E_∞-Rings and Modules in Kan Spectral Sheaves

by

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In loving memory of my grandmother, Qinfeng Huang.

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ABSTRACT

This thesis sets up the foundations of a theory of rings and modules on sheaves of spectra over topological spaces. The theory is based on Kan spectra, which is better behaved sheaf-theoretically, and a rigid smash product on Kan spectra is constructed, and is well-behaved enough for discussing E_∞-rings and their modules. Moreover, this thesis also develops localization on the homotopy category of sheaves of Kan spectra. Using the machinery of localization, the derived category of Kan spectral sheaves is defined and is compatible with the smash product. The main result of the thesis is building a symmetric monoidal structure on the derived category of modules over an E_∞-ring in Kan spectral sheaves.

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CHAPTER I

Introduction

The purpose of this thesis is to set up the foundations of spectral algebra in categories of sheaves of spectra over topological spaces. By spectral algebra, we mean the branch of mathematics which studies the analogues to the constructions in classical commutative algebra, e.g. localization and completion, in derived categories of spectra, or the *stable homotopy category*, and of their sheaves. Specifically, we provide "point-set" level constructions as well as methods of calculations to the following notions:

1. derived categories of sheaves of spectra, and derived functors on sheaves of spectra;

2. strict and derived Verdier duality for sheaves of spectra in the special case of locally closed embedding; and

3. operadic smash product in sheaves of spectra.

We emphasize that the approach using Kan's combinatorial spectra still remains the only one to-date which can provide a fully functional theory of sheaves of spectra; no such theory of sheaves based on May spectra has been successfully developed beyond the case of locally constant sheaves.

While this approach faces the challenge that Kan's spectra lack a smash product which is associative and commutative on the point-set level, a major achievement of this thesis is the definition of an operadic smash product on these spectra, as well as on their sheaves, which gives a well-behaved point-set model of the symmetric monoidal smash product in the derived categories. This provides a solid framework for the study of ring and module spectra, and for rigid constructions in the context of *spectral algebra on sheaves.*

More detailed introductions of some of the key notions mentioned above are in order.

1.1 Spectral algebra

Spectral algebra, as termed by M. Hopkins and J. Lurie, mainly studies, among other structures, *highly structured ring spectra.* Highly structured ring spectra were first termed "brave new rings" by F. Waldhausen, and extensively studied since by J. Peter May, A. Elmendorf, I. Kriz, M. Mandell, S. Schwede, B. Shipley, J. Smith, among others.

In the early development of the theory of spectra, for example the construction by M. Boardman in his PhD thesis (later written up by his student R. Vogt in [33]), and a later more extensive treatment by F. Adams [1], a stable homotopy category of spectra is constructed, and is found extremely useful in generalized cohomology theory. However, Boardman's and Adams' treatments have the same defect that, the underlying point-set models lack some very basic categorical properties, for example do not have all limits and colimits. Furthermore, in these models of spectra, the smash product is only associative and commutative after passing to the derived category, but not generally so on the point-set level. As a consequence, although the notions of ring and module spectra have already been introduced in [1] and have many useful applications, these models are inadequate for generalizing various constructions in commutative algebra—for example, while the stable homotopy category is triangulated, the cofiber (or "cone") of a morphism of module spectra need not be a module spectrum.

These defects are fixed by J. Peter May and his collaborators by giving "rigid" point-set models of the stable homotopy category. May arrives at the notion of what are now called Lewis–May spectra, or simply *May spectra* (see [18]); i.e. sequences of (based) topological spaces Z_n together with *homeomorphisms* $\rho_n : Z_n \cong \Omega Z_{n+1}$. Moreover, following the introduction of operads in [23], Elmendorf–Kriz–Mandell–May [10] models the stable homotopy category by $\mathbb{S}$-*modules*, which are spectra equipped with actions of the linear isometry operad $\mathscr{L}$, and constructs on them an *operadic smash product*, which is symmetric monoidal on the point-set level.

The category of $\mathbb{S}$-modules, together with alternative constructions of *symmetric spectra* [14] and *orthogonal spectra* [20], sets up the foundations of spectral algebra, which provides a deep homotopy theoretic generalization of classical commutative algebra.

In this thesis, we seek to further extend the foundations of spectral algebra to sheaves of spectra, or simply spectral sheaves*.*

1.2 Kan spectral sheaves

In addition to generalized cohomology, ordinary cohomology admits a generalization in another direction, namely, *abelian sheaf cohomology*, which reflects not only global but also local properties of topological spaces (or more generally, Grothendieck sites). Recall that, in generalized cohomology the coefficient of an abelian group A, or equivalently of the corresponding Eilenberg–MacLane spectrum HA, is replaced by a general spectrum; on the other hand, in sheaf cohomology one would instead replace the coefficient by a *sheaf* of abelian groups, or simply called abelian sheaf. It is thus natural to speculate a common generalization to both generalized cohomology and sheaf cohomology, via some notion of sheaves of spectra.

Naive attempt using sheaves of May spectra, which might seem the most natural at first, encounters technical difficulty of *sheafification*. Although a sheafification functor, or the left adjoint functor to the inclusion of sheaves into presheaves, must exist for sheaves of May spectra by the general categorical framework of Freyd–Kelly [11], its interaction with homotopy, even just on cell sheaves, seems very much intractable.

Another inadequacy of sheaves of May spectra could perhaps be seen heuristically by analogy with the case of abelian sheaves. The category of abelian sheaves has enough injectives, given by the *Godement resolution* [12], but it does not have enough *projectives* (=cofibrant replacement) for general base space. On the other hand, the category of May spectra, just like in the category of topological space, has a cofibrant approximation by cell spectra. It is therefore natural to consider left derived functors on sheaves of May spectra, while right derived functors, such as sheaf cohomology, are much more significant in abelian sheaves.

There have also been different approaches to generalized sheaf cohomology, the most successful ones of which include R. Thomason's approach [31] via *pre*sheaves of fibrant spectra. Thomason defines, under certain conditions, his version of generalized sheaf cohomology via the Bousfield–Friedlander Ω-spectra [4], with the use of the Godement resolutions, which only makes use of stalks and requires only filtered colimits and products. However, Thomason's approach is incapable of dealing with functors which cannot be determined on stalks, such as direct images of sheaves. In fact, since the category of Ω-spectra does not have arbitrary limits, it is already difficult to even discuss sheaves of Ω-spectra.

This then begs an alternative point-set model of the stable homotopy category which

1. has all limits and colimits,
2. is Eckmann–Hilton dual to May spectra, in the sense that all objects are *cofibrant* and have *fibrant* resolution, and
3. admits a sheafification that is easy to work with.

We propose the use of D. Kan's combinatorial spectra [16], or simply *Kan spectra*, which satisfies all the properties above. Originally introduced in 1960s, Kan spectra are obtained as direct stabilization of (based) simplicial sets, just as May spectra are to topological spaces. In the 70s, K. Brown [5] develops a fully functional theory of sheaves of Kan spectra, or simply *Kan spectral sheaves*, and defines generalized sheaf cohomology as well as direct image functors. Brown's category is later applied by Brown–Gersten [6] to algebraic K-theory, and by R. Piacenza [26] to local systems of Kan spectra, which is an alternate approach to *parametrized spectra* (see [21]).

In Brown's theory, the derived category of Kan spectral sheaves is constructed using the machinery of "category of fibrant objects", which is, roughly speaking, a half the structure of a model category. In [8], the foundation of the derived category of Kan spectral sheaves is revisited. The technique of [8] is now better understood via the language of *localization* (compare with *co*localization) of categories. In Chapter II, we give a detailed account of the results in [8], and apply the same methods to generalize to the case of *simplicial* Kan spectral sheaves.

The theory of derived category and right derived functors based on localization will be discussed carefully in Appendix A.2.

1.3 Locally closed Verdier duality

An immediate benefit of the foundational improvement from Brown's theory to localization is the ease of defining right derived functors, including the derived direct and inverse images functors commonly seen in abelian sheaf theory.

More specifically, in order to apply Brown's machinery of a category of fibrant objects to right-derive a functor, it is necessary for that functor to preserve weak equivalences of a certain subclass of flabby sheaves. For instance, the global section functor Γ only preserves weak equivalences of flabby sheaves which are (homotopically) *bounded below*, and thus Brown's theory only produces the generalized sheaf cohomology functor on the subcategory of the derived category of Kan spectral sheaves, on bounded-below objects.

On the other hand, the main result of [8] (see Corollary II.52) asserts the following:

> *A functor defined on the category of Kan spectral sheaves is right-derivable if and only if it preserves strong homotopy equivalences, in which case its right derived functor can be computed by the Godement resolution.*

This allows us to construct derived functors by first defining them on the point-set level and then applying the Godement resolution. Using this machinery, we construct, for a continuous map $f : X \to Y$ between locally compact Hausdorff spaces, the *direct image* functor f_* and the *proper direct image* functor $f_!$. Both of them are verified to preserve strong homotopies, and thus their derived functors Rf_* and $Rf_!$ are well-defined.

We also construct the inverse image functor f^*, which is *exact* in the sense that it preserves stalk-wise equivalences. In particular, as in the case of classical abelian sheaf theory, we have the adjunction (f^*, Rf_*).

Classically, *Verdier duality* states that the derived direct image functor $Rf_!$ has a *right* adjoint $f^!$. While the right adjoint in general does not exist on the point-set level, it does in the special case where $f = i$ is a *locally closed embedding*. This special case also generalizes easily to Kan spectral sheaves, as we may again define the functor $i^!$ first on the point-set level following the classical construction, and then pass to the derived category to obtain the right derived functor $Ri^!$. Note that in this case the functor $i_!$ is now exact, and as a result we have an adjunction $(i_!, Ri^!)$, which is a key special case of Verdier duality for Kan spectral sheaves.

This establishes a crucial part of the *six-functor formalism* on Kan spectral sheaves on locally closed subspaces of a locally compact Hausdorff topological space. In particular, on a *stratified space*, for instance a complex analytic variety, we can make sense of *constructible Kan spectral sheaves*, and in essence the "recollement" situation of [3]. We will discuss these results in Chapter III.

We conclude with the remark that the six-functor formalism makes possible the considerations of a *perverse t-structure* on constructible Kan spectral sheaves, whose heart provides *perverse Kan spectral sheaves* and thus opens up the possibility of *generalized intersection cohomology*.

1.4 Operadic smash product of Kan spectra

One challenge of Kan spectra, compared to other approaches to the stable homotopy category, particularly May spectra, is the substantial difficulty of defining a *symmetric monoidal smash product*. Such difficulty arises even when smashing a

Kan spectrum with a simplicial set. The reason is that, in contrast to the case of topological spaces, the simplicial suspension functor Σ does not commute with the smash product $\wedge$ of simplicial sets. Moreover, although there exists a comparison map

$$\Sigma(K \wedge T) \xleftarrow{\sim} (\Sigma K) \wedge T \tag{1.1}$$

which is also a weak equivalence, its direction is opposite to the desirable one to assemble a spectrum. Consequently, given a simplicial set K and a Kan spectrum Z, one already encounters an infinite staircase diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] \\
 & \Sigma(K \wedge \Sigma Z_{n-2}) \arrow[r] \arrow[d] & \Sigma^2(K \wedge T_{n-2}) \\
K \wedge \Sigma Z_{n-1} \arrow[r] \arrow[d] & \Sigma(K \wedge Z_{n-1}) & \\
K \wedge Z_n & &
\end{tikzcd} \tag{1.2}$$

as first observed by Kan [16] in construction of his "reduced product", which we call the *star product* and discuss in detail in Section 2.1.5.

Attempts to rescue the smash product, at least up to homotopy, include the works of Kan–Whitehead [17] and Burghelea–Deleanu [7]. In their approach, the smash product (which they call the "reduced join") is constructed, up to homotopy, from a combinatorial "bi-spectrum", which is, roughly speaking, a Kan spectrum which also has face and degeneracy operators indexed by negative integers. While it provides a model of the smash product of Kan spectra, it is too lax in nature to be useful in spectral algebra.

It is the main purpose of this thesis to take up the task of constructing, on the *point-set level, a workable smash product of Kan spectra, and extending to Kan spectral sheaves.* The development of such a theory occupies Chapter IV; we shall give a brief outline of our method.

Our construction is loosely analogous to that in [10]. As the first step, we seek the space (simplicial set) parametrizing various models of the smash product of Kan spectra; morally it should play the role of the linear isometry operad $\mathscr{L}$ for May spectra. Specifically, even though the comparison map (1.1) goes the "wrong" way, using various staircase diagrams (1.2), Adams' construction of smash product nonetheless applies, and produces different models of the smash product indexed by *Adams data* (which is originally termed by Adams as "handicrafted data"). On the other hand, while there is certainly no strict analogue to $\mathscr{L}$ in the simplicial world due to the rigidity of the simplex category Δ (for instance, the simplicial coordinates do not even admit permutations), we may nevertheless mimic the "universe extension functor" by inserting suspension coordinates into a Kan spectrum; such data of insertion are referred to as *slackening.* Combining these two data, we may form the categories of *slackened Adams data*, one for each "arity" of the smash product, which are shown in Section 4.1.2 to be contractible in the sense of [27].

Consequently, we obtain by taking the simplicial nerves a sequence of (weakly) contractible simplicial sets $\{\mathscr{A}(n)\}$. It is a curious discovery that this collection of such simplicial sets do not have an operad structure, but rather forms a *partial operad*, in the sense that composition maps are not defined entirely on the product simplicial sets but only certain subsets of them. The notion of partial operads were first explored in [24], and the little convex bodies operad there were superceded by Stein's operad [30]. In the spirit of May–Thomason [22], we construct in Section 4.1.3 a machinery of "rectifying" any such partial operad to an operad of the same weak homotopy type. This way, the partial operad $\mathscr{A}$ is rectified into a genuine E_∞-operad $\overline{\mathscr{A}}$.

However, for our purpose of constructing the (partial) operadic smash product on the categories of rings and modules based on Kan spectra, rectification of partial operads is largely unnecessary, since it suffices to consider the monad associated to the partial operad $\mathscr{A}$. We will, in Sections 4.1.4 and 4.2.1, construct the smash product via the two-sided bar construction of monads. This technique follows an unpublished preprint, which was a precursor to [10].

Finally, we will construct the derived categories of rings and modules, as well as the *derived smash product* of modules over a fixed ring spectrum. This uses the homotopy theory of (simplicial) Kan spectral sheaves developed in Chapter II, along with the techniques of derived categories and derived functors in the presence of *both* a localization *and* a colocalization. We will verify, in Sections 4.1.6 and 4.2.2, that the derived categories of Kan spectra and of their sheaves, together with the cofiber structure and the derived smash product, form *tensor triangulated categories* in the sense of [2]. The general categorical results for this construction is discussed in Appendix A.3. This concludes our journey of setting up the foundations of spectral algebra in sheaves of spectra.

CHAPTER II

Kan Spectra and Kan Spectral Sheaves

2.1 Kan spectra

2.1.1 Definition

Let Δ denote the simplex category, whose objects are nonempty finite linear orders $\mathbf{n} = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N}_0$ and whose morphisms are order-preserving functions $\rho : \mathbf{m} \to \mathbf{n}$. Consider the endofunctor

$$\Phi : \Delta \longrightarrow \Delta$$

sending each object $\mathbf{n}$ to $\mathbf{n+1}$ and each morphism $\rho : \mathbf{m} \to \mathbf{n}$ to the function

$$\Phi(\rho)(i) = \begin{cases} \rho(i) & \text{if } i \leq m, \\ n+1 & \text{if } i = m+1. \end{cases}$$

We define the *stable simplex category* Δ_{st} to be the strict colimit of the diagram

$$\Delta \longrightarrow \Delta \longrightarrow \cdots \tag{2.1}$$

in the category of small categories. We denote by $\Phi^{\infty-\ell}$ the canonical functor from the ℓth copy of Δ in Diagram (2.1) to Δ_{st}. Note that we could extend Diagram (2.1) infinitely to the left without changing the colimit, thus there is a canonical functor $\Phi^{\infty-\ell} : \Delta \to \Delta_{\mathrm{st}}$ for any $\ell \in \mathbb{Z}$.

We give a more explicit description of the stable simplex category Δ_{st}. Its objects are in bijection with $\mathbb{Z}$ via the correspondence $\Phi^{\infty-\ell}(\mathbf{n}) = \mathbf{n} - \ell$. Recall that morphisms in Δ are generated by the face maps $\delta_i^n : \mathbf{n-1} \to \mathbf{n}$ and the degeneracy maps $\sigma_i^n : \mathbf{n+1} \to \mathbf{n}$ for $0 \leq i \leq n$ which satisfy the usual simplicial relations. Since for any $m, n \in \mathbb{N}_0$ and $i, j \in \mathbb{Z}$ we have

$$\begin{aligned} \operatorname{Hom}_{\Delta_{\mathrm{st}}}\left(\Phi^{\infty-i}(\mathbf{m}), \Phi^{\infty-j}(\mathbf{n})\right) &= \varinjlim_{k \geq i,j} \operatorname{Hom}_{\Delta}\left(\Phi^{k-i}(m), \Phi^{k-j}(n)\right) \\ &= \varinjlim_{k \geq i,j} \operatorname{Hom}_{\Delta}(\mathbf{m}-\mathbf{i}+\mathbf{k}, \mathbf{n}-\mathbf{j}+\mathbf{k}), \end{aligned} \tag{2.2}$$

the morphisms in Δ_{st} are generated by the maps $\Phi^{\infty-k}(\delta_i^{n+k}) : \mathbf{n-1} \to \mathbf{n}$ and $\Phi^{\infty-k}(\sigma_i^{n+k}) : \mathbf{n+1} \to \mathbf{n}$ for $k \geq -n$ and $0 \leq i \leq n+k$. Note that $\Phi(\delta_i^n) = \delta_i^{n+1}$ and $\Phi(\sigma_i^n) = \sigma_i^{n+1}$, the maps are in fact independent of k. Thus, morphisms in Δ_{st} are generated by the *face maps* $d_i := \Phi^{\infty-k}(\delta_i^{n+k})$ and the *degeneracy maps* $s_i := \Phi^{\infty-k}(\sigma_i^{n+k})$ for $i \geq 0$, which again satisfy the usual simplicial relations. A composition of face (resp. degeneracy) maps is called an *iterated face* (resp. *degeneracy) map*.

Furthermore, for each $m, n \in \mathbb{Z}$ and $k \geq -n$, we may extend each function $\rho \in \operatorname{Hom}_{\Delta}(\mathbf{m}+\mathbf{k}, \mathbf{n}+\mathbf{k})$ to an order-preserving function $\Phi^{\infty-k}(\rho)$ by sending each $s > m+k$ to $n-m+s$. Then using (2.2), a morphism $f \in \operatorname{Hom}_{\Delta_{\mathrm{st}}}(\mathbf{m}, \mathbf{n})$ can be identified as an order-preserving function $f : \mathbb{N}_0 \to \mathbb{N}_0$ that is eventually bijective, that is, $f(s+1) = f(s)+1$ for s sufficiently large. In particular, f can be decomposed into a surjective map followed by an injective map just as in Δ, and again a surjective (resp. injective) map is precisely an iterated face (resp. degeneracy) map.

Let Set_* denote the category of based sets and based functions; the basepoint will be denoted by $*$.

Definition II.1. A (based) *stable simplicial set* is a functor $T : \Delta_{\mathrm{st}}^{\mathrm{op}} \to \mathsf{Set}_*$. The category of based stable simplicial set is the functor category $\Delta_{\mathrm{st}}^{\mathrm{op}}\text{-}\mathsf{Set}_*$. A *Kan spectrum* is a based stable simplicial set Z satisfying an additional finiteness property: for all $x \in Z(n)$, there exists $k \in \mathbb{Z}$ such that $d_i(x) = *$ for $i > k$.

For a Kan spectrum X, we say an element $x \in X(n)$ has *degree*

$$\deg(x) = \min\{k | d_i(x) = * \text{ for } i > k\} = \max\{i | d_i(x) \neq *\}.$$

The category $\mathscr{S}$ of Kan spectra is the full subcategory of $\Delta_{\mathrm{st}}^{\mathrm{op}}\text{-}\mathsf{Set}_*$ on Kan spectra. Note that $\mathscr{S}$ is a coreflective subcategory of $\Delta_{\mathrm{st}}^{\mathrm{op}}\text{-}\mathsf{Set}_*$, that is, the natural inclusion functor has a right adjoint given by passing to the based stable simplicial subset satisfying the required finiteness property. In particular, the category $\mathscr{S}$ has all (small) limits and colimits.

Example II.2. The *sphere spectrum* $\mathbb{S}$ is the free Kan spectrum generated by a single element $\alpha = \alpha_0 \in \mathbb{S}(0)$. Specifically, we have

$$\mathbb{S}(n) = \begin{cases} \{\alpha_n, *\} & \text{if } n \geq 0, \\ \{*\} & \text{if } n < 0, \end{cases}$$

with faces $d_i(\alpha_n) = \alpha_{n-1}$ $(n \geq 1)$ and degeneracies $s_i(\alpha_n) = \alpha_{n+1}$ $(n \geq 0)$.

Definition II.3. Let Z be a Kan spectrum. An element $z \in Z(n) \setminus *$ is *degenerate* if there exists an element $y \in Z(m)$ and an iterated degeneracy $s \in \mathrm{Hom}_{\Delta_{\mathrm{st}}}(\mathbf{n}, \mathbf{m})$ such that $z = s(y)$; z is *non-degenerate* otherwise.

Analogous to the Eilenberg–Zilber lemma in the case of simplicial sets, we have the following:

Lemma II.4 (Eilenberg–Zilber lemma for Kan spectra)**.** *Let Z be a Kan spectrum and $z \in Z(n) \setminus *$ an element. Then there exist a unique non-degenerate element $y \in Z(m)$ and a unique iterated degeneracy $s \in \mathrm{Hom}_{\Delta_{\mathrm{st}}}(\mathbf{n}, \mathbf{m})$ such that $z = s(y)$.*

The proof will be postponed to page 18, after we introduce an alternate description of Kan spectra (see Proposition II.9).

2.1.2 Loop and suspension. Spectrification

Given a based simplicial set $K : \Delta^{\mathrm{op}} \to \mathsf{Set}_*$, the *loop space* ΩK is the simplicial set given by

$$(\Omega K)(n) = \{x \in K(n+1) | d_{n+1}(x) = *\},$$

and $d_i : (\Omega K)(n) \to (\Omega K)(n-1)$ and $s_i : (\Omega K)(n) \to (\Omega K)(n+1)$ the restrictions of $d_i : K(n+1) \to K(n)$ and $s_i : K(n+1) \to K(n+2)$. The assignment $K \mapsto \Omega K$ defines a functor

$$\Omega : \Delta^{\mathrm{op}}\text{-}\mathsf{Set}_* \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*,$$

called the *loop functor*. Note that an element $x \in \Omega T(n)$ is non-degenerate in ΩT if and only if, when regarded as an element in $T(n+1)$, it is non-degenerate in T.

The loop functor has a left adjoint Σ, called the *suspension functor*. Explicitly, the suspension functor is given as follows. Let Δ_0 be the subcategory of Δ consisting of the same objects $\mathbf{n}$ and morphisms $\rho : \mathbf{m} \to \mathbf{n}$ such that $\rho(m-1) \leq n-1$, or equivalently, $\rho^{-1}(n)$ has at most one element m. In particular, any morphism of the form $\Phi(\rho)$ is contained in Δ_0. Define a functor

$$\Sigma_0 : \Delta^{\mathrm{op}}\text{-}\mathsf{Set}_* \longrightarrow \Delta_0^{\mathrm{op}}\text{-}\mathsf{Set}_*$$

by

$$\Sigma_0 K(n) := \begin{cases} * & \text{if } n = 0, \\ K(n-1) & \text{if } n \geq 1. \end{cases}$$

A morphism of the form $\Phi(\rho)$ acts on $\Sigma_0 K$ the same way as ρ does on K, and any other morphism acts trivially by sending all elements to $*$. Denote by $\iota : \Delta_0^{\mathrm{op}} \hookrightarrow \Delta^{\mathrm{op}}$

the opposite of the inclusion functor, then we define

$$\Sigma K \coloneqq \mathrm{Lan}_\iota(\Sigma_0 K) = \iota_\sharp(\Sigma_0 K)$$

as the left Kan extension along ι. To see that Σ is left adjoint to Ω, observe

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}(\Sigma K, L) = \mathrm{Hom}_{\Delta_0^{\mathrm{op}}\text{-}\mathsf{Set}_*}(\Sigma_0 K, \iota^* L) = \mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}(K, \Omega L).$$

Notice that Σ_0 raises the dimensions all non-degenerate simplices by one, and $\iota_\sharp$ freely adds in degenerate simplices to the last face of each simplex to form a simplicial set.

Proposition II.5 ([16, Prop. 2.3]). *Write $|?| : \Delta^{\mathrm{op}}\text{-}\mathsf{Set}_* \to \mathsf{Top}_*$ the geometric realization functor of simplicial sets, and $\Sigma : \mathsf{Top}_* \to \mathsf{Top}_*$ the usual suspension functor on spaces given smashing by $(S^1, 1)$. Then there is a canonical natural isomorphism of functors*

$$\Sigma|?| \cong |\Sigma(?)| : \Delta^{\mathrm{op}}\text{-}\mathsf{Set}_* \longrightarrow \mathsf{Top}_*,$$

from the category of based simplicial sets to the category of based spaces.

Proof. Since every based simplicial set is a colimit of Δ^n_+, and all three functors involved are left adjoints and thus commute with colimits, it suffices to observe that $\Sigma\left|\Delta^n_+\right|$ is naturally homeomorphic to $\left|\Sigma\left(\Delta^n_+\right)\right|$. $\square$

Corollary II.6. *For any based simplicial set K, there is a functorial inclusion*

$$|\Omega(K)| \hookrightarrow \Omega|K|.$$

Proof. The desired map is the adjoint to the composite

$$\Sigma|\Omega(K)| \xrightarrow{\cong} |\Sigma\Omega(K)| \xrightarrow{|\epsilon_K|} |K|,$$

where the first map is the isomorphism in Proposition II.5, and the second map is the geometric realization of the counit of the adjunction. $\square$

Proposition II.7. *The unit of adjunction*

$$\eta : \mathrm{Id} \longrightarrow \Omega\Sigma$$

is a natural isomorphism.

Proof. It suffices to observe that η preserves non-degenerate elements of each degree since both Ω and Σ do. $\square$

Now we consider the interaction between $\Delta^{\mathrm{op}}_{\mathrm{st}}\text{-}\mathsf{Set}_*$ and $\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*$. Given a based stable simplicial set $T : \Delta^{\mathrm{op}} \to \mathsf{Set}_*$ and $k \in \mathbb{Z}$, define a simplicial set $\Omega^{\infty+k}T$ by:

$$(\Omega^{\infty+k}T)(n) \coloneqq \{x \in T(n+k) | d_m(x) = * \text{ for } m > n\},$$

and set

$$d_i : (\Omega^{\infty+k}T)(n) \to (\Omega^{\infty+k}T)(n-1)$$
$$s_i : (\Omega^{\infty+k}T)(n) \to (\Omega^{\infty+k}T)(n+1)$$

to be the restrictions of $d_i : T(n+1) \to T(n)$ and $s_i : T(n+1) \to T(n+2)$, respectively. The assignment $T \mapsto \Omega^{\infty+k}T$ defines a functor

$$\Omega^{\infty+k} : \Delta^{\mathrm{op}}_{\mathrm{st}}\text{-}\mathsf{Set}_* \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*.$$

Lemma II.8. *The functor $\Omega^{\infty+k}$ has a left adjoint denoted $\Sigma^{\infty+k}$.*

In the special case $k = 0$, the functor Σ^{∞} is called the *shift suspension functor*.

We postpone the proof of Lemma II.8 to page 20, after the introduction of Kan prespectra and spectrification (see Proposition II.10).

From the definition, we have

$$\Omega^{\ell}\Omega^{\infty+k} = \Omega^{\infty+k+\ell} \tag{2.3}$$

and consequently $\Sigma^{\infty+k}\Sigma^{\ell} = \Sigma^{\infty+k+\ell}$. Using the functors $\Omega^{\infty+k}$, we give an equivalent description of the category $\mathscr{S}$ of Kan spectra. Consider the category $\mathscr{S}'$ whose objects are sequences of simplicial sets $\{Z_n\}_{n\in\mathbb{Z}}$ together with structure maps

$$\rho_n : Z_n \xrightarrow{\cong} \Omega Z_{n+1},$$

and whose morphisms from $\{Z_n\}$ to $\{T_n\}$ are sequences $\{f_n : Z_n \to T_n\}_{n\in\mathbb{Z}}$ of maps between simplicial sets commuting with the structure maps ρ_n.

Proposition II.9. *The category $\mathscr{S}$ is canonically equivalent to the category $\mathscr{S}'$.*

Proof. Given a Kan spectrum Z, the sequence $\{\Omega^{\infty-n}Z\}$ of simplicial sets gives an object in $\mathscr{S}'$ by (2.3), and this assignment is functorial.

Conversely, given an object $\{Z_n\}$ in $\mathscr{S}'$, define a set $Z(n)$ by

$$Z(n) = \varinjlim_{k} Z_k(n+k),$$

where the colimit is taken over the diagram

$$\cdots \hookrightarrow Z_k(n+k) \cong (\Omega Z_{k+1})(n+k) \hookrightarrow Z_{k+1}(n+k+1) \hookrightarrow \cdots .$$

Taking the maps $d_i : Z(n) \to Z(n-1)$ and $s_i : Z(n) \to Z(n+1)$ as the colimits of the face and degeneracy maps of the simplicial sets Z_k makes Z a based stable simplicial set. Furthermore, Z is indeed a Kan spectrum because each element in $Z_k(n+k)$, hence that in $Z(n)$, has only finitely many nonzero faces. Functoriality is clear.

It remains to see that these two functors are inverse to each other. Given any Kan spectrum Z, since

$$\left(\Omega^{\infty-k}Z\right)(n+k) = \{x \in Z(n) | d_m(x) = * \text{ for } m > n+k\},$$

so the colimit $\underrightarrow{\mathrm{colim}}_k \left(\Omega^{\infty-k}Z\right)(n+k)$ is precisely $Z(n)$. On the other hand, given any $\{Z_n\}$ in $\mathscr{S}'$, the set

$$(\Omega^{\infty-n}Z)(i) = \left\{ x \in \underrightarrow{\mathrm{colim}}_{k}\, Z_k(i-n+k) \middle| d_m(x) = * \text{ for } m > i \right\}$$

contains $Z_n(i)$ with $k = n$, and is contained in $Z_n(i)$ because $d_m(x) = *$ for $m > i$. $\square$

Given the perspective of Proposition II.9, we may prove the Eilenberg–Zilber lemma for Kan spectra:

Proof of Lemma II.4. For any element $x \in Z(n)$, let $d_i(x) = *$ for $i > k$. Then x may be regarded as an element in $(\Omega^{\infty+n-k}Z)(k) = Z_{k-n}(k)$. By the Eilenberg–Zilber lemma for simplicial sets, there exists a unique integer $0 < \ell < k$, a unique non-degenerate element $y \in Z_{k-n}(k-\ell)$, and a unique degeneracy $s \in \mathrm{Hom}_\Delta(\mathbf{k}, \mathbf{k}-\mathbf{l})$ such that $x = s(y)$. Note that y in turn can be regarded as an element in $Z(n-\ell)$, and s as an element in $\mathrm{Hom}_{\Delta_{\mathrm{st}}}(\mathbf{n}, \mathbf{n}-\mathbf{l})$. We may then take $m = n - \ell$.

While k is not necessarily unique, if x is regarded as an element in $Z_{k'-n}(k')$, and $x = s'(y')$ for some $y' \in Z_{k'-n}(k'-\ell')$, we may again regard x an an element in $Z_{N-n}(N)$, where $N = \max\{k, k'\}$. Then the uniqueness result of the Eilenberg–Zilber lemma implies that $y = y'$ and $s = s'$, and in particular $\ell = \ell'$ and hence m is unique. $\square$

In light of Proposition II.9, we will identify the categories $\mathscr{S}$ and $\mathscr{S}'$, and use the two descriptions of Kan spectra interchangeably.

Motivated by the description, we also consider the category $\mathscr{P}$ of *Kan prespectra.* A Kan prespectrum is a sequence of simplicial sets $\{Z_n\}_{n\in\mathbb{Z}}$ together with structure maps

$$\rho_n : Z_n \longrightarrow \Omega Z_{n+1},$$

or equivalently by adjunctions

$$\sigma_n : \Sigma Z_n \longrightarrow Z_{n+1}.$$

A morphism between Kan prespectra $\{Z_n\}$ and $\{T_n\}$ is a sequence $\{f_n : Z_n \to T_n\}_{n\in\mathbb{Z}}$ of maps between simplicial sets commuting with the structure maps ρ_n, or equivalently with σ_n.

Thus the category $\mathscr{S}$ of Kan spectra is a full subcategory of Kan prespectra $\mathscr{P}$. The inclusion functor is denoted

$$\mathrm{Ps} : \mathscr{S} \longrightarrow \mathscr{P},$$

called the *forgetful functor*.

Proposition II.10. *The forgetful functor* Ps *has a left adjoint*

$$\mathrm{Sp} : \mathscr{P} \longrightarrow \mathscr{S},$$

called the spectrification functor*. In addition, the composite functor* $\mathrm{Sp} \circ \mathrm{Ps}$ *is the identity functor.*

Proof. The functor Sp is given by

$$(\mathrm{Sp}(Z))_n = \varinjlim_k \Omega^k Z_{n+k},$$

where the colimit is taken over the diagram

$$\cdots \longrightarrow \Omega^k Z_{n+k} \xrightarrow{\Omega^k \rho_{n+k}} \Omega^k \Omega Z_{n+k+1} \longrightarrow \cdots .$$

$\mathrm{Sp}(Z)$ is indeed a Kan spectrum, since Ω commutes with sequential colimits.

To see the adjunction, let T be a Kan prespectrum and Z a Kan spectrum, and let $f : T \to \mathrm{Ps}\, Z$ be a morphism in $\mathscr{P}$. Fix $n \in \mathbb{Z}$, then for any $k \in \mathbb{Z}$, there is a morphism of simplicial sets

$$\Omega^k T_{n+k} \xrightarrow{\Omega^k f_{n+k}} \Omega^k Z_{n+k} \cong Z_n,$$

thus taking the colimit gives a morphism $\mathrm{Sp}(f)_n : \mathrm{Sp}(T)_n \to Z_n$. The commutativity of f_n with ρ_n shows that of $\mathrm{Sp}(f)_n$ with the structure maps, as well as that the association of $\mathrm{Sp}(f)_n$ to f_n is a bijection. □

Remark II.11. Equivalently, one may replace the index set $\mathbb{Z}$ by $\mathbb{N}_0$ and obtain yet another category $\mathscr{S}_0$ that is equivalent to $\mathscr{S}$, and another category $\mathscr{P}_0$ of "Kan prespectra". The inclusion $\mathscr{S}_0 \hookrightarrow \mathscr{P}_0$ again has a left adjoint, the "spectrification". However, although the natural inclusion $\mathscr{S}_0 \hookrightarrow \mathscr{S}'$ is an equivalence of categories (since both are equivalent to $\mathscr{S}$), the inclusion $\mathscr{P}_0 \hookrightarrow \mathscr{P}$ is not.

Now we construct the functor $\Sigma^{\infty+k}$ left adjoint to $\Omega^{\infty+k}$.

Proof of Lemma II.8. Given a simplicial set K, consider the Kan prespectrum T given by $T_n = \Sigma^{n+k} K$ with structure maps $\sigma_n : \Sigma T_n \to T_{n+1}$ the identity maps. For any Kan spectrum Z, any morphism

$$K \xrightarrow{f} \Omega^{\infty+k} Z = Z_{-k}$$

of simplicial sets induces morphisms

$$T_n = \Sigma^{n+k} K \xrightarrow{\Sigma^{n+k} f} \Sigma^{n+k} Z_{-k} \xrightarrow{\sigma} Z_n$$

of simplicial sets, which are compatible with the structure maps σ_n, and conversely any morphism $T \to \mathrm{Ps}\, Z$ of Kan prespectra arises this way. Therefore, we have bijections

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}(K, \Omega^{\infty+k} Z) \cong \mathrm{Hom}_{\mathscr{P}}(T, \mathrm{Ps}\, Z) \cong \mathrm{Hom}_{\mathscr{S}}(\mathrm{Sp}\, T, Z). \qquad \square$$

Example II.12. Recall the sphere spectrum from Example II.2; we claim that

$$\mathbb{S} = \Sigma^\infty S^0,$$

where $S^0 \cong *_+$ denotes the simplicial 0-sphere. To see this, note from definition that

$$\mathbb{S}_n(k) = \{x \in S(k-n) | d_m(x) = * \text{ for } m > k\} = \begin{cases} \{\alpha_{k-n}, *\} & \text{for } k \geq n \\ \{*\} & \text{for } k < n. \end{cases},$$

so $\mathbb{S}_n$ is the free simplicial set generated by a single element α_n in degree n; this is precisely S^n. On the other hand, note that $\Sigma^n S^0 = S^n$, so the Kan prespectrum T with $T_n = \Sigma^n S^0 = S^n$ is already a Kan spectrum, and is in fact precisely the sphere spectrum $\mathbb{S}$.

Definition II.13. For any $n \in \mathbb{Z}$, the *n-sphere spectrum* $\mathbb{S}^n$ is defined as $\Sigma^{\infty+n} S^0 \cong \Sigma^\infty S^n$.

2.1.3 Comparison with May spectra

The categories of Kan spectra and of May spectra are related by level-wise applying the geometric realization functor $|?|$ and the singular complex functor Sing; we denote the resulting functors by $\mathscr{L}$ and Sing, respectively, and again they form an adjoint pair.

For any May spectrum E, denote by $\mathrm{Sing}(E)$ the Kan (pre)spectrum given by $\{\mathrm{Sing}(E)_n = \mathrm{Sing}(E_n)\}$; this is in fact a Kan spectrum because

$$\Omega \,\mathrm{Sing}(E_n) \cong \mathrm{Sing}(\Omega E_n) \cong \mathrm{Sing}(E_{n+1})$$

since both Ω and Sing are right adjoints.

On the other hand, if Z is a Kan spectrum, then it follows from Corollary II.6 that the May prespectrum $|Z|$ is an inclusion prespectrum, namely, the structure maps $Z_n \to \Omega Z_{n+1}$ are inclusions of spaces. We denote by $\mathscr{L}(Z)$ the May spectrum obtained as the spectrification of $\{|Z_n|\}$. Since spectrification of May spectra is left adjoint, we have an adjoint pair $(\mathscr{L}, \mathrm{Sing})$. With this adjunction, we may "transfer" the notion of weak equivalences from May spectra.

Definition II.14. A morphism $f : Z \to T$ in $\mathscr{S}$ is a *weak equivalence* if the morphism $\mathscr{L}(f) : \mathscr{L}(Z) \to \mathscr{L}(T)$ of May spectra is a weak equivalence.

Lemma II.15. *For any Kan spectrum Z, the unit of adjunction*

$$\eta_Z : Z \longrightarrow \mathrm{Sing}(\mathscr{L}(Z)) \tag{2.4}$$

is a weak equivalence.

Proof. By definition, we must show that $\mathscr{L}\eta_Z : \mathscr{L}(Z) \to \mathscr{L}\,\mathrm{Sing}\,\mathscr{L}(Z)$ is a weak equivalence of May spectra. Since the composite

$$\mathscr{L}(Z) \xrightarrow{\mathscr{L}\eta} \mathscr{L}\,\mathrm{Sing}\,\mathscr{L}(Z) \xrightarrow{\eta\mathscr{L}} \mathscr{L}(Z)$$

is the identity which is a weak equivalence, it suffices to show that $\eta\mathscr{L}$ is a weak equivalence of May spectra. We claim that for any May spectrum E, the counit of adjunction

$$\epsilon_E : \mathscr{L}\,\mathrm{Sing}(E) \longrightarrow E$$

is a weak equivalence. In fact, this is the spectrification of the morphism of inclusion prespectra level-wise given by $|\,\mathrm{Sing}(E_n)| \to E_n$ which is known to be a weak equivalence of spaces. Thus our claim follows from that spectrification of inclusion spectra sends level-wise weak equivalence to weak equivalence. $\square$

Remark II.16. It is well known that $\mathrm{Sing}(E)_n = \mathrm{Sing}(E_n)$ is a (Kan) fibrant simplicial set; we say that $\mathrm{Sing}(E)$ is *level-wise fibrant*. We will introduce another notion of fibrancy for Kan spectra, which turns out to be equivalent to level-wise fibrancy (see Proposition II.18 below). Therefore, the unit of adjunction η is a "fibrant replacement functor".

2.1.4 Relative cell maps and anodyne extensions

Let Δ^n denote the standard n-simplex, which is the (unbased) simplicial set represented by $\mathbf{n}$. It has a unique non-degenerate element $\alpha \in \Delta^n(n)$ corresponding to the morphism $\mathrm{id}_{\mathbf{n}} \in \Delta(\mathbf{n}, \mathbf{n})$. We denote by $\partial\Delta^n$ the simplicial subset of Δ^n generated by all the non-degenerate elements except α, and by V^n_k the simplicial subset of $\partial\Delta^n$ generated by all the non-degenerate elements except $d_k(\alpha)$. For an unbased simplicial set K, we denote by K_+ the simplicial set with a disjoint basepoint attached.

A *cell* of Kan spectra is a morphism of the form

$$\Sigma^{\infty+\ell}\partial\Delta^n_+ \hookrightarrow \Sigma^{\infty+\ell}\Delta^n_+,$$

for some $n \in \mathbb{N}_0$ and $\ell \in \mathbb{Z}$, while an *anodyne cell* of Kan spectra is a morphism of the form

$$\Sigma^{\infty+\ell}V^n_{k+} \hookrightarrow \Sigma^{\infty+\ell}\Delta^n_+,$$

for some $n \in \mathbb{N}_0$, $0 \leq k \leq n$, and $\ell \in \mathbb{Z}$. Intuitively, a relative cell map (resp. anodyne extension) is obtained by repeatedly attaching (resp. anodyne) cells.

Definition II.17. Let $f : X \to Y$ be a morphism of Kan spectra.

- f is a *relative cell map* if there exists a diagram of Kan spectra

$$X = Y_{(-1)} \to Y_{(0)} \to \cdots \to Y_{(m)} \to Y_{(m+1)} \to \cdots \to \varinjlim_m Y_{(m)} = Y, \tag{2.5}$$

  where each map $Y_{(m)} \to Y_{(m+1)}$ is obtained as the pushout

$$\begin{tikzcd} \bigvee_{i\in I_m} \Sigma^{\infty+\ell_m(i)}\partial\Delta^{n_m(i)}_+ \arrow[r, "f_m"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & Y_{(m)} \arrow[d] \\ \bigvee_{i\in I_m} \Sigma^{\infty+\ell_m(i)}\Delta^{n_m(i)}_+ \arrow[r] & Y_{(m+1)} \end{tikzcd}$$

for some indexing set I_m, functions $n_m : I_m \to \mathbb{N}_0$ and $\ell_m : I_m \to \mathbb{Z}$, and some morphism f_m. A Kan spectrum X is a *cell spectrum* if the initial morphism $* \to X$ is a relative cell map.

- f is an *anodyne extension* if, in Diagram (2.5), each $Y_{(m)} \to Y_{(m+1)}$ is instead obtained as the pushout

$$\begin{tikzcd} \bigvee_{i\in I_m} \Sigma^{\infty+\ell_m(i)} V^{n_m(i)}_{k_m(i)+} \arrow[r, "f_m"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & Y_{(m)} \arrow[d] \\ \bigvee_{i\in I_m} \Sigma^{\infty+\ell_m(i)} \Delta^{n_m(i)}_{+} \arrow[r] & Y_{(m+1)} \end{tikzcd}$$

for some indexing set I_m, functions $n_m, k_m : I_m \to \mathbb{N}_0$ and $\ell_m : I_m \to \mathbb{Z}$ such that $k_m(i) \le n_m(i)$ for any $i \in I_m$, and some morphism f_m.

- f is a *fibration* if it has the right lifting property with respect to all anodyne cells, or equivalently, all anodyne extension. That is, for any commuting diagram

$$\begin{tikzcd} \Sigma^{\infty+\ell} V^n_{k+} \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Sigma^{\infty+\ell} \Delta^n_{+} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{2.6}$$

there exists a dashed arrow making the entire diagram commutative.

- A Kan spectrum Z is *fibrant* if the canonical map $Z \to *$ is a fibration.

Since all (anodyne) cells are monomorphisms,we see that all relative cell maps (as well as anodyne extensions) are monomorphisms. In fact, the converse is also true.

Proposition II.18.

(1) Every monomorphism of Kan spectra is a relative cell map. In particular, every Kan spectrum is a cell spectrum.

(2) Every anodyne extension is a weak equivalence.

(3) A morphism of Kan spectra $f : X \to Y$ is a fibration if and only if it is a level-wise fibration, that is, $f : X_n \to Y_n$ is a fibration of simplicial sets for any $n \in \mathbb{Z}$. In particular, a Kan spectrum Z is fibrant if and only if it is level-wise fibrant.

Proof. (1) Let $f : X \to Y$ be a monomorphism of Kan spectra, and we will explicitly construct a relative cell structure. For any $k \geq 0$, let $Y_{(k)}(n)$ be the set of elements x of the form $s(y)$ such that y is non-degenerate and $\deg(y) \leq k$ (recall that $\deg(y) = k$ if $d_k(y) \neq *$ and $d_i(y) = *$ for $i > k$). By Eilenberg–Zilber lemma II.4, this process is well-defined, and indeed every $Y_{(k)}$ is obtained from $Y_{(k-1)}$ by attaching cells $\Sigma^{\infty-\ell}\Delta^k_+$. Indeed, every element $y \in Y_{(k)}(n)$ of degree k corresponds to a morphism $\Delta^n_+ \longrightarrow (Y_{(k)})_{n-k}$, or equivalently by adjunction

$$\Sigma^{\infty-(n-k)}\Delta^k_+ \longrightarrow Y_{(k)},$$

and all of its faces necessarily has degree less than $k-1$.

(2) The image of an anodyne cell under the functor $\mathscr{L}$ is a weak equivalence and an inclusion of May spectra, so a pushout of such a map is also a weak equivalence and an inclusion. Since the left adjoint $\mathscr{L}$ commutes with pushouts and sequential colimits, it suffices to observe that, in a system of May spectra where the connecting maps are both weak equivalences and inclusions, the canonical map from each term into the sequential limit is a weak equivalence (and an inclusion).

(3) Using the adjunction $(\Sigma^{\infty+\ell}, \Omega^{\infty+\ell})$ to the diagram (2.6), we see that f is a

fibration if and only if, for any $\ell \in \mathbb{Z}$ and any commuting diagram

$$\begin{tikzcd} V^n_{k+} \arrow[r] \arrow[d, hook] & X_{-\ell} \arrow[d, "f_{-\ell}"] \\ \Delta^n_+ \arrow[r] \arrow[ur, dashed] & Y_{-\ell} \end{tikzcd}$$

there exists a dashed arrow making the entire diagram commutative. That is, $f_{-\ell}$ is a fibration of simplicial sets. □

Remark II.19. Note that if, instead of a cell of the form $\Sigma^{\infty-\ell}\Delta^n_+$, we attach a cell $\Sigma^{\infty-\ell-k}\Delta^{n+k}_+$ for some $k \geq 0$, then the resulting new Kan spectrum is isomorphic to the original one.

2.1.5 Function spectra. Star product with simplicial sets

Recall that, for any simplicial sets K and L, the smash product $K \wedge L$ of simplicial sets is given by, the pushout

$$\begin{tikzcd} K \vee L \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & K \times L \arrow[d] \\ * \arrow[r] & K \wedge L \end{tikzcd}$$

Note that $|K \wedge L| \cong |K| \wedge |L|$ since the left adjoint $|?|$ commutes with $\vee$ and pushout, and it commutes with $\times$ by Milnor's theorem.

For K fixed, the functor $K\wedge ?$ has a right adjoint $F(K, -)$ given by

$$F(K, L)_n = \mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}\left(K \wedge \Delta^n_+, L\right),$$

with the basepoint the constant map taking value at the basepoint in T.

Lemma II.20. *For any simplicial sets K and L, there is a canonical epimorphism of simplicial sets*

$$(\Sigma K) \wedge L \longrightarrow \Sigma(K \wedge L) \tag{2.7}$$

which is natural in both K and T.

Proof. In [16, Prop. 6.1] this map is constructed and shown to be natural. Indeed, to see the construction, it suffices to consider the special case $K = \Delta^m_+$ and $L = \Delta^n_+$, since every based simplicial set is a colimit of standard simplices, and the functor $\wedge$ commutes with colimits in each coordinate. Note that the non-degenerate $(m+n+1)$-simplices in

$$\Sigma\left(\Delta^m_+ \wedge \Delta^n_+\right) = \Sigma\left(\Delta^m \times \Delta^n\right)_+$$

corresponds to the non-degenerate $(m+n)$-simplices in $\left(\Delta^m \times \Delta^n\right)_+$, and therefore, corresponds to riffle (m,n)-shuffles. On the other hand, the non-degenerate $(m+n+1)$-simplices in

$$\Sigma\left(\Delta^m_+\right) \wedge \Delta^n_+$$

corresponds to the non-degenerate $(m+n+1)$-simplices in $\left(\Delta^{m+1} \times \Delta^n\right)_+$, that is, to riffle $(m+1,n)$-shuffles. Then the desired map is constructed by applying the last degeneracy s_{m+1} to the non-degenerate $(m+n+1)$-simplices in $\Sigma\left(\Delta^m_+\right) \wedge \Delta^n_+$ corresponding to all the $(m+1,n)$-shuffles except those ending with an element in the set of $(m+1)$-elements.

From the construction, it is also clear that the map is an epimorphism. Indeed, it is a quasi-fibrations with all fibers contractible, hence a weak equivalence. $\square$

Corollary II.21. *For any simplicial sets K and L, there is a canonical monomorphism of simplicial sets*

$$F(K,\Omega L) \longrightarrow \Omega F(K,L) \tag{2.8}$$

which is natural in both K and L.

Proof. Fix L and consider the functors $(\Sigma ?) \wedge L$ and $\Sigma(? \wedge L)$, which have right adjoints $\Omega F(L,?)$ and $F(L,\Omega ?)$, respectively. Then for any simplicial sets K and M, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}((\Sigma K)\wedge L, M) \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}(K, \Omega F(L,M)) \arrow[d, hook] \\
\mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}(\Sigma(K\wedge L), M) \arrow[r, "\cong"'] & \mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}(K, F(L,\Omega M))
\end{tikzcd}$$

here the left arrow is injective because (2.7) is an epimorphism, so the injectivity of the right arrow implies that (2.8) is a monomorphism. □

For a simplicial set K and a Kan prespectrum T, we define the *function prespectrum* $F_p(K,T)$ as follows. Put $F_p(K,T)_n \coloneqq F(K,T_n)$, equipped with structure maps

$$\rho_n : F(K,T_n) \xrightarrow{F(K,\rho_n)} F(K,\Omega T_{n+1}) \longrightarrow \Omega F(K,T_{n+1}),$$

where the second map is (2.8).

Lemma II.22. *For a simplicial set K, the function prespectrum functor $F_p(K,?)$: $\mathscr{P} \to \mathscr{P}$ has a left adjoint denoted by $K \cdot ?$.*

Proof. We construct the left adjoint explicitly. Given a Kan prespectrum T, we put

$$(2.9)\qquad (K\cdot T)_n \coloneqq \varinjlim \left[\begin{tikzcd}
 & & \vdots \arrow[d] \\
 & \Sigma(K\wedge \Sigma T_{n-2}) \arrow[r] \arrow[d] & \Sigma^2(K\wedge T_{n-2}) \\
K\wedge \Sigma T_{n-1} \arrow[r] \arrow[d] & \Sigma(K\wedge T_{n-1}) & \\
K\wedge T_n & &
\end{tikzcd}\right]$$

Since the left adjoint Σ commutes with colimits, we have

$$\Sigma(K \cdot T)_n \cong \underrightarrow{\mathrm{colim}} \left[\begin{tikzcd} & \vdots \arrow[d] \\ \Sigma(K \wedge \Sigma T_{n-2}) \arrow[r] \arrow[d] & \Sigma^2(K \wedge T_{n-1}) \\ \Sigma(K \wedge T_n) & \end{tikzcd} \right]$$

Since this diagram is a part of the colimit diagram defining $(K \cdot T)_{n+1}$, there is a canonical map

$$\Sigma(K \cdot T)_n \longrightarrow (K \cdot T)_{n+1};$$

these will be the structure maps for $(K \cdot T)$.

To see the adjunction, let $(K \cdot T) \to Z$ be a morphism of Kan prespectra. Then for any $n \in \mathbb{Z}$, there is a morphism

$$K \wedge T_n \longrightarrow (K \cdot T)_n \longrightarrow Z_n$$

of simplicial sets, which by adjunction induces a morphism

$$T_n \longrightarrow F(K, Z_n) = F_p(K, Z)_n.$$

It is easily check that these maps are compatible with the structure maps σ_n.

Conversely, let $T \to F_p(K, Z)$ be a morphism of Kan prespectra, and for any $n \in \mathbb{Z}$ we have a morphism

$$T_{n-k} \longrightarrow F_p(K, Z)_{n-k} = F(K, Z_{n-k})$$

of simplicial sets. Applying the functor Σ^k to its adjoint and post-composing with the structure map of Z, we obtain a morphism

$$\Sigma^k (K \wedge T_{n-k}) \longrightarrow \Sigma^k Z_{n-k} \xrightarrow{\sigma_{n-k}} Z_n$$

of simplicial sets. One checks that, for different values of k, these morphisms are compatible with Diagram (2.9), hence inducing a morphism $(K \cdot T)_n \to Z_n$. They are again compatible with the structure maps σ_n, hence defining a morphism $K \cdot T \to Z$ of Kan prespectra. □

Definition II.23. For a simplicial set K and a Kan spectrum Z, we define the *function spectrum*

$$F(K, Z) \coloneqq \mathrm{Sp}(F(K, \mathrm{Ps}\, Z))$$

as the spectrification of the function prespectrum, and define the *star product*

$$K \star Z \coloneqq \mathrm{Sp}(K \cdot (\mathrm{Ps}\, Z))$$

as the spectrification of the dot product prespectrum.

Lemma II.24. *If Z is a Kan spectrum, then we have an isomorphism*

$$\mathscr{L}(K \star Z) \cong |K| \wedge |\mathscr{L}(Z)|,$$

where the smash product on the right hand side is the smash product of May spectra by spaces.

Proof. First note that the left adjoint $|?|$ commutes with colimits, distributes over the smash product of simplicial sets, and commutes with suspension by Proposition II.5, and that the arrows in the diagram in (2.9) are all isomorphisms, we have a homeomorphism

$$|(K \cdot Z)_n| \cong |K| \wedge |Z_n|.$$

Since spectrification commutes with the left adjoint $\mathscr{L}$, we conclude that $\mathscr{L}(K \star Z)$ is precisely the spectrification of the prespectrum $\{|K| \wedge |Z_n\}$, which is simply $|K| \wedge \mathscr{L}(Z)$. □

Corollary II.25. *For any Kan spectrum Z, the functor $? \star Z$ preserves weak equivalences as well as monomorphisms.*

Proof. Note that $K \to K'$ is a weak equivalence if and only if $|K| \to |K'|$ is, and similarly a morphism of Kan spectra $Z \to Z'$ is a weak equivalence if and only if $\mathscr{L}(Z) \to \mathscr{L}(Z')$ is, so the fact that $? \star Z$ preserves weak equivalences between Kan spectra follows from Lemma II.24 and the analogous statement in May spectra.

To see that $? \star Z$ preserves monomorphism, note that for any based simplicial L the functor $? \wedge L$ preserves monomorphisms, so comparing diagram (2.9) gives the desired monomorphism. $\square$

Lemma II.26. *For any simplicial sets K and L and any Kan spectrum Z, there is a morphism*

$$(T_1 \wedge T_2) \star Z \xrightarrow{\cong} T_1 \star (T_2 \star Z)$$

which is natural in T_1, T_2 and Z.

Proof. This follows from associativity of the smash product between based simplicial sets. $\square$

Lemma II.27. *For any based simplicial set K, the functor $K\star ? : \mathscr{S} \to \mathscr{S}$ preserves directed colimits of spectra.*

Proof. Since both Sp is left adjoint to Ps, while $K \cdot ? : \mathscr{P} \to \mathscr{P}$ is the left adjoint to the functor $F_p(K, ?) : \mathscr{P} \to \mathscr{P}$ defined level-wise, hence both Sp and $K \cdot ?$ preserve all colimits.

It remains to argue that Ps also preserve directed colimits. Given a directed diagram $Z : D \to \mathscr{S}$, the colimit $\varinjlim^{\mathscr{P}} Z$ in $\mathscr{P}$ is formed level-wise, that is

$$\left[\varinjlim^{\mathscr{P}} Z\right](n) = \varinjlim Z(n)$$

Since the loop functor Ω commutes with directed colimits, if the structure morphisms

$$(Z(d))(n) \longrightarrow (\Omega(Z(d)))(n+1)$$

are isomorphisms for each object $d \in \operatorname{Obj} D$, then so is the induced morphism

$$\left[\varinjlim Z\right](n) \longrightarrow \Omega\left[\varinjlim Z\right](n+1). \qquad \square$$

2.1.6 Localization of Kan spectra

Definition II.28. A (*strong*) *homotopy* between morphisms $f, g : X \rightrightarrows Y$ of Kan spectra is a morphism

$$h : I_+ \star X \longrightarrow Y$$

such that the diagram

$$\begin{tikzcd} \{0\}_+ \star X \arrow[r, "i_{0+}\star\mathrm{id}"] \arrow[dr, "f"'] & I_+ \star X \arrow[d, "h"] & \{1\}_+ \star X \arrow[l, "i_{1+}\star\mathrm{id}"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes.

We write $\simeq$ for the equivalence relation generated by homotopies. It is a congruence relation on the category $\mathscr{S}$, that is, it is preserved under pre- and post-compositions, and thus we may form the quotient category $h\mathscr{S} := \mathscr{S}/\simeq$, which we call the (*strong*) *homotopy category* of Kan spectra. By a (*strong*) *homotopy equivalence*, we refer to an isomorphism in $h\mathscr{S}$, or a morphism in $\mathscr{S}$ whose image under the canonical (quotient) functor $\mathscr{S} \to h\mathscr{S}$ is an isomorphism.

Lemma II.29. *A homotopy equivalence between Kan spectra is a weak equivalence.*

Proof. By Lemma II.24, we see that a strong homotopy between morphisms of Kan spectra gives rise to, after applying realization functor $\mathscr{L}$, a strong homotopy between morphisms of May spectra. In particular, a strong homotopy equivalence of Kan spectra is a strong homotopy equivalence between May spectra. The claim then follows since the two categories of spectra have the same weak equivalences. $\square$

Lemma II.30. *If $f : X \to Y$ is a monomorphism of Kan spectra and also a weak equivalence, then there exists a monomorphism of Kan spectra $g : Y \to Z$ such that $g \circ f$ is an anodyne extension.*

Proof. By Proposition II.18, there exists a relative cell structure $Y_{(n)}$ such that $Y_{(-1)} = X$, $\varinjlim Y_{(n)} = Y$, and each $Y_{(m+1)}$ is obtained from $Y_{(m)}$ by attaching cells $\Sigma^{\infty-\ell}\Delta^n_+$. We will inductively construct subspectra $Y_{(m)} \hookrightarrow Z_{(m)}$ such that $X \hookrightarrow Z_{(m)}$ is an anodyne extension. Then putting $Z \coloneqq \varinjlim Z_{(m)}$ gives the desired inclusion $g : Y \to Z$ such that $g \circ f$ is an anodyne extension.

For the base case we put $Z_{(-1)} = Z$. Suppose $Z_{(m)}$ has been constructed. We may first apply the small object argument to attach all possible anodyne cells to $Z_{(m)}$ and assume $Z_{(m)}$ is fibrant. Now consider any possibly attaching map with boundary

$$\Sigma^{\infty-\ell}\partial\Delta^n_+ \longrightarrow Y_{(m)}$$

of a cell in Y with boundary in $Y_{(m)}$, which corresponds to a morphism of simplicial sets

$$\partial\Delta^n_+ \longrightarrow \left(Y_{(m)}\right)_\ell ;$$

note that by Remark II.19 we may choose ℓ large enough that $X_\ell \hookrightarrow Y_\ell$ is indeed an equivalence of simplicial sets. Then the composite

$$\partial\Delta^n_+ \longrightarrow \left(Y_{(m)}\right)_\ell \xhookrightarrow{\sim} \left(Z_{(m)}\right)_\ell$$

must have a filling $\Delta^n_+ \longrightarrow (Z_{(m)})_\ell$, which is equivalent to that the cell

$$\Sigma^{\infty-\ell}\Delta^n_+ \longrightarrow Z_{(m)}$$

is already in $Z_{(m)}$. Thus, for each cell to be attached to $Y_{(m)}$ to form $Y_{(m+1)}$, we attach to $Z_{(m)}$ via

$$\begin{tikzcd} \Sigma^{\infty-\ell}\left(\partial\Delta^n_+ \times I \vee \Delta^n_+ \times \{0\}\right) \arrow[r] \arrow[d, hook] & Z_{(m)} \\ \Sigma^{\infty-\ell}\left(\Delta^n_+ \times I\right) & \end{tikzcd}$$

This gives an anodyne extension, and therefore so is $X \hookrightarrow Z_{(m)}$. This process must terminate for set-theoretic reason, and the induction is complete. $\square$

Proposition II.31. *The fibrant Kan spectra in the homotopy category $h\mathscr{S}$ are local with respect to the class $\mathcal{W}$ of weak equivalences.*

Proof. Let $f : X \xrightarrow{\sim} Y$ be a weak equivalence, and E a fibrant Kan spectrum. First note that we may assume f is an inclusion by replacing Y by the mapping cylinder

$$\begin{tikzcd} \{1\}_+ \star X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \{1\}_+ \star Y \arrow[d] \\ I_+ \star X \arrow[r] & Mf \end{tikzcd}$$

and f by the morphism

$$X = \{0\}_+ \star X \longrightarrow Mf,$$

which is an inclusion by Corollary II.25. Note that by the canonical morphism $Mf \to Y$ is a homotopy equivalence by Lemma II.26.

To prove that f is surjective, we apply Lemma II.30 to form an anodyne extension $g \circ f$ for some inclusion $g : Y \hookrightarrow Z$. Since E is fibrant, the lifting problem

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d, hook, "g\circ f"', "\sim"] & E \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed, "\exists"'] & * \end{tikzcd}$$

has a solution $Z \to E$, and the composite $Y \to E$ gives the lift of any map $X \to E$.

To prove that f is injective, form the double mapping cylinder

$$Pf \coloneqq \varinjlim \left[\begin{tikzcd} \{1\}_+ \star X \arrow[r] \arrow[d] & \{1\}_+ \star Y \\ II_+ \star X & \\ \{1'\}_+ \star X \arrow[r] \arrow[u] & \{1'\}_+ \star Y \end{tikzcd} \right]$$

where II_+ is the based simplicial set $I_+ \vee_{\{0\}+} I_+$. Then there is an inclusion

$$Pf \hookrightarrow II_+ \star Y$$

and is also a weak equivalence since f is. We may now apply Lemma II.30 to extend any morphism $Pf \to E$ to $II_+ \star Y \to E$, which implies that any two maps $Y \rightrightarrows E$ are homotopic if their post-compositions with f are, as desired. $\square$

Theorem II.32. *The strong homotopy category $h\mathscr{S}$ of Kan spectra has a localization with respect to weak equivalences. Furthermore, the localization is given by the functor* $\operatorname{Sing}\mathscr{L}$ *together with the unit of adjunction* (2.4)*.*

Proof. The unit of adjunction in Lemma II.15 is an equivalence, and a Kan spectrum of the form $\operatorname{Sing}\mathscr{L}X$ is level-wise fibrant and hence fibrant by Proposition II.18, which is local in $h\mathscr{S}$ by Proposition II.31 above. $\square$

Corollary II.33. *The derived category $D\mathscr{S} = \mathcal{W}^{-1}\mathscr{S}$ of the category of Kan spectra with respect to weak equivalences exists, and it is equivalent to the usual stable homotopy category. Moreover, a functor $F : \mathscr{S} \to \mathscr{D}$ is right-derivable if and only if it preserves strong homotopies, and its right derived functor*

$$RF : D\mathscr{S} \longrightarrow D\mathscr{D},$$

can be computed as

$$RF(?) = F(\operatorname{Sing}\mathscr{L}(?)).$$

Proof. The existence of $D\mathscr{S}$ follows from the Theorem II.32 and the general results of localization (see Theorems A.7 and A.9). Moreover, since a morphism f in $\mathscr{S}$ is a weak equivalence if and only if $\mathscr{L}(f)$ is a weak equivalence in the category of May spectra, it follows that $D\mathscr{S}$ is equivalent to the derived category of the category of May spectra with respect to weak equivalences, which is precisely the usual stable homotopy category. Finally, A functor $F : \mathscr{S} \to \mathscr{D}$ preserves strong homotopies if and only if it induces a unique functor $\overline{F} : h\mathscr{S} \to \mathscr{D}$, which is always right-derivable as a consequence of Theorem A.9. $\square$

2.2 Kan spectral sheaves

2.2.1 Preliminaries

In this section, we fix a space $\mathscr{X}$. Recall that for a complete and cocomplete category $\mathscr{C}$, a $\mathscr{C}$-valued *presheaf* is a functor $\mathrm{Open}(\mathscr{X})^{\mathrm{op}} \to \mathscr{C}$, where $\mathrm{Open}(\mathscr{X})$ is the partially ordered set of open subsets of $\mathscr{X}$ ordered by inclusion.

Given a presheaf $\mathcal{F}$, an open set $U \subseteq \mathscr{X}$, and any covering $\{U_i \subseteq U\}$, there is a canonical morphism

$$\mathcal{F}(U) \longrightarrow \mathrm{eq}\left[\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)\right]. \tag{2.10}$$

The presheaf $\mathcal{F}$ is a *separated presheaf* if a *sheaf* if for any open set U and any covering $\{U_i\}$, the diagram (2.10) is a monomorphism, and $\mathcal{F}$ is a *sheaf* if (2.10) is an isomorphism. We denote by $\mathsf{Sh}_{\mathscr{C}}(\mathscr{X})$ (resp. $\mathsf{PSh}_{\mathscr{C}}(\mathscr{X})$) the category of $\mathscr{C}$-valued sheaves (resp. presheaves) on $\mathscr{X}$.

Our main objects of study are objects in $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ and $\mathsf{PSh}_{\mathscr{S}}(\mathscr{X})$, which we call *Kan spectral (pre)sheaves.*

Recall that the *stalk* of a (pre)sheaf $\mathcal{F}$ at $x \in \mathscr{X}$ is the colimit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U).$$

Definition II.34. A morphism $f : \mathcal{F} \to \mathcal{G}$ of Kan spectral sheaves is a *stalk-wise (weak) equivalence* if, for any $x \in \mathscr{X}$, the induced morphism

$$f_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

is a weak equivalence of Kan spectra.

Our goal is to construct the derived category of Kan spectral sheaves with respect to stalk-wise equivalences, as well as derived functors from the derived category. To do so, we will make use of the notions of the relative cell maps and anodyne extensions as in the case of Kan spectra.

2.2.2 Sheafification of Kan spectral sheaves

First recall the "plus-construction"* for presheaves. Let $\mathcal{F}$ be a presheaf on $\mathscr{X}$ in a complete and cocomplete category $\mathscr{C}$. Define another presheaf $\mathcal{F}^+$ on $\mathscr{X}$ by the formula

$$\mathcal{F}^+(U) := \varinjlim_{\{U_i \to U\}\text{ covering}} \left(\mathrm{eq} \left[\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right] \right), \tag{2.11}$$

where the colimit is taken over the category (opposite to that) of coverings of U under refinements.

It is well-known that (see for example, [29, Tag 00W1]) the double plus-construction is the sheafification for presheaves of (based) sets. More precisely, we have the following results:

Theorem ([29, Tag 00WB])**.** *Let $\mathcal{F}$ be a presheaf of sets, then*

* Not to be confused with the Quillen plus-construction.

1. *the presheaf* $\mathcal{F}^+$ *is always separated;*

2. *if* $\mathcal{F}$ *is separated, then the presheaf* $\mathcal{F}^+$ *is a sheaf, and the canonical morphism* $\mathcal{F} \to \mathcal{F}^+$ *is a monomorphism; and*

3. *if* $\mathcal{F}$ *is already a sheaf, then the sheaf* $\mathcal{F}^+$ *is isomorphic to* $\mathcal{F}$*.*

As a consequence, the functor

$$\begin{array}{ccc} \mathsf{PSh}_{\mathsf{Set}_*} * (\mathscr{X}) & \longrightarrow & \mathsf{Sh}_{\mathsf{Set}_*} * (\mathscr{X}) \\ \mathcal{F} & \longmapsto & (\mathcal{F}^+)^+ =: \mathcal{F}^\# \end{array}$$

is the sheafification *functor for presheaves of sets. That is, it takes a presheaf to a sheaf, and is left adjoint to the forgetful functor from sheaf on* $\mathscr{X}$ *to presheaves on* $\mathscr{X}$*.* □

Corollary II.35. *The same assertions above hold true with based sets replaced by based stable simplicial sets.*

Proof. Via the identification of the category of (pre)sheaves of based stable simplicial sets with the category of stable simplicial sheaves of based sets, we see that the (double) plus construction in $\mathsf{PSh}_{\Delta_{\mathrm{st}}^{\mathrm{op}}\text{-}\mathsf{Set}_*}(\mathscr{X})$ is the same as the object-wise (double) plus construction in $\Delta_{\mathrm{st}}^{\mathrm{op}}$-$\mathsf{PSh}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}_*}(\mathscr{X})$ across objects of Δ_{st}. □

Recall that the full inclusion $\mathscr{S} \hookrightarrow \Delta_{\mathrm{st}}^{\mathrm{op}}$-$\mathsf{Set}_*$ exhibits $\mathscr{S}$ as a coreflective subcategory of $\Delta_{\mathrm{st}}^{\mathrm{op}}$-$\mathsf{Set}_*$, and we denote by R the right adjoint to the inclusion.

Corollary II.36. *Consider the functor*

$$\begin{array}{ccccc} L & : & \mathsf{PSh}_{\mathscr{S}}(\mathscr{X}) & \longrightarrow & \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \\ & & \mathcal{F} & \longmapsto & [U \mapsto R(\mathcal{F}^+(U))]\,; \end{array}$$

then the sheafification *of Kan spectral presheaves is given by the functor* LL*.*

Proof. The canonical morphism $\mathcal{F} \to \mathcal{F}^+$ factors as

$$\mathcal{F} = R\mathcal{F} \longrightarrow R\mathcal{F}^+ = L\mathcal{F} \hookrightarrow \mathcal{F}^+,$$

and if $\mathcal{G}$ is a Kan spectral sheaf, then any presheaf morphism $\mathcal{F} \to \mathcal{G}$ uniquely factors as

$$\mathcal{F} \longrightarrow \mathcal{F}^+ \longrightarrow \mathcal{G}$$

which then further factors, by applying the functor R again, as

$$\mathcal{F} \longrightarrow L\mathcal{F} \longrightarrow \mathcal{G};$$

this is unique.

It remains to see that $LL\mathcal{F}$ is indeed a sheaf. First note that $L\mathcal{F}$ is separated since R commutes with the product and the equalizer and preserves monomorphisms, thus

$$\begin{aligned}(L\mathcal{F})(U) = R(\mathcal{F}^+(U)) \hookrightarrow R\,\mathrm{eq}\left[\prod_i \mathcal{F}^+(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}^+(U_i \cap U_j)\right] \\ \cong \mathrm{eq}\left[\prod_i (L\mathcal{F})(U_i) \rightrightarrows \prod_{i,j} (L\mathcal{F})(U_i \cap U_j)\right]\end{aligned}$$

for any cover $\{U_i \to U\}$. In particular, $(L\mathcal{F})^+$ is itself a sheaf; but then

$$\begin{aligned}(LL\mathcal{F})(U) = R(L\mathcal{F})^+(U) \cong R\,\mathrm{eq}\left[\prod_i (L\mathcal{F})^+(U_i) \rightrightarrows \prod_{i,j} (L\mathcal{F})^+(U_i \cap U_j)\right] \\ \cong \mathrm{eq}\left[\prod_i (LL\mathcal{F})(U_i) \rightrightarrows \prod_{i,j} (LL\mathcal{F})(U_i \cap U_j)\right].\end{aligned}$$

This completes the proof. □

Henceforth, we will slightly abuse notation and also denote by $(?)^+$ the functor L, and by $(?)^\#$ the sheafification LL.

Corollary II.37. *The category $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ is a reflective subcategory of the functor category $\mathsf{PSh}_{\mathscr{S}}(\mathscr{X})$, and in particular is itself complete and cocomplete.* □

Finally, we mention a well-known result, of which we will freely make use without reference.

Corollary II.38. *For any Kan spectral presheaf* $\mathcal{F}$ *and any point* $x \in X$*, we have an isomorphism of stalks*

$$\mathcal{F}_x \cong (\mathcal{F}^{\#})x.$$

Proof. This follows from that the left adjoint $(?)^{\#}$ preserves the colimit $(?)_x$. □

2.2.3 Relative cell sheaves and anodyne extensions

Let T be a based simplicial set, and U an open subset of $\mathscr{X}$. We denote by T_U the based simplicial sheaf which sends any open set U' to T if $U' \subseteq U$ and to the singleton otherwise. For an inclusion $\iota : U' \hookrightarrow U$ of open subsets of $\mathscr{X}$, we consider pushouts

$$\begin{tikzcd} \left(\partial\Delta^n_+\right)_{U'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \left(\partial\Delta^n_+\right)_U \arrow[d] \\ \left(\Delta^n_+\right)_{U'} \arrow[r] & \left(\Delta^n_+\right)_\iota \end{tikzcd}$$

and

$$\begin{tikzcd} \left(V^n_{k+}\right)_{U'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \left(V^n_{k+}\right)_U \arrow[d] \\ \left(\Delta^n_+\right)_{U'} \arrow[r] & \left(E^n_{k+}\right)_\iota \end{tikzcd}$$

in the category of based simplicial sheaves. A *cell* in Kan spectral sheaves is then a morphism of the form

$$\Sigma^{\infty+\ell}\left(\Delta^n_+\right)_\iota \hookrightarrow \Sigma^{\infty+\ell}\left(\Delta^n_+\right)_U,$$

for some $n \in \mathbb{N}_0$, $\ell \in \mathbb{Z}$, and inclusion ι of open sets of $\mathscr{X}$, while an *anodyne cell* is a morphism of the form

$$\Sigma^{\infty+\ell}\left(E^n_{k+}\right)_\iota \hookrightarrow \Sigma^{\infty+\ell}\left(\Delta^n_+\right)_U,$$

for some $n \in \mathbb{N}_0$, $0 \leq k \leq n$, $\ell \in \mathbb{Z}$, and inclusion ι. Here, the functor $\Sigma^{\infty+\ell}$ from based simplicial sheaves to Kan spectral sheaves are defined section-wise followed by sheafification. It has a right adjoint $\Omega^{\infty+\ell}$, which is defined section-wise, as the resulting presheaf is indeed a sheaf.

Definition II.39. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of Kan spectral sheaves.

- f is a *relative cell sheaf map* if there exists a cardinal α and a collection $\{\mathcal{G}_\gamma\}_{\gamma \leq \alpha}$ of sheaves such that

  - $\mathcal{G}_{(-1)} = \mathcal{F}$ and $\mathcal{G}_\alpha = \mathcal{G}$;
  - if β is a limiting ordinal, then $\mathcal{G}_\beta = \varinjlim_{\gamma<\beta} \mathcal{G}_\gamma$;
  - for any ordinal $\beta < \alpha$, there exists an inclusion $\iota_\beta : U'_\beta \hookrightarrow U_\beta$ of open sets of $\mathscr{X}$ and a pushout diagram

$$\begin{tikzcd} \Sigma^{\infty+\ell_\beta}\left(\Delta^{n_\beta}_+\right)_{\iota_\beta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G}_\beta \arrow[d] \\ \Sigma^{\infty+\ell_\beta}\left(\Delta^{n_\beta}_+\right)_{U_\beta} \arrow[r] & \mathcal{G}_{\beta+1} \end{tikzcd}$$

  A Kan spectral $\mathcal{F}$ is a *cell sheaf* if the initial morphism $* \to \mathcal{F}$ is a relative cell sheaf map.

- f is a *anodyne extension* if, in the construction above, each $Y_{(\beta+1)}$ is instead obtained as the pushout

$$\begin{tikzcd} \Sigma^{\infty+\ell_\beta}\left(E^{n_\beta}_{k_\beta+}\right)_{\iota_\beta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G}_\beta \arrow[d] \\ \Sigma^{\infty+\ell_\beta}\left(\Delta^{n_\beta}_+\right)_{U_\beta} \arrow[r] & \mathcal{G}_{\beta+1} \end{tikzcd}$$

It is clear that (anodyne) cells are monomorphisms, and anodyne cells are also weak equivalences. Since taking stalks commutes with the pushout, and monomorphisms and stalk-wise equivalences are detected by stalks, we have

Lemma II.40.

1. *Relative cell sheaf maps and anodyne extensions are monomorphisms.*

2. *Anodyne extensions are stalk-wise equivalences.* □

Proposition II.41. *Every monomorphism of Kan spectral sheaves is a relative cell sheaf map. In particular, every Kan spectral sheaf is a cell sheaf.*

Proof. Let $f : \mathcal{F} \to \mathcal{G}$ be a monomorphism of Kan spectral sheaves. We will inductively construct, for any ordinal β, a sheaf $\mathcal{G}_\beta$ together with a monomorphism $\mathcal{G}_\beta \to \mathcal{G}$. We start with $\mathcal{G}_{(-1)} = \mathcal{F}$, and if β is a limiting ordinal, we put $\mathcal{G}_\beta = \varinjlim_{\gamma<\beta} \mathcal{G}_\gamma$. Now suppose $\mathcal{G}_\beta$ has been constructed for some ordinal β; if $\mathcal{G}_\beta \to \mathcal{G}$ is an isomorphism, then we are done (and we may simply put $\mathcal{G}_{\beta+1} = \mathcal{G}_\beta$ from then on). Otherwise, there exists a point $x \in \mathscr{X}$ such that $(\mathcal{G}_\beta)_x \hookrightarrow \mathcal{G}_x$ is not an isomorphism. In particular, there exists an element $t \in \mathcal{G}_x(k)$ which is not in $(\mathcal{G}_\beta)_x(k)$, but all of whose faces are in $(\mathcal{G}_\beta)_x(k-1)$. Indeed, there exists an open set $U \subseteq \mathscr{X}$ such that t is represented by $\widetilde{t} \in \mathcal{G}(U)(k)$. Similarly, there exists an open subset $U' \subseteq U \subseteq \mathscr{X}$ such that all faces of t are represented by elements in $(\mathcal{G}_\beta)(U)(k-1)$. Denote by ι the inclusion $U' \subseteq U$, then we may define $\mathcal{G}_{\beta+1}$ to be the pushout

$$\begin{tikzcd}
\Sigma^{\infty+\ell_\beta}\left(\Delta_+^{n_\beta}\right)_{\iota_\beta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G}_\beta \arrow[d] \arrow[ddr, hook, bend left] & \\
\Sigma^{\infty+\ell_\beta}\left(\Delta_+^{n_\beta}\right)_{U_\beta} \arrow[r] \arrow[drr, bend right] & \mathcal{G}_{\beta+1} \arrow[dr, dashed] & \\
 & & \mathcal{G}
\end{tikzcd}$$

where the top horizontal arrow represents the faces of $\widetilde{t}$, the bottom curved arrow represents $\widetilde{t}$, and the right curved arrow is the inclusion from the inductive hypothesis. Note that we again have a monomorphism $\mathcal{G}_{\beta+1} \hookrightarrow \mathcal{G}$, since it is true at all stalks $y \in \mathscr{X}$.

Now we have $t \in (\mathcal{G}_{\beta+1})_x$. Iterating this process to a sufficiently large ordinal β exhausts all elements in any $\mathcal{G}_x(n)$ that is not in $\mathcal{F}_x(n)$, so we must have $\mathcal{G}_\beta = \mathcal{G}$ for some ordinal β. □

Lemma II.42. *If $f : \mathcal{F} \to \mathcal{G}$ is a monomorphism of Kan spectral sheaves and also a stalk-wise weak equivalence, then there exists a morphism of Kan spectral sheaves $g : \mathcal{G} \to \mathcal{H}$ such that $g \circ f$ is an anodyne extension.*

Proof. Using Proposition II.68, we express f as a relative cell sheaf map. We will inductively construct morphisms

$$g_\beta : \mathcal{G}_\beta \longrightarrow \mathcal{H}_\beta$$

such that $g_\beta \circ f$ is an anodyne extension. Again, we start with $\mathcal{G}_{(-1)} = \mathcal{F}$, and if β is a limit ordinal, we put $\mathcal{G}_\beta = \varinjlim_{\gamma<\beta} \mathcal{G}_\gamma$. Now suppose g_β has beeen constructed for some ordinal β; if $\mathcal{G}_\beta = \mathcal{G}$ then we are done. Otherwise, there exists an element $t \in \mathcal{G}_x(k)$ which is not in $(\mathcal{G}_\beta)_x(k)$ but all of whose faces are. Moreover, there exist open subsets $U' \subseteq U \subseteq \mathscr{X}$ such that t is represented by $\widetilde{t} \in \mathcal{G}(U)(k)$, and the faces of t are represented by elements in $(\mathcal{G}_\beta)(U)(k-1)$. Moreover, since f is a stalk-wise equivalence, we may assume, upon replacing U by a smaller neighborhood containing U', that $\widetilde{t}$ lifts, up to homotopy, to an element in $\mathscr{L}(\mathcal{H}_\beta(U'))$.

We claim that we may assume, without loss of generality, that the restriction map

$$\mathcal{H}_\beta(U) \longrightarrow \mathcal{H}_\beta(U')$$

is a fibration of Kan spectra. Indeed, we may attach, in ω steps, all possible non-isomorphic anodyne cells to $\mathcal{H}_\beta$ in each step. This is similar to, although not exactly the same as, the canonical factorization of morphisms into anodyne extensions followed by fibration (since sheafification is applied in every step of attaching), the small object argument still applies, and the resulting sheaf $\mathcal{H}_\beta$ has the property the restriction map along $\iota : U' \subseteq U$ is a fibration.

Now we construct the morphism $g_{\beta+1}$. By assumption, the composite

$$\Sigma^{\infty-\ell}\left(\Delta^n_+\right)_\iota \longrightarrow \mathcal{G}_\beta \longrightarrow \mathcal{G}$$

representing the faces of $\widetilde{t}$ extends to

$$\Sigma^{\infty-\ell}\left(\Delta^n_+\right)_U \longrightarrow \mathcal{G}.$$

By our assumptions, this means that the morphism

$$\Sigma^{\infty-\ell}\left(\Delta^n_+\right)_\iota \longrightarrow \mathcal{G}_\beta \longrightarrow \mathcal{H}_\beta$$

extends to

$$\Sigma^{\infty-\ell}\left(\Delta^n_+\right)_U \longrightarrow \mathcal{H}_\beta,$$

by adjunction and that restriction along ι is a fibration. But this means, in the process of attaching cells to $\mathcal{G}_\beta$ to construct $\mathcal{G}_{\beta+1}$, we may instead form a pushout of the form

$$\begin{tikzcd}
\Sigma^{\infty-\ell}\left(Q^n_+\right)_\iota \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \mathcal{H}_\beta \arrow[d] \\
\Sigma^{\infty-\ell}\left((\Delta^n \times I)_+\right)_U \arrow[r] & \mathcal{H}_{\beta+1}
\end{tikzcd}$$

where $(Q^n)_+$ is the based simplicial sheaf formed as the pushout

$$\begin{tikzcd}
\left(\left(\partial\Delta^n_+ \wedge I_+\right) \vee \left(\Delta^n_+ \wedge S^0\right)\right)_{U'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \left(\left(\partial\Delta^n_+ \wedge I_+\right) \vee \left(\Delta^n_+ \wedge S^0\right)\right)_U \arrow[d] \\
\left(\Delta^n_+ \wedge I_+\right)_{U'} \arrow[r] & \left(Q^n_+\right)_\iota
\end{tikzcd}$$

Note that this is an anodyne extension, and therefore so is $\mathcal{F} \hookrightarrow \mathcal{H}_\beta \hookrightarrow \mathcal{H}_{\beta+1}$. This process must terminate for set-theoretic reason, and the induction is complete. $\square$

2.2.4 Localization of Kan spectral sheaves

Definition II.43. Define functors

$$K \star_p ? : \mathsf{PSh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow \mathsf{PSh}_{\mathscr{S}}(\mathscr{X}), \qquad K \star ? : \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}),$$

by $(K \star_p \mathcal{F})(U) \coloneqq K \star (\mathcal{F}(U))$ for a spectral presheaf $\mathcal{F}$ on $\mathscr{X}$ and open set $U \subseteq \mathscr{X}$, and $K \star \mathcal{F} \coloneqq (K \star_p \mathcal{F})^{\#}$, where $\#$ denotes the sheafification functor.

Definition II.44 (Strong homotopy)**.** A *strong homotopy* $h : \varphi \simeq \psi$ between two morphisms $\varphi, \psi : \mathcal{F} \rightrightarrows \mathcal{G}$ in $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ (or more generally, in $\mathsf{PSh}_{\mathscr{S}}(\mathscr{X})$) is a morphism

$$h : I_+ \star \mathcal{F} \longrightarrow \mathcal{G},$$

so that $h \circ d_0 = \varphi$ and $h \circ d_1 = \psi$, where $d_i : \{i\}_+ \hookrightarrow I_+$.

A morphism $\varphi : \mathcal{F} \xrightarrow{\simeq} \mathcal{G}$ in $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ is a (*strong*) *homotopy equivalence* if there is a morphism $\psi : \mathcal{G} \to \mathcal{F}$ such that $\psi \circ \varphi \simeq \mathrm{id}_{\mathcal{F}}$ and $\varphi \circ \psi \simeq \mathrm{id}_{\mathcal{G}}$.

We write $\simeq$ for the equivalence relation generated by strong homotopies. It is again a congruence relation on the category $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$, and we may form the (*strong*) *homotopy category* of Kan spectral sheaves $h\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \coloneqq \mathsf{Sh}_{\mathscr{S}}(\mathscr{X})/{\simeq}$. By a (*strong*) *homotopy equivalence*, we refer to an isomorphism in $h\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$.

Lemma II.45. *A homotopy equivalence is a stalk-wise weak equivalence.*

Proof. By Lemma II.29, it suffices to show that a strong homotopy of Kan spectral sheaves induces strong homotopies on stalks.

A strong homotopy of Kan spectral sheaves

$$I_+ \star \mathcal{F} \longrightarrow \mathcal{G}$$

induces, on each open set $U \subseteq \mathscr{X}$, a strong homotopy of Kan spectra

$$I_+ \star (\mathcal{F}(U)) \longrightarrow (I_+ \star \mathcal{F})(U) \longrightarrow \mathcal{G}(U).$$

Passing to the stalk at $x \in \mathscr{X}$, on the level of Kan prespectra, we have a morphism

$$I_+ \cdot \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

since $I_+ \cdot ?$ is a left adjoint. Since $\mathcal{G}_x$ is itself a Kan spectrum, taking spectrification yields a strong homotopy

$$I_+ \star \mathcal{F}_x \longrightarrow \mathcal{G}_x,$$

as desired. □

Definition II.46 (Godement resolution)**.** Consider the space $\mathscr{X}_{dis}$ which is the discrete space on the underlying set of $\mathscr{X}$, and let p denote the natural map $\mathscr{X} \to \mathscr{X}_{dis}$. For any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$, we define

$$\mathbf{T}\mathcal{F} := p_* p^* \mathcal{F} = \prod_{x \in \mathscr{X}} x_* x^* \mathcal{F} = \prod_{x \in \mathscr{X}} x_* \mathcal{F}_x$$

where x_* denotes the skyscraper sheaf at x. $\mathbf{T}$ is a monad, and the *cosimplicial Godement resolution* is defined as the cosimplicial object $\mathrm{Gode}^\bullet \mathcal{F}$ associated to $\mathbf{T}$. Specifically, $\mathrm{Gode}^p \mathcal{F} = \mathbf{T}^{p+1}\mathcal{F}$ with standard coface and codegeneracy morphisms.

Note that the unit of the adjunction (x^*, x_*) provides a coaugmentation

$$\mathcal{F} \longrightarrow \mathrm{Gode}^\bullet \mathcal{F}$$

of cosimplicial Kan spectral sheaves, which is a cosimplicial homotopy equivalence.

For a cosimplicial Kan spectrum $Z^\bullet$, the cosimplicial realization is defined as the categorical end

$$\begin{aligned} |Z^\bullet| &:= \int_{\Delta^{in}} \prod_{m \in \mathbb{N}_0} F\left(\Delta^m_+, Z^m\right) \\ &\cong \mathrm{eq}\left[\prod_{m \in \Delta} F\left(\Delta^m_+, Z^m\right) \rightrightarrows \prod_{\phi \in \Delta^{in}(k,m)} F\left(\Delta^m_+, Z^k\right),\right] \end{aligned} \tag{2.12}$$

where Δ^{in} is the subcategory of Δ consisting of all injective morphisms.

Definition II.47 (Cosimplicial realization of sheaves)**.** Given a cosimplicial Kan spectral sheaf $\mathcal{F}^\bullet$ on $\mathscr{X}$, we define the *cosimplicial realization* $|\mathcal{F}^\bullet|$ of $\mathcal{F}^\bullet$ by

$$|\mathcal{F}^\bullet|(U) = |\mathcal{F}^\bullet(U)|.$$

Since the cosimplicial realization is a limit, and $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ is a reflective subcategory of $\mathsf{PSh}_{\mathscr{S}}(\mathscr{X})$, $|\operatorname{Gode}^\bullet \mathcal{F}|$ is itself a Kan spectral sheaf on $\mathscr{X}$. Applying cosimplicial realization to the coaugmentation $\mathcal{F} \to \operatorname{Gode}^\bullet \mathcal{F}$ induces a natural morphism

$$\mathcal{F} \longrightarrow |\operatorname{Gode}^\bullet \mathcal{F}|$$

of Kan spectral sheaves. We refer the target of this morphism as the *Godement resolution* of $\mathcal{F}$, and simply denote by $\operatorname{Gode} \mathcal{F}$.

Lemma II.48. *If $\mathcal{F}$ is a Kan spectral sheaf whose stalks are fibrant Kan spectra, then for any inclusion of open subsets $U' \subseteq U \subseteq \mathscr{X}$, the restriction map*

$$\operatorname{Gode} \mathcal{F}(U) \longrightarrow \operatorname{Gode} \mathcal{F}(U')$$

is a Kan fibration.

Proof. First note that $\mathbf{T}\mathcal{F}$ has the desired property, that is, its restriction maps are fibrations because

$$\mathbf{T}\mathcal{F}(U) = \prod_{x \in U} \mathcal{F}_x \cong \prod_{x \in U'} \mathcal{F}_x \times \prod_{x \in U \setminus U'} \mathcal{F}_x \longrightarrow \prod_{x \in U'} \mathcal{F}_x = \mathbf{T}\mathcal{F}(U')$$

is projection of product forgetting factors that are fibrant. Since fibrations/fibrant objects are characterized by a right lifting property, products of fibrant spectra are again fibrant, so the sheaf $\mathbf{T}\mathcal{F}$ is again section-wise fibrant and therefore stalk-wise fibrant. It thus follows that the sheaves $\operatorname{Gode}^p \mathcal{F}$ also have the desired property.

The lemma now reduces to the following claim: if $X^\bullet \to Y^\bullet$ is a degree-wise fibration of cosimplicial Kan spectra, then the cosimplicial realization (2.12) produces a fibration of Kan spectra $|X| \to |Y|$. Observe that for all $m, m' \in \Delta$, the morphism

$$F\left(\Delta_+^m, X^{m'}\right) \longrightarrow F\left(\Delta_+^m, Y^{m'}\right)$$

is a fibration by adjunction since the morphism

$$\Delta_+^m \star \Sigma^{\infty-\ell} V_{k+}^n \hookrightarrow \Delta_+^m \star \Sigma^{\infty-\ell} \Delta_+^n$$

is an anodyne extension. Now our claim follows since the right lifting property of the equalizers with respect to anodyne cells reduces to that of each product of function spectra. $\square$

Given a Kan spectral sheaf $\mathcal{F}$, we denote by $\operatorname{Sing}\mathscr{L}\mathcal{F}$ the sheafification of the presheaf

$$U \longmapsto \operatorname{Sing}\mathscr{L}(\mathcal{F}(U)).$$

Corollary II.49. *For any Kan spectral sheaf $\mathcal{F}$, the sheaf $\operatorname{Gode}\operatorname{Sing}\mathscr{L}\mathcal{F}$ has the property that, for any inclusion $U' \subseteq U$, the restriction map*

$$\operatorname{Gode}\operatorname{Sing}\mathscr{L}\mathcal{F}(U) \longrightarrow \operatorname{Gode}\operatorname{Sing}\mathscr{L}\mathcal{F}(U')$$

is a fibration of Kan spectra. In particular, the sheaf $\operatorname{Gode}\operatorname{Sing}\mathscr{L}\mathcal{F}$ is section-wise fibrant.

Proof. The presheaf $U \mapsto \operatorname{Sing}\mathscr{L}(\mathcal{F}(U))$ is section-wise Kan fibrant by Proposition II.18, and is section-wise weakly equivalent to $\mathcal{F}$ as a presheaf. Therefore, after sheafification, the sheaf $\operatorname{Sing}\mathscr{L}\mathcal{F}$ is stalk-wise fibrant, and the natural morphism $\mathcal{F} \to \operatorname{Sing}\mathscr{L}\mathcal{F}$ is a stalk-wise weak equivalence. The desired claim is now a consequence of Lemma II.48, and the statement regarding section-wise fibrancy follows from that $\mathcal{F}(\varnothing) = *$. $\square$

Recall that the *cohomological dimension* $\operatorname{cd}\mathscr{X}$ of $\mathscr{X}$ is the smallest integer $d \geq 0$ such that for any abelian sheaf $\mathcal{A}$ we have $H^n(\mathscr{X};\mathcal{A}) = 0$ for all $n > d$. Typical examples of spaces with finite cohomological dimensions include finite-dimensional manifolds and noetherian scheme of finite Krull dimension. Henceforth, we shall assume that $\mathscr{X}$ has finite cohomological dimension.

Proposition II.50. *The Kan spectral sheaves* $\operatorname{Gode}\operatorname{Sing}\mathscr{L}\mathcal{E}$ *in the homotopy category* $h\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ *are local with respect to the class* $\mathcal{W}$ *of stalk-wise weak equivalences.*

Proof. We follow the proof of Proposition II.31. Let $f : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$ be a stalk-wise equivalence, and $\mathcal{E}' \coloneqq \operatorname{Gode}\operatorname{Sing}\mathscr{L}\mathcal{E}$ for some Kan spectral sheaf $\mathcal{E}$. We may first replacing $\mathcal{G}$ by the mapping cylinder Mf, if necessary, to assume that f is an inclusion. Note that by the canonical morphism $Mf \to Y$ is a homotopy equivalence by Lemma II.26.

To prove that f is surjective, we apply Lemma II.42 to form an anodyne extension $g \circ f$ for some inclusion $g : \mathcal{G} \hookrightarrow \mathcal{H}$. Since $\mathcal{E}$ is fibrant, the lifting problem

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\forall"] \arrow[d, tail, "g\circ f"', "\sim"] & \mathcal{E}' \arrow[d, two heads] \\ \mathcal{H} \arrow[r] \arrow[ur, dashed, "\exists"'] & * \end{tikzcd}$$

has a solution $\mathcal{H} \to \mathcal{E}'$ by Corollary II.49, so the composite $\mathcal{G} \to \mathcal{E}'$ gives the lift of any map $\mathcal{F} \to \mathcal{E}'$.

To prove that f is injective, again form the double mapping cylinder

$$Pf \coloneqq \varinjlim \left[\begin{tikzcd} \{1\}_+ \star \mathcal{F} \arrow[r] \arrow[d] & \{1\}_+ \star \mathcal{G} \\ II_+ \star \mathcal{F} & \\ \{1'\}_+ \star \mathcal{F} \arrow[r] \arrow[u] & \{1'\}_+ \star \mathcal{G} \end{tikzcd} \right]$$

and there is an inclusion and stalk-wise equivalence

$$Pf \hookrightarrow II_+ \star \mathcal{G}.$$

We may now apply Lemma II.42 and Corollary II.49 again to extend any morphism $Pf \to \mathcal{E}'$ to $II_+ \star \mathcal{G} \to \mathcal{E}'$, which implies that, if any two maps $\mathcal{G} \rightrightarrows \mathcal{E}'$ are homotopic when post-composed with f, then they are themselves homotopic.

Finally, the natural morphism

$$\mathcal{F} \longrightarrow \mathrm{Gode}\,\mathrm{Sing}\,\mathscr{L}\mathcal{F} \tag{2.13}$$

being a stalk-wise weak equivalence is a consequence of [28, Thm. 4.14]. □

Theorem II.51. *The strong homotopy category $h\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ has a localization with respect to stalk-wise weak equivalences given by* (2.13)*.*

Proof. In light of Proposition II.50, it remains to show that the natural morphism $\mathcal{F} \to \mathrm{Gode}\,\mathcal{F}$ is a stalk-wise equivalence. □

Corollary II.52. *The derived category $D\mathscr{S} = \mathcal{W}^{-1}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ of the category of Kan spectral sheaves on $\mathscr{X}$ with respect to stalk-wise weak equivalences exists, and it is equivalent to K. Brown's [5] homotopy category $\mathcal{S}ta\mathcal{H}o(\mathscr{X})$ of sheaves of spectra on $\mathscr{X}$. Moreover, a functor $F : \mathscr{S} \to \mathscr{D}$ is right-derivable if and only if it preserves strong homotopies, and its right derived functor*

$$RF : D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow D\mathscr{D},$$

can be computed via the Godement resolution as

$$RF(?) = F(\mathrm{Gode}\,\mathrm{Sing}\,\mathscr{L}(?)).$$

Proof. The existence of $D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ follows from the Theorem II.51 and the general results of localization (see Theorems A.7 and A.9). Moreover, since our category of spectral sheaves and the notion of weak equivalences agree with the ones of Brown's, we conclude that their derived categories coincide. Finally, A functor $F : \mathscr{S} \to \mathscr{D}$ preserves strong homotopies if and only if it induces a unique functor $\overline{F} : h\mathscr{S} \to \mathscr{D}$, which is always right-derivable by the localization. $\square$

Remark II.53. In [8], the result above is generalized to the case of spectral sheaves on a site $\mathscr{C}$ be a site of finite cohomological dimension which also satisfies the following assumptions:

(A1) the site $\mathscr{C}$ is small and has enough points;

(A2) the site $\mathscr{C}$ has sub-canonical topology, and any subsheaf of any representable sheaf is representable;

(A3) for every sheaf of sets $\mathcal{F}$ on a site $\mathscr{C}$, p a point of $\mathscr{C}$, and $t \in \mathcal{F}_x$, there exists an object $U \in \operatorname{Obj}\mathscr{C}$ and and an injective morphism $\underline{U} \to \mathcal{F}$ such that p is in U and $t \in \mathcal{F}(U)$;

The (small) site associated to a topological space $\mathscr{X}$ obviously satisfy the assumptions (A1)–(A3) above, and it has finite cohomological dimension condition if and only if $\mathscr{X}$ does.

2.3 Simplicial Kan spectra and spectral sheaves

For future reference, we generalize Sections 2.1.6 and 2.2.4 to the case of simplicial objects in Kan spectra and Kan spectral sheaves, and we construct their localizations.

2.3.1 Bisimplicial sets

Recall that a *bisimplicial set* is a functor

$$X_{\bullet,\bullet} : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set},$$

and a morphism of bisimplicial sets is a natural transformation. A bisimplicial set X may be identified with a simplicial object in simplicial sets in two ways, namely "horizontally" and "vertically". Using these identifications, there are *total realization* functors

$$\begin{tikzcd} (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set} \arrow[r, shift left, "|?|^h_\bullet"] \arrow[r, shift right, "|?|^v_\bullet"'] \arrow[d, shift right, "\mathrm{Tot}^h"'] \arrow[d, shift left, "\mathrm{Tot}^v"] & \Delta^{\mathrm{op}}\text{-}\mathsf{Top} \arrow[d, "|?|"] \\ \Delta^{\mathrm{op}}\text{-}\mathsf{Set} \arrow[r, "|?|"'] & \mathsf{Top} \end{tikzcd}$$

Here, $|?|_\bullet$ is the geometric realization of bisimplicial sets in each simplicial degree, and Tot is the totalization of bisimplicial sets; in both cases the superscript h or v indicates whether the realization takes place across horizontal or vertical simplicial degree. It is well known that both totalizations are isomorphic to the *diagonal* functor

$$(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set} \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathsf{Set}$$

by sending each bisimplicial set $X_{\bullet,\bullet}$ to the simplicial set $X_\bullet$ with $X_n \coloneqq X_{n,n}$. It thus follows that both realizations are isomorphic to the composite

$$\begin{tikzcd} (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set} \arrow[d, "d"'] & \\ \Delta^{\mathrm{op}}\text{-}\mathsf{Set} \arrow[r, "|?|"'] & \mathsf{Top} \end{tikzcd}$$

which we call the *diagonal realization* and denote by $\|?\|$.

Definition II.54. A morphism of bisimplicial sets $f : X \to Y$ is a *diagonal (weak) equivalence* if the diagonal $d(f) : d(X) \to d(Y)$ is a weak equivalence of simplicial sets, or equivalently, if and only if the diagonal realization $\|f\| : \|X\| \to \|Y\|$ is a weak equivalence of topological spaces.

To each pair of simplicial sets K and L, there associates a bisimplicial set

$$(K \boxtimes L)_{m,n} := K_m \times L_n;$$

this forms an *external product* functor

$$? \boxtimes ? : \Delta^{\mathrm{op}}\text{-}\mathsf{Set} \times \Delta^{\mathrm{op}}\text{-}\mathsf{Set} \longrightarrow (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set}.$$

For example, for $m, n \in \mathbb{N}_0$, the bisimplicial set $\Delta^m \boxtimes \Delta^n$ is precisely the presheaf $\Delta^{m,n}$ represented by the object $(m, n) \in \Delta \times \Delta$.

A *bi-cell* has the form

$$\partial\Delta^{m,n} := (\partial\Delta^m \boxtimes \Delta^n) \cup (\Delta^m \boxtimes \partial\Delta^n) \hookrightarrow \Delta^m \boxtimes \Delta^n \cong \Delta^{m,n}, \tag{2.14}$$

while a *bi-anodyne* cell has either the form

$$V^{m,n}_{k,*} := (V^m_k \boxtimes \Delta^n) \cup (\Delta^m \boxtimes \partial\Delta^n) \hookrightarrow \Delta^m \boxtimes \Delta^n \cong \Delta^{m,n}, \tag{2.15}$$

or

$$V^{m,n}_{*,k} := (\Delta^m \boxtimes V^n_k) \cup (\partial\Delta^m \boxtimes \Delta^n) \hookrightarrow \Delta^m \boxtimes \Delta^n \cong \Delta^{m,n}. \tag{2.16}$$

We may now define the notions of relative bi-cell maps, bi-anodyne extensions, and fibrations analogously to the case of simplicial sets, and again we have the following analogous results:

Theorem II.55. *There is a model structure, called* diagonal model structure*, on the category of bisimplicial sets, where equivalences are diagonal weak equivalences, cofibrations are monomorphisms, and fibrations are morphisms with right lifting property with respect to all anodyne cells.*

Proof. The diagonal model structure is proven in [15]; see also [9, Thm. 5.5.7], which also establishes the Quillen equivalence. □

The following statements are direct consequences of the model structure.

Proposition II.56.

1. *Every monomorphism of bisimplicial sets is a relative bi-cell map. In particular, every bisimplicial set is a bi-cell bisimplicial set.*
2. *A morphism is a bi-anodyne extension if and only if it is both a monomorphism and a diagonal weak equivalence.*
3. *A morphism of bisimplicial sets has right lifting property with respect to all monomorphisms if and only if it is both a fibration and a diagonal weak equivalence.* □

The diagonal functor d has a left adjoint $d_!$ characterized by

$$d_!(\Delta^n) \coloneqq \Delta^{n,n}$$

in addition to preservation of colimits, and d also has a right adjoint defined by

$$d_*(K)_{m,n} \coloneqq \mathrm{Hom}_{\Delta^{\mathrm{op}}\text{-}\mathsf{Set}}(\Delta^m \times \Delta^n, K).$$

Theorem II.57 ([9, Thm. 5.5.7])**.** *Moreover, the pairs $(d_!, d)$ and (d, d_*) are both Quillen equivalences between the category of simplicial sets with the Quillen model structure, and the category bisimplicial sets with the diagonal model structure.* □

Corollary II.58. *The composition of units of adjunctions*

$$\mathrm{Id} \longrightarrow d_* d \longrightarrow d_* \circ \mathrm{Sing} \circ |?| \circ d \tag{2.17}$$

is a fibrant replacement in the category of bisimplicial sets.

Proof. Recall that $\mathrm{Id} \to \mathrm{Sing} \circ |?|$ is a fibrant replacement in Δ^{op}-Set, so the above composition, being a "derived unit of adjunction", is a weak equivalence. The fact that a bisimplicial set of the form $d_* \mathrm{Sing}\, |dX|$ is fibrant follows from that $|dX|$ as a topological space is fibrant and that each

$$|dV^n_k| \hookrightarrow |d\Delta^n|$$

is an anodyne extension in the category of topological spaces. □

There is also a notion of strong (geometric) homotopy arising from the interval object $\Delta^{1,1}$. Note that strong homotopy equivalence in this sense encodes both "horizontal" and "vertical" homotopies via the inclusions

$$\Delta^{1,0} \hookrightarrow \Delta^{1,1} \hookleftarrow \Delta^{0,1}.$$

In the standard way, this leads to the notions of strong homotopy equivalences and strong homotopy category $h(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})$-$\mathsf{Set}$ of bisimplicial sets.

The following are immediate:

Proposition II.59.

1. *The morphism*

$$X \cong \Delta^{0,0} \times X \longrightarrow \Delta^{1,1} \times X$$

is an acyclic cofibration.

2. *The inclusion*

$$\left(\partial\Delta^{m,n} \times \Delta^{1,1}\right) \cup \left(\Delta^{m,n} \times \Delta^{0,0}\right) \hookrightarrow \Delta^{m,n} \times \Delta^{1,1}$$

is an acyclic cofibration. □

Therefore, by arguments similar to those for Proposition II.31, we may show the following result.

Proposition II.60. *Fibrant bisimplicial sets are local with respect to diagonal weak equivalences in the strong homotopy category* $h(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})$ -Set. □

We end this section by remarking that all constructions above have the based variants, where the category Set is replaced by Set_*, bisimplicial sets X by the the based ones X_+ with a disjoint basepoint attached, and cartesian product $\times$ by the smash product $\wedge$.

2.3.2 Simplicial Kan spectra

Let $\mathscr{M}\mathscr{S}$ denote the category of May spectra. There are adjunctions

$$\begin{tikzcd} \Delta^{\mathrm{op}}\text{-}\mathscr{S} \arrow[r, shift left, "\mathscr{L}"] & \Delta^{\mathrm{op}}\text{-}\mathscr{M}\mathscr{S} \arrow[l, shift left, "\mathrm{Sing}"] \arrow[d, shift left, "|?|"] \\ & \mathscr{M}\mathscr{S} \arrow[u, shift left, "F_\bullet"] \end{tikzcd}$$

where $(\mathscr{L}, \mathrm{Sing})$ is defined degree-wise extending the adjunction between Kan spectra and May spectra, and the functors $|?|$ and $F_\bullet$ are defined as follows: For a simplicial May spectrum $Z_\bullet$, its *totalization* is defined as

$$|Z_\bullet| \coloneqq \int^{n\in\Delta} \left|\Delta^n_+\right| \wedge Z_n,$$

while for a May spectrum E, the *free simplicial May spectrum* is defined as

$$F_\bullet(X) \coloneqq F\left(|\Delta^\bullet_+|, X\right).$$

Here, the $|\Delta^n_+|$ is the topological standard n-simplex, $\wedge$ is the smash product of May spectra by based topological spaces, and F is the function May spectrum functor, so in particular $F(|\Delta^n_+|, ?)$ is right adjoint to $|\Delta^n_+| \wedge ?$.

We denote by $\|\mathscr{L}\|$ the composite $|?| \circ \mathscr{L}$, and by $\mathrm{Sing}_\bullet$ the composite $\mathrm{Sing} \circ F_\bullet$; it is clear that the pair $(\|\mathscr{L}\|, \mathrm{Sing}_\bullet)$ is an adjunction.

Definition II.61. A morphism $f_\bullet : X_\bullet \to Y_\bullet$ of simplicial Kan spectra is a *total (weak) equivalence* if $\|\mathscr{L}\|(f_\bullet)$ is a weak equivalence of May spectra.

Lemma II.62. *The following are weak equivalences of simplicial Kan spectra:*

1. *a degree-wise weak equivalence of simplicial Kan spectra, i.e., a morphism $f_\bullet : X_\bullet \to Y_\bullet$ such that f_n is a weak equivalence of Kan spectra for each n;*
2. *a simplicial homotopy equivalence of simplicial Kan spectra.*

Proof. These follow from the that the simplicial realization $|?|$ of simplicial May spectra realizes degree-wise weak equivalences and simplicial homotopy equivalences to weak equivalences, along with the observation that $\mathscr{L}$ preserves strong homotopy equivalences. $\square$

For each $k \in \mathbb{Z}$, there is a simplicial shift suspension spectrum functor

$$\Sigma^{\infty+k,\bullet} : (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set}_* \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathscr{S}$$

extending the functor Σ^∞ characterized in Lemma II.8 and constructed in the proof in page 20. Specifically, we have

$$\left(\Sigma^{\infty+k,\bullet} X_{\bullet,\bullet}\right)_n := \Sigma^{\infty+k}\left(X_{\bullet,n}\right)$$

for each $n \in \Delta$. It has right adjoint $\Omega^{\infty+k,\bullet}$ given by applying the functor $\Omega^{\infty+k}$ to each "vertical" simplicial degree.

A *cell* of simplicial Kan spectra has the form of $\Sigma^{\infty+\ell}$ applied to (2.14) for some $\ell \in \mathbb{Z}$, and an *anodyne cell* of simplicial Kan spectra has the form of $\Sigma^{\infty+\ell}$ applied to either (2.15) or (2.16). Again, these give rise to the notions of relative cell maps, anodyne extensions, and fibrations of simplicial Kan spectra, as well as cell and fibrant simplicial Kan spectra.

Analogously to Proposition II.18 and Lemma II.30 we have the following results:

Proposition II.63.

1. *A morphism of simplicial Kan spectra is a relative cell maps if and only if it is a monomorphism. In particular, all simplicial Kan spectra are cell.*

2. *An anodyne extension of simplicial Kan spectra is both a monomorphism and a total weak equivalence.*

3. *A morphism $f : X \to Y$ of simplicial Kan spectra is a fibration if and only if it is a level-wise fibration of bisimplicial sets, that is, if $\Omega^{\infty+\ell}$ is a fibration of bisimplicial sets for any $\ell \in \mathbb{Z}$.* □

Lemma II.64. *If $f_\bullet : X_\bullet \to Y_\bullet$ is a monomorphism of simplicial Kan spectra and also a total weak equivalence, then there exists a monomorphism of simplicial Kan spectra $g_\bullet : Y_\bullet \to Z_\bullet$ such that $g_\bullet \circ f_\bullet$ is an anodyne extension.* □

Recall from Sections 2.1.6 and 2.3.1 that both the category $\mathscr{S}$ of Kan spectra and the category $(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set}_*$ have localizations, with respect to weak equivalences and diagonal weak equivalences, by fibrant objects, which are given by (2.4) and (2.17), respectively. As a common generalization of these cases, we will show that the unit of adjunction

$$\eta : \mathrm{Id} \longrightarrow \mathrm{Sing}_\bullet \|\mathscr{L}\| \tag{2.18}$$

gives a localization in the category $\Delta^{\mathrm{op}}\text{-}\mathscr{S}$ of simplicial Kan spectra.

Lemma II.65. *The unit of adjunction* (2.18) *is a natural total weak equivalence, with targets fibrant simplicial Kan spectra.*

Proof. A similar argument to the proof of Lemma II.15 shows that the morphism (2.18) is a total weak equivalence. To see that $\mathrm{Sing}_\bullet \|\mathscr{L}\|(Z_\bullet)$ is fibrant for any simplicial Kan spectrum $Z_\bullet$, it suffices to solve the lifting problem

$$\begin{tikzcd}
\Sigma^{\infty+\ell}A \arrow[r] \arrow[d] & \mathrm{Sing}_\bullet \|\mathscr{L}\|(Z_\bullet) \\
\Sigma^{\infty+\ell}B \arrow[ur, dashed] &
\end{tikzcd}$$

with respect to any bi-anodyne cell $A \to B$ in bisimplicial sets, that is, of the form either (2.15) or (2.16). By the adjunction $(\|\mathscr{L}\|, \mathrm{Sing}_\bullet)$, this is equivalent to the lifting problem

$$\begin{tikzcd}
\|\mathscr{L}\| \left(\Sigma^{\infty+\ell}A\right) \arrow[r] \arrow[d] & \|\mathscr{L}\|(Z_\bullet) \\
\|\mathscr{L}\| \left(\Sigma^{\infty+\ell}B\right) \arrow[ur, dashed] &
\end{tikzcd}$$

which always has solution since the May spectrum $\|\mathscr{L}\|(Z_\bullet)$ is necessarily fibrant. $\square$

To discuss locality of fibrant simplicial Kan spectra, we need the notion of strong (geometric) homotopy. To this end, we consider the bisimplicial set $\Delta^{1,1}$ as the interval object. We also extend the star product II.23 to a functor

$$? \star ? : (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set}_* \times \Delta^{\mathrm{op}}\text{-}\mathscr{S} \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathscr{S}$$

as follows. Given a based bisimplicial set $K_{\bullet,\bullet}$ and a simplicial Kan spectrum $Z_\bullet$, define a simplicial Kan spectrum $K \star Z$ by

$$(K \star Z)_n \coloneqq K_{\bullet,n} \star Z_n.$$

Now the star product $J_+ \star ?$ with based bisimplicial set J_+ gives rise to the notions of strong homotopy, strong homotopy equivalences, and strong homotopy category $h\Delta^{\mathrm{op}}\text{-}\mathscr{S}$ of simplicial Kan spectra.

Note that strong homotopies contains both degree-wise strong homotopies in simplicial Kan spectra, as well as simplicial homotopies in simplicial objects in Kan spectra.

The following is analogous to Proposition II.31 and Theorem II.32.

Theorem II.66. *The strong homotopy category $h\Delta^{\mathrm{op}}\text{-}\mathscr{S}$ of simplicial Kan spectra has a localization with respect to total weak equivalences, given by the functor* $\mathrm{Sing}_\bullet \|\mathscr{L}\|$ *together with the unit of adjunction* (2.18)*.*

2.3.3 Simplicial Kan spectral sheaves

A *simplicial Kan spectra sheaf*, on a space $\mathscr{X}$, is a simplicial object in the category $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ of Kan spectral sheaves on $\mathscr{X}$, or equivalently a sheaf on $\mathscr{X}$ taking value in the category $\Delta^{\mathrm{op}}\text{-}\mathscr{S}$ of simplicial Kan spectra; it is the second viewpoint that we will mostly take. We denote by $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ the category of simplicial Kan spectral sheaves on $\mathscr{X}$. The category $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ is complete and cocomplete since the categories $\mathscr{S}$ and therefore $\Delta^{\mathrm{op}}\text{-}\mathscr{S}$ are, and it follows that there is again the sheafification functor of simplicial Kan spectral sheaves.

Definition II.67. A morphism $f_\bullet : \mathcal{F}_\bullet \to \mathcal{G}_\bullet$ of simplicial Kan spectral sheaves is a *stalk-wise total* (*weak*) *equivalence* if, for any $x \in \mathscr{X}$, the induced morphism

$$(f_\bullet)_x : (\mathcal{F}_\bullet)_x \longrightarrow (\mathcal{G}_\bullet)_x$$

is a total weak equivalence of simplicial Kan spectra.

We will construct the localization mimicking the method used in Section 2.2.4.

For a based bisimplicial set K and an open subset $U \subseteq \mathscr{X}$, we denote by K_U the based bisimplicial sheaf sending U' to K if $U' \subseteq U$ and to the singleton otherwise.

We form pushouts

$$\begin{tikzcd} (\partial\Delta_+^{m,n})_{U'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (\partial\Delta_+^{m,n})_U \arrow[d] \\ (\Delta_+^{m,n})_{U'} \arrow[r] & (\Delta_+^{m,n})_\iota \end{tikzcd}$$

as well as

$$\begin{tikzcd} \left(V_{k,*+}^{m,n}\right)_{U'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \left(V_{k,*+}^{m,n}\right)_U \arrow[d] \\ (\Delta_+^{m,n})_{U'} \arrow[r] & \left(E_{k,*+}^{n}\right)_\iota \end{tikzcd}$$

and

$$\begin{tikzcd} \left(V_{*,k+}^{m,n}\right)_{U'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \left(V_{*,k+}^{m,n}\right)_U \arrow[d] \\ (\Delta_+^{m,n})_{U'} \arrow[r] & \left(E_{*,k+}^{n}\right)_\iota \end{tikzcd}$$

A morphism in simplicial Kan spectral sheaf is called a *cell* if it has the form

$$\Sigma^{\infty+\ell,\bullet}\left(\Delta_+^{m,n}\right)_\iota \longrightarrow \Sigma^{\infty+\ell,\bullet}\left(\Delta_+^{m,n}\right)_U \tag{2.19}$$

and an *anodyne cell* if it has the form

$$\Sigma^{\infty+\ell,\bullet}\left(E_{k,*+}^{n}\right)_\iota \longrightarrow \Sigma^{\infty+\ell,\bullet}\left(\Delta_+^{m,n}\right)_U$$

or

$$\Sigma^{\infty+\ell,\bullet}\left(E_{*,k+}^{n}\right)_\iota \longrightarrow \Sigma^{\infty+\ell,\bullet}\left(\Delta_+^{m,n}\right)_U$$

where the functor $\Sigma^{\infty+\ell,\bullet}$ is applied section-wise followed by sheafification. As before, these give rise to the notions of relative cell sheaf maps and anodyne extensions. Both of these are monomorphisms, and anodyne extensions are furthermore stalk-wise total equivalences.

Analogously to Proposition II.41 and Lemma II.42, we have the following results.

Proposition II.68. *Every monomorphism of simplicial Kan spectral sheaves is a relative cell sheaf map.* □

Lemma II.69. *Every monomorphism which is also a stalk-wise total equivalence can be extended to an anodyne extension.* □

The star product constructed in Section 2.3.2 can be extended to a functor

$$? \star ? : (\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}})\text{-}\mathsf{Set}_* \times \Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$$

by applying the star product section-wise followed by sheafification. Now strong homotopies, strong homotopy equivalences, and the strong homotopy category can be defined using the functor $J_+ star?$, where J is the interval object of based bisimplicial sets considered in Section 2.3.2.

Furthermore, we may also extend the adjoint functors $(\|\mathscr{L}\|, \mathrm{Sing}_\bullet)$ to simplicial Kan spectral sheaves via section-wise construction and sheafification, and finally, the Godement resolution (Definition II.46) to simplicial Kan spectral sheaves. This gives, analogously to the localization (2.13) for Kan spectral sheaves, a natural morphism

$$\mathcal{F}_\bullet \longrightarrow \mathrm{Gode}\,\mathrm{Sing}_\bullet \|\mathscr{L}\|(\mathcal{F}_\bullet). \tag{2.20}$$

Like Corollary II.49, the simplicial Kan sheaves $\mathrm{Gode}\,\mathrm{Sing}_\bullet \|\mathscr{L}\|(\mathcal{F}_\bullet)$ have the property that the restriction map along any inclusion of open subsets of $\mathscr{X}$ is a fibration of simplicial Kan spectra.

We may finally state the main result, which is analogous to and can be proven using a similar argument to Theorem II.51.

Theorem II.70. *The simplicial Kan spectral sheaves of the form*

$$\mathrm{Gode}\,\mathrm{Sing}_\bullet \|\mathscr{L}\|(\mathcal{F}_\bullet)$$

are local with respect to stalk-wise total equivalences, and the strong homotopy category $h\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ has a localization given by (2.20). □

2.4 Connection to homotopy sheaves of spectra

In this section, we explore the connection between Kan spectral sheaves and *homotopy sheaves* of Kan spectra, which is a different notion of sheaves with the sheaf condition satisfied "up to homotopy". We will see that the localizations proven in Theorems II.51 and II.70 play the role of *homotopy sheafification*. We will make precise these ideas below.

To start, let $\mathscr{C}$ be a complete and cocomplete category, and consider a $\mathscr{C}$-valued presheaf on a topological space $\mathscr{X}$. Given an open set $U \subseteq \mathscr{X}$ and a covering $\mathcal{U} = \{U_i \subseteq U\}$, there is a diagram

(2.21)
$$\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots \substack{\longrightarrow\\ \vdots\\ \longrightarrow} \prod_{i_1,\dots,i_n} \mathcal{F}\left(U_{i_1} \cap \cdots \cap U_{i_n}\right) \substack{\longrightarrow\\ \vdots\\ \longrightarrow} \cdots$$

in $\mathscr{C}$, extending the equalizer diagram in (2.10); we denote this diagram by $\mathcal{F}(\mathcal{U})_\bullet$. The limit $\varprojlim \mathcal{F}(\mathcal{U})_\bullet$ of this diagram is precisely the equalize we saw in (2.10).

Now suppose $\mathscr{C}$ has a model structure, and denote by R a functorial fibrant replacement functor in $\mathscr{C}$. Then we can consider the *homotopy limit* of the diagram $\mathcal{F}(\mathcal{U})$, and it is well known that

$$\underleftarrow{\mathrm{Holim}}\, \mathcal{F}(\mathcal{U})_\bullet \cong \varprojlim R\mathcal{F}(\mathcal{U})_\bullet,$$

where the limit on the right-hand side is taken over the diagram obtained by fibrantly replacing each term $\mathcal{F}\left(U_{i_1} \cap \cdots \cap U_{i_n}\right)$ in the products in the diagram $\mathcal{F}(\mathcal{U})_\bullet$. Note that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & R\mathcal{F}(U) \arrow[d] \\
\varprojlim \mathcal{F}(\mathcal{U})_\bullet \arrow[r] & \underleftarrow{\mathrm{Holim}}\, \mathcal{F}(\mathcal{U})_\bullet
\end{tikzcd}$$

in the homotopy category $\mathsf{Ho}\mathscr{C}$.

Definition II.71. A $\mathscr{C}$-valued presheaf is called a ($\mathscr{C}$-valued) *homotopy sheaf* over $\mathscr{X}$ if, for any covering $\mathcal{U}$ of an open set $U \subseteq \mathscr{X}$, the canonical morphism

$$\mathcal{F}(U) \longrightarrow \underleftarrow{\mathrm{Holim}}\, \mathcal{F}(\mathcal{U})_{\bullet}$$

is an isomorphism in $\mathsf{Ho}\mathscr{C}$ (that is, represented by an equivalence is $\mathscr{C}$).

Note that this definition applies to more generally any category $\mathscr{C}$ with equivalences which has notions of a derived category and homotopy limits of shape (2.21).

Taking $\mathscr{C}$ to be the category Δ^{op}-Set of simplicial sets with the standard Quillen model structure, we arrive at a notion of *homotopy simplicial sheaves.* There is also based variant with $\mathscr{C} = \Delta^{\mathrm{op}}$-$\mathsf{Set}_*$.

We are most interested in the case of Kan spectra, as well as simplicial objects in them. Kan spectra admit a model structure which was first constructed by K. Brown.

Theorem II.72 ([5, Thm. 5])**.** *There is a model structure on the category $\mathscr{S}$ of Kan spectra where*

- *equivalences are the weak equivalences as defined in Definition II.14,*
- *cofibrations are monomorphisms, and*
- *fibrations are characterized by the right lifting property with respect to morphisms that are both equivalences and monomorphisms.* □

Remark II.73. By Proposition II.18 (3), the fibrations in Brown's model structure are precisely the fibrations as defined in Definition II.17, which are also the same as level-wise fibrations.

Given Brown's model structure, we can consider homotopy sheaves in Kan spectra, which we call *homotopy sheaves of Kan spectra.* The category of homotopy sheaves of Kan spectra is the full subcategory of the category $\mathsf{PSh}_{\mathscr{S}}(\mathscr{X})$ of presheaves of Kan spectra, consisting of all homotopy sheaves; we denote this full subcategory by $\mathsf{hoSh}_{\mathscr{S}}(\mathscr{X})$.

Lemma II.74. *Let $\mathcal{F}$ be a presheaf of Kan spectra over $\mathscr{X}$. Then the Godement resolution*

$$\operatorname{Gode}\operatorname{Sing}\mathscr{L}\mathcal{F}$$

as in (2.13) *is a homotopy sheaf of Kan spectra.*

Proof. Since the sheaf $\mathcal{F}'$ is section-wise fibrant by Corollary II.49, we have, for any covering $\mathcal{U}$ of any open set $U \subseteq \mathscr{X}$

$$\underleftarrow{\operatorname{Holim}}\, \mathcal{F}'(\mathcal{U})_\bullet \cong \underleftarrow{\lim}\, \mathcal{F}'(\mathcal{U})_\bullet \cong \mathcal{F}'(U)$$

(the second isomorphism follows from the sheaf condition of $\mathcal{F}'$). □

Corollary II.75. *Every presheaf of Kan spectra is stalk-wise equivalent to a homotopy sheaf of Kan spectra, which can be given by the Godement resolution in particular.* □

Now we turn to the case with $\mathscr{C} = \Delta^{\mathrm{op}}\text{-}\mathscr{S}$ being the category of simplicial Kan spectra. While there is no known model structure on simplicial Kan spectra, we do nonetheless have a notion of total equivalences (Definition II.61), with respect to which a derived category exists and can be constructed from the localization constructed in Theorem II.66. To define homotopy sheaves in simplicial Kan spectra, it still remains to define homotopy limits of the shape (2.21). We will do so "by hand".

Definition II.76. Let $f : X_\bullet \to Y_\bullet$ be a morphism of simplicial Kan spectra. The *mapping fiber* Ff of f is defined as the pullback

$$\begin{tikzcd} Ff \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(\Delta^{1,1}, Y_\bullet) \arrow[d] \\ X_\bullet \arrow[r, "f"'] & Y_\bullet \end{tikzcd}$$

where $F(\Delta^{1,1}, Y_\bullet)$ is the simplicial Kan spectrum whose nth level is the Kan spectrum

$$F\left((\Delta^{1,1})_{\bullet,n}, Y_n\right).$$

It is clear that the morphism $Ff \to X_\bullet$ is a total equivalence, and that the functor $F(\Delta^{1,1}, ?)$ is right adjoint to $\Delta^{1,1} \star ?$. Moreover, the canonical map $Ff \to Y$ is a fibration.

For a pair of morphisms $f, g : X_\bullet \rightrightarrows Y_\bullet$ of simplicial Kan spectra, we may define the double mapping fiber $F(f, g)$ as the limit of the diagram

$$\begin{tikzcd} & & X_\bullet \arrow[d, "g"] \\ & F(\Delta^{1,1}, Y_\bullet) \arrow[r] \arrow[d] & Y_\bullet \\ X_\bullet \arrow[r, "f"'] & Y_\bullet & \end{tikzcd}$$

More generally, for an n-tuple $(f_1, ..., f_n)$ of morphisms $X_\bullet \to Y_\bullet$, we define the n-fold mapping fiber $F(f_1, ..., f_n)$ as

$$X_\bullet \times_{f_1} F(\Delta^{1,1}, Y_\bullet) \times_{f_2} X_\bullet \times_{f_2} F(\Delta^{1,1}, Y_\bullet) \times_{f_3} \cdots \times_{f_{n-1}} F(\Delta^{1,1}, Y_\bullet) \times_{f_n} X_\bullet.$$

Using the n-fold mapping fibers, we will "fatten" the diagram (2.21), and use it to define homotopy limit of shape (2.21) and in turn homotopy sheaves of simplicial Kan spectra.

Let $\mathcal{F}$ be a presheaf of simplicial Kan spectra over $\mathscr{X}$, and $\mathcal{U}$ an open covering of an open set $U \subseteq \mathscr{X}$. Consider the diagram

(2.22)

$$F(r^1_{1,2}, r^2_{1,2}) \rightrightarrows F(r^1_{1,2,3}, r^2_{1,2,3}, r^3_{1,2,3}) \Rrightarrow F(r^1_{1,2,3,4}, r^2_{1,2,3,4}, r^3_{1,2,3,4}, r^4_{1,2,3,4}) \cdots$$

where the mapping fibers $F(r^1_{1,\ldots n}, \ldots, r^n_{1,\ldots n})$ are defined inductively as follows:

- $F(r^1_{1,2}, r^2_{1,2})$ is the double mapping fiber of

$$\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

- Suppose $F(r^1_{1,\ldots n-1}, \ldots, r^{n-1}_{1,\ldots n-1})$ is constructed, then $F(r^1_{1,\ldots n}, \ldots, r^n_{1,\ldots n})$ is the n-fold mapping fiber of

$$F(r^1_{1,\ldots n-1}, \ldots, r^{n-1}_{1,\ldots n-1}) \begin{array}{c}\longrightarrow \\ \vdots \\ \longrightarrow\end{array} \prod_{i_1,\ldots,i_n} \mathcal{F}\left(U_{i_1} \cap \cdots \cap U_{i_n}\right)$$

The limit of this diagram is denoted by

$$\underleftarrow{\mathrm{Holim}}\, \mathcal{F}(\mathcal{U}).$$

Definition II.77. A presheaf $\mathcal{F}_\bullet$ of simplicial Kan spectra is a *homotopy sheaf* over $\mathscr{X}$ if, for any covering $\mathcal{U}$ of an open set $U \subseteq \mathscr{X}$, the canonical morphism

$$\mathcal{F}(U) \longrightarrow \underleftarrow{\mathrm{Holim}}\, \mathcal{F}(\mathcal{U})_\bullet$$

is a total equivalence.

Note again that if $\mathcal{F}$ is section-wise fibrant, then the homotopy limit $\underleftarrow{\mathrm{Holim}}\, \mathcal{F}(\mathcal{U})_\bullet$ coincides with the ordinary limit $\underleftarrow{\lim}\, \mathcal{F}(\mathcal{U})_\bullet$. Therefore, we conclude the following result analogous to homotopy sheaves of Kan spectra.

Theorem II.78. *For any presheaf* $\mathcal{F}$ *of simplicial Kan spectra over* $\mathscr{X}$*, the Godement resolution*

$$\operatorname{Gode}\operatorname{Sing}_{\bullet}\|\mathscr{L}\|(\mathcal{F})$$

as in (2.20) *is a homotopy sheaf of simplicial Kan spectra. Thus, every presheaf of simplicial Kan spectra is stalk-wise equivalent to a homotopy sheaf of simplicial Kan spectra, which can be given by the Godement resolution in particular.*

CHAPTER III

Constructible Derived Category of Kan Spectral Sheaves

Unless otherwise specified, all spaces considered in this chapter are locally compact Hausdorff topological spaces. We will elaborate on the case of Kan spectral sheaves, but the constructions and the results in this section can be easily generalized to simplicial Kan spectral sheaves.

3.1 The proper direct image functor $f_!$

3.1.1 Definition

Let $\mathcal{F}$ be a Kan spectral sheaf (or more generally, a Kan spectral presheaf), $U \subseteq \mathscr{X}$ an open set. For $a \in \mathcal{F}(U)(n)$, we define the *support* of a to be the set

$$\operatorname{supp}(a) = \{x \in U : a_x \neq * \in \mathcal{F}_x(n)\},$$

where a_x denotes the image of a under the map $\mathcal{F}(U)(n) \to \mathcal{F}_x(n)$.

Lemma III.1. *The support of a is closed in U, hence locally (but not necessarily globally) closed in $\mathscr{X}$.*

Proof. If $a_x = *$ for some $x \in U$, then there exists an open neighborhood $W \subseteq U$ such that $\rho^U_W(a) = *$ by definition of stalks, and hence $W \cap \operatorname{supp}(a) = \varnothing$. $\square$

In this section, fix a map $f : \mathscr{X} \to \mathscr{Y}$ of spaces. Recall that f is *proper* if the preimage of any compact set over f is again compact. In locally compact Hausdorff spaces, this is equivalent to that f is a closed map with compact fibers.

Lemma III.2 (Properness is local on the target)**.** *Let $f : \mathscr{X} \to \mathscr{Y}$ be any map.*

1. *If f is proper and $V \subseteq \mathscr{Y}$ is any set, then $f^{-1}(V) \xrightarrow{f} V$ is also proper.*

2. *If $\mathscr{Y} = \bigcup_\alpha V_\alpha$ is an open cover, and each $f^{-1}(V_\alpha) \xrightarrow{f} V_\alpha$ is proper, then f is proper.*

Proof. 1. Clearly any fiber of $f|_V$ is also that of f and thus compact, so it suffices to show that $f|_V$ is closed. Any closed set in $f^{-1}(V)$ is an intersection with a closed set Q in $\mathscr{X}$, and

$$f|_V(Q \cap f^{-1}(V)) = f(Q) \cap V$$

is closed in V.

2. Clearly any fiber of f is also that of some f_α and thus compact, so it again suffices to show that f is closed. Let Q be any closed set in $\mathscr{X}$, then for each $i \in I$ the set $Q \cap f^{-1}(V_\alpha)$ is closed, and therefore

$$f|_{V_\alpha}(Q \cap f^{-1}(V_\alpha)) = f(Q) \cap V_\alpha$$

is closed in V_α. It thus follows that $f(Q)$ is also closed. $\square$

Definition III.3. Let $\mathcal{F}$ be a Kan spectral sheaf (or more generaly, a Kan spectral presheaf) on $\mathscr{X}$. For $V \subseteq \mathscr{Y}$ open and $n \in \mathbb{Z}$, define

$$f_!\mathcal{F}(V)(n) := \left\{a \in \mathcal{F}(f^{-1}(V))(n) : \operatorname{supp}(a) \xrightarrow{f} V \text{ is proper}^*\right\} \subseteq f_*\mathcal{F}(V)(n).$$

The main goal of this section is to construct the functor

$$f_! : \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow \mathsf{Sh}_{\mathscr{S}}(\mathscr{Y}).$$

Proposition III.4. *Suppose $\mathcal{F}$ is a Kan spectral presheaf on $\mathscr{X}$. For any fixed V, the sequence $\{f_!\mathcal{F}(V)(n)\}_{n\in\mathbb{Z}}$ of based sets defines a Kan spectrum $f_!\mathcal{F}(V)$.*

Proof. It suffices to show that $f_!\mathcal{F}(V)$ is a stable simplicial set, because then it would be a stable simplicial subset of $f_*\mathcal{F}(V)$, so it automatically satisfies the finiteness condition defining a Kan spectrum.

For any $n \in \mathbb{Z}$ and any $i \in \mathbb{N}_0$, we have the *i-face* and *i-degeneracy* maps

$$d_i : f_*\mathcal{F}(V)(n) \longrightarrow f_*\mathcal{F}(V)(n-1), \qquad s_i : f_*\mathcal{F}(V)(n) \longrightarrow f_*\mathcal{F}(V)(n+1),$$

respectively. We claim that, when restricted to $f_!\mathcal{F}(V)(n)$, these maps have image contained in $f_!\mathcal{F}(V)(n-1)$ and $f_!\mathcal{F}(V)(n-1)$, respectively, hence giving the i-face and i-degeneracy maps for $f_!\mathcal{F}(V)$. The simplicial relations are also satisfied because they hold for $f_*\mathcal{F}(V)$.

Let $a \in f_!\mathcal{F}(V)(n)$, and suppose $a_x = * \in \mathcal{F}_x(n)$ for some $x \in f^{-1}(V)$. Then the naturality of the colimit diagram defining the stalk $\mathcal{F}_x$ implies

$$(d_i a)_x = (ds_i)_x a_x = * \in \mathcal{F}_x(n-1);$$

where $(d_i)_x$ is the i-face map of the Kan spectrum $\mathcal{F}_x$, which must preserve the basepoint. Therefore, we have $\mathrm{supp}(d_i a) \subseteq \mathrm{supp}(a)$. By Lemma III.1, the closed inclusion $\mathrm{supp}(d_i a) \hookrightarrow \mathrm{supp}(a)$ is proper. It then follows that $d_i a \in f_!\mathcal{F}(V)(n-1)$. A similar argument also shows that $s_i a \in f_!\mathcal{F}(n+1)$. $\square$

Proposition III.5. *For a Kan spectral presheaf $\mathcal{F}$ on $\mathscr{X}$, the assignment $V \mapsto f_!\mathcal{F}(V)$ defines a presheaf $f_!\mathcal{F}$ on $\mathscr{Y}$.*

Proof. Let $W \subseteq V$ be open sets in $\mathscr{Y}$. Since $f_*\mathcal{F}$ is a sheaf, there is a restriction morphism $\rho^V_W : f_*\mathcal{F}(V) \to f_*(W)$ of Kan spectra. Again we show that this induces a morphism $\rho^V_W : f_!\mathcal{F}(V) \to f_!(W)$, then functoriality inherits from f_*.

Let $a \in f_!\mathcal{F}(V)(n)$. Note that the restriction morphism $\rho^{f^{-1}(V)}_{f^{-1}(W)}$ of the sheaf $\mathcal{F}$ sends each a to $\rho^{f^{-1}(V)}_{f^{-1}(W)}(n)a \in \mathcal{F}(f^{-1}(W))(n)$. Moreover, for any $x \in f^{-1}(W) \subseteq f^{-1}(V)$ we have $a_x = \left(\rho^{f^{-1}(V)}_{f^{-1}(W)}(n)a\right)_x$ by the definition of stalk, hence we have

$$\operatorname{supp}\left(\rho^{f^{-1}(V)}_{f^{-1}(W)}(n)a\right) = \operatorname{supp}(a) \cap f^{-1}(W).$$

It thus follows that $\operatorname{supp}\left(\rho^{f^{-1}(V)}_{f^{-1}(W)}(n)a\right) \xrightarrow{f} W$ is proper by Lemma III.2. $\square$

Proposition III.6. *For any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$, the Kan spectral presheaf $f_!\mathcal{F}$ is indeed a Kan spectral sheaf.*

Proof. We check that

$$f_!\mathcal{F}(V) \longrightarrow \prod_{\alpha} f_!\mathcal{F}(V_\alpha) \rightrightarrows \prod_{\alpha,\beta} f_!\mathcal{F}(V_\alpha \cap V_\beta)$$

is an equalizer diagram for any open cover $V = \bigcup_{\alpha \in A} V_\alpha$. Since $f_*\mathcal{F}$ is a sheaf on $\mathscr{Y}$, we have an equalizer diagram

$$f_*\mathcal{F}(V) \longrightarrow \prod_{\alpha} f_*\mathcal{F}(V_\alpha) \rightrightarrows \prod_{\alpha,\beta} f_*\mathcal{F}(V_\alpha \cap V_\beta),$$

and there is an obvious morphism, induced by inclusions, from the first diagram into the second one.

Recall that limits in $\Delta^{\mathrm{op}}_{\mathrm{st}}$-$\mathsf{Set}$ are constructed level-wise, and, since $\mathscr{S}$ is a coreflexive subcategory of $\Delta^{\mathrm{op}}_{\mathrm{st}}$-$\mathsf{Set}$, limits in $\mathscr{S}$ are the coreflector of the level-wise limits in $\Delta^{\mathrm{op}}_{\mathrm{st}}$-$\mathsf{Set}$, that is, taking the stable simplicial subsets satisfying the finiteness condition defining Kan spectra. Therefore, $f_*\mathcal{F}(V)(n)$ consists of the elements in

$$\operatorname{eq}\left[\prod_{\alpha} f_*\mathcal{F}(V_\alpha)(n) \rightrightarrows \prod_{\alpha,\beta} f_*\mathcal{F}(V_\alpha \cap V_\beta)(n)\right]$$

which in addition satisfy the finiteness condition, and we want to show that $f_!\mathcal{F}(V)(n)$ consists of those in

$$\mathrm{eq}\left[\prod_{\alpha} f_!\mathcal{F}(V_\alpha)(n) \rightrightarrows \prod_{\alpha,\beta} f_!\mathcal{F}(V_\alpha \cap V_\beta)(n)\right],$$

satisfying the finiteness condition. Moreover, it is clear that the second equalizer is contained in the first, so we only need to check that, if $a \in f_*\mathcal{F}(V)(n)$ is also in the second equalizer, then $\mathrm{supp}(a) \xrightarrow{f} V$ is proper, that is $a \in f_!\mathcal{F}(V)(n)$.

By the argument in Proposition III.5, we may conclude that for any $\alpha \in A$ we have

$$\mathrm{supp}\left(\rho^{f^{-1}(V)}_{f^{-1}(V_\alpha)}(n)a\right) = \mathrm{supp}(a) \cap f^{-1}(V_\alpha),$$

and hence

$$\mathrm{supp}(a) = \bigcup_{\alpha \in A} \mathrm{supp}\left(\rho^{f^{-1}(V)}_{f^{-1}(V_\alpha)}(n)a\right),$$

and $\mathrm{supp}(a) \xrightarrow{f} V$ is proper by Lemma III.2. $\square$

Theorem III.7. *The assignment $\mathcal{F} \mapsto f_!\mathcal{F}$ defines functors*

$$f_! : \mathsf{PSh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow \mathsf{PSh}_{\mathscr{S}}(\mathscr{Y})$$

and

$$f_! : \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \to \mathsf{Sh}_{\mathscr{S}}(\mathscr{Y}),$$

both of which will be called the proper direct image *functor. Moreover, in either case $f_!$ is a subfunctor of f_*, and the diagram of functors*

$$\begin{tikzcd}
\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \arrow[r, "f_!"] \arrow[d, hook] & \mathsf{Sh}_{\mathscr{S}}(\mathscr{Y}) \arrow[d, hook] \\
\mathsf{PSh}_{\mathscr{S}}(\mathscr{X}) \arrow[r, "f_!"] & \mathsf{PSh}_{\mathscr{S}}(\mathscr{Y})
\end{tikzcd}$$

commutes.

Proof. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of Kan spectral presheaves. We define $f_!\varphi$ to be the restriction of $f_*\varphi$, and we attempt to show that $f_*\varphi$ maps $f_!\mathcal{F}$ into $f_!\mathcal{G}$. In particular, it then follows that $f_!$ preserves identity and compositions.

Let $V \subseteq \mathscr{Y}$ be an open set, and $a \in f_!\mathcal{F}(V)(n) \subseteq \mathcal{F}(f^{-1}(V))(n)$. Recall that

$$f_*\varphi(n) : a \longmapsto \varphi_{f^{-1}(V)}(n)a \in \mathcal{G}(f^{-1}(V))(n),$$

so again if $a_x = * \in \mathcal{F}_x(n)$ for some $x \in f^{-1}(V)$, then the naturality of the colimit diagram defining the stalk map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ implies that

$$\left(\varphi_{f^{-1}(V)}a\right)_x = \varphi_x(n)(a_x) = * \in \mathcal{G}_x(n).$$

Therefore, we have $\mathrm{supp}(f_*\varphi(n)(a)) \subseteq \mathrm{supp}(a)$, and again by Lemma III.1 we conclude that $f_*\varphi(n)(a) \in f_!\mathcal{G}(n)$. □

3.1.2 Basic properties of $f_!$

We next collect some important properties of the proper direct image functor $f_!$.

Proposition III.8. *For maps $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$, there is an isomorphism of functors $(gf)_! \cong g_! \circ f_! : \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \to \mathsf{Sh}_{\mathscr{S}}(\mathscr{Z})$.*

Proof. This is true as in the cases of abelian sheaves and of based sets, since both $g_!(f_!(\mathcal{F}))$ and $(gf)_!(\mathcal{F})$ collect properly supported sections. □

Proposition III.9. *The functor $f_!$ is left exact.*

Proof. Since all finite limits can be computed in terms of pullbacks and terminal objects, it suffices to show that $f_!$ preserves these special finite limits.

The terminal object in $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ is simply the constant sheaf $*_{\mathscr{X}}$ on $\mathscr{X}$, whose section on any $U \subseteq \mathscr{X}$ open only consists of the singleton. Since f_* is a right adjoint, it preserves terminal object, so we have $f_*(*_{\mathscr{X}}) = *_{\mathscr{Y}}$. Since $f_!(*_{\mathscr{X}})$ is a subsheaf of $f_*(*_{\mathscr{X}}) = *_{\mathscr{Y}}$ which has no proper subsheaf, it follows that $f_!(*_{\mathscr{X}}) = *_{\mathscr{Y}}$ also.

Now suppose we are given a pullback diagram

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\psi'"] \arrow[d, "\varphi'"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F} \arrow[d, "\varphi"] \\
\mathcal{G} \arrow[r, "\psi"'] & \mathcal{H}
\end{tikzcd}$$

in $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$, then we want to show that

$$\begin{tikzcd}
f_!\mathcal{K} \arrow[r, "f_!\psi'"] \arrow[d, "f_!\varphi'"'] & f_!\mathcal{F} \arrow[d, "f_!\varphi"] \\
f_!\mathcal{G} \arrow[r, "f_!\psi"] & f_!\mathcal{H}
\end{tikzcd}$$

is a pullback diagram. Since pullback of sheaves can be detected on sections, it suffices to prove that for each $V \subseteq \mathscr{Y}$ open

$$\begin{tikzcd}
f_!\mathcal{K}(V) \arrow[r, "f_!\psi'(V)"] \arrow[d, "f_!\varphi'(V)"'] & f_!\mathcal{F}(V) \arrow[d, "f_!\varphi(V)"] \\
f_!\mathcal{G}(V) \arrow[r, "f_!\psi(V)"] & f_!\mathcal{H}(V)
\end{tikzcd}$$

is a pullback diagram. Again since f_* is a right adjoint, it preserves all limits, so we have a pullback diagram

$$\begin{tikzcd}
f_*\mathcal{K}(V) \arrow[r, "f_*\psi'(V)"] \arrow[d, "f_*\varphi'(V)"'] \arrow[dr, phantom, "\lrcorner"] & f_*\mathcal{F}(V) \arrow[d, "f_*\varphi(V)"] \\
f_*\mathcal{G}(V) \arrow[r, "f_*\psi(V)"'] & f_*\mathcal{H}(V)
\end{tikzcd}$$

By a similar argument as in Proposition III.6, it suffices to show that for each $n \in \mathbb{Z}$, and for each $e \in f_*\mathcal{K}(V)(n)$ which is also in

$$\varprojlim \left[\begin{tikzcd}
 & f_!\mathcal{F}(V)(n) \arrow[d, "f_*\varphi(V)(n)"] \\
f_!\mathcal{G}(V)(n) \arrow[r, "f_*\psi(V)(n)"] & f_!\mathcal{H}(V)(n)
\end{tikzcd} \right]$$

the map $\mathrm{supp}(e) \xrightarrow{f} V$ is proper, or $e \in f_!\mathcal{K}(V)(n)$. Note that from the assumptions we have in particular

$$a := f_*\psi'(V)(n)(e) \in f_!\mathcal{F}(V)(n),$$

so if $e_x = *$ for some $x \in f^{-1}(V)$, then

$$a_x = (f_*\psi'(V)(e))_x = (f_*\psi')\, e_x = *.$$

Thus we conclude that $\mathrm{supp}(e)$ is a closed subset of $\mathrm{supp}(a)$ which is proper over V, hence $\mathrm{supp}(e)$ is itself proper over V. □

3.1.3 Locally closed strict Verdier duality

If $i : \mathscr{X} \hookrightarrow \mathscr{Y}$ be a closed embedding, then the proper direct image functor $i_!$ is just i_*. Slighlty more generally, we consider a locally closed embedding $j : \mathscr{X} \hookrightarrow \mathscr{Y}$. For any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{Y}$, we denote by $\mathcal{F}_{\mathscr{X}}$ the sheaf $j_!j^*\mathcal{F}$, which is naturally a subsheaf of $\mathcal{F}$. If $k : W \subseteq \mathscr{X}$ a subset such that $j' : W \subseteq \mathscr{Y}$ is locally closed, then there is a sheaf morphism

$$\mathcal{F}_W = j'_!(j')^*\mathcal{F} = j_!k_!k^*j^*\mathcal{F} \longrightarrow j_!j^*\mathcal{F} = \mathcal{F}_{\mathscr{X}},$$

where the natural transformation $k_!k^* \to \mathrm{Id}$ arises from the composition of $k_! \hookrightarrow k_*$ followed by the counit of adjunction $k_*k^* \to \mathrm{Id}$.

Proposition III.10. *The functor $j_!$ is the* extension by zero *functor in the following sense: for any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$, we have*

$$(j_!\mathcal{F})_y = \begin{cases} \mathcal{F}_y & \text{if } y \in \mathscr{X}, \\ * & \text{if } y \notin \mathscr{X}, \end{cases}$$

As a consequence of the description of stalks above, we have $\mathcal{F}_{\mathscr{X}}(V) = \mathcal{F}(V)$ if $V \subseteq \mathscr{X}$, and $\mathcal{F}_{\mathscr{X}}(V) = *$ if $V \cap \mathscr{X} = \varnothing$.

Proof. Note that in this special case, we have

$$j_!\mathcal{F}(V) = \{a \in \mathcal{F}(\mathscr{X} \cap V) : \operatorname{supp}(a) \subseteq \mathscr{X} \cap V \text{ is closed in } V\} \subseteq j_*\mathcal{F}(V).$$

Suppose $y \in \mathscr{X}$, then any $a_y \in \mathcal{F}_y$ is represented by $a_V \in \mathcal{F}(V)$ for some $V \subseteq \mathscr{Y}$ open. By Lemma III.1 we have $\operatorname{supp}(a_V)$ is closed in V, therefore $a_V \in j_!\mathcal{F}(\mathscr{X} \cap V)$ and $a_y \in j_!\mathcal{F}(V)$. Conversely, if $y \notin \mathscr{X}$, then y has an open neighborhood $(\mathscr{Y} \setminus C) \cap V$ disjoint with $\mathscr{X}$, over which the section of $j_*\mathcal{F}$ consists of only the singleton $*$. Since $\operatorname{supp}(*) = \varnothing$ is closed in any open set, it follows that $(j_!\mathcal{F})_y = *$. □

Theorem III.11. *The functor $j_!$ preserves weak equivalences; in particular, it descends to a functor*

$$j_! : D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow D\mathsf{Sh}_{\mathscr{S}}(\mathscr{Y}),$$

namely, it is the same as its right derived functor. Moreover, $j_!$ is exact.

Proof. This follows from Theorem III.10 since weak equivalences are defined stalk-wise. Moreover, the exactness of $j_!$ follows from Theorem III.9 and the description of stalks of $j_!\mathcal{F}$, as right exactness can be detected stalk-wise because stalks preserve colimits. □

Proposition III.12. *The functor*

$$j_! : \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow \mathsf{Sh}_{\mathscr{S}}(\mathscr{Y})$$

is an embedding into the full subcategory of Kan spectral sheaves $\mathcal{G}$ on $\mathscr{Y}$ which have trivial stalks outside $\mathscr{X}$. Moreover, the inverse functor is precisely j^*.

Proof. First of all, we note that for any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$, the composition

$$j^*j_!\mathcal{F} \longrightarrow j^*j_*\mathcal{F} \longrightarrow \mathcal{F}$$

of j^* applied to the inclusion followed by the counit of adjunction is an isomorphism. In fact, for any $z \in \mathscr{Z}$, we have

$$(j^* j_! \mathcal{F})_z = (j_! \mathcal{F})_z = \mathcal{F}_z.$$

On the other hand, suppose $\mathcal{G}$ is a Kan spectral sheaf on $\mathscr{Y}$ that satisfies $\mathcal{G}_y = *$ for all $y \notin \mathscr{Z}$, then the unit of adjunction $\mathcal{G} \to j_* j^* \mathcal{G}$ factors as

$$\mathcal{G} \longrightarrow j_! j^* \mathcal{G} \longrightarrow j_* j^* \mathcal{G},$$

and the first morphism is an isomorphism, since for any $y \in \mathscr{Y}$ we have

$$(j_! j^* \mathcal{G})_y = \begin{cases} (i^* \mathcal{G})_y & \text{if } y \in \mathscr{Z}, \\ * & \text{if } y \notin \mathscr{Z}, \end{cases}$$

which coincides with $\mathcal{G}_y$. $\square$

Definition III.13. For a Kan spectral sheaf $\mathcal{G}$ on $\mathscr{Y}$, an open set $V \subseteq \mathscr{Y}$, and $n \in \mathbb{Z}$, we define

$$\mathcal{G}^{\mathscr{Z}}(V)(n) := \{a \in \mathcal{G}(V) : \operatorname{supp}(a) \subseteq \mathscr{Z}\}.$$

Proposition III.14. *The assignment $\left[V \mapsto \left\{\mathcal{G}^{\mathscr{Z}}(V)(n)\right\}\right]$ defines a Kan spectral sheaf $\mathcal{G}^{\mathscr{Z}}$ on $\mathscr{Y}$.*

Proof. The proof goes along the same lines as that of Proposition III.4 to Theorem III.6, as we only need to verify the containment of supports, which we did exactly the same in those proofs. $\square$

Definition III.15. For any Kan spectral sheaf $\mathcal{G}$ on $\mathscr{Y}$, we denote by $j^! \mathcal{G}$ the Kan spectral sheaf $j^* \mathcal{G}^{\mathscr{Z}}$ on $\mathscr{Z}$.

Clearly, if $j : \mathscr{Z} \hookrightarrow \mathscr{Y}$ is an open embedding, then $j^! = j^*$.

Proposition III.16. *The assignment* $\mathcal{G} \mapsto j^!\mathcal{G}$ *defines a functor*

$$j^! : \mathsf{Sh}_{\mathscr{S}}(\mathscr{Y}) \longrightarrow \mathsf{Sh}_{\mathscr{S}}(\mathscr{Z}),$$

called the exceptional inverse image*functor.*

Proof. Again we only need to check the containment of supports, which has been done in the proof of Theorem III.7. $\square$

Theorem III.17 (Locally closed Verdier duality)**.** *The functor* $j^!$ *is right adjoint to* $j_!$.

Proof. Clearly $(\mathcal{G}^{\mathscr{Z}})_y = *$ for all $y \notin \mathscr{Z}$, so by Proposition III.12

$$j_!j^!\mathcal{G}(V) = j_!j^*\mathcal{G}^{\mathscr{Z}}(V) = \mathcal{G}^{\mathscr{Z}}(V) \subseteq \mathcal{G}(V),$$

and we have an inclusion $j_!j^!\mathcal{G} \to \mathcal{G}$. Moreover, given any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{Z}$, the sheaf $j_!\mathcal{F}$ has satisfies $(j_!\mathcal{F})_y = *$ for $y \notin \mathscr{Z}$, so any morphism $j_!\mathcal{F} \to \mathcal{G}$ factors as

$$j_!\mathcal{F} \longrightarrow j_!j^!\mathcal{G} \longrightarrow \mathcal{G},$$

where the first morphism is the same as, by Proposition III.12 again, a map $\mathcal{F} \to j^!\mathcal{G}$ upon application of j^*. This shows that the desired adjunction. $\square$

Corollary III.18. *Let* $i : \mathscr{Z} \hookrightarrow \mathscr{Y}$ *and* $j : V \hookrightarrow \mathscr{Y}$ *be closed and open embeddings. Then both* $i_* = i_!$ *and* $j^! = j^*$ *have both left and right adjoints.* $\square$

3.2 Right derived functors

Fix a map $f : \mathscr{X} \to \mathscr{Y}$ of spaces.

3.2.1 The derived direct image Rf_*

Theorem III.19. *The direct image functor f_* preserves strong homotopy, and therefore the right derived functor*

$$Rf_* : D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow D\mathsf{Sh}_{\mathscr{S}}(\mathscr{Y})$$

exists.

Proof. There is an isomorphism of sheaves

$$I_+ \star f_*\mathfrak{F} \cong f_*\left(I_+ \star \mathfrak{F}\right)$$

natural in both K and $\mathfrak{F}$, since both are the sheafification of $I_+ \star_p f_*\mathfrak{F} \cong f_*(I_+ \star_p \mathfrak{F})$. It then follows that f_* preserves strong homotopy, and Theorem II.52 applies. □

Note that f_* has a left adjoint f^*, which preserves stalks and, therefore, weak equivalences. Hence Proposition A.10 applies:

Corollary III.20. *The direct image functor f_* preserves local sheaves.* □

Corollary III.21. *The derived direct image functor Rf_* has a left adjoint, which we again denote by f^*.* □

Theorem III.22. *For any maps $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$, there is an isomorphism of functors*

$$R(gf)_* \cong Rg_* \circ Rf_* : D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow D\mathsf{Sh}_{\mathscr{S}}(\mathscr{Z}).$$

Proof. For any Kan spectral sheaf $\mathfrak{F}$ on $\mathscr{X}$, the Godement resolution

$$\mathrm{Gode}\,\mathrm{Sing}\,\mathscr{L}\mathfrak{F}$$

is local, therefore so is the sheaf

$$Rf_*(\mathfrak{F}) \cong f_*(\mathrm{Gode}\,\mathrm{Sing}\,\mathscr{L}\mathfrak{F})$$

by Corollary III.20 (1). Hence, there are isomorphisms

$$
\begin{aligned}
R(g \circ f)_*(\mathcal{F}) &\cong (g \circ f)_*(\mathrm{Gode}\,\mathrm{Sing}\,\mathscr{L}\mathcal{F}) \\
&\cong g_*(f_*(\mathrm{Gode}\,\mathrm{Sing}\,\mathscr{L}\mathcal{F})) \cong Rg_*(Rf_*(\mathcal{F})).
\end{aligned}
$$

in $D\mathsf{Sh}_{\mathscr{S}}(\mathscr{Z})$. □

3.2.2 The derived proper direct image $Rf_!$

Lemma III.23. *For any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$ there is an isomorphism $(K \star_p \mathcal{F})_x \cong K \star (\mathcal{F}_x)$ at any $x \in \mathscr{X}$.*

Proof. Since stalk is a directed colimit, this follows from Lemma II.27. □

Proposition III.24. *For any Kan spectral presheaf $\mathcal{F}$ on $\mathscr{X}$ and any based simplicial set K, there is an isomorphism*

$$\theta_K : K \star_p f_!\mathcal{F} \longrightarrow f_!(K \star_p \mathcal{F})$$

of presheaves on $\mathscr{Y}$, natural in both K and $\mathcal{F}$, and the diagram of presheaves

$$
\begin{tikzcd}
K \star_p f_!\mathcal{F} \arrow[r, "\theta_K"] \arrow[d] & f_!(K \star_p \mathcal{F}) \arrow[d, hook] \\
K \star_p f_*\mathcal{F} \arrow[r, equal] & f_*(K \star_p \mathcal{F})
\end{tikzcd}
$$

commutes.

Proof. The morphism θ_K is the restriction of the bottom isomorphism. Moreover, it follows from Lemma III.23 that

$$f_*(K \star_p \mathcal{F})_y \cong (K \star_p f_*\mathcal{F})_y \cong K \star ((f_*\mathcal{F})_y),$$

for any $y \in \mathscr{Y}$, and therefore on any $V \subseteq \mathscr{Y}$ open, elements in $f_*(K \star_p \mathcal{F})(V)$ have proper support if and only if they correspond via $K \star ?$ to elements in $K \star_p (f_*(\mathcal{F})(V))$ with proper support. In other words, in the diagram above, the isomorphism at the bottom restricts to an isomorphism θ on top. □

Proposition III.25. *For any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$ and any based simplicial set K, there is a morphism*

$$\theta_K : K \star f_! \mathcal{F} \longrightarrow f_!(K \star \mathcal{F})$$

of sheaves on $\mathscr{Y}$ natural in both K and $\mathcal{F}$, and so that the diagram of presheaves

$$\begin{tikzcd} K \star_p f_! \mathcal{F} \arrow[r, "\theta_K"] \arrow[d] & f_!(K \star_p \mathcal{F}) \arrow[d] \\ K \star f_! \mathcal{F} \arrow[r, "\theta_K"] & f_!(K \star \mathcal{F}) \end{tikzcd}$$

commutes.

Proof. The presheaf morphism $K \star_p \mathcal{F} \to K \star \mathcal{F}$ from sheafification induces another presheaf morphism $f_!(K \star_p \mathcal{F}) \to f_!(K \star \mathcal{F})$. Since the target of this morphism is actually a sheaf by Proposition III.6, precomposition with the presheaf morphism $K \star_p f_! \mathcal{F} \to f_!(K \star_p \mathcal{F})$ in Proposition III.24 induces, by the universal property of sheafification, a natural morphism of Kan spectral sheaves

$$K \star f_! \mathcal{F} \longrightarrow f_!(K \star \mathcal{F}). \qquad \square$$

Theorem III.26. *The proper direct image functor $f_!$ preserves strong homotopy of Kan spectral sheaves, and therefore the right derived functor*

$$Rf_! : D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow D\mathsf{Sh}_{\mathscr{S}}(\mathscr{Y})$$

exists.

Proof. Given a strong homotopy $h : I_+ \star \mathcal{F} \to \mathcal{G}$ between sheaf morphisms $\varphi, \psi : \mathcal{F} \rightrightarrows \mathcal{G}$, we have a commutative diagram

$$\begin{tikzcd} \{0\}_+ \star \mathcal{F} \arrow[r] \arrow[d, equal] & I_+ \star \mathcal{F} \arrow[d, "h"] & \{1\}_+ \star \mathcal{F} \arrow[l] \arrow[d, equal] \\ \mathcal{F} \arrow[r, "\varphi"'] & \mathcal{G} & \mathcal{F} \arrow[l, "\psi"] \end{tikzcd}$$

We form the following diagram by applying $f_!$ and using Proposition III.25:

$$\begin{tikzcd}
\{0\}_+ \star f_!\mathcal{F} \arrow[rr] \arrow[dd, equal] \arrow[dr, "\theta_{\{0\}_+}"'] & & I_+ \star f_!\mathcal{F} \arrow[d, "\theta_{I_+}"] & & \{1\}_+ \star f_!\mathcal{F} . \arrow[ll] \arrow[dd, equal] \arrow[dl, "\theta_{\{1\}_+}"] \\
& f_!(\{0\}_+ \star \mathcal{F}) \arrow[r] \arrow[dl, equal] & f_!(I_+ \star \mathcal{F}) \arrow[d, "f_!h"] & f_!(\{1\}_+ \star \mathcal{F}) \arrow[l] \arrow[dr, equal] & \\
f_!\mathcal{F} \arrow[rr, "f_!\varphi"'] & & f_!\mathcal{G} & & f_!\mathcal{F} \arrow[ll, "f_!\psi"]
\end{tikzcd}$$

this diagram commutes by functoriality of $f_!$ and naturality of the morphism θ. Therefore, $f_!h \circ \theta$ is a strong homotopy between $f_!\varphi$ and $f_!\psi$. □

Corollary III.27. *With respect to the localization on $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$, the local sheaves are "$f_!$-acyclic" in the sense that*

$$Rf_!\mathcal{F} \cong Rf_!\mathcal{G}$$

for any weak equivalence $\mathcal{F} \to \mathcal{G} \in \operatorname{Mor} \mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$. In particular, the Godement resolutions are $f_!$-acyclic.

Proof. Since weak equivalences between local sheaves are automatically strong homotopy equivalences, this follows from Theorem III.26. □

Conjecture. *For maps $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$, there is an canonical isomorphism*

$$R(gf)_! \xrightarrow{\cong} Rg_! \circ Rf_!.$$

This possibly requires some analogue of the Grothendieck spectral sequence

$$E_2^{pq} = R^p g_! R^q f_! \implies R^{p+q}(gf)_!.$$

3.2.3 The derived exceptional inverse image $Rj^!$

For the rest of this section, fix a locally closed embedding $j : \mathscr{Z} \to \mathscr{Y}$.

Theorem III.28. *The exceptional inverse image functor $j^!$ preserves strong homotopy of Kan spectral sheaves. As a consequence, the right derived functor*

$$Rj^! : D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow D\mathsf{Sh}_{\mathscr{S}}(\mathscr{Y})$$

exists.

Proof. In fact, for any Kan spectral sheaf $\mathcal{G}$ on $\mathscr{Y}$, we have

$$j^!(I_+ \star_p \mathcal{G}) \cong I_+ \star_p j^! \mathcal{G}$$

since both $K \star_p ?$ and $j^! = j^*(?)^{\mathscr{X}}$ preserve stalks. From the universal property of sheafification, this isomorphism induces a sheaf morphism

$$I_+ \star j^! \mathcal{G} \longrightarrow j^!(I_+ \star \mathcal{G}). \qquad \square$$

Theorem III.29 (Derived locally closed Verdier duality)**.** *The derived direct image functor*

$$j_! : D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow D\mathsf{Sh}_{\mathscr{S}}(\mathscr{Y})$$

has as right adjoint the functor $Rj^!$, the derived exceptional inverse image functor.

Proof. In light of Corollary III.11 and Theorem III.28, Proposition A.10 again applies to the adjunction $(j_!, j^!)$ from Theorem III.17. $\square$

Also as a consequence of Proposition A.10, we also obtain the following result.

Proposition III.30. *The functor $j^!$ preserves local sheaves.* $\square$

3.2.4 Higher derived functors

In this subsection, we record the notion of generalized higher direct image functors and prove some of their basic facts from K. Brown's doctoral thesis.

Definition III.31. For a Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$, the q-th *homotopy group sheaf* $\underline{\pi}_q\mathcal{F}$ is the abelian sheaf on $\mathscr{X}$ defined as the sheafification of the *homotopy presheaf* ${}^p\underline{\pi}_q : U \mapsto \pi_q(\mathcal{F}(U))$.

For each $q \in \mathbb{Z}$, the assignment $\mathcal{F} \mapsto \underline{\pi}_q\mathcal{F}$ defines a functor

$$\underline{\pi}_q : \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \longrightarrow \mathsf{Sh}_{\mathsf{Ab}}(\mathscr{X}).$$

Lemma III.32. *For any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$ and any $x \in \mathscr{X}$, we have an isomorphism $(\underline{\pi}_q\mathcal{F})_x \cong \pi_q(\mathcal{F}_x)$ of Kan spectra.*

Proof. Since sheafification preserves stalks, and that homotopy groups commutes with directed colimits, we have

$$(\underline{\pi}_q\mathcal{F})_x \cong \varinjlim_{x\in U} \pi_q(\mathcal{F}(U)) \cong \pi_q\left(\varinjlim_{x\in U} \mathcal{F}(U)\right) = \pi_q(\mathcal{F}_x). \qquad \square$$

Corollary III.33. *A morphism in $\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ is a weak equivalence if and only if it induces isomorphisms on all homotopy group sheaves.*

Definition III.34. For a map $f : \mathscr{X} \to \mathscr{Y}$, the *generalized higher direct image* functors are defined by

$$R^q f_*(\mathcal{F}) = \underline{\pi}_{-q} Rf_*\mathcal{F}.$$

Similarly, the *generalized higher proper direct image* functors are defined by

$$R^q f_!(\mathcal{F}) = \underline{\pi}_{-q} Rf_!\mathcal{F}.$$

For the unique map $p_{\mathscr{X}} : \mathscr{X} \to *$ from $\mathscr{X}$ to a point, $(p_{\mathscr{X}})_* = \Gamma$ and $(p_{\mathscr{X}})_! = \Gamma_c$ are the global section (with proper support) functors, and their higher right derived functors are *generalized sheaf cohomology*

$$H^q(\mathscr{X}, \mathcal{F}) := \pi_{-q} R\Gamma(\mathscr{X}, \mathcal{F})$$

and *generalized compactly supported sheaf cohomology*

$$H^q_c(\mathscr{X}, \mathcal{F}) \coloneqq \pi_{-q} R\Gamma_c(\mathscr{X}, \mathcal{F}),$$

respectively.

Lemma III.35. *$R^q f_*(\mathcal{F})$ is the sheafification of the presheaf*

$$V \mapsto H^q(f^{-1}(V); \mathcal{F})$$

. Similarly, $R^q f_!(\mathcal{F})$ is the sheafification of the presheaf

$$V \mapsto H^q_c(f^{-1}(V); \mathcal{F})$$

.

Proof. Since

$$Rf_*\mathcal{F}(V) = \mathrm{Gode}\,\mathrm{Sing}\,\mathscr{L}\mathcal{F}(f^{-1}(V)) = R\Gamma(f^{-1}(V); \mathcal{F}),$$

taking π_{-q} shows that the presheaf $V \mapsto H^q(f^{-1}(V); \mathcal{F})$ is the homotopy presheaf of $R^q f_*(\mathcal{F})$. The assertion about $R^q f_!(\mathcal{F})$ is proven similarly. □

3.3 The constructible derived category

In this section, we consider the category of stratified spaces and stratified maps. Recall that a *stratification* of a space $\mathscr{X}$ is a locally finite partition

$$\mathscr{X} = \bigsqcup_{\lambda \in \Lambda} \mathscr{X}_\lambda,$$

such that

1. each $\mathscr{X}_\lambda$, called a *stratum*, is a connected submanifold and is locally closed in $\mathscr{X}$;

2. the closure $\overline{\mathscr{X}}_\lambda$ of any stratum $\mathscr{X}_\lambda$ is a union $\bigsqcup_\mu \mathscr{X}_\mu$ of strata.

A *stratified space* is a space $\mathscr{X}$ equipped with a stratification Λ. A stratified map $f : \mathscr{X} \to \mathscr{Y}$ is one such that the preimage of any stratum of $\mathscr{Y}$ is a union of strata of $\mathscr{X}$. Recall an abelian sheaf $\mathcal{F}$ on a space $\mathscr{X}$ is constructible if there exists some stratification Λ such that the restriction $\mathcal{F}|_{\mathscr{X}_\lambda}$ on each stratum $\mathscr{X}_\lambda$ of Λ is a local system of abelian groups.

Definition III.36. A Kan spectral sheaf $\mathcal{F}$ is *constructible* is all of its homotopy group sheaves $\underline{\pi}_q\mathcal{F}$ are constructible abelian sheaves.

By Corollary III.33, the notion of constructibility passes to the derived category $D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ of Kan spectral sheaves on $\mathscr{X}$. The full subcategory of $D\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ consisting of constructible Kan spectral sheaves is denoted by $D_c\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$, and called the *constructible derived category.*

Lemma III.37. *For a locally closed embedding $j : \mathscr{Z} \hookrightarrow \mathscr{X}$, the functor $j_!$ preserves constructibility.*

Proof. We show that the diagram

$$\begin{tikzcd} \mathsf{Sh}_{\mathscr{S}}(\mathscr{Z}) \arrow[r, "j_!"] \arrow[d, "\underline{\pi}_q"'] & \mathsf{Sh}_{\mathscr{S}}(\mathscr{X}) \arrow[d, "\underline{\pi}_q"] \\ \mathsf{Sh}_{\mathsf{Ab}}(\mathscr{Z}) \arrow[r, "j_!"'] & \mathsf{Sh}_{\mathsf{Ab}}(\mathscr{X}) \end{tikzcd}$$

commutes. For any Kan spectral sheaf $\mathcal{F}$ on $\mathscr{Z}$, we have ${}^p\underline{\pi}_q(j_*\mathcal{F}) = j_*({}^p\underline{\pi}_q\mathcal{F})$ since both of them send $U \subseteq \mathscr{X}$ to $\pi_q(\mathcal{F}(U \cap \mathscr{Z}))$. By inspection on supports, we also obtain ${}^p\underline{\pi}_q(j_!\mathcal{F}) = j_!({}^p\underline{\pi}_q\mathcal{F})$. Taking sheafification, we get a natural morphism $\underline{\pi}_q(j_!\mathcal{F}) \longrightarrow j_!(\underline{\pi}_q\mathcal{F})$. Since $\underline{\pi}_q$ preserves stalks by Lemma III.32 and $j_!$ commutes with stalks at points in $\mathscr{Z}$ (and have trivial stalks otherwise), we may detect stalk-wise to that this morphism is an isomorphism. $\square$

Proposition III.38. *For an open embedding $j : U \hookrightarrow \mathscr{X}$, the functor Rj_* preserves constructibility.*

Proof. By refining the stratification on $\mathscr{X}$, if necessary, we may assume that U is a union of strata U_α. For any constructible Kan spectral sheaf $\mathcal{F}$ on $\mathscr{X}$, the homotopy group sheaves $\underline{\pi}_q Rj_*\mathcal{F}$ are simply restrictions of the constructible abelian sheaves $\underline{\pi}_q\mathcal{F}$ onto U. They are constructible since their further restrictions onto each U_α are local systems of abelian sheaves. □

Corollary III.39. *For a locally closed embedding $W : \mathscr{Z} \hookrightarrow \mathscr{X}$, the functors Rj_* and $j_!$ both preserve constructibility.*

Proof. We may decompose j into $W \overset{k}{\hookrightarrow} \mathscr{Z} \overset{i}{\hookrightarrow} \mathscr{X}$, where k is an open embedding and i is a closed embedding (in fact, we may take $\mathscr{Z}$ to be the closure of W in $\mathscr{X}$). By Theorem III.22 and Proposition III.8, along with the observation that $i_* = i_!$ is exact, we have $Rj_* = i_! \circ Rk_*$ and $j_! = i_! \circ k_!$. Since three functors preserve constructibility, so do their compositions. □

CHAPTER IV

Rings and Modules in Kan Spectral Sheaves

4.1 Rings and modules in Kan spectra

4.1.1 Models of the smash product

Fix Kan spectra X and Y. An *Adams datum* (a, b) is a sequence $\{(a_n, b_n) : n \in \mathbb{N}_0\}$ of pairs of integers subject to the following conditions:

- $a_0 = b_0 = 0$;
- $a_n + b_n = n$;
- $a_n \leq a_{n+1} \leq a_n + 1$, so equivalently $b_n \leq b_{n+1} \leq b_n + 1$;
- $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$.

Given an Adams datum (a, b), we want to define a model $X \wedge_{(a,b)} Y$ of the smash product of X and Y labelled by (a, b), by mimicking the construction of $K \star (?)$, where K is a based simplicial set.

Define

$$X \wedge_{(a,b)} Y := \mathrm{Sp}\left(X \,{}^{p}\!\wedge_{(a,b)} Y\right),$$

where $X\,{}^{p}{\wedge}_{(a,b)}\,Y$ is the Kan prespectrum defined by

(4.1)

$$\left(X\,{}^{p}{\wedge}_{(a,b)}\,Y\right)_n := \underrightarrow{\operatorname{colim}} \left[\begin{tikzcd} & & \vdots \arrow[d] \\ & \Sigma(?_{n-1}) \arrow[r, "\sim"] \arrow[d] & \Sigma^2\left(X_{a_{n-2}} \wedge Y_{b_{n-2}}\right) \\ ?_n \arrow[r, "\sim"] \arrow[d] & \Sigma\left(X_{a_{n-1}} \wedge Y_{b_{n-1}}\right) & \\ X_{a_n} \wedge Y_{b_n} & & \end{tikzcd} \right]$$

where

$$?_n = \left(\Sigma^{a_n - a_{n-1}} X_{a_{n-1}}\right) \wedge \left(\Sigma^{b_n - b_{n-1}} Y_{b_{n-1}}\right) = \begin{cases} X_{a_{n-1}} \wedge \Sigma Y_{b_{n-1}}, \text{ or} \\ \Sigma X_{a_{n-1}} \wedge Y_{b_{n-1}}. \end{cases}$$

To obtain the structure map

$$\Sigma\left(X\,{}^{p}{\wedge}_{(a,b)}\,Y\right)_n \longrightarrow \left(X\,{}^{p}{\wedge}_{(a,b)}\,Y\right)_{n+1}, \tag{4.2}$$

note that Σ is a left adjoint and hence commutes with colimits, so

$$\Sigma\left(X\,{}^{p}{\wedge}_{(a,b)}\,Y\right)_n \cong \underrightarrow{\operatorname{colim}} \left[\begin{tikzcd} & & \vdots \arrow[d] \\ & \Sigma^2(?_{n-1}) \arrow[r, "\sim"] \arrow[d] & \Sigma^3\left(X_{a_{n-2}} \wedge Y_{b_{n-2}}\right) \\ \Sigma(?_n) \arrow[r, "\sim"] \arrow[d] & \Sigma^2\left(X_{a_{n-1}} \wedge Y_{b_{n-1}}\right) & \\ \Sigma\left(X_{a_n} \wedge Y_{b_n}\right) & & \end{tikzcd} \right].$$

Observe that the diagram above is a part of the diagram defining

$$\left(X\,{}^{p}\!\wedge_{(a,b)} Y\right)_{n+1} := \underrightarrow{\mathrm{colim}} \left[\begin{tikzcd} & & \vdots \arrow[d] \\ & \Sigma(?_n) \arrow[r, "\sim"] \arrow[d] & \Sigma^2\left(X_{a_{n-1}} \wedge Y_{b_{n-1}}\right) \\ ?_{n+1} \arrow[r, "\sim"] \arrow[d] & \Sigma\left(X_{a_n} \wedge Y_{b_n}\right) & \\ X_{a_{n+1}} \wedge Y_{b_{n+1}} & & \end{tikzcd} \right],$$

so there is a canonical map (4.2) as desired.

The construction $\wedge_{(a,b)}$ is functorial in both coordinates, that is, given maps $X \to X'$ and $Y \to Y'$ between Kan spectra, there is a canonical map

$$X \wedge_{(a,b)} Y \to X' \wedge_{(a,b)} Y'.$$

A *slackening datum* is a sequence $c = (c_n)$ of natural numbers subject to the following conditions:

- $c_0 = 0$;
- $c_n \leq c_{n+1} \leq c_n + 1$;
- $\lim_{n\to\infty}(n - c_n) = \infty$.

Given a slackening datum, we may form a Kan spectrum ${}^{p}X_{(c)}$ by

$$\left({}^{p}X_{(c)}\right)_n = \Sigma^{c_n} X_{n-c_n}$$

with the structure map

$$\Sigma\Sigma^{c_n} X_{n-c_n} \longrightarrow \Sigma^{c_{n+1}} X_{n+1-c_{n+1}}$$

which is either induced from the structure map $\Sigma X_{n-c_n} \to X_{n+1-c_n}$ in the case $c_{n+1} = c_n$, or is the identity map in the case $c_{n+1} = c_n + 1$. The spectrification $\mathrm{Sp}({}^pX_{(c)})$ is denoted by $X_{(c)}$.

Suppose c and c' are slackening data such that $c \geq c'$, that is, $c_n \geq c'_n$ for all n, then we have a map

$$X_{(c)} \longrightarrow X_{(c')},$$

obtained as the spectrification of the prespectrum map ${}^pX_{(c)} \longrightarrow {}^pX_{(c')}$ given levelwise by the iterated structure maps $\Sigma^{k+\ell} X_n \to \Sigma^{\ell} X_{n+k}$ of the Kan spectrum X. Therefore, given slackening data c, c', d and d', such that $c \geq c'$ and $d \geq d'$, and Kan spectra X and Y, there is a natural map

$$X_{(c)} \wedge_{(a,b)} Y_{(d)} \longrightarrow X_{(c')} \wedge_{(a,b)} Y_{(d')}. \tag{4.3}$$

Also note that certain vertical arrows in the diagrams defining the prespectrum $X_{(c)} \,{}^p\!\wedge_{(a,b)} Y_{(d)}$ are in fact equalities; more precisely, this occurs when

$$a_{n-k} = a_{n-k-1} + 1 \qquad \text{and} \qquad c_{a_{n-k}} = c_{a_{n-k-1}} + 1, \tag{4.4}$$

or

$$b_{n-k} = b_{n-k-1} + 1 \qquad \text{and} \qquad d_{b_{n-k}} = d_{b_{n-k-1}} + 1. \tag{4.5}$$

In such cases, the corner

$$\begin{tikzcd}
 & & \vdots \arrow[d] \\
 & \Sigma^k(?_{n-k}) \arrow[r, "\sim"] \arrow[d, equal] & \Sigma^{k+1}\left(\left(X_{(c)}\right)_{a_{n-k-1}} \wedge \left(Y_{(d)}\right)_{b_{n-k-1}}\right) \\
\cdots \arrow[r, "\sim"] & \Sigma^k\left(\left(X_{(c)}\right)_{a_{n-k}} \wedge \left(Y_{(d)}\right)_{b_{n-k}}\right) &
\end{tikzcd}$$

may be omitted when taking the colimit. A choice of such corners to omit therefore induces a map

$$X_{(c)} \wedge_{(a,b)} Y_{(d)} \longrightarrow X_{(c')} \wedge_{(a',b')} Y_{(d')},$$

where (a', b') and c' and d' are obtained from (a, b) and c and d by dropping the parts corresponding to the choice of omission. More precisely, in the case of (4.4), we have

$$\begin{cases} a'_i = a_i, b'_i = b_i, c'_{a_i} = c_{a_i}, d'_{b_i} = d_{b_i} & \text{if } i \leq n-k-1, \\ a'_{i-1} = a_i - 1, b'_{i-1} = b_i, c'_{a_{i-1}} = c_{a_{i-1}}, d'_{b_{i-1}} = d_{b_i} & \text{if } i \geq n-k. \end{cases}$$

The case (4.5) is similar. The general case is obtained by iterating this procedure.

Finally, for any slackening datum e' with $e' \leq e$, we get a map

$$\left(X_{(c)} \wedge_{(a,b)} Y_{(d)}\right)_{(e)} \longrightarrow \left(X_{(c')} \wedge_{(a',b')} Y_{(d')}\right)_{(e')}. \tag{4.6}$$

To make the situation more manageable, we need a more categorical description of the Adams data and the slackening data.

Lemma IV.1. *The following data are equivalent:*

- *an Adams datum;*
- *a monotone bijection*

  $$\mathbb{N}_0 \xrightarrow{\leq, \cong} \mathbb{N}_0 \amalg \mathbb{N}_0, \tag{4.7}$$

  where the coproduct is taken in the category of posets. □

On the other hand, a slackening datum c can be thought of as an inclusion $C \overset{\infty}{\hookrightarrow} \mathbb{N}_0$ with infinite complement. In fact, C is the set $\{n | c_{n+1} - c_n = 1\}$, whereas given such a C, the sequence (c_n) can be recovered by $c_n = \#(C \cap \mathbb{N}_{<n})$. Therefore,

a model $\left(X_{(c)} \wedge_{(a,b)} Y_{(d)}\right)_{(e)}$ indexed by a choice of an Adams datum and slackening data may be encoded by a diagram

$$\begin{tikzcd}
C \amalg D \arrow[d, hook, "\infty\amalg\infty"'] & \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, hook, "E", "\leq"'] & \mathbb{N}_0
\end{tikzcd} \tag{4.8}$$

Here the superscript E corresponds to the slackening datum e in the above sense.

Definition IV.2. The category $\mathscr{A}^{(2)}$ of *slackened Adams data* is defined as follows. The objects are diagrams (4.8), and a morphism is a commutative diagram

$$\begin{tikzcd}
& C \amalg D \arrow[dd, hook, "\infty\amalg\infty"] & \\
C' \amalg D' \arrow[ur, hook] \arrow[dd, hook, "\infty\amalg\infty"'] & & \\
& \mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, hook, "E", "\leq"'] & \mathbb{N}_0 \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[ur, hook, "\leq\amalg\leq"'] \arrow[r, hook, "E'", "\leq"'] & \mathbb{N}_0 \arrow[ur, hook, "\leq"] &
\end{tikzcd} \tag{4.9}$$

such that the arrows

$$\begin{tikzcd}
& C \amalg D \arrow[d, hook, "\infty\amalg\infty"] \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, hook, "\leq\amalg\leq"'] & \mathbb{N}_0 \amalg \mathbb{N}_0
\end{tikzcd}$$

are jointly surjective. (In (4.9), we decorate the horizontal inclusions with their complements.)

Regarding the elements not in the image of order inclusion $\mathbb{N}_0 \hookrightarrow \mathbb{N}_0$ as being "dropped", the joint surjectivity condition means precisely that only the elements labelled by the slackening data C and D may be dropped.

To summarize the discussion above, we have

Proposition IV.3. *There is a functor*

$$\big(?_{(?)} \wedge_{(?,?)} ?_{(?)}\big)_{(?)} : (\mathscr{A}^{(2)})^{\mathrm{op}} \times \mathscr{S} \times \mathscr{S} \longrightarrow \mathscr{S},$$

where $\mathscr{S}$ denotes the category of Kan spectra. In other words, every morphism (4.9) *gives rises to a map* (4.6) *of Kan spectra.* □

We will show in the next section that the category $\mathscr{A}^{(2)}$ is contractible. We denote by $\mathscr{A}(2)$ the nerve $N\mathscr{A}^{(2)}$ of the category $\mathscr{A}^{(2)}$; it is a contractible simplicial set.

We can similarly consider categories $\mathscr{A}^{(n)}$ encoding all models

$$\left(\bigwedge_{(a_1,\dots,a_n)} (X_i)_{(c_i)}\right)_{(e)}$$

of the n-fold smash product $X_1 \wedge \cdots \wedge X_n$ of the Kan spectra X_i. More precisely, it has as objects diagrams

$$\begin{tikzcd} C_1 \amalg \cdots \amalg C_n \arrow[d, hook, "\infty\amalg\cdots\amalg\infty"'] & \\ \mathbb{N}_0 \amalg \cdots \amalg \mathbb{N}_0 \arrow[r, hook, "\leq"'] & \mathbb{N}_0 \end{tikzcd} \tag{4.10}$$

and morphisms are again commutative cubes with a joint surjectivity condition, similarly to (4.9). By a similar strategy to the proof of contractibility of $\mathscr{A}^{(2)}$, we can inductively show, by forgetting one variable at a time, that these categories $\mathscr{A}^{(n)}$ are all contractible. We will denote by $\mathscr{A}(n)$ the nerve $N\mathscr{A}^{(n)}$, which are weakly contractible simplicial sets.

Analogously to Proposition IV.3, for any $n \in \mathbb{N}$ there is a functor

$$\left(\bigwedge_{(?,\dots,?)} (?)_{(?)}\right)_{(?)} : (\mathscr{A}^{(n)})^{\mathrm{op}} \times \mathscr{S}^{\times n} \longrightarrow \mathscr{S}$$

To simplify notation, we denote by $\bigwedge_\sigma(X_1, \dots, X_n)$ the image of $(\sigma, X_1, \dots, X_n)$ under the functor above, where $\sigma \in \mathscr{A}^{(n)}$ and $X_1, \dots, X_n \in \mathscr{S}$.

Definition IV.4. Define the *twisted half-smash product* functor

$$\mathscr{A}(n) \ltimes ? : \mathscr{S}^{\times n} \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathscr{S}$$

by setting

$$\mathscr{A}(n) \ltimes (X_1, ..., X_n)_m := \bigvee_{\substack{\sigma_0 \to \cdots \to \sigma_n \\ \in N_m \mathscr{A}^{(n)}}} \bigwedge_{\sigma_n} (X_1, ..., X_n),$$

where faces act by composition of arrows in $\sigma_0 \to \cdots \to \sigma_n$ and degeneracies act by insertion of identity arrows.

We furthermore extend the twisted half-smash product to simplicial Kan spectra

$$\mathscr{A}(n) \ltimes ? : (\Delta^{\mathrm{op}}\text{-}\mathscr{S})^{\times n} \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathscr{S}$$

by applying diagonal totalization to the resulting bisimiplicial Kan spectrum.

Observe that shuffles can be composed. More precisely, let $k \geq 0$ and $j_1, ..., j_k \geq 0$ be natural numbers, and put $j = j_1 + \cdots + j_k$. Then a j-fold shuffle $\mathbb{N}_0 \amalg \cdots \amalg \mathbb{N}_0 \xrightarrow{\leq, \cong} \mathbb{N}_0$ of $\mathbb{N}_0$ is equivalently a composite of j_s-fold shuffles $(1 \leq s \leq k)$ followed by a k-fold shuffle. However, there is no composition map

$$\mathscr{A}(k) \times \mathscr{A}(j_1) \times \cdots \times \mathscr{A}(j_k) \longrightarrow \mathscr{A}(j)$$

due to the jointly surjective condition on the morphisms. Nonetheless, such a composition map can be defined on a simplicial subset of the source, and is obtained as the nerve of the functor from the subcategory of

$$\mathscr{A}^{(k)} \times \mathscr{A}^{(j_1)} \times \cdots \times \mathscr{A}^{(j_k)}$$

consisting of all shuffles with compatible slacks, to the category $\mathscr{A}^{(j)}$. In general, given any tree T, there is a simplicial subset $\mathscr{A}(T) \subseteq \prod_{t \in T} \mathscr{A}(n_t)$ consisting of all shuffles with compatible slacks, and there are composition maps

$$\gamma_T : \mathscr{A}(T) \longrightarrow \mathscr{A}(n_T).$$

This exhibits a *partial operad* structure on the set $\{\mathscr{A}(T)\}$ labelled by all trees, and $\mathscr{A}$ will be called the *Adams partial operad*.

We will give precise definitions of trees and partial operads in Section 4.1.3, and develop a machinery to "rectify" them into an actual operad $\overline{\mathscr{A}}$ show that it is indeed an E_∞-operad.

However, our point is the rectification is largely unnecessary, as the partial operad structure $\{\mathscr{A}(T)\}_{T \text{ tree}}$ suffices for our purpose to describe the notion of E_∞-ring Kan spectra. We will follow the second approach in Section 4.1.4 to define ring and modules in Kan spectra, and construct the operadic smash product of modules.

4.1.2 The contractibility of the category $\mathscr{A}^{(2)}$

A functor is called a *homotopy equivalence* if it induces a homotopy equivalence on the classifying spaces, i.e., the geometric realization of the nerves. A category is said to be *contractible* if its classifying space is.

The goal of this section is to prove the following result.

Theorem IV.5. *The category $\mathscr{A}^{(2)}$ which encodes all models $\left(X_{(c)} \wedge_{(a,b)} Y_{(d)}\right)_{(e)}$ of the smash product $X \wedge Y$ is contractible.*

We denote by C, D and E the subsets of $\mathbb{N}_0$ corresponding to the slackening data c, d and e as in the previous section. As the first step, we will show that the data C, D and E can be forgotten without changing the homotopy type of the category.

In this section, we denote by $\mathscr{A}_0$ the category with objects the Adams data (4.7) and morphisms the commutative diagrams

$$\begin{tikzcd} \mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\leq,\cong"] \arrow[d, hook, "\leq\amalg\leq"'] & \mathbb{N}_0 \arrow[d, hook, "\leq"] \\ \mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\leq,\cong"'] & \mathbb{N}_0 \end{tikzcd}$$

We also denote by $\mathscr{A}_1$ the full subcategory of $\mathscr{A}^{(2)}$ consisting of objects with $E = \varnothing$, that is, objects of the form (4.8) with the bottom arrow a monotone bijection.

Note that there is an obvious forgetful functor $\mathscr{A}_1 \to \mathscr{A}_0$ forgetting C and D.

Lemma IV.6. *The inclusion $\mathscr{A}_1 \hookrightarrow \mathscr{A}^{(2)}$ is itself a left adjoint, and thus is a homotopy equivalence.*

Proof. The right adjoint to this inclusion functor is given by, for an arbitrary object of the form (4.8), restricting the bottom arrow to the monotone bijection onto the image. □

Proposition IV.7. *The forgetful functor $f : \mathscr{A}_1 \to \mathscr{A}_0$ is a homotopy equivalence.*

Proof. Fix an arbitrary object

$$a : \mathbb{N}_0 \amalg \mathbb{N}_0 \xrightarrow{\leq, \cong} \mathbb{N}_0$$

of $\mathscr{A}_0$. By Quillen's Theorem A [27], it suffices to show that the comma category $(f \downarrow a)$ is contractible.

Note that an object of $(f \downarrow a)$ has the form

$$\begin{tikzcd} C \amalg D \arrow[r, hook] & \mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\leq, \cong"] \arrow[d, hook] & \mathbb{N}_0 \arrow[d, hook] \\ & \mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\leq, \cong", "a"'] & \mathbb{N}_0 \end{tikzcd}$$

Consider the full subcategory $(f \downarrow a)_0$ of $(f \downarrow a)$ on objects

$$\begin{tikzcd} C \amalg D \arrow[r, hook] & \mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\leq, \cong"] \arrow[d, equal] & \mathbb{N}_0 \arrow[d, hook] \\ & \mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\leq, \cong", "a"'] & \mathbb{N}_0 \end{tikzcd}$$

Note that the joint surjectivity condition holds automatically, so the category $(f \downarrow a)_0$ is isomorphic to the product of two copies of the poset of subsets of $\mathbb{N}_0$; this poset is contractible since it admits an initial object $(\varnothing, \varnothing)$.

Therefore, it remains to show that the inclusion $(f \downarrow a)_0 \hookrightarrow (f \downarrow a)$ is a homotopy equivalence. Fix any object b of $(f \downarrow a)$, and observe that the comma category $\big(b \downarrow (f \downarrow a)_0\big)$ is isomorphic to the poset of pairs (C', D') of subsets of $\mathbb{N}_0$ with respect to inclusions, subject to the following two conditions:

- $C \subseteq C'$ and $D \subseteq D'$, and
- in both diagrams

$$\begin{tikzcd} C \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \mathbb{N}_0 \arrow[d, hook] \\ C' \arrow[r, hook] & \mathbb{N}_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \mathbb{N}_0 \arrow[d, hook] \\ D' \arrow[r, hook] & \mathbb{N}_0 \end{tikzcd}$$

  the solid arrows are jointly surjective.

We claim that this poset is directed, hence contractible. Indeed, given C' and C'' both containing C and making the left diagram above jointly surjective, then so is $C' \cap C''$, since elements not in the image of the vertical arrow lie in the image of both C' and C''. Similar holds for D, and thus for any two pairs (C', D') and (C'', D''), the pair $(C' \cap C'', D' \cap D'')$ maps to both of them. $\square$

Next, consider the monoid M of monotone inclusions $\mathbb{N}_0 \overset{\leq}{\hookrightarrow} \mathbb{N}_0$. There is an obvious forgetful functor $g : \mathscr{A}_0 \to M$ keeping the first copy of $\mathbb{N}_0$.

Proposition IV.8. *The monoid M is contractible.*

Proof. It is a general fact that, for a (small) category $\mathscr{C}$, the nerve $N\mathscr{C}$ is 2-coskeletal, therefore its classifying space $B\mathscr{C} = |N\mathscr{C}|$ has no higher homotopy groups $\pi_n(B\mathscr{C})$ for $n \geq 2$. It thus suffices to show that $\pi_1(BM)$ is trivial. Also recall [27, Prop. 1] that

$$\pi_1(B\mathscr{C}, c) = \mathrm{Aut}_{\mathscr{C}[(\mathrm{Arr}\,\mathscr{C})^{-1}]}(c)$$

where $\mathscr{C}[(\operatorname{Arr}\mathscr{C})^{-1}]$ denotes the groupoid obtained from $\mathscr{C}$ by formally inverting all arrows. In the case where $\mathscr{C} = M$ is a monoid, this is the Grothendieck group of M. Thus, we have reduced the proof to the following Lemma. □

Lemma IV.9. *The Grothendieck group KM of the monoid M is trivial.*

Proof. We show that every element $f \in M$ is conjugate to the identity.

Step 1: If f has no fixed points, then f is conjugate to the translation map $t(n) = n + 1$.

Monotonicity, injectivity, and fixed-point freeness together imply that $f(n) > n$, so the function $g(n) = f^n(1)$ is also a monotone inclusion It is clear that $gt = fg$.

Define a "suspension" operation σ on M by

$$(\sigma f)(n) = \begin{cases} 1 & n = 1, \\ f(n) & n > 1. \end{cases}$$

Step 2: If f is not an identity, then f is conjugate to the map $\sigma^m t$ for some $m \geq 0$.

Observe that f has only finitely many fixed-points. For if $f(m) = m$, then $f(n) = n$ for any $n \leq m$ by monotonicity and injectivity. Let $m < \infty$ be the number of fixed-points of f, then $f = \sigma^m g$. Since g is conjugate to t, we see that f is conjugate to $\sigma^m t$.

Step 3: For any $m \geq 0$, $\sigma^m t$ is conjugate to t^m.

This is clear since $\sigma^m t \circ t^m = t^m \circ t^m$.

Thus, we have shown that every non-identity element in M is conjugate to t^m for some m. It then remains to show that

Step 4: t is conjugate to the identity.

Denote by $\mathfrak{e}, \mathfrak{o} : \mathbb{N}_0 \to \mathbb{N}_0$ the inclusions to evens and odds, i.e., $\mathfrak{e}(n) = 2n$ and $\mathfrak{o}(n) = 2n + 1$. We will show that $\mathfrak{o}$ is conjugate to $\mathfrak{e}$, then our claim follows since

$$\mathfrak{o} \circ \mathrm{id} = t \circ \mathfrak{e}.$$

Let $q \in M$ be given by

$$q(n) = \begin{cases} 2n & n \text{ even}, \\ 2n+1 & n \text{ odd}, \end{cases}$$

then $\mathfrak{o}^2 = q \circ \mathfrak{o}$ and $\mathfrak{e}^2 = q \circ \mathfrak{e}$. This means that $\mathfrak{o}$, q and $\mathfrak{e}$ are pair-wise conjugate, and the proof is complete. $\square$

To finish the proof of the contractibility of $\mathscr{A}^{(2)}$, we are left to show

Proposition IV.10. *The forgetful functor $g : \mathscr{A}_0 \to M$ is a homotopy equivalence.*

Consider the comma category $(\mathbb{N}_0 \downarrow g)$ whose objects are diagrams

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, hook, "\leq"'] & \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "{\leq,\cong}"] & \mathbb{N}_0
\end{tikzcd}$$

and whose morphisms are commutative diagrams

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, hook, "\leq"'] \arrow[dd, bend right=60, "\leq"'] & & & \\
\mathbb{N}_0 \arrow[d, hook, "\leq"'] & \amalg & \mathbb{N}_0 \arrow[d, hook, "\leq"] \arrow[r, "{\leq,\cong}"] & \mathbb{N}_0 \arrow[d, hook, "\leq"] \\
\mathbb{N}_0 & \amalg & \mathbb{N}_0 \arrow[r, "{\leq,\cong}"'] & \mathbb{N}_0
\end{tikzcd}$$

By Quillen's Theorem A, Proposition IV.10 reduces to the contractibility of the comma category $(\mathbb{N}_0 \downarrow g)$. Also consider the subcategory $(\mathbb{N}_0 \downarrow g)_0$ of $(\mathbb{N}_0 \downarrow g)$ consisting of morphisms

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, equal] & & \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "{\leq,\cong}"] \arrow[d, equal] & \mathbb{N}_0 \arrow[d, hook, "\leq"] \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "{\leq,\cong}"'] & \mathbb{N}_0
\end{tikzcd}$$

Lemma IV.11. *Then inclusion $\iota : (\mathbb{N}_0 \downarrow g)_0 \hookrightarrow (\mathbb{N}_0 \downarrow g)$ is a left adjoint, thus in particular a homotopy equivalence.*

Proof. Define a functor $r : (\mathbb{N}_0 \downarrow g) \hookrightarrow (\mathbb{N}_0 \downarrow g)_0$ by

$$\left[\begin{tikzcd}
\mathbb{N}_0 \arrow[d, hook, "i"'] & \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "a", "\cong"'] & \mathbb{N}_0
\end{tikzcd}\right] \longmapsto \left[\begin{tikzcd}
\mathbb{N}_0 \arrow[d, equal, "i"'] & \\
\operatorname{Im} i \amalg \mathbb{N}_0 \arrow[r, "a", "\cong"'] & a^{-1}(\operatorname{Im} i) \cup a^{-1}(\mathbb{N}_0)
\end{tikzcd}\right]$$

Then the data of a morphism

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, equal] \arrow[dd, bend right=60, hook, "i"'] & \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\cong"] \arrow[d, hook, "\leq"] & \mathbb{N}_0 \arrow[d, hook, "\leq"] \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "a", "\cong"'] & \mathbb{N}_0
\end{tikzcd}$$

in $(\mathbb{N}_0 \downarrow g)$ is equivalent to the data of a morphism

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, equal] & \\
\mathbb{N}_0 \amalg \mathbb{N}_0 \arrow[r, "\cong"] \arrow[d, hook, "\leq"] & \mathbb{N}_0 \arrow[d, hook, "\leq"] \\
\operatorname{Im} i \amalg \mathbb{N}_0 \arrow[r, "a", "\cong"'] & a^{-1}(\operatorname{Im} i) \cup a^{-1}(\mathbb{N}_0)
\end{tikzcd}$$

in $(\mathbb{N}_0 \downarrow g)_0$. This means precisely that

$$\mathrm{Hom}_{(\mathbb{N}_0 \downarrow g)}(\iota x_0, x) \cong \mathrm{Hom}_{(\mathbb{N}_0 \downarrow g)_0}(x_0, rx),$$

so r is right adjoint to ι. $\square$

Thus, we have established that the category $\mathscr{A}^{(2)}$ is homotopy equivalent to the comma category $(\mathbb{N}_0 \downarrow g)_0$, which can be equivalently described as follows: its objects are inclusions $\mathbb{N}_0 \overset{\infty}{\hookrightarrow} \mathbb{N}_0$ with infinite complements, and morphisms between two such objects are commutative triangles

$$\begin{tikzcd} \mathbb{N}_0 \arrow[r, hook, "\infty"] \arrow[dr, hook, "\infty"'] & \mathbb{N}_0 \arrow[d, hook] \\ & \mathbb{N}_0 \end{tikzcd}$$

From this we see that there is an embedding of $(\mathbb{N}_0 \downarrow g)_0$ into the product $(\Delta_a)^\omega$ of countably many copies of the augmented simplex category Δ_a, given by sending each $c : \mathbb{N}_0 \overset{\infty}{\hookrightarrow} \mathbb{N}_0$ to

$$\prod_{i=0}^{\infty} [c_{i+1} - c_i - 2]$$

(with the convention that $[-1] = \varnothing$); the infinite complement condition is equivalent to that infinitely many coordinates lie in the usual simplex category Δ, namely, are nonempty. We denote by $\mathscr{C}$ the embedded image.

Lemma IV.12. *The classifying space $B\mathscr{C}$ has no homotopy groups, hence $\mathscr{C}$ is contractible.*

Proof. We prove that the simplicial homotopy groups of the nerve $N\mathscr{C}$ are trivial. Consider any generator of $\pi_k(N\mathscr{C})$ represented by a map $\mathrm{Sd}^N S^k \to N\mathscr{C}$, where Sd^N denotes the N-fold barycentric subdivision functor, and $S^k \cong \Delta^k/\partial\Delta^k$ is the simplicial k-sphere. Since $\mathrm{Sd}^N S^k$ has finitely many nondegenerate simplices, for each i there exists $m_i \in \mathbb{N}_0$ such that the image of $\operatorname{Sd}^N S^k$ is contained in the infinite product $\prod_{i=0}^{\infty} \Delta_{m_i}$ of finite-dimensional simplices. Since one can explicitly construct a simplicial contraction of this infinite product, it follows that this generator is simplicially null-homotopic. This shows that $\pi_k(N\mathscr{C}) = 0$. □

This concludes the proof of Proposition IV.10. Combining Lemma IV.6 and Propositions IV.7, IV.8, and IV.10, we conclude our proof of Theorem IV.5.

4.1.3 Rectification of operads

In this section, by a "tree" we shall mean a version of the concept of *planar tree*, where leaves are not counted as nodes, and the tree with a single node is not allowed. We give the precise definition below

Definition IV.13. A *tree* (T, n, k) consists of the following data:

- a finite poset $(T, \leq)$ of *branch nodes*,
- an *arity function* $n : T \to \mathbb{N}$,
- a *grafting function* k which sends each pair $s < t$ in T to $k(s,t) \in \{1, ..., n_t\}$;

Such data are subject to the following axioms:

- T has a unique maximal element;
- for any $s \leq t$ in T, the interval $[s,t]_T = \{r \in T : s \leq r \leq t\}$ is a total order;
- for any $s_1 \leq s_2 < t$ in T, we have $k(s_1, t) = k(s_2, t)$;
- for $s_1, s_2 \in T$, let $t = \sup(s_1, s_2)$; if $k(s_1, t) = k(s_2, t)$, then s_1 and s_2 are comparable.

Remark IV.14. Equivalently, a tree is a finite poset $(T, \leq)$ with a unique maximal element such that, for all $s \leq t \in T$, the interval $[s,t]_T$ is a total order, and that, to each $t \in T$ is equipped a natural number n_t and an injective function

$$\phi_t : \{s \in T | s < t \text{ and } \nexists x \in T \text{ with } s < x < t\}.$$

An element $i \in \{1, ..., n_t\}$ is thought of as a *bud*, each of which is either a *leaf* if it is not equal to some $k(s,t)$, or is grafted by a tree $T_{t,i}$:

Lemma IV.15. *For any $t \in T$ and $i \in \{1, ..., n_t\}$, the set*

$$T_{t,i} = \{s \in T : s \leq t,\, k(s,t) = i\},$$

endowed with restrictions of $\leq$, n and k, is again a tree. □

A subset T' of a tree T is a *subtree* if, when endowed with restrictions of $\leq$, n and k, it is itself a tree. We remark that being a subtree is a condition on the sub*set* T', with the rest of the tree data uniquely determined by those of T.

Definition IV.16. A morphism of trees is a function $f : T \to T'$ together with a bijection

$$\theta_s : \coprod_{t \in f^{-1}(s)} \left(\{1, ..., n_t\} \setminus \{k(x,t) | x < t,\, x \in f^{-1}(s)\}\right) \xrightarrow{\cong} \{1, ..., n_s\}$$

for each $s \in \operatorname{Im} f$ (in particular, it follows that $n_s = \sum_{t \in f^{-1}(s)} n_t - |f^{-1}(s)| + 1$), satisfying the following properties:

- if $s \leq t$ in T, then $f(s) \leq f(t)$;
- if $f(s) < f(t)$, then $s < t$;
- for any $s \in T'$, the subset $f^{-1}(s)$ is a subtree of T;

- for any $s < t$ in T such that $f(s) < f(t)$, we have

$$k(f(s), f(t)) = (\theta_{f(t)} \circ \iota)(k(s, t_0))$$

where $t_0 = \min([s,t] \cap f^{-1}(f(t)))$, and ι is the canonical inclusion of the set

$$\{1, ..., n_{t_0}\} \setminus \{k(x, t_0) | x < t,\, x \in f^{-1}(f(t))\}$$

into the coproduct.

In the above definition, the source of the bijection θ_s collects all leaves in the tree $f^{-1}(s)$, which we demand to biject with the buds of the grafting node s. More generally, we define the number of leaves of a tree T as

$$n_T \coloneqq \left| \coprod_{t \in T} (\{1, ..., n_t\} \setminus \{k(x,t) | x < t,\, x \in T\}) \right|.$$

Then the a morphism $f : T \to T'$ of trees necessarily satisfies that $n_{f^{-1}(s)} = n_T$ for any $s \in \operatorname{Im} f$.

A morphism of trees is injective, surjective, or bijective if the underlying function is. An injection is precisely an inclusion of a subtree, while a surjection $f : T \to T'$ can be regarded as grafting a tree $f^{-1}(s)$ to each branch node $s \in T'$. Every morphism of trees factors as a surjection (onto the image) follows by an injection.

We denote by $\operatorname{Aut}(T)$ the group of automorphisms, or bijective self-maps, of T. Note that there are embeddings $\operatorname{Aut}(T) \hookrightarrow \Sigma_{n_T}$ and $\operatorname{Aut}(T) \hookrightarrow \prod_{t \in T} \Sigma_{n_t}$, as an automorphism of a tree necessarily permutes the leaves, as well as the set of buds for each branch node.

We will make frequent use of a special kind of tree $[n]$, whose underlying poset is a singleton $\{r\}$, arity function is constant with value n, and grafting function is the empty function. It is clear that $\operatorname{Aut}([n]) \cong \Sigma_n$.

Let Π denote the category of trees and injections of trees, and consider a functor $\mathscr{D} : \Pi^{\mathrm{op}} \to \Delta^{\mathrm{op}}\text{-}\mathsf{Set}$. In particular, given any tree T and any $t \in T$, there is an injection $[n_t] \hookrightarrow T$ sending the unique node in $[n_t]$ to t, which induces a map $\mathscr{D}(T) \to \mathscr{D}([n_t])$.

Definition IV.17. A *domain* (in simplicial sets) is a functor $\Pi^{\mathrm{op}} \to \Delta^{\mathrm{op}}\text{-}\mathsf{Set}$ subject to the *Segal condition*: for any tree T, the canonical map

$$\delta_T : \mathscr{D}(T) \longrightarrow \prod_{t \in T} \mathscr{D}([n_t])$$

is an inclusion and a weak equivalence.

Lemma IV.18. *Let $\mathscr{D}$ be a domain. For any partition of a tree T into subtrees $T_1, ..., T_n$, namely, each T_i is a subtree of T such that every $t \in T$ belongs to exactly one T_i, the canonical map*

$$\mathscr{D}(T) \longrightarrow \prod_i \mathscr{D}(T_i)$$

is an inclusion and a weak equivalence.

Proof. The functoriality with respect to Π^{op} implies that the diagram

$$\begin{tikzcd} \mathscr{D}(T) \arrow[r] \arrow[dr, hook, "\sim"'] & \prod_i \mathscr{D}(T_i) \arrow[d, hook, "\sim"] \\ & \prod_{t\in T} \mathscr{D}([n_t]) \end{tikzcd}$$

This forces the top arrow to be an inclusion and a weak equivalence. □

The category Dom of domains is the full subcategory of the functor category $\Pi^{\mathrm{op}}\text{-}\Delta^{\mathrm{op}}\text{-}\mathsf{Set}$ on domains. We also consider the functor category $\Sigma\text{-}\Delta^{\mathrm{op}}\text{-}\mathsf{Set}$, where $\Sigma = \coprod_{n>0} \Sigma_n$ is the permutation groupoid, that is, it has as object set $\mathbb{N}_0$, and $\Sigma(n,n) = \Sigma_n$ and $\Sigma(m,n) = \varnothing$ for $m \neq n$.

Proposition IV.19. *There is an adjoint pair* (L,R) *between* Dom *and* $\Sigma\text{-}\Delta^{\mathrm{op}}\text{-}\mathsf{Set}$ *given by*

$$\begin{array}{rclcl} L : & \mathsf{Dom} & \longleftrightarrow & \Sigma\text{-}\Delta^{\mathrm{op}}\text{-}\mathsf{Set} & : R \\ & \mathscr{D} & \longmapsto & (\mathscr{D}([n]))_{n\in\mathbb{N}} & \\ & \left(\prod_{t\in T}\mathscr{C}(n_t)\right)_{T\ tree} & \longleftarrow & \mathscr{C} & \end{array}$$

Proof. Each $L(\mathscr{D})(n) = \mathscr{D}([n])$ is a Σ_n-simplicial set. On the other hand, given an injection $f : T \hookrightarrow T'$ of trees, we set the map $R(\mathscr{C})(f)$ to be the projection

$$\prod_{t\in T'}\mathscr{C}(n_t) \twoheadrightarrow \prod_{t\in T}\mathscr{C}(n_t),$$

and the maps δ_T are just the identities. The functoriality is clear, and the adjunction follows from $LR = \mathrm{Id}$, and that the maps δ_T define a natural transformation $\delta : \mathrm{Id} \to RL$ such that $L\delta = \mathrm{Id}_L$. $\square$

We now proceed to define monads Φ and $\widehat{\Phi}$ on Dom and $\Sigma\text{-}\Delta^{\mathrm{op}}\text{-}\mathsf{Set}$, respectively, such that Φ-algebras are precisely operads, and $\widehat{\Phi}$-algebras may be regarded as "partial operads".

Definition IV.20. For an object $\mathscr{C}$ in $\Sigma\text{-}\Delta^{\mathrm{op}}\text{-}\mathsf{Set}$, define for each $n \in \mathbb{N}$

$$\Phi(\mathscr{C})(n) \coloneqq \coprod_{\substack{T\ \mathrm{tree} \\ n_T = n}} \left(\left(\prod_{t\in T}\mathscr{C}(n_t) \right) \times_{\mathrm{Aut}(T)} \Sigma_{n_T} \right),$$

which is a (right) Σ_n-simplicial set.

It is convenient to write $\mathscr{C}(T)$ for the balanced product

$$\left(\prod_{t\in T}\mathscr{C}(n_t) \right) \times_{\mathrm{Aut}(T)} \Sigma_{n_T},$$

so $\Phi(\mathscr{C})(n) = \coprod_{n_T=n}\mathscr{C}(T)$.

Proposition IV.21. Φ *defines a monad on* $\Sigma\text{-}\Delta^{\mathrm{op}}\text{-}\mathsf{Set}$.

Proof. Functoriality is clear. The unit $\eta : \mathrm{Id} \to \Phi$ is defined by mapping, for each $\mathscr{C}$ in Σ-Δ^{op}-Set and each n, the simplicial set $\mathscr{C}(n)$ to

$$\mathscr{C}(n) \cong \mathscr{C}(n) \times_{\mathrm{Aut}([n])} \Sigma_n =: \mathscr{C}([n])$$

labeled by the tree $[n]$ (recall $\mathrm{Aut}([n]) \cong \Sigma_n$).

To define the multiplication $\mu : \Phi^2 \to \Phi$, fix $n \in \mathbb{N}$. For any choice of the following data

- a tree T with $n_T = n$,
- for each $t \in T$, a tree T_t with $n_{T_t} = n_t$,

there is a tree S obtained by "wreathing" trees T_t to T, and it satisfies $n_S = n_T = n$. This provides a map

$$\prod_{t \in T} \prod_{s \in T_t} \mathscr{C}(n_s) \longrightarrow \prod_{s \in S} \mathscr{C}(n_s),$$

which induces on the balanced products a map

$$\prod_{t \in T} \mathscr{C}(T_t) \longrightarrow \mathscr{C}(S).$$

Taking coproduct over the choices T_t gives

$$\coprod_{\substack{(T_t)_{t \in T} \text{ trees} \\ n_{T_t} = n_t}} \prod_{t \in T} \mathscr{C}(T_t) = \prod_{t \in T} \coprod_{\substack{T_t \text{ tree} \\ n_{T_t} = n_t}} \mathscr{C}(T_t) = \prod_{t \in T} \Phi(\mathscr{C})(n_t) \longrightarrow \coprod_{\substack{S \text{ tree} \\ n_S = n}} \mathscr{C}(S),$$

which again induces on balanced products a map

$$\Phi(\mathscr{C})(T) \longrightarrow \coprod_{\substack{S \text{ tree} \\ n_S = n}} \mathscr{C}(S).$$

Finally taking coproduct over all choices T gives the desired map

$$\Phi^2(\mathscr{C})(n) = \coprod_{\substack{T \text{ tree} \\ n_T = n}} \Phi(\mathscr{C})(T) \longrightarrow \coprod_{\substack{S \text{ tree} \\ n_S = n}} \mathscr{C}(S) = \Phi(\mathscr{C})(n).$$

Unitality and associativity are readily checked. $\square$

Corollary IV.22. *$R\Phi L$ defines a monad on* Dom.

Proof. The unit $\eta_{R\Phi L} : \mathrm{Id} \to R\Phi L$ is given by

$$\mathrm{Id} \xrightarrow{\delta} RL \xrightarrow{R\eta_\Phi L} R\Phi L$$

while the multiplication $\mu_{R\Phi L} : (R\Phi L)^2 \to R\Phi L$ is given by

$$(R\Phi L)^2 = R\Phi L R\Phi L = R\Phi^2 L \xrightarrow{\delta} R\Phi L.$$

Associativity and unitality of $R\Phi L$ follow from the corresponding properties for Φ.

□

Definition IV.23. For a domain $\mathscr{D}$, define for each tree T

$$\widehat{\Phi}(\mathscr{D})(T) = \int^{S \twoheadrightarrow T} \mathscr{D}(S)$$

where "$\twoheadrightarrow$" denotes a surjection of trees, and the coend is taken over the groupoid

$$\begin{tikzcd} S \arrow[r, two heads, "f'"] \arrow[d, "\cong"'] & T \\ S' \arrow[ur, two heads, "f"'] & \end{tikzcd}$$

of isomorphisms of trees surjective over T.

Lemma IV.24. *$\widehat{\Phi}(\mathscr{D})$ is a domain.*

Proof. Given an injection of trees $g : T' \hookrightarrow T$, we define

$$\widehat{\Phi}(\mathscr{D})(g) : \widehat{\Phi}(\mathscr{D})(T) \longrightarrow \widehat{\Phi}(\mathscr{D})(T')$$

as follows. Any surjection $f : S \twoheadrightarrow T$ restricts to a surjection $f|_{T'} : f^{-1}(T') \twoheadrightarrow T'$. Taking the coend of $\mathscr{D}(f^{-1}(T'))$ over all such f gives the desired morphism. Moreover, the Segal condition holds because the coend simplifies to a coproduct over isomorphism classes of maps $f : S \twoheadrightarrow T$. □

Proposition IV.25. $\widehat{\Phi}$ *defines a monad on* Dom.

Proof. Functoriality of $\widehat{\Phi}$ is again obvious. The unit $\widehat{\eta} : \mathrm{Id} \to \widehat{\Phi}$ is given by, for each $\mathscr{D}$ in Dom and each tree T, the canonical map

$$\delta_T : \mathscr{D}(T) = \int^{T \xrightarrow{\mathrm{id}} T} \longrightarrow \int^{f: S \twoheadrightarrow T} \mathscr{D}(S).$$

To define the multiplication $\widehat{\mu} : \widehat{\Phi}^2 \to \widehat{\Phi}$, fix a tree T, and choose a surjection $f : S \twoheadrightarrow T$. We define a map

$$\int^{f: S \twoheadrightarrow T} \int^{g: R \twoheadrightarrow S} \mathscr{D}(R) \longrightarrow \int^{f: S \twoheadrightarrow T} \mathscr{D}(S)$$

by sending $\mathscr{D}(R)$ into the part of single coend labeled by $R \xrightarrow{g} S \xrightarrow{f} T$. Associativity and unitality are readily checked. □

Lemma IV.26. *There is an isomorphism of functors $R\Phi = \widehat{\Phi}R$.*

Proof. Given a Σ-simplicial set $\mathscr{C}$ and a tree T, recall

$$(\widehat{\Phi}R\mathscr{C})(T) = \int^{S \twoheadrightarrow T} \prod_{s \in S} \mathscr{C}(n_s).$$

On the other hand,

$$(R\Phi\mathscr{C})(T) = \prod_{t \in T} \coprod_{\substack{S \text{ tree} \\ n_{S_t} = n_t}} \left(\prod_{s \in S_t} \mathscr{C}(n_s) \times_{\mathrm{Aut}\, S_t} \Sigma_{n_{S_t}} \right).$$

Here, we may wreath the trees S_t together along T to form a larger tree S with a surjection $S \twoheadrightarrow T$, and the product of coproducts may be rewritten as $\coprod_{S \twoheadrightarrow T} \prod_{t \in T}$. By comparing the two formulas with care, one concludes that $\widehat{\Phi}R\mathscr{C}$ and $R\Phi\mathscr{C}$ are naturally isomorphic. □

A map $\mathscr{D} \to \mathscr{D}'$ of domains is called a (tree-wise) weak equivalence if $\mathscr{D}(T) \xrightarrow{\sim} \mathscr{D}'(T)$ is a weak equivalence for any tree T, and a map $M \to N$ of monads on Dom is called a natural weak equivalence if $M\mathscr{D} \to N\mathscr{D}$ is a (tree-wise) weak equivalence for any domain $\mathscr{D}$.

Theorem IV.27. *There is a natural weak equivalence $\widehat{\Phi} \xrightarrow{\sim} R\Phi L$.*

Proof. The desired natural transformation is given by

$$\widehat{\Phi} \xrightarrow{\widehat{\Phi}\eta} \widehat{\Phi} RL = R\Phi L.$$

Note that the unit of adjunction $\eta : \mathscr{D} \to RL\mathscr{D}$ is precisely the map δ, which by the Segal condition is an inclusion and a weak equivalence at each tree T. Moreover, it is clear from definition that $\widehat{\Phi}$ preserves tree-wise weak equivalence, so this natural transformation is indeed a natural equivalence. □

Recall from [23] that, for a monad $M : C \to C$ and left and right M-functors $F : D \to C$ and $G : C \to E$, the *two-sided bar construction* $B_*(G, M, F)$ is the simplicial object in the functor category E^C given by $B_n(G, M, F) := GM^nF$. The face maps are given by right action on G, multiplication maps of M, and left action on F, respectively, and the degeneracy maps are given by unit maps of M.

Corollary IV.28. *The two-sided bar construction $B_*(\Phi L, \widehat{\Phi}, \mathscr{D})$ defines a simplicial Φ-algebra such that $RB_*(\Phi L, \widehat{\Phi}, \mathscr{D})$ is naturally simplicially weakly equivalent to $\mathscr{D}$.*

Proof. There are natural simplicial weak equivalences

$$R\mathscr{C}_* := RB_*(\Phi L, \widehat{\Phi}, \mathscr{D}) = B_*(R\Phi L, \widehat{\Phi}, \mathscr{D}) \xleftarrow{\sim} B_*(\widehat{\Phi}, \widehat{\Phi}, \mathscr{D}) \xrightarrow{\sim} \mathscr{D}.$$ □

Definition IV.29. The *rectification* of a Φ-algebra $\mathscr{D}$ is the simplicial Φ-algebra $B_*(\Phi L, \widehat{\Phi}, \mathscr{D})$.

Definition IV.30. The *Adams operad* $\overline{\mathscr{A}}$ is the rectification $B_*(\Phi L, \widehat{\Phi}, \mathscr{A})$ of the Adams partial operad $\mathscr{A}$ constructed at the end of Section 4.1.2.

Lemma IV.31. *For each tree T, the simplicial set $\mathscr{A}(T)$ is a retract of the space $\mathscr{A}(n_T)$, and is in particular weakly contractible.*

Proof. Denote by $\mathscr{A}^{(T)}$ for now the subcategory of $\prod_{t\in T}\mathscr{A}^{(n_t)}$, so that $\mathscr{A}(T) = N\mathscr{A}^{(T)}$. We will construct a split of the functor $\mathscr{A}^T \to \mathscr{A}^{(n_T)}$. Denote by $[k; j_1, ...j_k]$ the tree where the root has arity k and with its sth bud grafted by the tree $[j_s]$. It suffices to construct the split

$$\mathscr{A}^{(j)} \longrightarrow \mathscr{A}^{(k;j_1,...j_k)},$$

as the split in the general case can be obtained as the iteration of the functor above. Indeed, this split is immediate, as a shuffle j-fold shuffle is equivalent to a collection of k shuffles, each of which is j_s-fold ($1 \leq s \leq k$), and then followed by a k-fold shuffle. $\square$

The following is an immediate consequence of Corollary IV.28 and Lemma IV.31.

Corollary IV.32. *The Adams operad $\overline{\mathscr{A}}$ is an E_∞-operad.* $\square$

4.1.4 Operadic smash product in Kan spectra

Definition IV.33. An *E_∞-ring Kan spectrum*, or an *$\mathscr{A}$-algebra* in Δ^{op}-$\mathscr{S}$, is a simplicial Kan spectrum $X_\bullet$ together with morphisms

$$\mathscr{A}(n) \ltimes (\underbrace{X_\bullet, ..., X_\bullet}_{n \text{ times}}) \longrightarrow X_\bullet$$

for any $n \in \mathbb{N}$, satisfying the condition that, for any surjection of trees $\phi : S \twoheadrightarrow T$, the diagram

$$\begin{tikzcd}
\prod_{t\in T}\mathscr{A}(\phi^{-1}(t)) \ltimes (X_\bullet, ..., X_\bullet) \arrow[r, "\delta"'] \arrow[d, "\gamma"] & \prod_{t\in T}\mathscr{A}\left(\left[n_{\phi^{-1}(t)}\right]\right) \ltimes (X_\bullet, ..., X_\bullet) \arrow[d, "\theta"'] \\
\prod_{s\in S}\mathscr{A}(s) \ltimes (X_\bullet, ..., X_\bullet) \arrow[r, "\theta"] & X_\bullet
\end{tikzcd}$$

commutes. Here each product $\prod_{s\in S}\mathscr{A}(s)\ltimes(X_\bullet,...,X_\bullet)$ is interpreted as an iterated twisted half-smash product over the tree S, and the bottom and the right arrows are iterations of the morphisms θ, the top arrow is induced by the composition

$$\gamma_T : \mathscr{A}(T) \longrightarrow \mathscr{A}([n_T])$$

of the partial operad $\mathscr{A}$, and the left arrow is induced from the morphism

$$\delta_T : \mathscr{A}(T) \overset{\sim}{\hookrightarrow} \prod_{t\in T}\mathscr{A}([n_t]).$$

Define a monad $\mathcal{A}$ on the category Δ^{op}-$\mathscr{S}$ of simplicial Kan spectra by

(4.11)

$$\mathcal{A}X_\bullet := \operatorname{coeq}\left[\coprod_{\phi:S\twoheadrightarrow T}\prod_{t\in T}\mathscr{A}(\phi^{-1}(t))\ltimes(X_\bullet,...,X_\bullet)\rightrightarrows\coprod_{U}\prod_{u\in U}\mathscr{A}([n_u])\ltimes(X_\bullet,...,X_\bullet)\right]$$

where the top arrow identifies $S=U$ and applies the morphism δ of $\mathscr{A}$, while the bottom arrow identifies $T=U$ and applies the composition γ. Note that $n_S = n_T = n_U$, so the number of $X_\bullet$ remains unchanged; however, the factors $X_\bullet$ may be permuted according to the surjection ϕ.

An immediate comparison of definitions gives the following.

Proposition IV.34. *There is an isomorphism between the category of $\mathcal{A}$-algebras and the category of $\mathscr{A}$-algebras, which restricts to the identity functor on the underlying simplicial Kan spectra.* $\square$

Similarly, we may also define a functor

$$\mathcal{A}_{(1)} : (\Delta^{\mathrm{op}}\text{-}\mathscr{S})^2 \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathscr{S}$$

from the category of pairs of simplicial Kan spectra to the category of simplicial Kan spectra, by defining $\mathcal{A}_{(1)}(X_\bullet, Y_\bullet)$ to be the coequalizer of

$$(4.12)\qquad \coprod_{\phi:S\twoheadrightarrow T}\prod_{t\in T}\mathscr{A}(\phi^{-1}(t))\ltimes(X_\bullet,...,X_\bullet,Y_\bullet)\rightrightarrows\coprod_{U}\prod_{u\in U}\mathscr{A}([n_u])\ltimes(X_\bullet,...,X_\bullet,Y_\bullet),$$

where the coproduct in the source of the coequalizer is taken over surjections $\phi : S \twoheadrightarrow T$ which keeps the last coordinate fixed. Then there is a monad $\underline{\mathcal{A}}_{(1)}$ in the category $(\Delta^{\mathrm{op}}\text{-}\mathscr{S})^2$ of pairs of simplicial Kan spectra defined by

$$\underline{\mathcal{A}}_{(1)}(X_\bullet, Y_\bullet) \coloneqq \big(\mathcal{A}X_\bullet, \mathcal{A}_{(1)}(X_\bullet, Y_\bullet)\big)\,.$$

Definition IV.35. Let R be an E_∞-ring Kan spectrum, or equivalently an $\mathcal{A}$-algebra. A simplicial Kan spectrum M is an *R-module* if the pair (R, M) is an $\underline{\mathcal{A}}_{(1)}$-algebra.

Similarly, there is a monad $\underline{\mathcal{A}}_{(1,1)}$ in the category $(\Delta^{\mathrm{op}}\text{-}\mathscr{S})^3$ of triples of simplicial Kan spectra defined by

$$\underline{\mathcal{A}}_{(1,1)}(X_\bullet, Y_\bullet, Z_\bullet) \coloneqq \big(\mathcal{A}X_\bullet, \mathcal{A}_{(1)}(X_\bullet, Y_\bullet), \mathcal{A}_{(1)}(X_\bullet, Z_\bullet)\big)\,.$$

We also define a functor

$$\mathcal{A}_{(2)} : (\Delta^{\mathrm{op}}\text{-}\mathscr{S})^3 \longrightarrow \Delta^{\mathrm{op}}\text{-}\mathscr{S}$$

from the category of triples of simplicial Kan spectra to the category of simplicial Kan spectra by the formula

$$\coprod_{\phi : S \twoheadrightarrow T} \prod_{t \in T} \mathscr{A}(\phi^{-1}(t)) \ltimes (X_\bullet, ..., X_\bullet, Y_\bullet, Z_\bullet) \rightrightarrows \coprod_{U} \prod_{u \in U} \mathscr{A}([n_u]) \ltimes (X_\bullet, ..., X_\bullet, Y_\bullet, Z_\bullet)$$

again with ϕ keeping the last two coordinates fixed, and the functor

$$\underline{\mathcal{A}}_{(2)} : (\Delta^{\mathrm{op}}\text{-}\mathscr{S})^3 \to (\Delta^{\mathrm{op}}\text{-}\mathscr{S})^2$$

by

$$\underline{\mathcal{A}}_{(2)}(X_\bullet, Y_\bullet, Z_\bullet) \coloneqq \big(\mathcal{A}X_\bullet, \mathcal{A}_{(2)}(X_\bullet, Y_\bullet, Z_\bullet)\big)\,.$$

Note that $\underline{\mathcal{A}}_{(2)}$ is both a left-algebra over $\underline{\mathcal{A}}_{(1)}$ and a right-algebra over $\underline{\mathcal{A}}_{(1,1)}$.

Definition IV.36. Given an E_∞-Kan spectrum R and R-modules M and N, the *operadic smash product* $M \wedge_R N$ is the module over the E_∞-ring Kan spectrum $B(\mathcal{A}, \mathcal{A}, R)$ defined as the second variable of the two-sided bar construction

$$B\left(\underline{\mathcal{A}}_{(2)}, \underline{\mathcal{A}}_{(1,1)}, (R, M, N)\right).$$

More explicitly,

$$M \wedge_R N \coloneqq B\left(\mathcal{A}_{(2)}, \underline{\mathcal{A}}_{(1,1)}, (R, M, N)\right). \tag{4.13}$$

We remark that, while $M \wedge_R N$ is not a module over the original ring R, the ring $B(\mathcal{A}, \mathcal{A}, R)$ is nonetheless canonically weakly equivalent to R, and thus, as we shall see in the next section, the derived categories of modules over R and $B(\mathcal{A}, \mathcal{A}, R)$, respectively, are canonically equivalent.

4.1.5 Derived categories of rings and modules

Definition IV.37. A morphism of E_∞-ring Kan spectra, or equivalently $\mathcal{A}$-algebras in simplicial Kan spectra, is a *weak equivalence* if the morphism is, as a morphism of the underlying simplicial Kan spectra, a total weak equivalence (Definition II.61).

We will construct the derived category of $\mathcal{A}$-algebras with respect to weak equivalences using the techniques in Section A.3, by considering both a *fibrant approximation*

$$\mathrm{Id} \longrightarrow \mathrm{Sing}_\bullet \|\mathscr{L}\|$$

as well as *cell approximation*

$$Q \longrightarrow \mathrm{Id},$$

based on the discussion of simplicial Kan spectra in Section 2.3.2.

First of all, we say that a morphism of E_∞-ring Kan spectra is a fibration if it is a fibration as a morphism of the underlying simplicial Kan spectra.

Proposition IV.38. *If $X_\bullet$ is an $\mathcal{A}$-algebra, then so is* $\operatorname{Sing}_\bullet \|\mathscr{L}\|(X_\bullet)$.

Proof. In this proof, we may drop the subscript $_\bullet$ when it is clear from the context that the objects involved are simplicial. We will show that a morphism

$$\mathscr{A}(n) \ltimes (Y_1, ..., Y_n) \to Z$$

of simplicial Kan spectra induces a morphism

$$\mathscr{A}(n) \ltimes (\operatorname{Sing}_\bullet \|\mathscr{L}\|(Y_1), ..., \operatorname{Sing}_\bullet \|\mathscr{L}\|(Y_n)) \to \operatorname{Sing}_\bullet \|\mathscr{L}\|(Z).$$

By fixing an element in $\mathscr{A}(n)$, it suffices to construct, for each n-fold Adams datum a, a morphism

$$\bigwedge_a (\operatorname{Sing}_\bullet \|\mathscr{L}\|(X_1), ..., \operatorname{Sing}_\bullet \|\mathscr{L}\|(X_n))) \to \operatorname{Sing} \mathscr{L}(Z).$$

We elaborate on the case $n = 2$, and the cases for higher n are done similarly.

We start by constructing, for simplicial Kan prespectra X and Y, a natural canonical morphism

$$\operatorname{Sing}_\bullet \|X\| \wedge^p_{(a,b)} \operatorname{Sing}_\bullet \|Y\| \to \operatorname{Sing}_\bullet \left\| X \wedge^p_{(a,b)} Y \right\|, \tag{4.14}$$

where (a, b) is a fixed Adams datum, and $\wedge^p$ is the smash product on the prespectrum level performed degree-wise, and $\|?\|$ is the degree-wise (May) realization followed by the simplicial realization. Notice that, at level n, the source of (4.14) is the colimit of the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
(?)'_n \arrow[r, "\sim"] \arrow[d] & \Sigma^{1,\bullet} \left(\operatorname{Sing}_\bullet \left\|X_{a_{n-1},\bullet}\right\| \wedge \operatorname{Sing}_\bullet \left\|Y_{b_{n-1},\bullet}\right\|\right) \\
\operatorname{Sing}_\bullet \|X_{a_n,\bullet}\| \wedge \operatorname{Sing}_\bullet \|Y_{b_n,\bullet}\| &
\end{tikzcd}$$

where

$$(?)'_n = \begin{cases} \operatorname{Sing}_\bullet \left\|X_{a_{n-1},\bullet}\right\| \wedge \Sigma^{1,\bullet} \operatorname{Sing}_\bullet \left\|Y_{b_n,\bullet}\right\|, & \text{or} \\ \Sigma^{1,\bullet} \operatorname{Sing}_\bullet \left\|X_{a_{n-1},\bullet}\right\| \wedge \operatorname{Sing}_\bullet \left\|Y_{b_n,\bullet}\right\|, \end{cases}$$

depending on the Adams datum (a, b). Here, $X_{n,\bullet}$ denotes the bisimplicial set

$$X_{n,\bullet} : (k, \ell) \longmapsto \left(\Omega^{\infty-n} X_\ell\right)_k,$$

and $\Sigma^{1,\bullet}$ signifies the Kan suspension of bisimplicial sets with respect to the first simplicial coordinate. Using the natural canonical morphisms of based bisimplicial sets

$$\operatorname{Sing}_\bullet \|K\| \wedge \operatorname{Sing}_\bullet \|L\| \to \operatorname{Sing}_\bullet \|K \wedge L\|$$

and

$$\Sigma^{1,\bullet} \operatorname{Sing}_\bullet \|K\| \to \operatorname{Sing}_\bullet \left\|\Sigma^{1,\bullet} K\right\|$$

for based bisimplicial sets K and L, the diagram above maps to

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
\operatorname{Sing}_\bullet \left\|\Sigma^{1,\bullet}(?_n)\right\| \arrow[r, "\sim"] \arrow[d] & \operatorname{Sing}_\bullet \left\|\Sigma^{1,\bullet}(X_{a_{n-1},\bullet} \wedge Y_{b_{n-1},\bullet})\right\| \\
\operatorname{Sing}_\bullet \left\|X_{a_n,\bullet} \wedge Y_{b_n,\bullet}\right\| &
\end{tikzcd}$$

where

$$?_n = \begin{cases} X_{a_{n-1},\bullet} \wedge \Sigma^{1,\bullet} Y_{b_{n-1},\bullet}, & \text{or} \\ \Sigma^{1,\bullet} X_{a_{n-1},\bullet} \wedge Y_{b_{n-1},\bullet}. \end{cases}$$

Passing to the colimit, the source of (4.14) maps naturally and canonically to

$$\operatorname{Sing}_\bullet \left\| \underrightarrow{\operatorname{colim}} \left[\begin{tikzcd} & \vdots \arrow[d] \\ \Sigma^{1,\bullet}(?_n) \arrow[r, "\sim"] \arrow[d] & \Sigma^{1,\bullet}(X_{a_{n-1},\bullet} \wedge Y_{b_{n-1},\bullet}) \\ X_{a_n,\bullet} \wedge Y_{b_n,\bullet} & \end{tikzcd} \right] \right\|$$

which is the target of (4.14).

We next show that the canonical morphism (4.14) also passes to the spectrum level and gives a canonical morphism

$$\operatorname{Sing}_\bullet \|\mathscr{L}\|(X) \wedge_{(a,b)} \operatorname{Sing}_\bullet \|\mathscr{L}\|(Y) \to \operatorname{Sing}_\bullet \|\mathscr{L}\| \left(X \wedge_{(a,b)} Y\right) \tag{4.15}$$

for simplicial Kan spectra X and Y. Since $\|?\|$ commutes with $\wedge$ for (bi-)simplicial sets, it suffices to construct, for inclusion May prespectra Z and T, a natural canonical morphism

$$\operatorname{Sing} \circ \operatorname{Sp}(Z) \wedge^p_{(a,b)} \operatorname{Sing}_\bullet \circ \operatorname{Sp}(T) \to \operatorname{Sing} \circ \operatorname{Sp}\left(Z \wedge^p_{(a,b)} T\right). \tag{4.16}$$

Recall that the spectrification of an inclusion May prespectrum Z is given by

$$\operatorname{Sp}(Z)_n = \underrightarrow{\operatorname{colim}}_k \Omega^k Z_{n+k}.$$

At level n, the source of (4.16) is the colimit of the diagram

$$\begin{tikzcd} & \vdots \arrow[d] \\ (?)''_n \arrow[r, "\sim"] \arrow[d] & \Sigma(\operatorname{Sing} \underrightarrow{\operatorname{colim}}\, \Omega^{k'} Z_{a_{n-1}+k'} \wedge \operatorname{Sing} \underrightarrow{\operatorname{colim}}\, \Omega^{\ell'} T_{b_{n-1}+\ell'}) \\ \operatorname{Sing} \underrightarrow{\operatorname{colim}}\, \Omega^k Z_{a_n+k} \wedge \operatorname{Sing} \underrightarrow{\operatorname{colim}}\, \Omega^\ell T_{b_n+\ell} & \end{tikzcd}.$$

For each fixed pair (k, ℓ), we may choose m sufficiently large such that $a_{n+m} \geq a_n + k$ and $b_{n+m} \geq b_n + \ell$; this is possible since $\lim_n a_n = \lim_n b_n = \infty$. Then there are natural canonical morphisms

$$\Omega^k Z_{a_{n+k}} \wedge \Omega^\ell T_{b_{n+\ell}} \longrightarrow \Omega^{a_{n+m}-a_n} Z_{a_{n+m}} \wedge \Omega^{b_{n+m}-b_n} T_{b_{n+m}} \longrightarrow \Omega^{m,\bullet}(Z_{a_{n+m}} \wedge T_{b_{n+m}})$$

By taking the colimits appropriately, we conclude that the diagram above maps to

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
\mathrm{Sing}_\bullet \underrightarrow{\mathrm{colim}}\, \Omega^{m',\bullet}(?_{m+n}) \arrow[r, "\sim"] \arrow[d] & \Sigma^{1,\bullet}\left(\mathrm{Sing}_\bullet \underrightarrow{\mathrm{colim}}\, \Omega^{m''}(Z_{a_{n-1+m''},\bullet} \wedge T_{b_{n-1+m''},\bullet})\right) \\
\mathrm{Sing}_\bullet \underrightarrow{\mathrm{colim}}\, \Omega^{m}(Z_{a_{n+m},\bullet} \wedge T_{b_{n+m},\bullet}) &
\end{tikzcd}.$$

whose colimit is precisely the target of (4.16). This completes the constructions of (4.16), and thus of (4.15) by appropriately spectrifying $\wedge^p_{(a,b)}$. □

By a similar argument, one can show that:

Proposition IV.39. *If the pair (X, Y) is an $\underline{\mathcal{A}}_{(1)}$-algebra, then so is the pair*

$$\mathrm{Sing}_\bullet \|\mathscr{L}\|(X), \mathrm{Sing}_\bullet \|\mathscr{L}\|(Y)).$$ □

We also construct cell approximations of $\mathcal{A}$-algebras. Specifically, a *cell* in the category of $\mathcal{A}$-algebras is the free $\mathcal{A}$-algebra functor applied to a cell in the category of simplicial Kan spectra, namely the image of a morphism of the form (2.14) under the shift suspension functor $\Sigma^{\infty+\ell,\bullet}$. This leads to the notions of relative cell $\mathcal{A}$-algebra maps and cell $\mathcal{A}$-algebras.

Lemma IV.40. *For any simplicial Kan spectrum X, the free $\mathcal{A}$-algebra $\mathcal{A}X$ is a cell $\mathcal{A}$-algebra.*

Proof. Since the free $\mathcal{A}$-algebra functor $\mathcal{A}$ is a left adjoint to the forgetful functor from $\mathcal{A}$-algebras to simplicial Kan spectra, it commutes with colimits and thus, in particular, with attaching cells. Since X is a cell simplicial Kan spectrum by Proposition II.63, it follows that $\mathcal{A}X$ is a cell $\mathcal{A}$-algebra. $\square$

Proposition IV.41. *For any $\mathcal{A}$-algebra X, there exists a cell $\mathcal{A}$-algebra QX and a total weak equivalence $QX \xrightarrow{\sim} X$ of the underlying simplicial Kan spectra. Moreover, the assignment $X \mapsto QX$ is functorial, and the total equivalences form a natural transformation $Q \to \mathrm{Id}$.*

Proof. We put QX to be the (diagonal) realization of the two-sided bar-construction,

$$QX := |B_\bullet(\mathcal{A}, \mathcal{A}, X)|\,,$$

which is itself cell. This is clearly functorial in X, and the natural map $QX \to X$ is a total weak equivalence since it is the realization of the simplicial contraction $B_\bullet(\mathcal{A}, \mathcal{A}, X) \to X$. $\square$

Proposition IV.42. *The cell $\mathcal{A}$-algebras are colocal with respect to total weak equivalences between $\mathcal{A}$-algebras whose underlying simplicial Kan spectra are fibrant.*

Proof. This follows from the adjunction between the free $\mathcal{A}$-algebra functor and the forgetful functor from $\mathcal{A}$-algebras to simplicial Kan spectra, as well as the right lifting property of fibrant simplicial Kan spectra with respect to cells. $\square$

Theorem IV.43. *The derived category of $\mathcal{A}$-algebras exists, and is equivalent to the category whose objects are $\mathcal{A}$-algebras, and whose morphisms between X and Y are $\mathcal{A}$-algebra morphisms from the cell $\mathcal{A}$-algebra QX to the fibrant $\mathcal{A}$-algebra* $\mathrm{Sing}_\bullet \|\mathscr{L}\|(Y)$.

Proof. This is a consequence of Theorem A.13. $\square$

We shall also treat the category of $\underline{\mathcal{A}}_{(1)}$-algebras.

Definition IV.44. A morphism of $\underline{\mathcal{A}}_{(1)}$-algebras $(R, M) \to (R', M')$ is a *coordinate-wise weak equivalence* if both morphisms $R \to R'$ and $M \to M'$ are, as morphisms of the underlying simplicial Kan spectra, total weak equivalences (Definition II.61).

A similar theory of fibrant and cell algebras can also be introduced for $\underline{\mathcal{A}}_{(1)}$-algebras by working coordinate-wise. Consequently, we obtain the derived category $D\mathsf{Mod}$ of $\underline{\mathcal{A}}_{(1)}$-algebras with respect to coordinate-wise weak equivalences.

Definition IV.45. Let R be an E_∞-ring Kan spectrum. The *derived category of R-modules* $D(R)$ is the full subcategory of $D\mathsf{Mod}$ consisting of pairs (R, M) with the first coordinate being the fixed object R, and the second coordinate an R-module M.

Lemma IV.46. *If R and R' are weakly equivalent, then there is a canonical equivalence of categories between $D(R)$ and $D(R')$.*

Proof. The weak equivalence $R \xrightarrow{\sim} R'$ makes any R'-module M also an R-module, and furthermore $(R, M) \to (R', M)$ is a weak equivalence, or an isomorphism in $D\mathsf{Mod}$. $\square$

Corollary IV.47. *For any E_∞-ring Kan spectrum R, the formula 4.13 yields a well-defined smash product*

$$? \wedge^L_R ? : D(R) \times D(R) \longrightarrow D(R)$$

defined by

$$M \wedge^L_R N := QM \wedge_R QN,$$

where $QM \to M$ and $QN \to N$ are cell R-algebra approximations.

4.1.6 Symmetric monoidal structure

In this section, we prove the following theorem:

Theorem IV.48. *The category $D(R)$ has a symmetric monoidal structure $(\wedge_R^L, R)$, where $\wedge_R^L$ is the derived operadic smash product defined in IV.47, and S is the sphere spectrum.*

First of all, we note that the symmetry of the operadic smash product holds on the point-set level, that is, before passing to the derived category. That is, we have a natural isomorphism

$$M \wedge_R N \cong N \wedge_R M$$

for any E_∞-ring Kan spectrum R and R-modules M and N from the definitions of $\underline{\mathcal{A}}_{(2)}$ and $\underline{\mathcal{A}}_{(1,1)}$ via the balanced product over the symmetric groups. Thus, it suffices to construct the (left) unitor and associator on the point-set level, and to show that they are weak equivalences.

To construct the left unitor

$$R \wedge_R^L M \xrightarrow{\cong} M, \tag{4.17}$$

we construct, on the point-set level, a natural morphism

$$B\left(\underline{\mathcal{A}}_{(2)}, \underline{\mathcal{A}}_{(1,1)}, (R, R, M)\right) \longrightarrow B\left(\underline{\mathcal{A}}_{(1)}, \underline{\mathcal{A}}_{(1)}, (R, M)\right). \tag{4.18}$$

This suffices because the simplicial homotopy equivalence

$$B_\bullet\left(\underline{\mathcal{A}}_{(1)}, \underline{\mathcal{A}}_{(1)}, (R, M)\right) \xrightarrow{\simeq} (R, M),$$

yields, via the diagonal realization of multi-simplicial objects, to a weak equivalence

$$B\left(\underline{\mathcal{A}}_{(1)}, \underline{\mathcal{A}}_{(1)}, (R, M)\right) \xrightarrow{\sim} (R, M).$$

To this end, first recall from definition that

$$\begin{aligned}B_n\left(\underline{\mathcal{A}}_{(2)}, \underline{\mathcal{A}}_{(1,1)}, (R,R,M)\right) &= \underline{\mathcal{A}}_{(2)} \circ \underline{\mathcal{A}}^n_{(1,1)}(R,R,M) \\ &= \underline{\mathcal{A}}_{(2)} \circ \Big(\mathcal{A}\circ\mathcal{A}^{n-1}R, \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R,R), \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R,M)\Big),\end{aligned}$$

and

$$B_n\left(\underline{\mathcal{A}}_{(1)}, \underline{\mathcal{A}}_{(1)}, (R,M)\right) = \underline{\mathcal{A}}_{(1)} \circ \Big(\mathcal{A}^n R, \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R,M)\Big).$$

Lemma IV.49. *Let R' be an E_∞-ring Kan spectrum R' and M' an R'-module. We have natural morphisms*

$$\mathcal{A}_{(1)}(R',R') \longrightarrow \mathcal{A}R' \tag{4.19}$$

and

$$\mathcal{A}_{(2)}(R',R',M') \longrightarrow \mathcal{A}_{(1)}(R',M'). \tag{4.20}$$

Proof. The morphism (4.19) is the canonical morphism between the balanced products in the definitions of $\mathcal{A}$ and $\mathcal{A}_{(1)}$, via the embeddings $\Sigma_n \hookrightarrow \Sigma_{n+1}$ by permuting the first n labels. The morphism (4.20) is obtained similarly, using instead the definitions of the functors $\mathcal{A}_{(1)}$ and $\mathcal{A}_{(2)}$, and the embeddings $\Sigma_{n+1} \hookrightarrow \Sigma_{n+2}$. □

Thus, we obtain for each $n \geq 1$ natural morphisms

$$\begin{array}{c}\underline{\mathcal{A}}_{(2)} \circ \Big(\mathcal{A}\circ\mathcal{A}^{n-1}R, \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R,R), \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R,M)\Big) \\ \big\downarrow \\ \underline{\mathcal{A}}_{(2)} \circ \Big(\mathcal{A}^n R, \mathcal{A}^n R, \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R,M)\Big) \\ \big\downarrow \\ \underline{\mathcal{A}}_{(1)} \circ \Big(\mathcal{A}^n R, \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R,M)\Big)\end{array}$$

first using (4.19) with $(R', M') = \underline{\mathcal{A}}_{(1)}^{n-1}(R, M)$ and then (4.20) with $(R', M') = \underline{\mathcal{A}}_{(1)}^{n}(R, M)$. Taking the diagonal realization of the multi-simplicial objects, this yields the desired morphism (4.18).

Next we construct the associator

$$\alpha_{M,N,P} : \left(M \wedge_R^L N\right) \wedge_R^L P \xrightarrow{\cong} M \wedge_R^L \left(N \wedge_R^L P\right). \tag{4.21}$$

We first define the three-fold operadic smash product to be

$$M \wedge_R N \wedge_R P \coloneqq B\left(\mathcal{A}_{(3)}, \underline{\mathcal{A}}_{(1,1,1)}, (R, M, N, P)\right),$$

where $\underline{\mathcal{A}}_{(1,1,1)}$ is a monad in the category $(\Delta^{\mathrm{op}}\text{-}\mathscr{S})^4$ defined similarly to $\underline{\mathcal{A}}_{(1,1)}$, and $\mathcal{A}_{(3)} : (\Delta^{\mathrm{op}}\text{-}\mathscr{S})^4 \to \Delta^{\mathrm{op}}\text{-}\mathscr{S}$ is defined similarly to $\mathcal{A}_{(1)}$ and $\mathcal{A}_{(2)}$. We will compare it to the module

$$(M \wedge_R N) \wedge_{B(\mathcal{A},\mathcal{A},R)} P.$$

where P is regarded as a $B(\mathcal{A}, \mathcal{A}, R)$-module via the canonical map $B(\mathcal{A}, \mathcal{A}, R) \to R$. In fact, we will construct, on the point-set level, a natural morphism

(4.22)

$$(M \wedge_R N) \wedge_{B(\mathcal{A},\mathcal{A},R)} B\left(\mathcal{A}_{(1)}, \underline{\mathcal{A}}_{(1)}, (R, P)\right) \longrightarrow B\left(\mathcal{A}_{(3)}, \underline{\mathcal{A}}_{(1,1,1)}, (R, M, N, P)\right).$$

For each $n \geq 1$, the nth simplicial degree of the source of the morphism above is

$$\begin{aligned}
&\mathcal{A}_2 \circ \underline{\mathcal{A}}_{(1,1)}^{n}\Big(\mathcal{A}^{n+1}R, \mathcal{A}_2 \circ \underline{\mathcal{A}}_{(1,1)}^{n}(R, M, N), \mathcal{A}_1 \circ \underline{\mathcal{A}}_{(1)}^{n}(R, P)\Big) \\
&= \mathcal{A}_2 \circ \underline{\mathcal{A}}_{(1,1)}^{n} \circ \underline{\mathcal{A}}_{(2,1)}\Big(\mathcal{A}^{n}R, \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}_{(1)}^{n-1}(R, M), \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}_{(1)}^{n-1}(R, N), \\
&\qquad \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}_{(1)}^{n-1}(R, P)\Big),
\end{aligned}$$

while the nth simplicial degree of the target is

$$\mathcal{A}_3 \circ \underline{\mathcal{A}}_{(1,1,1)}^{n}(R, M, N, P).$$

Using the multiplication maps of the monad $\mathcal{A}$, we obtain the following.

Lemma IV.50. *There are natural transformations*

$$\underline{\mathcal{A}}_{(2)} \circ \underline{\mathcal{A}}_{(1,1)} \longrightarrow \underline{\mathcal{A}}_{(2)} \tag{4.23}$$

and

$$\underline{\mathcal{A}}_{(2)} \circ \underline{\mathcal{A}}_{(2,1)} \longrightarrow \underline{\mathcal{A}}_{(3)}. \tag{4.24}$$

□

By iterating (4.23) and (4.24), we obtain a natural morphism

$$\begin{array}{c}
\mathcal{A}_2 \circ \underline{\mathcal{A}}^n_{(1,1)} \circ \underline{\mathcal{A}}_{(2,1)} \Big(\mathcal{A}^n R, \underline{\mathcal{A}}^n_{(1,1)}(R, M, N), \underline{\mathcal{A}}^n_{(1)}(R, P) \Big) \\
\downarrow \\
\underline{\mathcal{A}}_{(3)} \Big(\mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R, M), \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R, N), \mathcal{A}_{(1)} \circ \underline{\mathcal{A}}^{n-1}_{(1)}(R, P) \Big) \\
\| \\
\underline{\mathcal{A}}_{(3)} \circ \underline{\mathcal{A}}^n_{(1,1,1)}(R, M, N, P)
\end{array}$$

These morphisms therefore assemble into the desired morphism (4.22).

It still remains to construct (4.21). To this end, it suffices to observe that there is a zig-zag of weak equivalences

$$\big(M \wedge^L_R N \big) \wedge^L_{B(\mathcal{A},\mathcal{A},R)} P \xrightarrow{\sim} M \wedge^L_R N \wedge^L_R P \xleftarrow{\sim} M \wedge^L_{B(\mathcal{A},\mathcal{A},R)} \big(N \wedge^L_R P \big)$$

of morphisms of the form (4.22). Since these morphisms are invertible upon passing to the derived category $D(R)$, we may define the associator (4.21) as the composite above.

Now we return to the proof of the symmetric monoidal structure $(D(R), \wedge^L_R, R)$.

Proof of Theorem IV.48. It remains to check that the unitor (4.17) and associator (4.21) satisfy the following three axioms (see [19, Chap. XI]):

(1) the triangle axiom for the unitors and associator;

(2) the pentagon axiom for the associator;

(3) the hexagon identity for associator and braiding.

To see the triangle axiom, consider the diagram

$$\begin{tikzcd}
(M \wedge_R R) \wedge_{B(\mathcal{A},\mathcal{A},R)} N \arrow[dr] & & \\
& M \wedge_R R \wedge_R N \arrow[r, "\sim"] & M \wedge_R N \\
M \wedge_{B(\mathcal{A},\mathcal{A},R)} (R \wedge_R N) \arrow[ur] & &
\end{tikzcd}$$

where the two arrows on the left are constructed from (4.22) and the arrow on the right is (4.18). Upon passing to the derived category $D(R)$, the two composites are homotopic to the $\lambda_M \wedge \mathrm{id}_N$ and $\mathrm{id}_M \wedge \lambda_N$, respectively, and the zig-zag on the left gives rise to the associator.

To see the pentagon axiom, we may define the four-fold operadic smash product to be

$$M_1 \wedge_R M_2 \wedge_R M_3 \wedge_R M_4 \coloneqq B\left(\mathcal{A}_4, \underline{\mathcal{A}}_{(1,1,1,1)}, (R, M_1, M_2, M_3, M_4)\right),$$

and construct a natural morphism from each of the vertex of the pentagon, each of which is an iteration of the two-fold operadic smash product on the ordered quadruple (M_1, M_2, M_3, M_4), to the four-fold product. One then observes that, every edge of the pentagon is formed as a zig-zag of these natural morphisms which, upon passing to $D(R)$, are weak equivalences. The fact that the pentagon diagram commutes amounts to that all iterations of the two-fold product is represented by the same object

$$M_1 \wedge^L_R M_2 \wedge^L_R M_3 \wedge^L_R M_4$$

in $D(R)$.

Finally, to see the hexagon diagram, we may similarly observe that, given the symmetry on the point-set level, all iterations of the two-fold operadic smash product on the ordered triple (M, N, P) are represented by the three-fold product $M \wedge_R^L N \wedge_R^L P$ in $D(R)$. $\square$

4.1.7 Tensor triangulated structure

Recall from [2] (also cf. [13] and [25]) that a *tensor triangulated category* is a triangulated category equipped with a symmetric monoidal product $\wedge$ which is exact in each variable. The goal of this section is to prove that the derived category $D(R)$, equipped with $\wedge_R^L$, is a tensor triangulated category.

We start by constructing the triangulated structure on the category $D(R)$. Recall that $D\mathscr{S}$ is equivalent to the stable homotopy category, and thus is triangulated. Specifically, it has as translation functor the derived suspension functor $L\Sigma$, and as distinguished triangles those weakly equivalent (isomorphic in the derived category) to some cofiber sequence

$$X_\bullet \xrightarrow{f_\bullet} Y_\bullet \longrightarrow (Cf)_\bullet \xrightarrow{\Sigma},$$

where $(Cf)_n = C(f_n)$ is the degree-wise cofiber of $f_\bullet$.

Lemma IV.51. *Let R be an E_∞-ring Kan spectrum. If $f : M \to N$ is a morphism of R-modules, then the cofiber Cf is also an R-module.*

Proof. To obtain the desired R-module structure, we construct a natural morphism

$$\underline{\mathcal{A}}_{(1)}(R, Cf) := \big(\mathcal{A}R, \mathcal{A}_{(1)}(R, Cf)\big) \longrightarrow (R, Cf) \tag{4.25}$$

of pairs of simplicial Kan spectra. Recall the cofiber in simplicial Kan spectra is constructed as the pushout

$$\begin{tikzcd} X_\bullet \arrow[r, "f_\bullet"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & Y_\bullet \arrow[d] \\ I_+ \star X_\bullet \arrow[r] & Cf_\bullet \end{tikzcd}$$

and the free R-module functor $(\mathcal{A}R, \mathcal{A}_{(1)}(R, Cf))$ is left adjoint to the forgetful functor $R\mathsf{Mod} \to \Delta^{\mathrm{op}}\text{-}\mathscr{S}$ and thus commutes with colimits, so we have a natural isomorphism

$$\big(\mathcal{A}R, \mathcal{A}_{(1)}(R, Cf)\big) \cong \big(\mathcal{A}R, C\mathcal{A}_{(1)}(R, f)\big)\,,$$

where the second coordinate on the right-hand side is the cofiber of the morphism

$$\mathcal{A}_{(1)}(R, M) \longrightarrow \mathcal{A}_{(1)}(R, N),$$

and then the functoriality of cofiber allows us to construct (4.25) as the dotted vertical arrow in the diagram

$$\begin{tikzcd} (\mathcal{A}R, \mathcal{A}_{(1)}(R, M)) \arrow[r] \arrow[d] & (\mathcal{A}R, \mathcal{A}_{(1)}(R, N)) \arrow[r] \arrow[d] & (\mathcal{A}R, \mathcal{A}_{(1)}(R, Cf)) \arrow[d, dashed] \\ (R, M) \arrow[r] & (R, N) \arrow[r] & (R, Cf) \end{tikzcd}$$

to make the entire diagram commute. It is readily checked that the morphism (4.25) is indeed a morphism of $\underline{\mathcal{A}}_{(1)}$-algebra, and the commutativity of the right square implies that the morphism $N \to Cf$ is a morphism of R-modules. □

Corollary IV.52. *Let R be an E_∞-ring Kan spectrum. If M is an R-module, then so is ΣM.* □

Theorem IV.53. *Let R be an E_∞-ring Kan spectrum. The category $D(R)$ has a triangulated structure where the translation functor the derived suspension $L\Sigma$, and a triangle*

$$M \longrightarrow N \longrightarrow P \xrightarrow{\Sigma}$$

is distinguished if it is distinguished in $\Delta^{\mathrm{op}}\text{-}\mathscr{S}$.

Proof. We check the axioms (TR1)–(TR4) in [32]. Observe that the axiom (TR1) holds by definition, while axioms (TR2) and (TR4) hold by forgetting to the category Δ^{op}-$\mathscr{S}$ of simplicial Kan spectra. Note that (TR3) follows from the other axioms, as observed for example by May [25]. □

Now we investigate the tensor triangulated structure $\wedge_R^L$ on $D(R)$.

Theorem IV.54. *Let R be an E_∞-ring Kan spectrum. The category $D(R)$ with symmetric monoidal structure as in Theorem IV.48 and triangulated structure as in Theorem IV.53 is a tensor triangulated category in the sense of [2].*

Proof. It suffices to show that, for any R-module M, the functor $M \wedge_R^L (?)$ preserves coproducts (that is, it is additive) as well as cofibers (that is, it is exact). Note the analogous statements on the level of simplicial sets, that is, for any based simplicial set K, the functor $K \wedge (?)$ preserves coproducts and cofiber since it is a left adjoint. By construction, the functors $\underline{\mathcal{A}}_{(1,1)}(R, M, ?)$ and $\mathcal{A}_{(2)}(R, M, ?)$ both preserve coproducts and cofibers. Finally, since both coproducts and cofibers for simplicial Kan spectra are formed as degree-wise colimits, they are also preserved by the bar construction and diagonal realization. □

4.2 Rings and modules in Kan spectral sheaves

In this section, we fix a space $\mathscr{X}$, and generalize the notions of E_∞-ring and modules to simplicial Kan spectral sheaves on $\mathscr{X}$. To avoid cumbersome notation, we will sometimes suppress the subscript $\bullet$ of the simplicial object when there is no ambiguity.

For simplicial Kan spectral sheaves $\mathcal{F}_1, ..., \mathcal{F}_n$ on $\mathscr{X}$, the *twisted half-smash prod-*

uct is defined as the sheafification of the presheaf

$$U \longmapsto \mathscr{A}(n) \ltimes (\mathcal{F}_1(U), ..., \mathcal{F}_n(U))$$

extending the twisted half-smash product for simplicial Kan spectra in Definition IV.4.

Definition IV.55. An *E_∞-ring Kan spectral sheaf*, or an *$\mathscr{A}$-algebra* in $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$, is a simplicial Kan spectral sheaf $\mathcal{F}_\bullet$ together with morphisms

$$\mathscr{A}(n) \ltimes (\underbrace{\mathcal{F}_\bullet, ..., \mathcal{F}_\bullet}_{n \text{ times}}) \longrightarrow \mathcal{F}_\bullet$$

for any $n \in \mathbb{N}$, satisfying the condition that, for any surjection of trees $\phi : S \twoheadrightarrow T$, the diagram

$$\begin{tikzcd}
\prod_{t\in T} \mathscr{A}(\phi^{-1}(t)) \ltimes (\mathcal{F}_\bullet, ..., \mathcal{F}_\bullet) \arrow[r, "\delta"'] \arrow[d, "\gamma"] & \prod_{t\in T} \mathscr{A}\left(\left[n_{\phi^{-1}(t)}\right]\right) \ltimes (\mathcal{F}_\bullet, ..., \mathcal{F}_\bullet) \arrow[d, "\theta"'] \\
\prod_{s\in S} \mathscr{A}(s) \ltimes (\mathcal{F}_\bullet, ..., \mathcal{F}_\bullet) \arrow[r, "\theta"] & \mathcal{F}_\bullet
\end{tikzcd}$$

commutes.

In other words, an E_∞-ring Kan spectral sheaf is a sheaf in the category of an E_∞-ring Kan spectra; compare with Definition IV.33.

We can construct monads $\mathcal{A}$, $\underline{\mathcal{A}}_{(1)}$, $\underline{\mathcal{A}}_{(1,1)}$, etc., and functors $\mathcal{A}_{(j)}$ and $\underline{\mathcal{A}}_{(j)}$ for $j \geq 1$ in a completely analogous way. Again, we have the following result.

Proposition IV.56. *There is an isomorphism between the category of $\mathcal{A}$-algebras and the category of $\mathscr{A}$-algebras, which restricts to the identity functor on underlying objects.* $\square$

Definition IV.57. Let $\mathcal{R}$ be an E_∞-ring Kan spectral sheaf, or equivalently an $\mathcal{A}$-algebra. A simplicial Kan spectral sheaf $\mathcal{M}$ is an $\mathcal{R}$-*module* if the pair $(\mathcal{R}, \mathcal{M})$ is an $\underline{\mathcal{A}}_{(1)}$-algebra.

Definition IV.58. Given an E_∞-Kan spectral sheaf $\mathcal{R}$ and $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$, the *operadic smash product* $\mathcal{M} \wedge_{\mathcal{R}} \mathcal{N}$ is the module over the E_∞-ring Kan spectral sheaf $B(\mathcal{A}, \mathcal{A}, \mathcal{R})$ defined as the second variable of the two-sided bar construction

$$B\left(\underline{\mathcal{A}}_{(2)}, \underline{\mathcal{A}}_{(1,1)}, (\mathcal{R}, \mathcal{M}, \mathcal{N})\right).$$

More explicitly,

$$\mathcal{M} \wedge_{\mathcal{R}} \mathcal{N} \coloneqq B\left(\mathcal{A}_{(2)}, \underline{\mathcal{A}}_{(1,1)}, (\mathcal{R}, \mathcal{M}, \mathcal{N})\right). \tag{4.26}$$

Again, while $\mathcal{M} \wedge_{\mathcal{R}} \mathcal{N}$ is not a module over the original ring $\mathcal{R}$, the ring $B(\mathcal{A}, \mathcal{A}, \mathcal{R})$ is nonetheless canonically weakly equivalent to $\mathcal{R}$, and thus, as we shall see next, the derived categories of modules over $\mathcal{R}$ and $B(\mathcal{A}, \mathcal{A}, \mathcal{R})$, respectively, are canonically equivalent.

4.2.1 Derived categories of rings and modules

We will again construct the derived category of $\mathcal{A}$-algebras (resp. $\underline{\mathcal{A}}_{(1)}$-algebras) with respect to weak equivalences using the techniques of fibrant approximation and cell approximation.

First recall from 2.3.3 that the category $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ of simplicial Kan spectral sheaves on $\mathscr{X}$ has a localization (2.20), with respect to stalk-wise total equivalences.

Proposition IV.59. *If* $\mathcal{F}$ *is an* $\mathcal{A}$*-algebra, then so is* $\operatorname{Gode} \mathcal{F}$*.*

Proof. First observe that if $\mathcal{F}$ is an $\mathcal{A}$-algebra in $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$, then for any $x \in \mathscr{X}$, the stalk $\mathcal{F}_x$ is an $\mathcal{A}$-algebra in $\Delta^{\mathrm{op}}\text{-}\mathscr{S}$. Indeed, the sheaf $\mathcal{A}\mathcal{F}$ has the same stalk as the presheaf $U \mapsto \mathcal{A}(\mathcal{F}(U))$, so we have a natural morphism

$$\mathcal{A}\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{A}\mathcal{F}(U) \longrightarrow \varinjlim_{U \ni x} \mathcal{F}(U) = \mathcal{F}_x,$$

using that each section $\mathcal{F}(U)$ of U is an $\mathcal{A}$-algebra in Δ^{op}-$\mathscr{S}$.

Now recall from the Godement resolution $\operatorname{Gode}\mathcal{F}$ is the realization of the cosimplicial object $\operatorname{Gode}^n \mathcal{F} = \mathbf{T}^{n+1}\mathcal{F}$, where

$$(\mathbf{T}\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x.$$

Since each $\mathcal{F}_x$ is an $\mathcal{A}$-algebra, it follows that there is a natural morphism

$$\mathcal{A}(\mathbf{T}\mathcal{F}) \longrightarrow \mathbf{T}\mathcal{F}$$

which is section-wise given by

$$\mathcal{A}\left(\prod_{x \in U} \mathcal{F}_x\right) \longrightarrow \prod_{x \in U} \mathcal{A}(\mathcal{F}_x) \longrightarrow \prod_{x \in U} \mathcal{F}_x = \mathbf{T}\mathcal{F}.$$

By induction, we conclude that each $\operatorname{Gode}^n \mathcal{F}$ is itself an $\mathcal{A}$-algebra.

It remains to show that, for any cosimplicial $\mathcal{G}^\bullet$ in Δ^{op}-$\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$, there is a canonical morphism

$$\mathcal{A}\left|\mathcal{G}^\bullet\right| \longrightarrow \left|\mathcal{A}\mathcal{G}^\bullet\right|,$$

for then the desired structure map of the $\mathcal{A}$-algebra $\operatorname{Gode}\mathcal{F}$ is the composition

$$\mathcal{A}\operatorname{Gode}\mathcal{F} = \mathcal{A}\left|\operatorname{Gode}^\bullet \mathcal{F}\right| \longrightarrow \left|\mathcal{A}\operatorname{Gode}^\bullet \mathcal{F}\right| \longrightarrow \left|\operatorname{Gode}^\bullet \mathcal{F}\right| = \operatorname{Gode}\mathcal{F}.$$

Recall that the realization of a cosimplicial sheaf is section-wise realization of the cosimplicial objects, while the functor $\mathcal{A}$ on sheaves are defined section-wise followed by sheafification, it suffices to show that there is a natural morphism

$$\mathcal{A}\left|Z^\bullet\right| \longrightarrow \left|\mathcal{A}Z^\bullet\right|$$

for each cosimplicial-simplicial Kan spectrum $Z^\bullet$, where $\mathcal{A}Z^\bullet$ is $\mathcal{A}$ applied to each cosimplicial degree of $Z^\bullet$, and the realization functor for cosimplicial Kan spectra is given in (2.12). Now the desired natural morphism above readily follows from the universal property of the equalizer. $\square$

By a similar argument, one can show that

Proposition IV.60. *If* $(\mathcal{F}, \mathcal{G})$ *is an* $\underline{\mathcal{A}}_{(1)}$*-algebra, then so is* $(\operatorname{Gode}\mathcal{F}, \operatorname{Gode}\mathcal{G})$. $\square$

Combining Propositions IV.38 and IV.59, we have:

Corollary IV.61. *The category of* E_∞*-ring Kan spectral sheaves has a fibrant approximation*

$$\mathrm{Id} \longrightarrow \operatorname{Gode}\operatorname{Sing}_\bullet \|\mathscr{L}\|.$$

Moreover, the category of $\underline{\mathcal{A}}_{(1)}$*-algebras in simplicial Kan spectral sheaves also has a fibrant approximation by applying the above coordinate-wise.* $\square$

For cell approximations of $\mathcal{A}$-algebras in Kan spectral sheaves, we define a *cell* to be the free $\mathcal{A}$-algebra functor applied to a cell (2.19) in the category of simplicial Kan spectral sheaves. This leads to, in the usual way, the notions of relative cell $\mathcal{A}$-algebra maps and cell $\mathcal{A}$-algebras.

By arguments similar to Propositions IV.41 and IV.42, and Theorem IV.43, we have the following analogous results:

Proposition IV.62. *For any* $\mathcal{A}$*-algebra* $\mathcal{F}$ *in* $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$*, there exists a cell* $\mathcal{A}$*-algebra* $Q\mathcal{F}$ *and a stalk-wise total equivalence* $Q\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ *of the underlying simplicial Kan spectral sheaves. Moreover, the assignment* $\mathcal{F} \mapsto Q\mathcal{F}$ *is functorial, and the total equivalences form a natural transformation* $Q \to \mathrm{Id}$. $\square$

Proposition IV.63. *The cell $\mathcal{A}$-algebras in $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ are colocal with respect to stalk-wise total equivalences between $\mathcal{A}$-algebras whose underlying spectral sheaves are of the form* $\operatorname{Gode}\operatorname{Sing}_\bullet \|\mathscr{L}\|(\mathcal{F})$. □

Theorem IV.64. *The derived category of $\mathcal{A}$-algebras in $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ exists, and is equivalent to the category whose objects are $\mathcal{A}$-algebras, and whose morphisms between $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{A}$-algebra maps from the cell $\mathcal{A}$-algebra $Q\mathcal{F}$ to the CE-fibrant $\mathcal{A}$-algebra* $\operatorname{Gode}\operatorname{Sing}_\bullet \|\mathscr{L}\|(\mathcal{G})$. □

The notions of fibrant and cell algebras, and coordinate-wise weak equivalences can also be introduced for $\underline{\mathcal{A}}_{(1)}$-algebras, and by a similar argument we obtain the derived category $D\mathsf{Mod}(\mathscr{X})$ of $\underline{\mathcal{A}}_{(1)}$-algebras in $\Delta^{\mathrm{op}}\text{-}\mathsf{Sh}_{\mathscr{S}}(\mathscr{X})$ with respect to coordinate-wise stalk-wise weak equivalences.

Definition IV.65. Let $\mathcal{R}$ be an E_∞-ring Kan spectral sheaf on $\mathscr{X}$. The *derived category of $\mathcal{R}$-modules* $D(\mathcal{R})$ is the full subcategory of $D\mathsf{Mod}(\mathscr{X})$ consisting of pairs $(\mathcal{R}, \mathcal{M})$ with the first coordinate fixed as $\mathcal{R}$, and the second coordinate an $\mathcal{R}$-module $\mathcal{M}$.

Analogously to Lemma IV.46, we have

Lemma IV.66. *If $\mathcal{R}$ and $\mathcal{R}'$ are stalk-wise weakly equivalent, then there is a canonical equivalence of categories between $D(\mathcal{R})$ and $D(\mathcal{R}')$.* □

As a consequence, we conclude the following result analogous to the case of spectra.

Corollary IV.67. *For any E_∞-ring Kan spectral sheaf $\mathcal{R}$ on $\mathscr{X}$, the formula* (4.26) *yields a well-defined smash product*

$$? \wedge_R^L ? : D(\mathcal{R}) \times D(\mathcal{R}) \longrightarrow D(\mathcal{R})$$

defined by

$$\mathcal{M} \wedge^L_{\mathcal{R}} \mathcal{N} := Q\mathcal{M} \wedge_{\mathcal{R}} Q\mathcal{N},$$

where $Q\mathcal{M} \to \mathcal{M}$ *and* $Q\mathcal{N} \to \mathcal{N}$ *are cell* $\mathcal{R}$*-algebra approximations.* □

4.2.2 Tensor triangulated structure

In this section, we will prove that for an E_∞-ring Kan spectral sheaf $\mathcal{R}$ on $\mathscr{X}$, the category $D(\mathcal{R})$, equipped with the smash product $\wedge^L_{\mathcal{R}}$, is a tensor triangulated category in the sense of [2].

We first observe that the argument in Section 4.1.6 can be easily generalized to sheaves with no change. Therefore, we have the following:

Theorem IV.68. *The category* $(D(\mathcal{R}), \wedge^L_{\mathcal{R}}, \mathcal{R})$ *is a symmetric monoidal category.* □

We next construct the triangulated structure on the category $D(\mathcal{R})$. The cofiber in simplicial Kan spectral sheaves is defined as the pushout

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G} \arrow[d] \\ I_+ \star \mathcal{F} \arrow[r] & \mathcal{C}f \end{tikzcd}$$

where the functor $\star$ is constructed section-wise followed by sheafification. Equivalently, $\mathcal{C}f$ is the sheafification of the presheaf which sends an open set $U \subseteq \mathscr{X}$ to the simplicial Kan spectrum $C(f(U))$. Consequently, the suspension $\Sigma\mathcal{F}$ of a simplicial Kan spectral sheaf $\mathcal{F}$ is the section-wise suspension followed by sheafification.

Following the argument for the analogous statements in the case of modules over an E_∞-ring Kan spectrum, we obtain the following results.

Proposition IV.69. *If* $f : \mathcal{F} \to \mathcal{G}$ *is a morphism of* $\mathcal{R}$*-modules, then* Cf *is an* $\mathcal{R}$*-module. In particular, if* $\mathcal{F}$ *is an* $\mathcal{R}$*-module, then so is* $\Sigma\mathcal{F}$*.* □

Theorem IV.70. *The category $D(\mathcal{R})$ has a triangulated structure where the translation functor is the derived suspension functor $L\Sigma$, and a triangle*

$$\mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P} \xrightarrow{\Sigma}$$

is distinguished if it is isomorphic to a some cofiber sequence

$$\mathcal{M}' \xrightarrow{f} \mathcal{N}' \longrightarrow \mathcal{C}f \xrightarrow{\Sigma} . \qquad \square$$

Theorem IV.71. *The category $D(\mathcal{R})$ with symmetric monoidal structure as in Theorem IV.68 and triangulated structure as in Theorem IV.70 is a tensor triangulated category in the sense of [2].* $\square$

APPENDIX

APPENDIX A

Derived Categories and Derived Functors via (Co)localization

A.1 Derived categories and derived functors

Let $\mathscr{C}$ be a category, and $\mathscr{E} \subset \operatorname{Mor}\mathscr{C}$ a class of morphisms in $\mathscr{C}$, called *equivalences.*

We review the definitions of derived categories and (right) derived functors.

Definition A.1. The (*strict*) *derived category* $D\mathscr{C} \coloneqq \mathscr{E}^{-1}\mathscr{C}$ of $\mathscr{C}$ with respect to $\mathscr{E}$, if one exists, is a category with a functor

$$\Phi : \mathscr{C} \longrightarrow D\mathscr{C}$$

which sends $\mathscr{E}$ to the class $\operatorname{Iso}(D\mathscr{C})$ of isomorphisms in $D\mathscr{C}$, such that for any functor $F : \mathscr{C} \to \mathscr{D}$ which sends $\mathscr{E}$ to $\operatorname{Iso}(\mathscr{D})$, there is a *unique* functor $F' : D\mathscr{C} \to \mathscr{D}$ such that $F = F' \circ \Phi$, that is, the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\Phi"'] & \mathscr{D} \\ D\mathscr{C} \arrow[ur, dashed, "\exists! F'"'] & \end{tikzcd}$$

of categories and functors commute strictly.

As usual, a strict derived category, if exists, is unique up to *isomorphism* of categories.

Remark A.2. From the 2-categorical point of view, the requirement of strict commutativity seems very restrictive. In fact, we could define derived categories in the 2-categorical sense by relaxing to commutativity up to unique natural isomorphism. Then, a 2-categorical derived category, if exists, is unique up to *equivalence* of categories.

In this thesis, all derived categories considered are strict. We remark that a derived category in the strict sense is a derived category in the 2-categorical sense, thus any 2-categorical derived category is naturally equivalent to a strict one.

Lemma A.3. *If a derived category $D\mathscr{C}$ of $\mathscr{C}$ exists, then the functor $\Phi : \mathscr{C} \to D\mathscr{C}$ is a bijection on objects.*

Proof. We define a category $\mathscr{D}$ as follows: $\operatorname{Obj}\mathscr{D}$ is the same as $\operatorname{Obj}\mathscr{C}$, and for any $x, y \in \operatorname{Obj}\mathscr{D}$, we define $\operatorname{Hom}_{\mathscr{D}}(x, y) \coloneqq \operatorname{Hom}_{D\mathscr{C}}(\Phi x, \Phi y)$. There is a functor $\Psi : \mathscr{C} \to \mathscr{D}$ which is identity on objects and Φ on morphisms. Then the functor Φ factorizes into

$$\mathscr{C} \xrightarrow{\Psi} \mathscr{D} \xrightarrow{\Gamma} D\mathscr{C},$$

where Γ is Φ on objects and identity on morphisms. Clearly Ψ makes $\mathscr{D}$ a derived category of $\mathscr{C}$, thus Γ is in fact an isomorphism of categories. $\square$

Therefore, without loss of generality, we can always choose $D\mathscr{C}$ in such a way that it has the same objects as $\mathscr{C}$.

Definition A.4. A functor $F : \mathscr{C} \to \mathscr{D}$ is *right-derivable* if the *left* Kan extension $\operatorname{Lan}_{\Phi} F$ of F along $\Phi : \mathscr{C} \to D\mathscr{C}$ exists. In this case, the functor $\operatorname{Lan}_{\Phi} F$ is called the (*total*) *right derived functor* and denoted RF.

More explicitly, there exists a natural transformation

$$\eta : F \longrightarrow RF \circ \Phi,$$

and for any functor $G : D\mathscr{C} \to \mathscr{D}$ and any natural transformation $\kappa : F \longrightarrow G \circ \Phi$, there exists a *unique* natural transformation $\lambda : RF \to G$ such that $(\lambda\Phi) \circ \eta = \kappa$.

If F already sends $\mathscr{E}$ to $\mathrm{Iso}(\mathscr{D})$, then we see from the definition of $D\mathscr{C}$ that $RF = F$.

A.2 Localization

Definition A.5. Let $\mathscr{C}$ be a category, $\mathscr{E} \subseteq \mathrm{Mor}\,\mathscr{C}$ a class of morphisms in $\mathscr{C}$.

- An *object* $z \in \mathscr{C}$ is *local* with respect to $\mathscr{E}$ if, for any morphism $f : x \to y$ in $\mathscr{E}$, the map

$$\mathrm{Hom}_{\mathscr{C}}(f, z) : \mathrm{Hom}_{\mathscr{C}}(y, z) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(x, z)$$

  is a bijection.

- A *localization* of a category $\mathscr{C}$ with respect to $\mathscr{E}$ and consists of an endofunctor $(?)' : \mathscr{C} \to \mathscr{C}$ and a natural transformation $\gamma : \mathrm{Id} \to (?)'$ such that for any $x \in \mathrm{Obj}\,\mathscr{C}$, the object x' is local with respect to $\mathscr{E}$, and the morphism $\gamma_x : x \to x'$ in $\mathscr{E}$.

In other words, an object is local if it satisfies a version of "co-Whitehead theorem" with respect to "equivalences" $\mathscr{E}$, and the localization is an approximation by local objects "from the right", or a "local resolution".

Lemma A.6. *If $f : x \to y$ is a morphism in $\mathscr{E}$ with both x and y local, then f is an isomorphism.*

Proof. Locality of x implies that there is a bijection

$$\mathrm{Hom}_{\mathscr{C}}(f,x) : \mathrm{Hom}_{\mathscr{C}}(y,x) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(x,x).$$

Let $g : y \to x$ be the pre-image of id_x, that is $g \circ f = \mathrm{id}_x$. Now locality of y implies that there is a bijection

$$\mathrm{Hom}_{\mathscr{C}}(f,y) : \mathrm{Hom}_{\mathscr{C}}(y,y) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(x,y).$$

Note $(f \circ g) \circ f = f \circ (g \circ f) = f = \mathrm{id}_y \circ f$, so both $f \circ g$ and id_y are mapped to f; it then follows that $f \circ g = \mathrm{id}_y$. □

Theorem A.7. *If $\mathscr{C}$ has a localization with respect to $\mathscr{E}$, then the derived category $D\mathscr{C} = \mathscr{E}^{-1}\mathscr{C}$ exists. Up to isomorphism, the category $D\mathscr{C}$ can be described as follows:*

- $\mathrm{Obj}\, D\mathscr{C} \coloneqq \mathrm{Obj}\,\mathscr{C}$;
- *for any $x, y \in \mathrm{Obj}\,\mathscr{C}$, we have*

$$\mathrm{Hom}_{D\mathscr{C}}(x,y) \coloneqq \mathrm{Hom}_{\mathscr{C}}(x',y').$$

Moreover, the canonical functor $\Phi : \mathscr{C} \to D\mathscr{C}$ is given by sending an object x to x' and a morphism f to f'.

Proof. We will verify that Φ satisfies the universal property in Definition A.1. First of all, Φ sends $\mathscr{E}$ to $\mathrm{Iso}(\mathscr{D})$ by Lemma A.6.

Now let $F : \mathscr{C} \to \mathscr{D}$ be a functor sending $\mathscr{E}$ to $\mathrm{Iso}(\mathscr{D})$, and we will construct a unique functor $F' : D\mathscr{C} \to \mathscr{D}$ with $F = F' \circ \Phi$. Consider any morphism $f \in \mathrm{Mor}\,\mathscr{C}$, then we must put

$$F'(f') \coloneqq F'(\Phi(f)) = F(f). \tag{A.1}$$

Next, for any object x of $\mathscr{C}$, we have commutative diagrams

$$\begin{tikzcd} x \arrow[r, "\gamma_x", "\sim"'] \arrow[d, "\gamma_x"', "\sim"] & x' \arrow[d, "\sim"', "\gamma_{(x')}"] \\ x' \arrow[r, "\sim", "(\gamma_x)'"'] & x'' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} F'(x) \arrow[r, "F'(\gamma_x)", "\cong"'] \arrow[d, "F'(\gamma_x)"', "\cong"] & F'(x') \arrow[d, "\cong"', "F'(\gamma_{(x')})"] \\ F'(x') \arrow[r, "\cong", "F'((\gamma_x)')"'] & F'(x'') \end{tikzcd}$$

so we must have

$$F'\left(\gamma_{(x')}\right) \coloneqq F'\left((\gamma_x)'\right) = F(\gamma_x). \tag{A.2}$$

Now let $g \in \operatorname{Hom}_{D\mathscr{C}}(x, y) \coloneqq \operatorname{Hom}_{\mathscr{C}}(x', y')$ be an arbitrary morphism. We have commutative diagrams

$$\begin{tikzcd} x' \arrow[r, "g"] \arrow[d, "\gamma_{(x')}"', "\sim"] & y' \arrow[d, "\sim"', "\gamma_{(y')}"] \\ x'' \arrow[r, "g'"'] & y'' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} F'(x') \arrow[r, "F'(g)"] \arrow[d, "F'(\gamma_{(x')})"', "\cong"] & F'(y') \arrow[d, "\cong"', "F'(\gamma_{(y')})"] \\ F'(x'') \arrow[r, "F'(g')"'] & F'(y'') \end{tikzcd}$$

which forces

$$F'(g) \coloneqq F'(\gamma_{(y')})^{-1} \circ F'(g') \circ F(\gamma_{(x')}) = F(\gamma_y)^{-1} \circ F(g) \circ F(\gamma_x). \tag{A.3}$$

We must also check the consistency between (A.3) and (A.1), that is, we must show that

$$F(\gamma_y)^{-1} \circ F(f') \circ F(\gamma_x) = F(f).$$

This follows from the commutative diagrams

$$\begin{tikzcd} x \arrow[r, "f", "\sim"'] \arrow[d, "\gamma_x"', "\sim"] & y \arrow[d, "\sim"', "\gamma_y"] \\ x' \arrow[r, "\sim", "f'"'] & y' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} F(x) \arrow[r, "F(f)", "\cong"'] \arrow[d, "F(\gamma_x)"', "\cong"] & F(y) \arrow[d, "\cong"', "F(\gamma_y)"] \\ F(x') \arrow[r, "\cong", "F(f')"'] & F(y') \end{tikzcd}$$

In particular, taking $x = y$ and $g = \mathrm{id}_x$, we see that $F'(x) = F(x)$ for any object x of $\mathscr{C}$. One readily checks that this definition makes F' a functor, and uniqueness follows since the definition is forced. $\square$

Remark A.8. In the above construction of the derived category $D\mathscr{C}$, we indeed have a bijection

$$\mathrm{Hom}_{D\mathscr{C}}(x,y) \coloneqq \mathrm{Hom}_{\mathscr{C}}(x',y') \cong \mathrm{Hom}_{\mathscr{C}}(x,y)$$

since y' is local. Therefore, when computing the hom-set in the derived categories, it suffices to "resolve" only in the second coordinate.

Theorem A.9. *For any functor $F : \mathscr{C} \to \mathscr{D}$, the* right derived functor *RF exists, and it can be computed by*

$$RF = F(?)'.$$

Proof. We begin by constructing the natural transformation η. For each object x of $\mathscr{C}$, we put

$$\eta_x : F(x) \xrightarrow{F(\gamma_x)} F(x') \xrightarrow{F(\gamma_{x'})} F(x'') = RF(x') = (RF \circ \Phi)(x).$$

Naturality of η follows from the naturality of γ and functoriality of F.

It remains to check the desired universality of η. Let $G : D\mathscr{C} \to \mathscr{D}$ be any functor, and $\kappa : F \to G \circ \Phi$ any natural transformation. That is, for any $x \in \mathrm{Obj}\,\mathscr{C}$, there is a natural morphism

$$\kappa_x : F(x) \longrightarrow (G \circ \Phi)(x) = G(x'),$$

and we must construct a natural morphism

$$\lambda_x : RF(x) = F(x') \longrightarrow G(x),$$

such that for any $x \in \mathrm{Obj}\,\mathscr{C}$, we have

$$\kappa_x = (\lambda\Phi)_x \circ \eta_x = \lambda_{x'} \circ F(\gamma_{(x')}) \circ F(\gamma_x).$$

To this end, consider the diagram

$$\begin{tikzcd}
F(x) \arrow[d, "F(\gamma_x)"', "\cong"] \arrow[dr, "\kappa_x"] & G(x) \arrow[d, "G(\gamma_x)", "\cong"'] \\
F(x') \arrow[ur, dashed, "\lambda_x"] \arrow[d, "F(\gamma_{(x')})"', "\cong"] \arrow[dr, "\kappa_{x'}"'] & G(x') \arrow[d, "G(\gamma_{(x')})", "\cong"'] \\
F(x'') \arrow[ur, dashed, "\lambda_{x'}"] & G(x'')
\end{tikzcd}$$

so we must put

$$\lambda_{x'} \coloneqq \kappa_x \circ F(\gamma_x)^{-1} \circ F(\gamma_{x'})^{-1}.$$

Moreover, the naturality assumption forces

$$\lambda_x \coloneqq G(\gamma_x)^{-1} \circ \lambda_{x'} \circ F(\gamma_{x'}) = G(\gamma_x)^{-1} \circ \kappa_x \circ F(\gamma_x)^{-1}.$$

Naturality of λ follows from the naturality of κ and functoriality of F and G. The equality $(\lambda\Phi) \circ \eta = \kappa$ is readily checked, and the uniqueness of λ follows since the definition is forced. $\square$

We end with a useful lemma concerning adjoint functors and their derived functors.

Lemma A.10. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories, with equivalences $\mathscr{E} \subset \operatorname{Mor}\mathscr{C}$ and $\mathscr{E}' \subset \operatorname{Mor}\mathscr{C}'$. Let*

$$F : \mathscr{C} \iff \mathscr{D} : G$$

be an adjunction. Suppose F preserves weak equivalences, so it induces to a functor $D\mathscr{C} \to D\mathscr{C}'$ which we again denote by F. Then:

(1) G preserves local objects.

(2) There is an adjunction

$$F : D\mathscr{C} \iff D\mathscr{C}' : RG.$$

Proof. For (1), let $z \in \operatorname{Obj} \mathscr{C}'$ be a local object, and let $x \to y$ be a morphism in $\mathscr{E}$, so $F(x) \to F(y)$ is in $\mathscr{E}'$. Then there are bijections

$$\operatorname{Hom}_{\mathscr{C}'}(x, G(z)) \cong \operatorname{Hom}_{\mathscr{C}}(F(x), z) \cong \operatorname{Hom}_{\mathscr{C}}(F(y), z) \cong \operatorname{Hom}_{\mathscr{C}'}(y, G(z)).$$

This proves the locality of $G(z)$. For (2), consider any $x \in \operatorname{Obj} \mathscr{C}$ and $z \in \operatorname{Obj} \mathscr{C}'$. Then z' is a local object, therefore so is $G(z')$ by (1). Hence, there are bijections

$$\begin{aligned} \operatorname{Hom}_{D\mathscr{C}}(F(x), z) &\cong \operatorname{Hom}_{\mathscr{C}}(F(x), z') \\ &\cong \operatorname{Hom}_{\mathscr{C}}(x, G(z')) \cong \operatorname{Hom}_{D\mathscr{C}'}(x, RG(z)). \end{aligned}$$

This proves the desired adjunction. $\square$

A.3 Localization and colocalization

In this section, we treat the case of a category with equivalences, which has both a "localization" and a "colocalization" in some sense, and we seek an explicit description of its derived category.

More precisely, let $\mathscr{C}$ be a category with equivalences $\mathscr{E}$, and suppose there are the following data:

- full subcategories $\mathscr{C}_L$ and $\mathscr{C}_R$ of $\mathscr{C}$,
- functors $(?)_L : \mathscr{C} \to \mathscr{C}_L$ and $(?)_R : \mathscr{C} \to \mathscr{C}_R$, and
- natural transformations $(?)_L \xrightarrow{\lambda} \mathrm{Id} \xrightarrow{\rho} (?)_R$, all of whose components lie in $\mathscr{E}$.

which satisfy the following axioms:

1. the objects of $\mathscr{C}_L$ are *colocal* with respect to morphisms in $\mathscr{E} \cap \operatorname{Mor} \mathscr{C}_R$, and
2. the objects of $\mathscr{C}_R$ are *local* with respect to morphisms in $\mathscr{E} \cap \operatorname{Mor} \mathscr{C}_L$.

With these data, we propose to define a category $\mathscr{D}$ as follows:

- $\operatorname{Obj}\mathscr{D} = \operatorname{Obj}\mathscr{C}$;
- for $x, y \in \operatorname{Obj}\mathscr{C}$, we have$\operatorname{Hom}_{\mathscr{D}}(x, y) = \operatorname{Hom}_{\mathscr{C}}(x_L, y_R)$.

It follows from the two-of-three property of $\mathscr{E}$ that both $(?)_L$ and $(?)_R$ sends $\mathscr{E}$ to $\mathscr{E}$.

Proposition A.11. *The classes* $\operatorname{Obj}\mathscr{D}$ *and* $\operatorname{Mor}\mathscr{D} \coloneqq \{\operatorname{Hom}_{\mathscr{D}}(x, y) | x, y \in \operatorname{Obj}\mathscr{D}\}$ *specified above form a category* $\mathscr{D}$*.*

Proof. We first declare that an identity morphism id_x in $\operatorname{Hom}_{\mathscr{D}}(x, x)$ is represented by the morphism $\rho_x \circ \lambda_x$ in $\operatorname{Hom}_{\mathscr{C}}(x_L, x_R)$.

Next we define the composition

$$\circ : \operatorname{Hom}_{\mathscr{D}}(y, z) \times \operatorname{Hom}_{\mathscr{D}}(y, z) \longrightarrow \operatorname{Hom}_{\mathscr{D}}(x, z)$$

for any objects $x, y, z \in \operatorname{Obj}\mathscr{D}$. Suppose given morphisms $[f] \in \operatorname{Hom}_{\mathscr{D}}(x, y)$ and $[g] \in \operatorname{Hom}_{\mathscr{D}}(y, z)$ be represented by morphisms $f : x_L \to y_R$ and $g : y_L \to z_R$ in $\mathscr{C}$. Then we have a commutative diagram

(A.4)
$$\begin{tikzcd}
x_L \arrow[d, "\sim"'] \arrow[drr, dashed, "\widetilde{f}"] \arrow[ddrrr, bend right, "f"'] \arrow[ddrrrrrr, dotted, "\overline{gf}"] & & & y_L \arrow[dl, "\rho_{(y_L)}"', "\sim"] \arrow[d, "\sim"] \arrow[ddrrr, bend left, "g"] & & & \\
x & & (y_L)_R \arrow[dr, "(\lambda_y)_R"', "\sim"] \arrow[ddrrr, bend left, "g_R"] & y \arrow[d, "\sim"] & & & z \arrow[d, "\sim"] \\
& & & y_R & & & z_R \arrow[dl, "(\rho_z)_R", "\sim"'] \\
& & & & & (z_R)_R &
\end{tikzcd}$$

where we decorate by $\sim$ morphisms in $\mathscr{E}$, and $\widetilde{f}$ is the unique lift of f along $(\lambda_y)_R$, whereas $\widetilde{gf}$ is the unique lift of $g_R \circ \widetilde{f}$ along $(\rho_z)_R$. We define $[g] \circ [f]$ to be the morphism in $\operatorname{Hom}_{\mathscr{D}}(x, z)$ represented by $\overline{gf} \in \operatorname{Hom}_{\mathscr{C}}(x_L, z_R)$.

To check the associativity axiom, comparing the commutative diagrams

(A.5)

$$\begin{tikzcd}
x_L \arrow[d, "\sim"'] \arrow[dr, dashed, "\widetilde{f}"] \arrow[ddrr, bend right, "f"'] \arrow[drrr, dotted, bend left, "\widetilde{gf}"] \arrow[ddrrrr, dotted, bend left, "\overline{gf}"] \arrow[ddrrrrrr, squiggly, bend left, "\overline{h(gf)}"] & & y_L \arrow[dl] \arrow[d] \arrow[ddrr, bend left, "g"] & & z_L \arrow[dl] \arrow[d] \arrow[ddrr, bend right, "h"'] & & \\
x & (y_L)_R \arrow[dr, "\sim"'] \arrow[ddrr, bend left, "g_R"'] & y \arrow[d] & (z_L)_R \arrow[dr, "\sim"] \arrow[ddrr, bend right, "h_R"'] & z \arrow[d] & & w \arrow[d, "\sim"] \\
& & y_R & & z_R \arrow[dl, "\sim"] & & w_R \arrow[dl, "\sim"] \\
& & & (z_R)_R & & (w_R)_R &
\end{tikzcd}$$

(where $\widetilde{gf}$ is the unique lift of $\overline{gf}$ along $(\lambda_z)_R$, and $\overline{h(gf)}$ is the unique lift of $h_R \circ \widetilde{gf}$ along $(\rho_w)_R$) and

(A.6)

$$\begin{tikzcd}
x_L \arrow[d, "\sim"'] \arrow[dr, dashed, "\widetilde{f}"] \arrow[ddrr, bend right, "f"'] \arrow[ddrrrrrr, squiggly, bend left, "\overline{(hg)f}"] & & y_L \arrow[dl] \arrow[d] \arrow[dr, dashed, "\widetilde{g}"'] \arrow[ddrr, bend left, "g"] \arrow[ddrrrr, dotted, bend left, "\overline{hg}"] & & z_L \arrow[dl] \arrow[d] \arrow[ddrr, bend right, "h"'] & & \\
x & (y_L)_R \arrow[dr, "\sim"'] \arrow[ddrr, bend left, "g_R"'] \arrow[ddrrrr, dotted, bend left, "(\overline{hg})_R"] & y \arrow[d] & (z_L)_R \arrow[dr, "\sim"] \arrow[ddrr, bend right, "h_R"'] & z \arrow[d] & & w \arrow[d, "\sim"] \\
& & y_R & & z_R \arrow[dl, "\sim"] & & w_R \arrow[dl, "\sim"] \\
& & & (z_R)_R & & (w_R)_R &
\end{tikzcd}$$

(where $\overline{(hg)f}$ is the unique lift of $(\overline{hg})_R \circ \widetilde{f}$ along $(\rho_w)_R$) gives that $\overline{h(gf)} = \overline{(hg)f}$. $\square$

Proposition A.12. *The assignment*

$$\begin{array}{rcccl}
\Phi & : & \mathrm{Hom}_{\mathscr{C}}(x,y) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(x,y) \cong \mathrm{Hom}_{\mathscr{C}}(x_L, y_R) \\
& & f & \longmapsto & \rho_y \circ f \circ \lambda_x
\end{array}$$

forms a functor $\Phi : \mathscr{C} \to \mathscr{D}$.

Proof. Φ sends identities to identities by definition. To see that it preserves composition, consider the commutative diagram

$$
\begin{tikzcd}
x_L \arrow[rr, "f_L", "\sim"'] \arrow[d, "\sim"'] \arrow[dr, dashed, "\widetilde{f}"'] & & y_L \arrow[d, "\sim"] \arrow[dl, "\rho_{(y_L)}"'] & & \\
x \arrow[rr, "f"] & & y \arrow[rr, "g"] \arrow[d, "\sim"] & & z \arrow[d, "\sim"] \\
& (y_L)_R \arrow[r, "\sim", "(\lambda_y)_R"'] \arrow[rr, bend right, "g_R"', shift right=8] & y_R \arrow[rr, "g_R", "\sim"'] & & z_R \arrow[dl, "\sim"', "(\rho_z)_R"] \\
& & & (z_R)_R &
\end{tikzcd}
\tag{A.7}
$$

and note that $\rho_y \circ \widetilde{f \circ \lambda_x} = \rho_{(y_L)} \circ f_L$ and thus

$$
\begin{aligned}
(\rho_z \circ g \circ \lambda_y)_R \circ (\rho_y \circ \widetilde{f \circ \lambda_x}) &= (\rho_z \circ g \circ \lambda_y)_R \circ \rho_{(y_L)} \circ f_L \\
&= (\rho_z)_R \circ g_R \circ (\lambda_y)_R \circ \rho_{(y_L)} \circ f_L \\
&= (\rho_z)_R \circ g_R \circ \rho_y \circ \lambda_y \circ f_L \\
&= (\rho_z)_R \circ \rho_z \circ g \circ f \circ \lambda_x.
\end{aligned}
$$

The uniqueness of lifting guarantees that the composition of $\Phi(f)$ and $\Phi(g)$ defined above is precisely $\Phi(g \circ f)$. $\square$

Theorem A.13. *The functor*

$$\Phi : \mathscr{C} \longrightarrow \mathscr{D}$$

exhibits $\mathscr{D}$ as the derived category $D\mathscr{C} = \mathscr{E}^{-1}\mathscr{C}$ of $\mathscr{C}$ with respect to $\mathscr{E}$.

Proof. Again, we will verify that Φ satisfies the universal property in in Definition A.1. First of all, Φ sends $\mathscr{E}$ to $\mathrm{Iso}(\mathscr{D})$. Indeed, give a morphism $f : x \to y$ in $\mathscr{E}$, consider the commutative diagram

$$
\begin{tikzcd}
x_L \arrow[r, "f_L", "\sim"'] \arrow[d, "\lambda_x"', "\sim"] & y_L \arrow[d, "\sim"', "\lambda_y"] \arrow[ddl, dashed] \\
x \arrow[r, "f", "\sim"'] \arrow[d, "\rho_x"', "\sim"] & y \arrow[d, "\sim"', "\rho_y"] \\
x_R \arrow[r, "\sim", "f_R"'] & y_R
\end{tikzcd}
$$

where the dashed arrow is th unique lift of $\rho_y \circ \lambda_y$ along f_R, or equivalently the unique lift of $\rho_x \circ \lambda_x$ along f_L, thus making the entire diagram commutative. It is easy to see by the composition defined in (A.4) that this dashed arrow is the inverse to f in $\mathscr{D}$.

Next let $F : \mathscr{C} \to \mathscr{A}$ be a functor which sends $\mathcal{E}$ to $\mathrm{Iso}(\mathscr{A})$, and we will construct a unique functor $\overline{F} : \mathscr{D} \to \mathscr{A}$ such that $F = \overline{F} \circ \Phi$. For any object $x \in \mathscr{C}$, we are forced to put

$$\overline{F}(\rho_x \circ \lambda_x) =: \overline{F}(\Phi(\mathrm{id}_x)) = \mathrm{id}_{F(x)},$$

so

$$\overline{F}(\rho_x) = \overline{F}(\lambda_x)^{-1}. \tag{A.8}$$

Also for any morphism $f : x \to y$ in $\mathscr{C}$, we are forced to put

$$\overline{F}(\rho_y \circ f \circ \lambda_x) =: \overline{F}(\Phi(f)) = F(f). \tag{A.9}$$

Moreover, for any object $x \in \mathscr{C}$, we have commutative diagrams

$$\begin{tikzcd}
(x_L)_L \arrow[r, "(\lambda_x)_L"] \arrow[d, "\lambda_{(x_L)}"'] & x_L \arrow[r, "(\rho_x)_L"] \arrow[d, "\lambda_x"'] & (x_R)_L \arrow[d, "\lambda_{(x_R)}"] \\
x_L \arrow[r, "\lambda_x"] \arrow[d, "\rho_{(x_L)}"'] & x \arrow[r, "\rho_x"] \arrow[d, "\rho_x"'] & x_R \arrow[d, "\rho_{(x_R)}"] \\
(x_L)_R \arrow[r, "(\lambda_x)_R"'] & x_R \arrow[r, "(\rho_x)_R"'] & (x_R)_R
\end{tikzcd}
\implies
\begin{tikzcd}
\overline{F}(x_L)_L \arrow[r, "\overline{F}(\lambda_x)_L"] \arrow[d, "\overline{F}\lambda_{(x_L)}"'] & \overline{F}x_L \arrow[r, "(\rho_x)_L"] \arrow[d, "\overline{F}\lambda_x"'] & \overline{F}(x_R)_L \arrow[d, "\overline{F}\lambda_{(x_R)}"] \\
\overline{F}x_L \arrow[r, "\overline{F}\lambda_x"] \arrow[d, "\overline{F}\rho_{(x_L)}"'] & \overline{F}x \arrow[r, "\overline{F}\rho_x"] \arrow[d, "\overline{F}\rho_x"'] & \overline{F}x_R \arrow[d, "\overline{F}\rho_{(x_R)}"] \\
\overline{F}(x_L)_R \arrow[r, "\overline{F}(\lambda_x)_R"'] & \overline{F}x_R \arrow[r, "\overline{F}(\rho_x)_R"'] & \overline{F}(x_R)_R
\end{tikzcd}$$

where the one on the left consists entirely of equivalences and the right of isomorphisms, so we are forced to put

$$\overline{F}\left(\lambda_{(x_L)}\right) = \overline{F}\left(\rho_x \circ \lambda_x \circ \lambda_{(x_L)}\right) =: \overline{F}\left(\Phi(\lambda_x)\right) = F(\lambda_x)$$

and

$$\overline{F}\left(\rho_{(x_R)}\right) = \overline{F}\left(\rho_{(x_R)} \circ \rho_x \circ \lambda_x\right) =: \overline{F}\left(\Phi(\rho_x)\right) = F(\rho_x).$$

Combining all these, we are forced to put, for any morphism $\overline{f} : x_L \to y_R$ in $\mathscr{C}$

$$\begin{aligned} F(\overline{f}) = \overline{F}\left(\Phi(\overline{f})\right) &= \overline{F}\left(\rho_{(y_R)} \circ \overline{f} \circ \lambda_{(x_L)}\right) \\ &= \overline{F}\left(\rho_{(y_R)}\right) \circ \overline{F}(\overline{f}) \circ \overline{F}\left(\lambda_{(x_L)}\right) \\ &= F(\rho_y) \circ \overline{F}(\overline{f}) \circ F(\lambda_x), \end{aligned}$$

which forces the definition

$$\overline{F}(\overline{f}) = F(\rho_y)^{-1} \circ F(\overline{f}) \circ F(\lambda_x)^{-1}. \tag{A.10}$$

It is clear that $\overline{F}$ defines a functor, hence this definition is compatible with (A.8), and the compatibility with (A.9) is easily checked. □

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BIBLIOGRAPHY

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