Stabilizations of $\mathbb{E}_\infty$ Operads and p-Adic Stable Homotopy Theory

by

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ABSTRACT

In this thesis, we study differential graded operads and p-adic stable homotopy theory.

We first construct a new class of differential graded operads, which we call the stable operads. These operads are, in a particular sense, stabilizations of $\mathbb{E}_\infty$ operads. We develop a homotopy theory of algebras over these stable operads and a theory of (co)homology operations for algebras over these stable operads. We note interesting properties of these operads, such as that, non-equivariantly, in each arity, they have (almost) trivial homology, whereas, equivariantly, these homologies sum to a certain completion of the generalized Steenrod algebra and so are highly non-trivial. We also justify the adjective "stable" by showing that, among other things, the monads associated to these operads are additive in the homotopy coherent, or ∞-, sense.

We then provide an application of our stable operads to p-adic stable homotopy theory. It is well-known that cochains on spaces yield examples of algebras over $\mathbb{E}_\infty$ operads. We show that in the stable case, cochains on spectra yield examples of algebras over our stable operads. Moreover, a result of Mandell says that, endowed with the $\mathbb{E}_\infty$ algebraic structure, cochains on spaces provide algebraic models of p-adic homotopy types. We show that, endowed with the algebraic structure codified by our stable operads, spectral cochains provide algebraic models for p-adic stable homotopy types.

CHAPTER 1

Introduction

In this work, we cover two main topics:

(1) Stable (dg) operads, which provide a new class of operads, and which are, in a particular sense, stable analogues of $\mathbb{E}_\infty$ operads.

(2) Algebraic models for p-adic stable homotopy types using these operads.

First, let us recall the notion of an $\mathbb{E}_\infty$ cochain operad and how it gives rise to algebraic models of p-adic homotopy types. Let $\mathcal{E}^\dagger$ be a model for the $\mathbb{E}_\infty$ cochain operad (throughout this work, we have used the symbol † to distinguish contexts with cochain complexes from contexts with chain complexes). Given a cochain complex X, the structure of an algebra over an $\mathcal{E}^\dagger$ encodes a homotopy coherent commutative, associative and unital multiplication. If we take the cohomology of the complex, killing the higher homotopies, we find that $\mathrm{H}^\bullet(X)$ inherits a (graded) commutative algebra structure in the traditional sense. In fact, the cohomology inherits even more structure. It posseses certain cohomology operations P^s, $s \in \mathbb{Z}$, which satisfy an instability condition, and as a result becomes an unstable module over $\mathcal{B}$, the algebra of generalized Steenrod operations.

A particular case of such $\mathcal{E}^\dagger$-algebras are the cochains $\mathrm{C}^\bullet(X)$ on spaces X. In this case, the algebra structure on the cohomology is given by the cup product, while the operations are the Steenrod operations. While the cochains, as a dg module, might not remember the homotopy type of a space, in [Man01], Mandell demonstrated that if we take cochains with coefficients in $\overline{\mathbb{F}}_p$, the cochains functor

$$\mathrm{C}^\bullet(-;\overline{\mathbb{F}}_p)\colon \mathsf{Spc}^{\mathrm{op}} \to \bar{\mathcal{E}}^\dagger\text{-}\mathsf{Alg}$$

as a functor to $\bar{\mathcal{E}}^\dagger$-algebras (where $\bar{\mathcal{E}}^\dagger$ is the $\mathbb{E}_\infty$ cochain operad over $\overline{\mathbb{F}}_p$), induces a full embedding of the homotopy category of spaces into the derived category of $\bar{\mathcal{E}}^\dagger$-algebras when we restrict to connected nilpotent p-complete spaces of finite p-type, which is to say the $\bar{\mathcal{E}}^\dagger$-algebras remember the homotopy types of such spaces. Thus, while rational homotopy types admit algebraic models via CDGAs, p-adically, we can take $\bar{\mathcal{E}}^\dagger$-algebras.

Now we move onto stable operads and stable homotopy types. First of all, we show that the operad $\mathcal{E}^\dagger$ possesses a stabilization map

$$\Psi \colon \Sigma\mathcal{E}^\dagger \to \mathcal{E}^\dagger$$

from its operadic suspension to itself. Upon iteration, via an inverse limit, we produce a new operad, denoted $\mathcal{E}^\dagger_{\mathrm{st}}$, which is our stable operad. In fact, we have a new class of operads, the stable operads, in the sense that we are able to perform the above construction for multiple models of the $\mathbb{E}_\infty$ operad (we have a stable Barratt-Eccles operad, a stable McClure-Smith operad, and also a stable Eilenberg-Zilber operad, though the latter is not an $\mathbb{E}_\infty$ operad). The stable operad $\mathcal{E}^\dagger_{\mathrm{st}}$ appears to be of independent interest outside of its application to p-adic stable homotopy theory which we discuss below; for example, due to its homotopy additivity which we also discuss below.

First, we demonstrate that one has homotopical control over $\mathcal{E}^\dagger_{\mathrm{st}}$ and the corresponding category of algebras $\mathcal{E}^\dagger_{\mathrm{st}}$-Alg in the following sense.

Theorem 1.1. *The monad* $\mathbf{E_{st}}^\dagger$ *associated to* $\mathcal{E}^\dagger_{\mathrm{st}}$ *preserves weak equivalences.*

Theorem 1.2. *The category* $\mathcal{E}^\dagger_{st}$-Alg *admits a Quillen semi-model structure where the weak equivalences and fibrations are the quasi-isomorphisms and degreewise epimorphisms.*

(See Definition 2.37 in the second chapter of the work for the definition of a Quillen semi-model structure, a weakening of the more well-known notion of a Quillen model structure.) Next, we develop a theory of cohomology operations for algebras over $\mathcal{E}^\dagger_{\mathrm{st}}$. We once again get operations P^s for $s \in \mathbb{Z}$, but they now no longer satisfy the instability condition.

Theorem 1.3. *We have the following:*

(i) The cohomologies of $\mathcal{E}^\dagger_{\mathrm{st}}$*-algebras possess natural operations* P^s *for* $s \in \mathbb{Z}$ *which satisfy the Adem relations. These aoperations do not, however, satisfy the instability condition seen in the unstable case.*

(ii) Given a cochain complex X*, we have a natural isomorphism:*

$$\mathrm{H}^\bullet(\mathbf{E}^\dagger_{\mathbf{st}}X) \cong \widehat{\mathcal{B}} \otimes \mathrm{H}^\bullet(X)$$

Here $\widehat{\mathcal{B}}$ is a certain completion, with respect to a filtration by excess, of the algebra $\mathcal{B}$. Note that, in the unstable case of $\mathcal{E}^\dagger$, in (ii) above, we would not only have to tensor with $\mathcal{B}$ to add in the operations, but also enforce the instability condition and also take a polynomial algebra to add in products. In the case of $\mathcal{E}_{\mathrm{st}}^{\dagger}$, the instability of the operations and the products disappear.

Next, we justify the "stable" in "stable operad". This should of course be a statement about homotopy coherent, or ∞-, additivity, and this is exactly what we demonstrate. In particular, we demonstrate that the monad $\mathbf{E}_{\mathbf{st}}^{\dagger}$ is homotopy coherent, or ∞-, additive, in the following sense.

Theorem 1.4. *We have the following:*

(i) For dg modules X and Y, we have a natural quasi-isomorphism:

$$\mathbf{E}_{\mathbf{st}}^{\dagger}(X \oplus Y) \sim \mathbf{E}_{\mathbf{st}}^{\dagger}(X) \oplus \mathbf{E}_{\mathbf{st}}^{\dagger}(Y)$$

(ii) More generally, for cofibrant $\mathcal{E}_{\mathrm{st}}^{\dagger}$-algebras A and B, we have a natural quasi-isomorphism:

$$A \amalg B \sim A \oplus B$$

Finally, we move onto the application to p-adic stable homotopy types. For this, we need to fix a model for spectra. We take the classical sequential model in the sense of Bousfield-Friedlander, with the exception that, rather than the ordinary suspension $- \wedge \mathbb{S}^1$, we use the Kan suspension of based simplicial sets. We then define an appropriate, and concrete in the sense that we get honest dg modules, notion of spectral cochains and then prove the following, providing another sense in which $\mathcal{E}_{\mathrm{st}}^{\dagger}$ is a stable analogue of $\mathcal{E}^{\dagger}$.

Theorem 1.5. *Given any spectrum E, the spectral cochains $\mathrm{C}^{\bullet}(E)$ naturally form an algebra over $\mathcal{E}_{\mathrm{st}}^{\dagger}$.*

Finally then, we get algebraic models for p-adic stable homotopy types in the following sense – in the statement, the cochains functor $\overline{\mathrm{C}}^{\bullet}$, one over $\overline{\mathbb{F}}_p$, is constructed from $\mathrm{C}^{\bullet}$ simply by tensoring with $\overline{\mathbb{F}}_p$, and similarly, the operad $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$, an operad over $\overline{\mathbb{F}}_p$, is constructed from $\mathcal{E}_{\mathrm{st}}^{\dagger}$ by tensoring with $\overline{\mathbb{F}}_p$.

Theorem 1.6. *The spectral cochains functor*

$$\overline{C}^{\bullet} \colon \mathsf{Sp}^{\mathrm{op}} \to \bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}$$

induces a full embedding of the stable homotopy category into the derived category of $\bar{\mathcal{E}}_{\mathrm{st}}$-algebras when we restrict to bounded below p-complete spectra of finite p-type.

We mentioned above that rational homotopy types can be modelled by commutative DGAs, and also that p-adic homotopy types can be modelled by $\mathbb{E}_\infty$ DGAs. It is also well-known that rational stable homotopy types can be modelled by chain complexes. Our result for p-adic stable homotopy types then completes the following picture.

Algebraic models for homotopy types	unstable	stable
rational	commutative DGAs	chain complexes
p-adic	$\mathbb{E}_\infty$ DGAs	$\bar{\mathcal{E}}_{\mathrm{st}}^\dagger$ DGAs

We can make a remark on the structure that is captured by the operad $\mathcal{E}_{\mathrm{st}}^\dagger$. In both the unstable and stable p-adic cases, the models arise via cochains, with coefficients in $\overline{\mathbb{F}}_p$. Let us consider just $\mathbb{F}_p$ cochains. Given a space X, the mod p cochains are given, as a spectrum, by $[\Sigma_+^\infty X, \mathrm{H}\mathbb{F}_p]$. This object carries the following structure: (i) it is an $\mathrm{H}\mathbb{F}_p$-module via "pointwise" scalar multiplication (ii) it possesses an action, via postcomposition, by $[\mathrm{H}\mathbb{F}_p, \mathrm{H}\mathbb{F}_p]$ (iii) it is a ring spectrum via the multiplication of $\mathrm{H}\mathbb{F}_p$. These manifest, respectively, in our work as follows: (i) the cochains can be modelled as a dg module (ii) the cohomology inherits an action by $\mathcal{B}/(1-P^0) \cong \mathcal{A}$ (iii) the cochains form an $\mathbb{E}_\infty$ algebra. In fact, we shall see that (ii) is a consequence of (iii). Now let us consider cochains on a spectrum E. The cochains are given, as a spectrum, by $[E, \mathrm{H}\mathbb{F}_p]$. This object carries the following structure: (i) it is an $\mathrm{H}\mathbb{F}_p$-module via "pointwise" scalar multiplication (ii) it possesses an action, via postcomposition, by $[\mathrm{H}\mathbb{F}_p, \mathrm{H}\mathbb{F}_p]$. It no longer posseses a ring structure as, although the multiplication of $\mathrm{H}\mathbb{F}_p$ is still present, to define a "pointwise" multiplication, one needs a diagonal map, which general spectra, unlike spaces and their suspension spectra, do not possess. The structure that is still present manifests, respectively, in our work as follows: (i) the cochains can be modelled as a dg module (ii) the cohomology inherits an action by $\widehat{\mathcal{B}}/(1-P^0) \cong \mathcal{A}$. We shall see that these operations in (ii) are a consequence of the $\mathcal{E}_{\mathrm{st}}^\dagger$-algebra structure, and so it is primarily these operations which this operad serves to encode.

To end this introduction, we list here some notations, terminology and conventions which are used throughout the work:

- For each integer $n \geq 0$, $[n]$ denotes the poset $0 \to 1 \to \cdots \to n$.
- For each integer $n \geq 0$, (n) denotes the set $\{1, \ldots, n\}$; (0) is the empty set.
- For $n \geq 0$, Σ_n denotes the symmetric group on n letters; Σ_0 is the trivial group, where the unique element is thought of as representing the unique isomorphism on the empty set.

- Given a simplicial set S, S_n^{nd} denotes the non-degenerate n-simplices of S.

- As is standard, the term *chain complex* will refer to a graded module equipped with a differential of degree -1 and the term *cochain complex* will refer to a graded module equipped with a differential of degree $+1$; in addition, we shall let *differential graded module* refer to either of these two possibilities. The phrase *differential graded* is often shortened to *dg*. The category of chain complexes over k is denoted by Ch_k, and the category of cochain complexes over k is denoted by Co_k; the symbol DG_k will denote either of these two possibilities.

- By default, all dg modules are unbounded.

- Given dg modules X and Y, the tensor product $X \otimes Y$ is defined, as usual, by letting $x \otimes y$ have degree $|x| + |y|$, and the differential follows the standard sign convention:

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^{|x|} x \otimes \partial y$$

  This tensor product on dg modules is always endowed with the symmetry $X \otimes Y \to Y \otimes X$ defined by:

$$x \otimes y \mapsto (-1)^{|x||y|}(y \otimes x)$$

  The internal hom $\mathrm{F}(X, Y)$ is defined, in degree n, as the collection of degree d graded module maps $f\colon X_\bullet \to Y_{\bullet+d}$ (no compatibility with the differnetial is required for these maps), and the differential on $\mathrm{F}(X, Y)$ is given by:

$$\partial f = \partial \circ f - (-1)^{|f|}(f \circ \partial)$$

- Given a dg module X and $n \in \mathbb{Z}$, we let $X[n]$ be the dg module defined by setting $X[n]_d = X_{d-n}$. Note that we do not change the sign of any differential. Note also that, if we let k denote a ground field, $X[1] \cong X \otimes k[1]$ and $X[-1] \cong \mathrm{F}(k[1], X)$.

- Let k denote a field. We shall often denote the chain or cochain complex

$$\cdots \leftarrow 0 \leftarrow \underset{\text{deg. } n}{k} \leftarrow 0 \leftarrow \cdots \qquad\qquad \cdots \to 0 \to \underset{\text{deg. } n}{k} \to 0 \to \cdots$$

  by $\mathbb{S}^n$ and refer to them as *sphere complexes*, and the chain or cochain complex

$$\cdots \leftarrow 0 \leftarrow \underset{\text{deg. } n-1}{k} \xleftarrow{\text{id}} \underset{\text{deg. } n}{k} \leftarrow 0 \leftarrow \cdots \qquad\qquad \cdots \to 0 \to \underset{\text{deg. } n-1}{k} \xrightarrow{\text{id}} \underset{\text{deg. } n}{k} \to 0 \to \cdots$$

  by $\mathbb{D}^n$, and refer to them as *disk complexes*.

- Let R denote a ring. A dg module over R is said to be *of finite type* if it is finitely generated in each degree; it is said to be *finite* if it is, in addition, bounded from above and below. In the case that $R = \mathbb{F}_p[\Sigma_n]$ for some prime p and $n \geq 0$, we can replace finitely generated by finitely presented as $\mathbb{F}_p[\Sigma_n]$ is Noetherian (see [BLS81]).

- All operads are dg operads, and are symmetric. The notations, in the sense of the typeface, for operads and their corresponding monads and free algebra functors will follow the following rule: if $\mathcal{P}$ denotes an operad, the corresponding monad and free algebra functor will both be denoted by $\mathbf{P}$.

- All (co)chains on spaces or spectra are normalized.

CHAPTER 2

Differential Graded Operads and Their Algebras

In this chapter, we cover general aspects of dg operads and their algebras. Throughout this chapter, as well as the rest of the work, we will have two versions of most results, depending on whether the differential has degree -1 or $+1$. For this reason, prior to discussing operads, we begin with some basic results regarding indexing, mostly to allow us to set in place some notations which will allow us to be precise and clear in our statements in later parts of this work. Throughout this chapter, k will denote an unspecified but fixed field.

Let Ch_k denote the category of chain complexes of k-modules, and Co_k the category of cochain complexes of k-modules.

Definition 2.1. Given a chain complex X over k, the *dual chain complex* $\mathrm{D}(X)$ of X is defined as

$$X^\vee := \mathrm{F}(X, k[0])$$

where F denotes the internal hom of chain complexes. Similarly, if Y is a cochain complex over k, the *dual cochain complex* $\mathrm{D}(Y)$ of Y is defined as

$$Y^\vee := \mathrm{F}(Y, k[0])$$

where, in this case, F denotes the internal hom of cochain complexes. These yield contravariant functors

$$\mathsf{Ch}_k \to \mathsf{Ch}_k \qquad \mathsf{Co}_k \to \mathsf{Co}_k$$

which are involutions up to natural isomorphism, and are both denoted by $(-)^\vee$.

Remark 2.2. Given a dg module X, we have that $X^\vee$ consists of graded module maps $X \to k[0]$, of any degree; in a fixed degree d, this amounts to module maps $X_{-d} \to k$ in the chain case, and $X^{-d} \to k$ in the cochain case. The negative sign here ensures that the induced differential on $X^\vee$ has the same degree as the differential of the original dg module X. Note that if X is concentrated in non-negative degrees, $X^\vee$ will be concentrated in non-positive degrees, and vice versa. $\qquad ||$

Definition 2.3. Given a chain complex X over k, the *associated cochain complex* $X^\dagger$ is that which is defined by setting:

$$(X^\dagger)^p := X_{-p}$$

Similarly, if Y is a cochain complex over k, the *associated chain complex* $Y^\dagger$ is that which is defined by the same formula:

$$(Y^\dagger)_p := Y^{-p}$$

These yield an inverse pair of functors

$$\mathsf{Ch}_k \to \mathsf{Co}_k \qquad \mathsf{Co}_k \to \mathsf{Ch}_k$$

both of which are denoted by $(-)^\dagger$.

Example 2.4. Given any space, the cochains on the space are constructed from the chains by first dualizing via $(-)^\vee$ and then reindexing via $(-)^\dagger$.

2.1 Differential Graded Operads and Their (Co)algebras

Definition 2.5. A *differential graded operad*, or *dg operad* for short, over k, is an operad in either Ch_k or Co_k.

More fully, a dg operad, denoted say $\mathcal{P}$, consists of the following data:

- A dg module $\mathcal{P}(n)$ for each $n \geq 0$.
- For each $n \geq 0$ and $k_1, \ldots, k_n \geq 0$, a map

  $$\gamma_{n,k_1,\ldots,k_n} \colon \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \cdots + k_n)$$

  where, given $n \geq 0$, $k_1, \ldots, k_n \geq 0$ and $l_{1,1}, \ldots, l_{1,k_1}, \ldots, l_{n,1}, \ldots, l_{n,k_n} \geq 0$, the following square commutes:

  $$\begin{tikzcd}
  \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \otimes \mathcal{P}(l_{1,1}) \otimes \cdots \otimes \mathcal{P}(l_{n,k_n}) \arrow[r] \arrow[d] & \mathcal{P}(k_1 + \cdots + k_n) \otimes \mathcal{P}(l_{1,1}) \otimes \cdots \otimes \mathcal{P}(l_{n,k_n}) \arrow[d] \\
  \mathcal{P}(n) \otimes \mathcal{P}(l_{1,1} + \cdots + l_{1,k_1}) \otimes \cdots \otimes \mathcal{P}(l_{n,1} + \cdots + l_{n,k_n}) \arrow[r] & \mathcal{P}(l_{1,1} + \cdots + l_{1,k_n})
  \end{tikzcd}$$

- A specified element $\iota \in \mathcal{P}(1)$ such that, for any $n \geq 0$, the maps

  $$\gamma_{1,1} \colon \mathcal{P}(1) \otimes \mathcal{P}(n) \to \mathcal{P}(n) \qquad \gamma_{n,1,\ldots,1} \colon \mathcal{P}(n) \otimes \mathcal{P}(1) \otimes \cdots \otimes \mathcal{P}(1) \to \mathcal{P}(n)$$

  satisfy: $\gamma_{1,1}(\iota \otimes f) = f$ and $\gamma_{n,1,\ldots,1}(f \otimes \iota \otimes \cdots \otimes \iota) = f$ for all $f \in \mathcal{P}(n)$.

- For each $n \geq 0$, a right action of Σ_n on $\mathcal{P}(n)$, such that for each $n \geq 0$, $k_1, \ldots, k_n \geq 0$, and $\sigma \in \Sigma_n$, $\tau_1 \in \Sigma_{k_1}$,. . . ,$\tau_n \in \Sigma_{k_n}$, the following diagrams commute:

$$\begin{tikzcd} \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \arrow[r, "\mathrm{id} \otimes \tau_1 \otimes \cdots \otimes \tau_n"] \arrow[d] & \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \arrow[d] \\ \mathcal{P}(k_1 + \cdots + k_n) \arrow[r, "\tau_1 \boxplus \cdots \boxplus \tau_n"'] & \mathcal{P}(k_1 + \cdots + k_n) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \arrow[r, "\sigma \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}"] \arrow[d] & \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \arrow[dd] \\ \mathcal{P}(n) \otimes \mathcal{P}(k_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathcal{P}(k_{\sigma^{-1}(n)}) \arrow[d] & \\ \mathcal{P}(k_1 + \cdots + k_n) \arrow[r, "{\sigma[k_1, \ldots, k_n]}"'] & \mathcal{P}(k_1 + \cdots + k_n) \end{tikzcd}$$

Here $\tau_1 \boxplus \cdots \boxplus \tau_n$ denotes an internal permutation of blocks and $\sigma[k_1, \ldots, k_n]$ denotes an external permutation of blocks, in each case of blocks of size $k_1, \ldots, k_n$.

Moreover, a map of dg operads, $F \colon \mathcal{P} \to \mathcal{Q}$ consists of the following data:

- For each $n \geq 0$, a dg map $F(n) \colon \mathcal{P}(n) \to \mathcal{Q}(n)$, such that the following hold:
  - For each $n \geq 0$ and $k_1, \ldots, k_n \geq 0$, the following square commutes:

$$\begin{tikzcd} \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \arrow[r] \arrow[d] & \mathcal{P}(k_1 + \cdots + k_n) \arrow[d] \\ \mathcal{Q}(n) \otimes \mathcal{Q}(k_1) \otimes \cdots \otimes \mathcal{Q}(k_n) \arrow[r] & \mathcal{Q}(k_1 + \cdots + k_n) \end{tikzcd}$$

  - $F(1)(\iota_{\mathcal{P}}) = \iota_{\mathcal{Q}}$.
  - For each $n \geq 0$, $F(n)$ is Σ_n-equivariant.

We thus have a category of operads in Ch_k, and a category of operads in Co_k, which we denote by $\mathsf{Op}(\mathsf{Ch}_k)$ and $\mathsf{Op}(\mathsf{Co}_k)$, respectively.

Remark 2.6. We will sometimes use an alternative characterization of the data of the composition maps γ in the structure of an operad. Namely, we can instead specify, for each n, m and $i = 1, \ldots, n$, maps:

$$\circ_i \colon \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1)$$

Given these maps, we get the maps $\mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \cdots + k_n)$ by sending $f \otimes g_1 \otimes \cdots \otimes g_n$ to $(((f \circ_n g_n) \circ_{n-1} g_{n-1}) \cdots) \circ_1 g_1$. See, for example, [Mar06] for details on the equivalence between these two pieces of data. $||$

We have seen dualization and reindexing operations, $(-)^\vee$ and $(-)^\dagger$, on dg modules. We now consider what effect these operations have on dg operads. When applied termwise to an operad, the former yields, under suitable finiteness hypotheses, a co-operad (and inversely a co-operad yields an operad upon dualization). Since we have no use for these co-operads in this work, we say no more on this. The reindexing operation however does yield another operad: an easy check shows that the reindexing operation is compatible with all the structure data in an operad. Thus we may make the following definition.

Definition 2.7. If $\mathcal{P}$ is an operad in $\mathsf{Op}(\mathsf{Ch}_k)$, the *associated operad in* $\mathsf{Op}(\mathsf{Co}_k)$, denoted $\mathcal{P}^\dagger$, is defined by setting:

$$\mathcal{P}^\dagger(n) := \mathcal{P}(n)^\dagger$$

Similarly, if $\mathcal{Q}$ is an operad in $\mathsf{Op}(\mathsf{Co}_k)$, the *associated operad in* $\mathsf{Op}(\mathsf{Ch}_k)$, denoted $\mathcal{Q}^\dagger$, is defined via the same formula:

$$\mathcal{Q}^\dagger(n) := \mathcal{Q}(n)^\dagger$$

These yield an inverse pair of functors

$$\mathsf{Op}(\mathsf{Ch}_k) \to \mathsf{Op}(\mathsf{Co}_k) \qquad \mathsf{Op}(\mathsf{Co}_k) \to \mathsf{Op}(\mathsf{Ch}_k)$$

both of which are denoted by $(-)^\dagger$.

Remark 2.8. We have introduced these simple definitions here as it will be important later to be careful about this reindexing when we consider which operads act on chains on spaces or spectra, and which act on cochains on spaces or spectra. $||$

We now wish to define algebras and coalgebras over dg operads. Before we can do this, we need to define endomorphism and coendomorphism operads.

Definition 2.9. Let X be a dg module over k. The *endomorphism operad* of X, denoted $\mathcal{E}nd_X$, is the dg operad defined by setting

$$\mathcal{E}nd_X(n) := \mathrm{F}(X^{\otimes n}, X)$$

where F denotes the internal hom of dg modules, and where the operadic structure data is given by the obvious composition maps. Moreover, the *coendomorphism operad* of X, denoted $\mathcal{C}o\mathcal{E}nd_X$, is the dg operad defined by setting

$$\mathcal{CoEnd}_X(n) := \mathrm{F}(X, X^{\otimes n})$$

and again, the operadic structure data is clear.

Now we can define algebras and coalgebras over operads.

Definition 2.10. Let $\mathcal{P}$ be a dg operad over k. An *algebra* over $\mathcal{P}$ is a dg module X equipped with an operad morphism $\mathcal{P} \to \mathcal{End}_X$. A *coalgebra* over $\mathcal{P}$ is a dg module X equipped with an operad morphism $\mathcal{P} \to \mathcal{CoEnd}_X$. Moreover, *maps of algebras and coalgebras* are dg maps which respect the algebra, respectively coalgebra, structure maps in the obvious sense.

Thus, given a dg operad $\mathcal{P}$, we have categories of $\mathcal{P}$-algebras and $\mathcal{P}$-coalgebras, which we denote by $\mathcal{P}$-Alg and $\mathcal{P}$-Coalg, respectively.

Remark 2.11. One can rephrase the definitions of algebras and coalgebras over operads as sequences of maps $\mathcal{P}(n) \otimes X^{\otimes n} \to X$ and $\mathcal{P}(n) \otimes X \to X^{\otimes n}$, respectively, satisfying suitable properties. ||

Example 2.12. For any dg operad $\mathcal{P}$ over k, there is an initial $\mathcal{P}$-algebra, namely $\mathcal{P}(0)$. The algebra structure maps are given by the composition maps γ in the case $k_1 = \cdots = k_n = 0$. Given any algebra X, the unique algebra map $\mathcal{P}(0) \to X$ is given by the algebra structure map for X in the case $n = 0$. Intuitively, this algebra contains the necessary "constant elements" and nothing else.

We now consider what effect the dualizing and reindexing constructions, $(-)^\vee$ and $(-)^\dagger$, have on algebras and coalgebras over dg operads.

Proposition 2.13. *Let $\mathcal{P}$ be a dg operad which is aritywise of finite type. We have the following:*

(i) If X is a $\mathcal{P}$-algebra of finite type, $X^\vee$ is canonically a $\mathcal{P}$-coalgebra.

(ii) If X is a $\mathcal{P}$-coalgebra, $X^\vee$ is canonically a $\mathcal{P}$-algebra.

Here, in (i), by finite type we mean finite type as a dg module.

Proof. (i): The coalgebra structure maps are given by the following composites:

$$\mathcal{P}(n) \otimes X^\vee \to \mathcal{P}(n) \otimes (\mathcal{P}(n) \otimes X^{\otimes n})^\vee \cong \mathcal{P}(n) \otimes \mathcal{P}(n)^\vee \otimes (X^\vee)^{\otimes n} \to (X^\vee)^{\otimes n}$$

Here the first map is induced by the dual of the algebra structure map for X and the third map is induced by the canonical pairing $\mathcal{P}(n) \otimes \mathcal{P}(n)^\vee \to k[0]$.

(ii): The algebra structure maps are given by the following composites:

$$\mathcal{P}(n) \otimes (X^\vee)^{\otimes n} \to \mathcal{P}(n) \otimes (X^{\otimes n})^\vee \to \mathcal{P}(n) \otimes (\mathcal{P}(n) \otimes X)^\vee \cong \mathcal{P}(n) \otimes \mathcal{P}(n)^\vee \otimes X^\vee \to X^\vee$$

Here the second map is induced by the dual of the coalgebra structure map for X and the fourth map by the canonical pairing $\mathcal{P}(n) \otimes \mathcal{P}(n)^\vee \to k[0]$. $\square$

Proposition 2.14. *Let $\mathcal{P}$ be a dg operad over k. The reindexing operator yields functors:*

$$(-)^\dagger \colon \mathcal{P}\text{-}\mathsf{Alg} \to \mathcal{P}^\dagger\text{-}\mathsf{Alg} \qquad (-)^\dagger \colon \mathcal{P}\text{-}\mathsf{Coalg} \to \mathcal{P}^\dagger\text{-}\mathsf{Coalg}$$

Proof. Simply apply $(-)^\dagger$ to the given structure maps, $\mathcal{P}(n) \otimes X^{\otimes n} \to X$ or $\mathcal{P}(n) \otimes X \to X^{\otimes n}$, and note that $(-)^\dagger$ commutes with tensor products. In short, the structure maps are exactly as they were, only the degree assignments change. $\square$

Finally, before moving on, we consider pullbacks of (co)algebra structures over dg operads. Given an algebra or coalgebra X over $\mathcal{P}$, and an operad morphism $\mathcal{Q} \to \mathcal{P}$, by precompositon of the map to the (co)endomorphism operad, X also becomes a (co)algebra over $\mathcal{Q}$. This gives rise to a pull back functor from $\mathcal{P}$-(co)algebras to $\mathcal{Q}$-(co)algebras.

Proposition 2.15. *Let $\mathcal{P}$ and $\mathcal{Q}$ be dg operads over k and $f \colon \mathcal{Q} \to \mathcal{P}$ an operad map. Then the induced pull back functors*

$$f^* \colon \mathcal{P}\text{-}\mathsf{Alg} \to \mathcal{Q}\text{-}\mathsf{Alg} \qquad f^* \colon \mathcal{P}\text{-}\mathsf{Coalg} \to \mathcal{Q}\text{-}\mathsf{Coalg}$$

have right adjoints.

Proof. This follows from the adjoint functor theorem. See [lGL18]. $\square$

We let

$$f_* \colon \mathcal{Q}\text{-}\mathsf{Alg} \to \mathcal{P}\text{-}\mathsf{Alg} \qquad f_* \colon \mathcal{Q}\text{-}\mathsf{Coalg} \to \mathcal{P}\text{-}\mathsf{Coalg}$$

denote the right adjoints provided by the result above.

2.2 Categorical Constructions With (Co)algebras Over Operads

We now consider categorical constructions with algebras over operads. First of all, the following result ensures that such constructions always exist.

Proposition 2.16. *Given any dg operad $\mathcal{P}$ over k, the categories $\mathcal{P}$-Alg and $\mathcal{P}$-Coalg are bicomplete.*

Proof. See [Rez96]. $\square$

Next, the following result tells us that filtered colimits of algebras can always be computed at the level of dg modules.

Proposition 2.17. *Let $\mathcal{P}$ be a dg operad over k. If we have a filtered diagram of $\mathcal{P}$-algebras, the colimit is that same as that in dg modules; more precisely, filtered colimits of $\mathcal{P}$-algebras are created in the category of dg modules.*

Proof. See [Rez96]. □

Next, we consider the case of coproducts of algebras in detail. Let $\mathcal{P}$ be a dg operad over k, and let $\mathbf{P}$ be the associated free algebra functor. Recall that:

$$\mathbf{P}(X) = \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{\Sigma_n} X^{\otimes n}$$

Proposition 2.18. *Given $\mathcal{P}$-algebras A and B, the coproduct $A \amalg B$ can be constructed as the dg module coequalizer*

$$\mathbf{P}(\mathbf{P}A \oplus \mathbf{P}B) \rightrightarrows \mathbf{P}(A \oplus B) \to A \amalg B$$

where one of the two parallel maps is given by the algebra structure maps $\mathbf{P}A \to A$, $\mathbf{P}B \to B$, and the other is the composite of the map $\mathbf{P}(\mathbf{P}A \oplus \mathbf{P}B) \to \mathbf{P}(\mathbf{P}(A \oplus B))$, induced by the inclusions $A \to A \oplus B$, $B \to A \oplus B$, and the monadic structure map $\mathbf{P}(\mathbf{P}(A \oplus B)) \to \mathbf{P}(A \oplus B)$.

Proof. See [Fre98]. The intuition here is that $\mathbf{P}(A \oplus B)$ freely adds in all the products that must exist in the coproduct, and then identification of the two parallel maps enforces the product relations which were already present in A and B. See Remark 2.19 below for more details. □

We can give the above coequalizer a more concrete form. Note that

$$\begin{aligned}
\mathbf{P}(\mathbf{P}A \oplus \mathbf{P}B) &= \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{\Sigma_n} (\mathbf{P}A \oplus \mathbf{P}B)^{\otimes n} \\
&= \bigoplus_{n\geq 0} \bigoplus_{i+j=n} \mathcal{P}(n) \otimes_{\Sigma_i \times \Sigma_j} (\mathbf{P}A)^{\otimes i} \otimes (\mathbf{P}B)^{\otimes j} \\
&= \bigoplus_{i,j\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} (\mathbf{P}A)^{\otimes i} \otimes (\mathbf{P}B)^{\otimes j}
\end{aligned}$$

Similarly,

$$\mathbf{P}(A \oplus B) = \bigoplus_{i,j\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} A^{\otimes i} \otimes B^{\otimes j}$$

Making these substitutions, the coproduct $A \amalg B$ can be constructed as the following dg module coequalizer:

$$\bigoplus_{i,j\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} (\mathbf{P}A)^{\otimes i} \otimes (\mathbf{P}B)^{\otimes j} \rightrightarrows \bigoplus_{i,j\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} A^{\otimes i} \otimes B^{\otimes j} \to A \amalg B$$

Here one of the two parallel maps is given by the maps $\mathbf{P}A \to A$ and $\mathbf{P}B \to B$ provided by the $\mathcal{P}$-algebra structures of A and B, and the other is given by operadic composition as follows:

$$\begin{aligned}&\rho|\sigma^1|a_1^1|\cdots|a_{r_1}^1|\cdots|\sigma^i|a_1^i|\cdots|a_{r_i}^i|\tau^1|b_1^1|\cdots|b_{s_1}^1|\cdots|\tau^j|b_1^j|\cdots|b_{s_j}^j \\ &\qquad\qquad \mapsto [\rho, \sigma^1, \ldots, \sigma^i, \tau^1, \ldots, \tau^j]|a_1^1|\cdots|a_{r_1}^1|\cdots|a_1^i|\cdots|a_{r_i}^i|b_1^1|\cdots|b_{s_1}^1|\cdots|b_1^j|\cdots|b_{s_j}^j\end{aligned}$$

Here, for brevity, we have placed vertical bars where $\otimes$'s are more customary. Moreover, the term $[\rho, \sigma^1, \ldots, \sigma^i, \tau^1, \ldots, \tau^j]$ denotes the operadic composition of the formal operations.

Remark 2.19. The intuition behind the above construction, analogous to that of free products of monoids, is that, given any elements $a_1, \ldots, a_i$ in A and $b_1, \ldots, b_j$ in B, and also an $(i+j)$-ary operation σ in $\mathcal{P}(i+j)$, in the coproduct, there ought to exist an element which represents the result of the action of σ on the tuple $(a_1, \ldots, a_i, b_1, \ldots, b_j)$. For this purpose, we add in the formal element $\sigma \otimes a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j$. If we permute the tuple via a pair of permutations in $\Sigma_i \times \Sigma_j$, the result ought to be equivalent to that of acting on the operation σ, and so we take the equivariant tensor product. We ought also to have products where the a's and b's don't occur in the order above, but these are taken care of by the permutations in Σ_n which are not in any of the products $\Sigma_i \times \Sigma_j$. ||

We now consider a special case of coproducts of algebras, that in which both summands are free, namely $\mathbf{P}X \amalg \mathbf{P}Y$. In this case, the idea is that, in the second term in the coequalizer, we may generate simply on X and Y, and then may also omit the first term, needing to impose no relations.

Proposition 2.20. *Let $\mathcal{P}$ be a dg operad over k and X, Y dg modules over k. Then we have a natural isomorphism:*

$$\mathbf{P}X \amalg \mathbf{P}Y \cong \bigoplus_{i,j\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} X^{\otimes i} \otimes Y^{\otimes j}$$

Proof. Since $\mathbf{P}$, as a map to $\mathcal{P}$-algebras, is a left adjoint, it preserves coproducts and so we have:

$$\mathbf{P}X \amalg \mathbf{P}Y \cong \mathbf{P}(X \oplus Y) \cong \bigoplus_{i,j\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} X^{\otimes i} \otimes Y^{\otimes j}$$

□

2.3 Cell Algebras

Given any dg operad over k, we have a notion of a cell algebra over this operad, defined as follows.

Definition 2.21. Let $\mathcal{P}$ be a dg operad over k. A *cell $\mathcal{P}$-algebra* is a $\mathcal{P}$-algebra A such that there exists a cotower of $\mathcal{P}$-algebras

$$A_0 \to A_1 \to A_2 \to \cdots$$

and a colimiting map from this cotower to A, such that:

- A_0 is the initial $\mathcal{P}$-algebra, namely $\mathcal{P}(0)$.
- For each $n \geq 0$, the map $A_n \to A_{n+1}$ fits into an algebra pushout square

$$\begin{tikzcd} \mathbf{P}M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_n \arrow[d] \\ \mathbf{P}\mathrm{C}M \arrow[r] & A_{n+1} \end{tikzcd}$$

  where M is a dg module which is degreewise free and has zero differentials.

Remark 2.22. The condition that the dg module M be degreewise free and have zero differentials is equivalent to that it be a sum of copies of the standard sphere complexes. Moreover, the cone on such a complex, denooted $\mathrm{C}M$ above, is then a sum of the standard disk complexes. $||$

More generally, we also define cell maps as follows.

Definition 2.23. Let $\mathcal{P}$ be a dg operad over k. A *cell map* $A \to B$ of $\mathcal{P}$-algebras is a map such that there exists a cotower of $\mathcal{P}$-algebras

$$A_0 \to A_1 \to A_2 \to \cdots$$

and a colimiting map from this cotower to B, such that:

- $A_0 = A$ and the map $A_0 \to B$ is the given map $A \to B$.
- For each $n \geq 0$, the map $A_n \to A_{n+1}$ fits into an algebra pushout square

$$\begin{tikzcd} \mathbf{P}M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_n \arrow[d] \\ \mathbf{P}\mathrm{C}M \arrow[r] & A_{n+1} \end{tikzcd}$$

where M is a dg module which is degreewise free and has zero differentials.

Remark 2.24. Looking at the definitions, we see that a $\mathcal{P}$-algebra is a cell algebra if and only if the unique map $\mathcal{P}(0) \to A$ is a cell map. $||$

We now describe well-known concrete models of cell algebras. Let $\mathcal{P}$ be a dg operad over k and let A be a cell $\mathcal{P}$-algebra. Let also

$$A_0 \to A_1 \to A_2 \to \cdots$$

be a cell filtration of A and fix some choices $M_1, M_2, \ldots$ for the dg modules which appear in the attachment squares above. For each $n \geq 0$, let $N_n = \oplus_{i \leq n} M_i$, where $N_0 = 0$, and let also $N = \oplus_{i \geq 1} M_i$. Then one can construct models for the A_n and A, denoted say by B_n and B, as follows. Let $\mathcal{P}^{\#}$ denote the operad in graded k-modules formed by forgetting the differentials present in $\mathcal{P}$. For $n \geq 0$, we have that, as a graded module

$$B_n = \mathbf{P}^{\#}(N_n[1]) = \bigoplus_{k \geq 0} \mathcal{P}(k) \otimes_{\Sigma_k} (N_n[1])^{\otimes k}$$

and the differentials of the B_n are induced inductively, via the Leibniz rule, the attachment maps $\mathbf{P} M_n \to A_{n+1}$, together with the operadic composition maps of $\mathcal{P}$. In the limit, we have that, as a graded module

$$B = \mathbf{P}^{\#}(N[1]) = \bigoplus_{k \geq 0} \mathcal{P}(k) \otimes_{\Sigma_k} (N[1])^{\otimes k}$$

and in this case the differential is of course induced by those of the B_n. The precise analogue of the statement that the B_n and B are models, respectively, for the A_n and A is the following well-known result (see, for example, [Man01] and [Fre09]).

Proposition 2.25. *Let $\mathcal{P}$ be a dg operad over k. Given the cell algebra A, its skeleta A_n and the algebras B_n and B defined above, there exists a diagram of isomorphisms of $\mathcal{P}$-algebras as follows*

$$\begin{array}{ccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \end{array}$$

and this diagram induces, in the limit, an isomorphism $A \to B$. $\square$

2.4 Enveloping Operads

Earlier, we considered coproducts of algebras over operads. We now consider the special case of the coproducts where one summand is free, namely the coproducts $A \amalg \mathbf{P}X$. In this case, we need only impose relations from the algebra structure in A. This leads us to first define a general construction on A, that of the enveloping operad of A, which, as the name suggests, also has relations to other notions, such as that of representations of A.

Definition 2.26. Let $\mathcal{P}$ be a dg operad over k and A a $\mathcal{P}$-algebra. The *enveloping operad* of A, denoted $\mathcal{U}^A$, is defined as follows. For each $j \geq 0$, the dg $k[\Sigma_j]$-module $\mathcal{U}^A(j)$ is defined to be the dg module coequalizer

$$\bigoplus_{i\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i} (\mathbf{P}A)^{\otimes i} \rightrightarrows \bigoplus_{i\geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i} A^{\otimes i} \to \mathcal{U}^A(j)$$

where one of the two parallel maps is induced by the $\mathcal{P}$-algebra structure map $\mathbf{P}A \to A$ of A and the other by the composition product of $\mathcal{P}$. Morever, the operadic structure maps of $\mathcal{U}^A$ are induced by those of $\mathcal{P}$.

Example 2.27. (i) In operadic degree 0, we have $\mathcal{U}^A(0) \cong A$, where the universal coequalizer map is given by the $\mathcal{P}$-algebra structure map of A.

(ii) In operadic degree 1, as usual, $\mathcal{U}^A(1)$ forms a unital associative algebra via the composition product $\mathcal{U}^A(1) \otimes \mathcal{U}^A(1) \to \mathcal{U}^A(1)$. By definition, this algebra $\mathcal{U}^A(1)$ is the *enveloping algebra* of A – see [GK94] and [Fre09].

Remark 2.28. In the construction of the $\mathcal{U}^A(j)$, note that the two parallel maps preserve the $i = 0$ summands, and moreover that, the coequalizer of just these two summands is simply $\mathcal{P}(j)$. It follows that, for each A and $j \geq 0$, $\mathcal{U}^A(j)$ is equipped with a canonical map $\mathcal{P}(j) \to \mathcal{U}^A(j)$; in fact, these assemble into an operad map $\mathcal{P} \to \mathcal{U}^A$. ||

Proposition 2.29. *Let $\mathcal{P}$ be a dg operad over k and A a $\mathcal{P}$-algebra. Then we have an equivalence*

$$\mathcal{U}^A\text{-}\mathsf{Alg} \simeq \mathcal{P}\text{-}\mathsf{Alg}_{A/}$$

between the category of $\mathcal{U}^A$-algebras and the category of $\mathcal{P}$-algebras under A.

Proof. We shall describe the correspondence between the objects of the two categories; for further details, see [GJ], [Man01] or [Fre09]. Let B be a $\mathcal{U}^A$-algebra and let $\mathbf{U}^A$ denote free $\mathcal{U}^A$-algebra functor, so that we are provided a map $\mathbf{U}^A B \to B$. To endow B with a $\mathcal{P}$-algebra structure we simply pull back across the canonical map $\mathcal{P} \to \mathcal{U}^A$ given in Remark 2.28. Moreover, in the case $j = 0$, inside $\mathbf{U}^A B$, we find the term $\mathcal{U}^A(0) = A$ as per Example 2.27. Thus we have a map $A \to B$, which one can check is a $\mathcal{P}$-algebra map.

Now suppose that B is instead a $\mathcal{P}$-algebra equipped with a $\mathcal{P}$-algebra map $A \to B$. We wish to construct a map $\mathbf{U}^A B \to B$. We have that $\mathcal{U}^A(j) \otimes_{\Sigma_j} B^{\otimes j}$ is given by the following coequalizer:

$$\bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} (\mathbf{P}A)^{\otimes i} \otimes B^{\otimes j} \rightrightarrows \bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} A^{\otimes i} \otimes B^{\otimes j} \to \mathcal{U}^A(j) \otimes_{\Sigma_j} B^{\otimes j}$$

Via the map $A \to B$, from the first term, we may pass to

$$\bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} (\mathbf{P}B)^{\otimes i} \otimes B^{\otimes j}$$

and from the second, to

$$\bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} B^{\otimes i} \otimes B^{\otimes j}$$

which we note is simply $\mathbf{P}B$. Moreover, upon taking the coequalizer, we now simply get B, so that we have an induced map $\mathcal{U}^A(j) \otimes_{\Sigma_j} B^{\otimes j} \to B$. Summing these up over $j \geq 0$, we get the desired map $\mathbf{U}^A B \to B$. Finally, one can check that this map does indeed satisfy the properties required in order to provide a $\mathcal{U}^A$-algebra structure for B, as desired. $\square$

Example 2.30. (i) An easy check of the definitions and universal properties demonstrates that the enveloping operad $\mathcal{U}^{\mathcal{P}(0)}$ of the initial $\mathcal{P}$-algebra $\mathcal{P}(0)$ is simply $\mathcal{P}$; that is, for each $j \geq 0$, we have $\mathcal{U}^{\mathcal{P}(0)}(j) = \mathcal{P}(j)$. This is what one ought to expect in view of Proposition 2.29 and the fact that $\mathcal{P}(0)$ is initial. .

(ii) In the case of a free algebra $\mathbf{P}X$, in forming the enveloping operad, we can simply generate on X rather than $\mathbf{P}X$ and dispense with the relations imposed by the parallel maps in the coequalizer, so that:

$$\mathcal{U}^{\mathbf{P}X}(j) \cong \bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i} X^{\otimes i}$$

Note that we then have

$$\mathbf{U}^{\mathbf{P}X} Y \cong \bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i \times \Sigma_j} X^{\otimes i} \otimes Y^{\otimes j}$$

which, by Proposition 2.20 is simply $\mathbf{P}X \amalg \mathbf{P}Y$, as we ought to expect in view of Proposition 2.29.

Proposition 2.31. *Let $\mathcal{P}$ be a dg operad over k and let A be a $\mathcal{P}$-algebra. Given any dg module X, we have a natural, in X, isomorphism of $\mathcal{P}$-algebras under A as follows:*

$$A \amalg \mathbf{P}X \cong \mathbf{U}^A X$$

The intuition here is that $\mathcal{U}^A(j)$ incorporates the relations needed to preserve the algebra structure of A, and no relations need be imposed for the free algebra $\mathbf{P}X$.

Proof. We present two arguments. On the one hand, we can replace the general coequalizer model for coproducts in Proposition 2.18 with a coequalizer of a pair of maps from $\mathbf{P}(\mathbf{P}A \otimes X)$ to $\mathbf{P}(A \oplus X)$, and then do a comparison of universal properties. This will demonstrate that we have an isomorphism of $\mathcal{P}$-algebras, and then we can also compare the canonical maps from A. Alternatively, we can argue as follows. By Proposition 2.29 and its proof, the right hand side, being the free $\mathcal{U}^A$-algebra on X, is the free $\mathcal{P}$-algebra under A on X. On the other hand, we can construct this left adjoint in steps:

$$\begin{tikzcd} \mathcal{P}\text{-}\mathsf{Alg}_{A/} \arrow[r] \arrow[r, phantom, shift right=2, "\top"] & \mathcal{P}\text{-}\mathsf{Alg} \arrow[r] \arrow[l, bend left, dashed] \arrow[r, phantom, shift right=2, "\top"] & \mathsf{DG}_k \arrow[l, bend left, dashed] \end{tikzcd}$$

Here the two right adjoints are forgetful, the first left adjoint freely constructs $\mathbf{P}X$, and the second then freely constructs $A \amalg \mathbf{P}X$, giving the desired natural isomorphism. □

Next, we describe a well-known concrete model for the enveloping operads of cell algebras. Let $\mathcal{P}$ be a dg operad over k and let A be a cell $\mathcal{P}$-algebra. Let also

$$A_0 \to A_1 \to A_2 \to \cdots$$

be a cell filtration of A and fix some choices $M_1, M_2, \dots$ for the dg modules which appear in the attachment squares above. For each $n \geq 0$, let $N_n = \oplus_{i \leq n} M_i$, where $N_0 = 0$, and let also $N = \oplus_{i \geq 1} M_i$. Then the models, for which we refer to [Man01], for the $\mathcal{U}^{A_n}(j)$ and $\mathcal{U}^A(j)$, for $n, j \geq 0$, are as follows.

Proposition 2.32. *Let $\mathcal{P}$ be a dg operad over k. Given the cell $\mathcal{P}$-algebra A and its skeleta A_n as above, for each $n, j \geq 0$, we have that, as a $k[\Sigma_j]$-module:*

$$\mathcal{U}^{A_n}(j) = \bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i} (N_n[1])^{\otimes i}$$

Similarly, for each $j \geq 0$, we have that, as a $k[\Sigma_j]$-module:

$$\mathcal{U}^A(j) = \bigoplus_{i \geq 0} \mathcal{P}(i+j) \otimes_{\Sigma_i} (N[1])^{\otimes i}$$

□

2.5 The Homotopy Theory of Operads and Their Algebras

In this section, we consider the homotopy theories, in the sense of Quillen model structures, of operads and their algebras. First, we recall model structures on dg operads over k. In this section, we shall let Op_k denote the category of dg operads over k, which is to say that Op_k denotes either of the two categories $\mathsf{Op}(\mathsf{Ch}_k)$ and $\mathsf{Op}(\mathsf{Co}_k)$, for both of which all that is said below will make sense. In fact, we do not have a model structure on all of Op_k. Instead, we shall restrict to two possible cases. We say that an operad $\mathcal{P}$ is *reduced* if $\mathcal{P}(0) = k[0]$, and moreover we require that maps of reduced operads be the identity in arity zero, which leads us to the category $\mathsf{Op}_k^{\mathrm{r}}$ of reduced operads. On the other hand, we say that an operad $\mathcal{P}$ is *null-reduced* if $\mathcal{P}(0) = 0$, and of course maps between null-reduced operads will necessarily be the identity in arity zero, which leads us to the category $\mathsf{Op}_k^{\mathrm{nr}}$ of null-reduced operads.

Proposition 2.33. *The categories $\mathsf{Op}_k^{\mathrm{r}}$ and $\mathsf{Op}_k^{\mathrm{nr}}$ of reduced and null-reduced operads, respectively, admit a Quillen model structure where the weak equivalences and fibrations are, respectively, the levelwise quasi-isomorphisms and levelwise epimorphisms.*

Proof. See [BM03] for the case of reduced operads and [Hin97], together with [Hin03], for the case of null-reduced operads. In either case, the idea is to pull back the projective model structure on dg modules across forgetful functors. □

As such, given an operad which is reduced or null-reduced, via the above model structure, we can speak of the cofibrancy of this operad.

Now let us consider model structures on algebras over operads. We of course would like the weak equivalences to simply be maps whose underlying dg module map is a quasi-isomorphism, as the weak equivalences of algebras ought to simply be the weak equivalences of dg modules which respect the algebraic structure that is now present. To get an actual model structure with these weak equivalences, the idea is once to more pull back the projective model structure on dg modules across the obvious forgetful functor. This leads us to the following definition.

Definition 2.34. Let $\mathcal{P}$ be a dg operad over k. We say that $\mathcal{P}$ is *admissible* if $\mathcal{P}$-Alg admits a model structure where the weak equivalences and fibrations are the quasi-isomorphisms and degreewise epimorphisms, respectively.

Proposition 2.35. *Let $\mathcal{P}$ be a dg operad over k. If $\mathcal{P}$ is a cofibrant reduced operad or a cofibrant null-reduced operad, it is admissible.*

Proof. See [BM03] for the reduced case and [Hin01] for the null-reduced case. □

Recall the adjoints f_* to operad algebra pull back maps in Proposition 2.15.

Proposition 2.36. *Let $\mathcal{P}$ and $\mathcal{Q}$ be dg operads over k, $f\colon \mathcal{Q} \to \mathcal{P}$ an operad map and*

$$\begin{tikzcd} \mathcal{P}\text{-}\mathsf{Alg} \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & \mathcal{Q}\text{-}\mathsf{Alg} \arrow[l, shift left, "f_*"] \end{tikzcd}$$

the induced adjunction. If $\mathcal{P}$ and $\mathcal{Q}$ are reduced and admissible, or null-reduced and admissible, this adjunction is a Quillen adjunction.

Proof. See [BM03] for the reduced case and [Hin01] for the null-reduced case. $\square$

In fact, we will be interested in a weakening of model structures, to semi-model structures, and a corresponding weakening of admissibility to semi-admissibility. We recall first the definition of semi-model categories. In short, a semi-model category is (almost) exactly a model category except that the factorization $\to = \overset{\sim}{\hookrightarrow}\twoheadrightarrow$ and the lifting property $(\overset{\sim}{\hookrightarrow}) \boxslash (\twoheadrightarrow)$ (which is to say, those that involve trivial cofibrations) are required to hold only in the case where the source is cofibrant.

Definition 2.37. A *Quillen semi-model category* is a category M, together with three specified classes of morphisms, $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$, such that the following hold:

- M is bicomplete.
- The class $\mathcal{W}$ satisfies 2-out-of-3.
- The classes $\mathcal{W}, \mathcal{C}$ and $\mathcal{F}$ are closed under retracts.

Given the above, we note that, by the first property, M must possess both an initial object $\emptyset$ and a final object $*$. We then define cofibrant objects and fibrant objects as per usual, where we note that the notions are invariant under choices of initial and final objects as they lead to maps which are retracts of one another. Having made these definitions, we also require the following (a prefix of "cof" means that the object is required to be cofibrant):

- We have the following liftings for maps $A \to B$, $X \to Y$:

$$\begin{tikzcd} \text{cof } A \arrow[d, hook, "\sim"'] \arrow[r, phantom, "\boxslash"] & X \arrow[d, two heads] \\ B & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[d, hook] \arrow[r, phantom, "\boxslash"] & X \arrow[d, two heads, "\sim"] \\ B & Y \end{tikzcd}$$

- We have the following factorizations for a map $A \to B$:

$$\text{cof } A \overset{\sim}{\hookrightarrow} \cdot \twoheadrightarrow B \qquad A \hookrightarrow \cdot \overset{\sim}{\twoheadrightarrow} B$$

Finally, we add in:

- Fibrations are closed under composition, products and base change.

Remark 2.38. Similar notions of semi-model categories have been considered by White [Whi17], Spitzweck [Spi01] and Fresse [Fre09]. $||$

With this definition, one can run through the the standard arguments for model categories to verify that we can still perform analogous constructions of the derived category and derived functors, with appropriate modifications. In particular, one can still construct the derived category via bifibrant replacements – see Theorem 2.13 in [Man01], and moreover, as for derived functors, the relevant result which we will need later is the following.

Proposition 2.39. *Let $L\colon \mathsf{E} \to \mathsf{M}$ and $R\colon \mathsf{M} \to \mathsf{E}$ be left and right adjoints between a semi-model category E and a model category M. Then we have the following:*

(i) If L preserves cofibrations between cofibrant objects and R preserves fibrations, then the left derived functor of L and the right derived functor of R exist and are adjoint. Moreover, L converts weak equivalences between cofibrant objects to weak equivalences, and the restriction of the left derived functor of L to the cofibrant objects is naturally isomorphic to the derived functor of the restriction of L.

(ii) Suppose that (i) holds and in addition for any cofibrant object A in E and any fibrant object Y in M, a map $A \to RY$ is a weak equivalence if and only if the adjoint $LA \to Y$ is a weak equivalence. Then the left derived functor of L and the right derived functor of R are inverse equivalences.

Moreover, we also have the following:

(iii) The hypothesis in (i) above is equivalent to each of the following:

- *L preserves cofibrations between cofibrant objects and acyclic cofibrations between cofibrant objects.*
- *R preserves fibrations and acyclic fibrations.*

Proof. These follow mostly by the standard model category theoretic arguments. See Theorems 2.14 and 2.15 in [Man01]. $\square$

We can now define semi-admissibility for dg operads.

Definition 2.40. Given a dg operad $\mathcal{P}$ over k, we say that it is *semi-admissible* if $\mathcal{P}$-Alg admits a semi-model structure where the weak equivalences and fibrations are the quasi-isomorphisms and degreewise epimorphisms, respectively.

We shall now give criteria for the admissibility and semi-admissibility of a dg operad over k, due to Mandell in [Man01] and Hinich in [Hin97].

Proposition 2.41. *Let $\mathcal{P}$ be a dg operad over k, $\mathbf{P}$ the associated free algebra functor. Then we have the following:*

(i) If, for any $\mathcal{P}$-algebra A, the natural map

$$A \to A \amalg \mathbf{P}(\mathbb{D}^n)$$

where $\mathbb{D}^n$ is a disk complex, is a quasi-isomorphism, $\mathcal{P}$ is admissible.

(ii) If the above condition holds for any cell $\mathcal{P}$-algebra, $\mathcal{P}$ is semi-admissible.

Moreover, in either case, the cofibrations are exactly the retracts of cell maps.

Proof. Let us define the weak equivalences, fibrations and cofibrations as we must. Given a map $f\colon A \to B$ of $\mathcal{P}$-algebras, we set the following:

- We say that it is a *weak equivalence* if it is a quasi-isomorphism.
- We say that it is a *fibration* if it is a degreewise epimorphism.
- We say that it is a *cofibration* if it has the left lifting property w.r.t those maps which are both weak equivalences and fibrations.

Now we proceed to verify the (semi-)model structure axioms. We know by Proposition 2.16 that $\mathcal{P}$-Alg is bicomplete. Moreover, 2-out-of-3 for weak equivalences is clear, just as in the case of dg modules. Closure under retracts for weak equivalences and fibrations once again follows from the analogous fact in the case of dg modules, and in the case of cofibrations follows from the fact that any class of morphisms defined by a lifting property is always closed under retracts. Thus, it remains to verify the lifting and factorization axioms.

Let $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ denote the classes of weak equivalences, cofibrations and fibrations. We shall first demonstrate the factorization axioms. First we demonstrate the half of the axiom which says that any map can be factored into a cofibration followed by a trivial fibration. To see this, consider an arbitrary $\mathcal{P}$-algebra map $f\colon A \to B$ and note that, by the standard small object argument, we have that we can factor f as $p \circ i$ where $i\colon A \to C$ is a cell map and $p\colon C \to B$ has the right lifting property w.r.t all maps of the form $\mathbf{P}M \to \mathbf{P}\mathrm{C}M$ where M is a dg module which is degreewise free and has zero differentials. Now, any cell map is necessarily a cofibration (as the maps $\mathbf{P}M \to \mathbf{P}\mathrm{C}M$ are cofibrations by adjointness and cofibrations are closed under pushouts and transfinite compositions as this holds for all classes of morphisms defined by a left lifting property). Thus i is a cofibration. Moreover, we claim that p is a trivial fibration. To see this, note that, by adjointness, as a dg map, p has the right lifting property with respect to all dg maps of the form $M \to \mathrm{C}M$. These maps, in the cases where M is a sphere complex reduce to the maps $\mathbb{S}^{n-1} \to \mathbb{D}^n$, which constitute the generating cofibrations in the standard cofibrantly generated projective model structure on dg modules. It follows that p is a trivial fibration of dg modules, and so of $\mathcal{P}$-algebras.

Now, before verifying the remaining model structure axioms, we demonstrate that the cofibrations are exactly the retracts of cell maps. For a dg module M which is degreewise free and has zero differentials, by adjointness, the map $\mathbf{P}M \to \mathbf{B}\mathrm{C}M$ has the left lifting property with respect to trivial fibrations (as above, in the cases where M is a sphere complex, these maps reduce to the generating cofibrations in the standard cofibrantly generated projective model structure on dg modules; moreover, for more general M, since M is degreewise free and has zero differentials, these maps are coproducts of the maps $\mathbb{S}^{n-1} \to \mathbb{D}^n$). Since left lifting properties are closed under retracts, pushouts and transfinite compositions, we have that all retracts of cell maps have the left lifting property against trivial fibrations, and so are cofibrations, by definition of cofibrations. Next, given a cofibration $f\colon A \to B$. As above, we may factor it as $f = pi$ where i is a cell map and p is a trivial fibration. As f is a cofibration, we have $f \boxslash p$, and so, by the standard retract argument, f must be a retract of i and thus is a retract of a cell map, as desired.

Next, note that if A is cofibrant, because, as above, cofibrations are retracts of cell maps, A must be a retract of a cell algebra and moreover, the map $A \to A \amalg \mathbf{P}(\mathbb{D}^n)$ must be a retract of a map $A' \to A' \amalg \mathbf{P}(\mathbb{D}^n)$ where A' is cell. Thus, the condition supposed in (ii) in fact holds not only for cell algebras, but for all cofibrant algebras.

Now let us consider the half of the factorization axiom which says that any map can be factored into a trivial cofibration followed by a fibration; the following proof works in all cases in the case of (i) and for those cases in which the source is cofibrant in the case of (ii). Once again, let $f\colon A \to B$ be an arbitrary $\mathcal{P}$-algebra map. We can factor f as follows:

$$A \xrightarrow{i} A \amalg \left(\coprod_{b \in B} \mathbf{P}(\mathbb{D}^{\deg(b)}) \right) \xrightarrow{p} B$$

Here i is the canonical inclusion of A. On the other hand, p, on the summand A is the given map f, while on a summand $\mathbf{P}(\mathbb{D}^{\deg(b)})$, is the map induced by the dg map $\mathbb{D}^{\deg(b)} \to B$ which sends the generator in degree $\deg(b)$ to b. It is clear then that p is fibration. Moreover, i is a cofibration as it is the coproduct of the identity on A and the cell maps $\mathcal{P}(0) \to \mathbf{P}(\mathbb{D}^{\deg(b)})$. Finally, i is also a weak equivalence. To see this, choose some ordering (according to some ordinal) on the elements $b \in B$ and write the coproduct as a composition:

$$A \to A \amalg \mathbf{P}(\mathbb{D}^{n_1}) \to A \amalg \mathbf{P}(\mathbb{D}^{n_1}) \amalg \mathbf{P}(\mathbb{D}^{n_2}) \to \cdots$$

Each map in this cotower is a quasi-isomorphism by the assumption in the proposition statement. If the cotower is finite, the resulting composite is of course a quasi-isomorphism. If it is infinite, the composite is once again a quasi-isomorphism because the forgetful functor to dg modules commutes with filtered colimits as per Proposition 2.17 and the result of course holds in dg modules.

Finally, it remains to verify the lifting properties. That $\mathcal{C} \boxslash \mathcal{W} \cap \mathcal{F}$ is immediate by the definition of cofibrations. Thus we need to demonstrate that $\mathcal{W} \cap \mathcal{C} \boxslash \mathcal{F}$; again, the following proof works in all cases in the case of (i) and for those cases in which the source is cofibrant in the case of (ii). Let $f \colon A \to B$ be a trivial cofibration. As above, factor f as follows:

$$A \xrightarrow{i} A \amalg \left(\coprod_{b \in B} \mathbf{P}(\mathbb{D}^{\deg(b)}) \right) \xrightarrow{p} B$$

By 2-out-of-3, p is now not only a fibration but a trivial fibration. Since, $\mathcal{C} \boxslash \mathcal{W} \cap \mathcal{F}$, we have $f \boxslash p$. Thus, by the standard retract argument, f is a retract of i. Since the identity on A obviously has the left lifting property against fibrations, and, by adjointness so do the maps $\mathcal{P}(0) \to \mathbf{P}(\mathbb{D}^{\deg(b)})$ (note that $\mathcal{P}(0)$ is the free $\mathcal{P}$-algebra on the zero complex), we have that f has the left lifting property against fibrations, as such a property is closed under coproducts and retracts.

This completes the proof, except that in case (ii), we also need to verify that the fibrations are closed under composition, products and base change, which is not automatic in semi-model categories. This holds because the forgetful functor to dg modules commutes with limits and because fibrations of algebras are detected at the dg module level. □

2.6 Operadic (De)suspensions

We now discuss notions of operadic suspension and desuspension, which will be important for us when we discuss stabilizations of $\mathbb{E}_\infty$ operads. We shall need to explicitly distinguish the case of chain complexes and cochain complexes, i.e., of chain operads and cochain operads.

If X is a chain complex and $n \in \mathbb{Z}$, we let $X[n]$ be the chain complex where $X[n]_d = X_{n-d}$. Note that we do not change the sign of any differential.

Definition 2.42. Given a chain operad $\mathcal{P}$ over k, it's *operadic suspension* $\Sigma\mathcal{P}$ is defined by setting

$$\Sigma\mathcal{P} := \mathcal{P} \otimes_k \mathcal{E}nd_{k[1]}$$

where $(\mathcal{P} \otimes \mathcal{E}nd_{k[1]})(n) = \mathcal{P}(n) \otimes \mathcal{E}nd_{k[1]}(n)$, the symmetric group Σ_n acts diagonally, the identity is the diagonal one and the composition structure maps are those which permute tensor factors and then apply the composition maps for $\mathcal{P}$ and $\mathcal{E}nd_{k[1]}$.

Remark 2.43. Intuition for the operadic suspension may be provided as follows. The chain complex $k[1]$ can be thought of as the k-linear 1-sphere. The endomorphism operad on it then as a dg operadic analogue of the 1-sphere, and so $\mathcal{P} \otimes \mathcal{E}nd_{k[1]}$ as a smash product with an analogue of the 1-sphere. ||

Note that we have

$$\mathcal{E}nd_{k[1]}(n)_d = \{\text{maps } k[1]^{\otimes n} \to k[1] \text{ in } \mathsf{Gr}_k \text{ of degree } d\}$$

where Gr_k denotes the category of $\mathbb{Z}$-graded k-modules. Thus $\mathcal{E}nd_{k[1]}(n)_d$ is zero if $d \neq 1 - n$ and is k otherwise, so that $\mathcal{E}nd_{k[1]}(n) = k[1 - n]$. Thus we have:

$$(\Sigma\mathcal{P})(n) \cong \mathcal{P}(n) \otimes k[1 - n] \cong \mathcal{P}(n)[1 - n]$$

Next, we consider operadic desuspensions of chain operads.

Definition 2.44. Given a chain operad $\mathcal{P}$ over k, it's *operadic desuspension* $\Sigma\mathcal{P}$ is defined by setting

$$\Sigma^{-1}\mathcal{P} := \mathcal{P} \otimes_k \mathcal{E}nd_{k[-1]}$$

where $(\mathcal{P} \otimes \mathcal{E}nd_{k[-1]})(n) = \mathcal{P}(n) \otimes \mathcal{E}nd_{k[-1]}(n)$, the symmetric group Σ_n acts diagonally, the identity is the diagonal one and the composition structure maps are those which permute tensor factors and then apply the composition maps for $\mathcal{P}$ and $\mathcal{E}nd_{k[-1]}$.

Remark 2.45. The intuition now is of course that we think of $k[-1]$ and $\mathcal{E}nd_{k[-1]}$ as analogues of the (-1)-sphere. ||

In this case, we find that:

$$(\Sigma^{-1}\mathcal{P})(n) \cong \mathcal{P}(n) \otimes_k k[n - 1] \cong \mathcal{P}(n)[n - 1]$$

Now we consider the case of cochain complexes and cochain operads. In this case, the roles of $k[1]$ and $k[-1]$ are swapped. If X is a cochain complex and $n \in \mathbb{Z}$, we again let $X[n]$ be the cochain complex where $X[n]_d = X_{n-d}$. Note again that we do not change the sign of any differential.

Definition 2.46. Given a cochain operad $\mathcal{P}$ over k, it's *operadic suspension* $\Sigma\mathcal{P}$ is defined by setting

$$\Sigma\mathcal{P} := \mathcal{P} \otimes_k \mathcal{E}nd_{k[-1]}$$

where $(\mathcal{P} \otimes \mathcal{E}nd_{k[-1]})(n) = \mathcal{P}(n) \otimes \mathcal{E}nd_{k[-1]}(n)$, the symmetric group Σ_n acts diagonally, the identity is the diagonal one and the composition structure maps are those which permute tensor factors and then apply the composition maps for $\mathcal{P}$ and $\mathcal{E}nd_{k[-1]}$.

Definition 2.47. Given a cochain operad $\mathcal{P}$ over k, it's *operadic desuspension* $\Sigma^{-1}\mathcal{P}$ is defined by setting

$$\Sigma^{-1}\mathcal{P} := \mathcal{P} \otimes_k \mathcal{E}nd_{k[1]}$$

where $(\mathcal{P} \otimes \mathcal{E}nd_{k[-1]})(n) = \mathcal{P}(n) \otimes \mathcal{E}nd_{k[1]}(n)$, the symmetric group Σ_n acts diagonally, the identity is the diagonal one and the composition structure maps are those which permute tensor factors and then apply the composition maps for $\mathcal{P}$ and $\mathcal{E}nd_{k[1]}$.

Earlier, we discussed a reindexing operator $(-)^\dagger$ between chain and cochain complexes. We now discuss how this construction behaves with respect to operadic (de)suspensions.

Proposition 2.48. *Let $\mathcal{P}$ be a chain or cochain operad over k. Then we have:*

$$(\Sigma\mathcal{P})^\dagger = \Sigma(\mathcal{P}^\dagger) \quad \textit{and} \quad (\Sigma^{-1}\mathcal{P})^\dagger = \Sigma^{-1}(\mathcal{P}^\dagger)$$

Proof. An easy direct check one degree at a time. Note that, for either identity, the notion of suspension on one side is that for chain operads, while on the other is for cochain operads. $\square$

2.7 On (Co)algebras Over (De)suspended Operads

Let $\mathcal{P}$ be a dg operad over k. We now wish to discuss the relation between (co)algebra structures over $\mathcal{P}$ and (co)algebra structures over the (de)suspensions $\Sigma^r\mathcal{P}$, where $r \in \mathbb{Z}$. Once again, we must explicitly distinguish between the chain and cochain cases.

Proposition 2.49. *Let $\mathcal{P}$ be a chain operad over k. Then, for each $r \in \mathbb{Z}$, we have functors as follows:*

$$\mathcal{P}\text{-}\mathsf{Alg} \xrightarrow{(\cdot)[r]} \Sigma^r\mathcal{P}\text{-}\mathsf{Alg} \qquad \mathcal{P}\text{-}\mathsf{Coalg} \xrightarrow{(\cdot)[-r]} \Sigma^r\mathcal{P}\text{-}\mathsf{Coalg}$$

On the other hand, if $\mathcal{P}$ is a cochain operad over k, then, we have functors as follows:

$$\mathcal{P}\text{-}\mathsf{Alg} \xrightarrow{(\cdot)[-r]} \Sigma^r\mathcal{P}\text{-}\mathsf{Alg} \qquad \mathcal{P}\text{-}\mathsf{Coalg} \xrightarrow{(\cdot)[r]} \Sigma^r\mathcal{P}\text{-}\mathsf{Coalg}$$

Proof. We first consider the chain case. Let X be an algebra over $\mathcal{P}$. We wish to show that $X[r]$ is an algebra over $\Sigma^r\mathcal{P}$. To do so, we must construct maps:

$$(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X[r]^{\otimes n} \to X[r]$$

These are as follows:

$$(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X[r]^{\otimes n} = \mathcal{P}(n)[r - rn] \otimes_{\Sigma_n} X^{\otimes n}[rn] = (\mathcal{P}(n) \otimes_{\Sigma_n} X^{\otimes n})[r] \to X[r]$$

Now suppose that X is a coalgebra over $\mathcal{P}$. We wish to show that $X[-r]$ is a coalgebra over $\Sigma^r\mathcal{P}$. To do so, we must construct maps:

$$(\Sigma^r\mathcal{P}(n)) \otimes_{\Sigma_n} X[-r] \to X[-r]^{\otimes n}$$

These are as follows:

$$\begin{aligned}(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X[-r] = \mathcal{P}(n)[r - rn] \otimes_{\Sigma_n} X[-r] = (\mathcal{P}(n) \otimes_{\Sigma_n} X)[-rn] \to X^{\otimes n}[-rn] \\ = X[-r]^{\otimes n}\end{aligned}$$

Now we consider the cochain case. Let X be an algebra over $\mathcal{P}$. We wish to show that $X[-r]$ is an algebra over $\Sigma^r\mathcal{P}$. To do so, we must construct maps:

$$(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X[-r]^{\otimes n} \to X[-r]$$

These are as follows:

$$(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X[-r]^{\otimes n} = \mathcal{P}(n)[rn - r] \otimes_{\Sigma_n} X^{\otimes n}[-rn] = (\mathcal{P}(n) \otimes_{\Sigma_n} X^{\otimes n})[-r] \to X[-r]$$

Finally, suppose that X is a coalgebra over $\mathcal{P}$. We wish to show that $X[r]$ is a coalgebra over $\Sigma^r\mathcal{P}$. To do so, we must construct maps:

$$(\Sigma^r\mathcal{P}(n)) \otimes_{\Sigma_n} X[r] \to X[r]^{\otimes n}$$

These are as follows:

$$(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X[r] = \mathcal{P}(n)[rn - r] \otimes_{\Sigma_n} X[r] = (\mathcal{P}(n) \otimes_{\Sigma_n} X)[rn] \to X^{\otimes n}[rn] = X[r]^{\otimes n}$$

$\square$

Now let us discuss free algebras over (de)suspended operads. Given a dg operad $\mathcal{P}$, and a (de)suspension $\Sigma^r\mathcal{P}$, we shall let $\Sigma^r\mathbf{P}$ denote the monad and free algebra functor associated to $\Sigma^r\mathcal{P}$. Note though that we are not (de)suspending the monad, but rather the operad.

Proposition 2.50. *Let $\mathcal{P}$ denote a chain operad over k. Then, for a chain complex X, we have:*

$$(\Sigma^r\mathbf{P})X = \mathbf{P}(X[-r])[r]$$

On the other hand, if $\mathcal{P}$ is a cochain operad over k, for a cochain complex X, we have:

$$(\Sigma^r\mathbf{P})X = \mathbf{P}(X[r])[-r]$$

Proof. We first consider the chain case. We have:

$$\begin{aligned}
(\Sigma^r\mathbf{P})X &= \bigoplus_{n\geq 0}(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X^{\otimes n} \\
&= \bigoplus_{n\geq 0}\mathcal{P}(n)[r-rn] \otimes_{\Sigma_n} X^{\otimes n} \\
&= \bigoplus_{n\geq 0}(\mathcal{P}(n) \otimes_{\Sigma_n} X^{\otimes n}[-rn])[r] \\
&= \left(\bigoplus_{n\geq 0}\mathcal{P}(n) \otimes_{\Sigma_n} X[-r]^{\otimes n}\right)[r] \\
&= \mathbf{P}(X[-r])[r]
\end{aligned}$$

Now we consider the cochain case. We have that:

$$\begin{aligned}
(\Sigma^r\mathbf{P})X &= \bigoplus_{n\geq 0}(\Sigma^r\mathcal{P})(n) \otimes_{\Sigma_n} X^{\otimes n} \\
&= \bigoplus_{n\geq 0}\mathcal{P}(n)[rn-r] \otimes_{\Sigma_n} X^{\otimes n} \\
&= \bigoplus_{n\geq 0}(\mathcal{P}(n) \otimes_{\Sigma_n} X^{\otimes n}[rn])[-r] \\
&= \left(\bigoplus_{n\geq 0}\mathcal{P}(n) \otimes_{\Sigma_n} X[r]^{\otimes n}\right)[-r] \\
&= \mathbf{P}(X[r])[-r]
\end{aligned}$$

□

CHAPTER 3

$\mathbb{E}_\infty$ Operads and p-Adic Homotopy Types

In this chapter, we shall introduce a particular class of dg operads, the $\mathbb{E}_\infty$ operads. The ground field will now be taken to be $\mathbb{F}_p$ for an unspecified but fixed prime p.

3.1 $\mathbb{E}_\infty$ Operads

We begin with the definition of $\mathbb{E}_\infty$ operads.

Definition 3.1. A chain operad $\mathcal{P}$ over $\mathbb{F}_p$ is said to be an $\mathbb{E}_\infty$ *operad* if, for each $n \geq 0$, we have that $\mathcal{P}(n)$ is zero in negative degrees, is quasi-isomorphic to $\mathbb{F}_p[0]$ and is $\mathbb{F}_p[\Sigma_n]$-free; if $\mathcal{P}$ is a cochain operad, we use the same term $\mathbb{E}_\infty$ *operad* for the same condition but with negative degrees replaced by positive degrees.

There exist many models of $\mathbb{E}_\infty$ operads over $\mathbb{F}_p$. We will now introduce two models, the former of which, the Barratt-Eccles operad, will usually be our favoured one. In spaces, by which we mean simplicial sets, there exists an operad $\mathcal{E}_{\text{spc}}$ where:

$$\mathcal{E}_{\text{spc}}(n) = \mathrm{E}\Sigma_n$$

Here $\mathrm{E}\Sigma_n$ denotes the total space of the universal Σ_n-bundle; in particular, in simplicial degree d, we have that $(\mathrm{E}\Sigma_n)_d = \Sigma_n^{\times(d+1)}$. This is the operad originally defined by Barratt and Eccles in [BE74]. We are interested in dg operads and so we take chains on this operad.

Definition 3.2. The *Barratt-Eccles chain operad*, denoted $\mathcal{E}$, is the dg operad over $\mathbb{F}_p$ defined by:

$$\mathcal{E}(n) = \mathrm{C}_\bullet(\mathrm{E}\Sigma_n)$$

Moreover, the *Barratt-Eccles cochain operad* is then the operad $\mathcal{E}^\dagger$.

Here $\mathrm{C}_\bullet$ denotes normalized chains, and we get a dg operad upon taking these because the normalized chains functor is symmetric monoidal. As in this case, all other (co)chains in this work shall be normalized. The chains here are of course taken with coefficients in $\mathbb{F}_p$, so that we get a dg operad over $\mathbb{F}_p$. In defining the Barratt-Eccles cochain operad, we are employing the reindexing operator in Definition 2.7. Note that the Barratt-Eccles chain operad is concentrated in non-negative degrees, while the Barratt-Eccles cochain operad is concentrated in non-positive degrees.

Proposition 3.3. *The Barratt-Eccles chain and cochain operads are $\mathbb{E}_\infty$ operads.*

Proof. We shall demonstrate the chain case, from which the cochain case of course obviously follows. The operad $\mathcal{E}$ is clearly concentrated in non-negative degrees. Moreover, as is standard, each $\mathrm{E}\Sigma_n$ is contractible, so that we have, for each $n \geq 0$, $\mathrm{H}_\bullet \mathcal{E}(n) = \mathrm{H}_\bullet(\mathrm{E}\Sigma_n) = \mathbb{F}_p[0]$, as desired. Finally, we can show that each $\mathcal{E}(n)$ is free over $\mathbb{F}_p[\Sigma_n]$. To see this, consider $\mathcal{E}(n)_d$, which is the free $\mathbb{F}_p$-module on tuples $(\sigma_0, \ldots, \sigma_d)$ of permutations of $(n) = \{1, \ldots, n\}$ where no two adjacent permutations coincide. We want an $\mathbb{F}_p[\Sigma_n]$-basis of this module. The symmetric group Σ_n acts, on the right, freely on the collection of such tuples. Let $t_1, \ldots, t_r$ be representatives of each orbit of this action. We claim that the t_i form an $\mathbb{F}_p[\Sigma_n]$-basis. To see this, first consider some linear combination:

$$t_1 \left(\sum a_i \sigma_i^1\right) + \cdots + t_r \left(\sum a_j \sigma_j^r\right)$$

Upon expanding out each of the summands

$$t_1 \left(\sum a_i \sigma_i^1\right), \ldots, t_r \left(\sum a_j \sigma_i^r\right)$$

no cancellations can occur between expansions as no two terms which come from different expansions can coincide (since the resulting tuples will lie in distinct orbits of the action by Σ_n), and no two terms from within a single expansion can coincide since, as already noted, the action of Σ_n on the the tuples $(\sigma_0, \ldots, \sigma_d)$ is free. Thus, we have $\mathbb{F}_p[\Sigma_n]$-linear independence of the t_i. It remains to show that these span all of $\mathcal{E}(n)_d$. Consider some arbitrary term $a_1 u_1 + \cdots + a_s u_s$ in $\mathcal{E}(n)_d$, where the $a_i \in \mathbb{F}_p$ and the u_i are tuples of permutations of (n). Each u_i lies in some orbit of the action of Σ_n on the tuples, say that of t_i, so that $u_i = t_i \sigma$ for some unique (due to freeness) $\sigma \in \Sigma_n$. Upon collecting like terms together, we will have an expression of this arbitrary term as an $\mathbb{F}_p[\Sigma_n]$-linear combination of the t_i, which completes the proof. $\square$

We now consider the homotopy theory of $\mathcal{E}$-algebras and $\mathcal{E}^\dagger$-algebras. Recall the notion of admissibility for operads as in Definition 2.34.

Proposition 3.4. *The Barrat-Eccles chain and cochain operads, $\mathcal{E}$ and $\mathcal{E}^\dagger$, are admissible.*

Proof. The case of $\mathcal{E}$ is demonstrated in [BF04] and the exact same proof works also in the cochain case. The crucial point behind the argument is the existence of a diagonal map $\mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$, which arises from the fact that $\mathcal{E}$ is constructed by taking chains on an operad in spaces and a diagonal exists at the level of spaces. $\square$

Thus, the categories $\mathcal{E}$-Alg and $\mathcal{E}^{\dagger}$-Alg admit Quillen model structures in which the weak equivalences and fibrations are those maps whose underlying dg maps are quasi-isomorphisms and degreewise epimorphisms, respectively.

Before moving on, we now describe another model for the $\mathbb{E}_\infty$ operad, which will be useful for us in constructing the $\mathbb{E}_\infty$ operad action on cochains on spaces, and which was defined originally by McClure and Smith in [McC03], so that we call it the McClure-Smith operad (see also [BF04], where it is called the surjection operad). In order to define this operad, we require some preliminaries. For each integer $n \geq 0$, let (n) denote the set $\{1, \ldots, n\}$, where $(0) := \varnothing$. Given a set map $f\colon (m) \to (n)$, we will often view it as, and denote it by, the indexed sequence $(f(1), \ldots, f(m))$.

Suppose given $n \geq 0$ and a surjection $f\colon (m) \to (n)$. Say that f is *degenerate* if there exists some $l \in (m)$ such that $f(l) = f(l+1)$, and otherwise *non-degenerate*; that is, call f degenerate exactly when the sequence $(f(1), \ldots, f(m))$ contains two equal adjacent entries. For each $n \geq 0$, let $\mathrm{S}(n)$ be the graded $\mathbb{F}_p$-module freely generated by maps $f\colon (-) \to (n)$, where, if the source is (m), the assigned degree is $m - n$. Let $\mathrm{N}(n)$ denote the sub graded module generated by the non-surjective maps and $\mathrm{D}(n)$ the sub graded module generated by the degenerate surjections. For each $n \geq 0$, set:

$$\mathcal{M}(n) := \mathrm{S}(n)/(\mathrm{N}(n) + \mathrm{D}(n))$$

Remark 3.5. Taken over $n \geq 0$, the above graded modules will be the underlying graded modules of the McClure-Smith chain operad. It is clear that $\mathcal{M}(n)$ is the graded $\mathbb{F}_p$-module freely generated by the non-degenerate surjections $f\colon (-) \to (n)$, where, as above, if the source is (m), the assigned degree is $m - n$. Moreover, $\mathcal{M}(n)_d$ is zero if $d < 0$, and otherwise is freely generated by the non-degenerate surjections $(n + d) \to (n)$. $||$

We now endow the $\mathcal{M}(n)$ with differentials. To keep track of signs, given a surjection $f\colon (m) \to (n)$ and a $i \in (m)$, we set:

$$\tau_f(i) = \#\{j \in (m) \mid f(j) < f(i) \text{ or } f(j) = f(i) \text{ and } j \leq i\}$$

The differential of $\mathcal{M}(n)$ is then defined as follows. Given a non-degenerate surjection $f\colon (m) \to (n)$, denoted also by $(f(1), \ldots, f(m))$, we set:

$$\partial(f) = \sum_{i=1}^{m} (-1)^{\tau_f(i) - f(i)} (f(1), \ldots, \widehat{f(i)}, \ldots, f(m))$$

Here, for $i \in (m)$, the term $(f(1), \dots, \widehat{f(i)}, \dots, f(m))$ on the righthand side denotes the the map $(m-1) \to (n)$ whose images are, in order, exactly those that appear in the sequence $(f(1), \dots, \widehat{f(i)}, \dots, f(m))$; if, upon omitting $f(i)$, this resulting map is no longer a surjection or is degenerate, that term is taken to be zero. The verification that this indeed defines a well-defined differential may be found in [McC03].

It remains to describe the operadic structure data. The identity in $\mathcal{M}(1)$ is the identity on (1) and the action of Σ_n on $\mathcal{M}(n)$ is by postcomposition of surjections onto (n). As for the composition maps, we shall specify this in the form of maps $\circ_r \colon \mathcal{M}(n) \otimes \mathcal{M}(m) \to \mathcal{M}(n+m-1)$, for $r = 1, \dots, n$, as per Remark 2.6. Let $f\colon (N) \to (n)$ and $g\colon (M) \to (m)$ be non-degenerate surjections. We need to define a composite $f \circ_r g$, which will be zero or a non-degenerate surjection $(N+M-1) \to (n+m-1)$. This composite can be described algorithmically as follows:

- In the sequence $(f(1), \dots, f(N))$, let t be the number of occurences of r. Let these occurences be given by $f(i_1), \dots, f(i_t)$.
- Fix a choice of $t+1$ entries $1 = j_0 \leq j_1 \leq \dots \leq j_{t-1} \leq j_t = M$ inside (M), where the first is 1 and the final M. In the sequence $(f(1), \dots, f(N))$, replace $f(i_1)$ by the subsequence $(g(j_0), \dots, g(j_1))$, $f(i_2)$ by the subsequence $(g(j_1), \dots, g(j_2))$, and so on, with the final replacement being that of $f(i_t)$ by the subsequence $(g(j_{t-1}), \dots, g(j_t))$. Note that the resulting sequence has length $N - t + M + t - 1 = N + M - 1$. Now alter this sequence as follows: (i) increase each entry $g(j)$ which has been entered by $r-1$ (ii) increase those entries $f(i)$ which remain and where $f(i) > r$ by $M-1$.
- The resulting sequence gives a map $f_{(j_0,\dots,j_t)}\colon (N+M-1) \to (n+m-1)$; if it is not a surjection or is a degenerate surjection, replace it with zero.
- The composite $f \circ_r g$ is then the sum of all the resulting maps $f_{(j_0,\dots,j_t)}$, the sum being taken over the tuples $(j_0, j_1, \dots j_t)$.

The verification that the above algorithmic procedure yields well-defined maps $\mathcal{M}(n) \otimes \mathcal{M}(m) \to \mathcal{M}(n+m-1)$ for $r = 1, \dots, n$, and yields an operad structure on the chain complexes $\mathcal{M}(n)$, may be found in [BF04]. As per Remark 2.6, the composition maps in the form

$$\mathcal{M}(n) \otimes \mathcal{M}(k_1) \otimes \dots \otimes \mathcal{M}(k_n) \to \mathcal{M}(k_1 + \dots k_n)$$

can be computed iteratively by applying, when given $f \otimes g_1 \otimes \dots \otimes g_n$, the above algorithmic procedure first to compute $f \circ_n g_n$, and then to compute $(f \circ_n g_n) \circ_{n-1} g_{n-1}$, and so on.

Definition 3.6. The *McClure-Smith chain operad* is the operad consisting of the chain complexes $\mathcal{M}(n)$ and structural data specified above. Moreover, the *McClure-Smith cochain operad* is then the operad $\mathcal{M}^\dagger$.

In defining the cochain operad, we are making use of the reindexing operator in Definition 2.7.

Proposition 3.7. *The McClure-Smith chain and cochain operads are* $\mathbb{E}_\infty$ *operads.*

Proof. We shall demonstrate the chain case, from which the cochain case of course obviously follows. For each $n \geq 0$, $\mathcal{M}(n)$ is concentrated in non-negative degrees because there cannot be a surjection $(m) \to (n)$ if $m < n$. Moreover, that $\mathcal{M}(n)$, for each $n \geq 0$, has the homology of a point is proven in [McC03]. Thus we need only show that $\mathcal{M}(n)$, for each $n \geq 0$, is $\mathbb{F}_p[\Sigma_n]$-free. Consider $\mathcal{M}(n)_d$, which is the free $\mathbb{F}_p$-module on non-degenerate surjections $(n+d) \to (n)$. We want an $\mathbb{F}_p[\Sigma_n]$-basis of this module. Consider all partitions, denoted by say $\mathcal{A}$, of $(n+d)$ which have exactly n pieces and which are such that no piece contains two adjacent entries. For each such partition $\mathcal{A}$, choose any surjection $f_{\mathcal{A}} \colon (n+d) \to (n)$ which identifies exactly those elements which lie within a single piece of the partition. We claim that the $f_{\mathcal{A}}$ form an $\mathbb{F}_p[\Sigma_n]$-basis. To see this, first consider some linear combination:

$$f_{\mathcal{A}_1}\left(\sum a_i \sigma_i^{\mathcal{A}_1}\right) + \cdots + f_{\mathcal{A}_r}\left(\sum a_j \sigma_j^{\mathcal{A}_r}\right)$$

Upon expanding out each of the summands

$$f_{\mathcal{A}_1}\left(\sum a_i \sigma_i^{\mathcal{A}_1}\right), \ldots, f_{\mathcal{A}_r}\left(\sum a_j \sigma_i^{\mathcal{A}_r}\right)$$

no cancellations can occur between expansions as no two terms which come from different expansions can coincide (since if two surjections coincide, the two partitions they induce on $(n+d)$ must coincide), and no two terms from within a single expansion can coincide since the action of Σ_n on the surjections $(n+d) \to (n)$ is free. Thus, we have $\mathbb{F}_p[\Sigma_n]$-linear independence of the $f_{\mathcal{A}}$. It remains to show that these span all of $\mathcal{M}(n)_d$. Consider an arbitrary element $a_1 f_1 + \cdots + a_r f_r$ where the $a_i \in \mathbb{F}_p$ and the f_i are non-degenerate surjections $(n+d) \to (n)$. Each f_i induces some partition, say $\mathcal{A}_i$, on $(n+d)$, and it is then clear that $f = f_{\mathcal{A}_i}\sigma$ for some unique $\sigma \in \Sigma_n$. Upon collecting like terms together, we will have an expression of our arbitrary element as an $\mathbb{F}_p[\Sigma_n]$-linear combination of the $f_{\mathcal{A}}$, which completes the proof. $\square$

Finally, we now describe a relation between the Barratt-Eccles and McClure smith operads. The two chain operads are related via a map:

$$\mathrm{TR} \colon \mathcal{E} \to \mathcal{M}$$

This map can be described algorithmically as follows:

- Consider some tuple $(\rho_0, \ldots, \rho_d) \in \mathcal{E}(n)_d = \mathrm{C}_d(\mathrm{E}\Sigma_n)$, where the ρ_i are permutations of (n).
- Let $r_0, \ldots, r_d$ be any positive integers such that $r_0 + \cdots + r_d = n + d$. Note that each r_i is necessarily $\leq n$; moreover, each $r_0 + \cdots + r_i$ is necessarily $\leq n + i$. Form a sequence $(\rho_0(1), \ldots, \rho_0(r_0))$ of length r_0 using the first r_0 entries of the sequence given by ρ_0. Next, form a sequence of length r_1 using the first r_1 entries of the sequence given by ρ_1, but skipping any entry which has already occured as a non-final entry of a previous sequence (there are $r_0 - 1$ such entries, and we have $n - (r_0 - 1) - r_1 = n - r_0 - r_1 + 1 \geq 0$). Now repeat this process to construct sequences of length $r_2, \ldots, r_d$.
- Concatenate the $d + 1$ sequences constructed in the previous point to construct an indexed sequence of length $r_0 + \cdots + r_d = n + d$. This yields a map $f_{(r_0,\ldots,r_d)} \colon (n + d) \to (n)$. If this map is not a surjection or is a non-degenerate surjection, replace it by zero.
- The image of $(\rho_0, \ldots, \rho_d)$ under TR_n is the sum $\sum f_{(r_0,\ldots,r_d)}$ over the tuples $(r_0, \ldots, r_d)$.

The verification that the above algorithmic procedure defines a well-defined map of operads may be found in [BF04]. Moreover, in the same work, it is shown that the map TR is onto in each operadic degree. Thus we see that the McClure-Smith chain operad $\mathcal{M}$ is a quotient of the Barratt-Eccles chain operad $\mathcal{E}$.

Finally, upon reindexing, we also get a map between the corresponding cochain operads:

$$\mathrm{TR}^\dagger \colon \mathcal{E}^\dagger \to \mathcal{M}^\dagger$$

This map is of course also onto in each operadic degree, so that the McClure-Smith cochain operad $\mathcal{M}$ is a quotient of the Barratt-Eccles cochain operad $\mathcal{E}$.

3.2 Products and Operations for the (Co)homologies of $\mathbb{E}_\infty$ DG Algebras

The algebraic structure encoded by an $\mathbb{E}_\infty$ operad is that of a homotopy coherent commutative, associative and unital multiplication, where the binary multiplication itself is encoded in the arity 2 part of the operad. Given an algebra A over an $\mathbb{E}_\infty$ operad, if we take the (co)homology of A, in a sense nullifying the higher homotopies, setting them to be the identities, we shall show that we get a multiplication which is graded-commutative, associative and unital in the traditional sense. Moreover, this product is not all the structure inherited by the (co)homology of A. There are also operations, denoted Q^s in the chain case and P^s in the cochain case, where $s \in \mathbb{Z}$ and where Q^s

and P^s are of degree s. These operations are very much a characteristic p phenomenon, as they arise from generalized power maps $x \mapsto x^p$, which are of course linear only in characteristic p. In discussing the products and these operations, we shall use the Barratt-Eccles chain and cochain operads as our models of the $\mathbb{E}_\infty$ operads. All the theory that we develop, however, could also be developed with the McClure-Smith chain and cochain operads.

Let us first consider the case of the chain operad $\mathcal{E}$. As mentioned above, the binary multiplication on the (co)homology of an $\mathcal{E}$-algebra will arise, unsurprisingly, from the arity 2 part of $\mathcal{E}$. For this reason, we first describe in detail the chain complex $\mathcal{E}(2)$ and set in place some convenient notations. By definition, we have that $\mathcal{E}(2) = \mathrm{C}_\bullet(\mathrm{E}\Sigma_2)$, where we recall that the chains are normalized. In degree $d \geq 0$, we have that $(\mathrm{E}\Sigma_2)_d = \Sigma_2^{\times(d+1)}$. Moreover, the degenerate simplices in $\mathrm{E}\Sigma_2$ correspond to those tuples of permutations which have repeated adjacent entries. Thus, if we let τ denote the only non-trivial permutation of $(2) = \{1, 2\}$, we find that, for $d \geq 0$:

$$(\mathrm{E}\Sigma_2)_d^{\mathrm{nd}} = \{(1, \tau, 1, \dots), (\tau, 1, \tau, \dots)\}$$

Here "nd" indicates "non-degenerate". Henceforth, for $d \geq 0$, we shall always let e_d denote the non-degenerate d-dimensional simplex $(1, \tau, 1, \dots)$ of $\mathrm{E}\Sigma_2$; the simplex $(\tau, 1, \tau, \dots)$ of $\mathrm{E}\Sigma_2$ is then $e_d\tau$. Thus, in each non-negative degree $d \geq 0$, we have that $\mathcal{E}(2)_d = \mathbb{F}_p\{e_d, e_d\tau\}$. Moreover, the differential $\mathcal{E}(2)_d \to \mathcal{E}(2)_{d-1}$ arises by making successive omissions of the entries of a given tuple. On the non-degenerate simplices $(1, \tau, 1, \dots), (\tau, 1, \tau, \dots)$, only the omission of the first and final entries result in a non-degenerate simplex, and so, we see that:

$$\partial(e_d) = \begin{cases} e_{d-1}(\tau - 1) & d \text{ is odd} \\ e_{d-1}(\tau + 1) & d \text{ is even} \end{cases}$$

That is, we have the following picture of $\mathcal{E}(2)$, where e_d, for $d \geq 0$, is the $\mathbb{F}_p[\Sigma_2]$-generator in degree d:

$$\mathcal{E}(2): \qquad \cdots \longleftarrow 0 \longleftarrow \underset{\deg 0}{\mathbb{F}_p[\Sigma_2]} \xleftarrow{\tau-1} \underset{\deg 1}{\mathbb{F}_p[\Sigma_2]} \xleftarrow{\tau+1} \underset{\deg 2}{\mathbb{F}_p[\Sigma_2]} \longleftarrow \cdots$$

In the cochain case, we instead have the following picture of $\mathcal{E}^\dagger(2)$, where e_d, for $d \geq 0$, is the $\mathbb{F}_p[\Sigma_2]$-generator now in degree $-d$:

$$\mathcal{E}^\dagger(2): \qquad \cdots \longrightarrow \underset{\deg -2}{\mathbb{F}_p[\Sigma_2]} \xrightarrow{\tau+1} \underset{\deg -1}{\mathbb{F}_p[\Sigma_2]} \xrightarrow{\tau-1} \underset{\deg 0}{\mathbb{F}_p[\Sigma_2]} \longrightarrow 0 \longrightarrow \cdots$$

In either case, we have precisely the standard $\mathbb{F}_p[\Sigma_2]$-free resolution of $\mathbb{F}_p$ (see [May70]).

We now describe the product structure on the (co)homology of an algebra over $\mathcal{E}$ or $\mathcal{E}^\dagger$.

Proposition 3.8. *Given an algebra A over $\mathcal{E}$, under the composite*

$$\mathrm{H}_\bullet(A)^{\otimes 2} \cong \mathbb{F}_p[0] \otimes \mathrm{H}_\bullet(A)^{\otimes 2} \cong \mathrm{H}_\bullet(\mathcal{E}(2)) \otimes \mathrm{H}_\bullet(A)^{\otimes 2} \cong \mathrm{H}_\bullet(\mathcal{E}(2) \otimes A^{\otimes 2}) \to \mathrm{H}_\bullet(A)$$

the homology $\mathrm{H}_\bullet(A)$ is a graded-commutative, unital and associative, graded algebra over $\mathbb{F}_p$. Similarly, given an algebra A over $\mathcal{E}^\dagger$, under the composite

$$\mathrm{H}^\bullet(A)^{\otimes 2} \cong \mathbb{F}_p[0] \otimes \mathrm{H}^\bullet(A)^{\otimes 2} \cong \mathrm{H}^\bullet(\mathcal{E}(2)) \otimes \mathrm{H}^\bullet(A)^{\otimes 2} \cong \mathrm{H}^\bullet(\mathcal{E}(2) \otimes A^{\otimes 2}) \to \mathrm{H}^\bullet(A)$$

the cohomology $\mathrm{H}^\bullet(A)$ is a graded-commutative, unital and associative, graded algebra over $\mathbb{F}_p$.

In the composites above, in either case, the second map sends $1 \otimes [a] \otimes [a']$ to $[e_0] \otimes [a] \otimes [a']$, and the third map is a Kunneth map. We see that, in either case, the total composite sends $[a] \otimes [a']$ to $[(e_0)_*(a, a')]$, where we use the notation $\sigma_*(c, c')$ to denote the image of $\sigma \otimes a \otimes a'$ under the map $\mathcal{E}(2) \otimes A^{\otimes 2} \to A$ in the chain case, and the map $\mathcal{E}^\dagger(2) \otimes A^{\otimes 2} \to A$ in the cochain case.

Proof. We shall outline the chain case; the cochain case is entirely analogous. To see associativity, note that:

$$[a] \cdot ([a'] \cdot [a'']) = [(e_0)_*(a, (e_0)_*(a', a''))] \qquad ([a] \cdot [a']) \cdot [a''] = [(e_0)_*((e_0)_*(a, a'), a'')]$$

The result now follows by noting that $e_0 \otimes (\mathrm{id}_{(1)}) \otimes e_0$ and $e_0 \otimes e_0 \otimes (\mathrm{id}_{(1)})$ have the same image under the composition maps $\mathcal{E}(2) \otimes \mathcal{E}(1) \otimes \mathcal{E}(2) \to \mathcal{E}(3)$ and $\mathcal{E}(2) \otimes \mathcal{E}(2) \otimes \mathcal{E}(1) \to \mathcal{E}(3)$, respectively.

To see unitality, first, let u denote the image of $(\mathrm{id}_{(0)})$ under the map $\mathcal{E}(0) \to A$, and then consider $[u] \in \mathrm{H}_\bullet(A)$ (note that u is a cycle since $(\mathrm{id}_{(0)})$ is of degree zero). Note that:

$$[u] \cdot [a] = [(e_0)_*((\mathrm{id}_{(0)})_*(1), a)] \qquad [a] \cdot [u] = [(e_0)_*(a, (\mathrm{id}_{(0)})_*(1))]$$

The result now follows by noting that $e_0 \otimes (\mathrm{id}_{(0)}) \otimes (\mathrm{id}_{(1)})$ and $e_0 \otimes (\mathrm{id}_{(1)}) \otimes (\mathrm{id}_{(0)})$ have the same image under the composition maps $\mathcal{E}(2) \otimes \mathcal{E}(0) \otimes \mathcal{E}(1) \to \mathcal{E}(1)$ and $\mathcal{E}(2) \otimes \mathcal{E}(1) \otimes \mathcal{E}(0) \to \mathcal{E}(1)$, respectively.

Finally, to see commutativity, note that:

$$\begin{aligned}
[a]\cdot[a'] &= [(e_0)_*(a,a'))] \\
&= (-1)^{|a||a'|}[(e_0\tau)_*(a',a)] \\
&= (-1)^{|a||a'|}[(e_0)_*(a',a)] \\
&= [a']\cdot[a]
\end{aligned}$$

Here the second equality comes from the fact that the map $\mathcal{E} \to \mathcal{End}_A$ respects the symmetric group actions and the third equality comes from the fact that, e_0 and $e_0\tau$ are homologically equivalent, as $\partial(e_1) = e_0\tau - e_0$. □

The above describes the product structure of the (co)homology of $\mathbb{E}_\infty$ dg algebras. We now describe the operations Q^s and P^s on these (co)homologies. Once again, we shall discuss the case of the chain operad $\mathcal{E}$ first. As mentioned above, these operations arise from generalized power maps $x \mapsto x^p$, which, unsuprisingly, arise from the arity p part of $\mathcal{E}$. For this reason, we shall first discuss in detail the complex $\mathcal{E}(p)$. Recall from above that $\mathcal{E}(2)$ is exactly the standard $\mathbb{F}_p[\Sigma_2]$-free resolution of $\mathbb{F}_p$. In arity p, for the purpose of the operations, we shall be interested in an $\mathbb{F}_p[C_p]$-resolution, where C_p is cyclic of order p. Let α be the permutation $1,2,\cdots,p \mapsto 2,3,\cdots,p,1$, and let C_p then be the group generated by α. Recall, as in [May70], that the desired resolution is given by the following chain complex, which we denote by W:

$$W: \qquad \cdots \longleftarrow 0 \longleftarrow \underset{\text{deg } 0}{\mathbb{F}_p[C_p]} \xleftarrow{\alpha-1} \underset{\text{deg } 1}{\mathbb{F}_p[C_p]} \xleftarrow{1+\alpha+\cdots+\alpha^{p-1}} \underset{\text{deg } 2}{\mathbb{F}_p[\Sigma_2]} \xleftarrow{\alpha-1} \underset{\text{deg } 3}{\mathbb{F}_p[\Sigma_2]} \longleftarrow \cdots$$

Now, since W is free and since $\mathcal{E}(p)$ is acylic in positive dimensions, the standard acylic models lemma tells us that there is a, unique up to homotopy, map

$$F\colon W \to \mathcal{E}(p)$$

which, on H_0 induces the isomorphism $\mathrm{H}_0(W) \to \mathrm{H}_0(\mathcal{E}(p))$ given by $1 \mapsto (\mathrm{id}_{(p)})$. In fact, we can inductively explicitly construct such a map as follows:

- In degree 0, set $F_0(1) = (1)$, where the righthand 1 denotes $\mathrm{id}_{(p)}$, and then set, necessarily and more generally, for $j = 0,1,\ldots,p-1$, set $F_0(\alpha^j) = (\alpha^j)$.
- In degree 1, set $F_1(1) = (1,\alpha)$, and then, necessarily and more generally, for $j = 0,1,\ldots,p-1$, $F_1(\alpha^j) = (\alpha^j,\alpha^{j+1})$. Note that $\partial(1,\alpha) = (\alpha)-(1) = (1)\cdot(\alpha-1)$, as desired.
- In degree 2, set $F_2(1) = (1,\alpha,\alpha^2)+(1,\alpha^2,\alpha^3)+\cdots+(1,\alpha^{p-1},1)$, and then set $F_2(\alpha^j)$ as one must for $j = 0,1,\ldots,p-1$. An easy computation shows that $\partial((1,\alpha,\alpha^2)+(1,\alpha^2,\alpha^3)+$

$\cdots + (1, \alpha^{p-1}, 1)) = (1, \alpha) + (\alpha, \alpha^2) + \cdots + (\alpha^{p-1}, 1) = (1, \alpha) \cdot (1 + \alpha + \cdots + \alpha^{p-1})$, as desired.

- In all remaining degrees, define the map inductively as follows: when making the definition in an odd degree d, we take the tuples occuring in $F_{d-1}(\alpha)$ and append 1 at the beginning of each one; when making the definition in an even degree d, we take the tuples occuring in each of the images $F_{d-1}(\alpha), \cdots, F_{d-1}(\alpha^{p-1})$ and append 1 at the beginning of each one. An induction demonstrates that, as in the cases above, the necessary relations hold so that we have a map $W \to \mathcal{E}(2)$, and one which obviously satisfies the specified condition in degree 0.

Henceforth, for $d \geq 0$, we shall let $f_d \in \mathcal{E}(p)_d$ denote the image of 1 under the map F_d as defined specified above.

Remark 3.9. In the case $p = 2$, we have $C_2 = \Sigma_2$, W coincides exactly with $\mathcal{E}(2)$, the map F is an isomorphism and the elements f_d are exactly the elements e_d which we defined in the previous section. ||

In the case of the cochain operad $\mathcal{E}^\dagger$, we replace W by $W^\dagger$, F by

$$F^\dagger \colon W^\dagger \to \mathcal{E}^\dagger(p)$$

and then, for $d \geq 0$, let $f_d \in \mathcal{E}^\dagger(p)^{-d}$ denote the image of 1 under $F^\dagger_{-d}$.

We can now construct the desired operations. We shall separate the cases $p = 2$ and $p > 2$. First, we consider the case $p = 2$, for which we recall that the elements f_d and e_d which we have defined above coincide.

Proposition 3.10. *Suppose that $p = 2$, where p is our fixed prime. Given an algebra A over $\mathcal{E}$, for each $s \in \mathbb{Z}$ and $[a] \in \mathrm{H}_q(A)$, by setting*

$$Q^s([a]) = \begin{cases} [(e_{s-q})_*(a, a)] & s \geq q \\ 0 & s < q \end{cases}$$

we get a well-defined graded map

$$Q^s \colon \mathrm{H}_\bullet(A) \to \mathrm{H}_\bullet(A)$$

which is linear over $\mathbb{F}_2$, of degree s and natural in A. Similarly, given an algebra A over $\mathcal{E}^\dagger$, for *each $s \in \mathbb{Z}$ and $[a] \in \mathrm{H}^q(A)$, by setting*

$$P^s([a]) = \begin{cases} [(e_{q-s})_*(a,a)] & s \leq q \\ 0 & s > q \end{cases}$$

we get a well-defined graded map

$$P^s \colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A)$$

which is linear over $\mathbb{F}_2$, of degree s and natural in A.

As we have done earlier, here we use the notation $\sigma_*(a,a)$ for the image of $\sigma \otimes a \otimes a$ under $\mathcal{E}(2) \otimes A^{\otimes 2} \to A$ in the chain case, or under $\mathcal{E}^\dagger(2) \otimes A^{\otimes 2} \to A$ in the cochain case. As such, we see that the maps $a \mapsto \sigma_*(a,a)$ are generalized squaring maps.

Proof. Let us first consider the case of the chain operad $\mathcal{E}$. To see that the Q^s are well-defined, we refer to [May70], where it is shown that if $a, a' \in A_q$ are homologous cycles, then $e_{q-s} \otimes a \otimes a$ and $e_{q-s} \otimes a' \otimes a'$ are homologus in $\mathrm{H}_\bullet(\mathcal{E}(2) \otimes_{\Sigma_2} A^{\otimes 2})$. The well-definedness then follows from noting that, in the case $s \geq q$, $Q^s([a])$ is precisely the image of this well-defined element of $\mathrm{H}_\bullet(\mathcal{E}(2) \otimes_{\Sigma_2} A^{\otimes 2})$ under the map $\mathrm{H}_\bullet(\mathcal{E}(2) \otimes_{\Sigma_2} A^{\otimes 2}) \to \mathrm{H}_\bullet(A)$.

To see linearity over $\mathbb{F}_2$, first, we have homogeneity, in the non-trivial case where $s \geq q$, as follows:

$$\begin{aligned} Q^s(\lambda[a]) &= Q^s([\lambda a]) \\ &= [e_{s-q}(\lambda a, \lambda a)] \\ &= \lambda^2[e_{s-q}(a,a)] \\ &= \lambda[e_{s-q}(a,a)] \\ &= \lambda Q^s([a]) \end{aligned}$$

As for additivity, first, let $[a],[b] \in \mathrm{H}_q(A)$ and suppose that $s \geq q$, as the case $s < q$ is clear. Consider $e_{s-q} \otimes (a+b) \otimes (a+b) - e_{s-q} \otimes a \otimes a - e_{s-q} \otimes b \otimes b$ as an element of $\mathcal{E}(2) \otimes_{\Sigma_2} A^{\otimes 2}$. We need to show that, under the map $\mathrm{H}_\bullet(\mathcal{E}(2) \otimes_{\Sigma_2} A^{\otimes 2}) \to \mathrm{H}_\bullet(A)$, the class of this element has image zero. We have $e_{s-q} \otimes (a+b) \otimes (a+b) - e_{s-q} \otimes a \otimes a - e_{s-q} \otimes b \otimes b = e_{s-q} \otimes a \otimes b + e_{s-q} \otimes b \otimes a$, and then the result follows by the following:

$$\begin{aligned}e_{s-q}\otimes a\otimes b+e_{s-q}\otimes b\otimes a &= e_{s-q}\otimes a\otimes b+e_{s-q}\tau\otimes a\otimes b\\ &= e_{s-q}(\tau+1)\otimes a\otimes b\\ &= \partial(e_{s-q}\otimes a\otimes b)\end{aligned}$$

To see that Q^s is homogeneous of degree s, simply note that, given $[a]\in \mathrm{H}_q(A)$, in $\mathcal{E}(2)\otimes A^{\otimes 2}$, $e_{s-q}\otimes a\otimes a$ has degree $(s-q)+q+q=q+s$.

Finally, we verify naturality. Let $f\colon A\to B$ be a map of algebras over $\mathcal{E}$. We need to show that, for each $s\in\mathbb{Z}$, the following square commutes:

$$\begin{tikzcd} \mathrm{H}_\bullet(A) \arrow[r, "Q^s"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright ?"] & \mathrm{H}_\bullet(A) \arrow[d, "f_*"] \\ \mathrm{H}_\bullet(B) \arrow[r, "Q^s"'] & \mathrm{H}_\bullet(B) \end{tikzcd}$$

This follows from the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{H}_\bullet(\mathcal{E}(2)\otimes_{\Sigma_2} A^{\otimes 2}) \arrow[r, "\mathcal{E}\text{-action}"] \arrow[d, "f_*"'] & \mathrm{H}_\bullet(A) \arrow[d, "f_*"] \\ \mathrm{H}_\bullet(\mathcal{E}(2)\otimes_{\Sigma_2} B^{\otimes 2}) \arrow[r, "\mathcal{E}\text{-action}"'] & \mathrm{H}_\bullet(B) \end{tikzcd}$$

This completes the proof in the case of the chain operad $\mathcal{E}$. The proof in the case of the cochain operad $\mathcal{E}^\dagger$ is entirely analogous; the only essential difference to note is that, in this case, e_{q-s} has degree $s-q$, as opposed to $q-s$. $\square$

Remark 3.11. As above, the complex $\mathcal{E}(2)$, in the case $p=2$, is as follows:

$$\cdots \longleftarrow 0 \longleftarrow \underset{\deg 0}{\mathbb{F}_2[\Sigma_2]} \xleftarrow{\tau+1} \underset{\deg 1}{\mathbb{F}_2[\Sigma_2]} \xleftarrow{\tau+1} \underset{\deg 2}{\mathbb{F}_2[\Sigma_2]} \longleftarrow \cdots$$

Note that the equivariant homology of $\mathcal{E}(2)$, which is to say the homology of $\mathcal{E}(2)/\Sigma_2$, is generated precisely by the e_i. Thus we see that homology operations arise from this equivariant homology. We can also anticipate this via an intuitive argument as follows. Suppose we generate the free algebra $\mathbf{E}\mathbb{F}_2[0]$ on the "point" $\mathbb{F}_2[0]$. Later, we will see that the homologies of free algebras are given by freely adding in products and operations, subject to certain properties. As $\mathbf{E}\mathbb{F}_2[0]$ is the free algebra on a point, from an intuitive standpoint we expect its homology $\mathrm{H}_\bullet(\mathbf{E}\mathbb{F}_2[0])$ to be a minimal object containing the operations and products, which is to say, we expect it to be an algebra of the operations and products thereof (this will be made precise later – see Remark 3.35). On the other hand, we have that $\mathrm{H}_\bullet(\mathbf{E}\mathbb{F}_2[0]) = \oplus_n \mathrm{H}_\bullet(\mathcal{E}(n)/\Sigma_n)$, so that its homology is precisely the sum of the equivariant homologies of the $\mathcal{E}(n)$. In fact, we will see that the homology of $\mathcal{E}(0)/\Sigma_0$ contributes a multiplicative unit, the homology of $\mathcal{E}(1)/\Sigma_1$ contains the generating point, the homology of $\mathcal{E}(2)/\Sigma_2$ contributes the operations, while the rest of the $\mathcal{E}(n)/\Sigma_n$ contribute the iterated operations as well as products of these operations. Similar considerations apply in the cochain case, where the complex is given by

$$\cdots \longrightarrow \underset{\deg -2}{\mathbb{F}_2[\Sigma_2]} \xrightarrow{\tau+1} \underset{\deg -1}{\mathbb{F}_2[\Sigma_2]} \xrightarrow{\tau+1} \underset{\deg 0}{\mathbb{F}_2[\Sigma_2]} \longrightarrow 0 \longrightarrow \cdots$$

and the homology of $\mathcal{E}^\dagger(2)/\Sigma_2$, which is concentrated in non-positive degrees, is generated again precisely by the e_i. $\quad ||$

Next, we describe the case $p > 2$. For this purpose we introduce the following notation for $q \in \mathbb{Z}$:

$$\nu(q) = \begin{cases} (-1)^{q/2} & q \text{ even} \\ (-1)^{(q-1)/2}((p-1)/2)! & q \text{ odd} \end{cases}$$

Proposition 3.12. *Suppose that $p > 2$, where p is our fixed prime. Given an algebra A over $\mathcal{E}$, for each $s \in \mathbb{Z}$ and $[a] \in \mathrm{H}_q(A)$, by setting*

$$Q^s([a]) = \begin{cases} (-1)^s \nu(q)[(f_{(2s-q)(p-1)})_*(a, \ldots, a)] & 2s \geq q \\ 0 & 2s < q \end{cases}$$

$$\beta Q^s([a]) = \begin{cases} (-1)^s \nu(q)[(f_{(2s-q)(p-1)-1})_*(a, \ldots, a)] & 2s > q \\ 0 & 2s \leq q \end{cases}$$

we get well-defined graded maps

$$Q^s \colon \mathrm{H}_\bullet(A) \to \mathrm{H}_\bullet(A) \qquad \beta Q^s \colon \mathrm{H}_\bullet(A) \to \mathrm{H}_\bullet(A)$$

which are linear over $\mathbb{F}_p$, of degrees $2s(p-1)$ and $2s(p-1) - 1$ respectively, and natural in A. Similarly, given an algebra A over $\mathcal{E}^\dagger$, for each $s \in \mathbb{Z}$ and $[a] \in \mathrm{H}^q(A)$, by setting

$$P^s([a]) = \begin{cases} (-1)^s \nu(-q)[(f_{(q-2s)(p-1)})_*(a, \ldots, a)] & 2s \leq q \\ 0 & 2s > q \end{cases}$$

$$\beta P^s([a]) = \begin{cases} (-1)^s \nu(-q)[(f_{(q-2s)(p-1)-1})_*(a, \ldots, a)] & 2s < q \\ 0 & 2s \geq q \end{cases}$$

we get well-defined graded maps

$$P^s\colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A) \qquad \beta P^s\colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A)$$

which are linear over $\mathbb{F}_p$*, of degrees* $2s(p-1)$ *and* $2s(p-1)+1$ *respectively, and natural in* A*.*

Once more, as earlier, here we use the notation $\sigma_*(a,\ldots,a)$ for the image of $\sigma\otimes a\otimes\cdots\otimes a$ under $\mathcal{E}(p)\otimes A^{\otimes p}\to A$ in the chain case, or under $\mathcal{E}^\dagger(p)\otimes A^{\otimes p}\to A$ in the cochain case. As such, we see that the maps $a\mapsto\sigma_*(a,\ldots,a)$ are generalized p^{th} power maps. Note also that βQ^s and βP^s are individual symbols in their own right, not composites of an individual β with the Q^s or P^s.

Proof. In both the chain and cochain case, the proof is similar to the $p=2$ case which we have already considered, and so we will be brief and mention only the chain case. Well-definedness follows because, as in [May70], we have that if $a, a'\in A_q$ are homologous cycles, then $f_{q-s}\otimes a\otimes\cdots\otimes a$ and $f_{q-s}\otimes a'\otimes\cdots\otimes a'$ are homologous in $\mathrm{H}_\bullet(\mathcal{E}(p)\otimes_{C_p} A^{\otimes p})$. Naturality follows just as in the $p=2$ case. The degrees follow from recalling that, for $d\geq 0$, f_d has degree d in the chain case and $-d$ in the cochain case. As for linearity over $\mathbb{F}_p$, homogeneity follows just as in the case $p=2$ case. For additivity, consider $[a],[b]\in\mathrm{H}_q(A)$. Note that $\Delta(a,b)=(a+b)^{\otimes p}-a^{\otimes p}-b^{\otimes p}$ is a sum of monomials each of which contains at least one a and at least one b as a factor, and moreover, that, given any such monomial, say m, each monomial $m\cdot\alpha^j$, where $j=0,\ldots,p-1$ and $\alpha^j\in C_p$, also occurs in $\Delta(a,b)$. Since C_p permutes such monomials freely we can choose some representatives of the C_p-orbits of such monomials and let c denote the sum of such representatives. Then we have that $\Delta(a,b)=(1+\alpha+\cdots+\alpha^{p-1})\cdot c$ (note that each α^j is an even permutation, so that no signs are required). Now the result follows by noting that, in $\mathcal{E}(p)\otimes_{C_p} A^{\otimes p}$, if i is odd, we have

$$\begin{aligned}\partial(f_{i+1}\otimes c) &= (f_i\cdot(1+\alpha+\cdots+\alpha^{p-1}))\otimes c\\ &= f_i\otimes((1+\alpha+\cdots+\alpha^{p-1})\cdot c)\\ &= f_i\otimes\Delta(a,b)\end{aligned}$$

whereas, if i is even, we have

$$\begin{aligned}\partial(f_{i+1}\cdot(\alpha-1)^{p-2}\otimes c) &= (f_i\cdot(\alpha-1)^{p-1})\otimes c\\ &= f_i\otimes((\alpha-1)^{p-1}\cdot c)\\ &= f_i\otimes((1+\alpha+\cdots+\alpha^{p-1})\cdot c)\\ &= f_i\otimes\Delta(a,b)\end{aligned}$$

where we have used that $(\alpha-1)^{p-1}=1+\alpha+\cdots+\alpha^{p-1}$ in $\mathbb{F}_p[C_p]$, which follows from the identiy $\binom{p-1}{i}=(-1)^i$ in $\mathbb{F}_p$. This completes the proof. $\square$

We have now developed operations for (co)homologies of algebras over $\mathcal{E}$ and $\mathcal{E}^\dagger$. Recall from Proposition 2.14 that if A is an algebra over one of these operads, then $A^\dagger$ is naturally an algebra over the other. Via this, we can relate the operations Q^s and P^s as follows.

Proposition 3.13. *If A is an algebra over $\mathcal{E}$, so that $A^\dagger$ is an algebra over $\mathcal{E}^\dagger$, then the operations Q^s on $\mathrm{H}_\bullet(A)$ and P^s on $\mathrm{H}^\bullet(A^\dagger)$, for $s \in \mathbb{Z}$, if $p = 2$, and also the βQ^s and βP^s if $p > 2$, are related as follows:*

$$Q^s = (P^{-s})^\dagger, \;\; \beta Q^s = (\beta P^{-s})^\dagger \qquad P^s = (Q^{-s})^\dagger, \;\; \beta P^s = (\beta Q^{-s})^\dagger$$

Moreover, the same relations hold if A is an algebra over $\mathcal{E}^\dagger$, so that $A^\dagger$ is an algebra over $\mathcal{E}$, and the operations $Q^s, \beta Q^s$ act on $\mathrm{H}_\bullet(A^\dagger)$ while the $P^s, \beta P^s$ act on $\mathrm{H}^\bullet(A)$.

Proof. This follows by an easy direct check of the definitions. For example, let us consider the case where A is an algebra over $\mathcal{E}^\dagger$ and $p = 2$. Given $[a] \in \mathrm{H}^q(A)$, we have that, for $s \leq q$, $P^s([a])$ is the image of $[e_{q-s} \otimes a \otimes a]$ under the map $\theta_{q+s}\colon \mathrm{H}^{q+s}(\mathcal{E}^\dagger(2) \otimes_{\Sigma_2} A^{\otimes 2}) \to \mathrm{H}^{q+s}(A)$. By definition of $A^\dagger$ and the $\mathcal{E}$-action on $A^\dagger$, we have that $[a] \in \mathrm{H}_{-q}(A^\dagger)$ and the aforementioned image is equivalent to the image of $[e_{q-s} \otimes a \otimes a]$ under the map $\theta_{-q-s}\colon \mathrm{H}_{-q-s}(\mathcal{E}(2) \otimes_{\Sigma_2} (A^\dagger)^{\otimes 2}) \to \mathrm{H}_{-q-s}(A^\dagger)$, which, noting that $-s \geq -q$, is exactly $Q^{-s}([a])$ $\square$

We now proceed to develop some basic properties of the above operations. As before, we shall separate the cases $p = 2$ and $p > 2$.

Proposition 3.14. *Suppose that $p = 2$, where p is our fixed prime. Given an algebra A over $\mathcal{E}$, we have the following on the operations Q^s:*

(i) For all a, b such that $a > 2b$, we have $Q^a Q^b = \sum_{i \in \mathbb{Z}} \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i$.

(ii) For any $c \in \mathrm{H}_\bullet(A)$, we have that $Q^s c = 0$ whenever $s < |c|$.

(iii) For any $c \in \mathrm{H}_\bullet(A)$, we have that $Q^s c = c^2$ if $s = |c|$.

(iv) Letting 1 denote the multiplicative unit of $\mathrm{H}_\bullet(A)$, we have that $Q^s 1 = 0$ for all $s \neq 0$.

Similarly, given an algebra A over $\mathcal{E}^\dagger$, we have the following on the operations P^s:

(v) For all a, b such that $a < 2b$, we have $P^a P^b = \sum_{i \in \mathbb{Z}} \binom{b-i-1}{a-2i} P^{a+b-i} P^i$.

(vi) For any $c \in \mathrm{H}^\bullet(A)$, we have that $P^s c = 0$ whenever $s > |c|$.

(vii) For any $c \in \mathrm{H}^\bullet(A)$, we have that $P^s c = c^2$ if $s = |c|$.

(viii) Letting 1 denote the multiplicative unit of $\mathrm{H}^\bullet(A)$, we have that $P^s 1 = 0$ for all $s \neq 0$.

In this case where $p = 2$, the relations in (i) and (v) are known as the *Adem relations*. The properties in (ii) and (vi), a condition on the action of the operations on the (co)homology, will henceforth be referred to as the *instability* of the operations. While trivial to prove from the definitions, this property of the action of the operations is an important one. It is what will distinguish these operations from the operations which we shall construct later for algebras over the stabilizations of $\mathcal{E}$ and $\mathcal{E}^\dagger$. This is also why we refer to this property as instability; later, we will see that it arises precisely because $\mathcal{E}(2)$ is zero in negative degrees and because $\mathcal{E}^\dagger(2)$ is zero in positive degrees. The properties in (iii)-(iv) and (vii)-(viii) relate the action of the operations to the product structures on the (co)homologies.

Proof. We shall discuss only the case of the chain operad $\mathcal{E}$. The case of the cochain operad $\mathcal{E}^\dagger$ then follows via Proposition 3.13.

(i): See [CLM76].

(ii): Immediate from the definition of the operations.

(iii): Let $c = [a]$. Since $s = |a|$, we have that:

$$Q^s c = [(e_{s-|a|})_*(a,a)] = [(e_0)_*(a,a)] = [a]^2 = c^2$$

(iv): If $s \leq -1$, this follows by (ii). Otherwise, letting $1 = [u]$ where u denotes the image of 1 under $\mathcal{E}(0) \to A$, we have that:

$$Q^s 1 = [(e_s)_*(u,u)] = [(e_s)_*((\mathrm{id}_{(0)})_*(1), (\mathrm{id}_{(0)})_*(1))]$$

The result now follows from noting that $\mathcal{E}(2) \otimes \mathcal{E}(0) \otimes \mathcal{E}(0) \to \mathcal{E}(0)$ maps $e_s \otimes (\mathrm{id}_{(0)}) \otimes (\mathrm{id}_{(0)})$, for $s \geq 1$, to an element of positive degree, and so to zero. $\square$

Finally, we mention the basic properties in the case $p > 2$.

Proposition 3.15. *Suppose that $p > 2$, where p is our fixed prime. Given an algebra A over $\mathcal{E}$, we have the following on the operations Q^s:*

(i) For all a, b, if $a > pb$, we have that $Q^aQ^b = \sum_{i\in\mathbb{Z}}(-1)^{a+i}\binom{(p-1)(i-b)-1}{pi-a}Q^{a+b-i}Q^i$ and $\beta Q^aQ^b = \sum_{i\in\mathbb{Z}}(-1)^{a+i}\binom{(p-1)(i-b)-1}{pi-a}\beta Q^{a+b-i}Q^i$, and, if $a \geq pb$, we have that $Q^a\beta Q^b = \sum_{i\in\mathbb{Z}}(-1)^{a+i}\left(\binom{(p-1)(i-b)}{pi-a}\beta Q^{a+b-i}Q^i - \binom{(p-1)(i-b)-1}{pi-a-1}Q^{a+b-i}\beta Q^i\right)$ and also that $\beta Q^a\beta Q^b = \sum_{i\in\mathbb{Z}}(-1)^{a+i}\binom{(p-1)(i-b)-1}{pi-a-1}\beta Q^{a+b-i}\beta Q^i$.

(ii) For any $c \in \mathrm{H}_\bullet(A)$, we have that $Q^s c = 0$ whenever $2s < |c|$, and that $\beta Q^s c = 0$ whenever $2s \leq |c|$.

(iii) For any $c \in \mathrm{H}_\bullet(A)$, we have that $Q^s c = c^p$ if $2s = |c|$.

(iv) Letting 1 denote the multiplicative unit of $\mathrm{H}_\bullet(A)$, we have that $Q^s 1 = 0$ for all $s \neq 0$ and $\beta Q^s 1 = 0$ for all s.

Similarly, given an algebra A over $\mathcal{E}^\dagger$, we have the following on the operations P^s:

(v) For all a, b, if $a < pb$, we have that $P^a P^b = \sum_{i \in \mathbb{Z}} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$ and $\beta P^a P^b = \sum_{i \in \mathbb{Z}} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} \beta P^{a+b-i} P^i$, and, if $a \leq pb$, we have that $P^a \beta P^b = \sum_{i \in \mathbb{Z}} (-1)^{a+i} \left(\binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i - \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i \right)$ and also that $\beta P^a \beta P^b = \sum_{i \in \mathbb{Z}} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi-1} \beta P^{a+b-i} \beta P^i$.

(vi) For any $c \in \mathrm{H}^\bullet(A)$, we have that $P^s c = 0$ whenever $2s > |c|$, and that $\beta P^s c = 0$ whenever $2s \geq |c|$.

(vii) For any $c \in \mathrm{H}^\bullet(A)$, we have that $P^s c = c^p$ if $2s = |c|$.

(viii) Letting 1 denote the multiplicative unit of $\mathrm{H}^\bullet(A)$, we have that $P^s 1 = 0$ for all $s \neq 0$ and $\beta P^s 1 = 0$ for all s.

In this case where $p > 2$, the relations in (i) and (v) are known as the *Adem relations*. The properties in (ii) and (vi), a condition on the action of the operations on the (co)homology, will henceforth be referred to as the *instability* of the operations. As in the case where $p = 2$, while trivial to prove from the definitions, this property of the action of the operations is an important one. It is what will distinguish these operations from the operations which we shall construct later for algebras over the stabilizations of $\mathcal{E}$ and $\mathcal{E}^\dagger$. This is also why we refer to this property as instability; later, we will see that it arises precisely because $\mathcal{E}(2)$ is zero in negative degrees and because $\mathcal{E}^\dagger(2)$ is zero in positive degrees. The properties in (iii)-(iv) and (vii)-(viii) relate the action of the operations to the product structures on the (co)homologies.

Proof. See [May70] for the Adem relations in (i) and (v). The proofs of (ii)-(iv) and (vi)-(viii) are analogous to the case where $p = 2$. $\square$

3.3 The Algebras $\mathcal{S}$ and $\mathcal{B}$ of Generalized Dyer-Lashof & Steenrod Operations

We are now going to define two algebras, one, $\mathcal{S}$, which arises from considerations with the chain operad $\mathcal{E}$, and another, $\mathcal{B}$, which arises from considerations with the cochain operad $\mathcal{E}^\dagger$. The two algebras will in fact be isomorphic (almost – one needs to negate the degrees of one algebra), so that consideration of one given that of the other is in a sense superfluous. However, for the purpose of clarity, it is convenient to define both. In fact, we have an algebra $\mathcal{S}$ and another $\mathcal{B}$ for each value of the prime p. In order to define these algebras, we need to discuss multi-indices and some associated definitions, both of which shall vary depending on whether $p = 2$ or $p > 2$.

First suppose that $p = 2$. In this case, a *multi-index* is a sequence $I = (i_1, \ldots, i_k)$ of integers $i_j \in \mathbb{Z}$, where $k \geq 0$ (if $k = 0$, we have the empty sequnce $()$). Given such a multi-index, we will associate to it either the formal string $Q^{i_1} \cdots Q^{i_k}$, or the formal string $P^{i_1} \cdots P^{i_k}$. Given the multi-index $I = (i_1, \ldots, i_k)$, we then set the following:

- If to the multi-index we associate the string $Q^{i_1} \cdots Q^{i_k}$, or if instead we associate to it the string $P^{i_1} \cdots P^{i_k}$, in either case, the *length* $l(I)$ is k; if $I = ()$, this is to be interpreted as 0.

- If to the multi-index we associate the string $Q^{i_1} \cdots Q^{i_k}$, or if instead we associate to it the string $P^{i_1} \cdots P^{i_k}$, in either case, the *degree* $d(I)$ is $i_1 + \cdots + i_k$; if $I = ()$, this is to be interpreted as 0.

- If to the multi-index we associate the string $Q^{i_1} \cdots Q^{i_k}$, or if instead we associate to it the string $P^{i_1} \cdots P^{i_k}$, in either case, the *excess* $e(I)$ is $i_1 - i_2 - \cdots - i_k$; if $I = ()$, in the former association, this is to be interpreted as $+\infty$, and in the latter, as $-\infty$.

- If to the multi-index we associate the string $Q^{i_1} \cdots Q^{i_k}$, we say that it is *admissible* if $i_j \leq 2i_{j+1}$ for each j; the empty multi-index $I = ()$ is also, by convention, taken to be admissible. On the other hand, if to the multi-index we associate the string $P^{i_1} \cdots P^{i_k}$, we use the same term *admissible* to mean that $i_j \geq 2i_{j+1}$ for each j; the empty multi-index $I = ()$ is again also, by convention, taken to be admissible.

Now suppose that $p > 2$. In this case, a *multi-index* is a sequence $I = (\varepsilon_1, i_1, \ldots, \varepsilon_k, i_k)$ of integers $i_j \in \mathbb{Z}$ and $\varepsilon_j \in \{0, 1\}$, where $k \geq 0$ (if $k = 0$, we have the empty sequnce $()$). Given such a multi-index, we associate to it the formal string $\beta^{\varepsilon_1} Q^{i_1} \cdots \beta^{\varepsilon_k} Q^{i_k}$, or the string $\beta^{\varepsilon_1} P^{i_1} \cdots \beta^{\varepsilon_k} P^{i_k}$, where β^1 here is to be intepreted as β and β^0 as an empty symbol. Given the multi-index $I = (\varepsilon_1, i_1, \ldots, \varepsilon_k, i_k)$, we then set the following:

- If to the multi-index we associate the string $\beta^{\varepsilon_1} Q^{i_1} \cdots \beta^{\varepsilon_k} Q^{i_k}$, or if instead we associate to it the string $\beta^{\varepsilon_1} P^{i_1} \cdots \beta^{\varepsilon_k} P^{i_k}$, in either case, the *length* $l(I)$ is k; if $I = ()$, this is to be interpreted as 0.

- If to the multi-index we associate the string $\beta^{\varepsilon_1} Q^{i_1} \cdots \beta^{\varepsilon_k} Q^{i_k}$, the *degree* $d(I)$ is $(2i_1(p - 1) - \varepsilon_1) + \cdots + (2i_k(p-1) - \varepsilon_k)$; if $I = ()$, this is to be interpreted as 0. If to the multi-index we associate the string $\beta^{\varepsilon_1}P^{i_1}\cdots\beta^{\varepsilon_k}P^{i_k}$, the *degree* $d(I)$ is $(2i_1(p-1)+\varepsilon_1)+\cdots+(2i_k(p-1)+\varepsilon_k)$; if $I=()$, again, this is to be interpreted as 0.

- If to the multi-index we associate the string $\beta^{\varepsilon_1}Q^{i_1}\cdots\beta^{\varepsilon_k}Q^{i_k}$, the *excess* $e(I)$ is $(2i_1-\varepsilon_1)-(2i_2(p-1)-\varepsilon_2)-\cdots-(2i_k(p-1)-\varepsilon_k)$; if $I=()$, this is to be interpreted as $+\infty$. If to the multi-index we associate the string $\beta^{\varepsilon_1}P^{i_1}\cdots\beta^{\varepsilon_k}P^{i_k}$, the *excess* $e(I)$ is $(2i_1+\varepsilon_1)-(2i_2(p-1)+\varepsilon_2)-\cdots-(2i_k(p-1)+\varepsilon_k)$; if $I=()$, this is to be interpreted as $-\infty$.

- If to the multi-index we associate the string $\beta^{\varepsilon_1}Q^{i_1}\cdots\beta^{\varepsilon_k}Q^{i_k}$, it is said to be *admissible* if $i_j\le pi_{j+1}-\varepsilon_{j+1}$ for each j; the empty multi-index $I=()$ is also, by convention, taken to be admissible. On the other hand, if to the multi-index we associate the string $\beta^{\varepsilon_1}P^{i_1}\cdots\beta^{\varepsilon_k}P^{i_k}$, we use the same term *admissible* to mean that $i_j\ge pi_{j+1}+\varepsilon_{j+1}$ for each j; the empty multi-index $I=()$ is again, by convention, taken to be admissible.

Remark 3.16. We can immediately see that the admissibility of a multi-index is related to the Adem relations. For example, consider the case of $p=2$, a multi-index $I=(i_1,\ldots,i_k)$ and associated string $Q^{i_1}\cdots Q^{i_k}$. Certainly when $k=2$, the term $Q^{i_1}Q^{i_2}$, interpeted as an iterated homology operation, admits an application of the Adem relations if and only if $i_1>2i_2$, which is to say if and only if (i_1,i_2) is not admissible. More generally, we will see below that the relations apply to $Q^{i_1}\cdots Q^{i_k}$ if and only if I is not admissible. As for the excess of a multi-index $I=(i_1,\ldots,i_k)$, we will see below that it is related to the instability of the operations Q^s on the homology $\mathrm{H}_\bullet(A)$ of an $\mathcal{E}$-algebra A. Note also that the the excess in this case where $p=2$ may also be written as $i_k-\sum_{j=2}^{k}(2i_j-i_{j-1})$, which gives a relation to the admissibility condition; moreover, it can also be written as $2i_1-d(I)$, giving a relation to the degree. $||$

Now we proceed to define the algebras $\mathcal{S}$ and $\mathcal{B}$.

Definition 3.17. If $p=2$, where p is our fixed prime, we set

$$\mathcal{S}:=\mathbf{F}\{Q^s\mid s\in\mathbb{Z}\}/I_{\text{Adem}}\qquad \mathcal{B}:=\mathbf{F}\{P^s\mid s\in\mathbb{Z}\}/I_{\text{Adem}}$$

where $\mathbf{F}\{Q^s\mid s\in\mathbb{Z}\}$ and $\mathbf{F}\{P^s\mid s\in\mathbb{Z}\}$ denote the free graded algebras over $\mathbb{F}_2$ on formal symbols Q^s and P^s, for $s\in\mathbb{Z}$, where Q^s and P^s have degree s, and, in either case, I_{Adem} denotes the two-sided ideal generated by the Adem relations. On the other hand, if $p>2$, we set

$$\mathcal{S}:=\mathbf{F}\{Q^s,\beta Q^s\mid s\in\mathbb{Z}\}/I_{\text{Adem}}\qquad \mathcal{B}:=\mathbf{F}\{P^s,\beta P^s\mid s\in\mathbb{Z}\}/I_{\text{Adem}}$$

where $\mathbf{F}\{Q^s,\beta Q^s\mid s\in\mathbb{Z}\}$ and $\mathbf{F}\{P^s,\beta P^s\mid s\in\mathbb{Z}\}$ denote the free graded algebras over $\mathbb{F}_p$ on formal symbols $Q^s,\beta Q^s$ and $P^s,\beta P^s$, for $s\in\mathbb{Z}$, where $Q^s,\beta Q^s,P^s,\beta P^s$ have degrees

$2s(p-1), 2s(p-1)-1, 2s(p-1), 2s(p-1)+1$ respectively, and, in either case, I_{Adem} denotes the two-sided ideal generated by the Adem relations. In either case, we call $\mathcal{S}$ the *algebra of generalized Dyer-Lashof operations* and $\mathcal{B}$ the *algebra of generalized Steenrod operations*.

Remark 3.18. In the above definition, the Adem relations are those which can be found in Proposition 3.14 and Proposition 3.15. Note that we have used the same symbol I_{Adem} for four different objects. Note also that the Adem relations apply per degree (that is, the ideal generated by them is a homogeneous one), so that $\mathcal{S}$ and $\mathcal{B}$ do indeed inherit the gradings of the free algebras. ||

Now we note that the two algebras are almost isomorphic. Recall that we have a reindexing construction $(-)^\dagger$ which negates degrees and which we have applied to dg modules, dg operads and (co)algebras over these operads. We can clearly also apply it to graded algebras, and we do so in the following result.

Proposition 3.19. *If $p = 2$, where p is our fixed prime, we have an isomorphism*

$$\mathcal{S} \cong \mathcal{B}^\dagger$$

where $Q^s \mapsto P^{-s}$. If $p > 2$, we again have such an isomorphism, where in this case $Q^s \mapsto P^{-s}$ and $\beta Q^s \mapsto \beta P^{-s}$.

Proof. We spell out the $p = 2$ case; the $p > 2$ case is analogous. Consider the correspondence $Q^s \mapsto P^{-s}$. Let $a > 2b$. We need to check that Q^aQ^b and $\sum_i \binom{i-b-1}{2i-a} Q^{a+b-i}Q^i$ map to the same elements. We have that Q^aQ^b maps to $P^{-a}P^{-b}$. Since (a, b) is non-admissible in the Q-sense, $(-a, -b)$ is non-admissible in the P-sense, that is, $-a < -2b$. Thus by the Adem relations we have that the image is:

$$\sum_i \binom{-b-i-1}{-a-2i} P^{-a-b-i}P^i$$

We can replace the summation variable from i to $-i$, giving us:

$$\sum_i \binom{i-b-1}{2i-a} P^{-(a+b-i)}P^{-i}$$

This is exactly the prescribed image of $\sum_i \binom{i-b-1}{2i-a} Q^{a+b-i}Q^i$, as desired. Thus we have a well-defined map $\mathcal{S} \to \mathcal{B}^\dagger$. To see that this is an isomorphism, note that we can define a map in the other direction as well in an analogous manner and then note that the two are inverse to one another. □

Remark 3.20. The above isomorphism allows transfer of results about one of $\mathcal{S}$ and $\mathcal{B}$ to results about the other. For future reference, we note the following about this isomorphism:

- It maps admissible monomials to admissible monomials.

- It negates the excess of a monomial.
- It respects the obvious filtrations by length on the two algebras.

||

Next, we make precise the relation between admissibility and the Adem relations.

Proposition 3.21. *The algebra $\mathcal{S}$ has an $\mathbb{F}_p$-basis given by the monomials Q^I where I is admissible; the algebra $\mathcal{B}$ has an $\mathbb{F}_p$-basis given by the monomials P^I where I is admissible.*

These bases are known as the *Cartan-Serre bases*. Note: as above, admissibility in the two cases here means different things, and moreover, its meaning varies between the cases $p = 2$ and $p > 2$.

Proof. See [CLM76]. □

Remark 3.22. Consider the case of $p = 2$ and the algebra $\mathcal{S}$. The relevant Adem relation reads

$$Q^a Q^b = \sum_i \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i$$

where $a > 2b$. For a term on the righthand size to have a non-zero coefficient, we must have, on the one hand, $2i \geq a$, which is to say $i \geq a/2$ (which then implies that $i > b$ and so $i \geq b+1$) and, on the other hand, $2i - a \leq i - b - 1$, which is to say $i \leq a - b - 1$. Moreover, for such i, we have $a + b - i < a + a/2 - i \leq a + a/2 - a/2 = a \leq 2i$. Thus, the terms which appear on the right-hand side are indeed admissible. ||

Now, by construction of $\mathcal{S}$ and $\mathcal{B}$, and the results of the previous section, we have that, if A is an algebra over $\mathcal{E}$, $\mathrm{H}_\bullet(A)$ is a module over $\mathcal{S}$, and if A is an algebra over $\mathcal{E}^\dagger$, $\mathrm{H}^\bullet(A)$ is a module over $\mathcal{B}$. Moreover, these modules satisfy certain special properties, which are extensions of the instability properties which we noted in Proposition 3.14 and 3.15 (they are upgrades of these instability properties to iterated operations). These properties make precise the relation between the excess of a multi-index and the instability of the operations.

Proposition 3.23. *Given an algebra A over $\mathcal{E}$, the iterated operations Q^I satisfy the following instability condition:*

$$Q^I c = 0 \quad \text{whenever} \quad e(I) < |c|$$

Similarly, given an algebra A over $\mathcal{E}^\dagger$, the iterated operations P^I satisfy the following instability condition:

$$P^I c = 0 \quad \text{whenever} \quad e(I) > |c|$$

Proof. We shall describe the case of the chain operad $\mathcal{E}$; the case of the cochain operad $\mathcal{E}^\dagger$ is analogous. First suppose that $p = 2$, where p is our fixed prime. Let $c \in \mathrm{H}_q(A)$ and consider the iterated operation $Q^{i_1} \cdots Q^{i_k}$. We have that $Q^{i_2} \cdots Q^{i_k} c \in \mathrm{H}_{q+i_2+\cdots+i_k}(A)$, and so, by Proposition 3.14, $Q^{i_1} \cdots Q^{i_k} c$ is zero whenever $i_1 < q + i_2 + \cdots + i_k$, or, put another way, whenever $i_1 - i_2 - \cdots - i_k < q$, which gives the required result. Now suppose that $p > 2$. We have that $\beta^{\varepsilon_2} Q^{i_2} \cdots \beta^{\varepsilon_k} Q^{i_k} c \in \mathrm{H}_{q+(2i_2(p-1)-\varepsilon_2)+\cdots+(2i_k(p-1)-\varepsilon_k)}(A)$, and so, by Proposition 3.15, $\beta^{\varepsilon_1} Q^{i_1} \cdots \beta^{\varepsilon_k} Q^{i_k} c$ is zero whenever $2i_1 - \varepsilon_1 < q + (2i_2(p-1) - \varepsilon_2) + \cdots + (2i_k(p-1) - \varepsilon_k)$, which is to say whenever $e(I) < q$, as desired. □

We make a definition to codify the above behaviour.

Definition 3.24. We say that an $\mathcal{S}$-module H is *unstable* if $Q^I h = 0$ is zero whenever $e(I) < |h|$. Similarly, we say that a $\mathcal{B}$-module H is *unstable* if $P^I h = 0$ is zero whenever $e(I) > |h|$.

Corollary 3.25. *Given an algebra A over $\mathcal{E}$, $\mathrm{H}_\bullet(A)$ is an unstable module over $\mathcal{S}$; similarly, given an algebra A over $\mathcal{E}^\dagger$, $\mathrm{H}^\bullet(A)$ is an unstable module over $\mathcal{B}$.*

Now, if A is an algebra over $\mathcal{E}$ which happens to be bounded below, say by the degree d, by the above, we will have that Q^I acts by zero for all I such that $e(I) < d$. Moreover, an analogous remark of course applies for algebras over $\mathcal{E}^\dagger$. For this reason, it will be convenient to introduce the following definitions and results.

Definition 3.26. For each $k \in \mathbb{Z}$, we set:

$$\mathcal{S}_{\geq k} := \mathbf{F}\{Q^s \mid s \in \mathbb{Z}\}/(I_{\text{Adem}} + I_{\text{exc}\,<\,k}) = \mathcal{S}/I_{<k}$$

$$\mathcal{B}_{\leq k} := \mathbf{F}\{P^s \mid s \in \mathbb{Z}\}/(I_{\text{Adem}} + I_{\text{exc}\,>\,k}) = \mathcal{S}/I_{>k}$$

Here, in the former case, $I_{\text{exc}\,<\,k}$ denotes the two-sided ideal of $\mathbf{F}\{Q^s \mid s \in \mathbb{Z}\}$ generated by monomials of excess $< k$, and simlarly, in the latter case, $I_{\text{exc}\,>\,k}$ denotes the two-sided ideal of $\mathbf{F}\{P^s \mid s \in \mathbb{Z}\}$ generated by monomials of excess $> k$. Moreover, we have $I_{<k} := (I_{\text{Adem}} + I_{\text{exc}\,<k})/I_{\text{Adem}}$ and $I_{>k} := (I_{\text{Adem}} + I_{\text{exc}\,>k})/I_{\text{Adem}}$.

Remark 3.27. Sometimes, we will use the notations $I_{\text{exc}\,\leq\,k}$, $I_{\leq k}$ and $\mathcal{S}_{>k}$ in place of $I_{\text{exc}\,<\,k+1}$, $I_{<k+1}$ and $\mathcal{S}_{\geq k+1}$, and similarly $I_{\text{exc}\,\geq\,k}$, $I_{\geq k}$ and $\mathcal{B}_{<k}$ in place of $I_{\text{exc}\,>\,k-1}$, $I_{>k-1}$ and $\mathcal{B}_{\leq k-1}$. ||

Proposition 3.28. *For each $k \in \mathbb{Z}$, the algebra $\mathcal{S}_{\geq k}$ has an $\mathbb{F}_p$-basis given by the monomials Q^I where I is admissible and $e(I) \geq k$. Similarly, For each $k \in \mathbb{Z}$, the algebra $\mathcal{B}_{\leq k}$ has an $\mathbb{F}_p$-basis given by the monomials P^I where I is admissible and $e(I) \leq k$.*

Proof. See [CLM76]. □

3.4 The (Co)homology of Free $\mathbb{E}_\infty$ DG Algebras

In this section, we describe the homology of free algebras over the operads $\mathcal{E}$ and $\mathcal{E}^\dagger$. The final result essentially says that the structure inherited by the (co)homology of $\mathcal{E}$-algebras and $\mathcal{E}^\dagger$-algebras is precisely the graded-commutative algebra structure and the operations which were described in the previous section, together with the conditions on these two pieces of data described in Propositions 3.14 and 3.15. That is, given a dg module X, the (co)homology $\mathrm{H}_\bullet(\mathbf{E}X)$ or $\mathrm{H}^\bullet(\mathbf{E}^\dagger X)$ will be constructed from $\mathrm{H}_\bullet(X)$ or $\mathrm{H}^\bullet(X)$, respectively, by freely adding in operations and products which satisfy the properties listed in Propositions 3.14 and 3.15. To make this precise, we first define the necessary free functors which we require.

First, let $\mathbf{O}_{\mathcal{S}}$ be the functor which sends a graded $\mathbb{F}_p$-module to the free $\mathcal{S}$-module on it; namely, $M \mapsto \mathcal{S} \otimes M$. Similarly, let $\mathbf{O}_{\mathcal{B}}$ be the functor $M \mapsto \mathcal{B} \otimes M$. Upon applying say the former to the homology of an input chain complex on which to generate the free algebra, we will have taken care of the operations (we use the notation $\mathbf{O}$ to indicate "operations"). However, we also want the action to be unstable. Thus, we define a functor $\mathbf{O}_{\mathcal{S},\text{un}}$, on graded $\mathbb{F}_p$-modules, as follows. Given a graded $\mathbb{F}_p$-module M, we set $\mathbf{O}_{\mathcal{S},\text{un}}M$ to be the quotient of $\mathbf{O}_{\mathcal{S}}M$ by the submodule generated by the terms $Q^I m$ where $e(I) < |m|$. That is, $\mathbf{O}_{\mathcal{S},\text{un}}M$ is the free unstable module over $\mathcal{S}$ on M. (We use the "un" to indicate "unstable".) Upon applying $\mathbf{O}_{\mathcal{S},\text{un}}$ to the homology of an input chain complex on which to generate the free algebra, we will have taken care of the operations and properties (i) and (ii) in Propositions 3.14 and 3.15. Similarly, we let $\mathbf{O}_{\mathcal{B},\textbf{un}}$ be the functor on graded $\mathbb{F}_p$-modules which sends M to the quotient of $\mathbf{O}_{\mathcal{B}}M$ by the submodule generated by the terms $P^I m$ where $e(I) > |m|$.

Next, let $\mathbf{P}_{\mathcal{S}}$ be the functor which sends an unstable $\mathcal{S}$-module to the free graded-commutative algebra on the underlying graded $\mathbb{F}_p$-module (we use $\mathbf{P}$ to indicate "products"). Note that $\mathbf{P}_{\mathcal{S}}H$ still carries an action by $\mathcal{S}$, defined via the following requirements: (i) on the unit, we require that $Q^s 1$ is 1 if $s = 0$ and 0 if $s \neq 0$, and also, if $p > 2$, that $\beta Q^s 1$ is zero for all s (ii) we require the following internal Cartan formulae to hold, where the second applies only in the case $p > 2$:

$$Q^s(xy) = \sum_{i+j=s} (Q^i x)(Q^j y)$$

$$\beta Q^s(xy) = \sum_{i+j=s} (\beta Q^i x)(Q^j y) + (-1)^{|x|}(Q^i x)(\beta Q^j y)$$

Similarly, let $\mathbf{P}_{\mathcal{B}}$ be the functor which sends an unstable $\mathcal{B}$-module to the free graded-commutative algebra on the underlying graded $\mathbb{F}_p$-module. Note again that $\mathbf{P}_{\mathcal{B}}H$ still carries an action by $\mathcal{B}$, defined via the following requirements: (i) on the unit, we require that $P^s 1$ is 1 if $s = 0$ and 0 if $s \neq 0$, and also, if $p > 2$, that $\beta P^s 1$ is zero for all s (ii) we require the following internal Cartan formulae to hold, where the second applies only in the case $p > 2$:

$$P^s(xy) = \sum_{i+j=s} (P^i x)(P^j y)$$

$$\beta P^s(xy) = \sum_{i+j=s} (\beta P^i x)(P^j y) + (-1)^{|x|}(P^i x)(\beta P^j y)$$

Upon applying say the former functor, namely $\mathbf{P}_{\mathcal{S}}$, to the homology of an input chain complex on which to generate the free algebra, having already applied $\mathbf{O}_{\mathcal{S},\text{un}}$ to add in the operations, we will have also added in the products, and taken care of (iv) in Propositions 3.14 and 3.15. We still require, however, the compatibility between the product structure and the operations described by (iii) in the same propositions. Thus, we define a functor $\mathbf{P}_{\mathcal{S},\text{pw}}$, on unstable $\mathcal{S}$-modules, as follows. If $p = 2$, given such a module H, we set $\mathbf{P}_{\mathcal{S},\text{pw}}H$ to be the quotient of $\mathbf{P}_{\mathcal{S}}H$ by the $\mathbb{F}_p$-ideal generated by the terms $Q^{|h|}h - h \cdot h$; if $p > 2$, we set $\mathbf{P}_{\mathcal{S},\text{pw}}H$ to be the quotient of $\mathbf{P}_{\mathcal{S}}H$ by the $\mathbb{F}_p$-ideal generated by the terms $Q^{|h|/2}h - h^p$ where h of course is required to lie in even degrees. (We use “pw” to indicate “powers”.) Similarly, we define a functor $\mathbf{P}_{\mathcal{B},\text{pw}}$, on unstable $\mathcal{B}$-modules, as follows. If $p = 2$, given such a module H, we set $\mathbf{P}_{\mathcal{B},\text{pw}}H$ to be the quotient of $\mathbf{P}_{\mathcal{B}}H$ by the $\mathbb{F}_p$-ideal generated by the terms $P^{|h|}h - h \cdot h$; if $p > 2$, we set $\mathbf{P}_{\mathcal{B},\text{pw}}H$ to be the quotient of $\mathbf{P}_{\mathcal{B}}H$ by the $\mathbb{F}_p$-ideal generated by the terms $P^{|h|/2}h - h^p$ where h of course is required to lie in even degrees. We have now taken into account all the necessary compatibilities between the products and operations, insofar as we can show that the operations remain well-defined, for note that we did not require the ideal to be an $\mathcal{S}$-submodule in the former case or a $\mathcal{B}$-submodule in the latter case. This is indeed the case however, as the following lemma demonstrates; as a result of it, we have that $\mathbf{P}_{\mathcal{S},\text{pw}}H$ and $\mathbf{P}_{\mathcal{B},\text{pw}}H$, where H is an appropriate input in either case, carry a well-defined action by $\mathcal{S}$ and $\mathcal{B}$, respectively.

Lemma 3.29. *Given an unstable $\mathcal{S}$-module H, the $\mathbb{F}_p$-ideal of $\mathbf{P}_{\mathcal{S}}H$ generated by the terms $Q^{|h|}h - h^2$, if $p = 2$, or by $Q^{|h|/2}h - h^p$ if $p > 2$, is an $\mathcal{S}$-submodule of $\mathbf{P}_{\mathcal{S}}H$. Similarly, given an unstable $\mathcal{B}$-module H, the $\mathbb{F}_p$-ideal of $\mathbf{P}_{\mathcal{B}}H$ generated by the terms $P^{|h|}h - h^2$, if $p = 2$, or by $P^{|h|/2}h - h^p$ if $p > 2$, is a $\mathcal{B}$-submodule of $\mathbf{P}_{\mathcal{B}}H$.*

Proof. We shall give the proof in the case of the unstable $\mathcal{S}$-module and where $p = 2$; the other cases are similar. Set $s := |h|$ and let $r \in \mathbb{Z}$ be arbitrary. We must show that the ideal contains:

$$Q^r(Q^s h - h \cdot h) = Q^r Q^s h - Q^r(h \cdot h)$$

Note that this suffices to show closure of the ideal under the action of $\mathcal{S}$ because iterated operations $Q^{i_1} \cdots Q^{i_k}$ act one at a time, and because the action of Q^r on any product of terms of the form

$(Q^s h - h \cdot h)$ will reduce to this case by the Cartan formula.

Suppose first that $r < 2s$. Then we have that $Q^r Q^s h = 0$ by the instability of the action of $\mathcal{S}$ on H, since $Q^s x$ has degree $2s$. Moreover, we have, by the Cartan formula, $Q^r(h \cdot h) = \sum_{i+j=r}(Q^i h) \cdot (Q^j h)$, and this is also zero since, for $Q^i h$ and $Q^j h$ to be non-zero, by instability once more, we require that $i, j \geq s$, but then $i + j \geq 2s$, so that the sum is in fact empty.

Now suppose that $r = 2s$. Then $Q^r Q^s h = Q^{2s} Q^s h$. On the other hand, by instability, the only non-zero term in $Q^r(h \cdot h) = \sum_{i+j=r}(Q^i h) \cdot (Q^j h)$ is that which has $i = j = s$, thus leaving us with $(Q^s h) \cdot (Q^s h)$. We are thus left with $Q^{2s} Q^s h - (Q^s h) \cdot (Q^s h)$, and this is of the form required to be a generator of the ideal since $Q^s h$ has degree $2s$.

Finally, suppose that $r > 2s$. By the Adem relations, we have that:

$$Q^r Q^s h = \sum_i \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i h$$

Given a term in the sum on the right-hand side, for it to be non-zero, we require that $2i \geq r$, as this value occurs in the binomial coefficient. On the other hand, by instability, we also require that $r + s - i \geq i + s$, which is to say, $2i \leq r$. Thus the only non-zero term which occurs is that which has $2i = r$. In particular, if r is odd, the sum is zero, and otherwise, setting $r = 2s + 2k$ for some $k \geq 1$, it is given by $Q^{2s+k} Q^{s+k} h$. On the other hand, we have that the non-zero terms in $Q^r(h \cdot h) = \sum_{i+j=r}(Q^i h) \cdot (Q^j h)$ are those given by $(i, j) = (s, r-s), (s+1, r-s-1), \ldots, (r-s, s)$. This sequence contains $r - 2s + 1$ terms. Thus, if r is odd, so that the number of terms is even, as we are in characteristic 2, the sum will amount to zero by symmetry. On the other hand, if r is even, there will remain one term, which, if $r = 2s + 2k$ where $k \geq 1$, will be the case of $(s + k, s + k)$, which is to say the sum will amount to $(Q^{s+k} h) \cdot (Q^{s+k} h)$. All told, if r is odd, the $Q^r Q^s h - Q^r(h \cdot h)$ amounts to zero, which is in the ideal, and if r is even and equal to $2s + 2k$, $k \geq 1$, it amounts to $Q^{2s+k} Q^{s+k} h - (Q^{s+k} h) \cdot (Q^{s+k} h)$, which is also in the ideal, since $Q^{s+k} h$ has degree $2s + k$. $\square$

We now combine the above functors to define the functors which will give the (co)homology of free algebras over $\mathcal{E}$ and $\mathcal{E}^\dagger$. We define the functor $\mathbf{Q}_{\mathcal{S}}$, on graded $\mathbb{F}_p$-modules, to be the composite:

$$\mathbf{Q}_{\mathcal{S}} := \mathbf{P}_{\mathcal{S},\text{pw}} \mathbf{O}_{\mathcal{S},\text{un}}$$

Similarly, we define the functor $\mathbf{Q}_{\mathcal{B}}$, on graded $\mathbb{F}_p$-modules, to be the composite:

$$\mathbf{Q}_{\mathcal{B}} := \mathbf{P}_{\mathcal{B},\text{pw}}\mathbf{O}_{\mathcal{B},\text{un}}$$

It is clear that, given a graded $\mathbb{F}_p$-module M, $\mathbf{Q}_{\mathcal{S}}M$ is the free object on M which carries an action by operations Q^s, and also the βQ^s if $p > 2$, and a product structure such that the properties in (i)-(iv) of Proposition 3.14 if $p = 2$, or those in (i)-(iv) of Proposition 3.15 if $p > 2$, are satisfied. Similarly, given a graded $\mathbb{F}_p$-module M, $\mathbf{Q}_{\mathcal{B}}M$ is the free object on M which carries an action by operations P^s, and also the βP^s if $p > 2$, and a product structure such that the properties in (v)-(viii) of Proposition 3.14 if $p = 2$, or those in (v)-(viii) of Proposition 3.15 if $p > 2$, are satisfied. Thus, if X is a chain complex, we have a natural map:

$$\varphi\colon \mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X) \to \mathrm{H}_\bullet(\mathbf{E}X)$$

On the other hand, if X is a cochain complex, we have a natural map:

$$\psi\colon \mathbf{Q}_{\mathcal{B}}(\mathrm{H}^\bullet X) \to \mathrm{H}^\bullet(\mathbf{E}^\dagger X)$$

Finally then, the precise result on the homologies of the free algebras over $\mathcal{E}$ and $\mathcal{E}^\dagger$ is as follows.

Proposition 3.30. *If X is a chain complex, we have a natural isomorphism:*

$$\mathrm{H}_\bullet(\mathbf{E}X) \cong \mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X)$$

If X is a cochain complex, we have a natural isomorphism:

$$\mathrm{H}^\bullet(\mathbf{E}^\dagger X) \cong \mathbf{Q}_{\mathcal{B}}(\mathrm{H}^\bullet X)$$

More specifically, the canonical maps φ and ψ above are isomorphisms.

Proof. See [CLM76]. □

We can also give a slighty different description of the functors $\mathbf{Q}_{\mathcal{S}}$ and $\mathbf{Q}_{\mathcal{B}}$, in which there is no need to enforce the p^{th} power relations when adding in products. Consider the case of the functor $\mathbf{Q}_{\mathcal{S}}$. Given a graded module M, if $\{c_i\}$ is a basis of M, then, by Proposition 3.21, $\mathbf{P}_{\mathcal{S}}\mathbf{O}_{\mathcal{S},\text{un}}M$ is the free graded-commutative algebra over $\mathbb{F}_p$ on the monomials $Q^I c_i$ where I is admissible and $e(I) \geq |c_i|$; note that I may be empty here. Moreover, $\mathbf{Q}_{\mathcal{S}}M$ is the quotient of this free algebra by the $\mathbb{F}_p$-ideal generated by the terms $Q^s c_i - c_i^p$ where $s = p$ if $p = 2$ and $2s = p$ if $p > 2$. The change in our alternative description will be that, if we modify the additon of the operations so as to omit those monomials where $e(I) = |c_i|$, we can replace $\mathbf{P}_{\mathcal{S},\text{pw}}$ with simply the free graded-commuative algebra functor $\mathbf{P}_{\mathcal{S}}$, and exactly analogous remarks apply in the case of $\mathbf{Q}_{\mathcal{B}}$. The precise statement is as follows.

Proposition 3.31. *If X is a chain complex, and $\{c_i\}$ is a homogeneous basis of $\mathrm{H}_\bullet(X)$, then we have that:*

$$\mathrm{H}_\bullet(\mathbf{E}X) \cong \mathbb{F}_p\{Q^I c_i \mid I \text{ admissible}, e(I) > |c_i|\}$$

Here the right-hand side denotes the free graded-commutative algebra over $\mathbb{F}_p$ on the monomials $Q^I c_i$ where I is admissible and $e(I) >$ deg(c_i). Similarly, if X is a cochain complex, and $\{c_i\}$ is a homogeneous basis of $\mathrm{H}^\bullet(X)$, then we have that:

$$\mathrm{H}^\bullet(\mathbf{E}^\dagger X) \cong \mathbb{F}_p\{P^I c_i \mid I \text{ admissible}, e(I) < |c_i|\}$$

The important differences here, in the chain case, are the change of the condition "$e(I) \geq |c_i|$" to "$e(I) > |c_i|$" and the omission of the quotient on the algebra. The idea is that the formation of this quotient of the algebra amounts exactly to the omission of the terms $Q^I c_i$ where $e(I) = |c_i|$ because, for example, the terms $Q^s c_i$ where the excess $s = |c_i|$ are set equivalent to the product $c_i \cdot c_i$ and, more generally, the terms $Q^{i_1} \cdots Q^{i_k} c_i$ where $i_1 - i_2 - \cdots - i_k = |c_i|$ (and so $i_1 = i_2 + \cdots i_k + |c_i|$) are set equivalent to the product $(Q^{i_2} \cdots Q^{i_k} c_i) \cdot (Q^{i_2} \cdots Q^{i_k} c_i)$. Put in other terms, this holds, in the $p = 2$ and chain case, because, due to (iii) in Proposition 3.14, in any degree, the lowest excess operations which act non-trivially serve only to square elements in that degree, and act trivially on all other degrees, so that they are represented exactly by the formal squares of elements.

Proof. We shall describe the case of a chain complex and where $p = 2$. The other cases are similar. Let M be a graded module with basis $\{c_i\}$. Let $\mathbf{F}M$ denote the free graded-commutative algebra over $\mathbb{F}_2$ on the monomials $Q^I c_i$ where I is admissible and $e(I) > |c_i|$. We then have an obvious map

$$\varphi\colon \mathbf{F}M \to \mathbf{Q}_{\mathcal{S}}M$$

and, by Proposition 3.30, it suffices to show that this map is an isomorphism, which amounts to demonstrating that it is bijective, as the compatibility with the rest of the data is automatic. Injectivity is clear as the extra relations imposed in $\mathbf{Q}_{\mathcal{S}}M$ are not between the generators present in $\mathbf{F}M$. Now let us demonstrate surjectivity. To do this, the only non-obvious thing which we need to show is that any monomial $Q^I c_i$ where $I = (i_1, \ldots, i_k)$ is admissible and $e(I) = |c_i|$ occurs in the image. We will do this by induction on the length $l(I)$ of I. We know that I cannot be empty since the empty multi-index has infinite excess. Suppose then that I has length 1, so that $I = (i_1)$ and $i_1 = |c_i|$. By the power relation imposed in constructing $\mathbf{Q}_{\mathcal{S}}M$, we have that

$$Q^{i_i} c_i = c_i \cdot c_i$$

and so we are done, as $c_i \cdot c_i$ certainly occurs in the image. Now suppose that we have demonstrated the result for all lengths $< k$ and let $I = (i_1, \ldots, i_k)$ where $e(I) = |e_i|$. Since $e(I) = i_1 - i_2 - \cdots - i_k = |c_i|$, we have that, in $\mathbf{Q}_{\mathcal{S}}M$:

$$Q^I c_i = (Q^{i_2} \cdots Q^{i_k} c_i) \cdot (Q^{i_2} \cdots Q^{i_k} c_i)$$

Thus, if $Q^{i_2} \cdots Q^{i_k} c_i$ occurs in the image, we are done. Note that

$$i_2 - i_3 - \cdots - i_k = i_k - \sum_{j=3}^{k}(2i_j - i_{j-1}) \geq i_k - \sum_{j=2}^{k}(2i_j - i_{j-1}) = i_1 - i_2 - \cdots - i_k$$

since I is admissible. Thus, the new multi-index $I' = (i_2, \ldots, i_k)$, which is obviously admissible as I is, is also such that $e(I') \geq |c_i|$. If in fact $e(I') > |e_i|$, we are done, as then $Q^{i_2} \cdots Q^{i_k} c_i$ of course lies in the image. If not, we have $e(I') = |c_i|$ and I' has length $< k$, so that we are done by the inductive hypothesis. $\square$

Next, we wish to describe certain refinements of Propositions 3.30 and 3.31. Let X be a chain complex. Note that $\mathbf{E}X$ is naturally filtered by setting

$$\mathrm{F}_t\mathbf{E}X = \bigoplus_{n \leq t} \mathcal{E}(n) \otimes_{\Sigma_n} X^{\otimes n}$$

for $t \geq 0$. We then also have filtration of $\mathrm{H}_\bullet(\mathbf{E}X)$ by setting

$$\mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X) = \mathrm{img}(\mathrm{H}_\bullet(\mathrm{F}_t\mathbf{E}X) \to \mathrm{H}_\bullet(\mathbf{E}X))$$

again for $t \geq 0$. Similarly, if X is a cochain complex, we have filtrations $\mathrm{F}_t\mathbf{E}^\dagger X$ and $\mathrm{F}_t\mathrm{H}^\bullet(\mathbf{E}^\dagger X)$ of $\mathbf{E}^\dagger X$ and $\mathrm{H}^\bullet(\mathbf{E}^\dagger X)$, respectively. In either case, the refinement will compute these filtration pieces of the (co)homology. Note that, with this filtration, $\mathrm{H}_\bullet(\mathbf{E}X)$ in the chain case, or $\mathrm{H}^\bullet(\mathbf{E}^\bullet X)$ in the cochain case, is a filtered algebra. To see this, say in the chain case, it suffices to consider an element of $\mathrm{H}_\bullet(\mathbf{E}X)$ of the form $[\sigma \otimes (x_1 \otimes \cdots \otimes x_n)] \in \mathrm{F}_n\mathrm{H}_\bullet(\mathbf{E}X)$, and another, say $[\tau \otimes (x'_1 \otimes \cdots \otimes x'_m)] \in \mathrm{F}_m\mathrm{H}_\bullet(\mathbf{E}X)$, of the same form, where $\sigma \in \mathcal{E}(n)$ and $\tau \in \mathcal{E}(m)$. We have that

$$\begin{aligned}[\sigma \otimes (x_1 \otimes \cdots \otimes x_n)] \cdot [\tau \otimes (x'_1 \otimes \cdots \otimes x'_m)] &= e_0([\sigma \otimes (x_1 \otimes \cdots \otimes x_n)], [\tau \otimes (x'_1 \otimes \cdots \otimes x'_m)]) \\ &= (e_0; \sigma, \tau) \otimes x_1 \otimes \cdots \otimes x_n \otimes x'_1 \otimes \cdots \otimes x'_m\end{aligned}$$

where $(e_0; \sigma, \tau)$ is the image of $e_0 \otimes \sigma \otimes \tau$ under the operadic composition map $\mathcal{E}(2) \otimes \mathcal{E}(n) \otimes \mathcal{E}(m) \to \mathcal{E}(n+m)$. We thus see that the product lies in $\mathrm{F}_{n+m}\mathrm{H}_\bullet(\mathbf{E}X)$, as desired.

Now, note that, given a graded module M, $\mathbf{Q}_{\mathcal{S}}M$ and $\mathbf{Q}_{\mathcal{B}}M$ also carry natural filtrations. To see this, say in the case of $\mathbf{Q}_{\mathcal{S}}M$, first note that the free algebra, say $\mathcal{F}$, on the symbols Q^s, and also the symbols βQ^s if $p > 2$, has a filtration by length, where, for $t \geq 0$, we set $\mathrm{F}_t\mathcal{F}$ to comprise sums of those monomials Q^I where $p^{l(I)} \leq t$, which is to say where $l \leq \lfloor \log_p t \rfloor$ (we choose this indexing of the filtration by length because, in the action on $\mathbf{E}X$, each operation in Q^I will amount to an application, for some n, of the map $\mathcal{E}(p) \otimes \mathcal{E}(n) \otimes \cdots \otimes \mathcal{E}(n) \to \mathcal{E}(pn)$ and so will multiply the arity of the operad tensor factor by p). Next, via the images in $\mathcal{S}$ of these filtration pieces, we get a filtration of $\mathcal{S}$. Now note that the images, tensored with M, of these, give a filtration of $\mathcal{S} \otimes M$ and then that, in turn, the images of these give a filtration of $\mathbf{O}_{\mathcal{S},\text{un}}M$. Finally, we get a filtration of $\mathbf{P}_{\mathcal{S}}\mathbf{O}_{\mathcal{S},\text{un}}M$ by simply requiring that it be a filtered algebra and then, via images, we get a filtration of $\mathbf{Q}_{\mathcal{S}}M$. Similar remarks apply in the case of $\mathbf{Q}_{\mathcal{B}}M$.

Now consider once more the canonical maps

$$\varphi\colon \mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X) \to \mathrm{H}_\bullet(\mathbf{E}X)$$

$$\psi\colon \mathbf{Q}_{\mathcal{B}}(\mathrm{H}^\bullet X) \to \mathrm{H}^\bullet(\mathbf{E}^\dagger X)$$

defined above, where in the first case X is a chain complex, and in the second a cochain complex. We claim that these maps are in fact compatible with the filtrations which we have just described above.

Lemma 3.32. *The maps φ and ψ respect the filtrations defined above, in that they induce maps*

$$\varphi_t\colon \mathrm{F}_t\mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X) \to \mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X)$$

$$\psi_t\colon \mathrm{F}_t\mathbf{Q}_{\mathcal{B}}(\mathrm{H}^\bullet X) \to \mathrm{F}_t\mathrm{H}^\bullet(\mathbf{E}^\dagger X)$$

for each $t \geq 0$.

Proof. We shall outline the case in which X is chain complex and $p = 2$; the other cases are analogous. First note that $\mathrm{F}_0\mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X)$ is simply the free $\mathbb{F}_2$-module on the unit 1 of the algebra. On the other hand, $\mathrm{F}_0\mathrm{H}_\bullet(\mathbf{E}X)$ is simply $\mathrm{H}_\bullet(\mathcal{E}(0)) \cong \mathbb{F}_2[0]$. Moreover, the unit 1 is sent to to the generator of $\mathrm{H}_\bullet(\mathcal{E}(0))$, so that we have the desired result.

Next, in $\mathrm{F}_1\mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X)$, we also have the homology classes in $\mathrm{H}_\bullet(X)$ and products of these classes. On the other hand, $\mathrm{F}_1\mathrm{H}_\bullet(\mathbf{E}X) = \mathrm{H}_\bullet(\mathcal{E}(0)) \oplus \mathrm{H}_\bullet(\mathcal{E}(1) \otimes X)$. Given $c = [x] \in \mathrm{H}_\bullet(X)$, it is mapped to $[(\mathrm{id}_{(1)}) \otimes x]$, which lies in $\mathrm{H}_\bullet(\mathcal{E}(1) \otimes X)$, and so we have the desired result.

Now consider $\mathrm{F}_t\mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X)$ for some $t \geq 2$. We have $\mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X) = \bigoplus_{n \leq t} \mathrm{H}_\bullet(\mathcal{E}(n) \otimes_{\Sigma_n} X^{\otimes n})$.

It suffices to demonstrate the result for those elements of $\mathrm{F}_t\mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X)$ which are of the form $(Q^{I_1}c_1)\cdots(Q^{I_k}c_k)$ where the multi-indices $I_1,\dots,I_k$ are such that $2^{l(I_1)}+\cdots+2^{l(I_k)}\le t$ and the c_i are homogeneous. Consider first $Q^{I_k}c_k$. As above, c_k is mapped to $[(\mathrm{id}_{(1)})\otimes x_k]\in\mathrm{H}_\bullet(\mathcal{E}(1)\otimes X)$, where x_k is a representative cycle of c_k. Let $I_k=(i_{k,1},\dots,i_{k,l(I_k)})$. We have that $Q^{i_{k,l(I_k)}}c$ maps to

$$[e_{i_{k,l(I_k)}-\deg(c_k)}((\mathrm{id}_{(1)})\otimes x_k,(\mathrm{id}_{(1)})\otimes x_k)]$$

where $e_{i_{k,l(I_k)}-\deg(c_k)}\in\mathcal{E}(2)$ and, if $i_{k,l(I_k)}-\deg(c_k)<0$, we understand this element to be zero. Now, the representative $e_{i_{k,l(I_k)}-\deg(c_k)}((\mathrm{id}_{(1)})\otimes x_k,(\mathrm{id}_{(1)})\otimes x_k)$ denotes the image under

$$\mathcal{E}(2)\otimes\mathbf{E}X\otimes\mathbf{E}X\to\mathbf{E}X$$

of $e_{i_{k,l(I_k)}-\deg(c_k)}\otimes((\mathrm{id}_{(1)})\otimes x_k)\otimes((\mathrm{id}_{(1)})\otimes x_k)$ and this image is precisely

$$(e_{i_{k,l(I_k)}-\deg(c_k)};(\mathrm{id}_{(1)}),(\mathrm{id}_{(1)}))\otimes(x_k\otimes x_k)\in\mathcal{E}(2)\otimes_{\Sigma_2}X^{\otimes 2}$$

where $(e_{i_{k,l(I_k)}-\deg(c_k)};(\mathrm{id}_{(1)}),(\mathrm{id}_{(1)}))$ is the image of $e_{i_{k,l(I_k)}-\deg(c_k)}\otimes(\mathrm{id}_{(1)})\otimes(\mathrm{id}_{(1)})$ under the operadic composition map:

$$\mathcal{E}(2)\otimes\mathcal{E}(1)\otimes\mathcal{E}(1)\to\mathcal{E}(2)$$

Thus, we see that $Q^{I_k}c_k$ maps into $\mathrm{H}_\bullet(\mathcal{E}(2)\otimes_{\Sigma_2}X^{\otimes 2})\subseteq\mathrm{F}_2\mathrm{H}_\bullet(\mathbf{E}X)$. Now, we may repeat this procedure to apply each of the operations in Q^{I_k}, where at each stage we have an application of one of the operadic composition maps

$$\mathcal{E}(2)\otimes\mathcal{E}(n)\otimes\mathcal{E}(n)\to\mathcal{E}(2n)$$

so that the arity of operadic tensor factor is doubled while the original element of $\mathrm{H}_\bullet X$, or in fact its representative x_k, goes along for the ride, being repeated so as to double the length of the other tensor factor. All told, upon inducting on the r in $i_{k,r}$, an arbitrary entry of $I_k=(i_{k,1},\dots,i_{k,l(I_k)})$, we find that $Q^{I_k}c_k$ maps into

$$\mathrm{H}_\bullet\left(\mathcal{E}(2^{l(I_k)})\otimes_{\Sigma_{2^{l(I_k)}}}X^{\otimes 2^{l(I_k)}}\right)$$

which is contained in $\mathrm{F}_{2^{l(I_k)}}\mathrm{H}_\bullet(\mathbf{E}X)$. By the same argument, we have more generally, for each $r=1,\dots,k$, that $Q^{I_r}c_r$ maps into

$$\mathrm{H}_\bullet\left(\mathcal{E}(2^{l(I_r)})\otimes_{\Sigma_{2^{l(I_r)}}}X^{\otimes 2^{l(I_r)}}\right)$$

which is contained in $\mathrm{F}_{2^{l(I_r)}}\mathrm{H}_\bullet(\mathbf{E}X)$. Thus, we have that the product $(Q^{I_1}c_1)\cdots(Q^{I_k}c_k)$ maps into $\mathrm{F}_{2^{l(I_1)}+\cdots+2^{l(I_k)}}\mathrm{H}_\bullet(\mathbf{E}X)$, and so, because $2^{l(I_1)}+\cdots+2^{l(I_k)} \leq t$, we have that it maps into $\mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X)$, as desired. □

Now we can provide the refinement of Proposition 3.30.

Proposition 3.33. *If X is a chain complex, for each $t \geq 0$, the map*

$$\varphi_t \colon \mathrm{F}_t\mathbf{Q}_{\mathcal{S}}(\mathrm{H}_\bullet X) \to \mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X)$$

defined above is an isomorphism. Similarly, if X is a cochain complex, for each $t \geq 0$, the map

$$\psi_t \colon \mathrm{F}_t\mathbf{Q}_{\mathcal{B}}(\mathrm{H}^\bullet X) \to \mathrm{F}_t\mathrm{H}^\bullet(\mathbf{E}^\dagger X)$$

defined above is an isomorphism.

Proof. See [CLM76]. □

Using this result, we can also refine the description of the (co)homologies in Proposition 3.31.

Proposition 3.34. *If X is a chain complex and $\{c_i\}$ is a basis of $\mathrm{H}_\bullet(X)$, then, for each $t \geq 0$, we have that:*

$$\mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X) \cong \langle\{(Q^{I_1}c_1)\cdots(Q^{I_k}c_k) \mid k \geq 0, I \textit{ admissible}, e(I) > |c_i|, p^{l(I_1)}+\cdots+p^{l(I_k)} \leq t\}\rangle$$

Here the right-hand side denotes an $\mathbb{F}_p$-submodule of the free graded-commutative algebra over $\mathbb{F}_p$ on the monomials $Q^I c_i$ where I is admissible, $e(I) > |c_i|$. Similarly, if X is a cochain complex and $\{c_i\}$ is a basis of $\mathrm{H}^\bullet(X)$, then, for each $t \geq 0$, we have that

$$\mathrm{F}_t\mathrm{H}^\bullet(\mathbf{E}^\dagger X) \cong \langle\{(P^{I_1}c_1)\cdots(P^{I_k}c_k) \mid k \geq 0, I \textit{ admissible}, e(I) < |c_i|, p^{l(I_1)}+\cdots+p^{l(I_k)} \leq t\}\rangle$$

Proof. This follows by a deduction from Proposition 3.33 almost identical to the deduction of Proposition 3.31 from 3.30. □

Remark 3.35. We can now revisit Remark 3.11 and make the ideas there more precise. Suppose that $p = 2$ and let us work with chain complexes. In that remark, we mentioned that we expect $\mathbf{E}\mathbb{F}_2[0]$ to be an algebra of operations and products thereof. By Proposition 3.31, we see that

$$\mathrm{H}_\bullet(\mathbf{E}\mathbb{F}_2[0]) \cong \mathbb{F}_2\{Q^I \mid I \text{ admissible}, e(I) > 0\}$$

which is exactly an algebra of operations and products (where, as we have seen, the admissibility and excess conditions are direct consequences of, and so are to be interpreted here as manifestations of, the properties in Proposition 3.14 that the operations and products satisfy). In Remark 3.11, we also mentioned that we will later be able to identify the contributions to the above algebra from each piece in the direct sum decomposition $\mathrm{H}_\bullet(\mathbf{E}\mathbb{F}_2[0]) = \oplus_n \mathrm{H}_\bullet(\mathcal{E}(n)/\Sigma_n)$. Proposition 3.34 allows us to do this. Looking at the case $t = 0$ in that proposition, we see that the homology of $\mathcal{E}(0)/\Sigma_0$ contributes precisely the multiplicative unit (or, more accurately, the $\mathbb{F}_2$-submodule generated by the multiplicative unit). Looking at the case $t = 1$, we see that the homology of $\mathcal{E}(1)/\Sigma_1$ contains the generating point. Looking at the case $t = 2$, we see that the homology of $\mathcal{E}(2)/\Sigma_2$ contributes the operations. Finally, we see that the remaining $\mathcal{E}(n)/\Sigma_n$ contribute the iterated operations as well as products thereof – and we can clearly read off exactly in which summand any such contribution arises. Moreover, we can easily make analogous observations in the case of the homology $\mathrm{H}_\bullet(\mathbf{E}X)$ of the free algebra $\mathbf{E}X$ on a general chain complex X. $||$

3.5 The Eilenberg-Zilber Operad

We now introduce another operad, the Eilenberg-Zilber operad, as always in both a chain and a cochain version. Let $\mathsf{Gr}_{\mathbb{F}_p}$ denote the category with objects the $\mathbb{Z}$-graded $\mathbb{F}_p$-modules and morphisms the homogeneous maps (maps with some fixed degree). Let also Spc denote the category of spaces, by which we mean simplicial sets. Fix some $n \geq 0$. For each $d \in \mathbb{Z}$, we set:

$$\mathcal{Z}(n)_d := \{\text{nat trans, as functors } \mathsf{Spc} \to \mathsf{Gr}_{\mathbb{F}_p}, \mathbf{C}_\bullet(-) \to \mathbf{C}_\bullet(-)^{\otimes n} \text{ of deg } d\}$$

Thus $\mathcal{Z}(n)_d$ consists of the degree d, n-ary co-operations on chains; note that, as always throughout this work, the chains here are normalized. Given such a natural transformation $\alpha = \{\alpha_S \colon \mathbf{C}_\bullet(S) \to \mathbf{C}_\bullet(S)^{\otimes n}\}$ of degree d, we get a natural transformation $\partial\alpha$ of degree $d-1$ by setting, for a simplicial set S and a non-degenerate simplex $s \in S$, $(\partial\alpha)_S(s) = \partial\alpha_S(s) - (-1)^d \alpha_S \partial(s)$. An easy check shows that this gives us a differential:

$$\partial \colon \mathcal{Z}(n)_d \to \mathcal{Z}(n)_{d-1}$$

The operad identity in $\mathcal{Z}(1)$ is the identity transformation. The symmetric group Σ_n acts on $\mathcal{Z}(n)$ by permuting the tensor factors. Finally, the operad composition maps are analogous to those that occur in co-endomorphism operads, defined in the previous chapter.

Definition 3.36. The *Eilenberg-Zilber chain operad* is the operad $\mathcal{Z}$ defined above; the *Eilenberg-Zilber cochain operad* is the operad $\mathcal{Z}^\dagger$.

Remark 3.37. An easy check shows that, since we are working over a field and since co-operations on chains and operations on cochains are determined by their values on the standard simplices, we can equivalently define the Eilenberg-Zilber cochain operad by setting

$$\mathcal{Z}^{\dagger}(n)_d = \{\text{nat trans, as functors } \mathsf{Spc}^{\mathrm{op}} \to \mathsf{Gr}_{\mathbb{F}_p}, \mathrm{C}^{\bullet}(-)^{\otimes n} \to \mathrm{C}^{\bullet}(-) \text{ of deg } d\}$$

and setting the rest of the operad data in a manner analogous to that above for the chain operad. Thus $\mathcal{Z}^{\dagger}(n)_d$ consists of the degree d, n-ary operations on cochains. ||

In [HS87], Hinich and Schechtman demonstrated that, for each n, $\mathcal{Z}(n)$, has the homology of a point. It is not, however, $\mathbb{E}_\infty$, as it fails to be zero in negative degrees and is not Σ-free. It does however admit a map, in fact an embedding

$$\mathrm{AW}\colon \mathcal{M} \to \mathcal{Z}$$

from the McClure-Smith operad, which we shall now describe. Once we have done so, we of course will also have a map:

$$\mathrm{AW}^{\dagger}\colon \mathcal{M}^{\dagger} \to \mathcal{Z}^{\dagger}$$

First, we need some definitions.

Definition 3.38. Let V be a finite linearly ordered set. An *overlapping partition* $\mathcal{A}$ of V with m pieces is a collection of subsets $A_1, \dots, A_m$ of L with the following properties:

- If $i < j$, then each element of A_i is $\leq$ each element of A_j.
- For $i < m$, $A_i \cap A_{i+1}$ has exactly one element.

Definition 3.39. Suppose given $n \geq 0$ and a surjection $f\colon (m) \to (n)$. We then get a natural transformation, over simplicial sets, $\langle f \rangle \colon \mathrm{C}_{\bullet}(-) \to \mathrm{C}_{\bullet}(-)^{\otimes n}$, where given a simplicial set S and $\sigma\colon \Delta_p \to S$, we have that

$$\langle f \rangle(\sigma) = \sum_{\mathcal{A}} \bigotimes_{i=1}^{n} \sigma| \amalg_{f(j)=i} A_j \tag{3.40}$$

where $\mathcal{A}$ runs through the overlapping partitions of $[p] = \{0, 1, \dots, p\}$ with m pieces. We call these natural transformations, the *sequence co-operations*. Note that since $\sum_{\mathcal{A}} \bigotimes_{i=1}^{n} \sigma| \amalg_{f(j)=i} A_j$ has degree $\sum_{i=1}^{n}(|f^{-1}(i)| - 1) = m - n$, the sequence operation $\langle f \rangle$ is homogeneous of degree $m - n$.

For the following lemma, recall that we say that a surjection $(m) \to (n)$ is degenerate if it maps two adjacent entries in the source to the same entry in the target.

Lemma 3.41. *Suppose given $n \geq 0$ and a surjection $f\colon (m) \to (n)$. If f is degenerate, $\langle f \rangle$ is the zero transformation.*

Proof. If the surjection is degenerate, one of the tensor factors in the righthand side of (3.40) receives a repeated coordinate and so is zero as the chains are normalized. □

By the above lemma, for each $n \geq 0$, we have a map:

$$\mathrm{AW}_n \colon \mathcal{M}(n) \to \mathcal{Z}(n)$$

As in [BF04], we use the notation "AW_n" because the sequence operations generalize the classical Alexander-Whitney diagonal operation.

Proposition 3.42. *We have the following:*

(i) The maps $\mathrm{AW}_n \colon \mathcal{M}(n) \to \mathcal{Z}(n)$ *are injective and are chain maps.*

(ii) Together, the AW_n *yield an operad map* $\mathrm{AW} \colon \mathcal{M} \to \mathcal{Z}$.

Proof. See [McC03]. □

Thus, as claimed earlier, the McClure-Smith operad $\mathcal{M}$ embeds into the Eilenberg-Zilber operad $\mathcal{Z}$. Moreover, as we mentioned earlier, we then of course also get an embedding $\mathrm{AW}^\dagger \colon \mathcal{M}^\dagger \to \mathcal{Z}^\dagger$ of cochain operads. As we already had maps, in fact quotient maps, $\mathrm{TR} \colon \mathcal{E} \to \mathcal{Z}$ and $\mathrm{TR}^\dagger \colon \mathcal{E}^\dagger \to \mathcal{M}^\dagger$, we now have the following sequences of maps:

$$\mathcal{E} \to \mathcal{M} \to \mathcal{Z} \qquad \mathcal{E}^\dagger \to \mathcal{M}^\dagger \to \mathcal{Z}^\dagger$$

3.6 Cochains On Spaces as $\mathbb{E}_\infty$ DG Algebras and the Steenrod Algebra $\mathcal{A}$

In this section, with the help of the Eilenberg-Zilber operad as an intermediary, we shall endow cochains on spaces with an $\mathbb{E}_\infty$ algebra structutre. This will also allow us to construct the Steenrod operations.

Proposition 3.43. *Given any simplicial set* S, $\mathrm{C}^\bullet(S;\mathbb{F}_p)$ *is naturally an* $\mathcal{E}^\dagger$*-algebra.*

Proof. The chains $\mathrm{C}_\bullet(S;\mathbb{F}_p)$ admit an obvious $\mathcal{Z}$-coalgebra structure. To be precise, the map

$$\mathcal{Z}(n) \otimes \mathrm{C}_\bullet(S;\mathbb{F}_p) \to \mathrm{C}_\bullet(S;\mathbb{F}_p)^{\otimes n}$$

sends $\alpha \otimes s$ to $\alpha(s)$. Thus, by Propositions 2.13 and 2.14, the cochains $\mathrm{C}^\bullet(S;\mathbb{F}_p)$ are naturally a $\mathcal{Z}^\dagger$-algebra. The $\mathcal{E}^\dagger$-algebra structure now arises by pull back across the map $\mathcal{E}^\dagger \to \mathcal{Z}^\dagger$ constructed in the previous section. □

By the general results in earlier sections regarding $\mathcal{E}^\dagger$ algebras, we get the following corollary.

Corollary 3.44. *For any simplicial set S, $\mathrm{H}^\bullet(S;\mathbb{F}_p)$ is a graded-commutative algebra over $\mathbb{F}_p$ and, if $p=2$, possesses operations P^s, of degree s, for $s \in \mathbb{Z}$, while if $p>2$, possesses operations $\beta^\varepsilon P^s$, of degree $2s(p-1)+\varepsilon$, for $s \in \mathbb{Z}$ and $\varepsilon \in \{0,1\}$.* □

Remark 3.45. In the case $p = 2$, the operations P^s are exactly the Steenrod squares Sq^s. The construction of the Steenrod operations by this algebraic method was first carried out in [May70]. Moreover, the product yielding the graded-commutative algebra structure of the cohomology is exactly the cup product. ||

In the case of cochains on spaces, the operations P^s and βP^s satisfy certain important properties which do not hold in general. These are as follows.

Proposition 3.46. *Given a simplicial set S, the operation P^0 acts by the identity on $\mathrm{H}^\bullet(S;\mathbb{F}_p)$.*

Proof. We shall outline the proof in the case $p = 2$; the case $p > 2$ can be demonstrated by analogous, though more laboursome, arguments. By Proposition 3.13, the operation P^0 on $\mathrm{H}^\bullet(\mathrm{C}^\bullet(S))$ arises by application of the reindexing operator $(-)^\dagger$ to the operation Q^0 on $\mathrm{H}_\bullet(\mathrm{C}^\bullet(S)^\dagger) = \mathrm{H}_\bullet(\mathrm{C}_\bullet(S)^\vee)$; here $(-)^\vee$ denotes the dualization operator. Let us consider the map:

$$\mathcal{E}(2) \otimes \mathrm{C}_\bullet(S)^\vee \otimes \mathrm{C}_\bullet(S)^\vee \to \mathrm{C}_\bullet(S)^\vee \tag{3.47}$$

Let $[\alpha]$ be a cohomology class in degree d. Then, in $\mathrm{C}_\bullet(S)^\vee$, α lies in degree $-d$. We need to show that the class of the image, under the above map, of $e_{0-(-d)} \otimes \alpha \otimes \alpha = e_d \otimes \alpha \otimes \alpha$ is exactly $[\alpha]$. In fact, we shall show that the image, call it β, itself is exactly α. Note that, in $\mathrm{C}_\bullet(S)^\vee$, β lies in degree $d-d-d=-d$, and so it is a degree d cochain, and thus it acts on degree d chains. As in the proof of Proposition 3.43, we have a map dual to the one above as follows:

$$\mathcal{E}(2) \otimes \mathrm{C}_\bullet(S) \to \mathrm{C}_\bullet(S) \otimes \mathrm{C}_\bullet(S) \tag{3.48}$$

Via this map, the action of β on a degree d simplex s can be described as follows: under (3.48), we take the image of $e_d \otimes s$, and then we apply $\alpha \otimes \alpha$. Now, as the map $\mathcal{E} \to \mathcal{Z}$ factors as $\mathcal{E} \to \mathcal{M} \to \mathcal{Z}$, the map (3.48) factors as:

$$\mathcal{E}(2) \otimes \mathrm{C}_\bullet(S) \to \mathcal{M}(2) \otimes \mathrm{C}_\bullet(S) \to \mathrm{C}_\bullet(S) \otimes \mathrm{C}_\bullet(S) \tag{3.49}$$

An easy check shows that the map $\mathcal{E}(2) \to \mathcal{M}(2)$ sends e_d to the surjection $(d+2) \to (2)$ given by the sequence $(1212\cdots)$. We now need to apply the sequence co-operation, as defined in Definition 3.39, corresponding to this surjection. Let $v_0, \dots, v_d$ denote the vertices of s. As required by the sequence co-operation, we need to subdivide these vertices to form an overlapping partition, as defined in Definition 3.38, containing $d+2$ pieces. We then need to form the tensors which occur in the righthand side of (3.40) and then apply $\alpha \otimes \alpha$. Since α is non-zero only on degree d simplices, if either tensor factor is a proper face, the final result will be zero. Thus we need only consider those overlapping partitions which lead to the tensor $s \otimes s$.

We claim that there is exactly one overlapping partition which yields the tensor $s \otimes s$, namely, that given by $v_0, v_0v_1, v_1v_2, \ldots, v_{d-1}v_d, v_d$. Note that this partition has exactly $d+2$ terms, as desired. Note also that this does indeed yield the tensor $s \otimes s$ in the righthand side of (3.40). To see that it is the only possible such overlapping partition, note first that no piece can have more than 3 entries as than, according to the surjection $(1212\cdots)$, any vertex interior to this piece will be fed to only one of the tensor factors in (3.40). As such, any piece of the overlapping partition has length at most 2. As such, if we include the repetitions due to the overlaps, the total number of vertices which occur in the partition will be exactly $2(d+2)-x$, where x is the number of pieces with length 1. If we desire the tensor $s \otimes s$, the total such number must be $2(d+1)$, and so x must be 2. The first piece must be simply v_0, as otherwise the second tensor factor won't receive this vertex, and similarly, the final piece must be simply v_d, as otherwise the first tensor factor won't receive this vertex. It follows that the partition must be exactly the partition considered above. Finally then, we have

$$\beta(s) = (\alpha \otimes \alpha)(s \otimes s) = \alpha(s)^2 = \alpha(s)$$

so that $\beta = \alpha$, as desired. □

Next, we wish to show that, on cochains on spaces, the operations P^s and βP^s for negative s act by zero. A direct proof like that above is once again possible. However, we can be more brief via the following result.

Proposition 3.50. *In the algebra $\mathcal{B}$, we have that, for $s > 0$, $P^{-s}(P^0)^s = 0$ and, if $p > 2$, $\beta P^{-s}(P^0)^s$.*

Proof. This follows by the Adem relations and an induction. See [Man01]. □

Proposition 3.51. *Given a simplicial set S, for $s < 0$, the operations P^s and βP^s act by zero on* $\mathrm{H}^\bullet(S)$.

Proof. This follows immediately from Propositiona 3.46 and 3.50. □

We now recall the Steenrod algebra $\mathcal{A}$, comprising operations on the mod p cohomology of spaces, and use the above results to relate $\mathcal{B}$ to $\mathcal{A}$. We take the following as our definition of the Steenrod algebra.

Definition 3.52. If $p = 2$, where p is our fixed prime, we set

$$\mathcal{A} := \mathbf{F}\{P^s \mid s \geq 0\}/(I_{\text{Adem}}, 1 - P^0)$$

where $\mathbf{F}\{P^s \mid s \in \mathbb{Z}\}$ denotes the free graded algebra over $\mathbb{F}_2$ on formal symbols P^s, for $s \geq 0$, where P^s has degree s, and I_{Adem} denotes the two-sided ideal generated by the Adem relations. On the other hand, if $p > 2$, we set

$$\mathcal{A} := \mathbf{F}\{P^s, \beta P^s \mid s \geq 0\}/(I_{\text{Adem}}, 1 - P^0)$$

where $\mathbf{F}\{P^s, \beta P^s \mid s \geq 0\}$ denotes the free graded algebra over $\mathbb{F}_p$ on formal symbols $P^s, \beta P^s$, for $s \geq 0$, where $P^s, \beta P^s$ have degrees $2s(p-1), 2s(p-1)+1$ respectively, and I_{Adem} denotes the two-sided ideal generated by the Adem relations. In either case, we call $\mathcal{A}$ the *Steenrod algebra*.

Remark 3.53. In the above definition of $\mathcal{A}$, the Adem relations are to be understood as those in Propositions 3.14 and 3.15 except where the summation index is restricted so as to yield only operations of non-negative degree. ||

The Steenrod algebra has a basis, the Cartan-Serre basis, similar to the Cartan-Serre basis which we described earlier for $\mathcal{B}$.

Proposition 3.54. *The Steenrod algebra $\mathcal{A}$ has an $\mathbb{F}_p$-basis given by the monomials P^I where I is admissible and, if $p = 2$, $I = (i_1, \ldots, i_k)$ satisfies $i_j > 0$ for each j, and if $p > 2$, $I = (\varepsilon_1, i_1, \ldots, \varepsilon_k, i_k)$ satisfies, once again, $i_j > 0$ for each j.*

Proof. See [Mil58]. □

Next, note that we have an algebra map

$$\mathcal{B} \to \mathcal{A}$$

given by first passing from $\mathcal{B}$ to $\mathbf{F}\{P^s \mid s \geq 0\}/I_{\text{Adem}}$ by sending P^s to P^s if $s \geq 0$, and otherwise to 0. That this is well-defined follows by an easy check which says that if $a < 2b$, the non-zero summands in the Adem relation for $P^a P^b$ must all possess a negative degree operation so long as one of a and b is negative. Now, this map $mathcalB \to \mathcal{A}$ is clearly surjective and clearly kills the two-sided ideal of $\mathcal{B}$ generated by $1 - P^0$. In fact, this ideal is precisely what it annihilates, as the following result shows.

Proposition 3.55. *The map $\mathcal{B} \to \mathcal{A}$ yields an isomorphism $\mathcal{B}/(1 - P^0) \cong \mathcal{A}$.*

Proof. This follows by a comparison of the Cartan-Serre bases. See [Man01]. □

Remark 3.56. By Proposition 3.46, we have that the $\mathcal{E}^\dagger$-action on cochains on spaces yields an action on the cohomology of a space by $\mathcal{B}/(1-P^0)$, and so the proposition above yields an action by the Steenrod algebra on the cohomologies of spaces. $||$

3.7 Algebraic Models of p-Adic Homotopy Types

In the previous section, we saw that cochains on spaces are $\mathbb{E}_\infty$ dg algebras. In this section, we describe how, when endowed with this algebraic structure, the cochains provide algebraic models for p-adic homotopy types. Let Spc denote the category of spaces, by which we mean simplicial sets. So far, we have considered the cochain functor

$$\mathrm{C}^\bullet \colon \mathsf{Spc}^{\mathrm{op}} \to \mathsf{Co}_{\mathbb{F}_p}$$

where the coefficients lie in $\mathbb{F}_p$. In order to model p-adic homotopy types however, it is necessary to take coefficients in the algebraic closure $\overline{\mathbb{F}}_p$. We let

$$\overline{\mathrm{C}}^\bullet \colon \mathsf{Spc} \to \mathsf{Co}_{\overline{\mathbb{F}}_p}$$

denote the cochains functor with coefficients in $\overline{\mathbb{F}}_p$. Moreover, we let $\bar{\mathcal{E}}^\dagger$ denote the Barrat-Eccles cochain operad, though with coefficients taken in $\overline{\mathbb{F}}_p$. With coefficients in $\overline{\mathbb{F}}_p$, the relations which we saw in previous sections between the Barratt-Eccles, McClure-Smith and Eilenberg-Zilber operads continue to hold, with the exact same proofs. In particular, given a simplicial set S, $\overline{\mathrm{C}}^\bullet(S)$ is naturally an algebra over $\bar{\mathcal{E}}^\dagger$. We can thus lift our $\overline{\mathbb{F}}_p$-cochains functor above to a functor as follows:

$$\overline{\mathrm{C}}^\bullet \colon \mathsf{Spc} \to \bar{\mathcal{E}}^\dagger\text{-}\mathsf{Alg}$$

Next, note that, by exactly the same proof as in Proposition 3.4, the operad $\bar{\mathcal{E}}^\dagger$ is admissible, so that the category $\bar{\mathcal{E}}^\dagger$-$\mathsf{Alg}$ admits a Quillen model structure where the weak equivalences and fibrations are, respectively, the quasi-isomorphisms and surjections. In particular, we can then construct the homotopy category of $\bar{\mathcal{E}}^\dagger$ in the usual fashion. Equipped with this, we can now make precise the idea that the cochains, as $\mathbb{E}_\infty$ dg algebras, and with coefficients in $\overline{\mathbb{F}}_p$, yield algebraic models for p-adic homotopy types.

Proposition 3.57. *The cochains functor*

$$\overline{\mathrm{C}}^\bullet \colon \mathsf{Spc} \to \bar{\mathcal{E}}^\dagger\text{-}\mathsf{Alg}$$

admits a left derived functor from the homotopy category of spaces to the homotopy category of

$\bar{\mathcal{E}}^{\dagger}$-algebras, and, when restricted to the connected nilpotent p-complete spaces of finite p-type, this derived functor is a full embedding.

Proof. See [Man01]. The idea is to show that this holds when we restrict to the Eilenberg-MacLane spaces $\mathrm{K}(\mathbb{Z}/p^i, n)$ and $\mathrm{K}(\mathbb{Z}_p^{\wedge}, n)$ and then induct up Postnikov towers for the general case. $\square$

CHAPTER 4

Stabilizations of $\mathbb{E}_\infty$ Operads

In this chapter, we shall construct stable analogues of the Eilenberg-Zilber, McClure-Smith and Barratt-Eccles operads. Note that only the latter two constitute stabilizations of $\mathbb{E}_\infty$ operads, as the Eilenberg-Zilber operad is not $\mathbb{E}_\infty$, as already noted earlier. In order to construct actions of the latter two on spectral cochains, however, as we will do later, it is convenient to also have a stable analogoue of the Eilenberg-Zilber operad. Now, prior to constructing these stabilizations, we first discuss some basic constructions on simplicial sets, which we shall also need in later chapters. As in the previous chapter, at the outset we let p denote an unspecified but fixed prime, and, when considering the aforementioned operads, the ground field will be taken to be $\mathbb{F}_p$.

4.1 Cones, Kan Suspensions and Moore Loop Spaces

As is standard, we let Δ denote the simplex category. As we have done earlier, we let Spc denote the category of spaces, by which we mean simplicial sets. We also let Spc_* denote the category of based spaces, by which we mean based simplicial sets. For each $d \geq 0$, we let Δ_d denote the standard d-simplex. Given a based simplicial set, there exists more than one possible choice for a suspension functor. The most obvious one is perhaps $- \wedge \mathbb{S}^1$, where $\mathbb{S}^1 = \Delta_1/\partial\Delta_1$, but we will use a different one, the Kan suspension, which is weakly equivalent to $- \wedge \mathbb{S}^1$. Similarly, rather than $\mathrm{F}(\mathbb{S}^1, -)$ for loopings, we will use a different, but weakly equivalent, looping functor, the Moore loopings. We reserve the standard suspension and loops notations for the Kan suspensions and Moore loopings:

$$\Sigma\colon \mathsf{Spc}_* \rightleftarrows \mathsf{Spc}_*\colon \Omega$$

In order to define the Kan suspension, we first consider a cone functor, first in the unbased case and then the based case. Let S be a simplicial set. Then the (unreduced) cone on S, denoted $\widehat{\mathrm{C}}(S)$, is defined to be the simplicial set given by the colimit

$$\widehat{\mathrm{C}}(X) := \operatorname*{colim}_{\Delta_d \to S} \Delta_{d+1}$$

where the indexing category is the simplex category of S (the category of maps from the standard simplices to S). Using this, we can define the cones on based simplicial sets.

Definition 4.1. Let S be a based simplicial set. Then the (reduced) *cone* on S, denoted $\mathrm{C}(S)$ is defined to be the pushout

$$\begin{tikzcd}
\widehat{\mathrm{C}}(\Delta_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \widehat{\mathrm{C}}(S) \arrow[d] \\
* \arrow[r] & \mathrm{C}(S)
\end{tikzcd}$$

where the upper map is induced by the map $\Delta_0 \to S$ classifying the basepoint of S and the pushout is formed in Spc_*.

Unravelling the above definition, and in particular computing the pushout, the cones on based simplicial sets can be given the following more explicit description (see Chapter 3, Section 5 in [GJ09]). Let S be a based simplicial set as above. In degree n, we find that:

$$\mathrm{C}(S)_d = S_d \vee S_{d-1} \vee \cdots \vee S_0$$

Moreover, the action of the simplicial operators is as follows. Consider some map $\theta\colon [d] \to [e]$ in Δ. We want a function $S_e \vee S_{e-1} \vee \cdots \vee S_0 \to S_d \vee S_{d-1} \vee \cdots \vee S_0$. Let $i \in \{0, 1, \dots, e\}$. Our function will be a based one, so that we need to define, for each such i, a map $S_i \to S_d \vee S_{d-1} \vee \cdots \vee S_0$. Consider the last $i+1$ elements $[e]$. If the preimage under θ of these elements is empty, our map is just the constant one at the basepoint. Otherwise, we form the restricted map with source the preimage of the final $i+1$ elements of $[e]$ and target these final $i+1$ elements of $[e]$ and then reindex so that we have a map

$$\theta(i)\colon [j] \to [i] \tag{4.2}$$

for some $j \in \{0, 1, \dots, d\}$. The desired map is then $\theta(i)^*\colon S_i \to S_j$ followed by the inclusion into $S_d \vee S_{d-1} \vee \cdots \vee S_0$.

In the following result, in which we compute the non-degenerate simplices in cones, and in later results, given a simplicial set S, we let S_d^{nd} denote the collection of non-degenerate d-simplices of S. Note that, for any simplicial set S, $S_0^{\mathrm{nd}} = S_0$. Note also that, if S is a simplicial set, and S_+ the corresponding disjointly based simplicial set, we have:

$$\mathrm{C}(S_+)_d = S_d \amalg S_{d-1} \amalg \cdots \amalg S_0 \amalg *$$

Proposition 4.3. *Let S be a based simplicial set. We have:*

$$\mathrm{C}(S)_d^{\mathrm{nd}} = \begin{cases} S_d^{\mathrm{nd}} \amalg S_{d-1}^{\mathrm{nd}} & \text{if } d \geq 2 \\ S_1^{\mathrm{nd}} \amalg (S_0 \smallsetminus *) & \text{if } d = 1 \\ S_0 & \text{if } d = 0 \end{cases}$$

In particular, for a disjointly based S_+, where S is now an unbased simplicial set, we have:

$$\mathrm{C}(S_+)_d^{\mathrm{nd}} = \begin{cases} S_p^{\mathrm{nd}} \amalg S_{p-1}^{\mathrm{nd}} & \text{if } d \geq 1 \\ S_0 \amalg * & \text{if } d = 0 \end{cases}$$

Proof. First suppose that $d \geq 2$. Let $s \in \mathrm{C}(S)_d = S_d \vee S_{d-1} \vee \cdots \vee S_0$ and suppose that $s \in S_i$ for some $i \leq d-2$. Consider $s^0 \colon [d] \to [d-1]$, which can be pictured as:

$$\begin{matrix} 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 0 & 1 & 2 & \cdots & d-1 \end{matrix}$$

Using notation as in (4.2), consider $s^0(i)$ (that is, the map formed by restricting to the preimage of the final $i+1$ elements and then reindexing). Since $i+1 \leq d-1$, the first two elements of the above list are ignored. Thus we see that, upon the reindexing, $s^0(i)$ is just the identity on $[i]$. This shows that $s_0 \colon \mathrm{C}(S)_{d-1} \to \mathrm{C}(S)_d$, i.e., $s_0 \colon S_{d-1} \vee S_{d-2} \vee \cdots \vee S_0 \to S_d \vee S_{d-1} \vee \cdots \vee S_0$, is just the inclusion on the summands $S_{d-2}, \ldots, S_0$, and this shows that our s above is degenerate.

Next, consider any degeneracy operator $s_k \colon \mathrm{C}(S)_{d-1} \to \mathrm{C}(S)_d$, $0 \leq k \leq d-1$. We show that, in the summands S_d and S_{d-1} of $\mathrm{C}(S)_d$, only the basepoint and degenerates can occur as images. Let $s \in \mathrm{C}(S)_{d-1}$. Consider $s^k(i) \colon [j] \to [i]$. For the image $s_k(s)$ to lie in S_d or S_{d-1}, we need j to be $d-1$ or d. Suppose that it is d. Then i must have been $d-1$ and we see that $s^k(i)$ is just s^k again, so that the image $s_k(s)$ will certainly be degenerate. Now suppose that j is $d-1$. Then i must have been $d-2$ and k must have been ≥ 1, so that $s^k(i)$ will be some degeneracy operator $s^l \colon [d-1] \to [d-2]$, and so $s_k(s)$ once again will be degenerate.

Next, we need to show that every s in S_d or S_{d-1} which was degenerate as a simplex in S_d or S_{d-1} is degenerate as a simplex in $\mathrm{C}(S)_d$. This follows from the previous argument because, in the case $j = d$ above, $s^k(i)$ could have been any degeneracy operator $s^k \colon [d] \to [d-1]$, $0 \leq k \leq d-1$, and in the case $j = d-1$, $s^k(i)$ could have been any degeneracy operator $s^l \colon [d-1] \to [d-2]$, $0 \leq l \leq d-2$ (that is, any such l can be acheived with an appropriate choice for k).

This completes the proof for the case $d \geq 2$. For $d = 1$, the argument is analogous: we don't have to bother with the $i \leq d-2$ cases, and the rest is the same except that we can't have any degeneracy operator mapping into the summand S_0 since there is no S_{-1} summand in degree 0. Finally, the $d = 0$ result holds because 0-simplices are always non-degenerate.

The case of a disjointly based S_+ follows immediately from the general case upon noting that $(S_+)_0 \smallsetminus * = S_0$. □

Proposition 4.4. *For any $d \geq 0$, we have an isomorphism of based simplicial sets*

$$\Delta_{d+1} \xrightarrow{\cong} \mathbf{C}(\Delta_{d+})$$

where Δ_{d+1} is based at 0.

Proof. The map is as follows. Consider some $\theta\colon [e] \to [d+1]$ in $(\Delta_{d+1})_e$. We have that $\mathbf{C}(\Delta_{d+})_e = (\Delta_d)_e \amalg \cdots \amalg (\Delta_d)_0 \amalg *$. If θ doesn't map anything to the final $d+1$ elements of $[d+1]$, that is, if it maps everything to 0, then we send it to $*$. Otherwise, we get some new map $\theta(d)\colon [j] \to [d]$ (the notation here is as in (4.2)), for some $j \in \{0, 1 \ldots, d\}$ and θ is mapped to this element of $\mathbf{C}(\Delta_{d+})_e$. An easy check shows that this does indeed define a map, in fact an isomorphism, of based simplicial sets. □

Now we proceed to discuss Kan suspensions of based simplicial sets. First, note that, given any based simplicial set S, we have a canonical inclusion map

$$i\colon S \to \mathbf{C}(S) \tag{4.5}$$

which, in degree d, is just the inclusion $S_d \to S_d \vee S_{d-1} \vee \cdots \vee S_0$ of the S_d summand (this is a map of based simplicial sets because the simplicial operators act on the wedge sums "summand-wise").

Definition 4.6. Given a based simplicial set S, its *Kan suspension*, denoted ΣS, is defined by setting

$$\Sigma S := \mathbf{C}(S)/S$$

where the inclusion $S \to \mathbf{C}(S)$ is as above.

Thus, given a based simplicial set S and $d \geq 0$, we have:

$$(\Sigma S)_d \cong \begin{cases} S_{d-1} \vee \cdots \vee S_0 & \text{if } d \geq 1 \\ * & \text{if } d = 0 \end{cases}$$

In particular, for the case of a disjointly based S_+, where S is now an unbased simplicial set, we have:

$$(\Sigma S_+)_d \cong \begin{cases} S_{d-1} \amalg \cdots \amalg S_0 \amalg * & \text{if } d \geq 1 \\ * & \text{if } d = 0 \end{cases}$$

Remark 4.7. The Kan suspension of based simplicial sets can also be described as a left Kan extension, capturing the idea which might be expressed as "formally shift up by 1 all simplices and then freely add in the degeneracies". To make this precise, let $\mathrm{sh}\colon \Delta \to \Delta$ denote the shift functor which sends $[d]$ to $[d+1]$ and a map $\theta\colon [d] \to [e]$ to the map $[d+1] \to [e+1]$ which sends 0 to 0 and is a shifted copy of θ on the final $d+1$ entries. Let Δ^{sh} denote the subcategory of Δ which is the image of this shift functor; this is the subcategory of Δ comprising the same objects, except $[0]$, and arrows only those which map 0, and only 0, to 0. Now, given a based simplicial set S, if we formally set $(\Sigma S)_d = S_{d-1}$ for $d \geq 1$, given a map $[d] \to [e]$ in Δ^{sh}, we can act on ΣS, yielding a map $(\Sigma S)_e \to (\Sigma S)_d$, by first dropping the $0 \mapsto 0$ portion of the map, reindexing to get a map $[d-1] \to [e-1]$, and then acting as in S. That is, ΣS, as defined here, gives us a functor $\Delta^{\text{sh}} \to \mathsf{Set}$. The Kan suspension then is exactly the left Kan extension of this functor $\Delta^{\text{sh}} \to \mathsf{Set}$ along the inclusion $\Delta^{\text{sh}} \hookrightarrow \Delta$. ||

Remark 4.8. We say here a few words regarding the relation between the Kan suspension to the more usual smash suspension $- \wedge \mathbb{S}^1$, where $\mathbb{S}^1 = \Delta_1/\partial\Delta_1$. Recall that a weak equivalence of simplicial sets is a map which, under geometric realization, maps to a weak homotopy equivalence of topological spaces. The relation between the two suspensions is that, for based simplicial sets S, there is a natural weak equivalence:

$$S \wedge \mathbb{S}^1 \to \Sigma S$$

For a proof, see Proposition 2.17 in [Ste15]. ||

We now wish to compute the non-degenerates in Kan suspensions. To do so, we first compute, more generally, non-degenerates in quotients.

Proposition 4.9. *Let S be a based simplicial set and A a non-empty based sub simplicial set of S. We have:*

$$(S/A)_d^{\text{nd}} = \begin{cases} S_d^{\text{nd}} \smallsetminus A_d & \text{if } d \geq 1 \\ (S_0 \smallsetminus A_0) \amalg * & \text{if } d = 0 \end{cases}$$

Proof. The case $d = 0$ is obvious. Let then $d \geq 1$. Consider an arbitrary degeneracy operator

$$s_j\colon S_{d-1}/A_{d-1} \to S_d/A_d\colon \begin{cases} s \mapsto s_j(s) & \text{if } s \in S_{d-1} \smallsetminus A_{d-1} \text{ and } s_j(s) \in S_d \smallsetminus A_d \\ s \mapsto * & \text{if } s \in S_{d-1} \smallsetminus A_{d-1} \text{ and } s_j(s) \in A_d \\ a \mapsto * & \text{if } a \in A_{d-1} \end{cases}$$

We see that if $s \in S_d$ was degenerate in S, say $s = s_j(s')$, this will still hold in S/A (note that s' will necessarily not lie in S_{d-1}). This shows that $(S/A)_d^{\text{nd}} \subseteq S_d^{\text{nd}} \smallsetminus A_d$. Moreover, it is clear that if

$s \in S_d$ wasn't the image of any degeneracy operator s_j before, it won't be again, except possibly if $s \in A_d$. Thus $S_d^{\mathrm{nd}} \setminus A_d \subseteq (S/A)_d^{\mathrm{nd}}$. $\square$

Proposition 4.10. *Let S be a based simplicial set. We have:*

$$(\Sigma S)_d^{\mathrm{nd}} \cong \begin{cases} S_{d-1}^{\mathrm{nd}} & \text{if } d \geq 2 \\ S_0 \setminus * & \text{if } d = 1 \\ * & \text{if } d = 0 \end{cases}$$

In particular, for a disjointly based S_+, where S is now an unbased simplicial set, we have:

$$(\Sigma S_+)_d^{\mathrm{nd}} \cong \begin{cases} S_{d-1}^{\mathrm{nd}} & \text{if } d \geq 1 \\ * & \text{if } d = 0 \end{cases}$$

Proof. The first part follows from Propositions 4.3 and 4.9. For the second part, note that $(S_+)_{d-1}^{\mathrm{nd}} = S_{d-1}^{\mathrm{nd}}$ for $d \geq 2$ and $(S_+)_0 \setminus * = S_0$. $\square$

Next, we record some simple facts and a definition regarding Kan suspensions which will be needed later.

Proposition 4.11. *The Kan suspension Σ preserves monomorphisms.*

Proof. This is immediate from the fact that induced maps act "wedge-wise". $\square$

Definition 4.12. Given a based simplicial set S and a simplex $s\colon \Delta_d \to S$ of S, of dimension d, let $\Sigma s\colon \Delta_{d+1} \to \Sigma S$ denote the corresponding simplex of dimension $d+1$, given by inclusion into the first wedge summand, of ΣX. We call Σs the *suspension* of s.

Proposition 4.13. *Let S and T be based simplicial sets, $f\colon S \to T$ a based map and s a simplex of S. We have that $(\Sigma f)(\Sigma s) = \Sigma(f(s))$.*

Proof. This follows immediately from the fact that Σf acts "wedge-wise". $\square$

Proposition 4.14. *Let S be a based simplicial set and s a d-simplex of S. Then we have:*

$$d_i(\Sigma s) = \begin{cases} \Sigma(d_{i-1}s) & i = 1, \ldots, d+1 \\ * & i = 0 \end{cases}$$

Note that the Σ's here are used in the sense of Definition 4.12, not as summation symbols.

Proof. Let $s \in S_d$ and consider some $d_i\colon S_d \to S_{d-1}$, $d^i\colon [d-1] \to [d]$. Consider the map $[d] \to [d+1]$ achieved by adjoining $0 \mapsto 0$ at the beginning (that is, we send 0 to 0 and otherwise, i, for $i \geq 1$, to $d^i(i-1)+1$) and note that this is exactly d^{i+1}. By definition of the action of simplicial operators on cones and suspensions, we have that $d_{i+1}(\Sigma s) = \Sigma(d_i s)$. For the d_0 case, again, this follows from the definition of the action of the simplicial operators on cones and suspensions. $\square$

Finally, we note a fact about the chains on Kan suspensions. Recall our convention that all (co)chains are normalized, and moreover, in the case of based simplicial sets, they are of course reduced.

Proposition 4.15. *Let S be a based simplicial set. We have a natural isomorphism of chain complexes:*

$$\Phi\colon \mathbf{C}_\bullet(\Sigma S) \xrightarrow{\cong} \mathbf{C}_\bullet(S)[1]$$

The chains here may be taken to have any desired coefficients.

Proof. This follows from Propositions 4.10 and 4.14. $\square$

Finally, we discuss Moore loop spaces, which constitute the loops functor which is right adjoint to the Kan suspension defined above. For more detail on this loops functor, see, for example, Chapter 2, Section 6 of [Wu10].

Definition 4.16. Let S be a based simplicial set. The *Moore loop space* of S is defined by setting, for each $d \geq 0$:

$$(\Omega S)_d := \{s \in S_{d+1} \mid d_1 \cdots d_{d+1}(s) = *, d_0(s) = *\}$$

We of course also need actions of the simplicial operators $d_i^d\colon (\Omega S)_d \to (\Omega S)_{d-1}$ and $s_i^d\colon (\Omega S)_d \to (\Omega S)_{d+1}$, and for these, we apply d_{i+1}^{d+1} and s_{i+1}^{d+1}; one can check that the simplicial identities do indeed hold.

Remark 4.17. On the action of the simplicial operators, more generally, given a map $\theta\colon [d] \to [e]$ in Δ, to act on an element of $(\Omega X)_e$, we abut $0 \mapsto 0$ at the beginning to get a map $[d+1] \to [e+1]$ and then act. $||$

Prior to discussing the adjunction with the Kan suspension, as we did with cones and suspensions, we compute the non-degenerates in Moore loop spaces.

Proposition 4.18. *Let S be a based simplicial set. Given any $d \geq 0$, we have that:*

$$(\Omega S)_d^{\mathrm{nd}} \cong \begin{cases} S_{d+1}^{\mathrm{nd}} \cap (\Omega S)_d & \text{if } d \geq 1 \\ (S_1^{\mathrm{nd}} \cup *) \cap (\Omega S)_0 & \text{if } d = 0 \end{cases}$$

Proof. Let $s \in (\Omega S)_d \subseteq S_{d+1}$. If $d = 0$, the inclusion $(S_1^{\mathrm{nd}} \cup *) \cap (\Omega S)_0 \subseteq (\Omega S)_0^{\mathrm{nd}}$ is immediate since 0-dimensional simplices are always non-degenerate. Suppose then that $d \geq 1$, and that s is non-degenerate as a $(d+1)$-dimensional simplex of S. If s is degenerate as a d-dimensional simplex of ΩS, we have that $s = (s_i^{d-1})^{\Omega S}(s')$ for some $s' \in (\Omega S)_{d-1} \subseteq S_d$ and $0 \leq i \leq d-1$. Since $(s_i^d)^{\Omega S}(s') = (s_{i+1}^d)^S(s')$, this contradicts the fact that s is non-degenerate as a $(d+1)$-dimensional simplex of S. Thus we have that $S_{d+1}^{\mathrm{nd}} \cap (\Omega S)_d \subseteq (\Omega S)_d^{\mathrm{nd}}$, for all $d \geq 1$.

Now suppose that s is non-degenerate as a d-dimensional simplex of ΩS. First consider the case where $d = 0$. If s is degenerate as a 1-simplex of S, then we have $s = (s_0^0)^S(s')$ for some $s' \in S_0$, and so, because $s \in (\Omega S)_0$, we have that $* = (d_0^1)^S s = (d_0^1)^S (s_0^0)^S(s') = s'$, so that also $s = *$. Thus we have that $(\Omega S)_0^{\mathrm{nd}} \subseteq (S_1^{\mathrm{nd}} \cup *) \cap (\Omega S)_0$. Now consider the case where $d \geq 1$. If S is degenerate as a $(d+1)$-dimensional simplex of S, then $s = (s_i^d)^S(s')$ for some $s' \in S_d$ and $0 \leq i \leq d$. Suppose that $i = 0$. Then, because $s \in (\Omega S)_d$, we have that $* = (d_0^{d+1})^S s = (d_0^{d+1})^S (s_0^d)^S(s') = s'$, so that also $s = *$. As $d \geq 1$, this contradicts the assumption that s is non-degenerate in ΩS, giving us $(\Omega S)_d^{\mathrm{nd}} \subseteq S_{d+1}^{\mathrm{nd}} \cap (\Omega S)_d$. Now suppose that $i \geq 1$. Then we have that $s' = (d_{i+1}^{d+1})^S (s_i^d)^S(s') = (d_{i+1}^{d+1})^S(s) = (d_i^d)^{\Omega S}(s)$, so that $s' \in (\Omega S)_{d-1}$, and moreover, $s = (s_i^d)^S(s') = (s_{i-1}^{d-1})^{\Omega S}(s')$. This contradicts the assumption that s is non-degenerate in ΩS, and so we have $(\Omega S)_d^{\mathrm{nd}} \subseteq (S_{d+1}^{\mathrm{nd}} \cup *) \cap (\Omega S)_d$, as desired. $\square$

Proposition 4.19. *We have the following:*

(i) The Kan suspensions and Moore loop spaces constitute an adjunction as follows:

$$\mathsf{Spc}_* \underset{\Omega}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Spc}_*$$

(ii) For all based simplicial sets S, the unit $S \to \Omega\Sigma S$ is an isomorphism.

(iii) For all based simplicial sets S, the counit $\Sigma\Omega S \to S$ is a monomorphism.

Proof. (i): The necessary verifications are straightforward; for a written account, see Proposition 2.14 in [Ste15] – our loop functor is dual to the one used there, but an entirely analogous argument carries through.

(ii): To demonstrate this, we explicitly describe the unit of adjunction. It is given by maps $S \to \Omega\Sigma S$. We have $(\Omega\Sigma S)_d = \{s \in (\Sigma S)_{d+1} \mid d_0(s) = d_1 \cdots d_{d+1}(s) = *\} = \{s \in S_d \vee \cdots \vee S_0 \mid d_0(s) = d_1 \cdots d_{d+1}(s) = *\}$. Using the definition of the action of the simplicial operators on suspensions, we find that on each $S_d, \ldots, S_0$, the action by d_0 is the identity, so that the elements that go to $*$ under d_0 are exactly those in S_d. Moreover, the condition $d_1 \cdots d_{d+1}(s) = *$ is automatic for all simplices since $(\Sigma T)_0 = *$ for any T (one can also directly check that, for $s \in S_d$, $d_1 \cdots d_{d+1}(s) = d_0 \cdots d_d(s)$). Thus $(\Omega\Sigma S)_d = S_d$. One can check that the unit of adjunction is then just the identity on S_d and hence an isomorphism.

(iii): It suffices (by, for example, the Eilenberg-Zilber lemma expressing degenerate simplices uniquely as iterated degeneracies of non-degenerate simplices) to show that the counit preserves non-degenerate simplices and that it is injective when restricted to the non-degenerate simplices. In dimension $d = 0$, this is clear since $(\Sigma T)_0 = *$ for any T. Let $d \geq 1$. We have that:

$$(\Sigma\Omega S)_d = (\Omega S)_{d-1} \wedge \cdots \wedge (\Omega S)_0$$

By Proposition 4.10, the non-degenerate simplices are exactly the elements which lie in the first summand, $(\Omega S)_{d-1}$ (excluding the basepoint if $d = 1$). Moreover, an easy check shows that the counit, restricted to this summand, is simply the inclusion into S_d. This map is then certainly injective on the non-degenerate simplices. It remains to show that the non-degenerate simplices are preserved, and this follows by Proposition 4.18, which tells us that a non-degenerate element in $(\Omega S)_{d-1}$ is necessarily non-degenerate in S_d (except possibly in the case $d = 1$, where the element may also be the basepoint, but as just mentioned above, in the case $d = 1$, the basepoint is to be excluded). $\square$

4.2 The Stable Eilenberg-Zilber Operad

We are now ready, having covered the preliminaries in the previous section, to construct a stable analogoue of the Eilenberg-Zilber operad. In order to stabilize this operad, we first need introduce basepoints. Thus, we alter the Eilenberg-Zilber operad slightly, and consider instead the following operad, consisting of co-operations on chains on based simplicial sets.

Definition 4.20. The *reduced Eilenberg-Zilber chain operad*, denoted $\mathcal{Z}_*$, is the operad constructed in the same manner as the Eilenberg-Zilber chain operad $\mathcal{Z}$ except that the chains are to be taken on based simplicial sets (and so are of course reduced, as well as being normalized as always). Thus, for example, we have the following:

$$\mathcal{Z}_*(n)_d := \{\text{nat trans, as functors } \mathsf{Spc}_* \to \mathsf{Gr}_{\mathbb{F}_p}, \mathsf{C}_\bullet(-) \to \mathsf{C}_\bullet(-)^{\otimes n} \text{ of deg } d\}$$

The *reduced Eilenberg-Zilber cochain operad* is then defined to be $\mathcal{Z}_*^\dagger$.

Now we show that the operad $\mathcal{Z}_*$ admits a stabilization map

$$\Psi\colon \Sigma\mathcal{Z}_* \to \mathcal{Z}_*$$

where Σ here denotes the operadic suspension of chain operads, found in Definition 2.42. In order to define this map, we need to first specify maps $\mathcal{Z}_*(n)[1-n] \to \mathcal{Z}_*(n)$. Consider some natural transformation α in $\mathcal{Z}_*(n)[1-n]_d$. This is a natural transformation $\mathbf{C}_\bullet(-) \to \mathbf{C}_\bullet(-)^{\otimes n}$ of degree $d+n-1$ over based simplicial sets. By precomposition with the Kan suspension, we get a natural transformation $\mathbf{C}_\bullet(\Sigma S) \to \mathbf{C}_\bullet(\Sigma S)^{\otimes n}$ over based simplicial sets X. From Proposition 4.15, we have the natural isomorphism $\Phi\colon \mathbf{C}_\bullet(\Sigma S) \to \mathbf{C}_\bullet(S)[1]$. Thus, by pre and postcomposition, we get a natural transformation $\Phi\alpha\Phi^{-1}\colon \mathbf{C}_\bullet(S)[1] \to (\mathbf{C}_\bullet(S)[1])^{\otimes n}$. Thus, for each S we have, for any $i \in \mathbb{Z}$, a map:

$$\begin{array}{lc}
\mathbf{C}_\bullet(S)_i & \\
\quad\downarrow \mathrm{id} & \\
\mathbf{C}_\bullet(S)[1]_{i+1} \xrightarrow{\Phi\alpha\Phi^{-1}} & \bigoplus_{j_1+\cdots+j_n=i+1+d+n-1} \mathbf{C}_\bullet(S)[1]_{j_1} \otimes \cdots \otimes \mathbf{C}_\bullet(S)[1]_{j_n} \\
& \downarrow \mathrm{id} \\
& \bigoplus_{j_1+\cdots+j_n=i+d+n} \mathbf{C}_\bullet(S)_{j_1-1} \otimes \cdots \otimes \mathbf{C}_\bullet(S)_{j_n-1} \\
& \downarrow \\
& \bigoplus_{j_1+\cdots+j_n=i+d} \mathbf{C}_\bullet(S)_{j_1} \otimes \cdots \otimes \mathbf{C}_\bullet(S)_{j_n} \\
& \downarrow \mathrm{id} \\
& (\mathbf{C}_\bullet(S)^{\otimes n})_{i+d}
\end{array}$$

This gives us a natural transformation $\mathbf{C}_\bullet(-) \to \mathbf{C}_\bullet(-)^{\otimes n}$ of degree d, which is to say an element, say α', of $\mathcal{Z}_*(n)_d$. This correpondence $\alpha \mapsto \alpha'$ gives us a map $(\Sigma\mathcal{Z}_*)(n) \to \mathcal{Z}_*(n)$. Moreover, one can check that this map is a chain map, and then that, assembling over n, we get an operad map $\Psi\colon \Sigma\mathcal{Z}_* \to \mathcal{Z}_*$, as desired. Upon iterating this construction, we have maps

$$\Sigma^{k+1}\mathcal{Z}_* \to \Sigma^k\mathcal{Z}_*$$

for each $k \geq 0$. We shall be somewhat loose in our notation and denote these also by Ψ.

By Proposition 2.48, the operadic suspension Σ and the reindexing operator $(-)^\dagger$ commute, so that all of the above applies also to the reduced Eilenberg-Zilber cochain operad, in that we have a stabilization map

$$\Psi\colon \Sigma\mathcal{Z}_*^\dagger \to \mathcal{Z}_*^\dagger$$

for which we use the same symbol Ψ, and, upon iteration, more generally, a map

$$\Sigma^{k+1}\mathcal{Z}_* \to \Sigma^k \mathcal{Z}_*$$

for each $k \geq 0$. With these maps in hand, we can now define the stable Eilenberg-Zilber chain and cochain operads.

Definition 4.21. The *stable Eilenberg-Zilber chain operad*, denoted $\mathcal{Z}_{\text{st}}$, is the operad defined by as follows:

$$\mathcal{Z}_{\text{st}} := \varprojlim(\cdots \xrightarrow{\Psi} \Sigma^2\mathcal{Z}_* \xrightarrow{\Psi} \Sigma\mathcal{Z}_* \xrightarrow{\Psi} \mathcal{Z}_*)$$

Similarly, the *stable Eilenberg-Zilber cochain operad*, denoted $\mathcal{Z}_{\text{st}}^\dagger$, is the operad defined by as follows:

$$\mathcal{Z}_{\text{st}}^\dagger := \varprojlim(\cdots \xrightarrow{\Psi} \Sigma^2\mathcal{Z}_*^\dagger \xrightarrow{\Psi} \Sigma\mathcal{Z}_*^\dagger \xrightarrow{\Psi} \mathcal{Z}_*^\dagger)$$

Remark 4.22. As the reindexing operator $(-)^\dagger$ commutes with the operadic suspension, as noted above, and clearly also with the inverse limit, there is no ambiguity of notation in writing $\mathcal{Z}_{\text{st}}^\dagger$ to denote the stable Eilenberg-Zilber cochain operad, in that we can also construct it simply by applying $(-)^\dagger$ to the $\mathcal{Z}_{\text{st}}$. ||

Consider the chain complex $\mathcal{Z}_{\text{st}}(n)$ in operadic degree n of the Eilenberg-Zilber chain operad. Since limits of operads are formed termwise, $\mathcal{Z}_{\text{st}}(n)$ is equivalent to the limit, in chain complexes, of the diagram:

$$\cdots \xrightarrow{\Psi} \Sigma^2\mathcal{Z}_*(n) \xrightarrow{\Psi} \Sigma\mathcal{Z}_*(n) \xrightarrow{\Psi} \mathcal{Z}_*(n)$$

That is, it is the limit of:

$$\cdots \xrightarrow{\Psi} \mathcal{Z}_*(n)[2-2n] \xrightarrow{\Psi} \mathcal{Z}_*(n)[1-n] \xrightarrow{\Psi} \mathcal{Z}_*(n)$$

In degree $d \in \mathbb{Z}$ then, we have:

$$\mathcal{Z}_{\text{st}}(n)_d \subseteq \prod_{k\geq 0} \mathcal{Z}_*(n)[k-kn]_d = \prod_{k \geq 0} \mathcal{Z}_*(n)_{d+kn-n} \tag{4.23}$$

More specifically, an element of $\mathcal{Z}_{\text{st}}(n)$ in degree d is a sequence $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ where α_0 is a degree d chain co-operation $\mathbf{C}_\bullet(-) \to \mathbf{C}_\bullet(-)^{\otimes n}$, α_1 is a degree $d+n-1$ chain co-operation $\mathbf{C}_\bullet(-) \to \mathbf{C}_\bullet(-)^{\otimes n}$ such that $\Psi(\alpha_1) = \alpha_0$, and so on. Similarly, the cochain complex $\mathcal{Z}_{\text{st}}^\dagger(n)$ is the limit, in cochain complexes, of the diagram

$$\cdots \xrightarrow{\Psi} \mathcal{Z}_*^\dagger(n)[2n-2] \xrightarrow{\Psi} \mathcal{Z}_*^\dagger(n)[n-1] \xrightarrow{\Psi} \mathcal{Z}_*^\dagger(n)$$

and, recalling what was said in Remark 3.37, an element of $\mathcal{Z}_{\mathrm{st}}^{\dagger}(n)$ in degree d is a sequence $(\alpha_0, \alpha_1, \alpha_2, \dots)$ where α_0 is a degree d cochain operation $\mathrm{C}^{\bullet}(-)^{\otimes n} \to \mathrm{C}^{\bullet}(-)$, α_1 is a degree $d - n + 1$ cochain operation $\mathrm{C}^{\bullet}(-)^{\otimes n} \to \mathrm{C}^{\bullet}(-)$ such that $\Psi(\alpha_1) = \alpha_0$, and so on.

4.3 The Stable McClure-Smith Operad

In this section, as we did for the Eilenberg-Zilber operad, we stabilize the McClure-Smith operad, as always in both a chain form and a cochain form. First, we discuss the chain complex version. As with the Eilenberg-Zilber chain operad, we first show that $\mathcal{M}$ admits a stabilization map:

$$\Psi \colon \Sigma\mathcal{M} \to \mathcal{M}$$

Here we have once again used the symbol Ψ, just as in the stabilization of $\mathcal{Z}_*$. The context will always make it clear which map we intend by this symbol. Now, to define this map, we need to define, for each $n \geq 0$, a map $(\Sigma\mathcal{M})(n) \to \mathcal{M}(n)$, which is to say a map $\mathcal{M}(n)[1-n] \to \mathcal{M}(n)$. Consider a non-degenerate surjection $f \in \mathcal{M}(n)[1-n]_d$. This is a non-degenerate surjection $f \colon (m) \to (n)$ where $m - n = d + n - 1$ and so $m = d + 2n - 1$. We define $\Psi(f)$ algorithmically as follows:

- If $(f(1), \dots, f(n))$ is not a permutation of $(1, \dots, n)$, $\Psi(f)$ is zero.
- If $(f(1), \dots, f(n))$ is a permutation of $(1, \dots, n)$, $\Psi(f)$ is represented by the map $(n+d) \to (n)$ given by the sequence $(f(n), \dots, f(d+2n-1))$.

Proposition 4.24. *The above algorithmic procedure yields an operad map $\Psi \colon \Sigma\mathcal{M} \to \mathcal{M}$.*

Proof. See [BF04]. $\square$

As for the cochain operad $\mathcal{M}^{\dagger}$, by Proposition 2.48, the operadic suspension Σ and the reindexing operator $(-)^{\dagger}$ commute, so that we also have a stabilization map

$$\Psi \colon \Sigma\mathcal{M}^{\dagger} \to \mathcal{M}^{\dagger}$$

for which we once more use the same symbol Ψ.

Definition 4.25. The *stable McClure-Smith chain operad*, denoted $\mathcal{M}_{\mathrm{st}}$, is the operad defined as follows:

$$\mathcal{M}_{\mathrm{st}} := \varprojlim(\cdots \xrightarrow{\Psi} \Sigma^2\mathcal{M} \xrightarrow{\Psi} \Sigma\mathcal{M} \xrightarrow{\Psi} \mathcal{M})$$

The *stable McClure-Smith cochain operad*, denoted $\mathcal{M}_{\mathrm{st}}^{\dagger}$, is the operad defined as follows:

$$\mathcal{M}_{\mathrm{st}}^{\dagger} := \varprojlim(\cdots \xrightarrow{\Psi} \Sigma^2\mathcal{M}^{\dagger} \xrightarrow{\Psi} \Sigma\mathcal{M}^{\dagger} \xrightarrow{\Psi} \mathcal{M}^{\dagger})$$

Remark 4.26. As per Proposition 2.48, the reindexing operator $(-)^\dagger$ commutes with the operadic suspension. As it clearly also commutes with the inverse limit, there is no ambiguity of notation in writing $\mathcal{M}_{\mathrm{st}}^\dagger$ to denote the stable Eilenberg-Zilber cochain operad, in that we can also construct it simply by applying $(-)^\dagger$ to the $\mathcal{M}_{\mathrm{st}}$. ||

Next, we wish to compare the stabilization maps for the McClure-Smith and Eilenberg-Zilber operads. First, note that, given a surjection $f\colon (m) \to (n)$, the sequence co-operations, as defined in Definition 3.39, yield a natural transformation $\mathbf{C}_\bullet(-) \to \mathbf{C}_\bullet(-)^{\otimes n}$ not only over simplicial sets, but also over based simplicial sets. To see this, let S be a based simplicial set and let $\Delta_0 \to S$ classify the basepoint $*_S \in S_0$ of S. Then, since every piece of an overlapping partition of $[0]$ is just $\{0\}$, the only case in which each tensor factor in (3.40) will be non-degenerate is if the input surjection is the identity on (n). In this case the image is $*_S \otimes \cdots \otimes *_S$, and this tensor is zero in $\mathbf{C}_\bullet(X)^{\otimes n}$ as the chains are reduced chains. As such, an easy check shows us that we then have an operad map

$$\mathrm{AW}\colon \mathcal{M} \to \mathcal{Z}_*$$

which we denote by the same symbol, AW, as earlier. Upin applying $(-)^\dagger$, we also get a map between the corresponding cochain operads:

$$\mathrm{AW}^\dagger\colon \mathcal{M}^\dagger \to \mathcal{Z}_*^\dagger$$

Proposition 4.27. *The following squares commute:*

$$\begin{tikzcd} \Sigma\mathcal{M} \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{AW}"'] & \mathcal{M} \arrow[d, "\mathrm{AW}"] \\ \Sigma\mathcal{Z}_* \arrow[r, "\Psi"'] & \mathcal{Z}_* \end{tikzcd} \qquad \begin{tikzcd} \Sigma\mathcal{M}^\dagger \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{AW}^\dagger"'] & \mathcal{M}^\dagger \arrow[d, "\mathrm{AW}^\dagger"] \\ \Sigma\mathcal{Z}_*^\dagger \arrow[r, "\Psi"'] & \mathcal{Z}_*^\dagger \end{tikzcd}$$

Proof. We shall give a proof in the case of the chain operads; the case of the cochain operads of course then is an immediate consequence. Let $n \geq 0$, $d \in \mathbb{Z}$ and let f be a non-denerate surjection in $(\Sigma\mathcal{M})(n)_d = \mathcal{M}(n)[1-n]_d = \mathcal{M}(n)_{d+n-1}$. Then f is a non-degenerate surjection $(d+2n-1) \to (n)$. Suppose first that $(f(1), \ldots, f(n))$ is not a permutation of $(1, \ldots, n)$. Then, upon applying $\mathrm{AW} \circ \Psi$, we get zero. We then wish to show that application of $\Psi \circ \Sigma\mathrm{AW}$ also yields zero. Thus we wish to show that the transformation

$$\Psi(\langle f \rangle)\colon \mathbf{C}_\bullet(S) \to \mathbf{C}_\bullet(S)^{\otimes n}$$

ranging over arbitrary based simplicial sets S, is zero. Consider some e-dimensional simplex s of S, classified by say $\sigma\colon \Delta_e \to S$. This yields a $(e+1)$-simplex Σs of ΣS, classifed by some

$\tau\colon \Delta_{e+1} \to \Sigma S$. Then we have

$$\Psi(\langle f \rangle)(\sigma) = \sum_{\mathcal{A}} \bigotimes_{i=1}^{n} \tau | \amalg_{f(j)=i} A_j \tag{4.28}$$

where $\mathcal{A}$ ranges over the overlapping partitions of $[e+1]$ with $d+2n-1$ pieces. Consider a tensor factor $\tau| \amalg_{f(j)=i} A_j$. This simplex is classified by the composite

$$\Delta_{e'} \longrightarrow \Delta_{e+1} \xrightarrow{\tau} \Sigma S$$

where the first map is induced by some map $\theta\colon [e'] \to [e+1]$ which is itself formed by including $\amalg_{f(j)=i} A_j$ into $[e+1]$ and then reindexing the source. The actual simplex, in $(\Sigma S)_{e'}$, is given by the image of the identity on $[e']$ under this composite. Now, the first map in the composite sends $\mathrm{id}_{[e']}$ to θ. Then, recalling the definition of the action of simplicial operators on Kan suspensions, the second map sends θ to $\theta(e)^* s$ where $\theta(e)\colon [e''] \to [e]$ (the notation here is as in (4.2)) is the map constructed from $\theta\colon [e'] \to [e+1]$ by restricting the target to the final $e+1$ entries (that is, all but the first entry 0), the source to the corresponding preimage, and then reindexing both the source and target. By Proposition 4.10, this simplex $\theta(e)^*(s) \in (\Sigma S)_{e'}$, is non-degenerate exactly when the map $\theta\colon [e'] \to [e+1]$ fixes 0 and maps no other entry to 0. Thus, for the tensor factor $\tau| \amalg_{f(j)=i} A_j$ to be non-zero, it is necessary that the first entry of the first A_j in the disjoint union to be 0. In particular, for $\Psi(\langle f \rangle)(\sigma)$ to be non-zero, we need this to be true of all the tensor factors. Thus the only possibly non-zero terms in the sum in the righthand side of (4.28) are those corresponding to overlapping partitions $\mathcal{A}$ for which at least the first n pieces of $\mathcal{A}$ begin with 0. Now, by our assumption that $(f(1), \ldots, f(n))$ is not a permutation of $(1, \ldots, n)$, we have that there is a repeat in the first n entries. Thus, in a term $\bigotimes_{i=1}^{n} \tau| \amalg_{f(j)=i} A_j$, where $\mathcal{A}$ is such that the first n pieces begin with 0, there is some tensor factor for which the map $\theta\colon [e'] \to [e+1]$ maps both 0 and 1 to 0. This tensor factor then is zero and so the entire tensor is zero. Thus we see that the entire sum in (4.28) is necessarily zero, as desired.

Now suppose that $(f(1), \ldots, f(n))$ is a permutation of $(1, \ldots, n)$. Consider $(\mathrm{AW} \circ \Psi)(f)$. We have that $\Psi(f)$ is a map $(n+d) \to (n)$ and $\mathrm{AW}(\Psi(f))$ is a sequence co-operation

$$\langle \Psi(f) \rangle \colon \mathbf{C}_\bullet(S) \to \mathbf{C}_\bullet(S)^{\otimes n}$$

ranging over based simplicial sets S. Let us apply this co-operation to some e-dimensional simplex s of S, classified by say $\sigma\colon \Delta_e \to S$. To compute the result, we need to choose overlapping partitions of $[e]$ with $n+d$ pieces, and the assignment of tensor factors to these pieces is given by the sequence

$(f(n), \ldots, f(d+2n-1))$. On the other hand, consider instead the image of f under $\Psi \circ \Sigma\mathrm{AW}$. This image is a sequence co-operation

$$\Psi(\langle f \rangle)\colon \mathbf{C}_\bullet(S) \to \mathbf{C}_\bullet(S)^{\otimes n}$$

again ranging over based simplicial sets. In this case, to compute the action of this co-operation on the simplex s, we first pass to the suspended $(e+1)$-dimensional simplex Σs of ΣS, and then we need to choose overlapping partitions of $[e+1]$ with $d+2n-1$ pieces. As above, in this latter case, we need only consider those overlapping partitions in which the first n pieces begin with zero, as all others lead to a zero tensor. In such overlapping partitions, in particular, the first $n-1$ pieces must be just $\{0\}$. Moreover, we can restrict further and also require that the n^{th} piece contain both 0 and 1, as if the n^{th} piece is simply $\{0\}$, then the $(n+1)^{\text{th}}$ piece will necessarily also begin with 0 and so, since, by assumption, the sequence $(f(1), \ldots, f(n))$ already contains each of $1, \ldots, n$, a tensor factor will be degenerate and so the tensor will be zero. All told, we consider only overlapping partitions such that the first $n-1$ pieces are $\{0\}$ and the n^{th} piece contains both 0 and 1. In particular, we are left to choose only the final $(d+2n-1)-(n-1) = n+d$ pieces, and in fact, the choice of these amounts exactly to that of an overlapping partition of $[e]$ with $n+d$ pieces: to see the bijection, given such a partition of $[e]$, add 1 to each entry, and then add 0 at the beginning of the first piece. Next, given such data which encodes both kinds of overlapping partitions, in either case the assignment of tensor factors is the same as, in the case of $\langle \Psi(f) \rangle$ it is given, by the definition of Ψ, by $(f(n), \ldots, f(d+2n-1))$, and, in the case of $\Psi(\langle f \rangle)$, the assignment of tensor factors for the final $n+d$ pieces is given by the same subsequence $(f(n), \ldots, f(d+2n-1))$. Finally, the actual simplices which occur as tensor factors coincide because, in the case of $\langle \Psi(f) \rangle$, we take restrictions, or faces, of s, whereas in the case of $\Psi(\langle f \rangle)$, there is an extra coordinate, a 0 at the beginning, for each restriction, but we take restrictions of Σs, and, in forming these restrictions, as per the definition of the action of simplicial operators on Kan suspensons, we first remove the first coordinate, the extra 0, and then take the corresponding restriction of s, so that, all told, we get the same restricted simplex, as desired. $\square$

As a result of Proposition 4.27, we have commutative diagrams as follows:

$$\begin{tikzcd}
\cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{M} \arrow[r, "\Psi"] \arrow[d, "\Sigma^2\mathrm{AW}"] & \Sigma\mathcal{M} \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{AW}"] & \mathcal{M} \arrow[d, "\mathrm{AW}"] \\
\cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{Z}_* \arrow[r, "\Psi"] & \Sigma\mathcal{Z}_* \arrow[r, "\Psi"] & \mathcal{Z}_*
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{M}^\dagger \arrow[r, "\Psi"] \arrow[d, "\Sigma^2\mathrm{AW}^\dagger"] & \Sigma\mathcal{M}^\dagger \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{AW}^\dagger"] & \mathcal{M}^\dagger \arrow[d, "\mathrm{AW}^\dagger"] \\
\cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{Z}_*^\dagger \arrow[r, "\Psi"] & \Sigma\mathcal{Z}_*^\dagger \arrow[r, "\Psi"] & \mathcal{Z}_*^\dagger
\end{tikzcd}$$

From these, we get induced maps

$$\mathrm{AW}_{\mathrm{st}} \colon \mathcal{M}_{\mathrm{st}} \to \mathcal{Z}_{\mathrm{st}} \qquad \mathrm{AW}_{\mathrm{st}}^\dagger \colon \mathcal{M}_{\mathrm{st}}^\dagger \to \mathcal{Z}_{\mathrm{st}}^\dagger \tag{4.29}$$

between the stable Eilenberg-Zilber and stable McClure-Smith, chain and cochain, operads.

4.4 The Stable Barratt-Eccles Operad

We have now constructed stabilizations of both the Eilenberg-Zilber and McClure-Smith operads. In this section, we present a third and final stabilization, that of the Barratt-Eccles operad. Once more, we begin by constructing a stabilization map

$$\Psi \colon \Sigma\mathcal{E} \to \mathcal{E}$$

which will yet again be denoted by Ψ just as in the previous two stabilizations (the context will always make it clear which map we intend by this symbol). Now, in order to define this map, we need to define, for each $n \geq 0$, a map $(\Sigma\mathcal{E})(n) \to \mathcal{E}(n)$, which is to say a map $\mathcal{E}(n)[1-n] \to \mathcal{E}(n)$. Consider a tuple $(\rho_0, \ldots, \rho_{d+n-1})$ in $\mathcal{E}(n)[1-n]_d = \mathcal{E}(n)_{d+n-1}$, where each ρ_i is a permutation in Σ_n. Then we define the image $\Psi((\rho_0, \ldots, \rho_{d+n-1}))$ algorithmically as follows:

- If $(\rho_0(1), \ldots, \rho_{n-1}(1))$ is not a permutation of $(1, \ldots, n)$, $\Psi((\rho_0, \ldots, \rho_{d+n-1}))$ is zero.
- If $(\rho_0(1), \ldots, \rho_{n-1}(1))$ is a permutation of $(1, \ldots, n)$, $\Psi((\rho_0, \ldots, \rho_{d+n-1}))$ is the tuple $(\rho_{n-1}, \ldots, \rho_{d+n-1}) \in \mathcal{E}(n)_d$.

Proposition 4.30. *The above algorithmic procedure yields an operad map $\Psi \colon \Sigma\mathcal{E} \to \mathcal{E}$.*

Proof. See [BF04]. □

The above will take care of the Barratt-Eccles chain operad. As for the cochain operad $\mathcal{E}^\dagger$, just as we noted for the previous two stabilizations, by Proposition 2.48, the operadic suspension Σ and the reindexing operator $(-)^\dagger$ commute, so that we also have a stabilization map

$$\Psi \colon \Sigma\mathcal{E}^\dagger \to \mathcal{E}^\dagger$$

for which we once more use the same symbol Ψ.

Definition 4.31. The *stable Barratt-Eccles chain operad*, denoted $\mathcal{E}_{\text{st}}$, is the operad defined as follows:

$$\mathcal{E}_{\text{st}} := \varprojlim(\cdots \xrightarrow{\Psi} \Sigma^2\mathcal{E} \xrightarrow{\Psi} \Sigma\mathcal{E} \xrightarrow{\Psi} \mathcal{E})$$

The *stable Barratt-Eccles cochain operad*, denoted $\mathcal{E}_{\text{st}}^{\dagger}$, is the operad defined as follows:

$$\mathcal{E}_{\text{st}}^{\dagger} := \varprojlim(\cdots \xrightarrow{\Psi} \Sigma^2\mathcal{E}^{\dagger} \xrightarrow{\Psi} \Sigma\mathcal{E}^{\dagger} \xrightarrow{\Psi} \mathcal{E}^{\dagger})$$

Remark 4.32. As the reindexing operator $(-)^{\dagger}$ commutes with the operadic suspension, as noted above, and clearly also with the inverse limit, there is no ambiguity of notation in writing $\mathcal{E}_{\text{st}}^{\dagger}$ to denote the stable Eilenberg-Zilber cochain operad, in that we can also construct it simply by applying $(-)^{\dagger}$ to the $\mathcal{E}_{\text{st}}$. $||$

Next, we wish to compare the stabilization maps for the Barratt-Eccles and McClure-Smith operads.

Proposition 4.33. *The following squares commute:*

$$\begin{tikzcd} \Sigma\mathcal{E} \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{TR}"'] & \mathcal{E} \arrow[d, "\mathrm{TR}"] \\ \Sigma\mathcal{M} \arrow[r, "\Psi"'] & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} \Sigma\mathcal{E}^{\dagger} \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{TR}^{\dagger}"'] & \mathcal{E}^{\dagger} \arrow[d, "\mathrm{TR}^{\dagger}"] \\ \Sigma\mathcal{M}^{\dagger} \arrow[r, "\Psi"'] & \mathcal{M}^{\dagger} \end{tikzcd}$$

Proof. See [BF04] for the case of the chain operads; the case of the cochain operads then follows immediately. $\square$

Combining the above commutative squares with the ones earlier which compared the stabilization maps for the McClure-Smith and Eilenberg-Zilber operads, we have the following commutative diagrams:

$$\begin{tikzcd} \cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{E} \arrow[r, "\Psi"] \arrow[d, "\Sigma^2\mathrm{TR}"] & \Sigma\mathcal{E} \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{TR}"] & \mathcal{E} \arrow[d, "\mathrm{TR}"] \\ \cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{M} \arrow[r, "\Psi"] \arrow[d, "\Sigma^2\mathrm{AW}"] & \Sigma\mathcal{M} \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{AW}"] & \mathcal{M} \arrow[d, "\mathrm{AW}"] \\ \cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{Z}_* \arrow[r, "\Psi"] & \Sigma\mathcal{Z}_* \arrow[r, "\Psi"] & \mathcal{Z}_* \end{tikzcd}$$

$$
\begin{tikzcd}
\cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{E}^\dagger \arrow[r, "\Psi"] \arrow[d, "\Sigma^2\mathrm{TR}^\dagger"] & \Sigma\mathcal{E}^\dagger \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{TR}^\dagger"] & \mathcal{E}^\dagger \arrow[d, "\mathrm{TR}^\dagger"] \\
\cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{M}^\dagger \arrow[r, "\Psi"] \arrow[d, "\Sigma^2\mathrm{AW}^\dagger"] & \Sigma\mathcal{M}^\dagger \arrow[r, "\Psi"] \arrow[d, "\Sigma\mathrm{AW}^\dagger"] & \mathcal{M}^\dagger \arrow[d, "\mathrm{AW}^\dagger"] \\
\cdots \arrow[r, "\Psi"] & \Sigma^2\mathcal{Z}_*^\dagger \arrow[r, "\Psi"] & \Sigma\mathcal{Z}_*^\dagger \arrow[r, "\Psi"] & \mathcal{Z}_*^\dagger
\end{tikzcd}
$$

From this, our canonical maps in (4.29) now extend to the following sequences of maps:

$$\mathcal{E}_{\mathrm{st}} \xrightarrow{\mathrm{TR}_{\mathrm{st}}} \mathcal{M}_{\mathrm{st}} \xrightarrow{\mathrm{AW}_{\mathrm{st}}} \mathcal{Z}_{\mathrm{st}} \qquad \mathcal{E}_{\mathrm{st}}^\dagger \xrightarrow{\mathrm{TR}_{\mathrm{st}}^\dagger} \mathcal{M}_{\mathrm{st}}^\dagger \xrightarrow{\mathrm{AW}_{\mathrm{st}}^\dagger} \mathcal{Z}_{\mathrm{st}}^\dagger$$

CHAPTER 5

Working With the Stable Operads and Their Algebras

In this chapter, we shall begin to do some work with the stable operads which we constructed in the previous chapter. As we mentioned earlier, it is the stable Barratt-Eccles and stable McClure-Smith operads which constitute stabilizations of $\mathbb{E}_\infty$ operads. Henceforth, we will work with the stable Barratt-Eccles operad as our preferred stabilization, though all that we say holds also for the stable McClure-Smith operad. As in the previous chapter, the ground field will be $\mathbb{F}_p$, for an unspecified but fixed prime p.

5.1 The (Co)homology of the Stable Operads

To begin, we have a result regarding the stabilization maps for the Barratt-Eccles operad, which allows us below to compute the non-equivariant homology of the stable Barratt-Eccles operad, and which will be useful also for other purposes later.

Proposition 5.1. *For each $n \geq 0$, the towers*

$$\cdots \to (\Sigma^2\mathcal{E})(n) \to (\Sigma\mathcal{E})(n) \to \mathcal{E}(n) \qquad \cdots \to (\Sigma^2\mathcal{E}^\dagger)(n) \to (\Sigma\mathcal{E}^\dagger)(n) \to \mathcal{E}^\dagger(n)$$

satisfy the Mittag-Leffler condition. In fact, if $n \geq 1$, the maps in the towers are onto.

Proof. We shall give a proof of the case of the chain operad; the case of the cochain operad follows by reindexing. First, suppose that $n = 0$. In this case, the Mittag-Leffler property holds because $(\Sigma^k\mathcal{E})(0)$ is simply $\mathbb{F}_p[k]$, and so the stabilization maps are then necessarily zero maps. Now suppose that $n \geq 1$. We will prove surjectivity of the map $(\Sigma\mathcal{E})(n) \to \mathcal{E}(n)$; the surjectivity of the remaining maps is entirely analogous. Let $d \geq 0$. Then $\mathcal{E}(n)_d$ is generated by tuples $(\rho_0, \ldots, \rho_d)$ where the ρ_i are permutations in Σ_n. On the other hand, $(\Sigma\mathcal{E})(n)_d = \mathcal{E}(n)[1-n]_d = \mathcal{E}(n)_{d+n-1}$ is generated by tuples $(\rho'_0, \ldots, \rho'_{d+n-1})$ where the ρ'_i are once again permutations in Σ_n. Given a particular tuple $(\rho_0, \ldots, \rho_d)$ in $\mathcal{E}(n)_d$, we can of course find permutations $\rho'_1, \ldots, \rho'_{n-1}$ in Σ_n such that $(\rho'_1(1), \ldots, \rho'_{n-1}(1), \rho_0(1))$ is a permutation of $(1, \ldots, n)$. Then, by definition of the stabilization map, we have that $(\rho_0, \ldots, \rho_d)$ is the image of $(\rho'_1, \ldots, \rho'_{n-1}, \rho_0, \ldots, \rho_d)$, and so we have the desired surjectivity. $\square$

The above result allows us to compute the non-equivariant (co)homology of the stable Barratt-Eccles operad. The result is that, non-equivariantly, the operads are simply zero (except that the unit is present in arity 1). Later, we shall contrast this with a result which shows that the equivariant (co)homologies, on the other hand, are highly non-trivial.

Proposition 5.2. *We have the following:*

$$\mathrm{H}_\bullet \mathcal{E}_{\mathrm{st}}(n) \cong \begin{cases} 0 & n \neq 1 \\ \mathbb{F}_p[0] & n = 1 \end{cases} \qquad \mathrm{H}^\bullet \mathcal{E}^\dagger_{\mathrm{st}}(n) \cong \begin{cases} 0 & n \neq 1 \\ \mathbb{F}_p[0] & n = 1 \end{cases}$$

Proof. We shall give a proof of the case of the chain operad; the case of the cochain operad follows by reindexing. By definition, $\mathcal{E}_{\mathrm{st}}(n)$ is the limit of the following tower:

$$\cdots \to (\Sigma^2 \mathcal{E})(n) \to (\Sigma \mathcal{E})(n) \to \mathcal{E}(n)$$

By Proposition 5.1, this tower satisfies the Mittag-Leffler condition, and so, for each $d \in \mathbb{Z}$, we have an induced short exact seqeuence as follows:

$$0 \to \lim_k{}^1 \mathrm{H}_{d+1}((\Sigma^k \mathcal{E})(n)) \to \mathrm{H}_d(\mathcal{E}_{\mathrm{st}}(n)) \to \lim_k \mathrm{H}_d((\Sigma^k \mathcal{E})(n)) \to 0$$

Moreover, as $\mathcal{E}$ is $\mathbb{E}_\infty$, $\mathrm{H}_{d+1}((\Sigma^k \mathcal{E})(n))$ is simply $\mathbb{F}_p$ if $d+1 = k - kn$, and zero otherwise, the tower comprising the $\mathrm{H}_{d+1}((\Sigma^k \mathcal{E})(n))$ clearly satisfies the Mittag-Leffler condition itself, so that the induced map

$$\mathrm{H}_d(\mathcal{E}_{\mathrm{st}}(n)) \to \lim_k \mathrm{H}_d((\Sigma^k \mathcal{E})(n))$$

is in fact an isomorphism, for each $d \in \mathbb{Z}$. The result now follows immediately from the fact that $\mathcal{E}$ is $\mathbb{E}_\infty$ and $(\Sigma^k \mathcal{E})(n) = \mathcal{E}(n)[k - kn]$. $\square$

5.2 The Homotopy Theory of Algebras Over the Stable Operads

In this section, we describe how one can do homotopy theory, in the sense of Quillen (semi-)model structures, with algebras over the stable Barratt-Eccles operad. In order to do this, we wish first to demonstrate that the corresponding monad preserves weak equivalences.

Proposition 5.3. *The monads $\mathbf{E}_{\mathbf{st}}$ and $\mathbf{E}^\dagger_{\mathbf{st}}$ associated to the stable Barratt-Eccles chain and cochain operads preserve quasi-isomorphisms.*

Proof. We shall demonstrate the case of the chain operad; the case of the cochain operad is entirely analogous. First, recall that the monad $\mathbf{E}$, associated to the unstable Barratt-Eccles chain operad, preserves quasi-isomorphisms, which follows immediately from the fact that, for each $n \geq 0$, $\mathcal{E}(n)$ is $\mathbb{F}_p[\Sigma_n]$-free. For each $k \geq 0$, let $\Sigma^k\mathbf{E}$ denote the monad associated to the operadic suspension $\Sigma^k\mathcal{E}$. For exactly the same reason as for $\mathbf{E}$, each $\Sigma^k\mathbf{E}$ also preserves quasi-isomorphisms.

Next, we show that $\mathbf{E}_{\mathbf{st}}$ preserves quasi-isomorphisms between finite chain complexes. Here by "finite chain complex" we mean a complex which is bounded and finitely generated in each degree, or, equivalently, bounded and finitely presented in each degree (to see the equivalence, note that $\mathbb{F}_p[\Sigma_n]$ is Noetherian, as per [BLS81]) . Given any chain complex X, we have that:

$$\mathbf{E}_{\mathbf{st}}(X) = \bigoplus_{n\geq 0} \mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} X^{\otimes n}$$

Fix some $n \geq 0$. Because, in each degree, each $(\Sigma^k\mathcal{E})(n)$ is of finite dimension over $\mathbb{F}_p[\Sigma_n]$, and because, if X is finite over $\mathbb{F}_p$, we have that $X^{\otimes n}$ is finite over $\mathbb{F}_p[\Sigma_n]$, we have that we can commute the tensor product and inverse limit to conclude that:

$$\mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} X^{\otimes n} = (\lim_k\, (\Sigma^k\mathcal{E})(n)) \otimes_{\Sigma_n} X^{\otimes n} = \lim_k\, ((\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n})$$

Moreover, given a map $f\colon X \to Y$ between finite complexes X and Y, we can write, for each $n \geq 0$, the induced map $\mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} X^{\otimes n} \to \mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} Y^{\otimes n}$ as the map induced on inverse limits by the maps $(\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n} \to (\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} Y^{\otimes n}$. If f if a quasi-isomorphism, each of the latter maps $(\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n} \to \Sigma^k\mathcal{E}(n) \otimes_{\Sigma_n} Y^{\otimes n}$ is also a quasi-isomorphism. Thus we have the following diagram of quasi-isomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & (\Sigma^2\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n} \arrow[r] \arrow[d, "\sim"] & (\Sigma\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n} \arrow[r] \arrow[d, "\sim"] & \mathcal{E}(n) \otimes_{\Sigma_n} X^{\otimes n} \arrow[d, "\sim"] \\
\cdots \arrow[r] & (\Sigma^2\mathcal{E})(n) \otimes_{\Sigma_n} Y^{\otimes n} \arrow[r] & (\Sigma\mathcal{E})(n) \otimes_{\Sigma_n} Y^{\otimes n} \arrow[r] & \mathcal{E}(n) \otimes_{\Sigma_n} Y^{\otimes n}
\end{tikzcd}$$

and the map $\mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} X^{\otimes n} \to \mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} Y^{\otimes n}$ is the map induced on the limits of the towers by the vertical arrows in this diagram. Now, it follows easily from Proposition 5.1 that both the upper and lower towers satisfy the Mittag-Leffler condition. As such, for each $d \geq 0$, the vertical arrows induce a map of short exact sequences as follows:

$$\begin{tikzcd}
0 \arrow[r] & \lim^1_k \mathrm{H}_{d+1}((\Sigma^k \mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n}) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_d(\mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} X^{\otimes n}) \arrow[r] \arrow[d] & \lim_k \mathrm{H}_d((\Sigma^k \mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \lim^1_k \mathrm{H}_{d+1}((\Sigma^k \mathcal{E})(n) \otimes_{\Sigma_n} Y^{\otimes n}) \arrow[r] & \mathrm{H}_d(\mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} Y^{\otimes n}) \arrow[r] & \lim_k \mathrm{H}_d((\Sigma^k \mathcal{E})(n) \otimes_{\Sigma_n} Y^{\otimes n}) \arrow[r] & 0
\end{tikzcd}$$

Thus, by the five lemma, the map induced on the limits is itself a quasi-isomorphism. Moreover, taking the direct sum of these maps for $n \geq 0$, it follows that the map

$$\mathbf{E}_{\mathbf{st}}(X) = \bigoplus_{n \geq 0} \mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} X^{\otimes n} \to \bigoplus_{n \geq 0} \mathcal{E}_{\mathrm{st}}(n) \otimes_{\Sigma_n} Y^{\otimes n} = \mathbf{E}_{\mathbf{st}}(Y)$$

is a quasi-isomorphism as desired.

It remains to show that $\mathbf{E}_{\mathbf{st}}$ preserves quasi-isomorphisms between not necessarily finite chain complexes. To deduce this from the case of finite complexes, recall that any monad associated to an operad preserves filtered colimits (see [Rez96]) and also that filtered colimits of complexes are exact. Next, given any chain complex X, note that

$$X = \operatorname*{colim}_{S \subseteq_{\mathrm{fin}} X} S$$

where $S \subseteq_{\mathrm{fin}} X$ denotes the category of finite subcomplexes of X, the category of which is clearly filtered. Let $f \colon X \to Y$ be a quasi-isomorphism, where X and Y are arbitrary. Because $\mathbf{E}_{\mathbf{st}}$ preserves filtered colimits, we have a map

$$\operatorname*{colim}_{S \subseteq_{\mathrm{fin}} X} \mathbf{E}_{\mathbf{st}}(S) \to \operatorname*{colim}_{T \subseteq_{\mathrm{fin}} Y} \mathbf{E}_{\mathbf{st}}(T)$$

induced by f and we need to show that this map is a quasi-isomorphism; note that these colimits can be taken to be in complexes due to Proposition 2.17. We wish to use the fact that filtered colimits of complexes are exact, but are unable to do so at the moment because there are no induced maps between the summands in the colimits; in fact, the indexing categories for the colimits are not even the same. We remedy this as follows. It is standard that, over $\mathbb{F}_p$, as over any field, any chain complex can be written as a direct sum $(\bigoplus_{i \in I} \mathbb{S}^{n_i}) \oplus (\bigoplus_{j \in J} \mathbb{D}^{n_j})$, where $\mathbb{S}^n$ and $\mathbb{D}^n$ denote the standard sphere and disk complexes. Note that, given the complex $(\bigoplus_{i \in I} \mathbb{S}^{n_i}) \oplus (\bigoplus_{j \in J} \mathbb{D}^{n_j})$, the homology is given exactly by the spherical summands $\bigoplus_{i \in I} \mathbb{S}^{n_i}$. We now split the proof into two cases.

Case 1: Suppose that $X = (\bigoplus_{i \in I} \mathbb{S}^{n_i})$, $Y = (\bigoplus_{i \in I} \mathbb{S}^{n_i}) \oplus (\bigoplus_{j \in J} \mathbb{D}^{n_j})$ and f is the inclusion $X \to Y$, which is obviously a quasi-isomorphism. Note that every subcomplex T of Y is necessarily a sum of the summands in $(\bigoplus_{i \in I} \mathbb{S}^{n_i}) \oplus (\bigoplus_{j \in J} \mathbb{D}^{n_j})$. For each finite subcomplex T of Y, let S_T

denote the finite subcomplex of X which contains only the spherical summands which occur in T. We thus have that, for each finite subcomplex T of Y, f restricts to a map $i_T \colon S_T \to T$ and that this map is itself a quasi-isomorphism. Moreover, we clearly have that

$$X = \operatorname*{colim}_{T \subseteq_{\text{fin}} Y} S_T$$

as the change of index category simply causes some repeats in the summands. We have thus decomposed the map $f \colon X \to Y$ into the map induced on colimits by the maps i_T:

$$X = \operatorname*{colim}_{T \subseteq_{\text{fin}} Y} S_T \to \operatorname*{colim}_{T \subseteq_{\text{fin}} Y} T = Y$$

Moreover, the map $\mathbf{E}_{\mathbf{st}} X \to \mathbf{E}_{\mathbf{st}} Y$ induced by f is then decomposed as the following:

$$\mathbf{E}_{\mathbf{st}} X = \operatorname*{colim}_{T \subseteq_{\text{fin}} Y} \mathbf{E}_{\mathbf{st}}(S_T) \to \operatorname*{colim}_{T \subseteq_{\text{fin}} Y} \mathbf{E}_{\mathbf{st}}(T) = \mathbf{E}_{\mathbf{st}} Y$$

Finally, this map induced on colimits is a quasi-isomorphism by what we have shown above in the case of finite complexes and the exactness of filtered colimits.

Case 2: Now consider general X and Y and a quasi-isomorphism $f \colon X \to Y$. Let $X = (\bigoplus_{i \in I_1} \mathbb{S}^{n_i}) \oplus (\bigoplus_{j \in J_1} \mathbb{D}^{n_j})$ and let $Y = (\bigoplus_{i \in I_2} \mathbb{S}^{n_i}) \oplus (\bigoplus_{j \in J_2} \mathbb{D}^{n_j})$. Since f is a quasi-isomorphism, it follows that f must restrict to an isomorphism $\bigoplus_{i \in I_1} \mathbb{S}^{n_i} \to \bigoplus_{i \in I_2} \mathbb{S}^{n_i}$. We then get the following commutative square:

$$\begin{tikzcd}
\bigoplus_{i \in I_1} \mathbb{S}^{n_i} \arrow[r, "\cong"] \arrow[d, "\subseteq"'] & \bigoplus_{i \in I_2} \mathbb{S}^{n_i} \arrow[d, "\subseteq"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Upon applying $\mathbf{E}_{\mathbf{st}}$ to this square, having already established Case 1, we get the desired result.

$\square$

Next, in order to construct a Quillen semi-model structure for algebras over the stable Barratt-Eccles operad, our goal is to show that the operads $\mathcal{E}_{\text{st}}$ and $\mathcal{E}_{\text{st}}^{\dagger}$ are semi-admissible. To demonstrate semi-admissibillity, due to Proposition 2.41, we shall be interested in the coproducts $A \amalg \mathbf{E}_{\text{st}}(\mathbb{D}^n)$ and $A \amalg \mathbf{E}_{\text{st}}^{\dagger}(\mathbb{D}^n)$ for cell algebras A. Due to Proposition 2.31, this leads us to consider the enveloping operads $\mathcal{U}^A$ for cell algebras A. In particular, we shall wish to show that each term $\mathcal{U}^A(j)$ is sufficiently nice in that, as a functor on left $\mathbb{F}_p[\Sigma_j]$-complexes, $\mathcal{U}^A(j) \otimes_{\mathbb{F}_p[\Sigma_j]} -$ preserves quasi-isomorphisms between finite complexes. We now begin to demonstrate these facts about $\mathcal{E}_{\text{st}}$

and $\mathcal{E}_{\mathrm{st}}^{\dagger}$. First of all, we need a few lemmas. For the following lemmas, given a ring R and a dg right R-module C, let us say that C is *finitely flat* if, as a functor on finite dg left R-modules, $C \otimes_R -$ preserves quasi-isomorphisms.

Lemma 5.4. *For each $n \geq 0$, $\mathcal{E}_{st}(n)$ and $\mathcal{E}_{st}^{\dagger}(n)$ are finitely flat over $\mathbb{F}_p[\Sigma_n]$.*

Proof. The argument is exactly that which occured in the proof of Proposition 5.3, noting that, for finite complexes, we can commute tensors past the limits which appear in the construction of the stable operads, that the resulting towers satisfy the Mittag-Leffler condition due to Proposition 5.1 and then that the map induced on the limits is itself a quasi-isomorphism by the standard $\lim^1$ argument which we used in the proof of Proposition 5.3. $\square$

Lemma 5.5. *Let $m, n \geq 0$. Given a finitely flat dg right $\mathbb{F}_p[\Sigma_{m+n}]$-module M and a finite dg left $\mathbb{F}_p[\Sigma_m]$-module N, $M \otimes_{\mathbb{F}_p[\Sigma_m]} N$ is finitely flat over $\mathbb{F}_p[\Sigma_n]$.*

Proof. Given a finite dg left $\mathbb{F}_p[\Sigma_n]$-module P, we have a natural isomorphism

$$(M \otimes_{\mathbb{F}_p[\Sigma_m]} N) \otimes_{\mathbb{F}_p[\Sigma_n]} P \cong M \otimes_{\mathbb{F}_p[\Sigma_{m+n}]} (\mathbb{F}_p[\Sigma_{m+n}] \otimes_{\mathbb{F}_p[\Sigma_m] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\Sigma_n]} (N \otimes_{\mathbb{F}_p} P))$$

and from this the result follows immediately, noting that $\mathbb{F}_p[\Sigma_{m+n}]$ is flat over $\mathbb{F}_p[\Sigma_m] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\Sigma_n]$, and that $\mathbb{F}_p[\Sigma_{m+n}] \otimes_{\mathbb{F}_p[\Sigma_m] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\Sigma_n]} (N \otimes_{\mathbb{F}_p} P)$ is a finite complex over $\mathbb{F}_p[\Sigma_{m+n}]$ given the finiteness of N and P. $\square$

Lemma 5.6. *Let R be a ring and let $i\colon C \to D$ be a map of dg right R-modules which is split as a morphism of graded right R-modules. Then, if any two of C, D and D/C are finitely flat, so is the third.*

Proof. We shall consider the case of chain complexes, the case of cochain complexes differing mostly only in some notations. Given any chain complex P of left R-modules, the sequence

$$0 \to C \otimes_R P \to D \otimes_R P \to (D/C) \otimes_R P \to 0$$

is exact, as tensor is always right exact and the given retraction $r\colon D \to C$ gives an induced retraction $r \otimes_R \mathrm{id}_P\colon D \otimes_R P \to C \otimes_R P$ (at the level of graded modules). This yields a long exact sequence in homology:

$$\cdots \to \mathrm{H}_n(C \otimes_R P) \to \mathrm{H}_n(D \otimes_R P) \to \mathrm{H}_n((D/C) \otimes_R P) \to \cdots$$

Given any quasi-isomorphism $P \to Q$ between finite chain complexes of left R-modules, we get a morphism of these long exact sequences:

$$\begin{array}{ccccccccc}\cdots & \longrightarrow & \mathrm{H}_n(C \otimes_R P) & \to & \mathrm{H}_n(D \otimes_R P) & \rightarrowtail & \mathrm{H}_n((D/C) \otimes_R P) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}_n(C \otimes_R Q) & \to & \mathrm{H}_n(D \otimes_R Q) & \rightarrowtail & \mathrm{H}_n((D/C) \otimes_R Q) & \longrightarrow & \cdots \end{array}$$

The result now follows by the five lemma. □

We can now demonstrate the result regarding the finite flatness of the terms of the enveloping operads.

Proposition 5.7. *Let A be a cell $\mathcal{E}_{\text{st}}$-algebra or a cell $\mathcal{E}_{\text{st}}^{\dagger}$-algebra. Let also $\mathcal{U}^A$ denote the associated enveloping operad. Then, for all $j \geq 0$, $\mathcal{U}^A(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$.*

Proof. We shall demonstrate the result in the case of the chain operad $\mathcal{E}_{\text{st}}$, the case of the cochain operad being entirely analogous. Let

$$A_0 \to A_1 \to A_2 \to \cdots$$

be a cell filtration of A and fix some choices $M_1, M_2, \ldots$ for the chain complexes which appear in the attachment squares. For each $n \geq 0$, let $N_n = \oplus_{i \leq n} M_i$, where $N_0 = 0$, and let also $N = \oplus_{i \geq 0} M_i$. As in Section 2.4 of the second chapter, we have that, for each $j \geq 0$, as a graded right $\mathbb{F}_p[\Sigma_j]$-module:

$$\mathcal{U}^A(j) = \bigoplus_{i \geq 0} \mathcal{E}_{\text{st}}(i+j) \otimes_{\Sigma_i} (N[1])^{\otimes i}$$

The differential on $\mathcal{U}^A(j)$, we recall, is given by the Leibniz rule, the attachment maps and the operadic composition. Moreover, for each $n \geq 0$, and again for each $j \geq 0$, as a graded right $\mathbb{F}_p[\Sigma_j]$-module:

$$\mathcal{U}^{A_n}(j) = \bigoplus_{i \geq 0} \mathcal{E}_{\text{st}}(i+j) \otimes_{\Sigma_i} (N_n[1])^{\otimes i}$$

Thus we see that the operad $\mathcal{U}^A$ is filtered by the operads $\mathcal{U}^{A_n}$. Now, as A_0 is the initial $\mathcal{E}_{\text{st}}$-algebra $\mathcal{E}_{\text{st}}(0)$, we have $\mathcal{U}^{A_0} = \mathcal{U}^{\mathcal{E}_{\text{st}}(0)} = \mathcal{E}_{\text{st}}$ (see Example 2.30). The terms of the operad $\mathcal{U}^{A_1}$ then arise from the terms of $\mathcal{U}^{A_0}$ by attachment of cells; more generally, for $n \geq 1$, the terms of the operad $\mathcal{U}^{A_n}$ arise from the terms of $\mathcal{U}^{A_{n-1}}$ by attachment of cells. This allows us to define, for $n \geq 1$, a filtration on the terms of the operad $\mathcal{U}^{A_n}$ as follows. Fix such an n. For any $j \geq 0$, we let $\mathrm{F}_m \mathcal{U}^{A_n}(j)$, where $m \geq 0$, denote the sub graded module of $\mathcal{U}^{A_n}(j)$ generated by the elements $\sigma \otimes a_1 \otimes \cdots \otimes a_i$ where at most m of the factors $a_1, \ldots, a_i \in N_n[1]$ project to a non-zero element in $M_n[1]$ (which constitutes the "most recently added cells"); note that, since, when computing the differential of $\sigma \otimes a_1 \otimes \cdots \otimes a_i$ via the Leibniz rule, if $a_r \in M_n[1]$, we map it to the corresponding element of $\oplus_{i\geq 0}\mathcal{E}_{\mathrm{st}}(i) \otimes_{\Sigma_i} (N_{n-1}[1])^{\otimes i}$ via the attachment map $M_n \to A_{n-1}$, we have that the differential preserves the sub graded module $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ so that we in fact have a sub chain complex. Now, given $n \geq 1$ and $j \geq 0$, as graded right $\mathbb{F}_p[\Sigma_j]$-modules, note that we have that

$$\mathcal{U}^{A_n}(j) = \bigoplus_{i\geq 0} \mathcal{E}_{\mathrm{st}}(i+j) \otimes_{\Sigma_i} (N_n[1])^{\otimes i} = \bigoplus_{i\geq 0} \bigoplus_{l=0}^{i} \mathcal{E}_{\mathrm{st}}(i+j) \otimes_{\Sigma_{i-l}\times\Sigma_l} N_{n-1}[1]^{\otimes(i-l)} \otimes M_n[1]^{\otimes l}$$

and, for any $m \geq 0$, the submodule $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is then given by:

$$\mathrm{F}_m\,\mathcal{U}^{A_n}(j) = \bigoplus_{i\geq 0} \bigoplus_{l=0}^{\min(i,m)} \mathcal{E}_{\mathrm{st}}(i+j) \otimes_{\Sigma_{i-l}\times\Sigma_l} N_{n-1}[1]^{\otimes(i-l)} \otimes M_n[1]^{\otimes l}$$

Thus we see that, for $m \geq 1$, the inclusions $\mathrm{F}_{m-1}\,\mathcal{U}^{A_n}(j) \to \mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ are, at the level of the underlying graded modules, split monomorphisms. We also have that:

$$\begin{aligned}
\mathrm{F}_m\,\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\,\mathcal{U}^{A_n}(j) &\cong \bigoplus_{i\geq m} \mathcal{E}_{\mathrm{st}}(i+j) \otimes_{\Sigma_{i-m}\times\Sigma_m} N_{n-1}[1]^{\otimes(i-m)} \otimes M_n[1]^{\otimes m} \\
&\cong \left(\bigoplus_{i\geq m} \mathcal{E}_{\mathrm{st}}(i+j) \otimes_{\Sigma_{i-m}} N_{n-1}[1]^{\otimes(i-m)}\right) \otimes_{\Sigma_m} M_n[1]^{\otimes m} \\
&= \left(\bigoplus_{i\geq 0} \mathcal{E}_{\mathrm{st}}(i+m+j) \otimes_{\Sigma_i} N_{n-1}[1]^{\otimes i}\right) \otimes_{\Sigma_m} M_n[1]^{\otimes m} \\
&= \mathcal{U}^{A_{n-1}}(m+j) \otimes_{\Sigma_m} M_n[1]^{\otimes m}
\end{aligned}$$

Moreover, recalling that when we compute the differential of $\sigma \otimes a_1 \otimes \cdots \otimes a_i$ via the Leibniz rule, if $a_r \in M_n[1]$, we map it to the corresponding element of $\oplus_{i\geq 0}\mathcal{E}_{\mathrm{st}}(i) \otimes_{\Sigma_i} N_{n-1}[1]^{\otimes i}$ via the attachment map $M_n \to A_{n-1}$, and so in particular we map to zero in the quotient $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\,\mathcal{U}^{A_n}(j)$, we see that the isomorphism

$$\mathrm{F}_m\,\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\,\mathcal{U}^{A_n}(j) \cong \mathcal{U}^{A_{n-1}}(m+j) \otimes_{\Sigma_m} M_n[1]^{\otimes m}$$

is in fact one of chain complexes, not only of graded modules.

Now we prove the desired result by an induction. We shall show that, for each $m, j, n \geq 0$, $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$, and we will do this by inducting on n. In the case $n = 0$, as noted above, we have that $\mathcal{U}^{A_0} = \mathcal{E}_{\mathrm{st}}$, and moreover that $\mathrm{F}_m\,\mathcal{U}^{A_0}(j) = \mathcal{E}_{\mathrm{st}}(j)$ for all $m, j \geq 0$. The required flatness then follows by Lemma 5.4. Suppose now that, for some $n \geq 1$, we have that $\mathrm{F}_m\,\mathcal{U}^{A_{n-1}}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$ for all $m, j \geq 0$. We wish to show that $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$ for all $m, j \geq 0$. We shall do this by inducting over m. By definition of the filtration piece F_0, we have that, for each $j \geq 0$, $\mathrm{F}_0\,\mathcal{U}^{A_n}(j) = \mathcal{U}^{A_{n-1}}(j) = \mathrm{colim}_m\,\mathrm{F}_m\,\mathcal{U}^{A_{n-1}}(j)$ which, by invoking the inductive hypothesisis for the induction over n and passing to the colimit, we see is finitely flat over $\mathbb{F}_p[\Sigma_j]$. Next, suppose that for some $m \geq 1$, $\mathrm{F}_{m-1}\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$. As above, we have that:

$$\mathrm{F}_m\,\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\,\mathcal{U}^{A_n}(j) \cong \mathcal{U}^{A_{n-1}}(m+j) \otimes_{\Sigma_m} M_n[1]^{\otimes m}$$

Now, by invoking the inductive hypothesis for the induction over n and passing to the colimit, we see that $\mathcal{U}^{A_{n-1}}(j+m) = \mathrm{colim}_{m'}\,\mathrm{F}_{m'}\,\mathcal{U}^{A_{n-1}}(m+j)$ is finitely flat over $\mathbb{F}_p[\Sigma_{m+j}]$. Moreover, by Lemma 5.5, we have that $\mathcal{U}^{A_{n-1}}(m+j) \otimes_{\Sigma_m} M_n[1]^{\otimes m}$ is then finitely flat over $\mathbb{F}_p[\Sigma_j]$ so long as M_n is finite. In fact, this holds for arbitrary M_n as a non-finite M_n can be written as a filtered colimit of its finite subcomplexes and both the tensor product $\mathcal{U}^{A_{n-1}}(m+j) \otimes_{\Sigma_m} -$ and the tensor power $(-)^{\otimes m}$ commute with filtered colimits. Next, recalling that the inclusion $\mathrm{F}_{m-1}\,\mathcal{U}^{A_n}(j) \to \mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is split at the level of the underlying graded modules, we may now invoke the inductive hypothesis for the induction over m and apply Lemma 5.6 to conclude that $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$, as desired. This completes the induction over m so that we have that $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$ for all $m, j \geq 0$. Moreover, this conclusion then completes the induction over n so that we have that $\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$ for all $m, j, n \geq 0$. Finally then, if we fix a $j \geq 0$, upon passing to the colimit, we have that $\mathcal{U}^{A_n}(j) = \mathrm{colim}_m\mathrm{F}_m\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$, and then, passing to the colimit again, we have the desired result that $\mathcal{U}^{A}(j) = \mathrm{colim}_n\,\mathcal{U}^{A_n}(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$, which completes the proof. □

We now use the above result to achieve our original goal, which was to show that the operads $\mathcal{E}_{\mathrm{st}}$ and $\mathcal{E}_{\mathrm{st}}^{\dagger}$ are semi-admissible.

Proposition 5.8. *The Barratt-Eccles chain and cochain operads, $\mathcal{E}_{st}$ and $\mathcal{E}_{st}^{\dagger}$, are semi-admissible.*

Proof. We shall demonstrate the case of the chain operad, the case of the cochain operad being entirely analogous. By Proposition 2.41, it suffices to show that, if A is a cell $\mathcal{E}_{\mathrm{st}}$-algebra, then for each $n \in \mathbb{Z}$, the canonical map

$$A \to A \amalg \mathbf{E}_{\mathbf{st}}(\mathbb{D}^n)$$

is a quasi-isomorphism. We note that, by Proposition 2.31, as an algebra under A, we have:

$$A \amalg \mathbf{E}_{\mathbf{st}}(\mathbb{D}^n) \cong \mathbf{U}^A(\mathbb{D}^n) = \bigoplus_{j\geq 0} \mathcal{U}^A(j) \otimes_{\Sigma_j} (\mathbb{D}^n)^{\otimes j} = A \oplus \left(\bigoplus_{j \geq 1} \mathcal{U}^A(j) \otimes_{\Sigma_j} (\mathbb{D}^n)^{\otimes j} \right)$$

Now, for $j \geq 1$, $(\mathbb{D}^n)^{\otimes j}$ has zero homology and is finite. Moreover, by Proposition 5.7, $\mathcal{U}^A(j)$ is finitely flat over $\mathbb{F}_p[\Sigma_j]$, so that this zero homology is preserved by the tensor, which gives us the desired result. $\square$

Thus, by Proposition 2.41, we have the following.

Corollary 5.9. *The categories of algebras* $\mathcal{E}_{\mathrm{st}}$-$\mathsf{Alg}$ *and* $\mathcal{E}_{\mathrm{st}}^{\dagger}$-$\mathsf{Alg}$ *possess a Quillen semi-model structure where:*

- *The weak equivalences are the quasi-isomorphisms.*
- *The fibrations are the surjective maps.*
- *The cofibrations are retracts of relative cell complexes, where the cells are the maps* $\mathbf{E}_{\mathbf{st}}M \to \mathbf{E}_{\mathbf{st}}\mathrm{C}M$ *in the chain case, and the maps* $\mathbf{E}_{\mathbf{st}}^{\dagger}M \to \mathbf{E}_{\mathbf{st}}^{\dagger}\mathrm{C}M$ *in the cochain case, where* M *is a degreewise free complex with zero differentials.*

5.3 (Co)homology Operations for Algebras Over the Stable Operads I

In the case of the unstable Barratt-Eccles operad, earlier, we demonstrated that if A is an algebra over this operad, that its (co)homology inherits a product structure as well as certain operations. We now make analogous considerations with the stable Barratt-Eccles operad. We will see that the products disappear, whereas the operations, the Q^s in the case of the chain operad, and the P^s in the case of the cochain operad, remain, though they no longer satisfy the instability property which we saw earlier. In fact, in the next two sections, we will see that the Q^s and P^s, respectively, and their iterations, do not account for all the operations that exist in the (co)homology of algebras over $\mathcal{E}_{\mathrm{st}}$ and $\mathcal{E}_{\mathrm{st}}^{\dagger}$. Instead, rather than the algebras of operations $\mathcal{S}$ and $\mathcal{B}$ which we saw earlier, one gets certain completions $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$, which we define in the next section, and which contain certain infinite sums. Nevertheless, the action of the infinite sums, in any given instance, we shall see reduces to an action by elements of $\mathcal{S}$ and $\mathcal{B}$, and so we spend some time in this section making explicit just this latter case. We do this also because it is illustrative to do so in the sense of seeing exactly how it is that the products and instablity of the operations disappear. We shall restrict ourselves in this section alone to the case $p = 2$; analogous explicit considerations in the $p > 2$ case are also possible, though more cumbersome.

Now, to begin, recall that the products, and also the operations as $p = 2$ here, in the case of the unstable operad were defined with the help of the arity 2 part of the operad. As such, our first goal is to examine the arity 2 part of the stable Barratt-Eccles operad. Recall that, the complexes $\mathcal{E}(2)$

and $\mathcal{E}^\dagger(2)$ were exactly the standard $\mathbb{F}_2[\Sigma_2]$-free resolutions of $\mathbb{F}_2$, namely:

$$\mathcal{E}(2): \qquad \cdots \longleftarrow 0 \longleftarrow \underset{\deg 0}{\mathbb{F}_2[\Sigma_2]} \xleftarrow{1+\tau} \underset{\deg 1}{\mathbb{F}_2[\Sigma_2]} \xleftarrow{1+\tau} \underset{\deg 2}{\mathbb{F}_2[\Sigma_2]} \longleftarrow \cdots$$

$$\mathcal{E}^\dagger(2): \qquad \cdots \longrightarrow \underset{\deg -2}{\mathbb{F}_2[\Sigma_2]} \xrightarrow{1+\tau} \underset{\deg -1}{\mathbb{F}_2[\Sigma_2]} \xrightarrow{1+\tau} \underset{\deg 0}{\mathbb{F}_2[\Sigma_2]} \longrightarrow 0 \longrightarrow \cdots$$

Here, τ denotes the non-trivial permutation of $\{1,2\}$. Moreover, earlier, we used the notation e_d for the element $(1,\tau,1,\tau,\dots)$ in both $\mathcal{E}(2)$, $\mathcal{E}^\dagger(2)$, where in the former it had degree d while in the latter it had degree $-d$. Now let us see what we get in our stable situation.

Proposition 5.10. *The chain complex $\mathcal{E}_{\mathrm{st}}(2)$, and the cochain complex $\mathcal{E}^\dagger_{\mathrm{st}}(2)$, are as follows:*

$$\mathcal{E}_{\mathrm{st}}(2): \qquad \cdots \longleftarrow \mathbb{F}_2[\Sigma_2] \xleftarrow{1+\tau} \mathbb{F}_2[\Sigma_2] \xleftarrow{1+\tau} \mathbb{F}_2[\Sigma_2] \xleftarrow{1+\tau} \mathbb{F}_2[\Sigma_2] \longleftarrow \cdots$$

$$\mathcal{E}^\dagger_{\mathrm{st}}(2): \qquad \cdots \longrightarrow \mathbb{F}_2[\Sigma_2] \xrightarrow{1+\tau} \mathbb{F}_2[\Sigma_2] \xrightarrow{1+\tau} \mathbb{F}_2[\Sigma_2] \xrightarrow{1+\tau} \mathbb{F}_2[\Sigma_2] \longrightarrow \cdots$$

Proof. We shall prove the case of the chain operad, the case of the cochain operad being entirely analogous. We have a description of $\mathcal{E}(2)$ above. Moreover, for each $k \geq 0$, we have that $(\Sigma^k\mathcal{E})(2) = \mathcal{E}(2)[-k]$. Now, given as input a tuple $(\rho_0,\dots,\rho_d)$ of permutations $\rho_i \in \Sigma_2$, for some $d \geq 0$, by definition, the stabilization map $(\Sigma^{k+1}\mathcal{E})(2) \to (\Sigma^k\mathcal{E})(2)$ simply drops the first entry of the tuple. It follows that, if we write the complexes $(\Sigma^k\mathcal{E})(2)$ vertically, the tower $\cdots \to (\Sigma^2\mathcal{E})(2) \to (\Sigma\mathcal{E})(2) \to \mathcal{E}(2)$ looks as follows:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathbb{F}_2[\Sigma_2] \arrow[r, "\tau"] \arrow[d, "1+\tau"] & \mathbb{F}_2[\Sigma_2] \arrow[r, "\tau"] \arrow[d, "1+\tau"] & \mathbb{F}_2[\Sigma_2] \arrow[d, "1+\tau"] \\
\cdots \arrow[r] & \mathbb{F}_2[\Sigma_2] \arrow[r, "\tau"] \arrow[d, "1+\tau"] & \mathbb{F}_2[\Sigma_2] \arrow[r, "\tau"] \arrow[d, "1+\tau"] & \mathbb{F}_2[\Sigma_2] \arrow[d] \\
\cdots \arrow[r] & \mathbb{F}_2[\Sigma_2] \arrow[r, "\tau"] \arrow[d, "1+\tau"] & \mathbb{F}_2[\Sigma_2] \arrow[r] \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & \mathbb{F}_2[\Sigma_2] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & \vdots & \vdots & \vdots
\end{tikzcd}$$

The inverse limit is then clearly the desired complex. $\square$

Remark 5.11. We saw earlier, in Proposition 5.2, that the non-equivariant homology of $\mathcal{E}_{st}(2)$ is zero. On the other hand, Proposition 5.10 above and an easy calculation shows that the equivariant homology of $\mathcal{E}_{st}$, by which we mean the homology of $\mathcal{E}_{st}(2)/\Sigma_2$, consists of exactly a unique $\mathbb{F}_2$ generator in each degree:

$$\mathrm{H}_\bullet(\mathcal{E}_{st}(2)/\Sigma_2) : \qquad \cdots \qquad \mathbb{F}_2 \qquad \mathbb{F}_2 \qquad \mathbb{F}_2 \qquad \mathbb{F}_2 \qquad \cdots$$

For comparison, in the unstable case, via the description of $\mathcal{E}(2)$ in the third chapter and another easy calculation, we have that the homology of $\mathcal{E}(2)/\Sigma_2$, consists of exactly a unique $\mathbb{F}_2$ generator in each non-negative degree: non-negative degree:

$$\mathrm{H}_\bullet(\mathcal{E}(2)/\Sigma_2) : \qquad \cdots \qquad 0 \qquad \mathbb{F}_2 \qquad \mathbb{F}_2 \qquad \mathbb{F}_2 \qquad \cdots$$

We have seen before that it is precisely these generators which lead to the operations Q^s, and, moreover, we shall see below that it is precisely the existence of generators in also the negative degrees which will eliminate the instability property of the operations in the case of the stable operad. Furthermore, entirely analogous remarks hold in the case of the cochain operads. $||$

Just as we did with the unstable Barratt-Eccles operad, we set in place some standardized notations to work with the stable Barratt-Eccles operad. For each $d \in \mathbb{Z}$, when working with the stable Barratt-Eccles chain operad, we shall let e_d, a degree d element of $\mathcal{E}_{st}(2)$, be as follows:

- In the case $d = 0$, we set e_0 to be the following infinite tuple of tuples:

  $$(\cdots, (1, \tau, 1), (\tau, 1), (1))$$

  In the case $d = 1$, we set e_1 to be the following infinite tuple of tuples:

  $$(\cdots, (1, \tau, 1, \tau), (\tau, 1, \tau), (1, \tau))$$

  More generally, for any $d \geq 0$, e_d is the infinite tuple of tuples constructed as follows: begin with the tuple $(1, \tau, 1, \tau, \dots)$ containing $d + 1$ entries, and then alternately append either 1 or τ at the beginning, starting with τ.

- In the case $d = -1$, we set e_{-1} to be the following infinite tuple of tuples:

  $$(\cdots, (1, \tau, 1), (\tau, 1), (1), 0)$$

In the case $d = -2$, we set e_{-2} to be the following infinite tuple of tuples:

$$(\cdots, (1, \tau, 1), (\tau, 1), (1), 0, 0)$$

More generally, for $d < 0$, e_d is the infinite tuple of tuples constructed as follows: begin with $|d|$ zeros, and then proceed as in e_0.

Similarly, when we are working with the stable Barratt-Eccles cochain operad, we let e_d denote the very same elements in $\mathcal{E}^\dagger(2)$, but which now have degree $-d$.

Remark 5.12. Our use of the notation e_d here is in fact consistent with our use of the same notation earlier, in the following sense. By construction of the stable Barratt-Eccles operad, we have canonical maps

$$\mathcal{E}_{\mathrm{st}}(2) \to \mathcal{E}(2) \qquad \mathcal{E}_{\mathrm{st}}^\dagger(2) \to \mathcal{E}^\dagger(2)$$

and, for $d \geq 0$, these map the element of $\mathcal{E}_{\mathrm{st}}(2)$ or $\mathcal{E}_{\mathrm{st}}^\dagger(2)$ denoted by e_d to the element of $\mathcal{E}(2)$ or $\mathcal{E}^\dagger(2)$ denoted also by e_d. Moreover, in the case $d < 0$, these maps kill the element e_d. $||$

Having made explicit the arity 2 part of the stable Barratt-Eccles operad, we can now consider the disappearance of products. The precise statement regarding this will be that of the computation of the (co)homology of free algebras, to appear below. Here, however, we can make a remark regarding exactly how it is that the products disappear. Recall that $\mathcal{E}(2)$ is non-negative and that its homology is $\mathbb{F}_2[0]$. Similarly, $\mathcal{E}^\dagger(2)$ is non-positive and its cohomology is $\mathbb{F}_2[0]$. Moreover, in either case, the degree 0 (co)homology generator, denoted e_0, was precisely what led to the existence of products in the (co)homologies of the corresponding algebras. Here, however, we see that $\mathcal{E}_{\mathrm{st}}(2)$ extends also into negative degrees, and that $\mathcal{E}_{\mathrm{st}}^\dagger(2)$ extends also into positive degrees, and in particular, as we can explicitly see from Proposition 5.10, or as we already saw in Proposition 5.2, the (co)homology is zero; in fact, the degree zero element e_0 is no longer a cycle. It is this disappearance of the degree 0 (co)homology generator which leads to the disappearance of the products.

Remark 5.13. One can also consider the disappearance of products in an iterative manner, as follows. We have seen that the (co)homologies of algebras over $\mathcal{E}$ or $\mathcal{E}^\dagger$ have a product. By an entirely analogous construction, or with the help of Proposition 2.50, one can show that the (co)homologies of algebras over $\Sigma\mathcal{E}$ or $\Sigma\mathcal{E}^\dagger$ have a shifted product where the product of a degree a element with a degree b element lies in degree $a + b - 1$ in the chain case, and in degree $a + b + 1$ in the cochain case. Similarly, the (co)homologies of algebras over $\Sigma^2\mathcal{E}$ or $\Sigma^2\mathcal{E}^\dagger$ have a shifted product where the product of a degree a element with a degree b element lies in degree $a + b - 2$ in the chain case, and in degree $a + b + 2$ in the cochain case. This continues, and eventually, in the limit, in the case of the stabilizations $\mathcal{E}_{\mathrm{st}}$ and $\mathcal{E}_{\mathrm{st}}^\dagger$, the product disappears. $||$

Having considered the products, we now consider the operations, which, as we mentioned above, do continue to exist, though now will no longer satisfy the instability property which we saw earlier.

Proposition 5.14. *Given an algebra A over $\mathcal{E}_{\mathrm{st}}$, for each $s \in \mathbb{Z}$ and $[a] \in \mathrm{H}_q(A)$, by setting*

$$Q^s([a]) = [(e_{s-q})_*(a,a)]$$

we get a well-defined graded map

$$Q^s \colon \mathrm{H}_\bullet(A) \to \mathrm{H}_\bullet(A)$$

which is linear over $\mathbb{F}_2$, of degree s and natural in A. Similarly, given an algebra A over $\mathcal{E}_{\mathrm{st}}^\dagger$, for each $s \in \mathbb{Z}$ and $[a] \in \mathrm{H}^q(A)$, by setting

$$P^s([a]) = [(e_{q-s})_*(a,a)]$$

we get a well-defined graded map

$$P^s \colon \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A)$$

which is linear over $\mathbb{F}_2$, of degree s and natural in A.

As we did earlier, here we use the notation $\sigma_*(a,a)$ for the image of $\sigma \otimes a \otimes a$ under $\mathcal{E}_{\mathrm{st}}(2) \otimes A^{\otimes 2} \to A$ in the chain case, or under $\mathcal{E}_{\mathrm{st}}^\dagger(2) \otimes A^{\otimes 2} \to A$ in the cochain case. Note the similarity with Proposition 3.10; the key difference is that, here, the e_d exist also in negative degrees in the chain case and also in positive degrees in the cochain case, so that we are not forced to set the operations to be zero when $s < q$ in the chain case, or when $s > q$ in the cochain case (we could of course also have defined e_d to be zero in the unstable case).

Proof. In either case, linearity, the degree and naturality follow exactly as in the proof of the unstable version of this result in Proposition 3.10. It is only the well-definedness which we need to be careful about. We shall demonstrate it in the case of the chain operad, the case of the cochain operad being entirely analogous. This follows from the following two facts, which we shall demonstrate: (i) given a cycle a in A, $e_d \otimes a \otimes a$, for any d, is a cycle in $\mathcal{E}_{\mathrm{st}}(2) \otimes_{\Sigma_2} A^{\otimes 2}$ (ii) if a and a' are homologous cycles in A, $e_d \otimes a \otimes a$ and $e_d \otimes a' \otimes a'$ are homologous cycles in $\mathcal{E}_{\mathrm{st}}(2) \otimes_{\Sigma_2} A^{\otimes 2}$. Consider (i) first. This follows from the following identities, which hold in $\mathcal{E}_{\mathrm{st}}(2) \otimes_{\Sigma_2} A^{\otimes 2}$:

$$\begin{aligned}\partial(e_d \otimes a \otimes a) &= (e_{d-1} \cdot (1+\tau)) \otimes a \otimes a \\ &= e_{d-1} \otimes ((1+\tau) \cdot a \otimes a) \\ &= e_{d-1} \otimes a \otimes a + e_{d-1} \otimes (\tau \cdot a \otimes a) \\ &= e_{d-1} \otimes a \otimes a + e_{d-1} \otimes a \otimes a = 0\end{aligned}$$

Next, consider (ii). We need to show that $e_d \otimes a \otimes a - e_d \otimes a' \otimes a'$ is a boundary in $\mathcal{E}_{\mathrm{st}}(2) \otimes_{\Sigma_2} A^{\otimes 2}$. By assumption, we know that $a - a'$ is a boundary in A; let $a - a' = \partial b$. The desired result then follows from the following easily verifiable identity:

$$\partial(e_d \otimes a \otimes b + e_d \otimes b \otimes a' + e_{d-1} \otimes b \otimes b) = e_d \otimes a \otimes a - e_d \otimes a' \otimes a'$$

□

The above result gives us a stable analogue of Proposition 3.10, which constructed operations in the unstable case. We now consider what is an appropriate analogue of Proposition 3.14, which described certain fundamental properties of the operations in the unstable case. We may ignore (iii), (iv) in the chain case and (vii), (viii) in the cochain case, as those involve the products, which we have seen disappear in our stable situation here. Moreover, we can also ignore (ii) in the chain case and (vi) in the cochain case, as these give the instability property of the operations, which we have seen above disappeared due to the existence of the e_d now also in negative dimensions in the chain case, or also in positive dimensions in the cochain case. Thus, we are left to consider only the Adem relations. These do indeed hold, though we do not verify them here (they follow, for example, from the computation of the (co)homologies of free algebras given in the next section).

Remark 5.15. Above, in Remark 5.13, we showed how one can see the disappearance of the products in an iterative manner. We can also see the disappearance of the instability of the operations in an iterative manner, as follows. We have seen that the operations in the case of algebras over $\mathcal{E}$ and $\mathcal{E}^\dagger$ satisfy instability. By an analogous construction, or with the help of Proposition 2.50, one can show that one also has operations in the case of algebras over $\Sigma\mathcal{E}$ and $\Sigma\mathcal{E}^\dagger$ and moreover, these satisfy a shifted instability condition, which says, in the chain case, that $Q^s[a]$ is zero so long as $s < |a| - 1$, and in the cochain case, that $P^s[a]$ is zero so long as $s > |a| + 1$. Similarly, one also has operations in the case of algebras over $\Sigma^2\mathcal{E}$ or $\Sigma^2\mathcal{E}^\dagger$, and these satisfy the shifted instability condition which says that, in the chain case, $Q^s[a]$ is zero so long as $s < |a| - 2$, and in the cochain case, that $P^s[a]$ is zero so long as $s > |a| + 2$. This continues, and eventually, in the limit, in the case of the stabilizations $\mathcal{E}$ and $\mathcal{E}^\dagger$, the instability disappears. ||

Finally, note that if A is an algebra over $\mathcal{E}$ or $\mathcal{E}^\dagger$, by pull back across the canonical maps

$$\mathcal{E}_{\text{st}} \to \mathcal{E} \qquad \mathcal{E}_{\text{st}}^\dagger \to \mathcal{E}^\dagger$$

A is also an algebra over $\mathcal{E}_{\text{st}}$ or $\mathcal{E}_{\text{st}}^\dagger$, respectively. The following result compares the operations that then result.

Proposition 5.16. *If A is an algebra over $\mathcal{E}$ or $\mathcal{E}^\dagger$, the operations on its (co)homology as an algebra over $\mathcal{E}$ or $\mathcal{E}^\dagger$ coincide with the operations of the same name on its (co)homology as an algebra over $\mathcal{E}_{\text{st}}$ or $\mathcal{E}_{\text{st}}^\dagger$, respectively.*

Proof. This follows immediately from what was said in Remark 5.12. □

5.4 The Completions $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$

In the case of algebras over the unstable operads $\mathcal{E}$ and $\mathcal{E}^\dagger$, we found that their (co)homologies inherited actions by the algebras of operations $\mathcal{S}$ and $\mathcal{B}$, respectively. In the case of algebras over $\mathcal{E}_{\text{st}}$ and $\mathcal{E}_{\text{st}}^\dagger$, at least when $p = 2$, we have seen actions by the same algebras in the previous section; an important difference, as we noted, is that the actions are no longer unstable, in the sense of Definition 3.24. However, as we already mentioned in the pervious section, in the stable case, these operations do not account for all operations. Instead, one needs to allow certain infinite sums of the Q^I's or P^I's, leading to certain completions $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$ of $\mathcal{S}$ and $\mathcal{B}$, respectively. In this section, we shall give a precise construction of these completions, and then, in the next section, we shall see that, in the case of a free algebra on a complex X, in the chain case, the homology of the algebra is precisely the free $\widehat{\mathcal{S}}$-module on $\mathrm{H}_\bullet(X)$, whereas in the cochain case, the cohomology of the algebra is precisely the free $\widehat{\mathcal{B}}$-module on $\mathrm{H}^\bullet(X)$.

We shall first construct the completion $\widehat{\mathcal{S}}$. We begin with a construction of the underlying graded module of $\widehat{\mathcal{S}}$. To define this graded module, consider functions:

$$f\colon \{\text{admissible multi-indices}\} \to \mathbb{F}_p$$

We have an addition and a scalar multiplication for such functions, computed pointwise. We think of such a function as a possibly infinite sum, and so use the suggestive notation

$$\sum_{I \text{ admissible}} a_I Q^I$$

where $a_I = f(I)$. Our graded module will consist of such sums, with particular finiteness properties in relation to the length and excess of multi-indices. Specifically, the underlying graded module of

$\widehat{\mathcal{S}}$ is defined by setting that, in degree $d \in \mathbb{Z}$, the graded piece $\widehat{\mathcal{S}}_d$ is to consist of the sums

$$\sum_{I \text{ admissible}} a_I Q^I$$

with the following requirements:

- For all I, if $a_I \neq 0$, $d(I) = d$.
- The set of lengths $\#\{l(I) \mid a_I \neq 0\}$ is bounded above, or, equivalently, finite.
- For any $k \in \mathbb{Z}$, $\#\{I \mid a_I \neq 0, e(I) > k\}$ is finite.

Remark 5.17. Since, given any non-empty multi-index I of degree d, we have $e(I) = 2i_1 - d(I) = 2i_1 - d$, where $I = (i_1, \ldots, i_k)$, in the $p = 2$ case and $e(I) = 2pi_1 - 2\varepsilon_1 - d(I) = 2pi_1 - 2\varepsilon_1 - d \geq 2pi_1 - 2 - d$, where $I = (\varepsilon_1, i_1, \ldots, \varepsilon_k, i_k)$, in the $p > 2$ case, we can rephrase the third condition as saying that, given any $k \in \mathbb{Z}$, there may exist at most finitely many I with $a_I \neq 0$ which are non-empty and are such that the entry i_1 is larger than k. We may then also, imprecisely though suggestively, package the condition as "$i_1 \to -\infty$". $||$

Note that we have an obvious embedding of graded modules:

$$\mathcal{S} \hookrightarrow \widehat{\mathcal{S}}$$

An example of an element, one in degree 0, which is present in the completion $\widehat{\mathcal{S}}$ but not in $\mathcal{S}$, in the case $p = 2$, is the following infinite sum:

$$\sum_{k \geq 0} Q^{-k} Q^k$$

In fact, as the following proposition demonstrates, all elements of $\widehat{\mathcal{S}}$ which are not in $\mathcal{S}$ share the features of this example which say that the initial entries of the multi-indices tend to $-\infty$ while the final entries tend to $+\infty$.

Proposition 5.18. *Let $\sum a_I Q^I$ be an element of $\widehat{S}$. We have the following:*

(i) Given any $k \in \mathbb{Z}$, for all but finitely many I, the initial entry is less than k.

(ii) Given any $k \in \mathbb{Z}$, for all but finitely many I, the final entry is greater than k.

Here, in the case $p > 2$, where multi-indices take the form $(\varepsilon_1, i_1, \ldots, \varepsilon_r, i_r)$, where the i_j lie in $\mathbb{Z}$ while the ε_j lie in $\{0, 1\}$, the first entry is taken to be i_1, and the final entry, i_r, which is to say we disregard the ε_j for this particular purpose.

Proof. (i): Let the given element lie in degree d. When $p = 2$, the result follows by the fact that $e(I) = 2i_1 - d$ for any $I = (i_1, \dots, i_r)$ of degree d and that, in the sum, there can only be finitely many elements of excess above a given bound. When $p > 2$, the result follows in a similar fashion, using instead the identity $e(I) = 2pi_1 - 2\varepsilon_1 - d$ for any $I = (\varepsilon_1, i_1, \dots, \varepsilon_r, i_r)$ of degree d.

(ii): Let the given element lie in degree d. Of course there can be only one length one monomial which occurs in the sum. Consider then monomials of length $r \geq 2$. First consider the case where $p = 2$. Given a multi-index $I = (i_1, \dots, i_r)$, by admissibility, we have $i_1 \leq 2i_2 \leq 2^2 i_3 \leq \cdots \leq 2^{r-1} i_r$. Put another way, we have that $i_j \leq 2^{r-j} i_r$ for $j = 1, \dots, r$. Thus, we have

$$\begin{aligned} d &= i_1 + (i_2 + \cdots + i_r) \\ &\leq i_1 + (2^{r-2} i_r + 2^{r-3} i_r + \cdots + i_r) \\ &= i_1 + C i_r \end{aligned}$$

where $C = 1 + 2 + \cdots + 2^{r-2} > 0$. Thus we have $i_r \geq \frac{1}{C}(d - i_1)$ and so the result follows by part (i). Now consider the case $p > 2$. In this case, given a multi-index $(\varepsilon_1, i_1, \dots, \varepsilon_r, i_r)$, admissibility gives us that $i_j \leq pi_{j+1} - \varepsilon_{j+1}$ for each $j = 1, \dots, r-1$, and so, in particular, $i_j \leq pi_{j+1}$ for each $j = 1, \dots, r-1$. It follows that $i_j \leq p^{r-j}(i_r)$ for $j = 1, \dots, r$. Moreover, we have that $d(I) = 2(p-1)(i_1 + \cdots + i_k) - \varepsilon_1 - \cdots - \varepsilon_r \geq 2(p-1)(i_1 + \cdots + i_k) - r$. The argument now is analogous to the one above for the $p = 2$ case. $\square$

We now wish to endow our graded module $\widehat{S}$ with an algebra structure. To do so, however, we first need some technical lemmas regarding the Adem relations. Given any multi-index I, via the Cartan-Serre basis provided by Proposition 3.21, we know that Q^I can be written uniquely as a sum

$$\sum a_K Q^K$$

where each K is admissible. We shall call this the *admissible monomials expansion* of Q^I. Note that since the Adem relations either annihilate a monomial or preserve its length, any K for which a_K is non-zero must have the same length as I.

Lemma 5.19. *Let I be a non-empty multi-index. If K is a multi-index which appears in the admissible monomials expansion of Q^I, then the following hold:*

$$\text{(initial entry of } K) \leq \text{(initial entry of } I) \qquad \text{(final entry of } K) \geq \text{(final entry of } I)$$

As before, here we follow our convention that, in the case $p > 2$, where multi-indices take the form $(\varepsilon_1, i_1, \dots, \varepsilon_r, i_r)$, where the i_j lie in $\mathbb{Z}$ while the ε_j lie in $\{0, 1\}$, the first entry is taken to be

i_1, and the final entry, i_r, which is to say we disregard the ε_j for this particular purpose.

Proof. We shall give a proof of the case where $p = 2$; the case where $p > 2$ follows by a similar proof, upon appropriate modifications. Let $I = (i_1, \ldots, i_n)$. In the case $n = 1$, we already have admissibility and so the result is trivial. Consider the case where $n = 2$. Let $I = (a, b)$. If $a \geq 2b$, $Q^a Q^b$ is admissible and so the result is trivial. Suppose then that $a > 2b$. Then, by the Adem relations, we have that the admissible monomials expansion is:

$$\sum_i \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i$$

(Recall, as in Remark 3.22, that we have seen that the terms on the right-hand side are indeed admissible.) The terms on the right-hand side which appear are those with index i satisfying $a/2 \leq i \leq a-b-1$. Thus the maximum first entry, say $k_{\text{max init}}$, of the multi-indices $(a+b-i, i)$ which occurs satisfies $k_{\text{max init}} \leq a + b - a/2 = a/2 + b < a/2 + a/2 = a$, giving us the desired result. On the other hand, the minimum second entry, say $k_{\text{min final}}$, of the multi-indices $(a+b-i, i)$ which occurs satisfies $k_{\text{min final}} \geq a/2 > b$, once again giving us the desired result.

Now let us consider the case $n \geq 3$. We have that there exists a finite sequence of terms, say $T_1, \ldots, T_r, r \geq 1$, in the free algebra $\mathcal{F}$ over $\mathbb{F}_2$ on the Q^i, $i \in \mathbb{Z}$, which is such that $T_1 = Q^{i_1} \cdots Q^{i_n}$, $T_r = \sum Q^K$ is the admissible monomials expansion of $Q^{i_1} \cdots Q^{i_n}$, and, for each $j \geq 2$, T_j is constructed from T_{j-1} by taking some monomial summand Q^J and replacing a sub-monomial $Q^a Q^b$ of Q^J with the equivalent $\sum_i \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i$ provided by the Adem relations. Now, if the move which is made in transitioning from T_{j-1} to T_j is applied to a sub-monomial $Q^a Q^b$ where Q^a is not the initial entry of the corresponding monomial Q^J, there is no change made to any initial entry in any monomial summand. If, on the other hand, the move is applied to a sub-monomial $Q^a Q^b$ where Q^a is the initial entry of Q^j, by the argument in the $n = 2$ case above, the maximum of all the initial entries in the resulting multi-indices in T_j is bounded above by the original such maximum in T_{j-1}. Thus, by a simple induction, we have that the maximum of the initial entries of all the multi-indices appearing in $\sum Q^K$ is indeed bounded above by i_1. By an entirely analogous argument, considering instead the cases where Q^b is, or is not, the final entry of Q^J, we have that the minimum of the final entries of all the multi-indices appearing in $\sum Q^K$ is indeed bounded below by i_k. $\square$

Lemma 5.20. *Let I be a multi-index. If K is a multi-index which appears in the admissible monomials expansion of Q^I, then $e(K) \leq e(I)$.*

Proof. The case of an empty I is trivial, so suppose that it is non-empty. Suppose that $p = 2$. Let $I = (i_1, \ldots, i_n)$ and $K = (k_1, \ldots, k_n)$, where $n \geq 1$. We can write $e(I) = 2i_1 - d(I)$ and $e(K) = 2k_1 - d(K)$. The Adem relations preserve degree, so that $d(I) = d(K)$. The result then follows by Lemma 5.19. Now suppose that $p > 2$. Let $I = (\varepsilon_1, i_1, \ldots, \varepsilon_n, i_n)$ and $K = (\varepsilon_1', k_1, \ldots, \varepsilon_n', k_n)$, where $n \geq 1$. We can write $e(I) = 2pi_1 - 2\varepsilon_1 - d(I)$ and $e(K) = 2pk_1 - 2\varepsilon_1' - d(K)$. The Adem relations preserve degree, so that $d(I) = d(K)$. Moreover, an examination of the Adem relations in Proposition 3.15 shows that, if $\varepsilon_1 = 1$, then $\varepsilon_1' = 1$, so that $\varepsilon_1 \leq \varepsilon_1'$ and so $-2\varepsilon_1 \geq -2\varepsilon_1'$. The result now follows by Lemma 5.19. $\square$

Lemma 5.21. *Let I and J be admissible multi-indices. If K is a multi-index which appears in the admissible monomials expanasion of $Q^I Q^J$, then $e(K) \leq e(J)$.*

Proof. We shall give a proof of the case where $p = 2$; the case where $p > 2$ follows by a similar proof, upon appropriate modifications. The proof will be via three inductions.

Consider the case when I has length 1. Let $I = (a)$. We will prove this case by induction on the length of J. If J has length 0, it is empty, the monomial in question is Q^a, which is already admissible, and $e(J) = +\infty$, so that we have the desired result. Now suppose J has length 1. Let $J = (b)$. If $a \leq 2b$, the monomial in question, $Q^a Q^b$, is already admissible, and the excess is $a - b$, which is bounded above by $e(J) = b$ since $a \leq 2b$. On the other hand, if $a > 2b$, the admissible monomials expansion is given by

$$\sum_i \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i$$

where $a/2 \leq i \leq a - b - 1$. The excess of a generic term on the right-hand side is given by $a + b - 2i$ and this is bounded above by $a + b - 2(a/2) = b = e(J)$, giving us the desired result. Now suppose that we have the desired result for J of length $< n$, where $n \geq 2$. Consider $Q^a Q^{j_1} \cdots Q^{j_n} = (Q^a Q^{j_1} \cdots Q^{j_{n-1}}) Q^{j_n}$. Let $\sum Q^K$ be the admissible monomials expansion of $Q^a Q^{j_1} \cdots Q^{j_{n-1}}$. By the induction hypothesis, for each K, we have $e(K) \leq e(J) + j_n$. We now have $Q^a Q^{j_1} \cdots Q^{j_n} = \sum Q^K Q^{j_n}$. By Proposition 5.20, for a given K, any multi-index which appears in the admissible monomials expansion of the term $Q^K Q^{j_n}$ has excess bounded above by $e(K, j_n) = e(K) - j_n \leq e(J) + j_n - j_n = e(J)$, giving us the desired result. We have thus established, by induction on the length of J, the case in which I has length 1.

Now consider the case where J has length 1. Let $J = (b)$. We will prove this case by induction on the length of I. If I has length zero, it is empty, the monomial in question is Q^b, which is already admissible and so the desired result is trivial. Suppose that I has length 1. Let $I = (a)$. The monomial in question is then $Q^a Q^b$ and the desired result follows by exactly the same argument as the one above which was already made for this monomial. Now suppose that we have the desired result for I of length $< n$, where $n \geq 2$. Consider $Q^{i_1} \cdots Q^{i_n} Q^b = Q^{i_1}(Q^{i_1} \cdots Q^{j_{n-1}} Q^b)$. Let

$\sum Q^K$ be the admissible monomials expansion of $Q^{i_1} \cdots Q^{j_{n-1}} Q^b$. By the induction hypothesis, for each K, we have $e(K) \leq b$. We now have $Q^{i_1} \cdots Q^{i_n} Q^b = \sum Q^{i_1} Q^K$. By the result established by the previous induction, that of the case in which I has length 1, we have that, for a given K, any multi-index which appears in the admissible monomials expansion of the term $Q^{i_1} Q^K$ has excess bounded above by $e(K) \leq b$, giving us the desired result. We have thus established, by induction on the length of I, the case in which J has length 1.

We will now prove the general statement in the proposition by induction on the length of J. First suppose that J has length 0. Then the monomial in question is Q^I, the admissible monomials expansion is also simply Q^I (as I is assumed to be admissible), and we have $e(I) \leq e(J)$ for any I since $e(J) = +\infty$. If J has length 1, we have the desired result by the second of the two previous inductions. Now suppose that we have the desired result for J of length $< n$, where $n \geq 2$. Consider $Q^I Q^{j_1} \cdots Q^{j_n} = (Q^I Q^{j_1} \cdots Q^{j_{n-1}}) Q^{j_n}$. Let $\sum Q^K$ be the admissible monomials expansion of $Q^I Q^{j_1} \cdots Q^{j_{n-1}}$. By the induction hypothesis, for each K, we have $e(K) \leq e(J) + j_n$. We now have $Q^I Q^{j_1} \cdots Q^{j_n} = \sum Q^K Q^{j_n}$. By Lemma 5.20, for a given K, any multi-index which appears in the admissible monomials expansion of the term $Q^K Q^{j_n}$ has excess bounded above by $e(K, j_n) = e(K) - j_n \leq e(J) + j_n - j_n = e(J)$, giving us the desired result. We have thus established, by induction on the length of J, the completely general case. $\square$

We are now ready to equip our graded module $\widehat{\mathcal{S}}$ with an algebra structure. For each $d_1, d_2 \in \mathbb{Z}$, we must construct maps:

$$\widehat{\mathcal{S}}_{d_1} \otimes \widehat{\mathcal{S}}_{d_2} \to \widehat{\mathcal{S}}_{d_1+d_2}$$

Consider two infinite sums, the product of which

$$\left(\sum_I a_I Q^I \right) \cdot \left(\sum_I b_I Q^I \right)$$

we wish to construct, where we suppose that the only a_I and b_I which are non-zero are those for which the degree is d_1, d_2 respectively. Given any two admissible I and J, let

$$Q^I Q^J = \sum_{K \text{ admissible}} c_K^{I,J} Q^K$$

be the admissible monomials expansion of $Q^I Q^J$; note that only finitely many of the $c_K^{I,J}$ may be non-zero. We then set:

$$\left(\sum_I a_I Q^I \right) \cdot \left(\sum_I b_I Q^I \right) := \sum_K \left(\sum_{I,J} a_I b_J c_K^{I,J} \right) Q^K \tag{5.22}$$

Proposition 5.23. *The product on $\widehat{\mathcal{S}}$ as above is well-defined and equips $\widehat{\mathcal{S}}$ with an algebra structure over $\mathbb{F}_p$.*

Proof. We first show that the righthand side of (5.22) is well-defined as an infinite sum. To do this, we need to ensure that the sum $\sum_{I,J} a_I b_J c_K^{I,J}$ is finite for any given K. Fix such a K, say K_0. Let $d = d(K_0)$ and $e = e(K_0)$. By definition of $\widehat{\mathcal{S}}$ as a graded module, we know that, for all but finitely many I, we have that $a_I = 0$ or $e(I) < e(K_0) + d_2$. Note that given such an I, it is necessarily non-empty. Now, for such an I, where $e(I) < e(K) + d_2$, we have that, for any J for which $b_J \neq 0$, $e(IJ) = e(I) - d_2 < e(K) + d_2 - d_2 = e(K)$, where IJ denotes the concatenation of I and J. Thus, by Lemma 5.20, there are only finitely many I for which $a_I \neq 0$ and $c_{K_0}^{I,J} \neq 0$, where the latter amounts to saying that Q^{K_0} appears in the admissible monomials expansion of $Q^I Q^J$. Fix such an I, say I_0. We know that, for all but finitely many J, we have that $b_J = 0$ or $e(J) < e(K)$. Thus, by Lemma 5.21, there are only finitely many J for which $b_J \neq 0$ and $c_{K_0}^{I_0,J} \neq 0$, where the latter amounts to saying that Q^{K_0} appears in the admissible monomials expansion of $Q^{I_0} Q^J$. All told, we have demonstrated that, for any given K, there are only finitely many terms in both the infinite sums

$$\sum_I a_I Q^I \qquad \text{and} \qquad \sum_I b_I Q^I$$

which make a non-zero contribution to the coefficient

$$\sum_{I,J} a_I b_J c_K^{I,J}$$

of Q^K. Thus the product is indeed well-defined, at least as infinite sum.

Next, it is moreover an element of $\widehat{\mathcal{S}}_{d_1+d_2}$ for the following reasons: (i) the degree condition is satisfied because the Adem relations preserve degree (ii) the length condition is satisfied because the Adem relations either annihilate an element or preserve its length (iii) the excess condition is satisfied by the same argument as above, which showed not only well-definedness as an infinite sum, but more strongly that all but finitely many pairings of the Q^I and Q^J yield monomials with an associated excess below any given bound.

Finally, because any given coefficient arises from a product of finite sums, it is clear that the requisite associativity, identity and bilinearity follow from the fact that the definition yields the product of $\mathcal{S}$ when restricted to finite sums and that these properties do indeed hold for the product of $\mathcal{S}$. $\square$

We have now constructed $\widehat{\mathcal{S}}$ as a graded algebra over $\mathbb{F}_p$. Moreover, the embedding

$$\mathcal{S} \hookrightarrow \widehat{\mathcal{S}}$$

is now clearly one of algebras. It remains to make precise in what sense $\widehat{\mathcal{S}}$ is a completion of $\mathcal{S}$. For $k \geq 0$, recall the quotients $\mathcal{S}_{>k}$ of $\mathcal{S}$ defined as per Definition 3.26 and Remark 3.27. Each of these algebras is filtered by length where $\mathrm{F}_t\mathcal{S}_{>k}$ consists of those monomials Q^I satisfying the length bound $p^{l(I)} \leq t$. Note that, for any $k \leq l$, we have a canonical map

$$\mathcal{S}_{>k} \to \mathcal{S}_{>l}$$

and that this map is a filtered map, in that, for each $t \geq 0$, we have induced maps:

$$\mathrm{F}_t\mathcal{S}_{>k} \to \mathrm{F}_t\mathcal{S}_{>l}$$

Next, note that we can also filter $\widehat{\mathcal{S}}$ similarly by length by setting, for $t \geq 0$, $\mathrm{F}_t\widehat{\mathcal{S}}$ to comprise the sums $\sum_{I \text{ admissible}} a_I Q^I$ which satisfy the degree, length and excess requirements in the definition of $\widehat{\mathcal{S}}$ and which also satisfy, more specifically regarding length, the bound $p^{l(I)} \leq t$ for any I where $a_I \neq 0$. For each $k \geq 0$, we have a map

$$\widehat{\mathcal{S}} \to \mathcal{S}_{>-k}$$

which projects an infinite sum to the sub-sum of the elements of excess $> -k$, of which there are finitely many by definition of $\widehat{\mathcal{S}}$. Moreover, this map is a filtered map in that we have, for each $t \geq 0$, an induced map:

$$\mathrm{F}_t\widehat{\mathcal{S}} \to \mathrm{F}_t\mathcal{S}_{>-k}$$

These maps are compatible with the maps $\mathrm{F}_t\mathcal{S}_{>k} \to \mathrm{F}_t\mathcal{S}_{>l}$ above in that they yield a left cone on the tower

$$\cdots \to \mathrm{F}_t\mathcal{S}_{>-2} \to \mathrm{F}_t\mathcal{S}_{>-1} \to \mathrm{F}_t\mathcal{S}_{>0}$$

and so yield a map $\mathrm{F}_t\widehat{\mathcal{S}} \to \lim_{k\geq 0} \mathrm{F}_t\mathcal{S}_{>-k}$. The precise statement then regarding in what sense $\widehat{\mathcal{S}}$ is a completion of $\mathcal{S}$ is the following.

Proposition 5.24. *The map $\mathrm{F}_t\widehat{\mathcal{S}} \to \lim_{k\geq 0} \mathrm{F}_t\mathcal{S}_{>-k}$ constructed above is an isomorphism of graded modules. Moreover, as graded modules, we have that:*

$$\widehat{\mathcal{S}} \cong \operatorname*{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_t\mathcal{S}_{>-k}$$

Proof. By Proposition 3.28, $\mathcal{S}_{>-k}$ has a basis given by the admissible monomials of excess strictly larger than $-k$. Moreover, each $\mathrm{F}_t\mathcal{S}_{>-k}$ then has a basis given by the admissible monomials Q^I satisfying both $e(I) > -k$ and the length bound $p^{l(I)} \leq t$. Moreover, the map $\mathrm{F}_t\mathcal{S}_{>-k-1} \to \mathrm{F}_t\mathcal{S}_{>-k}$ simply kills those basis elements with excess exactly $-k-1$. The first part of the result now follows by an easy verification of the necessary universal property. The second part then follows by noting that, under this established isomorphism, given $t \leq t'$, the map induced on the limits by the natural maps $\mathrm{F}_t\mathcal{S}_{>-k} \to \mathrm{F}_{t'}\mathcal{S}_{>-k}$ corresponds exactly to the inclusion $\mathrm{F}_t\widehat{\mathcal{S}} \to \mathrm{F}_{t'}\widehat{\mathcal{S}}$ $\square$

We have now completed all of our desired goals regarding the algebra $\widehat{\mathcal{S}}$. We now consider the algebra $\widehat{\mathcal{B}}$. Having done the work to construct $\widehat{\mathcal{S}}$, the isomorphism $\mathcal{B} \cong \mathcal{S}^\dagger$ in Proposition 3.19 will allow for an expedited construction of $\widehat{\mathcal{B}}$. As in the case of $\widehat{\mathcal{S}}$, we first construct the underlying graded module of $\widehat{\mathcal{B}}$. To do this, we once again consider functions

$$f\colon \{\text{admissible multi-indices}\} \to \mathbb{F}_p$$

where now we recall that the notion of admissibility has a different meaning. Once more, we think of such a function as a possibly infinite sum, and this time use the suggestive notation

$$\sum_{I \text{ admissible}} a_I P^I$$

where $a_I = f(I)$. Our graded module will consist of such sums, with a particular finiteness properties in relation to the length and excess of multi-indices. Specifically, we define the underlying graded module of $\widehat{\mathcal{B}}$ by setting that, in degree $d \in \mathbb{Z}$, $\widehat{\mathcal{B}}_d$ is to consist of the sums

$$\sum_{I \text{ admissible}} a_I P^I$$

with the following requirements:

- For all I, if $a_I \neq 0$, $d(I_i) = d$.
- The set of lengths $\#\{l(I) \mid a_I \neq 0\}$ is bounded above, or, equivalently, finite.
- For any $k \in \mathbb{Z}$, $\#\{I \mid a_I \neq 0, e(I) < k\}$ is finite.

Note the change in the excess condition, as compared to the definition of the underlying graded module of $\widehat{\mathcal{S}}$.

Remark 5.25. Since, given any non-empty multi-index I of degree d, we have $e(I) = 2i_1 - d(I) = d$, where $I = (i_1, \dots, i_k)$, in the $p = 2$ case, and $e(I) = 2pi_1 + 2\varepsilon_1 - d(I) = 2pi_1 + 2\varepsilon_1 - d \leq 2pi_1 + 2 - d$, where $I = (\varepsilon_1, i_1, \dots, \varepsilon_k, i_k)$, in the $p > 2$ case, we can rephrase the third condition as saying that, given any $k \in \mathbb{Z}$, there may exist at most finitely many I with $a_I \neq 0$ which are non-empty and are such that the entry i_1 is smaller than k. We may then also imprecisely though suggestively package the condition as "$i_1 \to +\infty$". $||$

We now have a graded module $\widehat{\mathcal{B}}$ and an obvious embedding of graded modules:

$$\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$$

An example of an element, one in degree 0, which is present in the completion $\widehat{\mathcal{B}}$ but not in $\mathcal{B}$ is the following infinite sum:

$$\sum_{k \geq 0} P^k P^{-k}$$

In fact, as the following proposition demonstrates, all elements of $\widehat{\mathcal{B}}$ which are not in $\mathcal{B}$ share the features of the above example which say that the initial entries of the multi-indices tend to $+\infty$ while the final entries tend to $-\infty$.

Proposition 5.26. *Let $\sum a_I P^I$ be an element of $\widehat{B}$. We have the following:*

(i) Given any $k \in \mathbb{Z}$, for all but finitely many I, the initial entry is greater than k.

(ii) Given any $k \in \mathbb{Z}$, for all but finitely many I, the final entry is less than by k.

Here, as before, in the case $p > 2$, where multi-indices take the form $(\varepsilon_1, i_1, \ldots, \varepsilon_r, i_r)$, where the i_j lie in $\mathbb{Z}$ while the ε_j lie in $\{0, 1\}$, the first entry is taken to be i_1, and the final entry, i_r, which is to say we disregard the ε_j for this particular purpose.

Proof. The proof is exactly that of Proposition 5.18, with the appropriate modifications. □

We now equip $\widehat{\mathcal{B}}$ with an algebra structure. We could develop analogues of all the technical results which we proved in the process of constructing the algebra structure of $\widehat{\mathcal{S}}$. However, as mentioned earlier, the isomorphism $\mathcal{B} \cong \mathcal{S}^\dagger$ in Proposition 3.19 provides a shorter route. Recall, as per Remark 3.20, that this isomorphism maps admissible monomials to admissible monomials, preserves length and negates the excess. It follows that it extends to an isomorphism of graded modules $\widehat{\mathcal{B}} \cong \widehat{\mathcal{S}}^\dagger$ such that the following square commutes:

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook] \arrow[d, "\cong"'] & \widehat{\mathcal{B}} \arrow[d, "\cong"] \\ \mathcal{S}^\dagger \arrow[r, hook] & \widehat{\mathcal{S}}^\dagger \end{tikzcd}$$

Now, to define the product structure of $\widehat{\mathcal{B}}$, we simply transfer the corresponding structure on $\widehat{\mathcal{S}}^\dagger$ across the right vertical isomorphism. It then immediately follows that the isomorphism $\widehat{\mathcal{B}} \cong \widehat{\mathcal{S}}^\dagger$ is now one of algebras, and that the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ is now also one of algebras, so that the entire commutative square above is now one in the category of algebras. Moreover, because the isomorphism between $\mathcal{B}$ and $\mathcal{S}^\dagger$ maps admissible monomials to admissible monomials (and so admissible monomial expansions to admissible monomial expansions), an easy check shows that the products

$$\left(\sum_I a_I P^I\right) \cdot \left(\sum_I b_I P^I\right)$$

in $\widehat{\mathcal{B}}$ admit a definition by expansion of the terms $P^I P^J$ entirely analogous to that for the products in $\widehat{\mathcal{S}}$.

We have thus constructed the desired algebra $\widehat{\mathcal{B}}$. It remains to make precise in what sense it is a completion of $\mathcal{B}$; the final result, presented below, is of course analogous to the case of $\mathcal{S}$ and $\widehat{\mathcal{S}}$. For $k \geq 0$, recall the quotients $\mathcal{B}_{<k}$ of $\mathcal{B}$ defined as per Definition 3.26 and Remark 3.27. Each of these algebras is filtered by length where $\mathrm{F}_t\mathcal{B}_{<k}$ consists of those monomials P^I satisfying the length bound $p^{l(I)} \leq t$. Note that, for any $k \leq l$, we have a canonical map

$$\mathcal{B}_{<l} \to \mathcal{B}_{<k}$$

and that this map is a filtered map, in that, for each $t \geq 0$, we have induced maps:

$$\mathrm{F}_t\mathcal{B}_{<l} \to \mathrm{F}_t\mathcal{B}_{<k}$$

Next, note that we can also filter $\widehat{\mathcal{B}}$ similarly by length by setting, for $t \geq 0$, $\mathrm{F}_t\widehat{\mathcal{B}}$ to comprise the sums $\sum_{I \text{ admissible}} a_I P^I$ which satisfy the degree, length and excess requirements in the definition of $\widehat{\mathcal{B}}$ and which also satisfy, more specifically regarding length, the bound $p^{l(I)} \leq t$ for any I where $a_I \neq 0$. For each $k \geq 0$, we have a map

$$\widehat{\mathcal{B}} \to \mathcal{B}_{<k}$$

which projects an infinite sum to the sub-sum of the elements of excess $< k$, of which there are finitely many by definition of $\widehat{\mathcal{B}}$. Moreover, this map is a filtered map in that we have, for each $t \geq 0$, an induced map:

$$\mathrm{F}_t\widehat{\mathcal{B}} \to \mathrm{F}_t\mathcal{B}_{<k}$$

These maps are compatible with the maps $\mathrm{F}_t\mathcal{B}_{<l} \to \mathrm{F}_t\mathcal{B}_{<k}$ above in that they yield a left cone on

the tower

$$\cdots \to \mathrm{F}_t\mathcal{B}_{<2} \to \mathrm{F}_t\mathcal{B}_{<1} \to \mathrm{F}_t\mathcal{B}_{<0}$$

and so yield a map $\mathrm{F}_t\widehat{\mathcal{B}} \to \lim_{k\geq 0}\mathrm{F}_t\mathcal{B}_{<k}$. The precise statement then regarding in what sense $\widehat{\mathcal{B}}$ is a completion of $\mathcal{B}$ is the following.

Proposition 5.27. *The map $\mathrm{F}_t\widehat{\mathcal{B}} \to \lim_{k\geq 0}\mathrm{F}_t\mathcal{B}_{<k}$ constructed above is an isomorphism of graded modules. Moreover, as graded modules, we have that:*

$$\widehat{\mathcal{B}} \cong \operatorname*{colim}_{t\geq 0}\lim_{k\geq 0}\mathrm{F}_t\mathcal{B}_{<k}$$

Proof. The proof is entirely analogous to that of Proposition 5.24, with the appropriate modifications. □

5.5 The (Co)homology of Free Algebras Over the Stable Operads

In the previous section, we constructed the algebras $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$. In this section, we shall show that, if X is a chain complex, the homology of the free $\mathcal{E}_{\mathrm{st}}$-algebra on X is precisely the free $\widehat{\mathcal{S}}$-module on $\mathrm{H}_\bullet(X)$, namely $\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X)$, and similarly that, if X is a cochain complex, the cohomology of the free $\mathcal{E}_{\mathrm{st}}^\dagger$-algebra on X is precisely the free $\widehat{\mathcal{B}}$-module on $\mathrm{H}^\bullet(X)$, namely $\widehat{\mathcal{B}} \otimes \mathrm{H}^\bullet(X)$. In order to compare $\mathrm{H}_\bullet(\mathbf{E}_{\mathbf{st}}X)$ and $\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X)$ when X is a chain complex, or $\mathrm{H}^\bullet(\mathbf{E}_{\mathbf{st}}^\dagger X)$ and $\widehat{\mathcal{B}} \otimes \mathrm{H}^\bullet(X)$ when X is a cochain complex, we shall introduce intermediating constructions. Thus, we define a functor A on chain complexes by setting:

$$\mathrm{A}(X) := \operatorname*{colim}_{t\geq 0}\lim_{k\geq 0}\mathrm{F}_t\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$$

Similarly, we define a functor B on cochain complexes by setting:

$$\mathrm{B}(X) := \operatorname*{colim}_{t\geq 0}\lim_{k\geq 0}\mathrm{F}_t\mathrm{H}^\bullet((\Sigma^k\mathbf{E}^\dagger)X)$$

We study these functors by studying the steps in their construction, one at a time. Let X be a chain complex. Let $\{c_i\}$ be a basis of $\mathrm{H}_\bullet(X)$. By Propositions 3.31 and 2.50, we have that, for each $k \geq 0$, $\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$ is isomorphic to a shift up by k of the free graded-commutative algebra over $\mathbb{F}_p$ on the terms Q^Ic_i where I is admissible and $e(I) > |c_i| - k$. Let $\mathcal{F}_k$ denote this latter object. Then, via the maps $\Sigma^{k+1}\mathcal{E} \to \Sigma^k\mathcal{E}$, we have a commutative diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}_2 \arrow[r] \arrow[d, "\cong"] & \mathcal{F}_1 \arrow[r] \arrow[d, "\cong"] & \mathcal{F}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{H}_\bullet((\Sigma^2\mathbf{E})X) \arrow[r] & \mathrm{H}_\bullet((\Sigma\mathbf{E})X) \arrow[r] & \mathrm{H}_\bullet(\mathbf{E}X)
\end{tikzcd}$$

Now, let X instead be a cochain complex, and let $\{c_i\}$ be a basis of $\mathrm{H}^\bullet(X)$. By the same Propositions 3.31 and 2.50, we have that, for each $k \geq 0$, $\mathrm{H}^\bullet((\Sigma^k \mathbf{E}^\dagger)X)$ is isomorphic to a shift down by k of the free graded-commutative algebra over $\mathbb{F}_p$ on the terms $P^I c_i$ where I is admissible and $e(I) < |c_i| + k$. Let $\mathcal{G}_k$ denote this object. Then, via the maps $\Sigma^{k+1}\mathcal{E} \to \Sigma^k \mathcal{E}$, we have a commutative diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}_2 \arrow[r] \arrow[d, "\cong"] & \mathcal{G}_1 \arrow[r] \arrow[d, "\cong"] & \mathcal{G}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{H}^\bullet((\Sigma^2 \mathbf{E}^\dagger)X) \arrow[r] & \mathrm{H}^\bullet((\Sigma \mathbf{E}^\dagger)X) \arrow[r] & \mathrm{H}^\bullet(\mathbf{E}^\dagger X)
\end{tikzcd}$$

Proposition 5.28. *Let X be a dg module. For each $k \geq 0$, the maps $\mathcal{F}_{k+1} \to \mathcal{F}_k$ and $\mathcal{G}_{k+1} \to \mathcal{G}_k$ kill any product, including the empty product, which is to say the multiplicative unit.*

One can sum up this proposition as "the instability of products".

Proof. We shall give a proof of the chain version, which is to say of the maps $\mathcal{F}_{k+1} \to \mathcal{F}_k$, as the cochain version is analogous. Let us first deal with non-empty products. Recall our notation e_d, for $d \geq 0$, for particular elements of $\mathcal{E}(2)$, where e_d has degree d. Let e_d^k denote the corresponding element of $(\Sigma^k \mathcal{E})(2) = \mathcal{E}(2)[-k]$. Note that e_d^k has degree $d - k$. In particular e_0^k has degree $-k$, and, since $\Sigma^k \mathcal{E}$ is zero below degree k, the stabilization map $\Sigma^{k+1}\mathcal{E} \to \Sigma^k \mathcal{E}$ must map e_0^{k+1} to 0. Now, fix $k \geq 0$ and consider some product $(Q^I c) \cdot (Q^{I'} c')$ in $\mathcal{F}_{k+1}$. Let $[\eta]$ and $[\xi]$, respectively, be the images of $Q^I c$ and $Q^{I'} c'$ under the isomorphism $\mathcal{F}_{k+1} \to \mathrm{H}_\bullet((\Sigma^{k+1}\mathbf{E})X)$. Then, by definition of the products in the homologies, $(Q^I c) \cdot (Q^{I'} c')$ maps, under this same isomorphism, to $[e_0^{k+1}(\eta, \xi)]$, by which we mean the class of the image of $e_0^{k+1} \otimes \eta \otimes \xi$ under the map $(\Sigma^{k+1}\mathcal{E})(2) \otimes ((\Sigma^{k+1}\mathbf{E})X)^{\otimes 2} \to (\Sigma^{k+1}\mathbf{E})X$. Now, since e_0^{k+1} maps to 0 under $\Sigma^{k+1}\mathcal{E} \to \Sigma^k \mathcal{E}$, as noted above, we have that the product $(Q^I c) \cdot (Q^{I'} c')$ maps to zero under the composite $\mathcal{F}_{k+1} \to \mathrm{H}_\bullet((\Sigma^{k+1}\mathbf{E})X) \to \mathrm{H}_\bullet((\Sigma^k \mathbf{E})X)$, and this proves the required result for the map $\mathcal{F}_{k+1} \to \mathcal{F}_k$, except in the case of the empty product. For the empty product, the argument is similar: one notes that the multiplicative unit in $\mathcal{F}_{k+1}$ corresponds to the generator in degree $k+1$ of $(\Sigma^{k+1}\mathcal{E})(0)$, and this generator is mapped to zero in $(\Sigma^k \mathcal{E})(0)$ since the latter is zero above degree k. $\square$

We can also consider filtered versions of the above diagrams. Consider again the case where X is a chain complex and $\{c_i\}$ is a basis of $\mathrm{H}_\bullet(X)$. By Propositions 3.34 and 2.50, given $t \geq 0$, $\mathrm{F}_t \mathrm{H}_\bullet((\Sigma^k \mathbf{E})X)$ is a shift up by k of the $\mathbb{F}_p$-submodule of $\mathcal{F}_k$ generated by the products $(Q^{I_1} c_1) \cdots (Q^{I_k} c_r)$, where $r \geq 0$, I is admissible, $e(I) > |c_i| - k$ and $p^{l(I_1)} + \cdots + p^{l(I_r)} \leq t$. As usual, we let $\mathrm{F}_t \mathcal{F}_k$ denote this filtration piece of $\mathcal{F}_k$. We then have a commutative diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{F}_t\mathcal{F}_2 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{F}_1 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{F}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{F}_t\mathrm{H}_\bullet((\Sigma^2\mathbf{E})X) \arrow[r] & \mathrm{F}_t\mathrm{H}_\bullet((\Sigma\mathbf{E})X) \arrow[r] & \mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X)
\end{tikzcd}$$

Now consider again the case where X is a cochain complex and $\{c_i\}$ is a basis of $\mathrm{H}^\bullet(X)$. By Propositions 3.34 and 2.50, given $t \geq 0$, $\mathrm{F}_t\mathrm{H}^\bullet((\Sigma^k\mathbf{E}^\dagger)X)$ is a shift up down by k of the $\mathbb{F}_p$-submodule of $\mathcal{G}_k$ generated by the products $(P^{I_1}c_1)\cdots(P^{I_k}c_r)$, where $r \geq 0$, I is admissible, $e(I) < |c_i| + k$ and $p^{l(I_1)} + \cdots + p^{l(I_r)} \leq t$. As usual, we let $\mathrm{F}_t\mathcal{G}_k$ denote this filtration piece of $\mathcal{G}_k$. We then have a commutative diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{F}_t\mathcal{G}_2 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{G}_1 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{G}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{F}_t\mathrm{H}^\bullet((\Sigma^2\mathbf{E}^\dagger)X) \arrow[r] & \mathrm{F}_t\mathrm{H}^\bullet((\Sigma\mathbf{E}^\dagger)X) \arrow[r] & \mathrm{F}_t\mathrm{H}^\bullet(\mathbf{E}^\dagger X)
\end{tikzcd}$$

Next, we make precise in what sense the products, due to their instability as described in Proposition 5.28 above, disappear in the limits. If X is a chain complex, $\{c_i\}$ is a basis of $\mathrm{H}_\bullet(X)$ and $\mathcal{F}_k$ is as above, we let $\mathcal{F}_k^+$ be the submodule of $\mathcal{F}_k$ generated by the monomials Q^Ic_i where $e(I) > |e_i| - k$; that is, it is submodule which omits all products, including the empty product, and is also a free graded module over $\mathbb{F}_p$ on the monomials Q^Ie_i where I is admissible and $e(I) > |e_i| - k$. Just like $\mathcal{F}_k$, $\mathcal{F}_k^+$ is filtered, where $\mathrm{F}_t\mathcal{F}_k^+$ denotes the submodule generated by the monomials Q^Ic_i which satisfy $e(I) > |e_i| - k$ and also the additional requirement that $p^{l(I)} \leq t$. The inclusion $\mathcal{F}_k^+ \hookrightarrow \mathcal{F}_k$ is then clearly a filtered one, in that, for each $t \geq 0$, we have an induced inclusion $\mathrm{F}_t\mathcal{F}_k^+ \hookrightarrow \mathrm{F}_t\mathcal{F}_k$. We now have commutative diagrams as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}_2^+ \arrow[r] \arrow[d, hook] & \mathcal{F}_1^+ \arrow[r] \arrow[d, hook] & \mathcal{F}_0^+ \arrow[d, hook] \\
\cdots \arrow[r] & \mathcal{F}_2 \arrow[r] \arrow[d, "\cong"] & \mathcal{F}_1 \arrow[r] \arrow[d, "\cong"] & \mathcal{F}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{H}_\bullet((\Sigma^2\mathbf{E})X) \arrow[r] & \mathrm{H}_\bullet((\Sigma\mathbf{E})X) \arrow[r] & \mathrm{H}_\bullet(\mathbf{E}X)
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{F}_t\mathcal{F}_2^+ \arrow[r] \arrow[d, hook] & \mathrm{F}_t\mathcal{F}_1^+ \arrow[r] \arrow[d, hook] & \mathrm{F}_t\mathcal{F}_0^+ \arrow[d, hook] \\
\cdots \arrow[r] & \mathrm{F}_t\mathcal{F}_2 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{F}_1 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{F}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{F}_t\mathrm{H}_\bullet((\Sigma^2\mathbf{E})X) \arrow[r] & \mathrm{F}_t\mathrm{H}_\bullet((\Sigma\mathbf{E})X) \arrow[r] & \mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X)
\end{tikzcd}$$

On the other hand, if X is a cochain complex, $\{c_i\}$ a basis of $\mathrm{H}^\bullet(X)$ and $\mathcal{G}_k$ is as above, we let $\mathcal{G}_k^+$ be the submodule of $\mathcal{G}_k$ generated by the monomials $P^I c_i$ where $e(I) < |e_i| + k$; that is, it is submodule which omits all products, including the empty product, and is also a free graded module over $\mathbb{F}_p$ on the monomials $P^I e_i$ where I is admissible and $e(I) < |e_i| + k$. Just like $\mathcal{G}_k$, $\mathcal{G}_k^+$ is filtered, where $\mathrm{F}_t\mathcal{G}_k^+$ denotes the submodule generated by the monomials $P^I c_i$ which satisfy $e(I) < |e_i| + k$ and also the additional requirement that $p^{l(I)} \leq t$. The inclusion $\mathcal{G}_k^+ \to \mathcal{G}_k$ is then clearly a filtered one, in that, for each $t \geq 0$, we have an induced inclusion $\mathrm{F}_t\mathcal{G}_k^+ \to \mathrm{F}_t\mathcal{G}_k$. We now have commutative diagrams as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}_2^+ \arrow[r] \arrow[d, hook] & \mathcal{G}_1^+ \arrow[r] \arrow[d, hook] & \mathcal{G}_0^+ \arrow[d, hook] \\
\cdots \arrow[r] & \mathcal{G}_2 \arrow[r] \arrow[d, "\cong"] & \mathcal{G}_1 \arrow[r] \arrow[d, "\cong"] & \mathcal{G}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{H}^\bullet((\Sigma^2\mathbf{E}^\dagger)X) \arrow[r] & \mathrm{H}^\bullet((\Sigma\mathbf{E}^\dagger)X) \arrow[r] & \mathrm{H}^\bullet(\mathbf{E}^\dagger X)
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{F}_t\mathcal{G}_2^+ \arrow[r] \arrow[d, hook] & \mathrm{F}_t\mathcal{G}_1^+ \arrow[r] \arrow[d, hook] & \mathrm{F}_t\mathcal{G}_0^+ \arrow[d, hook] \\
\cdots \arrow[r] & \mathrm{F}_t\mathcal{G}_2 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{G}_1 \arrow[r] \arrow[d, "\cong"] & \mathrm{F}_t\mathcal{G}_0 \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathrm{F}_t\mathrm{H}^\bullet((\Sigma^2\mathbf{E}^\dagger)X) \arrow[r] & \mathrm{F}_t\mathrm{H}^\bullet((\Sigma\mathbf{E}^\dagger)X) \arrow[r] & \mathrm{F}_t\mathrm{H}^\bullet(\mathbf{E}^\dagger X)
\end{tikzcd}$$

Proposition 5.29. *If X is a chain complex, $\{c_i\}$ is a basis of $\mathrm{H}_\bullet(X)$ and $t \geq 0$, the induced maps*

$$\lim_{k\geq 0} \mathcal{F}_k^+ \to \lim_{k\geq 0} \mathrm{H}_\bullet((\Sigma^k\mathbf{E})X) \qquad \lim_{k\geq 0} \mathrm{F}_t\mathcal{F}_k^+ \to \lim_{k\geq 0} \mathrm{F}_t\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$$

are isomorphisms. Similarly, if X is a cochain complex, $\{c_i\}$ is a basis of $\mathrm{H}^\bullet(X)$ and $t \geq 0$, the induced maps

$$\lim_{k\geq 0} \mathcal{G}_k^+ \to \lim_{k\geq 0} \mathrm{H}^\bullet((\Sigma^k\mathbf{E}^\dagger)X) \qquad \lim_{k\geq 0} \mathrm{F}_t\mathcal{G}_k^+ \to \lim_{k\geq 0} \mathrm{F}_t\mathrm{H}^\bullet((\Sigma^k\mathbf{E}^\dagger)X)$$

are isomorphisms.

Proof. Consider the case of the map $\lim_k \mathcal{F}_k^+ \to \lim_k \mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$; the other cases are analogous. We need to show that the induced map $\lim_k \mathcal{F}_k^+ \to \lim_k \mathcal{F}_k$ is an isomorphism, as we already know that the map $\lim_k \mathcal{F}_k \lim_k \mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$ is an isomorphism. Injectivity is obvious from the standard description of inverse limits of towers via infinite tuples and, moreover, surjectivity follows from this description and the instability of products described in Proposition 5.28. $\square$

Thus, given a chain complex X, we have a description of $\lim_{k\geq 0} \mathrm{F}_t\mathbf{H}_\bullet((\Sigma^k\mathbf{E})X)$ and so, upon taking the colimit over t, a description of $\mathrm{A}(X)$. Similarly, given a cochain complex X, we have a description of $\lim_{k\geq 0} \mathrm{F}_t\mathrm{H}^\bullet((\Sigma^k\mathbf{E}^\dagger)X)$ for each $k \geq 0$, and so, upon taking the colimit over t, a description of $\mathrm{B}(X)$. This allows us to demonstrate the following a priori unexpected properties of A and B.

Proposition 5.30. *The functors* A *and* B *are additive.*

Proof. We shall demonstrate the case of A; the case of B is entirely analogous. Let X and Y be chain complexes, and let $\{c_i\}$ and $\{d_j\}$ be basses of their homologies $\mathrm{H}_\bullet(X)$ and $\mathrm{H}_\bullet(Y)$, respectively. We wish to show that the canonical map

$$\mathrm{A}(X) \oplus \mathrm{A}(Y) \to \mathrm{A}(X \oplus Y)$$

is an isomorphism. Now, by definition, we have:

$$\mathrm{A}(X) = \operatorname*{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_t\mathbf{H}_\bullet((\Sigma^k\mathbf{E})X)$$

Moreover, by Proposition 5.29, for each $k, t \geq 0$, $\lim_{k\geq 0} \mathrm{F}_t\mathbf{H}_\bullet((\Sigma^k\mathbf{E})X)$ is isomorphic to $\lim_{k\geq 0} \mathrm{F}_t\mathcal{F}_k^+$, where $\mathrm{F}_t\mathcal{F}_k^+$ is the free graded module on the monomials $Q^I c_i$ which satisfy $e(I) > |e_i| - k$ and $p^{l(I)} \leq t$. It follows that, in degree say $d \in \mathbb{Z}$, $\mathrm{A}(X)$ is isomorphic to the module which consists of infinite sums

$$\sum a_{I_i,c_i}(Q^{I_i}c_i)$$

by which we mean functions $f\colon \{(I_i, c_i) \mid I_i \text{ admissible}\} \to \mathbb{F}_p$, where $f(I_i, c_i) = a_{I_i,c_i}$, satisfying the following requirements:

- For all (I_i, c_i), if $a_{I_i,c_i} \neq 0$, $d(I_i) + |c_i| = d$.
- The set of lengths $\#\{l(I_i) \mid a_{I_i,c_i} \neq 0\}$ is bounded above, or, equivalently, finite.
- For any $k \geq 0$, $\#\{(I_i, c_i) \mid a_{I_i,c_i} \neq 0, e(I_i) > |c_i| - k\}$ is finite.

Similarly, $\mathrm{A}(Y)$ is isomorphic to the module which consists of infinite sums

$$\sum a_{I_j,d_j}(Q^{I_j}d_j)$$

satisfying the following requirements:

- For all (I_j, d_j), if $a_{I_j,d_j} \neq 0$, $d(I_j) + |d_j| = d$.
- The set of lengths $\#\{l(I_j) \mid a_{I_j,d_j} \neq 0\}$ is bounded above, or, equivalently, finite.

- For any $k \geq 0$, $\#\{(I_j, d_j) \mid a_{I_j,d_j} \neq 0, e(I_j) > |d_j| - k\}$ is finite.

Moreover, as $\mathrm{H}_\bullet(X \oplus Y) = \mathrm{H}_\bullet(X) \oplus \mathrm{H}_\bullet(Y)$ has basis $\{c_i\} \amalg \{d_j\}$, $\mathrm{A}(X \oplus Y)$ is isomorphic to the module which consists of infinite sums

$$\sum a_{I_i,c_i}(Q^{I_i}c_i) + \sum a_{I_j,d_j}(Q^{I_j}d_j)$$

satisfying the following requirements:

- For all (I_i, c_i) and (I_j, d_j), if $a_{I_i,d_i} \neq 0$, $d(I_i) + |c_i| = d$, and if $a_{I_j,d_j} \neq 0$, $d(I_j) + |d_j| = d$.
- Both sets of lengths $\#\{l(I_i) \mid a_{I_i,c_i} \neq 0\}$ and $\#\{l(I_j) \mid a_{I_j,d_i} \neq 0\}$ are bounded above, or, equivalently, finite.
- For any $k \geq 0$, both sets $\#\{(I_i, c_i) \mid a_{I_i,c_i} \neq 0, e(I_i) > |c_i| - k\}$ and $\#\{(I_j, d_j) \mid a_{I_j,d_j} \neq 0, e(I_j) > |d_j| - k\}$ is finite.

An easy check shows that, under the identifications provided by the above isomorphisms, the canonical map $\mathrm{A}(X) \oplus \mathrm{A}(Y) \to \mathrm{A}(X \oplus Y)$ corresponds to the obvious inclusion of sets of infinite sums. The map is then clearly an isomorphism, as desired. $\square$

We now show how $\mathrm{A}(X)$ and $\mathrm{B}(X)$ intermediate between $\mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X)$, $\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X)$ and $\mathrm{H}^\bullet(\mathbf{E}^\dagger_{\mathrm{st}}X)$, $\widehat{\mathcal{B}} \otimes \mathrm{H}^\bullet(X)$, respectively. First, let X be a chain complex. Via the canonical maps $\mathcal{E}_{\mathrm{st}} \to \Sigma^k\mathcal{E}$, we get canonical maps $\mathbf{E}_{\mathbf{st}}X \to (\Sigma^k\mathbf{E})X$, and as a result also maps $\mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}_{\mathbf{st}}X) \to \mathrm{F}_t\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$. These maps are compatible with the maps $\mathrm{F}_t\mathrm{H}_\bullet((\Sigma^{k+1}\mathbf{E})X) \to \mathrm{F}_t\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$, so that we get an induced map $\mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}_{\mathbf{st}}X) \to \lim_{k\geq 0} \mathrm{F}_t\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$. Upon taking colimits over t, we get an induced natural, in X, map:

$$\Phi_1 \colon \mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X) \to \mathrm{A}(X)$$

Now consider $\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X)$. We construct a natural map:

$$\Phi_2 \colon \widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X) \to \mathrm{A}(X)$$

This map is defined as follows; the idea is that the term $\lim_k \mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$ which arises in the construction of $\mathrm{A}(X)$ inherits an action by the term $\mathcal{S}_{>-k}$ which arises in the construction of $\widehat{\mathcal{S}}$. Recall, as we saw in Propostion 5.24, that we have $\widehat{\mathcal{S}} = \mathrm{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_t\mathcal{S}_{>-k}$. For each d_1, d_2, we need to specify a map $\widehat{\mathcal{S}}_{d_1} \otimes \mathrm{H}_{d_2}(X) = \mathrm{colim}_{t\geq 0}(\mathrm{F}_t\widehat{\mathcal{S}}_{d_1} \otimes \mathrm{H}_{d_2}(X)) \to \mathrm{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_t\mathrm{H}_{d_1+d_2}((\Sigma^k\mathbf{E})X)$. Fix some $t_0 \geq 0$. For each $k \geq 0$, consider the following composite:

$$\mathrm{F}_{t_0}\widehat{\mathcal{S}}_{d_1} \otimes \mathrm{H}_{d_2}(X) \to \mathrm{F}_{t_0}(\mathcal{S}_{>d_2-k})_{d_1} \otimes \mathrm{F}_1\mathrm{H}_{d_2}((\Sigma^k\mathbf{E})X) \to \mathrm{F}_{t_0+1}\mathrm{H}_{d_1+d_2}((\Sigma^k\mathbb{E})X)$$

Here the first map is induced by the canonical map $\mathrm{F}_{t_0}\widehat{\mathcal{S}}_{d_1} \to \mathrm{F}_{t_0}(\mathcal{S}_{>d_2-k})_{d_1}$ (where, if $d_2 - k \geq 1$, we first pass to $\mathrm{F}_{t_0}(\mathcal{S}_{>0})_{d_1}$ and then to $\mathrm{F}_{t_0}(\mathcal{S}_{>d_2-k})_{d_1}$) and the unit of adjunction $X \to (\Sigma^k\mathbf{E})X$. Moreover, the second map is that which is provided by Propositions 3.30, 3.33 and 2.50. The above composites are compatible with the maps $\mathrm{F}_{t_0+1}\mathrm{H}_{d_1+d_2}((\Sigma^{k+1}\mathbf{E})X) \to \mathrm{F}_{t_0+1}\mathrm{H}_{d_1+d_2}((\Sigma^k\mathbf{E})X)$, so that we get an induced map

$$\mathrm{F}_{t_0}\widehat{\mathcal{S}}_{d_1} \otimes \mathrm{H}_{d_2}(X) \to \lim_{k\geq 0} \mathrm{F}_{t_0+1}\mathrm{H}_{d_1+d_2}((\Sigma^k\mathbb{E})X) \hookrightarrow \mathrm{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_t\mathrm{H}_{d_1+d_2}((\Sigma^k\mathbf{E})X)$$

and then finally, we note that the above maps are compatible with the inclusions $\mathrm{F}_t\widehat{\mathcal{S}}_{d_1} \hookrightarrow \mathrm{F}_{t+1}\widehat{\mathcal{S}}_{d_1}$, so that we get the desired map

$$\widehat{\mathcal{S}}_{d_1} \otimes \mathrm{H}_{d_2}(X) \to \operatorname*{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_t\mathrm{H}_{d_1+d_2}((\Sigma^k\mathbf{E})X)$$

and this completes the construction of Φ_2.

On the other hand, if X is a cochain complex, by entirely analogous constructions, we get natural maps

$$\Phi_1 \colon \mathrm{H}^\bullet(\mathbf{E}^\dagger_{\mathrm{st}}X) \to \mathrm{B}(X)$$

$$\Phi_2 \colon \widehat{\mathcal{B}} \otimes \mathrm{H}^\bullet(X) \to \mathrm{B}(X)$$

which we also denote by Φ_1 and Φ_2 (the context will always make it clear which maps we intend by these symbols).

Proposition 5.31. *For finite dg modules X, the maps Φ_1 and Φ_2, in both the chain and cochain cases, defined above are isomorphisms.*

Recall that by a finite dg module over $\mathbb{F}_p$ we mean one which is bounded and of finite dimension in each degree. In order to demonstrate this result, we first need the following lemma.

Lemma 5.32. *For any chain complex X, the towers*

$$\cdots \to (\Sigma^2\mathbf{E})X \to (\Sigma\mathbf{E})X \to \mathbf{E}X$$

$$\cdots \to \mathrm{H}_\bullet((\Sigma^2\mathbf{E})X) \to \mathrm{H}_\bullet((\Sigma\mathbf{E})X) \to \mathrm{H}_\bullet(\mathbf{E}X)$$

satisfy the Mittag-Leffler condition. Moreover, for each $t \geq 0$, the towers

$$\cdots \to \mathrm{F}_t(\Sigma^2\mathbf{E})X \to \mathrm{F}_t(\Sigma\mathbf{E})X \to \mathrm{F}_t\mathbf{E}X$$

$$\cdots \to \mathrm{F}_t\mathrm{H}_\bullet((\Sigma^2\mathbf{E})X) \to \mathrm{F}_t\mathrm{H}_\bullet((\Sigma\mathbf{E})X) \to \mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}X)$$

also satisfy the Mittag-Leffler condition. Similarly, if X is a cochain complex, the towers

$$\cdots \to (\Sigma^2\mathbf{E}^\dagger)X \to (\Sigma\mathbf{E}^\dagger)X \to \mathbf{E}^\dagger X$$

$$\cdots \to \mathrm{H}^\bullet((\Sigma^2\mathbf{E}^\dagger)X) \to \mathrm{H}^\bullet((\Sigma\mathbf{E}^\dagger)X) \to \mathrm{H}^\bullet(\mathbf{E}^\dagger X)$$

satisfy the Mittag-Leffler condition. Moreover, for each $t \geq 0$, the towers

$$\cdots \to \mathrm{F}_t(\Sigma^2\mathbf{E}^\dagger)X \to \mathrm{F}_t(\Sigma\mathbf{E}^\dagger)X \to \mathrm{F}_t\mathbf{E}^\dagger X$$

$$\cdots \to \mathrm{F}_t\mathrm{H}^\bullet((\Sigma^2\mathbf{E}^\dagger)X) \to \mathrm{F}_t\mathrm{H}^\bullet((\Sigma\mathbf{E}^\dagger)X) \to \mathrm{F}_t\mathrm{H}^\bullet(\mathbf{E}^\dagger X)$$

also satisfy the Mittag-Leffler condition.

Proof. We shall demonstrate the case of the chain operad; the case of the cochain operad is entirely analogous. In fact, we shall show that, for each $n \geq 0$ and any chain complex X, the towers

$$\cdots \to (\Sigma^2\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n} \to (\Sigma\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n} \to \mathcal{E}(n) \otimes_{\Sigma_n} X^{\otimes n}$$

$$\cdots \to \mathrm{H}_\bullet((\Sigma^2\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n}) \to \mathrm{H}_\bullet((\Sigma\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n}) \to \mathrm{H}_\bullet(\mathcal{E}(n) \otimes_{\Sigma_n} X^{\otimes n})$$

satisfy the Mittag-Leffler condition, from which the desired results follow immediately. The Mittag-Leffler property for the first of the two towers above follows immediately from the Mittag-Leffler property of the tower $\cdots \to (\Sigma^2\mathcal{E})(n) \to (\Sigma\mathcal{E})(n) \to \mathcal{E}(n)$, which was established in Proposition 5.1. Consider then the second tower. Let $\{c_i\}$ denote a basis of $\mathrm{H}_\bullet(X)$, and recall, as in the previous section, that, by Propositions 3.31 and 2.50, we have that, for each $k \geq 0$, $\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$ is isomorphic to a shift up by k of the free graded-commutative algebra over $\mathbb{F}_p$ on the terms $Q^I c_i$ where I is admissible and $e(I) > |c_i| - k$. Let $\mathcal{F}_k$ denote this object, as earlier. By Proposition 3.34, we have that $\mathrm{H}_\bullet((\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n})$ is the $\mathbb{F}_p$-submodule of $\mathcal{F}_k$ generated by the terms $(Q^{I_1}c_1)\cdots(Q^{I_r}c_r)$ where $r \geq 0$, I is admissible, $e(I) > |c_i| - k$ and $p^{l(I_1)} + \cdots + p^{l(I_k)} = n$. Now, given a fixed k, the map $\mathrm{H}_\bullet((\Sigma^{k+1}\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n}) \to \mathrm{H}_\bullet((\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n})$ sends any $Q^I c$ which satisfies, not only that $e(I) > |c| - k - 1$, but rather $e(I) > |c| - k$, to itself, so that all monomials $Q^I c$ in $\mathcal{F}_k$ occur in the image. On the other hand, by Proposition 5.28, all products are killed, and in particular, no products occur in the image. We have thus identified the image of the map $\mathrm{H}_\bullet((\Sigma^{k+1}\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n}) \to \mathrm{H}_\bullet((\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n})$, and, moreover, essentially the same argument shows that all maps $\mathrm{H}_\bullet((\Sigma^{k'}\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n}) \to \mathrm{H}_\bullet((\Sigma^k\mathcal{E})(n) \otimes_{\Sigma_n} X^{\otimes n})$, for $k' \geq k+1$, have this same image, which gives us the desired result. $\square$

Proof of Proposition 5.31. We shall demonstrate the chain case; the cochain case is analogous. Let

X be a finite chain complex. We first consider the map of Φ_1. We have:

$$\begin{aligned}
\mathrm{F}_t\mathbf{E}_{\mathrm{st}}X &= \bigoplus_{n\le t}\mathcal{E}_{\mathrm{st}}(n)\otimes_{\Sigma_n}X^{\otimes n}\\
&= \bigoplus_{n\le t}\lim_{k\ge 0}(\Sigma^k\mathcal{E}(n))\otimes_{\Sigma_n}X^{\otimes n}\\
&\cong \bigoplus_{n\le t}\lim_{k\ge 0}(\Sigma^k\mathcal{E}(n)\otimes_{\Sigma_n}X^{\otimes n})\\
&\cong \lim_{k\ge 0}(\bigoplus_{n\le t}\Sigma^k\mathcal{E}(n)\otimes_{\Sigma_n}X^{\otimes n})\\
&= \lim_{k\ge 0}\mathrm{F}_t(\Sigma^k\mathbf{E})X
\end{aligned}$$

Here the first isomorphism holds because X is finite. By Lemma 5.32, we have that

$$\mathrm{F}_t\mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X)\cong\lim_{k\ge 0}\mathrm{F}_t\mathrm{H}_\bullet((\Sigma^k\mathbf{E})X)$$

and then, upon taking colimits over $t\ge 0$, we get that $\Phi_1\colon \mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X)\to \mathrm{A}(X)$ is an isomorphism, as desired. Now consider the map Φ_2. It is standard that, over $\mathbb{F}_p$, as over any field, any chain complex can be written as a direct sum $(\bigoplus_{i\in I}\mathbb{S}^{n_i})\oplus(\bigoplus_{j\in J}\mathbb{D}^{n_j})$, where $\mathbb{S}^n$ and $\mathbb{D}^n$ denote the standard sphere and disk complexes. As X is finite, both I and J here must be finite. Because, by Proposition 5.30, A is additive, and because $\widehat{\mathcal{S}}\otimes\mathrm{H}_\bullet(-)$ is of course also additive, we need only demonstrate the case where X is a sphere complex $\mathbb{S}^n$ or a disk complex $\mathbb{D}^n$. The case of the disk complex follows immediately from the fact that the $\Sigma^k\mathbf{E}$ preserve quasi-isomorphisms (as the operads $\Sigma^k\mathcal{E}$ are Σ-free). Thus, it remains to consider the case where $X=\mathbb{S}^n$ for some $n\in\mathbb{Z}$. Let c_n denote a generator of $\mathrm{H}_\bullet(X)$ in degree n. By Proposition 5.24, we have that:

$$\widehat{\mathcal{S}}\otimes\mathrm{H}_\bullet(X)=\operatorname*{colim}_{t\ge 0}\lim_{k\ge 0}\mathrm{F}_t\mathcal{S}_{>-k}\otimes\mathbb{F}_p\{c_n\}$$

By Proposition 3.28, it follows that $\widehat{\mathcal{S}}\otimes\mathrm{H}_\bullet(X)$, in degree say $d\in\mathbb{Z}$, is isomorphic to the module which consists of infinite sums

$$\sum a_I(Q^I c_n)$$

by which we mean functions $f\colon\{I\mid I\text{admissible}\}\to\mathbb{F}_p$, where $f(I)=a_I$, satisfying the following requirements:

- For all I, if $a_I\ne 0$, $d(I)+n=d$.
- The set of lengths $\#\{l(I)\mid a_I\ne 0\}$ is bounded above, or, equivalently, finite.
- For any $k\ge 0$, $\#\{I\mid a_I\ne 0, e(I)>-k\}$ is finite.

On the other hand, as in the proof of Proposition 5.30, $\mathrm{A}(X)$, in degree d, is isomorphic to the module which consists of infinite sums

$$\sum a_I(Q^I c_n)$$

satisfying the following requirements:

- For all I, if $a_I \neq 0$, $d(I) + n = d$.
- The set of lengths $\#\{l(I) \mid a_I \neq 0\}$ is bounded above, or, equivalently, finite.
- For any $k \geq 0$, $\#\{I \mid a_I \neq 0, e(I) > n - k\}$ is finite.

The only difference occurs in the third condition; however, the two conditions are obviously equivalent, and an easy check shows that the map Φ_2, under these isomorphisms, corresponds to simply the identity, and so is an isomorphism. □

Finally, we are now able to compute the (co)homologies of free algebras over the stable Barratt-Eccles chain and cochain operads.

Proposition 5.33. *Given a chain complex X, we have a natural isomorphism:*

$$\mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X) \cong \widehat{S} \otimes \mathrm{H}_\bullet(X)$$

Similarly, given a cochain complex X, we have a natural isomorphism:

$$\mathrm{H}^\bullet(\mathbf{E}^\dagger_{\mathrm{st}}X) \cong \widehat{B} \otimes \mathrm{H}^\bullet(X)$$

Proof. We shall demonstrate the chain case, the cochain case being analogous. Consider the maps

$$\Phi_1 \colon \mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X) \to \mathrm{A}(X)$$

$$\Phi_2 \colon \widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X) \to \mathrm{A}(X)$$

which we recall are natural in X. We will demonstrate the desired result by showing that Φ_1 and Φ_2 are both injective and that they have the same image inside $\mathrm{A}(X)$. It is standard that, over $\mathbb{F}_p$, as over any field, any chain complex is, up to isomorphism, a direct sum $(\bigoplus_{i\in I}\mathbb{S}^{n_i}) \oplus (\bigoplus_{j\in J}\mathbb{D}^{n_j})$, where $\mathbb{S}^n$ and $\mathbb{D}^n$ denote the standard sphere and disk complexes. Thus, by naturality, it suffices to show that we have the aforementioned injectivity and coincidence of images for the complexes $(\bigoplus_{i\in I}\mathbb{S}^{n_i}) \oplus (\bigoplus_{j\in J}\mathbb{D}^{n_j})$. Given such a complex $(\bigoplus_{i\in I}\mathbb{S}^{n_i}) \oplus (\bigoplus_{j\in J}\mathbb{D}^{n_j})$, the inclusion $\bigoplus_{i\in I}\mathbb{S}^{n_i} \to (\bigoplus_{i\in I}\mathbb{S}^{n_i}) \oplus (\bigoplus_{j\in J}\mathbb{D}^{n_j})$ is a quasi-isomorphism. Note that $\mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X)$

preserves quasi-isomorphisms by Proposition 5.3, $\mathrm{A}(X)$ preserves quasi-isomorphisms because the $\Sigma^k\mathcal{E}$ are Σ-free and $\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet X$ preserves quasi-isomorphisms for obvious reasons. It follows, by naturality again, that it suffices to demonstrate that the result for a complex X of the form $\bigoplus_{i\in I}\mathbb{S}^{n_i}$. Let X be such a complex and let then $\{c_i\}$ be a basis for $\mathrm{H}_\bullet(X)$. Letting λ be an ordinal in bijection with I, construct a filtration

$$X_0 \to X_1 \to \cdots \to X_\alpha \to \cdots$$

of X, where each X_α is finite, and where, for each $\alpha \in \lambda$, $\{c_i\}_{i\in\alpha}$ gives a basis of $\mathrm{H}_\bullet(X_\alpha)$. Because $\mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X)$ and $\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X)$ commute with filtered colimits, and because, by Proposition 5.31, Φ_1 and Φ_2 are isomorphisms when the input is finite, we can factor Φ_1 and Φ_2 as follows:

$$\begin{tikzcd}
\mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X) \arrow[r, "\cong"] & \mathrm{colim}_\alpha \mathrm{H}_\bullet(\mathbf{E}_{\mathrm{st}}X_\alpha) \arrow[dr, "\cong"] & & \\
 & & \mathrm{colim}_\alpha \mathrm{A}(X_\alpha) \arrow[r] & \mathrm{A}(X) \\
\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X) \arrow[r, "\cong"'] & \mathrm{colim}_\alpha(\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(X_\alpha)) \arrow[ur, "\cong"'] & &
\end{tikzcd}$$

It follows that the images of Φ and Ψ coincide, as desired. As for injectivity, by the above factorizations, it suffices to demonstrate injectivity of the map:

$$\mathrm{colim}_\alpha \mathrm{A}(X_\alpha) \longrightarrow \mathrm{A}(X)$$

As in the proof of Proposition 5.30, $\mathrm{A}(X)$, in degree d, is isomorphic to the module which consists of infinite sums

$$\sum a_{I_i,c_i}(Q^{I_i}c_i)$$

by which we mean functions $f\colon \{(I_i, c_i) \mid I_i \text{ admissible}\} \to \mathbb{F}_p$, where $f(I_i, c_i) = a_{I_i,c_i}$, satisfying the following requirements:

- For all (I_i, c_i), if $a_{I_i,c_i} \neq 0$, $d(I_i) + |c_i| = d$.
- The set of lengths $\#\{l(I_i) \mid a_{I_i,c_i} \neq 0\}$ is bounded above, or, equivalently, finite.
- For any $k \geq 0$, $\#\{(I_i, c_i) \mid a_{I_i,c_i} \neq 0, e(I_i) > |c_i| - k\}$ is finite.

In the above, i varies over all of λ. For each α, under the above isomorphism, $\mathrm{A}(X_\alpha)$ can then be identifed with the subset of $\mathrm{A}(X)$ comprising the sums which satisfy the additional requirement that $a_{I_i,c_i} = 0$ if $i \notin \alpha$. It follows that the map $\mathrm{colim}_\alpha \mathrm{A}(X_\alpha) \to \mathrm{A}(X_\alpha)$ is injective, with image consisting of the sums which satisfy the additional requirement that there exists some $\alpha \in \lambda$ such that $a_{I_i,c_i} = 0$ for all $i \notin \alpha$. □

Remark 5.34. We saw in Proposition 5.2 that the non-equivariant homologies of the stable operads $\mathcal{E}_{\mathrm{st}}$ and $\mathcal{E}_{\mathrm{st}}^\dagger$ are simply zero (except the unit in arity 1). On the other hand, the equivariant homologies, summed up, $\oplus_n \mathrm{H}_\bullet(\mathcal{E}_{\mathrm{st}}(n)/\Sigma_n)$ in the chain case and $\oplus_n \mathrm{H}^\bullet(\mathcal{E}_{\mathrm{st}}^\dagger(n)/\Sigma_n)$ in the cochain case, yield $\mathrm{H}_\bullet(\mathbf{E}_{\mathbf{st}}\mathbb{F}_p[0])$ and $\mathrm{H}^\bullet(\mathbf{E}_{\mathbf{st}}^\dagger\mathbb{F}_p[0])$, respectively. By Proposition 5.33 above, these are exactly $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$. Thus, while the non-equivariant homologies are (almost) zero, the equivariant homologies are highly non-trivial objects. $\|$

Remark 5.35. By Proposition 5.33 above, we see that the (co)homology of a free algebra over $\mathcal{E}_{\mathrm{st}}$ or $\mathcal{E}_{\mathrm{st}}^\dagger$ is a module over $\widehat{\mathcal{S}}$ or $\widehat{\mathcal{B}}$, respectively. We saw earlier that in the unstable case we get modules over $\mathcal{S}$ and $\mathcal{B}$, and in fact, unstable such modules. One might wonder why $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$ were not seen in the unstable case. We note here that, if we impose the instability condition ($Q^I x$ is zero for $e(I) < |x|$ in the case of actions by $\mathcal{S}$ or $\widehat{\mathcal{S}}$, and $P^I x$ is zero for $e(I) > |x|$ in the case of actions by $\mathcal{B}$ and $\widehat{\mathcal{B}}$), that actions by $\mathcal{S}$ and $\widehat{\mathcal{S}}$, and those by $\mathcal{B}$ and $\widehat{\mathcal{B}}$, amount to the same thing. More precisely, an unstable action by $\mathcal{S}$, or by $\mathcal{B}$, extends uniquely to an unstable action by $\widehat{\mathcal{S}}$, or by $\widehat{\mathcal{B}}$, respectively. This is because, in any infinite sum in $\widehat{\mathcal{S}}$, the excess decreases to $-\infty$ and so, on any given x, all but finitely many terms must act by zero; similarly, in any infinite sum in $\widehat{\mathcal{B}}$, the excess increases to $+\infty$ and so, on any given x, again, all but finitely many terms must act by zero. In particular, in the case of the unstable operads, we could equivalently have said that we get unstable modules over $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$. $\|$

5.6 (Co)homology Operations for Algebras Over the Stable Operads II

In Section 5.3, we constructed (co)homology operations for algebras over the stable Barratt-Eccles operad. We did this however only for $p = 2$. In this section, we shall provide operations for all p, and moreover, we shall show that, not only do we have the operations which lie in $\mathcal{S}$ and $\mathcal{B}$, but in fact also have certain infinite sums of the Q^I and P^I, namely those which lie in $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{B}}$, respectively.

Proposition 5.36. *Given an algebra A over $\mathcal{E}_{\mathrm{st}}$, $\mathrm{H}_\bullet(A)$ is naturally an algebra over $\widehat{\mathcal{S}}$. Similarly, given an algebra A over $\mathcal{E}_{\mathrm{st}}^\dagger$, $\mathrm{H}^\bullet(A)$ is naturally an algebra over $\widehat{\mathcal{B}}$.*

We have not mentioned it explicitly in the statement of the theorem, but, in the case $p = 2$, the operations here inside $\mathcal{S}$ and $\mathcal{B}$ of course coincide with those constructed in Section 5.3.

Proof. As usual, we will just outline the case of the chain operad. The map describing the action of $\widehat{\mathcal{S}}$ is the following composite:

$$\widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(A) \xrightarrow{\cong} \mathrm{H}_\bullet(\mathbf{E}_{\mathbf{st}}A) \to \mathrm{H}_\bullet(A)$$

Here the first map is the isomorphism provided by Proposition 5.33. The second map is that which arises by applying $\mathrm{H}_\bullet(-)$ to the structure map $\alpha\colon \mathbf{E}_{\mathbf{st}}A \to A$ which describes the $\mathcal{E}_{\mathrm{st}}$-algebra structure of A. The properties required by an $\widehat{\mathcal{S}}$-action follows from those required by an $\mathcal{E}_{\mathrm{st}}$-action. For example, associativity can be derived as follows. Letting $m\colon \mathbf{E}_{\mathbf{st}}\mathbf{E}_{\mathbf{st}} \Rightarrow \mathbf{E}_{\mathbf{st}}$ denote the monadic multiplication, we have the following commutative square:

$$\begin{tikzcd} \mathbf{E}_{\mathbf{st}}\mathbf{E}_{\mathbf{st}}A \arrow[r, "\mathbf{E}_{\mathbf{st}}\alpha"] \arrow[d, "m_A"'] & \mathbf{E}_{\mathbf{st}}A \arrow[d, "\alpha"] \\ \mathbf{E}_{\mathbf{st}}A \arrow[r, "\alpha"'] & A \end{tikzcd}$$

Upon applying $\mathrm{H}_\bullet(-)$, and invoking Proposition 5.33, we get the following commutative square:

$$\begin{tikzcd} \widehat{\mathcal{S}} \otimes \widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(A) \arrow[r] \arrow[d] & \widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(A) \arrow[d] \\ \widehat{\mathcal{S}} \otimes \mathrm{H}_\bullet(A) \arrow[r] & \mathrm{H}_\bullet(A) \end{tikzcd}$$

A diagram chase now yields associativity of the $\widehat{\mathcal{S}}$-action. □

5.7 The Homotopy Coherent, or ∞-, Additivity of the Stable Operads

As our final results in this chapter, we wish to demonstrate the homotopy coherent, or ∞-, additivity of stable Barratt-Eccles operad, justifying the adjective "stable", in three successively more general forms. First, we will show that given free algebras $\mathbf{E}_{\mathbf{st}}X$, $\mathbf{E}_{\mathbf{st}}Y$ in the chain case, or $\mathbf{E}^{\dagger}_{\mathbf{st}}X$, $\mathbf{E}^{\dagger}_{\mathbf{st}}Y$ in the cochain case, the algebra coproducts $\mathbf{E}_{\mathbf{st}}X \amalg \mathbf{E}_{\mathbf{st}}Y$ and $\mathbf{E}^{\dagger}_{\mathbf{st}}X \amalg \mathbf{E}^{\dagger}_{\mathbf{st}}Y$ are naturally quasi-isomorphic to the direct sums $\mathbf{E}_{\mathbf{st}}X \oplus \mathbf{E}_{\mathbf{st}}Y$ and $\mathbf{E}^{\dagger}_{\mathbf{st}}X \oplus \mathbf{E}^{\dagger}_{\mathbf{st}}Y$, respectively. As $\mathbf{E}_{\mathbf{st}}$ and $\mathbf{E}^{\dagger}_{\mathbf{st}}$ are left adjoints as functors from dg modules to algebras, and so preserve colimits, we can also phrase this result as saying that $\mathbf{E}_{\mathbf{st}}$ and $\mathbf{E}^{\dagger}_{\mathbf{st}}$, as monads on dg modules, are homotopy additive. Next, we shall generalize this result and show that, for cofibrant algebras A and B, over either of $\mathcal{E}_{\mathrm{st}}$ and $\mathcal{E}^{\dagger}_{\mathrm{st}}$, the coproduct $A \amalg B$ is naturally quasi-isomorphic to $A \oplus B$. Here the cofibrancy is in the sense of the Quillen semi-model structures provided by Corollary 5.9. Finally, we shall generalize this one step further and show that if, given a diagram of algebras $A \leftarrow C \rightarrow B$, if A and B are cofibrant and $C \rightarrow A$ is a cofibration, then $A \amalg_C B$ is naturally quasi-isomorphic to $A \oplus_C B$.

We begin with the homotopy additivity of the monads $\mathbf{E}_{\mathbf{st}}$ and $\mathbf{E}^{\dagger}_{\mathbf{st}}$.

Proposition 5.37. *If X and Y are chain complexes, we have a natural quasi-isomorphism:*

$$\mathbf{E}_{\mathbf{st}}(X \oplus Y) \sim \mathbf{E}_{\mathbf{st}}(X) \oplus \mathbf{E}_{\mathbf{st}}(Y)$$

Similarly, if X and Y are cochain complexes, we have a natural quasi-isomorphism:

$$\mathbf{E}_{\mathbf{st}}^{\dagger}(X \oplus Y) \sim \mathbf{E}_{\mathbf{st}}^{\dagger}(X) \oplus \mathbf{E}_{\mathbf{st}}^{\dagger}(Y)$$

Proof. As usual, we shall demonstrate the case of the chain operad; the case of the cochain operad is entirely analogous. Given the chain complexes X and Y, we have a canonical map

$$\mathbf{E}_{\mathbf{st}}(X) \oplus \mathbf{E}_{\mathbf{st}}(Y) \to \mathbf{E}_{\mathbf{st}}(X \oplus Y)$$

and we claim that this map is a quasi-isomorphism. If X and Y are finite (where, as before, by a finite complex over $\mathbb{F}_p$ we mean one which is bounded and of finite dimension in each degree), Proposition 5.31 gives us a computation of the necessarily homologies via the functor A, and then the result follows by the additivity of A as per Proposition 5.30. Now fix X to be some finite complex, say X_0, and consider $\mathbf{E}_{\mathbf{st}}(X_0) \oplus \mathbf{E}_{\mathbf{st}}(-)$ and $\mathbf{E}_{\mathbf{st}}(X_0 \oplus -)$ as endofunctors on chain complexes. Due to Proposition 2.17, both of these functors preserve filtered colimits. Since any complex Y can be written as a filtered colimit of its finite subcomplexes, by naturality and exactness of filtered colimits of complexes, we have that the map $\mathbf{E}_{\mathbf{st}}(X_0) \oplus \mathbf{E}_{\mathbf{st}}(Y) \to \mathbf{E}_{\mathbf{st}}(X_0 \oplus Y)$ is a quasi-isomorphism for all Y. Now repeat this argument, fixing instead Y to be some, this time arbitrary, complex and considering the terms as functors of X, to get the desired general result. $\square$

As we noted above, Propostion 5.37 can be regarded as a statement about coproducts of free algebras. We now consider, more generally, coproducts of cofibrant algebras. As we saw in the second chapter, coproducts of cell algebras may be computed via enveloping operads, and so we shall be led to consider once more these enveloping operads. First, however, we have the following lemma.

Lemma 5.38. *For each $j \geq 2$ and each non-trivial partition $j = j_1 + \cdots + j_k$, we have that:*

$$\mathcal{E}_{\mathrm{st}}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \sim 0 \qquad \mathcal{E}_{\mathrm{st}}^{\dagger}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \sim 0$$

Here non-trivial means that the partition is not indiscrete. Moreover, by $\mathcal{E}_{\mathrm{st}}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$, we mean $\mathcal{E}_{\mathrm{st}}(j) \otimes_{\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}} (\mathbb{S}^0)^{j_1} \otimes \cdots \otimes (\mathbb{S}^0)^{j_k}$, where $\mathbb{S}^0$ is the complex $\mathbb{F}_p[0]$, and similarly in the case of the cochain operad.

Proof. By Proposition 5.37, we have that the canonical map

$$\mathbf{E}_{\mathbf{st}}(\mathbb{S}^0) \oplus \cdots \oplus \mathbf{E}_{\mathbf{st}}(\mathbb{S}^0) \to \mathbf{E}_{\mathbf{st}}(\mathbb{S}^0) \amalg \cdots \amalg \mathbf{E}_{\mathbf{st}}(\mathbb{S}^0)$$

where $\mathbb{S}^0$ denotes the complex $\mathbb{F}_p[0]$, and where we take k copies in the case of partitions of size k, is a quasi-isomorphism. The result then follows by the formula for coproducts in Proposition 2.20. $\square$

Now, given an algebra A over $\mathcal{E}_{\text{st}}$ or $\mathcal{E}_{\text{st}}^{\dagger}$, and the associated enveloping operad $\mathcal{U}^A$, recall, as in Remark 2.28, that, we have a canonical map $\mathcal{E}_{\text{st}} \to \mathcal{U}^A$, or $\mathcal{E}_{\text{st}}^{\dagger} \to \mathcal{U}^A$ in the cochain case.

Lemma 5.39. *Given a cofibrant algebra A over $\mathcal{E}_{\text{st}}$, for each $j \geq 1$ and any partition $j = j_1 + \cdots + j_k$, the canonical map*

$$\mathcal{E}_{st}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathcal{U}^A(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$$

is a quasi-isomorphism. Similarly, given a cofibrant algebra A over $\mathcal{E}_{\text{st}}^{\dagger}$, for each $j \geq 1$ and any partition $j = j_1 + \cdots + j_k$, the canonical map

$$\mathcal{E}_{st}^{\dagger}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathcal{U}^A(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$$

is a quasi-isomorphism.

Proof. As usual, we consider only the chain case, as the cochain case is analogous. Moreover, without loss of generality, we may take A to be a cell $\mathcal{E}$-algebra. Let

$$A_0 \to A_1 \to A_2 \to \cdots$$

be a cell filtration of A and fix some choices $M_1, M_2, \ldots$ for the chain complexes which appear in the attachment squares. For each $n \geq 0$, let $N_n = \oplus_{i \leq n} M_i$, where $N_0 = 0$, and let also $N = \oplus_{i \geq 0} M_i$. As per Proposition 2.32, we have that, for each $j \geq 0$, as a graded right $\mathbb{F}_p[\Sigma_j]$-module:

$$\mathcal{U}^A(j) = \bigoplus_{i \geq 0} \mathcal{E}_{\text{st}}(i+j) \otimes_{\Sigma_i} (N[1])^{\otimes i} \tag{5.40}$$

The differential on $\mathcal{U}^A(j)$, we recall, is given by the Leibniz rule, the attachment maps and the operadic composition. Moreover, for each $n \geq 0$, and again for each $j \geq 0$, as a graded right $\mathbb{F}_p[\Sigma_j]$-module:

$$\mathcal{U}^{A_n}(j) = \bigoplus_{i \geq 0} \mathcal{E}_{\text{st}}(i+j) \otimes_{\Sigma_i} (N_n[1])^{\otimes i} \tag{5.41}$$

Thus we see that the operad $\mathcal{U}^A$ is filtered by the operads $\mathcal{U}^{A_n}$. Now, as A_0 is the initial $\mathcal{E}_{\text{st}}$-algebra $\mathcal{E}_{\text{st}}(0)$, we have $\mathcal{U}^{A_0} = \mathcal{U}^{\mathcal{E}_{\text{st}}(0)} = \mathcal{E}_{\text{st}}$ (see Example 2.30). The terms of the operad $\mathcal{U}^{A_1}$ then arise from the terms of $\mathcal{U}^{A_0}$ by attachment of cells; more generally, for $n \geq 1$, the terms of the operad $\mathcal{U}^{A_n}$ arise from the terms of $\mathcal{U}^{A_{n-1}}$ by attachment of cells. Recall, as in the proof of Proposition 5.7, that this yields filtrations $\mathrm{F}_m\mathcal{U}^{A_n}$ on the $\mathcal{U}^{A_n}$.

Now, for each $j \geq 0$, the map $\mathcal{E}_{\mathrm{st}}(j) \to \mathcal{U}^A(j)$ corresponds to the inclusion of the $i = 0$ summand in (5.40), and, similarly, for each $n \geq 0$, the map $\mathcal{E}_{\mathrm{st}}(j) \to \mathcal{U}^{A_n}(j)$ corresponds to the inclusion of the $i = 0$ summand in (5.41). It follows that the map $\mathcal{E}_{\mathrm{st}}(j) \to \mathcal{U}^A(j)$ factors through $\mathcal{U}^{A_n}(j)$ for each $n \geq 0$ and, moreover, the maps $\mathcal{E}_{\mathrm{st}}(j) \to \mathcal{U}^{A_n}(j)$ factor through $\mathrm{F}_m\mathcal{U}^{A_n}(j)$ for each $m \geq 0$. We shall now prove the desired result via an induction. Specifically, we shall show that, for each $m, n \geq 0$, the map

$$\mathcal{E}_{\mathrm{st}}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathrm{F}_m\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$$

is a quasi-isomorphism for all $j \geq 1$ and any partition $j = j_1 + \cdots + j_k$. The desired result then follows by passage to colimits. We will prove this statement by an induction on n. In the case $n = 0$, we have that $\mathrm{F}_m\mathcal{U}^{A_0}(j) = \mathcal{E}_{\mathrm{st}}(j)$ for each $m \geq 0$ and $j \geq 1$, so that the result is obvious. Next suppose that, for some $n \geq 1$, the property holds for $\mathrm{F}_m\mathcal{U}^{A_{n-1}}$ holds for all $m \geq 0$. We shall show that this same property holds for $\mathrm{F}_m\mathcal{U}^{A_n}$ for $m \geq 0$, by an induction over m. We have that, for each $j \geq 1$, $\mathrm{F}_0\mathcal{U}^{A_n}(j) = \mathcal{U}^{A_{n-1}}(j) = \mathrm{colim}_m\,\mathrm{F}_m\mathcal{U}^{A_{n-1}}(j)$ which, by invoking the inductive hypothesisis for the induction over n and passing to the colimit, we see satisfies the required property, recalling the exactness of filtered colimits of complexes. Next, suppose that the required property holds for $\mathrm{F}_{m-1}\mathcal{U}^{A_n}(j)$ for some $m \geq 1$. Fix some $j \geq 1$ and a partition $j = j_1 + \cdots + j_k$. We wish to show that the map $\mathcal{E}_{\mathrm{st}}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathrm{F}_m\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ is a quasi-isomorphism. Since we can factor this map as

$$\mathcal{E}_{\mathrm{st}}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathrm{F}_{m-1}\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathrm{F}_m\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$$

it suffices, due to the inductive hypothesis for the induction over m, to show that the second map $\mathrm{F}_{m-1}\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathrm{F}_m\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ is a quasi-isomorphism. As in the proof of Proposition 5.7, we have an exact sequence

$$0 \to \mathrm{F}_{m-1}\mathcal{U}^{A_n}(j) \to \mathrm{F}_m\mathcal{U}^{A_n}(j) \to \mathrm{F}_m\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\mathcal{U}^{A_n}(j) \to 0$$

which is split as the level of graded modules at the lefthand end. As in the proof of Lemma 5.6, we thus have an induced exact sequence as follows:

$$(5.42)\quad \begin{aligned} 0 &\to \mathrm{F}_{m-1}\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \mathrm{F}_m\mathcal{U}^{A_n}(j)/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \\ &\to (\mathrm{F}_m\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\mathcal{U}^{A_n}(j))/\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to 0 \end{aligned}$$

By the long exact sequence in homology, it suffices to show that the righthand term, namely $(\mathrm{F}_m\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\mathcal{U}^{A_n}(j))/\Sigma_{j_1}\times\cdots\times\Sigma_{j_k}$, has zero homology. As in the proof of Proposition 5.7, we have that:

$$\mathrm{F}_m\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\mathcal{U}^{A_n}(j)\cong\mathcal{U}^{A_{n-1}}(m+j)\otimes_{\Sigma_m}M_n[1]^{\otimes m}$$

It follows that $(\mathrm{F}_m\mathcal{U}^{A_n}(j)/\mathrm{F}_{m-1}\mathcal{U}^{A_n}(j))/\Sigma_{j_1}\times\cdots\times\Sigma_{j_k}$ is isomorphic to:

$$\mathcal{U}^{A_{n-1}}(m+j)\otimes_{\Sigma_m\times\Sigma_{j_1}\times\cdots\times\Sigma_{j_k}}M_n[1]^{\otimes m}\otimes(\mathbb{S}^0)^{j_1}\otimes\cdots\otimes(\mathbb{S}^0)^{j_k}$$

By the inductive hypothesis for the induction over n, and by writing $M_n[1]$ as a filtered colimit of its finite subcomplexes if it isn't finite, we have that this term is isomorphic to

$$\mathcal{E}_{\mathrm{st}}(m+j)\otimes_{\Sigma_m\times\Sigma_{j_1}\times\cdots\times\Sigma_{j_k}}M_n[1]^{\otimes m}\otimes(\mathbb{S}^0)^{j_1}\otimes\cdots\otimes(\mathbb{S}^0)^{j_k}$$

and since $m+j\geq 2$, this has zero homology by Lemma 5.38. This completes the induction over m and then also the induction over n. $\square$

Proposition 5.43. *Let A and B be cofibrant algebras over $\mathcal{E}_{\mathrm{st}}$ or $\mathcal{E}_{\mathrm{st}}^{\dagger}$. Then we have a natural quasi-isomorphism:*

$$A\amalg B\sim A\oplus B$$

Proof. We shall demonstrate the case of the chain operad; the case of the cochain operad is analogous. Without loss of generality, we may take A and B to be cell $\mathcal{E}$-algebras. Let

$$A_0\to A_1\to A_2\to\cdots$$

be a cell filtration of A and fix some choices $M_1^A, M_2^A,\ldots$ for the chain complexes which appear in the attachment squares. Let also

$$B_0\to B_1\to B_2\to\cdots$$

be a cell filtration of B and fix some choices $M_1^B, M_2^B,\ldots$ for the chain complexes which appear in the attachment squares. Recall then the formulae for $\mathcal{U}^{A_n}(j)$ and $\mathcal{U}^{B_n}(j)$, for example in the proof of Lemma 5.39 above, and also their filtration pieces $\mathrm{F}_m\mathcal{U}^{A_n}(j)$ and $\mathrm{F}_m\mathcal{U}^{B_n}(j)$. It suffices to show that the canonical map

$$A_n\oplus B_n\to A_n\amalg B_n$$

is a quasi-isomorphism for each $n\geq 0$. In the case $n=0$, we get a map $\mathcal{E}_{\mathrm{st}}(0)\oplus\mathcal{E}_{\mathrm{st}}(0)\to\mathcal{E}_{\mathrm{st}}(0)\amalg\mathcal{E}_{\mathrm{st}}(0)\cong\mathcal{E}_{\mathrm{st}}(0)$ (this isomorphism holds since $\mathcal{E}_{\mathrm{st}}(0)$ is initial) and this map is necessarily a quasi-isomorphism since $\mathcal{E}_{\mathrm{st}}(0)$ has zero homology as per Proposition 5.2. Suppose then that, for some $n\geq 1$, the map $A_{n-1}\oplus B_{n-1}\to A_{n-1}\amalg B_{n-1}$ is a quasi-isomorphism. Note that A_n, B_n,

$A_n \amalg B_n$, may be identified, respectively, with $\mathcal{U}^{A_n}(0)$, $\mathcal{U}^{B_n}(0)$, $\mathcal{U}^{A_n \amalg B_n}(0)$. Moreover, we have filtration pieces $\mathrm{F}_m \mathcal{U}^{A_n}(0), \mathrm{F}_m \mathcal{U}^{B_n}(0)$ and $\mathrm{F}_m \mathcal{U}^{A_n \amalg B_n}(0)$, and the map $A_n \oplus B_n \to A_n \amalg B_n$ is a filtered map, in that we have induced maps as follows:

$$\mathrm{F}_m \mathcal{U}^{A_n}(0) \oplus \mathrm{F}_m \mathcal{U}^{B_n}(0) \to \mathrm{F}_m \mathcal{U}^{A_n \amalg B_n}(0)$$

We thus get an induced map of the strongly convergent spectral sequences associated to these filtrations. Recalling, from the proof of Proposition 5.7, the computations of the associated graded pieces corresponding to the filtrations on the $\mathcal{U}^{A_n}(j)$, we have that the map on the E^1-terms consists of the maps:

$$\begin{aligned}\Big(\mathcal{U}^{A_{n-1}}(m) \otimes_{\Sigma_m} (M_n^A[1])^{\otimes m}\Big) \oplus \Big(\mathcal{U}^{B_{n-1}}(m) \otimes_{\Sigma_m} (M_n^B[1])^{\otimes m}\Big)& \\ \longrightarrow \mathcal{U}^{A_{n-1} \amalg B_{n-1}}(m) \otimes_{\Sigma_m} (M_n^A[1] \oplus M_n^B[1])^{\otimes m}&\end{aligned}$$

If $m = 0$, this map reduces to $A_{n-1} \oplus B_{n-1} \to A_{n-1} \amalg B_{n-1}$ and so is a quasi-isomorphism by the inductive hypothesis. Suppose then that $m \geq 1$. Then, by Lemma 5.39, it suffices to show that the map

$$\begin{aligned}\Big(\mathcal{E}_{\mathrm{st}}(m) \otimes_{\Sigma_m} (M_n^A[1])^{\otimes m}\Big) \oplus \Big(\mathcal{E}_{\mathrm{st}}(m) \otimes_{\Sigma_m} (M_n^B[1])^{\otimes m}\Big)& \\ \longrightarrow \mathcal{E}_{\mathrm{st}}(m) \otimes_{\Sigma_m} (M_n^A[1] \oplus M_n^B[1])^{\otimes m}&\end{aligned}$$

is a quasi-isomorphism. If $m = 1$, this is obvious, so suppose that $m \geq 2$. We have that

$$\mathcal{E}_{\mathrm{st}}(m) \otimes_{\Sigma_m} (M_n^A[1] \oplus M_n^B[1])^{\otimes m} \cong \bigoplus_{l=0}^{m} \mathcal{E}_{\mathrm{st}}(m) \otimes_{\Sigma_{m-l} \times \Sigma_l} (M_n^A[1])^{\otimes (m-l)} \otimes (M_n^B[1])^{\otimes l}$$

By Lemma 5.38, the only two summands which have non-zero homology are those corresponding to $l = 0, m$, and so we once more have a quasi-isomorphism, as desired. It follows that the aforementioned map of spectral sequences is an isomorphism from E^2 onwards, and so we have that the map $A_n \oplus B_n \to A_n \amalg B_n$ is a quasi-isomorphism, completing the induction. $\square$

Next, we provide a relative form of Proposition 5.43, which is to say, we consider pushouts of algebras. The claim will be that, for algebras A, B and C over $\mathcal{E}_{\mathrm{st}}$ or $\mathcal{E}_{\mathrm{st}}^{\dagger}$, under suitable circumstances, we have that $A \amalg_C B \sim A \oplus_C B$. We shall compute the pushout via a bar construction. Given a diagram $A \leftarrow C \to B$ of algebras over $\mathcal{E}_{\mathrm{st}}$ or $\mathcal{E}_{\mathrm{st}}^{\dagger}$, the bar construction

$\beta_\bullet(A, C, B)$ is the simplicial $\mathcal{E}_{\text{st}}$-algebra that is given in simplicial degree n as follows:

$$\beta_n(A, C, B) := A \amalg \underbrace{C \amalg \cdots \amalg C}_{n \text{ factors}} \amalg B$$

Given this simplicial algebra, we can consider its normalization $\mathrm{N}(\beta_\bullet(A, C, B))$, which is once again an algebra over $\mathcal{E}_{\text{st}}$ or $\mathcal{E}_{\text{st}}^\dagger$ via the shuffle map (see [KM95] for the normalization of a simplicial algebra and its algebra structure). Now, regarding $A \amalg_C B$ as a constant simplicial algebra, the canonical maps $A \amalg B \to A \amalg_C B$ and $C \to A \amalg_C B$ induce a map of simplicial algebras $\beta_\bullet(A, C, B) \to A \amalg_C B$ and therefore a map of algebras on their normalizations, $\mathrm{N}(\beta_\bullet(A, C, B)) \to A \amalg_C B$.

Proposition 5.44. *Given a diagram $A \leftarrow C \to B$ of algebras over $\mathcal{E}_{st}$ or $\mathcal{E}_{st}^\dagger$, if each of A and B and C are cofibrant and $C \to B$ is a cofibration, then the canonical map*

$$\mathrm{N}(\beta_\bullet(A, C, B)) \to A \amalg_C B$$

is a quasi-isomorphism.

In order to prove this result, we first need two lemmas.

Lemma 5.45. *Given a diagram $A \leftarrow C \to B$ of algebras over $\mathcal{E}_{\text{st}}$ or $\mathcal{E}_{\text{st}}^\dagger$, if each of A, B and C is cofibrant, the normalization $\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B)}(j))$ is finitely flat as a dg right $\mathbb{F}_p[\Sigma_j]$-module.*

Proof. Given a dg left $\mathbb{F}_p[\Sigma_j]$-module X, we have a natural isomorphism:

$$\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B)}(j)) \otimes_{\Sigma_j} X \cong \mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B)}(j) \otimes_{\Sigma_j} X)$$

The required flatness now follows from Proposition 5.7. □

Lemma 5.46. *Let $n \geq 0$, P and Q be dg right $\mathbb{F}_p[\Sigma_n]$-modules which are finitely flat, and let $P \to Q$ be a quasi-isomorphism. Then, for all dg $\mathbb{F}_p$-modules X, the induced map*

$$P \otimes_{\Sigma_n} X^{\otimes n} \to Q \otimes_{\Sigma_n} X^{\otimes n}$$

is a quasi-isomorphism.

Proof. We shall show this for finite X; the case of a general X then follows since any X is a filtered colimit of its finite subcomplexes and filtered colimits commute with finite tensor powers and tensor products. In fact, we will show that, for any finite dg right $\mathbb{F}_p[\Sigma_n]$-module Z, the natural map

$$P \otimes_{\Sigma_n} Z \to Q \otimes_{\Sigma_n} Z$$

is a quasi-isomorphism. To see this, let $Z_{\text{cof}} \to Z$ be a cofibrant approximation of Z, in the projective Quillen model structure on dg right $\mathbb{F}_p[\Sigma_n]$-modules, and then consider the following commutative square:

$$\begin{tikzcd}
P \otimes_{\Sigma_n} Z \arrow[r] & Q \otimes_{\Sigma_n} Z \\
P \otimes_{\Sigma_n} Z_{\text{cof}} \arrow[r] \arrow[u] & Q \otimes_{\Sigma_n} Z_{\text{cof}} \arrow[u]
\end{tikzcd}$$

The bottom horizontal map is then a map

$$P \otimes^{\mathbb{L}}_{\Sigma_n} Z \to Q \otimes^{\mathbb{L}}_{\Sigma_n} Z$$

between the derived tensor products. To be precise, the derived functors are those of the functors $P \otimes_{\Sigma_n} -$ and $Q \otimes_{\Sigma_n} -$. Since, however, the two possible derived tensor products, achieved upon fixing one or the other variable, are naturally isomorphic, the above map can be identified with the image of $P \to Q$ under the derived functor of $- \otimes^{\mathbb{L}}_{\Sigma_n} Z$. As such, as $P \to Q$ is a quasi-isomorphism, the above map is also quasi-isomorphism. Now, if we can show that the vertical maps are also quasi-isomorphisms, we will have the desired result. The proofs for the two are identical, so we will describe just the one for the lefthand vertical map. Also, we shall consider the case of chain complexes; the case of cochain complexes is analogous. Since Z is bounded below, we can take Z_{cof} to also be bounded below. Suppose that Z is bounded above at degree d, in that $Z_{d'}$ is zero for $d' > d$. Consider the truncation $\tau_{\leq d+1} Z_{\text{cof}}$ which is to say the complex which coincides with Z_{cof} up to, and including, degree $d+1$, but is zero thereafter. More generally, we consider the truncations $\tau_{\leq d+i} Z_{\text{cof}}$ for $i \geq 1$. Then we can write Z_{cof} as the colimit of:

$$\tau_{d+1} Z_{\text{cof}} \to \tau_{d+2} Z_{\text{cof}} \to \cdots$$

Moreover, since Z is zero above degree d, we have maps $\tau_{\leq d+i} Z_{\text{cof}} \to Z$, each of which is a quasi-isomorphism. Thus we get the following diagram:

$$\begin{tikzcd}
\tau_{\leq d+1} Z_{\text{cof}} \arrow[r] \arrow[d, "\sim"] & \tau_{\leq d+2} Z_{\text{cof}} \arrow[r] \arrow[d, "\sim"] & \tau_{\leq d+3} Z_{\text{cof}} \arrow[r] \arrow[d, "\sim"] & \cdots \\
Z \arrow[r] & Z \arrow[r] & Z \arrow[r] & \cdots
\end{tikzcd}$$

If we tensor this diagram with P, since P is finitely flat, the vertical arrows remain quasi-isomorphisms. Moreover, the map induced on the colimits of the two cotowers in the resulting diagram is exactly the arrow $P \otimes_{\Sigma_n} Z_{\mathrm{cof}} \to P \otimes_{\Sigma_n} Z$, so that we have our desired result, as sequential colimits are exact. $\square$

Proof of Proposition 5.44. As usual, we shall prove only the case of the chain operad, the case of the cochain operad being analogous. Moreover, we may assume without loss of generality that A is a cell algebra and that $C \to B$ is a relative cell map. We shall in fact prove the more general fact that the map

$$\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B)}(j)) \to \mathcal{U}^{A \amalg_C B}(j)$$

is a quasi-isomorphism for each $j \geq 0$. The desired result is the case $j = 0$. Let

$$B_0 \to B_1 \to B_2 \to \cdots$$

be a factorization of $C \to B$ as a relative cell map, so that $B_0 = C$, and fix some choices $M_1, M_2, \ldots$ for the chain complexes which appear in the attachment squares. By passage to colimits, it suffices to show that, for all $n \geq 0$, the map

$$\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B_n)}(j)) \to \mathcal{U}^{A \amalg_C B_n}(j)$$

is a quasi-isomorphism for each $j \geq 0$. Now, in the case $n = 0$, we get the map $\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,C)}(j)) \to \mathcal{U}^{A}(j)$. By a standard argument, the map of simplicial algebras $\beta_\bullet(A,C,C) \to A$ is a homotopy equivalence, and so, upon forming the arity j parts of the enveloping operads, we have that the map $\mathcal{U}^{\beta_\bullet(A,C,C)}(j) \to \mathcal{U}^{A}(j)$ is a homotopy equivalence of simplicial dg right $\mathbb{F}_p[\Sigma_j]$-modules and, as simplicial homotopies induce chain homotopies on normalizations, upon taking normalizations, we get a chain homotopy equivalence of dg right $\mathbb{F}_p[\Sigma_j]$-modules. In particular, the map is a quasi-isomorphism, as desired. Now suppose that, for some $n \geq 1$, the map $\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B_{n-1})}(j)) \to \mathcal{U}^{A \amalg_C B_{n-1}}(j)$ is a quasi-isomorphism for each $j \geq 0$. Recall the filtrations on the enveloping operads of cell algebras, for example, as in the proof of Lemma 5.39. The simplicial map $\mathcal{U}^{\beta_\bullet(A,C,B_n)}(j) \to \mathcal{U}^{A \amalg_C B_n}(j)$ is in fact a filtered map, in that, for each $m \geq 0$, we have an induced map $\mathrm{F}_m\mathcal{U}^{\beta_\bullet(A,C,B_n)}(j) \to \mathrm{F}_m\mathcal{U}^{A \amalg_C B_n}(j)$. We now take the normalization and consider the induced map on the strongly convergent spectral sequences associated to these filtrations. Recalling, from the proof of Proposition 5.7, the computations of the associated graded pieces corresponding to the filtrations on the enveloping operads, we have that the map on the E^1-terms consists of the following maps:

$$\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B_{n-1})}(m+j)) \otimes_{\Sigma_m} M_n[1]^{\otimes m} \to \mathcal{U}^{A \amalg_C B_{n-1}}(m+j) \otimes_{\Sigma_m} M_n[1]^{\otimes m}$$

By Lemma 5.45, Proposition 5.7, the inductive hypothesis and Lemma 5.46, this map is a quasi-isomorphism for all $m \geq 0$. It follows that the map of spectral sequences is an isomorphism from E^2 onwards, and so the map $\mathrm{N}(\mathcal{U}^{\beta_\bullet(A,C,B_n)}(j)) \to \mathcal{U}^{A \amalg_C B_n}(j)$ is a quasi-isomorphism. This completes the induction. $\square$

Finally, with the help of the computation of the pushout in Proposition 5.44, we can now prove the desired result.

Proposition 5.47. *Given a diagram $A \leftarrow C \to B$ of algebras over $\mathcal{E}_{st}$ or $\mathcal{E}_{st}^{\dagger}$, if each of A, B and C are cofibrant, and $C \to B$ is a cofibration, then we have that:*

$$A \amalg_C B \sim A \oplus_C B$$

Proof. Let $\beta_\bullet^{\mathrm{dg}}(A, C, B)$ denote the bar construction in dg modules, so that, in simplicial degree n, we have:

$$\beta_n^{\mathrm{dg}}(A, C, B) := A \oplus \underbrace{C \oplus \cdots \oplus C}_{n \text{ factors}} \oplus B$$

Then we have a composite quasi-isomorphism:

$$\mathrm{N}(\beta_\bullet^{\mathrm{dg}}(A, C, B)) \xrightarrow{\sim} \mathrm{N}(\beta_\bullet(A, C, B)) \xrightarrow{\sim} A \amalg_C B$$

Here the first map is a quasi-isomorphism by Proposition 5.43, and the second is a quasi-isomorphism by Proposition 5.44. Moreover, since cofibrations of algebras, being retracts of relative cell maps, are necessarily cofibrations of complexes (in the standard projective Quillen model structure on complexes), we have a natural quasi-isomorphism $\mathrm{N}(\beta_\bullet^{\mathrm{dg}}(A, C, B)) \sim A \oplus_C B$, and this gives us the desired result. $\square$

CHAPTER 6

Algebraic Models of p-Adic Stable Homotopy Types

In this chapter, we develop an application of the stable operads to p-adic stable homotopy theory. As in the previous chapter, we will fix the stable Barratt-Eccles operad $\mathcal{E}_{\text{st}}$ as a model for the stable operad; all that we say, however, also applies to the stable McClure-Smith operad. We shall see that cochains on spectra, appropriately defined, yield algebras over the stable Barratt-Eccles operad and moreover that, endowed with this structure, the cochains yield algebraic models for p-adic stable homotopy types, where p here is a fixed but unspecified prime, as in previous chapters.

6.1 Spectra and Their Model Structure

We begin by fixing what it is that we mean by a spectrum. We adopt the following definition.

Definition 6.1. A *spectrum* E is a sequence of based simplicial sets $E_0, E_1, E_2, \ldots$ together with a collection of maps $\rho_n \colon \Sigma E_n \to E_{n+1}$, where the suspension is the Kan suspension, or equivalently, maps $\sigma_n \colon E_n \to \Omega E_{n+1}$, where the loop space is the Moore loop space. A *map of spectra* $f \colon E \to F$ is given by a collection of maps $f_n \colon E_n \to F_n$ which are compatible with the structure maps.

We thus have a category of spectra, and we denote this category by Sp.

Definition 6.2. We set the following:

- Given a spectrum E, for $i \in \mathbb{Z}$, the i^{th} *stable homotopy group* of E is the colimit $\pi_i^{\text{st}}(E) := \text{colim}_{k \geq 0} \pi_{i+k}(|E_k|)$; here, given $k \geq 0$, the map $\pi_{i+k}(|E_k|) \to \pi_{i+k+1}(|E_{k+1}|)$ is that which sends the class of a based map $\mathbb{S}^{i+k}_{\text{top}} \to |E_k|$ to the class of the composite $\mathbb{S}^{i+k+1}_{\text{top}} \cong \Sigma \mathbb{S}^{i+k}_{\text{top}} \to |\Sigma E_k| \to |E_{k+1}|$ where we use the fact that, upon geometric realization, the Kan suspension can be identified, up to natural isomorphism, with the usual topological suspension, as shown for example in Proposition 2.16 in [Ste15].

- Given a map $f \colon E \to F$ of spectra, we call it a *stable weak homotopy equivalence* if the induced maps $\pi_i^{\text{st}}(E) \to \pi_i^{\text{st}}(F)$ are isomorphisms for each $i \in \mathbb{Z}$.

Proposition 6.3. *On* Sp*, there is a model structure such that:*

(i) A map $f\colon E \to F$ *is a weak equivalence if and only if it is a stable weak homotopy equivalence.*

(ii) A map $f\colon E \to F$ *is a cofibration if and only if the map* $f_0\colon E_0 \to F_0$ *and the maps* $E_{n+1} \amalg_{\Sigma E_n} \Sigma F_n \to F_{n+1}$*, for* $n \geq 0$*, are cofibrations of based simplicial sets; they are, in particular, levelwise cofibrations.*

(iii) A map $f\colon E \to F$ *is a fibration if and only if it has the right lifting property with respect to maps which are both cofibrations and weak equivalences; they are, in particular, levelwise fibrations.*

Proof. All except that the cofibrations are levelwise monomorphisms is immediate from Theorem 2.29 in [Ste15]. To see that cofibrations are monomorphisms, let $f\colon E \to F$ be a cofibration of spectra E and F. By definition, f_0 is a monomorphism, and moreover, so is $E_1 \amalg_{\Sigma E_0} \Sigma F_0 \to F_1$. Consider the following pushout square:

$$\begin{tikzcd}
\Sigma E_0 \arrow[r, "\Sigma f_0"] \arrow[d, "\rho_0"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Sigma F_0 \arrow[d] \\
E_1 \arrow[r, "i"'] & E_1 \amalg_{\Sigma E_0} \Sigma F_0
\end{tikzcd}$$

By Proposition 4.11, Σf_0 is a monomorphism. Since in simplicial sets, pushouts of monomorphisms are once again monomorphisms, we see that the map i is a monomorphism, and thus so is the composite $E_1 \to E_1 \amalg_{\Sigma E_0} \Sigma F_0 \to F_1$, which is exactly f_1. Repeating this argument, we see by induction that each $f_n\colon E_n \to F_n$ is a monomorphism. $\square$

Remark 6.4. As per [BF78, §2.5] and [Ste15], there is a Quillen equivalence

$$\mathsf{Sp}\colon \mathsf{Sp} \rightleftarrows \mathsf{CSp}\colon \mathrm{Ps}$$

between our category of spectra and CSp, the category of Kan's combinatorial spectra, equipped with the model structure of Brown in [Bro73]. $||$

Definition 6.5. We set the following:

- Given a spectrum E, say that it is an Ω-*spectrum* if each E_n is a Kan complex and the maps σ_n are weak homotopy equivalences of based simplicial sets.

- Given a spectrum E, say that it is a *strict Ω-spectrum* if each E_n is a Kan complex and the maps σ_n are isomorphisms of based simplicial sets.

- Given a spectrum E, say that it is *Σ-cofibrant* if the maps ρ_n are cofibrations of based simplicial sets.

Remark 6.6. Given an Ω-spectrum E, we clearly have that $\pi_i^{\text{st}}(E) \cong \pi_i(E_0)$ for $i \geq 0$, and $\pi_i^{\text{st}}(E) \cong \pi_0(E_{|i|})$ for $i < 0$. $||$

Proposition 6.7. *We have the following:*

(i) The fibrant spectra are exactly the Ω-spectra. The cofibrant spectra are exactly the Σ-cofibrant spectra. The strict Ω-spectra are bifibrant.

(ii) A map between Ω-spectra is a weak equivalence if and only if it is a levelwise weak homotopy equivalence of based simplicial sets.

(iii) A map between strict Ω-spectra is a fibration if and only if it is levelwise fibration of based simplicial sets.

Proof. (i): The case of fibrant objects follows from Theorem 2.29 in [Ste15]. For the cofibrant objects, note that a map $f\colon E' \to E$ is a cofibration if $E_0' \to E_0$ and $E'_{n+1} \coprod_{\Sigma E'_n} \Sigma E_n \to E_{n+1}$ are cofibrations. Taking the E'_n to be $*$, we are left with the map $* \to E_0$ and the maps $\Sigma E_n \to E_{n+1}$. The former is of course always a cofibration of based simplicial sets. Now consider the case of strict Ω-spectra. If each map $E_n \to \Omega E_{n+1}$ is an isomorphism, the maps $\Sigma E_n \to E_{n+1}$ are monomorphisms as they may be written as $\Sigma E_n \to \Sigma\Omega E_{n+1} \to E_{n+1}$ where the first map is an isomorphism and the second is a monomorphism by part (iii) of Proposition 4.19.

(ii): See [Ste15].

(iii): Let $E \to F$ be a levelwise fibration between strict Ω-spectra E and F. Consider the adjunction between spectra and Kan's combinatorial spectra in Remark 6.4 above. It is immediate from the definitions (in, e.g., [Ste15]) that if E is a strict Ω-spectra, then the unit of adjunction $E \to \mathrm{Ps}\,\mathrm{Sp}E$ is an isomorphism. Thus the induced map $\mathrm{Ps}\,\mathrm{Sp}E$ is a levelwise fibration. By Proposition 3.18 in [Ste15], given a map f between combinatorial spectra, $\mathrm{Ps}(f)$ is a levelwise fibration if and only if it is a fibration. Thus $\mathrm{Ps}\,\mathrm{Sp}E$ is a fibration. It follows that the map $E \to F$ is itself a fibration, as desired. $\square$

Next, we give a few examples of spectra. The first is that of suspension spectra, which are spectra freely generated on a space.

Definition 6.8. Given a based simplicial set S, the *suspension spectrum* $\Sigma^\infty S$ is defined by setting $(\Sigma^\infty S)_n = \Sigma^n S$, and the structure maps, in the form via suspensions, are identities. More generally, for each $n \geq 0$, we define $\Sigma^{\infty-n}X$ by setting $(\Sigma^{\infty-n}X)_m = \Sigma^{m-n}X$ for $m \geq n$, and $*$ for $m < n$ and the structure maps to be the obvious ones in suspension form. If $n = 0$, we recover $\Sigma^\infty S$.

Remark 6.9. Given a based simplicial set X, the structure maps $\rho_n \colon \Sigma(\Sigma^\infty X)_n \to (\Sigma^\infty X)_{n+1}$ for the corresponding suspension spectrum Σ^∞ are identities. It follows that the structure maps $\sigma_n \colon (\Sigma^\infty X)_n \to \Omega(\Sigma^\infty)_{n+1X}$ are components of the unit of the (Σ, Ω)-adjunction and so are, by Proposition 4.19, isomorphisms. The component simplicial sets $\Sigma^n X$, however, are not necessarily Kan complexes, even if X is (e.g., consider the case of the 0-sphere), and so the suspension spectrum is not necessarily a strict Ω-spectrum, or an Ω-spectrum at all for that matter. ||

The second example is that of Eilenberg-MacLane spectra, associated to abelian groups. Let A be any abelian group. Then we set the Eilenberg-MacLane space $\mathrm{K}(A, n)$, for each $n \geq 0$, as a simplicial set, to be the simplicial set whose d-simplices are given by the cocycles $\mathrm{Z}^n(\Delta_d; A)$. Let us now consider the simplicial sets $\mathrm{K}(A, n)$ as a based simplicial set, with zero as the basepoint. We claim that the based simplicial sets $\mathrm{K}(A, 0), \mathrm{K}(A, 1), \mathrm{K}(A, 2), \dots$ assemble into a spectrum. To see this, we need structure maps:

$$\mathrm{K}(A, n) \to \Omega\mathrm{K}(A, n+1)$$

These are as follows. Let α be an n-cocycle on Δ_d. Then, we may act on the chains on Δ_{d+1} by stipulating that, given a simplex $[n+1] \to [d+1]$ in Δ_{d+1}, if no entry maps to zero, we send it to zero, or if exactly one entry maps to zero, we drop 0 from both the source and target and then reindex to get a map $[n] \to [d]$, a simplex in Δ_d, and then act by α. (We needn't concern ourselves with the case of maps $[n+1] \to [d+1]$ which send more than one entry to zero as those yield degenerate simplices of Δ_{d+1}.) An easy check shows that this action on chains defines an $(n+1)$-cocycle β on Δ_{d+1},that this cocycle lies in $\Omega\mathrm{K}(A, n+1)$ and moreover that the assignment $\alpha \mapsto \beta$ yields a map of based simplicial sets $\mathrm{K}(A, n) \to \Omega\mathrm{K}(A, n+1)$, as desired.

Definition 6.10. Given an abelian group A, the *Eilenberg-MacLane spectrum* $\mathrm{H}A$ is the spectrum where $(\mathrm{H}A)_n = \mathrm{K}(A, n)$ with the structure maps as above.

Remark 6.11. Combining Proposition 6.12 below and Remark 6.6 above, we have that $\pi_0^{\mathrm{st}}(\mathrm{H}A) \cong \pi_0(\mathrm{K}(A, 0)) \cong A$, whereas $\pi_i^{\mathrm{st}}(\mathrm{H}A) \cong \pi_i(\mathrm{K}(A, 0)) \cong *$ for $i > 0$ and $\pi_i^{\mathrm{st}}(\mathrm{H}A) \cong \pi_0(\mathrm{K}(A, |i|)) \cong *$ for $i < 0$. ||

Proposition 6.12. *Given any abelian group A, the Eilenberg-MacLane spectrum $\mathrm{H}A$ is a strict Ω-spectrum, and so is bifibrant.*

Proof. We only need to verify that $\mathrm{H}A$ is a strict Ω-spectrum as the bifibrancy then follows by Proposition 6.7. That each $(\mathrm{H}A)_n$ is a Kan complex follows by the fact that it is the underlying simplicial set of a simplicial group. It then remains to show that the structure maps $\mathrm{K}(A, n) \to \Omega\mathrm{K}(A, n+1)$ above are isomorphisms. First, note that the graded pieces are in fact abelian groups and that the structure maps are clearly maps of abelian groups. Thus, for injectivity, we can simply check that only zero maps to zero. Consider some n-cocycle on Δ_p which maps to the zero $(n+1)$-cocycle on Δ_{p+1}. Given $[n] \to [p]$, abut $0 \mapsto 0$. On the resulting map $[n+1] \to [p+1]$ the image cocycle acts by the original one. Thus the original one must be zero. This demonstrates injectivity. Now we show surjectivity. Consider some $(n+1)$-cocycle on Δ_{p+1}. We define an n-cocycle on Δ_p as follows: given any $[n] \to [p]$, abut $0 \mapsto 0$ and then act by the given cocycle. This is indeed a cocycle: given $[n+1] \to [p]$, if we take faces then abut $0 \mapsto 0$ we get the same as first abutting to $[n+2] \to [p+1]$ and then taking the faces d_i for $1 \leq i \leq n+2$, so that we must be getting the same final result as if we acted upon d_0 of the abutment to $[n+2] \to [p+1]$ by the original cocycle, but then this will map nothing to 0 (since in the abutment only 0 mapped to 0), and thus the final result will be zero, as desired. Now we note that our given $(n+1)$-cocycle on Δ_{p+1} is exactly the image of this newly constructed n-cocycle on Δ_p: it certainly is if exactly one entry maps to 0; moreover, if no entry maps to 0, it must be mapped to 0 by our cocycle due to the $d_0 = *$ condition, and we can ignore the cases where more than one entry maps to zero since we are taking normalized cochains. $\square$

Finally, we consider shifts of Eilenberg-MacLane spectra. Let A be any abelian group. Then, as above, we have the Eilenberg-MacLane spectrum $\mathrm{H}A$. Given any $k \in \mathbb{Z}$, we define a spectrum $\Sigma^k\mathrm{H}A$, where if $k = 0$, the construction will recover $\mathrm{H}A$.

Definition 6.13. Given an abelian group A and $k \in \mathbb{Z}$, the *shifted Eilenberg-MacLane spectrum* $\Sigma^k\mathrm{H}A$ is the spectrum where $(\Sigma^k\mathrm{H}A)_n = \mathrm{K}(A, k+n)$.

Here, for $n < 0$, we interpret $\mathrm{K}(A, n)$ to be $*$, by which we mean the based Δ_0. A unifying formula which holds in all cases is achieved by setting $(\Sigma^k\mathrm{H}A)_{n,d} = \mathrm{Z}^{k+n}(\Delta_d; A)$.

Remark 6.14. Case by case considerations similar to those in Remark 6.11 show that $\pi_k^{\mathrm{st}}(\Sigma^k\mathrm{H}A) \cong \pi_k(\mathrm{K}(A, k)) \cong A$ whereas $\pi_i^{\mathrm{st}}(\Sigma^k\mathrm{H}A) \cong *$ for $i \neq k$. $||$

The structure maps for shifted Eilenberg-MacLane spectra are as in the case of the (unshifted) Eilenberg-MacLane spectra. As a result, the following result follows immediately from Proposition 6.12 above.

Proposition 6.15. *Given any abelian group A and $k \in \mathbb{Z}$, the shifted Eilenberg-MacLane spectrum $\Sigma^k\mathrm{H}A$ is a strict Ω-spectrum, and so is bifibrant.* $\square$

6.2 Spectral Cochains as $\mathcal{E}_{\mathrm{st}}^{\dagger}$-Algebras and the Relation Between $\widehat{\mathcal{B}}$ and $\mathcal{A}$

Our goal in this chapter is to construct algebraic models for p-adic stable homotopy types. In Chapter 3, via Proposition 3.57, we found algebraic models for p-adic unstable homotopy types, and the algebraic objects were constructed via cochains. We thus wish to define notions of spectral chains and spectral cochains for our model of spectra.

Let E be a spectrum, with structure maps $\rho_n \colon \Sigma E_n \to E_{n+1}$. If we apply the mod p chains functor, we get maps $\mathrm{C}_\bullet(E_n; \mathbb{F}_p)[1] \to \mathrm{C}_\bullet(\Sigma E_n; \mathbb{F}_p) \to \mathrm{C}_\bullet(E_{n+1}; \mathbb{F}_p)$ where the first map is that which is in Proposition 4.15. Equivalently, we have a map:

$$\mathrm{C}_\bullet(E_n; \mathbb{F}_p) \to \mathrm{C}_\bullet(E_{n+1}; \mathbb{F}_p)[-1]$$

Moreover, upon applying the dualization operator $(-)^\vee$ from the second chapter, we get maps $\mathrm{C}_\bullet(E_{n+1}; \mathbb{F}_p)^\vee[1] = \mathrm{C}_\bullet(E_{n+1}; \mathbb{F}_p)[-1]^\vee \to \mathrm{C}_\bullet(E_n; \mathbb{F}_p)^\vee$, and so, upon applying the reindexing operator $(-)^\dagger$ from the second chapter, and moving the shift from the source to the target, we get maps:

$$\mathrm{C}^\bullet(E_{n+1}; \mathbb{F}_p)[-1] \to \mathrm{C}^\bullet(E_n; \mathbb{F}_p)$$

We now define our notion of spectral chains and cochains.

Definition 6.16. Let E be a spectrum. The *chains on E with coefficients in $\mathbb{F}_p$*, denoted $\mathrm{C}_\bullet(E; \mathbb{F}_p)$, are defined as follows:

$$\mathrm{C}_\bullet(E; \mathbb{F}_p) := \mathrm{colim}(\mathrm{C}_\bullet(E_0; \mathbb{F}_p) \to \mathrm{C}_\bullet(E_1; \mathbb{F}_p)[-1] \to \mathrm{C}_\bullet(E_2; \mathbb{F}_p)[-2] \to \cdots)$$

The *cochains on E with coefficients in $\mathbb{F}_p$*, denoted $\mathrm{C}^\bullet(E; \mathbb{F}_p)$, are defined as follows:

$$\mathrm{C}^\bullet(E; \mathbb{F}_p) := \lim(\cdots \to \mathrm{C}^\bullet(E_2; \mathbb{F}_p)[-2] \to \mathrm{C}^\bullet(E_1; \mathbb{F}_p)[-1] \to \mathrm{C}^\bullet(E_0; \mathbb{F}_p))$$

Remark 6.17. We can be describe the spectral (co)chains more explicitly. Let E be a spectrum, with structure maps $\rho_n \colon \Sigma E_n \to E_{n+1}$. Given a simplex d-simplex x in E_n, we have a corresponding $(d+1)$-simplex Σx in ΣE_n (the notation Σx here is as in Definition 4.12), and thus, upon applying ρ, a $(d+1)$-simplex $\rho(\Sigma x) \in E_{n+1}$. Pictorially:

$$x \in (E_n)_d \rightsquigarrow \Sigma x \in (\Sigma E_n)_{d+1} \rightsquigarrow \rho(\Sigma x) \in (E_{n+1})_{d+1}$$

We then clearly have that:

$$\mathrm{C}_\bullet(E;\mathbb{F}_p)=\bigoplus_{n\geq 0}\mathrm{C}_\bullet(E_n;\mathbb{F}_p)[-n]\Big/(x-\rho(\Sigma x))$$

To be even more explicit, an easy check, using Proposition 4.14, shows that $\mathrm{C}_d(E;\mathbb{F}_p)$ is the free $\mathbb{F}_p$-module on $\amalg_{e-n=d}(E_n)_e$, modulo the submodule generated by the basepoint, the degenerate simplices and the terms $x-\rho(\Sigma x)$ where $x\in(E_n)_e$ for some n,e such that $e-n=d$ and so then $\rho(\Sigma x)\in(E_{n+1})_{e+1}$, and we also have that $(e+1)-(n+1)=d$. To keep the bookkeeping precise, in the future, given an element $x\in\amalg_{e-n=d}(E_n)_e$, we shall let $[n,e,x]$ denote the corresponding element of $\mathrm{C}_d(E;\mathbb{F}_p)$. Moreover, on the cochains: an easy degreewise check shows that the internal hom complex functor $\mathrm{F}(-,\mathbb{F}_p[0])$ converts the colimit appearing in the definition of the chains into a limit, and it then follows that the cochains are exactly the cochain complex formed by application of $(-)^\dagger\circ(-)^\vee$ to the chains, where $(-)^\vee$ and $(-)^\dagger$ are as in the second chapter. ||

Remark 6.18. We can relate the (co)chains to those on combinatorial spectra. We have that the (co)chains on E as defined above are exactly the (co)chains, in the usual sense, on the associated combinatorial spectrum $\mathrm{Ps}(E)$, where Ps is as in Remark 6.4. ||

Now, we wish to show that the spectral cochains defined above yield algebras over the stable Barratt-Eccles cochain operad $\mathcal{E}_{\mathrm{st}}^\dagger$. We first show that chains on spectra naturally form coalgebras over the stable Eilenberg-Zilber chain operad $\mathcal{Z}_{\mathrm{st}}$.

Proposition 6.19. *Let E be a spectrum. The chains $\mathrm{C}_\bullet(E;\mathbb{F}_p)$ naturally form a coalgebra over the stable Eilenberg-Zilber chain operad $\mathcal{Z}_{\mathrm{st}}$.*

Proof. We wish to produce a coaction of the stable Eilenberg-Zilber operad on the chains $\mathrm{C}_\bullet(E)$. Fix $k\geq 0$. We then want a map:

$$\mu\colon\ \mathcal{Z}_{\mathrm{st}}(k)\otimes\mathrm{C}_\bullet(E)\to\mathrm{C}_\bullet(E)^{\otimes k}$$

To do so, we will construct a bilinear map $\bar{\mu}\colon\ \mathcal{Z}_{\mathrm{st}}(k)\times\mathrm{C}_\bullet(E)\to\mathrm{C}_\bullet(E)^{\otimes k}$ which preserves the degrees and then check that the resulting map on the tensor product commutes with the differentials. We first produce a map $\bar{\bar{\mu}}\colon\ \mathcal{Z}_{\mathrm{st}}(k)\times\mathrm{S}_\bullet(E)\to\mathrm{C}_\bullet(E)^{\otimes k}$ where $\mathrm{S}_\bullet(E)=\mathbb{F}_p[\amalg_{n,d}(E_n)_d]$ (of which $\mathrm{C}_\bullet(E)$ is a quotient, as per Remark 6.17). Let $\alpha=(\alpha_0,\alpha_1,\dots)\in\mathcal{Z}_{\mathrm{st}}(k)$ be of degree d and let $x\in(E_n)_e$, which is of degree $e-n$ in $\mathrm{S}_\bullet(E)$. We set $\bar{\bar{\mu}}(\alpha,x):=\alpha_n(x)$; or more precisely, we set $\bar{\bar{\mu}}(\alpha,x)$ to be the image of x under the composite:

$$\mathrm{S}_\bullet(E_n)\longrightarrow\mathrm{C}_\bullet(E_n)\xrightarrow{\alpha_n}\mathrm{C}_\bullet(E_n)^{\otimes k}\longrightarrow\mathrm{C}_\bullet(E)^{\otimes k}$$

(Here $\mathrm{S}_\bullet(E_n) = \mathbb{F}_p[\amalg_e (E_n)_e]$.) Next we extend this definition to all of $\mathcal{Z}_{\mathrm{st}}(k) \times \mathrm{S}_\bullet(E)$ by linearity in the second variable, recalling that $\mathrm{S}_\bullet(E)$ is the free $\mathbb{F}_p$-module on $\coprod_{n,e}(E_n)_e$. The map is then clearly bilinear in both variables. We need to check that $\bar{\bar{\mu}}(\alpha, e)$ is of degree $d+e-n$. Recall that α_n is of degree $d + n(k-1)$. As an element of $\mathrm{S}_\bullet(E_n)$, x has degree e, and then, after application of the first map in the composite above, degree e once more. Upon application of α_n, we get an element of degree $e + d + n(k-1) = d + e - n + nk$, and then, since the final map reduces degree by n in each tensor factor, we get an element of degree $d + e - n + nk - nk = d + e - n$ as desired. (More precisely: prior to application of the final map, we have a sum of terms of the form $t_1 \otimes \cdots \otimes t_k \in (E_n)_{d_1} \otimes \cdots \otimes (E_n)_{d_k}$ where $d_1 + \cdots + d_k = d + e - n + nk$, and then after application of the final map, such terms have degree $(d_1 - n) + \cdots + (d_k - n) = (d_1 + \cdots + d_k) - nk = d + e - n + nk - nk = d + e - n$.) Thus $\bar{\bar{\mu}}(\alpha, x)$ lies in $(\mathrm{C}_\bullet(E)^{\otimes k})_{d+e-n}$, as desired.

Now we show that our map $\mathcal{Z}_{\mathrm{st}}(k) \times \mathrm{S}_\bullet(E) \to \mathrm{C}_\bullet(E)^{\otimes k}$ descends to a map $\mathcal{Z}_{\mathrm{st}}(k) \times \mathrm{C}_\bullet(E) \to \mathrm{C}_\bullet(E)^{\otimes k}$. Suppose first that $x \in (E_n)_e$ is degenerate, the basepoint or a degeneracy of a basepoint. Then, it will be killed by the first of the three maps in the composite above, and so will be killed by $\bar{\bar{\mu}}$. Next, we need $[n, e, x]$ and $[n+1, e+1, \rho_n(\Sigma x)]$ (the notations here are as in Remark 6.17) to be identified. That is, we need $\alpha_{n+1}(\rho_n(\Sigma x))$ to be equal to $\alpha_n(x)$. Or, more precisely, we need the image of x under

$$\mathrm{S}_\bullet(E_n) \longrightarrow \mathrm{C}_\bullet(E_n) \xrightarrow{\alpha_n} \mathrm{C}_\bullet(E_n)^{\otimes k} \longrightarrow \mathrm{C}_\bullet(E)^{\otimes k} \tag{6.20}$$

to coincide with the image of $\rho_n(\Sigma x)$ under:

$$\mathrm{S}_\bullet(E_{n+1}) \longrightarrow \mathrm{C}_\bullet(E_{n+1}) \xrightarrow{\alpha_{n+1}} \mathrm{C}_\bullet(E_{n+1})^{\otimes k} \longrightarrow \mathrm{C}_\bullet(E)^{\otimes k} \tag{6.21}$$

Consider the following diagram, in which the square commutes by naturality of α_{n+1}:

$$\begin{tikzcd}
\mathrm{C}_\bullet(\Sigma E_n) \arrow[r, "(\alpha_{n+1})_{\Sigma E_n}"] \arrow[d, "\mathrm{C}_\bullet(\rho_n)"'] & \mathrm{C}_\bullet(\Sigma E_n)^{\otimes k} \arrow[d, "\mathrm{C}_\bullet(\rho_n)^{\otimes k}"] & \\
\mathrm{C}_\bullet(E_{n+1}) \arrow[r, "(\alpha_{n+1})_{E_{n+1}}"'] & \mathrm{C}_\bullet(E_{n+1})^{\otimes k} \arrow[r] & \mathrm{C}_\bullet(E)^{\otimes k}
\end{tikzcd}$$

Start with Σx at the topleft corner. Applying the sequence of maps given by $\downarrow, \rightarrow, \rightarrow$, we get the desired image of $\rho_n(\Sigma x)$. On the other hand, applying $\rightarrow$, because $\alpha_n = \Psi(\alpha_{n+1})$, we get, upon altering the degree assignment, $\alpha_n(x)$ (note that $\mathrm{C}_\bullet(\Sigma E_n)^{\otimes k}$ and $\mathrm{C}_\bullet(E_n)^{\otimes k}$ are exactly equal, except for the degrees); or more precisely, we get the image of x under the first two of the three maps in the composite (6.20). Letting this image be $\sum[\Sigma t_1] \otimes \cdots \otimes [\Sigma t_k]$, where the t_i are simplices of E_n of degrees say d_i (and Σt_i denote the corresponding suspended simplices as in Definition 4.12), the image under the entire composite is then $\sum[n, d_1, t_1] \otimes \cdots \otimes [n, d_k, t_k]$. Now, go back to $\sum[\Sigma t_1] \otimes \cdots \otimes [\Sigma t_k]$ and apply $\downarrow$. We get $\sum[\rho_n(\Sigma t_1)] \otimes \cdots \otimes [\rho_n(\Sigma t_k)]$. Next, applying $\rightarrow$, we get $\sum[n+1, d_1+1, \rho_n(\Sigma t_1)] \otimes \cdots \otimes [n+1, d_n+1, \rho_n(\Sigma t_k)]$. Because the square commutes, we have that the desired image of $\rho_n(\Sigma e)$ under the composite in (6.21) is this sum $\sum[n+1, d_1+1, \rho_n(\Sigma t_1)] \otimes \cdots \otimes [n+1, d_n+1, \rho_n(\Sigma t_k)]$. But in $\mathrm{C}_\bullet(E)$, we have $[n+1, d_i+1, \rho_n(\Sigma t_i)] = [n, d_i, t_i]$, so that this desired image is $\sum[n, d_1, t_1] \otimes \cdots \otimes [n, d_k, t_k]$, and this was exactly also the desired image of x under the composite (6.20), so that the two coincide, as was needed.

By the above, we see that $\bar{\bar{\mu}} \colon \mathcal{Z}_{\mathrm{st}}(k) \times \mathrm{S}_\bullet(E) \to \mathrm{C}_\bullet(E)^{\otimes k}$ descends to a bilinear map $\bar{\mu} \colon \mathcal{Z}_{\mathrm{st}}(k) \times \mathrm{C}_\bullet(E) \to \mathrm{C}_\bullet(E)^{\otimes k}$ which preserves the degrees, and we let the associated map $\mathcal{Z}_{\mathrm{st}}(k) \otimes \mathrm{C}_\bullet(E) \to \mathrm{C}_\bullet(E)^{\otimes k}$ be μ. It remains to check that μ commutes with the differentials. Consider $\alpha \in \mathcal{Z}_{\mathrm{st}}(k)_d$ and $x \in (E_n)_e$. We have $\partial(\alpha \otimes [n, e, x]) = \partial\alpha \otimes [n, e, x] + (-1)^d \alpha \otimes \partial[n, e, x] = (\partial\alpha_0, \partial\alpha_1, \dots) \otimes [n, e, x] + (-1)^d \alpha \otimes (\sum_i [n, e, d_i(x)])$ which under μ has image

$$\begin{aligned}(\partial\alpha_n)(x) + (-1)^d \alpha_n(\sum_i d_i(x)) &= \partial_{\mathrm{C}_\bullet(E)^{\otimes k}}(\alpha_n(x)) - (-1)^d \alpha_n(\sum_i d_i(x)) + (-1)^d \alpha_n(\sum_i d_i(x)) \\ &= \partial_{\mathrm{C}_\bullet(E)^{\otimes k}}(\alpha_n(x)) \\ &= \partial_{\mathrm{C}_\bullet(E)^{\otimes k}}(\mu(\alpha, x))\end{aligned}$$

as desired.

We now have the coaction maps μ, and one can verify that the compatibility conditions required for an operad coaction are indeed satisfied.

Next, we demonstrate the functoriality. Let E and F be spectra and $f \colon E \to F$ a map of spectra. We then have an induced map $f_* \colon \mathrm{C}_\bullet(E) \to \mathrm{C}_\bullet(F)$ of chain complexes as above, and we need to check that this map is compatible with the coaction of $\mathcal{Z}_{\mathrm{st}}$. Consider $[n, e, x]$ in $\mathrm{C}_\bullet(E)$. Applying f_* and then $\mu^F \colon \mathcal{Z}_{\mathrm{st}}(k) \otimes \mathrm{C}_\bullet(F) \to \mathrm{C}_\bullet(F)^{\otimes k}$, we get $\alpha_n(f_n(x))$, which by naturality of α_n is equal to $f_n^{\otimes k}(\alpha_n(x))$, and this is exactly the result upon instead first applying μ^E and then $f_*^{\otimes k}$. This gives us induced maps and preservation of identity and composition is clear. $\square$

Proposition 6.22. *Given a spectrum E, the cochain complex $\mathrm{C}^\bullet(E)$ is naturally an algebra over the stable Barratt-Eccles cochain operad $\mathcal{E}_{\mathrm{st}}^\dagger$.*

Proof. By Propositions 6.19, the chains $\mathrm{C}_\bullet(E)$ are a coalgebra over $\mathcal{Z}_{\mathrm{st}}$. By Proposition 2.13, if we apply $(-)^\vee$ to the chains, we get an algebra over $\mathcal{Z}_{\mathrm{st}}$. Thus, by Proposition 2.14, if we apply $(-)^\dagger \circ (-)^{\mathrm{st}}$ to the chains, yielding the cochains as per Remark 6.17, we get an algebra over $\mathcal{Z}_{\mathrm{st}}^\dagger$. Finally, we get an $\mathcal{E}_{\mathrm{st}}^\dagger$-algebra structure by pulling across the map $\mathcal{E}_{\mathrm{st}}^\dagger \to \mathcal{Z}_{\mathrm{st}}^\dagger$ constructed in the fourth chapter. $\square$

As a result of the above, we can interpret cochains on spectra as a functor to algebras over $\mathcal{E}_{\mathrm{st}}^\dagger$:

$$\mathrm{C}^\bullet \colon \mathsf{Sp}^{\mathrm{op}} \to \mathcal{E}_{\mathrm{st}}^\dagger\text{-}\mathsf{Alg}$$

Remark 6.23. We can in fact view the action of the stable operad on spectral cochains in an iterative manner, as follows. By definition, we have that

$$\mathrm{C}_\bullet(E) := \operatorname{colim}(\mathrm{C}_\bullet(E_0) \to \mathrm{C}_\bullet(E_1)[-1] \to \mathrm{C}_\bullet(E_2)[-2] \to \cdots)$$

$$\mathrm{C}^\bullet(E) := \lim(\cdots \to \mathrm{C}^\bullet(E_2)[-2] \to \mathrm{C}^\bullet(E_1)[-1] \to \mathrm{C}^\bullet(E_0))$$

As in Proposition 3.43 and its proof, we have that $\mathrm{C}_\bullet(E_0;\mathbb{F}_p)$ and $\mathrm{C}^\bullet(E_0;\mathbb{F}_p)$ form, respectively, a coalgebra over $\mathcal{E}$ and an algebra over $\mathcal{E}^\dagger$. Thus, by Proposition 2.49, we have that the second terms, $\mathrm{C}_\bullet(E_0;\mathbb{F}_p)[-1]$ and $\mathrm{C}^\bullet(E_0;\mathbb{F}_p)[-1]$ form, respectively, a coalgebra over $\Sigma\mathcal{E}$ and an algebra over $\Sigma\mathcal{E}^\dagger$. Similarly, we have that the third terms, $\mathrm{C}_\bullet(E_0;\mathbb{F}_p)[-2]$ and $\mathrm{C}^\bullet(E_0;\mathbb{F}_p)[-2]$ form, respectively, a coalgebra over $\Sigma^2\mathcal{E}$ and an algebra over $\Sigma^2\mathcal{E}^\dagger$. In the limit, we get (co)algebra structures over the stable operads. ||

Now, by Proposition 6.22 and the work in the previous chapter, the cohomologies $\mathrm{H}^\bullet(E;\mathbb{F}_p)$ inherit operations P^s, and also βP^s in the case $p > 2$. As in the case of spaces, as shown below, they satisfy an important property which does not hold in general.

Proposition 6.24. *Given a spectrum E, the operation P^0 acts by the identity on* $\mathrm{H}^\bullet(E;\mathbb{F}_p)$.

Proof. We shall prove the $p = 2$ case; the $p > 2$ case is analogous. The operation P^0 is computed via images under the map $\mathcal{E}_{\mathrm{st}}^\dagger(2) \otimes \mathrm{C}^\bullet(E)^{\otimes 2} \to \mathrm{C}^\bullet(E)$. Let us use the notation e_d^{un} for that element of $\mathcal{E}^\dagger(2)$ which we called e_d in Section 3.2 of the third chapter, and let us use the notation e_d^{st} for that element of $\mathcal{E}_{\mathrm{st}}^\dagger(2)$ which we called e_d in Section 5.3 of the fifth chapter. By Proposition 5.10 and its proof, we have that:

$$e_d^{\mathrm{st}} = \begin{cases} (e_d^{\mathrm{un}}, \tau e_{d+1}^{\mathrm{un}}, e_{d+2}^{\mathrm{un}}, \tau e_{d+3}^{\mathrm{un}}, \dots) & d \geq 0 \\ (0, \dots, 0, e_0^{\mathrm{un}}, \tau e_1^{\mathrm{un}}, e_2^{\mathrm{un}}, \tau e_3^{\mathrm{un}}, \dots) & d < 0 \end{cases} \tag{6.25}$$

Here in the second case there are $|d|$ zeros. Consider some cocycle α in $\mathrm{C}^d(E)$. Let $\beta \in \mathrm{C}^d(E)$ be the image of $e_d^{\mathrm{st}} \otimes \alpha \otimes \alpha$ under $\mathcal{E}_{\mathrm{st}}^\dagger(2) \otimes \mathrm{C}^\bullet(E)^{\otimes 2} \to \mathrm{C}^\bullet(E)$. By definition of P^0, $P^0\alpha$ is given by the class of β. Thus we need to show that $\beta = \alpha$. Consider a simplex $x \in (E_n)_e$, where $e - n = d$. As the $\mathcal{E}_{\text{st}}^{\dagger}$-action on cochains is dual to a $\mathcal{E}_{\text{st}}$-coaction on chains, $\beta(x)$ is given by $(\alpha \otimes \alpha)(y)$ where y is the image of $e_d^{\text{st}} \otimes x$ under $\mathcal{E}_{\text{st}}(2) \otimes \mathbf{C}_\bullet(E) \to \mathbf{C}_\bullet(E)^{\otimes 2}$. By the definition of the $\mathcal{E}_{\text{st}}$-coaction on chains and by (6.25) above, we have that, since x lies in E_n, computation of y reduces to a computation of the image of $e_e^{\text{un}} \otimes x \otimes x$ under the unstable coaction map $\mathcal{E}(2) \otimes \mathbf{C}_\bullet(E_n) \to \mathbf{C}_\bullet(E_n)^{\otimes 2}$. As in the proof of the unstable case in Proposition 3.46, y is precisely $x \otimes x$. As such $\beta(x) = (\alpha \otimes \alpha)(x \otimes x) = \alpha(x)^2 = \alpha(x)$, and so $\beta = \alpha$, as desired. $\square$

Now, we have seen that spectral cochains yield algebras over $\mathcal{E}_{\text{st}}^{\dagger}$. Due to this, and Propositions 5.33 and 6.24, we consider now the quotient $\widehat{\mathcal{B}}/(1 - P^0)$. First, we construct a map:

$$\widehat{\mathcal{B}} \to \mathcal{A}$$

Recall that the Steenrod algebra $\mathcal{A}$ is defined to be the quotient of $\mathbf{F}\{P^s \mid s \geq 0\}$, or $\mathbf{F}\{P^s, \beta P^s \mid s \geq 0\}$, by the ideal generated by the Adem relations and $1 - P^0$. Given an arbitrary element $\sum a_I P^I$ of $\widehat{\mathcal{B}}$ we map it to the class of the subsum consisting of those multi-indices in which all entries are non-negative; this sum is finite by Proposition 5.26. That this is a map of algebras follows by the following lemma.

Lemma 6.26. *Given admissible multi-indices I and J, if either of them contains a negative entry, than all terms in the admissible monomials expansion of $P^I P^J$ must contain a negative entry, or, equivalently by admissibility, must have a negative final entry.*

Proof. We shall outline the case where $p = 2$; the $p > 2$ case is analogous. If J contains a negative entry, or, equivalently by admissibility, has a negative final entry, the result follows by the obvious analogue for $\mathcal{B}$ of Lemma 5.19. Assume then that it is I that has a negative final entry. We shall prove the result by inducting on the length of J. If J has length zero, it is empty and the result is trivial. Suppose that J has length one, and say that it is equal to (b), for some $b \in \mathbb{Z}$. Consider $P^I P^b$. We shall prove the result in this case by an induction on the length of I. As I is required to contain an negative entry, it cannot have zero length. Suppose that I has length one, and say that it is equal to (a), where we must have $a < 0$. If $a \geq 2b$, the result is obvious as then $b < 0$. Suppose that $a < 2b$. Then the admissible monomials expansion of $P^a P^b$ is given by the Adem relations:

$$\sum \binom{b-i-1}{a-2i} P^{a+b-i} P^i$$

In order for the binomial coefficient to be non-zero, we must have $i \leq a/2$, and so we see that the final entry i in the multi-index $(a + b - i, i)$ is always negative, as desired. Now suppose that we have the result for terms $P^I P^b$ where I has length $< n$, for some $n \geq 2$. Given an admissible multi-index $I = (i_1, \dots, i_n)$ of length n, where $i_n < 0$, we have $P^I P^b = P^{i_1}(P^{i_2} \cdots P^{i_n} P^b)$. Upon first forming the admissible monomials expansion of $P^{i_2} \cdots P^{i_n} P^b$, the inductive hypothesis for the induction on the length of I and Lemma 5.19 give us the desired result. Now let us return to the induction on the length of J. We have demonstrated the result in the cases where J has length zero or one. Now suppose that we have the result for terms $P^I P^J$ where J has length $< n$, for some $n \geq 2$. Given an admissible multi-index $J = (j_1, \dots, j_n)$ of length n, we have $P^I P^J = (P^I P^{j_1} \cdots P^{j_{n-1}}) P^{j_n}$. The result now follows by the inductive hypothesis and by the case where J has length one. $\square$

Proposition 6.27. *We have the following:*

(i) The left ideal of $\widehat{\mathcal{B}}$ generated by $1 - P^0$ coincides with the two-sided ideal and the above map $\widehat{\mathcal{B}} \to \mathcal{A}$ induces an algebra isomorphism:

$$\widehat{\mathcal{B}}/(1 - P^0) \cong \mathcal{A}$$

(ii) The following sequence is exact:

$$0 \longrightarrow \widehat{\mathcal{B}} \xrightarrow{1-P^0} \widehat{\mathcal{B}} \longrightarrow \mathcal{A} \longrightarrow 0$$

In the above sequence, the map denoted by $1 - P^0$ is right multiplication by $1 - P^0$. Note that, as shown in [Man01], both of these statements hold true if we replace $\widehat{\mathcal{B}}$ with $\mathcal{B}$.

Proof. We demonstrate these facts in reverse order, beginning with (ii). As in [Man01], for each $k \geq 0$, we have an exact sequence as follows:

$$0 \longrightarrow \mathcal{B}_{\leq k} \xrightarrow{1-P^0} \mathcal{B}_{\leq k} \longrightarrow \mathcal{A}_{\leq k} \longrightarrow 0$$

For each $t \geq 0$, we can consider the sequence

$$0 \longrightarrow \mathrm{F}_t\mathcal{B}_{\leq k} \xrightarrow{1-P^0} \mathrm{F}_{t+1}\mathcal{B}_{\leq k} \longrightarrow \mathrm{F}_{t+1}\mathcal{A}_{\leq k} \longrightarrow 0$$

and this sequence is itself exact: surjectivity at the righthand end is clear, injectivity at the lefthand end follows from the exactness of the previous sequence, and exactness in the middle follows by examination of the bases provided by Proposition 3.28, just as in the case of the previous sequence. Upon taking limits over k, since the maps $\mathrm{F}_t\mathcal{B}_{\leq k+1} \to \mathrm{F}_t\mathcal{B}_{\leq k}$ are clearly onto, so that the corresponding $\lim^1$ vanishes, we get an exact sequence as follows:

$$0 \longrightarrow \lim_{k \geq 0} \mathrm{F}_t\mathcal{B}_{\leq k} \longrightarrow \lim_{k \geq 0} \mathrm{F}_{t+1}\mathcal{B}_{\leq k} \longrightarrow \lim_{k \geq 0} \mathrm{F}_{t+1}\mathcal{A}_{\leq k} \longrightarrow 0$$

Upon taking colimits over t, by Proposition 5.27, we get an exact sequence as follows:

$$0 \longrightarrow \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{B}} \longrightarrow \operatorname*{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_{t+1}\mathcal{A}_{\leq k} \longrightarrow 0$$

Now, $\operatorname{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_{t+1}\mathcal{A}_{\leq k}$ can be given a description via potentially infinite sums just as $\operatorname{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_{t+1}\mathcal{B}_{\leq k}$ was described via the potentially infinite sums in $\widehat{\mathcal{B}}$. However, due to Proposition 5.26, any such sum must necessarily be finite, which is to say $\operatorname{colim}_{t\geq 0} \lim_{k\geq 0} \mathrm{F}_{t+1}\mathcal{A}_{\leq k} \cong \mathcal{A}$ and so we have an exact sequence:

$$0 \longrightarrow \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{B}} \longrightarrow \mathcal{A} \longrightarrow 0$$

Finally, unravelling the identifications which we have made, an easy check shows that the maps in this sequence are exactly those in the proposition statement. This completes the proof of (ii).

Now let us consider (i). By the exactness in (ii), the kernel of the map $\widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$ coincides with the image of the map $\widehat{\mathcal{B}} \to \widehat{\mathcal{B}}$ which we have denoted by $1 - P^0$. This latter image is the left ideal of $\widehat{\mathcal{B}}$ generated by $1 - P^0$. By definition of the map $\widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$, its kernel clearly contains the two-sided ideal of $\widehat{\mathcal{B}}$ generated by $1 - P^0$, and so we have that this two-sided ideal is contained in the aforementioned left ideal, from which it follows that these two ideals coincide. Finally, we have the isomorphism $\widehat{\mathcal{B}}/(1 - P^0) \cong \mathcal{A}$ because the map $\widehat{\mathcal{B}} \to \mathcal{A}$ is onto and the kernel, as just established, is precisely the two-sided ideal generated by $1 - P^0$. $\square$

6.3 Change of Coefficients from $\mathbb{F}_p$ to $\overline{\mathbb{F}}_p$

Hitherto, we have worked with coefficients in $\mathbb{F}_p$, whether it was with the stable operads or with the spectral cochains. In order to construct algebraic models of p-adic stable homotopy types however, we must pass to the algebraic closure $\overline{\mathbb{F}}_p$. We describe the necessary modifications in this section. First, we define a new operad, one over $\overline{\mathbb{F}}_p$.

Definition 6.28. The operad $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$, an operad in cochain complexes over $\overline{\mathbb{F}}_p$, is defined as follows:

$$\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}(n) := \mathcal{E}^{\dagger}_{\mathrm{st}}(n) \otimes_{\mathbb{F}_p[\Sigma_n]} \overline{\mathbb{F}}_p[\Sigma_n]$$

We now have three tasks to complete, tasks which we completed in the case of the operad $\mathcal{E}_{\mathrm{st}}$: (i) show that one can do homotopy theory over $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$ (ii) compute the cohomology of free algebras over $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$ and develop a theory of cohomology operations (iii) demonstrate homotopy additivity properties of $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$. Let us first consider (i). Our goal is to show that, just like $\mathcal{E}_{\mathrm{st}}$, the monad associated to $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$ preserves quasi-isomorphisms, and moreover, that $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$ is semi-admissible. As usual, we denote the monad, and also the free algebra functor, associated to the operad $\bar{\mathcal{E}}_{\text{st}}^{\dagger}$, by $\overline{\mathbf{E}}_{\text{st}}^{\dagger}$.

Proposition 6.29. *We have the following:*

(i) The monad corresponding to the operad $\bar{\mathcal{E}}_{st}^{\dagger}$ preserves quasi-isomorphisms.

(ii) The operad $\bar{\mathcal{E}}_{st}^{\dagger}$ is semi-admissible, which is to say the category of algebras $\bar{\mathcal{E}}_{\text{st}}^{\dagger}$-$\mathsf{Alg}$ possesses a Quillen semi-model structure where:

- *The weak equivalences are the quasi-isomorphisms.*
- *The fibrations are the surjective maps.*
- *The cofibrations are retracts of relative cell complexes, where the cells are the maps $\overline{\mathbf{E}}_{\text{st}}^{\dagger}M \to \overline{\mathbf{E}}_{\text{st}}^{\dagger}\mathrm{C}M$ where M is a degreewise free $\overline{\mathbb{F}}_p$-complex with zero differentials.*

Proof. (i): Given a cochain complex X over $\mathbb{F}_p$, an easy check shows that:

$$\overline{\mathbf{E}}_{\text{st}}^{\dagger}(\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} X) \cong \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} (\mathbf{E}_{\text{st}}^{\dagger}X) \tag{6.30}$$

As in the proof of Proposition 5.3, it suffices to consider quasi-isomorphisms which are monomorphisms. Given a monomorphism $f\colon X \to Y$ of $\overline{\mathbb{F}}_p$-complexes, by an appropriate choice of bases, we can find $\mathbb{F}_p$-complexes $\underline{X}$ and $\underline{Y}$ together with a commutative square as follows:

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "f"'] & \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \underline{X} \arrow[d] \\ Y \arrow[r, "\cong"'] & \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \underline{Y} \end{tikzcd}$$

The result now follows by Proposition 5.3 and (6.30) above.

(ii): Let A be a cell $\bar{\mathcal{E}}_{\text{st}}^{\dagger}$-algebra. As in the proof of Proposition 5.8, it suffices to show that $\mathcal{U}^A(j) \otimes_{\overline{\mathbb{F}}_p[\Sigma_j]} -$ preserves the quasi-isomorphism $0 \to (\mathbb{D}^d)^{\otimes j}$, where $j \geq 1$. In the proof of the semi-admissibility of $\mathcal{E}_{\text{st}}^{\dagger}$, we showed that tensoring with the enveloping operad pieces preserves quasi-isomorphisms between finite $\mathbb{F}_p[\Sigma_j]$-complexes. By an analogous argument, in the case here, we have that tensoring with the enveloping operad pieces preserves those quasi-isomorphisms between $\overline{\mathbb{F}}_p[\Sigma_j]$-complexes which are of the form $f \otimes_{\mathbb{F}_p[\Sigma_j]} \overline{\mathbb{F}}_p[\Sigma_j]$ where $f\colon X \to Y$ is a quasi-isomorphism between finite $\mathbb{F}_p[\Sigma_j]$-complexes. As the quasi-isomorphism $0 \to (\mathbb{D}^d)^{\otimes j}$, where $j \geq 1$, is of this form, we have the desired result. □

The semi-model structure constructed above yields also the derived category of $\bar{\mathcal{E}}_{\text{st}}^{\dagger}$-algebrs. We now move onto the second task (ii), that of computing the cohomologies of free $\bar{\mathcal{E}}_{\text{st}}^{\dagger}$-algebras, and developing a theory of cohomology operations. The former is achieved via the following result and Proposition 5.33.

Proposition 6.31. *Given an $\overline{\mathbb{F}}_p$-complex X, we have that:*

$$\mathrm{H}^\bullet(\overline{\mathbf{E}}^\dagger_{\mathrm{st}}X) \cong \widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \mathrm{H}^\bullet(X)$$

Note that the tensor is over $\mathbb{F}_p$, not over $\overline{\mathbb{F}}_p$.

Proof. Choose an $\mathbb{F}_p$-complex $\underline{X}$ such that $\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \underline{X}$. We then have that:

$$\begin{aligned}
\mathrm{H}^\bullet(\overline{\mathbf{E}}^\dagger_{\mathrm{st}}X) &\cong \mathrm{H}^\bullet(\overline{\mathbf{E}}^\dagger_{\mathrm{st}}(\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \underline{X})) \\
&\cong \mathrm{H}^\bullet(\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathbf{E}^\dagger_{\mathrm{st}}(\underline{X})) \\
&\cong \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathrm{H}^\bullet(\mathbf{E}^\dagger_{\mathrm{st}}\underline{X}) \\
&\cong \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \mathrm{H}^\bullet(\underline{X}) \\
&\cong \widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \mathrm{H}^\bullet(X)
\end{aligned}$$

□

Now we consider cohomology operations for $\bar{\mathcal{E}}^\dagger_{\mathrm{st}}$-algebras.

Proposition 6.32. *Given an algebra A over $\bar{\mathcal{E}}^\dagger_{\mathrm{st}}$, the cohomology $\mathrm{H}^\bullet(A)$ possesses an $\mathbb{F}_p$-linear action by $\widehat{\mathcal{B}}$.*

Note that the operations are $\mathbb{F}_p$-linear, as opposed to $\overline{\mathbb{F}}_p$-linear.

Proof. As per Proposition 6.31, we have an isomorphism $\mathrm{H}^\bullet(\bar{\mathbf{E}}^\dagger_{\mathbf{st}}A) \cong \widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \mathrm{H}^\bullet(A)$. The $\mathbb{F}_p$-linear action of $\widehat{\mathcal{B}}$ is then via the composite

$$\widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \mathrm{H}^\bullet(A) \xrightarrow{\cong} \mathrm{H}^\bullet(\bar{\mathbf{E}}^\dagger_{\mathbf{st}}A) \longrightarrow \mathrm{H}^\bullet(A)$$

where the second map is that which we achieve by applying $\mathrm{H}^\bullet(-)$ to the algebra structure map $\bar{\mathbf{E}}^\dagger_{\mathbf{st}}A \to A$ of A. □

Next, we consider task (iii), that of the homotopy additivity of $\bar{\mathcal{E}}^\dagger_{\mathrm{st}}$.

Proposition 6.33. *We have the following:*

(i) Given $\overline{\mathbb{F}}_p$-complexes X and Y, we have a natural quasi-isomorphism:

$$\overline{\mathbf{E}}^\dagger_{\mathrm{st}}(X \oplus Y) \sim \overline{\mathbf{E}}^\dagger_{st}(X) \oplus \overline{\mathbf{E}}^\dagger_{st}(Y)$$

(ii) Given cofibrant $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$-algebras A and B, we have a natural quasi-isomorphism:

$$A \amalg B \sim A \oplus B$$

(iii) Given a diagram $A \leftarrow C \rightarrow B$ of $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$-algebras, if each of A, B and C are cofibrant, and $C \rightarrow B$ is a cofibration, then we have that:

$$A \amalg_C B \sim A \oplus_C B$$

Proof. (i): Given the $\overline{\mathbb{F}}_p$-complexes X and Y, we have a canonical map:

$$\overline{\mathbf{E}}_{\mathrm{st}}^{\dagger}(X) \oplus \overline{\mathbf{E}}_{\mathrm{st}}^{\dagger}(Y) \rightarrow \overline{\mathbf{E}}_{\mathrm{st}}^{\dagger}(X \oplus Y)$$

Upon choosing bases for X and Y, we have $\mathbb{F}_p$-complexes $\underline{X}$ and $\underline{Y}$ such that $X \cong \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \underline{X}$ and $Y \cong \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \underline{Y}$. We of course then also have a basis for $X \oplus Y$ and an isomorphism $X \oplus Y \cong \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} (X \oplus Y)$. It follows that the above canonical map can be constructed by tensoring the map

$$\mathbf{E}_{\mathrm{st}}^{\dagger}(\underline{X}) \oplus \mathbf{E}_{\mathrm{st}}^{\dagger}(\underline{Y}) \rightarrow \mathbf{E}_{\mathrm{st}}^{\dagger}(\underline{X} \oplus \underline{Y})$$

with $\overline{\mathbb{F}}_p$. The result now follows by Proposition 5.37.

(ii): Note that the obvious analogue of Lemma 5.38 holds for $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$, by the same proof, using (i) above. With this, all arguments for the case of $\mathcal{E}_{\mathrm{st}}^{\dagger}$ carry through also for $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$.

(iii): Note that an analogue of Lemma 5.45 holds for $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$; this is the analogue where one replaces the finite $\mathbb{F}_p[\Sigma_j]$-modules with those of the form $\overline{\mathbb{F}}_p[\Sigma_j] \otimes_{\mathbb{F}_p[\Sigma_j]} M$ where M is a finite $\mathbb{F}_p[\Sigma_j]$-module. Note also that the obvious analogue of Lemma 5.46 also holds for $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$. With this, all the remaining arguments in the case of $\mathcal{E}_{\mathrm{st}}^{\dagger}$ carry through also in our case here. $\square$

We have now completed the transition from coefficients in $\mathbb{F}_p$ to coefficients in $\overline{\mathbb{F}}_p$ at the level of the operad. Next, we consider spectral cochains with coefficients in $\overline{\mathbb{F}}_p$.

Definition 6.34. Given a spectrum E, we set the following:

$$\overline{\mathbf{C}}^{\bullet}(E) := \mathbf{C}^{\bullet}(E) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$$

Of couse, one would hope that the cochains $\overline{\mathbf{C}}^{\bullet}(E)$ yield algebras over $\bar{\mathcal{E}}_{st}^{\dagger}$. The following result confirms this.

Proposition 6.35. *Given a spectrum E, $\overline{\mathrm{C}}^{\bullet}(E)$ is naturally an algebra over $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$.*

Proof. In general, if X if an $\mathbb{F}_p$-complex which is an $\mathcal{E}^{\dagger}_{\mathrm{st}}$-algebra via a structure map $\mathbf{E}^{\dagger}_{\mathrm{st}}X \to X$, then $X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is an $\overline{\mathbb{F}}_p$-complex which is an $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$-algebra, via the structure map:

$$\overline{\mathbf{E}}^{\dagger}_{\mathrm{st}}(X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \cong (\mathbf{E}^{\dagger}_{\mathrm{st}}X) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \to X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$$

□

We have now completed the transition of all previous material to coefficients in $\overline{\mathbb{F}}_p$.

6.4 An Adjoint to Spectral Cochains

Consider the spectral cochains functor:

$$\overline{\mathrm{C}}^{\bullet} \colon \mathsf{Sp}^{\mathrm{op}} \to \bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}$$

We shall construct an adjoint to this spectral cochains functor. That is, we will construct an adjoint functor:

$$\mathrm{U} \colon \bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg} \to \mathsf{Sp}^{\mathrm{op}}$$

We define U by setting, given an $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$-algebra A, the following in spectral degree $n \geq 0$ and simplicial degree $d \geq 0$:

$$\mathrm{U}(A)_{n,d} := \bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\mathsf{Alg}(A, \overline{\mathrm{C}}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+})) = \bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\mathsf{Alg}(A, \mathbf{C}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$$

With A and n fixed, $\mathrm{U}(A)_{n,d}$ is clearly contravariantly functorial in d, so that we have a simplicial set $\mathrm{U}(A)_n$; moreover, it becomes a based simplicial set upon endowing it with the zero map as a basepoint. We now want maps $\mathrm{U}(A)_n \to \Omega\mathrm{U}(A)_{n+1}$ or $\Sigma\mathrm{U}(A)_n \to \mathrm{U}(A)_{n+1}$. In dimension p, this means a map $\Sigma\mathrm{U}(A)_{n,p} \to \mathrm{U}(A)_{n+1,p+1}$, such that d_0 and $d_1 \cdots d_{p+1}$, applied to simplices in the image, map to $*$. Thus we want a map

$$\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}(A, \mathbf{C}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \to \bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}(A, \mathbf{C}^{\bullet}(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$$

which is such that the algebra maps which lie in the image of this map satisfy the property that they yield the zero map upon postcomposition either with the map $\mathbf{C}^{\bullet}(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \to \mathbf{C}^{\bullet}(\Sigma^{\infty-n-1}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ induced by d_0, or instead with the map $\mathbf{C}^{\bullet}(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \to \mathbf{C}^{\bullet}(\Sigma^{\infty-n-1}\Delta_{0+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ induced by $d_1 \cdots d_{d+1}$. We first note that we have an isomorphism of differential graded $\mathbb{F}_p$-modules

$$\mathsf{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+}) \to \mathsf{C}_\bullet(\Sigma\Delta_{d+})$$

of degree $n+1$ (note that here in the source we are taking chains on a spectrum while in the target we are taking chains on a based simplicial set). This isomorphism is given by sending $[n, e, x]$ (see Remark 6.17 for this notation) in Δ_{d+}, which is of degree $e-n$, to $[\Sigma x]$ in $\Sigma\Delta_{d+}$, which is of degree $e+1$; that this is an isomorphism follows from Proposition 4.10. Next, note that we have an isomorphism of differential graded $\mathbb{F}_p$-modules

$$\mathsf{C}_\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \to \mathsf{C}_\bullet(\Delta_{d+1,+})$$

again of degree $n+1$. This isomorphism is given by sending $[n+1, e, x]$ in $\Delta_{d+1,+}$, which is of degree $e-n-1$, to $[x]$ in $\Delta_{d,+}$, which is of degree e. Now, using Proposition 4.4 and the canonical map $\mathsf{C}(X) \to \Sigma X$, we have a canonical map $\Delta_{d+1} \to \Sigma\Delta_{d+}$, yielding a map $\Delta_{d+1,+} \to \Sigma\Delta_{d+}$, and so, using the above isomorphisms, we get a composite map

$$\mathsf{C}_\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \to \mathsf{C}_\bullet(\Delta_{d+1,+}) \to \mathsf{C}_\bullet(\Sigma\Delta_{d+}) \to \mathsf{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+})$$

which is a map of chain complexes since the degree is $(n+1)+0-(n+1)=0$. Moreover, we claim that it is a map of $\mathcal{E}_{\mathrm{st}}$-coalgebras. Once we have this, by dualization, tensoring with $\overline{\mathbb{F}}_p$, and postcomposition, we get the desired map:

$$\mathcal{E}_{\mathrm{st}}\text{-}\mathsf{Alg}(A, \mathsf{C}^\bullet(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \to \mathcal{E}_{\mathrm{st}}\text{-}\mathsf{Alg}(A, \mathsf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$$

To see that the map is an $\mathcal{E}_{\mathrm{st}}$-coalgebra map, consider some element $[n+1, e, x]$ in the source $\mathsf{C}_\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+})$, where x is some map $[e] \to [d+1]$. Let x' be the corresponding map $[e'] \to [d]$ by restricting to the preimage of the final $d+1$ elements (if this preimage is empty, or if it is all of $[e]$, we have $*$). The corresponding element of $\mathsf{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+})$ is $[n+1, e', x']$ (we have $n+1$ here instead of n since in the third map in the composition, the isomorphism maps to the $(n+1)^{\text{st}}$ level, instead of the n^{th} level, of the spectrum $\mathsf{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+})$). Thus an element $\alpha = (\alpha_0, \alpha_1, \dots)$ of $\mathcal{E}_{\mathrm{st}}(k)$ coacts on both the element in the source and the element in the target target by α_{n+1}, yielding $\alpha_{n+1}(e)$ and $\alpha_{n+1}(e')$. Thus what we want is for the following square to commute:

$$\begin{tikzcd}
\mathbf{C}_\bullet(\Delta_{d+1,+}) \arrow[r] \arrow[d, "\alpha_{n+1}"'] & \mathbf{C}_\bullet(\Sigma\Delta_{d+})\arrow[d, "\alpha_{n+1}"] \\
\mathbf{C}_\bullet(\Delta_{d+1,+})^{\otimes k} \arrow[r] & \mathbf{C}_\bullet(\Sigma\Delta_{d+})^{\otimes k}
\end{tikzcd}$$

This commutativity follows from the naturality of α_{n+1}.

Next, we check the required condition above that postcomposing maps in the image with the map $\mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \to \mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d,+})$ induced by d_0, or with the map $\mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \to \mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{0+})$ induced by $d_1 \cdots d_{d+1}$, gives the zero map. That is, first, given a map

$$A \to \mathbf{C}^\bullet(\Sigma^{\infty-n}\Delta_{d+}) \to \mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+})$$

we want the composite

$$A \to \mathbf{C}^\bullet(\Sigma^{\infty-n}\Delta_{d+}) \to \mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \to \mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d,+})$$

where the final map is given by d_0, to be zero. Here the composite of the latter two maps is the dual of the following composite:

$$\begin{tikzcd}
\mathbf{C}_\bullet(\Sigma^{\infty-n-1}\Delta_{d+}) \arrow[d] & & & \\
\mathbf{C}_\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \arrow[r] & \mathbf{C}_\bullet(\Delta_{d+1,+}) \arrow[r] & \mathbf{C}_\bullet(\Sigma\Delta_{d+}) \arrow[r] & \mathbf{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+})
\end{tikzcd}$$

Start with some q-simplex $[e] \to [d]$. Then it gets postcomposed to $[e] \to [d+1]$ where the image doesn't contain 0, then we get this same map again but with a different degree, then we restrict to those entries which don't map to 0 and so get back the original $[e] \to [d]$ which in the suspension is killed (mapped to $*$) and thus we will get zero in A at the end of the composition.

On the other hand, if we had postcomposed instead with the map $\mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \to \mathbf{C}^\bullet(\Sigma^{\infty-n-1}\Delta_{0+})$ induced by $d_1 \cdots d_{d+1}$, the vertical map above alone would change, and, proceeding as in the analysis above, we would start with the identity on $[0]$ and this will map under the map induced by $d_1 \cdots d_{d+1}$ to the inclusion $[0] \to [\mathrm{d}+1]$ mapping 0 to 0, and this will be killed by the map $\mathbf{C}_\bullet(\Delta_{d+1,+}) \to \mathbf{C}_\bullet(\Sigma\Delta_{d+})$ (see the definition of the map to the cone in Proposition 4.4).

We have now constructed maps $\mathrm{U}(A)_{n,d} \to \mathrm{U}(A)_{n+1,d+1}$, i.e., maps $(\mathrm{U}(A)_n)_d \to (\Omega\mathrm{U}(A)_{n+1})_d$. Moreover, one can readily check that these maps commute with the simplicial operators, so that we have the desired simplicial set maps $\mathrm{U}(A)_n \to \Omega\mathrm{U}(A)_{n+1}$.

We now have a spectrum $\mathrm{U}(A)$ associated to A. We next show that this construction is functorial in $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$-algebras A. This is easily seen from the fact that

$$\mathrm{U}(A)_{n,d} = \bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}(A, \mathsf{C}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$$

as, given a map $A \to B$, we get an induced map from $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}(B, \mathsf{C}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$ to $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}(A, \mathsf{C}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$ by precomposition. Considering these for a fixed n but variable d, we get a simplicial set map $\mathrm{U}(B)_n \to \mathrm{U}(A)_n$ since the simplicial operators act by postcomposition and so commute with the precomposition maps $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}(B, \mathsf{C}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \to \bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}(A, \mathsf{C}^{\bullet}(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$. Moreover, one can immediately verify that these simplicial set maps are compatible with the structure maps of the spectra $\mathrm{U}(A)$ and $\mathrm{U}(B)$. Thus we have a functor:

$$\mathrm{U}\colon \bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg} \to \mathsf{Sp}^{\mathrm{op}}$$

Proposition 6.36. *The functor* U *is left adjoint to the cochains functor on spectra, so that we have an adjunction:*

$$\begin{tikzcd} \mathsf{Sp}^{\mathrm{op}} \arrow[r, shift left, "\overline{\mathsf{C}}^{\bullet}"] \arrow[r, phantom, "\top"] & \bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg} \arrow[l, shift left, "\mathrm{U}"] \end{tikzcd}$$

Proof. Let E be a spectrum and A a $\mathcal{E}_{\mathrm{st}}$-algebra. We wish to construct the natural isomorphism between $\mathsf{Sp}(E, \mathrm{U}(A))$ and $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}(A, \overline{\mathsf{C}}^{\bullet}(E))$. This requires verifications which are not obvious but not too difficult, though they are rather lengthy. We shall provide here the part of the correspondence which yields an $\bar{\mathcal{E}}_{\mathrm{st}}^{\dagger}$algebra map $g = \{g_n\}\colon A \to \overline{\mathsf{C}}^{\bullet}(E)$ when given a spectrum map $f = \{f_n\}\colon E \to \mathrm{U}(A)$. Fix such a spectrum map $f = \{f_n\}\colon E \to \mathrm{U}(A)$. We want an algebra map $g = \{g_n\}\colon A \to \overline{\mathsf{C}}^{\bullet}(E)$. Consider $a \in A$, of degree say n. We desire a map $g_n(a)\colon \bar{\mathsf{C}}_n(E) \to \mathbb{F}_p$. Consider some $[m, e, x]$ in $(E_m)_e$ where $e - m = n$. We have an element $f_m(x) \in \mathrm{U}(A)_{m,e}$, which is to say a map $A \to \overline{\mathsf{C}}^{\bullet}(\Sigma^{\infty-m}\Delta_{e+})$, and so, taking the image of a, a map $\bar{\mathsf{C}}_n(\Sigma^{\infty-m}\Delta_{e+}) \to \mathbb{F}_p$. Let $g(a)([m, e, x])$ be the image under this map of $[m, e, \mathrm{id}_{[e]}]$ (note that this cell is of degree $e - m = n$); that is, $g_n(a)([m, e, x]) = f_m(x)(a)([m, e, \mathrm{id}_{[e]}])$. Linearity of g_n follows from that of $f_m(x)$. Next, we must check that the differentials are preserved. Fix some $a \in A^n$. We have two $(n+1)$-cochains $g_{n+1}(\partial a), \partial g_n(a)\colon \bar{\mathsf{C}}_{n+1}(E) \to \mathbb{F}_p$ and we desire that these two to be the same. Consider some $[m, e, x]$ where $x \in (E_m)_e$ and $e - m = n + 1$. The latter cochain first forms $\partial[m, e, x] = \sum_i [m, e-1, d_i(x)]$ and then sends this to $\sum_i f_m(d_i(x))(a)([m, e-1, \mathrm{id}_{[e-1]}])$. On the other hand, the former cochain sends it to $f_m(x)(\partial a)([m, e, \mathrm{id}_{[e]}])$. Now, since f_m is a map of simplicial sets, we have $\sum_i f_m(d_i(x))(a)([m, e-1, \mathrm{id}_{[e-1]}]) = \sum_i (d_i f_m(x))(a)([m, e-1, \mathrm{id}_{[e-1]}])$.

For each i, the map $d_i f_m(e)$ is the composite:

$$A \xrightarrow{f_m(x)} \bar{\mathbf{C}}^\bullet(\Sigma^{\infty-m}\Delta_{e+}) \xrightarrow{d_i} \bar{\mathbf{C}}^\bullet(\Sigma^{\infty-m}\Delta_{e-1,+})$$

It follows that $\sum_i (d_i f_m(x))(a)([m, e-1, \mathrm{id}_{[e-1]}]) = \sum_i (f_m(x))(a)([m, e-1, d_i])$. On the other hand, since $f_m(x)$ is a map of cochain complexes, we have that $f_m(x)(\partial a) = \partial f_m(x)(a)$. It follows that

$$\begin{aligned} f_m(x)(\partial a)([m, e, \mathrm{id}_{[e]}]) &= (\partial f_m(x)(a))([m, e, \mathrm{id}_{[e]}]) \\ &= f_m(x)(a)(\partial[m, e, \mathrm{id}_{[e]}]) \\ &= f_m(x)(a)\left(\sum_i [m, e-1, d_i]\right) \\ &= \sum_i f_m(x)(a)([m, e-1, d_i]) \end{aligned}$$

Thus, as desired, the two cochains coincide. Next, we must check that our map $g = \{g_n\}\colon A \to \mathbf{C}^\bullet(E)$ respects the actions by $\bar{\mathcal{E}}^\dagger_{\mathrm{st}}$. Let $\alpha = (\alpha_0, \alpha_1, \dots) \in \bar{\mathcal{E}}^\dagger_{\mathrm{st}}(k)$. Consider some $a_1, \dots, a_k \in A$, and assume without loss of generality (due to linearity), that the a_i are homogeneous, say of degrees $n_1, \dots, n_k$. If we first act by α and then apply g, we get a cochain whose image at $[m, e, x]$, where $e - m = n_1 + \dots + n_k$, is $f_m(x)(\alpha(a_1, \dots, a_k))([m, e, \mathrm{id}_{[e]}])$. Since $f_m(x)$ is a map of algebras, this is equivalent to $\alpha(f_m(x)(a_1), \dots, f_m(x)(a_k))([m, e, \mathrm{id}_{[e]}])$. On the other hand, if we apply g first and then act by α, we first get cochains $g_{n_1}(a_1), \dots, g_{n_k}(a_k)$ and then the cochain $\alpha(g_{n_1}(a_1), \dots, g_{n_k}(a_k))$. Let the coaction of α on $[m, e, \mathrm{id}_{[e]}]$ be $\sum [m, e_1, \theta_1] \otimes \dots \otimes [m, e_k, \theta_k]$, where θ_i is a map $[e_i] \to [e]$. Then, considering the map $\Delta_e \to E_m$ corresponding to the same $x \in (E_m)_e$ as above, and using naturality of α_m, we find that the coaction of α on $[m, e, x]$ is given by $\sum [m, e_1, \theta_1^* x] \otimes \dots \otimes [m, e_k, \theta_k^* x]$. Now, by definition of the $\bar{\mathcal{E}}^\dagger_{\mathrm{st}}$-action on spectral cochains, if we evaluate the cochain $\alpha(g_{n_1}(a_1), \dots, g_{n_k}(a_k))$ at $[m, e, x]$, we get

$$\alpha(g_{n_1}(a_1), \dots, g_{n_k}(a_k))([m, e, x]) = (g_{n_1}(a_1) \otimes \dots \otimes g_{n_k}(a_k))(\alpha \cdot [m, e, x])$$

which amounts to:

$$(g_{n_1}(a_1) \otimes \dots \otimes g_{n_k}(a_k)) \left(\sum [m, e_1, \theta_1^* x] \otimes \dots \otimes [m, e_k, \theta_k^* x]\right)$$

On the other hand, we have:

$$\alpha(f_m(x)(a_1), \dots, f_m(x)(a_k))([m, e, \mathrm{id}_{[e]}]) = (f_m(x)(a_1) \otimes \dots \otimes f_m(x)(a_k))(\alpha \cdot [m, e, \mathrm{id}_{[e]}])$$

which is to say

$$(f_m(x)(a_1) \otimes \cdots \otimes f_m(x)(a_k)) \left(\sum [m, e_1, \theta_1] \otimes \cdots \otimes [m, e_k, \theta_k]\right)$$

and this amounts to:

$$\sum f_m(x)(a_1)([m, e_1, \theta_1]) \otimes \cdots \otimes f_m(x)(a_k)([m, e_k, \theta_k])$$

In either case, we only have to worry about summands where $e_i - m = n_i$ for each i. In this case, the former becomes:

$$\sum (f_m(\theta_1^* x)(a_1)[m, q_1, \mathrm{id}_{[q_1]}] \otimes \cdots \otimes f_m(\theta_k^* x)(a_k)[m, q_k, \mathrm{id}_{[q_k]}]$$

which is to say:

$$\sum ((\theta_1^* f_m)(x)(a_1)[m, e_1, \mathrm{id}_{[e_1]}] \otimes \cdots \otimes (\theta_k^* f_m)(x)(a_k)[m, e_k, \mathrm{id}_{[e_k]}]$$

Now, for each i, $\theta_i^* f_m(x)$ is the composite:

$$A \xrightarrow{f_m(x)} \overline{\mathrm{C}}^{\bullet}(\Sigma^{\infty - m}\Delta_{e+}) \xrightarrow{\theta_i} \overline{\mathrm{C}}^{\bullet}(\Sigma^{\infty - m}\Delta_{e_i,+})$$

It follows that, for each i, $((\theta_1^* f_m)(x)(a_1)[m, e_i, \mathrm{id}_{[e_i]}] = f_m(x)(a_1)[m, e_i, \theta_i]$. Thus the two cochains coincide, as desired. $\square$

Now we consider homotopical properties of the above spectral cochains adjunction.

Proposition 6.37. *The spectral cochains adjunction*

$$\mathsf{Sp}^{\mathrm{op}} \underset{\mathrm{U}}{\overset{\overline{\mathrm{C}}^{\bullet}}{\rightleftarrows}} \bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}$$

is a Quillen adjunction.

Note that here on the righthand side we have a Quillen semi-model category, as opposed to a Quillen model category. By a Quillen adjunction, we mean one which satisfies the conditions in Proposition 2.39 (iii).

Proof. We first demonstrate that $\overline{\mathrm{C}}^{\bullet}$ preserves fibrations, which is to say that $\overline{\mathrm{C}}^{\bullet}$ sends a cofibration $i\colon E \to F$ of spectra to an epimorphism. Since i is a cofibration, we have that $i_0\colon E_0 \to F_0$ and, for $n \geq 0$, the maps

$$E_{n+1} \amalg_{\Sigma E_n} \Sigma F_n \to F_{n+1}$$

are cofibrations of based simplicial sets, which is to say that they are injective in each simplicial degree. In particular, by Proposition 6.3, each $i_n\colon E_n \to F_n$, for $n \geq 0$, is a monomorphism. Since monomorphisms of simplicial sets preserve non-degenerate simplices, each i_n, for $n \geq 0$, preserves non-degenerate simplices, and of course also the basepoints and their degeneracies. Thus, upon taking chains $\mathrm{C}_\bullet(-)$, we get a sequential colimit of monomorphisms, which is once again a monomorphism since sequential colimits are exact. As we are over a field, we have a split inclusion, so that, upon dualizing, reindexing and tensoring, we have the result for the cochains $\bar{\mathrm{C}}^\bullet(-)$.

Next, we shall show that U preserves cofibrations. Given a cofibration $A \to B$ of algebras, we wish to show that $\mathrm{U}(A) \to \mathrm{U}(B)$ is a cofibration in the opposite category of spectra. We know that all cofibrations of algebras may be written as retracts of cell maps. As U is a left adjoint and so preserves colimits, we need only show that U maps the cofibrations $\bar{\mathbf{E}}^\dagger_{\mathbf{st}}M \to \bar{\mathbf{E}}^\dagger_{\mathbf{st}}\mathrm{C}M$ to cofibrations. Here M is an $\overline{\mathbb{F}}_p$-complex with zero differentials. In fact, since for such M the map $M \to \mathrm{C}M$ decomposes as a direct sum of maps $\mathbb{S}^n \to \mathbb{D}^{n+1}$ for various n, we need only consider this case. We shall show, more generally, that if $X \to Y$ is an inclusion of complexes where X and Y are of finite type (by which we mean they are finite dimensional in each degree), then $\mathrm{U}\bar{\mathbf{E}}^\dagger_{\mathbf{st}}Y \to \mathrm{U}\bar{\mathbf{E}}^\dagger_{\mathbf{st}}X$ is a fibration of spectra.

To begin, we claim that, given any complex X of finite type, $\mathrm{U}\bar{\mathbf{E}}^\dagger_{\mathbf{st}}X$ is a strict Ω-spectrum. First, note that, in each spectral degree n and simplicial degree d, we have that:

$$\begin{aligned}(\mathrm{U}\bar{\mathbf{E}}^\dagger_{\mathbf{st}}X)_{n,d} &= \bar{\mathcal{E}}^\dagger_{\mathrm{st}}\text{-}\mathsf{Alg}(\bar{\mathbf{E}}^\dagger_{\mathbf{st}}X, \mathrm{C}^\bullet(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)\\ &\cong \mathsf{Co}_{\overline{\mathbb{F}}_p}(X, \mathrm{C}^\bullet(\Sigma^{\infty-n}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)\end{aligned}$$

As maps of complexes are closed under addition, we have that, for each $n \geq 0$, $(\mathrm{U}\bar{\mathbf{E}}^\dagger_{\mathbf{st}}X)_n$ is the underlying simplicial set of a simplicial abelian group, and so a Kan complex. Next, we will show that the maps $\mathrm{U}(A)_n \to \Omega\mathrm{U}(A)_{n+1}$ are bijections in each simplicial degree d. To see this, first note that, since X is of finite type, we may dualize to find that

$$(\mathrm{U}\bar{\mathbf{E}}^\dagger_{\mathbf{st}}X)_{n,p} \cong \mathsf{Ch}_{\overline{\mathbb{F}}_p}(\mathrm{C}_\bullet(\Sigma^{\infty-n}\Delta_{p+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathrm{D}X) \cong \mathsf{Ch}_{\mathbb{F}_p}(\mathrm{C}_\bullet(\Sigma^{\infty-n}\Delta_{p+}), \mathrm{D}X)$$

where $\mathrm{D}X$ is the chain complex given by $(\mathrm{D}X)_e = \mathrm{Hom}_{\overline{\mathbb{F}}_p}(X_e, \overline{\mathbb{F}}_p)$. Under this isomorphism, the map $\mathrm{U}(A)_n \to \Omega\mathrm{U}(A)_{n+1}$ is given by sending a complex map $\mathrm{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+}) \to \mathrm{D}X$ to the composite

$$\mathrm{C}_\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \longrightarrow \mathrm{C}_\bullet(\Delta_{d+1,+}) \longrightarrow \mathrm{C}_\bullet(\Sigma\Delta_{d+}) \longrightarrow \mathrm{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+}) \longrightarrow \mathrm{D}X$$

where the first three maps are standard maps defined as part of the definition of U. Consider a map $\mathrm{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+}) \to \mathrm{D}X$ which is non-zero. Then there exists some simplex $\theta\colon [e] \to [d]$ in Δ_{d+} which is not mapped to zero. Consider now the map $\theta'\colon [e+1] \to [d+1]$ which maps 0 to 0 and maps i to $\theta(i-1)+1$ for $i \geq 1$. This gives a simplex in $\Delta_{d+1,+}$ and so an element of the source of the composite above. Upon applying the first map, we get θ' again, and then upon applying the second map, we get the original $\theta\colon [e] \to [d]$ but in dimension $e+1$, then the original θ and then finally a non-zero element in X by our assumption above. This shows that $\mathrm{U}(X)_n \to \Omega\mathrm{U}(X)_{n+1}$ is injective in each simplicial degree. It remains to demonstrate surjectivity. The proof is similar. Suppose given a map $f\colon \mathrm{C}_\bullet(\Sigma^{\infty-n-1}\Delta_{d+1,+}) \to \mathrm{D}X$ and suppose that it satisfies the "$d_0 = d_1 \cdots d_{n+1} = *$" condition required for membership in $\Omega\mathrm{U}(X)_{n+1,d}$. We then need to define a map $g\colon \mathrm{C}_\bullet(\Sigma^{\infty-n}\Delta_{d+}) \to \mathrm{D}X$. Given $\theta\colon [e] \to [d]$, we map it to $f(\theta')$ where θ' is defined as above. One can check directly that this is indeed a map of complexes, and we see that, upon precomposition with the first three maps above, we get the original map f since, as before, $\theta' \mapsto \theta$ under the composite of the first three maps. This completes the proof that $\mathrm{U}\bar{\mathbf{E}}^{\dagger}_{\mathbf{st}}X$ is a strict Ω-spectrum.

Now, invoking Proposition 6.7 (iii), it remains to show that if $X \to Y$ is an inclusion of complexes where X and Y are of finite type, then $\mathrm{U}\bar{\mathbf{E}}^{\dagger}_{\mathbf{st}}Y \to \mathrm{U}\bar{\mathbf{E}}^{\dagger}_{\mathbf{st}}X$ is a levelwise fibration of spectra. Thus, for each $n \geq 0$, we desire lifts of the following squares:

(6.38)
$$\begin{tikzcd} \Lambda^i_d \arrow[r] \arrow[d] & (\mathrm{U}\bar{\mathbf{E}}^{\dagger}_{\mathbf{st}}Y)_n \arrow[d] \\ \Delta_d \arrow[r] \arrow[ur, dashed] & (\mathrm{U}\bar{\mathbf{E}}^{\dagger}_{\mathbf{st}}X)_n \end{tikzcd}$$

Via the earlier identifications, this amounts to a lift of the square:

(6.39)
$$\begin{tikzcd} \Lambda^i_d \arrow[r] \arrow[d] & (\mathrm{V}(\mathrm{D}Y))_n \arrow[d] \\ \Delta_d \arrow[r] \arrow[ur, dashed] & (\mathrm{V}(\mathrm{D}X))_n \end{tikzcd}$$

Here V is the functor $\mathsf{Ch}_{\overline{\mathbb{F}}_p} \to \mathsf{Sp}$ given by setting

$$\mathsf{V}(Z)_{n,d} := \mathsf{Ch}_{\overline{\mathbb{F}}_p}(\overline{\mathbf{C}}_\bullet(\Sigma^{\infty-n}\Delta_{d+}), Z)$$

which we note is a right adjoint to the spectral chains functor taken as a functor to simply chain complexes, forgetting the coalgebra structure. Considering the composite adjunction

$$
\begin{tikzcd}
\mathsf{Set}_\Delta \arrow[r, shift left, "\Sigma^{\infty-n}"] \arrow[r, phantom, "\perp"] & \mathsf{Sp} \arrow[l, shift left, "(-)_n"] \arrow[r, shift left, "\overline{\mathrm{C}}_\bullet"] \arrow[r, phantom, "\perp"] & \mathsf{Ch}_{\overline{\mathbb{F}}_p} \arrow[l, shift left, "\mathrm{V}"]
\end{tikzcd}
$$

we see that the above lifting problem is equivalent to one of the following form in chain complexes:

$$
\begin{tikzcd}
\overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Lambda^i_{d+}) \arrow[r] \arrow[d] & \mathrm{D}Y \arrow[d, "\mathrm{D}f"] \\
\overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Delta_{d+}) \arrow[r] \arrow[ur, dashed] & \mathrm{D}X
\end{tikzcd}
$$

Now, up to shifts, we have that $\overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Delta_{d+}) \cong \overline{\mathrm{C}}_\bullet(\Delta_{d+})$ and $\overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Lambda^i_{d+}) \cong \overline{\mathrm{C}}_\bullet(\Lambda^i_{d+})$. As a result, $\overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Delta_{d+})$ and $\overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Lambda^i_{d+})$ are acyclic chain complexes. Moreover, $\overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Lambda^i_{d+}) \to \overline{\mathrm{C}}_\bullet(\Sigma^{\infty-n}\Delta_{d+})$ is clearly an inclusion between free complexes. Thus the lefthand vertical map is a trivial cofibration in the standard projective model structure on chain complexes. On the other hand, since $X \to Y$ was an inclusion of complexes, $\mathrm{D}Y \to \mathrm{D}X$ is an epimorphism. Thus the lift exists, as desired. $\square$

As a result of the above, we have a derived adjunction:

$$\mathsf{hSp}^{\mathrm{op}} \rightleftarrows \mathsf{h}\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}$$

6.5 Resolvability Theorems

In this section, we study the derived cochains adjunction

$$\mathsf{hSp}^{\mathrm{op}} \rightleftarrows \mathsf{h}\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}$$

constructed in the previous section. We wish to show that, when restricted to the bounded below p-complete spectra of finite p-type, the map $\mathsf{hSp}^{\mathrm{op}} \to \mathsf{h}\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}$ is fully faithful. This is equivalent, for formal reasons, to showing that, for any such spectrum E, the "unit" of the derived adjunction $E \to (\text{der U}) \circ (\text{der } \overline{\mathrm{C}}^\bullet(-))(E)$ ("der" indicates a derived functor) is an isomorphism. For this reason, we make the following definition.

Definition 6.40. A spectrum E is said to be *resolvable* if the unit of the derived spectral cochains adjunction for this spectrum is an isomorphism.

To begin, we wish to prove that $\mathrm{H}\mathbb{F}_p$, and more generally $\Sigma^n\mathrm{H}\mathbb{F}_p$, $n \in \mathbb{Z}$, is resolvable. In order to do this, due to Proposition 6.15, we need not perform any replacement of $\Sigma^n\mathrm{H}\mathbb{F}_p$, but do need to perform a cofibrant replacement of $\overline{\mathrm{C}}^\bullet(\Sigma^n\mathrm{H}\mathbb{F}_p)$. We will do this by constructing a cell model for $\overline{\mathrm{C}}^\bullet(\Sigma^n\mathrm{H}\mathbb{F}_p)$. Fix $n \in \mathbb{Z}$. Intuitively, one expects that $\Sigma^n\mathrm{H}\mathbb{F}_p$ on the spectral side ought to correspond to $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$, or something similar, on the algebraic side. As per Proposition 6.32, we have an operation P^0 on the cohomology of the free algebra $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$. As per Proposition 6.24, P^0 always acts by the identity on spectral cochains. Moreover, we shall see that this is the only special circumstance which we need to take into account, in that we shall be able to construct our cell model for $\overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ by forcing this operation P^0 to be the identity.

First, recall that $\overline{\mathrm{C}}^{\bullet}(\mathrm{H}\mathbb{F}_p)$ is given by applying $-\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p$ to the following:

$$\lim(\cdots \to \mathrm{C}^{\bullet}(\mathrm{K}(\mathbb{F}_p,2))[-2] \to \mathrm{C}^{\bullet}(\mathrm{K}(\mathbb{F}_p,1))[-1] \to \mathrm{C}^{\bullet}(\mathrm{K}(\mathbb{F}_p,0)))$$

Thus, in particular $\overline{\mathrm{C}}^{0}(\mathrm{H}\mathbb{F}_p)$ is given by applying $-\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p$ to the following:

$$\lim(\cdots \to \mathrm{C}^{2}(\mathrm{K}(\mathbb{F}_p,2)) \to \mathrm{C}^{1}(\mathrm{K}(\mathbb{F}_p,1)) \to \mathrm{C}^{0}(\mathrm{K}(\mathbb{F}_p,0)))$$

Next, recall that, for each $m \geq 0$, $\mathrm{K}(\mathbb{F}_p,m)$ is the based simplicial set whose d-simplices are given by $\mathrm{Z}^m(\Delta_d;\mathbb{F}_p)$ (note that this is $*$ when $d<m$ and $\mathbb{F}_p$ when $d=m$). For each $m \geq 0$, we have a canonical fundamental class given by the cocycle k_m in $\mathrm{C}^m\mathrm{K}(\mathbb{F}_p,m)$ which sends $\alpha \in \mathrm{Z}^m(\Delta_m;\mathbb{F}_p)$ to $\alpha(\mathrm{id}_{[m]})$. Upon unravelling the definition of the structure maps for Eilenberg-MacLane spectra, we find that, for each $m \geq 0$, the map $\mathrm{C}^{m+1}(\mathrm{K}(\mathbb{F}_p,m+1)) \to \mathrm{C}^{m}(\mathrm{K}(\mathbb{F}_p,m))$ sends k_{m+1} to k_m. As such, we have a well-defined canonical element $(\cdots,k_2,k_1,k_0)$ in the inverse limit and so a well-defined canonical element $h_0=(\cdots,k_2,k_1,k_0)\otimes 1$ in $\overline{\mathrm{C}}^{0}(\mathrm{H}\mathbb{F}_p)$. More generally, with n as above, consider again $\overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$, which is given by applying $-\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p$ to the following:

$$\lim(\cdots \to \mathrm{C}^{\bullet}(\mathrm{K}(\mathbb{F}_p,n+2))[-2] \to \mathrm{C}^{\bullet}(\mathrm{K}(\mathbb{F}_p,n+1))[-1] \to \mathrm{C}^{\bullet}(\mathrm{K}(\mathbb{F}_p,n)))$$

We have that, in particular, $\overline{\mathrm{C}}^{n}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ is given by applying $-\otimes_{\mathbb{F}_p}\overline{\mathbb{F}}_p$ to the following:

$$\lim(\cdots \to \mathrm{C}^{n+2}(\mathrm{K}(\mathbb{F}_p,n+2)) \to \mathrm{C}^{n+1}(\mathrm{K}(\mathbb{F}_p,n+1)) \to \mathrm{C}^{n}(\mathrm{K}(\mathbb{F}_p,n)))$$

Once again, upon unravelling the definition of the structure maps for generalized Eilenberg-MacLane spectra, we find that, for each $m \geq n$, the map $\mathrm{C}^{m+1}(\mathrm{K}(\mathbb{F}_p,m+1)) \to \mathrm{C}^{m}(\mathrm{K}(\mathbb{F}_p,m))$ sends k_{m+1} to k_m. As such, we have a well-defined canonical element $(\cdots,k_{n+2},k_{n+1},k_n)$ in the inverse limit and so a well-defined canonical element $h_n=(\cdots,k_{n+2},k_{n+1},k_n)\otimes 1$ in $\overline{\mathrm{C}}^{n}(\Sigma^n\mathrm{H}\mathbb{F}_p)$. Note that, for each n, h_n is a cocycle, because each k_m, $m \geq 0$, is a cocycle.

Now, we shall construct our cell model for $\overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ by attaching a cell to the free algebra $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$ to set P^0 to act by the identity. Let i_n denote the degree n cocycle of $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$ given by the tensor $\mathrm{id} \otimes 1$. Let also p_n be a representative of the class $(1 - P^0)[i_n]$, and then denote also by the same symbol the map $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \to \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$ induced by the map $\overline{\mathbb{F}}_p[n] \to \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \colon 1 \mapsto p_n$. Now define an $\bar{\mathcal{E}}^{\dagger}_{\mathrm{st}}$-algebra J_n via the following pushout diagram:

$$\begin{tikzcd}
\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \arrow[r, "p_n"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \arrow[d] \\
\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \arrow[r] & J_n
\end{tikzcd}$$

This algebra J_n is our putative cell model for $\overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$. In order to show that it is indeed a model for these cochains in an appropriate sense, we construct a comparison map $J_n \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$. First, let $f \colon \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ denote the map induced by the map $\overline{\mathbb{F}}_p[n] \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p) \colon 1 \mapsto h_n$. Next, let q_n denote a degree $n+1$ element in $\overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ which is such that $\partial(q_n)$ is a representative of $(1 - P^0)[h_n]$ (such an element q_n exists since, as per Proposition 6.24, $(1 - P^0)[h_n]$ is zero). Denote by g the map $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ induced by the map $\mathrm{C}\overline{\mathbb{F}}_p[n] \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ which sends the degree n and $n+1$ generators, respectively, to p_n and q_n. Now, by checking the images of i_n, we have that the following square commutes:

$$\begin{tikzcd}
\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \arrow[r, "p_n"] \arrow[d] & \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \arrow[d, "f"] \\
\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \arrow[r, "g"'] & \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)
\end{tikzcd}$$

As such, we get an induced map:

$$a \colon J_n \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$$

The following result now makes precise that J_n is a cell model for $\overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$.

Proposition 6.41. *For each $n \in \mathbb{Z}$, the map $a \colon J_n \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ above is a quasi-isomorphism.*

Proof. Consider the composite:

$$\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \oplus_{\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]} \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \longrightarrow J_n \xrightarrow{a} \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$$

By Proposition 5.47, the first map is a quasi-isomorphism, and so it suffices to demonstrate that the composite, say c, is a quasi-isomorphism. Consider now instead the following composite:

$$\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \xrightarrow{b} \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \oplus_{\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]} \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \xrightarrow{c} \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$$

Here b is the canonical map from the first summand in the pushout. We claim that, upon taking cohomology, both b and $c \circ b$ are onto and have the same kernel. It suffices to do this as then c is clearly necessarily a quasi-isomorphism. Let ι denote the map $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \to \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n]$ and consider the following exact sequence:

$$0 \to \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \xrightarrow{p_n - \iota} \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \oplus \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \longrightarrow \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \oplus_{\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]} \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \to 0$$

By Proposition 6.31, we can identify the cohomology of $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$ with $\widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p[n]$, and by Propositions 6.31 and 6.29, we can identify the cohomology of $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \oplus \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n]$ also with $\widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p[n]$. Moreover, under this identification, the map corresponding to $p_n - b$ sends 1 to $1 - P^0$ and so, more generally, becomes right multiplication by $1 - P^0$. Noting that this map is injective (which follows from the fact that the Adem relations preserve length), it follows from the long exact sequence in cohomology that, on cohomology, the map b is onto with kernel the left ideal of $\widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p[n]$ generated by $1 - P^0$, which we note, by Proposition 6.27, coincides with the two-sided ideal generated by $1 - P^0$.

Now consider the composite $c \circ b$. Upon identifying once more the cohomology of $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$ with $\widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p[n]$, we have a map $\widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p[n] \to \mathrm{H}^{\bullet}(\overline{\mathrm{C}}^{\bullet}(\Sigma^n \mathrm{H}\mathbb{F}_p))$. By Propositions 6.24 and 6.27, we get an induced map:

$$(\widehat{\mathcal{B}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p[n])/(1 - P^0) \cong \mathcal{A} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p[n] \to \mathrm{H}^{\bullet}(\overline{\mathrm{C}}^{\bullet}(\Sigma^n \mathrm{H}\mathbb{F}_p))$$

Noting that 1 is mapped to the fundamental class $[h_n]$, by the standard calculation of the cohomology of Eilenberg-MacLane spectra, we have that this map is an isomorphism. As such, just as with b, at the level of cohomology, $c \circ b$ is onto with kernel the two-sided ideal generated by $1 - P^0$, and this completes the proof. $\square$

Having constructed our cofibrant replacement of $\overline{\mathrm{C}}^{\bullet}(\Sigma^n \mathrm{H}\mathbb{F}_p)$, we now need to consider how this replacement transforms under application of U. For this purpose, we have the following result.

Proposition 6.42. *We have the following:*

(i) $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \cong \Sigma^n \mathrm{H}\overline{\mathbb{F}}_p$ *and, under the identification,* $\mathrm{U}p_n$ *induces on* π^{st}_n *the map* $1 - \Phi$ *where* Φ *is the Frobenius automorphism of* $\overline{\mathbb{F}}_p$.

(ii) $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \sim *$ *or, more specfically,* $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n]$ *is a contractible Kan complex in each spectral degree.*

Proof. (i): In spectral degree m and simplicial degree d, we have:

$$\begin{aligned}
(\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n])_{m,d} &= \overline{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}(\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n], \overline{\mathbf{C}}^{\bullet}(\Sigma^{\infty-m}\Delta_{d+})) \\
&\cong \mathsf{Co}_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p[n], \overline{\mathbf{C}}^{\bullet}(\Sigma^{\infty-m}\Delta_{d+})) \\
&\cong \mathsf{Z}^n(\overline{\mathbf{C}}^{\bullet}(\Sigma^{\infty-m}\Delta_{d+})) \\
&= \mathsf{Z}^n(\mathbf{C}^{\bullet}(\Sigma^{\infty-m}\Delta_{d+}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \\
&\cong \mathsf{Z}^n(\mathbf{C}^{\bullet}(\Delta_{d+})[-m] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \\
&\cong \mathsf{Z}^{n+m}(\Delta_d; \overline{\mathbb{F}}_p) \\
&= (\Sigma^n \mathrm{H}\overline{\mathbb{F}}_p)_{m,d}
\end{aligned}$$

One can readily verify directly that the action of the simplicial operators coincide and that so do the spectral structure maps.

By Proposition 6.15 and Remark 6.6, we can compute the n^{th} stable homotopy group of $\Sigma^n\mathrm{H}\overline{\mathbb{F}}_p$ via the n^{th} unstable homotopy group of the space in spectral degree 0. Moreover, we find that, under the identification $\pi_n^{\mathrm{st}}(\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]) \cong \pi_n^{\mathrm{st}}(\Sigma^n\mathrm{H}\overline{\mathbb{F}}_p) \cong \overline{\mathbb{F}}_p$, an element $\lambda \in \overline{\mathbb{F}}_p$ corresponds to the class of the map

$$\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \longrightarrow \overline{\mathbf{C}}^{\bullet}(\Sigma^{\infty}\Delta_{n+}) \cong \overline{\mathbf{C}}^{\bullet}(\Delta_{n+})$$

which sends i_n to the cochain α which sends $\iota_n \in (\Delta_n)_n$ to λ. To act by $\mathrm{U}p_n$, we precompose with $p_n\colon \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \to \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]$. Thus, we need to compute the image of i_n under the following composite:

$$\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \xrightarrow{p_n} \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \longrightarrow \overline{\mathbf{C}}^{\bullet}(\Sigma^{\infty}\Delta_{n+}) \cong \overline{\mathbf{C}}^{\bullet}(\Delta_{n+})$$

To do this, we need to act by $1 - P^0$ on $\alpha \in \overline{\mathbf{C}}^{\bullet}(\Sigma^{\infty}\Delta_{n+})$. If we unravel the definition of the action of $1 - P^0$ on the cochains on the spectrum $\Sigma^{\infty}\Delta_{n+}$, we find that this action reduces to the action of $1 - P^0$ on the cochains on the space Δ_n and, moreover, in the proof of Proposition 3.46, we saw that the action of P^0 on the cochains on a space sends a cochain β to a cochain β' such that $\beta'(s) = \beta(s)^p$ for all simplices s. Thus, under the composite above, i_n maps to a cochain which sends ι_n to $1 - \lambda^p$, as desired.

(ii): By Proposition 6.29 (i), as $\mathrm{C}\overline{\mathbb{F}}_p$ is acyclic, the canonical map $\overline{\mathcal{E}}^{\dagger}_{\mathrm{st}}(0) \to \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n]$ is a quasi-isomorphism. Since $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n]$ is a cell algebra, by the semi-model categories result and Proposition 6.37, we have that $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \to \mathrm{U}\overline{\mathcal{E}}^{\dagger}_{\mathrm{st}}(0)$ is a weak equivalence of spectra. As $\overline{\mathcal{E}}^{\dagger}_{\mathrm{st}}(0)$ is the initial algebra, by the definition of U, we have that $\mathrm{U}\overline{\mathcal{E}}^{\dagger}_{\mathrm{st}}(0) = *$. Moreover, we saw in the proof of Proposition 6.37 that $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}X$ is a strict Ω-spectrum, and so a fibrant spectrum, when X

is a complex of finite type, so that $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n]$ is fibrant spectrum. Thus, by Proposition 6.7 (ii), $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \to \mathrm{U}\overline{\mathcal{E}}^{\dagger}_{\mathrm{st}}(0)$ is a levelwise weak equivalence, from which the desired result immediately follows. $\square$

We can now demonstrate the resolvability of $\Sigma^n\mathrm{H}\mathbb{F}_p$.

Proposition 6.43. *For each $n \in \mathbb{Z}$, $\Sigma^n\mathrm{H}\mathbb{F}_p$ is resolvable.*

Proof. Consider again the pushout square

$$\begin{tikzcd}
\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \arrow[r, "p_n"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \arrow[d] \\
\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \arrow[r] & J_n
\end{tikzcd}$$

and the map $J_n \to \overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p)$ which we constructed above. Upon applying U to the pushout square, as U is a left adjoint and maps to the opposite category of spectra, we get a pullback square of spectra as follows:

$$\begin{tikzcd}
\mathrm{U}J_n \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near end] & \mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n] \arrow[d, two heads] \\
\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \arrow[r, "\mathrm{U}p_n"'] & \mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n]
\end{tikzcd}$$

Here the vertical maps are fibrations because $\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\overline{\mathbb{F}}_p[n] \to \overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}\mathrm{C}\overline{\mathbb{F}}_p[n]$ is a cofibration between cell algebras and because, by Proposition 6.37, U maps cofibrations between cofibrant algebras to fibrations of spectra. By Proposition 6.41, the unit of the derived adjunction is represented by the composite

$$\Sigma^n\mathrm{H}\mathbb{F}_p \to \mathrm{U}\overline{\mathrm{C}}^{\bullet}(\Sigma^n\mathrm{H}\mathbb{F}_p) \to \mathrm{U}J_n$$

so that we need to show that this map is weak equivalence. We saw in the proof of Proposition 6.37 that $\mathrm{U}\overline{\mathbf{E}}^{\dagger}_{\mathbf{st}}X$ is a strict Ω-spectrum, and so a fibrant spectrum, when X is a complex of finite type. This implies that all spectra in the square above are fibrant. By Proposition 6.42 and the long exact sequence in stable homotopy groups, we have that $\pi_i^{\mathrm{st}}(\mathrm{U}J_n)$ is $\mathbb{F}_p$ when $i = n$, and zero otherwise. Thus, it suffices to show that the map $\Sigma^n\mathrm{H}\mathbb{F}_p \to \mathrm{U}J_n$, say η, is an isomorphism on π_n^{st}. This amounts to showing that the map

$$\mathsf{hSp}(\Sigma^\infty\mathbb{S}^n, \Sigma^n\mathrm{H}\mathbb{F}_p) \to \mathsf{hSp}(\Sigma^\infty\mathbb{S}^n, \mathrm{U}J_n) = \mathsf{hSp}^{\mathrm{op}}(\mathrm{U}J_n, \Sigma^\infty\mathbb{S}^n) \cong \mathsf{h}\overline{\mathcal{E}}^{\dagger}_{\mathrm{st}}\text{-}\mathsf{Alg}(J_n, \overline{\mathrm{C}}^{\bullet}(\Sigma^\infty\mathbb{S}^n))$$

induced by η and the derived adjunction isomorphism is bijective, or equivalently, injective. For each $\lambda \in \mathbb{F}_p$, consider the map $\sigma_\lambda\colon \Sigma^\infty \mathbb{S}^n \to \Sigma^n \mathrm{H}\mathbb{F}_p$ given by the map $\mathbb{S}^n \to \mathrm{K}(\mathbb{F}_p, n)$ which sends the unique non-denenerate n-simplex to the n-cocycle α on Δ_n defined by $\alpha(\mathrm{id}_{[n]}) = \lambda$. The images of these maps under the localization functor $\gamma_{\mathsf{Sp}}\colon \mathsf{Sp} \to \mathsf{hSp}$ give the p distinct maps in $\mathsf{hSp}(\Sigma^\infty \mathbb{S}^n, \Sigma^n \mathrm{H}\mathbb{F}_p)$. Fix $\lambda \in \mathbb{F}_p$ and consider σ_λ. Unravelling the definition of the above map, the image of $\gamma_{\mathsf{Sp}}(\sigma_\lambda)$ is computed as follows: form the composite $J_n \to \overline{\mathrm{C}}^\bullet(\Sigma^n \mathrm{H}\mathbb{F}_p) \to \overline{\mathrm{C}}^\bullet(\Sigma^\infty \mathbb{S}^n)$, where the second map is σ_λ and the first map is the adjoint of η, and then take the image of this map under the localization functor $\gamma_{\overline{\mathcal{E}}^\dagger_{\mathrm{st}}\text{-}\mathsf{Alg}}\colon \overline{\mathcal{E}}^\dagger_{\mathrm{st}}\text{-}\mathsf{Alg} \to \mathsf{h}\overline{\mathcal{E}}^\dagger_{\mathrm{st}}\text{-}\mathsf{Alg}$. We have that, for different values of λ, the maps $\overline{\mathrm{C}}^\bullet(\Sigma^n \mathrm{H}\mathbb{F}_p) \to \overline{\mathrm{C}}^\bullet(\Sigma^\infty \mathbb{S}^n)$ differ on cohomology, and thus so must the composite maps $J_n \to \overline{\mathrm{C}}^\bullet(\Sigma^n \mathrm{H}\mathbb{F}_p) \to \overline{\mathrm{C}}^\bullet(\Sigma^\infty \mathbb{S}^n)$, and as a result the images of these maps under $\gamma_{\overline{\mathcal{E}}^\dagger_{\mathrm{st}}\text{-}\mathsf{Alg}}$ must be distinct. Thus the above map is injective, and so bijective, as desired. $\square$

Above, we have demonstrated the resolvability of certain Eilenberg-MacLane spectra. We now demonstrate results which will also allow us to induct up Postnikov towers for more general resolvability results.

Proposition 6.44. *Let E be a spectrum and suppose that it can be described as the inverse limit of a diagram*

$$\cdots \to E_2 \to E_1 \to E_0$$

such that:

- *Each map $E_{n+1} \to E_n$ is a fibration and E_0 is fibrant.*
- *The canonical map* $\operatorname{colim} \overline{\mathrm{H}}^\bullet E_n \to \overline{\mathrm{H}}^\bullet E$ *is an isomorphism.*

Then E is resolvable whenever each of the E_n is resolvable.

Proof. Suppose that the E_n are resolvable. We can factor maps of $\overline{\mathcal{E}}^\dagger_{\mathrm{st}}$-algebras into relative cell inclusions followed by trivial fibrations. Applying this to the cotower $\overline{\mathrm{C}}^\bullet E_0 \to \overline{\mathrm{C}}^\bullet E_1 \to \overline{\mathrm{C}}^\bullet E_2 \to \cdots$ we get a diagram of $\bar{\mathcal{E}}^\dagger_{\mathrm{st}}$-algebras as follows:

$$\begin{tikzcd}
\overline{\mathbb{F}}_p \arrow[r, hook] & A_0 \arrow[r, hook] \arrow[d, two heads, "\sim"] & A_1 \arrow[r, hook] \arrow[d, two heads, "\sim"] & \cdots \\
& \overline{\mathrm{C}}^\bullet E_0 \arrow[r] & \overline{\mathrm{C}}^\bullet E_1 \arrow[r] & \cdots
\end{tikzcd}$$

Set $A = \operatorname{colim} A_n$. Then, by the assumption that $\overline{\mathrm{H}}^\bullet E \cong \operatorname{colim} \overline{\mathrm{H}}^\bullet E_n$, we have that the canonical map $A \to \overline{\mathrm{C}}^\bullet E$ is a quasi-isomorphism. Applying U, we have that $\mathrm{U}A$ is the inverse limit of the $\mathrm{U}A_n$ and have a commutative diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & E_1 \arrow[r] \arrow[d, "\sim"] & E_0 \arrow[d, "\sim"] \\
\cdots \arrow[r] & \mathrm{U}A_1 \arrow[r] & \mathrm{U}A_0
\end{tikzcd}$$

Here the vertical maps are the composites $E_n \to \mathrm{U}\overline{\mathbf{C}}^\bullet E_n \to \mathrm{U}A_n$ which are quasi-isomorphisms as the E_n are resolvable and these composites represent components of the unit of the derived adjunction. Moreover, since U maps cofibrations between cofibrant algebras to fibrations of spectra, each map in the bottom row is a fibration and each of the $\mathrm{U}A_n$ are fibrant. As weak equivalences between fibrant spectra are simply levelwise weak equivalences (see Proposition 6.7 (ii)), by the usual argument for inverse limits of weak equivalences of spaces along towers of fibrations, we find that the induced map on limits $E \to \mathrm{U}A$ is a weak equivalence. This map is the composite $E \to \mathrm{U}\overline{\mathbf{C}}^\bullet E \to A$ and so represents the unit of the derived adjunction, which is thus an isomorphism, and so E is resolvable, as desired. □

Next, we wish to consider resolvability of fibre products.

Proposition 6.45. *Let E be a spectrum and suppose that it can be written as a fibre product*

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E_2 \arrow[d] \\
E_1 \arrow[r] & F
\end{tikzcd}$$

such that:

- *E_1, E_2 and F are fibrant and E_1, E_2 are of finite p-type.*
- *The righthand vertical map $E_2 \to F$ is a fibration.*
- *There exists an N such that, for $n > N$, $(E_1)_n, (E_2)_n$ are connected and F_n is simply connected.*

Then E is resolvable whenever each of E_1, E_2, F is resolvable.

Proof. Suppose that E_1, E_2 and F are resolvable. For the diagram $\overline{\mathbf{C}}^\bullet(E_1) \leftarrow \overline{\mathbf{C}}^\bullet(F) \to \overline{\mathbf{C}}^\bullet(E_2)$, we take a cofibrant approximation:

$$\begin{tikzcd}
B \arrow[d, "\sim"] & A \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\
\overline{\mathbf{C}}^\bullet(E_1) & \overline{\mathbf{C}}^\bullet(F) \arrow[l] \arrow[r] & \overline{\mathbf{C}}^\bullet(E_2)
\end{tikzcd}$$

Suppose that we can show that the map $B \amalg_A C \to \overline{\mathbf{C}}^\bullet(E)$ is a quasi-isomorphism. Then the unit of the derived adjunction for $E_1 \times_F E_2$ is represented by the map $E \to \mathrm{U}(B \amalg_A C)$. We have the following commutative diagram:

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr] \arrow[dd] & & E_2 \arrow[dr] \arrow[dd] & \\
& \mathrm{U}(B \amalg_A C) \arrow[rr] \arrow[dd] & & \mathrm{U}C \arrow[dd] \\
E_1 \arrow[rr] \arrow[dr] & & F \arrow[dr] & \\
& \mathrm{U}B \arrow[rr] & & \mathrm{U}A
\end{tikzcd}$$

If E_1, E_2, F are resolvable, each of the maps $E_1 \to \mathrm{U}B$, $E_2 \to \mathrm{U}C$ and $F \to \mathrm{U}A$ is a weak equivalence. Combining this with the fact that U maps quasi-isomorphisms between cofibrant algebras to weak equivalences and using 2-out-of-3, we conclude that $E \to \mathrm{U}(B \amalg_A C)$ is a weak equivalence, as desired.

Thus it remains to show that the composite

$$B \amalg_A C \to \overline{\mathbf{C}}^\bullet(E_1) \amalg_{\overline{\mathbf{C}}^\bullet(F)} \overline{\mathbf{C}}^\bullet(E_2) \to \overline{\mathbf{C}}^\bullet(E)$$

is a quasi-isomorphism. Recall that the pushout may be computed via the bar construction

$$\mathrm{Bar}_n(B, A, C) = B \amalg \underbrace{A \amalg \cdots \amalg A}_{n \text{ factors}} \amalg C$$

in that, by Proposition 5.44, the induced map $\mathrm{N}(\mathrm{Bar}_\bullet(B, A, C)) \to B \amalg_A C$ from the normalization is a quasi-isomorphism. We wish to relate this pushout to the fibre product. We first construct cochains on the fibre product via a cobar construction. The cobar construction is defined as follows:

$$\mathrm{Cobar}^n(E_1, F, E_2) = E_1 \times \underbrace{F \times \cdots \times F}_{n \text{ factors}} \times E_2$$

This gives a cosimplicial spectrum with coface maps induced by diagonal maps and codegeneracies by projections. Applying cochains, we get a simplicial cochain complex:

$$\overline{\mathbf{C}}^\bullet(\mathrm{Cobar}^\bullet(E_1, F, E_2))$$

Considering E as a constant cosimplicial spectrum, we have an induced map

$$\mathrm{N}(\overline{\mathbf{C}}^\bullet(\mathrm{Cobar}^\bullet(E_1, F, E_2))) \to \overline{\mathbf{C}}^\bullet(E)$$

and we claim that this is a quasi-isomorphism. Expressing spectral cochains as an inverse limit of space level cochains, we have that the map $\overline{\mathbf{C}}^\bullet((\mathrm{Cobar}^\bullet(E_1, F, E_2))) \to \overline{\mathbf{C}}^\bullet(E)$ is an inverse limit of the maps:

$$\overline{\mathbf{C}}^\bullet(\mathrm{Cobar}^\bullet((E_1)_n, F_n, (E_2)_n))[-n] \to \overline{\mathbf{C}}^\bullet(E_n)[-n]$$

As in the proof of Lemma 5.2 in [Man01], for sufficiently large n, upon normalization, these maps are quasi-isomorphisms. Moreover, the maps forming the inverse limit tower are epimorphisms since E_1, E_2, F are fibrant. Thus, by a $\lim^1$ argument, we have that the map $\overline{\mathbf{C}}^\bullet((\mathrm{Cobar}^\bullet(E_1, F, E_2))) \to \overline{\mathbf{C}}^\bullet(E)$ between the inverse limits is also a quasi-isomorphism.

Now we relate the bar and cobar constructions. Using the various projection maps on $E_1 \times (F \times \cdots \times F) \times E_2$, we have maps:

$$\begin{aligned} B \amalg (A \amalg \cdots \amalg A) \amalg C \to \overline{\mathbf{C}}^\bullet(E_1) \amalg (\overline{\mathbf{C}}^\bullet(F) \amalg \cdots \amalg \overline{\mathbf{C}}^\bullet(F)) \amalg \overline{\mathbf{C}}^\bullet(E_2) \\ \to \overline{\mathbf{C}}^\bullet(E_1 \times (F \times \cdots \times F) \times E_2) \end{aligned}$$

These maps are quasi-isomorphisms because if we postcompose with the map

$$\overline{\mathbf{C}}^\bullet(E_1 \times (F \times \cdots \times F) \times E_2) \to \overline{\mathbf{C}}^\bullet(E_1 \amalg (F \amalg \cdots \amalg F) \amalg E_2)$$

induced by the canonical map

$$E_1 \amalg (F \amalg \cdots \amalg F) \amalg E_2 \to E_1 \times (F \times \cdots \times F) \times E_2$$

(given by a matrix with identity maps along the diagonal and zero maps elsewhere) and make the identification $\overline{\mathbf{C}}^\bullet(E_1 \amalg (F \amalg \cdots \amalg F) \amalg E_2) \cong \overline{\mathbf{C}}^\bullet(E_1) \amalg (\overline{\mathbf{C}}^\bullet(F) \amalg \cdots \amalg \overline{\mathbf{C}}^\bullet(F)) \amalg \overline{\mathbf{C}}^\bullet(E_2)$, we get a quasi-isomorphism by definition of A, B and C, and because the canonical map $E_1 \amalg (F \amalg \cdots \amalg F) \amalg E_2 \to E_1 \times (F \times \cdots \times F) \times E_2$ is a weak equivalence of spectra by the usual argument (coproducts and products of fibrant spectra are weakly equivalent). Now, it follows that we get a quasi-isomorphism of simplicial $\bar{\mathcal{E}}^\dagger_{\mathrm{st}}$-algebras

$$\mathrm{Bar}_\bullet(B, A, C) \to \overline{\mathbf{C}}^\bullet(\mathrm{Cobar}^\bullet(E_1, F, E_2))$$

and so a quasi-isomorphism:

$$\mathrm{N}(\mathrm{Bar}_\bullet(B, A, C)) \to \mathrm{N}(\overline{\mathbf{C}}^\bullet(\mathrm{Cobar}^\bullet(E_1, F, E_2)))$$

Finally, we can make use of this by noting that we have a commutative square as follows

$$\begin{tikzcd}
\mathrm{N}(\mathrm{Bar}_\bullet(B, A, C)) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{N}(\overline{\mathbf{C}}^\bullet(\mathrm{Cobar}^\bullet(E_1, F, E_2))) \arrow[d, "\sim"] \\
B \amalg_C A \arrow[r] & \overline{\mathbf{C}}^\bullet(E)
\end{tikzcd}$$

where the bottom map is the aforementioned composite $B \amalg_A C \to \overline{\mathbf{C}}^\bullet(E_1) \amalg_{\overline{\mathbf{C}}^\bullet(F)} \overline{\mathbf{C}}^\bullet(E_2) \to \overline{\mathbf{C}}^\bullet(E)$, and so we are done. $\square$

We can now extend our resolvability result to include further Eilenberg-MacLane spectra.

Proposition 6.46. *The Eilenberg-MacLane spectra $\Sigma^n \mathrm{H}A$, for $n \in \mathbb{Z}$, with $A = \mathbb{Z}/p^m$ for some $m \geq 1$ or $A = \mathbb{Z}_p^\wedge$ are resolvable.*

Proof. For $m \geq 1$ and $n \in \mathbb{Z}$, recall that we have well-known commutative squares as follows:

$$\begin{tikzcd}
\mathrm{K}(\mathbb{Z}/p^m, n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \mathrm{PK}(\mathbb{Z}/p, n+1) \arrow[d, two heads] \\
\mathrm{K}(\mathbb{Z}/p^{m-1}, n) \arrow[r] & \mathrm{K}(\mathbb{Z}/p, n+1)
\end{tikzcd}$$

(Here P denotes a path space, and, given the description of the Eilenberg-MacLane spaces before, these maps can be given entirely combinatorial descriptions.) An easy check shows that the maps in these squares in fact assemble together to yield maps of the Eilenberg-MacLane spectra, so that, for $m \geq 1$ and $n \in \mathbb{Z}$, we have commutative squares as follows:

$$\begin{tikzcd}
\Sigma^n \mathrm{H}\mathbb{Z}/p^{m+1} \arrow[r] \arrow[d] & \mathrm{P}\Sigma^{n+1}\mathrm{H}\mathbb{Z}/p \arrow[d, two heads] \\
\Sigma^n \mathrm{H}\mathbb{Z}/p^m \arrow[r] & \Sigma^{n+1}\mathrm{H}\mathbb{Z}/p
\end{tikzcd}$$

Moreover, the conditions of Proposition 6.45 are satisfied, so that, by induction, we have the desired result for $\mathbb{Z}/p^m$ for $m \geq 1$. Next, Proposition 6.44 gives us the case of $\Sigma^n \mathrm{H}\mathbb{Z}_p^\wedge$ using the following tower:

$$\Sigma^n \mathrm{H}\mathbb{Z}_p^\wedge = \lim(\cdots \to \Sigma^n \mathrm{H}\mathbb{Z}/p^m \to \cdots \to \Sigma^n \mathrm{H}\mathbb{Z}/p)$$

□

We are now finally able to provide the desired algebraic models of p-adic stable homotopy types.

Proposition 6.47. *All bounded below, p-complete spectra of finite p-type are resolvable. As a result, the cochains functor*

$$\overline{\mathsf{C}}^{\bullet} \colon \mathsf{Sp}^{\mathrm{op}} \to \overline{\mathcal{E}}_{\mathrm{st}}^{\dagger}\text{-}\mathsf{Alg}$$

induces a full embedding of the homotopy category of spectra into the derived category of $\overline{\mathcal{E}}_{\mathrm{st}}^{\dagger}$-algebras when we restrict to bounded below, p-complete spectra of finite p-type.

Proof. This follows by our resolvability results above, namely Propositions 6.46, 6.45 and 6.44, and the fact that bounded below, p-complete spectra of finite p-type admit Postnikov towers in which the fibres are $\Sigma^n \mathrm{H}A$, for $n \in \mathbb{Z}$, with either $A = \mathbb{Z}/p^m$ for some $m \geq 1$ or $A = \mathbb{Z}_p^{\wedge}$. □

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