is allowed to increase. In the following context, based on the technique of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation, we obtain the explicit expressions of the optimal reinsurance-investment strategy and the corresponding minimum probability of drawdown for both cases.

For convenience, we denote

$$\hat{f}(q,\pi) = \left[(\mu - r)\pi + a\eta q\right] h_u + \frac{1}{2}\left(\sigma^2\pi^2 + 2\sigma b\rho q\pi + b^2q^2\right) h_{uu}.$$

Differentiating $\hat{f}$ with respect to q and π, respectively, yields

$$\begin{cases} \dfrac{\partial^2 \hat{f}}{\partial q^2} = b^2 h_{uu}, \qquad \dfrac{\partial^2 \hat{f}}{\partial \pi^2} = \sigma^2 h_{uu}, \\ \dfrac{\partial^2 \hat{f}}{\partial q \partial \pi} = \rho\sigma b h_{uu}. \end{cases}$$

*If $m \ge u_s$, then condition (*iii*) is moot, and we only require equality in condition (*vi*).

It is not difficult to see that the Hessian matrix of $\hat{f}$ is positive definite, and thus the minimizer of $\hat{f}$ is obtained at

$$\begin{cases} \hat{q}(u) = \dfrac{\rho b(\mu - r) - a\eta\sigma}{\sigma b^2(1-\rho^2)} \cdot \dfrac{h_u}{h_{uu}}, \\ \hat{\pi}(u) = \dfrac{\rho a\eta\sigma - (\mu - r)b}{\sigma^2 b(1-\rho^2)} \cdot \dfrac{h_u}{h_{uu}}. \end{cases} \tag{7}$$

If Theorem 3.1 (i) holds, we must have $\frac{h_u}{h_{uu}} \leq 0$. In the following context, we assume that $0 < \rho < 1$ since the optimal results for the case of $-1 < \rho < 0$ can be derived along the same lines. Then because of the constraints of the optimal strategy and the fact of $\frac{(\mu-r)b}{\rho a\sigma} / \frac{\rho(\mu-r)b}{a\sigma} = \frac{1}{\rho^2} > 1$, we need to discuss the optimization problem in the following three cases:

$$\begin{cases} \text{Case 1}: & \dfrac{\rho(\mu-r)b}{a\sigma} < \eta < \dfrac{(\mu-r)b}{\rho a\sigma} & (ie, \hat{q}(u) \geq 0, \hat{\pi}(u) \geq 0), \\ \text{Case 2}: & \eta \geq \dfrac{(\mu-r)b}{\rho a\sigma} & (ie, \hat{q}(u) \geq 0, \hat{\pi}(u) \leq 0), \\ \text{Case 3}: & \eta \leq \dfrac{\rho(\mu-r)b}{a\sigma} & (ie, \hat{q}(u) \leq 0, \hat{\pi}(u) \geq 0). \end{cases}$$

Here, we just present the proof of Case 1 in detail since the analysis of the other two cases is technically similar.

Case 1: $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$

In this case, we have $\hat{q}(u) > 0$ and $\hat{\pi}(u) > 0$. If $0 \leq \hat{q}(u) \leq 1$ holds, inserting $(q(u), \pi(u)) = (\hat{q}(u), \hat{\pi}(u))$ into Equation (6) and putting $\mathcal{A}^v h(u, m) = 0$ yield

$$\frac{1}{\xi_{11}(u)} = \frac{h_u}{h_{uu}} = \frac{2[ru + a(\theta - \eta)](1-\rho^2)}{\Delta} < 0 \tag{8}$$

with

$$\Delta = \left(\frac{\mu - r}{\sigma}\right)^2 - \frac{2\rho(\mu - r)a\eta}{\sigma b} + \left(\frac{a\eta}{b}\right)^2 > 0. \tag{9}$$

Substituting $\frac{h_u}{h_{uu}}$ in Equation (8) back into (7), it follows that

$$\begin{cases} \hat{q}(u) = \dfrac{2[ru + a(\theta - \eta)]}{\Delta} \cdot \dfrac{\rho(\mu - r)b - \sigma a\eta}{\sigma b^2}, \\ \hat{\pi}(u) = \dfrac{2[ru + a(\theta - \eta)]}{\Delta} \cdot \dfrac{\sigma\rho a\eta - (\mu - r)b}{\sigma^2 b}. \end{cases} \tag{10}$$

Let

$$u_1 = \frac{1}{r}\left[a(\eta - \theta) + \frac{\Delta\sigma b^2}{2\left(\rho b(\mu - r) - \sigma a\eta\right)}\right]. \tag{11}$$

It is not difficult to verify that $u_1 < u_s$. Besides, we can see from Equation (10) that $\hat{q}(u)$ and $\hat{\pi}(u)$ are decreasing functions with respect to u. Thus, when $\max(\alpha m, u_1) \leq u \leq u_s$, we have $0 \leq \hat{q}(u) \leq 1$ and $\hat{\pi}(u) \geq 0$, and hence $q^*(u) = \hat{q}(u)$ and $\pi^*(u) = \hat{\pi}(u)$. On the other hand, when $u < u_1$, it is easy to see that $\hat{q}(u) > 1$. So, we have to choose $q^*(u) = 1$, and derive the corresponding

$$\tilde{\pi}(u) = \frac{\mu - r}{\sigma^2}\frac{h_u}{h_{uu}} - \frac{b\rho}{\sigma}.$$

Therefore, if $\tilde{\pi}(u) \geq 0$, substituting $(q(u), \pi(u)) = (1, \tilde{\pi}(u))$ into Equation (6) and letting $\mathcal{A}^v h(u, m) = 0$ yield

$$\frac{1}{\xi_{12}(u)} = \frac{h_u}{h_{uu}} = \frac{-(ru + B) + \sqrt{(ru + B)^2 - 4AC}}{2A} < 0, \tag{12}$$

in which $A, B,$ and C are defined by

$$\begin{cases} A = -\dfrac{(\mu - r)^2}{2\sigma^2} < 0, \quad B = a\theta - \dfrac{b\rho(\mu - r)}{\sigma}, \\ C = \dfrac{b^2(1-\rho^2)}{2} > 0. \end{cases} \tag{13}$$

Then, it is not difficult to show that

$$\tilde{\pi}(u) = -\frac{\mu - r}{\sigma^2} \cdot \frac{-(ru + B) + \sqrt{(ru+B)^2 - 4AC}}{2A} - \frac{\rho b}{\sigma}. \tag{14}$$

Note that $\tilde{\pi}(u)$ is also a decreasing function with respect to u. Let

$$\tilde{u}_1 = \frac{1}{r}\left[\frac{b(\mu - r)}{2\rho\sigma} - a\theta\right],$$

and then we have $\tilde{\pi}(\tilde{u}_1) = 0$. With some calculations in Appendix B.1, we prove that $\tilde{u}_1 > u_1$. Therefore, we can come to the conclusion that when $\alpha m \le u < \max(\alpha m, u_1)$, we have $\tilde{\pi}(u) > 0$, and thus $q^*(u) = 1$ and $\pi^*(u) = \tilde{\pi}(u)$.

To summarize, we give the optimal reinsurance-investment strategy and the corresponding minimum probability of drawdown for the case of $m \ge u_s$ with $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$ in the following theorem.

Theorem 2. *Suppose that* $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$*. Let* $\xi_{11}(u)$ *and* $\xi_{12}(u)$ *be given in Equations (8) and (12), respectively;* $\hat{q}(u)$*,* $\hat{\pi}(u)$*, and* $\tilde{\pi}(u)$ *be given in Equations (10) and (14), respectively;* u_1 *be given in Equation (11); A, B, and C be given in Equation (13); and* g_{1i} $(i = 1, 2)$ *be given in Appendix A.1. If* $u_s \le m$*, the minimum probability of drawdown for the surplus process (2) is given by*

$$\phi(u, m) = \begin{cases} 1 - \dfrac{g_{11}(u, m)}{g_{12}(u_s, m)}, & \alpha m \le u < \max(\alpha m, u_1), \\ 1 - \dfrac{g_{12}(u, m)}{g_{12}(u_s, m)}, & \max(\alpha m, u_1) \le u \le u_s \le m, \end{cases}$$

and the corresponding optimal reinsurance-investment strategy is

$$(q^*(u), \pi^*(u)) = \begin{cases} (1, \tilde{\pi}(u)), & \alpha m \le u < \max(\alpha m, u_1), \\ (\hat{q}(u), \hat{\pi}(u)), & \max(\alpha m, u_1) \le u \le u_s. \end{cases} \tag{15}$$

Proof. See Appendix B.2. ∎

In the next theorem, the optimal results for the case of $m < u_s$ with $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$ are presented.

Theorem 3. *Suppose that* $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$*. Let* $\xi_{11}(u)$ *and* $\xi_{12}(u)$ *be given in Equations (8) and (12), respectively;* $\hat{q}(u)$*,* $\hat{\pi}(u)$ *and* $\tilde{\pi}(u)$ *be given in Equations (10) and (14), respectively;* u_1 *be given in Equation (11); A, B, and C be given in Equation (13); and* g_{1i}, f_{1i} $(i = 1, 2)$ *be given in Appendix A.1. Then, for* $m < u_s$*,*

(i) *if* $\max(\alpha m, u_1) \le m < u_s$*, the minimum probability of drawdown for the surplus process (2) is given by*

$$\phi(u, m) = \begin{cases} 1 - k_{12}(m) \cdot \dfrac{g_{11}(u, m)}{g_{12}(u_s, u_s)}, & \alpha m \le u < \max(\alpha m, u_1), \\ 1 - k_{12}(m) \cdot \dfrac{g_{12}(u, m)}{g_{12}(u_s, u_s)}, & \max(\alpha m, u_1) \le u \le m < u_s, \end{cases}$$

where

$$k_{12}(m) = \exp\left\{\int_m^{u_s} -f_{12}(y)dy\right\};$$

(ii) if $\alpha m \le m < \max(\alpha m, u_1)$*, the minimum probability of drawdown for the surplus process (2) is given by*

$$\phi(u,m) = 1 - k_{11}(m) \cdot \frac{g_{11}(u,m)}{g_{12}(u_s,u_s)}$$

for any $u \in [\alpha m, m]$*, where*

$$k_{11}(m) = \exp\left\{ \left(-\int_m^{u_1} f_{11}(y) - \int_{u_1}^{u_s} f_{12}(y) \right) dy \right\}.$$

Also, the corresponding optimal reinsurance-investment strategy has the form

$$(q^*(u), \pi^*(u)) = \begin{cases} (1, \tilde{\pi}(u)), & \alpha m \le u < \max(\alpha m, u_1), \\ (\hat{q}(u), \hat{\pi}(u)), & \max(\alpha m, u_1) \le u \le m < u_s. \end{cases} \tag{16}$$

Proof. See Appendix B.3. ■

Remark 2. Setting $\alpha = 0$ in Theorem 3.2, then drawdown is the same as ruin for the ruin level 0. It is not difficult to find that the optimal reinsurance-investment strategy is identical to the one when minimizing the probability of ruin before drawdown happens. Besides, since the relationship between αm and u_1 is uncertain, the optimal strategy depends not only on the value of surplus wealth u but also on m and α.

By the same way, we can get the optimal results for the other two cases as follows:

Case 2: $\eta \ge \frac{(\mu - r)b}{\rho a \sigma}$

In this case, $\hat{q}(u) \ge 0$ and $\hat{\pi}(u) \le 0$, and thus we have to choose $\pi^*(u) = 0$, based on which we obtain

$$\overline{q}(u) = -\frac{a\eta}{b^2} \frac{h_u}{h_{uu}} > 0. \tag{17}$$

If $0 \le \overline{q}(u) \le 1$, we get $q^*(u) = \overline{q}(u)$, and

$$\frac{1}{\xi_{21}(u)} = \frac{h_u}{h_{uu}} = \frac{2b^2 (ru + a(\theta - \eta))}{a^2 \eta^2} < 0. \tag{18}$$

Thus, bringing Equation (18) back into Equation (17) yields

$$\overline{q}(u) = -\frac{2 (ru + a(\theta - \eta))}{a\eta}. \tag{19}$$

We denote

$$u' = \frac{(a\eta - 2a\theta)}{2r}, \tag{20}$$

and then it is not difficult to see that $\overline{q}(u') = 1$. In particular, when $\max(\alpha m, u') \le u \le u_s$, we have $0 \le \overline{q}(u) \le 1$. However, when $u < u'$, we have to choose $q^*(u) = 1$ and it then follows that

$$\frac{1}{\xi_{22}(u)} = \frac{h_u}{h_{uu}} = -\frac{b^2}{2(ru + a\theta)} < 0. \tag{21}$$

We give the optimal results for both the cases of $u_s \le m$ and $u_s > m$ with $\eta \ge \frac{(\mu - r)b}{\rho a \sigma}$ in Theorem 3.4.

Theorem 4. *Suppose that $\eta \geq \frac{(\mu-r)b}{\rho a\sigma}$. Let $\xi_{21}(u)$ and $\xi_{22}(u)$ be given in Equations (18) and (21), respectively; $\overline{q}(u)$ and u' be given in Equations (19) and (20), respectively; and g_{2i}, $f_{2i}(i=1,2)$ be given in Appendix A.2. If $u_s \leq m$, then for any $u \in [\alpha m, u_s]$, the minimum probability of drawdown for the surplus process (2) is given by*

$$\phi(u,m) = \begin{cases} 1 - \dfrac{g_{21}(u,m)}{g_{22}(u_s,m)}, & \alpha m \leq u < \max(\alpha m, u'), \\ 1 - \dfrac{g_{22}(u,m)}{g_{22}(u_s,m)}, & \max(\alpha m, u') \leq u \leq u_s \leq m; \end{cases}$$

For $m < u_s$,

(i) if $\max(\alpha m, u') \leq m < u_s$, then for any $u \in [\alpha m, m]$, the minimum probability of drawdown for the surplus process (2) is given by

$$\phi(u,m) = \begin{cases} 1 - k_{22}(m) \cdot \dfrac{g_{21}(u,m)}{g_{22}(u_s,u_s)}, & \alpha m \leq u < \max(\alpha m, u'), \\ 1 - k_{22}(m) \cdot \dfrac{g_{22}(u,m)}{g_{22}(u_s,u_s)}, & \max(\alpha m, u') \leq u \leq m < u_s, \end{cases}$$

where

$$k_{22}(m) = \exp\left\{ \int_m^{u_s} -f_{22}(y)dy \right\};$$

(ii) if $\alpha m \leq m < \max(\alpha m, u')$, then for any $u \in [\alpha m, m]$, the minimum probability of drawdown for the surplus process (2) is given by

$$\phi(u,m) = 1 - k_{21}(m) \cdot \frac{g_{21}(u,m)}{g_{22}(u_s,u_s)},$$

where

$$k_{21}(m) = \exp\left\{ \left(- \int_m^{u'} f_{21}(y) - \int_{u'}^{u_s} f_{22}(y) \right) dy \right\}.$$

Also, the corresponding optimal reinsurance-investment strategy has the form

$$(q^*(u), \pi^*(u)) = \begin{cases} (1,0), & \alpha m \leq u < \max(\alpha m, u'), \\ \left(\overline{q}(u), 0\right), & \max(\alpha m, u') \leq u \leq \min(m, u_s). \end{cases}$$

Case 3: $\eta \leq \frac{\rho(\mu-r)b}{a\sigma}$

In this case, we have $\hat{q}(u) \leq 0$ and $\hat{\pi}(u) \geq 0$. Thus, we have to choose $q^*(u) = 0$ based on which we obtain

$$\overline{\pi}(u) = -\frac{\mu - r}{\sigma^2}\frac{h_u}{h_{uu}} > 0.$$

Then we get $\pi^*(u) = \overline{\pi}(u)$ and the corresponding

$$\frac{1}{\xi_{31}(u)} = \frac{h_u}{h_{uu}} = \frac{2\sigma^2(ru + a(\theta - \eta))}{(\mu - r)^2} < 0. \tag{22}$$

Therefore, it follows that

$$\overline{\pi}(u) = -\frac{2(ru + a(\theta - \eta))}{\mu - r}. \tag{23}$$

We conclude the optimal results for $\eta \leq \frac{\rho(\mu-r)b}{a\sigma}$ in the following Theorem 3.5.

Theorem 5. *Suppose that $\eta \leq \frac{\rho(\mu-r)b}{a\sigma}$. Let $\xi_{31}(u)$ be given in Equation (22), $\overline{\pi}(u)$ be given in (23), and g_{31}, f_{31} be given in Appendix A.3. For any $u \in [\alpha m, \min(m, u_s)]$, the minimum probability of drawdown for the surplus process (2) is given by*

$$\phi(u,m) = \begin{cases} 1 - \dfrac{g_{31}(u,m)}{g_{31}(u_s,m)}, & \text{if } \alpha m \leq u \leq u_s \leq m, \\ 1 - k_{31}(m) \cdot \dfrac{g_{31}(u,m)}{g_{31}(u_s,u_s)}, & \text{if } \alpha m \leq u \leq m < u_s, \end{cases}$$

where

$$k_{31}(m) = \exp\left\{ - \int_m^{u_s} f_{31}(y) dy \right\}.$$

Finally, the corresponding optimal reinsurance-investment strategy has the form

$$(q^*(u), \pi^*(u)) = \left(0, \overline{\pi}(u)\right).$$

3.3 | Reaching the safe level

In this subsection, we examine the behavior of the optimally controlled surplus process, and show that the optimal strategy will never achieve the safe level u_s with positive probability in finite time. Here, we only present the proof for Case 1 with $\alpha m < u_1 < u_s$, then the similar results in other cases can be obtained along the same lines.

Proposition 1. *Suppose that $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$ and $\alpha m < u_1 < u_s$. Let $\hat{U}_t^*$ be the optimally controlled wealth starting at u. Define the hitting times*

$$\tau_s^* = \inf\{t \geq 0 : \hat{U}_t^* \geq u_s\},$$

and

$$\tau_\alpha^* = \inf\{t \geq 0 : \hat{U}_t^* \leq \alpha m\}.$$

Then, for any $u \in (\alpha m, \min(m, u_s))$, we have $\mathbb{P}^{u,m}(\tau_s^* < \tau_\alpha^*) = 0$.

Proof. Because we are only interested in whether the safe level can be reached before drawdown occurs, we may extend the domain of (q^*, π^*) to $\mathbb{R} \times \mathbb{R}$ and set

$$\begin{cases} q^*(u) = \dfrac{[ru + a(\theta-\eta)]}{\Delta} \cdot \dfrac{\rho(\mu-r)b - \sigma a\eta}{\sigma b^2}, \\ \pi^*(u) = \dfrac{[ru + a(\theta-\eta)]}{\Delta} \cdot \dfrac{\sigma\rho a\eta - (\mu-r)b}{\sigma^2 b} \end{cases}$$

for $u < \alpha m$. Define $\mathbf{b}$ and $\mathbf{s}$ on $\mathbb{R}$ by

$$\mathbf{b}(u) = ru + (\mu-r)\pi^*(u) + (\theta - \eta + q^*(u)\eta)\, a,$$

and

$$\mathbf{s}(u) = \sqrt{\sigma^2(\pi^*(u))^2 + 2\sigma b\rho q^*(u)\pi^*(u) + b^2(q^*(u))^2}.$$

One can show that $\mathbf{b}(u) = 0$ for $u < \alpha m$. Next, define the scale function p on $\mathbb{R}$ by

$$p(u) = \int_{\alpha m}^{u} \exp\left(-2 \int_{\alpha m}^{y} \frac{\mathbf{b}(z)}{\mathbf{s}^2(z)}\, dz \right) dy,$$

and define the function v on $\mathbb{R}\times\mathbb{R}$ by

$$v(u,m)=\int_{\alpha m}^{u}p'(y)\int_{\alpha m}^{y}\frac{2dz}{p'(z)\mathbf{s}^2(z)}\,dy=\int_{\alpha m}^{u}\frac{2\,(p(u)-p(y))}{p'(y)\mathbf{s}^2(y)}\,dy. \tag{24}$$

Now, we want to show that $v(-\infty,m)=v(u_s,m)=\infty$. First, from $\mathbf{b}(u)=0$ for $u<\alpha m$, it follows that $p(-\infty)=\int_{\alpha m}^{-\infty}1dy=-\infty$. Thus, the expression in (5.74) on page 348 of Karatzas and Shreve[22] implies that $v(-\infty,m)=\infty$. Next, note that we have

$$-\frac{2\mathbf{b}(u)}{\mathbf{s}^2(u)}=\begin{cases}\xi_{12}(u)=\dfrac{2A}{-(ru+B)+\sqrt{(ru+B)^2-4AC}}, & \text{if } \alpha m<u<u_1,\\ \xi_{11}(u)=\dfrac{\Delta}{2[ru+a(\theta-\eta)](1-\rho^2)}, & \text{if } u_1\le u<u_s.\end{cases}$$

Besides, according to Equation (24), we know that

$$\begin{aligned}v(u_s,m)&=\int_{\alpha m}^{u_s}\frac{2\,(p(u_s)-p(x))}{p'(x)\mathbf{s}^2(x)}\,dx\\ [4mm]&=\int_{\alpha m}^{u_1}\frac{2\,(p(u_s)-p(x))}{p'(x)\mathbf{s}^2(x)}\,dx+\int_{u_1}^{u_s}\frac{2\,(p(u_s)-p(x))}{p'(x)\mathbf{s}^2(x)}\,dx.\end{aligned} \tag{25}$$

It is not difficult to prove that the first integral in Equation (25) is positive. Let $d=\frac{\Delta}{2r(1-\rho^2)}$, then for $x\in(u_1,u_s)$, it follows that

$$\begin{aligned}p(u_s)-p(x)&=\int_{x}^{u_s}\exp\left(-2\int_{\alpha m}^{y}\frac{\mathbf{b}(z)}{\mathbf{s}^2(z)}\,dz\right)dy\\&=\int_{x}^{u_s}\exp\left(\int_{\alpha m}^{u_1}\xi_{12}(z)\,dz+\int_{u_1}^{y}\xi_{11}(z)dz\right)dy\\&=\exp\left(\int_{\alpha m}^{u_1}\xi_{12}(z)\,dz\right)\cdot\frac{-1}{d+1}\frac{(x-u_s)^{d+1}}{(u_1-u_s)^d},\end{aligned}$$

and

$$\frac{2}{p'(x)\mathbf{s}^2(x)}=\exp\left(-\int_{\alpha m}^{u_1}\xi_{12}(z)\,dz\right)\cdot\frac{d}{r}\frac{(u_1-u_s)^d}{(x-u_s)^(d+2)}.$$

Thus, we have

$$\begin{aligned}\textstyle\int_{u_1}^{u_s}\frac{2\,(p(u_s)-p(x))}{p'(x)\mathbf{s}^2(x)}\,dx&=\textstyle\int_{u_1}^{u_s}\frac{d}{r(d+1)}\cdot\frac{1}{u_s-x}\,dx\\&=\infty.\end{aligned}$$

Therefore, we get $v(u_s,m)=\infty$. It follows from Feller's test for explosions (theorem 5.5.29 on page 348 of Karatzas and Shreve[22]) that $\mathbb{P}^{u,m}(\tau_s^*<\tau_\alpha^*)=0$ for $u\in(\alpha m,\min(m,u_s))$. ■

Remark 3. We can see from Equation (10) that the insurer would rather retain more of its insurance risk and invest less amount in the risky asset as the surplus gets closer to the safe level u_s. In fact, both the drift and the volatility of the optimally controlled surplus process approach 0 as the surplus approaches u_s. Thus, it is to be expected that the safe level might not be reachable, and Proposition 1 confirms our intuition.

Let $\tau=\tau_\alpha^*\wedge\tau_s^*$ denote the first hitting time of αm or u_s when the initial surplus u lies in $(\alpha m,\min(m,u_s))$. Since $v(u_s,m)=\infty$, then from proposition 5.5.32 on page 350 of Karaztas and Shreve,[22] we can deduce that $0<P(\tau<\infty)<1$. Furthermore, in combination with Proposition 1, we can see that either drawdown occurs with probability $\phi(u,m)=P(\tau<\infty)$ or the optimal controlled surplus value lies strictly between αm and u_s, for all time, with probability of

$1-\phi(u,m)$. The similar conclusion is also derived in the works of Bayraktar and Zhang,[23] Angoshtari et al,[5] and Han et al.[6,7]

3.4 Comparisons of optimal strategies

In this section, we would like to investigate some special cases of our risk model, that is, $\pi=0$, $\rho=0$ or $r=0$ in Equation (2), and compare the optimal strategies derived in Section 3.2 with those for different risk models.

Firstly, settting $\pi=0$ in Equation (2), that is, the insurer only purchases proportional reinsurance and invests all its surplus in risk-free bond, then the surplus process in Equation (2) can be reduced to

$$d\hat{U}_t^{\pi_0} = \left[r\hat{U}_t^{\pi_0} + (\theta-\eta+\eta q_t)a\right] dt + bq_t\, dB_{2t}. \tag{26}$$

It is not difficult to show that the optimal reinsurance strategy for the process (26) is given by

$$q_{\pi_0}^*(u) = \begin{cases} 1, & \alpha m \le u < \max(\alpha m, u'), \\ \overline{q}(u), & \max(\alpha m, u') \le u \le \min(m, u_s), \end{cases} \tag{27}$$

where $\overline{q}(u)$ and u' are given in Equations (17) and (20), respectively.

Theorem 6. *When $u\in[\alpha m, \min(m,u_s)]$, the inequality $q^*(u)\le q_{\pi_0}^*(u)$ holds for any $\eta>0$ with $q^*(u)$ given in Theorems 3 to 5 and $q_{\pi_0}^*(u)$ given in Equation (27).*

Proof. See Appendix B.4. ■

Remark 4. We can see from Theorem 3.6 that the retention level for the case without investment is always no less than the one with investment at all levels of surplus $\alpha m\le u\le \min(m,u_s)$. This conclusion is somehow relevant to the assumption of $\mu>r$. Because of the high return from the risky investment, there is more chance to increase its profit by investing its surplus into the risky assets and then the insurer can optimally buy more reinsurance to reduce the risk. In particular, if we assume $\mu\le r$ in the risk model (2), it is not difficult to find that the insurer chooses investing nothing into the risky asset, and thus the retention level for the case with investment is always equal to the one without investment.

Secondly, setting $\rho=0$ in (2), that is, the Brownian motions B_{1t} and B_{2t} are independent, then the surplus process in Equation (2) can be simplified into

$$d\hat{U}_t^{\rho_0} = \left[r\hat{U}_t^{\rho_0} + (\mu-r)\pi_t + (\theta-\eta+\eta q_t)a\right] dt + \sigma\pi_t\, dB_{1t} + bq_t d\, B_{2t}. \tag{28}$$

For convenience, we denote

$$\Delta_{\rho_0} = \left(\frac{\mu-r}{\sigma}\right)^2 + \left(\frac{a\eta}{b}\right)^2 > 0,$$

and

$$u_1^{\rho_0} = \frac{1}{r}\left(a(\eta-\theta) - \frac{b^2\Delta_{\rho_0}}{2a\eta}\right). \tag{29}$$

By using the same method in Section 3.2, one can show that the optimal reinsurance-investment strategy to minimize the probability of drawdown for the surplus process (28) is given by

$$\left(q_{\rho_0}^*(u), \pi_{\rho_0}^*(u)\right) = \left(-\frac{2a\eta\,[ru+a(\theta-\eta)]}{b^2\Delta_{\rho_0}}, -\frac{2(\mu-r)\,[ru+a(\theta-\eta)]}{\sigma^2\Delta_{\rho_0}}\right) \tag{30}$$

for $\max(\alpha m, u_1^{\rho_0})\le u\le \min(u_s,m)$, and

$$\left(q_{\rho_0}^*(u), \pi_{\rho_0}^*(u)\right) = \left(1, \frac{-(ru+a\theta)+\sqrt{(ru+a\theta)^2+(\frac{\mu-r}{\sigma})^2 b^2}}{\mu-r}\right) \tag{31}$$

for $\alpha m\le u<\max(\alpha m, u_1^{\rho_0})$.

Theorem 7. *When $u \in [\alpha m, \min(m, u_s)]$, we have the following relations for the optimal reinsurance-investment strategies $(q^*(u), \pi^*(u))$ and $(q^*_{\rho_0}(u), \pi^*_{\rho_0}(u))$ given in Theorems 3 to 5 and Equations 30, (31), respectively:*

(i) if $\eta \le \frac{(\mu-r)b}{a\sigma}$, we have $q^*(u) \le q^*_{\rho_0}(u)$ for any $u \in [\alpha m, \min(m, u_s)]$, if $\eta > \frac{(\mu-r)b}{a\sigma}$, we have $q^*(u) \ge q^*_{\rho_0}(u)$ for any $u \in [\alpha m, \min(m, u_s)]$;
(ii) if $\eta \le \frac{\rho(\mu-r)b}{a\sigma}$, we have $\pi^*(u) \ge \pi^*_{\rho_0}(u)$ for any $u \in [\alpha m, \min(m, u_s)]$, if $\frac{\rho(\mu-r)b}{a\sigma} < \eta \le \frac{(\mu-r)b}{a\sigma}$, there exists a unique u_0 such that $\pi^*(u) \ge \pi^*_{\rho_0}(u)$ for any $u \in [\max(\alpha m, u_0), \min(m, u_s)]$ and $\pi^*(u) < \pi^*_{\rho_0}(u)$ for any $u \in [\alpha m, \max(\alpha m, u_0))$, and if $\eta > \frac{(\mu-r)b}{a\sigma}$, we have $\pi^*(u) \le \pi^*_{\rho_0}(u)$ for any $u \in [\alpha m, \min(m, u_s)]$.

Proof. See Appendix B.5 (also see Figures 1 and 2 for details). ∎

Remark 5. Note that the relationship between the optimal strategies for the two cases of $\rho > 0$ and $\rho = 0$ is uncertain. Also, from Figures 1 and 2, we can see clearly that for the case of $\rho > 0$, the value of η has a greater impact on the optimal strategy. Additionally, if we assume the correlation coefficient $-1 < \rho < 0$ in Equation (2), then $\hat{q}(u)$ and $\hat{\pi}(u)$ given in Equation (10) are always positive, and thus the optimal reinsurance-investment strategy can be obtained along the same lines as in Case 1. We find that the comparison results given in Theorem 3.7 will be totally opposite under the assumption of $-1 < \rho < 0$.

Finally, setting $r = 0$ in Equation (2), that is, the insurer only purchases proportional reinsurance and invests its surplus in risky asset, then the surplus process in Equation (2) can be reduced to

$$d\hat{U}^{r_0}_t = \left[\mu\pi_t + (\theta - \eta + \eta q_t)a\right] dt + \sqrt{\sigma^2\pi_t^2 + 2\sigma b\rho q_t\pi_t + b^2 q_t^2}d\, B_t. \tag{32}$$

For convenience, we denote

$$\Delta_{r_0} = \left(\frac{\mu}{\sigma}\right)^2 - \frac{2\rho\mu a\eta}{\sigma b} + \left(\frac{a\eta}{b}\right)^2 > 0.$$

To keep things simple, we constrain $q(u) \in (0, \infty)$. For $q(u) \in [0, 1]$, the insurer has a proportional reinsurance cover; and for $q(u) \in (1, \infty)$, it may be thought of as acquiring new business. Then we have the following results.

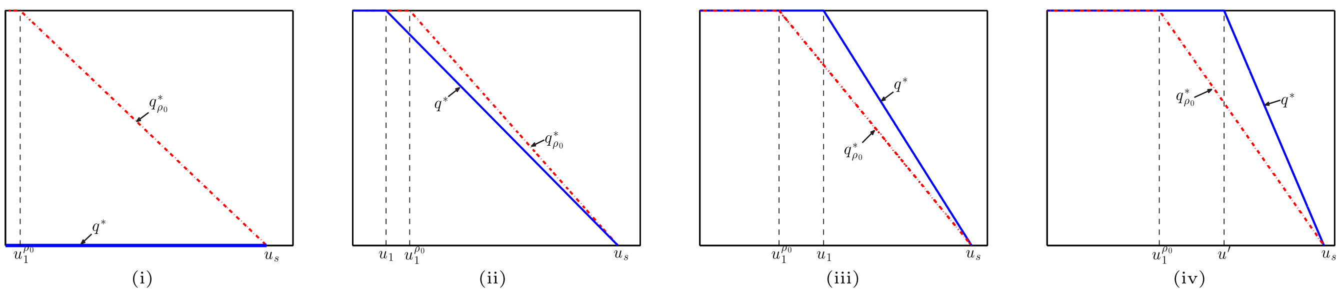

FIGURE 1 The relationship between q^* and $q^*_{\rho_0}$ [Colour figure can be viewed at wileyonlinelibrary.com]

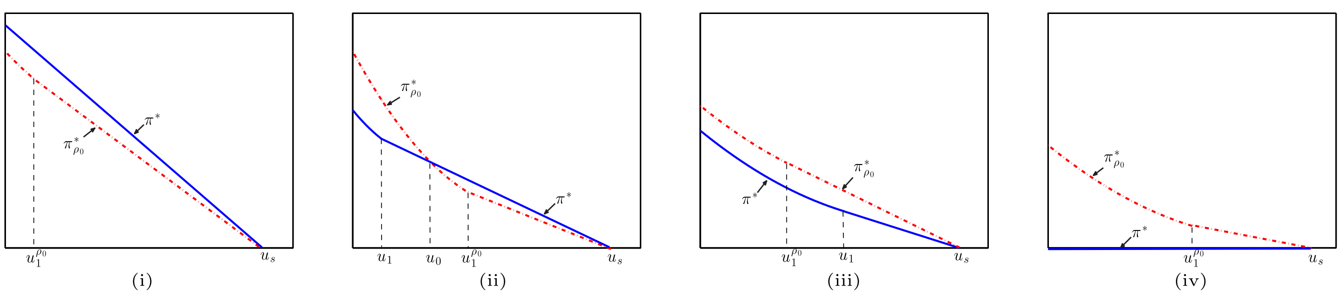

FIGURE 2 The relationship between π^* and $\pi^*_{\rho_0}$ [Colour figure can be viewed at wileyonlinelibrary.com]

When $\eta \leq \frac{\rho\mu b}{a\sigma}$, the optimal reinsurance-investment strategy is

$$\left(q_{r_0}^*, \pi_{r_0}^*\right) = \left(0, \frac{2a(\eta-\theta)}{\mu}\right). \tag{33}$$

When $\frac{\rho\mu b}{a\sigma} < \eta < \frac{\mu b}{\rho a\sigma}$, the optimal reinsurance-investment strategy is

$$\left(q_{r_0}^*, \pi_{r_0}^*\right) = \left(\frac{2a(\theta-\eta)\,(\rho\mu b - \sigma a\eta)}{\sigma b^2\Delta_{r_0}}, \frac{2a(\theta-\eta)\,(\sigma\rho a\eta - \mu b)}{\sigma^2 b\Delta_{r_0}}\right). \tag{34}$$

When $\eta \geq \frac{\mu b}{\rho a\sigma}$, the optimal reinsurance-investment strategy is

$$\left(q_{r_0}^*, \pi_{r_0}^*\right) = \left(\tfrac{2a(\eta-\theta)}{a\eta}, 0\right). \tag{35}$$

Theorem 8. *Suppose that the admissible reinsurance strategy $q(u) \in [0, \infty)$. Then we have the following relations for the optimal reinsurance-investment strategies $(q^*(u), \pi^*(u))$ and $(q_{r_0}^*, \pi_{r_0}^*)$ given in Theorems 3-5 and Eqs.* 33-(35)*, respectively:On the one hand, if $r < (1-\rho^2)\mu$, then the inequality $\frac{\rho\mu b}{a\sigma} < \frac{(\mu-r)b}{\rho a\sigma}$ holds, thus it follows that*

(i) When $\eta \leq \frac{\rho(\mu-r)b}{a\sigma}$, $q^*(u) = q_{r_0}^* = 0$ for all $u \in [\alpha m, \max(m, u_s)]$. Let

$$u_1^{r_0} = \frac{a(\eta-\theta)}{\mu},$$

then we have $\pi^*(u) > \pi_{r_0}^*$ for $u < u_1^{r_0}$, and $\pi^*(u) \leq \pi_{r_0}^*$ for $u \geq u_1^{r_0}$.
(ii) When $\frac{\rho(\mu-r)b}{a\sigma} < \eta \leq \frac{\rho\mu b}{a\sigma}$, then $q^*(u) \geq q_{r_0}^*$ holds for all $u \in [\alpha m, \max(m, u_s)]$. Let

$$u_2^{r_0} = \frac{a(\eta-\theta)}{r} + \frac{a(\eta-\theta)\Delta\sigma^2 b}{r\mu(\sigma\rho a\eta - (\mu-r)b)},$$

then we can see that $\pi^*(u) > \pi_{r_0}^*$ for $u < u_2^{r_0}$, and $\pi^*(u) \leq \pi_{r_0}^*$ for $u \geq u_2^{r_0}$.
(iii) When $\frac{\rho\mu b}{a\sigma} < \eta \leq \frac{(\mu-r)b}{\rho a\sigma}$, we define

$$\left\{ u_3^{r_0} = \tfrac{1}{r}\left(a(\eta-\theta) + \frac{2a(\theta-\eta)(\sigma\rho a\eta - \mu b)\Delta}{(\sigma\rho a\eta - (\mu-r)b)\,\Delta_{r_0}}\right), [5mm] u_4^{r_0} = \tfrac{1}{r}\left(a(\eta-\theta) + \frac{2a(\theta-\eta)(\rho\mu b - \sigma a\eta)\Delta}{(\rho(\mu-r)b - \sigma a\eta)\,\Delta_{r_0}}\right).\right.$$

It is easy to verify that $u_3^{r_0} < u_4^{r_0}$. Thus, we have $q^*(u) > q_{r_0}^*$ and $\pi^*(u) \geq \pi_{r_0}$ for $u \leq u_3^{r_0}$, $q^*(u) \geq q_{r_0}^*$ and $\pi^*(u) < \pi_{r_0}$ for $u_3^{r_0} < u \leq u_4^{r_0}$, and $q^*(u) < q_{r_0}^*$ and $\pi^*(u) < \pi_{r_0}$ for $u > u_4^{r_0}$.
(iv) When $\frac{(\mu-r)b}{\rho a\sigma} < \eta \leq \frac{\mu b}{\rho a\sigma}$, we always have $\pi^*(u) < \pi_{r_0}^*$ since $\pi^*(u) = 0$. Define

$$u_5^{r_0} = \frac{1}{r}\left(a(\eta-\theta) + \frac{a^2\eta(\eta-\theta)(\rho\mu b - \sigma a\eta)}{\sigma b^2\Delta_{r_0}}\right).$$

Thus, one can show that $q^*(u) > q_{r_0}^*$ for $u < u_5^{r_0}$, and $q^*(u) \leq q_{r_0}^*$ for $u \geq u_5^{r_0}$.
(v) When $\eta > \frac{\mu b}{\rho a\sigma}$, $\pi^*(u) = \pi_{r_0}^* = 0$, and the inequality $q^*(u) < q_{r_0}^*$ always holds. On the other hand, if $r \geq (1-\rho^2)\mu$, then the inequality $\frac{\rho\mu b}{a\sigma} \geq \frac{(\mu-r)b}{\rho a\sigma}$ holds, thus if follows that:

(i) When $\eta \leq \frac{\rho(\mu-r)b}{a\sigma}$, we have $q^*(u) = q_{r_0}^* = 0$ for all $u \in [\alpha m, \max(m, u_s)]$, and $\pi^*(u) > \pi_{r_0}^*$ for $u < u_1^{r_0}$, $\pi^*(u) \leq \pi_{r_0}^*$ for $u \geq u_1^{r_0}$.
(ii) When $\frac{\rho(\mu-r)b}{a\sigma} < \eta \leq \frac{(\mu-r)b}{\rho a\sigma}$, since $q_{r_0}^* = 0$, we have $q^*(u) \geq q_{r_0}^*$ for all $u \in [\alpha m, \max(m, u_s)]$, and $\pi^*(u) > \pi_{r_0}^*$ for $u < u_2^{r_0}$, $\pi^*(u) \leq \pi_{r_0}^*$ for $u \geq u_2^{r_0}$.
(iii) When $\frac{(\mu-r)b}{\rho a\sigma} < \eta \leq \frac{\rho\mu b}{a\sigma}$, it is easy to verify that $q^*(u) \geq q_{r_0}^*$ and $\pi^*(u) \leq \pi_{r_0}^*$ for all $u \in [\alpha m, \max(m, u_s)]$.
(iv) When $\frac{\rho\mu b}{a\sigma} < \eta \leq \frac{\mu b}{\rho a\sigma}$, we have $\pi^*(u) < \pi_{r_0}^*$ for all $u \in [\alpha m, \max(m, u_s)]$, and $q^*(u) > q_{r_0}^*$ for $u < u_5^{r_0}$, $q^*(u) \leq q_{r_0}^*$ for $u \geq u_5^{r_0}$.
(v) When $\eta > \frac{\mu b}{\rho a\sigma}$, $\pi^*(u) = \pi_{r_0}^* = 0$ and the inequality $q^*(u) < q_{r_0}^*$ always holds.

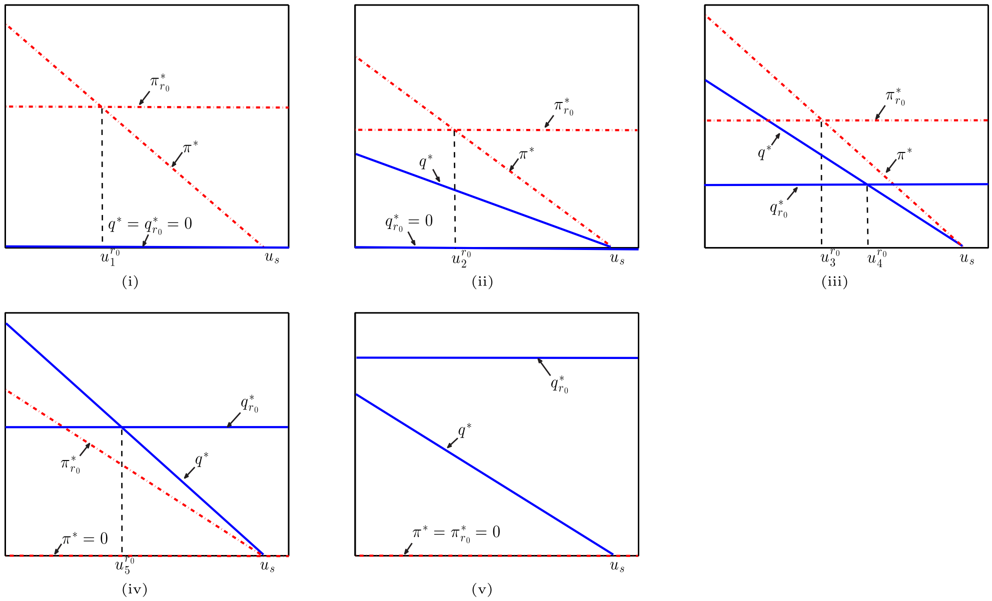

FIGURE 3 The relationship between (q^*, π^*) and $(q^*_{r_0}, \pi^*_{r_0})$ for $r < (1-\rho^2)\mu$ [Colour figure can be viewed at wileyonlinelibrary.com]

Remark 6. Note that when $r > 0$, the safe level is finite and the optimal reinsurance-investment strategy approaches zero as the surplus approaches u_s. It is to be expected since when the surplus is large enough, the interest earned can largely cover the shortfall between premiums and expensive reinsurance purchase, then the insurer can optimally transfers more claims to the reinsurer and invest less amount into the risky asset for the case of $r > 0$ to avoid the drawdown. However, when $r = 0$, the safe level goes to ∞, and the optimal strategy $(q^*_{r_0}, \pi^*_{r_0})$ is independent of the surplus u. Therefore, there should exist some u^{r_0} such that $q^*(u) \le q^*_{r_0}$ and $\pi^*(u) \le \pi^*_{r_0}$ for any $u \ge u^{r_0}$ expect for the case of $q^*_{r_0}$ or $\pi^*_{r_0}$ equals to 0 (see Figure 3 for details).

4 | MINIMIZING THE EXPECTED TIME TO REACH A GIVEN CAPITAL LEVEL IN THE SAFE-REGION

In this section, we suppose that the initial surplus u is larger than u_s, then according to the analysis at the end of Section 2, the insurer never needs to face the probability of drawdown in this region, but may be interested in the criterion of reaching a given capital level as quickly as possible (see Problem 2.2). Therefore, in the following context, we restrict our attention to solve the optimization problem of minimizing the expected time to reach κ.

Via a verification theorem similar to Theorem 3.1 (eg, Luo et al[16]), if we find a smooth, decreasing, convex solution $\tilde{h}(u)$ for the following boundary-value problem (BVP), then this solution equals the value function φ. For $u_s < u \le \kappa$, recall from the definition of the differential operator $\mathcal{A}^v$ in (6), we have

$$\min_{v \in D} \mathcal{A}^v \tilde{h}(u) + 1 = 0 \tag{36}$$

with $\tilde{h}(\kappa) = 0$. In Section 4.1, the optimal reinsurance-investment strategy and the corresponding value function are derived explicitly. Then we compare the optimal strategy with those for two special cases of the risk model (2) in Section 4.2.

Remark 7. If $\alpha m \le u < \kappa \le u_s$, combining with Remark 1, we can see that the minimum probability that the controlled surplus process reaches the drawdown level before reaching the upper goal κ is always positive. Hence $\varphi(u) = \inf_{v \in D} \mathbb{E}^u(\tau_\kappa) = \infty$. Therefore, it is not applicable to make an optimal decision by minimizing the expected time to reach a goal in the danger-zone except for the risk model with cheap reinsurance as in Liang and Bai[17] and Luo et al.[16]

4.1 | The expected time to reach a given upper goal

For convenience, we denote

$$\tilde{f}(q,\pi) = \left[(\mu - r)\pi + a\eta q\right]\tilde{h}_u + \frac{1}{2}(\sigma^2\pi^2 + 2\sigma b\rho q\pi + b^2q^2)\tilde{h}_{uu}.$$

Similarly, we can see that the Hessian matrix of $\tilde{f}$ is positive definite, and thus the minimizer of $\tilde{f}$ is obtained at

$$\begin{cases} \hat{q}_1(u) = \dfrac{\rho b(\mu - r) - a\eta\sigma}{\sigma b^2(1-\rho^2)} \cdot \dfrac{\tilde{h}_u}{\tilde{h}_{uu}}, \\ \hat{\pi}_1(u) = \dfrac{\rho a\eta\sigma - (\mu - r)b}{\sigma^2 b(1-\rho^2)} \cdot \dfrac{\tilde{h}_u}{\tilde{h}_{uu}}. \end{cases} \tag{37}$$

Due to the constraints of the optimal strategy, we still need to discuss the optimization problem in three different cases as mentioned in Section 3.2.

Here, we discuss Case 1 firstly. In this case, $\hat{q}_1(u) > 0$, $\hat{\pi}_1(u) > 0$. If $0 \le \hat{q}_1(u) \le 1$ holds, then replacing $(q(u), \pi(u))$ in Equation (36) with $(\hat{q}_1(u), \hat{\pi}_1(u))$ yields

$$[ru + a(\theta - \eta)]\tilde{h}_u - \frac{\Delta}{2(1-\rho^2)}\frac{\tilde{h}_u^2}{\tilde{h}_{uu}} + 1 = 0 \tag{38}$$

with Δ be given in Equation (9). Recall u_s of Equation (4), then the general solution to Equation(38) has the form

$$\tilde{h}(u,\kappa) = -\frac{2(1-\rho^2)}{\Delta + 2(1-\rho^2)r}\ln r(u - u_s) + C_1, \tag{39}$$

where C_1 is a constant to be determined. From Equation (39), we have $\frac{\tilde{h}_u}{\tilde{h}_{uu}} = u_s - u$, then substituting $\frac{\tilde{h}_u}{\tilde{h}_{uu}}$ back into Equation (37) yields

$$\begin{cases} \hat{q}_1(u) = \dfrac{\rho(\mu - r)b - \sigma a\eta}{\sigma b^2(1-\rho^2)} \cdot (u_s - u), \\ \hat{\pi}_1(u) = \dfrac{\sigma\rho a\eta - (\mu - r)b}{\sigma^2 b(1-\rho^2)} \cdot (u_s - u). \end{cases} \tag{40}$$

Let

$$u_2 = u_s - \frac{b^2\sigma(1-\rho^2)}{\rho b(\mu - r) - a\eta\sigma}.$$

It is clear that $u_2 > u_s$. Note that $\hat{q}_1(u)$ and $\hat{\pi}_1(u)$ are both increasing functions with respect to u. Thus, when $u_s < u \le \min(u_2, \kappa)$, we have $0 < \hat{q}_1(u) \le 1$ and $\hat{\pi}_1(u) > 0$, and hence $q^*(u) = \hat{q}_1(u)$ and $\pi^*(u) = \hat{\pi}_1(u)$. On the other hand, when $\min(u_2, \kappa) < u \le \kappa$, we have $\hat{q}(u) > 1$, then we have to choose $q^*(u) = 1$, and derive the corresponding

$$\tilde{\pi}(u) = \frac{\mu - r}{\sigma^2}\frac{h_u}{h_{uu}} - \frac{b\rho}{\sigma}. \tag{41}$$

If $\tilde{\pi}(u) \ge 0$, substituting $(q(u), \pi(u)) = (1, \tilde{\pi}(u))$ into Equation (36) yields

$$\left(ru + a\theta - \frac{b\rho(\mu - r)}{\sigma}\right)\tilde{h}_u - \frac{(\mu - r)^2}{2\sigma^2}\frac{\tilde{h}_u^2}{\tilde{h}_{uu}} + \frac{1}{2}b^2(1+\rho^2)\tilde{h}_{uu} + 1 = 0. \tag{42}$$

Since the function $\tilde{h}$ satisfies the following boundary conditions

$$\tilde{h}(\kappa) = 0, \quad \tilde{h}_u(u_2) = \frac{1}{B_1 - r} \cdot \frac{1}{u_2 - u_s},$$

then by using the Matlab ODE solver "ode45," the solution to the differential Eq. 42 can be numerically approximated. Besides, setting $\theta = \eta$ in Equation (36), we can verify that $\tilde{\pi}(u)$ given in Equation (41) is nonnegative and $\tilde{h}(u)$ is indeed our value function along the same lines as in Liang and Bai.[17] Here, to derive the explicit expressions for the optimal strategy and the value function with the assumption of $\eta > \theta$, we release the constraint of $q(u) \in [0, 1]$ to $q(u) \in (0, \infty)$ in the rest of this section. Similar to the Case 1, optimal results for the other two cases can be derived explicitly. Then, we have

Theorem 9. *Let Δ be given in Equation (9); $(\hat{q}(u), \hat{\pi}(u))$ be given in Equation (40). Then for any $u \in (u_s, \kappa]$, the value function φ for minimizing the expected time to reach a given capital level κ is given as follows:*

(i) if $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$, we have

$$\varphi(u) = -\frac{2(1-\rho^2)}{\Delta + 2(1-\rho^2)r} \ln \frac{u - u_s}{\kappa - u_s}, \tag{43}$$

and the corresponding optimal reinsurance-investment strategy is given by

$$(q^*(u), \pi^*(u)) = (\hat{q}(u), \hat{\pi}(u))\,; \tag{44}$$

(ii) if $\eta \geq \frac{(\mu-r)b}{\rho a\sigma}$, we have

$$\varphi(u) = -\frac{2b^2}{a^2\eta^2 + 2b^2 r} \ln \frac{u - u_s}{\kappa - u_s},$$

and the corresponding optimal reinsurance-investment strategy is given by

$$(q^*(u), \pi^*(u)) = \left(\frac{a\eta}{b^2}(u - u_s), 0\right);$$

(iii) if $\eta \leq \frac{\rho(\mu-r)b}{a\sigma}$, we have

$$\varphi(u) = -\frac{2\sigma^2}{(\mu-r)^2 + 2\sigma^2 r} \ln \frac{u - u_s}{\kappa - u_s},$$

and the corresponding optimal reinsurance-investment strategy is given by

$$(q^*(u), \hat{\pi}^*(u)) = \left(0, \tfrac{\mu-r}{\sigma^2}(u - u_s)\right).$$

Proof. See Appendix B.6. ∎

Remark 8. We can see from Theorem 4.1 that the optimal reinsurance-investment strategy depends on how far surplus is above the boundary u_s. Note that when $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$, substituting the optimal strategy in Equation (40) back into the surplus process (2) yields

$$d\hat{U}_t^* = (\hat{U}_t^* - u_s)\left(r + \frac{\Delta}{1-\rho^2}\right)dt + (\hat{U}_t^* - u_s)\sqrt{\frac{\Delta}{1-\rho^2}}dB_t, \qquad 0 \leq t < \tau_\kappa^*, \tag{45}$$

where $\tau_\kappa^* := \inf\{t > 0 : \hat{U}_t^* = \kappa\}$, which is a linear stochastic stochastic differential equation. Note that the solution to Equation (45) is

$$\hat{U}_t^* = (\hat{U}_0^* - u_s)\exp\left\{\left(r + \frac{\Delta}{2(1-\rho^2)}\right)\, t + \sqrt{\frac{\Delta}{1-\rho^2}}\, B_t\right\} + u_s. \tag{46}$$

It, then, follows that

$$\varphi(\hat{U}_t^*) - \varphi(\hat{U}_0^*) = -t - \frac{2\sqrt{\Delta(a-\rho^2)}}{2r(1-\rho^2)+\Delta} B_t, \quad 0 \le t < \tau_\kappa^*, \tag{47}$$

that is, under the optimal reinsurance-investment strategy, the process $\varphi(\hat{U}_t^*) - \varphi(\hat{U}_0^*)$ follows a simple Brownian motion with a drift coefficient equals to -1. From this, with boundary condition $\varphi(\kappa) = 0$, it is easy to recover the value function (43) from Equation (47) by evaluating the expected value of Equation (47) at $t = \tau_\kappa^*$, which then gives $\mathbb{E}_u(\tau_\kappa^*) = \varphi(u)$. Besides, it is clearly from Equation (46) that the low bound u_s is inaccessible from above, ensuring that the insurer will stay in the safe-region forever, almost surely. The conclusion holds for the other two cases. Browne[24] also observed such a similar phenomenon when minimizing the time to reach a goal for a general consumption function.

4.2 | Comparisons of optimal strategies

In this subsection, under the criterion of minimizing the expected time to reach a given capital level, we compare the optimal strategy $(q^*(u), \pi^*(u))$ with those for two special cases where $\pi = 0$ or $\rho = 0$ in Equation (2). Note that when $r = 0$, the safe-level approaches ∞, thus there does not exist a κ such that $\kappa > u_s$.

When $\pi = 0$ in Equation (2), it is not difficult to show that the optimal reinsurance strategy is given by

$$q_{\pi_0}^*(u) = \frac{a\eta}{b^2}(u - u_s) \tag{48}$$

for all $u \in (u_s, \kappa]$. Then, we have

Theorem 10. *When $u \in (u_s, \kappa]$, the inequality $q^*(u) \le q_{\pi_0}^*(u)$ holds for any $\eta > 0$ with $q^*(u)$ given in Theorems 9 and $q_{\pi_0}^*(u)$ given in Equation (48).*

Proof. See Appendix B.7. ■

Remark 9. We can come to the conclusion that the retention level for the case without investment is always no less than the one with investment at all levels of surplus $u_s < u \le \kappa$. This result is kind of reasonable. Because of the assumption of $r < \mu$, the wealth increases relatively slowly in the case without investment, and thus the insurer has to optimally purchase less reinsurance to achieve the goal as quickly as possible.

When $\rho = 0$ in Equation (2), one can show that the optimal strategy is given by

$$(q_{\rho_0}^*(u), \pi_{\rho_0}^*(u)) = \left(\frac{a\eta}{b^2}(u - u_s), \frac{(\mu - r)}{\sigma^2}(u - u_s) \right) \tag{49}$$

for all $u \in (u_s, \kappa]$. Then, we can derive the following results.

Theorem 11. *When $u \in (u_s, \kappa]$, the inequalities $q^*(u) \le q_{\rho_0}^*(u)$ and $\pi^*(u) \le \pi_{\rho_0}^*(u)$ hold for any $\eta > 0$ with $(q^*(u), \pi^*(u))$ given in Theorems 9 and $(q_{\rho_0}^*(u), \pi_{\rho_0}^*(u))$ given in Equation (49).*

Proof. See Appendix B.8. ■

Remark 10. As noted earlier when $\rho > 0$, the claim process and the price process of the risky asset are positive correlated, which implies that the insurer holds a greater risk in the financial market (see the volatility part in Equation (2)). Thus, it is to be expected that the insurer always optimally invests smaller amount into the risky-asset but purchases more reinsurance. This result is different from Theorem 3.7, which illustrates the intuition that when minimizing the probability of drawdown in the danger-zone, the insurer would rather find a trade-off between the profit and the risk. But in the safe-region, since the low bound u_s is inaccessible under the optimal strategy, then the insurer may pay more attention to increasing the profit instead of reducing the risk.

5 | NUMERICAL EXAMPLES

In this section, we present several examples to show the effect of different parameters on the optimal strategies and the associated value functions. Besides, some comparisons are also made to investigate the relationship between the optimal results for different risk models. Here, we only consider the case of $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$.

5.1 | The influence of η and σ on the optimal results

In Example 5.1, we investigate the influence of the reinsurer's safety loading η and the volatility σ on the optimal reinsurance-investment strategy and the corresponding minimum probability of drawdown.

It is easy to see from Table 1 that a greater value of η yields greater values of q^* and π^*, which illustrates the intuition that when the reinsurance premium increases, the insurer would rather retain a greater share of each claim by purchasing less reinsurance. Meanwhile, when the reinsurance premium keeps increasing, to avoid drawdown, the insurer optimally invests larger amount in the risky asset to increase its profit. Such decisions in turn make drawdown more likely.

Table 2 shows that q^* increases but π^* decreases as σ increases. It makes sense because a greater value of σ implies a greater risk of the risky asset. To reduce the risk, the insurer optimally invests less amount into the risky asset. Further, since the wealth increases relatively slowly with less investment, the insurer optimally keeps more retention of each claim, and thus drawdown is certainly more likely to occur.

We present Example 5.2 to illustrate the impact of the parameters η and σ on the optimal reinsurance-investment strategy and the corresponding minimum expected time to reach the upper boundary κ.

Example 1. We set $\mathbb{E}Y = 1$, $\mathbb{E}Y^2 = 2$, $\lambda = 3$, $r = 0.05$, $u = 17$, $\theta = 0.12$, $\rho = 0.4$, $\mu = 1$, and $\kappa = 21$. We set $\sigma = 2$ and $\eta \in [0.2, 0.4]$ in Table 3, which implies that the largest safe level $u_s = 16.8$. Hence, $u = 17$ and $\kappa = 21$ are always larger than u_s. We also set $\eta = 0.32$ and $\sigma \in [2, 4]$ in Table 4.

Example 2. We set $\mathbb{E}Y = 2$, $\mathbb{E}Y^2 = 1$, $\lambda = 3$, $r = 0.05$, $u = 4$, $\theta = 0.12$, $\rho = 0.4$, $\mu = 1$, $\alpha = 0.2$, and $m = 15$. We set $\sigma = 2$ and $\eta \in [0.2, 0.4]$ in Table 1, which implies that the smallest safe level $u_s = 4.8$. Hence, $u = 4$ is always smaller than u_s. We also set $\eta = 0.32$ and $\sigma \in [2, 4]$ in Table 2.

TABLE 1 Values of $q^*(u)$, $\pi^*(u)$,and $\phi(4, 15)$

η	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40
$q^*(u)$	0.0093	0.0331	0.0678	0.1120	0.1642	0.2229	0.2864	0.3532	0.4219	0.4915	0.5608
$\pi^*(u)$	0.0783	0.1875	0.2855	0.3712	0.4443	0.5047	0.5527	0.5892	0.6151	0.6313	0.6390
$\phi(4, 15)$	0.0693	0.2597	0.3983	0.4931	0.5597	0.6076	0.6436	0.6710	0.6923	0.7090	0.7222

TABLE 2 Values of $q^*(u)$, $\pi^*(u)$,and $\phi(4, 15)$

σ	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$q^*(u)$	0.2864	0.3498	0.4098	0.4652	0.5154	0.5604	0.6002	0.6351	0.6657	0.6923	0.7154
$\pi^*(u)$	0.5527	0.4887	0.4280	0.3720	0.3212	0.2758	0.2356	0.2003	0.1694	0.1425	0.1191
$\phi(4, 15)$	0.6436	0.6671	0.6847	0.6979	0.7079	0.7157	0.7216	0.7263	0.7299	0.7328	0.7350

TABLE 3 Values of $q^*(u)$, $\pi^*(u)$,and $\varphi(17, 21)$

η	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40
$q^*(u)$	0.3258	0.4247	0.4950	0.5368	0.5500	0.5346	0.4907	0.4181	0.3171	0.1874	0.0292
$\pi^*(u)$	2.7379	2.4044	2.0850	1.7795	1.4881	1.2106	0.9471	0.6976	0.4622	0.2407	0.0332
$\varphi(17, 21)$	1.7227	1.8620	2.0225	2.2124	2.4451	2.7422	3.1419	3.7197	4.6511	6.4913	14.0724

TABLE 4 Values of $q^*(u)$, $\pi^*(u)$,and $\varphi(17, 21)$

σ	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
$q^*(u)$	0.4907	0.5326	0.5676	0.5972	0.6266	0.6446	0.6638	0.6808	0.6959	0.7094	0.7215
$\pi^*(u)$	0.9471	0.7442	0.5929	0.4776	0.3880	0.3173	0.2606	0.2147	0.1771	0.1460	0.1201
$\varphi(17, 21)$	3.1419	3.4207	3.6549	3.8492	4.0087	4.1388	4.2444	4.3298	4.3986	4.4538	4.4979

We can see from Table 3 that as η increases, the retention level q^* first increases and then decreases after reaching a certain level. Besides, we find that a great value of η yields a smaller value of π^*. These observations are kind of reasonable. When η is not large enough, the insurer would rather retain a greater share of each claim by purchasing less reinsurance as the reinsurance premium increases. However, a greater value of η also implies a higher safe level. The expression (40) shows that the optimal reinsurance-investment strategy depends on how far above the surplus is from the boundary u_s. Thus, π^* decreases as η increases because of the reducing difference between the surplus u and the boundary u_s. This illustration also holds for q^* when η is large enough.

Table 4 shows that π^* decreases as σ increases. It is to be expected since the insurer would rather invest less amount into risky asset when the risk becomes larger. However, we can see that q^* increases as σ increases, which illustrates the intuition that when the low bound u_s is inaccessible under the optimal strategy (see Remark 8), the insurer optimally purchases less reinsurance to achieve the goal as quickly as possible. Moreover, from Tables 3 and 4, it is not difficult to find that when the insurer becomes more alert to invest into the risky asset, it eventually leads to a greater value of the expected time to achieve the upper boundary κ, which is also kind of reasonable.

5.2 | Comparison

In Example 5.3, under the criterion of minimizing the probability of drawdown, we compare the optimal results given in Theorem 3.2 with those for special cases where $\pi = 0$, $\rho = 0$ or $r = 0$ in Equation (2).

Example 3. We set $\mathbb{E}Y = 1$, $\mathbb{E}Y^2 = 2$, $\lambda = 3$, $\theta = 0.12$, $\eta = 0.32$, $r = 0.05$, $\rho = 0.4$, $\mu = 1$ and $\sigma = 2$, then the safe level $u_s = 12$. We also set $\alpha = 0.2$ and $m = 15$, which implies that the drawdown level equals 3. Note that $m > u_s$, thus, minimizing the probability of drawdown reduces to one of minimizing the probability of ruin with ruin level 3.

It is not difficult to see that the optimal reinsurance-investment strategy decreases as u increases. Meanwhile, the corresponding minimum probability of drawdown ϕ is also a decreasing function with respect to u. These observations are kind of reasonable. When the surplus reaches the safe level, the company can buy full reinsurance and invest all the surplus in the risk-free asset to earn interest rate. Then the surplus of the insurance company will never decrease, and thus drawdown cannot happen.

Note that the relationship between the optimal strategies shown in Table 5 is consistent with the one presented in Theorems 3.6 and 3.7. We find that the value of the minimum probability of drawdown with investment is always smaller than the one without investment. Meanwhile, the minimum drawdown probability for $\rho > 0$ is always larger than the one for $\rho = 0$. They are natural consequences, since, when the surplus can be invested in the risky asset, the insurer has more choices to avoid the drawdown. But as noted earlier when $\rho > 0$, the insurer holds a greater risk in the financial market, then drawdown is certainly more likely to occur.

In particular, setting $r = 0$ in Equation (2), the safe-level approaches to ∞ and the optimal reinsurance-investment strategy is independent of u. By using $(q^*_{r_0}, \pi^*_{r_0})$ in Equation (33), we have $\left(q^*_{r_0}, \pi^*_{r_0}\right) = (0.3809, 0.8343)$. Since $u_3^{r_0} = -12.1512$ and $u_4^{r_0} = -9.2831$ are both negative, it is to be expected from Theorem 3.8 that the inequalities $q^* < q^*_{r_0}$ and $\pi < \pi^*_{r_0}$ always hold.

In Example 5.4, under the criterion of minimizing the expected time to reach a given capital level, we compare the optimal results given in Theorem 4.1 with those for special cases where $\pi = 0$ or $\rho = 0$ in Equation (2).

Example 4. We set $\mathbb{E}Y = 1$, $\mathbb{E}Y^2 = 2$, $\lambda = 3$, $\theta = 0.12$, $\eta = 0.32$ and $r = 0.05$, which implies that the safe level $u_s = 12$. We also set $\rho = 0.4$, $\mu = 1$, $\sigma = 2$, and $\kappa = 21$.

Table 6 shows that the optimal reinsurance-investment strategy increases as u increases, which illustrates the intuition that when the wealth approaches the upper boundary κ, the insurer optimally purchases less reinsurance and invests

TABLE 5 Values of the optimal results in different cases

u	3	4	5	6	7	8	9	10	11	12
$q^*(u)$	0.3221	0.2864	0.2506	0.2148	0.1790	0.1432	0.1074	0.0716	0.0358	0
$\pi^*(u)$	0.6218	0.5527	0.4836	0.4146	0.3455	0.2764	0.2073	0.1382	0.0691	0
$\phi(u,15)$	1.0000	0.6436	0.3905	0.2193	0.1109	0.0481	0.0164	0.0036	0.0003	0
$q^*_{\rho_0}(u)$	0.3797	0.3375	0.2953	0.2531	0.2110	0.1688	0.1266	0.0844	0.0422	0
$\pi^*_{\rho_0}(u)$	0.5636	0.5010	0.4384	0.3758	0.3131	0.2505	0.1879	0.1253	0.0626	0
$\phi_{\rho_0}(u,15)$	1.0000	0.5687	0.2999	0.1433	0.0598	0.0205	0.0052	0.0007	0.0000	0
$q^*_{\pi_0}(u)$	0.9375	0.8333	0.7292	0.6250	0.5208	0.4167	0.3125	0.2083	0.1042	0
$\phi_{\pi_0}(u,15)$	1.0000	0.7418	0.5287	0.3576	0.2252	0.1279	0.0671	0.0221	0.0038	0

TABLE 6 Values of the optimal results in different cases

u	12	13	14	15	16	17	18	19	20	21
$q^*(u)$	0	0.0981	0.1963	0.2944	0.3925	0.4907	0.5888	0.6869	0.7851	0.8832
$\pi^*(u)$	0	0.1894	0.3788	0.5683	0.7577	0.9471	1.1365	1.3260	1.5154	1.7048
$\varphi(u,21)$	∞	11.7448	8.0397	5.8724	4.3346	3.1419	2.1673	1.3433	0.6296	0
$q^*_{\rho_0}(u)$	0	0.1600	0.3200	0.4800	0.6400	0.8000	0.9600	1.1200	1.2800	1.4400
$\pi^*_{\rho_0}(u)$	0	0.2375	0.4750	0.7125	0.9500	1.1875	1.4250	1.6625	1.9000	2.1375
$\varphi_{\rho_0}(u,21)$	∞	9.1699	6.2771	4.5850	3.3843	2.4531	1.6922	1.0488	0.4916	0
$q^*_{\pi_0}(u)$	0	0.1600	0.3200	0.4800	0.6400	0.8000	0.9600	1.1200	1.2800	1.4400
$\varphi_{\pi_0}(u,21)$	∞	17.3283	11.8618	8.6641	6.3950	4.6355	3.1977	1.9820	0.9289	0

larger amount in the risky asset to obtain more chances of reaching the upper goal. Then it is to be expected that the minimum expected time decreases as the surplus gets closer to the given capital level κ.

Similarly, we can see that the relationship between the optimal strategies is coincide with the results given in Theorems 4.2 and 4.3. Besides, it is not difficult to find that the minimum expected time with investment is always smaller than the one without investment, but the value for $\rho > 0$ is always larger than the one for $\rho = 0$. This phenomenon can be explained by the same reasons as in Example 5.3.

6 | CONCLUSIONS

In this paper, we assumed that the insurer can purchase proportional reinsurance and invest its surplus in a financial market consisting one risky asset and one risk-free asset. Under two different criteria, the optimization problems are fully solved in two complementary regions, and the value functions and the associated optimal reinsurance-investment strategies are derived explicitly for the risk models with noncheap reinsurance. It is worthwhile to mention that we investigate the behavior of the surplus process and show how the optimally controlled surplus acts under the optimal risk control policies. Further, we provide several special cases of our risk model and derive some interesting observations during the comparison between the optimal strategies.

Although the literature on the optimal reinsurance is increasing rapidly, there are still many interesting problems that deserve to be investigated. For the further research, we may introduce the model uncertainty (ambiguity) into an insurer's controlled surplus process and solve the optimal robust reinsurance-investment strategy under the same criteria in this paper. It would also be interesting to consider other forms of reinsurance and include more complicated investment controls, such as with borrowing constraints. Besides, most researchers only focus on the wealth management of an insurer and ignore the interest of a reinsurer. Actually, the reinsurer also aims to minimize the probability of drawdown or minimize the expected time to reach a upper goal. Thus, one may investigate the optimization problem for a general insurance company which holds shares of an insurance company and a reinsurance company in a continuous model.

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ORCID

Zhibin Liang https://orcid.org/0000-0002-3152-8081

REFERENCES

1. Browne S. Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. *Math Oper Res*. 1995;20(4):937-958.
2. Zhang X, Zhou M, Guo J. Optimal combinational quota-share and excess-of-loss reinsurance policies in a dynamic setting. *Appl Stoch Models Bus Ind*. 2007;23(1):63-71.
3. Liang Z, Yuen KC. Optimal dynamic reinsurance with dependent risks: variance premium principle. *Scand Actuar J*. 2016;2016(1):18-36.
4. Liang X, Young VR. Minimizing the probability of ruin: optimal per-loss reinsurance. *Insur Math Econ*. 2018;82:181-190.
5. Angoshtari B, Bayraktar E, Young VR. Optimal investment to minimize the probability of drawdown. *Stochastics*. 2016;88(6):946-958.
6. Han X, Liang Z, Yuen KC. Optimal proportional reinsurance to minimize the probability of drawdown under thinning-dependence structure. *Scand Actuar J*. 2018;2018(10):863-889.
7. Han X, Liang Z, Zhang C. Optimal proportional reinsurance with common shock dependence to minimise the probability of drawdown. *Ann Actuar Sci*. 2019;13(2):268-294.
8. Angoshtari B, Bayraktar E, Young VR. Minimizing the probability of lifetime drawdown under constant consumption. *Insur Math Econ*. 2016;69:210-223.
9. Chen X, Landriault D, Li B, Li D. On minimizing drawdown risks of lifetime investments. *Insur Math Econ*. 2015;65:46-54.
10. Grossman SJ, Zhou Z. Optimal investment strategies for controlling drawdowns. *Math Financ*. 1993;3(3):241-276.
11. Cvitanić J, Karatzas I. On portfolio optimization under drawdown constraints. *IMA Lect Notes Math Appl*. 1995;65:77-88.
12. Elie R, Touzi N. Optimal lifetime consumption and investment under a drawdown constraint. *Financ Stoch*. 2008;12(3):299-330.
13. Heath D, Orey S, Pestien V, Sudderth W. Minimizing or maximizing the expected time to reach zero. *SIAM J Control Optim*. 1987;25(1):195-205.
14. Bayraktar E, Young VR. Optimal investment strategy to minimize occupation time. *Ann Oper Res*. 2010;176(1):389-408.
15. Frostig E. The expected time to ruin in a risk process with constant barrier via martingales. *Insur Math Econ*. 2005;37(2):216-228.
16. Luo S, Wang M, Zeng X. Optimal reinsurance: minimize the expected time to reach a goal. *Scand Actuar J*. 2016;2016(8):1-22.
17. Liang X, Bai L. Minimizing expected time to reach a given capital level before ruin. *J Ind Manag Optim*. 2017;13(4):1771-1791.
18. Bai L, Cai J, Zhou M. Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. *Insur Math Econ*. 2013;53(3):664-670.
19. Grandell J. *Aspects of Risk Theory*. New York, NY: Springer; 1991.
20. Han X, Liang Z, Young VR. Optimal reinsurance to minimize the probability of drawdown under the mean-variance premium principle. Working paper, University of Michigan; 2018.
21. Taksar MI, Markussen C. Optimal dynamics reinsurance policies for large insurance portfolios. *Financ Stoch*. 2003;7(1):97-121.
22. Karatzas I, Shreve SE. *Brownian Motion and Stochastic Calculus*. 2nd ed. New York, NY: Springer; 1991.
23. Bayraktar E, Zhang Y. Minimizing the probability of lifetime ruin under ambiguity aversion. *SIAM J Control Optim*. 2015;53(1):58-90.
24. Browne S. Survival and growth with a liability: optimal portfolio strategies in continuous time. *Math Oper Res*. 1997;22(2):468-493.

SUPPORTING INFORMATION

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APPENDIX A. AUXILIARY FUNCTIONS

A.1. The functions g_{1i} and f_{1i} $(i = 1, 2)$ are given by

$$\begin{cases} g_{11}(u, m) = \int_{\alpha m}^{u} \exp\left\{ \int_{\alpha m}^{y} \xi_{12}(w)dw \right\} dy, \\ g_{12}(u, m) = \int_{\alpha m}^{\alpha m \vee u_1} \exp\left\{ \int_{\alpha m}^{y} \xi_{12}(w)dw \right\} dy \\ + \int_{\alpha m \vee u_1}^{u} \exp\left\{ \left(\int_{\alpha m}^{\alpha m \vee u_1} \xi_{12}(w) + \int_{\alpha m \vee u_1}^{y} \xi_{11}(w) \right) dw \right\} dy, \end{cases}$$

and

$$f_{1i}(y) = \begin{cases} \alpha \left[\dfrac{1}{g_{1i}(y, y)} + \xi_{11}(\alpha y) \right], & if u_1 \leq \alpha m, \\ \alpha \left[\dfrac{1}{g_{1i}(y, y)} + \xi_{12}(\alpha y) \right], & if\ \alpha m < u_1. \end{cases}$$

A.2. The functions g_{2i} and f_{2i} (i=1,2) are given by

$$\begin{cases} g_{21}(u, m) = \int_{\alpha m}^{u} \exp\left\{ \int_{\alpha m}^{y} \xi_{22}(w)dw \right\} dy, \\ g_{22}(u, m) = \int_{\alpha m}^{\alpha m \vee u'} \exp\left\{ \int_{\alpha m}^{y} \xi_{22}(w)dw \right\} dy \\ \qquad + \int_{\alpha m \vee u'}^{u} \exp\left\{ \left(\int_{\alpha m}^{\alpha m \vee u'} \xi_{22}(w) + \int_{\alpha m \vee u'}^{y} \xi_{21}(w) \right) dw \right\} dy, \end{cases}$$

and

$$f_{2i}(y) = \begin{cases} \alpha \left[\dfrac{1}{g_{2i}(y, y)} + \xi_{21}(\alpha y) \right], & if\ u' \leq \alpha m, \\ \alpha \left[\dfrac{1}{g_{2i}(y, y)} + \xi_{22}(\alpha y) \right], & if\ \alpha m < u'. \end{cases}$$

A.3. The functions g_{31} and f_{31} are given by

$$g_{31}(u, m) = \int_{\alpha m}^{u} \exp\left\{ \int_{\alpha m}^{y} \xi_{32}(w)dw \right\} dy,$$

and

$$f_{31}(y) = \alpha \left[\frac{1}{g_{31}(y, y)} + \xi_{31}(\alpha y) \right].$$

APPENDIX B. PROOFS

B.1. The proof of $u_1 < \tilde{u}_1$

Proof. Note that

$$
\begin{aligned}
\tilde{u}_1 - u_1 &= \frac{1}{r}\left[\frac{b(\mu-r)}{2\rho\sigma} - a\eta - \frac{b^2(\mu-r)^2 - 2\rho\sigma b(\mu-r)a\eta + \sigma^2 a^2\eta^2}{2\left(\rho\sigma b(\mu-r) - \sigma^2 a\eta\right)}\right] \\
&= \frac{-b(\mu-r)\sigma^2 a\eta + 2\rho^2\sigma^2 b(\mu-r)a\eta - \rho\sigma^3 a^2\eta^2}{2r\rho\sigma\left(\rho\sigma b(\mu-r) - \sigma^2 a\eta\right)} \\
&= \frac{\sigma^2 b(\mu-r)a\eta(\rho^2-1) - \rho\sigma^3 a^2\eta^2}{2r\rho\sigma^2\left(\rho b(\mu-r) - \sigma a\eta\right)}.
\end{aligned} \tag{B1}
$$

In Case 1, the inequality

$$
\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}
$$

holds, it then follows that the denominator of (B1) is nonpositive. Obviously, the numerator of the Equation (B1) is negative. Thus, we have $u_1 < \tilde{u}_1$. This completes our proof. ■

B.2. The proof of Theorem 3.2

Proof. When $m \geq u_s$, h solves the BVP

$$
[ru + a(\theta-\eta)]\, h_u + \min_{(q,\pi)}\left\{((\mu-r)\pi_t + \eta a q_t)\, h_u + \frac{1}{2}\left(\sigma^2\pi_t^2 + 2\sigma b\rho q_t\pi_t + b^2 q_t^2\right) h_{uu}\right\} = 0 \tag{B2}
$$

with boundary conditions

$$
h(\alpha m, m) = 1, \quad h(u_s, m) = 0. \tag{B3}
$$

We guess the solution to Equation (B2) has the following form

$$
h(u,m) = \begin{cases} h_1(u,m) = c_1 \int_{\alpha m}^{u} \exp\left\{\int_{\alpha m}^{y} \xi_{12}(w)dw\right\} dy + c_2, & \alpha m \leq u < \max(\alpha m, u_1), \\ h_2(u,m) = c_3 \int_{u_1}^{u} \exp\left\{\int_{u_1}^{y} \xi_{11}(w)dw\right\} dy + c_4, & \max(\alpha m, u_1) \leq u \leq u_s \leq m. \end{cases}
$$

The rest of the work is to determine the constants c_i $(i = 1,2,3,4)$ by using the boundary conditions in Equation (B3) and the smooth-fit conditions, that is,

$$
h_1(u_1, m) = h_2(u_1, m), \quad \frac{\partial h_1(u_1,m)}{\partial u} = \frac{\partial h_2(u_1,m)}{\partial u}.
$$

By some calculation, we derive

$$
\begin{cases} c_1 = -\dfrac{1}{g_{12}(u_s,m)}, & c_2 = 1, \\ c_3 = -\dfrac{\exp\int_{\alpha m}^{u_1}\xi_{11}(w)dw}{g_{12}(u_s,m)}, & c_4 = 1 - \dfrac{g_{11}(u_1,m)}{g_{12}(u_s,m)}. \end{cases}
$$

It is straightforward to show that h satisfies Conditions (i), (ii), (iv), (v) and (vi) of Theorem 3.1. Condition (iii) is moot because $m \geq u_s$. Thus, we have $\phi = h$, and (q^*, π^*) given by (15) is the optimal reinsurance-investment strategy. This completes our proof. ■

B.3. The proof of Theorem 3.3

Proof. When $m < u_s$, h solves the same equation given in Equation (B2), and needs to satisfy the following boundary conditions

$$\begin{cases} h(\alpha m, m) = 1, \quad h(u_s, u_s) = 0, \\ h_m(m, m) = 0. \end{cases} \tag{B4}$$

We present the proof for the case of $m \in [\max(\alpha m, u_1), u_s]$ only. Then the proof for $m \in [\alpha m, \max(\alpha m, u_1))$ can be derived similarly. For simplicity, we assume that $\alpha m < u_1$, similar results can be obtained for $\alpha m \geq u_1$.

A general solution to the BVP is

$$h(u,m) = \begin{cases} h_3(u,m) = d_1(m) \displaystyle\int_{\alpha m}^{u} \exp\left\{ \int_{\alpha m}^{y} \xi_{12}(w)dw \right\} dy + d_2(m), & \alpha m \leq u < u_1, \\ h_4(u,m) = d_3(m) \displaystyle\int_{u_1}^{u} \exp\left\{ \int_{u_1}^{y} \xi_{11}(w)dw \right\} dy + d_4(m), & u_1 \leq u \leq m < u_s, \end{cases}$$

where the functions $d_i(m)$ $(i = 1, 2, 3, 4)$ are to be determined according to the boundary conditions (B4) and the smooth-fit conditions, that is,

$$\begin{cases} h_3(u_1, m) = h_4(u_1, m), \quad \dfrac{\partial h_4(m,m)}{\partial m} = 0, \\ \dfrac{\partial h_3(u,m)}{\partial u} = \dfrac{\partial h_4(u,m)}{\partial u}. \end{cases}$$

By solving the system directly, it gives

$$\begin{cases} d_1(m) = -\dfrac{1}{g_{12}(u_s, u_s)} \cdot \exp\left\{ -\displaystyle\int_m^{u_s} \alpha \left[\frac{1}{g_{12}(y,y)} + \xi_{12}(\alpha y) \right] dy \right\}, \\ d_2(m) = 1, \quad d_3(m) = d_1(m) \exp \displaystyle\int_{\alpha m}^{u_1} \xi_{12}(w)dw, \\ d_4(m) = 1 + d_1(m) g_{11}(u_1, m). \end{cases}$$

For $u_1 \leq u \leq m < u_s$, we can see that

$$\begin{aligned} h_m(u,m) &= d_1(m) \left[\alpha \left(\frac{1}{g_{12}(m,m)} + \xi_{12}(\alpha m) \right) g_{12}(u,m) - \alpha \xi_{12}(\alpha m) g_{12}(u,m) - \alpha \right] \\ &= -\alpha\, d_1(m) \left[1 - \frac{g_{12}(u,m)}{g_{12}(m,m)} \right] \geq 0. \end{aligned}$$

It then follows that $h_m(m, m) = 0$ for $u = m$. Besides, we can verify that $h_m(u, m)$ decreases but $1 - h(u, m)$ increases as u increases, thus condition *(iii)* is satisfied. Moreover, it is not difficult to show that h also satisfies Conditions (i), (ii), (iv), (v), and (vi) of Theorem 3.1. Therefore, we have $\phi = h$ with the optimal strategy (q^*, π^*) given in Equation (16). This completes our proof. ■

B.4. The proof of Theorem 3.6

Proof. When $\frac{\rho(\mu - r)b}{a\sigma} < \eta < \frac{(\mu - r)b}{\rho a \sigma}$, from Equations (11) and (20), we have

$$\begin{aligned} u_1 - u' &= \frac{1}{2r} \left(a\eta + \frac{\Delta \sigma b^2}{\rho b(\mu - r) - \sigma a \eta} \right) \\ &= \frac{(\mu - r)b}{2r\sigma\left(\rho b(\mu - r) - \sigma a \eta\right)} ((\mu - r)b - a\eta\rho\sigma) < 0. \end{aligned}$$

Thus, according to the expressions in Equations (15) and (27), it follows that

$$q^*(u) - q^*_{\pi_0}(u) = 2\,[ru + a(\theta - \eta)]\left(\frac{\rho b(\mu - r) - \sigma a\eta}{\Delta\sigma b^2} + \frac{1}{a\eta}\right)$$
$$= 2\,[ru + a(\theta - \eta)]\cdot\frac{(\mu - r)\,((\mu - r)b - a\eta\rho\sigma)}{\Delta\sigma^2 ba\eta} < 0$$

for any $u \in (\max(\alpha m, u'), \min(m, u_s)]$. Also, it is easy to show that $q^*(u) \le q^*_{\pi_0}(u)$ for any $u \in (\alpha m, \max(\alpha m, u'))$.

Besides, when $\eta \ge \frac{(\mu-r)b}{\rho a\sigma}$, we have $q^*(u) = q^*_{\pi_0}(u)$ because of $\pi^*(u) = 0$; and when $\eta \le \frac{\rho(\mu-r)b}{a\sigma}$, $q^*(u) \le q^*_{\pi_0}(u)$ because of $q^*(u) = 0$. Therefore, the inequality $q^*(u) \le q^*_{\pi_0}(u)$ holds for any $\eta > 0$. This completes our proof. ■

B.5. The proof of Theorem 3.7

Proof. (i) When $\eta \le \frac{\rho(\mu-r)b}{a\sigma}$, since $q^*(u) = 0$, we have $q^*(u) \le q^*_{\rho_0}(u)$ for any $u \in [\alpha m, \min(m, u_s)]$. Besides, if $\alpha m \le u \le \max(\alpha m, u_1^{\rho_0})$, comparing $\pi^*(u)$ in Equation (23) with $\pi^*_{\rho_0}(u)$ in (31), the inequality

$$\pi^*(u) - \pi^*_{\rho_0}(u) = -\frac{(ru + a\theta - 2a\eta) + \sqrt{(ru + a\theta)^2 + \left(\frac{\mu-r}{\sigma}\right)^2 b^2}}{(\mu - r)} > 0$$

is equivalent to $u < u_0^{\rho_0}$ with

$$u_0^{\rho_0} = \frac{1}{r}\left(a(\eta - \theta) - \frac{(\mu - r)^2 b^2}{4a\eta\sigma^2}\right).$$

It is not difficult to prove $u_0^{\rho_0} > u_1^{\rho_0}$, thus we have $\pi^*(u) > \pi^*_{\rho_0}(u)$ for any $u \in [\alpha m, \max(\alpha m, u_1^{\rho_0})]$. If $\max(\alpha m, u_1^{\rho_0}) < u \le \min(m, u_s)$, we can easily get $\pi^*(u) > \pi^*_{\rho_0}(u)$ from Equations (23) and (30). Therefore, the inequality $\pi^*(u) > \pi^*_{\rho_0}(u)$ holds for all levels of surplus $u \in (\alpha m, \min(m, u_s))$.

(ii) When $\frac{\rho(\mu-r)b}{a\sigma} < \eta \le \frac{(\mu-r)b}{a\sigma}$, from Equations (11) and (29), we have

$$u_1 - u_1^{\rho_0} = \frac{\Delta\sigma b^2}{2r\,(\rho b(\mu - r) - \sigma a\eta)} + \frac{b^2\Delta_{\rho_0}}{2ra\eta}$$
$$= \frac{\rho b^3(\mu - r)}{2ra\eta\,(\rho b(\mu - r) - \sigma a\eta)}\left(\left(\frac{\mu - r}{\sigma}\right)^2 - \left(\frac{a\eta}{b}\right)^2\right) \le 0.$$

Thus, according to the expressions in Equations (10) and (30), it follows that

$$llq^*(u) - q^*_{\rho_0}(u) = 2\,[ru + a(\theta - \eta)]\left(\frac{\rho(\mu - r)b - \sigma a\eta}{\sigma b^2\Delta} + \frac{a\eta}{b^2\Delta_{\rho_0}}\right)$$
$$= \frac{2\,[ru + a(\theta - \eta)]\,\rho b(\mu - r)}{\sigma b^2\Delta\Delta_{\rho_0}}\left(\left(\frac{\mu - r}{\sigma}\right)^2 - \left(\frac{a\eta}{b}\right)^2\right) \le 0,$$

and

$$\pi^*(u) - \pi^*_{\rho_0}(u) = 2\,[ru + a(\theta - \eta)]\left(\frac{\sigma\rho a\eta - (\mu - r)b}{\sigma^2 b\Delta} + \frac{\mu - r}{\sigma^2\Delta_{\rho_0}}\right)$$
$$= \frac{2\,[ru + a(\theta - \eta)]\,\rho\sigma a\eta}{\sigma^2 b\Delta\Delta_{\rho_0}}\left(\left(\frac{a\eta}{b}\right)^2 - \left(\frac{\mu - r}{\sigma}\right)^2\right) \ge 0$$

for any $u \in (\max(\alpha m, u_1^{\rho_0}), \min(m, u_s)]$. Besides, it is not difficult to find that $q^*(u) = q^*_{\rho_0}(u) = 1$ and

$$\begin{aligned}
\pi^*(u) - \pi^*_{\rho_0}(u) &= \frac{\sqrt{(ru+B)^2 - 4AC} - \sqrt{(ru+a\theta)^2 + (\frac{\mu-r}{\sigma})^2 b^2}}{\mu - r} \\
&= \frac{(ru+B)^2 - 4AC - (ru+a\theta)^2 - (\frac{\mu-r}{\sigma})^2 b^2}{(\mu - r)\left(\sqrt{(ru+B)^2 - 4AC} + \sqrt{(ru+a\theta)^2 + (\frac{\mu-r}{\sigma})^2 b^2}\right)} \\
&= \frac{-2(ru+a\theta)b\rho}{\sigma\left(\sqrt{(ru+B)^2 - 4AC} + \sqrt{(ru+a\theta)^2 + (\frac{\mu-r}{\sigma})^2 b^2}\right)} < 0
\end{aligned}$$

for any $u \in (\alpha m, \max(\alpha m, u_1))$. Thus, when $\frac{\rho(\mu-r)b}{a\sigma} < \eta \le \frac{(\mu-r)b}{a\sigma}$, according to the monotonicity and continuity of the optimal reinsurance-investment strategy, we can see that the inequality $q^*(u) \le q^*_{\rho_0}(u)$ holds for all levels of surplus $u \in (\alpha m, \min(m, u_s))$; and there exists a unique $u_0 \in [u_1, u_1^{\rho_0}]$ such that $\pi^*(u) \ge \pi^*_{\rho_0}(u)$ for any $u \in [\max(\alpha m, u_0), \min(m, u_s)]$, and $\pi^*(u) < \pi^*_{\rho_0}(u)$ for any $u \in [\alpha m, \max(\alpha m, u_0))$.

(iii) When $\frac{(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$, along the same lines as in (ii), we have $q^*(u) \ge q^*_{\rho_0}(u)$ and $\pi^*(u) \le \pi^*_{\rho_0}(u)$ for all $u \in [\alpha m, \min(m, u_s)]$.

(iv) When $\eta \ge \frac{(\mu-r)b}{\rho a\sigma}$, $\pi^*_{\rho_0}(u)$ is always no less than $\pi^*(u)$ because of $\pi^*(u) = 0$. Recalling the expressions of $q^*(u)$ and u' in Equations (19) and (20), it follows that

$$u' - u_1^{\rho_0} = -\frac{a\eta}{2r} + \frac{b^2\Delta_{\rho_0}}{2ra\eta} > 0,$$

and

$$q^*(u) - q^*_{\rho_0}(u) = -2(ru + a(\theta - \eta))\left(\frac{1}{a\eta} - \frac{a\eta}{b^2\Delta_{\rho_0}}\right) > 0$$

for any $u \in [\max(\alpha m, u'), \min(m, u_s)]$. Thus, it is not difficult to see that the inequality $q^*(u) \ge q^*_{\rho_0}(u)$ holds for any $u \in (\alpha m, \min(m, u_s))$. This completes our proof. ■

B.6. The proof of Theorem 4.1

Proof. Here, we only present the proof for the case of $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$. Let $\tilde{h}$ equal to the right-hand side of Equation (43), then we can see that $\tilde{h}$ solves Equation (36) with the boundary condition $\tilde{h}(\kappa) = 0$. Besides, it is readily verified that $\tilde{h}_u < 0$ and $\tilde{h}_{uu} > 0$ for all $u_s < u \le \kappa$. Therefore, the function $\tilde{h}$ is indeed the value function φ and (q^*, π^*) in (44) is the associated optimal reinsurance-investment strategy. The proof for the other two cases, that is, $\eta \ge \frac{(\mu-r)b}{\rho a\sigma}$ and $\eta \le \frac{\rho(\mu-r)b}{a\sigma}$ can be derived similarly. This completes our proof. ■

B.7. The proof of Theorem 4.2

Proof. When $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$, we have

$$\begin{aligned}
q^*(u) - q^*_{\pi_0}(u) &= \left(\frac{\rho(\mu-r)b - \sigma a\eta}{\sigma b^2(1-\rho^2)} + \frac{a\eta}{b^2}\right)(u_s - u) \\
&= \frac{\rho\left((\mu-r)b - \rho a\eta\sigma\right)}{\sigma b^2(1-\rho^2)} \cdot (u_s - u) < 0
\end{aligned}$$

for any $u \in (u_s, \kappa]$. Also, it is straightforward to show that the inequality $q^*(u) \le q^*_{\pi_0}(u)$ holds for both the cases of $\eta \ge \frac{(\mu-r)b}{\rho a\sigma}$ and $\eta \le \frac{\rho(\mu-r)b}{a\sigma}$. This completes our proof. ■

B.8. The proof of Theorem 4.3

Proof. When $\frac{\rho(\mu-r)b}{a\sigma} < \eta < \frac{(\mu-r)b}{\rho a\sigma}$, according to the expressions in Equations (40) and (49), it follows that

$$q^*(u) - q^*_{\rho_0}(u) = \frac{\rho\,((\mu-r)b - a\eta\sigma\rho)}{\sigma b^2(1-\rho^2)} \cdot (u_s - u) < 0,$$

and

$$\pi^*(u) - \pi^*_{\rho_0}(u) = \frac{\rho\,(a\eta\sigma - (\mu-r)b\rho)}{\sigma^2 b(1-\rho^2)} \cdot (u_s - u) < 0$$

for any $u \in (u_s, \kappa]$. Besides, when $\eta \geq \frac{(\mu-r)b}{\rho a\sigma}$ or $\eta \leq \frac{\rho(\mu-r)b}{a\sigma}$, it is not difficult to verify that inequalities $\pi^*_{\rho_0}(u) \geq \pi^*(u)$ and $q^*_{\rho_0}(u) \geq q^*(u)$ always hold. This completes our proof. ■