Relatively Dominated Representations

by

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DEDICATION

To 大舅公, for having planted the first seeds of the discipline in my young mind.

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PREFACE

This thesis is about certain relatively hyperbolic discrete subgroups of semisimple Lie groups.

If those are not words that inspire familiarity, Chapter 0 contains some background and context which will hopefully be helpful. That chapter is written with the curious reader who may or may not be a mathematician, but is at least not too daunted by things like matrices and vector spaces, in mind.

For the mathematical reader who has some familiarity with at least some of those words, Chapter 1 provides a more traditional introduction, including, towards its end, an outline of the contents of the rest of the thesis and how they are organized.

To our world today and its pressing challenges, this thesis, in the best and worst tradition of Hardy's apology, contributes just about nothing, beyond a fleeting glimpse of a beautiful edifice of abstract thought, and the real but ephemeral possibility of a more thorough understanding of the structure of space, broadly construed. Those are not entirely nothing, nevertheless, and I hope Chapter 0 will help bring that message to an at least marginally larger audience.

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ABSTRACT

Convex cocompact subgroups of rank-one semisimple Lie groups such as PSL(2,R) form a structurally stable class of quasi-isometrically embedded discrete subgroups which are naturally associated to negatively-curved geometric structures. Anosov representations give a higher-rank analogue of convex cocompactness which shares many of its good geometric and dynamical properties, and have become important objects of study in higher Teichmueller theory.

In this thesis we introduce relatively dominated representations as a relativization of Anosov representations, or in other words a higher-rank analogue of geometric finiteness—a controlled weakening of convex cocompactness to allow for isolated failures of hyperbolicity.

We prove that groups admitting relatively dominated representations must be relatively hyperbolic, that these representations induce limit maps with good properties, provide examples, and draw connections to work of Kapovich–Leeb which also introduces higher-rank analogues of geometric finiteness.

CHAPTER 1

Background for a Generalish Reader

For most of human experience, “geometry” has meant, and for many continues to mean, Euclidean geometry in two or three dimensions—the geometry that we see on a piece of paper, or that we instinctively and viscerally move through.

The historical roots of this thesis lie in developments from the 19$^{\text{th}}$ century which significantly broadened the scope of geometry and enriched humanity’s study of spatial structure.

1.1 Fundamental groups of manifolds

In the late 19$^{\text{th}}$ century, the branch of topology was born, with the appearance of Poincaré’s *Analysis Situs* being a particular historical landmark. Topology concerns geometric properties, such as connectedness, compactness, or the number of holes, that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing.

Many of these properties can be succinctly and precisely encoded by suitable algebraic objects such as homotopy, homology or cohomology groups—indeed, this is one of the major contributions of Poincaré’s *Analysis Situs*. One of the simplest of these algebraic objects, which already carries considerable information about the topology of a space X, is the **fundamental group** $\pi_1 X$, which roughly speaking describes “essential holes” in X and—via the group operation—how they interact; elements of $\pi_1 X$ represent classes of loops around these “holes”.

In many applications the underlying spaces we are dealing with are smooth (n-)manifolds—that is, spaces which locally have the topology of (n-dimensional) Euclidean space, although at larger scales they may have quite different properties, and which have well-defined tangent spaces. Fundamental groups of smooth manifolds, and other groups which are their spiritual relatives, will be one of the fundamental players in this thesis.

1.2 Discrete subgroups of Lie groups

In a different direction, the parallel postulate from Euclidean geometry had (finally) been discovered to be independent, and non-Euclidean geometry was born, in at least two different flavors: hyperbolic and spherical. In Euclidean geometry, given a line ℓ and a point p not on ℓ, there is exactly one line through p that does not intersect ℓ; we call this the "line through p parallel to ℓ". In hyperbolic geometry, given a line ℓ and a point p not on ℓ, there are infinitely many "lines through p parallel to ℓ", in this same sense; in spherical geometry, such parallel lines do not exist.

In order to provide a unifying framework for these various flavors of geometry, Felix Klein proposed what subsequently became known as the Erlangen program. This emphasized characterizing geometries in terms of their symmetries, using ideas from projective geometry and the then-nascent theory of groups.

When we are dealing with smooth manifolds, a different unifying framework is provided by Riemannian geometry, which starts with a Riemannian metric on the manifold—a gadget to measure infinitesimal lengths and angles. Different choices of Riemannian metrics on what is topologically our familiar 2-dimensional space, for example, yield geometric spaces which locally have Euclidean, hyperbolic, or spherical geometry.

When we have a manifold X with a Riemannian metric g, Myers and Steenrod proved in 1939 that the group of transformations $\mathrm{Isom}(X)$ of the manifold X that preserve the metric g is a Lie group—a group which is itself a (finite-dimensional) smooth manifold, where the smooth manifold structure and group structure are compatible with each other (the group operation and the inverse map are themselves smooth maps.) When we have a subgroup Γ of $\mathrm{Isom}(X)$, i.e. a collection of metric-preserving transformations of X which themselves form a group, we can consider the quotient space X/Γ obtained by identifying points in M taken to one another by transformations in Γ. When Γ is (torsionfree and) discrete—i.e. when we can find a collection of small balls, one around each element of Γ, each of which does not contain any other element of Γ—this quotient X/Γ is itself a manifold.

In this sense, **discrete subgroups** Γ **of a Lie group** $\mathrm{Isom}(X)$ lead to manifolds which locally have the geometry of X; the subgroup Γ describes how this geometry shows up on our manifold. This is one of the principal reasons such subgroups are objects of geometric interest.

1.3 Geometric structures and holonomy representations

We can go the other way—starting from a manifold M which locally has the geometry of another ("model") manifold X, we can obtain a discrete subgroup Γ of $G := \mathrm{Isom}(X)$—by considering the holonomy representation.

To describe what this is, we first consider the developing map $\varphi : \tilde{M} \to X$. Here $\tilde{M}$ is the universal cover of M, a topological "unrolling" of M: for example, if we imagine taking a infinite flat piece of paper and identifying points which are exactly 1 meter apart vertically or horizontally, the resulting quotient manifold is a torus (a donut); taking the universal cover of the torus involves exactly undoing this process, so that the universal cover of the torus is the 2-dimensional plane.

Intuitively, the developing map φ describes what one might see if one takes this "unrolled" version $\tilde{M}$ of M and looks at where in the model manifold X each part goes. One has some freedom here to decide where and how to plunk down the first bit, but then the subsequent bits are entirely determined, at least up to global symmetries of X. (To describe this more precisely uses the language of real analytic manifolds and maps and analytic continuation; the map φ is then well-defined up to composition by elements of G.)

Now, a loop γ in the fundamental group $\pi_1 M$ corresponds to a path in $\tilde{M}$ with endpoints $\tilde{\gamma}(a)$ and $\tilde{\gamma}(b)$ which are identified once we undo the "unrolling" and go back from $\tilde{M}$ to M. As we travel in $\tilde{M}$ from $\tilde{\gamma}(a)$ to $\tilde{\gamma}(b)$, the view around us changes, in a way which reflects or describes a symmetry of the space M. In terms of the topological space $\tilde{M}$, this "change of view" is described by a covering transformation T_γ. If we look through the lens of the geometric model X, the "change in view" is described by an isometry $\rho(\gamma)$. The relation between the two may be described precisely, using the developing map, by the equation

$$\varphi \circ T_\gamma = \rho(\gamma) \circ \varphi.$$

One can check that if we have two loops $\gamma_1, \gamma_2 \in \pi_1 M$ that we travel along one after the other, the isometry corresponding to the composition $\gamma_1\gamma_2$ is the same as what we get by composing the isometries $\rho(\gamma_1)$ and $\rho(\gamma_2)$, or in other words $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$, and similarly verify the other conditions needed so that the elements $\{\rho(\gamma) \mid \gamma \in \pi_1 M\} \subset \mathrm{Isom}(X) = G$ form a group. In other words $\rho : \pi_1 M \to X$ given by $\gamma \mapsto \rho(\gamma)$ is a map of groups, i.e. a representation.

This representation is well-defined up to conjugation by elements of G—this corresponds to globally changing the geometry by a symmetry of X—and is called a **holonomy representation**.

The data of the developing map completely determines the holonomy representation, up to conjugation as described above. Conversely, the data of the holonomy representation does not always completely determine the developing map, although it does in some cases. The data of the developing map and/or the holonomy representation together determine a (G, X)-structure on M: this may be described, informally, as "instructions", in terms of group elements in G, for how to dress the "naked" topological object M with geometric "clothing" cut from the cloth of X.

These are not things that this thesis is directly concerned about, but are a substantial part of the background motivation. In terms of the objects introduced earlier, the holonomy representation gives us a way to obtain a discrete subgroup Γ of $G = \mathrm{Isom}(X)$ starting from a manifold M which has been given, locally, the geometry of X.

1.4 The joys of negative curvature

Hyperbolic geometry can be wildly different from the Euclidean geometry we are more familiar with, in ways which may not be apparent from the somewhat clinical description in §1.2.

In Euclidean geometry, the volumes of balls grow polynomially in their radius; in hyperbolic geometry, they grow exponentially. In a sense that can be made precise, most of the volume of a ball in Euclidean space is near its center; by contrast, almost all of the volume of ball in hyperbolic space is near its boundary.

In Euclidean geometry, two geodesics (straight lines) going off in different directions from the same point diverge at a linear rate; in hyperbolic geometry, they diverge exponentially. A golfer playing on a hyperbolic golf course would never hit the ball into the hole: an error of a few fractions of a degree from a hundred yards out would lead to the ball landing off by miles.

These and related characteristics of hyperbolic geometry lead to dynamical systems associated to hyperbolic manifolds, such as the geodesic flow on a hyperbolic manifold, enjoying good statistical properties. A flow is a way of describing motion in a space; a geodesic flow is a flow where the motion proceeds along geodesics at unit speed, as determined by the Riemannian metric. The geodesic flow on a hyperbolic manifold is mixing: any two points, no matter close at the start, eventually have essentially statistically independent trajectories. This makes it difficult to accurately predict individual trajectories in the presence of any uncertainty about initial conditions; on the other hand, average behavior for the system can be reliably and precisely predicted.

Such statistical properties are more broadly characteristic of chaotic dynamical systems, many of which also share other characteristics with geodesic flows on hyperbolic manifolds.

Not unrelatedly, hyperbolic geometry is structurally stable: in senses that can be made precise, small perturbations to the geometry do not essentially change the nature and properties of the geometry, in a way which is not true for Euclidean geometry.

For one thing, quasigeodesic segments, i.e. slight perturbations of geodesic segments—the more general analogue of "straight line segments"—, remain uniformly close to geodesic segments with the same endpoints: this is known as the Morse lemma.

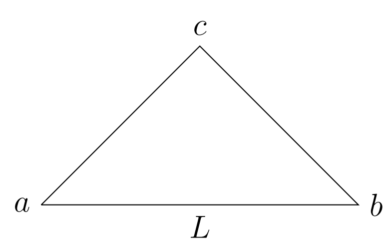


Figure 1.1: Failure of the Morse lemma in the Euclidean plane: the broken path acb is a $(\sqrt{2}, 0)$-quasigeodesic independent of the length L of the geodesic ab, but c is $\frac{1}{2}L$ far from ab.

For another, the geodesic flow g^t on a hyperbolic manifold M is structurally stable, meaning a flow ϕ^t which is a small perturbation of g^t looks essentially the same as g^t—there is a differentiable map $\psi : T^1M \to T^1M$ on the unit tangent bundle to M which conjugates ϕ^t and g^t, i.e. $\psi \circ \phi^t = g^t \circ \psi$.

Geodesic flows are particular examples of dynamical systems, and it is in fact more broadly true that the notion of structural stability for dynamical systems is closely related to notions of hyperbolicity in dynamics which are inspired by the properties of geodesic flows on hyperbolic manifolds.

Riemannian geometry contains a notion of curvature of space, which is completely determined by Riemannian metric. From the perspective of Riemannian geometry, hyperbolic space is negatively curved, and indeed in a precise sense has constant negative curvature everywhere.

Many of the properties of hyperbolic geometry—in particular the stability properties described above—continue to hold for more general negatively-curved geometries, especially when the curvature is pinched, i.e. does not go arbitrarily close to or far away from zero.

Euclidean geometry is flat: in the same precise sense as above, its curvature is zero everywhere. There are also many natural examples, in the worlds of Riemannian geometry and Lie groups, of non-positively curved spaces, i.e. spaces with mixed negative and zero curvatures. These spaces do not have the same stability properties as negatively-curved ones, although to the extent that they do have negative curvature in spots, they may continue to exhibit some weakened versions of these properties.

Very roughly speaking, the question that this thesis investigates is one particular instance of to what extent desirable properties of hyperbolic geometries and their associated dynamical systems may be seen to persist in more general nonpositively-curved settings.

We next turn to describing this instance in slightly more detail.

1.5 Teichmüller theory, classical and higher

Compact orientable surfaces without boundary (2-manifolds) are classified topologically by a single number, their genus. Genus-0 surfaces are spheres; genus-1 surfaces are tori; a genus-2 surface is a torus with an additional handle (see figure) and so on.

We can try to put a constant-curvature Riemannian metric on one of these surfaces: the uniformization theorem tells us that, depending on the genus, this metric will necessarily be spherical (i.e. have constant positive curvature; this happens in the genus-0 case), Euclidean (this happens in the genus-1 case), or hyperbolic (this happens for all higher genera.)

Moreover, there are an infinite number of possible hyperbolic metrics on a genus-g surface Σ_g, for any $g \geq 2$; in fact, there is a whole $(6g-6)$-dimensional ball of them. One way to see

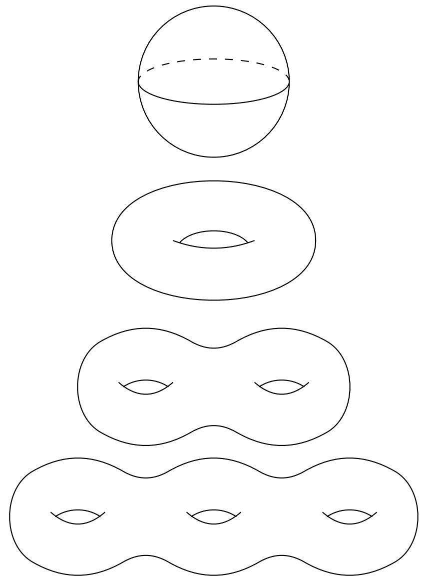


Figure 1.2: Classification of surfaces: surfaces of genus 0, 1, 2 and 3, or in the notation above Σ_0, Σ_1, Σ_2, and Σ_3. Of these, the last two admit hyperbolic metrics. TikZ finesse on last 3 diagrams courtesy of Salman Siddiqi.

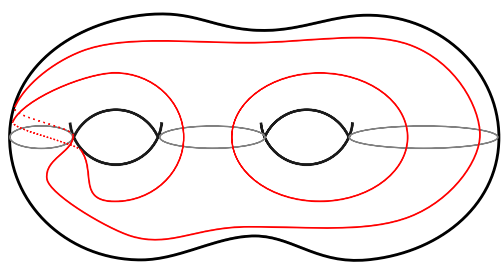


Figure 1.3: The grey curves form a pants decomposition, here assigned lengths 1, 2 and 3; the red curves are drawn to indicate twist parameters growing across the pants curves. This data specifies a hyperbolic metric on the Σ_2 shown here.

this is to start by picking $3g-3$ disjoint closed curves on a genus-g surface, i.e. a so-called pants decomposition, because the complementary regions are topologically similar to pairs of pants. One can then check that, to specify a hyperbolic metric on Σ_g, it suffices to specify, for each of these $3g-3$ curves, the length the curve and a "twist" parameter which describes how the complementary regions on either side of the curve are glued together.

Moreover, any choice of these lengths and twists gives rise to a hyperbolic metric, and different metrics correspond to different values of these parameters. Thus there is in fact a space of hyperbolic metrics on Σ_g that is topologically $\mathbb{R}^{6g-6}$; one can, furthermore, define a geometry on this space of geometries in a way which reflects aspects of the individual geometries which make up the space. The resulting space $\mathrm{Teich}(\Sigma_g)$, as a topological and geometric object, is called Teichmüller space, and is the eponymous object of study of ("classical", i.e. dating from the 1940s) Teichmüller theory.

Recall from §1.3 that a hyperbolic metric on Σ_g gives rise to a holonomy representation $\rho : \pi_1\Sigma_g \to \mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$. In this case, the holonomy representation also completely determines the hyperbolic metric, and hence we can write $\mathrm{Teich}(\Sigma_g)$ as a subspace of the space of representations $\chi(\pi_1\Sigma_g, \mathrm{PSL}(2,\mathbb{R}))$. In fact, $\mathrm{Teich}(\Sigma_g)$ forms an entire connected component of this representation space, one with especially nice topology as noted above.

In the 1980s, Nigel Hitchin, using analytic tools, showed that the representation spaces $\chi(\pi_1\Sigma_g, \mathrm{PSL}(n,\mathbb{R}))$, where $n \geq 3$, similarly contain connected components with nice topology, and these components contain the image of the Teichmüller space $\mathrm{Teich}(\Sigma_g)$—viewed as a subspace of the space of representations into $\mathrm{PSL}(2,\mathbb{R})$ as above—under a natural irreducible representation $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$. In 2006, François Labourie showed in [Lab06] that every representation in one of these components is discrete and faithful.

Inspired by this and subsequent results, such connected components of representation spaces have come to be called higher Teichmüller spaces, and the study of representations which lie—or might lie—in these spaces is loosely known as higher Teichmüller theory.

Labourie, in order to study the geometry of individual representations in the Hitchin components, came up with the notion of Anosov representations. These are certain representations of negatively-curved groups such as $\pi_1\Sigma_g$ into isometry groups of non-positively curved spaces such as $\mathrm{PSL}(n,\mathbb{R})$ which, by and large, still exhibit negatively-curved behavior.

Still roughly speaking, the aim of this thesis is to develop a relativized version of this notion: a class of representations of non-positively curved groups—with controlled instances of flat behavior—into groups such as $\mathrm{PSL}(n,\mathbb{R})$, which still has good geometric and dynamical properties.

1.6 Geometric group theory

To describe more fully what "negative curvature" or "non-positive curvature" means in the context of groups, we will take a minor detour into geometric group theory.

Groups are algebraic objects that were defined and built to study symmetry, and in that sense serve a geometric purpose. The basic philosophy of geometric group theory turns this idea on its head a little, and says that groups, especially infinite but finitely-generated ones, are naturally geometric objects, and the study of their geometry can yield insight into their algebraic properties.

A precursor of this philosophy can be seen in Max Dehn's study of the algorithmic properties of fundamental groups of surfaces from the 1920s. By essentially using the geometry of the associated surfaces, he showed that these groups have solvable word and conjugacy problems, among other things. Gromov took inspiration from this work and subsequent generalizations to define the notion of word-hyperbolic groups in [Gro87]: a finitely-generated group Γ is word-hyperbolic if any

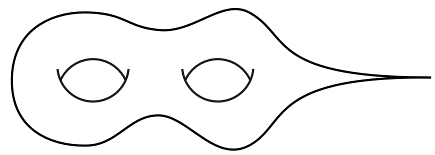


Figure 1.4: A cusped hyperbolic surface

Cayley graph of Γ, with the naturally-associated word metric, has thin triangles.

Thin triangles are a characteristic feature of hyperbolic—or, more generally, negatively-curved—geometry; Gromov's great insight was that picking out and requiring one such characteristic feature of negative geometry is a sufficient characterization of negative curvature in groups.

It is, and remains, much less clear what a good general notion of non-positive curvature in groups would be. One notion, describing a particular sort of non-positive curvature rather than the general case, is relative hyperbolicity: a finitely-generated group Γ is hyperbolic relative to a collection $\mathcal{P}$ of subgroups if, when we build a certain auxiliary space $X = X(\Gamma, \mathcal{P})$ given by adding things to the parts of a Cayley graph of Γ corresponding to the subgroups in $\mathcal{P}$, X is hyperbolic in the sense of having thin triangles.

Prototypical examples of relatively-hyperbolic groups include fundamental groups of finite-volume hyperbolic manifolds with punctures, which are hyperbolic relative to subgroups of loops around the puncture/s:

1.7 Aside: applications

We end this chapter by taking another minor detour, this one slightly further afield, to briefly explore possible applications of the spread of ideas described above outside mathematics.

1.7.1 Negatively-curved space-time and limiting cases

Einstein's theory of general relativity models the universe that we live in as a four-dimensional geometric object, space-time. Massive objects create regions of negative curvature in the surrounding space-time—in the words of John Wheeler, "matter tells space-time how to curve"–and this accounts for the phenomenon of gravitational attraction. In some sense, negative curvature is a generic case. This was one of the first applications of the geometric ideas described above outside of mathematics.

Non-positive curvature is a limiting case of the more generic case of negative curvature, and may be useful even in the study of the generic case in this way.

1.7.2 Poincaré and mixed-curvature embeddings

In many data analysis applications, data sets—including, for instance, sets of images—may already come naturally embedded in a parameter space In other cases, for example with lexical databases or semantic networks, there may not be such an intrinsic embedding, but it may still be useful to embed the data set in some geometric space: given such an embedding, one can measure distance between data points using a metric on the underlying space.

Because many parameter space are intrinsically finite-dimensional vector spaces, it has been natural to think of the Euclidean (L^2) metric or other metrics (such as the L^1 or sup metric) which come from vector space norms. In some cases, however, for example when the data sets in questions exhibit a large amount of branching, it may make more sense to use a hyperbolic metric, i.e. to think of embedding our data set into a hyperbolic or other negatively-curved space. This helps because the exponential growth of balls in hyperbolic space means there is inherently more room to accommodate the branching.

This is the underlying insight behind recent work in [NK17] on Poincaré embeddings of large data sets into hyperbolic space, which achieved state-of-the-art performance on certain natural language processing tasks. Similar methods have also recently been applied to other machine learning tasks such as image classification [KMU$^+$19] and recommendation systems [TTZ$^+$20].

In some applications, on the other hand, where the data is inherently cyclic in nature, it may make more sense to embed data sets on a sphere; see e.g. [MHW$^+$19].

It is natural to wonder about combining these, for complex data sets, and indeed this was done in [GSGR19]. Whereas [GSGR19] studied embeddings into products of different constant-curvature spaces, embeddings into different mixed-curvature spaces, such as higher-rank symmetric spaces, provide an alternative avenue for exploration which may also be of interest.

1.7.3 Quantiative comparisons between geometric objects

Given a collection of geometric objects—for instance shapes of bones, or organs as sensed by appropriate medical imaging devices—we may want to compare "how far apart they are". Making such quantitative comparisons between geometric objects is the natural province of Teichmüller theory and its relatives. In [KH15], ideas from Teichmüller theory are used to build a metric on spaces of primate bone and teeth shapes, with applications to identifying evolutionary patterns.

There have also been similar applications of Teichmüller theory and its surrounding ideas to medical imaging, see e.g. [WDG$^+$09].

Higher Teichmüller theory is not yet as developed as classical Teichmüller theory; in particular, there is so far only the beginning of a geometric theory there. Nevertheless, it may one day also—possibly—offer tools for quantitative analysis of geometric objects of interest in applications.

CHAPTER 2

Introduction

Given a rank-one semisimple Lie group G such as $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{SO}(1,3)$, the notion of convex cocompactness, first introduced in the setting of Kleinian groups acting on $\mathbb{H}^3$, gives us a stable class of subgroups with good geometric and dynamical properties.

When G is instead a higher-rank semisimple Lie group, such as $\mathrm{SL}(d,\mathbb{R})$ with $d \geq 3$, Anosov subgroups are, at present, the best analogue of convex cocompact ones. These were originally defined in [Lab06], as a tool to study the dynamics and geometry of individual Hitchin representations, and further developed in [GW12]. There have subsequently been many other equivalent characterizations: see for instance [KLP16], [GGKW17], and [BPS19].

In rank one, the class of convex cocompact subgroups form part of the strictly larger class of geometrically finite subgroups, which may be understood as convex cocompactness with the possible addition of certain degenerate "cuspidal" ends with controlled geometry. Geometrically finite groups continue to have many of the good properties of convex cocompact groups, modulo mild degeneracy at the cusps which may need controlled by additional hypotheses.

In prior work [KL18], Kapovich and Leeb proposed relativized versions of the Anosov condition, which may be considered to be higher-rank analogues of geometric finiteness. In this paper we propose another, inspired by the characterization in [BPS19] and making use of the theory of relatively hyperbolic groups.

Below, all of our groups Γ will be finitely-generated, and, to avoid unnecessary additional technicalities, torsion-free.

The condition on representations which we wish to define is given in terms of singular values and subspaces, and in terms of a modified word-length: given a matrix $A \in \mathrm{GL}(d,\mathbb{R})$, let $\sigma_i(A)$ (for $1 \leq i \leq d$) denote the i^{th} singular value of A.

Fix Γ a finitely-generated torsion-free group and a finite collection $\mathcal{P}$ of finitely-generated subgroups satisfying certain conditions (RH) (described in Definition 5.1) which are automatic if Γ is hyperbolic relative to $\mathcal{P}$. We will designate the subgroups in $\mathcal{P}$ and their conjugates "peripheral".

Given Γ and $\mathcal{P}$ as above, we will say that the images of peripheral subgroups under a representation $\rho : \Gamma \to \mathrm{GL}(d,\mathbb{R})$ are well-behaved if they satisfy certain conditions which essentially ensure their images are parabolic, plus mild technical conditions governing the behaviors of limits of Cartan projections. All of these conditions are described precisely in Definition 5.2.

Let X be a cusped space for $(\Gamma, \mathcal{P})$ as constructed in [GM08] (see §2 for definitions.) Write d_c to denote the metric on X, and $|\cdot|_c := d_c(\mathrm{id}, \cdot)$. These are defined in [GM08] in the case where Γ is hyperbolic relative to $\mathcal{P}$, but the same construction can be done and continues to make sense in the more general case of Γ a torsion-free finitely-generated group and $\mathcal{P}$ a malnormal finite collection of finitely-generated subgroups.

Given Γ a finitely-generated torsion-free subgroup and a collection $\mathcal{P}$ of finitely-generated subgroups satisfying (RH), we will say a representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is **1-dominated relative to** $\mathcal{P}$ (Definition 5.3), if there exists constants $C, \mu > 0$ such that (D^-) for all $\gamma \in \Gamma$, $\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq Ce^{\mu|\gamma|_c}$, and the images of peripheral subgroups under ρ are well-behaved.

Examples of relatively-dominated representations include geometrically-finite hyperbolic holonomies and geometrically-finite convex projective holonomies in the sense of [CM14a]; we also remark that in the case $\mathcal{P} = \varnothing$, we recover the [BPS19] definition of dominated reprsentations.

1-relatively dominated representations are discrete and faithful, and send non-peripheral elements to proximal images. Their orbit maps are quasi-isometric embeddings of the relative Cayley graph, i.e. the Cayley graph with the metric induced from the cusped space $X \supset \mathrm{Cay}(\Gamma)$.

In the setting of Anosov representations, [KLP18] proved that if Γ is finitely-generated and $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is such that there exist constants $C, \mu > 0$ so that $\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq Ce^{\mu|\gamma|}$ for all $\gamma \in \Gamma$, then ρ is (P_1)-Anosov, and in particular Γ must be word-hyperbolic. An alternative proof of this appears in [BPS19] and was the original inspiration for this work. Here we can prove a relative analogue to this hyperbolicity theorem:

Theorem 2.1 (Theorem 6.9)**.** *If $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is 1-dominated relative to $\mathcal{P}$, and Γ contains non-peripheral elements, then Γ must be hyperbolic relative to $\mathcal{P}$.*

Moreover, given a 1-relatively dominated representation, we have limit maps from the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ with many of the good properties of Anosov limit maps:

Theorem 2.2 (Theorem 7.2)**.** *Given $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ 1-dominated relative to $\mathcal{P}$, we have well-defined, Γ-equivariant, continuous maps $\xi : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)$ and $\xi^* : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)^*)$ which are dynamics-preserving, compatible and transverse.*

A key technical input into the proofs of these theorems is a powerful generalization of the Oseledec theorem recently formulated in [QTZ19]; we will use a slightly modified version of this result, whose proof is discussed in Appendix B.

Our approach is different from that of [KL18]—the latter really focuses on the geometry of the symmetric space whereas we look more at the intrinsic geometry associated to the relatively hyperbolic group—but we show that the resulting notions are closely related:

Theorem 2.3 (Theorems 9.4 and 9.12)**.** *(a) If $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is relatively dominated, then $\rho(\Gamma)$ is relatively RCA (in the sense of [KL18]) with uniformly regular peripherals.*

(b) If $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is such that $\rho(\Gamma)$ is relatively RCA with uniformly regular and undistorted peripherals satisfying an additional technical condition, then ρ is relatively dominated.

The rest of this paper is organized as follows: we start by reviewing relevant background facts on relatively hyperbolic groups in §3.1 and on singular value decompositions in §3. We then give the definition of relatively dominated representations, as well as noting some immediate properties, in §5. §6.1 proves a key transversality property, §6.2 the relative hyperbolicity theorem, and §7 the existence of the limit maps. §8 briefly discusses examples. §9 describes links between the notion of relatively dominated representations introduced here and notions in [KL18]; finally, §10 discusses extending the definition in §5 to more general semisimple Lie groups and parabolic subgroups.

Appendix A collects various linear algebra lemmas which are used throughout, especially in the later sections; Appendix B contains a proof of the generalization of the Oseledets theorem alluded to above.

CHAPTER 3

Preliminaries

3.0 Hyperbolic spaces and groups

A metric space $X = (X, d)$ is **geodesic** if given any two points $x, y \in X$, there is a rectifiable path in X whose length is equal to $d(x, y)$. Given any $\delta \geq 0$, we say that a geodesic metric space X is δ**-hyperbolic** if every geodesic triangle in X is δ-thin, i.e. given a geodesic triangle xyz, the side xy is contained in the union of the δ-neighborhoods of yz and zx.

Example 3.1. The hyperbolic plane $\mathbb{H}^2$ is $(\log 2)$-hyperbolic, by the following geometric argument (for a primer on the geometry of the hyperbolic plane, see [Kat92] or [CFKP97]):

- Given any geodesic triangle xyz and a side xy, we can find an ideal triangle $\xi\eta\zeta$ in $\mathbb{H}^2$ such that the bi-infinite geodesic $\xi\eta$ contains xy as a subsegment, and the triangle xyz is contained inside $\xi\eta\zeta$.

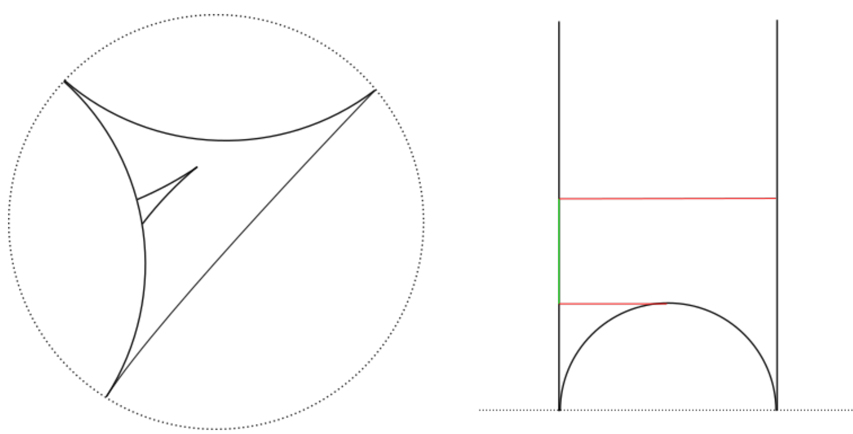


Figure 3.1: A triangle is contained in an ideal triangle; an ideal triangle with illustrative vertical and horizontal paths (green and red, resp.)

- Any two ideal triangles in $\mathbb{H}^2$ are isometric, for $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ acts triply-transitively on the boundary of $\mathbb{H}^2$.

- Consider the upper half-space model of $\mathbb{H}^2$ and let $\xi\eta\zeta$ be the ideal triangle with vertices at 0, 1, and ∞. What remains is now an explicit computation: any point on the edge between 0 and 1, or on one of the vertical edges with $y \leq 1$, is within $\frac{1}{2}$ of one of the other sides just by considering a horizontal path; any point on the vertical edge with $y \geq 2$ is within $\frac{1}{2}$ of the other vertical edge, again by considering a horiontal path. Finally, any point on the vertical edge with $1 < y < 2$ is within $\frac{1}{2}(\log 2 + 1)$ of one of the other edges, by considering a piecewise linear path that is first vertical until $y = 1$ or $y = 2$, whichever is closer, and then horiontal.

More generally, hyperbolic n-space is $(\log 2)$-hyperbolic, because every geodesic triangle sits in some totally geodesic $\mathbb{H}^2$ subspace.

Suppose we have a finitely-generated group Γ with a finite generating set S. We assume S is symmetric, i.e. whenever $s \in S$, $s^{-1} \in S$ as well. The graph (in the sense of graph theory) $\mathrm{Cay}(\Gamma, S)$ whose vertices are the elements of Γ, and where there is an edge between g and h if and only if $g = hs$ for some $s \in S$, is called the **Cayley graph** of Γ with respect to the generating set S.

We can make $\mathrm{Cay}(\Gamma, S)$ a metric space by defining each edge to have length 1, and considering the path metric d_S, i.e. for any $g, h \in \Gamma$, $d_S(g, h)$ is the number of edges in the shortest path between g and h in the graph $\mathrm{Cay}(\Gamma, S)$.

d_S is a left-invariant metric on $\mathrm{Cay}(\Gamma, S)$, which we will call the **word metric** on Γ associated to S. With the word metric, $\mathrm{Cay}(\Gamma, S)$ becomes a geometric object associated to the finitely-generated group Γ.

This geometric object depends on the choice of generating set S, but any two word metrics d_S and d_T corresponding to different finite generating sets for the same group Γ are **quasi-isometric**, meaning that there exist constants $k \geq 1$ and $c \geq 0$ such that

$$\frac{1}{k}\, d_S(g, h) - c \leq d_T(g, h) \leq k\, d_S(g, h) + c$$

for all $g, h \in \Gamma$. The quasi-isometry condition is a "coarsely biLipschitz condition": if $c = 0$, it is precisely a k-biLipschitz condition; where $c > 0$, the result is essentially k-biLipschitz, except at small scales as determined by the constant c.

We can then say that a finitely-generated group is a well-defined geometric object, independent of a choice of generating set, up to quasi-isometry. The study of groups as geometric objects in this sense is a key idea in geometric group theory. Slightly more generally, geometric group theory aims to study their groups through their geometric actions. A key instance and tool here is the following

Lemma 3.2 (Milnor-Švarc)**.** *Given Γ a finitely-generated group and X a geodesic metric space, if $\Gamma \curvearrowright X$ properly discontinuously, cocompactly and isometrically, then Γ is quasi-isometric to X.*

An important class of finitely-generated groups with many nice properties is given by the **δ-hyperbolic groups**, which are defined as groups which act properly discontinuously, cocompactly and isometrically on some δ-hyperbolic space.

A group which is δ-hyperbolic for some $\delta > 0$ is also called **word-hyperbolic**, on account of the role of the word metric, or **Gromov-hyperbolic**, in recognition of Gromov's seminal work in establishing and studying the notion of δ-hyperbolicity. We may similarly speak of Gromov-hyperbolic spaces, if we wish to de-emphasize the particular value of the constant δ.

For any fixed δ, δ-hyperbolicity is not a quasi-isometry invariant: a space Y which is quasi-isometric to a δ-hyperbolic space X may not itself be δ-hyperbolic, for the same δ. However, Gromov-hyperbolicity is a quasi-isometry invariant, because the Morse lemma holds in Gromov-hyperbolic spaces: given a δ-hyperbolic space $X = (X, d)$ and constants $k \geq 1$, $c \geq 0$, there exists $\lambda > 0$ (depending only on δ, k and c) such that any image of a geodesic segment under a (k, c)-quasi-isometry (also called a (k, c)-quasigeodesic), i.e. a map $f : X \to X$ satisfying

$$\frac{1}{k} d\left(f(x), f(y)\right) - c \leq d(x, y) \leq k d\left(f(x), f(y)\right) + c,$$

is within λ in Hausdorff distance of any geodesic segment between the same endpoints, i.e. given any geodesic segment and any (k, c)-quasigeodesic segment between the same endpoints, any point on the quasigeodesic segment is within λ of some point of some point on the geodesic segment.

Now if Y is (k, c)-quasi-isometric to X, and X is δ-hyperbolic, let $f : X \to Y$ be a (k, c)-quasi-isometry, and let $\lambda = \lambda(\delta, k, c)$ be the constant produced by the Morse lemma. Given a geodesic triangle xyz in Y, $f(xyz)$ is a quasigeodesic triangle in X. Let $f(x)\, f(y)\, f(z)$ be a geodesic triangle in X with the same vertices. It is δ-thin, and so $f(xyz)$ is $(\delta + \lambda)$-thin, and hence xyz must be $(k(\delta + \lambda) + c)$-thin.

Example 3.3. The fundamental group $\Gamma = \pi_1 \Sigma_g$ of a closed surface of genus $g \geq 2$ is word-hyperbolic, by the Milnor-Švarc lemma with $X = \mathbb{H}^2$, the hyperbolic plane.

More generally, fundamental groups of closed hyperbolic n-manifolds are word-hyperbolic, by the same argument with $X = \mathbb{H}^n$.

Example 3.4. A nonabelian free group F_r with $r \geq 2$ is word-hyperbolic by the Milnor-Švarc lemma with X a convex subset of $\mathbb{H}^2$ given e.g. by representing F_r as the fundamental group of a hyperbolic surface with geodesic boundary: the lifts of the boundary geodesics cut out X.

Note that the world of hyperbolic groups can be quite large. For instance, Kapovich in [Kap05], §8 and Canary–Stover–Tsouvalas in [CST19] have built non-linear word-hyperbolic groups. These are generally built starting from superrigid rank-one lattices, i.e. lattices in $\mathrm{Sp}(1, n)$ or F_4^{-20}. Indeed, such lattices are word-hyperbolic, but have many surprising properties.

The notion of word-hyperbolicity broadly captures the idea of negative curvature in finitely-generated groups; many features of negatively-curved Riemannian manifolds, for instance, have analogues in the world of Gromov-hyperbolic spaces and word-hyperbolic groups (see e.g. [BH99], Chapters III.H, III.Γ.2 and III.Γ.3).

One salient feature of the quintessential negatively-curved space $\mathbb{H}^n$ is that it has a boundary $\partial\mathbb{H}^n$ which is topologically a $(n-1)$-sphere. This may be generalized to the world of Gromov-hyperbolic spaces and groups as follows: let $X = (X, d)$ be a Gromov-hyperbolic metric space; the **Gromov boundary** of X is

$$\partial X := \{\text{unit speed geodesic rays in } X\}/\sim,$$

where $\gamma_1 \sim \gamma_2$ if $d(\gamma_1(t), \gamma_2(t))$ is uniformly bounded above for all $t \geq 0$. If γ is a unit speed geodesic ray in X, we denote its equivalence class in X by $[\gamma]$.

∂X comes equipped with a natural topology, which may be defined as follows: for any three points $a, b, c \in X$, define

$$(a, b)_c := d(a, c) + d(b, c) - d(a, b).$$

Choose a(n arbitrary base)point $o \in X$. For any $\xi \in \partial X$ and any $r > 0$, define

$$V(\xi, r) := \left\{ [\gamma] \in \partial X : \text{ there exists } \gamma_1 \text{ with } [\gamma_1] = \xi \text{ and } \liminf_{t\to\infty} (\gamma_1(t), \gamma(t))_o \geq r \right\}.$$

One should think of $V(\xi, r)$ as the set of "endpoints at infinity" of a cone of geodesic rays emanating from o. We define the topology on ∂X by declaring the collection $\{V(\xi, r) : \xi \in \partial X, r > 0\}$ to be a basis; the resulting topology is independent of the choice of o.

The Gromov boundary is well-defined, as a topological object, up to quasi-isometry:

Theorem 3.5. *Let X_0 and X_1 be geodesic metric spaces and $f : X_0 \to X_1$ be a quasi-isometry. If X_0 is Gromov hyperbolic, then f extends uniquely to a homeomorphism $\partial f : \partial X_0 \to \partial X_1$.*

Proof. See e.g. [BH99], Theorem III.Γ.3.9. □

When $X = \mathbb{H}^n$, one may check that the Gromov boundary ∂X, according to the definitions here, is exactly (homeomorphic to) the usual boundary $\partial\mathbb{H}^n$, and has the topology of a $(n-1)$-sphere.

3.1 Relatively hyperbolic groups

Compared to negative curvature, the idea of non-positive curvature in groups is more finicky and hard to capture as succinctly. The next notion we introduce is one particular manifestation of it that has some good properties and is a good generalization of a certain type or instance of non-positive curvature.

Relative hyperbolicity is a group-theoretic notion—originally suggested by Gromov in [Gro87], and further developed by Bowditch [Bow12], Farb [Far98], Groves–Manning [GM08], and others—of non-positive curvature inspired by the geometry of cusped hyperbolic manifolds and free products.

The geometry of a relatively hyperbolic group is akin to the geometry of a cusped hyperbolic manifold in that it is negatively-curved outside of certain regions, which, like the cusps in a cusped hyperbolic manifold, can be more or less separated from each other.

There are various ways to make this intuition precise, resulting in various equivalent characterizations of relatively hyperbolic groups. We will use a definition of Bowditch, in the tradition of Gromov:

Consider a finite-volume cusped hyperbolic manifold with an open neighborhood of each cusp removed: call the resulting truncated manifold M. The universal cover $\tilde{M}$ of such a M is hyperbolic space with a countable set of horoballs removed. The universal cover $\tilde{M}$ is not Gromov-hyperbolic; distances along horospheres that bound removed horoballs are distorted. If we glue the removed horoballs back in to the universal cover, however, the resulting space will again be hyperbolic space.

We can do a similar thing from a group-theoretic perspective: the Cayley graph of the fundamental group $\pi_1 M$ is not word-hyperbolic, because the cusp subgroups fail to quasi-isometrically embed into hyperbolic space. However, we can glue in metric graphs quasi-isometric to horoballs ("combinatorial horoballs") along the subgraphs of the Cayley graph corresponding to these cusp subgroups, and the resulting space (a "cusped space" or "augmented space") will again be quasi-isometric to hyperbolic space. We then say that $\pi_1 M$ is hyperbolic *relative to* its cusp subgroups.

More precisely (and more generally), let Γ be a finitely generated group and $S = S^{-1}$ a finite generating set. We consider the following construction:

Definition 3.6 ([GM08], Definition 3.1)**.** *Given a subgraph Λ of the Cayley graph $\mathrm{Cay}(\Gamma, S)$, the* **combinatorial horoball** *based on Λ, denoted $\mathcal{H} = \mathcal{H}(\Lambda)$, is the 1-complex*[1] *formed as follows:*

- *the vertex set $\mathcal{H}^{(0)}$ is given by $\Lambda^{(0)} \times \mathbb{Z}_{\geq 0}$*
- *the edge set $\mathcal{H}^{(1)}$ consists of the following two types of edges:*
  1. *If $k \geq 0$, and v and $w \in \Lambda^{(0)}$ are such that $0 < d_\Lambda(v, w) \leq 2^k$, then there is a ("horizontal") edge connecting (v, k) to (w, k)*
  2. *If $k \geq 0$ and $v \in \Lambda^{(0)}$, there is a ("vertical") edge joining (v, k) to $(v, k+1)$.*

$\mathcal{H}$ is metrized by assigning length 1 to all edges.

[1] Groves-Manning combinatorial horoballs are actually defined as 2-complexes; the definition here is really of a 1-skeleton of a Groves-Manning horoball. For metric purposes only the 1-skeleton matters.

Example 3.7. The combinatorial horoball over $\mathbb{Z}^d$ is quasi-isometric to a horoball in $\mathbb{H}^{d+1}$, via the map sending $(\vec{v}, n)$ in $\mathbb{Z}^d \times \mathbb{N}$ to $(\vec{v}, e^n)$ in the upper half-space.

To see this, we note that the distance between i and $n + i$ in upper half-space model is at most $2\log n + 1$, since the piecewise linear path with three segments—a vertical one from i to ni, a horizontal one from ni to $ni + n$ and a vertical one from $ni + n$ to $i + n$—has length $2\log n + 1$.

On the other hand, the distance between i and $n+i$ is at least $2\log n$, since because the length of half of the geodesic is at least the vertical displacement, which is (in the hyperbolic metric) $\geq \log n$.

More generally, the distance between i and $a + bi$, for $a \geq b$ is less than $2\log a - \log b + 1$ and at least $2\log a - \log b$ by the same arguments; if $a < b$, the piecewise linear path from i to bi to $a + bi$ has length $\leq \log b + 1$, and the geodesic must cover at least the vertical displacement $\log b$.

In all of these cases, the length of the hyperbolic geodesic is within bounded additive error of the geodesic distance between the corresponding endpoints in the combinatorial horoball. It is easy to check from the formula that the map is quasi-surjective onto the horoball based at ∞ passing through i: any point in this horoball is within distance $\frac{d}{2}$, in the hyperbolic metric, of the image of the combinatorial horoball.

Next let $\mathcal{P}$ be a finite collection of finitely-generated subgroups of Γ, and suppose S is a **compatible generating set**, i.e. for each $P \in \mathcal{P}$, $S \cap P$ generates P.

Definition 3.8 ([GM08], Definition 3.12)**.** *Given* $\Gamma, \mathcal{P}, S$ *as above, the* **cusped space** $X(\Gamma, \mathcal{P}, S)$ *is the simplicial metric graph*

$$\mathrm{Cay}(\Gamma, S) \cup \bigcup \mathcal{H}(\gamma P)$$

where the union is taken over all left cosets of elements of $\mathcal{P}$*, i.e. over* $P \in \mathcal{P}$ *and (for each* P*)* γP *in a collection of representatives for left cosets of* P*.*

Here the induced subgraph of $\mathcal{H}(tP)$ *on the* $tP \times \{0\}$ *vertices is identified with (the induced subgraph of)* $tP \subset \mathrm{Cay}(\Gamma, S)$ *in the natural way.*

Definition 3.9. Γ *is hyperbolic relative to* $\mathcal{P}$ *if and only if for any compatible generating set* S*, the cusped space* $X(\Gamma, \mathcal{P}, S)$ *is* δ*-hyperbolic (where the hyperbolicity constant* δ *may depend on* S*.)*

We will also call $(\Gamma, \mathcal{P})$ *a* **relatively hyperbolic structure**.

The following terminology will be useful further below:

Definition 3.10. $\mathrm{Cay}(\Gamma, S)$ *considered as a subspace of* $X(\Gamma, \mathcal{P}, S)$*—i.e. with the metric inherited from* $X(\Gamma, \mathcal{P}, S)$*—will be called the* **relative Cayley graph**.

Below, with a fixed choice of Γ, $\mathcal{P}$ and S as above, for $\gamma, \gamma' \in \Gamma$, $d(\gamma, \gamma')$ will denotes the distance between γ and γ' in the Cayley graph with the word metric, and $|\gamma| := d(\mathrm{id}, \gamma)$ denotes word length in this metric. Similarly, $d_c(\gamma, \gamma')$ denotes distance in the corresponding cusped space and $|\gamma|_c := d_c(\mathrm{id}, \gamma)$ denotes cusped word-length.

Example 3.11. For $\Gamma = \pi_1 M$ the fundamental group of a geometrically-finite hyperbolic n-manifold, $\mathcal{P}$ the collection of cusp subgroups, and S any compatible generating set, the cusped space $X(\Gamma, \mathcal{P}, S)$ is quasi-isometric to a convex subspace C of $\mathbb{H}^n$ obtained by removing funnels corresponding to lifts of boundary components; C may also be described as the convex hull of the limit set of Γ. Hence Γ is hyperbolic relative to $\mathcal{P}$.

Proof of quasi-isometry. This may be verified directly using hyperbolic geometry: the quasi-isometric embedding of the Cayley graph is still given by the orbit map. This sends the ends of each coset γP of a cusp subgroup P to a single point $\xi \in \partial\mathbb{H}^n$, and we may extend the orbit map to a quasi-isometric embedding of the combinatorial horoball over γP (the 0-simplices of which we address as elements of $P \times \mathbb{Z}_{\geq 0}$) to a quasi-horoball based at ξ as follows:

- for each $p \in \gamma P$, let $\eta_p : [0, \infty) \to \mathbb{H}^n$ be the geodesic ray from the image of p to ξ;
- send (p, n) to $\eta_p(\lambda n)$ with $\lambda = e^2/2$ (the normalization constant needed so that the exponential decay factor between levels of the combinatorial horoballs matches the exponential decay factor between their images in $\mathbb{H}^n$.)

Call this map ϕ. To check that this is indeed a quasi-isometry, we invoke the following argument of Cannon and Cooper:

Lemma 3.12 ([CC92], Lemma 4.2)**.** *Given two spaces X, Y with path metrics d_X, d_Y, $\phi : X \to Y$ is a quasi-isometry if it satisfies the following three conditions:*

(i) (quasi-onto) for some $\epsilon > 0$, $Y \subset N(\phi(X), \epsilon)$ (the ϵ-neighborhood of $\phi(X)$);

(ii) (Lipschitz) for some $L > 0$ and all $x_1, x_2 \in X$, $d_Y(\phi(x_1), \phi_2(x)) \leq L d_X(x_1, x_2)$; and

(iii) (uniformly non-collapsing) for each $R > 0$ there exists an $r > 0$ such that if $d_X(x_1, x_2) > r$ then $d_Y(\phi(x_1), \phi_2(x_2)) > R$.

Let $p_1, \ldots, p_k$ be parabolic fixed points belonging to different conjugacy classes, and $N_1, \ldots, N_k$ be a system of disjoint horoballs based at $p_1, \ldots, p_k$ (resp.) in $\mathbb{H}^n$ such that $\bigcup\{\gamma N_i : \gamma \in \Gamma; i = 1, \ldots, n\} =: \mathcal{N}$ fills out a family of disjoint open horoballs in $\mathbb{H}^n$, and

$$\phi(\Gamma) \subset \mathbb{H}^n \setminus \mathcal{N} =: \mathcal{Q}.$$

($\mathcal{Q}$ is the "thick part", or "truncated hyperbolic space".)

To verify condition (i) here: let y be a point of $C \subset \mathbb{H}^n$. Then either there exists some $i \in \{1, \ldots, n\}$ and $\gamma \in \Gamma$ such that $y \in \gamma N_i$, or $y \in \mathcal{Q}$. In the latter case,

$$d_{\mathbb{H}^n}(y, \phi(X^{(0)})) \leq \operatorname{diam}(\mathcal{Q}/\Gamma) < \infty.$$

In the former case, consider the horoball γN_i, which has center $\gamma p_i =: p$. As noted in Example 3.7, y is within distance δ of a vertex of the combinatorial horoball for γP_i, where $\gamma P_i \gamma^{-1}$ is the maximal parabolic subgroup of Γ fixing p, where δ may be chosen independent of i and H.

Hence, condition (i) of the Lemma is satisfied with $\epsilon \geq \max\{\operatorname{diam}(\mathcal{Q}/\Gamma), \delta + 1\} < \infty$.

For condition (ii): by Milnor-Švarc, ϕ is a quasi-isometry between the Cayley graph and the truncated hyperbolic space $\mathbb{H}^n \setminus N$. As noted in Example 3.7, ϕ is a quasi-isometry between the system of combinatorial horoballs and the system of horoballs N. In both cases, in fact, it is not difficult to show that the quasi-isometry in question is Lipschitz, in the latter case with uniform constants across the entire system of horoballs. This, together with the triangle inequality, gives us that ϕ is Lipschitz as a map from all of X to $\mathbb{H}^n$.

For condition (iii): suppose, on the contrary, that there exists $R > 0$ such that for every positive integer m, there exist points $x_m, w_m \in X$ such that $d(x_m, w_m) \geq m$, but $d(\phi(x_m), \phi(w_m)) \leq R$.

Since ϕ is a quasi-isometry between the system of combinatorial horoballs and the system of horoballs removed from hyperbolic space, there exists $r_0 > 0$ such that if x and w are points in the same combinatorial horoball,and $d(x, w) > r_0$, then $d(\phi(x), \phi(w)) > R$.

Suppose $m \geq (L+1)r_0$, where L is the Lipschitz constant from (ii); without loss of generality suppose $L \geq 1$. Choose a geodesic path from x_m to w_m in X. If this geodesic path has a connected subpath of length at least Lr_0 in a combinatorial horoball, then by the previous paragraph $d(x_m, w_m) \geq R$. Otherwise the geodesic path has a connected subpath of length at least r_0 with both endpoints in the Cayley graph. Then, by the same computation as in Example 3.7,

$$d(\phi(x_m), \phi(w_m)) \geq 2 \log d_{\mathcal{Q}}(\phi(x_m), \phi(w_m)) \geq L \log |x_m^{-1} w_m| \geq \frac{L}{2}(|x_m^{-1} w_m|_c - 1) \geq \frac{L}{2}(r_0 - 1).$$

In particular, if we suppose (without loss of generality—choose r_0 to be larger if not) $\frac{L}{2}(r_0 - 1) \geq R$, then we have a contradiction.

This verifies the hypotheses of Lemma 3.12, and hence ϕ is a quasi-isometry as desired. □

Example 3.13 ([Bow12], Theorem 7.11)**.** If Γ is a word-hyperbolic group, and $\mathcal{H}$ is a malnormal collection of proper, quasiconvex subgroups, then Γ is hyperbolic relative to $\mathcal{P}$. Here **malnormal** means that for all $\gamma \in \Gamma$ and $P, P' \in \mathcal{P}$, $\gamma P \gamma^{-1} \cap P' = 1$ unless $\gamma \in P = P'$.

For instance if $\Gamma = \pi_1 \Sigma_g$ is the fundamental group of a closed surface of genus $g \geq 2$, Γ is hyperbolic relative to the infinite cyclic group generated by any hyperbolic element whose axis projects to a simple closed curve on Σ_g.

We remark that for a fixed relatively hyperbolic structure $(\Gamma, \mathcal{P})$, any two cusped spaces, corresponding to different compatible generating sets S, are quasi-isometric ([Gro13], Corollary 6.7): in particular, the notion above is well-defined independent of the choice of generating set S. There is a natural action of Γ on the cusped space $X = X(\Gamma, \mathcal{P}, S)$; with respect to this action, the quasi-isometry between two cusped spaces $X(\Gamma, \mathcal{P}, S_i)$ $(i = 1, 2)$ is Γ-equivariant.

In particular, this gives us a notion of a boundary associated to the data of a relatively hyperbolic group Γ and its peripheral subgroup $\mathcal{P}$:

Definition 3.14. *For Γ hyperbolic relative to $\mathcal{P}$, the* **Bowditch boundary** $\partial(\Gamma, \mathcal{P})$ *is defined as the Gromov boundary $\partial_\infty X$ of any cusped space $X = X(\Gamma, \mathcal{P}, S)$.*

By the remarks above, this is well-defined up to homeomorphism, independent of the choice of compatible generating set S ([Bow12], §9.)

Example 3.15. For $\Gamma = \pi_1 M$ the fundamental group of a geometrically-finite hyperbolic n-manifold, and $\mathcal{P}$ the collection of cusp subgroups, $\partial(\Gamma, \mathcal{P})$ is, up to homeomorphism, the $(n-1)$-sphere.

Example 3.16. Consider $\Gamma = \pi_1 \Sigma_2$ represented as a geometrically-finite Kleinian group with an accidental parabolic, obtained from a Fuchsian representation by pinching the curve γ separating the two genera into a node in one of the two conformal boundary components:

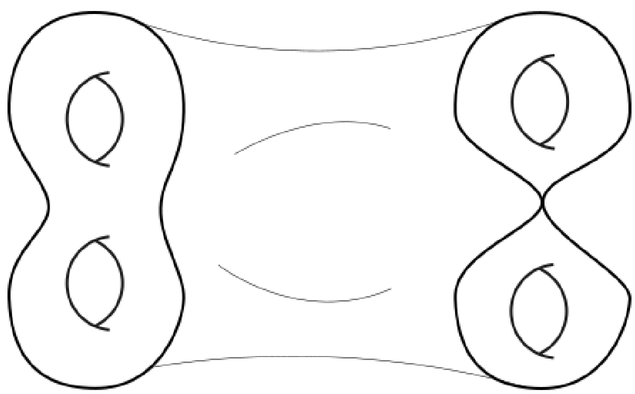


Figure 3.2: Schematic illustration of geometrically-finite quasi-Fuchsian with a accidental parabolic

As a hyperbolic group, Γ has Gromov boundary $\partial_\infty \Gamma \cong S^1$; Γ is also relatively hyperbolic relative to the infinite cyclic subgroup $P = \langle \gamma \rangle$, and the Bowdtich boundary $\partial(\Gamma, P)$ may be obtained from $\partial_\infty \Gamma$ by identifying all pairs of endpoints of axes of conjugates of γ. ([Man15], Theorem 1.3)

The cusped space $X = X(\Gamma, P)$ here is quasi-isometric to the hyperbolic plane with axes of conjugates of γ collapsed to points.

3.1.1 A Bowditch–Yaman criterion for relative hyperbolicity

The Bowditch criterion [Bow98] states, roughly speaking, that we can show a group Γ is hyperbolic by exhibiting an action of Γ on a metric space satisfying certain properties which are characteristic of the action of a hyperbolic group on its Gromov boundary. Moreover, if the hypotheses are satisfied, the space (and action) we produce is naturally identified with the Gromov boundary of the group (and the action of the group thereon.)

Using the Bowditch boundary and generalizing Bowditch's arguments, Asli Yaman proved an analogue of Bowditch's criterion for relatively hyperbolic groups:

Definition 3.17. *If M is a compact metric space, $\Gamma \curvearrowright M$ is a* **convergence group action** *if the induced action on the space $M^{(3)}$ of distinct triples is properly discontinuous.*

$\Gamma \curvearrowright M$ is a **geometrically-finite** *convergence group action if every point in M is either a conical limit point or a bounded parabolic point.*

($x \in M$ is a **conical limit point** *if there exists a sequence $(g_i) \subset \Gamma$ and $a, b \in M$ ($a \neq b$) such that $g_i x \to a$ and $g_i y \to b$ for any $y \in M \setminus \{x\}$.*

$H \leq \Gamma$ is **parabolic** *if it is infinite, fixes some point of M, and contains no infinite-order element with fixed locus of size 2. Such H have unique fixed points in M, called* **parabolic points***. A parabolic point $x \in M$ is* **bounded** *if $(M \setminus \{x\})/\operatorname{Stab}_\Gamma(x)$ is compact. A parabolic subgroup is* **maximal** *if it is not a proper subgroup of any larger parabolic subgroup.)*

Example 3.18. Let Γ be the fundamental group of a finite-volume cusped hyperbolic surface S. The universal cover $\tilde{S}$ of S is isometric to the hyperbolic plane $\mathbb{H}^2$, and we may identify the Gromov boundary $\partial\tilde{S}$ with $\partial\mathbb{H}^2$. In this way Γ acts on $M := \partial\mathbb{H}^2$.

Then, for this group action, each parabolic fixed point—parabolic in the sense of hyperbolic geometry—is a bounded parabolic point in the sense of Definition 3.17, with stabilizer given by the corresponding cusp stabilizer group.

A hyperbolic fixed point $x = \gamma^+$ is a conical limit point: here we can take $a = x = \gamma^+$, $b = \gamma^-$, and $g_i := \gamma^{-i}$.

More generally, any point $x \in M$ which is not a lift of a cusp is the endpoint of a geodesic γ in $\mathbb{H}^2$ whose projection to S returns to the compact core C infinitely often. Now take $a = x$ and b to be the other endpoint of γ. Fixing a basepoint $o \in \mathbb{H}^2$, let g_i be a sequence of elements such that the orbit points $g_i \cdot o$ are in C and uniformly close to γ, and $g_i \cdot o \cdot a$. Then $g_i \cdot x \to a$, and for any $y \neq x$, $g_i \cdot y \to b$ (after taking a subsequence of the g_i as needed, to extract a limit.) Hence we see that x is also a conical limit point.

Theorem 3.19 ([Yam06], Theorem 0.1)**.** *Suppose that M is a non-empty, perfect, compact metric space, and $\Gamma \curvearrowright M$ as a geometrically-finite convergence group.*

Suppose also that the stabiliser of each bounded parabolic point is finitely generated.

Then Γ is hyperbolic relative to the collection $\mathcal{P}$ of its maximal parabolic subgroups, and M is equivariantly homeomorphic to $\partial(\Gamma, \mathcal{P})$.

Building on earlier work of Tukia, Gerasimov has shown in [Ger09] that geometric finiteness, as well as the finite generation of the parabolic stabilisers, can be characterized using the induced group action on the space of distinct pairs. Putting these together, we obtain

Theorem 3.20. *Suppose Γ is finitely-generated, M is a non-empty, perfect, compact metrizable space, and $\Gamma \curvearrowright M$ is such that the induced action on $M^{(3)}$ is properly discontinuous and the induced action on $M^{(2)}$ is cocompact.*

Then Γ is hyperbolic relative to the maximal parabolic subgroups of the action $\Gamma \curvearrowright M$.

We will use this in §6.2 to prove that groups admitting relatively dominated representations must be relatively hyperbolic.

3.1.2 Geodesics in the cusped space

Let Γ be a finitely-generated group, $\mathcal{P}$ be a malnormal finite collection of finitely-generated subgroups, and let $S = S^{-1}$ be a compatible finite generating set as above. Let $X = X(\Gamma, \mathcal{P}, S)$ be the cusped space, and $\mathrm{Cay}(\Gamma) = \mathrm{Cay}(\Gamma, S)$ the Cayley graph.

We emphasize that none of the results in this or the next subsection requires Γ to be relatively hyperbolic, although the motivation for the constructions involved comes from relative hyperbolicity. This will be useful below, in the proof of the relative hyperbolicity theorem (Theorem 6.9.)

We start by pointing out a family of preferred geodesics in the combinatorial horoballs:

Lemma 3.21 ([GM08], Lemma 3.10)**.** *Let $\mathcal{H}(\Gamma)$ be a combinatorial horoball. Suppose that $x, y \in \mathcal{H}(\Gamma)$ are distinct vertices. Then there is a geodesic $\gamma(x,y) = \gamma(y,x)$ between x and y which consists of at most two vertical segments and a single horizontal segment of length at most 3.*

We will call any such geodesic a **preferred geodesic**. We have the following estimate going between uncusped and cusped lengths in a peripheral subgroup:

Proposition 3.22. *Suppose γ is a word contained in a single peripheral subgroup.*

Then $\frac{2}{\log 2} \log|\gamma| \leq |\gamma|_c \leq \frac{2}{\log 2}\log|\gamma| + 1$, or equivalently $\frac{1}{\sqrt{2}}\sqrt{2}^{|\gamma|_c} \leq |\gamma| \leq \sqrt{2}^{|\gamma|_c}$, where $|\gamma|$ refers to wordlength in the peripheral subgroup, not in the ambient group.

Proof. Let γ be a peripheral element of Γ which can be written as a word of word-length L in the generators of $S \cap P$.

There is always a path in the cusped space X from id to γ which consists of going up $\lfloor \log_2 L \rfloor$, going across 1, and then going down $\lfloor \log_2 L \rfloor$, and so the cusped word-length is certainly bounded from above by $2 \log_2 L + 1 = \frac{2}{\log 2} \log L + 1$.

Conversely, any path in X of cusped length at most $2 \log_2 L - 1$ with a single horizontal segment of (cusped) length ℓ corresponds to a word of word-length at most $\ell \cdot 2^{\log_2 L - \frac{\ell+1}{2}} = 2^{-\frac{\ell+1}{2}} \ell L < L$ whenever $\ell \geq 1$.

Note that any path in X which has two distinct endpoints in $\mathrm{Cay}(\Gamma) \subset X$ must contain at least one horizontal edge. By Lemma 3.21, there is always a geodesic in the cusped space from id to γ consisting of at most two vertical segments and a single horizontal segment.

Hence the cusped word-length is bounded from below by $2 \log_2 L = \frac{2}{\log 2} \log L$, as desired. □

Given a path $\gamma : I \to \mathrm{Cay}(\Gamma)$ in the Cayley graph such that $\gamma(I \cap \mathbb{Z}) \subset \Gamma$, we can consider γ as a **relative path** (γ, H), where H is a subset of I consisting of a disjoint union of finitely many subintervals $H_1, \ldots, H_n$ occurring in this order along I, such that each $\eta_i := \gamma|_{H_i}$ is a maximal subpath lying in a closed combinatorial horoball B_i, and $\gamma|_{I \smallsetminus H}$ contains no edges of $\mathrm{Cay}(\Gamma)$ labelled by a peripheral generator.

Similarly, a path $\hat{\gamma} : \hat{I} \to X$ in the cusped space with endpoints in $\mathrm{Cay}(\Gamma) \subset X$ may be considered as a relative path $(\hat{\gamma}, \hat{H})$, where $\hat{H} = \coprod_{i=1}^{n} \hat{H}_i$, $\hat{H}_1, \ldots, \hat{H}_n$ occur in this order along $\hat{I}$, each $\hat{\eta}_i := \hat{\gamma}|_{\hat{H}_i}$ is a maximal subpath in a closed combinatorial horoball B_i, and $\hat{\gamma}|_{\hat{I} \smallsetminus \hat{H}}$ lies inside the Cayley graph. Below, we will consider only geodesics and quasigeodesic paths $\hat{\gamma} : \hat{I} \to X$ where all of the $\hat{\eta}_i$ are preferred geodesics (in the sense of Lemma 3.21.)

We will refer to the η_i and $\hat{\eta}_i$ as **peripheral excursions** (see Figure 3.4 for an illustrated example). We remark that the η_i, or any other subpath of γ in the Cayley graph, may be considered as a word and hence a group element in Γ; this will be used without further comment below.

Given a path $\hat{\gamma} : \hat{I} \to X$ whose peripheral excursions are all preferred geodesics, we may replace each excursion $\hat{\eta}_i = \hat{\gamma}|_{\hat{H}_i}$ into a combinatorial horoball with a geodesic path (or, more precisely, a path with geodesic image) $\eta_i = \pi \circ \hat{\eta}_i$ in the Cayley (sub)graph of the corresponding peripheral subgroup connecting the same endpoints, by omitting the vertical segments of the preferred geodesic $\hat{\eta}_i$ and replacing the horizontal segment with the corresponding segment at level 0, i.e. in the Cayley graph.[2] We call this the "project" operation, since it involves "projecting" paths inside combinatorial horoballs onto the boundaries of those horoballs. This produces a path $\gamma = \pi \circ \hat{\gamma} : \hat{I} \to \mathrm{Cay}(\Gamma)$.

Below, given any path α in the Cayley graph with endpoints $g, h \in \Gamma$, or any path $\hat{\alpha}$ in the cusped space with endpoints in $g, h \in X$, we write $\ell(\alpha)$ to denote $d(g, h)$ i.e. distance measured

[2] As a parametrized path this has constant image on the subintervals of $\hat{H}_i$ corresponding to the vertical segments, and travels along the projected horizontal segment at constant speed.

according to the word metric in $\mathrm{Cay}(\Gamma)$, and $\ell_c(\hat{\alpha}))$ to denote $d_c(g,h)$, where d_c denotes distance in the cusped space.

The following observation will be used many times below. It is likely well-known, but we could not find it in the literature.

Proposition 3.23. *Given a geodesic $\hat{\gamma} : \hat{J} \to X$ with endpoints in $\mathrm{Cay}(\Gamma) \subset X$ and whose peripheral excursions are all preferred geodesics, let $\gamma = \pi \circ \hat{\gamma} : \hat{J} \to \mathrm{Cay}(\Gamma)$ be its projected image.*

Given any subinterval $[a,b] \subset \hat{J}$, consider the subpath $\gamma|_{[a,b]}$ as a relative path $(\gamma|_{[a,b]}, H)$ where $H = (H_1, \ldots, H_n)$, and write $\eta_i := \gamma|_{H_i}$; then we have the biLipschitz equivalence

$$\frac{1}{3} \leq \frac{d_c(\gamma(a), \gamma(b))}{\ell(\gamma|_{[a,b]}) - \sum_{i=1}^n \ell(\eta_i) + \sum_{i=1}^n \hat{\ell}(\eta_i)} \leq \frac{2}{\log 2} + 1 < 4$$

where $\hat{\ell}(\eta_i) := \max\{\log(\ell(\eta_i)), 1\}$.

Proof. If $\gamma|_{[a,b]}$ lies in a single peripheral excursion, then this follows from the fact that the projection operation replaces excursions with geodesic paths in the Cayley graph and from Proposition 3.22.

More generally, since we start with a geodesic in the cusped space, we have

$$d_c(\gamma(a), \gamma(b)) \leq \ell_c(\gamma|_{[a,b] \setminus H}) + \sum_{i=1}^n \ell_c(\eta_i). \tag{3.1}$$

Here $\gamma|_{[a,b]\setminus H}$ is a disjoint union of subpaths $\gamma_1, \ldots, \gamma_k$ of γ with endpoints in Γ, and $\ell_c(\gamma|_{[a,b]\setminus H}) := \sum_{i=1}^k \ell_c(\gamma_i)$, where $\ell_c(\gamma_i)$ denotes cusped distance between the endpoints of the subpath γ_i.

If the endpoints of our subpath do not lie in the middle of a (projected) peripheral excursion, we can promote the inequality (3.1) to an equality

$$d_c(\gamma(a), \gamma(b)) = \ell_c(\gamma|_{[a,b] \setminus H}) + \sum_{i=1}^n \ell_c(\eta_i). \tag{2.1'}$$

Now suppose one of our endpoints, say b, does lie in the middle of a projected peripheral excursion, say η_n. (The case where a lies in the middle of an excursion will be similar.) This is the special case which will take the remaining time:

Let b^- be such that $\hat{\gamma}(b^-)$ is the endpoint of η_n between $\gamma(a)$ and $\gamma(b)$. The infinite vertical ray into the combinatorial horoball from $\gamma(b)$ hits the image of $\hat{\gamma}$ at the point $\hat{\gamma}(b)$. We remark that, by the properties of the project operation, $\gamma(a) = \hat{\gamma}(a)$ and $\gamma(b^-) = \hat{\gamma}(b^-)$.

Note $\hat{\gamma}|_{[a,b]}$ is a geodesic, so by the triangle inequality

$$\begin{aligned} d_c(\gamma(a),\gamma(b)) + d_c(\hat{\gamma}(b),\gamma(b)) &\geq d_c(\gamma(a),\hat{\gamma}(b)) \\ &= d_c(\gamma(a),\gamma(b^-)) + d_c(\gamma(b^-),\hat{\gamma}(b)) \end{aligned} \tag{3.2}$$

Moreover, $[\gamma(b),\hat{\gamma}(b)]$ consists of a single vertical segment, (an isometric translate of) which is a subpath of $\hat{\gamma}|_{[b^-,b]}$, so $d_c(\gamma(b),\hat{\gamma}(b)) \leq d_c(\gamma(b^-),\hat{\gamma}(b))$. Combining these observations with (3.2), we obtain

$$d_c(\gamma(a),\gamma(b)) + d_c(\hat{\gamma}(b),\gamma(b)) \geq d_c(\gamma(a),\gamma(b^-)) + d_c(\gamma(b^-),\hat{\gamma}(b))$$

so

$$\begin{aligned} d_c(\gamma(a),\gamma(b)) &\geq d_c(\gamma(a),\gamma(b^-)) + d_c(\gamma(b^-),\hat{\gamma}(b)) - d_c(\hat{\gamma}(b),\gamma(b)) \\ &\geq d_c(\gamma(a),\gamma(b^-)). \end{aligned}$$

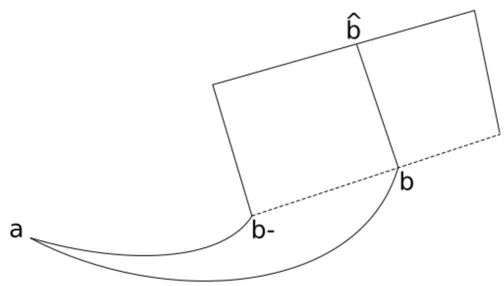


Figure 3.3: Solid lines here indicated geodesics in X, dotted lines indicate projected geodesics

On the other hand, again applying the triangle equality (and multiplying both sides by $\frac{1}{2}$) we have

$$\frac{1}{2}\left(d_c(\gamma(b^-),\gamma(b)) - d_c(\gamma(a),\gamma(b^-))\right) \leq \frac{1}{2}d_c(\gamma(a),\gamma(b)).$$

Adding together these inequalities, we obtain

$$\frac{3}{2}d_c(\gamma(a),\gamma(b)) \geq \frac{1}{2}\left(d_c(\gamma(a),\gamma(b^-)) + d_c(\gamma(b^-),\gamma(b))\right).$$

Now apply (3.1) to $\gamma|_{[a,b^-]}$, where we have equality, and remark that $d_c(\gamma(b^-),\gamma(b)) = \ell_c(\eta_n)$

by the properties of the project operation, so that we may rewrite this inequality as

$$\begin{aligned} d_c(\gamma(a),\gamma(b)) &\geq \frac{1}{3}\left(d_c(\gamma(a),\gamma(b^-)) + d_c(\gamma(b^-),\gamma(b))\right) \\ &= \frac{1}{3}\left(\ell(\gamma|_{[a,b]\setminus H}) + \sum_{i=1}^{n-1} \ell_c(\eta_i) + \ell_c(\eta_n)\right) \end{aligned}$$

and so we have $d_c(\gamma(a),\gamma(b)) \geq \frac{1}{3}\left(\ell_c(\gamma|_{[a,b]\setminus H}) + \sum_{i=1}^n \ell_c(\eta_i)\right)$.

By the definition of the cusped metric and of a relative path,

$$\ell_c(\gamma|_{[a,b]\setminus H}) = \ell(\gamma|_{[a,b]\setminus H}) = \ell(\gamma|_{[a,b]}) - \sum_{i=1}^{n} \ell(\eta_i).$$

By Proposition 3.22, for each i between 1 and n,

$$\frac{2}{\log 2}\log \ell(\eta_i) \leq \ell_c(\eta_i) \leq \frac{2}{\log 2}\log \ell(\eta_i) + 1.$$

Hence, writing $L := \ell(\gamma|_{[a,b]}) - \sum_{i=1}^n \ell(\eta_i) + \sum_{i=1}^n \hat{\ell}(\eta_i)$, we have

$$\frac{1}{3}L \leq d_c(\gamma(a),\gamma(b)) \leq \frac{2}{\log 2}L + n \leq \left(\frac{2}{\log 2} + 1\right) L$$

as desired. □

In particular, we note the following very coarse equivalence statement:

Corollary 3.24. *For any sequence of elements $(\gamma_n) \subset \Gamma$, $|\gamma_n|_c \to \infty$ if and only if $|\gamma_n| \to \infty$.*

3.1.3 Reparametrizing projected geodesics

Given a geodesic segment $\hat{\gamma}$ in the cusped space with endpoints in $\mathrm{Cay}(\Gamma)$, we can take its projection $\gamma = \pi \circ \hat{\gamma} : \hat{I} \to \mathrm{Cay}(\Gamma)$ and then reparametrize it in such a way that the increments correspond, approximately, to linear increments in cusped distance. Slightly more generally we will find it useful to consider paths in $\mathrm{Cay}(\Gamma)$ that "behave metrically like quasi-geodesics in the relative Cayley graph", in the following sense:

Definition 3.25. *Given any path $\gamma : I \to \mathrm{Cay}(\Gamma)$ such that I has integer endpoints and $\gamma(I \cap \mathbb{Z}) \subset \Gamma$, define the* **depth** *$\delta(n) = \delta_\gamma(n)$ of a point $\gamma(n)$ (for any $n \in I \cap \mathbb{Z}$) as*

(a) the smallest integer d such that at least one of $\gamma_r(n-d)$, $\gamma_r(n+d)$ is well-defined (i.e. $\{n-d, n+d\} \cap I \neq \varnothing$) and not in the same peripheral coset as $\gamma(n)$, **or**

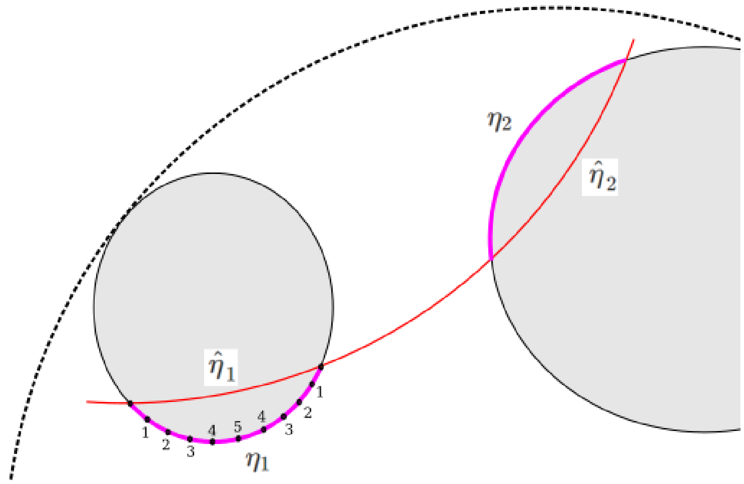


Figure 3.4: The red path $\hat{\gamma}$ in the cusped space may be considered as a relative path $(\hat{\Gamma}, \hat{H})$ as described above. The peripheral excursions $\hat{\eta}_1$ and $\hat{\eta}_2$ are the parts of $\hat{\gamma}$ lying inside the horoballs (colored grey); their projections η_1 and η_2 are the corresponding fuchsia paths with the same endpoints. Depths of some of the vertices along η_1 are labelled as an illustrative example.

(b) if no such integer exists, $\min\{\sup I - n, n - \inf I\}$.

Definition 3.26. *Given constants* $\underline{v}, \bar{v} > 0$*, an* $(\underline{v}, \bar{v})$**-metric quasigeodesic path** *is a path* $\gamma : I \to \mathrm{Cay}(\Gamma)$ *with* $\gamma(I \cap \mathbb{Z}) \subset \Gamma$ *such that for all integers* $m, n \in I$*,*

(i) $|\gamma(n)^{-1}\gamma(m)|_c \geq \underline{v}^{-1}|m - n| - \underline{v}$,

(ii) $|\gamma(n)^{-1}\gamma(m)|_c \leq \bar{v}(|m - n| + \min\{\delta(m), \delta(n)\}) + \bar{v}$*, and*

(iii) if $\gamma(n)^{-1}\gamma(n+1) \in P$ *for some* $P \in \mathcal{P}$*, we have* $\gamma(n)^{-1}\gamma(n+1) = p_{n,1} \cdots p_{n,\ell(n)}$ *where each* $p_{n,i}$ *is a peripheral generator of* P*, and*

$$2^{\delta(n)-1} \leq \ell(n) = |\gamma(n)^{-1}\gamma(n+1)| \leq 2^{\delta(n)+1}.$$

We can now make more precise our assertion about reparametrizing projected geodesic segments:

Proposition 3.27. *Given a cusped space* $X = X(\Gamma, \mathcal{P}, S)$*, for any projected geodesic* $\gamma = \pi \circ \hat{\gamma} : I \to \mathrm{Cay}(\Gamma)$ *with at least one end not inside a peripheral coset, we have a reparametrization of its image* $\gamma_r : I_r \to \mathrm{Cay}(\Gamma)$ *which is a* $(6, 20)$*-metric quasigeodesic path. (In fact, we can improve the inequalities slightly so that for all integers* $m, n \in I_r$*,*

(i) $|\gamma_r(n)^{-1}\gamma_r(m)|_c \geq \frac{1}{6}|m - n|$*, and*

(ii) $|\gamma_r(n)^{-1}\gamma_r(m)|_c \le 8(|m-n| + \min\{\delta(m),\delta(n)\}) + 20.$*)*

Proof. We define the reparametrization as follows:

- Outside of the peripheral excursions, parametrize by arc-length in $\mathrm{Cay}(\Gamma)$.
- Within an infinite but not bi-infinite peripheral excursion, the first letter is left alone, the next two are multiplied together, then the next four multiplied together, and so on.
- Within a finite peripheral excursion of cusped length E, do this from both ends simultaneously, and do some rounding as necessary. More precisely, to each natural number n we associate an ordered partition of positive integers as follows:
  - If $n = 1 + 2 + \cdots + 2^{k-1} + 2^k + 2^{k-1} + \cdots + 1$ for some $k \in \mathbb{Z}_{\ge 0}$, that is the associated ordered partition (e.g. $22 = 1+2+4+8+4+2+1$, so $(1,2,4,8,4,2,1)$ is the ordered partition associated to 22.) Call these numbers n_k. Note $n_k = 3 \cdot 2^k - 2$.
  - If $n \in (n_k, n_{k+1})$, associate to n the ordered partition $(1, 2, \ldots, 2^k + (n - n_k), 2^{k-1}, \ldots, 1)$. Note the middle term will be between $2^k + 1$ and $2^k + (n_{k+1} - n_k - 1) = 2^k + 3 \cdot 2^k - 1 = 2^{k+2} - 1$ in this case.

    For example, $n = 17 \in (n_2, n_3) = (10, 22)$, and so the ordered partition for 17 is given by $(1, 2, 4+7, 2, 1) = (1, 2, 11, 2, 1)$

  Then take the ordered partition $(a_1, \ldots, a_l)$ associated to E, and if $\gamma(s) = \gamma_r(s_r)$ is the start of the peripheral excursion, define $\gamma_r(s_r + j) = \gamma(m + \sum_{i=1}^{j} a_i)$ for $1 \le j \le l$.

To verify that this satisfies the desired criteria, we remark that the reparametrization does not modify cusped length outside of the peripheral excursions; inside a peripheral excursion of length E, the sum of any j consecutive numbers inside the partition associated to E is at least

$$1 + \cdots + 2^{j-1} = 2^j - 1$$

if j is no more than half the length of the partition; if j is greater than this threshold, this sum is still bounded below by

$$1 + \cdots + 2^{\ell_2 - 1} = 2^{\ell_2} - 1 \ge 2^{j/2} - 1,$$

where ℓ_2 is the floor of half the length of the partition, since the sum must contain a sum of ℓ_2 consecutive numbers inside the partition.

Thus, by Proposition 3.22, the cusped length of the part of the peripheral excursion associated to this part of the reparametrization is no less than $2\log_2(2^{j/2} - 1) \ge j - 1$. Considering separately what happens for small values of j, we may further replace this lower bound with $j/2$.

Proposition 3.23 then gives us

$$d_c(\gamma_r(n), \gamma_r(m)) \geq \frac{1}{3}\left(\ell(\gamma_r|_{[n,m]\setminus I_\eta}) + \ell\left(\gamma_r|_{I_\eta}\right)\right) \geq \frac{1}{6}|m-n|$$

This suffices to verify (i).

To verify (ii), we recall that, if $w_{m,n} := \gamma_r(m)^{-1}\gamma_r(n)$ is a peripheral word of length $\ell(w_{m,n})$, its cusped length is between $2\log_2 \ell(w_{m,n})$ and $2\log_2 \ell(w_{m,n}) + 1$ (see Proposition 3.22.)

By construction $\ell(w_{m,m+1}) \leq 2^{\delta(m)+1}$, so $|w_{m,m+1}|_c \leq 2\delta(m) + 3$, and more generally,

$$|w_{m,n}|_c \leq 2\log_2(2^{\delta(m)+1} + \cdots + 2^{\delta(n)+1}) + 1$$

and, writing $\delta = \min\{\delta(m), \delta(n)\}$, this latter is bounded above by

$$\begin{aligned} 2\log_2\left(2^{\delta+1} + \cdots + 2^{\delta+1+|m-n|}\right) + 1 &\leq 2\log_2\left(2^{\delta+1} \cdot (2^{|m-n|+1} - 1)\right) + 1 \\ &\leq 2(\delta + |m-n|) + 5. \end{aligned}$$

This, again in conjunction with Proposition 3.23, which yields

$$d_c(\gamma_r(n), \gamma_r(m)) \leq 4\left(\ell(\gamma_r|_{[n,m]\setminus I_\eta}) + \ell(\gamma_r|_{I_\eta})\right) \leq 8\left(|m-n| + \min\{\delta(m), \delta(n)\}\right) + 20,$$

suffices to prove the Proposition. □

3.2 Singular value decompositions

The condition on representations which we will define is given in terms of singular values and subspaces: given a matrix $g \in \mathrm{GL}(d, \mathbb{R})$, let $\sigma_i(g)$ (for $1 \leq i \leq d$) denote its i^{th} singular value.

Measuring these requires specifying a norm on $\mathbb{R}^d$, although the conditions below are independent (up to possibly changing the constants) of this choice of norm. Below we will assume we have fixed a norm coming from an inner product on $\mathbb{R}^d$; by viewing the symmetric space $\mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$ as a space of (homothety classes of) inner products on $\mathbb{R}^d$, this is equivalent to choosing a basepoint $o \in \mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$ (and then arbitrarily fixing a scaling).

Furthermore, write $U_i(g)$ to denote the span of the i largest axes in the image of the unit sphere in $\mathbb{R}^d$ under g, and $S_i(g) := U_i(g^{-1})$ (the letters come from "Unstable" and "Stable"; these names are inspired by ideas from dynamics.) Note $U_i(g)$ is well-defined if and only if we have a singular-value gap $\sigma_i(g) > \sigma_{i+1}(g)$.

More precisely, given any $g \in \mathrm{GL}(d, \mathbb{R})$, we may write $g = KAL$, where K and L are orthogonal matrices and A is a diagonal matrix with nonincreasing entries down the diagonal. A

is uniquely determined, and we may define $\sigma_i(g) = A_{ii}$. $U_i(g)$ is given by the span of the first i columns of K, which is well-defined as long as $\sigma_i(g) > \sigma_{i+1}(g)$.

Example 3.28. Let $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$.

Then $g = KAL$ where

$$K = \begin{pmatrix} 0 & 1 & 0 \\ \frac{T_-}{\sqrt{T_-^2+1}} & 0 & \frac{T_+}{\sqrt{T_+^2+1}} \\ \frac{1}{\sqrt{T_-^2+1}} & 0 & \frac{1}{\sqrt{T_+^2+1}} \end{pmatrix},$$

$$A = \begin{pmatrix} \sqrt{nT_- - 1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{1}{nT_- - 1}} \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & -\frac{T_-}{\sqrt{T_-^2+1}} & \frac{1}{\sqrt{T_-^2+1}} \\ 1 & 0 & 0 \\ 0 & -\frac{T_+}{\sqrt{T_+^2+1}} & \frac{1}{\sqrt{T_+^2+1}} \end{pmatrix},$$

where $T_\pm = n - \frac{1}{2}(n \pm \sqrt{n^2+4} = \frac{1}{2}(n \mp \sqrt{n^2+4})$. Note $T_+T_- = \frac{1}{4}(n - (n^2+4)) = -1$ and $T_- \approx n$.

$\sigma_1(g) = \sqrt{\frac{n^2+n\sqrt{n^2+4}+2}{2}} \approx n$, $\sigma_2(g) = 1$, and $\sigma_3(g) \approx \frac{1}{n}$.

$U_1(g) = \mathbb{R} \cdot \left(0, \frac{T_-}{\sqrt{T_-^2+1}}, \frac{1}{\sqrt{T_-^2+1}}\right)^T \approx \mathbb{R} \cdot (0, 1, 1/n)$ and $U_2(g)$ is spanned by $U_1(g)$ and $\mathbb{R} \cdot (1,0,0)^T$.

We remark that, for $g \in \mathrm{SL}(d, \mathbb{R})$, this singular-value decomposition is a (particular choice of) Cartan decomposition. We will occasionally write (given $g = KAL$ as above)

$$a(g) := (\log A_{11}, \ldots, \log A_{dd}) = (\log \sigma_1(g), \ldots, \log \sigma_d(g)).$$

Note that the norm $\|a(g)\| = \sqrt{(\log \sigma_1(g))^2 + \cdots + (\log \sigma_d(g))^2}$ is equal to the distance $d(o, g \cdot o)$ in the associated symmetric space $\mathrm{SL}(d, \mathbb{R}) / \mathrm{SO}(d)$ (see e.g. formula (7.3) in [BPS19].)

CHAPTER 4

Dominated Representations, d'après Bochi–Potrie–Sambarino

In this chapter, we give an expository overview of the theory of dominated representations, which we are subsequently generalizing.

Definition 4.1. *Let Γ be a finitely-generated group and write $|\cdot|$ to denote a(ny) word metric on Γ. A representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is said to be P_1*-**dominated** *if there exist constants $C \geq 1$ and $\mu > 0$ such that*

$$\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq Ce^{\mu|\gamma|}$$

for all $\gamma \in \Gamma$.

It is immediate from the definition (cf. Proposition 5.5 later) that P_1-dominated representations are discrete and have finite kernel; it follows from a standard formula for distance in the symmetric space in terms of singular values (cf. Proposition 5.9) that if ρ is P_1-dominated, then the orbit map $o \mapsto \gamma \cdot o$ gives a quasi-isometric embedding of Γ into the symmetric space $\mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$.

Moreover, although Γ was only assumed to be finitely-generated, it can in fact be proven that

Theorem 4.2 ([BPS19], Theorem 3.2; [KLP18], Theorem 1.4)**.** *If a group Γ admits a P_1-dominated representation into $\mathrm{GL}(d, \mathbb{R})$, then Γ is word-hyperbolic.*

Bochi–Potrie–Samabarino in [BPS19] proved that P_1-dominated representations are P_1-Anosov, in the sense of [Lab06] and [GW12], and vice versa; in other words, Definition 4.1 provides one of many equivalent characterizations for this class of representations, representing one of the many different approaches to them.

One feature of P_1-Anosov representations into $\mathrm{GL}(d, \mathbb{R})$ is that they come with nice limit maps $\xi : \partial_\infty\Gamma \to \mathbf{P}(\mathbb{R}^d)$ and $\xi^* : \partial_\infty\Gamma \to \mathbf{P}(\mathbb{R}^d)^*$—specifically, these are continuous, $\rho(\Gamma)$-equivariant, compatible, dynamics-preserving and transverse (see Chapter 7 for definitions of some of these), and may be defined in terms of limits of unstable spaces as

$$\xi(x) = \lim_{n\to\infty} U_1(\rho(\gamma_n))$$
$$\xi^*(x) = \lim_{n\to\infty} U_{d-1}(\rho(\gamma_n))$$

for any sequence $(\gamma_n) \subset \Gamma$ with $\gamma_n \to x$. These limit maps encode the long-term dynamics of $\rho(\Gamma)$ on the projective and dual projective spaces. Moreover, the existence of such limit maps with all of the properties described above, together with some other mild conditions, provides alternative characterizations of the class of Anosov representations; see [GGKW17] and [KLP16].

Another key property of Anosov representations $\rho : \Gamma \to G$ is *stability*: Anosov representations form an open subset of the space of representations $\mathrm{Hom}(\Gamma, G)$. In other words, small perturbations of Anosov representations remain Anosov. This was originally proven in [Lab06] and [GW12]; Bochi–Potrie–Sambarino provide an alternative proof, using ideas from hyperbolic dynamics, in [BPS19].

A representation $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{R})$, or into some other rank-one semisimple Lie group, is P_1-dominated if and only if $\rho(\Gamma)$ is convex cocompact, if and only if the orbit map $\gamma \mapsto \rho(\gamma) \cdot o$ is a quasi-isometric embedding $\Gamma \to \mathbb{H}^2$ for some (any) basepoint o.

Definition 4.3. *A discrete subgroup $\Gamma \leq \mathrm{SL}(2, \mathbb{R})$ is* **convex cocompact** *if Γ preserves a convex subspace C of the symmetric space—in this case $\mathbb{H}^2$—and acts cocompactly on C.*

Convex cocompact subgroups—i.e. images of convex cocompact representations—include uniform lattices, where C is all of the symmetric space, and also, for $\mathrm{SL}(2, \mathbb{R})$, (lifts of) holonomies of complete hyperbolic surfaces with funnels (but no cusps.)

The definition of convex cocompactness still makes sense for representations into higher-rank Lie groups, but does not yield new examples, due to the following

Theorem 4.4 ([KL06], [Qui05])**.** *If G is a (real, connected, noncompact, linear) higher-rank semisimple Lie group and $\Gamma \leq G$ is, up to finite index, a Zariski-dense convex cocompact discrete subgroup, then Γ is a product of convex cocompact subgroups of rank-one Lie groups and uniform lattices of higher-rank Lie groups.*

On the other hand, the class of representations where orbit maps give quasi-isometric embeddings is in some sense too large; in particular, it is not open:

Proposition 4.5 (Guichard, [GGKW17] Proposition A.1)**.** *Let Γ be a free group on two generators. There is a continuous family $\{\rho_t\}_{t\in[0,1]}$ of representations $\rho_t : \Gamma \to \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ such that*

- *ρ_0 is a quasi-isometric embedding;*
- *for any $t \notin \mathbb{Q}$, the group $\rho_t(\Gamma)$ is dense in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ (for the real topology).*

The class of Anosov (equivalently, dominated) representations furnishes a stable class of quasi-isometric embeddings and admits a fair range of examples, for instance Hitchin representations [Lab06], convex projective holonomies [Ben04], and so on.

CHAPTER 5

Relatively Dominated Representations

Recall that Γ is a finitely-generated group, which we assume to be torsion-free.

Let $\mathcal{P}$ be a finite collection of finitely-generated proper infinite subgroups; call all conjugates of these subgroups **peripheral**. A element of Γ is called peripheral if it belongs to any peripheral subgroup, and non-peripheral otherwise. Below we will write $\mathcal{P}^\Gamma$ to denote the set of all conjugates of groups in $\mathcal{P}$, $\bigcup \mathcal{P} := \bigcup_{P \in \mathcal{P}} P$ and $\bigcup \mathcal{P}^\Gamma := \bigcup_{Q \in \mathcal{P}^\Gamma} Q$ to denote the set of peripheral elements.

Let S be a compatible generating set, and let $X = X(\Gamma, \mathcal{P}, S)$ be the corresponding cusped space (see Definitions 3.6 and 3.8 above.) As above, let d_c denote the metric on X, and $|\cdot|_c := d_c(\mathrm{id}, \cdot)$ denote the cusped word-length.

For most of the arguments below we will also impose further conditions on $\mathcal{P}$:

Definition 5.1. *We say that a finite collection $\mathcal{P}$ of finitely-generated proper infinite subgroups satisfies (RH) if*

- *(malnormality) $\mathcal{P}$ is malnormal, i.e. for all $\gamma \in \Gamma$ and $P, P' \in \mathcal{P}$, $\gamma P \gamma^{-1} \cap P' = 1$ unless $\gamma \in P = P'$;*
- *(non-distortion) there exists $\nu > 0$ such that for any infinite-order non-peripheral element $\gamma \in \Gamma$, $|\gamma^n|_c \geq \nu |n|$;*
- *(local-to-global) there exist $\underline{\upsilon}, \bar{\upsilon} > 0$ and a constant $L > 0$ so that if $p = p_1 \ldots p_n$ is a geodesic word in $P \in \mathcal{P}$, $n > L$ and $\gamma p_1 \cdots p_L$ is a projected geodesic, then γp is an $(\underline{\upsilon}, \bar{\upsilon})$-metric projected quasigeodesic.*

We remark that all of these conditions hold automatically if Γ is hyperbolic relative to $\mathcal{P}$: malnormality follows for torsion-free Γ from [Osi06], Theorem 1.4; non-distortion follows from [Osi06], Theorem 1.14; the local-to-global condition is a particular case of the much more general local-to-global properties that hold due to the hyperbolicity of the cusped space X when Γ is relatively hyperbolic.

We introduce first a few technical conditions controlling what happens on the images of peripheral subgroups, and then the main notion we are defining:

Definition 5.2. *Given Γ and $\mathcal{P}$ as above, and a representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$, we say that the peripheral subgroups have* **well-behaved images under** *ρ if the following conditions are satisfied:*

- *(upper domination) there exist constants $C_1, \mu_1 > 0$ such that $\sigma_1(\rho(\eta)) \leq C_1 e^{\mu_1 |\eta|_c}$ for every peripheral element $\eta \in \bigcup \mathcal{P}$*
- *(unique limits) for each $P \in \mathcal{P}$, there exists $\xi_\rho(P) \in \mathbf{P}(\mathbb{R}^d)$ and $\xi^*_\rho(P) \in \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ such that for every sequence $(\eta_n) \subset P$ with $\eta_n \to \infty$, we have $\lim_{n\to\infty} U_1(\rho(\eta_n)) = \xi_\rho(P)$ and $\lim_{n\to\infty} U_{d-1}(\rho(\eta_n)) = \xi^*_\rho(P)$.*
- *(quadratic gaps) for every $\underline{\upsilon}, \bar{\upsilon} > 0$, there exists $C' \geq 0$ such that if $\eta \in P$ for some $P \in \mathcal{P}$, then, for any $\gamma \in \Gamma$, if $\gamma\eta$ ($\eta\gamma$, respectively) is an $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path then $\frac{\sigma_1}{\sigma_2}(\rho(\gamma\eta)) \geq C'|\eta|^2 = C' e^{|\eta|_c}$ ($\frac{\sigma_1}{\sigma_2}(\rho(\eta\gamma)) \geq C'|\eta|^2$, resp.);*
- *(uniform transversality) for every $P, P' \in \mathcal{P}$ and $\gamma \in \Gamma$, $\xi(P) \neq \xi(\gamma P' \gamma^{-1})$. Moreover, for every $\underline{\upsilon}, \bar{\upsilon} > 0$, there exists $\delta_0 > 0$ such that for all $P, P' \in \mathcal{P}$ and $g, h \in \Gamma$ such that there exists a bi-infinite $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path $\eta g h \eta'$ where η' is in P' and η is in P, we have $\sin \angle(g^{-1}\xi(P), h\, \xi^*(P')) > \delta_0$.*

We remark that the unique limits condition corresponds to the "tied-up horoballs" condition in [KL18], and the quadratic gaps condition is analogous to the uniform gap summation property that appears in [GGKW17].

Definition 5.3. *Fix Γ and $\mathcal{P}$ as above, with $\mathcal{P}$ satisfying (RH), and fix constants $\underline{C}, \underline{\mu} > 0$. A representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is* **1-almost dominated relative to** *$\mathcal{P}$ with lower domination constants $(\underline{C}, \underline{\mu})$, if it satisfies*

(D^-) for all $\gamma \in \Gamma$, $\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq \underline{C} e^{\underline{\mu}|\gamma|_c}$.

A 1-almost dominated representation ρ is **1-dominated relative to** *$\mathcal{P}$ with lower domination constants $(\underline{C}, \underline{\mu})$ if in addition the images of peripheral subgroups under ρ are well-behaved.*

Below we will sometimes refer to (D^-) as the lower domination inequality. We will sometimes suppress $\mathcal{P}$ and refer to 1-relatively dominated representations.

We further remark that many of the conditions in Definition 5.2 can be weakened or omitted if we assume relative hyperbolicity of the source group, together with the existence and transversality of limit maps: see Theorem 9.12, and associated definitions in that section, for a precise statement. We conjecture that it may further be possible that the uniform transversality hypothesis in Definition 5.2 can be made to follow from relative hyperbolicity and (D^-) as well.

5.1 Dual representations

Given $\rho : \Gamma \to \mathrm{GL}(V)$ with $V = \mathbb{R}^d$ as above (and the implicit choice of the standard basis, which fixes an identification $V \cong V^*$), we may define the dual representation $\rho^* : \Gamma \to \mathrm{GL}(V^*) \cong \mathrm{GL}(V)$ by $\rho^*(\gamma) = \rho(\gamma^{-1})^T$.

The following observations will be useful later:

Proposition 5.4. *If $\rho : \Gamma \to \mathrm{GL}(V)$ is 1-dominated relative to $\mathcal{P}$ with lower domination constants $(\underline{C}, \underline{\mu})$, then so is $\rho^* : \Gamma \to \mathrm{GL}(V)$.*

Furthermore, for all $\gamma \in \Gamma$, $U_1(\rho^*(\gamma)) = (U_{d-1}(\rho(\gamma)))^\perp$ and $U_{d-1}(\rho^*(\gamma)) = (U_1(\rho(\gamma)))^\perp$.

Proof. We have (D$^-$) since $\frac{\sigma_1}{\sigma_2}(\rho^*(\gamma)) = \frac{\sigma_1}{\sigma_2}(\rho(\gamma^{-1})) \geq \underline{C} e^{-\underline{\mu}|\gamma^{-1}|_c} = \underline{C} e^{-\underline{\mu}|\gamma|_c}$.

We can similarly get the quadratic gaps condition, since $\frac{\sigma_1}{\sigma_2}(\rho^*(\gamma\eta)) = \frac{\sigma_1}{\sigma_2}(\rho(\eta^{-1}\gamma^{-1})$ and $\frac{\sigma_1}{\sigma_2}(\rho^*(\eta\gamma)) = \frac{\sigma_1}{\sigma_2}(\rho(\gamma^{-1}\eta^{-1})$

Now if write the singular value decomposition $\rho(\gamma) = KAL$, then $\rho^*(\gamma) = (K^{-1})^T(A^{-1})^T(L^{-1})^T = KA^{-1}L$.

Recalling A has diagonal entries in non-increasing order, A^{-1} has diagonal entries in non-decreasing order; hence $U_1(\rho^*(\gamma))$ is the line spanned by the last column of K, which is $(U_{d-1}(\rho(\gamma)))^\perp$. Similarly, $U_{d-1}(\rho^*(\gamma))$ is the hyperplane spanned by the all but the first column of K; this is $(U_1(\rho(\gamma)))^\perp$.

Now the unique limits condition for ρ^* follows from the unique limits condition for ρ, since

$$\lim_{n\to\infty} U_1(\rho^*(\eta_n)) = \lim_{n\to\infty} (U_{d-1}(\rho(\eta_n)))^\perp = \xi^*_\rho(P)^\perp$$

and similarly

$$\lim_{n\to\infty} U_{d-1}(\rho^*(\eta_n)) = \lim_{n\to\infty} (U_1(\rho(\eta_n)))^\perp = \xi_\rho(P)^\perp$$

Similarly, the uniform transversality condition for ρ^* follows from the uniform transversality condition for ρ, due to the above identifications. □

5.2 Discreteness, faithfulness, proximal elements

Discreteness and faithfulness are straightforward consequences of the singular value gap growing coarsely with cusped word-length:

Proposition 5.5. *If $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is 1-almost relatively dominated, then ρ is discrete and faithful.*

Proof. Given any sequence of distinct elements $(\gamma_n) \subset \Gamma$, we must have $|\gamma_n|_c \to \infty$ since there are finitely many group elements γ satisfying $|\gamma|_c \leq N$ for each N.

(D^-) then gives $\log \frac{\sigma_1}{\sigma_2}(\rho(\gamma_n)) \geq \log \underline{C} + \underline{\mu}|\gamma_n|_c \to \infty$ for a $(1, \underline{C}, \underline{\mu})$-relatively almost dominated representation. Hence we cannot have $\rho(\gamma_n) \to \mathrm{id}$, which proves that ρ is discrete and has finite kernel. Since by assumption Γ is torsion-free, we may further conclude that ρ is faithful. □

Using in addition the property that our peripheral subgroups $\mathcal{P}$ satisfy (RH)—or, in particular, non-distortion—, we further obtain

Proposition 5.6. *Suppose $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is 1-almost relatively dominated. For any non-peripheral $\gamma \in \Gamma$, $\rho(\gamma)$ must be proximal.*

Proof. Recall the relation between the eigenvalues and singular values given by

$$\log |\lambda_i(\rho(\gamma))| = \lim_{n\to\infty} \frac{1}{n} \log \sigma_i(\rho(\gamma^n))$$

(see e.g. [Ben97], §2.5.) Suppose $\rho : \Gamma \to G$ is $(1, \underline{C}, \underline{\mu})$-almost relatively dominated.

Non-distortion implies there exists $\nu > 0$ such that $|\gamma^n|_c \geq \nu n$ for any non-peripheral γ, and (D^-) then implies $\log \frac{\sigma_1}{\sigma_2}(\gamma^n) \geq \underline{\mu}\nu n + \log \underline{C}$; hence we obtain

$$\log \left| \frac{\lambda_1}{\lambda_2} \right| (\rho(\gamma)) = \lim_{n\to\infty} \frac{1}{n} \log \frac{\sigma_1}{\sigma_2}(\gamma^n) \geq \underline{\mu}\nu > 0.$$

Hence $\rho(\gamma)$ is proximal, as desired. □

5.3 Relative quasi-isometric embedding

We can extend the upper domination hypothesis on the peripherals to a more general upper domination inequality (D^+). Using the upper and lower domination inequalities ($D^{\pm}$), we can then demonstrate that orbit maps are quasi-isometric embeddings of the relative Cayley graph, that is the Cayley graph with the extrinsic metric from the cusped space.

Proposition 5.7. *Suppose $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is 1-dominated relative to $\mathcal{P}$ with lower domination constants $(\underline{C}, \underline{\mu})$. Then there exists $\bar{C} > 1$ and $\bar{\mu} \geq \underline{\mu}$ such that for all $\gamma \in \Gamma$,*

$$\sigma_1(\rho(\gamma)) \leq \bar{C}^{\frac{1}{2}} e^{\frac{1}{2}\bar{\mu}|\gamma|_c}.$$

Since $\frac{\sigma_1}{\sigma_d}(\rho(\gamma)) = \sigma_1(\rho(\gamma)) \cdot \sigma_1(\rho(\gamma^{-1}))$, this immediately yields

Corollary 5.8 (D⁺). *For $\rho : \Gamma \to \mathrm{GL}(d,\mathbb{R})$ a 1-relatively dominated representation, let $\bar{C}$, and $\bar{\mu}$ be as in Proposition 5.7. We have*

$$\frac{\sigma_1}{\sigma_d}(\rho(\gamma)) \leq \bar{C} e^{\bar{\mu}|\gamma|_c}$$

for all $\gamma \in \Gamma$.

We will sometimes refer to (D⁺) as the upper domination inequality. Below, we will speak of relatively dominated representations with domination constants $(\underline{C}, \underline{\mu}, \bar{C}, \bar{\mu})$.

Proof of Proposition 5.7. We already know the related but weaker inequality $\sigma_1(\rho(\gamma)) \leq e^{\mu_2|\gamma|}$ from Γ being finitely-generated, where we may take $e^{\mu_2} = \max_{s\in S} \|s\|$ for our finite generating set S (which we used to build our cusped space).

More generally, given a word γ, we consider it as a relative path (γ, H) (see §3.1.2) where $H = H_1 \coprod \cdots \coprod H_n$, and suppose $\eta = (\eta_1, \ldots, \eta_n)$ where $\eta_i = \gamma|_{H_i}$ are the maximal peripheral excursions. Then we have

$$\begin{aligned}
\|\rho(\gamma)\| &\leq \|\rho(\gamma \setminus \eta)\| \cdot \prod_{i=1}^{n} \|\rho(\eta_i)\| \\
&\leq e^{\mu_2 \cdot \ell(\gamma\setminus\eta)} \cdot C_1^n e^{\mu_1 \sum_{i=1}^n |\eta_i|_c} \\
&\leq C_1^{|\gamma|_c} e^{\max\{\mu_2,\mu_1\}\cdot|\gamma|_c}
\end{aligned}$$

where $\|\rho(\gamma \setminus \eta)\|$ is to be interpreted as a product of $\|\rho(\gamma_i))\|$, where each γ_i is a maximal connected component of $\gamma \setminus \eta$ as a path; $\ell(\gamma \setminus \eta)$ is the sum of lengths of these paths (see §3.1.2.) C_1 and μ_1 here are the constants from the upper domination condition in Definition 5.2.

Here the second inequality follows from the first paragraph of the proof for individual non-peripheral pieces, and the upper domination hypothesis in Definition 5.2 for peripheral pieces, together with the equality (2.1') (from the proof of Proposition 3.23.)

In particular, writing $\bar{C}^{\frac{1}{2}} = C_1$ and $\frac{1}{2}\bar{\mu} = \max\{\mu_2, \mu_1\}$, we have the Proposition. $\square$

Proposition 5.9. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a representation which is 1-dominated relative to $\mathcal{P}$ with lower domination constants $(\underline{C}, \underline{\mu})$.*

Then the orbit maps $\gamma \mapsto \rho(\gamma) \cdot o$ are equivariant quasi-isometric embeddings of the relative Cayley graph $\mathrm{Cay}(\Gamma, S) \subset X(\Gamma, \mathcal{P}, S)$ into the symmetric space $G/K = \mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$.

Proof. By construction, the orbit map is equivariant, i.e. $\rho(\gamma_2\gamma_1) \cdot o = \rho(\gamma_2) \cdot (\rho(\gamma_1) \cdot o)$.

Viewing G/K as a space of inner products on $\mathbb{R}^d$, we recall the distance formula at the end of §3.2:

$$d_{G/K}(o, g \cdot o) = \sqrt{\sum (\log \sigma_i(g))^2}$$

for any $g \in \mathrm{SL}(d, \mathbb{R})$, where the o denotes the basepoint corresponding to our choice of inner product (see the beginning of this section.)

Now Proposition 5.7 implies $(\log \sigma_i(\rho(\gamma)))^2 \leq (\log \sigma_1(\rho(\gamma)))^2 \leq \frac{1}{4}\left(\log \bar{C} + \bar{\mu}|\gamma|_c\right)^2$ for $1 \leq i \leq d$, and so

$$\sqrt{\sum_{i=1}^{d} (\log \sigma_i(\rho(\gamma)))^2} \leq \frac{\sqrt{d}}{2}\left(\log \bar{C} + \bar{\mu}|\gamma|_c\right)$$

for all $\gamma \in \Gamma$. On the other hand, we have

$$\begin{aligned}\sqrt{\sum_{i=1}^{d} (\log \sigma_i(\rho(\gamma)))^2} &\geq \frac{1}{2}\left(|\log \sigma_1(\rho(\gamma))| + |\log \sigma_d(\rho(\gamma))|\right) \\ &\geq \frac{1}{2} \log \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq \frac{1}{2} \log \underline{C} + \frac{\underline{\mu}}{2}|\gamma|_c.\end{aligned}$$

Combining the two immediately yields that the orbit map into G/K is a quasi-isometric embedding *with respect to the cusped metric*. ☐

5.4 Upper domination and almost-unipotence

We remark that our upper domination hypothesis on the peripherals is equivalent to an "almost-unipotence" hypothesis on the eigenvalues of peripheral elements. This statement itself is not used further on, but one of the key steps in the proof (Corollary 5.12) will be.

Proposition 5.10. *Suppose $\rho : \Gamma \to G$ is an almost-dominated representation relative to $\mathcal{P}$ and its peripheral images have unique limit points.*

Its peripheral images satisfy the upper domination condition if and only if for every peripheral element $\eta \in P \in \mathcal{P}$, all of the eigenvalues of $\rho(\eta)$ have norm 1 (we will call this latter hypothesis quasi-unipotence of the peripherals.)

Proof. The forward implication follows from the observation that

$$\log |\lambda_i(\rho(\eta))| = \lim_{n \to \infty} \frac{1}{n} \log \sigma_i(\rho(\eta^n)).$$

In particular, $\log \frac{|\lambda_1|}{|\lambda_d|}(\rho(\eta)) = \lim_{n\to\infty} \frac{1}{n} \log \frac{\sigma_1}{\sigma_d}(\rho(\eta^n)) \leq \lim_{n\to\infty} \frac{1}{n}(2\bar{\mu}\log n + \log \bar{C} + 2\bar{\mu}) = 0$ from the upper domination inequality (D^+) and because $|\eta|_c \leq 2\log n + 1$; hence $|\lambda_1| = |\lambda_d|$, and since by hypothesis $|\lambda_1| \geq |\lambda_d| \geq \cdots \geq |\lambda_d|$, we may conclude that all of the eigenvalues have the same norm. Note that this part does not use the unique limits hypothesis.

The backward implication takes a little more work, and we start by observing the following structural results on the peripheral subgroups that are a consequence of the lower domination inequality (D^-) and the unique limits condition:

Lemma 5.11. *Suppose $\rho : \Gamma \to G$ is almost-dominated relative to $\mathcal{P}$ with peripherals satisfying the unique limit and quasi-unipotence hypotheses.*

Then, given any $P \in \mathcal{P}$, $\rho(P)$ is a discrete subgroup of a semidirect product $U_P \rtimes K_P < G = \mathrm{SL}^{\pm}(d, \mathbb{R})$ where U_P is a unipotent Lie group and K_P is a compact Lie group. (In particular, $\rho(P)$ is virtually nilpotent.)

Proof. Since $\rho(\Gamma)$ is relatively dominated, $\rho(P)$ has a unique limit point, and hence is a subgroup of some parabolic subgroup of G. By the Levi decomposition we may write this parabolic subgroup as $U_P \rtimes L_P < G$ where U_P is a unipotent Lie group and L_P is a semisimple Lie group.

Moreover, given any $\rho(\eta) = (u(\eta), l(\eta)) \in U_P \rtimes L_P$, $l(\rho)$ cannot have eigenvalues of norm different from 1, otherwise $l(\eta)$ would be proximal and then so would $\rho(\eta)$, but this is not possible by the quasi-unipotence hypothesis. Following the argument in [KL18], Theorem 5.12, it now follows that the Zariski closure of $\overline{\rho(P)}^Z < L_P$ must be a compact subgroup $K_P < G$; otherwise, by the main result of [Pra94], $\overline{\rho(P)}^Z$ will contain a proximal element, contradicting the previous assertion.

That $\rho(P)$ is virtually nilpotent then follows e.g. from Gromov's polynomial growth theorem, since $\rho(P)$ is a finitely-generated discrete subgroup of a nilpotent-by-compact Lie group and hence has polynomial growth (see also the Appendix to [KL18] for another proof of this part.) □

Corollary 5.12. *Suppose $\rho : \Gamma \to G$ is dominated relative to $\mathcal{P}$. Given any $\eta \in P \in \mathcal{P}$, there exists a unipotent matrix $u(\eta) = u_\rho(\eta) \in G$ with the same singular values as $\rho(\eta)$.*

Moreover, if $\eta = h_1 \cdots h_n$ where $h_i \in P$, then we may write $u(\eta) = \hat{u}_1 \cdots \hat{u}_n$ where each $\hat{u}_i$ is a unipotent element in U_P with the same singular values as $\rho(h_i)$.

Proof. Given any $(u, k) \in \rho(P) < U_P \rtimes K_P$, we note that $(u, k) \cdot (\mathrm{id}, k^{-1}) = (u, \mathrm{id})$ and (u, k) have the same singular values (for K_P consists of orthogonal matrices, and multiplying by an orthogonal matrix does not change the singular values); hence, if $\rho(\eta) = (u, k)$ we may take $u(\eta) = (u, \mathrm{id})$.

Given $\eta = h_1 \cdots h_n$, we write $\rho(h_i) = (u_i, k_i) \in U_P \rtimes K_P$, we have

$$\rho(\eta) = \left(u_1 u_2^{k_1} u_3^{k_1 k_2} \cdots u_n^{k_1 \cdots k_{n-1}}, k_1 \cdots k_n\right)$$

(where we write $u^k := kuk^{-1}$ to denote conjugation), and we observe as above that this has the same singular values as $u_1 u_2^{k_1} \cdots u_n^{k_1 \cdots k_{n-1}} = u(\eta)$. Moreover, each $u_i^{k_1 \cdots k_{i-1}}$ is in U_P since K_P normalizes U_P, and $u_i^{k_1 \cdots k_{i-1}}$ has the same singular values as u_i. Hence we may take $\hat{u}_i = u_i^{k_1 \cdots k_{i-1}}$. □

We may now demonstrate the upper domination condition.

Given $\eta \in P$, write it as a word $h_1 \cdots h_n$ where the h_i are generators of P.

Consider the associated unipotent product $u(\eta) = \hat{u}_1 \cdots \hat{u}_n$ given by Corollary 5.12, and write $\rho(\hat{u}_i) = \exp v_i$ where the v_i are strictly upper triangular matrices. Then

$$u(\eta) = \exp(v_1 + \cdots + v_n + z)$$

where, by the Baker–Campbell–Hausdorff formula (see e.g. [Tho82], Theorem 1), z is a sum of nested commutators of the v_i, with combinatorially computable (universal) coefficients (see e.g. [Gol56].) Specifically, since nested commutators of the v_i with length greater than d are zero, z is a sum of nested commutators of length at most d, and there are at most dn^d of these.

Moreover, since $(v_1 + \cdots + v_n + z)^{d+1} = v_i^{d+1} = z^{d+1} = 0$, the exponential map is in fact a polynomial of degree (at most) d in this case. Since P is finitely-generated, this tells us that the entries of $\rho(h_i)$, and hence its operator norm, are bounded above by polynomials of degree d^2 with coefficients depending on the operator norms of the generator images.

Finally, we note that the finiteness of $\mathcal{P}$ means that we can take some uniform choice of constants in the above. Thus there exists some polynomial q of degree at most d^2, depending only on our representation ρ, and hence some constant $C_1 > 0$ depending only on q, such that

$$\sigma_1(\rho(\eta)) = \sigma_1(u(\eta)) \leq q(|\eta|) \leq C_1 |\eta|^{d^2}$$

(for all $|\eta| \geq 1$.) By Proposition 3.22,

$$C_1 |\eta|^{d^2} \leq C_1 \sqrt{2}^{d^2 |\gamma|_c} = C_1 e^{\mu_1 \cdot |\gamma|_c}$$

where $\mu_1 := \frac{d^2}{2} \log 2$. □

CHAPTER 6

Relative Domination Implies Relative Hyperbolicity

6.1 Existence and transversality of limits

For the rest of this paper, let Γ be a finitely generated group, $\mathcal{P}$ be a finite collection of subgroups of Γ satisfying (RH), and $S = S^{-1}$ be a compatible finite generating set. For the next three sections (§§5, 6, and 7), fix $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ a representation which is 1-dominated relative to $\mathcal{P}$ with domination constants $(\underline{C}, \underline{\mu}, \bar{C}, \bar{\mu})$.

The goal of this section is to establish the following existence and transversality result, which will be very useful in the following sections:

Definition 6.1. *Let $I \subset \mathbb{R}$ be an interval (finite or infinite, open or closed) and let $\alpha : I \to \mathrm{Cay}(\Gamma)$ be a path with $\alpha(I \cap \mathbb{Z}) \subset \Gamma$.*

We define the sequence

$$\begin{aligned} x_\alpha &= (\dots A_{a-1}, \dots, A_{b-1}, \dots) \\ &:= (\cdots, \rho(\alpha(a)^{-1}\alpha(a-1)), \dots, \rho(\alpha(b)^{-1}\alpha(b-1)), \cdots) \in \mathrm{GL}(d, \mathbb{R})^{I \cap \mathbb{Z}} \end{aligned}$$

and call this the **matrix sequence associated to** α.

We say that α (or x_α) is **based at** id *if $0 \in I$ and $\alpha(0) = \mathrm{id}$.*

Proposition 6.2. *Let $\gamma = \pi \circ \hat{\gamma}$ be a bi-infinite $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path γ based at* id*, and let $x = x_\gamma = (A_k)_{k \in \mathbb{Z}}$ be the matrix sequence associated to γ. Then*

(i) the following limits

$$\begin{aligned} E^u(x) &:= \lim_{n \to \infty} U_1(A_{-1} \cdots A_{-n}) \\ E^s(x) &:= \lim_{n \to \infty} S_{d-1}(A_{n-1} \cdots A_0) \end{aligned}$$

exist and form a splitting $E^u(x) \oplus E^s(x)$ of $\mathbb{R}^d$, and *(ii) there is a uniform bound $s_{\min}$ (depending only on the quasigeodesic and domination constants) on the minimal separation $s(E^u(x), E^s(x)) := \sin\angle(E^u(x), E^s(x))$ between these linear subspaces.*

To prove this we will use the following theorem, which is a mild modification of a recent result of Quas–Thieullen–Zarrabi [QTZ19], which in turn is a vast generalization of the characterization of linear cocycles with dominated splittings given in Bochi–Gourmelon [BG09]:

Theorem 6.3. *Let $(A_k)_{k\in\mathbb{Z}} \subset \mathrm{GL}(d,\mathbb{R})$ be a sequence of matrices such that there exists constants $C \geq 1$ and $\mu, \mu' \geq 0$, with $\frac{1}{\mu}\log 3C > 1$, such that the following axioms are satisfied:*

- *(SVG-BG) for all $k \in \mathbb{Z}$ and all $n \geq 0$,*

$$\frac{\sigma_2}{\sigma_1}(A_{k+n-1}\cdots A_k) \leq Ce^{-n\mu}$$

- *(EC) for all $k \in \mathbb{Z}$ and all $n \geq 0$,*

$$d(S_{d-1}(A_{k+n-1}\cdots A_k), S_{d-1}(A_{k+n}\cdots A_k)) \leq Ce^{-n\mu},$$
$$d(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-(n+1)})) \leq Ce^{-n\mu}.$$

- *(FI)$_{back}$: for all $k \leq 0$ and $n, m \geq 0$*

$$\frac{\sigma_1(A_{k+n-1}\cdots A_{k-m})}{\sigma_1(A_{k+n-1}\cdots A_k)\cdot\sigma_1(A_{k-1}\cdots A_{k-m})} \geq C^{-1}e^{-m\mu'}$$

Then

(i) for each $k \in \mathbb{Z}$ in the sequence we have a splitting $E^u \oplus E^s$ of $\mathbb{R}^d$ given by

$$E^u(k) := \lim_{n\to\infty} U_1(A_{k-1}\cdots A_{k-n})$$
$$E^s(k) := \lim_{n\to\infty} S_{d-1}(A_{k+n-1}\cdots A_k)$$

which is equivariant in the sense that $A_k E^*(k) = E^*(k+1)$ for all $k \in \mathbb{Z}$ and $* \in \{u, s\}$;

(ii) moreover, for all $k \leq 0$, we have a uniform lower bound $s_{\min} = s_{\min}(C, \mu, \mu')$ on the gap $s(E^u(k), E^s(k)) := \sin\angle(E^u(k), E^s(k))$ given by

$$s(E^u(k), E^s(k)) \geq s_{\min} := \frac{2}{3}(3e)^{-2r}\exp\left(-\frac{3/2}{1-e^{-\mu}}\right)C^{-(1+2r)},$$

where $r := \frac{\mu'}{\mu}$.

We will defer the proof of this result to Appendix B and focus on showing how to obtain Proposition 6.2 given the Theorem. We remark that we may assume, without loss of generality, that our constants are such that the additional hypothesis $\frac{1}{\mu}\log 3C > 1$ specified in Theorem 6.3 is satisfied; if they are not, we can make C larger or μ smaller and the other required axioms will continue to hold with these adjusted constants.

Before beginning the argument, we remark that a number of linear algebra results, which will be used throughout this and subsequent proofs, are collected in Appendix A. We note that Lemma A.1, in particular, will be used many times below to control unstable spaces of products of matrices.

We start by establishing the following

Lemma 6.4. *Given $\underline{\upsilon}, \bar{\upsilon} > 0$, there exist constants $C \geq 1$ and $\mu > 0$, depending only on the representation and $\underline{\upsilon}, \bar{\upsilon}$, such that for any matrix sequence $x = x_\gamma$ associated to a bi-infinite $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path γ based at* id,

$$\begin{aligned} d(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-(n+1)})) &\leq Ce^{-n\mu} \\ d(S_{d-1}(A_{k+n-1}\cdots A_k), S_{d-1}(A_{k+n}\cdots A_k)) &\leq Ce^{-n\mu}. \end{aligned}$$

In other words, such sequences x_γ satisfy (EC), with constants depending only on the representation and the quasigeodesic constants. It then follows, using the triangle inequality, that the limits exist, and in fact convergence to the limits is uniform:

Corollary 6.5. *Given $x = x_\gamma = (A_k)$ a matrix sequence associated to bi-infinite $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path γ based at* id*, the limits*

$$E^u(x) := \lim_{n\to\infty} U_1(A_{-1}\cdots A_{-n}) \quad \text{and} \quad E^s(x) := \lim_{n\to\infty} S_{d-1}(A_{n-1}\cdots A_0)$$

exist, and

$$\begin{aligned} d(U_1(A_{-1}\cdots A_{-n}), E^u(x)) &\leq \frac{C}{1-e^{-\mu}} \cdot e^{-n\mu} \\ d(E^s(x), S_{d-1}(A_n\cdots A_0)) &\leq \frac{C}{1-e^{-\mu}} \cdot e^{-n\mu} \end{aligned}$$

where C, μ are the constants from Lemma 6.4.

To prove Lemma 6.4 it will be useful to more closely examine the parts of matrix sequences inside the peripheral subgroups. For this purpose, we recall the notions of peripheral excursion and depth from §3.1, now used for matrix sequences coming from paths in Γ:

Definition 6.6. *Given I an interval in $\mathbb{Z}$ and a sequence $x = x_\alpha = (A_k) \in \mathrm{GL}(d,\mathbb{R})^I$ associated to some path $\gamma : I \to \mathrm{Cay}(\Gamma)$, a* **peripheral excursion** *in x is a subsequence $(A_k) \in \mathrm{GL}(d,\mathbb{R})^J$ where $J \subset I$ is a subinterval and $\gamma|_J$ is a peripheral excursion in the sense of §3.1.2.*

The **depth** *of a matrix $A_k = \rho(\gamma(k)^{-1}\gamma(k-1))$ inside a peripheral excursion is the depth of $\gamma(k)^{-1}\gamma(k-1)$ in the sense of Definition 3.25.*

Proof of Lemma 6.4. We presently restrict our attention to $(A_{k-n})_{n>0}$, in order to study more carefully the limit giving $E^u(k)$.

We now derive two inequalities, each of which works to give us the bound we want in a different case. On the one hand, we have

$$\begin{aligned} &d(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-n-1})) \\ &\quad \leq \frac{\sigma_1}{\sigma_d}(A_k{-}n-1)\cdot\frac{\sigma_2}{\sigma_1}(A_{k-1}\cdots A_{k-n}) \\ &\quad \leq \frac{\sigma_1}{\sigma_d}(\rho(\gamma(k-n)^{-1}\gamma(k-n-1)))\cdot\frac{\sigma_2}{\sigma_1}(\rho(\gamma(k)^{-1}\gamma(k-n))) \end{aligned}$$

by Lemma A.1. By Corollary 5.8 and Definition 3.26,

$$\frac{\sigma_1}{\sigma_d}(\rho(\gamma(k-n)^{-1}\gamma(k-n-1))) \leq e^{\bar{\mu}\cdot\bar{\upsilon}(\delta(A_{k-n-1})+6)} = \bar{C}e^{6\bar{\mu}\bar{\upsilon}}e^{\bar{\mu}\bar{\upsilon}\cdot\delta(A_{k-n-1})};$$

by Definition 3.26 and the lower domination inequality (D$^-$),

$$\frac{\sigma_2}{\sigma_1}(\rho(\gamma(k)^{-1}\gamma(k-n))) \leq \underline{C}^{-1}e^{-\underline{\mu}\underline{\upsilon}}e^{\underline{\mu}\underline{\upsilon}n}$$

where $\underline{C}$ and $\underline{\mu}$ are the domination constants. Hence, writing $C_2 = \bar{C}\underline{C}^{-1}e^{6\bar{\mu}\bar{\upsilon}+\underline{\mu}\underline{\upsilon}}$, $\mu_2 = \bar{\mu}\bar{\upsilon}$, and $\mu_0 = \underline{\mu}\underline{\upsilon}$,

$$d(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-n-1})) \leq C_2 e^{\mu_2\cdot\delta(A_{k-n-1})}\cdot e^{-\mu_0 n} \tag{6.1}$$

This will turn out to give us the inequality we want when the depth $\delta(A_{k-n-1})$ is relatively small compared to n.

Alternatively, suppose a matrix lies in a peripheral excursion starting at $k-n_0$. Write $D := A_{k-1}\cdots A_{k-n_0}$ to denote the word prior to the excursion, and, for any integer n with A_{k-n} belonging to the peripheral excursion, $E(n-n_0) := A_{k-n_0-1}\cdots A_{k-n}$, so that we have the decomposition $A_{k-1}\cdots A_{k-n} = DE(n-n_0)$.

We break $E(n-n_0)^{-1}E(n+1-n_0) = A_{k-n-1}$ up into smaller chunks

$$A_{k-n-1} = A_{k-n-1,1}\cdots A_{k-n-1,\ell(k-n-1)} = \rho\left(p_{k-n-1,1}\cdots p_{k-n-1,\ell(k-n-1)}\right)$$

corresponding to single unbunched peripheral generators (as in property (iii) of Definition 3.26.)

For brevity, we write $F_j := A_{k-n-1,j}$ in the next inequality, and also adopt the convention $F_0 = \mathrm{id}$. Now we have

$$\begin{aligned}
&d(U_1(DE(n-n_0)), U_1(DE(n+1-n_0))) \\
&\quad \le \sum_{j=1}^{\ell(k-n-1)} d(U_1(DE(n-n_0)F_0\cdots F_{j-1}), U_1(DE(n-n_0)F_0\cdots F_j)) \\
&\quad \le \sum_{j=1}^{\ell(k-n-1)} \frac{\sigma_1}{\sigma_d}(F_j)\cdot\frac{\sigma_2}{\sigma_1}(DE(n-n_0)F_0\cdots F_{j-1}) \\
&\quad \le \bar{C}e^{\bar{\mu}} \sum_{j=1}^{\ell(k-n-1)} \frac{\sigma_2}{\sigma_1}(DE(n-n_0)F_0\cdots F_{j-1}) =: RHS_1
\end{aligned}$$

where we have used the triangle inequality $\ell(k-n-1)$ times, applied Lemma A.1 to each of the resulting terms, and then used Corollary 5.8 with the bound on the size of single generators; then, using the quadratic gaps condition (which bounds from below the first singular value gap for images of words ending in peripheral excursions)

$$\begin{aligned}
&d(U_1(DE(n-n_0)), U_1(DE(n+1-n_0))) \le RHS_1 \\
&\quad \le \sum_{j=1}^{\ell(k-n-1)} \frac{\sigma_1}{\sigma_d}(F_j)\cdot\frac{\sigma_2}{\sigma_1}(DE(n-n_0)F_0\cdots F_{j-1}) \\
&\quad \le \bar{C}e^{\bar{\mu}} \sum_{j=1}^{\ell(k-n-1)} \frac{\sigma_2}{\sigma_1}(\rho(\gamma(k)^{-1}\gamma(k-n_0)\cdot\gamma(k-n_0)^{-1}\gamma(k-n)p_{k-n-1,1}\cdots p_{k-n-1,j})) \\
&\quad \le \bar{C}e^{\bar{\mu}}\cdot\frac{1}{C'} \sum_{j=0}^{\ell(k-n-1)} |\gamma(k-n_0)^{-1}\gamma(k-n)p_{k-n-1,1}\cdots p_{k-n-1,j})|^{-2} =: RHS_2
\end{aligned}$$

and finally using the metric quasigeodesic lower bound and Proposition 3.23, we obtain

$$\begin{aligned}
&d(U_1(DE(n-n_0)), U_1(DE(n+1-n_0))) \le RHS_2 \\
&\quad \le \bar{C}e^{\bar{\mu}}\cdot\frac{1}{C'} \sum_{j=0}^{\ell(k-n-1)} \left(2^{\underline{\upsilon}^{-1}(n-n_0)-\underline{\upsilon}} + j\right)^{-2} \\
&\quad \le \frac{2^{1+\underline{\upsilon}}\bar{C}e^{\bar{\mu}}}{C'}\exp\left(-\frac{\log 2}{\underline{\upsilon}}(n-n_0)\right) \le C_3\exp\left(-\frac{\log 2}{\underline{\upsilon}}\cdot\delta(A_{k-n})\right) \qquad (6.2)
\end{aligned}$$

where $C_3 := \frac{2^{1+\underline{\upsilon}}\bar{C}e^{\bar{\mu}}}{C'}$; at the end we have used the general inequality

$$\sum_{j=0}^{b}(M+j)^{-2} = \sum_{j=M}^{M+b} j^{-2} \leq \int_{M-1}^{M+b} x^{-2}\,dx = \frac{1}{M-1} - \frac{1}{M+b} \leq \frac{2}{M}.$$

This second inequality will serve us when the depth $\delta(A_{k-n})$ is relatively large compared to n.

For $n > 0$ where the depth $\delta(A_{k-n-1}) \leq \frac{\mu_0}{2\mu_2}n$ (including all n where $\delta(A_{k-n}) = 0$, i.e. A_n is nonperipheral), it follows from (6.1) that

$$\begin{aligned} d\left(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-(n+1)})\right) &\leq C_2 e^{\mu_2(\delta(A_{k-n-1}))} \cdot C_0^{-1} e^{-\mu_0 n} \\ &\leq C_2 e^{\mu_2 \cdot \frac{\mu_0}{2\mu_2}n} e^{-\mu_0 n} = C_2 e^{-\frac{\mu_0}{2}n} \end{aligned}$$

For $n > 0$ where the depth $\delta(A_{k-n}) > \frac{\mu_0}{2\mu_2}n$, we have, from (6.2),

$$d\left(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-n-1})\right) \leq C_3 \exp\left(-\frac{\mu_0 \log 2}{2\underline{\upsilon}\mu_2}n\right)$$

and so **we have the desired inequalities for our Lemma, with** $C = \max\{C_2, C_3\}$ **and** $\mu = \frac{\mu_0}{2} \cdot \min\{1, \frac{\log 2}{\underline{\upsilon}\mu_2}\}$**.**

For $(A_{k+n})_{n\geq 0}$ and the limit giving $E^s(k)$, we may argue similarly, or alternatively we may consider the reversed dual sequence ${}^\iota x^* = (B_k)_{k\in\mathbb{Z}}$ given by

$$B_k := \rho^*(\gamma(-k-1)^{-1}\gamma(-k-2)) = (A_{-k-1}^{-1})^T \tag{6.3}$$

where ρ^* is the dual representation, which is also 1-relatively dominated (Proposition 5.4.)

By Proposition 5.4, we have

$$\begin{aligned} U_1(B_{-k-1}\cdots B_{-k-n}) &= U_1(\rho^*(\gamma_r(-k)^{-1}\gamma_r(n-k-1))) \\ &= \left(U_{d-1}(\rho(\gamma_r(k)^{-1}\gamma_r(k+n-2)))\right)^\perp \\ &= \left(S_{d-1}(\rho(\gamma_r(k+n-2)^{-1}\gamma_r(k)))\right)^\perp \\ &= (S_{d-1}(A_{k+n-1}\cdots A_k))^\perp \end{aligned}$$

Then we have

$$\begin{aligned} d(S_{d-1}(A_{k+n-1}\cdots A_k), S_{d-1}(A_{k+n}\cdots A_0)) &= d(U_1(B_{-k}\cdots B_{-k-n}), U_1(B_{-k}\cdots B_{-k-n-1})) \\ &\leq Ce^{-\mu n}. \end{aligned}$$

where in the last step we have used the argument above for the $E^u(-k)$ limit for ${}^\iota x^*$, $\square$

Proof of Proposition 6.2. By Corollary 6.5, the limits $E^u(x)$ and $E^s(x)$ exist, and the sequence $x = x_\gamma$ satisfies axiom (EC) in the statement of Theorem 6.3, with constants depending only on the domination and quasigeodesic constants.

From the upper and lower domination inequalities (D^-) and the metric quasigeodesic properties in Definition 3.26), $x = x_\gamma$ satisfies axiom (SVG-BG) in the statement of Theorem 6.3, with constants $C = \underline{C}e^{-\underline{\mu}\underline{\upsilon}}$ and $\mu = \underline{\mu}\underline{\upsilon}$.

Step 1: bounded-depth sequences.

Definition 6.7. *We say a sequence $x = (A_k)_{k\in\mathbb{Z}}$ has* **bounded depth** Δ *in the backward direction (in the forward direction, respectively) if $\delta(A_k) \leq \Delta$ for all $k \leq 0$ (for all $k \geq 0$, resp.)*

Equivalently, for x_{γ_r}, our (sub)path $\gamma|_{\mathbb{Z}_{\leq 0}}$ (or $\gamma|_{\mathbb{Z}_{\geq 0}}$, respectively) has peripheral excursions of bounded cusped length.

Proposition 6.8. *Given $\Delta \in \mathbb{Z}_{\geq 0}$, there exists $s_{\min}(\Delta)$ (which also depends on the quasigeodesic and domination constants) such that for any $x = x_{\gamma_r}$ with bounded depth Δ in the backward direction or in the forward direction, $s(E^u(x), E^s(x)) \geq s_{\min}(\Delta)$*

Proof. If $x = x_{\gamma_r}$ has bounded depth Δ in the backward direction, then x satisfies the axiom $(\mathrm{FI})_{\mathrm{back}}$ from the inequalities

$$\begin{aligned}\frac{\sigma_1(A_{k+n-1}\cdots A_{k-m})}{\sigma_1(A_{k+n-1}\cdots A_k)\cdot\sigma_1(A_{k-1}\cdots A_{k-m})} &\geq \frac{\sigma_1(A_{k+n-1}\cdots A_k)\cdot\sigma_d(A_{k-1}\cdots A_{k-m})}{\sigma_1(A_{k+n-1}\cdots A_k)\cdot\sigma_1(A_{k-1}\cdots A_{k-m})}\\ &\geq \frac{1}{C_2}e^{-\mu_2(\Delta+m)} = \frac{e^{-\mu_2\Delta}}{C_2}e^{-\mu_2 m};\end{aligned}$$

these inequalities follow from the general inequalities $\sigma_1(A)\cdot\sigma_1(B) \geq \sigma_1(AB) \geq \sigma_1(A)\cdot\sigma_d(B)$ and Corollary 5.8 and Definition 3.26, with C_2, μ_2 as in the proof of Lemma 6.4.

Thus if $x = x_{\gamma_r}$ has bounded depth Δ in the backward direction, it satisfies $(\mathrm{FI})_{\mathrm{back}}$ with $D = C_2 e^{\mu_2\Delta}$ and $\mu' = \mu_2$. In particular, Theorem 6.3 gives us $s(E^u(x), E^s(x)) \geq s_{\min}(\Delta)$ for some $s_{\min}(\Delta)$ depending also on the quasigeodesic and domination constants, and we obtain the Proposition for such sequences.

If $x = x_{\gamma_r} = (A_k)_{k\in\mathbb{Z}}$ has bounded depth Δ in the forward direction but not the backward direction, consider again the reversed dual sequence ${}^\iota x^* = (B_k)_{k\in\mathbb{Z}}$ defined above in (6.3).

The sequence ${}^\iota x^*$ has bounded depth in the backward direction, hence Proposition 6.8 we have $s(E^u({}^\iota x^*), E^s({}^\iota x^*)) \geq s_{\min}(\Delta)$.

But now, by Proposition 5.4, $E^u({}^\iota x^*) = E^s(x)^\perp$ since

$$\begin{aligned} E^u({}^\iota x^*) &= \lim_{n\to\infty} U_1(B_{-1}\cdots B_{-n}) \\ &= \lim_{n\to\infty} U_1(\rho^*(\gamma_r(0)^{-1}\gamma_r(n-2))) \\ &= \lim_{n\to\infty} (U_{d-1}(\rho(\gamma_r(0)^{-1}\gamma_r(n-2))))^\perp \\ &= \Big(\lim_{n\to\infty} S_{d-1}(\rho(\gamma_r(n-2)^{-1}\gamma_r(0)))\Big)^\perp = E^s(x)^\perp \end{aligned}$$

and similarly $E^s({}^\iota x^*) = E^u(x)^\perp$. Hence we have $s(E^u(x), E^s(x)) \geq s_{\min}(\Delta)$ as desired. □

Step 2: unbounded-depth sequences. If our sequence $x = x_{\gamma_r}$ does not have bounded depth in either the backward or forward directions, then the subpaths in both directions (i.e. both $\gamma|_{\mathbb{Z}_{\leq 0}}$ and $\gamma|_{\mathbb{Z}_{\geq 0}}$) contain arbitrarily long peripheral excursions.

Define $P^\pm \in \mathcal{P}$ and infinite peripheral excursions $p^\pm_\infty$ as follows:

- if γ is eventually peripheral in the forward (backward, respectively) direction, let p^+_∞ (p^-_∞, resp.) be the maximal infinite peripheral excursion of the form $\gamma|_{\geq N}$ for some $N \in \mathbb{Z}_{\geq 0}$ ($\gamma|_{\geq N}$ for some $N \in \mathbb{Z}_{\leq 0}$, resp.), and let P^+ (P^-, resp.) be the peripheral subgroup in which p^+_∞ (p^-_∞, resp.) lies.
- If γ is not eventually peripheral in the forward (backward, resp.) direction: by the finiteness of $|\mathcal{P}|$ and since the peripheral subgroups are finitely-generated, in this direction we can find $P^+ \in \mathcal{P}$ (P^-, resp.) and a sequence of increasingly longer peripheral excursions $p^\pm_n$ in $P^\pm$. By a diagonal argument these converge to an infinite peripheral excursion $p^\pm_\infty$ into $P^\pm$ (respectively.)

Let L be the constant from the local-to-global condition in Definition 5.1 and T_2 be the threshold such that

$$\frac{C}{1-e^{-\mu}} e^{-\mu T_2} \leq \frac{\delta_0}{8}$$

where $C, \mu > 0$ are the constants from Lemma 6.4, δ_0 is the constant from the uniform transversality condition, and define $T := \max\{T_2, L\}$.

Consider, in each direction, the first peripheral excursions into $P^\pm$ of depth at least T which (i.e. whose reparametrized projections) agree with $p^\pm_\infty$ up to length T. Take a sequence x' where we replace these peripheral excursions with $p^\pm_\infty$ (resp.) By construction and by the local-to-global condition, these are uniform metric projected quasigeodesics in both directions (starting from 0.) From the uniform transversality condition, we have $s(E^u(x'), E^s(x')) \geq \delta_0$.

Next we wish to use (EC) (more precisely, Corollary 6.5) and the choice of T to say that

$$d(E^u(x), E^u(x')) \leq \frac{\delta_0}{4} \qquad\qquad d(E^s(x), E^s(x')) \leq \frac{\delta_0}{4}.$$

To verify (EC) for x', remark that our construction—in particular the choice of T—together with the local-to-global condition give us that we have geodesic rays in both directions, and hence (EC) still follows from Lemma 6.4.

Hence $s(E^u(x), E^s(x)) \geq \frac{\delta_0}{2} > 0$ and we have a splitting.

To obtain the minimum gap: from Proposition 6.8 (i.e. step 1 above), we have a minimum gap $s(N)$ for any sequence of bounded depth N in either direction; from step 2, we have a minimum gap $\delta_0/2$ for sequences of unbounded depth. Suppose $s(N) \to 0$ as $N \to \infty$. Then we may choose an infinite sequence of matrix sequences $x^{(m)}$, each associated to a (reparametrized) $(\underline{\upsilon}, \bar{\upsilon})$-metric projected quasigeodesic of bounded depth d_m, with $d_m \to \infty$, such that the gap between $E^u(x^{(m)})$ and $E^s(x^{(m)})$ is bounded above by $\frac{1}{m}$.

Up to subsequence, these converge to some infinite sequence x which is associated to a reparametrized $(\underline{\upsilon}, \bar{\upsilon})$-metric projected quasigeodesic with zero gap between $E^u(x)$ and $E^s(x)$; but this is a contradiction whether x has unbounded or bounded depth.

Hence, by our compactness argument, we may choose our minimum gap to be

$$\min\{\delta_0/2, \inf_{N\in\mathbb{N}} s(N)\} > 0.$$

□

6.2 Relative domination implies relative hyperbolicity

Recall that Γ is a torsion-free finitely-generated group. We will presently prove the following

Theorem 6.9. *If $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is 1-dominated relative to $\mathcal{P} \neq \varnothing$ with domination constants $(\underline{C}, \underline{\mu}, \bar{C}, \bar{\mu})$, and $\Gamma \neq \bigcup \mathcal{P}^\Gamma$ (i.e. Γ contains non-peripheral elements), then Γ must be hyperbolic relative to $\mathcal{P}$.*

We remark that the statement is still true if $\mathcal{P} = \varnothing$—that is precisely the result from [BPS19].

The proof of Theorem 6.9 will use the criterion for relative hyperbolicity given in Theorem 3.20. To do so we will find a compact, perfect metric space on which Γ acts as a geometrically finite convergence group, and verify that the maximal parabolic subgroups are precisely the peripheral subgroups. Below, we construct such a space Λ_{rel}, verify it has the required properties, check that the action of Γ on the space of distinct triples $\Lambda_{rel}^{(3)}$ is properly discontinuous and the action of Γ

on the space of distinct pairs $\Lambda_{rel}^{(2)}$ is cocompact, and finally characterize the maximal parabolic subgroups.

We remark that the outline of the argument is adapted from that of [BPS19], §3. In particular, a statement describing north-south dynamics (Lemma 3.13 in [BPS19], Lemma 6.16 here), resulting from a quantitative transversality result (Corollary 6.14), is a key intermediate proposition. Here the geodesics we consider are located not in the group but in the associated cusped space, and this necessitates the new tools introduced in the previous section for the proof of the transversality result. There are also differences in the proofs due to the convergence action of the group being geometrically-finite rather than uniform; among other things, this, through our assumption that Γ contains both peripheral and non-peripheral elements, simplifies the proof of perfectness (Proposition 6.17.)

We fix some notation for the below. Fix $\ell_0 \in \mathbb{N}$ such that $\underline{C}e^{-\underline{\mu}\ell_0} < 1$. We will write, for brevity, $\Xi_\rho(\gamma) := U_1(\rho(\gamma))$ and $\Xi_\rho^*(\gamma) := S_{d-1}(\rho(\gamma)^{-1}) = U_{d-1}(\rho(\gamma))$, for $\gamma \in \Gamma$. We recall that these were defined in §3.2. Given $\xi, \zeta \in \mathbf{P}(\mathbb{R}^d)$ or $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$, $d(\xi,\zeta)$ will denote distance between ξ and ζ in the relevant Grassmannian.

6.2.1 The limit set

We will construct a candidate space Λ_{rel} for the compact metric space M required in Theorem 3.19, as follows:

$$\Lambda_{rel} := \bigcap_{n \geq \ell_0} \overline{\{\Xi_\rho(\gamma) : |\gamma|_c \geq n\}}.$$

We remark that any $\xi \in \Lambda_{rel}$ can be written as a limit $\lim_{n\to\infty} \Xi_\rho(\gamma_n)$ where $|\gamma_n|_c \to \infty$.

Remark 6.10. Λ_{rel} is closely related to Benoist's limit set from [Ben97]: at least in the case where $\rho(\Gamma)$ is Zariski-dense, Λ_{rel} is the natural projection of Benoist's limit set to the projective space.

It is fairly immediate that

Proposition 6.11. *Λ_{rel} is compact, non-empty, and $\rho(\Gamma)$-invariant.*

Proof. Λ_{rel} is compact and non-empty since it is a decreasing intersection of non-empty closed subsets of a Grassmannian, which is a compact space.

To show Λ_{rel} is $\rho(\Gamma)$-invariant, we fix $\eta \in \Gamma$ and $\xi \in \Lambda_{rel}$, and choose a sequence $(\gamma_n) \subset \Gamma$ such that $|\gamma_n|_c \to \infty$ and $\Xi(\gamma_n) \to \xi$. $\Xi(\eta\gamma_n)$ is well-defined whenever $|\gamma_n| \geq \ell_0 - |\eta|$, and by ($\mathrm{D}^-$) and Lemma A.1(A.2) we have

$$d(\rho(\eta)\,\Xi_\rho(\gamma_n), \Xi_\rho(\eta\gamma_n)) \leq \frac{\sigma_1}{\sigma_d}(\rho(\eta)) \cdot \underline{C}e^{-\underline{\mu}|\gamma_n|_c} \to 0$$

as $n \to \infty$, and so $\Xi_\rho(\eta\gamma_n) \to \rho(\eta)\xi$ as $n \to \infty$, and in particular $\rho(\eta)\xi \in \Lambda_{rel}$. □

6.2.2 Dynamics on the limit set

Recall that $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is a 1-relatively dominated representation with domination constants $(\underline{C}, \underline{\mu}, \bar{C}, \bar{\mu})$.

We start this section with the following comparability lemma, which follows from Corollary 5.8 and related estimates:

Lemma 6.12. *There exist constants $\nu \in (0,1)$, $c_0 > 1$ and $c_1 > 1$, depending only on the domination constants $\underline{C}, \underline{\mu} > 0$, such that for any $\gamma, \eta \in \Gamma$ satisfying $|\gamma|_c, |\eta|_c \geq \ell_0$ (with ℓ_0 as above), then*

$$d_c(\gamma, \eta) \geq \nu(|\gamma|_c + |\eta|_c) - c_0 - c_1 |\log d(\Xi_\rho(\gamma), \Xi_\rho(\eta))|.$$

Proof. Consider $\gamma, \eta \in \Gamma$ with cusped word length at least ℓ_0. Assume without loss of generality that $|\gamma|_c \leq |\eta|_c$. Applying Lemma A.1(A.1) to $A = \rho(\eta)$ and $B = \rho(\eta^{-1}\gamma)$, and using the relatively dominated condition and Corollary 5.8, we obtain

$$\begin{aligned} d\left(\Xi_\rho(\eta), \Xi_\rho(\gamma)\right) &\leq \frac{\sigma_1}{\sigma_d}\left(\rho(\eta^{-1}\gamma)\right) \cdot \frac{\sigma_2}{\sigma_1}(\rho(\eta)) \\ &\leq \bar{C} e^{\bar{\mu}|\eta^{-1}\gamma|_c} \cdot \underline{C} e^{-\underline{\mu}|\eta|_c} \end{aligned}$$

where $\bar{C}, \bar{\mu}$ are the constants from Corollary 5.8. Equivalently, after taking logarithms and isolating the $d_c(\gamma, \eta)$ term,

$$\begin{aligned} d_c(\gamma, \eta) = |\eta^{-1}\gamma|_c &\geq \bar{\mu}^{-1}\left(\underline{\mu}|\eta|_c - \log \bar{C} - \log \underline{C} + \log d(\Xi_\rho(\eta), \Xi_\rho(\gamma))\right) \\ &\geq \underline{\mu}\bar{\mu}^{-1}|\eta|_c - \bar{\mu}^{-1}\left(\log \bar{C} + \log \underline{C}\right) - \bar{\mu}^{-1}\left|\log d(\Xi_\rho(\eta), \Xi_\rho(\gamma))\right| \end{aligned}$$

and since $|\eta|_c \geq (|\gamma|_c + |\eta|_c)/2$, we obtain the lemma. □

In particular, applying this to projected geodesic rays, we obtain

Lemma 6.13. *If $(\gamma_n)_{n=0}^\infty, (\eta_n)_{n=0}^\infty$ are two projected geodesic sequences in Γ with $\gamma_0 = \eta_0 = \mathrm{id}$ such that $\lim_{n\to\infty} \Xi_\rho(\gamma_n) \neq \lim_{n\to\infty} \Xi_\rho(\eta_n)$, then $(\ldots, \eta_2, \eta_1, \mathrm{id}, \gamma_1, \gamma_2, \ldots)$ is a metric quasigeodesic, with quasigeodesic constants depending only on $d\left(\lim_{n\to\infty} \Xi_\rho(\gamma_n), \lim_{n\to\infty} \Xi_\rho(\eta_n)\right)$*

Proof. Given the hypotheses, it follows from Corollary 6.5 that the limits $\xi_\rho(\gamma) := \lim_{n\to\infty} \Xi_\rho(\gamma_n)$, $\xi_\rho(\eta) := \lim_{n\to\infty} \Xi_\rho(\eta_n)$ and $\xi_\rho^*(\eta) := \lim_{n\to\infty} \Xi_\rho^*(\eta_n)$ exist.

The previous Lemma applied to the pairs of elements (γ_n, η_n), together with Proposition 3.27, yields that the sequence $(\ldots, \eta_n, \ldots \eta_0, \mathrm{id}, \gamma_0, \ldots, \gamma_n, \ldots)$ is a metric quasigeodesic path, with constants depending on $\epsilon := d(\xi_\rho(\gamma_n), \xi_\rho(\eta_n))$, $\nu \in (0,1)$, c_0, c_1 from Lemma 6.12, and ℓ_0 from above.

More precisely, Proposition 3.27 verifies the metric quasigeodesic inequalities for any subpath restricted to one side of id, i.e. containing only elements γ_i or η_j.

For subpaths containing both some η_l and some γ_k, we have

$$d_c(\gamma_k, \eta_l) \leq d_c(\gamma_k, \mathrm{id}) + d_c(\mathrm{id}, \eta_l) \leq 8(k+l) + 40$$

from the triangle inequality and Proposition 3.27. For the lower bound here: write

$$c := \max\{2\ell_0, c_0 + c_1 \log(3/\epsilon)\},$$

and note that we have

$$d_c(\gamma_k, \eta_l) = |\eta_l^{-1}\gamma_k|_c \geq \nu(|\eta_l|_c + |\gamma_k|_c) - c \geq \frac{\nu}{6}(l+k) - c$$

from Lemma 6.12 and Proposition 3.27 when both $|\gamma_k|_c, |\eta_l|_c > \ell_0$. In the case $|\eta_l|_c \leq \ell_0$ we have

$$\begin{aligned} d_c(\gamma_k, \eta_l) &\geq d_c(\gamma_k, \mathrm{id}) - d_c(\eta_l, \mathrm{id}) \geq |\gamma_k|_c - \ell_0 \\ &\geq (|\gamma_k|_c + |\eta_l|_c) - 2\ell_0 \end{aligned}$$

and an analogous argument produces the same lower bound when $|\gamma_k|_c \leq \ell_0$. □

We may combine this with Proposition 6.2 to obtain

Corollary 6.14. *If $(\gamma_n)_{n=0}^\infty, (\eta_n)_{n=0}^\infty$ are two projected geodesic sequences in Γ with $\gamma_0 = \eta_0 = \mathrm{id}$ such that $\lim_{n\to\infty} \Xi_\rho(\gamma_n) \neq \lim_{n\to\infty} \Xi_\rho(\eta_n)$, then $\lim_{n\to\infty} \Xi_\rho(\gamma_n)$ is transverse to $\lim_{n\to\infty} \Xi_\rho^*(\eta_n)$.*

Proof. Given the hypotheses, it follows from Corollary 6.5 that the limits $\xi_\rho(\gamma) := \lim_{n\to\infty} \Xi_\rho(\gamma_n)$, $\xi_\rho(\eta) := \lim_{n\to\infty} \Xi_\rho(\eta_n)$ and $\xi_\rho^*(\eta) := \lim_{n\to\infty} \Xi_\rho^*(\eta_n)$ exist. Since γ_n and η_n piece together to form a metric quasigeodesic path (Lemma 6.13), Proposition 6.2 then yields the desired conclusion. □

We then use this together with a compactness argument to prove a finite transversality result:

Lemma 6.15. *For every $\epsilon > 0$, there exist $\ell_1 \geq \ell_0$ and $\delta > 0$ such that for all $\gamma, \eta \in \Gamma$ with*

(i) $|\gamma|_c, |\eta|_c > \ell_1$, and

(ii) $d(\Xi_\rho(\gamma), \Xi_\rho(\eta)) > \epsilon$,

we have $\angle(\Xi_\rho(\gamma), \Xi_\rho^*(\eta)) > \delta$.

Proof. The proof will proceed by contradiction. Assume there exist $\epsilon > 0$ and sequences $\ell_j \to \infty$, $\delta_j \to 0$ such that for each j there exist $\gamma_j, \eta_j \in \Gamma$ with $|\gamma_j|_c, |\eta_j|_c > \ell_j$ and $d\left(\Xi_\rho(\gamma_j), \Xi_\rho(\eta_j)\right) > \epsilon$, but $\angle(\Xi_\rho(\gamma_j), \Xi^*_\rho(\eta_j)) \leq \delta_j$.

Consider the γ_j and η_j as projected geodesics. By a diagonal argument, these converge, up to subsequence, to some (infinite words) $\gamma := g_1 \cdots g_n \cdots$ and $\eta := h_1 \cdots h_m \cdots$. Reparametrizing as needed, we may assume that these are $(6, 20)$-metric quasigeodesic paths (these constants being the ones obtained in Proposition 3.27.)

By Corollary 6.14, the limits $\xi_\rho(x_\gamma)$ and $\xi^*_\rho(x_\eta)$ exist, and $\angle(\xi_\rho(x_\gamma), \xi^*_\rho(x_\eta)) > 0$.

This gives us a contradiction, since, by construction, $\angle(\xi_\rho(x_\gamma), \xi^*_\rho(x_\eta)) = 0$. □

Using this last version of transversality, we then have the following statement describing a sort of North-South dynamics:

Lemma 6.16. *Given $\epsilon, \epsilon' > 0$, there exists $\ell > \ell_0$ such that for any $\eta \in \Gamma$ with $|\eta|_c > \ell$ and any $\xi \in \Lambda_{rel}$ with $d(\xi, \Xi_\rho(\eta^{-1})) > \epsilon$, we have*

$$d(\rho(\eta)\xi, \Xi_\rho(\eta)) \leq \epsilon'.$$

Proof. Let $\ell_1 \geq \ell_0$ and $\delta > 0$ be given by Lemma 6.15, with our given $\epsilon > 0$. Choose $\ell > \ell_1$ such that $\underline{C}e^{-\underline{\mu}\ell} < \epsilon' \sin\delta$.

Fix $\eta \in \Gamma$ and $\xi \in \Lambda_{rel}$ such that $|\eta|_c > \ell$ and $d(\xi, \Xi_\rho(\eta^{-1})) > \epsilon$. Choose a sequence $(\gamma_n) \subset \Gamma$ such that $|\gamma_n|_c \to \infty$ and $\Xi_\rho(\gamma_n) \to \xi$. Without loss of generality assume for each n we have $|\gamma_n|_c > \ell_1$ and

$$d(\Xi_\rho(\gamma_n), \Xi_\rho(\eta^{-1})) > \epsilon.$$

It then follows from Lemma 6.15 that

$$\angle(\Xi_\rho(\gamma_n), \Xi^*_\rho(\eta^{-1})) > \delta$$

and then, by Lemma A.2 with $A = \rho(\eta)$ and $P = \Xi(\gamma_n)$, we obtain

$$\begin{aligned} d(\rho(\eta)\,\Xi_\rho(\gamma_n), \Xi_\rho(\eta)) &\leq \frac{\sigma_2}{\sigma_1}(\rho(\eta))\frac{1}{\sin\angle\left(\Xi_\rho(\gamma_n), \Xi^*_\rho(\eta^{-1})\right)} \leq \frac{\sigma_2}{\sigma_1}(\rho(\eta))\frac{1}{\sin\delta} \\ &\leq \frac{\underline{C}e^{-\underline{\mu}\ell}}{\sin\delta} < \epsilon' \end{aligned}$$

and letting $n \to \infty$ we have $d(\rho(\eta)\xi, \Xi_\rho(\eta)) \leq \epsilon'$ as desired. □

6.2.3 Perfectness

Proposition 6.17. *Λ_{rel} is perfect, that is every point in Λ_{rel} is an accumulation point of other points in Λ_{rel}.*

Proof. We first claim that $|\Lambda_{rel}| \geq 3$. By assumption we have non-peripheral and hence (by Lemma 5.6) biproximal elements, and also peripheral elements. The proximal elements give us at least two distinct points $\xi^{\pm}$ in Λ_{rel}; the peripheral elements give us at least one point ξ_P in Λ_{rel}.

We claim that the peripheral point ξ_P is not fixed by any non-peripheral element of Γ, and in particular is distinct from the proximal limit points $\xi^{\pm}$. To see this, suppose $\gamma \in \Gamma$ is non-peripheral and fixes ξ_P. Then $\xi_{\gamma P \gamma^{-1}} = \xi_P$, which violates the transversality hypothesis in Definition 5.2.

Hence $|\Lambda_{rel}| \geq 3$.

Now let b_1 be a point in Λ_{rel}, and let $\epsilon' > 0$. We will show that the $2\epsilon'$-neighborhood of b_1 contains another element of Λ_{rel}.

Choose b_2, b_3 to be two distinct points of $\Lambda_{rel} \setminus \{b_1\}$. Let $\epsilon := \frac{1}{2}\min_{i \neq j} d(b_i, b_j)$. Let $\ell > \ell_0$ be given by Lemma 6.16, depending on ϵ and ϵ'. Choose $\eta \in \Gamma$ such that $|\eta|_c > \ell$ and $d(\Xi_\rho(\eta), b_1) < \epsilon'$. Consider $\Xi_\rho(\eta^{-1})$ as a linear subspace of $\mathbb{R}^d$; it can be ϵ-close to at most one of the spaces b_1, b_2, b_3. In other words, there are different indices $i, j \in \{1, 2, 3\}$ such that

$$d(b_i, \Xi_\rho(\eta^{-1})) > \epsilon$$

and similarly for b_j. In particular, by Lemma 6.16,

$$d(\rho(\eta)b_i, b_1) \leq d(\rho(\eta)b_i, \Xi_\rho(\eta)) + \epsilon' < 2\epsilon'.$$

By Γ-invariance, the spaces $\rho(\eta)b_i$ and $\rho(\eta)b_j$ are in Λ_{rel}; but at most one of them can be equal to b_1. □

6.2.4 Geometrically-finite convergence group action

We first prove that Γ acts on Λ_{rel} as a convergence group, that is to say

Proposition 6.18. *The natural induced action of Γ on the space $\Lambda_{rel}^{(3)}$ of distinct triples is properly discontinuous.*

Proof. We will pick out a distinguished family of compact sets of $\Lambda_{rel}^{(3)}$, and use these to prove proper discontinuity of the action. Given $T = (P_1, P_2, P_3) \in \Lambda_{rel}^{(3)}$ a triple of distinct points, define $|T| = |(P_1, P_2, P_3)| := \min_{i \neq j} d(P_i, P_j)$, where d is a(ny) Riemannian metric on the Grassmannian.

For every $\delta > 0$, $\left\{T \in \Lambda^{(3)}_{rel} : |T| \geq \delta\right\}$ is a compact subset of $\Lambda^{(3)}_{rel}$, and conversely every compact subset of $\Lambda^{(3)}_{rel}$ is contained in a subset of that form.

We will now establish that, given $\delta > 0$, there exists $\ell \in \mathbb{N}$ such that if $T \in \Lambda^{(3)}_{rel}$ satisfies $|T| > \delta$ and $\eta \in \Gamma$ satisfies $|\eta|_c > \ell$, then $|\rho(\eta)T| < \delta$. This will suffice to establish the proposition, since it implies that given any compact subset $\Lambda^{(3)}_{rel}$, all but finitely many words (those of length at most ℓ) must move the compact subset off itself.

Given $\delta > 0$, let ℓ be given by Lemma 6.16 with $\epsilon = \epsilon' = \frac{\delta}{2}$.

Now consider $(\xi_1, \xi_2, \xi_3) \in \Lambda^{(3)}_{rel}$ such that $|T| > \delta$, and $\eta \in \Gamma$ such that $|\eta|_c > \ell$. Note that $d(\Xi(\eta^{-1}), \xi_i) > \frac{\delta}{2}$ for at least two of the lines ξ_1, ξ_2, ξ_3—say, without loss of generality, ξ_1 and ξ_2.

Lemma 6.16 yields $d(\rho(\eta)\xi_i, \Xi_\rho(\eta)) < \frac{\delta}{2}$ for $i = 1, 2$, and so

$$|\rho(\eta)T| \leq d(\rho(\eta)\xi_1, \rho(\eta)\xi_2) < \delta,$$

as desired. □

We then prove that Γ in fact acts on Λ_{rel} as a geometrically finite convergence group. By Theorem 3.20, to demonstrate geometric finiteness it suffices to show cocompactness on the space of distinct pairs. For this we will use an expansivity argument:

Proposition 6.19. *The natural induced action of* Γ *on the space* $\Lambda^{(2)}_{rel}$ *of distinct pairs is cocompact.*

Proof. As with the case of distinct triples above, for every $\delta > 0$, $\left\{T \in \Lambda^{(2)}_{rel} : |T| \geq \delta\right\}$ is compact subset of $\Lambda^{(2)}_{rel}$, and conversely every compact subset of $\Lambda^{(2)}_{rel}$ is contained in a subset of that form. Here, analogously to above, $|T| := d(\xi_1, \xi_2)$.

We will now prove the following statement: there exists $\epsilon > 0$ such that for every $T = (\xi_1, \xi_2) \in \Lambda^{(2)}_{rel}$, there exists $\gamma \in \Gamma$ such that $|\rho(\gamma)T| \geq \epsilon$. This suffices to establish the Proposition.

Choose $\epsilon = \frac{1}{2}s_{\min}$, where $s_{\min}$ is the minimum gap from Proposition 6.2 for metric *geodesic* sequences given our domination constants. If $|T| \geq \epsilon$ then we may take $\gamma = \mathrm{id}$, so we may suppose that $|T| < \epsilon$.

Choose $(6, 20)$-metric quasigeodesic paths (the constants are from Proposition 3.27) $(\gamma_i = g_1 \cdots g_{|\gamma_i|})$, $(\eta_i = h_1 \cdots h_{|\eta_i|}) \subset \Gamma$ such that $\Xi_\rho(\gamma_i) \to \xi_1$, $\Xi_\rho(\eta_i) \to \xi_2$, and consider the sequence of matrices $(\ldots, A_{-1}, A_0, A_1, \ldots)$ given by $A_i = \rho(g^{-1}_{i+1})$ for $i \geq 0$ and $A_i = \rho(h_{|i|})$ for $i < 0$.

By Lemma 6.13, $(\ldots, \eta_2, \eta_1, \mathrm{id}, \gamma_1, \gamma_2, \ldots) =: x$ is a metric quasigeodesic.

If the sequence for ξ_1 is not eventually peripheral, then we may find an increasing sequence of $i_m > 0$ such that the shifted sequences

$$\sigma^{i_m} x := \left(\sigma^{i_m} A_n := A_{n+i_m}\right)_{n \in \mathbb{Z}}$$

converge (as $m \to \infty$) to a metric geodesic sequence $\sigma^\infty x = (B_n)_{n\in\mathbb{Z}}$, i.e. $B_n = \lim_{m\to\infty} \sigma^{i_m} A_n$ for each $n \in \mathbb{Z}$. By construction, for any given N we can find m_0 so that $\sigma^{i_m} A_n = B_n$ whenever $|n| \le N$ and $m \ge m_0$.

By Proposition 6.2, $\sin\angle\left(E^u(\sigma^\infty x), E^s(\sigma^\infty x)\right) > 2\epsilon$. Moreover, by Corollary 6.5, for all m large enough given the quasigeodesic constants, $\sin\angle\left(E^*(\sigma^{i_m} x), E^*(\sigma^\infty x)\right) < \frac{\epsilon}{2}$ for $* \in \{u, s\}$, so that

$$\sin\angle\left(E^u(\sigma^{i_m} x), E^s(\sigma^{i_m} x)\right) > \epsilon.$$

Since the endpoints of $\sigma^{i^m} x$ are given by acting on the endpoints of x by $A_{i_m-1}\cdots A_0 = \rho(g_1\cdots g_{i_m})^{-1} = \rho(\gamma_{i_m}^{-1})$, this establishes that $|\rho(\gamma_{i_m}^{-1})(\xi_1, \xi_2)| \ge \epsilon$, as desired.

We argue similarly if the sequence for ξ_2 is not eventually peripheral.

If the sequences for both ξ_1 and ξ_2 are eventually peripheral, there is a positive lower bound on the (infimum of the) distance between these (over all shifts, as above): if not, we can find $P, P' \in \mathcal{P}$ and a sequence of words $w_n \to \infty$ not starting with a letter from P such that $d(\xi(P), w_n\xi(P')) < 2^{-n}$. Up to a subsequence, the w_n converge to some infinite geodesic such that $\lim_{n\to\infty} \Xi_\rho(w_n) = \xi(P)$; but now observe that this infinite geodesic cannot be eventually peripheral in both directions—these limit points are all distinct by hypothesis—, and by the arguments above neither can it be not eventually peripheral. We conclude, by contradiction, that said lower bound must in fact exist. ☐

6.2.5 Peripherals are maximal parabolics

Lemma 6.20. *For any non-peripheral $\gamma \in \Gamma$, $\lim_{n\to\infty} \Xi_\rho(\gamma^n)$ is the top eigenline of $\rho(\gamma)$.*

Proof. Recall that $\rho(\gamma)$ is necessarily proximal (Proposition 5.6), so that the top eigenline is well-defined.

To show $\lim_{n\to\infty} \Xi_\rho(\gamma^n)$ is the top eigenline of $\rho(\gamma)$, we may apply Lemma A.2 with $A = \rho(\gamma^n)$ and L the top eigenline; then $d(L, \Xi_\rho(\gamma^n)) \le C_\gamma e^{-\mu_\gamma n}$ for positive constants C_γ, μ_γ depending only on $\rho(\gamma)$; in particular, as $n \to \infty$, this bound goes to zero, so that $\lim_{n\to\infty} \Xi_\rho(\gamma^n) = L$ as desired. ☐

Proposition 6.21. *The maximal parabolic subgroups of Γ are precisely (conjugates of) peripheral subgroups.*

Proof. Suppose H is a maximal parabolic subgroup.

Observe that H cannot contain non-peripheral elements. Indeed, suppose $\gamma \in \Gamma$ is non-peripheral. From Lemma 5.6 and 6.20, $\rho(\gamma)$ is proximal, and $\lim_{n\to\infty} \Xi_\rho(\gamma^n)$ is the top eigenline of $\rho(\gamma)$. Similarly, $\rho(\gamma^{-1})$ is proximal, and $\lim_{n\to\infty} \Xi_\rho(\gamma^{-n})$ is the bottom eigenline of $\rho(\gamma)$. These are distinct (by proximality), and are both fixed by γ, so $\gamma \notin H$.

Hence every $\gamma \in H$ is peripheral.

Now, from the unique limits hypothesis in Definition 5.2, for any peripheral subgroup P, $\lim_{n\to\infty} \Xi_\rho(\eta_n) = \xi_\rho(P)$ for any sequence $(\eta_n) \subset P$, and so P fixes $\xi_\rho(P)$. By Lemma 6.16, P fixes no other point $\beta \in \Lambda_{rel}$: any such β is at some definite distance $\epsilon(\beta) > 0$ from $\xi(P)$, and hence by Lemma 6.16, sufficiently long words in P must move β off of itself. Hence every peripheral subgroup P is parabolic, and extends to some maximal parabolic subgroup $\hat{P}$.

Suppose $\hat{P} \smallsetminus P \neq \varnothing$, so that $\hat{P}$ also contain some non-identity element q of some other peripheral subgroup $Q \neq P$. By the torsionfree assumption, $\hat{P} \cap Q$ contains arbitrarily large powers of q. By the same argument as in the previous paragraph, this implies that $Q \subset \hat{P}$. But this contradicts the first part of the uniform transversality hypothesis which stipulates that $\xi_\rho(P) \neq \xi_\rho(Q)$.

Hence we must have $\hat{P} = P$, i.e. the maximal parabolic subgroups are exactly the peripheral subgroups, as desired. □

It follows from the above that the parabolic points in Λ_{rel} are precisely the peripheral fixed points.

6.2.6 Summary of argument

Proof of Theorem 6.9. Consider a representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ which is 1-dominated relative to a prescribed collection of peripheral subgroups $\mathcal{P}$, such that Γ contains at least one non-peripheral element.

ρ induces an action of Γ on the space of lines $\mathbf{P}(\mathbb{R}^d)$. Consider $\Lambda_{rel} \subset \mathbf{P}(\mathbb{R}^d)$. It is non-empty, compact and Γ-invariant (Proposition 6.11), and perfect (Proposition 6.17.)

The diagonal action of Γ on $\Lambda_{rel}^{(3)}$ is properly discontinuous (Proposition 6.18) and the diagonal action on $\Lambda_{rel}^{(2)}$ is cocompact (Proposition 6.19.)

Moreover the maximal parabolic groups are precisely the peripheral subgroups; by Theorem 3.20 and since conical limit points cannot be parabolic these are all bounded, and in particular the stabiliser of each bounded parabolic point is finitely-generated (Proposition 6.21.)

We summarize all of this in a statement that will be used again in the next section:

Proposition 6.22. *Given a representation $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ which is 1-dominated relative to $\mathcal{P}$, $\rho(\Gamma)$ acts on Λ_{rel} as a geometrically-finite convergence group, with $\mathcal{P}^\Gamma$ as the set of maximal parabolic subgroups.*

Hence, by Theorem 3.20, Γ is hyperbolic relative to $\mathcal{P}$. □

CHAPTER 7

Limit Maps

In this section, we prove that a relatively dominated representation $\rho : (\Gamma, \mathcal{P}) \to \mathrm{GL}(d, \mathbb{R})$ gives us a pair of limit maps from the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ into projective space and its dual.

In the case where $\mathcal{P} = \varnothing$, this recovers the limit maps from the Gromov boundary of the group into projective space and its dual that we obtain for an Anosov representation.

Definition 7.1. *Suppose Γ is hyperbolic relative to $\mathcal{P}$, and we have a pair of continuous maps $\xi : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)$ and $\xi^* : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^{d*})$.*

ξ and ξ^* are said to be **compatible** *if $\xi(\eta) \subset \theta(\eta)$ as linear subspaces for all $\eta \in \partial(\Gamma, \mathcal{P})$.*

ξ and ξ^* are said to be **transverse** *if $\xi(\eta) \oplus \theta(\eta') = \mathbb{R}^d$ for all $\eta \neq \eta'$.*

Given $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ such that $\rho(P)$ is a parabolic subgroup of $\mathrm{GL}(d, \mathbb{R})$ for each $P \in \mathcal{P}$, ξ and ξ^* are said to be **dynamics-preserving** *if*

(i) $\xi(\gamma^+) = (\rho(\gamma))^+$ and $\xi^*(\gamma^+)^\perp = (\rho^*(\gamma))^+$. for all nonperipheral $\gamma \in \Gamma$, where $\gamma^+ := \lim_{n\to\infty} \gamma^n \in \partial(\Gamma, \mathcal{P})$ and $\rho(\gamma)^+$ is the attracting eigenline for $\rho(\gamma)$, and

(ii) If $\partial P \in \partial(\Gamma, P)$ is the unique point associated to $P \in \mathcal{P}$, then $\xi(\partial P)$ is the parabolic fixed point associated to $\rho(P)$.

Theorem 7.2. *Given $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ 1-dominated relative to $\mathcal{P}$, we have well-defined, $\rho(\Gamma)$-equivariant, continuous maps $\xi_\rho : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)$ and $\xi^*_\rho : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^{d*})$ which are dynamics-preserving, compatible, and transverse.*

Proof. Recall that if $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is 1-dominated relative to $\mathcal{P}$, then Γ is hyperbolic relative to $\mathcal{P}$ by Theorem 6.9. Moreover, as noted in Proposition 6.22, $\rho(\Gamma) \curvearrowright \Lambda_{rel}$ as a geometrically-finite convergence group, with $\mathcal{P}^\Gamma$ as the set of maximal parabolic subgroups.

Yaman's criterion (Theorem 3.19) then gives us an equivariant homeomorphism

$$\xi_\rho : \partial(\Gamma, \mathcal{P}) \to \Lambda_{rel} \subset \mathbf{P}(\mathbb{R}^d).$$

By looking at the action of $\rho(\Gamma)$ on the dual vector space (recall §5.1 and in particular Proposition 5.4), we similarly obtain an equivariant homeomorphism

$$\xi_\rho^* : \partial(\Gamma, \mathcal{P}) \to \Lambda_{rel}^* \subset \mathrm{Gr}_{d-1}(\mathbb{R}^d).$$

Equivariance then combines with the other properties of our limit set Λ_{rel} to imply that ξ_ρ and ξ_ρ^* are dynamics-preserving. Here we state the arguments for ξ_ρ; via the dual representation ρ^* they also imply the claim for ξ_ρ^*.

For non-peripheral elements γ, the attracting eigenline $\rho(\gamma)^+$ is contained in Λ_{rel} (Lemma 6.20). Every point in $\mathbf{P}(\mathbb{R}^d)$—outside a hyperplane given by the orthogonal complement of $\rho(\gamma)^+$—is attracted to $\rho(\gamma)^+$ under the action of $\rho(\gamma)$. By the transversality properties of Λ_{rel}, there exist points of Λ_{rel} outside of this hyperplane, since said hyperplane is equal to the attracting hyperplane of $\rho^*(\gamma^{-1})$, and by Corollary 6.14 any point of Λ_{rel} other than $\rho(\gamma^{-1})^+$ is transverse to this.

Hence, by equivariance, we have that $\xi_\rho(\gamma^n\zeta) = \rho(\gamma^n)\xi_\rho(\zeta) \to \rho(\gamma)^+$ as $n \to \infty$, for an open set of $\zeta \in \Lambda_{rel}$, and so $\xi_\rho(\gamma^+) = \xi_\rho\left(\lim_{n\to\infty} \gamma^n\right) = \rho(\gamma)^+$.

For peripheral elements $\eta \in P$, the associated limit line $\xi_\rho(P)$ is contained in Λ_{rel} by the unique limits assumption. Since ξ is a homeomorphism, there is some $\zeta \in \partial(\Gamma, \mathcal{P})$ such that $\xi_\rho(\zeta) = \rho(\eta)^+$. By equivariance, $\xi_\rho(\eta^n\zeta) = \rho(\eta^n)\xi_\rho(\zeta) \to \rho(\eta)^+$ as $n \to \infty$. Hence $\xi_\rho(\eta^+) = \xi_\rho\left(\lim_{n\to\infty} \eta^n\right) = \rho(\eta)^+$.

To verify that ξ_ρ and ξ_ρ^* are compatible and transverse, we will show that ξ_ρ, ξ_ρ^* satisfy

$$\xi_\rho(x) = \lim_{n\to\infty} \Xi_\rho(\gamma_n) \qquad \xi_\rho^*(x) = \lim_{n\to\infty} \Xi_\rho^*(\gamma_n)$$

for $(\gamma_n) \in \Gamma$ any projected geodesic in Γ such that $\gamma_n \to x$, and Ξ_ρ and Ξ_ρ^* as in §6.2.

To see this, we note that if $x = \gamma^+ \in \partial(\Gamma, \mathcal{P})$ is a proximal limit point, then $\xi_\rho(x)$ is the top eigenline of $\rho(\gamma)$ since ξ is dynamics-preserving, and by Lemma 6.20 this is equal to $\lim_{n\to\infty} \Xi_\rho(\gamma^n)$. If $x = \partial P \in \partial(\Gamma, \mathcal{P})$ is a parabolic limit point, then by the dynamics-preserving property $\xi_\rho(x) = \xi_\rho(\eta^+)$ for any $\eta \in P$, and by the unique limits hypothesis $\xi_\rho(x) = \xi_\rho(\eta^+) = \lim_{n\to\infty} \Xi_\rho(\eta_n)$ for any sequence $\eta_n \to \infty$ in P.

More generally, given $x \in \partial(\Gamma, \mathcal{P})$ that is not a peripheral fixed point, suppose (γ_n) is a sequence (along a metric quasigeodesic path) such that no γ_n ends in a peripheral letter and $\gamma_n \to x$. Pick any peripheral element $\eta \in \bigcup \mathcal{P}$.

Then, writing $x_n := \lim_{m\to\infty} \gamma_n\eta^m$, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\lim_{m\to\infty} \gamma_n\eta^m = \lim_{n\to\infty} \gamma_n = x$ (once n and m are large enough, by Lemma 6.12 the sequences involved may be taken to be uniform quasigeodesics.)

By continuity, $\xi_\rho(x) = \lim_{n\to\infty} \xi_\rho(x_n)$; we then have

$$\xi_\rho(x) = \lim_{n\to\infty} \xi_\rho(x_n) = \lim_{n\to\infty} \lim_{m\to\infty} \Xi_\rho(\gamma_n\eta^m) = \lim_{n\to\infty} \Xi_\rho(\gamma_n)$$

where the last equality follows from Corollary 6.5 (because the $\gamma_n\eta^m$ may be taken to be uniform quasigeodesics) and the triangle inequality:

$$\begin{aligned} d(\Xi(\gamma_n), \xi(x)) &\leq d\left(\Xi(\gamma_n), \Xi(\gamma_n\eta^m)\right) + d\left(\Xi(\gamma_n\eta^m), \xi(x_n)\right) + d\left(\xi(x), \xi(x_n)\right) \\ &\leq \hat{C}e^{-\hat{\mu}n} + \hat{C}e^{-\hat{\mu}m} + d\left(\xi(x), \xi(x_n)\right) \end{aligned}$$

and all of the terms that appear in the last line can be made arbitrarily small by taking (m and then) n sufficiently large.

We have written the argument above for ξ_ρ; the argument for ξ_ρ^* is entirely analogous.

The compatibility of ξ_ρ and ξ_ρ^* then follows since $\Xi_\rho(\gamma_n) \subset \Xi_\rho^*(\gamma_n)$ for all n by definition; the transversality of ξ_ρ and ξ_ρ^* follows from Corollary 6.14. □

Remark 7.3. We may alternatively prove this by defining the limit maps using

$$\xi_\rho(x) = \lim_{n\to\infty} \Xi_\rho(\gamma_n) \qquad\qquad \xi_\rho^*(x) = \lim_{n\to\infty} \Xi_\rho^*(\gamma_n)$$

for $(\gamma_n) \in \Gamma$ any projected geodesic in Γ such that $\gamma_n \to x$, as in [GGKW17], and directly showing, using arguments similar to those above and earlier in the paper, that these maps satisfy the desired properties. From the analysis above these will turn out to be equivalent to the limit maps supplied by Yaman's criterion.

CHAPTER 8

Examples

For a start, we observe that dominated representations are relatively dominated relative to $\mathcal{P} = \varnothing$, since in that case we have $|\cdot|_c = |\cdot|$. We will now show that geometrically finite subgroups of $\mathrm{SO}(1,d)$ and geometrically finite convex projective holonomies, in the sense of [CM14a], give examples of relatively dominated representations.

8.1 In rank one

In rank one, the relatively dominated condition coincides with the more classical notion of geometric finiteness. Here we will illustrate the particular example of geometrically finite real hyperbolic manifold holonomies; the arguments for the more general case are similar.

Example 8.1. Let M be a geometrically finite hyperbolic d-manifold, $\Gamma = \pi_1 M$, and $\rho : \Gamma \to \mathrm{PSO}(d,1) \subset \mathrm{PSL}(d+1,\mathbb{R})$ be its holonomy representation.

In this case we know that Γ is hyperbolic relative to the cusp stabilizers $\mathcal{P}$, and that the relative Cayley graph quasi-isometrically embeds into $\mathbb{H}^d$ (see Example 3.11.)

The quasi-isometric embedding of the relative Cayley graph immediately gives us both lower and upper domination inequalities (D$^{\pm}$), since $\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) = \frac{1}{2}\frac{\sigma_1}{\sigma_{d+1}}(\rho(\gamma))$ for any $\gamma \in \Gamma$, and there exists a basepoint $o \in \mathbb{H}^d$ so that $d(o, \rho(\gamma)\cdot o) = \log \frac{\sigma_1}{\sigma_{d+1}}(\rho(\gamma))$ for all $\gamma \in \Gamma$.

The unique limits condition is satisfied since each cusp stabilizer is parabolic; the quadratic gaps condition is satisfied in the peripherals since, by a direct computation (see Example 3.7 and Proposition 3.22),

$$\left|\log \frac{\sigma_1}{\sigma_2}(\rho(\eta)) - 2\log n\right| = |d(o, \rho(\eta)\cdot o) - 2\log n| \leq C_\eta$$

for any parabolic element η, where C_η is a constant depending on η. Conjugation changes this by a fixed additive constant, and we may take a uniform choice of such constant. The quadratic gaps condition is then satisfied in full, due to the following argument:

Definition 8.2. *We say that $\rho : (\Gamma, \mathcal{P}) \to \mathrm{PGL}(d, \mathbb{R})$ admits* **good limit maps** *if $\xi_\rho : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)$ given by $\lim_{n\to\infty} \gamma_n \mapsto \lim_{n\to\infty} \Xi_\rho(\gamma_n)$ and*

$\xi^*_\rho : \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)^*$ given by $\lim_{n\to\infty} \gamma_n \mapsto \lim_{n\to\infty} \Xi^*_\rho(\gamma_n)$ are well-defined, continuous, $\rho(\Gamma)$-equivariant, compatible, dynamics-preserving and transverse.

We note that in our case ρ admits good limit maps, with the image of ξ_ρ being, up to conjugation in $\mathrm{PSL}(d + 1, \mathbb{R})$, the limit set in the boundary of the Beltrami–Klein projective ball model of hyperbolic d-space in $\mathbf{P}(\mathbb{R}^{d+1})$, and the image of ξ^*_ρ consisting of hyperplanes tangent to the boundary.

Proposition 8.3. *Suppose $\rho : (\Gamma, \mathcal{P}) \to \mathrm{PGL}(d, \mathbb{R})$ admits good limit maps, and the quadratic gaps condition is satisfied for peripheral elements $\eta \in \bigcup \mathcal{P}$.*

Then the peripherals satisfy the quadratic gaps condition in full.

Proof. Given a geodesic $\gamma\eta$ where η is peripheral, Lemma A.3 gives us

$$\frac{\sigma_1}{\sigma_2}(\rho(\gamma\eta)) \geq \delta^2 \cdot \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \cdot \frac{\sigma_1}{\sigma_2}(\rho(\eta)),$$

where $\delta := \sin \angle(\Xi(\eta), \Xi^*(\gamma^{-1}))$; we then obtain the quadratic gaps condition for $\gamma\eta$ by using the transversality of the limit maps to obtain a uniform positive lower bound on δ and observing that $\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq 1$. More precisely: suppose no such δ exists; then we have a sequence of metric quasi-geodesics $\gamma_n\eta_n$, with η_n peripheral such that $\sin \angle(\Xi(\eta_n), \Xi * (\gamma_n^{-1})) \geq 2^{-n}$. Up to subsequence, these converge to some bi-infinite metric quasigeodesic $\gamma_\infty\eta_\infty$ with $\sin \angle(\xi(\eta_\infty), \xi^*(\gamma_\infty^{-1})) = 0$; but this is in contradiction with the transversality of the limit maps. □

The uniform transversality condition is also satisfied due to good limit maps, by the following

Proposition 8.4. *Suppose $\rho : (\Gamma, \mathcal{P}) \to \mathrm{PGL}(d, \mathbb{R})$ admits good limit maps. Then the uniform transversality hypothesis from Definition 5.2 is satisfied.*

Proof. By the transversality of the limit maps, $\gamma(g^{-1}v_1(P), hW_{d-1}(P')) > 0$. To obtain the *uniform* version of this hypothesis, suppose we have sequences $(\gamma_n), (\eta_n) \subset \Gamma$ and peripheral subgroups P, P' such that $\angle(\gamma_n^{-1}v_1(P'), \eta_n W_{d-1}(P)) < 2^{-n}$. Up to a subsequence, the γ_n^{-1} converge to some infinite (projected quasi-)geodesic $\gamma^{-1} : \mathbb{N} \to \Gamma$, and the η_n to some infinite (projected quasi-)geodesic $\eta : \mathbb{N} \to \Gamma$ and $\angle(\xi_\rho(\gamma^{-1}), \xi^*_\rho(\eta)) = 0$; but this contradicts transversality. □

8.2 A higher rank example

In higher rank, we have holonomies of geometrically-finite convex projective n-manifolds, in the sense of [CM14a]:

Definition 8.5 ([CM14a], Définition 1.5 and Théorème 1.3)**.** *Let $\Omega \subset \mathbf{P}(\mathbb{R}^{d+1})$ be a strictly convex domain with C^1 boundary. A finitely-generated discrete subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$ is* **geometrically finite** *if the 1-neighborhood of the convex core $\overline{C(\Lambda_\Gamma)/\Gamma} \subset \Omega/\Gamma$ is of finite volume.*

Proposition 8.6. *Let M be a d-manifold and write $\Gamma = \pi_1 M$. Suppose $\rho : \Gamma \to \mathrm{PGL}(d+1, \mathbb{R})$ is a geometrically-finite convex projective holonomy representation. Then ρ is 1-dominated relative to its cusp stabilizers.*

Proof. Let $\Omega := \tilde{M}$; this is a strictly convex domain in $\mathbf{P}(\mathbb{R}^{d+1})$ with C^1 boundary, and hence δ-hyperbolic given the Hilbert metric. Γ is hyperbolic relative to its cusp stabilizers $\mathcal{P}$, and acts on its limit set $\Lambda_\Gamma \subset \partial\Omega$ of accumulation points as a geometrically-finite convergence group ([CM14a], Théorème 1.9.)

In fact Λ_Γ, as well as the dual limit set $\Lambda_\Gamma^* \subset \mathbf{P}(\mathbb{R}^{d+1})^*$, may be equivariantly identified with $\partial(\Gamma, \mathcal{P})$, giving us continuous, compatible, dynamics-preserving limit maps; in particular $\xi_\rho^*(x)$ is tangent to $\partial\Omega$ at $\xi_\rho(x)$. This gives us the unique limits condition. Since $\partial\Omega$ is strictly convex and C^1, these limit maps are transverse. This gives us, via Proposition 8.4, the uniform transversality condition.

By [CLT15], Theorem 0.5, all of the peripheral elements $\eta \in \bigcup \mathcal{P}$ have image $\rho(\eta)$ projectively equivalent to an element in the holonomy of a hyperbolic cusp; in particular (cf. Example 8.1), we have quadratic gaps in the peripheral subgroups, and hence, by Proposition 8.3, the quadratic gaps condition in full.

We now claim that the orbit map is a relative quasi-isometric embedding from (Γ, d_c) into (Ω, d_Ω), where d_Ω denotes the Hilbert metric on Ω, and $d_\Omega(o, \gamma \cdot o) = \log \frac{\sigma_1}{\sigma_{d+1}}(\rho(\gamma))$ for all $\gamma \in \Gamma$.

To establish this, we observe that

- since the cusps are projectively equivalent, and hence isometric, to hyperbolic cusps, we have a system of disjoint horoballs $\mathcal{N}$ of Ω, with boundaries the images of cusp stabilizers, which is quasi-isometric to our system of combinatorial horoballs following the computations in Examples 8.1 / 3.7;
- the cocompact action of $\rho(\Gamma)$ on the compact core of $\tilde{M}$ as a geometrically-finite convex projective manifold gives, by the Milnor-Švarc lemma, a quasi-isometry from $\mathrm{Cay}(\Gamma)$ with the word metric to the truncated domain $\Omega \setminus \mathcal{N}$.

Then we may apply the same argument as in Example 3.11, using Lemma 3.12, to obtain our relative quasi-isometric embedding.

Finally, by [CM14b], Proposition 7.2, Corollaire 7.3 and Lemme 7.6, there exists $\epsilon = \epsilon(\rho) > 0$ such that $\log \frac{\lambda_1}{\lambda_2}(\rho(\gamma)) \geq \epsilon \log \frac{\lambda_1}{\lambda_{d+1}}(\rho(\gamma))$ for all non-peripheral $\gamma \in \Gamma$: more precisely, Lemme 7.6

bounds the ratio $\log \frac{\lambda_1}{\lambda_2}(\rho(\gamma)) \cdot \left(\log \frac{\lambda_1}{\lambda_{d+1}}(\rho(\gamma))\right)^{-1}$ from below by an auxiliary quantity $\frac{1}{2}\chi(\gamma)$ (half the top Lyapunov exponent for the Hilbert geodesic flow corresponding to $\rho(\gamma)$); Proposition 7.2 and Corollaire 7.3 together give us $\epsilon > 0$ (coming from the Hölder regularity of the boundary $\partial\Omega$) such that $\frac{1}{2}\chi(\gamma) > \left(1 + \frac{1}{\epsilon}\right)^{-1}$

We may then show that there exists $\epsilon' = \epsilon'(\rho) > 0$ such that $\log \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) > \epsilon' \log \frac{\sigma_1}{\sigma_{d+1}}(\rho(\gamma)) + \hat{C}_\rho$ where $\hat{C}_\rho$ is some constant depending only on the representation; this last inequality. which suffices to establish the lower domination inequality (D^-), will follow from the inequality with the eigenvalue gaps, together with results of [AMS95] and [Ben97] (as tied together in [GGKW17], Theorem 4.12):

Specifically, by [CM14a], Théorème 7.28, we may assume that ρ is strongly irreducible and Zariski-dense. Then [GGKW17], Theorem 4.12 states that there is a finite subset $F \subset \Gamma$ such that for any $\gamma \in \Gamma$ there exists $f \in F$ such that

$$\log \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq \log \frac{\lambda_1}{\lambda_2}(\rho(\gamma f)) - C_\rho$$

where C_ρ is some constant depending only on ρ, and similarly

$$\log \frac{\lambda_1}{\lambda_{d+1}}(\rho(\gamma f)) \geq \log \frac{\sigma_1}{\sigma_{d+1}}(\rho(\gamma)) - C_\rho,$$

and putting all of these inequalities together we obtain

$$\begin{aligned}\log \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) &\geq \log \frac{\lambda_1}{\lambda_2}(\rho(\gamma f)) - C_\rho \geq \epsilon \log \frac{\lambda_1}{\lambda_{d+1}}(\rho(\gamma f)) - C_\rho \\ &\geq \epsilon \log \frac{\sigma_1}{\sigma_{d+1}}(\rho(\gamma)) - (\epsilon + 1)C_\rho\end{aligned}$$

as desired. □

CHAPTER 9

Relation to Kapovich–Leeb

In [KL18], Kapovich and Leeb develop a number of possible relative analogues of Anosov representations. Here we describe how some of these are related to the notion of relatively dominated subgroups described here.

The definitions in [KL18] are formulated in terms of discrete subgroups $\Gamma \leq G$ of semisimple Lie groups G; we reformulate them in terms of discrete and faithful representations, and in the specific case of $G = \mathrm{SL}(d, \mathbb{R})$.

We also remark that the choice of a model Weyl chamber τ_{mod} in [KL18] is equivalent to the choice of a Cartan projection / set of roots, and in particular all of the definitions below are formulated in the specific case of the first and last simple roots $\left\{\log \frac{\sigma_1}{\sigma_2}, \log \frac{\sigma_{d-1}}{\sigma_d}\right\}$.

Below, given a representation $\rho : \Gamma \to G$, we let Λ_Γ denote the limit set of $\rho(\Gamma) \subset G$ in the flag variety $G/P_{1,d-1}$ corresponding to our chosen set of simple roots: a point in $G/P_{1,d-1}$ corresponds to a pair $(\xi, \xi^*) \in \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^d)^*$ such that the line corresponding to ξ is contained in the hyperplane represented by ξ^*. More specifically, Λ_Γ is the closure of the set of accumulation points $(\xi, \xi^*) = \lim_{n\to\infty} \left(\Xi_\rho(\gamma_n)), \Xi^*_\rho(\gamma_n))\right)$ for sequences $\gamma_n \to \infty$.

9.1 Relatively dominated implies relatively RCA

Definition 9.1 ([KL18], Definition 7.6)**.** $\rho : \Gamma \to G = \mathrm{SL}(d, \mathbb{R})$ *is* **relatively RCA** *if*

- *(regularity)* $\log \frac{\sigma_1}{\sigma_2}(\rho(\gamma_n)) \to \infty$ *for all sequences* $(\gamma_n)_{n\in\mathbb{N}}$ *going to infinity in* Γ.
- *(convergence) every point in* Λ_Γ *is either a conical limit point or a bounded parabolic point, and the stabilizers of the bounded parabolic points are finitely generated.*
- *(antipodality)* Λ_Γ *is antipodal, i.e. every pair of points in the limit set (has a pair of lifts which) can be joined by a bi-infinite geodesic in the symmetric space* $\mathrm{SL}(d, \mathbb{R})/\mathrm{SO}(d)$.

We remark that, roughly speaking, the relatively dominated condition (Definition 5.3) may be seen as strengthening the regularity hypothesis while weakening the convergence and antipodality hypotheses. There is also a more subtle distinction involving the role of the intrinsic geometry of Γ, which we elaborate on more in the next subsection.

We also remark that projecting $\Lambda_\Gamma \subset \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^d)^*$ to the first coordinate yields the limit set Λ_{rel} from §6.2 above.

Definition 9.2 ([KL18], Definition 7.1)**.** *A subgroup $\Gamma \leq G$ is* **relatively asymptotically embedded** *if it satisfies the regularity and antipodality conditions (as in the previous Definition), and admits a relatively hyperbolic structure $(\Gamma, \mathcal{P})$ such that there exists a Γ-equivariant homeomorphism $\partial_\infty(\Gamma, \mathcal{P}) \to \Lambda_\Gamma$.*

Theorem 9.3 ([KL18], Theorem 7.8)**.** *ρ is relatively RCA if and only if $\rho(\Gamma)$ is relatively asymptotically embedded.*

In particular, if $\rho : \Gamma \to G$ is relatively RCA then Γ is relatively hyperbolic. Below, we will use the notions of relative RCA and relative asymptotic embeddedness interchangeably.

Theorem 9.4. *If $\rho : \Gamma \to G$ is relatively dominated, then $\rho(\Gamma)$ is relatively asymptotically embedded.*

Proof. Regularity is immediate from the lower domination inequality (D^-) and the quasi-equivalence of $|\gamma|_c$ and $\|a(\rho(\gamma)\|$ (Proposition 5.9.)

Antipodality follows from transversality: given two points $\xi_\pm$ in the limit set, consider the associated hyperplanes $\theta_\pm$; then, by transversality we have a decomposition $\mathbb{R}^d = \xi_+ \oplus (\theta_+ \cap \theta_-) \oplus \xi_-$, which gives a bi-infinite geodesic joining the simplices associated to $(\xi_\pm, \theta_\pm)$ in the associated flag variety $G/P_{1,d-1}$—concretely, pick a diagonal matrix $A \in \mathrm{SL}(d, \mathbb{R})$ respecting that decomposition, and consider the bi-infinite geodesic $\exp(tA)$.

Asymptotic embeddedness follows from Theorem 7.2 on the limit maps: more precisely, we can combine both limit maps from that Theorem into a single limit map (ξ, ξ^*) into the flag manifold corresponding to our choice of τ_{mod}, and this single limit map gives us our asymptotic embedding. □

9.2 Uniform regularity and distortion, and equivalence of notions

Definition 9.5 ([KL18], §4.4.1)**.** *Γ is* **uniformly regular** *if there exist constants $\mu, c > 0$ such that $\log \frac{\sigma_1}{\sigma_2}(\rho(\gamma_n)) \geq \mu \|a(\rho(\gamma_n))\| - c$ for all $(\gamma_n) \subset \Gamma$ going to infinity.*

Definition 9.6. *Suppose Γ is hyperbolic relative to $\mathcal{P}$ and we have a representation $\rho : \Gamma \to G$. We say Γ (or any subgroup $H \leq \Gamma$) is* **relatively undistorted by** *ρ if ρ induces (via any orbit map) a quasi-isometric embedding of the relative Cayley (sub)graph (cf. Proposition 5.9) into the symmetric space, i.e. the cusped word-length $|\gamma|_c$ and the norm $\|a(\rho(\gamma))\|$ are quasi-equivalent for all $\gamma \in \Gamma$ (resp., for all $\gamma \in H$).*

Remark 9.7. Uniform regularity does not necessarily entail undistortedness: e.g. consider a hyperbolic mapping torus $\Gamma \subset \mathrm{SO}^+(1,3) \subset \mathrm{SL}(4,\mathbb{R})$ which is abstractly isomorphic to $\pi_1\Sigma_g \rtimes \mathbb{Z}$; the fiber groups (abstractly isomorphic to $\pi_1\Sigma_g$) are exponentially distorted. Γ, being a geometrically finite subgroup of $\mathrm{SO}^+(1,3)$, is uniformly regular (and undistorted by the inclusion map); the fiber groups, being exponentially distorted subgroups, are not quasi-isometrically embedded and hence not undistorted by the inclusion map. However, they remain uniformly regular, since this is a condition purely on the Cartan projections and independent of word-length.

Definition 9.8. *We say $\rho : \Gamma \to G$ is* **relatively uniform RCA and undistorted** *if it satisfies the convergence and antipodality conditions, and moreover $\rho(\Gamma)$ is uniformly regular and Γ is relatively undistorted by ρ.*

Theorem 9.9 ([KL18], Theorem 8.25)**.** *ρ is relatively uniform RCA and undistorted if and only if it is relatively asymptotically embedded with uniformly regular peripheral subgroups and Γ is relatively undistorted by ρ.*

Remark 9.10. We can in fact strengthen Theorem 9.4 to say that if $\rho : \Gamma \to G$ is relatively dominated, then $\rho(\Gamma)$ is relatively uniform RCA and undistorted, since, via Proposition 5.9, (D^-) is precisely the uniform regularity and undistortedness condition.

Remark 9.11. In the non-relative case, uniform regularity and undistortedness (URU) is equivalent to RCA [KLP16]. The proof goes through the notion of Morse subgroups and in particular requires some version of a higher-rank Morse lemma.

Theorem 9.12. *If $\rho : \Gamma \to G$ is such that $\rho(\Gamma)$ is relative uniform RCA and undistorted with peripherals also satisfying the quadratic gaps condition, then ρ is relatively dominated.*

Proof. Relative uniform RCA implies relative hyperbolicity of the source group (via Theorem 9.3); this immediately gives us (RH).

As noted in Remark 9.10, (D^-) is exactly the uniform regularity and undistortedness condition.

It remains to check that the hypotheses in Definition 5.2 are satisfied. The quadratic gaps condition has been assumed. Upper domination follows from [KL18], Corollary 5.13. Unique limits follow from the relative asymptotic embedding; by Proposition 8.4, so does uniform transversality. □

CHAPTER 10

Extending the Definition

As above let Γ be a finitely-generated group which is hyperbolic relative to some finite collection $\mathcal{P}$ of finitely-generated subgroups satisfying (RH) (Definition 5.1.)

We say that a representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is 1-relatively dominated (with domination constants $(\underline{C}, \underline{\mu}, \bar{C}, \bar{\mu})$) if it is the composition of a 1-relatively dominated representation $\hat{\rho} : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ (with the same domination constants) with the natural projection map $\pi : \mathrm{GL}(d, \mathbb{R}) \to \mathrm{PGL}(d, \mathbb{R})$, or more generally if we can find a group $\hat{\Gamma}$, a 2-to-1 homomorphism $f : \hat{\Gamma} \to \Gamma$, and a 1-dominated representation $\hat{\rho} : \hat{\Gamma} \to \mathrm{GL}(d, \mathbb{R})$ such that $\pi \circ \hat{\rho} = \rho \circ f$ (cf. [BPS19], Remark 3.4.)

Alternatively, we can continue to use Definitions 5.2 and 5.3, since *ratios* of singular values remain unchanged under the reductions considered here, and we can continue to work with the same symmetric space and flag spaces.

By considering the associated representations to $\mathrm{GL}(d, \mathbb{R})$, we have that Γ is hyperbolic relative to $\mathcal{P}$ in these cases as well (Theorem 6.9) and we have associated continuous, equivariant, dynamics-preserving, transverse limit maps (Theorem 7.2.) By considering the associated representations, or by working directly with the hypotheses in Definitions 5.2 and 5.3, the results from §5.2 and 5.3 continue to hold.

More generally, we may use the following standard fact from the representation theory of semisimple Lie groups:

Theorem 10.1 (cf. [GW12], Proposition 4.3 and Remark 4.12)**.** *Given G a semisimple Lie group with finite center and P a parabolic subgroup of G, there exists a finite dimensional irreducible representation $\phi = \phi_{G,P} : G \to \mathrm{SL}(V)$ such that $\phi(P)$ is the stabilizer (in $\phi(G)$) of a line in V.*

ϕ induces maps $\beta : G/P \to \mathbf{P}(V)$ and $\beta^* : G/Q \to \mathbf{P}(V^*)$, where Q is the opposite parabolic to P.

Moreover, if P is non-degenerate, then $\ker \phi = Z(G)$ and ϕ is an immersion.

For a construction, we refer the reader to [GW12], §4 (see also [BCLS15], Theorem 2.12 and Corollary 2.13.) The irreducible representation $\phi_{G,P}$ is called a Plücker representation in [BCLS15], or a Tits representation in [BPS19].

We now make the following

Definition 10.2. *Given Γ a finitely-generated group and $\mathcal{P}$ a finite collection of finitely-generated proper infinite subgroups satisfying (RH), G a semisimple Lie group with finite center and P a non-degenerate parabolic subgroup of G, we say that a representation $\rho : \Gamma \to G$ is P-dominated relative to $\mathcal{P}$ (with domination constants $(\underline{C}, \underline{\mu}, \bar{C}, \bar{\mu})$) if $\phi_{G,P} \circ \rho : \Gamma \to \mathrm{SL}(V)$ is 1-dominated relative to $\mathcal{P}$ (with the same constants).*

Given a P-relatively dominated representation $\rho : \Gamma \to G$, by applying Theorem 6.9 to $\phi_{G,P} \circ \rho : \Gamma \to \mathrm{SL}(V)$, we have that Γ is hyperbolic relative to $\mathcal{P}$ in these cases as well. By Theorem 7.2, $\phi_{G,P} \circ \rho$ has associated continuous, equivariant, dynamics-preserving, transverse limit maps of $\partial(\Gamma, \mathcal{P})$ into $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$; we may compose these with β^{-1} and $(\beta^*)^{-1}$ to obtain limit maps of $\partial(\Gamma, \mathcal{P})$ into the flag varieties G/P and G/Q. We may argue similarly to see that the results from §5.2 and 5.3 continue to hold.

As a particular case of this, suppose $G = \mathrm{SL}(d, \mathbb{R})$ and $P = P_k$ is the stabilizer of a k-plane in G. Then we may explicitly take $V = \bigwedge^k \mathbb{R}^d$ and $\phi_{G,P} : \mathrm{SL}(d, \mathbb{R}) \to \mathrm{SL}(V)$ to be the map given by the action of $\mathrm{SL}(d, \mathbb{R})$ on the exterior product V coming from the natural action $\mathrm{SL}(d, \mathbb{R}) \curvearrowright \mathbb{R}^d$.

We note, very briefly, that

$$\sigma_1(\bigwedge^k \rho(\gamma)) = \sigma_1 \cdots \sigma_k(\rho(\gamma)),$$
$$\sigma_2(\bigwedge^k \rho(\gamma)) = \sigma_1 \cdots \sigma_{k-1}\sigma_{k+1}(\rho(\gamma)),$$

and moreover $U_1(\bigwedge^k \rho(\gamma)) = U_k(\rho(\gamma))$ (in the sense that they represent the same k-dimensional subspace of $\mathbb{R}^d$) and

$$S_{D-1}(\bigwedge{}^k \rho(\gamma)) = U_{D-1}(\wedge^k \rho(\gamma^{-1})) = \langle \theta \in \mathrm{Gr}_k(\mathbb{R}^d) : \theta \not\pitchfork S_{d-k}(\rho(\gamma)) \rangle$$

(where $D := \binom{d}{k} = \dim \bigwedge^k \mathbb{R}^d$) and hence we may also equivalently and more directly define P_k-relatively dominated representations as in §5, replacing $\frac{\sigma_1}{\sigma_2}$ with $\frac{\sigma_k}{\sigma_{k+1}}$ as appropriate, and similarly replacing projective space and its dual with the appropriate Grassmannians.

APPENDIX A

Linear Algebraic Lemmas

We collect in this appendix various lemmas of quantitative linear algebra which are used in the proofs above and below, especially in sections 6.2 and 7. They appear in the order in which they are used above. These are elementary; many of them appear, with proof, in Appendix A of [BPS19].

Recall that, given $\xi, \eta \in \mathbf{P}(\mathbb{R}^d)$ or $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$, or more generally $\mathrm{Gr}_p(\mathbb{R}^d)$ for some p between 1 and d, $d(\xi, \eta)$ will denote distance between ξ and η in the relevant Grassmannian.

Below, we say that $A \in \mathrm{GL}(d, \mathbb{R})$ is P_p**-proximal** if $\sigma_{p+1}(A) > \sigma_p(A)$. Recall that $U_p(A)$ is well-defined once A is P_p-proximal.

Lemma A.1 ([GGKW17], Lemma 5.8; [BPS19], Lemmas A.4, A.5)**.** *Given $A, B \in \mathrm{GL}(d, \mathbb{R})$ with A, AB and BA P_p-proximal, we have*

$$d(U_p(A), U_p(AB)) \leq \frac{\sigma_1}{\sigma_d}(B)\frac{\sigma_{p+1}}{\sigma_p}(A) \tag{A.1}$$

$$d(BU_p(A), U_p(BA)) \leq \frac{\sigma_1}{\sigma_d}(B)\frac{\sigma_{p+1}}{\sigma_p}(A) \tag{A.2}$$

Lemma A.2 ([BPS19], Lemma A.6)**.** *Given any P_p-proximal $A \in \mathrm{GL}(d, \mathbb{R})$, and any p-dimensional subspace $P \subset \mathbb{R}^d$, we have*

$$d(A(P), U_p(A)) \leq \frac{\sigma_{p+1}}{\sigma_p}(A)\frac{1}{\sin \angle(P, S_{d-p}(A))}.$$

Lemma A.3 ([BPS19], Lemma A.7)**.** *Let $A, B \in \mathrm{GL}(d, \mathbb{R})$. Suppose that A and AB are P_p-proximal, and let $\alpha := \angle(U_p(B), S_{d-p}(A))$. Then*

$$\sigma_p(AB) \geq (\sin \alpha)\, \sigma_p(A)\sigma_p(B)$$

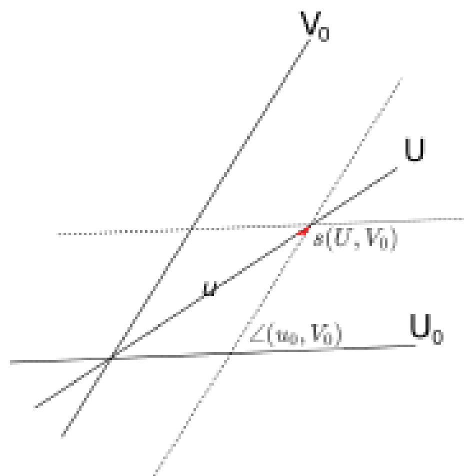
$$\sigma_{p+1}(AB) \leq (\sin \alpha)^{-1}\sigma_{p+1}(A)\sigma_{p+1}(B)$$

Lemma A.4 (cf. [QTZ19], Lemma A.24)**.** *If U_0 and V_0 are complementary vector subspaces, and U is the graph of $\Theta : U_0 \to V_0$, then we have*

$$s(U, V_0) \geq \frac{s(U_0, V_0)}{\| \operatorname{id} \oplus \Theta \|}.$$

Proof. Choose a vector $u \in U$ achieving the minimum gap $s(U, V_0)$. Scale it so that if we decompose u into its U_0 and V_0 components, its U_0 component u_0 is a unit vector. From the law of sines,

$$\frac{1}{s(U, V_0)} = \frac{\|u\|}{\sin \angle(u_0, V_0)} \leq \frac{\|u\|}{s(U_0, V_0)} \leq \frac{\| \operatorname{id} \oplus \Theta \|}{s(U_0, V_0)}$$



whence we have the desired inequality (see also illustration above.) □

Lemma A.5 (cf. [QTZ19], Lemma A.24)**.** *If U_0 and V_0 are complementary vector subspaces, and U is the graph of $\Theta : U_0 \to V_0$, then we have*

$$s(U, V_0) \leq \frac{1}{\| \operatorname{id} \oplus \Theta \|}.$$

Proof. Pick a vector $u \in U$ so that if we decompose u into its U_0 and V_0 components, its U_0 component u_0 is a unit vector, and $\|u\| = \| \operatorname{id} \oplus \Theta \|$. By the law of sines,

$$\frac{\| \operatorname{id} \oplus \Theta \|}{\sin \angle(u_0, V_0)} = \frac{\|u\|}{\sin \angle(u_0, V_0)} = \frac{1}{\sin \angle(u, V_0)}$$

but now

$$\| \operatorname{id} \oplus \Theta \| \leq \frac{\| \operatorname{id} \oplus \Theta \|}{\sin \angle(u_0, V_0)} = \frac{1}{\sin \angle(u, V_0)} \leq \frac{1}{s(U, V_0)},$$

whence the desired inequality. □

APPENDIX B

A Local Version of Quas–Thieullen–Zarrabi

The purpose of this appendix is to prove Theorem 6.3:

Theorem B.1 (Theorem 6.3)**.** *Let $(A_k)_{k\in\mathbb{Z}} \subset \mathrm{GL}(d,\mathbb{R})$ be a sequence of matrices such that there exists constants $C \geq 1$ and $\mu, \mu' \geq 0$, with $\frac{1}{\mu}\log 3C > 1$, such that the following axioms are satisfied:*

- *(SVG-BG) for all $k \in \mathbb{Z}$ and all $n \geq 0$,*

$$\frac{\sigma_2}{\sigma_1}(A_{k+n-1}\cdots A_k) \leq Ce^{-n\mu}$$

- *(EC) for all $k \in \mathbb{Z}$ and all $n \geq 0$,*

$$d(S_{d-1}(A_{k+n-1}\cdots A_k), S_{d-1}(A_{k+n}\cdots A_k)) \leq Ce^{-n\mu},$$
$$d(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-(n+1)})) \leq Ce^{-n\mu}.$$

- *(FI)$_{back}$: for all $k \leq 0$ and $n, m \geq 0$*

$$\frac{\sigma_1(A_{k+n-1}\cdots A_{k-m})}{\sigma_1(A_{k+n-1}\cdots A_k)\cdot\sigma_1(A_{k-1}\cdots A_{k-m})} \geq C^{-1}e^{-m\mu'}$$

Then

(i) for each $k \in \mathbb{Z}$ in the sequence we have a splitting $E^u \oplus E^s$ of $\mathbb{R}^d$ given by

$$E^u(k) := \lim_{n\to\infty} U_1(A_{k-1}\cdots A_{k-n})$$
$$E^s(k) := \lim_{n\to\infty} S_{d-1}(A_{k+n-1}\cdots A_k)$$

which is equivariant in the sense that $A_kE^*(k) = E^*(k+1)$ for all $k \in \mathbb{Z}$ and $* \in \{u, s\}$;

(ii) moreover, for all $k \leq 0$, we have a uniform lower bound $s_{min} = s_{min}(C, \mu, \mu')$ on the gap $s(E^u(k), E^s(k)) := \sin \angle(E^u(k), E^s(k))$ given by

$$s(E^u(k), E^s(k)) \geq s_{min} := \frac{2}{3}(3e)^{-2r} \exp\left(-\frac{3/2}{1-e^{-\mu}}\right) C^{-(1+2r)},$$

where $r := \frac{\mu'}{\mu}$.

This statement is a mild generalization of a specific case of the main result of [QTZ19]; it is the particular statement which is needed above. In particular, here we are working with finite-dimensional real vector spaces, and hence many of the technical difficulties in [QTZ19], which works in the more general case of Banach spaces, are significantly lightened.

We further remark, as noted earlier in the Introduction, that the main result of [QTZ19] is itself a generalization of the Oseledec multiplicative ergodic theorem; for background on this important result in dynamical systems, we refer the reader to [Fil19].

We also deal only with the specific case where the singular value gap/s are at $p = 1$ and $p = d - 1$, and $A_k \in \mathrm{GL}(d, \mathbb{R})$; these assumptions are natural in the application we have here.

Remark B.2. We can also follow the arguments of [QTZ19] to obtain a domination statement:

(iii) there exists $n_{\min}$ depending only on C, μ, μ' such that for all $k \leq 0$ and $n \geq n_{\min}$ with $k + n \leq 0$,

$$\frac{\left\|A_{n+k-1} \cdots A_k|_{E^s(k)}\right\|}{\mathbf{m}(A_{n+k-1} \cdots A_k|_{E^u(k)})} \leq \frac{16C}{9s_{min}^2} e^{-n\mu}.$$

where $\mathbf{m}(A)$ denotes the bottom singular value of $A \in \mathrm{GL}(d, \mathbb{R})$. We will not include the proof here, since we do not use this conclusion above.

We introduce some notation which will be useful below: write

- $A(k, n)$ for the product $A_{k+n-1} \cdots A_k$,
- $\sigma_i(k, n)$ as shorthand for $\sigma_i(A(k, n))$,
- $\tilde{U}(k, n) := U_1(A(k - n, n)) = A(k - n, n)U(k - n, n)$ and $V(k, n) = S_{d-1}(k, n)$.

We remark that, with these notations, we have

- $U(k, n) \perp V(k, n)$;
- $E^u(k) = \lim_{n \to \infty} \tilde{U}(k, n)$ and $E^s(k) = \lim_{n \to \infty} V(k, n)$.

B.1 Existence and equivariance of limits

It is immediate from (EC) that the limits $E^u(k)$ and $E^s(k)$ exist. In fact, we have the following uniform convergence estimates:

Lemma B.3. *For every $k, N \in \mathbb{Z}$,*

$$
\begin{aligned}
d(V(k,N), E^s(k)) &\leq \frac{Ce^{-N\mu}}{1-e^{-\mu}} \\
d(\tilde{U}(k,N), E^u(k)) &\leq \frac{Ce^{-N\mu}}{1-e^{-\mu}}.
\end{aligned}
$$

Proof. Immediate from by the triangle inequality and (EC). □

Equivariance follows from using Lemma A.1, whence

$$
\begin{aligned}
E^u(k) &= A_{k-1} \cdots A_0 \cdot \lim_{n\to\infty} U_1\left(A_{-1} \cdots A_{-n}\right) \\
&= A_{k-1} \cdots A_0 \cdot E^u(0) \qquad \text{for } k > 0 \\
E^u(0) &= A_{-1} \cdots A_k \cdot E^u(k) \\
\text{i.e. } E^u(k) &= A_k^{-1} \cdots A_{-1}^{-1} \cdot E^u(0) \qquad \text{for } k < 0
\end{aligned}
$$

and similarly

$$
\begin{aligned}
E^s(0) &= A_0^{-1} \cdots A_{k-1}^{-1} \cdot \lim_{n\to\infty} U_{d-1}\left(A_k^{-1} \cdots A_{k+n-1}^{-1}\right) \\
\text{i.e. } E^s(k) &= A_{k-1} \cdots A_0 \cdot E^s(0) \qquad \text{for } k > 0 \\
E^s(0) &= A_{-1} \cdots A_k \cdot E^s(k) \\
\text{i.e. } E^s(k) &= A_k^{-1} \cdots A_{-1}^{-1} \cdot E^s(0) \qquad \text{for } k < 0
\end{aligned}
$$

B.2 Proof of splitting

The proof will involve, essentially, carefully refined versions of arguments that can be used to give the Raghunathan estimates [Rag79]. Here we formulate these arguments in a series of lemmas, then assemble them into a proof of statement (ii), from which (i) follows.

We follow the argument in [QTZ19] §3, writing things out more concretely for our specific finite-dimensional, invertible case. We have supplied specific references to the corresponding / closely analogous lemmas in [QTZ19], in the hope that the reader interested in also reading the result there may find these helpful.

For the next five lemmas (through Lemma B.8, fix N sufficiently large so that

$$\sum_{n\geq N} e^{-n\mu} = \frac{e^{-N\mu}}{1-e^{-\mu}} \leq \frac{1}{3C} \tag{B.1}$$

The following lemma tells us that whenever m and n are sufficiently large, $A(k,n)$ expands vectors in $U(k,m)$ at least $\frac{2}{3}\sigma_1(k,n)$. More precisely, we have

Lemma B.4 ([QTZ19], Lemma 3.3). *For every $n,m \geq N$ and $k \in \mathbb{Z}$, we have*

$$\forall u \in U(k,m) \quad \|A(k,n)u\| \geq \frac{2}{3}\sigma_1(k,n) \cdot \|u\|$$

Proof. From (EC) and our choice of N, we have, arguing as in the proof of Lemma B.3,

$$d(U(k,n),U(k,m)) = d(V(k,n),V(k,m)) \leq \frac{1}{3}.$$

Given any unit vector $u \in U(k,m)$, write $u = v + w$ where $v \in U(k,n)$ and $w \in V(k,n) \perp v$. By the properties of the singular-value decomposition, $A(k,n)u = A(k,n)v + A(k,n)w$ is still an orthogonal decomposition, and we have

$$\begin{aligned}\|A(k,n)u\| \geq \|A(k,n)v\| &= \sigma_1(k,n) \cdot \cos\angle\left(U(k,m),U(k,n)\right) \\ &= \sigma_1(k,n)\sqrt{1-d(U(k,n),U(k,m))^2} \geq \sigma_1(k,n) \cdot \frac{2}{3}\end{aligned}$$

as desired. □

Recall $s(V,W)$ denotes the minimal gap $\inf\{\sin\langle(v,W) : v \in V, \|v\| = 1\}$ between the subspaces. We now use the (FI) hypothesis to prove a lemma which states that whenever m and n are sufficiently large, we have a lower bound on the gap between the approximate fast space and the slow space. More precisely, we have

Lemma B.5 ([QTZ19], Lemma 3.4). *For all $k \leq 0$ and $m \geq N$,*

$$s(A(k-N,N)U(k-N,m), E^s(k)) \geq \frac{2}{3}C^{-1}e^{-N\mu'}$$

Proof. Write $W_{k,m} := A(k-N,N)U(k-N,m)$.

Let $w \in W_{k,m}$ be a unit vector, and (given any $n \geq N$) write $w = w_1 + w_2$ where $w_1 \in U(k,n)$ and $w_2 \in V(k,n)$. Since $w = A(k-N,N)u$ for some $u \in U(k-N,m)$, we have, from Lemma

B.4 and the properties of the singular-value decomposition,

$$\|A(k,n)w\| = \|A(k-N,N+n)u\| \geq \frac{2}{3}\cdot\sigma_1(k-N,N+n)\|u\| \quad \text{and}$$
$$\|w\| \leq \sigma_1(k-N,N)\|u\|$$

so

$$\|A(k,n)w\| \geq \frac{2}{3}\cdot\frac{\sigma_1(k-N,N+n)}{\sigma_1(k-N,N)}\|w\|$$

On the other hand we also have

$$\|A(k,n)w_1\| = \sigma_1(k,n)\|w_1\| \qquad \text{and} \qquad \|A(k,n)w_2\| \leq \sigma_2(k,n)\|w_2\|$$

or, together,

$$\|A(k,n)w\| \leq \sigma_1(k,n)\left(\|w_1\| + \frac{\sigma_2}{\sigma_1}(k,n)\|w_2\|\right)$$

Combining the two estimates of $\|A(k,n)w\|$ we obtain

$$\begin{aligned}\|w_1+w_2\| = \|w\| &\leq \frac{3}{2}\frac{\sigma_1(k-N,N)}{\sigma_1(k-N,N+n)}\|A(k,n)w\| \\ &\leq \frac{3}{2}\frac{\sigma_1(k-N,N)\cdot\sigma_1(k,n)}{\sigma_1(k-N,N+n)}\left(\|w_1\| + \frac{\sigma_2}{\sigma_1}(k,n)\|w_2\|\right).\end{aligned}$$

By property $(\mathrm{FI})_{\mathrm{back}}$ we have

$$\frac{\sigma_1(k-N,N+n)}{\sigma_1(k-N,N)\cdot\sigma_1(k,n)} \geq C^{-1}e^{-N\mu'}$$

and using this and (SVG-BG) on the last inequality we further obtain

$$\begin{aligned}\|w_1+w_2\| = \|w\| &\leq \frac{3}{2}Ce^{N\mu'}\left(\|w_1\| + Ce^{-n\mu}\|w_2\|\right) \\ &= \frac{3}{2}Ce^{N\mu'}\|w_1\|\left(1 + Ce^{-n\mu}\frac{\|w_2\|}{\|w_1\|}\right).\end{aligned}$$

Now we claim that $\frac{\|w_2\|}{\|w_1\|} = \frac{\sqrt{1-\|w_1\|^2}}{\|w_1\|} = \sqrt{\|w_1\|^{-2}-1}$ is uniformly bounded above by some upper bound B that depends only on the constants C and μ'. If not, $\|w_1\|$ gets arbitrarily close to zero; in particular, it can be made smaller than $(3Ce^{N\mu'})^{-1}$. Then $1 = \|w\| \leq \frac{1}{2} + \frac{3}{2}C^2e^{N\mu'-n\mu} < 1$ for all large enough n, which is a contradiction.

With this upper bound in hand, we then have

$$s(W_{k,m}, V(k,n)) \geq \frac{\|w_1\|}{\|w\|} \geq \frac{2}{3} C^{-1} e^{-N\mu'} \left(1 + Ce^{-n\mu} B\right)^{-1}.$$

and we conclude by letting $n \to \infty$, since $\lim_{n\to\infty} V(k,n) = E^s(k)$. □

This does not quite suffice, since as $N \to \infty$ these lower bounds go to zero, and so *a priori* we could still have the minimal gap between the fast and slow spaces collapsing to zero. Onwards we push ... The idea is to do some kind a multiplicative block analysis, using Lemma B.5 to control each block, and using the subsequent lemma/s to control the remaining exponential terms. This we will achieve using, on the one hand, a lemma which controls expansion on the slow spaces:

Lemma B.6 ([QTZ19], Lemma 3.5)**.** *For all $n \geq N$ and $k \leq 0$,*

$$\left\| A(k,n)\big|_{E^s(k)} \right\| \leq \frac{2}{3} \cdot \sigma_1(k,n) \cdot e^{-(n-N)\mu}.$$

Proof. Let $w \in E^s(k)$, and write $w = w_1 + w_2$ where $w_1 \in U(k,n)$ and $w_2 \in V(k,n)$. Note we have

$$d(V(k,n), E^s(k)) \leq \sum_{m \geq n} Ce^{-m\mu} \leq \frac{Ce^{-n\mu}}{1 - e^{-\mu}} \leq \frac{1}{3} e^{-(n-N)\mu}$$

by Lemma B.3 and our choice of N. Then

$$\begin{aligned}
\|A(k,n)w_1\| &\leq \sigma_1(k,n)\|w_1\| \leq \sigma_1(k,n) \cdot \frac{1}{3} e^{-(n-N)\mu} \|w_2\| \\
\|A(k,n)w_2\| &\leq \sigma_2(k,n)\|w_2\|
\end{aligned}$$

and putting these two together we obtain

$$\begin{aligned}
\|A(k,n)w\| &\leq \|A(k,n)w_1\| + \|A(k,n)w_2\| \\
&\leq \sigma_1(k,n) \left(\frac{1}{3} e^{-(n-N)\mu} + \frac{\sigma_2}{\sigma_1}(k,n) \right) \|w_2\| \\
&\leq \sigma_1(k,n) \left(\frac{1}{3} e^{-(n-N)\mu} + Ce^{-n\mu} \right) \|w\| \\
&\leq \frac{2}{3} e^{-(n-N)\mu} \cdot \sigma_1(k,n) \|w\|
\end{aligned}$$

as desired. □

On the other hand, we have the following lemma which gives us some control on the slow space components of images of approximate fast spaces

Lemma B.7 ([QTZ19], Lemma 3.7)**.** *Let N be sufficiently large.*

(i) Given $w \in \mathbb{R}^d$ a unit vector, write $w = w_1 + w_2$ where $w_1 \in U(k - nN, nN)$ and $w_2 \in E^s(k - nN)$. Then we have $\|w_2\| \leq \frac{3}{2}$

(ii) The operator $\Gamma_{-n} : U(k - nN, nN) \to E^s(k - nN)$ whose graph is

$$W_{n+1} := A(k - (n+1)N, N)\, U(k - (n+1)N, (n+1)N)$$

satisfies $\|\Gamma_{-n}\| \leq \frac{9}{4} C e^{N\mu'}$.

Proof. By Lemma B.3 and our choice of N, $d(V(k - nN, nN), E^s(k - nN)) < \frac{1}{3}$; basic trigonometry then implies $\frac{\|w_2|_{U(k-nN,nN)}\|}{\|w_2|_{V(k-nN,nN)}\|} \leq \frac{1/3}{\sqrt{1-(1/3)^2}} = \frac{1}{2\sqrt{2}}$.

Hence, from the orthogonal decomposition $w_2 = w_2|_{V(k-nN,nN)} + w_2|_{U(k-nN,nN)}$, we get

$$\|w_2\| \leq \left(1 + \frac{1}{2\sqrt{2}}\right) \left\|w_2|_{V(k-nN,nN)}\right\|$$

and since $w_2|_{V(k-nN,nN)} = w|_{V(k-nN,nN)}$ we have $\|w_2|_{V(k-nN,nN)}\| \leq 1$, so in fact

$$\|w_2\| \leq 1 + \frac{1}{2\sqrt{2}} < \frac{3}{2}$$

For (ii): applying Lemma A.5 to the operator $\Gamma_{-n} : U(k - nN, nN) \to E^s(k - nN)$ gives us

$$\|\operatorname{id} \oplus \Gamma_{-n}\| \leq \frac{1}{s(W_{n+1}, E^s(k - nN))} \leq \frac{3}{2} C e^{N\mu'}$$

where the last inequality follows from Lemma B.5, which gives $s(W_{n+1}, E^s(k-nN)) \geq \frac{2}{3} C^{-1} e^{-N\mu'}$.

Now we observe that $\Gamma_{-n} = q_{-n} \circ (\operatorname{id} \oplus \Gamma_{-n})$ where q_{-n} is projection to $E^s(k - nN)$ parallel to $U(k - nN, nN)$. We observe that we may rewrite statement (i) as the assertion that $\|q_{-n}\| \leq \frac{3}{2}$. We put all of this together to obtain

$$\|\Gamma_{-n}\| \leq \|q_{-n}\| \|\operatorname{id} \oplus \Gamma_{-n}\| \leq \frac{9}{4} C e^{N\mu'}$$

as desired. □

Now we can put everything together:

Lemma B.8 ([QTZ19], Lemma 3.8)**.** *For every* $n \geq 1$,

$$s\left(\tilde{U}(k, nN), E^s(k)\right) \geq \frac{2}{3} C^{-1} e^{-N\mu'} \prod_{j=1}^{n-1} \left(1 + \frac{3}{2} D e^{N\mu'} j^{-3}\right)^{-1}.$$

Proof. From Lemma A.4 we have

$$s\left(\tilde{U}(k, nN), E^s(k)\right) \geq \frac{s(\tilde{U}(k, N), E^s(k))}{\| \operatorname{id} \oplus \Xi_n \|}$$

where $\Xi_n : \tilde{U}(k, N) \to E^s(k)$ is such that $\tilde{U}(k, nN)$ is the graph of Ξ_n. Since $\tilde{U}(k, N) = A(k - N, N)\, U(k - N, N)$, we have $s(\tilde{U}(k, N), E^s(k)) \geq \frac{2}{3} C^{-1} e^{-N\mu'}$ from Lemma B.5 and it remains to bound $\| \operatorname{id} \oplus \Xi_n \|$.

Write $A_{-n} := A(k - nN, N) = \begin{bmatrix} a_{-n} & 0 \\ c_{-n} & d_{-n} \end{bmatrix}$ where

$$\begin{aligned} a_{-n} &: U(k - nN, nN) \to U(k - (n-1)N, (n-1)N), \\ c_{-n} &: U(k - nN, nN) \to E^s(k - (n-1)N), \\ d_{-n} &: E^s(k - nN) \to E^s(k - (n-1)N) \end{aligned}$$

and the 0 in the upper-right corner comes from the equivariance of the slow spaces; here we adopt the notational convention $U(k, 0) := \tilde{U}(k, N)$.

Then $A^n_{-n} := A_{-1} \cdots A_{-n} = A(k - nN, nN) := \begin{bmatrix} a^n_{-n} & 0 \\ c^n_{-n} & d^n_{-n} \end{bmatrix}$. Now we have

$$A^{n+1}_{-(n+1)} = \begin{bmatrix} a^n_{-n} & 0 \\ c^n_{-n} & d^n_{-n} \end{bmatrix} \begin{bmatrix} a_{-(n+1)} & 0 \\ c_{-(n+1)} & d_{-(n+1)} \end{bmatrix}$$

and examining in particular the bottom-left entry of this product, we have

$$c^{n+1}_{-(n+1)} = c^n_{-n} a_{-(n+1)} + d^n_{-n} c_{-(n+1)}.$$

Since $a^{n+1}_{-(n+1)} = a^n_{-n} a_{-(n+1)}$,

$$c^{n+1}_{-(n+1)} (a^{n+1}_{-(n+1)})^{-1} = c^n_{-n} (a^n_{-n})^{-1} + d^n_{-n} c_{-(n+1)} (a_{-(n+1)})^{-1} (a^n_{-n})^{-1}. \tag{B.2}$$

Now, firstly, we observe that $\Xi_n = c^n_{-n}(a^n_{-n})^{-1}$, since from the block structure of A^n_{-n} we see that $c^n_{-n}(a^n_{-n})^{-1}$ maps from $U(k, 0) = \tilde{U}(k, N)$ to $E^s(k)$ with graph $A(k - nN, nN)\, U(k - nN, nN) = \tilde{U}(k, nN)$.

Secondly, we write $c_{-(n+1)}(a_{-(n+1)})^{-1} =: \Gamma_{-n} : U(k-nN, nN) \to E^s(k-nN)$ (see observation 1 below), and note that (B.2) combined with the triangle inequality, give us (writing id for the identity on the appropriate complementary subspace, so that $\mathrm{id} \oplus \Xi_n$ is a linear endomorphism of $\mathbb{R}^d$)

$$\|\mathrm{id} \oplus \Xi_{n+1}\| \leq \|\mathrm{id} \oplus \Xi_n\| \left(1 + \frac{\|d_{-n}^n\| \, \|\Gamma_{-n}\| \, \|(a_{-n}^n)^{-1}\|}{\|\mathrm{id} \oplus \Xi_n\|}\right). \tag{B.3}$$

To bound the last quantity that appears, we observe that

1. Γ_{-n} is precisely the operator from Lemma B.7(ii): $c_{-(n+1)}(a_{-(n+1)})^{-1}$ maps from $U(k-nN, nN)$ to $E^s(k-nN)$ with graph $A(k-(n+1)N, N)\, U(k-(n+1)N, (n+1)N)$.

   Hence, from Lemma B.7, $\|\Gamma_{-n}\| \leq \frac{9}{4} C e^{N\mu'}$

2. We have

$$\frac{\|(a_{-n}^n)^{-1}\|}{\|\mathrm{id} \oplus \Xi_n\|} \leq (\sigma_1(k-nN, nN))^{-1}$$

   since $(a_{-n}^n)^{-1} = \left(A_{-n}^n|_{U(k-nN,nN)}\right)^{-1} \circ (\mathrm{id} \oplus \Xi_n)$ (easier to see by writing a_n^{-n} as composition of $\left(A_{-n}^n|_{U(k-nN,nN)}\right)^{-1}$ with projection onto $\tilde{U}(k, N)$ parallel to $E^s(k)$) and

$$\|\left(A_{-n}^n|_{U(k-nN,nN)}\right)^{-1}\| = (\sigma_1(k-nN, nN))^{-1}.$$

3. From Lemma B.6,

$$\|d_{-n}^n\| \leq \frac{2}{3} \cdot \sigma_1(k-nN, nN) e^{-(n-1)N\mu}$$

Combining the bounds from these three observations, we obtain

$$\begin{aligned}\frac{\|d_{-n}^n\| \, \|\Gamma_{-n}\| \, \|(a_{-n}^n)^{-1}\|}{\|\mathrm{id} \oplus \Xi_n\|} &\leq \frac{2}{3} \cdot \frac{\sigma_1(k-nN, nN)}{\sigma_1(k-nN, nN)} \cdot e^{-(n-1)N\mu} \cdot \frac{9}{4} C e^{N\mu'} \\ &= \frac{3}{2} C \cdot e^{-(n-1)N\mu} e^{N\mu'}\end{aligned}$$

Since $\Xi_1 \equiv 0$, $\|\mathrm{id} \oplus \Xi_1\| = 1$. We then use this together with (B.3), as in [QTZ19], to obtain the iterative bound

$$\|\mathrm{id} \oplus \Xi_n\| \leq \prod_{j=0}^{n-2} \left(1 + \frac{3}{2} C e^{N(\mu' - j\mu)}\right)$$

and we are done. ∎

An elementary argument, done in [QTZ19], gives us control over the infinite product that appears as we take $n \to \infty$:

Lemma B.9 ([QTZ19], Lemma 3.10)**.** *Fix constants C, N, μ' and $\mu > 0$. Then*

$$\prod_{j=0}^{\infty} \left[1 + \frac{3}{2} C^{-1} e^{N(\mu' - j\mu)}\right] \leq \exp\left(\frac{3}{2} C^{-1} e^{-N\mu'} \cdot (1 - e^{-N\mu})^{-1}\right) < \infty.$$

Proof. Write $a_j := \frac{3}{2} C^{-1} e^{N\mu'} e^{-jN\mu}$. If $\mu > 0$, then $\sum_{j=1}^{\infty} a_j$ converges, and hence so does our infinite product $\prod_{j=1}^{\infty}(1 + a_j)$.

In particular, we have $\prod_{j=1}^{\infty}(1 + a_j) = \exp\left(\sum_{j=1}^{\infty} \log(1 + a_j)\right) \leq \exp(\sum_{j=1}^{\infty} a_j)$ since $a_j > 0$. Now observe $\sum_{j=1}^{\infty} a_j = \frac{3}{2} C^{-1} e^{N\mu'} \sum_{j=0}^{\infty} e^{-jN\mu}$, and $\sum_{j=0}^{\infty} e^{-jN\mu} = (1 - e^{-N\mu})^{-1}$. ∎

Now for the final assembly:

Proof of splitting. From Lemma B.8 and Lemma B.9, we have

$$\begin{aligned} s(\tilde{U}(k, nN), E^s(k)) &\geq \frac{2}{3} C^{-1} e^{-N\mu'} \prod_{j=0}^{n-2} \left[1 + \frac{3}{2} C e^{N(\mu' - j\mu)}\right]^{-1} \\ &\geq \frac{2}{3} C^{-1} \exp\left(-\frac{3}{2} C^{-1} e^{-N\mu'} (1 - e^{-N\mu})^{-1} - N\mu'\right). \end{aligned}$$

Now recall that N satisfies (B.1), i.e. $N \geq \frac{1}{\mu}(\log 3C - \log(1 - e^{-\mu})) > \frac{1}{\mu} \log 3C$. Pick $N \leq \frac{2}{\mu} \log 3C$, which implies that $e^{-N\mu'} \geq (3C)^{-2r}$ where $r := \frac{\mu'}{\mu}$. Such a choice of N exists from our hypothesis that $\frac{1}{\mu} \log 3C > 1$. Then

$$\begin{aligned} s(\tilde{U}(k, nN), E^s(k)) &\geq \frac{2}{3} C^{-1} \exp\left(-\frac{3}{2} \frac{C^{-(1+r)} 3^{-r}}{1 - e^{-N\mu}} - 2r \log 3C\right) \\ &\geq \frac{2}{3} 3^{-2r} C^{-(1+2r)} \exp\left(-\frac{3^{1-r} C^{-(1+r)}}{2(1 - e^{-N\mu})}\right) \\ &\geq \frac{2}{3} 3^{-2r} \exp\left(-\frac{3/2}{1 - e^{-\mu}}\right) C^{-(1+2r)}. \end{aligned}$$

Finally, using the fact that $\tilde{U}(k, nN) \to E^u(k)$ as $n \to \infty$, we are done. ∎

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