Functoriality and the Moduli of Sections, With Applications to Quasimaps

by

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To ψ, ϕ, and their cocreator

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Abstract

Motivated by Gromov-Witten theory, this thesis is about moduli of maps from curves to algebraic stacks, the obstruction theories of those moduli, and the functoriality of the stacks and their obstruction theories. The first part discusses the moduli of sections $\mathfrak{S}$ of a map $Z \to C$ from an artin stack Z to a family of twisted curves C over a base algebraic stack. The existence and basic properties of $\mathfrak{S}$ are due to Hall-Rydh; the new result in this thesis is that $\mathfrak{S}$ has a canonical obstruction theory (not necessarily perfect), generalizing known constructions on Deligne-Mumford substacks of $\mathfrak{S}$. We also work out basic functoriality properties of $\mathfrak{S}$ and its obstruction theory.

The second part proves an abelianization formula for the quasimap I-function. That is, if Z is an affine l.c.i. variety with an action by a complex reductive group G such that the quotient $Z/\!\!/_\theta G$ is a smooth projective variety, we relate the quasimap I-functions of $Z/\!\!/_\theta G$ and $Z/\!\!/_\theta T$ where T is a maximal torus of G. With the mirror theorems of Ciocane-Fontantine and Kim, this computes the genus-zero Gromov-Witten invariants of $Z/\!\!/_\theta G$ in good cases.

Chapter 1

Introduction

This thesis is about moduli of maps from complex curves to algebraic stacks, the obstruction theories of those moduli, and functoriality of these objects. The study of these objects is motivated by the quest to compute Gromov-Witten invariants.

1.0.1 What are Gromov-Witten invariants?

This section is written for the non-mathematician and it is not rigorous. I will dodge the question three times before telling you the actual answer. The dodges will tell you why both physicists and mathematicians are interested in computing Gromov-Witten invariants. The first dodge is snarky: Gromov-Witten invariants are numbers, like 12 or 27 or -64 or $2/3$. They are numbers associated to a specific *target*, or geometric space, usually a space with at least four real dimensions. Say we have a geometric space, call it X. The Gromov-Witten invariants of X are some numbers that tell me about how shapes like the ones in Figure 1.1 can fit inside X.

The second dodge is physics-y: Gromov-Witten invariants describe particle interactions in some model of space-time. The story begins with the long-standing quest of theoretical physicists to unify quantum physics and general relativity. One possible solution is string theory, which models fundamental particles with strings vibrating in space-time. These strings curl up in 6 extra dimensions that are present at every point in space-time (for a total of $6 + 4 = 10$ dimensions in our universe). The extra 6 dimensions are the geometric space X described above, and the shape of X determines how particles can interact in our universe. These particle interactions are geometrically modeled by the curves in Figure 1.1, and numerically they are equal to the Gromov-Witten invariants of X.

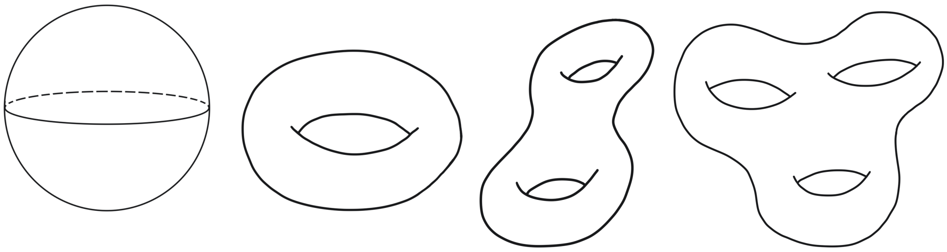


Figure 1.1: Some complex curves
These surfaces are called *complex curves* because they have one complex dimension (two real dimensions). The leftmost surface is just a (hollow) ball, and the one following is a (hollow) donut. The number of "holes" in the surface is its *genus*: from left to right, these curves have genus 0, 1, 2, and 3.

A priori, no one knows which X is the shape of our universe at its microscopic level. For a while, many people were wanting to compute Gromov-Witten invariants for lots of different X's, hoping to match the numbers for one special X to experimentally measured particle interactions. However, it is outside the reach of current science to measure the required particle interactions. Hence the physical interest in computing Gromov-Witten invariants is somewhat dated.

The last dodge is mathy: Gromov-Witten invariants are psuedo-answers to enumerative questions. An example of an *enumerative question* is the following:

How many points are contained in the intersection of two distinct lines in $\mathbb{C}^2$?

The answer to this particular question is, of course, either 1 (if the lines cross) or 0 (if they are parallel). One gets a "better" answer if one modifies the question:

How many points are contained in the intersection of two distinct lines in $\mathbb{P}^2$?

Now the answer is always 1: two parallel lines in $\mathbb{C}^2$ meet at infinity in $\mathbb{P}^2$.

Instead of counting points in specified intersections of lines, Gromov-Witten invariants count shapes like in Figure 1.1, called *curves*, containing certain points in X. For example, the answer to the question

$$\text{How many smooth conics in } \mathbb{P}^2 \text{ contain 5 generic points?} \tag{1.1}$$

is a Gromov-Witten invariant of $X = \mathbb{P}^2$ (you should picture the "smooth conic" here like the ball in Figure 1.1). But in general, Gromov-Witten invariants are not exactly equal to the answers to these curve-counting (or *enumerative*) questions. Indeed, these enumerative questions should always have positive whole numbers for answers, like 12 or 27, but Gromov-Witten invariants can be negative numbers or fractions. Instead, Gromov-Witten invariants answer a modified question whose answer is better behaved: I say the answer is better behaved because Gromov-Witten invariants exhibit a wealth of algebraic structures. These structures are interesting and beautiful in their own right, but they also make it possible to write down all the Gromov-Witten invariants of X (in many special but interesting cases) with one explicit formula, called a *generating series* (see (1.2) for an example of a generating series).

Finally, the actual answer: Gromov-Witten invariants are integrals on the *moduli of stable maps* from curves to X. The moduli space in question parametrizes functions from shapes like in Figure 1.1 to high-dimensional geometric spaces X. In symbols, a Gromov-Witten invariant is

$$\int_{[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} ev_1^*\gamma_1 \cup \ldots \cup ev_n^*\gamma_n$$

(where $\gamma_1, \ldots, \gamma_n$ are classes in $H^*(X, \mathbb{C})$). You should think of the notation $ev_1^*\gamma_1 \cup \ldots \cup ev_n^*\gamma_n$ as analogous to the "5 points" in the enumerative question (1.1). From my perspective, the interesting part is $[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$. In this notation, X is the same as $\mathbb{P}^2$ in the question (1.1), g is related to the word "smooth," β is related to the word "conic," and n is equal to 5 (the number of constraints in our counting problem). The notation $\mathcal{M}_{g,n}(X,\beta)$ is a highly abstracted kind of space called the *moduli of stable maps to* X. This space does not come with a way to do integrals; you have to define integration separately, and that choice of how to integrate is called the *virtual fundamental class*, notated $[\ldots]^{vir}$.

My thesis studies the spaces $\mathcal{M}_{g,n}(X,\beta)$ (and related ones called *moduli of quasimaps*) and their virtual fundamental classes. I derive some very general results about these moduli spaces, and then I use these results to compute explicit formulas for generating functions of Gromov-Witten invariants (see (1.2)).

1.0.2 Context for this thesis: quasimaps

Since the first mirror theorems of Givental [Giv98] and Lian-Liu-Yau [LLY97] in the 90s, there have been many approaches to computing Gromov-Witten invariants. The most relevant for this thesis is the strategy of *quasimaps to a GIT quotient* introduced by Ciocan-Fontanine, Kim, and Maulik in [CK10; CKM14] (building on ideas of Marian-Oprea-Panharipande in [MOP11]). The idea here is that if $Z/\!\!/_\theta G$ is a GIT quotient of an l.c.i. variety Z by a reductive group G, one should study not maps to $Z/\!\!/_\theta G$, but maps to the stack quotient $[Z/G]$. The latter maps are allowed to hit the unstable locus in Z, and they remember how they hit that locus—if $Z/\!\!/_\theta G$ is projective space, maps to the corresponding stack quotient are like rational maps, but they carry more information about the basepoints.

The moduli of all maps to $[Z/G]$ is very ill-behaved. It is an algebraic stack with infinite stabilizers, not in general separated or quasi-compact. But it contains the familiar Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}(Z/\!\!/_\theta G, \beta)$ as an open substack. It also contains other compactifications of maps from smooth source curves to $[Z/G]$; among these are the quasimap moduli spaces. The quasimap moduli spaces are indexed by a rational positive stability parameter ϵ and they stabilize for ϵ sufficiently small or large. When ϵ is large they recover $\overline{\mathcal{M}}_{g,n}(Z/\!\!/_\theta G, \beta)$. When ϵ is small, one hopes that they are easier to describe explicitly and hence easier to compute with. Wall-crossing theorems relate the quasimap invariants for changing values of ϵ and hence provide a strategy for computing Gromov-Witten invariants. The literature contains several wall-crossing theorems (the papers [CK14b; CK16; Zho19] are most relevant to this thesis), but relatively few computations of quasimap invariants for small ϵ.

The questions studied in this thesis aim to better understand the quasimap moduli spaces and compute quasimap invariants for small ϵ for a large class of targets.

1.0.3 Main results of this thesis

This thesis contains two parts. Aside from the fact that key results in the first part are used in the second, the two parts are independent. Each part contains a complete introduction to its contents. Here I will just informally summarize the results as they relate to the quasimap context explained above.

Part 1

The quasimap moduli spaces are certain moduli stacks of maps to the stack quotient $[Z/G]$. The first part of this thesis generalizes this setup slightly, replacing the moduli of maps to a stack X with the moduli of sections $\mathfrak{S}$ of a map $Y \to C$, where Y is an algebraic stack and C is a family of twisted curves over an algebraic stack U. (Note that the moduli of sections of $C \times X \to C$ recovers the moduli of maps to X.) The reason for using the moduli of sections is that the results developed here are also used to study the p-fields problem in my joint work [CJW19], but we will not discuss p-fields in this thesis. The quasimap moduli spaces are certain separated substacks of the Deligne-Mumford locus of $\mathfrak{S}$, but this thesis directly studies the entire (nonseparated, unbounded, algebraic) stack $\mathfrak{S}$. The main result is the following.

Theorem. *The algebraic stack $\mathfrak{S}$ has a canonical relative obstruction theory over U.*

By obstruction theory, I mean a morphism $\phi : E \to \mathbb{L}_{\mathfrak{S}/U}$ in the derived category of quasi-coherent sheaves on $\mathfrak{S}$, such that ϕ induces an isomorphism in cohomology in degrees 0 and 1 and a surjection in degree -1. The restriction of the canonical obstruction theory in the above Theorem recovers the standard obstruction theory on the quasimap moduli spaces when the base U is chosen to be the moduli of principal G-bundles on twisted curves. The main application of the above Theorem is that it also defines a *canonical* obstruction theory on the quasimap moduli spaces when the base U is just the moduli of twisted curves. (Previous constructions of an obstruction theory for $\mathfrak{S}$ relative to the moduli of twisted curves used a mapping cone construction, and hence were not canonical.) This is key for the computation in the second part.

The first part also proves some functoriality lemmas about $\mathfrak{S}$ and its canonical obstruction theory. This functoriality is also used in the second part of this thesis (as well as in [CJW19]). While these properties seem well-known, at least individually, the coherent presentation here seems quite useful in applications. As preparation for this functoriality and the Theorem above, I summarize the definitions and properties of quasicoherent sheaves on algebraic stacks, including their derived categories and also the cotangent complex. In addition, as a warmup to the proof of the Theorem above, this first part proves a mildly new result about the deformation theory of algebraic stacks.

Part 2

The second part deals directly with quasimap moduli spaces, though only with very special ones. The goal here is to actually compute the quasimap invariants for small ϵ, thereby completing the strategy outlined above for computing Gromov-Witten invariants. Previous to this thesis, the computation had been done for "abelian" targets $Z /\!\!/_\theta T$ for Z a vector space and T a torus, and a few "non-abelian" targets $Z /\!\!/_\theta G$ for non-abelian groups G (including all flavors of flag varieties). The invariants of these non-abelian examples demonstrated a phenomenon common to many contexts where it is possible to define invariants of a reductive group G: they were related to invariants of $Z /\!\!/_\theta T$ for a maximal torus $T \subset G$ via an *abelianization formula*. The second part of this thesis proves that formula in complete generality.

To state the abelianization formula, one encodes the quasimap invariants of $Z /\!\!/_\theta G$ (for small ϵ) in a generating function $I^{Z/\!\!/G}$ called the I-function. The coefficient $I_\beta^{Z/\!\!/G}$ of this generating series records invariants of degree β. The abelianization theorem may now be stated as follows.

Theorem (Rough statement)**.** *The I-functions of $Z /\!\!/_\theta G$ and $Z /\!\!/_\theta T$ satisfy*

$$I_\beta^{Z/\!\!/G} = \left[\sum_{\tilde{\beta} \to \beta} \left(\prod_\alpha \frac{\prod_{k=-\infty}^{\tilde{\beta}(\alpha)} (c_1(L_\alpha) + kz)}{\prod_{k=-\infty}^{0} (c_1(L_\alpha) + kz)} \right) I_{\tilde{\beta}}^{Z/\!\!/T}(z) \right] \tag{1.2}$$

where the sum is over all degrees $\tilde{\beta}$ mapping to β and the product ranges over all roots α of G.

The second part of the thesis proves this Theorem when Z admits a certain nice embedding into a vector space, including the case where Z is equal to a vector space. In this latter situation (Z is a vector space), an explicit formula for the quasimap I-function of $Z /\!\!/_\theta T$ is known, and in this case the above Theorem provides an explicit formula for the quasimap invariants of $Z /\!\!/_\theta G$. I include an example of using the Theorem to write down an I-function of a target whose invariants were previously unknown.

Chapter 2

Moduli of Sections

2.1 Introduction

This chapter is the product of my quest to understand what I consider the basic objects of Gromov-Witten theory: the moduli of prestable maps $\mathfrak{M}(\mathcal{X})$ to an algebraic stack target $\mathcal{X}$ and its obstruction theory. As explained in the introduction to this dissertation, I was led to consider the entire moduli of prestable maps because of questions arising from quasimap theory. However, this leads to many difficulties as both the target and resulting moduli are artin stacks. Fortunately, the theory of algebraic stacks is now developed enough to let us work with $\mathfrak{M}(\mathcal{X})$: for example, the stack $\mathfrak{M}(\mathcal{X})$ is shown to be algebraic and locally finite type in [HR19], and the theory of derived categories of quasicoherent sheaves on algebraic stacks, including a cotangent complex, is worked out in [HR17; LO08; Ols07a]. One could view this chapter as a routine check that $\mathfrak{M}(\mathcal{X})$ and its obstruction theory behave as we expect.

2.1.1 Moduli of sections and its obstruction theory

We recall the moduli of sections of an algebraic stack and its canonical obstruction theory. Consider a tower of algebraic stacks over $\mathbb{C}$

$$Z \to \mathcal{C} \xrightarrow{\pi} \mathfrak{U}$$

where $\pi : \mathcal{C} \to \mathfrak{U}$ is a flat finitely-presented family of connected, nodal, twisted curves (defined as in [AV02, Sec 4]) and $Z \to \mathfrak{U}$ is locally finitely presented, quasi-separated, and has affine stabilizers. We define the *moduli of sections* $\underline{\mathrm{Sec}}(Z/\mathcal{C})$ to be the stack whose fiber over $T \to \mathfrak{U}$ is $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \times_{\mathfrak{U}} T, Z)$. By [HR19, Thm 1.3], the stack $\underline{\mathrm{Sec}}(Z/\mathcal{C})$ is algebraic and the canonical morphism $\underline{\mathrm{Sec}}(Z/\mathcal{C}) \to \mathfrak{U}$ is locally finitely presented, quasi-separated, and has affine stabilizers. It has a universal curve $\mathcal{C}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})}$ and a universal section $n : \mathcal{C}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})} \to Z$.

Example 2.1.1. Let Y be an algebraic stack, locally finitely presented, quasi-separated, and with affine stabilizers over $\mathbb{C}$, and let $Y_{\mathfrak{U}}$ denote its base change to $\mathfrak{U}$. By [HR19, Thm 1.2] there is an algebraic stack $\underline{\mathrm{Hom}}_{\mathfrak{U}}(\mathcal{C}, Y_{\mathfrak{U}})$ whose fiber over $T \to \mathfrak{U}$ is $\mathrm{Hom}_{\mathfrak{U}}(\mathcal{C} \times_{\mathfrak{U}} T, Y_{\mathfrak{U}})$. On the other hand, we have the moduli of sections $\underline{\mathrm{Sec}}(\mathcal{C} \times_{\mathfrak{U}} Y_{\mathfrak{U}}/\mathcal{C})$. By Lemma 2.3.1, the projection $\mathcal{C} \times_{\mathfrak{U}} Y_{\mathfrak{U}} \to Y_{\mathfrak{U}}$ induces an isomorphism $\underline{\mathrm{Sec}}(\mathcal{C} \times_{\mathfrak{U}} Y_{\mathfrak{U}}/\mathcal{C}) \to \underline{\mathrm{Hom}}_{\mathfrak{U}}(\mathcal{C}, Y_{\mathfrak{U}})$

We now define a canonical construction for families of sections. If $X \to Y$ is a morphism of algebraic stacks, let $\mathbb{L}_{X/Y}$ denote the cotangent complex (see Section 2.2.3), and if $\mathcal{C} \to \mathfrak{U}$ is a family of curves, let $\omega^\bullet = \omega_{\mathcal{C}/\mathfrak{U}}[1]$ denote the dualizing object in the derived category, with trace map tr (see Section 2.2.2). To a diagram of algebraic stacks

$$\begin{array}{ccc} & & Z \\ & \overset{f}{\nearrow} & \downarrow \\ K & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ B & \longrightarrow & \mathfrak{U} \end{array}$$

where the square is fibered, we associate a morphism of complexes

$$\phi_{B/\mathfrak{U}} : \mathbb{E}_{B/\mathfrak{U}} := R\pi_*(f^*\mathbb{L}_{Z/\mathcal{C}} \otimes \omega^\bullet) \to \mathbb{L}_{B/\mathfrak{U}} \tag{2.1}$$

defined as follows. There is a morphism α in the derived category $\mathrm{D}_{\mathrm{qc}}(K)$ induced by the canonical arrows

$$Lf^*\mathbb{L}_{Z/\mathcal{C}} \to \mathbb{L}_{K/\mathcal{C}} \xleftarrow{\sim} \pi^*\mathbb{L}_{B/\mathfrak{U}} \tag{2.2}$$

where left-pointing arrow is an isomorphism since π is flat. There is also an arrow

$$\beta : R\pi_*(L\pi^*\mathbb{L}_{B/\mathfrak{U}} \otimes \omega^\bullet) \xleftarrow{\sim} \mathbb{L}_{B/\mathfrak{U}} \otimes R\pi_*\omega^\bullet \xrightarrow{id \otimes tr} \mathbb{L}_{B/\mathfrak{U}} \tag{2.3}$$

where the left-pointing arrow is the projection formula, an isomorphism in this context;

and the right-pointing arrow is induced by the trace map (see Section 2.2.2). We set

$$\phi_{B/\mathfrak{U}} = \beta \circ R\pi_*(\alpha \otimes \omega^\bullet).$$

2.1.2 Main results

Setting $B = \underline{\mathrm{Sec}}(Z/\mathcal{C})$ defines a canonical morphism $\phi_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}}$ of complexes in the unbounded derived category of quasi-coherent sheaves $\mathrm{D}_{\mathrm{qc}}(\underline{\mathrm{Sec}}(Z/\mathcal{C})_{\text{lis-et}})$ (see Section 2.2.1 for more about this category). The main result of this chapter is the following.

Theorem 2.1.2. *The canonical morphism $\phi_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}}$ is an obstruction theory on $\underline{\mathrm{Sec}}(Z/\mathcal{C})$.*

By "obstruction theory," we mean that ϕ induces an isomorphism of cohomology sheaves in degrees 0 and 1 and a surjection in degree -1.

If $\mathcal{M} \subset \underline{\mathrm{Sec}}(Z/\mathcal{C})$ is Deligne-Mumford and the restriction of $\mathbb{E}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}}$ to $\mathcal{M}$ is perfect, then by [BF97] one can use ϕ to construct a virtual cycle on $\mathcal{M}$ (see [AP19] for some discussion of the situation when $\mathcal{M}$ is not Deligne-Mumford). The obstruction theory in Theorem 2.1.2 recovers the standard obstruction theory on the moduli spaces of stable maps studied in [AGV08; Beh97], as well as the obstruction theory on the moduli of sections studied in [CL12; CL18].

The most obvious difference between Theorem 2.1.2 and the cited results is that Theorem 2.1.2 is a statement about 3-term obstruction theories on a stack that may not be Deligne-Mumford. However, even when it is restricted to the cited Deligne-Mumford situations, the proof of Theorem 2.1.2 uses a slightly different strategy than what is in the literature, and most importantly it includes a certain critical detail, which we will explain momentarily. We note that there are other approaches to constructing a virtual cycle on the moduli of stable maps, including [LT98] and [Wis11].

We now explain the difference between our approach to Theorem 2.1.2 and those in the literature, including an informal explanation of our added detail (this is explained more fully in Section 2.4.3). It is well-known (see Lemma 2.4.7) that ϕ is an obstruction theory if and only if, given any $f : T \to \underline{\mathrm{Sec}}(Z/\mathcal{C})$ from a scheme T and a square-zero ideal I on T, the maps

$$\mathrm{Ext}^i(Lf^*\mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}}, I) \to \mathrm{Ext}^i(Lf^*\mathbb{E}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}}, I) \tag{2.4}$$

induced by ϕ are isomorphisms for $i = -1, 0$, and "injective on obstructions" for $i = 1$.

(In other words, $\mathbb{E}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}}$ captures the deformation theory of $\underline{\mathrm{Sec}}(Z/\mathcal{C})$ over $\mathfrak{U}$, see Theorem 2.1.3.) The standard approach to Theorem 2.1.2 is to prove an equivalence of certain categories of deformations, and this equivalence implies that the Ext groups in (2.4) are isomorphic for $i = -1, 0$, but it does not explain why the map ϕ is an isomorphism.

Our proof of Theorem 2.1.2 adds the critical detail that ϕ itself induces isomorphisms of the desired Ext groups. The strategy is to use the "fundamental theorem" of [Ill71, Thm III.1.2.3] [Ols06, Thm 1.1], which is an isomorphism

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \simeq \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I) \tag{2.5}$$

between the category of square-zero extensions $\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I)$ of $\mathcal{X}$ and a Picard category determined by the cotangent complex (see Section 2.4.1). This fundamental theorem lets us explicitly relate the well-known equivalence of deformation categories mentioned in the previous paragraph and the morphism ϕ.

As a warmup to the proof of Theorem 2.1.2, we use the fundamental theorem (2.5) to prove that the cotangent complex governs deformation theory as expected. That is, suppose we are given a solid diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{Z} \end{array} \tag{2.6}$$

of algebraic stacks where $\mathcal{X}'$ is a square-zero extension of $\mathcal{X}$. Then one can define a category $\underline{\mathrm{Def}}(f)$ of dotted arrows making the entire diagram (2.6) commute. We prove the following theorem.

Theorem 2.1.3. *Consider a solid diagram* (2.6) *of algebraic stacks where $\mathcal{X} \to \mathcal{X}'$ is a square-zero extension of Deligne-Mumford stacks with ideal sheaf I, and f and g are representable.*

1. *There is an obstruction $o(f) \in Ext^1(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ whose vanishing is necessary and sufficient for the category $\underline{\mathrm{Def}}(f)$ to be nonempty.*

2. *If $o(f) = 0$, then the set of isomorphism classes of $\underline{\mathrm{Def}}(f)$ is a torsor under $Ext^0(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$.*

3. *If $o(f)=0$, then the automorphisms of any element of $\underline{\mathrm{Def}}(f)$ are canonically isomorphic to $Ext^{-1}(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$.*

The hypothesis that $\mathcal{X}$ is Deligne-Mumford is necessary for our method of proof, which uses that (2.5) comes from an isomorphism of Picard stacks on the small étale site of $\mathcal{X}$ when $\mathcal{X}$ is Deligne-Mumford [Ols06, Rmk 1.3]. In fact, we prove the analog of Theorem 2.1.3 at the level of Picard stacks and obtain Theorem 2.1.3 as a corollary. This theorem is a generalization of [Pom15, Thm 3.4]. If in addition $\mathcal{Y}$ and $\mathcal{Z}$ are Deligne-Mumford, Theorem 2.1.3 is just a special case of [Ill71, Prop III.2.2.4]; if $\mathcal{Z}$ is a scheme and $\mathcal{Y}$ is an algebraic stack, then it is a special case of [Ols06, Thm 1.5] (but as stated, Theorem 2.1.3 does not follow from either of these).

2.1.3 Organization of the chapter

Section 1 summarizes the definitions and properties of (derived) categories of quasicoherent sheaves on algebraic stacks, with special attention to the cotangent complex and duality for twisted curves. The results in this section are all well known.

Section 2 lists functoriality properties of the moduli spaces $\underline{\mathrm{Sec}}(Z/\mathcal{C})$ and their canonical obstruction theories ϕ. These properties seem to be well known. In the final section they are applied to explain why the moduli space and its obstruction theory are equivariant when a group acts on (3.16). This section was originally the appendix of a joint work with Qile Chen and Felix Janda [CJW19].

Section 3 is the core of this chapter. It reviews the Fundamental Theorem (2.6) and then uses it to prove (a more refined version of) Theorem 2.1.3. Section 2.4.3 proves the analog of the criterion [BF97, Thm 4.5] in the artin setting and discusses the role of the morphism ϕ in a little more detail. Section 2.4.4 proves Theorem 2.1.2. Finally, Section 2.4.5 gives some applications of Theorem 2.1.2. The first is to write down the cotangent complex of the moduli of principal G-bundles on an arbitrary family of curves, for G an affine group. The second application is our original motivation for proving Theorem 2.1.2, namely to relate different virtual cycles on the moduli space of quasimaps. This result is used in the second part of the thesis to relate our computations there to the ones in the literature.

Section 4 addresses the functoriality of the fundamental theorem (2.5), which is absolutely critical for our proof of Theorem 2.1.2. This functoriality is hinted at in [Ols06, Sec 2.33].

2.2 Background

2.2.1 Sheaves on algebraic stacks

Let $\mathcal{X}$ be an algebraic stack. In order to get a good notion of a cotangent complex for $\mathcal{X}$, we must work with sheaves on the lisse-étale site of $\mathcal{X}$ (see Section 2.2.3). The difficulty with working in this topos $\mathcal{X}_{\text{lis-et}}$ is that it is not functorial: a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks does not in general induce a morphism of topoi from $\mathcal{X}_{\text{lis-et}}$ to $\mathcal{Y}_{\text{lis-et}}$, because the inverse image functor f^{-1} may not be exact [Beh03, Warning 5.3.12]. One solution is to replace quasi-coherent sheaves on $\mathcal{X}_{\text{lis-et}}$ with an equivalent category that behaves functorially. This is a category of quasicoherent sheaves in a topos $X^{\bullet}_{\text{et}}$ on the strictly simplicial space associated to a smooth cover $X \to \mathcal{X}$ of $\mathcal{X}$ by an algebraic space. Because of this extra step, standard constructions and results for morphisms of quasi-coherent sheaves (insomuch as they come from morphisms of topoi) do not automatically apply to arbitrary morphisms of algebraic stacks (at least not in the lisse-étale topos).

In this section we briefly summarize some results about the topoi $\mathcal{X}_{\text{lis-et}}$ and $X^{\bullet}_{\text{et}}$, their associated derived categories of quasi-coherent sheaves, and functors between these categories, including those induced by morphisms of stacks.This section and the next are partial summary of [HR17, Sec 1], which in turn relies on [LO08; Ols07a].

Equivalence of topoi

Let $\mathcal{X}$ be an algebraic stack and $X \to \mathcal{X}$ a smooth cover by an algebraic space. These data define three ringed topoi:

1. the lisse-étale topos $\mathcal{X}_{\text{lis-et}}$ on $\mathcal{X}$ [LM00, 12.1] (see also [Stacks, Tag 0787])
2. the strictly simplicial topos $X^{+}_{\bullet,\text{lis-et}}$ defined in [Ols07a, Sec 2.1, 4.2], which we will denote $X^{\bullet}_{\text{lis-et}}$
3. the strictly simplicial topos $X^{+}_{\bullet,\text{et}}$ defined in [Ols07a, Sec 2.1, 4.2], which we will denote $X^{\bullet}_{\text{et}}$

We will use $\mathscr{O}$ to denote the ring object in each topos, sometimes with a subscript indicating the topos. For each topos we define several categories of sheaves:

1. categories of sheaves of $\mathscr{O}$-modules, for which we will not need notation

2. the corresponding unbounded derived categories, denoted $\mathrm{D}(\mathcal{X}_{\text{lis-et}})$ and so forth

3. categories of quasi-coherent sheaves, denoted $\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$ etc. (see [LM00, Def 13.2.2] or [Ols16, Def 9.1.14] for $\mathcal{X}_{\text{lis-et}}$; see [Ols07a, Def 6.9] for the strictly simplicial topoi)

4. the corresponding unbounded derived categories, denoted $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ etc. This is the full subcategory of $\mathrm{D}(\mathcal{X}_{\text{lis-et}})$ with cohomology in $\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$ ($\mathrm{D}_{\mathrm{qc}}(X^{\bullet}_{\text{lis-et}})$ and $\mathrm{D}_{\mathrm{qc}}(X^{\bullet}_{\text{et}})$ are defined similarly).

As in [LO08, Ex 2.2.5] there are morphisms of topoi

$$\begin{array}{ccc} X^{\bullet}_{\text{lis-et}} & \xrightarrow{r} & X^{\bullet}_{\text{et}} \\ \downarrow{\scriptstyle \epsilon} & & \\ \mathcal{X}_{\text{lis-et}} & & \end{array}$$

The functors r_* and r^* relating $\mathrm{QCoh}(T)$ and $\mathrm{QCoh}(X^{\bullet}_{\text{et}})$ are exact and inverse equivalences, and the same holds for the derived categories (see also [Ols07a, Prop 6.12]). Likewise, ϵ^* is exact and defines an equivalence of $\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$ and $\mathrm{QCoh}(T)$ with inverse equivalence ϵ_*, and ϵ^* and $R\epsilon_*$ are inverse equivalences for the derived categories.

The upshot of these functors is that we can compare (derived) categories of quasi-coherent sheaves on $\mathcal{X}_{\text{lis-et}}$ and $X^{\bullet}_{\text{et}}$ as follows. We define a restriction functor

$$\varpi_X^* : \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{QCoh}(X^{\bullet}_{\text{et}}) \tag{2.7}$$

to be the composition $r_* \circ \epsilon^*$. It is exact and hence naturally extends to a functor of the derived categories, and moreover it preserves $\mathscr{O}$. As a functor of quasi-coherent sheaves, ϖ_X^* has an inverse $(\varpi_X)_* = \epsilon_* \circ r^*$. Likewise, as a functor of derived categories of quasi-coherent sheaves, ϖ_X^* has an inverse $(R\varpi_X)_* = R\epsilon_* \circ r^*$.

Remark 2.2.1. To develop the theory of sheaves on $\mathcal{X}_{\text{lis-et}}$, it is probably better to replace $X^{\bullet}_{\text{lis-et}}$ with the restricted topos $\mathcal{X}_{\text{lis-et}}|_{X^{\bullet}}$, and to replace the strictly simplicial topos $X^{\bullet}_{\text{et}} = X^{+}_{\bullet,\text{et}}$ with its non-strict version (usually denoted $X_{\bullet,\text{et}}$). See, for example, [Ols16, Sec 9.2.3]. We have chosen our approach so that we can directly cite results appearing in [Ols07a], [HR17], and [Ols06].

Independence of cover

Let $\mathcal{X}$ be an algebraic stack. If $U \to \mathcal{X}$ and $V \to \mathcal{X}$ are two smooth covers by algebraic spaces, then we know the (derived) categories of quasicoherent sheaves on $U^\bullet_{\text{et}}$ and $V^\bullet_{\text{et}}$ are equivalent because they are both equivalent to the corresponding category on $\mathcal{X}_{\text{lis-et}}$. However we can also compare these categories directly.

Lemma 2.2.2. *Let $U \xrightarrow{f} V \to \mathcal{X}$ be smooth surjective morphisms with U, V algebraic spaces. Then $R(\varpi_V)_*(f^\bullet)^* = R(\varpi_U)_*$ and $\varpi_U^* = (f^\bullet)^*\varpi_V^*$.*

A corollary of this lemma is that $(f^\bullet)^*$ is an equivalence of the (derived) categories of quasi-coherent sheaves on $U^\bullet_{\text{et}}$ and $V^\bullet_{\text{et}}$.

Proof. The second equality follows from the first. For the first, it suffices to show that this square commutes:

$$\begin{array}{ccc}
\mathrm{QCoh}(U^\bullet_{\text{lis-et}}) & \xleftarrow[r^*]{} & \mathrm{QCoh}(U^\bullet_{\text{et}}) \\
\uparrow{\scriptstyle (f^\bullet)^*} & & \uparrow{\scriptstyle (f^\bullet)^*} \\
\mathrm{QCoh}(V^\bullet_{\text{lis-et}}) & \xleftarrow[r^*]{} & \mathrm{QCoh}(V^\bullet_{\text{et}})
\end{array}$$

If $T \xrightarrow{t} U^n$ is smooth, then $T \xrightarrow{f\circ t} V^n$ is also smooth. Hence if $\mathcal{F}$ is a sheaf in $\mathrm{QCoh}(V^\bullet_{\text{et}})$, then the restriction of either pullback to T is the sheaf $(f \circ t)^*\mathcal{F}$ on the small étale site of T (see the proof of [Ols16, Prop 9.1.18]. □

Deligne-Mumford setting

If $\mathcal{X}$ is a Deligne-Mumford stack, then we can also define the small étale site of $\mathcal{X}$ as in [LM00, Def 12.1] (see also [Ols16, Sec 9.1.16]). In this case, given a smooth cover $X \to \mathcal{X}$ by an algebraic space, there is a commutative diagram of morphisms of topoi

$$\begin{array}{ccc}
X^\bullet_{\text{lis-et}} & \xrightarrow{r} & X^\bullet_{\text{et}} \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon_{\text{et}}} \\
\mathcal{X}_{\text{lis-et}} & \xrightarrow{r_{\mathcal{X}}} & \mathcal{X}_{\text{et}}
\end{array} \tag{2.8}$$

where $(r^*_{\mathcal{X}}, (r_{\mathcal{X}})_*)$ are exact functors and inverse equivalences of (derived) categories of quasicoherent sheaves on $\mathcal{X}_{\text{lis-et}}$ and $\mathcal{X}_{\text{et}}$ (see [Ols16, Prop 9.1.18] for the underived statement). Similarly, the pair $(\epsilon^*_{\text{et}}, R(\epsilon_{\text{et}})_*)$ are inverse equivalences of $\mathrm{D}_{\text{qc}}(\mathcal{X}_{\text{et}})$ and $\mathrm{D}_{\text{qc}}(X^\bullet_{\text{et}})$. To see that (2.8) commutes, it suffices to show $r^*\epsilon^*_{\text{et}} = \epsilon^* r^*_{\mathcal{X}}$. This follows as in the proof of Lemma 2.2.2.

Standard functors

Let $\mathcal{X}$ be an algebraic stack. There is a (derived) tensor operation $\otimes$ on each of the categories $\mathrm{D}(\mathcal{X}_{\text{lis-et}})$, $\mathrm{D}(\mathcal{X}^\bullet_{\text{lis-et}})$ and $\mathrm{D}(X^\bullet_{\text{et}})$, that preserves the subcategories of complexes with quasi-coherent cohomology [Stacks, Tag 06YU]. Each category also has an internal hom functor which we denote $R\underline{Hom}$ [Stacks, Tag 08J7].[1]

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The functors f^* and Lf^* are only defined on $\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$ and $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$. To define them, find a commuting diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array} \tag{2.9}$$

with $X \to \mathcal{X}$ and $Y \to \mathcal{Y}$ smooth covers by algebraic spaces. Let $(f^\bullet_*, (f^\bullet)^{-1})$ denote the morphism of topoi from $X^\bullet_{\text{et}}$ to $Y^\bullet_{\text{et}}$. We define $Lf^* : \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ to be the functor

$$Lf^* = R(\varpi_X)_* L(f^\bullet)^* \varpi_X^* \tag{2.10}$$

and we define f^* similarly. Following [HR17, Sec 1.3], we define $Rf_* : \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}})$ to be the right adjoint of Lf^*. When f is quasi-compact and quasi-separated, this is the functor

$$Rf_* = R(\varpi_X)_* Rf^\bullet_* \varpi_X^*. \tag{2.11}$$

where $Rf^\bullet_*$ is the usual derived functor of $f^\bullet_*$. In this qcqs case, Rf_* agrees with the usual right derived functor of $f_* : \mathrm{QCoh}(\mathcal{X}_{\text{lis-et}}) \to \mathrm{QCoh}(\mathcal{Y}_{\text{lis-et}})$ on complexes in ${\mathrm{D}_{\mathrm{qc}}}^+(\mathcal{X}_{\text{lis-et}})$ [HR17, Lem 1.2], or on all complexes in $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ if f is *concentrated* [HR17, Thm 2.6].

Independence of cover

A priori, the definitions of f^* and Lf^* depend on the choice of diagram (2.9). Any two

[1] One can also define internal hom $R\underline{Hom}_{\mathrm{qc}}(\mathcal{F}, -)$ for the category $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ as the right adjoint to the functor $- \otimes \mathcal{F}$ as in [HR17, Sec 1.2]. We will not use this functor, which agrees with $R\underline{Hom}$ if $\mathcal{F}$ is perfect.

such diagrams are dominated by a third. So suppose we have a commuting diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & \mathcal{Y} \end{array} \tag{2.12}$$

where horizontal maps are smooth and X_i, Y_i are algebraic spaces. Let f_i^*, Lf_i^* denote the functors defined using $X_i \to Y_i$. A straightforward computation using Lemma 2.2.2 shows that

$$f_1^* = f_2^* \quad \text{and} \quad Lf_1^* = Lf_2^*. \tag{2.13}$$

A consequence of this observation is that if $f, g : \mathcal{X} \to \mathcal{Y}$ are two 1-morphisms and $\gamma : f \to g$ is a 2-morphism, then $f^* = g^*$ and $Lf^* = Lg^*$. This is because γ induces a commuting diagram of covers as in (2.12).

Deligne-Mumford setting

If $\mathcal{X}$ and $\mathcal{Y}$ are Deligne-Mumford then we also have a morphism of topoi $\mathcal{X}_{\text{et}} \to \mathcal{Y}_{\text{et}}$ and hence a functor $Lf_{\text{et}}^* : \mathrm{D}(\mathcal{Y}_{\text{et}}) \to \mathrm{D}(\mathcal{X}_{\text{et}})$ sending $\mathrm{D}_{\text{qc}}(\mathcal{Y}_{\text{et}})$ to $\mathrm{D}_{\text{qc}}(\mathcal{X}_{\text{et}})$ (note that now Lf_{et} is defined on the entire unbounded derived category). The commuting equivalences of (2.8) imply

$$Lf^* = (r_{\mathcal{X}})^* Lf_{\text{et}}^* (r_{\mathcal{Y}})_* \tag{2.14}$$

on $\mathrm{D}_{\text{qc}}(\mathcal{Y}_{\text{et}})$. Similarly, when f is quasi-compact and quasi-separated, we have

$$Rf_* = (r_{\mathcal{X}})^* R(f_{\text{et}})_* (r_{\mathcal{Y}})_*$$

on complexes in ${\mathrm{D}_{\text{qc}}}^+(\mathcal{X}_{\text{lis-et}})$.

Remark 2.2.3. We can also allow $X \to \mathcal{X}$ to be a smooth cover by a Deligne-Mumford stack, so that $X^\bullet$ is a simplicial Deligne-Mumford stack. As in (2.7) we have an equivalence of $\mathrm{D}_{\text{qc}}(\mathcal{X}_{\text{lis-et}})$ and $\mathrm{D}_{\text{qc}}(X_{\text{et}}^\bullet)$, and we can use this equivalence as in (2.10) to define (derived) pullback for $\mathrm{D}_{\text{qc}}(\mathcal{X}_{\text{lis-et}})$. By (2.13) this definition of derived pullback agrees with the one constructed previously.

2.2.2 Some properties

We present several properties of the functors in Section 2.2.1 that will be used in this chapter.

Symmetric monoidal categories

Amongst the definitions of Sections 2.2.1 and 2.2.1, there are several adjoint pairs of functors of symmetric monoidal categories, where the left adjoint is strong monoidal. Such pairs satisfy many familiar formulae for purely formal reasons.

Loosely speaking, a category is symmetric monoidal if it has an associative, commutative operation $\otimes$ with unit $\mathscr{O}$ (see [Hal, Appendix A] for a precise definition). A functor between two such categories is called strong monoidal if it preserves $\otimes$ and $\mathscr{O}$. We have several instances of this formalism:

- All the categories $\mathrm{D}(\mathcal{X}_{\text{lis-et}})$, $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$, $\mathrm{D}(X^\bullet_{\text{lis-et}})$, $\mathrm{D}_{\mathrm{qc}}(X^\bullet_{\text{lis-et}})$, $\mathrm{D}(X^\bullet_{\text{et}})$, and $\mathrm{D}_{\mathrm{qc}}(X^\bullet_{\text{et}})$ are symmetric monoidal with operation $\otimes$ and unit $\mathscr{O}$
- The functors ϖ_X^* and $(R\varpi_X)_*$ defined in Section 2.2.1 are strong monoidal. Likewise the functors ϵ_{et}^* and $R(\epsilon_{\text{et}})_*$ defined in (2.8) are strong monoidal.
- If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then Lf^* is strong monoidal (this uses the previous item).

We note that inverse equivalences form adjoint pairs (see eg [Mac71, Sec IV.4]).

Let $\mathscr{C}$, $\mathscr{D}$ be symmetric monoidal categories with $L, R : \mathscr{C} \rightleftarrows \mathscr{D}$ a left-right adjoint pair of functors, with L strong monoidal. Suppose we have internal hom functors $R\underline{Hom}$ on both $\mathscr{C}$ and $\mathscr{D}$.[2] Then given $B \in \mathscr{D}$ and $A \in \mathscr{C}$ there is a functorial isomorphism

$$R\underline{Hom}(A, R(B)) \xrightarrow{\sim} R(R\underline{Hom}(L(A), B)), \tag{2.15}$$

see [FHM03, (3.4)]. This leads to the following morphism which we will use repeatedly.

Lemma 2.2.4. *There is a morphism*

$$R\underline{Hom}(A, B) \to R(R\underline{Hom}(L(A), L(B))) \tag{2.16}$$

functorial in the adjoint pair (L, R), in A, and in B; and it is an isomorphism if L is fully faithful.

[2]Warning: If one wishes to apply what follows to $Lf^* : \mathrm{D}_{\mathrm{qc}}(\mathcal{Y}_{\text{lis-et}}) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$, then one must use the internal hom $R\underline{Hom}_{\mathrm{qc}}$ for these derived categories, which does not agree with $R\underline{Hom}$ in general; see footnote 1 on p. 15. We will only apply the following discussion in other contexts.

Proof. The morphism (2.16) is equal to the composition

$$R\underline{Hom}(A,B) \to R\underline{Hom}(A, RL(B)) \simeq R(R\underline{Hom}(L(A), L(B))$$

where the first arrow is induced by the unit $B \to RL(B)$ of the adjunction and the isomorphism is (2.15). This unit is an isomorphism if L is fully faithful. To see that (2.16) is functorial in (L, R), use the Yoneda embedding and check that the maps

$$\mathrm{Hom}(T, R\underline{Hom}(A,B)) \to \mathrm{Hom}(T, R(R\underline{Hom}(L(A), L(B))))$$

are induced by the set maps

$$\mathrm{Hom}(T \otimes A, B) \to \mathrm{Hom}(L(T \otimes A), L(B))$$

given by L. See Claim 2.4.10 in Section 2.4.4 for a detailed example of this kind of argument. □

Projection formula and duality for twisted curves

We will study families of twisted nodal curves as defined in [AV02, Sec 4] (see also [Ols07b]); note that such families are preserved by base change. From here on we will simply call these *families of twisted curves*.

If $\pi : \mathcal{C} \to \mathcal{M}$ is a family of twisted curves with $\mathcal{M}$ an algebraic stack, then the morphism π is *concentrated* in the sense of [HR17, Def 2.4] (this follows from [DG13, Thm 1.4.2]; see also [HR17, 14]). Then by [HR17, Cor 4.12], the projection formula holds for π; that is, the natural map

$$R\pi_*\mathcal{F} \otimes \mathcal{G} \to R\pi_*(\mathcal{F} \otimes \pi^*\mathcal{G}) \tag{2.17}$$

is a quasi-isomorphism for all $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{C}_{\text{lis-et}})$ and $\mathcal{G} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{M}_{\text{lis-et}})$. Likewise, if $f : \mathcal{N} \to \mathcal{M}$ is any morphism of algebraic stacks, with $\pi' : \mathcal{C} \times_{\mathcal{M}} \mathcal{N} \to \mathcal{N}$ and $f' : \mathcal{C} \times_{\mathcal{M}} \mathcal{N} \to \mathcal{C}$, then by [HR17, Cor 4.13] the natural base change map

$$Lf^*R\pi_*\mathcal{F} \to R\pi'_*L(f')^*\mathcal{F} \tag{2.18}$$

is also an isomorphism.

Lemma 2.2.5. *Let $\pi : \mathcal{C} \to \mathcal{M}$ be a family of twisted curves.*

1. *The map π has cohomological dimension ≤ 1.*

2. *If $\mathcal{M}$ is a Noetherian algebraic space, then $R\pi_*$ takes perfect complexes to perfect complexes.*

See e.g. [HR17, Def 2.1] for the definition of cohomological dimension.

Proof. To prove (1), let $\mathcal{F} \in \mathrm{QCoh}(\mathcal{C})$. We must show $R^i\pi_*\mathcal{F} = 0$ for $i > 1$. If $\mathcal{M}$ is a scheme, one can see this as follows: the map π factors through the coarse moduli space $\underline{\pi} : C \to \mathcal{M}$ which is a scheme. If p denotes the map $\mathcal{C} \to C$, then we compute $R\pi_*\mathcal{F} = R\underline{\pi}_* p_*\mathcal{F}$ and so the result follows from the usual computation of cohomology for representable curves and the fact that C is a good moduli space. Now the general case can be reduced to the case when $\mathcal{M}$ is a scheme using base change (2.18).

Compare our proof of (2) with [Stacks, Tag 08EV]. Since we may check the conclusions locally, by base change (2.18) it suffices to consider the case when $\mathcal{M} = M$ is a Noetherian scheme. Let $\mathscr{P} \in \mathrm{D}_{\mathrm{qc}}(\mathcal{C})$ be a perfect complex. Then by [Stacks, Tag 08G8, 0DJJ], $\mathscr{P}$ is pseudo-coherent and has finite tor dimension.

Applying [Stacks, Tag 08E8] étale-locally we see that $\mathscr{P}$ has coherent cohomology sheaves. By [Stacks, Tag 015J], there is a spectral sequence

$$R^p\pi_*H^q(\mathscr{P}) \implies R^{p+q}\pi_*(\mathscr{P})$$

Observe that if $\mathcal{F} \in \mathrm{QCoh}(\mathcal{C})$ is coherent, then so is $R^i\pi_*\mathcal{F}$ (this follows from [Stacks, Tag 0205] and the fact that pushing forward to the coarse moduli space of $\mathcal{C}$ preserves coherentness). So since π is concentrated, the spectral sequence implies that $R\pi_*\mathscr{P}$ has coherent cohomology.

To show that $R\pi_*\mathscr{P}$ is perfect, by [Stacks, Tag 08G8, 0DJJ, 08E8, 08EA] it remains to show that for any $\mathcal{F} \in \mathrm{QCoh}(M)$, the complex $R\pi_*(\mathscr{P}) \otimes \mathcal{F}$ has vanishing cohomology outside a finite range. But by the projection formula (2.17), we have

$$H^i(R\pi_*(\mathscr{P}) \otimes \mathcal{F}) = H^i(R\pi_*(\mathscr{P} \otimes \pi^*\mathcal{F}))$$

and the right hand side for i outside some finite range, since $\mathscr{P}$ is perfect and π is concentrated.

□

We will use the following version of Grothendieck duality. Compare with [Stacks, Tag 0E6N] for families of representable curves on schemes; especially compare the proof of Proposition 2.2.6 with the argument in [Stacks, Tag 0E61]. We use $\pi^!$ to denote the right adjoint to $R\pi_*$, which exists by [HR17, Thm 4.14].

Proposition 2.2.6. *For every family $\mathcal{C} \to \mathcal{M}$ of twisted curves on an algebraic stack $\mathcal{M}$ that is quasi-separated and locally finite type, there is a pair $(\omega^\bullet_{\mathcal{M}}, tr_{\mathcal{M}})$ with $\omega^\bullet_{\mathcal{M}} = \omega_{\mathcal{M}}[1]$ where $\omega_{\mathcal{M}} \in \mathrm{QCoh}(\mathcal{C}_{\text{lis-et}})$ is locally free and $tr_{\mathcal{M}} : R\pi_*\omega^\bullet_{\mathcal{M}} \to \mathscr{O}_{\mathcal{M}}$, such that the following hold:*

1. *The pair is functorial in the following sense. Given a fiber square*

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{N}} & \xrightarrow{f_{\mathcal{C}}} & \mathcal{C}_{\mathcal{M}} \\ \downarrow & & \downarrow \\ \mathcal{N} & \xrightarrow{f} & \mathcal{M} \end{array} \tag{2.19}$$

   there is a canonical isomorphism

$$f_{\mathcal{C}}^*\omega^\bullet_{\mathcal{M}} \xrightarrow{\sim} \omega^\bullet_{\mathcal{N}} \tag{2.20}$$

   such that the following square commutes

$$\begin{array}{ccc} Lf^*R\pi_*\omega^\bullet_{\mathcal{M}} & \xrightarrow{Lf^*tr_{\mathcal{M}}} & \mathscr{O}_{\mathcal{N}} \\ \downarrow{\scriptstyle (2.18)} & & \uparrow{\scriptstyle tr_{\mathcal{N}}} \\ R\pi_*Lf_{\mathcal{C}}^*\omega^\bullet_{\mathcal{M}} & \xrightarrow{(2.20)} & R\pi_*\omega^\bullet_{\mathcal{N}} \end{array} \tag{2.21}$$

2. *If $\mathcal{M}$ is a quasi-compact and quasi-separated Deligne-Mumford stack, then $\omega^\bullet_T = \pi^!\mathscr{O}_T$ and tr_T is the counit of the $(R\pi_*, \pi^!)$ adjunction.*

For a general base $\mathcal{M}$ we do not know if our construction of $(\omega^\bullet_{\mathcal{M}}, tr_{\mathcal{M}})$ agrees with the right adjoint to pushforward.

Proof. The idea is to take (2) as the definition of the pair $(\omega^\bullet_{\mathcal{M}}, tr_{\mathcal{M}})$ when $\mathcal{M}$ is a quasi-compact and quasi-separated algebraic space and show that this construction "glues" as desired. The key observation is that we have a base change property for $\pi^!$ in this setting, as we now explain (see also [Stacks, Tag 0AA5] and [Nee17]).

Suppose we have a square (2.19) with $\mathcal{N}$ and $\mathcal{M}$ Deligne-Mumford stacks. Then the isomorphism of left adjoints in (2.18) implies that the right adjoints are also isomorphic; that is, there is a canonical isomorphism

$$R(f_{\mathcal{C}})_*\pi^!_{\mathcal{N}}(K) \xrightarrow{\sim} \pi^!_{\mathcal{M}} \circ Rf_*(K) \tag{2.22}$$

for any $K \in \mathrm{D}_{\mathrm{qc}}(\mathcal{M})$. This lets us define the functorial base change map

$$Lf^*_{\mathcal{C}}\pi^!_{\mathcal{M}}(K) \to \pi^!_{\mathcal{N}}Lf^*(K) \tag{2.23}$$

as the adjoint to the composition

$$\pi^!_{\mathcal{M}} \to \pi^!_{\mathcal{M}}Rf_*Lf^*(K) \xleftarrow[\sim]{(2.22)} R(f_{\mathcal{C}})_*\pi^!_{\mathcal{N}}Lf^*. \tag{2.24}$$

The proof of the following lemma uses the notion of a *generating set* for a derived category of quasi-coherent sheaves: we say that $\{P_i\}_{i\in I} \subset \mathrm{D}_{\mathrm{qc}}(\mathcal{M})$ generates $\mathrm{D}_{\mathrm{qc}}(\mathcal{M})$ if, for any $\alpha : A \to B$ in $\mathrm{D}_{\mathrm{qc}}(\mathcal{M})$, the morphisms

$$\mathrm{Hom}(P_i[n], \alpha) : \mathrm{Hom}(P_i[n], A) \to \mathrm{Hom}(P_i[n], B)$$

are isomorphisms for each i if and only if α is an isomorphism. Here $P_i[n]$ denotes the shift operator applied to P_i.

Lemma 2.2.7. *If $\mathcal{N}$ and $\mathcal{M}$ are (possibly infinite) disjoint unions of quasi-compact, quasi-separated, and locally finite type Deligne-Mumford stacks, then* (2.23) *is an isomorphism.*

Proof. By the argument in [Stacks, Tag 0E9S], it suffices to prove the lemma in the case when f is quasi-affine. By [HR17, Thm A], the category $\mathrm{D}_{\mathrm{qc}}((\mathcal{C}_{\mathcal{M}})_{\mathrm{et}})$ is generated by a collection $\{P_i\}_{i\in I}$ of perfect complexes. (The cited theorem gives a single compact generator for the derived category of each connected component of $\mathcal{C}_{\mathcal{M}}$, since these components are quasi-compact and quasi-separated; after extending by 0 we get an infinite collection of perfect complexes that generate $\mathrm{D}_{\mathrm{qc}}((\mathcal{C}_{\mathcal{M}})_{\mathrm{et}})$.)

Our first claim is that $\{Lf^*_{\mathcal{C}}(P_i)\}_{i\in I}$ is a collection of perfect complexes that generate $\mathrm{D}_{\mathrm{qc}}((\mathcal{C}_{\mathcal{N}})_{\mathrm{et}})$. The complexes $Lf^*_{\mathcal{C}}(P_i)$ are perfect by [Stacks, Tag 08H6]. To see that they generate $\mathrm{D}_{\mathrm{qc}}((\mathcal{C}_{\mathcal{N}})_{\mathrm{et}})$, let $\alpha : A \to B$ be a morphism in $\mathrm{D}_{\mathrm{qc}}((\mathcal{C}_{\mathcal{N}})_{\mathrm{et}})$ such that the induced map $\mathrm{Hom}(Lf^*_{\mathcal{C}}P_i[n], \alpha)$ is an isomorphism for each P_i. Since f is quasi-affine, by [HR17, Cor 2.8], to show that α is an isomorphism it suffices to show that $Rf_*\alpha$ is an isomorphism. But this is something we can detect with the P_i. In fact there is a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}(P_i[n], Rf_*A) & \longrightarrow & \mathrm{Hom}(P_i[n], Rf_*B) \\ \| & & \| \\ \mathrm{Hom}(Lf_{\mathcal{C}}^*P_i[n], A) & \longrightarrow & \mathrm{Hom}(Lf_{\mathcal{C}}^*P_i, B) \end{array}$$

where the equalities are adjunction. This completes the proof of the first claim.

Now to show that (2.23) is an isomorphism, it suffices to show that the induced maps

$$\mathrm{Hom}(Lf_{\mathcal{C}}^*(P_i[n]),\ Lf_{\mathcal{C}}^* \circ \pi_{\mathcal{M}}^!(K)) \to \mathrm{Hom}(Lf_{\mathcal{C}}^*(P_i[n]),\ \pi_{\mathcal{N}}^! \circ Lf^*(K))$$

are isomorphisms for each $n \in \mathbb{Z}$ and for each P_i. Writing $P = P_i[n]$ and omitting subscripts on f and π, we have a diagram

$$\begin{array}{ccccc} \mathrm{Hom}(Lf^*(P), Lf^* \circ \pi^!(K)) & \longrightarrow & \mathrm{Hom}(Lf^*(P), \pi^! \circ Lf^*(K)) & & \\ {\scriptstyle Lf^*}\uparrow & & \| & & \\ \mathrm{Hom}(P, \pi^!(K)) & \xrightarrow{(2.24)} & \mathrm{Hom}(P, Rf_* \circ \pi^! \circ Lf^*(K)) & = & \mathrm{Hom}(Lf^*(P), \pi^! \circ Lf^*(K)) \\ \| & & & & \| \\ \mathrm{Hom}(R\pi_*(P), K) & \xrightarrow{Lf^*} & \mathrm{Hom}(Lf^* \circ R\pi_*(P), Lf^*(K)) & \xleftarrow{(2.18)} & \mathrm{Hom}(R\pi_* \circ Lf^*(P), Lf^*(K)) \end{array}$$

where the equalities are adjunctions, the top square commutes by definition of (2.23) via adjunction, and the bottom square commutes by the discussion in [Stacks, Tag 0AA5]. But the maps labeled Lf^* are isomorphisms because Lf^*P and $R\pi_*P$ are perfect—this uses Lemma 2.2.5 and [FHM03, Prop 3.12]. □

Now we can use a gluing argument to construct $(\omega_{\mathcal{M}}, tr_{\mathcal{M}})$ for an arbitrary family of twisted curves. First, for $\mathcal{M}$ a disjoint union of quasi-compact and quasi-separated Deligne-Mumford stacks locally of finite type, the right adjoint $\pi^!$ exists by [HR17, Thm 4.14]. Set $\omega_{\mathcal{M}}^\bullet = \pi^!(\mathscr{O}_{\mathcal{M}})$ and define $tr_{\mathcal{M}} : R\pi_*\omega_{\mathcal{M}}^\bullet \to \mathscr{O}_{\mathcal{M}}$ to be the counit of the adjunction. By Lemma 2.2.8 below, the complex $\omega_{\mathcal{M}}^\bullet$ is a locally free sheaf in degree -1. So part (2) of Proposition 2.2.6 is satisfied by definition. Moreover, part (1) is satisfied when $\mathcal{N}$ and $\mathcal{M}$ are disjoint unions of qcqs DM stacks locally of finite type: the isomorphism (2.20) is the base change map (2.23) (which is an isomorphism by Lemma 2.2.7), and the commuting diagram (2.21) follows from [Stacks, Tag 0E5L] which is completely formal.

Hence we may define $(\omega_{\mathcal{M}}^\bullet, tr_{\mathcal{M}})$ when $\mathcal{M}$ is an algebraic stack as follows. Let

$M \to \mathcal{M}$ be a smooth cover by a disjoint union of qcqs Deligne-Mumford stacks and let $\mathcal{C}_M = \mathcal{C} \times_{\mathcal{M}} M$. Let $\mathcal{C}_M^\bullet$ be the simplicial Deligne-Mumford stack that is the coskeleton of this cover (see Remark 2.2.3), and likewise let $M^\bullet$ be the simplicial stack induced from $M \to \mathcal{M}$. We have a map $\mathcal{C}_M^\bullet \to M^\bullet$ which at every level is a family of twisted curves on a disjoint union of qcqs DM stacks (this uses [Stacks, Tag 075S]). Hence we have a quasi-coherent sheaf $\omega_{M^\bullet}^\bullet$ on $(\mathcal{C}_M^\bullet)_{\text{et}}$ defined by the collection $\omega_{M^n}^\bullet$ and the canonical isomorphisms (2.20) (note that these compose as desired by [Stacks, Tag 0ATR]). Likewise by [Stacks, Tag 0DL9] we have a morphism $R\pi_*(\omega_{M^\bullet}^\bullet) \to \mathscr{O}_{M^\bullet}$ in $\mathrm{D}_{\text{qc}}((\mathcal{C}_M^\bullet)_{\text{et}})$ defined by the trace maps tr_{M^n} and the functoriality diagrams (2.21): the required negative exts vanish since $\omega_{M^\bullet}^\bullet$ is locally free in degree -1, so its pushforward is perfect in $[-1, 0]$ by Lemma 2.2.5. (The reference [Stacks, Tag 0DL9] constructs a morphism in the simplicial topos but we may restrict it to the strictly simplicial topos.) Hence using the equivalences of Section 2.2.1 we have a pair $(\omega_{\mathcal{M}}^\bullet, tr_{\mathcal{M}})$, where $\omega_{\mathcal{M}}^\bullet = R\varpi_*\omega_{M^\bullet}^\bullet$ is locally free in degree -1. A priori this pair depends on our choice of cover M, but the next paragraph shows that pairs constructed from different covers are canonically isomorphic in the sense of (1).

Finally we check that part (1) of Proposition 2.2.6 holds when $\mathcal{M}$ and $\mathcal{N}$ in (2.19) are quasi-separated algebraic stacks. Choose a diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array}$$

with $M \to \mathcal{M}$ and $N \to \mathcal{N}$ smooth covers by disjoint unions of qcqs DM stacks. Now we have a canonical map $Lf_{et}^*\omega_{N^\bullet}^\bullet \to \omega_{M^\bullet}^\bullet$ in $\mathrm{D}_{\text{qc}}(\mathcal{C}_M^\bullet)$ (using (2.21) and [Stacks, Tag 0ATR, Tag 0ATS]). From the definition of f^* on $\mathrm{QCoh}(\mathcal{X}_{\text{lis-et}})$, we see that (2.20) is satisfied. Likewise, by inserting several copies of $\varpi_{\mathcal{M}}$, $\varpi_{\mathcal{N}}$, $R(\varpi_{\mathcal{M}})_*$, and $R(\varpi_{\mathcal{N}})_*$ in (2.21), we see that this diagram holds for $(\omega_{\mathcal{M}}^\bullet, tr_{\mathcal{M}})$ as well.

□

Lemma 2.2.8. *Let $\mathcal{C} \to \mathcal{M}$ be a family of twisted curves on a quasi-compact, quasi-separated Deligne-Mumford stack $\mathcal{M}$ that is locally finite type. Then $\pi^!\mathscr{O}_{\mathcal{M}}$ is represented by a locally free sheaf in degree -1.*

Proof. In this proof, a superscript ! will always denote right adjoint to pushforward. By Lemma 2.2.7 it suffices to prove the statement when $\mathcal{M} = T$ is an affine scheme.

Let C be the coarse moduli space of $\mathcal{C}$. Let $U \to C$ be an étale map from an affine scheme U so that we have a diagram

$$
\begin{array}{ccccc}
V & \xrightarrow{\sigma} & [V/G] & \xrightarrow{\tau} & U \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{C} & \xrightarrow{p} & C \\
 & & & \searrow^{\pi} & \downarrow{\scriptstyle q} \\
 & & & & T
\end{array}
\qquad (\rho = \tau \circ \sigma : V \to U)
\tag{2.25}
$$

where the square is fibered (see [Ols07b, Prop 2.2]). If $V = \mathrm{Spec}(A)$ then $U = \mathrm{Spec}(A^G)$ where A^G is the ring of G-invariants. Note that all spaces appearing in the diagram are qcqs Deligne-Mumford stacks, and hence have compactly generated derived categories, and hence right adjoints to pushforward exist for all maps by [FHM03, Thm 8.4] (using [Stacks, Tag 0944]). Moreover since everything is Deligne-Mumford we may work with étale sites and all maps induce morphisms of topoi. Because $R\pi_* = Rq_*Rp_*$, we get a similar equality of right adjoints, and

$$\pi^! \mathscr{O}_T = p^! q^! \mathscr{O}_T.$$

Now by [Stacks, Tag 0E6P, 0E6R] we know $q^! \mathscr{O}_T$ is locally free in degree -1. In particular it is dualizable, so by [FHM03, Thm 8.4] we see that

$$\pi^! \mathscr{O}_T = p^* \omega_C^\bullet \otimes p^! \mathscr{O}_C.$$

So to prove the lemma it suffices to show that $p^! \mathscr{O}_C$ is perfect in degree 0.

By argument used to prove Lemma 2.2.7 shows that the base change map (analog of 2.23) for the fibered square in (2.25) is an isomorphism, so it suffices to show that $\tau^! \mathscr{O}_U$ is perfect in degree 0. To compute this we observe again that we have an equality

$$\rho^! \mathscr{O}_U = \sigma^* \tau^! \mathscr{O}_U \otimes \sigma^! \mathscr{O}_{[V/G]}$$

so it suffices to show that $\rho^! \mathscr{O}_U$ is $\mathscr{O}_V$. Indeed, the isomorphism $\rho^! \mathscr{O}_U \simeq \mathscr{O}_V$ implies that $\sigma^* \tau^! \mathscr{O}_U$ is dualizable, hence perfect by [HR17, Lem 4.3(3)], and then the same isomorphism also forces $\sigma^* \tau^! \mathscr{O}_U$ to be concentrated in degree 0.

Finally we compute $\rho^! \mathscr{O}_U$. To do so we apply [FHM03, Thm 4.14(3)], which says that $\rho^! \mathscr{O}_U$ is the sheaf corresponding to the A-module

$$R\underline{Hom}_{A^G}(A, A^G).$$

We compute below that $R\underline{Hom}_{A^G}(A, A^G) \simeq A$ as A-modules, so $\rho^! \mathscr{O}_U \simeq \mathscr{O}_V$ as desired. Let $R = A^G$. By [Ols07b, Prop 2.2] there are two possibilities for A:

1. $A = R[x]/(x^r - t)$ for some $t \in R$ and $G = \langle \zeta \rangle$ is the group of r^{th} roots of unity with generator ζ, and G acts by $\zeta \cdot p(x) = p(\zeta x)$
2. $A = R[x, y]/(xy - t, x^r - u, y^r - v)$ for some $t, u, v \in R$ and $G = \langle \zeta \rangle$ is the group of r^{th} roots of unity with generator ζ, and G acts by $\zeta \cdot p(x, y) = p(\zeta x, \zeta^{-1} y)$

Case 1: As an R-module, A is isomorphic to the sum of its G-eigenspaces, each of which is a free R-module:

$$A = R \oplus xR \oplus x^2 R \oplus \ldots \oplus x^{r-} R \tag{2.26}$$

Let $e_i^\vee \in \mathrm{Hom}_R(A, R)$ denote the projection to the $x^i R$-factor in (2.26). Then considering $\mathrm{Hom}_R(A, R)$ as an A-module, x^i sends $e_{r-1}^\vee$ to $e_{r-i-1}^\vee$ for $i = 1, \ldots, r-1$. This shows that the A-module homomorphism

$$A \xrightarrow{1 \mapsto e_{r-1}^\vee} \mathrm{Hom}_R(A, R)$$

is surjective. Since it is an isomorphism of R-modules, it is an isomorphism.

Case 2: In this case we may assume that $R = \mathbb{C}[u, v, t]/(uv - t^r)$ as every situation is the base change of this one (arguing as in Lemma 2.2.7). This computation is similar to the one in [Nir09, Lem 3.2].

As R-modules, we again have an eigenspace decomposition

$$A = R \oplus A_1 \oplus A_2 \oplus \ldots \oplus A_{r-1}$$

where ζ acts on A_i by multiplication by ζ^i. By Lemma 2.2.9 we have the following resolution of A by free R-modules:

$$\ldots \xrightarrow{d_3} R^{\oplus 2r-2} \xrightarrow{d_2} R^{\oplus 2r-2} \xrightarrow{d_1} R^{\oplus 2r-1} \xrightarrow{d_0} A \to 0 \tag{2.27}$$

Let $\{f_i, g_i\}_{i=1}^{r-1}$ denote a free basis for $R^{\oplus 2r-2}$ and let e denote the additional basis element of $R^{\oplus 2r-1}$. Then the maps d_i are defined as follows:

$$\begin{array}{lllll} d_0: & e_0 \mapsto 0 & d_i,\ i \text{ odd}: & f_i \mapsto vf_i - t^i g_{r-i} & d_i,\ i>0 \text{ even}: \quad f_i \mapsto uf_i + t^i g_{r-i} \\ & f_i \mapsto x^i & & g_{r-i} \mapsto ug_{r-i} - t^{r-i} f_i & \qquad\qquad\qquad\quad g_{r-i} \mapsto t^{r-i} f_i + vg_{r-i} \\ & g_i \mapsto y^i & & & \end{array}$$

Now $R\underline{Hom}_R(A, R)$ is computed by the chain complex

$$0 \to R^{\oplus 2r-1} \xrightarrow[d_1^\vee]{\begin{bmatrix} v & -t^i \\ -t^{r-i} & u \end{bmatrix}} R^{\oplus 2r-2} \xrightarrow[d_2^\vee]{\begin{bmatrix} u & t^i \\ t^{r-i} & v \end{bmatrix}} R^{\oplus 2r-2} \to \dots,$$

where we have written out the matrices for the maps resolving A_i for $i = 1, \dots, r-1$ (see (2.28)). Now the argument in Lemma 2.2.9 shows that the cohomology of this complex vanishes in positive degrees. In degree 0, we find $\ker(d_1^\vee)$ is equal to the image of $d_0^\vee$. Certainly $e_0^\vee \in \ker(d_1^\vee)$; on the other hand, this element generates $\ker(d_1^\vee)$ as an A-module. Indeed, we have

$$x^i \cdot e_0^\vee = uf_{r-i}^\vee + t^i g_i^\vee \qquad\qquad y^{r-i} \cdot e_0^\vee = t^{r-i} f_{r-i}^\vee + vg_i^\vee$$

which are precisely generators of $\ker(d_1^\vee) = \ker(d_3^\vee)$ which is equal to the image of $d_2^\vee$. This shows that the A-module homomorphism $A \to \mathrm{Hom}_R(A, R)$ induced by sending 1 to $e_0^\vee$ is surjective. It is injective because it is injective as a map of R-modules: the relations between x^i and y^{r_i} are given by the image of d_1 (on the i^{th} eigenspace), and the relations between $uf_{r-i}^\vee + t^i g_i^\vee$ and $t^{r-i} f_{r-i}^\vee + vg_i^\vee$ are given by the image of $d_1^\vee$ (on the $(r-i)^{th}$ eigenspace), and these are the same matrix, both equal to

$$\begin{bmatrix} v & -t^{r-i} \\ -t^i & u \end{bmatrix}.$$

This shows that $A \simeq \mathrm{Hom}_R(A, R)$ as desired. □

Lemma 2.2.9. *The complex* (2.27) *is a free resolution of A over R.*

Proof. We restrict (2.27) to each of the eigenspaces of A. We must show for $i =$

$1, \ldots, r-1$ that the complex

$$\cdots \xrightarrow[d_3]{\begin{bmatrix} v & -t^{r-i} \\ -t^i & u \end{bmatrix}} R^2 \xrightarrow[d_2]{\begin{bmatrix} u & t^{r-i} \\ t^i & v \end{bmatrix}} R^2 \xrightarrow[d_1]{\begin{bmatrix} v & -t^{r-i} \\ -t^i & u \end{bmatrix}} R^2 \xrightarrow[d_0]{\substack{f_i \mapsto x^i \\ g_{r-i} \mapsto y^{r-i}}} A_i \to 0 \tag{2.28}$$

is a resolution of A_i. We write $\{f = f_i, g = g_{r-i}\}$ for the free basis of each R^2. Clearly this is a chain complex. We check that it is acyclic.

Degree 0: Let $p, q \in R = \mathbb{C}[u, v, t]/(uv - t^r)$ be polynomials and suppose $pf + qg \in \ker(d_0)$. To show it is in the image of d_1 we find $a, b \in R$ such that

$$p = av - bt^{r-i} \qquad q = bu - t^i a. \tag{2.29}$$

Using the relation $uv = t^r$, we may uniquely write

$$p = p_u + p_v + p_0 \qquad q = q_u + q_v + q_0 \tag{2.30}$$

with $p_u, q_u \in u\mathbb{C}[t, u]$, $p_v, q_v \in v\mathbb{C}[t, v]$, and $p_0, q_0 \in \mathbb{C}[t]$. Since $pf + qg \in \ker(d_0)$, we know

$$px^i + qy^{r-i} = 0 \qquad \text{in } A.$$

Substituting the expressions (2.30) for p, q, writing $p_v = v(p_v/v)$ and $q_u = u(q_u/u)$, and applying the relations $xy = t, u = x^r$, and $v = y^r$ to eliminate xy-terms, we see that

$$p_u = -t^{r-i}(q_u/u) \qquad q_v = -t^i(p_v/v) \qquad p_0 = q_0 = 0. \tag{2.31}$$

Now set $a = p_v/v$ and $b = q_u/u$. Then (2.31) implies that (2.29) is satisfied.

Degree i odd: For simplicity of notation we work with $i = 1$. Let $p, q \in R$ and suppose that $pf + qg \in \ker(d_1)$. We show it is in the image of d_2 by finding $a, b \in R$ such that

$$p = au + bt^{r-i} \qquad q = at^i + bv. \tag{2.32}$$

Decompose p and q as in (2.30). Since $pf + qg \in \ker(d_1)$ we know $p(vf - t^i g) + g(ug - t^{r-i}f) = 0$ and hence

$$vp = t^{r-i}q \qquad uq = t^i p. \tag{2.33}$$

We get the relations

$$t^{r-i}q_v = v(p_v + p_0) \qquad t^i p_u = u(q_u + q_0)$$

by equating the v-parts of the first relation in (2.33) (resp. the u-parts of the second relation). Setting $a = p_u/u$ and $b = q_v/v$ we see that (2.32) is satisfied.

Degree $i > 0$ even: The argument in this case is parallel to the one used in the case where i is odd. □

Pushing forward stacks

Let $\mathcal{C} \to \mathcal{C}'$ be a continuous morphism of sites inducing a morphism of topoi (f^{-1}, f_*) and let $\mathcal{P}$ be a stack on $\mathcal{C}'$. Then there is a pushforward stack $f_*\mathcal{P}$ on $\mathcal{C}'$ whose fibers are given by the familiar rule $f_*\mathcal{P}(U) = \mathcal{P}(f(U))$ for $U \in \mathcal{C}$.

If $X^\bullet$ is a simplicial space and $\varpi : X^\bullet \to \mathcal{X}$ is an augmentation to a Deligne-Mumford stack, then we have a morphism of topoi $X^\bullet_{\text{et}} \to \mathcal{X}_{\text{et}}$ that does not come from a morphism of sites. However, if $\mathcal{P}$ is a stack on $X^\bullet_{\text{et}}$, we can still define the pushforward $\pi_*\mathcal{P}$ as a stack on $\mathcal{X}$. If $U \to \mathcal{X}$ is an étale map from a scheme, let $U^\bullet = U \times_{\mathcal{X}} X^\bullet$, and set

$$(\varpi_*\mathcal{P})(U) = \operatorname{Hom}_{X^\bullet_{\text{et}}}(U^\bullet, \mathcal{P}) \tag{2.34}$$

where the right hand side is the categorical hom for stacks on $X^\bullet_{\text{et}}$. Given morphisms $f : U \to V$, $u \in \varpi_*\mathcal{P}(U)$, and $v \in \varpi_*\mathcal{P}(V)$, an arrow from u to v over f is a commuting diagram

$$\begin{array}{ccccc} U & \longleftarrow & U^\bullet & \xrightarrow{u} & \mathcal{P} \\ \downarrow{\scriptstyle f} & & \downarrow & \nearrow{\scriptstyle v} & \\ V & \longleftarrow & V^\bullet & & \end{array}$$

where the square is fibered. This defines a stack $\varpi_*\mathcal{P}$ on $\mathcal{X}_{\text{et}}$.

We will apply this construction to Picard stacks. We begin with some general observations. First, if $\mathcal{P}$ is a Picard stack, so is $\varpi_*\mathcal{P}$ (see [73, Sec XVIII.1.4.9]). Second, let $\mathcal{P} \to \mathcal{Q}$ be a morphism of Picard stacks on $X^\bullet_{\text{et}}$ with kernel $\mathcal{K}$, meaning that there is a fiber diagram

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow{e} & \mathcal{P} \end{array}$$

where $\bullet$ is the constant sheaf with all its fibers equal to a single point. Then $\varpi_*\mathcal{K}$ is the kernel of $\varpi_*\mathcal{P} \to \varpi_*\mathcal{Q}$.

Finally, we wish to make the functor ϖ_* more explicit for a certain kind of Picard stack. Let $\mathcal{C}$ be a site. Recall from [73, Sec XVIII.1.4.11] the functor ch from the bounded derived category $D^{[-1,0]}(\mathcal{C})$ to the category of Picard stacks on $\mathcal{C}$ (in the latter category, arrows are isomorphism classes of morphisms of stacks). Given a two-term complex $\mathcal{F}^\bullet = \mathcal{F}^{-1} \to \mathcal{F}^0$ with $\mathcal{F}^{-1}$ injective and $U \in \mathcal{C}$, the objects of $ch(\mathcal{F})(U)$ are $\mathcal{F}^0$ and an arrow from x to y is an element $f \in \mathcal{F}^{-1}(U)$ with $df = y - x$. For general $\mathcal{F}$, this construction defines a prestack, and $ch(\mathcal{F})$ is the associated stack. In any case the isomorphism classes of objects of $ch(\mathcal{F}^\bullet)$ are the sheaf $H^0(\mathcal{F}^\bullet)$, and the automorphsims of the identity element are the sheaf $H^{-1}(\mathcal{F}^\bullet)$.

A rule for pushing forward $ch(\mathcal{F}^\bullet)$ along a morphism of sites was given in [73, Construction XVIII.1.4.19]. The same formula works for pushing forward along an augmentation $X^\bullet \to \mathcal{X}$ from a simplicial algebraic space to a Deligne-Mumford stack.

Lemma 2.2.10. *Let $\mathcal{F}$ be an object of $D^{[-1,\infty]}(X^\bullet_{et})$. Then there is a natural isomorphism*

$$\varpi_* ch(\tau_{\leq 0}\mathcal{F}) \simeq ch(\tau_{\leq 0} R\varpi_*\mathcal{F}) \tag{2.35}$$

Proof. We unwind the definitions of each side. By [73, Prop XVIII.1.4.15] we may assume that $\mathcal{F}$ is a complex of injectives. Then $R\pi_*\mathcal{F} = \pi_*\mathcal{F}$ is still a complex of injectives, so the fiber of the right hand side over a scheme U with an étale map to $\mathcal{X}$ is the quotient of $\pi_* \ker(\mathcal{F}^0 \to \mathcal{F}^1)(U)$ by $(\pi_*\mathcal{F}^{-1})(U)$. If $f : U \to V$ is a morphism of schemes étale over $\mathcal{X}$, then an arrow from $u \in \pi_* \ker(\mathcal{F}^0 \to \mathcal{F}^1)(U)$ to $v \in \pi_* \ker(\mathcal{F}^0 \to \mathcal{F}^1)(V)$ is an arrow $x \to f^*y$ given by a morphism in $\pi_*\mathcal{F}^{-1}(U)$.

By definition, the fiber of the left hand side is $\mathrm{Hom}_{X^\bullet_{\mathrm{et}}}(U^\bullet, ch(\tau_{\leq 0}\mathcal{F}))$, which is equal to $\mathrm{Hom}_{U^\bullet_{\mathrm{et}}}(U^\bullet, ch(\tau_{\leq 0}\mathcal{F})|_{U^\bullet})$ by Lemma 2.3.1. To compute $ch(\tau_{\leq 0}\mathcal{F})|_{U^\bullet}$, note that this is in particular a stack on each U^n (with some compatibility data). We can check directly that $(\tau_{\leq 0}\mathcal{F})|_{U^n} = (\tau_{\leq 0}\mathcal{F}|_{U^n})$ (using [Stacks, Tag 03F3] and the fact that $\mathcal{F}$ is a complex of injectives), so we see that the fiber of the left hand side of (2.35) is $\mathrm{Hom}_{U^\bullet_{\mathrm{et}}}(U^\bullet, ch(\tau_{\leq 0}\mathcal{F}|_{U^\bullet}))$. Writing $\pi^*\mathscr{O}_U = \mathscr{O}_{U^\bullet}$ for the sheaf represented by $U^\bullet$, this groupoid is the quotient of $\mathrm{Hom}_{U^\bullet}(\pi^*\mathscr{O}_U, \ker(\mathcal{F}^0 \to \mathcal{F}^1)|_{U^\bullet})$ by $\mathrm{Hom}_{U^\bullet_{\mathrm{et}}}(\pi^*\mathscr{O}_U, \mathcal{F}^{-1}|_{U^\bullet})$. This is equivalent via adjunction to the fiber in the previous paragraph.

Likewise, given $f : U \to V$, an arrow lying above f in the right hand side is a diagram

$$\begin{array}{ccc} ch(\tau_{\leq 0}\mathcal{F}|_{U^\bullet}) & \longrightarrow & ch(\tau_{\leq 0}\mathcal{F}|_{V^\bullet}) \\ \uparrow\downarrow & & \downarrow\uparrow \\ U^\bullet & \xrightarrow{f^\bullet} & V^\bullet \end{array}$$

that 2-commutes. One can see that these arrows are the same as those on the right hand side. □

2.2.3 Cotangent complex

In this section we define the cotangent complex of an arbitrary morphism of algebraic stacks. The first step is to define the sheaf of differentials of an algebraic space over an algebraic stack.

Differentials of an algebraic space relative to an algebraic stack

Let $X \to \mathcal{Y}$ be a morphism from an algebraic space X to an algebraic stack $\mathcal{Y}$. Fix $Y \to \mathcal{Y}$ a smooth cover by an algebraic space and let $Y_X = X \times_{\mathcal{Y}} Y$ be the fiber product (an algebraic space). Let $Y^\bullet$ and $Y_X^\bullet$ be the coskeletons of $Y \to \mathcal{Y}$ and $Y_X \to \mathcal{X}$, respectively. Let ϵ_Y denote the morphism of the topoi X_{et} and $Y^\bullet_{X,\text{et}}$ that was denoted ϵ_{et} in (2.8). Then let

$$\epsilon_Y^* \Omega^1_{X/\mathcal{Y}} = \Omega^1_{Y_X^\bullet/Y^\bullet} \quad \text{in } \mathrm{D}_{\mathrm{qc}}(Y^\bullet_{X,\text{et}})$$

and define

$$\Omega^1_{X/\mathcal{Y}} = R(\epsilon_Y)_*(\epsilon_Y^*\Omega^1_{X/\mathcal{Y}}) \quad \text{in } \mathrm{D}_{\mathrm{qc}}(X_{\text{et}}).$$

Here $\Omega^1_{Y_X^\bullet/Y^\bullet}$ is the sheaf of relative differentials for a morphism of ringed topoi, defined as in [Stacks, Tag 04BQ]. It is cartesian by the base change property of differentials. A priori the definition of $\Omega^1_{X/\mathcal{Y}}$ depends on the choice of Y, but part 1 of the following lemma shows that the sheaves arising from any two choices of Y are canonically isomorphic. See also [LM00, (17.5.6)].

Lemma 2.2.11. *The sheaf $\Omega_{X/\mathcal{Y}}$ has the following properties.*

1. *Given a commuting diagram*

$$\begin{array}{ccccc} X' & \longrightarrow & \mathcal{Y}' & \longleftarrow & Y' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow & & \big\downarrow \\ X & \longrightarrow & \mathcal{Y} & \longleftarrow & Y \end{array} \tag{2.36}$$

with X' and X algebraic spaces; $\mathcal{Y}'$ and $\mathcal{Y}$ algebraic stacks; and $Y \to \mathcal{Y}$, $Y' \to \mathcal{Y}'$ smooth covers by algebraic spaces, there is a canonical morphism

$$df : Lf^*\Omega^1_{X/\mathcal{Y}} \to \Omega^1_{X'/\mathcal{Y}'}$$

that is an isomorphism if the square with X' and $\mathcal{Y}'$ is fibered. The morphism df is functorial with respect to vertical compositions of diagrams (2.36)

2. *Given morphisms*

$$X' \xrightarrow{f} X \to \mathcal{Y} \tag{2.37}$$

with X' and X algebraic spaces and $\mathcal{Y}$ an algebriac stack, there is an exact sequence

$$f^*\Omega^1_{X/\mathcal{Y}} \to \Omega^1_{X'/\mathcal{Y}} \to \Omega_{X'/X} \to 0.$$

This sequence is exact on the left if f is smooth and it is functorial in the triple (2.37)*.*

The functoriality of the morphisms *df* mentioned in part (1) of this lemma is analogous to the chain rule in elementary calculus.

Proof. For (1), observe that there is a commuting diagram

$$\begin{array}{ccccc} Y'_{X'} & \longrightarrow & X' \times_X Y_X & \longrightarrow & Y_X \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ Y' & \longrightarrow & \mathcal{Y}' \times_{\mathcal{Y}} Y & \longrightarrow & Y \end{array} \tag{2.38}$$

with the left square fibered, and the right square is also fibered if the left square of (2.36) is fibered. Let $f^\bullet : (Y'_{X'})^\bullet \to (Y_X)^\bullet$ be the induced morphism of strictly simplicial spaces. Then there is a canonical map

$$d(f^\bullet) : (f^\bullet)^*\Omega^1_{Y_X^\bullet/Y^\bullet} \to \Omega^1_{Y'_{X'}{}^\bullet/Y'^\bullet}$$

which is is an isomorphism if both squares in (2.38) are fibered, because the map at each level comes from the canonical morphism of differentials for a fibered square of algebraic spaces (use [Stacks, Tag 05ZC]). Using (2.14), this defines the required canonical morphism. Its functoriality follows from [Stacks, Tag 05Z7].

For (2), choose a smooth cover $Y \to \mathcal{Y}$. Then we have a diagram of fibered squares

$$\begin{array}{ccccc} Y_{X'} & \xrightarrow{f^\bullet} & Y^\bullet_X & \longrightarrow & Y^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & X & \longrightarrow & \mathcal{Y} \end{array}$$

which leads to an exact sequence

$$(f^\bullet)^*\Omega^1_{Y^\bullet_X/Y^\bullet} \to \Omega^1_{Y^\bullet_{X'}/Y^\bullet} \to \Omega_{Y^\bullet_{X'}/Y^\bullet_X} \to 0$$

which is exact on the left if $Y^n_{X'} \to Y^n_X$ is smooth for each n. Exactness may be checked level-wise, where these claims follow from [Stacks, Tag 05Z8] and [Stacks, Tag o6BI]. Using (2.14), this defines the required exact sequence. Functoriality follows from the functoriality in part (1). □

Cotangent complex of algebraic stacks

Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Fix $Y \to \mathcal{Y}$ and $X \to \mathcal{X} \times_{\mathcal{Y}} Y$ smooth covers by algebraic spaces. Let $X^\bullet$, $Y^\bullet$, and $Y^\bullet_{\mathcal{X}}$ be the coskeletons of $X \to \mathcal{X}$, $Y \to \mathcal{Y}$, and $\mathcal{X} \times_{\mathcal{Y}} Y \to \mathcal{X}$, respectively. Let

$$\varpi_X^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}} = [\mathbb{L}_{X^\bullet/Y^\bullet} \to \Omega^1_{X^\bullet/Y^\bullet_{\mathcal{X}}}] \tag{2.39}$$

be the object of $D(X^\bullet_{\mathrm{et}})$ that, on X^n, is equal to

$$\mathbb{L}_{X^n/Y^n} \xrightarrow{\partial} \Omega^1_{X^n/Y^n_{\mathcal{X}}}.$$

Here, the map δ is the augmentation $\mathbb{L}_{X^n/Y^n} \to \Omega^1_{X^n/Y^n}$ followed by the canonical map of differentials for algebraic spaces over algebraic stacks that was defined in Lemma 2.2.11, and (2.39) is read as a complex with $\Omega^1_{X^\bullet/Y^\bullet_{\mathcal{X}}}$ in degree 1. Note that $\mathbb{L}_{X^\bullet/Y^\bullet}$ and $\Omega^1_{X^n/Y^n_{\mathcal{X}}}$ may not individually define objects of $\mathrm{D}_{\mathrm{qc}}(X^\bullet_{\mathrm{et}})$ (their cohomology sheaves may fail to be cartesian). However, by [Ols07a, Lem 8.3], the complex (2.39) is in

$\mathrm{D_{qc}}(X^\bullet_{\text{et}})$. Set

$$\mathbb{L}_{\mathcal{X}/\mathcal{Y}} = R(\varpi_X)_*(\varpi_X)^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \quad \text{in } \mathrm{D_{qc}}(\mathcal{X}_{\text{lis-et}}).$$

Given a commuting diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array} \tag{2.40}$$

of algebraic stacks, there is a canonical morphism

$$df : Lf^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \to \mathbb{L}_{\mathcal{X}'/\mathcal{Y}'} \tag{2.41}$$

defined as follows. Choose a diagram of algebraic spaces

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

with smooth surjective maps $X' \to \mathcal{X}'$, $X \to \mathcal{X}$, $Y' \to \mathcal{Y}'$, and $Y \to \mathcal{Y}$, and such that $X \to Y \times_{\mathcal{Y}} \mathcal{X}$ and $X' \to Y' \times_{\mathcal{Y}'} \mathcal{X}'$ are smooth and surjective. Then df is given by the morphisms

$$\begin{array}{ccc} L(f^\bullet)^*\mathbb{L}_{X^\bullet/Y^\bullet} & \xrightarrow{d(f^\bullet)} & \mathbb{L}_{X'^\bullet/Y'^\bullet} \\ \downarrow & & \downarrow \\ L(f^\bullet)^*\Omega^1_{X^\bullet/Y^\bullet_{\mathcal{X}}} & \xrightarrow{d(f^\bullet)} & \Omega^1_{X'^\bullet/Y'^\bullet_{\mathcal{X}'}} \end{array}$$

where the top map df is the canonical map of cotangent complexes and the bottom is the functoriality morphism of Lemma 2.2.11. The square defines a morphism of complexes by functoriality of the functoriality morphism.

From [Ols07a, Sec 8] we have the following properties of $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$.

1. The object $\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in \mathrm{D_{qc}}(\mathcal{X}_{\text{lis-et}})$ and the functoriality morphism df are independent of the choices involved, up to canonical isomorphism.

2. If the square (2.40) is fibered and either $\mathcal{X} \to \mathcal{Y}$ or $\mathcal{Y}' \to \mathcal{Y}$ is flat, then (2.41) is an isomorphism.

3. If $\mathcal{Y} \to \mathcal{Z}$ is another morphism of algebraic stacks, then there is a distinguished

triangle

$$Lg^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \to .$$

We close with a lemma on the functoriality of the cotangent complex.

Lemma 2.2.12. *Suppose we have the following 2-commuting diagram of algebraic stacks.*

$$\begin{array}{ccccc}
\mathcal{X} & & \xrightarrow{h} & & \mathcal{Z} \\
\downarrow{\scriptstyle \pi} & \searrow^{f} & & \nearrow^{g} & \downarrow{\scriptstyle \pi} \\
 & & \mathcal{Y} & & \\
 & & \downarrow{\scriptstyle \pi} & & \\
\mathcal{A} & & \longrightarrow & & \mathcal{C} \\
 & \searrow_{k} & & \nearrow & \\
 & & \mathcal{B} & &
\end{array}$$

1. *There is a commuting diagram in* $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$

$$\begin{array}{ccc}
Lh^*\mathbb{L}_{\mathcal{Z}/\mathcal{C}} & \xrightarrow{\quad dh \quad} & \mathbb{L}_{\mathcal{X}/\mathcal{A}} \\
 & {\scriptstyle Lf^*dg}\searrow \quad \nearrow{\scriptstyle df} & \\
 & Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{B}} &
\end{array}$$

   where the arrow $Lh^*\mathbb{L}_{\mathcal{Z}/\mathcal{C}} \to Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{B}}$ *also incorporates an identification* $Lh^* = Lf^* \circ Lg^*$.

2. *There is a morphism of distinguished triangles*

$$\begin{array}{ccccccc}
L\pi^*f^*\mathbb{L}_{\mathcal{B}/\mathcal{C}} & \longrightarrow & L\pi^*\mathbb{L}_{\mathcal{A}/\mathcal{C}} & \longrightarrow & L\pi^*\mathbb{L}_{\mathcal{A}/\mathcal{B}} & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
Lg^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} & \longrightarrow & \mathbb{L}_{\mathcal{X}/\mathcal{Z}} & \longrightarrow & \mathbb{L}_{\mathcal{X}/\mathcal{Y}} & \longrightarrow &
\end{array} \tag{2.42}$$

   where the arrow $L\pi^*Lk^*\mathbb{L}_{\mathcal{B}/\mathcal{C}} \to Lg^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}$ *also incorporates an identification* $L\pi^* \circ Lk^* = Lf^* \circ L\pi^*$.

Proof. To prove (1), choose smooth covers of all the stacks so that they fit into a commuting diagram in the category of algebraic spaces as follows.

$$\begin{array}{ccccc}
X & & \xrightarrow{h} & & Z \\
\downarrow{\scriptstyle \pi} & \searrow^{f} & & \nearrow^{g} & \downarrow{\scriptstyle \pi} \\
 & & Y & & \\
 & & \downarrow{\scriptstyle \pi} & & \\
A & & \longrightarrow & & C \\
 & \searrow_{k} & & \nearrow & \\
 & & B & &
\end{array}$$

Ensure moreover that $Z \to C_{\mathcal{Z}} := C \times_{\mathcal{C}} \mathcal{Z}$, $Y \to B_{\mathcal{Y}} := B \times_{\mathcal{B}} \mathcal{Y}$, and $X \to A_{\mathcal{X}} := A \times_{\mathcal{A}} \mathcal{X}$ are smooth and surjective. Then we get a commuting diagram

$$\begin{array}{ccccc} L(h^\bullet)^*\mathbb{L}_{Z^\bullet/C^\bullet} & \xrightarrow{L(f^\bullet)^*d(g^\bullet)} & L(f^\bullet)^*\mathbb{L}_{Y^\bullet/B^\bullet} & \xrightarrow{d(f^\bullet)} & \mathbb{L}_{X^\bullet/A^\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ L(h^\bullet)^*\Omega^1_{Z^\bullet/C^\bullet_{\mathcal{Z}}} & \xrightarrow{L(f^\bullet)^*d(g^\bullet)} & L(f^\bullet)^*\Omega^1_{Y^\bullet/B^\bullet_{\mathcal{Y}}} & \xrightarrow{d(f^\bullet)} & \Omega^1_{X^\bullet/A^\bullet_{\mathcal{X}}} \end{array}$$

The composition of the top arrows is the canonical morphism of cotangent complexes for $h^\bullet$, and likewise the composition of the bottom arrows is $d(h^\bullet)$ by Lemma 2.2.11.

To prove (2), form two commutative diagrams as in [Ols07a, (8.10.4)]: one for the triangle $\mathcal{XYZ}$ and one for the triangle $\mathcal{ABC}$. Ensure that the first diagram maps to the second. Then we get two copies of the diagram in [Ols07a, (8.10.5)]. The pullback of the diagram for $\mathcal{ABC}$ maps to the diagram for $\mathcal{XYZ}$: for the exact sequences of cotangent complexes (viewed as simplicial modules), this follows from [Ill71, (III.2.1.1.6)]; and for the exact sequences of sheaves of differentials it follows from Lemma 2.2.11. □

2.3 Functoriality

Let Z, $\mathcal{C}$, $\mathfrak{U}$, $\mathbb{E}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})}$, and $\phi_{\underline{\mathrm{Sec}}(Z/\mathcal{C})}$ be as in Section 2.1.1. This section was originally the appendix of the joint work [CJW19] with Qile Chen and Felix Janda; I thank my coauthors for helping me to polish it.

2.3.1 Properties of the moduli

Suppose we have morphisms of algebraic stacks

$$Z \to W \to \mathcal{C} \xrightarrow{\pi} \mathfrak{U} \tag{2.43}$$

where both $Z \to \mathfrak{U}$ and $W \to \mathfrak{U}$ are locally finitely presented, quasi-separated, and have affine stabilizers. We now prove some canonical isomorphisms of moduli of sections. The following observation will be useful.

Lemma 2.3.1. *Fix a diagram of algebraic stacks with morphisms as below, so that* *the square is fibered (and includes a 2-morphism α):*

$$\begin{array}{ccccc} & & F & \xrightarrow{z} & D \\ & & \downarrow{\scriptstyle y} & & \downarrow \\ A & \xrightarrow{x} & B & \longrightarrow & C \end{array}$$

Then the arrow $F \to D$ induces an equivalence of groupoids

$$\mathrm{Hom}_B(A, F) \xrightarrow{\sim} \mathrm{Hom}_C(A, D).$$

Proof. The domain is precisely the fiber product

$$\begin{array}{ccc} \mathrm{Hom}_B(A, F) & \longrightarrow & \mathrm{Hom}(A, F) \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{x} & \mathrm{Hom}(A, B) \end{array}$$

There is an analogous fibered square defining $\mathrm{Hom}_C(A, D)$; this second fibered square factors, yielding a fiber product

$$\begin{array}{ccc} \mathrm{Hom}_C(A, D) & \longrightarrow & \mathrm{Hom}(A, B) \times_{\mathrm{Hom}(A,C)} \mathrm{Hom}(A, D) \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{x} & \mathrm{Hom}(A, B) \end{array}$$

But there is an isomorphism $\mathrm{Hom}(A, F) \to \mathrm{Hom}(A, B) \times_{\mathrm{Hom}(A,C)} \mathrm{Hom}(A, D)$ sending f to $(y \circ f, z \circ f, \alpha)$. By the universal property of fiber products of groupoids, the induced morphism $\mathrm{Hom}_B(A, F) \to \mathrm{Hom}_C(A, D)$ is an isomorphism. □

In the context of (3.16), on $\underline{\mathrm{Sec}}(W/\mathcal{C})$ we have the universal curve (pullback of $\mathcal{C}$) and universal section, denoted $n\colon \mathcal{C}_{\underline{\mathrm{Sec}}(W/\mathcal{C})} \to W$.

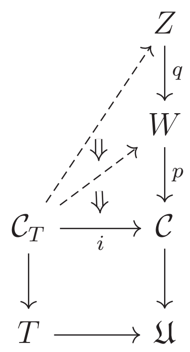
Lemma 2.3.2. *Let n^*Z denote the fiber product $\mathcal{C}_{\underline{\mathrm{Sec}}(W/\mathcal{C})} \times_W Z$. Then there is a canonical isomorphism*

$$\underline{\mathrm{Sec}}(n^*Z/\mathcal{C}_{\underline{\mathrm{Sec}}(W/\mathcal{C})}) \cong \underline{\mathrm{Sec}}(Z/\mathcal{C})$$

of stacks over $\mathfrak{U}$.

Proof. The canonical morphism $\Phi\colon \underline{\mathrm{Sec}}(n^*Z/\mathcal{C}_{\underline{\mathrm{Sec}}(W/\mathcal{C})}) \to \underline{\mathrm{Sec}}(Z/\mathcal{C})$ is a morphism of categories fibered in groupoids over $\mathfrak{U}$, so to show Φ is an equivalence, it suffices to study the induced map on fibers over a scheme $T \to \mathfrak{U}$.

We compute the fiber of $\mathcal{F} := \underline{\mathrm{Sec}}(n^*Z/\mathcal{C}_{\underline{\mathrm{Sec}}(W/\mathcal{C})})$. The fiber of $\mathcal{F}$ over an arrow $T \to \underline{\mathrm{Sec}}(W/\mathcal{C})$ is $\mathrm{Hom}_{\mathcal{C}_{\underline{\mathrm{Sec}}(W/\mathcal{C})}}(\mathcal{C}_T, n^*Z)$; by Lemma 2.3.1 this is equivalent to $\mathrm{Hom}_W(\mathcal{C}_T, Z)$. Hence $\mathcal{F}(T \to \mathfrak{U})$ is the groupoid of dotted arrows



Specifically, an object of $\mathcal{F}(\mathcal{T})$ is a tuple (z, w, τ, ω) where $z\colon \mathcal{C}_T \to Z$ and $w\colon \mathcal{C}_T \to W$ are 1-morphisms, and $\tau\colon q \circ z \to w$ and $\omega\colon p \circ w \to i$ are 2-morphisms. An arrow in $\mathcal{F}(T)$ from $(z_1, w_1, \tau_1, \omega_1)$ to $(z_2, w_2, \tau_2, \omega_2)$ is a pair of 2-morphisms $\alpha\colon w_1 \to w_2$ and $\beta\colon z_1 \to z_2$ such that $\omega_1 = \omega_2 \circ p(\alpha)$ and $\alpha \circ \tau_1 = \tau_2 \circ q(\beta)$.

Now let $\mathcal{G}$ be the usual construction of the fiber product for the diagram[3]

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\pi_2} & \underline{\mathrm{Sec}}(Z/\mathcal{C}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Sec}}(W/\mathcal{C}) & \xrightarrow{id} & \underline{\mathrm{Sec}}(W/\mathcal{C}) \end{array}$$

The map Φ factors as $\mathcal{F} \xrightarrow{\Phi'} \mathcal{G} \xrightarrow{\pi_2} \underline{\mathrm{Sec}}(Z/\mathcal{C})$. Of course π_2 is an equivalence of stacks over $\mathfrak{U}$. On the other hand, we claim that Φ' induces the literal identity map from $\mathcal{F}(T)$ to $\mathcal{G}(T)$. By definition, an object of the fiber of F is a tuple $(w, \omega; z, \zeta; \tau)$, where $w\colon \mathcal{C}_T \to W$ and $z\colon \mathcal{C}_T \to Z$ are 1-morphisms, $\omega\colon p \circ w \to i$ and $\zeta\colon p \circ q \circ z \to i$ are 2-morphisms, and $\tau\colon q \circ z \to w$ is a 2-morphism such that $\zeta = \omega \circ p(\tau)$. The final condition determines ζ from the other data, and hence these objects are literally the same as the objects of $\mathcal{F}$. Arrows in these two groupoids are also literally the same.

□

Lemma 2.3.3. *Let $Z \to \mathcal{C} \to \mathfrak{U}$ be as above. Suppose $Z' \to \mathcal{C}' \to \mathfrak{U}'$ is another tower*

[3] Compare with [Stacks, Tag 06N7].

of the same type, and suppose we have a commuting diagram of fibered squares

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathfrak{U}' & \xrightarrow{f} & \mathfrak{U} \end{array}$$

Then there is a canonical isomorphism

$$\underline{\mathrm{Sec}}(Z'/\mathcal{C}') \cong \underline{\mathrm{Sec}}(Z/\mathcal{C}) \times_{\mathfrak{U}} \mathfrak{U}' \tag{2.44}$$

of stacks over $\mathfrak{U}'$.

Proof. Let $\mathcal{F} = \underline{\mathrm{Sec}}(Z/\mathcal{C}) \times_{\mathfrak{U}} \mathfrak{U}'$. First, observe that a slight extension of the argument in [Stacks, Tag 06N7] shows that $\mathcal{F}$ is indeed fibered in groupoids over $\mathfrak{U}'$. So to show that the canonical map $\Phi\colon \underline{\mathrm{Sec}}(Z'/\mathcal{C}) \to \mathcal{F}$ is an equivalence, it suffices to show it is an equivalence on the fiber over arbitrary $x\colon T \to \mathfrak{U}'$.

The fiber $\mathcal{F}(T)$ has for objects tuples (a, α, n, ν) where (letting $C_a = C \times_{\mathfrak{U},a} T$) $a\colon T \to U$ and $n\colon C_a \to Z$ are 1-morphisms and $\alpha\colon f \circ x \to a$ and ν are 2-morphisms (ν witnesses the commutativity of a triangle, one of whose sides is n). A morphism in $\mathcal{F}(T)$ from (a, α, n, ν) to (b, β, m, μ) is a tuple (τ, σ) where $\tau\colon a \to b$ and $\sigma\colon n \to m \circ c_\tau$ are 2-morphisms (here $c_\tau\colon C_a \to C_b$ is the morphism induced by τ), such that (1) $\beta^{-1} \circ \tau \circ \alpha$ is the identity, and (2) the 2-cell with faces σ, ν, μ, and one other face determined by τ is commutative.

The fiber $\underline{\mathrm{Sec}}(Z'/\mathcal{C}')$ is by Lemma 2.3.1 equal to $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_T, Z)$. This groupoid has for objects pairs (n, ν) where $n\colon \mathcal{C}_T \to Z$ is a 1-morphism and ν is a 2-morphism witnessing the commutativity of the triangle over $\mathcal{C}$. A morphism from (n, ν) to (m, μ) is a 2-morphism $\sigma\colon n \to m$ such that the 2-cell with σ, ν, and μ commutes.

Let $\Phi_T\colon \underline{\mathrm{Sec}}(Z'/\mathcal{C}') \to \mathcal{F}(T)$ be the restriction of Φ to the fiber. Then Φ_T sends (n, ν) to $(f \circ x, id, n, \nu)$ and σ to (id, σ). The map Φ_T is essentially surjective because α induces an isomorphism from an object in the image of Φ_T to (a, α, n, ν). It is fully faithful because if $\beta = \alpha = id$, then condition (2) forces $\tau = id$. □

Lemma 2.3.4. *There is a natural morphism* $\underline{\mathrm{Sec}}(Z/\mathcal{C}) \to \underline{\mathrm{Sec}}(W/\mathcal{C})$. *If* $Z \to W$ *is a closed embedding, then so is* $\underline{\mathrm{Sec}}(Z/\mathcal{C}) \to \underline{\mathrm{Sec}}(W/\mathcal{C})$.

Proof. Let $\mathfrak{S}' = \underline{\mathrm{Sec}}(Z/\mathcal{C})$ and $\mathfrak{S} = \underline{\mathrm{Sec}}(W/\mathcal{C})$. Since $\mathfrak{S}' \to \mathfrak{U}$ is already locally of finite type, by [Stacks, Tag 04XV] it suffices to show that $\iota \colon \mathfrak{S}' \to \mathfrak{S}$ is universally closed and a monomorphism. The monomorphism property is immediate using the characterization in [Stacks, Tag 04ZZ]. By Lemma 2.3.3, to show that ι is universally closed it suffices to prove that it is a closed map, and since we already know that ι is a monomorphism, it suffices to show that $\iota(\mathfrak{S}')$ is closed in $\mathfrak{S}$. Now the set $\iota(\mathfrak{S}')$ consists of points whose $\pi_{\mathfrak{S}}$-fibers map completely into Z. Hence, $\mathfrak{S} \setminus \iota(\mathfrak{S}') = \pi_{\mathfrak{S}}(n^{-1}(W \setminus Z))$ where $n : \mathcal{C}_{\mathfrak{S}} \to W$ Since $\pi_{\mathfrak{S}}$ is flat, and hence open, this implies that $\iota(\mathfrak{S}')$ is closed, which finishes the proof of the lemma. $\square$

2.3.2 Properties of the obstruction theory

The morphism (2.1) is functorial in several senses. The proofs all use the following technical lemma, which is a reformulation of [Abr+, Lem 4.1].

Lemma 2.3.5. *Suppose we have a commuting diagram*

$$\begin{array}{ccccc} & & & \nearrow & Z \\ & & & & \downarrow \\ K' & \xrightarrow{\mu_K} & K & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ B' & \xrightarrow{\mu_B} & B & \longrightarrow & \mathfrak{U} \end{array}$$

where the squares are fibered. Then there is a commuting circuit

$$\begin{array}{ccccc} & L\mu_B^*\mathbb{E}_{B/\mathfrak{U}} & \xrightarrow{L\mu_B^*\phi_{B/\mathfrak{U}}} & L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} & \\ \swarrow{\sim} & & & & \| \\ R\pi_*((L\mu_K^* Lf^*\mathbb{L}_{Z/\mathcal{C}}) \otimes \omega^\bullet) & & & & L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \\ \searrow{F} & & & & \nearrow{\beta} \\ & R\pi_*((L\mu_K^* L\pi^*\mathbb{L}_{B/\mathfrak{U}}) \otimes \omega^\bullet) & \xrightarrow{\sim} & R\pi_*((L\pi^* L\mu_B^*\mathbb{L}_{B/\mathfrak{U}}) \otimes \omega^\bullet) & \end{array}$$

where β is the composition of (the inverse of) the projection morphism and the trace map as in (2.3).

Proof. We will demonstrate this circuit as the composition of three. The first circuit is the perimeter of the following commuting diagram

$$
\begin{array}{ccccc}
L\mu_B^* R\pi_*(Lf^*\mathbb{L}_{Z/\mathcal{C}} \otimes \omega^\bullet) & \longrightarrow & L\mu_B^* R\pi_*(\mathbb{L}_{K/\mathcal{C}} \otimes \omega^\bullet) & \longleftarrow & L\mu_B^* R\pi_*(L\pi^*\mathbb{L}_{B/U} \otimes \omega^\bullet) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
R\pi_*((L\mu_K^* Lf^*\mathbb{L}_{Z/\mathcal{C}}) \otimes \omega^\bullet) & \longrightarrow & R\pi_*((L\mu_K^*\mathbb{L}_{K/\mathcal{C}}) \otimes \omega^\bullet) & \xleftarrow{\sim} & R\pi_*((L\mu_K^* L\pi^*\mathbb{L}_{B/\mathfrak{U}}) \otimes \omega^\bullet)
\end{array}
$$

where the bottom row induces the the arrow F in the circuit above. The horizontal arrows are canonical maps of cotangent complexes and the vertical arrows are the composition of natural isomorphisms.

The second commuting circuit is the one in [Stacks, Tag 0B6B]. Unfortunately, as our derived functors do not come from morphisms of ringed spaces, it is not immediately obvious that the reference applies. In fact commutativity of this circuit can be shown for general symmetric monoidal categories, and the details are worked out in an updated version of [Hal, Appendix A] communicated to me privately.

$$
\begin{array}{ccccc}
L\mu_B^* R\pi_*(L\pi^*\mathbb{L}_{B/\mathfrak{U}} \otimes \omega^\bullet) & \xleftarrow{p} & L\mu_B^*(\mathbb{L}_{B/\mathfrak{U}} \otimes R\pi_*\omega^\bullet) & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle t} & & \\
R\pi_*(L\mu_K^*(L\pi^*\mathbb{L}_{B/\mathfrak{U}} \otimes \omega^\bullet)) & & L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \otimes L\mu_B^* R\pi_*\omega^\bullet & & \\
\downarrow{\scriptstyle t} & & & \searrow^{b} & \\
R\pi_*((L\mu_K^* L\pi^*\mathbb{L}_{B/\mathfrak{U}}) \otimes \omega^\bullet) & \xrightarrow{\sim} R\pi_*((L\pi^* L\mu_B^*\mathbb{L}_{B/\mathfrak{U}}) \otimes \omega^\bullet) & \xleftarrow{p} & & L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \otimes R\pi_* L\mu_K^*\omega^\bullet
\end{array}
$$

The third commuting circuit is the boundary of the diagram

$$
\begin{array}{ccc}
L\mu_B^*(\mathbb{L}_{B/\mathfrak{U}} \otimes R\pi_*\omega^\bullet) & \longrightarrow & L\mu_B^*(\mathbb{L}_{B/\mathfrak{U}} \otimes \mathscr{O}) \\
\downarrow{\scriptstyle t} & & \| \\
L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \otimes L\mu_B^* R\pi_*\omega^\bullet & \longrightarrow & L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \otimes L\mu_B^*\mathscr{O} \\
\downarrow{\scriptstyle b} & & \| \\
L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \otimes R\pi_*\mu_k^*\omega^\bullet & \longrightarrow & L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \otimes \mathscr{O}
\end{array}
$$

where the horizontal arrows are all induced by the trace map. The commutativity of the top square is the natural transformation of functors $L\mu_B^*(\mathbb{L}_{B/\mathfrak{U}} \otimes \bullet) \to L\mu_B^*(\mathbb{L}_{B/\mathfrak{U}}) \otimes L\mu_B^*(\bullet)$, and the commutativity of the bottom is functoriality of the pair $(\omega^\bullet, tr)$ (Proposition 2.2.6). □

The first functoriality lemma is designed to be used with Lemma 2.3.2.

Lemma 2.3.6. *Suppose we have a commuting diagram of algebraic stacks*

$$\begin{array}{ccccc} & & & & Z \\ & & & \nearrow^{f_Z} & \downarrow q \\ & & & & W \\ & & & \nearrow^{f_W} & \downarrow \\ K_Z & \xrightarrow{\mu_K} & K_W & \longrightarrow & \mathcal{C}_{\mathfrak{U}} \\ \downarrow & & \downarrow & & \downarrow \\ B_Z & \xrightarrow{\mu_B} & B_W & \longrightarrow & \mathfrak{U} \end{array} \tag{2.45}$$

where the squares are fibered. Then there is a morphism of distinguished triangles

$$\begin{array}{ccccccc} L\mu_B^*\mathbb{E}_{B_W/\mathfrak{U}} & \longrightarrow & \mathbb{E}_{B_Z/\mathfrak{U}} & \xrightarrow{d} & R\pi_*(Lf_Z^*\mathbb{L}_{Z/W} \otimes \omega^\bullet) & \longrightarrow & \\ \downarrow{\scriptstyle L\mu_B^*\phi_{B_W/\mathfrak{U}}} & & \downarrow{\scriptstyle \phi_{B_Z/\mathfrak{U}}} & & \downarrow & & \\ L\mu_B^*\mathbb{L}_{B_W/\mathfrak{U}} & \longrightarrow & \mathbb{L}_{B_Z/\mathfrak{U}} & \longrightarrow & \mathbb{L}_{B_Z/B_W} & \longrightarrow & \end{array} \tag{2.46}$$

where the distinguished triangle in the bottom row is the canonical one and the arrow labeled d is induced by differentiation.

Proof. By Lemma 2.2.12 we have morphisms of distinguished triangles

$$\begin{array}{ccccccc} Lf_Z^*q^*\mathbb{L}_{W/\mathcal{C}} & \longrightarrow & Lf_Z^*\mathbb{L}_{Z/\mathcal{C}} & \longrightarrow & Lf_Z^*\mathbb{L}_{Z/W} & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ L\mu_K^*\mathbb{L}_{K_W/\mathcal{C}} & \longrightarrow & \mathbb{L}_{K_Z/\mathcal{C}} & \longrightarrow & \mathbb{L}_{K_Z/K_W} & \longrightarrow & \\ \sim\uparrow & & \sim\uparrow & & \sim\uparrow & & \\ L\pi^*L\mu_B^*\mathbb{L}_{B_W/\mathfrak{U}} & \longrightarrow & L\pi^*\mathbb{L}_{B_Z/\mathfrak{U}} & \longrightarrow & L\pi^*\mathbb{L}_{B_Z/B_W} & \longrightarrow & \end{array}$$

where all arrows are induced by differentiation except for the two in the leftmost column, which also incorporate various commutations of derived pullback functors. We tensor the above diagram with $\omega^\bullet$ and apply $R\pi_*$ (here π is the projection $K_Z \to B_Z$).

The bottom row of the resulting diagram is the top row of the diagram

$$\begin{array}{ccccccc}
R\pi_*(L\pi^* L\mu_B^* \mathbb{L}_{B_W/\mathfrak{U}} \otimes \omega^\bullet) & \longrightarrow & R\pi_*(L\pi^* \mathbb{L}_{B_Z/\mathfrak{U}} \otimes \omega^\bullet) & \longrightarrow & R\pi_*(L\pi^* \mathbb{L}_{B_Z/B_W} \otimes \omega^\bullet) & \longrightarrow \\
\sim\uparrow & & \sim\uparrow & & \sim\uparrow & \\
L\mu_B^* \mathbb{L}_{B_W/\mathfrak{U}} \otimes R\pi_*\omega^\bullet & \longrightarrow & L\pi^* \mathbb{L}_{B_Z/\mathfrak{U}} \otimes R\pi_*\omega^\bullet & \longrightarrow & L\pi^* \mathbb{L}_{B_Z/B_W} \otimes R\pi_*\omega^\bullet & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
L\mu_B^* \mathbb{L}_{B_W/\mathfrak{U}} & \longrightarrow & \mathbb{L}_{B_Z/\mathfrak{U}} & \longrightarrow & \mathbb{L}_{B_Z/B_W} & \longrightarrow
\end{array}$$

where the top level of vertical arrows are the projection morphism (2.17) and the bottom vertical arrows are induced by the trace morphism. By Lemma 2.3.5 we may replace the leftmost column with $L\mu_B^* \phi_{B_W/\mathfrak{U}}$, completing the proof. □

Corollary 2.3.7. *Let $\phi_{B_Z/B_W} : \mathbb{E}_{B_Z/B_W} \to \mathbb{L}_{B_Z/B_W}$ be the morphism* (2.1) *constructed for the diagram below, so in particular $\mathbb{E}_{B_Z/B_W} = R\pi^* L g^* \mathbb{L}_{Z \times_W K_W / K_W}$.*

$$\begin{array}{ccc}
 & & Z \times_W K_W \\
 & \overset{g}{\nearrow} & \downarrow \\
K_Z & \longrightarrow & K_W \\
\downarrow & & \downarrow \\
B_Z & \longrightarrow & B_W
\end{array}$$

If $Z \to W$ or $K_W \to \mathcal{C}$ is flat, then there is a morphism of distinguished triangles

$$\begin{array}{ccccccc}
L\mu_B^* \mathbb{E}_{B_W/\mathfrak{U}} & \longrightarrow & \mathbb{E}_{B_Z/\mathfrak{U}} & \longrightarrow & \mathbb{E}_{B_Z/B_W} & \longrightarrow \\
\downarrow{\scriptstyle L\mu_B^* \phi_{B_W/\mathfrak{U}}} & & \downarrow{\scriptstyle \phi_{B_Z/\mathfrak{U}}} & & \downarrow{\scriptstyle \phi_{B_Z/B_W}} & \\
L\mu_B^* \mathbb{L}_{B_W/\mathfrak{U}} & \longrightarrow & \mathbb{L}_{B_Z/\mathfrak{U}} & \longrightarrow & \mathbb{L}_{B_Z/B_W} & \longrightarrow
\end{array} \tag{2.47}$$

where the bottom triangle is the canonical one.

Proof. We must show that there is a commuting square

$$\begin{array}{ccc}
R\pi_*(L f_Z^* \mathbb{L}_{Z/W} \otimes \omega^\bullet) & \xrightarrow{\sim} & \mathbb{E}_{B_Z/B_W} \\
\downarrow & & \downarrow \\
\mathbb{L}_{B_Z/B_W} & = & \mathbb{L}_{B_Z/B_W}
\end{array} \tag{2.48}$$

where the left vertical arrow is the third column of (2.46). Let $Z_K = Z \times_W K_W$, and

g and h be morphisms

$$K_Z \xrightarrow{g} Z_K \xrightarrow{h} Z.$$

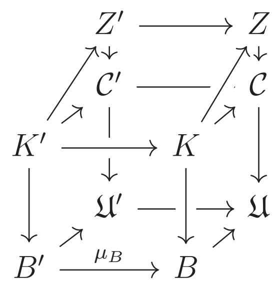
such that $h \circ g = f_Z$ (up to 2-morphism). By Lemma 2.2.12 (1) there is a commuting diagram

$$\begin{array}{ccccc} Lf_Z^*\mathbb{L}_{Z/W} & \xrightarrow{\gamma} & Lg^*Lh^*\mathbb{L}_{Z/W} & \xrightarrow{\sim} & Lg^*\mathbb{L}_{Z_K/K_W} \\ \downarrow & & & & \downarrow \\ \mathbb{L}_{K_Z/K_W} & & = & & \mathbb{L}_{K_Z/K_W} \end{array}$$

where γ is an isomorphism and the map labeled $\sim$ is an isomorphism because of the flatness assumption. From here it is straightforward to obtain the diagram (2.48). □

Our second functoriality lemma is designed to be used with Lemma 2.3.3 (see also [Abr+, Lem 4.1]).

Lemma 2.3.8. *Suppose we have a commuting diagram of algebraic stacks*



where all vertical squares in the bottom level are fibered (but the square with Z', Z, and $\mathcal{C}'$ need not be). Then there is a commuting diagram

$$\begin{array}{ccc} L\mu_B^*\mathbb{E}_{B/\mathfrak{U}} & \xrightarrow{L\mu_B^*\phi_{B/\mathfrak{U}}} & L\mu_B^*\mathbb{L}_{B/\mathfrak{U}} \\ \downarrow F & & \downarrow \\ \mathbb{E}_{B'/\mathfrak{U}'} & \xrightarrow{\phi_{B'/\mathfrak{U}'}} & \mathbb{L}_{B'/\mathfrak{U}'} \end{array} \qquad (2.49)$$

where the rightmost vertical arrow is the canonical one. If moreover the square with Z', Z, and $\mathcal{C}'$ is fibered and either $Z \to \mathcal{C}$ or $\mathcal{C}' \to \mathcal{C}$ is flat, then F is an isomorphism.

Proof. The proof is analogous to that of Lemma 2.3.6. In particular, if we let μ_Z denote the map $Z' \to Z$, then the map F consists of the canonical map $L\mu_Z^*\mathbb{L}_{Z/\mathcal{C}} \to \mathbb{L}_{Z'/\mathcal{C}'}$ and some isomorphisms. □

2.3.3 Equivariance of the moduli and its obstruction theory

In this section we use the functoriality properties of the moduli of sections and its obstruction theory to explain why these objects are equivariant when a group acts on (3.16). Let G be a flat, separated affine group scheme, finitely presented over $\mathbb{C}$. In this section we study the universal family

$$\begin{array}{ccc} & \nearrow & Z \\ & & \downarrow \\ \mathcal{C}_{Sec} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Sec}}(Z/\mathcal{C}) & \longrightarrow & \mathfrak{U} \end{array} \tag{2.50}$$

in the presence of a G-action. We use the definitions of G-stacks and equivariant morphisms in [Rom05a].

Lemma 2.3.9. *Given equivariant data for the column $Z \to \mathcal{C} \to \mathfrak{U}$ in* (2.50)*, it is possible to make the entire diagram equivariant.*

Proof. We have a commuting diagram

$$\begin{array}{ccccc} & \nearrow & G \times Z & \longrightarrow & Z \\ & & \downarrow & & \downarrow \\ G \times \mathcal{C}_{Sec} & \longrightarrow & G \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ G \times \underline{\mathrm{Sec}}(Z/\mathcal{C}) & \longrightarrow & G \times \mathfrak{U} & \longrightarrow & \mathfrak{U} \end{array}$$

where the left part of the diagram is just the product of G with (2.50), and the right part is the equivariance of the right column of (2.50)—in particular, the horizontal arrows are all action by G. Since all the squares are cartesian, by the universal property of $\underline{\mathrm{Sec}}(Z/\mathcal{C})$, this diagram factors canonically through the original one (2.50). This is the desired equivariance. □

Thanks to the functoriality of our constructions, the equivariant nature of $\underline{\mathrm{Sec}}(Z/\mathcal{C})$ is not difficult to understand.

Lemma 2.3.10. *There is a natural isomorphism*

$$[\underline{\mathrm{Sec}}(Z/\mathcal{C})/G] \simeq \underline{\mathrm{Sec}}([Z/G]/[\mathcal{C}/G]).$$

Under the resulting canonical map $\underline{\mathrm{Sec}}(Z/\mathcal{C}) \to \underline{\mathrm{Sec}}(Z/\mathcal{C})/G \xrightarrow{\sim} \underline{\mathrm{Sec}}([Z/G]/[\mathcal{C}/G])$*, the candidate obstruction theory on* $\underline{\mathrm{Sec}}([Z/G]/[\mathcal{C}/G])$ *pulls back to the candidate obstruction theory on* $\underline{\mathrm{Sec}}(Z/\mathcal{C})$*. In particular, the latter candidate obstruction theory is equivariant.*

Proof. We have a fiber diagram

$$\begin{array}{ccc} Z & \longrightarrow & [Z/G] \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & [\mathcal{C}/G] \\ \downarrow & & \downarrow \\ \mathfrak{U} & \longrightarrow & [\mathfrak{U}/G] \end{array}$$

where the horizontal maps are fppf covers (see [Rom05a, Theorem 4.1]). Hence by Lemma 2.3.8 we have the following commuting diagram where the square is fibered.

$$\begin{array}{ccc} & \nearrow & [\underline{\mathrm{Sec}}(Z/\mathcal{C})/G] \\ & & \downarrow F \\ \underline{\mathrm{Sec}}(Z/\mathcal{C}) & \xrightarrow{\rho_S} & \underline{\mathrm{Sec}}([Z/G]/[\mathcal{C}/G]) \\ \downarrow q & & \downarrow p \\ \mathfrak{U} & \xrightarrow{\rho_{\mathfrak{U}}} & [\mathfrak{U}/G] \end{array} \tag{2.51}$$

The map ρ_S is equivariant and hence factors as depicted. In fact, the outer trapezoid is also fibered, since it is a commuting diagram of G-torsors. Again, both horizontal maps and the diagonal map are fppf covers, so by descent the map labeled F is an isomorphism. Setting $X = [\underline{\mathrm{Sec}}(Z/\mathcal{C})/G] \simeq \underline{\mathrm{Sec}}([Z/G]/[\mathcal{C}/G])$, Lemma 2.3.8 also gives us a commuting square

$$\begin{array}{ccc} L\rho_S^*\mathbb{E}_{X/[\mathfrak{U}/G]} & \xrightarrow{\sim} & \mathbb{E}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}} \\ \downarrow L\rho_S^*\phi & & \downarrow \phi \\ L\rho_S^*\mathbb{L}_{X/[\mathfrak{U}/G]} & \xrightarrow{\sim} & \mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}} \end{array}$$

where both horizontal arrows are isomorphisms. This shows the equivariance of $\phi_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}}$. $\square$

For some applications (e.g. localization) it is useful to have an *absolute* obstruction theory $\phi_{abs}\colon \mathbb{E}_{abs} \to \mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})}$ on $\underline{\mathrm{Sec}}(Z/\mathcal{C})$. This is defined as a (shifted) mapping cone fitting in the following diagram:

$$\begin{array}{ccccccc} \mathbb{E}_{abs} & \longrightarrow & \mathbb{E}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}} & \xrightarrow{F} & q^*\mathbb{L}_{\mathfrak{U}}[1] & \longrightarrow & \\ \Big\downarrow{\scriptstyle \phi_{abs}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\| & & \\ \mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})} & \longrightarrow & \mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}} & \longrightarrow & q^*\mathbb{L}_{\mathfrak{U}}[1] & \longrightarrow & \end{array} \tag{2.52}$$

The morphism q was defined in (2.51). The triangle in the bottom row is the canonical one, and the arrow labeled "F" is the composition of ϕ with the canonical map of cotangent complexes.

Lemma 2.3.11. *The absolute obstruction theory $\phi_{abs}\colon \mathbb{E}_{abs} \to \mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})}$ is naturally equivariant.*

Proof. In the notation of (2.51), Lemma 2.2.12 gives us a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})} & \longrightarrow & \mathbb{L}_{\underline{\mathrm{Sec}}(Z/\mathcal{C})/\mathfrak{U}} & \longrightarrow & q^*\mathbb{L}_{\mathfrak{U}}[1] \\ \Big\uparrow & & \Big\uparrow & & \Big\uparrow \\ L\rho_S^*\mathbb{L}_X & \longrightarrow & L\rho_S^*\mathbb{L}_{X/[\mathfrak{U}/G]} & \longrightarrow & L\rho_S^* Lp^*\mathbb{L}_{[\mathfrak{U}/G]}[1] \end{array}$$

where all arrows are the canonical ones, except for the rightmost vertical arrow. Combining this morphism of distinguished triangles with (2.51) and (2.52) shows that the entire diagram (2.52) may be pulled back from $[\underline{\mathrm{Sec}}(Z/\mathcal{C})/G]$, and in particular is equivariant. $\square$

2.4 Obstruction theory

2.4.1 The fundamental theorem

We recall the "fundamental theorem" of [Ill71, Thm III.1.2.3] and its generalization in [Ols06, Thm 1.1, Rmk 1.3]. Given a representable morphism of algebraic stacks $x : \mathcal{X} \to \mathcal{Y}$ and a quasicoherent sheaf I on $\mathcal{X}$, recall from [Ols06, Sec 2.2] the category

$\underline{\text{Exal}}_{\mathcal{Y}}(\mathcal{X}, I)$: objects are (x', j, ϵ, ι) where $j : \mathcal{X} \hookrightarrow \mathcal{X}'$ is a square-zero extension of algebraic stacks, with $x' : \mathcal{X}' \to \mathcal{Y}$ a 1-morphism and $\epsilon : x \to x' \circ j$ a 2-morphism, and $\iota : I \to \ker(\mathscr{O}_{\mathcal{X}'} \to \mathscr{O}_{\mathcal{X}})$ is an isomorphism of sheaves. An arrow from $(x'_1, j_1, \epsilon_1, \iota_1)$ to $(x'_2, j_2, \epsilon_2, \iota_2)$ is a pair (ψ, ϕ) with $\psi : \mathcal{X}'_1 \to \mathcal{X}'_2$ a morphism of stacks and $\phi : x'_1 \to x'_2 \circ \psi$ a 2-morphism such that

(i) we have $j_2 = \psi \circ j_1$ and if ρ is the isomorphism

$$\ker(\mathscr{O}_{\mathcal{X}'_2} \to \mathscr{O}_{\mathcal{X}}) \to \ker(\mathscr{O}_{\mathcal{X}'_1} \to \mathscr{O}_{\mathcal{X}})$$

induced by ψ, then $\rho \circ \iota_2 = \iota_1$

(ii) we have $j_1^*(\phi) \circ \epsilon_1 = \epsilon_2$.

A consequence of condition (ii) is that $\underline{\text{Exal}}_{\mathcal{Y}}(\mathcal{X}, I)$ is a groupoid.

When $\mathcal{X}$ is Deligne-Mumford, the category $\underline{\text{Exal}}_{\mathcal{Y}}(\mathcal{X}, I)$ has a more refined version: by [Ols06, Sec 2.14], it is the value on $\mathcal{X}$ of a Picard stack $\underline{\underline{\text{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I)$ on $\mathcal{X}_{et}$. If $U \to \mathcal{X}$ is an étale map from a scheme, then the fiber $\underline{\underline{\text{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I)(U)$ is the groupoid $\underline{\text{Exal}}_{\mathcal{Y}}(U, I|_U)$. In fact, by [Ols06, Sec 2.26] there is an isomorphism of Picard stacks

$$\underline{\underline{\text{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I) \simeq \text{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]), \tag{2.53}$$

where for two complexes $F \in D^{[-\infty,a]}$ and $G \in D^{[b,a-1]}$, we define

$$\text{Ext}^{0/-1}(F, G) := ch(\tau_{\leq 0} R\underline{\text{Hom}}(F, G))$$

(see Section 2.2.2 for the definition of the functor ch). The isomorphism (2.53) is described in Section 2.5.2.

For us, the key property of the isomorphism (2.53) will be its functoriality under pullback and basechange as stated in the next two lemmas.

Lemma 2.4.1. *Suppose we have maps*

$$\mathcal{Z} \xrightarrow{f} \mathcal{W} \xrightarrow{g} \mathcal{Y}$$

with $\mathcal{Z}$ a Deligne-Mumford stack and f and $g \circ f$ representable. Then given a quasi-coherent sheaf $I \in \text{QCoh}(\mathcal{Z})$, there is a commuting diagram of Picard stacks on

$\mathcal{Z}_{\text{et}}$

$$\begin{array}{ccc} \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, I[1]) & \xrightarrow{A} & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{Y}}, I[1]) \\ {\scriptstyle (2.53)}\uparrow & & \uparrow{\scriptstyle (2.53)} \\ \underline{\underline{\mathrm{Exal}}}_{\mathcal{W}}(\mathcal{Z}, I) & \xrightarrow{B} & \underline{\underline{\mathrm{Exal}}}_{\mathcal{Z}}(\mathcal{X}, I) \end{array} \tag{2.54}$$

where A is induced by the canonical map $\mathbb{L}_{\mathcal{Z}/\mathcal{Y}} \to \mathbb{L}_{\mathcal{Z}/\mathcal{W}}$ and B is induced by composition with g.

Lemma 2.4.1 is a special case of [Ols06, (2.33.3)], but that result is stated only at the level of isomorphism classes of global objects and the proof is omitted. We will prove Lemma 2.4.1 in Section 2.5.3.

For the second functoriality lemma, suppose we have a fiber square

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{a} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{W} & \longrightarrow & \mathcal{Y} \end{array} \tag{2.55}$$

where $\mathcal{Z}$ and $\mathcal{X}$ are Deligne-Mumford stacks, the map $\mathcal{W} \to \mathcal{Y}$ is flat, and $\mathcal{X} \to \mathcal{Y}$ is a representable morphism. Then given a quasi-coherent sheaf $I \in \mathrm{QCoh}(\mathcal{X})$, there is a morphism of Picard stacks

$$\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I) \to \underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, a^*I) \tag{2.56}$$

sending $\mathcal{X}' \to \mathcal{Y}$ to the pullback $\mathcal{Z}' := \mathcal{X}' \times_{\mathcal{Y}} \mathcal{W} \to \mathcal{W}$ (observe that, since (2.55) is fibered, we have an induced map $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ with the desired kernel). The morphism (2.56) extends to a morphism of Picard stacks.

Lemma 2.4.2. *Given the fiber square* (2.55) *with $\mathcal{Z}$ and $\mathcal{X}$ Deligne-Mumford stacks and $\mathcal{X} \to \mathcal{Y}$ a representable morphism, and given a quasi-coherent sheaf $I \in \mathrm{QCoh}(\mathcal{X})$, there is a commuting diagram of Picard stacks on $\mathcal{X}_{\text{et}}$*

$$\begin{array}{ccccc} \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]) & \xrightarrow{C} & a_*\mathrm{Ext}^{0/-1}(a^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, a^*I[1]) & \xleftarrow[\sim]{D} & a_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, a^*I[1]) \\ {\scriptstyle (2.53)}\uparrow & & & & \uparrow{\scriptstyle (2.53)} \\ \underline{\underline{\mathrm{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I) & & \xrightarrow{E} & & a_*\underline{\underline{\mathrm{Exal}}}_{\mathcal{W}}(\mathcal{Z}, a^*I) \end{array} \tag{2.57}$$

where the arrow C is (2.16), *D is induced by the canonical map of cotangent complexes* *(an isomorphism in this case), and E is the map* (2.56).

We will prove Lemma 2.4.2 in Section 2.5.3.

2.4.2 Proof of Theorem 2.1.3

In this section we prove Theorem 2.1.3. In fact, we will prove a more refined version: this is Theorem 2.4.4.

Consider a solid diagram of algebraic stacks where $\mathcal{X} \hookrightarrow \mathcal{X}'$ is a square-zero extension of Deligne-Mumford stacks and f, g are representable

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ {\scriptstyle q}\downarrow & \nearrow & \downarrow{\scriptstyle r} \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{Z} \end{array} \tag{2.58}$$

with a fixed 2-morphism $\gamma : r \circ f \to g \circ q$. Define $\underline{\mathrm{Def}}(f)$ to be the category of *lifts* of f: an object of $\underline{\mathrm{Def}}(f)$ is a triple (k, ϵ, δ) such that $k : \mathcal{X}' \to \mathcal{Y}$ is a 1-morphism, and $\epsilon : f \to k \circ q$ and $\delta : r \circ k \to g$ are 2-morphisms satisfying $q^*(\delta) \circ r(\epsilon) = \gamma$. A morphism from $(k_1, \epsilon_1, \delta_1)$ to $(k_2, \epsilon_2, \delta_2)$ is a natural transformation $\tau : k_1 \to k_2$ such that $q^*(\tau) \circ \epsilon_1 = \epsilon_2$ and $\delta_1 = \delta_2 \circ r(\tau)$.

The category $\underline{\mathrm{Def}}(f)$ has a more refined version: it is the value on $\mathcal{X}$ a stack $\underline{\underline{\mathrm{Def}}}(f)$ on the small étale site $\mathcal{X}_{\mathrm{et}}$ of $\mathcal{X}$, defined as follows. The diagram (2.58) defines an element (g, q, γ, id) of $\underline{\mathrm{Exal}}_{\mathcal{Z}}(\mathcal{X}, I)$ which via restriction defines a section of the stack $\underline{\underline{\mathrm{Exal}}}_{\mathcal{Z}}(\mathcal{X}, I)$ over $\mathcal{X}$. We define $\underline{\underline{\mathrm{Def}}}(f)$ to be the fiber product

$$\begin{array}{ccc} \underline{\underline{\mathrm{Def}}}(f) & \longrightarrow & \underline{\underline{\mathrm{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I) \\ \downarrow & & \downarrow{\scriptstyle \underline{\underline{R}}} \\ \mathcal{X} & \xrightarrow{(g,q,\gamma,id)} & \underline{\underline{\mathrm{Exal}}}_{\mathcal{Z}}(\mathcal{X}, I) \end{array} \tag{2.59}$$

where the arrow R is induced by composition with $r : \mathcal{Y} \to \mathcal{Z}$ (it is the same as the map B in (2.54)). It is perhaps intuitive that the global sections of $\underline{\underline{\mathrm{Def}}}(f)$ are the category $\underline{\mathrm{Def}}(f)$, but we check this carefully in the next lemma.

Lemma 2.4.3. *If $u : U \to \mathcal{X}$ is an object of $\mathcal{X}_{\mathrm{et}}$, then the fiber $\underline{\underline{\mathrm{Def}}}(f)(U)$ is equivalent to the category $\underline{\mathrm{Def}}(f \circ u)$ for any extension $U \to U'$ over $q : \mathcal{X} \to \mathcal{X}'$.*

In other words, the category $\underline{\underline{\mathrm{Def}}}(f \circ u)$ is the category of lifts for the outer square of the diagram

$$\begin{array}{ccccc} U & \xrightarrow{u} & \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle u^*q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ U' & \longrightarrow & \mathcal{X}' & \xrightarrow{g} & \mathcal{Z} \end{array} \tag{2.60}$$

where the left square is fibered and $U' \to \mathcal{X}'$ is flat. The extension $U \to U'$ exists and is unique [Ols06, Thm 1.4].

Proof of Lemma 2.4.3. Let $u : U \to \mathcal{X}$ be an object of $\mathcal{X}_{\mathrm{et}}$. Then the fiber $\underline{\underline{\mathrm{Def}}}(f)(U)$ is the fiber product of the diagram of groupoids

$$\begin{array}{ccc} & & \underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I_U) \\ & & \downarrow{\scriptstyle \underline{R}} \\ \mathcal{X} & \xrightarrow{u^*g} & \underline{\mathrm{Exal}}_{\mathcal{Z}}(U, I_U) \end{array}$$

where the object $u^*g = u^*(g, q, \gamma, id)$ is given by the outer square of (2.60). An object of $\underline{\underline{\mathrm{Def}}}(f)(U)$ is an object $x \in \underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I_U)$ together with an arrow from $\underline{R}(x)$ to u^*g. From the universal property of fiber products, there is a canonical map

$$\Phi : \underline{\underline{\mathrm{Def}}}(f \circ u) \to \underline{\underline{\mathrm{Def}}}(f)(U)$$

which sends the object $(k, \epsilon, \delta) \in \underline{\underline{\mathrm{Def}}}(f \circ u)$ to the object $(k, u^*q, \epsilon, id) \in \underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I_U)$ and the arrow δ.

The functor Φ is fully faithful. Indeed, given two objects $(k_i, \epsilon_i, \delta_i) \in \underline{\underline{\mathrm{Def}}}(f \circ u)$, an arrow between their images in $\underline{\underline{\mathrm{Def}}}(f)(U)$ is given by a pair (ψ, ϕ) that defines a morphism between $(k_1, u^*q, \epsilon_1, id)$ and $(k_2, u^*q, \epsilon_2, id)$. In particular, $\psi : U' \to U'$ is a 1-morphism and $\phi : k_1 \to k_2$ is a 2-morphism. Furthermore the diagram

$$\begin{array}{ccc} r(k_1) & \xrightarrow{r(\phi)} & r(k_2) \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} \\ g & = & g \end{array}$$

is required to commute. Referring to the description of ψ and ϕ in Section 2.4.1, we see that condition (i) forces $\psi = id$ and (ii) reduces to $q^*(\phi) \circ \epsilon_1 = \epsilon_2$. So we see that an arrow between the images in $\underline{\underline{\mathrm{Def}}}(f)(U)$ is precisely an arrow ϕ between the original objects in $\underline{\mathrm{Def}}(f \circ u)$.

The functor Φ is also essentially surjective. Suppose we have an object of $\underline{\underline{\mathrm{Def}}}(f)(U)$ given by $(k, j, \epsilon, \iota) \in \underline{\mathrm{Exal}}_{\mathcal{Y}}(U, I_U)$ with (ψ, ϕ) the isomorphism to $(g \circ u', u^*q, u^*\gamma, id)$. Then this same object is isomorphic to the image of $(k \circ \psi^{-1}, \epsilon, (\psi^{-1})^*(\phi))$ under Φ via the pair (ψ, id). □

The definition of $\underline{\underline{\mathrm{Def}}}(f)$ allows for the following stronger version of Theorem 2.1.3.

Theorem 2.4.4. *Consider a solid diagram* (2.58) *of algebraic stacks where $\mathcal{X} \to \mathcal{X}'$ is a square-zero extension of Deligne-Mumford stacks with ideal sheaf I.*

1. *There is an obstruction $o(f) \in Ext^1(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ whose vanishing is necessary and sufficient for the set* $\mathrm{Def}(f)$ *to be nonempty.*

2. *If $o(f) = 0$, then $\underline{\underline{\mathrm{Def}}}(f)$ is a torsor for $\mathrm{Ext}^{0/-1}(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ on $\mathcal{X}_{et}$.*

Remark 2.4.5. To recover the statement of Theorem 2.1.3, we may take the values on $\mathcal{X}$ of the objects in Theorem 2.4.4 (2). When $\mathrm{Def}(f)$ is not empty, we obtain an isomorphism of groupoids

$$\mathrm{Ext}^{0/-1}(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)(\mathcal{X}) \times \underline{\mathrm{Def}}(f) \xrightarrow{\sim} \underline{\mathrm{Def}}(f) \times \underline{\mathrm{Def}}(f) \tag{2.61}$$

induced by the action and projection to the second factor. Restricting to isomorphism classes of objects, we get a bijection

$$\mathrm{Ext}^0(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \times \mathrm{Def}(f) \xrightarrow{\sim} \mathrm{Def}(f) \times \mathrm{Def}(f),$$

i.e., the set $\mathrm{Def}(f)$ is a torsor under $\mathrm{Ext}^0(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$. Likewise, if $f' \in \underline{\mathrm{Def}}(f)$ is any object and $e \in \mathrm{Ext}^0(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ is the trivial extension, then (2.61) yields a group isomorphism

$$\mathrm{Aut}(e) \times \mathrm{Aut}(f') \xrightarrow{\sim} \mathrm{Aut}(f') \times \mathrm{Aut}(f').$$

Restricting this isomorphism to the subgroup $\mathrm{Aut}(e) \times \{id\}$ yields an isomorphism $\mathrm{Aut}(e) \simeq \mathrm{Aut}(f')$. By the description of ch in Section 2.2.2, we see that $\mathrm{Aut}(f')$ is isomorphic to $\mathrm{Ext}^{-1}(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$.

Proof of Theorem 2.4.4. We employ the well-known strategy of studying the distinguished triangle

$$f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}}. \tag{2.62}$$

Applying Lemma 2.4.1 to the maps

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$$

we get a commuting diagram

$$\begin{array}{ccc} \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]) & \longrightarrow & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I[1]) \\ \downarrow \sim & & \downarrow \sim \\ \underline{\underline{\mathrm{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I) & \xrightarrow{\underline{\underline{R}}} & \underline{\underline{\mathrm{Exal}}}_{\mathcal{Z}}(\mathcal{X}, I) \end{array} \tag{2.63}$$

where the arrow $\underline{\underline{R}}$ is as in (2.59). When we restrict (2.63) to isomorphism classes of objects over $\mathcal{X}$, we get the commuting square in the diagram below.

$$\begin{array}{ccccc} \mathrm{Ext}^1(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I) & \longrightarrow & \mathrm{Ext}^1(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I) & \xrightarrow{ob} & \mathrm{Ext}^1(f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \\ \downarrow \sim & & \alpha \downarrow \sim & & \\ \mathrm{Exal}_{\mathcal{Y}}(\mathcal{X}, I) & \xrightarrow{R} & \mathrm{Exal}_{\mathcal{Z}}(\mathcal{X}, I) & & \end{array} \tag{2.64}$$

A consequence of Lemma 2.4.3 is that $\mathrm{Def}(f)$ is nonempty if and only if the fiber of R over $[g] := (g, q, \gamma, id)$ is nonempty. From the long exact sequence for $\mathrm{Ext}^i(-, I)$ applied to (2.62), we see that this happens if and only if the image of $[g]$ in $\mathrm{Ext}^1(f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ is 0. We define

$$o(f) = ob(\alpha^{-1}([g])). \tag{2.65}$$

In particular, the obstruction group is the image of ob.

If $\mathrm{Def}(f)$ is not empty, then by Lemma 2.4.6 below $\underline{\underline{\mathrm{Def}}}(f)$ is a torsor under the kernel of the morphism of Picard stacks

$$\underline{\underline{R}} : \underline{\underline{\mathrm{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I) \to \underline{\underline{\mathrm{Exal}}}_{\mathcal{Z}}(\mathcal{X}, I).$$

By [Ols06, Lem 2.29], using the distinguished triangle

$$R\underline{Hom}(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I[1]) \xrightarrow{F} R\underline{Hom}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I[1]) \to R\underline{Hom}(f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I[1]) \to$$

induced from (2.62), this kernel is canonically isomorphic to $ch((\tau_{\leq -1}Cone(\tau_{\leq 0}F))[-1])$.

But a direct computation shows that

$$\tau_{\leq -1} Cone(\tau_{\leq 0} F)) = \tau_{\leq -1} Cone(F) = \tau_{\leq 0} Cone(F[-1])$$

so we get that this kernel is canonically isomorphic to $\mathrm{Ext}^{0/-1}(f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$. □

Lemma 2.4.6. *Let $f : P \to Q$ be a morphism of Picard stacks on a site with underlying category $\mathcal{X}$, and let K denote the kernel. Let $q : \mathcal{X} \to Q$ be a section and $F = P \times_{Q,q} \mathcal{X}$ the fiber product. If $F(\mathcal{X})$ is not empty, then F is a K-torsor on $\mathcal{X}$.*

Proof. Let $\alpha : P \times P \to P$ denote the action. From the definitions we obtain a factorization of the composition $K \times F \to P \times P \xrightarrow{\alpha} P$ through F. Since $F(\mathcal{X})$ is not empty, to show that $\alpha : K \times F \to F$ makes F a K-torsor it remains to check that the map

$$K \times_{\mathcal{X}} F \xrightarrow{(\alpha, pr_2)} F \times_{\mathcal{X}} F$$

is an isomorphism. For this note that the map

$$F \times_{\mathcal{X}} F \xrightarrow{(\alpha(\cdot,(\cdot)^{-1}), pr_2)} K \times_{\mathcal{X}} F$$

is an inverse. □

2.4.3 A criterion for an obstruction theory

Let $\mathcal{Y} \to \mathcal{Z}$ be a morphism of algebraic stacks. We say a morphism $\phi : E \to L_{\mathcal{Y}/\mathcal{Z}}$ in the derived category of $\mathcal{Y}$ is an *obstruction theory* if the induced morphisms of cohomology sheaves satisfy $h^{-1}(\phi)$ is a surjection and $h^0(\phi)$, $h^1(\phi)$ are isomorphisms. On the other hand, for every diagram (2.58) we have induced homomorphisms

$$\Phi_i : \mathrm{Ext}^i(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \to \mathrm{Ext}^i(Lf^*E, I).$$

The following lemma is a well-known extension of [BF97, Thm 4.5] to the situation when $\mathcal{Y} \to \mathcal{Z}$ is an arbitrary morphism of algebraic stacks (see eg [AP19, Cor 8.5] and [Pom15, Thm 3.5]).

Lemma 2.4.7. *The following conditions are equivalent.*

1. *The morphism ϕ is an obstruction theory.*

2. For every diagram (2.58) *with $\mathcal{X}$ a scheme, the following hold:*

a) the element $\Phi_1(o(f)) \in \mathrm{Ext}^1(Lf^*E, I)$ vanishes if and only if $\underline{\mathrm{Def}}(f)$ is nonempty

b) if $\Phi_1(o(f)) = 0$ then Φ_0 and Φ_{-1} are isomorphisms

In fact, in condition (2) it suffices to check only diagrams (2.58) *with $\mathcal{X}$ an affine scheme.*

We emphasize a difference in wording between our criterion and that found in [BF97, Thm 4.5], [AP19, Cor 8.5], and [Pom15, Thm 3.5]: in place of condition (2b), the references cited require only that $\underline{\mathrm{Def}}(f)$ be a torsor under $\mathrm{Ext}^0(Lf^*E, I)$ (and automorphisms are isomorphic to $\mathrm{Ext}^{-1}(Lf^*E, I)$). This condition implies that for $i = 0, -1$ the groups $\mathrm{Ext}^i(Lf^*E, I)$ and $\mathrm{Ext}^i(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ are isomorphic, but not that Φ_i is an isomorphism. This is clearly not strong enough to force ϕ to be an obstruction theory. For instance, take $\mathcal{Y}$ a smooth scheme with $\mathbb{L}_{\mathcal{Y}} = \Omega^1_{\mathcal{Y}}$ the locally free sheaf of differentials in degree 0. Let $E = \Omega^1_{\mathcal{Y}}$ and let $\phi : E \to \mathbb{L}_{\mathcal{Y}}$ be the zero map. Then for any diagram (2.58), the map Φ_1 is injective (in fact $\mathrm{Ext}^1(Lf^*\mathbb{L}_{\mathcal{Y}}, I) = 0$ since $Lf^*\mathbb{L}_{\mathcal{Y}}$ is a projective module in degree 0) and the groups $\mathrm{Ext}^0(Lf^*E, I)$ and $\mathrm{Ext}^0(Lf^*\mathbb{L}_{\mathcal{Y}}, I)$ are isomorphic. But ϕ is not an obstruction theory.

We emphasize this difference because in our application of interest (the moduli of stable maps), it is relatively easy to check that $\underline{\mathrm{Def}}(f)$ is a torsor under $\mathrm{Ext}^0(Lf^*E, I)$ (and that automorphisms are isomorphic to $\mathrm{Ext}^{-1}(Lf^*E, I)$) using Theorem 2.4.4. It is much harder to show condition (2b), namely that the Φ_i are isomorphisms. This is the reason for our lengthy argument in Section 2.4.4.

The following proof of Lemma 2.4.7 is not new. Bhargav Bhatt explained part of the argument to me, and other parts are taken from [BF97, Thm 4.5].

Proof of Lemma 2.4.7. The following argument is well-known. Let C be the mapping cone of $E \to \mathbb{L}_{\mathcal{Y}/\mathcal{Z}}$. From the long exact sequence of cohomology sheaves for the resulting triangle we see that condition (1) is equivalent to

(1') $h^i(C) = 0$ for $i \geq -1$.

We will show that (1') and (2) are equivalent.

Assume (1'). Then $h^i(Lf^*C)$ also vanish for $i \geq -1$, so a spectral sequence [Stacks, Tag 07AA] for $\mathrm{Ext}^i(-, I)$ implies $\mathrm{Ext}^i(Lf^*C, I) = 0$ for $i \geq -1$ and any I. On the other hand, the distinguished triangle

$$Lf^*E \to Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to Lf^*C \to$$

gives rise to a long exact sequence

$$\longrightarrow \operatorname{Ext}^i(Lf^*C, I) \longrightarrow \operatorname{Ext}^i(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \longrightarrow \operatorname{Ext}^i(Lf^*E, I) \longrightarrow \operatorname{Ext}^{i+1}(Lf^*C, I) \longrightarrow$$

We conclude that Φ_1 is injective and Φ_0 and Φ_{-1} are isomorphisms. Combined with Theorem 2.4.4, this proves (2) (with $\mathcal{X}$ an arbitrary scheme).

Now assume (2) holds for every diagram (2.58) with $\mathcal{X}$ an affine scheme. By Theorem 2.4.4, this implies that for any diagram (2.58), with $\mathcal{X}$ an affine scheme, the map Φ_1 is injective on the span of obstruction elements $o(f)$. If $\mathcal{X} \to \mathcal{Y}$ is in fact a *smooth* cover by a scheme, then I claim Φ_1 is injective, i.e., every element of $\operatorname{Ext}^1(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I)$ is equal to $o(f)$ for some diagram (2.58). From the definition of the obstruction in (2.65), this is equivalent to the map *ob* being surjective. This follows from the long exact sequence

$$\to \operatorname{Ext}^1(\mathbb{L}_{\mathcal{X}/\mathcal{Z}}, I) \xrightarrow{ob} \operatorname{Ext}^1(Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}}, I) \to \operatorname{Ext}^2(\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, I) \to$$

induced from the distinguished triangle $\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}$, since $\mathbb{L}_{\mathcal{X}/\mathcal{Y}} = \Omega^1_{\mathcal{X}/\mathcal{Y}}[0]$ is a projective module in degree 0.

Likewise the assumption (2b) is stronger than it initially appears, since the maps Φ_0 and Φ_{-1} depend only on f and I and not the whole diagram (2.58). In fact, for any morphism $f : \mathcal{X} \to \mathcal{Y}$ and $\mathscr{O}_X$-module I, we can form a diagram (2.58) with $\mathcal{X}' \to \mathcal{Z}$ the trivial extension of $\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}$ by I. Then the category $\underline{\mathrm{Def}}(f)$ contains the trivial extension of $f : \mathcal{X} \to \mathcal{Y}$ and in particular is not empty. So $o(f) = 0$ for this diagram, and hence by assumption Φ_0 and Φ_{-1} are isomorphisms (independent of the value of $o(f)$).

Now let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth cover by an affine scheme. To prove (1'), it suffices to show that $h^i(Lf^*C) = 0$ for $i \geq -1$. Let I be an $\mathscr{O}_{\mathcal{X}}$-module. The long exact sequence of $\operatorname{Ext}^i(-, I)$ for the distinguished triangle $Lf^*E \to Lf^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to Lf^*C \to$, together with the observations of the above paragraphs that Φ_0 and Φ_{-1} are isomorphisms and Φ_1 is injective, implies that $\operatorname{Ext}^i(Lf^*C, I) = 0$ for every $i \leq 1$. On the other hand, by

[Stacks, Tag 07AA] there is a spectral sequence whose second page is

$$\mathrm{Ext}^i(h^{-j}(Lf^*C), I) \implies \mathrm{Ext}^{i+j}(Lf^*(C), I).$$

A priori we know Lf^*E and $Lf^*L_{\mathcal{Y}/\mathcal{Z}}$, and hence Lf^*C are in $D^{\leq 1}(\mathcal{X})$, so $h^i(Lf^*C) = 0$ for $i \geq 2$. Then by the above spectral sequence, the group $\mathrm{Ext}^0(h^1(Lf^*C), I)$ is equal to $\mathrm{Ext}^{-1}(Lf^*C, I)$ which vanishes for every I. This forces $h^1(Lf^*C)$ to vanish. Inductively applying the same argument to $\mathrm{Ext}^0(Lf^*C, I)$ and then $\mathrm{Ext}^1(Lf^*C, I)$ shows that $h^0(Lf^*C)$ and $h^{-1}(Lf^*C)$ vanish as well. □

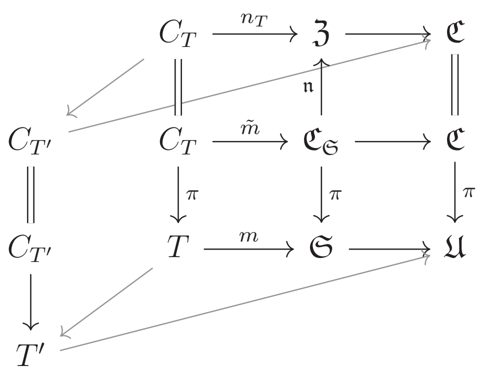
2.4.4 Proof of Theorem 2.1.2

We use Lemma 2.4.7. The idea for our argument was inspired by the arguments in [BF97, Prop 6.2] and [Abr+, Prop 4.2] and discussions with Bhargav Bhatt. We must check several compatibilities of morphisms, which for readability we present as a series of claims. Granting these claims, the crux of the argument is in the final step (Claim 2.4.13 below). The statements of the claims and the proof of Claim 2.4.13 (the final claim) can all be read with only the background presented so far in this thesis, but the proofs of the first four claims use the technical material in Section 2.5 (which contains the definitions of the objects and maps in Lemmas 2.4.1 and 2.4.2). To begin, fix a solid commuting diagram

$$\begin{array}{ccc} T & \xrightarrow{m} & \mathfrak{S} \\ \downarrow & & \downarrow \\ T' & \longrightarrow & \mathfrak{S} \end{array} \tag{2.66}$$

with $T \to T'$ a square zero extension of affine schemes with ideal sheaf I.

Claim 2.4.8. *There is a commuting diagram of stacks*



(2.67)

The horizontal levels of this diagram define categories $\underline{\mathrm{Def}}(m)$ *and* $\underline{\mathrm{Def}}(n_T)$ *that are canonically equivalent.*

Proof. Let γ be the 2-morphism witnessing the commutativity of (2.66). By Lemma 2.4.4 we may assume $\gamma = id$ (precisely, $\underline{\mathrm{Def}}(m)$ is canonically equivalent to another category $\underline{\mathrm{Def}}(m')$ with $\gamma = 1$). This yields the commuting diagram (2.67).

There is a functor $\underline{\mathrm{Def}}(n_T) \to \underline{\mathrm{Def}}(m)$ as follows. From an element (k, ϵ, δ) of $\underline{\mathrm{Def}}(n_T)$ we get an arrow $k_\delta : T' \to \mathfrak{S}$ determined by k and δ, making the resulting triangle over $\mathfrak{U}$ strictly commutative. The 2-morphism ϵ determines a 2-morphism (also denoted ϵ) from m to the composition $T \to T' \xrightarrow{k_\delta} \mathfrak{S}$. Hence our functor sends the object (k, ϵ, δ) to the object (k_δ, ϵ, id). This functor is fully faithful. To see that it is essentially surjective, let (k, ϵ, δ) be an object in $\underline{\mathrm{Def}}(m)$, and apply Lemma using $k : T' \to \mathfrak{S}$ for $f : \mathcal{X} \to \mathcal{Y}$. We get $k' : T' \to \mathfrak{S}$ and $\tau : k \to k'$, and in fact τ is an arrow from $(k', q^*(\tau^{-1}) \circ \epsilon, id)$ to (k, ϵ, δ) in $\underline{\mathrm{Def}}(m)$. □

Claim 2.4.9. *The diagram* (2.67) *leads to a commuting diagram of Picard stacks*

$$\begin{array}{ccccc}
\pi_*\mathrm{Ext}^{0/-1}(Ln_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}}, \pi^*I) & \longrightarrow & \pi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{C_T/\mathfrak{Z}}, \pi^*I[1]) & \xrightarrow{A} & \pi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{C_T/\mathfrak{C}}, \pi^*I[1]) \\
{\scriptstyle F}\uparrow & & {\scriptstyle A\circ D^{-1}\circ C}\uparrow & & {\scriptstyle D^{-1}\circ C}\uparrow \\
\mathrm{Ext}^{0/-1}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I) & \longrightarrow & \mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{S}}, I[1]) & \xrightarrow{A} & \mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{U}}, I[1])
\end{array} \tag{2.68}$$

where the arrows are labeled as in Lemmas 2.4.1 and 2.4.2, and the terms in the leftmost column are the kernels of the top and bottom horizontal maps.

Proof. By 2.2.12, diagram (2.67) leads to the following morphisms of distinguished triangles.

$$\begin{array}{ccccc} Ln_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}} & \longrightarrow & \mathbb{L}_{C_T/\mathfrak{C}} & \longrightarrow & \mathbb{L}_{C_T/\mathfrak{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ L\tilde{m}^*\mathbb{L}_{\mathfrak{C}_{\mathfrak{S}}/\mathfrak{C}} & \longrightarrow & \mathbb{L}_{C_T/\mathfrak{C}} & \longrightarrow & \mathbb{L}_{C_T/\mathfrak{C}_{\mathfrak{S}}} \\ \sim\uparrow & & \sim\uparrow & & \sim\uparrow \\ \pi^*Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}} & \longrightarrow & \pi^*\mathbb{L}_{T/\mathfrak{U}} & \longrightarrow & \pi^*\mathbb{L}_{T/\mathfrak{S}} \end{array} \tag{2.69}$$

This leads to the following commuting diagram of Picard stacks where the terms in the leftmost column are kernels.

$$\begin{array}{ccccc} \pi_*\mathrm{Ext}^{0/-1}(Ln_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}}, \pi^*I) & \longrightarrow & \pi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{C_T/\mathfrak{Z}}, \pi^*I[1]) & \xrightarrow{A} & \pi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{C_T/\mathfrak{C}}, \pi^*I[1]) \\ \uparrow & & A\uparrow & & \| \\ \pi_*\mathrm{Ext}^{0/-1}(L\tilde{m}^*\mathbb{L}_{\mathfrak{C}_{\mathfrak{S}}/\mathfrak{C}}, \pi^*I) & \longrightarrow & \pi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{C_T/\mathfrak{C}_{\mathfrak{S}}}, \pi^*I[1]) & \xrightarrow{A} & \pi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{C_T/\mathfrak{C}}, \pi^*I[1]) \\ \downarrow\sim & & D\downarrow\sim & & D\downarrow\sim \\ \pi_*\mathrm{Ext}^{0/-1}(\pi^*Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, \pi^*I) & \longrightarrow & \pi_*\mathrm{Ext}^{0/-1}(\pi^*\mathbb{L}_{T/\mathfrak{S}}, \pi^*I[1]) & \xrightarrow{A} & \pi_*\mathrm{Ext}^{0/-1}(\pi^*\mathbb{L}_{T/\mathfrak{U}}, \pi^*I[1]) \\ \uparrow & & C\uparrow & & C\uparrow \\ \mathrm{Ext}^{0/-1}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I) & \longrightarrow & \mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{S}}, I[1]) & \xrightarrow{A} & \mathrm{Ext}^{0/-1}(\mathbb{L}_{T/\mathfrak{U}}, I[1]) \end{array} \tag{2.70}$$

The top three rows come directly from applying the functor $\pi_*\mathrm{Ext}^{0/-1}(-, \pi^*I[1])$ to (2.69), applying [Ols06, Lem 2.29] as in the proof of Theorem 2.4.4, and the fact that π_* preserves kernels for Picard stacks. The bottom row likewise comes from a morphism of distinguished triangles

$$\begin{array}{ccccc} R\pi_*R\underline{Hom}(\pi^*\mathbb{L}_{T/\mathfrak{S}}, \pi^*I[1]) & \longrightarrow & R\pi_*R\underline{Hom}(\pi^*\mathbb{L}_{T/\mathfrak{U}}, \pi^*I[1]) & \longrightarrow & R\pi_*R\underline{Hom}(\pi^*Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, \pi^*I[1]) \\ \uparrow & & \uparrow & & \uparrow \\ R\underline{Hom}(\mathbb{L}_{T/\mathfrak{S}}, I[1]) & \longrightarrow & R\underline{Hom}(\mathbb{L}_{T/\mathfrak{U}}, I[1]) & \longrightarrow & R\underline{Hom}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I[1]) \end{array}$$

and an application of Lemma 2.2.10. This morphism of triangles is induced by (2.16). Now (2.68) is the perimeter of (2.70).

□

Claim 2.4.10. *The arrow F in* (2.68) *is quasi-isomorphic to the map induced by ϕ. More precisely (and more generally), there is a commuting square*

$$\begin{array}{ccc} R\underline{Hom}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I) & \xrightarrow{F'} & \pi_*R\underline{Hom}(Ln_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}}, \pi^*I) \\ \| & & \downarrow\sim \\ R\underline{Hom}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I) & \xrightarrow{R\underline{Hom}(Lm^*\phi, I)} & R\underline{Hom}(Lm^*E, I) \end{array} \tag{2.71}$$

where $ch(\tau_{\leq 0}F') = F$.

The crux of the argument is to show that there is a commuting diagram

$$\begin{array}{ccc}
R\pi_* R\underline{Hom}(Ln_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}}, \pi^* I) & \xrightarrow{\sim} & R\underline{Hom}(R\pi_*(Ln_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}} \otimes \omega^\bullet), I) \\
\uparrow & & \uparrow \\
R\pi_* R\underline{Hom}(\pi^* Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, \pi^* I) & \xrightarrow{\sim} & R\underline{Hom}(R\pi_*(\pi^* Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}} \otimes \omega^\bullet), I) \\
{\scriptstyle (2.15)}\Vert & & \uparrow \\
R\underline{Hom}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, \pi_*\pi^* I) & & \\
\uparrow & & \\
R\underline{Hom}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I) & = & R\underline{Hom}(Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I)
\end{array} \tag{2.72}$$

where the top vertical arrows are induced by the composition

$$Ln_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}} \xrightarrow{dn_T} \tilde{L}m^*\mathbb{L}_{C_{\mathfrak{S}}/\mathfrak{C}} \underset{\sim}{\xleftarrow{d\pi}} \pi^* Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}},$$

the bottom left vertical arrow is induced by the unit $I \to \pi_*\pi^* I$ of the adjunction, and the bottom right vertical arrow is induced by the projection formula and trace map as in (2.3). Granting this, define F' to be the composition of the left vertical arrows. To see that $ch(\tau_{\leq 0}F') = F$, use the decomposition of F' in (2.70) and Lemma 2.2.10 (which also holds for π a morphism of algebraic stacks). Moreover, by Lemma 2.3.5, the composition of the right vertical arrows in (2.72) is quasi-isomorphic to $R\underline{Hom}(Lm^*\phi, I)$.

Lemma 2.4.11. *There is a commuting diagram* (2.72) *with vertical arrows as specified.*

Proof. Note that we may choose the horizontal arrows in whatever (functorial) way is convenient for us, and so we can focus on the bottom square. We use the Yoneda embedding: let $A = Lm^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}$ and apply $\mathrm{Hom}(B, -)$ to the bottom square of (2.72). The commutativity of the resulting diagram

$$\begin{array}{ccc}
{}^1\,\mathrm{Hom}(B, R\pi_* R\underline{Hom}(\pi^* A, \pi^* B)) & \xrightarrow{\sim} & {}^2\,\mathrm{Hom}(B, R\underline{Hom}(R\pi_*(\pi^* A \otimes \omega^\bullet), I)) \\
\uparrow & & \uparrow \\
{}^3\,\mathrm{Hom}(B, R\underline{Hom}(A, I)) & = & {}^4\,\mathrm{Hom}(B, R\underline{Hom}(A, I))
\end{array} \tag{2.73}$$

follows from the commutativity of three diagrams that we describe below. The gray numbers are to aid the reader in assembling the diagrams. We will use the fact that if (L, R) is an adjoint pair of functors and $Y \to R(Z)$ is a morphism, then there is a commuting diagram

$$\begin{array}{ccc} \operatorname{Hom}(X, Y) & \xrightarrow{\operatorname{Hom}(id, f)} & \operatorname{Hom}(X, R(Z)) \\ \downarrow & & \Big\| a \\ \operatorname{Hom}(L(X), L(Y)) & \xrightarrow{\operatorname{Hom}(id, a(f))} & \operatorname{Hom}(L(X), Z) \end{array} \tag{2.74}$$

where the left vertical arrow is induced by the functor L and a stands for adjunction.

We show that (2.73) is equivalent to the diagram

$$\begin{array}{ccc} {}^{5}\operatorname{Hom}(\pi^*(B \otimes A), \pi^* I) & = & {}^{6}\operatorname{Hom}(\pi^*(B \otimes A) \otimes \omega^\bullet, \pi^* I \otimes \omega^\bullet) \\ a\Big\| & & \downarrow\wr \\ \operatorname{Hom}(B \otimes A, \pi_* \pi^* I) & & {}^{7}\operatorname{Hom}(B \otimes R\pi_*(\pi^* A \otimes \omega^\bullet), I) \\ \uparrow & & \uparrow \\ {}^{8}\operatorname{Hom}(B \otimes A, I) & = & {}^{9}\operatorname{Hom}(B \otimes A, I) \end{array} \tag{2.75}$$

In this diagram, the top equality is just the functor $- \otimes \omega^\bullet$. The bottom vertical arrows are induced by the unit and counit of the respective adjunctions. The top right vertical arrow is defined to be the composition

$$\begin{aligned} {}^{6}\operatorname{Hom}(\pi^*(B \otimes A) \otimes \omega^\bullet, \pi^* I \otimes \omega^\bullet) & \xrightarrow{\sim} \operatorname{Hom}(\pi^*(B \otimes A) \otimes \omega^\bullet, \pi^! I) \\ & = \operatorname{Hom}(R\pi_*(\pi^*(B \otimes A) \otimes \omega^\bullet), I) \\ & = \operatorname{Hom}(R\pi_*(\pi^* B \otimes (\pi^* A \otimes \omega^\bullet)), I) \\ & \xrightarrow[\sim]{(2.17)} {}^{7}\operatorname{Hom}(B \otimes R\pi_*(\pi^* A \otimes \omega^\bullet), I) \end{aligned}$$

The first isomorphism is induced by the map

$$\pi^* I \otimes \omega^\bullet \to \pi^! I$$

defined in [FHM03, (5.8)]. It is an isomorphism by [FHM03, Thm 8.4] using the results of [HR17]—that $\pi^!$ preserves coproducts follows as in [Nee96, Ex 5.2], and we are working on an affine scheme T which is in particular quasi-compact and quasi-separated.

The first equality is $(R\pi_*, \pi^!)$ adjunction. Diagram (2.75) commutes as follows. The composition ${}^{8} \to {}^{5} \to {}^{6}$ is just the maps of Hom sets induced by $\pi^*(-) \otimes \omega^\bullet$ (using (2.74)). Now commutativity is a direct computation using (2.74) and the diagram

$$
\begin{array}{ccccccc}
 & & R\pi_*(\pi^* I \otimes \omega^\bullet) & \xleftarrow[\sim]{(2.17)} & I \otimes R\pi_*\omega^\bullet & \xrightarrow{id\otimes tr} & I \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & R\pi_*(\pi^*(B\otimes A)\otimes\omega^\bullet) & \xleftarrow[\sim]{(2.17)} & (B\otimes A)\otimes R\pi_*\omega^\bullet & \xrightarrow{id\otimes tr} & B\otimes A \\
 & \swarrow\!\!= & & & \| & \nearrow_{id\otimes tr} & \\
R\pi_*(\pi^* B \otimes (\pi^* A \otimes \omega^\bullet)) & \xleftarrow[\sim]{(2.17)} & B \otimes R\pi_*(\pi^* A \otimes \omega^\bullet) & \xleftarrow[\sim]{(2.17)} & B \otimes (A \otimes R\pi_*\omega^\bullet) & &
\end{array}
$$

where the bottom left cell is associativity of the projection formula (follows from associativity of $\otimes$ and the definitions in [Hal, Appendix A].

To see that (2.73) is equivalent to (2.75), observe that we have commuting diagrams

$$
\begin{array}{ccc}
\mathrm{Hom}(\pi^* B, R\underline{Hom}(\pi^* A, \pi^* I)) & \overset{a}{=\!=\!=} & \mathrm{Hom}(\pi^* B \otimes \pi^* A, \pi^* I) \\
\|{}^{a} & & \| \\
{}^{1}\,\mathrm{Hom}(B, R\pi_* R\underline{Hom}(\pi^* A, \pi^* I)) & & {}^{5}\,\mathrm{Hom}(\pi^*(B\otimes A), \pi^* I) \\
\|{}^{\mathrm{Hom}(id,(2.15))} & & \|{}^{a} \\
\mathrm{Hom}(B, R\underline{Hom}(A, R\pi_*\pi^* I)) & \overset{a}{=\!=\!=} & \mathrm{Hom}(B\otimes A, R\pi_*\pi^* I) \\
\uparrow & & \uparrow \\
{}^{3}\,\mathrm{Hom}(B, R\underline{Hom}(A, I)) & \overset{a}{=\!=\!=} & {}^{8}\,\mathrm{Hom}(B\otimes A, I)
\end{array}
\tag{2.76}
$$

and

$$
\begin{array}{ccc}
{}^{7}\,\mathrm{Hom}(B \otimes R\pi_*(\pi^* A \otimes \omega^\bullet), I) & \overset{a}{=\!=\!=} & {}^{2}\,\mathrm{Hom}(B, R\underline{Hom}(R\pi_*(\pi^* A \otimes \omega^\bullet), I)) \\
\uparrow & & \uparrow \\
{}^{9}\,\mathrm{Hom}(B\otimes A, I) & \overset{a}{=\!=\!=} & {}^{4}\,\mathrm{Hom}(B, R\underline{Hom}(A, I))
\end{array}
\tag{2.77}
$$

and we can choose the isomorphism $1 \to 2$ in (2.73) to be the composition

$$
1 \xrightarrow[\sim]{\text{in } (2.76)} 5 \xrightarrow[\sim]{\text{in } (2.75)} 7 \xrightarrow[\sim]{\text{in } (2.77)} 2.
$$

The top square of (2.76) commutes by a direct computation using (2.74); see also [FHM03, 5]. The diagram (2.77) commutes by the naturality of this adjunction.

□

Claim 2.4.12. *The diagram* (2.67) *leads to a commuting diagram of Picard stacks*

$$\begin{array}{ccccc} \pi_*\underline{\underline{\mathrm{Def}}}(n_T) & \longrightarrow & \pi_*\underline{\underline{\mathrm{Exal}}}_{\mathfrak{Z}}(C_T, \pi^* I) & \xrightarrow{B} & \pi_*\underline{\underline{\mathrm{Exal}}}_{\mathfrak{C}}(C_T, \pi^* I) \\ {\scriptstyle G}\uparrow & & {\scriptstyle B\circ E}\uparrow & & {\scriptstyle E}\uparrow \\ \underline{\underline{\mathrm{Def}}}(m) & \longrightarrow & \underline{\underline{\mathrm{Exal}}}_{\mathfrak{S}}(T, I) & \xrightarrow{B} & \underline{\underline{\mathrm{Exal}}}_{\mathfrak{U}}(T, I) \end{array} \tag{2.78}$$

where the arrows B and E are as in Lemmas 2.4.1 and 2.4.2, and the terms in the leftmost column are fibers of the top and bottom horizontal maps.

Proof. From the bottom two (fibered) squares of (2.67), we get a diagram

$$\begin{array}{ccc} \pi_*\underline{\underline{\mathrm{Exal}}}_{\mathfrak{Z}}(C_T, \pi^* I) & \xrightarrow{B} & \pi_*\underline{\underline{\mathrm{Exal}}}_{\mathfrak{C}}(C_T, \pi^* I) \\ {\scriptstyle B}\uparrow & & \| \\ \pi_*\underline{\underline{\mathrm{Exal}}}_{\mathfrak{C}_{\mathfrak{S}}}(C_T, \pi^* I) & \xrightarrow{B} & \pi_*\underline{\underline{\mathrm{Exal}}}_{\mathfrak{C}}(C_T, \pi^* I) \\ {\scriptstyle E}\uparrow & & \uparrow{\scriptstyle E} \\ \underline{\underline{\mathrm{Exal}}}_{\mathfrak{S}}(T, I) & \xrightarrow{B} & \underline{\underline{\mathrm{Exal}}}_{\mathfrak{U}}(T, I) \end{array} \tag{2.79}$$

From the defintions of E and B, it is easy to check that this diagram commutes. Moreover, the element of $\underline{\mathrm{Exal}}_{\mathfrak{U}}(T, I)$ defined by m and the corresponding horizontal square in (2.67) maps to the element of $\underline{\mathrm{Exal}}_{\mathfrak{C}}(C_T, \pi^* I)$ defined by n_T and its horizontal square. So by definition 2.59 of $\underline{\underline{\mathrm{Def}}}$ and universal properties of fiber products, we get (2.78).

□

Claim 2.4.13. *Condition (2) in Lemma 2.4.7 holds.*

Proof. First, observe that by Claim 2.4.8, the arrow G in (2.78) is an isomorphism of Picard stacks. To complete the proof we study the commuting cube formed by mapping the right square of (2.68) to the right square of (2.78) via (2.53). Call this diagram of Picard stacks $\mathcal{D}$.

To prove (2a), restrict the diagram $\mathcal{D}$ to isomorphism classes of objects over T. As in (2.64) we extend this diagram by the obstruction maps, obtaining a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Exal}_{\mathfrak{C}}(C_T, \pi^*I) & \xleftarrow[\sim]{(2.53)} & \mathrm{Ext}^1(\mathbb{L}_{\mathcal{C}_T/\mathfrak{C}}, \pi^*I) & \xrightarrow{ob} & \mathrm{Ext}^1(n_T^*\mathbb{L}_{\mathfrak{Z}/\mathfrak{C}}, \pi^*I) \\
{\scriptstyle E}\uparrow & & {\scriptstyle D^{-1}\circ C}\uparrow & & {\scriptstyle \Phi_1'}\uparrow \\
\mathrm{Exal}_{\mathfrak{U}}(T, I) & \xleftarrow[\sim]{(2.53)} & \mathrm{Ext}^1(\mathbb{L}_{T/\mathfrak{U}}, I) & \xrightarrow{ob} & \mathrm{Ext}^1(m^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I)
\end{array}$$

We construct the right (commutative) square as in Claim 2.4.9. By defintion of E and commutativity of the diagram, the map labeled Φ_1' sends $o(m)$ to $o(n_T)$. On the other hand, Φ_1' is quasi-isomorphic to Φ_1 by Claim 2.4.10, where Φ_1 is defined as in Lemma 2.4.7. By Theorem 2.4.4, the element $o(n_T)$ (resp $o(m)$) vanishes if and only if $\mathrm{Def}(n_T)$ (resp $\mathrm{Def}(m)$) is nonempty. Since G is an isomorphism we have $\mathrm{Def}(n_T) = \mathrm{Def}(m)$ and we see that (2a) holds.

To prove (2b), recall the fact that a morphism $f : A \to B$ of complexes in $D^{[-1,0]}(T)$ is a quasi-isomorphism if and only if the induced map of Picard stacks $ch(f) : ch(A) \to ch(B)$ is an isomorphism (see eg [73, Prop XVIII.1.4.15]). Hence by Claim 2.4.10 it suffices to show that the map F in (2.68) is an isomorphism. As in the proof of Lemma 2.4.7 this property depends only on f and I and is independent of the square-zero extension T'. So we may check it from the diagram $\mathcal{D}$ which induces a commuting square of kernels like so:

$$\begin{array}{ccc}
\pi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{C_T/\mathfrak{Z}}, \pi^*I) & \xrightarrow[\sim]{(2.53)} & \pi_*\ker\left(\pi_*\underline{\underline{\mathrm{Exal}}}_{\mathcal{Z}}(C_T, \pi^*I) \to \pi_*\underline{\underline{\mathrm{Exal}}}_{\mathfrak{C}}(C_T, \pi^*I)\right) \\
{\scriptstyle F}\uparrow & & {\scriptstyle G'}\uparrow \\
\mathrm{Ext}^{0/-1}(m^*\mathbb{L}_{\mathfrak{S}/\mathfrak{U}}, I) & \xrightarrow[\sim]{(2.53)} & \ker\left(\underline{\underline{\mathrm{Exal}}}_{\mathfrak{S}}(T, I) \to \underline{\underline{\mathrm{Exal}}}_{\mathfrak{U}}(T, I)\right)
\end{array}$$

The map G' is just a special case of the map G when the outer square of (2.58) corresponds to the trivial extension. In particular it is an isomorphism. So F is an isomorphism as well.

□

2.4.5 Applications

We give two applications of Theorem 2.1.2.

Moduli of principal bundles

Let G be an affine algebraic group over $\mathbb{C}$, locally finitely presented, quasi-compact, and quasi-separated. Let $\pi : C \to \mathfrak{U}$ be a family of twisted curves. We describe the cotangent complex for the moduli of principal G-bundles $\mathfrak{S}$ on C.

Proposition 2.4.14. *The moduli space* $\mathfrak{S}$ *is smooth and the canonical obstruction theory* (2.1) *is a quasi-isomorphism.*

Proof. Let BG denote the Artin stack that is a quotient of a point by G. The stack $\mathfrak{S}$ may be described as a moduli of sections $\underline{\mathrm{Sec}}(C \times BG/C)$, and hence by Theorem 2.1.2 the morphism

$$\phi : R\pi_*(\mathfrak{n}^*\mathbb{L}_{BG} \otimes \omega^\bullet) \to \mathbb{L}_{\mathfrak{S}}$$

is an isomorphism in degrees 0 and 1 and surjective in degree -1. The cotangent complex of BG is a locally free sheaf in degree 1, so $R\pi_*(\mathfrak{n}^*\mathbb{L}_{BG} \otimes \omega^\bullet)$ is perfect in [0,1]. Hence $h^{-1}(\mathbb{L}_{\mathfrak{S}}) = 0$ while $h^0(\mathbb{L}_{\mathfrak{S}})$ is locally free. Then if $S \to \mathfrak{S}$ is a smooth cover by a scheme, we see from the construction of the cotangent complex that $h^{-1}(\mathbb{L}_S) = 0$ and $h^0(\mathbb{L}_S)$ is locally free. A standard argument using Theorem 2.1.3 shows that $\mathfrak{S}$ is formally smooth; since $\mathfrak{S}$ is also locally finitely presented, S is smooth. This implies that $\mathfrak{S}$ is smooth and hence ϕ is a quasi-isomorphism. □

Moduli of quasimaps

Fix a complex reductive group G. Let $\mathfrak{M} = \mathfrak{M}^{\mathrm{tw}}_{g,n}$ denote the moduli space of prestable orbifold curves of genus g with n markings, and let $\mathfrak{C}$ be its universal curve. Denote by $\mathfrak{B} = \underline{\mathrm{Sec}}(\mathfrak{C} \times BG/\mathfrak{C})$ the moduli stack of prestable orbicurves together with a principal G-bundle, and let $\mathfrak{P} \to \mathfrak{C}_{\mathfrak{B}}$ be the universal principal bundle over its universal curve. Now let Y be an affine l.c.i. variety with an action by G. Let θ be a character of G with $Y^{ss}_\theta = Y^s_\theta$ smooth and having finite G-stabilizers. Fix $\epsilon > 0$ and $\beta \in \mathrm{Hom}(\mathrm{Pic}([Y/G]), \mathbb{Q})$.

With this data, [CCK15] defines a moduli space of ϵ-stable quasimaps $\overline{\mathcal{M}}^\epsilon_{g,n}([Y/G], \beta)$ as an open substack of $\underline{\mathrm{Sec}}(\mathfrak{P} \times_G Y/\mathfrak{C}_{\mathfrak{B}})$ (see [CKM14] for the situation when Y^s_θ has trivial G-stabilizers). As such, it has a canonical obstruction theory relative to $\mathfrak{C}_{\mathfrak{B}}$). However it is often convenient (as in Chapter 3 of this thesis) to realize $\overline{\mathcal{M}}^\epsilon_{g,n}([Y/G], \beta)$ as an open substack of $\underline{\mathrm{Sec}}(\mathfrak{C} \times [Y/G]/\mathfrak{C})$, whence it inherits a canonical obstruction theory relative to $\mathfrak{M}$. We rigorously explain why this is equivalent.

The moduli spaces are identified by Lemma 2.3.2. We now investigate the obstruction theories (see also the discussion in [CK10, Section 5.3]). Setting $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{\epsilon}_{g,n}([Y/G], \beta)$, we have the following analog of (2.45).

$$
\begin{array}{ccccc}
 & & & & \mathfrak{C} \times [Y/G] \\
 & & & \nearrow & \downarrow q \\
 & & \mathfrak{P} \times_G Y & & \mathfrak{C} \times BG \\
 & \overset{u}{\nearrow} & \downarrow & \nearrow & \downarrow \\
\mathfrak{C}_{\overline{\mathcal{M}}} & \longrightarrow & \mathfrak{C} & \longrightarrow & \mathfrak{C} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathcal{M}} & \xrightarrow{\mu} & \mathfrak{B} & \longrightarrow & \mathfrak{M}
\end{array}
\tag{2.80}
$$

By Corollary 2.3.7 we have the following morphism of distinguished triangles:

$$
\begin{array}{ccccccc}
L\mu^*\mathbb{E}_{\mathfrak{B}/\mathfrak{M}} & \longrightarrow & \mathbb{E}_{\overline{\mathcal{M}}/\mathfrak{M}} & \longrightarrow & \mathbb{E}_{\overline{\mathcal{M}}/\mathfrak{B}} & \longrightarrow & \\
\downarrow{\scriptstyle L\mu^*\phi_{\mathfrak{B}/\mathfrak{M}}} & & \downarrow{\scriptstyle \phi_{\overline{\mathcal{M}}/\mathfrak{M}}} & & \downarrow{\scriptstyle \phi_{\overline{\mathcal{M}}/\mathfrak{B}}} & & \\
L\mu^*\mathbb{L}_{\mathfrak{B}/\mathfrak{M}} & \longrightarrow & \mathbb{L}_{\overline{\mathcal{M}}/\mathfrak{M}} & \longrightarrow & \mathbb{L}_{\overline{\mathcal{M}}/\mathfrak{B}} & \longrightarrow &
\end{array}
$$

By Proposition 2.4.14, the complex $L\mu^*\mathbb{E}_{\mathfrak{B}/\mathfrak{M}}$ is perfect in $[-1, 1]$ and $\phi_{\mathfrak{B}/\mathfrak{M}}$ is a quasi-isomorphism. Then argument of [KKP03, Prop 3] shows that $\phi_{\overline{\mathcal{M}}/\mathfrak{M}^{\mathrm{tw}}_{g,n}}$ and $\phi_{\overline{\mathcal{M}}/\mathfrak{B}}$ induce the same virtual class on $\overline{\mathcal{M}}$.

We have proved the following lemma.

Lemma 2.4.15. *The stack of ϵ-stable quasimaps $\overline{\mathcal{M}}^{\epsilon}_{g,n}([Y/G], \beta)$ is canonically isomorphic to an open substack of* $\underline{\mathrm{Sec}}(\mathfrak{C} \times [Y/G]/\mathfrak{C})$. *Moreover, the restriction of* (2.1) *to this substack is a perfect obstruction theory, and it induces the same virtual fundamental class as the perfect obstruction theory of [CKM14, 4.4.1] and [CCK15, Sec 2.4.5].*

2.5 Functoriality and the fundamental theorem

The goal of this section is to define the isomorphism (2.53) and prove its functoriality as in Lemmas 2.4.1 and 2.4.2. The strategy is to reduce to statements about simplicial algebraic spaces, which are in turn interpreted as general statements about ringed topoi. Hence we begin by studying the setting of ringed topoi in Section 2.5.1, and then in Section 2.5.2 we define (2.53) and in Section 2.5.3 we prove Lemmas 2.4.1 and 2.4.2.

2.5.1 Illusie's Theorem

Let $\mathcal{C}$ be a ringed site with $A \to B$ a morphism of sheaves of rings on $\mathcal{C}$, and let I be a sheaf of B-modules. This section discusses a canonical isomorphism

$$\beta : \underline{\underline{\mathrm{Exal}}}_A(B, I) \to \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \tag{2.81}$$

where the subscript on $\mathrm{Ext}_B^{0/-1}$ indicates that it uses internal hom for the category of B-modules on $\mathcal{C}$. This isomorphism is referred to as "Illusie's Theorem" in [Ols06, Appendix A], where it is proven in Theorem A.7 to be an isomorphism (see also [Ill71, Sec III.1.2.2]). We refer the reader to [Ols06, Thm A.7] for the definition of β; here we will discuss its functoriality under changing the rings A and B, the B-module I, and the topos $\mathcal{C}$.

Review of Exal

We review the category of algebra extensions and its functoriality. While our discussion is at the level of Picard categories, everything we say here holds for the relevant Picard stacks as well. As defined in [Ill71, Sec III.1.1.5], the objects of the Picard category $\underline{\mathrm{Exal}}_A(B, I)$ are A-algebra maps $E \to B$ with kernel I. We write these objects as short exact sequences

$$0 \to I \to E \to B \to 0. \tag{2.82}$$

If $I \to J$ is a morphism of B-modules, there is an induced morphism

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_A(B, J) \tag{2.83}$$

sending (3.53) to

$$0 \to J \to E \oplus_I J \to B \to 0;$$

i.e., the object $E \oplus_I J$ is defined by a pushout diagram (this is [Ill71, Equ III.1.1.5.2]). If $B' \to B$ is a morphism of A-algebras, then there is an induced morphism

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_A(B', I_{B'}) \tag{2.84}$$

sending (3.53) to

$$0 \to I \to E \times_B B' \to B' \to 0;$$

i.e., the object $E \times_B B'$ is defined by a fiber product diagram (this is [Ill71, Equ III.1.1.5.3]). Here, $I_{B'}$ denotes the sheaf I considered as a B'-module. If $A' \to A$ is a morphism of rings, there is an induced morphism

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_{A'}(B, I) \tag{2.85}$$

sending (3.53) to the same exact sequence (but $E \to B$ is now a morphism of A'-algebras) (this is [Ill71, Equ III.1.1.5.4]).

Finally, let $\mathcal{C}' \to \mathcal{C}$ be a continuous morphism of sites inducing a morphism of topoi (a_*, a^{-1}). Then we have an induced morphism

$$\underline{\mathrm{Exal}}_A(B, I) \to \underline{\mathrm{Exal}}_{a^{-1}A}(a^{-1}B, a^{-1}I) \tag{2.86}$$

sending (3.53) to its image under a^{-1}. We are using that a^{-1} is an exact functor.

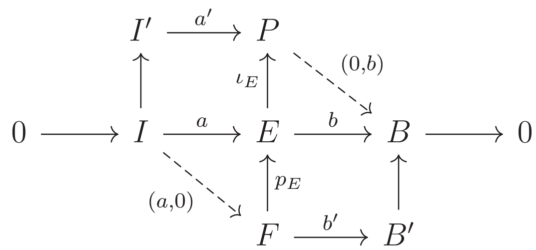
Lemma 2.5.1. *The maps* (2.83)*,* (2.84)*,* (2.85)*, and* (2.86) *commute pairwise.*

Proof. The most involved pair to check is (2.84) and (2.83). We work it out in detail and offer a few words about the remaining pairs at the end of the proof. When we say these commute, we mean that if $B' \to B$ is a morphism of rings and $I \to I'$ is a morphism of B-modules, then the diagram

$$\begin{array}{ccc} \underline{\mathrm{Exal}}_A(B, I) & \longrightarrow & \underline{\mathrm{Exal}}_A(B', I) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Exal}}_A(B, I') & \longrightarrow & \underline{\mathrm{Exal}}_A(B', I') \end{array} \tag{2.87}$$

is 2-commutative.

Given an element (3.53) of $\underline{\mathrm{Exal}}_A(B, I)$, we have a diagram



(2.88)

where $P = I' \oplus_I E$ and $F = E \times_B B'$. Set $Q = I' \oplus_I F$ and $G = P \times_B B'$. Then

$$0 \to I' \to Q \to B' \to 0$$

is the image of (3.53) under the composition $^{\to}\downarrow$ in (2.87), and likewise G defines the image under the composition $\downarrow_{\to}$. An arrow from Q to G in the groupoid $\underline{\mathrm{Exal}}_A(B', I')$ is given by four dashed arrows so that this diagram commutes:

$$\begin{array}{ccccccc} I & \longrightarrow & I' & \dashrightarrow & P & & \\ & \searrow & & \swarrow\!\!\!\!\nearrow\!\!\searrow & & \searrow & \\ & & F & \dashrightarrow & B' & \longrightarrow & B \end{array} \tag{2.89}$$

(To check commutativity, it suffices to check that the quadrilaterals $I'IFP$ and $I'IFB'$ and the perimeter commute.) The required collection of dotted arrows is given by $a' : I' \to P$, $0 : I' \to B'$, $\iota_E \circ p_E : F \to P$, and $b' : F \to B'$.

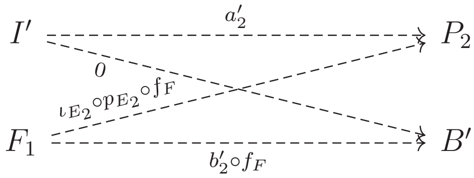
To show that the resulting arrows in $\underline{\mathrm{Exal}}_A(B', I')$ define a natural transformation, suppose we are given an arrow

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{a_1} & E_1 & \xrightarrow{b_1} & B & \longrightarrow & 0 \\ & & & \searrow^{a_2} & \downarrow f & \nearrow_{b_2} & & & \\ & & & & E_2 & & & & \end{array}$$

in $\underline{\mathrm{Exal}}_A(B, I)$. Let $f_P : P_1 \to P_2$ and $f_F : F_1 \to F_2$ be the maps induced by f, where P_i and F_i are defined as in (2.88). Likewise let Q_i and G_i be the images of E_i in $\underline{\mathrm{Exal}}_A(B', I')$ under the maps in (2.87). We must compare two maps from Q_1 to G_2 in $\underline{\mathrm{Exal}}_A(B', I')$. Such maps are given by diagrams of the form (2.89) with F replaced by F_1 and P replaced by P_2. In the situation at hand, one of the maps from Q_1 to G_2 is given by the diagram

$$\begin{array}{ccc} I' & \overset{f_P \circ a'_1}{\dashrightarrow} & P_2 \\ & {\scriptstyle 0:\ I' \dashrightarrow B'} \quad {\scriptstyle f_P \circ \iota_{E_1} \circ p_{E_1}:\ F_1 \dashrightarrow P_2} & \\ F_1 & \underset{b'_1}{\dashrightarrow} & B' \end{array}$$

and the other is given by the diagram



These are easily seen to consist of the same morphisms.

This completes the proof that (2.84) and (2.83) commute. Of the remaining pairs, most of the checks are trivial (in particular, the analog of (2.87) is strictly commutative). Only the pairs ((2.83), (2.86)) and ((2.84), (2.86)) are nontrivial. For these, one uses that a^{-1} is exact and hence preserves finite limits and colimits. □

Functoriality of Illusie's theorem

For each map of $\underline{\underline{\mathrm{Exal}}}$ stacks defined in the previous section, we see what it looks like as a map of $\mathrm{Ext}^{0/-1}$ stacks via the isomorphism β defined in (2.81).

Lemma 2.5.2. *The isomorphism* (2.81) *is functorial as follows.*

1. *Let $A \to B$ be a map of sheaves of rings on $\mathcal{C}$. If $I \to J$ is a morphism of B-modules, there is a commuting diagram*

$$\begin{array}{ccc} \underline{\underline{\mathrm{Exal}}}_A(B, I) & \xrightarrow{\beta} & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \\ \downarrow{\scriptstyle (2.83)} & & \downarrow \\ \underline{\underline{\mathrm{Exal}}}_A(B, J) & \xrightarrow{\beta} & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, J[1]) \end{array}$$

where the right vertical arrow is induced by functoriality of $R\underline{Hom}$.

2. *If there is a commuting square of rings*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

then there is a commuting square

$$
\begin{array}{ccc}
\underline{\underline{\mathrm{Exal}}}_A(B,I) & \xrightarrow{\beta} & & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \\
\downarrow & & & \downarrow \\
\underline{\underline{\mathrm{Exal}}}_{A'}(B',I_{B'}) & \xrightarrow{\beta} & \mathrm{Ext}_{B'}^{0/-1}(\mathbb{L}_{B'/A'}, I_{B'}[1]) = & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B'/A'}\otimes B, I[1])
\end{array}
\tag{2.90}
$$

where the left vertical arrow is (2.85) *followed by* (2.84) *and the right vertical arrow is induced by the canonical map* $\mathbb{L}_{B'/A'} \otimes B \to \mathbb{L}_{B/A}$*. The equality is induced by* (2.15)*.*

3. *Let* $\mathcal{C} \to \mathcal{C}'$ *be a continuous morphism of ringed sites inducing a flat morphism of topoi* (a^{-1}, a_*)*. Let* A *and* $B = \mathscr{O}_{\mathcal{C}}$ *be sheaves of rings on* $\mathcal{C}$*. Then if* I *is a sheaf of* B*-modules, there is a commuting diagram*

$$
\begin{array}{ccc}
\underline{\underline{\mathrm{Exal}}}_A(B,I) & \xrightarrow{\beta} & & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A}, I[1]) \\
\downarrow{\scriptstyle (2.86)} & & & \downarrow{\scriptstyle (2.16)} \\
a_*\underline{\underline{\mathrm{Exal}}}_{a^{-1}A}(a^{-1}B, a^{-1}I) & \xrightarrow{a_*\beta} & a_*\mathrm{Ext}_{a^{-1}B}^{0/-1}(a^{-1}\mathbb{L}_{B/A}, a^{-1}I[1]) \xrightarrow{F} & a_*\mathrm{Ext}_{La^*B}^{0/-1}(La^*\mathbb{L}_{B/A}, La^*I[1])
\end{array}
\tag{2.91}
$$

where the map F *is given by* (2.16) *for the adjoint functors* $(-\otimes^{\mathbf{L}}_{a^{-1}B} a^*B,\ (-)_{a^{-1}B})$*.*

Proof. We summarize the definition of β; see [Ols06, Thm A.7] for more details. Let $P^\bullet$ be the simplicial A-algebra given by the standard free resolution of the A-algebra B [Stacks, Tag 08SR]. The morphism β is defined to be a composition

$$
\underline{\underline{\mathrm{Exal}}}_A(B,I) \xrightarrow{\beta_1} \underline{\underline{\mathrm{Exal}}}_A(P^\bullet,I) \xrightarrow{\beta_2} \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A},I) \xrightarrow{\beta_3} \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A}\otimes B,I) \xrightarrow{\beta_4} \mathrm{Ext}^{0/-1}(\mathbb{L}_{B/A}, I[1])
$$

Here, $\underline{\underline{\mathrm{Ext}}}(\Omega^\bullet, I)$ denotes the Picard stack on $\mathcal{C}$ whose fiber on $U \in \mathcal{C}$ is the category of simplicial $\mathscr{O}_{\mathcal{C}}$-module extensions of $\Omega^\bullet$ by I (viewed as a simplicial module); see [Ols06, Sec A.1]. The map β_1 is given by the map (2.84) applied to the augmentation $P^\bullet \to B$, the morphism β_2 is given by taking differentials, β_3 is given by tensoring with B, and β_4 is the functorial isomorphism in [Ols06, Prop A.3].

Proof of (1). The desired functoriality follows from a commuting diagram

$$\begin{array}{ccccccccc} \underline{\underline{\mathrm{Exal}}}_A(B,I) & \xrightarrow{\beta_1} & \underline{\underline{\mathrm{Exal}}}_A(P^\bullet,I) & \xrightarrow{\beta_2} & \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A},I) & \xrightarrow{\beta_3} & \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A}\otimes B,I) & \xrightarrow{\beta_4} & \mathrm{Ext}^{0/-1}(\mathbb{L}_{B/A},I[1]) \\ \downarrow{\scriptstyle (2.83)} & & \downarrow{\scriptstyle (2.83)} & & \downarrow & & \downarrow & & \downarrow \\ \underline{\underline{\mathrm{Exal}}}_A(B,J) & \xrightarrow{\beta_1} & \underline{\underline{\mathrm{Exal}}}_A(P^\bullet,J) & \xrightarrow{\beta_2} & \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A},J) & \xrightarrow{\beta_3} & \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A}\otimes B,J) & \xrightarrow{\beta_4} & \mathrm{Ext}^{0/-1}(\mathbb{L}_{B/A},J[1]) \end{array} \tag{2.92}$$

The square with β_1 commutes by Lemma 2.5.1. The square with β_2 commutes because differentials commute with colimits [Stacks, Tag 031G]. The square with β_3 commutes because tensor product is a left adjoint and so commutes with colimits, and the square with β_4 commutes by the naturality in [Ols06, Prop A.3].

Proof of (2). The desired functoriality follows from two commuting diagrams. First we have

$$\begin{array}{ccccccc} \underline{\underline{\mathrm{Exal}}}_A(B,I) & \xrightarrow{\beta_1} & \underline{\underline{\mathrm{Exal}}}_A(P^\bullet,I) & \xrightarrow{\beta_2} & \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A},I) & \xrightarrow{\beta_3} & \underline{\underline{\mathrm{Ext}}}(\Omega_{P^\bullet/A}\otimes B,I) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{\underline{\mathrm{Exal}}}_{A'}(B',I) & \xrightarrow{\beta_1} & \underline{\underline{\mathrm{Exal}}}_{A'}((P^\bullet)',I) & \xrightarrow{\beta_2} & \underline{\underline{\mathrm{Ext}}}(\Omega_{(P^\bullet)'/A'},I) & \xrightarrow{\beta_3} & \underline{\underline{\mathrm{Ext}}}(\Omega_{(P^\bullet)'/A'}\otimes B',I) \end{array} \tag{2.93}$$

which we claim commutes. Here $(P^\bullet)'$ is the simplicial A'-algebra that is the standard resolution of B'. The two left vertical arrows are given by (2.85) and (2.84); the next two vertical arrows are given by the analog of (2.84) for the $\underline{\underline{\mathrm{Ext}}}$ stacks; and the final vertical arrow is the canonical map of cotangent complexes. The first square commutes by Lemma 2.5.1. The square with β_2 also commutes essentially by definition of the maps.

We check that the square with β_3 commutes. To clarify the situation, write $Y = \Omega_{P^\bullet/A}$ and $Y' = \Omega_{(P')^\bullet/A'}$, so we have a map $Y' \to Y$. Also drop the bullets on $P^\bullet$ and $(P^\bullet)'$ We must show this square commutes:

$$\begin{array}{ccc} \underline{\underline{\mathrm{Ext}}}(Y,I) & \xrightarrow{\otimes_P B} & \underline{\underline{\mathrm{Ext}}}(Y\otimes_P B,I) \\ \downarrow{\scriptstyle (2.84)} & & \downarrow{\scriptstyle (2.84)} \\ \underline{\underline{\mathrm{Ext}}}(Y',I) & \xrightarrow{\otimes_{P'} B'} & \underline{\underline{\mathrm{Ext}}}(Y'\otimes_{P'} B',I) \end{array}$$

Let $0 \to I \to X \to Y \to 0$ be an object of $\underline{\underline{\mathrm{Ext}}}(Y,I)$ over some $U \in \mathcal{C}$ that is in the image of β_2, so both X and Y are P-modules. Unwinding the definitions, we see that we need a morphism

$$(X \times_Y Y') \otimes_{P'} B' \to (X \otimes_P B) \times_{Y \otimes_P B} (Y' \otimes_{P'} B') \tag{2.94}$$

of B'-modules compatible with maps from I and with maps to $Y' \otimes_{P'} B'$ on both the source and target. To give (2.94), it suffices to give a morphism

$$X \times_Y Y' \to (X \otimes_P B) \times_{Y \otimes_P B} (Y' \otimes_{P'} B')$$

of P'-modules, but this is evident. It is straightforward to check that the resulting map (2.94) equalizes maps from I and maps to $Y' \otimes_{P'} B'$.

The second diagram comprising (2.90) is as follows.

$$\begin{array}{ccccc}
\underline{\underline{\mathrm{Ext}}}_B(\Omega_{P^\bullet/A} \otimes B, I) & \xrightarrow{\beta_4} & \mathrm{Ext}^{0/-1}_B(\mathbb{L}_{B/A}, I[1]) & & \\
\downarrow & & \downarrow & & \\
\underline{\underline{\mathrm{Ext}}}_{B'}((\Omega_{P^\bullet/A} \otimes B)_{B'}, I_{B'}) & \xrightarrow{\beta_4} & \mathrm{Ext}^{0/-1}_{B'}((\mathbb{L}_{B/A})_{B'}, I_{B'}[1]) & \overset{(2.15)}{=\!=} & \mathrm{Ext}^{0/-1}_B((\mathbb{L}_{B/A})_{B'} \otimes_{B'} B, I[1]) \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\underline{\mathrm{Ext}}}_{B'}(\Omega_{(P^\bullet)'/A'} \otimes B', I_{B'}) & \xrightarrow{\beta_4} & \mathrm{Ext}^{0/-1}_{B'}(\mathbb{L}_{B'/A'}, I_{B'}[1]) & \overset{(2.15)}{=\!=} & \mathrm{Ext}^{0/-1}_B(\mathbb{L}_{B'/A'} \otimes_{B'} B, I[1])
\end{array}$$

The composition of the left vertical arrows is equal to the right vertical arrow in (2.93). The diagonal arrow is induced by the counit of tensor-hom adjunction, and the equalities labeled (2.15) are just the defining property of internal hom. The two bottom right vertical arrows are induced by the canonical map $\mathbb{L}_{B/A} \to \mathbb{L}_{B'/A'}$.

The bottom right square commutes by functoriality of (2.15). The bottom left square commutes by definition of the map $\mathbb{L}_{B/A} \to \mathbb{L}_{B'/A'}$ and functoriality of β_4. The last map in the triangle is defined so that it is commutative (note the analogy with (2.16)). Finally the top left square commutes by a direct local computation, using the definition of β_4 and the fact that

$$(R\underline{Hom}_B(\mathbb{L}_{B/A}, I))_{B'} \to R\underline{Hom}_{B'}((\mathbb{L}_{B/A})_{B'}, I_{B'})$$

is locally given by applying the functor $(-)_{B'}$ to the complexes $\mathbb{L}_{B/A}$ and I.

Proof of (3). The map F is $ch\tau_{\leq 0}Ra_*$ applied to the morphism

$$R\underline{Hom}_{a^{-1}B}(a^{-1}\mathbb{L}_{B/A}, a^{-1}I) \to \left(R\underline{Hom}_{a^*B}(a^*\mathbb{L}_{B/A}, a^*I)\right)_{a^{-1}B}$$

of Lemma 2.2.4.

Let $A' = a^{-1}A$, $B' = a^{-1}B$, and $P = P^\bullet$, and let P' denote the standard resolution of B' as an A'-algebra. The desired commuting square comes from two commuting diagrams. First, we have

$$\begin{array}{ccccccc}
\underline{\underline{\underline{\mathrm{Exal}}}}_A(B,I) & \xrightarrow{\beta_1} & \underline{\underline{\underline{\mathrm{Exal}}}}_A(P,I) & \xrightarrow{\beta_2} & \underline{\underline{\underline{\mathrm{Ext}}}}(\Omega_{P/A},I) & \xrightarrow{\beta_3} & \underline{\underline{\underline{\mathrm{Ext}}}}(\Omega_{P/A}\otimes B,I) \\
\downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle a^{-1}} & & \downarrow \\
a_*\underline{\underline{\underline{\mathrm{Exal}}}}_{A'}(B',a^{-1}I) & \xrightarrow{\beta_1} & a_*\underline{\underline{\underline{\mathrm{Exal}}}}_{A'}(P',a^{-1}I) & \xrightarrow{\beta_2} & a_*\underline{\underline{\underline{\mathrm{Ext}}}}(\Omega_{P'/A'},a^{-1}I) & \xrightarrow{\beta_3} & a_*\underline{\underline{\underline{\mathrm{Ext}}}}(\Omega_{P'/A'}\otimes B',a^{-1}I)
\end{array}$$

The commutativity of these squares follows, from left to right, from [Stacks, Tag 08SV] and Lemma 2.5.1, from [Stacks, Tag 08TQ], and from [Stacks, Tag 07A4]. Second, we have

$$\begin{array}{ccccc}
\underline{\underline{\underline{\mathrm{Ext}}}}(\Omega_{P/A}\otimes B,I) & \xrightarrow{\beta_4} & \mathrm{Ext}_B^{0/-1}(\mathbb{L}_{B/A},I[1]) & \searrow^{(2.16)} & \\
\downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle a^{-1}} & & \\
a_*\underline{\underline{\underline{\mathrm{Ext}}}}(\Omega_{P'/A'}\otimes B',a^{-1}I) & \xrightarrow{\beta_4} & a_*\mathrm{Ext}^{0/-1}_{a^{-1}B}(a^{-1}\mathbb{L}_{B/A},a^{-1}I[1]) & \xrightarrow{F} & a_*\mathrm{Ext}^{0/-1}_{a^*B}(a^*\mathbb{L}_{B/A},a^*I[1])
\end{array}$$

In the triangle, all maps are induced by (2.2.4), and commutativity of this triangle is just the functoriality of that morphism in the adjoint functors (Lemma 2.2.4). The square commutes by direct local computation, using the definition of β_4 and the fact that

$$R\underline{Hom}_{\mathscr{O}_\mathcal{C}}(\mathbb{L}_{B/A},I) \to Ra_*R\underline{Hom}_{a^{-1}B}(a^{-1}\mathbb{L}_{B/A},a^{-1}I)$$

is locally given by applying the functor a^{-1} to the complexes $\mathbb{L}_{B/A}$ and I (this local description follows from the proof of Lemma 2.2.4). □

2.5.2 Description of (2.53)

Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks. Let $Y \to \mathcal{Y}$ be a smooth cover by a scheme with $\varpi_Y : Y^\bullet \to \mathcal{Y}$ the associated simplicial algebraic space and $\varpi_X : X^\bullet \to \mathcal{X}$ its pullback to $\mathcal{X}$. We will sometimes omit the subscripts on the augmentation maps π. In this section, the functors $(R\varpi_*, \varpi^*)$ will generally relate $\mathcal{X}_{\mathrm{et}}$ and $X^\bullet_{\mathrm{et}}$, as opposed to the lisse-étale sites (but recall that these are equivalent as in (2.8)). Choose $I \in \mathrm{QCoh}(\mathcal{X}_{\mathrm{et}})$. The isomorphism (2.53) is defined in [Ols06, Sec 2.26] to be the following composition of morphisms of Picard stacks on $\mathcal{X}_{\mathrm{et}}$:

$$\underline{\underline{\underline{\mathrm{Exal}}}}_\mathcal{Y}(\mathcal{X},I) \xrightarrow{\alpha} \varpi_*\underline{\underline{\underline{\mathrm{Exal}}}}_{f^{-1}\mathscr{O}_{Y^\bullet}}(\mathscr{O}_{X^\bullet},\varpi^*I) \xrightarrow{\varpi_*\beta} \varpi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{X^\bullet/Y^\bullet},\varpi^*I[1]) \xleftarrow{\gamma} \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{X}/\mathcal{Y}},I[1]).$$

Here we understand $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ to be an object in $\mathcal{X}_{\text{et}}$ via restriction. The objects and maps in this composition are defined as follows.

The only stack we have not yet defined is the Picard stack $\underline{\underline{\text{Exal}}}_{f^{-1}\mathscr{O}_{Y^\bullet}}(\mathscr{O}_{X^\bullet}, \varpi^* I)$ on $X^\bullet_{et}$. Recall that an object of $X^\bullet_{et}$ is given by an étale map $U \to X^n$ with U a scheme. To such an object, this stack associates the groupoid $\underline{\text{Exal}}_{f^{-1}}(\mathscr{O}_U, \varpi^* I|_U)$.

We describe α on a fiber over V, where $V \to \mathcal{X}$ is an étale map from a scheme. Let $V^\bullet = V \times_{\mathcal{X}} X^\bullet$. Given an extension V' of V by $I|_V$, pullback defines an extension $V'^\bullet$ of $V^\bullet$ by $\varpi_V^* I$. The map α sends the extension $V \to V'$ to the exact sequence of $f^{-1}\mathscr{O}_{Y^\bullet}$-modules

$$0 \to \varpi_V^* I \to \mathscr{O}_{V'^\bullet} \to \mathscr{O}_{V^\bullet} \to 0.$$

By definition (2.34), this is an element of $\pi_* \underline{\underline{\text{Exal}}}_{f^{-1}\mathscr{O}_{Y^\bullet}}(\mathscr{O}_{X^\bullet}, \varpi^* I)(V)$. The map α is an isomorphism by [Ols06, Sec 2.26].

The map $\beta : \underline{\underline{\text{Exal}}}_{f^{-1}\mathscr{O}_{Y^\bullet}}(\mathscr{O}_{X^\bullet}, \varpi^* I) \to \text{Ext}^{0/-1}(L_{X^\bullet/Y^\bullet}, I[1])$ was defined in (2.81).

The arrow γ is induced by (2.16) via Lemma 2.2.10. Note that it is an isomorphism since the restriction map ϖ^* is fully faithful (in fact, an equivalence of categories as in Section 2.2.1).

Theorem 2.5.3 ([Ols06, (2.26.3)]). *The morphism* (2.53) *is an isomorphism.*

We note that the proof of Lemma 2.4.1 shows that (2.53) is independent of the choice of cover $Y \to \mathcal{Y}$.

2.5.3 Proofs of Lemmas 2.4.1 and 2.4.2

We are finally ready to prove the functoriality of (2.53). We first describe an amalgamation of the three diagrams in Lemma 2.5.2 that will be used in both functoriality lemmas.

Let $\mathcal{C} \to \mathcal{C}'$ be a continuous morphism of ringed sites inducing a flat morphism of topoi (a^{-1}, a_*). Let A and $B = \mathscr{O}_{\mathcal{C}}$ be sheaves of rings on $\mathcal{C}$, let I be a sheaf of B-modules, and let A' be a sheaf of rings on $\mathcal{C}'$ such that there is a commuting diagram

$$\begin{array}{ccc} a^{-1}A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ a^{-1}B & \longrightarrow & a^*B \end{array}$$

Then we obtain the commuting diagram in Figure 2.1. In that figure, the left square is Lemma 2.5.2 part 3 and the right square is Lemma 2.5.2 part 2. The middle square is Lemma 2.5.2 part 1, using the unit $a^{-1}I \to (a^*I)_{a^{-1}B}$ of the tensor-hom adjunction. The triangle commutes by definition of the map F.

Proof of Lemma 2.4.1. Construct a diagram

$$
\begin{array}{ccccc}
U & \xrightarrow{r} & V & & \\
\downarrow{\scriptstyle \rho} & & \downarrow & & \\
Z & \longrightarrow & \bullet & \longrightarrow & Y \\
\downarrow{\scriptstyle \varpi} & & \downarrow & & \downarrow \\
\mathcal{Z} & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{Y}
\end{array}
$$

where $U, V, Z,$ and Y are algebraic spaces with $Y \to \mathcal{Y}$ and $V \to \bullet$ smooth and surjective and all squares are fibered. Let p denote the map $Z \to Y$ and let $\varpi' = \varpi \circ \rho$. Then commutativity of (2.54) is equivalent to commutativity of the following diagram.

$$
\begin{array}{ccccc}
\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, I[1]) & & \xrightarrow{\quad A \quad} & & \mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{Y}}, I[1]) \\
{\scriptstyle \sim}\downarrow{\scriptstyle \gamma} & & & \swarrow & {\scriptstyle \sim}\downarrow{\scriptstyle \gamma} \\
\varpi'_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{U^\bullet/V^\bullet}, \varpi'^*I[1]) & \longrightarrow & \varpi'_*\mathrm{Ext}^{0/-1}(\rho^*\mathbb{L}_{Z^\bullet/Y^\bullet}, \varpi'^*I[1]) & \xleftarrow{\sim} & \varpi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{Z^\bullet/Y^\bullet}, \varpi^*I[1]) \\
{\scriptstyle \varpi'_*\beta}\uparrow & & & & {\scriptstyle \varpi_*\beta}\uparrow \\
\varpi'_*\underline{\underline{\mathrm{Exal}}}_{r^{-1}\mathscr{O}_{V^\bullet}}(\mathscr{O}_{U^\bullet}, \varpi'^*I) & \longrightarrow & \varpi'_*\underline{\underline{\mathrm{Exal}}}_{\rho^{-1}p^{-1}\mathscr{O}_{Y^\bullet}}(\rho^{-1}\mathscr{O}_{Z^\bullet}, \varpi'^*I) & \longleftarrow & \varpi_*\underline{\underline{\mathrm{Exal}}}_{p^{-1}\mathscr{O}_{Y^\bullet}}(\mathscr{O}_{Z^\bullet}, \varpi^*I) \\
{\scriptstyle \alpha}\uparrow & & & & {\scriptstyle \alpha}\uparrow \\
\underline{\underline{\mathrm{Exal}}}_{\mathcal{W}}(\mathcal{Z}, I) & & \xrightarrow{\quad B \quad} & & \underline{\underline{\mathrm{Exal}}}_{\mathcal{Y}}(\mathcal{Z}, I)
\end{array}
\tag{2.95}
$$

The trapezoid commutes by definition of the canonical map $\mathbb{L}_{\mathcal{Z}/\mathcal{Y}} \to \mathbb{L}_{\mathcal{Z}/\mathcal{W}}$. The commutativity of the middle square is the diagram in Figure 2.1, reflected left-to-right, with $a = \rho$, $A = p^{-1}\mathscr{O}_{Y^\bullet}$, $A' = r^{-1}\mathscr{O}_{V^\bullet}$, and $B = \mathscr{O}_{Z^\bullet}$. In particular, in the triangle, all of the maps are induced by (2.16), and the triangle commutes by Lemma 2.2.4. Hence the arrow

$$\varpi'_*\mathrm{Ext}^{0/-1}(\rho^*\mathbb{L}_{Z^\bullet/Y^\bullet}, \varpi'^*I[1]) \leftarrow \varpi_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{Z^\bullet/Y^\bullet}, \varpi^*I[1])$$

is an equivalence because

$$\rho^* : \mathrm{QCoh}(Z^\bullet_{\mathrm{et}}) \to \mathrm{QCoh}(U^\bullet_{\mathrm{et}})$$

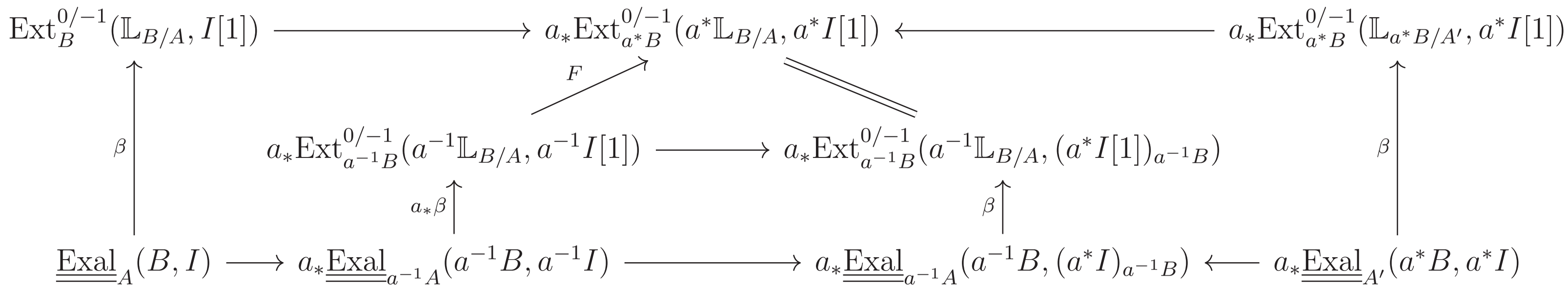


Figure 2.1: A commuting diagram

is fully faithful (since ϖ_U^* and ϖ_V^* are).

It remains to check that the bottom square commutes. We do this by direct computation. It suffices to work with global sections. For this, let $i : \mathcal{Z} \hookrightarrow \mathcal{Z}'$ be an element of $\underline{\underline{\mathrm{Exal}}}_{\mathcal{W}}(\mathcal{Z}, I)$. We have the following commuting diagram, where all squares are fibered:

$$\begin{array}{ccccccc} U & \xrightarrow{i} & U' & \longrightarrow & V & & \\ \downarrow{\scriptstyle \rho} & & \downarrow & & \downarrow & & \\ Z & \xrightarrow{i} & Z' & \longrightarrow & \bullet & \longrightarrow & Y \\ \downarrow{\scriptstyle \varpi} & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z} & \xrightarrow{i} & \mathcal{Z}' & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{Y} \end{array} \tag{2.96}$$

The map α sends $i : \mathcal{Z} \hookrightarrow \mathcal{Z}'$ to the extension

$$0 \to \varpi'^* I \xrightarrow{m} i^{-1}\mathscr{O}_{U'^\bullet} \to \mathscr{O}_{U^\bullet} \to 0$$

of $r^{-1}\mathscr{O}_{V^\bullet}$-modules, and the maps (2.85) and (2.84) send this to the extension

$$0 \to \varpi'^* I \xrightarrow{(m,0)} i^{-1}\mathscr{O}_{U'^\bullet} \times_{\mathscr{O}_{U^\bullet}} \rho^{-1}\mathscr{O}_{Z^\bullet} \to \mathscr{O}_{U^\bullet} \to 0 \tag{2.97}$$

of $\rho^{-1}p^{-1}\mathscr{O}_{Y^\bullet}$-modules.

On the other hand, the map B sends $i : \mathcal{Z} \hookrightarrow \mathcal{Z}'$ to the same extension, now as an element of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{Z}, I)$. The image of this under $\rho^{-1} \circ \alpha$ is

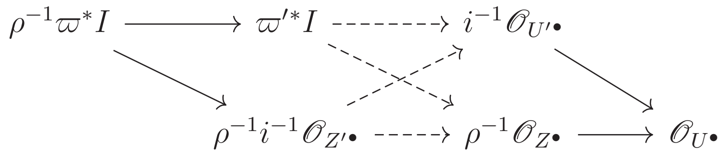
$$0 \to \rho^{-1}\varpi^* I \to \rho^{-1} i^{-1}\mathscr{O}_{Z'^\bullet} \xrightarrow{n} \rho^{-1}\mathscr{O}_{Z^\bullet} \to 0$$

an extension of $\rho^{-1}p^{-1}\mathscr{O}_{Y^\bullet}$-modules. Finally the map (2.83) sends this extension to

$$0 \to \varpi'^* I \to \varpi'^* I \oplus_{\rho^{-1}\varpi^* I} \rho^{-1} i^{-1}\mathscr{O}_{Z'^\bullet} \xrightarrow{(0,n)} \rho^{-1}\mathscr{O}_{Z^\bullet} \to 0 \tag{2.98}$$

also an extension of $\rho^{-1}p^{-1}\mathscr{O}_{Y^\bullet}$-modules.

To show that the bottom square commutes, we must find a morphism from (2.98) to (2.97) in the groupoid $\underline{\mathrm{Exal}}_{\rho^{-1}p^{-1}\mathscr{O}_{Y^\bullet}}(\rho^{-1}\mathscr{O}_{Z^\bullet}, \varpi'^* I)$ and show that it is functorial. Such a morphism is given by a collection of dotted arrows making the following diagram commute.



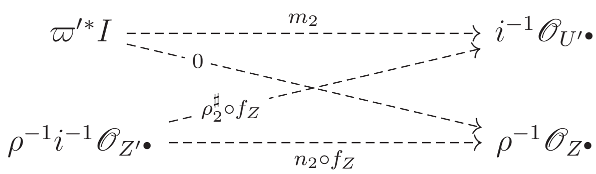
(2.99)

We choose arrows

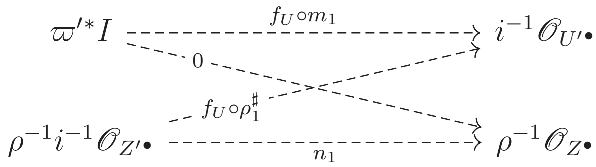
$$m : \varpi'^*I \to i^{-1}\mathscr{O}_{U'^\bullet} \qquad\qquad n : \rho^{-1}i^{-1}\mathscr{O}_{Z'^\bullet} \to \rho^{-1}\mathscr{O}_{Z^\bullet}$$
$$0 : \varpi'^*I \to \rho^{-1}\mathscr{O}_{Z^\bullet} \qquad\qquad \rho'^\sharp : \rho^{-1}i^{-1}\mathscr{O}_{Z'^\bullet} \to i^{-1}\mathscr{O}_{U'^\bullet}.$$

Commutativity of the resulting diagram follows from commutatitivity of (2.96).

We claim that this morphism is natural for arrows coming from $\underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, I)$. If we are given an arrow f from $i_1 : \mathcal{Z} \to \mathcal{Z}_1$ to $i_2 : \mathcal{Z} \to \mathcal{Z}_2$ inducing maps $f_U : i_1^1\mathscr{O}_{U_1^\bullet} \to i_2^{-1}\mathscr{O}_{U_2^\bullet}$ and $f_Z : \rho^{-1}i_1^{-1}\mathscr{O}_{Z_1^\bullet} \to \rho^{-1}i_1^{-1}\mathscr{O}_{Z_2^\bullet}$, then this naturality is equivalent to the fact that the maps in the criss-cross diagrams



and



coincide. $\square$

Proof of Lemma 2.3.8. Construct a diagram

$$\begin{array}{ccccc} Z & \longrightarrow & X \times_{\mathcal{X}} \mathcal{Z} & \longrightarrow & X \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle q} \\ W & \longrightarrow & Y \times_{\mathcal{Y}} \mathcal{W} & \longrightarrow & Y \end{array}$$

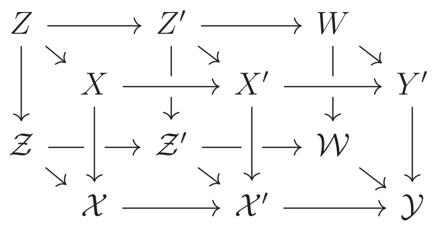
where the squares are fibered and $Y \to \mathcal{Y}$ is a smooth cover by a scheme, $X = Y \times_{\mathcal{Y}} \mathcal{X}$, and $W \to Y \times_{\mathcal{Y}} \mathcal{W}$ is a smooth cover by a scheme. Also use a to denote the map $Z \to X$. Then commutativity of (2.57) is equivalent to commutativity of the diagram in Figure 2.2.

In Figure 2.2, the top left square commutes by functoriality of (2.16) in the adjoint functors. The top right square commutes by definition of the canonical map of cotangent complexes. The middle rectangle is the diagram in Figure 2.1 with a the map $Z \to X$, $A = q^{-1}\mathscr{O}_{Y^\bullet}$, $B = \mathscr{O}_{X^\bullet}$, and $A' = p^{-1}\mathscr{O}_{W^\bullet}$. We have suppressed various squares commuting the maps a and ϖ.

It remains to check that the bottom square of the diagram in Figure 2.2 commutes. This we do by direct computation, using the diagram in Figure 2.1 to factor the map

$$\varpi_* \underline{\underline{\mathrm{Exal}}}_{q^{-1}\mathscr{O}_{Y^\bullet}}(\mathscr{O}_{X^\bullet}, \varpi^* I) \to a_* \varpi_* \underline{\underline{\mathrm{Exal}}}_{p^{-1}\mathscr{O}_{W^\bullet}}(\mathscr{O}_{Z^\bullet}, \varpi^* a^* I).$$

It suffices to work with global sections. Let $i : \mathcal{X} \hookrightarrow \mathcal{X}'$ be an element of $\underline{\mathrm{Exal}}_{\mathcal{Y}}(\mathcal{X}, I)$. Then we have a commuting diagram



where the front, bottom, and back squares are fibered (six squares in all). The map $a^{-1} \circ \alpha$ sends $i : \mathcal{X} \hookrightarrow \mathcal{X}'$ to the extension

$$0 \to a^{-1}\varpi^* I \to a^{-1} i^{-1} \mathscr{O}_{X'^\bullet} \to a^{-1}\mathscr{O}_{X^\bullet} \to 0$$

of $a^{-1}q^{-1}\mathscr{O}_{Y^\bullet}$-algebras, and the map (2.83) sends this extension to the extension

$$0 \to a^*\varpi^* I \to a^*\varpi^* I \oplus_{z^{-1}\varpi^* I} a^{-1} i^{-1} \mathscr{O}_{X^\bullet} \to a^{-1}\mathscr{O}_{X^\bullet} \to 0. \tag{2.100}$$

On the other hand, the map E sends $i : \mathcal{X} \hookrightarrow \mathcal{X}'$ to $\mathcal{Z} \hookrightarrow \mathcal{Z}'$, which under α corresponds to the extension

$$0 \to \varpi^* a^* I \to i^{-1}\mathscr{O}_{Z'^\bullet} \to \mathscr{O}_{Z^\bullet} \to 0$$

of $p^{-1}\mathscr{O}_{W^\bullet}$-algebras. After applying $\varpi^* a^* = a^* \varpi^*$ and the morphisms (2.85) and

$$
\begin{array}{ccccc}
\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, I[1]) & \xrightarrow[(2.16)]{C} & a_*\mathrm{Ext}^{0/-1}(a^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, a^*I[1]) & \xleftarrow[\sim]{D} & a_*\mathrm{Ext}^{0/-1}(\mathbb{L}_{\mathcal{Z}/\mathcal{W}}, a^*I[1]) \\
\gamma\big\downarrow\wr & & \big\downarrow(2.16) & & \gamma\big\downarrow\wr \\
\varpi_*\mathrm{Ext}^{0/-1}_{\mathscr{O}_{X^\bullet}}(\mathbb{L}_{X^\bullet/Y^\bullet}, \varpi^*I[1]) & \xrightarrow{(2.16)} & \varpi_*a_*\mathrm{Ext}^{0/-1}_{a^*\mathscr{O}_X}(a^*\mathbb{L}_{X^\bullet/Y^\bullet}, a^*\varpi^*I[1]) & \xleftarrow{\sim} & a_*\varpi_*\mathrm{Ext}^{0/-1}_{\mathscr{O}_{Z^\bullet}}(\mathbb{L}_{Z^\bullet/W^\bullet}, \varpi^*a^*I[1]) \\
\varpi_*\beta\big\uparrow & & & & \big\uparrow a_*\varpi_*\beta \\
\varpi_*\underline{\underline{\mathrm{Exal}}}_{q^{-1}\mathscr{O}_{Y^\bullet}}(\mathscr{O}_{X^\bullet}, \varpi^*I) & & \longrightarrow & & a_*\varpi_*\underline{\underline{\mathrm{Exal}}}_{p^{-1}\mathscr{O}_{W^\bullet}}(\mathscr{O}_{Z^\bullet}, \varpi^*a^*I) \\
\alpha\big\uparrow & & & & \big\uparrow\alpha \\
\underline{\underline{\mathrm{Exal}}}_{\mathcal{Y}}(\mathcal{X}, I) & & \xrightarrow{E} & & a_*\underline{\mathrm{Exal}}_{\mathcal{W}}(\mathcal{Z}, a^*I)
\end{array}
$$

Figure 2.2: A commuting diagram

(2.84), this becomes the extension

$$0 \to a^*\varpi^* \to i^{-1}\mathscr{O}_{Z'^\bullet} \times_{\mathscr{O}_{Z^\bullet}} a^{-1}\mathscr{O}_{X^\bullet} \to a^{-1}\mathscr{O}_{X^\bullet} \to 0 \tag{2.101}$$

of $a^{-1}q^{-1}\mathscr{O}_{Y^\bullet}$-algebras. As in the proof of Lemma 2.4.1, one can write down a functorial (necessarily iso)morphism between (2.100) and (2.101).

□

Chapter 3

Quasimaps From $\mathbb{P}^1$ to GIT Quotients

3.1 Introduction

Let Z be an affine l.c.i. variety and let G be a connected complex reductive algebraic group acting on Z with maximal torus T. A character θ of G defines a linearization of the trivial bundle on Z. From this data, we get two GIT quotients with a rational map between them: $Z/\!\!/T \dashrightarrow Z/\!\!/G$. The *abelian-nonabelian correspondence* is a conjectured relationship [CKS08, Conj 3.7.1] between the genus-zero Gromov-Witten invariants of $Z/\!\!/G$ and those of $Z/\!\!/T$. This paper proves a correspondence of their quasimap I-functions. In certain cases, our I-function correspondence implies [CKS08, Conj 3.7.1].

The abelian-nonabelian correspondence was observed for Grassmannians in the physics paper of Hori-Vafa [HV], and they conjectured that it should extend to complete intersections in flag varieties. In mathematics, the cohomology of $X/\!\!/G$ and $X/\!\!/T$ was studied by Ellingsrud-Stromme [ES89] and by Martin [Mar]: classes in $H^*(Z/\!\!/G, \mathbb{Q})$ can be "lifted" to classes in $H^*(Z/\!\!/T, \mathbb{Q})$, not necessarily uniquely, and the Poincaré pairings are related. The full correspondence of the genus-zero Gromov-Witten invariants is conjectured in [CKS08], and when $Z/\!\!/G$ is Fano with a sufficiently nice torus action, this correspondence is shown to be equivalent to a correspondence of small J-functions (generating series that parameterize invariants with one insertion). After applying the mirror result of [CK14b], the full abelian-nonabelian correspondence for these "nice" targets is equivalent to a correspondence of small I-functions.

3.1.1 Statement of the main result

We adopt the setup in [CK14b, Section 2.1], which in turn uses the GIT setup of [Kin94, Section 2]. Let Z, G, T, and θ be as stated. Then we have sets $Z^s(G)$ and $Z^{ss}(G)$ of θ-stable and semistable points in Z. We assume that $Z^s(G) = Z^{ss}(G)$ is not empty, that G acts on $Z^s(G)$ freely, and that $Z/\!\!/_\theta G := Z^s(G)/G$ is projective. We also assume that each of these statements holds with T in place of G. Hence $V/\!\!/_\theta G$ and $V/\!\!/_\theta T$ are smooth varieties.[4] We fix the character θ for all of this paper, and we will generally write $Z/\!\!/G := Z/\!\!/_\theta G$ and $Z/\!\!/T := Z/\!\!/_\theta T$.

Recall that there is a G-equivariant embedding $Z \to X$ into a G-representation X so that Z; we assume that this embedding can be chosen so that X satisfies all assumptions listed above ($X^s(G) = X^s s(G)$ is not empty, G acts on $X^s(G)$ freely, and $X/\!\!/_\theta G$ is projective; and each of these holds with T in place of G) and moreover $Z^s(G) = X^s(G) \cap Z$. These assumptions hold, for example, if $Z/\!\!/G$ is a complete intersection in $X/\!\!/G$, defined say by the zero locus of a homogeneous vector bundle.

The small quasimap I-function of $Z/\!\!/G$, defined in [CKM14], has the form

$$I^{Z/\!\!/G}(z) = 1 + \sum_{\beta \neq 0} q^\beta I_\beta^{Z/\!\!/G}(z) \tag{3.1}$$

where β is in $\mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ with $\mathrm{Pic}^G(Z)$ the group of G-equivariant line bundles on X, q^β is a formal variable, and the coefficients $I_\beta^{Z/\!\!/G}(z)$ are formal series in z and z^{-1} with coefficients in $H^*(X/\!\!/G)$.

To state the main theorem, we make two observations. First, the rational map $Z/\!\!/T \dashrightarrow Z/\!\!/G$ may be stated more precisely via the diagram

$$\begin{array}{ccc} Z^s(G)/T & \overset{j}{\hookrightarrow} & Z^s(T)/T \\ \big\downarrow{\scriptstyle g} & & \\ Z^s(G)/G & & \end{array} \tag{3.2}$$

Second, there is an inclusion

$$\chi(G) \to \mathrm{Pic}^G(Z) \tag{3.3}$$

[4] By [FJR18, Prop 3.1.2], the variety $V/\!\!/G$ is equal to $\mathbb{P}(V \oplus \mathbb{C})/\!\!/G$ for some choice of line bundle and linearization on $\mathbb{P}(V \oplus \mathbb{C})$ (and similarly for the $V/\!\!/T$).

sending the character ξ to

$$\mathscr{L}_\xi := Z \times \mathbb{C}_\xi \tag{3.4}$$

where $\mathbb{C}_\xi$ is the representation with character ξ.

Theorem 3.1.1. *The I-functions of $Z/\!\!/G$ and $Z/\!\!/T$ satisfy*

$$g^* I_\beta^{Z/\!\!/G}(z) = j^* \left[\sum_{\tilde{\beta}\to\beta} \left(\prod_\alpha \frac{\prod_{k=-\infty}^{\tilde{\beta}(\alpha)} (c_1(\mathscr{L}_\alpha) + kz)}{\prod_{k=-\infty}^{0} (c_1(\mathscr{L}_\alpha) + kz)} \right) I_{\tilde{\beta}}^{Z/\!\!/T}(z) \right], \tag{3.5}$$

where the sum is over all $\tilde{\beta}$ mapping to β under the natural map $\mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ *and the product is over all roots α of G.*

In Theorem 3.1.1, the quotient of infinite products is to be interpreted as follows. When $\tilde{\beta}(\alpha)$ is nonnegative, it is equal to the product

$$\prod_{k=1}^{\tilde{\beta}(\alpha)} (c_1(L_\alpha) + kz)$$

and when $\tilde{\beta}(\alpha)$ is negative, it is defined to be

$$\left(\prod_{k=\tilde{\beta}(\alpha)+1}^{0} (c_1(L_\alpha) + kz) \right)^{-1}.$$

Since g^* is injective, the equality (3.5) completely determines $I^{Z/\!\!/G}$. When Z is a vector space, combining (3.5) with Givental's formula for a twisted toric I-function [Giv98] yields a closed formula for the I-function of $Z/\!\!/G$.

Previous to this work, Theorem 3.1.1 was known for $Z/\!\!/G$ equal to a flag variety or the Hilbert scheme of n points in $\mathbb{C}^2$ [BCK05] [BCK08] [CKP12]. Since the posting of this paper, a proof of this result for quiver flag varieties has also appeared [Kal]. Finally, a result analogous to Theorem 3.1.1 was proved in the symplectic category by Gonzalez-Woodward using the language of stable gauged maps [GW]. It is not clear how the theories of gauged maps and quasimaps are related.

We remark that Theorem 3.1.1 completes a provisional result in [IIM17].

3.1.2 Supplementary results

In addition to Theorem 3.1.1, this chapter contains several ancilliary results, including examples and applications.

First, the geometric nature of our proof makes it useful in other contexts. That is, the bulk of the proof of Theorem 3.1.1 is a careful analysis of certain moduli spaces of maps from $\mathbb{P}^1$ to $[Z/G]$ and $[Z/T]$; the geometry of these moduli spaces is summarized in Proposition 3.3.1. This geometry is used in [Wen] to compute quasimap I-functions in K-theory.

Next, we prove the standard extension of Theorem 3.1.1 to the equivariant and twisted theories (Corollaries (3.5.1) and (3.5.2)). Essentially the same proof works in these settings, with only minor modifications. We also use the theory developed in [CK16] to write down a big I-function for $Z/\!\!/_\theta G$ when Z is a vector space (Corollary 3.5.4). This recovers, for example, an explicit big I-function for the Grassmannian $Gr(k, n)$.

Finally we explain the extent to which Theorem 3.1.1 implies abelianization for J-functions. In fact, with the reconstruction result in [CKS08] and the mirror result in [CK14b], the equivariant version of Theorem 3.1.1 implies the full abelian-nonabelian correspondence for projective Fano quotients $Z/\!\!/G$ with "nice" torus actions (Corollary 3.5.6). We use this result to explicitly compute an equivariant twisted small J-function for a Grassmann bundles on a Grassmannian variety (Theorem 3.5.7).

3.1.3 Conventions and notation

We work over $\mathbb{C}$ and all group actions are left actions. A variety is an irreducible separated finite type scheme over $\mathbb{C}$. Fix an affine variety X and another variety G that is a complex reductive algebraic group over $\mathbb{C}$ and fix $T \subset G$ a maximal torus. Let $N_G(T)$ be the normalizer of T in G, and let $W = N_G(T)/T$ denote the Weyl group. The characters of G are $\chi(G)$.

3.1.4 Organization of the chapter

Section 3.2 is mostly an informal discussion of ϵ-stable quasimaps to GIT quotients. Its purpose is to give some examples of the theory originally introduced by Ciocane-Fontanine, Kim, and Maulik in a series of papers: [CKM14], [CK14b], and [CK10]. The historical and mathematical context of the theory can be found in the original papers and in the survey articles [CK14a] and [Kim12]. However, in Section 3.2.3 we introduce some notation and conventions unique to our treatment of quasimap graph spaces, and Section 3.2.4 introduces some preliminary abelian-nonabelian correspondences.

In Section 3.3 we prove Proposition 3.3.1 about the geometry of the quasimap fixed loci. In Section 3.4 we prove Theorem 3.1.1. Finally in Section 3.5 we explain how the proof of Theorem 3.1.1 extends to the equivariant and twisted theories. We also explain how (and when) Theorem 3.1.1 implies [CKS08, Conj 3.7.1] and we explicitly write down an equivariant twisted small J-function for a Grassmann bundle on a Grassmannian variety. Section 3.6 contains a few general lemmas.

This chapter is a combination of the survey [Weba] and the paper [Webb].

3.2 Motivation and background

3.2.1 Motivation for quasimap theory

Gromov-Witten theory begins with the study of maps from *smooth* genus-g curves with n marks to a projective target, for example to $\mathbb{P}^n$. The set of such maps having a fixed degree d forms a moduli space, denoted (in this example) $\mathcal{M}_{g,n}(\mathbb{P}^n, d)$. The idea behind Gromov-Witten invariants is to define numerical invariants of $\mathbb{P}^n$ that are integrals of classes on this moduli space. Unfortunately, $\mathcal{M}_{g,n}(\mathbb{P}^n, d)$ is not compact; in order to define Gromov-Witten invariants of $\mathbb{P}^n$, we must replace $\mathcal{M}_{g,n}(\mathbb{P}^n, d)$ with a compactification. The most common compactification is contained in the moduli of prestable maps to $\mathbb{P}^n$ (the space denoted $\underline{\mathrm{Sec}}(\mathfrak{C} \times \mathbb{P}^n/\mathfrak{C})$ in the notation of Section 2.1.1). This compactification is the moduli space of Kontsevich-stable maps $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^n, d)$. These stable map moduli spaces and their virtual cycles are the central objects of Gromov-Witten theory.

The Kontsevich moduli space compactifies $\mathcal{M}_{g,n}(\mathbb{P}^n, d)$ by allowing the source curve to be nodal, but there are other ways to compactify. One may see hints of these other ways by looking at possible limits of families of maps in $\mathcal{M}_{g,n}(\mathbb{P}^n, d)$. Let C be equal to $\mathbb{P}^1$ with homogeneous coordinates $[x : y]$ and markings at $[1 : 1]$ and $[2 : 1]$. Define

$$\phi : \mathbb{C}^* \times \mathbb{P}^1 \to \mathbb{P}^2 \qquad \phi_a(x, y) = [ax^2 : xy : y^2] \quad \text{for} \quad a \in \mathbb{C}^*. \tag{3.6}$$

This is a family in $\mathcal{M}_{0,2}(\mathbb{P}^2, 2)$ with base $\mathbb{C}^*$. To extend it over the origin to a flat family over $\mathbb{C}$, we need to define a map ϕ_0. The natural choice seems to be

$$\phi_0(s,t) = [0 : xy : y^2], \tag{3.7}$$

but this has a basepoint (is undefined) at $y = 0$. To recover the limit in $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^3, 2)$, we resolve the rational map $\phi : \mathbb{A}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ (given by (3.6) and (3.7)) by blowing up this basepoint, adding an extra rational curve in the fiber over 0. This produces a morphism $\widetilde{\phi}$ from the blowup to $\mathbb{P}^2$, and $\widetilde{\phi}_0$ has a source curve that is two copies of $\mathbb{P}^1$, glued at a node. The limiting map has degree 1 on each copy. This is depicted in Figure 3.1.

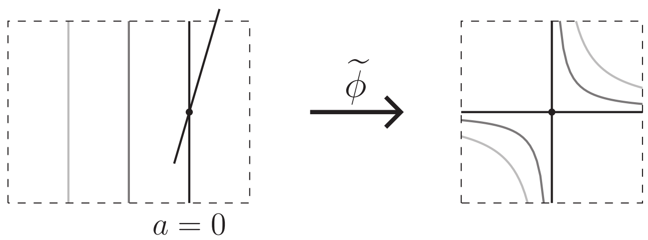


Figure 3.1: A family of stable maps extending ϕ
The morphism $\widetilde{\phi}$ maps from $\mathrm{Bl}_{a=y=0}\mathbb{A}^1 \times \mathbb{P}^1$ on the left to $\mathbb{P}^2$ on the right, depicted here in the chart of $\mathbb{P}^2$ where the middle coordinate is nonzero. The varying shades of gray show the fibers of this family of stable maps. The fiber over $a = 0$ is a map from a nodal curve.

However, what happens if we compactify $\mathcal{M}_{0,2}(\mathbb{P}^2, 2)$ by allowing basepoints? That is, what if we take $\phi_0(x,y) = [0 : xy : y^2]$ as a rational map from $\mathbb{P}^1$ to $\mathbb{P}^2$ to be the limit of (3.6)? Indeed, this rational map is a *stable quasimap*.

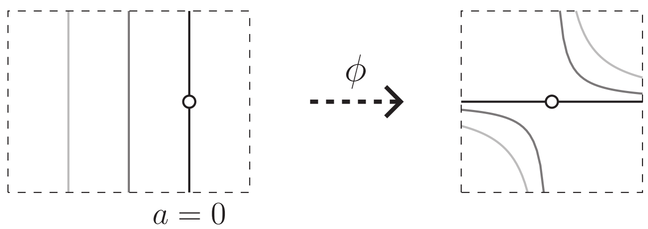


Figure 3.2: A family of stable quasimaps extending ϕ
The rational map ϕ takes $\mathbb{A}^1 \times \mathbb{P}^1$ on the left to $\mathbb{P}^2$ on the right. The varying shades of gray show the fibers of this family of quasimaps. In particular, the fiber over $a = 0$ is a rational map of degree 1 with a basepoint of length 1 (see Definition 3.2.7).

With the right definitions, stable quasimaps form a moduli space that compactifies $\mathcal{M}_{g,n}(\mathbb{P}^n, d)$ and is just as well-behaved as the Kontsevich moduli spaces. This compactification is a substack not of prestable maps to $\mathbb{P}^n$, but of prestable maps to the stack quotient $[\mathbb{C}^{n+1}/\mathbb{C}^*]$ (the space denoted $\underline{\mathrm{Sec}}(\mathfrak{C} \times [\mathbb{C}^{n+1}/\mathbb{C}^*]/\mathfrak{C})$ in the notation of Section 2.1.1). In fact, stability for quasimaps depends on a postive rational parameter ϵ, giving us a whole collection of moduli spaces, each a substack of the moduli of prestable maps to $[\mathbb{C}^{n+1}/\mathbb{C}^*]$. This collection has the following advantages:

1. When $\epsilon > 2$, the quasimap moduli space is equal to the familiar Kontsevich moduli space, and its invariants are Gromov-Witten invariants.

2. When ϵ is sufficiently small, certain quasimap invariants (the genus-0 invariants) are easier to compute. A generating function for these explicit invariants is called the *quasimap I-function*.

3. One can "cross the wall," relating quasimap invariants for differing values of ϵ, thereby (roughly) expressing genus-0 Gromov-Witten invariants in terms of the quasimap I-function.

These statements are a heuristic only; for careful statements and their proofs, see [CK; CK14b; CK17]. For certain targets, invariants with small ϵ and $g = 0$ were first studied by Givental [Giv96], while the strategy in (3) above for computing Gromov-Witten invariants was first employed by Bertram [Ber00].

3.2.2 Maps to quotients in algebraic geometry

When a reductive algebraic group G acts on an affine variety Z, we'd like to take the quotient, producing an algebraic object Z/G. Unfortunately, it is somewhat rare for there to be a scheme Z/G with all the desired properties of a quotient (including that Z/G is a *good geometric quotient* and a *categorical quotient*). Therefore, defining a quotient either requires us to modify the original data Z and G, or to leave the category of schemes. Two types of algebraic quotients that arise in quasimap theory are *geometric invariant theory (GIT) quotients* and *stack quotients.*

GIT quotients

We briefly summarize the key definitions of GIT for affine Z when G acts with no kernel, as found in [Kin94]. Fix a character θ of G. Given a nonnegative integer n, a function $f \in \Gamma(Z, \mathscr{O}_Z)$ is a *relative invariant of weight* θ^n if for every $x \in Z$ and

$g \in G$ we have $f(g \cdot x) = \theta(g)^n f(x)$. We use $\Gamma(Z, \mathscr{O}_Z)^{G,\theta^n}$ to denote relative invariants of weight θ^n.

Definition 3.2.1. A point $x \in Z$ is *θ-semistable* if there exists an integer $n \geq 1$ and a relative invariant f of weight θ^n such that $f(x) \neq 0$. If moreover the dimension of the orbit $G \cdot x$ is equal to the dimension of G and the G-action on $\{y \in Z \mid f(y) \neq 0\}$ is closed, then x is *θ-stable*.

The upshot of these definitions is that we get an open locus $Z^{ss}_\theta(G) \subset Z$ of semistable points with respect to θ, and a smaller locus $Z^s_\theta(G) \subset Z^{ss}_\theta(G)$ of stable points. We define the locus of *unstable points* to be $Z^{us}_\theta(G) = Z \setminus Z^{ss}_\theta(G)$. The sets $Z^s_\theta(G)$ and $Z^{ss}_\theta(G)$ can be computed with [Kin94, Prop 2.5]. We include the group G in our notation here because later we will vary it. The GIT quotient is defined to be

$$Z /\!\!/_\theta G = \mathrm{Proj}\left(\bigoplus_{n>0} \Gamma(Z, \mathscr{O}_Z)^{G,\theta^n}\right) \tag{3.8}$$

where the Proj is taken relative to the affine quotient $Spec(\Gamma(Z, \mathscr{O}_Z)^G)$. In particular, when $\Gamma(Z, \mathscr{O}_Z)^G$ is $\mathbb{C}$, then the GIT quotient is projective. The definition (3.8) may be expressed informally as $Z /\!\!/_\theta G = Z^s s(G)/G$. In this thesis we will assume $Z^s s_\theta(G) = Z^s_\theta(G)$, so we will often write

$$Z /\!\!/_\theta G = Z^s(G)/G.$$

Principal bundles and associated fiber bundles

Let G be a reductive algebraic group. A principal G-bundle on a scheme C is a scheme $\pi : \mathscr{P} \to C$ with π faithfully flat and locally finitely presented, together with an action $\mu : G \times \mathscr{P} \to \mathscr{P}$ leaving π invariant such that the map

$$G \times \mathscr{P} \xrightarrow{(\mu, pr_2)} \mathscr{P} \times_C \mathscr{P}$$

is an isomorphism. With our assumptions, a principal G-bundle is locally trivial in the étale topology (see e.g. [Ols16, Rmk 4.5.7]).

We close this section with an important example in which a desirable quotient does exist as a scheme, namely the *associated fiber bundle* to a principal G-bundle. Let $\mathscr{P} \to S$ be a principal G-bundle with a left G-action and let Z be an affine G-variety.

Then $\mathscr{P} \times Z$ carries a natural G-action defined on closed points by

$$g \cdot (p, x) = (gp, gx).$$

for $(p, x) \in \mathscr{P} \times Z$. We define the *associated fiber bundle* to be the quotient

$$\mathscr{P} \times_G Z := (\mathscr{P} \times Z)/G$$

where the right-hand side is defined to be the algebraic space that satisfies the universal property of quotients. In fact, the right hand side exists as a scheme when S has finite type, and it is a good geometric quotient (this uses affineness of Z; see for example the discussion in [Bri96, Sec 3]). It has a natural map to $\mathscr{P}/G = S$, with closed fibers isomorphic to Z. In particular, when Z is a vector space and G acts linearly on Z, the associated fiber bundle $\mathscr{P} \times_G Z$ is the total space of a vector bundle on S.

In fact, there is a bijection between principal GL_r-bundles and rank-r vector bundles on S given by sending $\mathscr{P}$ to $\mathscr{P} \times_G \mathbb{C}^r$, where $\mathbb{C}^r$ is a GL_r-module via left multiplication. We call $\mathscr{P} \times_G \mathbb{C}^r$ the *vector bundle associated to* $\mathscr{P}$, and $\mathscr{P}$ the *underlying principal bundle of* $\mathscr{P} \times_G \mathbb{C}^r$.

Stack quotients

Whereas the GIT quotient "forgets" the unstable locus Z^{us}, this information is retained in the stack quotient. The objects of the stack quotient $[Z/G]$ over a scheme S are diagrams

$$\begin{array}{ccc} \mathscr{P} & \longrightarrow & Z \\ \downarrow & & \\ S & & \end{array}$$

where $\mathscr{P}$ is a principal G-bundle (locally trivial in the étale topology) and $\mathscr{P} \to Z$ is a G-equivariant map. Morphisms in this category are given by fiber diagrams.

The stack quotient $[Z/G]$ and the GIT quotient $Z/\!\!/_\theta G$ are closely related. In a moment we will define quasimaps to $Z/\!\!/G$ to be certain "stable" maps to the stack quotient $[Z/G]$. Motivation for this definition comes from the previous section and the next example.

Example 3.2.2 (GIT vs stack quotient)**.** Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1}$ by

$$\lambda \cdot (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n). \tag{3.9}$$

Then if $\theta : \mathbb{C}^* \to \mathbb{C}^*$ is the identity, we have $(\mathbb{C}^{n+1})^{ss} = \mathbb{C}^{n+1} \setminus \{0\}$ and $\mathbb{C}^{n+1} /\!\!/_\theta \mathbb{C}^* = \mathbb{P}^n$. A map from a scheme S to $\mathbb{P}^n$ is given by a line bundle $\mathcal{L}$ on S and $n+1$ sections of $\mathcal{L}$ that do not simultaneously vanish.

On the other hand, the stack $[\mathbb{C}^{n+1}/\mathbb{C}^*]$ "remembers the origin." By definition, a map from a scheme S to $[\mathbb{C}^{n+1}/\mathbb{C}^*]$ is a principal $\mathbb{C}^*$-bundle $\mathscr{P}$ on S and an equivariant map $\mathscr{P} \to \mathbb{C}^{n+1}$. As we will see in Example 3.2.4, this equivariant map is equivalent to a section of $\mathscr{P} \times_{\mathbb{C}^*} \mathbb{C}^{n+1} = \mathcal{L}^{\oplus n+1} \to S$ where $\mathcal{L}$ is the line bundle associated to $\mathscr{P}$. Comparing this to the data of a map to $\mathbb{C}^{n+1} /\!\!/ \mathbb{C}^*$, we see that the only difference is that now, the sections of $\mathcal{L}$ are allowed to vanish simultaneously—i.e., we allow the map to "hit the origin."

Quasimaps to GIT quotients

Loosely speaking, a prestable quasimap to $Z /\!\!/_\theta G$ is a map to the stack quotient $[Z/G]$ that recovers a rational map to the GIT quotient.

Definition 3.2.3. A prestable *quasimap* to $Z /\!\!/ G$ is data $(C, x_1, \dots, x_n, \mathscr{P}, \tilde{u})$ where

- C is a curve with at worst nodal singularities and marked points $x_1, \dots, x_n$
- $\varrho : \mathscr{P} \to C$ is a principal G-bundle with a left G-action
- $\tilde{\sigma} : \mathscr{P} \to Z$ is a G-equivariant map with $\varrho(\tilde{\sigma}^{-1}(Z^{us}))$ a finite set that is disjoint from nodes and markings.

The *genus* of the quasimap is the genus of C.

Note that the principal bundle $\mathscr{P} \to C$ and morphism $\tilde{\sigma} : \mathscr{P} \to Z$ define a morphism from C to the stack quotient $[Z/G]$. It is helpful to replace $\tilde{\sigma}$ with an associated section of $\mathscr{P} \times_G Z \to C$, which we will typically denote σ. In fact this is equivalent data, as explained in the example below.

Example 3.2.4. If $\varrho : \mathscr{P} \to C$ is a principal G-bundle, then equivariant maps $\mathscr{P} \to Z$ are in bijection with sections of the fiber bundle $\mathscr{P} \times_G Z \to C$. We sketch this bijection.

From the universal property of fiber products, we have a natural bijection between $\mathrm{Hom}(\mathscr{P}, Z)$ and $\mathrm{Sec}(\mathscr{P}, \mathscr{P} \times Z)$ where Sec denotes the space of sections. Letting G act on morphisms by conjugation, this is a G-equivariant bijection, so we have

$$\mathrm{Hom}_G(\mathscr{P}, Z) \cong \mathrm{Sec}_G(\mathscr{P}, \mathscr{P} \times Z).$$

On the other hand, we sketch a bijection

$$\mathrm{Sec}(C, \mathscr{P} \times_G Z) \cong \mathrm{Sec}_G(\mathscr{P}, \mathscr{P} \times Z)$$

by interpreting an element of $\mathrm{Sec}(C, \mathscr{P} \times_G Z)$ as a map to the stack quotient $[(\mathscr{P} \times Z)/G]$. Such a map is by definition a principal G-bundle $\mathscr{Q}$ on C and an equivariant map to $\mathscr{P} \times Z$; to say that it is a section means that composition with projection to C is the identity:

$$\begin{array}{ccccc} \mathscr{Q} & \longrightarrow & \mathscr{P} \times Z & \longrightarrow & \mathscr{P} \\ \downarrow & & & & \downarrow \\ C & & \!\!=\!=\!=\!=\!=\!=\!\! & & C \end{array}$$

Then the induced map $\mathscr{Q} \to \mathscr{P}$ is a morphism of principal bundles, hence an isomorphism. After identifying $\mathscr{P}$ with $\mathscr{Q}$ this way, the map $\mathscr{Q} \to \mathscr{P} \times Z$ becomes a G-equivariant section $\mathscr{P} \to \mathscr{P} \times Z$. The reader is invited to find the inverse to this correspondence.

With this we can define the *degree* of a quasimap. The degree of $(C, \mathscr{P}, \tilde{\sigma})$ is the homomorphism $\beta \in \mathrm{Hom}(\mathrm{Pic}^G Z), \mathbb{Z})$ given by

$$\beta(\mathscr{L}) = \deg_C(\sigma^*(\mathscr{P} \times_G \mathscr{L})) \qquad \mathscr{L} \in \mathrm{Pic}^G(Z). \tag{3.10}$$

Here, $\mathrm{Pic}^G(X)$ is the group of G-equivariant line bundles on Z.

Example 3.2.5. For two GIT quotients $Z/\!\!/G$, we'll write down all quasimaps from $\mathbb{P}^1$ to $Z/\!\!/G$ as vectors of homogeneous polynomials.

1. If $Z = \mathbb{C}^{n+1}$ and $G = \mathbb{C}^*$ with action (3.9), and if θ is the identity character of $\mathbb{C}^*$, then $Z/\!\!/_\theta G$ is $\mathbb{P}^n$. A quasimap to $\mathbb{P}^n$ of degree $d \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ with source curve $\mathbb{P}^1$ is
   - A principal $\mathbb{C}^*$-bundle $\mathscr{P}$ such that $\deg_{\mathbb{P}^1}(\mathscr{P} \times_{\mathbb{C}^*} \mathbb{C}_\theta) = d$. Therefore $\mathscr{P}$ is the underlying principal bundle of $\mathscr{O}_{\mathbb{P}^1}(d)$.

- A section of $\mathscr{P} \times_{\mathbb{C}^*} \mathbb{C}^{n+1}$. Here the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ is given by (3.9), so that this vector bundle is $n+1$ copies of the associated bundle, i.e., $\mathscr{O}(d)^{\oplus n+1}$.

Therefore, a prestable quasimap to $\mathbb{P}^n$ of degree $d \in \mathbb{Z}$ is given by a vector

$$(\; p_1(x,y), \; p_2(x,y), \; \ldots, \; p_{n+1}(x,y) \;)$$

of homogeneous polynomials of degree d that are not all zero. In particular (3.7) is a prestable quasimap of degree 2.

2. If $Z = M_{k\times n}$ is $k\times n$ matrices over $\mathbb{C}$ and $G = GL_k$ acts on Z by left multiplication and θ is the determinant character, then $Z/\!\!/_\theta G$ is the Grassmannian $Gr(k,n)$. A quasimap to $Gr(k,n)$ of degree $d \in \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$ with source curve $\mathbb{P}^1$ is

   - A principal GL_k-bundle $\mathscr{P}$ on $\mathbb{P}^1$ such that $\deg_{\mathbb{P}^1}(\mathscr{P} \times_{GL_k} \mathbb{C}_\theta) = d$. If E is the rank-k vector bundle associated to $\mathscr{P}$, then $\mathscr{P} \times_{GL_k} \mathbb{C}_\theta$ is the determinant bundle of E. From Grothendieck's classification of principal bundles [Gro57], we have $E = \oplus_{i=1}^k \mathscr{O}(d_i)$ for some d_i with $\sum_{i=1}^k d_i = d$.
   - A section of $\mathscr{P} \times_{GL_k} M_{k\times n}$, i.e., of $E^{\oplus n}$.

   So a prestable quasimap to $Gr(k,n)$ of degree d is given by a matrix of polynomials

$$[p_{ij}(x,y)]_{1\le i\le k,\, 1\le j\le n} \tag{3.11}$$

   where $p_{ij}(x,y)$ is homogeneous of degree d_i. Because $M^{us}_{k\times n}$ is matrices of low rank, to define a prestable quasimap the matrix (3.11) must have low rank on a finite set.

Moduli of quasimaps

As explained in Section 2.4.5, prestable quasimaps are an open subset of the moduli of sections $\underline{\mathrm{Sec}}(\mathfrak{C} \times [Z/G]/\mathfrak{C})$ over $\mathfrak{M}_{g,n}$, where $\mathfrak{C}$ is the universal curve on $\mathfrak{M}_{g,n}$. We can further impose a stability condition given by a rational number ϵ to cut out a separated, Deligne-Mumford substack of the moduli. We will not use these ϵ-stable quasimap spaces in this thesis. However, for the sake of exposition, in this section we informally discuss the moduli problem encoded by the quasimap spaces, including ϵ-stability.

Let $\mathfrak{M}_{g,n}(Z/\!\!/G,\beta)$ be the moduli stack of prestable quasimaps to $Z/\!\!/G$ of genus g, degree β, and n marks. Objects in this category over a base S are families of prestable quasimaps on S; i.e., they are diagrams

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\tilde{\sigma}} & V \\ \downarrow & & \\ \mathcal{C} & & \end{array}$$

where $\mathcal{C} \to S$ is a flat family of genus-g nodal curves (not necessarily stable) on S, $\mathscr{P} \to \mathcal{C}$ is a principal G-bundle, $\tilde{\sigma} : \mathscr{P} \to Z$ is G-equivariant, and geometric fibers over S are prestable quasimaps of degree β. An isomorphism between objects in $\mathfrak{M}_{g,n}(Z/\!\!/G,\beta)(S)$—i.e., of quasimap families—is a commuting diagram

$$\begin{array}{ccccc} \mathscr{P}' & \xrightarrow{\sim} & \mathscr{P} & \xrightarrow{\tilde{\sigma}} & Z \\ \downarrow & & \downarrow & & \\ \mathcal{C}' & \xrightarrow{\sim} & \mathcal{C} & & \end{array} \qquad (\tilde{\sigma}' : \mathscr{P}' \to Z)$$

where $\mathcal{C}' \xrightarrow{\sim} \mathcal{C}$ commutes with the maps to S and the square is fibered. The stack $\mathfrak{M}_{g,n}(Z/\!\!/G,\beta)$ is not Deligne-Mumford, as some prestable quasimaps have non-finite automorphism groups. We see the familiar offenders from stable map theory: for example, a degree-0 map sending $\mathbb{P}^1$ to a point in $\mathbb{P}^2$ is invariant under the entire automorphism group of $\mathbb{P}^1$. However, there are new examples as well. For instance, $[x^2 : 3x^2 : x^2]$ defines a prestable quasimap of degree 2 from $\mathbb{P}^1$ to $\mathbb{P}^2$ which is invariant under $[x : y] \mapsto [x : ty]$.

Example 3.2.6. Let's find isomorphisms between the quasimaps in Example 3.2.5 that do not come from automorphisms of the source curve—in other words, we compute those isomorphisms of quasimaps from a fixed $\mathbb{P}^1$ to $\mathbb{P}^n$ and $Gr(k,n)$. Quasimaps from a fixed $\mathbb{P}^1$ will play an important role in this chapter.

1. Let Z, G, and θ be as in Example 3.2.5 part 1. From that example, a prestable quasimap from $\mathbb{P}^1$ to $\mathbb{P}^n$ of degree d is given by a vector of homogeneous degree-d polynomials $(p_i(x,y))_{i=1,\ldots,n+1}$. Then a quasimap isomorphism is an element $\alpha \in \mathrm{Aut}(\mathscr{O}(d)) = \mathbb{C}^*$, and it sends $(p_i(x,y))$ to $(\alpha p_i(x,y))$.

2. Let Z, G, and θ be as in Example 3.2.5 part 2. From that example, a prestable

quasimap from $\mathbb{P}^1$ to $Gr(k,n)$ of degree d is n sections of a vector bundle $\oplus_{i=1}^{k}\mathscr{O}(d_i)$ of degree d, which may be denoted by a $k\times n$ matrix $(p_{ij}(x,y))$ of polynomials where $p_{ij}(x,y)$ is homogeneous of degree d_i. Then a quasimap isomorphism is an element A of $\mathrm{Aut}(E)=\mathrm{Hom}(\oplus\mathscr{O}(d_i),\oplus\mathscr{O}(d_i))^\times$ which we may identify with a $k\times k$ matrix $(a_{\ell i}(x,y))$ of polynomials where $a_{\ell i}(x,y)$ is homogeneous of degree $d_\ell - d_i$. Such an isomorphism acts on a the quasimap $(p_{ij}(x,y))$ by matrix multiplication. (Notice that if $d_1 \geq d_2 \geq \ldots \geq d_k$, then $(a_{\ell i}(x,y))$ will be block upper triangular.)

We now define the ϵ stability conditions that identify certain open subsets of $\mathfrak{M}_{g,n}(Z/\!\!/G,\beta)$ containing $\mathcal{M}_{g,n}(Z/\!\!/G,\beta)$.

Definition 3.2.7. Let $(C,x_1,\ldots,x_n,\mathscr{P},\tilde{\sigma})$ be a prestable quasimap to $Z/\!\!/_\theta G$ and let $L=\mathscr{P}\times_G\mathbb{C}_\theta$. This quasimap is ϵ-stable if

1. On every component C' of C we have

$$2g_{C'}-2+n_{C'}+\epsilon\deg(L|_{C'})>0$$

   where $g_{C'}$ is the genus of C' and $n_{C'}$ is the number of marked points and nodes on C', and

2. For every $x\in C$ we have

$$\ell(x)\leq 1/\epsilon$$

   where $\ell(x)$, called the *length* of x, is the order of contact of $\sigma(C)$ with $\mathscr{P}\times_G Z^{us}$, and σ is the section of $\mathscr{P}\times_G Z\to C$ determined by $\tilde{\sigma}$. See [CKM14, Def 7.1.1] for more on the definition of length. For $x\in C$, the length $\ell(x)$ is nonzero if and only if $\sigma(x)$ is in $\mathscr{P}\times_G Z^{us}$, and in this case we say x is a *basepoint* of the quasimap.

The first condition (1) is sometimes stated as "$\omega_C(\sum x_i)\otimes L^\epsilon$ is ample." A family of prestable quasimaps is ϵ-stable if every geometric fiber is ϵ-stable. The moduli space of ϵ-stable quasimaps is denoted $\overline{\mathcal{M}}^{\epsilon}_{g,n}(Z/\!\!/G,\beta)$. The next theorem says that moduli spaces of ϵ-stable quasimaps are sufficiently well behaved for defining enumerative theories analogous to Gromov-Witten theory.

Theorem 3.2.8. *[CKM14, Thm 7.1.6] The moduli space $\overline{\mathcal{M}}^{\epsilon}_{g,n}(Z/\!\!/G,\beta)$ is a proper separated Deligne-Mumford stack of finite type.*

As explained in Section 3.2.1, the benefit of these spaces, at least in genus 0, is that when ϵ is greater than 2, the space $\overline{\mathcal{M}}^{\epsilon}_{0,n}(Z/\!\!/G, \beta)$ is the familiar space of stable maps $\overline{\mathcal{M}}_{0,n}(Z/\!\!/G, \beta)$. On the other hand, when ϵ is sufficiently small, the moduli spaces $\overline{\mathcal{M}}^{\epsilon}_{0,n}(Z/\!\!/G, \beta)$ do not depend on ϵ; this is called the 0+-stable quasimap moduli space. The space $\overline{\mathcal{M}}^{0+}_{0,n}(Z/\!\!/G, \beta)$ is more "computable" (see Example 3.2.17). Hence the wall-crossing theorem of [CK14b], which translates between invariants of $\overline{\mathcal{M}}^{\epsilon}_{0,n}(Z/\!\!/G, \beta)$ for differing values of ϵ, gives a way to relate the Gromov-Witten invariants of $Z/\!\!/G$ to invariants that are more computable.

We close with an example that illustrates how ϵ-stability cuts out separated substacks of $\mathfrak{M}_{g,n}(Z/\!\!/G, \beta)$.

Example 3.2.9. In Section 3.2.1 we described two possible limits of the family (3.6) of quasimaps to $\mathbb{P}^2$. One limit was the stable map $\widetilde{\phi}$. Because $\widetilde{\phi}$ has no basepoints, it satisfies condition (2) in Definition 3.2.7 for each ϵ. However, on the component with no marks—call it C'—we have

$$2g_{C'} - 2 + n_{C'} + \epsilon \deg(L|_{C'}) = -1 + \epsilon,$$

which is possible only for $\epsilon > 1$. So this map is ϵ stable for $\epsilon > 1$.

The second limit was the rational map $\phi_0(x, y) = [0 : xy : y^2]$. This quasimap satisfies condition (1) of Definition 3.2.7 for every $\epsilon > 0$. However, we have $\ell([1 : 0]) = 1$, so that this map is ϵ-stable only when $\epsilon \leq 1$.

Hence, the family (3.6) in $\overline{\mathcal{M}}^{\epsilon}_{g,n}(Z/\!\!/G, \beta)$ has the stable map $\widetilde{\phi}$ for a limit when $\epsilon > 1$, and the rational map ϕ_0 for a limit when $\epsilon \leq 1$.

3.2.3 Quasimap graph spaces and the I-function

Here we develop in detail the quasimap moduli spaces needed to define the I-function of [CKM14].

Graph quasimaps

The quasimap I-function is defined using a variant of the quasimap moduli spaces called *quasimap graph spaces.* This adds the data of an isomorphism between $\mathbb{P}^1$ and a component of the source curve. We review the definition of these objects for arbitrary genus (and no markings), and then we rigorously show how this definition simplifies in the case when $g = 0$. These simpler spaces (with $g = n = 0$) are the only ones needed to define the I-function.

Fix for the duration of this paper a copy of the projective line with projective coordinates $[u : v]$ and denote it $\mathbb{P}^1$. That is, the notation $\mathbb{P}^1$ in this paper will always refer to this particular copy of the projective line, with these coordinates. A priori we have the following definition from [CKM14, Def 7.2.1].

Definition 3.2.10. Let k be an algebraically closed field. A stable *genus-g geometric graph quasimap to* $Z/\!\!/_\theta G$ over k is a tuple $(C, \mathscr{P}, \sigma, \mathbf{x})$ where

- C is a connected genus-g nodal projective curve over k
- $\mathscr{P} \to C$ is a principal G-bundle
- σ is a section of the associated fiber bundle $\mathscr{P} \times_G Z$
- $\mathbf{x} : C \to \mathbb{P}^1$ is an isomorphism on one component of C and contracts the rest of the curve.

These data must satisfy the following conditions:

1. Let C_0 denote the component of C where $\mathbf{x}$ is an isomorphism, let $\widetilde{C} = \overline{(C \setminus C_0)}$ denote the closure of the complement, and let $p_1, \ldots, p_n$ denote the nodes of C. Then $\omega_{\widetilde{C}}(\sum p_i) \otimes (\mathscr{P} \times_G \mathbb{C}_\theta)^\epsilon$ is ample for every rational $\epsilon > 0$
2. The set of points $p \in C$ such that $\sigma(p) \notin Z^s$ is finite and disjoint from the p_i.

(Note that stability depends on the character θ used to construct the GIT quotient $Z/\!\!/_\theta G$.) We will only work with genus-0 quasimaps; in this situation, Definition 3.2.10 simplifies as follows. If $\widetilde{C}$ is nonempty, then it contains some component C_1 with only one node p_1. On this component the bundle $\omega_{C_1}(p_1) \otimes (\mathscr{P} \times_G \mathbb{C}_\theta)^\epsilon$ has negative degree for $\epsilon < 1$ and hence is not ample. So we see that Definition 3.2.10 is the following, which we will use as our definition of a quasimap.

Definition 3.2.11. Let k be an algebraically closed field. A stable *genus-0 geometric graph quasimap to* $Z/\!\!/_\theta G$ over k is a tuple $(C, \mathscr{P}, \sigma, \mathbf{x})$ where

- C is a smooth genus-0 projective curve over k
- $\mathscr{P} \to C$ is a principal G-bundle

- σ is a section of the associated fiber bundle $\mathscr{P} \times_G Z$
- $\mathbf{x} : C \to \mathbb{P}^1$ is an isomorphism.

Moreover, the set of points $p \in C$ such that $\sigma(p) \notin Z^s_\theta$ must be finite.

Again, stability depends on the character θ; since we have fixed θ once and for all in this paper, we will generally omit it from the notation. The set $\{p \in C \mid \sigma(p) \notin Z^s\}$ is called the *base locus* of the quasimap. Because we work exclusively with the quasimaps in Definition 3.2.11, we call them simply "quasimaps."

The *degree* of a quasimap $(C, \mathscr{P}, \sigma, \mathbf{x})$ is defined as in (3.10); that is, it is the homomorphism $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ given by

$$\beta(L) = \deg_C(\sigma^*(\mathscr{P} \times_G \mathscr{L})) \qquad \mathscr{L} \in \mathrm{Pic}^G(Z).$$

Let $T \subset G$ be a maximal torus. From the morphisms $\chi(G) \to \chi(T)$ and $\mathrm{Pic}^G(Z) \to \mathrm{Pic}^T(Z)$ and the inclusion (3.3), we have the following diagram, crucial for understanding how degree works in the abelian-nonabelian correspondence:

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z}) & \xrightarrow{r_{\mathrm{Pic}}} & \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z}) \\ \downarrow \ell_T & & \downarrow \ell \\ \mathrm{Hom}(\chi(T), \mathbb{Z}) & \xrightarrow{r_\chi} & \mathrm{Hom}(\chi(G), \mathbb{Z})) \end{array} \tag{3.12}$$

The maps are all given by restriction of homomorphisms.

We have one more defintion that depends on the character θ.

Definition 3.2.12. The classes $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ that are realized as the class of some stable quasimap to $Z /\!\!/_\theta G$ are called the *θ-effective classes* of (Z, G).

The θ-effective classes form a semigroup.

Remark 3.2.13. When Z is a vector space, and when $r_\chi = r_{\mathrm{Pic}}$ is restricted to θ-effective classes in both the source and target, it has finite fibers. This is shown in the proof of [CKM14, Thm 3.2.5].

It will be convenient to work with specific representatives of quasimaps. Let $U = \{v \neq 0\}$ and $V = \{u \neq 0\}$ be the distinguished affine subsets of $\mathbb{P}^1_k$. Then any morphism $\tau : U \setminus 0 \to G$ defines a transition function that can be used to glue the two trivial bundles $U \times G$ and $V \times G$. We denote the resulting principal G-bundle by $\mathscr{P}_\tau$.

In particular, any cocharacter τ of G defines a principal G-bundle $\mathscr{P}_\tau$ on $\mathbb{P}^1$, or more generally on any $\mathbb{P}^1_k$ by pullback. If $\mathscr{T}$ is a principal T-bundle, then the map $\tau \mapsto \mathscr{T}_\tau$ is W-equivariant with respect to the actions defined in (3.24) and Section 3.2.4

Any k-quasimap is isomorphic to one of the form $(\mathbb{P}^1_k, \mathscr{P}_\tau, \sigma, id)$ for some transition function τ. Note that the fiber bundle $\mathscr{P}_\tau \times_G Z$ is given by gluing the trivial bundles $U \times Z$ and $V \times Z$ via τ. Moreover a quasimap $(\mathbb{P}^1_k, \mathscr{P}_\tau, \sigma, id)$ is completely determined by the maps $\sigma_U : U \to Z$ and $\sigma_V : V \to Z$ of σ, where σ_U is the composition

$$U \xrightarrow{\sigma|_U} (\mathscr{P}_\tau \times_G Z)|_U = U \times Z \xrightarrow{pr_2} Z$$

and σ_V is defined similarly. Hence we have

$$\tau(u)\sigma_U(u) = \sigma_V(u) \qquad \text{for every } u \in U \setminus \{0\}.$$

Two quasimaps $(\mathbb{P}^1_k, \mathscr{P}_\tau, \sigma, id)$ and $(\mathbb{P}^1_k, \mathscr{P}_\omega, \rho, id)$ are isomorphic if and only if there are functions $\phi_U : U \to G$ and $\phi_V : V \to G$ such that

$$\begin{aligned} \phi_V(u^{-1})\tau(u) &= \omega(u)\phi_U(u) && \text{in } T, \text{ for every } u \in U \setminus \{0\} \\ \phi_U(u)\sigma_U(u) &= \rho_U(u) && \text{in } Z, \text{ for every } u \in U \\ \phi_V(v)\sigma_V(v) &= \rho_V(v) && \text{in } Z, \text{ for every } v \in V. \end{aligned} \tag{3.13}$$

Moduli space

As in [CKM14], quasimaps to $Z/\!\!/G$ of fixed degree have a good notion of a family.

Definition 3.2.14. A *family* of degree-β quasimaps to $Z/\!\!/G$ over a base scheme S is a tuple $(C, \mathscr{P}, \sigma, \mathbf{x})$ where

- $C \to S$ is a connected nodal genus-0 projective curve
- $\mathscr{P} \to C$ is a principal G-bundle
- σ is a section of $\mathscr{P} \times_G Z$
- $\mathbf{x} : C \to \mathbb{P}^1_S$ is an morphism

such that geometric fibers of $(C, \mathscr{P}, \sigma, \mathbf{x})$ are geometric quasimaps to $Z/\!\!/G$ of degree β.

A priori from this definition the curve $C \to S$ could be a nontrivial family of nodal curves. In fact, the condition that the geometric fibers of $\mathbf{x}$ are isomorphisms forces $\mathbf{x}$ to be an isomorphism. One can show this directly, or one can note that $(C, \mathbf{x})$ defines a family in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 1)$. This latter moduli space is isomorphic to the Grassmannian $Gr(\mathbb{P}^1, \mathbb{P}^1)$ according to [FP97, 5], which of course is represented by a point. So its universal family is trivial.

Definition 3.2.15. An isomorphism of families of stable quasimaps $(C, \mathscr{P}, \sigma, \mathbf{x})$ and $(C, \mathscr{P}', \sigma', \mathbf{x}')$ on S is a commuting diagram

$$\begin{array}{ccccc} & & \mathscr{P}' & \xrightarrow{\sim} & \mathscr{P} \\ & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \xleftarrow{\mathbf{x}'} & \mathcal{C}' & \xrightarrow[f]{\sim} & \mathcal{C} \end{array} \qquad \text{(with } \mathbf{x}: \mathcal{C} \to \mathbb{P}^1\text{)} \tag{3.14}$$

such that the square is fibered and $f^*\sigma = \sigma'$.

Let $QG_\beta(Z/\!\!/G)$ denote the groupoid of stable degree-β quasimaps to $Z/\!\!/G$. We will denote it simply QG_β when the target $Z/\!\!/G$ is understood. The space QG_β is called a *quasimap graph space* in analogy with Gromov-Witten theory, and it is equal to the space $\mathrm{Qmap}_{0,0}(Z/\!\!/G, \beta; \mathbb{P}^1)$ from [CKM14].

Theorem 3.2.16. *[CKM14, Theorem 7.2.2] The moduli space $QG_\beta(Z/\!\!/G)$ is a proper separated Deligne-Mumford stack of finite type.*

We observe that since the families of curves parametrized by QG_β are trivial,

$$\text{the universal curve on } QG_\beta(Z/\!\!/G) \text{ is trivial.} \tag{3.15}$$

Hence, we may interpret the moduli spaces $QG_\beta(Z/\!\!/G)$ in terms of the moduli of sections of Section 2.1.1: fixing the trivial family of curves $\mathbb{P}^1 \to pt$, the stack $QG_\beta(Z/\!\!/G)$ is an open substack of the moduli of sections $\underline{\mathrm{Sec}}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1)$ (see also Section 2.4.5). Note that from the tower of morphisms

$$\mathbb{P}^1 \times [Z/G] \to \mathbb{P}^1 \times BG \to \mathbb{P}^1 \tag{3.16}$$

we get a morphism of moduli of sections

$$\underline{\mathrm{Sec}}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1) \to \underline{\mathrm{Sec}}(\mathbb{P}^1 \times BG/\mathbb{P}^1) \tag{3.17}$$

where the latter space is the moduli of principal bundles on $\mathbb{P}^1$, denoted $\mathfrak{B}un_G$ in [CKM14, Sec 7.2]. The restriction of the forgetful map (3.17) to $QG_\beta(Z/\!\!/G)$ is denoted μ in [CKM14, Sec 7.2], and the morphism $QG_\beta(Z/\!\!/G) \to \mathcal{M}_{0,0}(\mathbb{P}^1, 1) = pt$ is denoted ν. By the argument in Section 2.4.5 it is equivalent to work over the point pt or over $\mathfrak{B}un_G$. This observation will be important when we choose a perfect obstruction theory on $QG_\beta(Z/\!\!/G)$ in the next section.

As an example, we explicitly describe the space $QG_\beta(\mathbb{P}^n)$.

Example 3.2.17. We have seen (Example 3.2.5) that a quasimap from $\mathbb{P}^1$ to $\mathbb{P}^n$ of degree d is a nonzero element of $\Gamma(\mathbb{P}^1, \mathscr{O}(d))^{\oplus n+1}$. However, two such sections define isomorphic quasimaps exactly when they differ by a complex scalar (Example 3.2.6). Hence, we naively expect

$$QG_d(\mathbb{P}^n) = \left(\, \Gamma(\mathbb{P}^1, \mathscr{O}(d))^{\oplus n+1} \;\setminus\; \{0\} \,\right) / \; \mathbb{C}^* \cong \mathbb{P}^N,$$

where $N = dn + d + n$. Indeed, this space carries a tautological family of quasimaps as follows. On $\mathbb{P}^N$ we have the trivial family of curves $\mathbb{P}^N \times \mathbb{P}^1$, and on this family the vector bundle $V = \mathscr{O}_{\mathbb{P}^N}(1)^{\oplus n+1} \otimes \mathscr{O}_{\mathbb{P}^1}(d)$. An element of $\mathbb{P}^N \times \mathbb{P}^1$ may be written $(\sigma, \mathbf{x})$ where $\sigma \in \Gamma(\mathbb{P}^1, \mathscr{O}(d))^{\oplus n+1}$ is a vector of $n+1$ degree-d homogeneous polynomials in two variables and $\mathbf{x} = (x, y)$. The tautological quasimap is given by the section of V sending $(\sigma, \mathbf{x})$ to $\sigma(\mathbf{x})$.

The tautological family on $\mathbb{P}^N$ defines a map $F : \mathbb{P}^N \to QG_d(\mathbb{P}^n)$ which is a bijection on closed points by Examples 3.2.5 and 3.2.6. It can be shown that $QG_d(\mathbb{P}^n)$ is a smooth algebraic space. Since $\mathbb{P}^N$ is smooth as well, it follows from Lemma 3.6.2 that F is an isomorphism.

Perfect obstruction theory

We describe an absolute perfect obstruction theory on QG_β. Let $C_{QG_\beta} \cong QG_\beta \times \mathbb{P}^1$ denote the universal curve with projection π to QG_β. Let $u : C_{QG_\beta} \to [Z/G]$ denote the universal map (equivalent data to the universal section of the Z-fiber bundle on QG_β). Since $QG_\beta(Z/\!\!/G)$ is an open substack of $\underline{\mathrm{Sec}}(\mathbb{P}^1 \times [Z/G]/\mathbb{P}^1)$, by Theorem 2.1.2 it has a canonical obstruction theory relative to the base pt of the family $\mathbb{P}^1$—that is, a canonical absolute obstruction theory

$$\phi : \mathbb{E}_{QG_\beta} := R\pi_* u^*(\mathbb{L}_{[Z/G]} \otimes \omega^\bullet) \to \mathbb{L}_{QG_\beta} \tag{3.18}$$

We note that since the cotangent complex $\mathbb{L}_{[Z/G]}$ is perfect, there is a canonical isomorphism

$$\mathbb{E}_{QG_\beta} \simeq (R\pi_* u^* \mathbb{T}_{[Z/G]})^\vee$$

given by [FHM03, (4.1)] (it is an isomorphism by [FHM03, Thm 4.4] and Proposition 2.2.6 part 2). A priori, this ϕ may not agree with the ν-relative theory defined in [CKM14, Sec 7.2], which is defined via a mapping cone construction to be compatible with the canonical μ-relative theory. However, the argument in Lemma 2.3.6 shows that (3.18) is in fact compatible with the μ-relative theory in [CKM14], implying that ϕ is a *perfect* obstruction theory and that it induces the same virtual cycle as the μ- and ν-relative theories in [CKM14]. We have proved the following lemma.

Lemma 3.2.18. *The arrow* (3.18) *is an absolute perfect obstruction theory on* QG_β *inducing the same virtual cycle as the one used in [CKM14].*

For this paper it is important that the absolute obstruction theory in Lemma 3.2.18 is constructed canonically, rather than as a (non-unique) mapping cone, so that it is functorial under abelianization.

Quasimap I-function

Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by

$$\lambda \cdot [u : v] = [\lambda u : v], \qquad \lambda \in \mathbb{C}^*. \tag{3.19}$$

This induces an action on QG_β, via

$$\lambda \cdot (C, \mathscr{P}, \sigma, \mathbf{x}) = (C, \mathscr{P}, \sigma, \lambda \circ \mathbf{x}). \tag{3.20}$$

For quasimaps of the form $(\mathbb{P}^1_k, \mathscr{P}_\tau, \sigma, id)$ we can write this action in another way. Observe that we have the diagram

$$\begin{tikzcd} & (\lambda^{-1})^*\mathscr{P}_\tau \arrow[r, "\sim"] \arrow[d] & \mathscr{P}_\tau \arrow[d] \\ \mathbb{P}^1_k \arrow[r, equal] & \mathbb{P}^1_k \arrow[r, "\lambda^{-1}"] & \mathbb{P}^1_k \arrow[ll, bend left, "\lambda"] \end{tikzcd}$$

which implies

$$\lambda \cdot (\mathbb{P}^1_k, \mathscr{P}_\tau, \sigma, id) = (\mathbb{P}^1_k, \mathscr{P}_{\tau\circ\lambda^{-1}}, \sigma \circ \lambda^{-1}, id). \tag{3.21}$$

In terms of the moduli of sections, the action described on QG_β comes from the $\mathbb{C}^*$-equivariant structure on the tower of morphisms (3.16) given by letting $\mathbb{C}^*$ act on $\mathbb{P}^1$ via (3.19). By Lemmas 2.3.9 and 2.3.10 this equivariant tower induces $\mathbb{C}^*$-actions on QG_β and C_{QG_β} making π and u equivariant. It also induces a canonical $\mathbb{C}^*$-equivariant structure on the perfect obstruction theory (3.18).

We define the *fixed locus* of QG_β under the $\mathbb{C}^*$-action as in [CKL17, Sec 3]. Its closed points are geometric quasimaps $(\mathbb{P}^1_k, \mathscr{P}, \sigma, \mathbf{x})$ such that $\lambda \cdot (\mathbb{P}^1_k, \mathscr{P}, \sigma, \mathbf{x})$ is isomorphic to $(\mathbb{P}^1_k, \mathscr{P}, \sigma, \mathbf{x})$ for every $\lambda \in \mathbb{C}^*$ (see eg [AHR19, Prop 5.23]). The I-function of $Z/\!\!/G$ is defined in terms of localization residues at certain fixed loci (see [CKM14, Sec 7.3]).

A fixed quasimap must have all its base points at $[0 : 1]$ or $[1 : 0]$. Indeed, If a graph quasimap $(\mathcal{C}, \mathscr{P}, \tilde{u}, \phi)$ over Spec($\mathbb{C}$) is $\mathbb{C}^*$-fixed, then for every $\lambda \in \mathbb{C}^*$ we have a diagram (3.14) with $(\mathcal{C}', \mathscr{P}', \tilde{u}', \phi') = (\mathcal{C}, \mathscr{P}, \tilde{u}, \lambda \circ \phi)$ and ϕ an isomorphism. Then the map $\mathcal{C}' \to \mathcal{C}$ in (3.14) must be $\phi^{-1} \circ \lambda \circ \phi$. But $\phi^{-1} \circ \lambda \circ \phi$ must fix basepoints of $\tilde{u}$, for every λ. This means that basepoints of $\tilde{u}$ have to be $\phi^{-1}([0 : 1])$ or $\phi^{-1}([1 : 0])$. We can use this information to identify components of the fixed locus: the lengths of these basepoints $\ell([0 : 1])$ and $\ell([1 : 0])$ are constant in families, so specifying these integers specifies a component of the fixed locus. Let $F_\beta(Z/\!\!/G)$ denote the component of the fixed locus of $QG_\beta(Z/\!\!/G)$ corresponding to quasimaps that have a unique basepoint at $[0 : 1]$ (it may not be connected). We will omit the space $Z/\!\!/G$ from the notation when there is no danger of confusion.

Moreover, for a fixed quasimap the resulting map

$$\mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\} \to Z/\!\!/G \tag{3.22}$$

must be constant. Let $ev_\bullet : F_\beta \to Z/\!\!/G$ send a quasimap to the point in $Z/\!\!/G$ that is the image of the constant map (3.22). More precisely $ev_\bullet$ may be defined as follows. Let $(\mathcal{C}, \mathscr{P}, \tilde{\sigma}, \phi)$ be an object of F_β lying over S. Recall from Example 3.2.4 that $\tilde{\sigma}$ defines a section σ of $\mathscr{P} \times_G X \to \mathcal{C}$; since $[1 : 0]$ is not a basepoint, $\sigma([1 : 0])$ is in $\mathscr{P} \times_G Z^s$. So define $ev_\bullet$ to send $(\mathcal{C}, \mathscr{P}, \tilde{\sigma}, \phi)$ to the morphism $S \to Z/\!\!/G$ apparent in the following diagram:

$$\begin{array}{ccc} & \mathscr{P} \times_G Z \hookleftarrow \mathscr{P} \times_G Z^s \longrightarrow Z^s/G = Z/\!\!/G \\ & \sigma \uparrow \downarrow & \\ S \times \{[1:0]\} \longrightarrow & \mathcal{C} & \end{array}$$

Finally we can define the I-function of $Z/\!\!/_\theta G$ as a formal power series in the q-adic completion of the semigroup ring generated by the semigroup of θ-effective classes on (Z, G).

Definition 3.2.19. The (small) I-function of $Z/\!\!/_\theta G$ is

$$I^{Z/\!\!/G}(z) = 1 + \sum_{\beta \neq 0} q^\beta I_\beta^{Z/\!\!/G}(z) \qquad \text{where} \qquad I_\beta^{Z/\!\!/G}(z) = (ev_\bullet)_* \left(\frac{[F_\beta]^{vir}}{e_{\mathbb{C}^*}(N_{F_\beta}^{vir})} \right) \tag{3.23}$$

and the sum is over all θ-effective classes of (Z, G).

3.2.4 Abelianization and the Weyl group

Action of the Weyl group

Heuristically, an abelian/nonabelian correspondence relates data of G to data of T and the Weyl group W. In this section we explain the action of W on several objects of interest.

This group acts on T, characters of T, dual characters, and cocharacters of T in the usual ways; i.e., if $w \in N_G(T)$ and $\xi \in \chi(T)$ and $\tau : \mathbb{C}^* \to T$ is a cocharacter, then

$$\begin{gathered} w \cdot t = wtw^{-1} \quad \text{for } t \in T \\ w \cdot \xi(t) = \xi(w^{-1} \cdot t) \quad \text{for } t \in T \\ w \cdot \tilde{\alpha}(\xi) = \tilde{\alpha}(w^{-1} \cdot \xi) \quad \text{for } \tilde{\alpha} \in \mathrm{Hom}(\chi(T), \mathbb{Z}) \\ (w \cdot \tau)(t) = w \cdot (\tau(t)) \quad \text{for } t \in T. \end{gathered} \tag{3.24}$$

Let Z be a G-scheme. Then W also acts on the quotient Z/T. For $w \in N_G(T)$ and $t \in T$, $z \in Z$ we compute

$$w(tz) = (wtw^{-1})(wz). \tag{3.25}$$

and if $w \in T$ then the objects z and wz are equivalent in Z/T. This means that if

Z/T is a scheme (e.g. a GIT quotient) then we get a well-defined action of W on Z/T.

The Weyl group also acts on the stack quotient $[Z/T]$ by [Rom05b, Rmk 2.4], but this action is more subtle. It remembers the fact that as computed in (3.25), the morphism $w : Z \to Z$ is not a morphism of T-schemes. Rather, the maps $Z \to [Z/T]$ given by tw and wt are isomorphic via a natural transformation equal to the commutator of t and w (i.e., the diagram in [Rom05b, Rmk 2.4] has a nontrivial 2-morphism.) Another way of expressing this phenomenon is to say that the map $w : Z \to Z$ is *twisted-equivariant* for the homomorphism $a : T \to T$ defined by $a(t) = wtw^{-1}$; that is,

$$w(tz) = a(t)w(z).$$

This twisted equivariance also manifests itself in the Weyl actions that we describe below.

On maps to $[Z/T]$. When we study the action of W on maps to $[Z/T]$, this twist a manifests itself in concrete ways. We explain how W acts on a map $S \to [Z/T]$ for a scheme S. Such a morphism is given by a fiber diagram (solid square)

$$\begin{array}{ccccc} \mathscr{T} & \longrightarrow & Z & \xrightarrow{w} & Z \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & [Z/T] & \overset{w}{\dashrightarrow} & [Z/T] \end{array} \tag{3.26}$$

where $\mathscr{T} \to Z$ is a (T-equivariant) morphism of principal T-bundles. For w and S-point of $N_G(T)$, the dashed arrow is the automorphism of $[Z/T](S)$ defined by w as in [Rom05b, Rmk 2.4]. The point is that the right square in (3.26) is NOT a pullback of principal T-bundles since the map w is only twisted equivariant. It becomes a pullback of principal T-bundles only when we twist the action on the source Z by a^{-1}. For example, when Z is a point we get an action on principal bundles: define $w\mathscr{T}$ to be the T-bundle with the same underlying space as $\mathscr{T}$ and with T-action $\cdot_w$ given by

$$t \cdot_w x = (w^{-1}tw) \cdot x \qquad \text{for } t \in T,\ x \in \mathscr{T}. \tag{3.27}$$

Here $\cdot$ denotes the usual action of T on $\mathscr{T}$. The identity map $\mathscr{T} \to w\mathscr{T}$ is an a-equivariant isomorphism.

When Z is nontrivial, the T-equivariant morphism $f : T \to Z$ in (3.26) is also part of the data of the morphism $S \to [Z/T]$, and in fact it is equivalent to the section of $\mathscr{T} \times_T Z$ defined by the quotient of $\mathscr{T} \xrightarrow{(id,f)} \mathscr{T} \times Z$. We see that the action of w on this map is given by

$$w \cdot (\mathscr{T}, \sigma : S \to \mathscr{T} \times_T Z) = (w\mathscr{T}, \varpi \circ \sigma : S \to w\mathscr{T} \times_T Z) \tag{3.28}$$

where $\varpi : \mathscr{T} \times_T Z \to w\mathscr{T} \times_T Z$ is the isomorphism coming from the a-equivariant map

$$\begin{gathered} \mathscr{T} \times Z \xrightarrow{\varpi} w\mathscr{T} \times Z \\ (x, z) \mapsto (x, wz). \end{gathered} \tag{3.29}$$

On $\mathrm{Pic}^T(Z)$**.** Finally, W acts on $\mathrm{Pic}^T(Z)$, by which we mean the group of line bundles on $[Z/T]$, or equivalently T-equivariant line bundles on Z. Since W acts on $[Z/T]$ we get an action on $\mathrm{Pic}^T(Z)$ by sending $\mathscr{L} \in Pic([Z/T])$ to the pullback $(w^{-1})^*\mathscr{L}$. We can make this more concrete in terms of T-equivariant line bundles on Z. Over the fiber square defining $(w^{-1})^*\mathscr{L}$, we have a fiber square of T-torsors

$$\begin{array}{ccc} (w^{-1})^*\mathscr{L}_1 & \longrightarrow & \mathscr{L}_1 \\ \downarrow & & \downarrow \\ wZ & \xrightarrow{\quad w \quad} & Z \end{array}$$

where wZ is Z with T-action twisted as in (3.27), so that the map $w : wZ \to Z$ is T-equivariant, and $\mathscr{L}_1$ is a T-equivariant line bundle on Z. From this one sees that

$$w \cdot \mathscr{L}_\xi = \mathscr{L}_{w\xi}, \tag{3.30}$$

i.e., the map $\chi(T) \to \mathrm{Pic}^T(Z)$ defined in (3.3) is W-equivariant. More generally, we have a pullback square of principal T-bundles

$$\begin{array}{ccc} (w^{-1})^*\mathscr{L}_1 & \longrightarrow & \mathscr{L}_1 \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\quad w^{-1} \quad} & Z \end{array} \tag{3.31}$$

where the horizontal maps are a-equivariant (notice that here we take the usual T-action on Z).

The action on $\mathrm{Pic}^T(Z)$ defines an action on $\mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$ analogous to (3.24).

Principal bundles on $\mathbb{P}^1$

We recall Grothendieck's classification of principal G-bundles on $\mathbb{P}^1$, which may be read as an abelian/nonabelian correspondence theorem.

Let $\mathfrak{B}un_G(\mathbb{P}^1)$ denote the moduli space of principal bundles on $\mathbb{P}^1$; that is, if S is a $\mathbb{C}$-scheme then an S-point of $\mathfrak{B}un(\mathbb{P}^1)$ is a principal G-bundle on $\mathbb{P}^1_S$. There is a natural map

$$\psi : \mathfrak{B}un_T(\mathbb{P}^1) \to \mathfrak{B}un_G(\mathbb{P}^1)$$

defined by sending a principal T-bundle $\mathscr{T}$ to $\mathscr{T} \times_T G$. (The space $\mathscr{T} \times_T G$ has a left G-action via multiplication on the right by g^{-1}.) Since $\mathfrak{B}un_T(\mathbb{P}^1)$ is equivalent to the stack of maps from $\mathbb{P}^1$ to $[\bullet/T]$, the discussion in Section 3.2.4 defines an action of W on $\mathfrak{B}un_T(\mathbb{P}^1)$.

Theorem 3.2.20 (Abelian/Nonabelian Correspondence for Principal Bundles)**.** *The map ψ is invariant under the action of W and the induced map $[\mathfrak{B}un_T(\mathbb{P}^1)/W] \to \mathfrak{B}un_G(\mathbb{P}^1)$ is a bijection on k-points, for k an algebraically closed field.*

Proof. For the invariance of ψ, let S be a scheme and let $\mathscr{T}$ be a principal T-bundle on $\mathbb{P}^1_S$. Let $w : S \to N_G(T)$ be an S-point of $N_G(T)$. Then the principal G-bundles $\mathscr{T} \times_T G$ and $w\mathscr{T} \times_T G$ are isomorphic via a map ϖ analogous to the one in (3.29). The remainder of the theorem is just a restatement of Grothendieck's classification theorem [Gro57] for principal bundles on $\mathbb{P}^1$ (see also [MS02, 393]). □

Because the isomorphism class of a principal T-bundle on $\mathbb{P}^1_k$ is completely determined by its degree, we may detect the isomorphism class of a principal G-bundle as follows. Let $\mathscr{P}$ be a principal G-bundle and let $\mathscr{T}$ be a principal T-bundle such that $\mathscr{T} \times_G G \cong \mathscr{P}$. We define a homomorphism $\tilde{\alpha} \in \mathrm{Hom}(\chi(T), \mathbb{Z})$ by

$$\tilde{\alpha}(\xi) = \deg_{\mathbb{P}^1}(\mathscr{T} \times_T \mathbb{C}_\xi) \qquad \xi \in \chi(T).$$

By the above discussion, knowing the Weyl-orbit of $\tilde{\alpha}$ in $\mathrm{Hom}(\chi(T), \mathbb{Z})$ is equivalent to knowing the isomorphism class of $\mathscr{P}$ as a principal G-bundle.

Cohomology of $X/\!\!/G$ and $X/\!\!/T$

We recall an abelian-nonabelian correspondence for cohomology of $X/\!\!/G$ and $X/\!\!/T$. Recall the diagram (3.2) where j is an open embedding and g is a fiber bundle with fiber G/T. The following result is well-known, but we could not find a reference in our setting.

Proposition 3.2.21. *The pullback g^* in diagram* (3.2) *induces an isomorphism*

$$g^* : A_*(Z^s(G)/G) \otimes \mathbb{Q} \xrightarrow{\sim} (A_*(Z^s(G)/T) \otimes \mathbb{Q})^W. \tag{3.32}$$

An analogous statement holds for cohomology rings with coefficients in $\mathbb{Q}$.

Proof. We prove the statement for chow groups. Compare the statement and its proof with [Bri98, Thm 10]. Notice that g factors through $Z^s(G)/T \to Z^s(G)/B$ where $B \subset G$ is a borel subgroup containing T. Since $Z^s(G)/T \to Z^s(G)/B$ is a fiber bundle with affine fibers, pullback induces an isomorphism on chow groups, and we are reduced to showing (3.32) with B in place of T.

Let R be the symmetric algebra over $\mathbb{Q}$ of the character group $\chi(T)$ of T and let R_+^W be the ring of W-invariants of positive degree. Recall the characteristic homomorphism that sends a character of T to the associated line bundle on G/T. Let

$$\phi : R/R_+^W \xrightarrow{\sim} A_*(G/B) \otimes \mathbb{Q}$$

be the (W-equivariant) isomorphism induced by the characteristic homomorphism (see for example [Bri98, 22]), and let $\{c_i\}_{i=1}^N$ be elements of R with images $[c_i]$ in R/R_+^W such that $\{\phi([c_i])\}_{i=1}^N$ is a basis for $A_*(G/B)$. Then the polynomials c_i also define classes $\tilde{c}_i$ in $A^*(Z^s(G)/B) \otimes \mathbb{Q}$ (via an analogous characteristic homomorphism induced by mapping characters to associated line bundles), and the restrictions of these classes to every fiber of g are a basis.

Thus we may apply the algebraic Leray-Hirsch theorem [EG97, Lem 6] to the G/B-fiber bundle g. Let $\{b_j\}$ be a basis for $A_*(Z^s(G)/G) \otimes \mathbb{Q}$. This theorem says that the map $\phi : A_*(Z^s(G)/G)_{\mathbb{Q}} \otimes A_*(G/B)_{\mathbb{Q}} \to A_*(Z^s(G)/B)_{\mathbb{Q}}$ given by

$$\sum b_j \otimes \phi([c_i]) \mapsto \sum g^*(b_j) \cup \tilde{c}_i$$

is an isomorphism. Because g is W-invariant, a direct computation shows that this map is W-equivariant. Now we take W-invariants of both sides. On the left side, if $\sum b_j \otimes \phi([c_i])$ is W-invariant, then because the classes b_j are all W-invariant, the classes $[c_i] \in R/R_+^W$ are as well. Hence if they are nonzero, they have degree zero. The result follows. $\square$

3.3 Abelianization for fixed loci in graph spaces

The goal of this section is to "pull back" diagram (3.2) to the $\mathbb{C}^*$-fixed loci in the quasimap moduli spaces. That is, we prove the following.

Proposition 3.3.1. *Let $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ be effective. For every $\tilde{\beta} \in r_{\mathrm{Pic}}^{-1}(\beta)$, there is*

- *a parabolic subgroup $P_{\ell_T(\tilde{\beta})} \subset G$, and*
- *a morphism $\psi_{\tilde{\beta}} : F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G) \to F_\beta(Z/\!\!/G)$ whose image we denote $F_{\tilde{\beta}}(Z/\!\!/G)$,*

fitting into the following commutative diagram:

$$\begin{array}{ccccc} F_{\tilde{\beta}}(Z/\!\!/G) & \xleftarrow{\psi_{\tilde{\beta}}} & F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G) & \overset{h}{\hookrightarrow} & F_{\tilde{\beta}}(Z/\!\!/T) \\ \downarrow i & & \downarrow & & \downarrow ev_\bullet \\ Z^s(G)/P_{\ell_T(\tilde{\beta})} & \xleftarrow{p} & Z^s(G)/T & \overset{j}{\hookrightarrow} & Z^s(T)/T \\ & \searrow^{f} & \downarrow g & & \\ & & Z^s(G)/G & & \end{array} \tag{3.33}$$

Here, the two squares are fibered, the vertical arrows in the top row are all closed embeddings, and the composition $f \circ i$ is the evaluation map $ev_\bullet$.

3.3.1 Definitions of $P_{\ell_T(\tilde{\beta})}$ and $\psi_{\tilde{\beta}}$

We may identify cocharacters with dual characters of T as follows. A dual character $\tilde{\alpha} \in \mathrm{Hom}(\chi(T), \mathbb{Z})$ determines a cocharacter $\tau_{\tilde{\alpha}}$ via the rule

$$\xi(\tau_{\tilde{\alpha}}(t)) = t^{-\tilde{\alpha}(\xi)} \qquad \text{for any} \qquad \xi \in \chi(T). \tag{3.34}$$

The negative sign in the exponential appears so that $\mathscr{T}_{\tau_{\tilde{\alpha}}}$ has degree $\tilde{\alpha}$. One can check that this identification of cocharacters and dual characters is W-equivariant under the actions defined in (3.24). To lighten the notation we will write $\mathscr{T}_{\tilde{\alpha}}$ for $\mathscr{T}_{\tau_{\tilde{\alpha}}}$ and $\mathscr{P}_{\tilde{\alpha}}$ for the associated principal G-bundle.

The construction of the parabolic subgroup $P_{\ell_T(\tilde{\beta})}$ uses the "dynamic method" (see for example [CGP10, Sec 2.1]). If $\tilde{\alpha} = \ell_T(\tilde{\beta})$ is a dual character and $\tau_{\tilde{\alpha}}$ the cocharacter defined in (3.34), then the dynamic method defines a (reduced) parabolic subgroup with closed points given by

$$P_{\tilde{\alpha}} = \{g \in G \mid \lim_{t\to 0} \tau_{\tilde{\alpha}}(t)^{-1} g \tau_{\tilde{\alpha}}(t) \text{ exists in } G\}. \tag{3.35}$$

The subgroup $P_{\tilde{\alpha}}$ clearly contains T. It has a natural inclusion into $\mathrm{Aut}(\mathscr{T}_\tau \times_T G)$, given by sending $g \in P_{\tilde{\alpha}}$ to the automorphism that is multiplication by g on $V \subset \mathbb{P}^1$. The dynamic method also produces a canonical Levi subgroup $L_{\tilde{\alpha}} \subset P_{\tilde{\alpha}}$, equal to the centrilizer of $\tau_{\tilde{\alpha}}$:

$$L_{\tilde{\alpha}} = \{g \in G \mid \tau_{\tilde{\alpha}}(t)^{-1} g \tau_{\tilde{\alpha}}(t) = g\}$$

In fact this is the unique Levi sugbroup of $P_{\tilde{\alpha}}$ containing T (see [CF, Prop 12.3.1].

Let

$$F^0 = F_{\tilde{\beta}}(Z/\!\!/T) \times_{X^s(T)/T} X^s(G)/T,$$

so F^0 is the open substack of $F_{\tilde{\beta}}(Z/\!\!/T)$ where $ev_\bullet$ lands in $Z^s(G)/T$. We define

$$\begin{aligned} \psi_{\tilde{\beta}} : F^0 &\to F_\beta(Z/\!\!/G) \\ (C, \mathscr{T}, \sigma, \mathbf{x}) &\mapsto (C, \mathscr{T} \times_T G, \sigma, \mathbf{x}). \end{aligned} \tag{3.36}$$

To process this definition, it may help to note that σ is a section of

$$\mathscr{T} \times_T Z = (\mathscr{T} \times_T G) \times_G Z.$$

A priori $\psi_{\tilde{\beta}}$ is a map to $QG_\beta(Z/\!\!/G)$; it is straightforward to check that it factors through $F_\beta(Z/\!\!/G)$. One uses the fact that isomorphisms of principal T-bundles induce isomorphisms of associated G-bundles.

3.3.2 Properties of F_β

The discussion of this section applies to both $F_\beta(Z/\!\!/G)$ and $F_\beta(Z/\!\!/T)$, so we omit the group from the notation. Our goal is to prove the following.

Lemma 3.3.2. *The stack F_β is a proper separated algebraic space of finite type. If Z smooth, then F_β is smooth and $[F_\beta]^{\mathrm{vir}}$ is the usual fundamental class of F_β.*

The content of this lemma is the statement about smoothness and being an algebraic space. We need smoothness of F_β because we have to work with its geometric points: we do not know of an analog of Grothendieck's classification theorem for principal G bundles on $\mathbb{P}^1_S$ when S is an arbitrary scheme. Fortunately, when F_β is smooth and proper, its geometric points (and their isotropy groups) characterize it up to isomorphism (see Lemma 3.6.2).

Proof. One sees from the definition of torus fixed loci in [CKL17, Sec 3] that F_β is a closed substack of QG_β, so all properties except smoothness and being an algebraic space follow from Theorem 3.2.16.

To check that F_β is an algebraic space, let $(\mathbb{P}^1_S, \mathscr{P}, \sigma, id)$ be a quasimap in F_β over a scheme S, and let ϕ be an automorphism of it, i.e., ϕ is an automorphism of $\mathscr{P}$ such that the induced automorphism of $\mathscr{P} \times_G Z$ fixes σ. If $U \to \mathbb{P}^1_S$ is an étale chart where $\mathscr{P}$ is trivial, then σ is given by a map $\sigma_U : U \to Z$ and ϕ is given by $\phi_U : U \to G$, and these data satisfy

$$\phi_U(u)\sigma_U(u) = \sigma_U(u)$$

for each $u \in U$. This means $\phi_U(u)$ is in the stabilizer $G_{\sigma_U(u)}$. Because the quasimap is stable, the group $G_{\sigma_U(u)}$ is trivial on an open subset of U. Hence ϕ_U is the identity, and ϕ is trivial.

Now we show that F_β is smooth. Let $F = F_\beta$. By [CKL17, Sec 3], the composition

$$(\mathbb{E}_{QG_\beta}|_F)^{\mathrm{fix}} \xrightarrow{(\phi|_F)^{\mathrm{fix}}} (\mathbb{L}_{QG_\beta}|_F)^{\mathrm{fix}} \to \mathbb{L}_F$$

is a perfect obstruction theory for F. We will show that $(\mathbb{E}_\nu|_F)^{\mathrm{fix}}$ has the property that h^{-1} vanishes and h^0 is locally free, and hence by [BF97, Prop 5.5] the stack F is smooth and $[F]^{\mathrm{vir}}$ is the usual fundamental class.

If Z is smooth, then there are vector bundles E_1, E_2 on $[Z/G]$ fitting into a distinguished triangle

$$E_1 \to E_2 \to \mathbb{T}_{[Z/G]} \to .$$

This can be seen, for example, from the description of the cotangent complex in Section

2.2.3 using the smooth cover $Z \to [Z/G]$. Hence we have a distinguished triangle

$$\mathbb{E}_{QG_\beta} \to (R\pi_* u^* E_2)^\vee \to (R\pi_* u^* E_1)^\vee \to . \tag{3.37}$$

We use the following fact.

Lemma 3.3.3. *If E is a vector bundle on $[Z/G]$, then $(R\pi_* u^* E)|_F^{\text{fix}}$ is quasi-isomorphic to a locally free sheaf in degree 0.*

Proof. We have $(R^1\pi_* u^* E)|_p = H^1(C_p, u^*E|_{C_p})$ for any closed point $p \in QG_\beta$. Because p is fixed, there is a $\mathbb{C}^*$-equivariant map identifying C_p with the target $\mathbb{P}^1$. So it has coordinates $[u : v]$ where the $\mathbb{C}^*$-action scales u. Since the linearization on u^*E is trivial, and $H^1(C_p, u^*E|_{C_p})$ has a basis of monomials in u, v where each variable has degree at least 1, we see that this representation has no fixed part. So the fiber of $(R^1\pi_* u^* E)|_F^{\text{fix}}$ vanishes at every closed point, and by Nakayama's lemma this sheaf is zero.

On the other hand, by [CKM14, Claim p.36] the complex $R\pi_* u^* E$ has a global resolution by vector bundles, so we may write $(R\pi_* u^* E)|_F^{\text{fix}} = [T^0 \xrightarrow{f} T^1]$ where the T_i are vector bundles. We have shown that f is surjective. Hence $T^\bullet$ is quasi-isomorphic to its truncation $\tau_{\leq 0} T^\bullet = \ker(f)$, a locally free sheaf in degree 0. □

Applying this lemma to the complexes in (3.37) and using the fact that dual commutes with fix and restriction to F for these complexes, we see that we have a distinguished triangle

$$\mathbb{E}_{QG_\beta}|_F^{\text{fix}} \to \widetilde{E}_2 \to \widetilde{E}_1 \to$$

with $\widetilde{E}_i$ locally free sheaves in degree 0. From the long exact sequence in cohomology, we see that $h^{-1}(\mathbb{E}_{QG_\beta}|_F^{\text{fix}})$ is zero and $h^0(\mathbb{E}_{QG_\beta}|_F^{\text{fix}})$ is the kernel of a map of locally free sheaves, hence locally free.

□

3.3.3 Proof of Proposition 3.3.1 when Z is a vector space

In this section we assume that Z is a vector space X and that G acts linearly on X, and we prove Proposition 3.3.1 in this case. When X is a vector space, the inclusion (3.3) is an isomorphism and the maps ℓ and ℓ_T are identity maps. Hence, in this section, we will write α for the degree of a quasimap to $X /\!/ G$ and $\tilde{\alpha}$ for the degree of a quasimap to $X /\!/ T$ (generally assuming that $r(\tilde{\alpha}) = \alpha$).

Let $\mathcal{B}$ be a basis for X where T acts diagonally, and let $\xi_1, \dots, \xi_r$ be the corresponding torus weights. The following lemma tells us explicitly what geometric quasimaps in $F_{\tilde{\alpha}}(X/\!\!/T)$ look like.

Lemma 3.3.4. *A geometric quasimap in $F_{\tilde{\alpha}}(X/\!\!/T)$ is isomorphic to one of the form $(\mathbb{P}^1_k, \mathscr{T}_{\tilde{\alpha}}, (c_i u^{\tilde{\alpha}(\xi_i)}), id)$ where the c_i are complex numbers, and conversely any quasimap in $QG_{\tilde{\alpha}}(X/\!\!/T)$ of this form is fixed. Moreover, $ev_\bullet$ sends this quasimap to the class of the vector $(c_1, \dots, c_r)$ in X.*

We remark that the vector of monomials

$$(c_i u^{\tilde{\alpha}(\xi_i)})_{i=1}^r$$

denotes a section of $\mathscr{T}_{\tilde{\alpha}} \times_G X = \oplus_{i=1}^r \mathscr{O}(\tilde{\alpha}(\xi_i)$ on $\mathbb{P}^1_k$ (with coordinates u, v) in the standard way.

Proof. After isomorphism, we may assume a geometric quasimap in $QG_{\tilde{\alpha}}(X/\!\!/T)$ is isomorphic to one of the form $(\mathbb{P}^1_k, \mathscr{T}_{\tilde{\alpha}}, \sigma, id)$.

Now assume this quasimap is in $F_{\tilde{\alpha}}(X/\!\!/T)$. We first show that σ_V must be constant. Combining (3.13) and (3.21), we see that for each $\lambda \in \mathbb{C}^*$ we have a morphism $\phi_V^\lambda : V \to T$ satisfying

$$\phi_V^\lambda(v)\sigma_V(\lambda^{-1}v) = \sigma_V(v) \quad \text{for all} \quad v \in V \text{ and } \lambda \in \mathbb{C}^*. \tag{3.38}$$

In the basis $\mathcal{B}$, the morphism $\phi_V : V \cong \mathbb{A}^1 \to X$ is given by a vector of polynomials in v, and (3.38) says that each coordinate polynomial must be homogeneous in v. Moreover, (3.38) says that $\sigma_V(\lambda^{-1}v)$ is in the T-orbit $T\sigma_V(v)$ for all λ, so since T-orbits on X^s are closed and $v = 0$ is not a basepoint, we have $\sigma_V(0) \in T\sigma_V(v)$ as well. This is not possible if some coordinate polynomial vanishes at $v = 0$ but is not itself identically zero; hence each must be constant.

On the other hand, the section σ in $\Gamma(\mathbb{P}^1_k, \oplus_{i=1}^r \mathscr{O}(\tilde{\alpha}(\xi_i))$ is represented by a vector of homogeneous polynomials in u and v. Since σ_V is constant, as a vector of polynomials σ must have the desired form.

Conversely we check that a quasimap $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, (c_i u^{\tilde{\alpha}(\xi_i)}), id)$ is fixed. By (3.13) and (3.21), for each $\lambda \in \mathbb{C}^*$ we must find morphisms $\phi_V^\lambda : V \to T$ and $\phi_U^\lambda : U \to T$

satisfying

$$\begin{aligned} \phi_V^\lambda(u^{-1})\tau_{\tilde\alpha}(u) &= \tau_{\tilde\alpha}(\lambda^{-1}u)\phi_U^\lambda(u) && \text{in } T, \text{ for every } u \in U \setminus \{0\} \\ \phi_U^\lambda(u)\sigma_U(u) &= \sigma_U(\lambda^{-1}u) && \text{in } X, \text{ for every } u \in U \\ \phi_V^\lambda(v)\sigma_V(v) &= \sigma_V(\lambda^{-1}v) && \text{in } X, \text{ for every } v \in V. \end{aligned}$$

One checks using (3.34) that $\phi_U^\lambda(u) = \tau_{\tilde\alpha}(\lambda)$ and $\phi_V^\lambda(v) = 1$. □

Our first application of Lemma 3.3.4 is as follows. The stack $QG_\alpha(X/\!\!/T) = \sqcup_{\tilde\alpha\mapsto\alpha} QG_{\tilde\alpha}(X/\!\!/T)$ is still of finite type (this follows from Remark 3.2.13; let $F_\alpha(X/\!\!/T) \subset QG_\alpha(X/\!\!/T)$ be the fixed locus. There is a natural rational map $\psi : QG_\alpha(X/\!\!/T) \dashrightarrow QG_\alpha(X/\!\!/G)$ given by the disjoint union of the maps $\psi_{\tilde\alpha}$ defined in (3.36), and by Grothendieck's classification (Theorem 3.2.20) the map ψ is surjective. The restriction of ψ to $F_\alpha(X/\!\!/T)$ factors through $F_\alpha(X/\!\!/G)$; however it is not at all clear that the restricted rational map $\psi : F_\alpha(X/\!\!/T) \dashrightarrow F_\alpha(X/\!\!/G)$ is still surjective. In other words, if a quasimap is fixed as a map to $[X/G]$, is it also fixed as a map to $[X/T]$? This is the content of the following lemma.

Lemma 3.3.5. *The rational map $\psi : F_\alpha(X/\!\!/T) \dashrightarrow F_\alpha(X/\!\!/G)$ sending $(C, \mathscr{T}, \sigma, \mathbf{x})$ to $(C, \mathscr{T} \times_T G, \sigma, \mathbf{x})$ is surjective.*

Proof. Because the F_α are proper, the image of ψ is closed, and it suffices to check (essential) surjectivity on $\mathbb{C}$-points. Let $(\mathbb{P}^1, \mathscr{P}_{\tilde\alpha}, \sigma, id)$ be an element of $F_\alpha(X/\!\!/G)$ over $\mathrm{Spec}(\mathbb{C})$. We find an automorphism ϕ of $\mathscr{P}_{\tilde\alpha}$ sending σ to a section ρ with ρ_V a constant function. By Lemma 3.3.4 the quasimap $(\mathbb{P}^1, \mathscr{T}_{\tilde\alpha}, \rho, id)$ is in $F_{\tilde\alpha}(X/\!\!/T)$, and $(\mathbb{P}^1, \mathscr{P}_{\tilde\alpha}, \sigma, id)$ is in its essential image.

To define ϕ_V, let $\iota : G \hookrightarrow X^s(G)$ be the morphism defined by $i(g) = g\sigma_V(0)$. This is a closed embedding as follows: since $X^s(G) \to X/\!\!/G$ is a principal G-bundle, the map $G \times X^s(G) \to X^s(G) \times_{X/\!\!/G} X^s(G)$ is an isomorphism. On the other hand by [Stacks, Tag 02XE] there is a fiber square

$$\begin{array}{ccc} X^s(G) \times_{X/\!\!/G} X^s(G) & \longrightarrow & X/\!\!/G \\ \downarrow & & \downarrow \\ X^s(G) \times X^s(G) & \longrightarrow & X/\!\!/G \times X/\!\!/G \end{array}$$

so since $X/\!\!/G$ is a separated scheme, the composition $G \times X^s(G) \to X^s(G) \times X^s(G)$ is a closed embedding.

We claim that $\sigma_V : V \to X^s(G)$ factors through this image. Granting this, we define $\phi_V = \iota^{-1}\sigma_V$, or in other words,

$$\phi_V(v)\sigma_V(0) = \sigma_V(v). \tag{3.39}$$

To see that σ_V factors through the image of ι, note that because this quasimap is fixed, $\sigma_V(V \setminus \{0\})$ is contained in a single G-orbit. Because G-orbits are closed, $\sigma_V(0)$ must also be in this orbit.

Now define

$$\phi_U(u) = \tau(u)^{-1}\phi_V(u^{-1})\tau(u) \qquad \text{for } u \in U \setminus \{0\}. \tag{3.40}$$

We show that ϕ_U extends to all of U. Embed G as a closed subgroup in some GL_n. There is a commuting diagram

$$\begin{array}{ccccccc} U \setminus \{0\} & \xrightarrow{\phi_U} & G & \hookrightarrow & \mathbb{A}^{n^2} & \xhookrightarrow{j_0} & \mathbb{P}^{n^2} \\ & & & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det_{\mathbb{P}}} \\ & & & & \mathbb{C} & \xhookrightarrow{i_0} & \mathbb{P}^1 \end{array}$$

where $\mathbb{A}^{n^2}$ is the ring of $n \times n$ matrices, and all hooked arrows are embeddings with $j_0(x_1, \ldots, x_{n^2}) = [x_1 : \ldots x_{n^2} : 1]$ and $i_0(x) = [x : 1]$. The composition arrow $U \setminus \{0\} \xrightarrow{\phi_U} G \hookrightarrow \mathbb{A}^N$ extends to one $\tilde{F} : U \to \mathbb{P}^N$.

We show $\tilde{F}$ factors through G. First notice that the morphism $\det \circ \phi_V$ from $V \simeq \mathbb{A}^1$ to $\mathbb{C}^*$ must be constant, say equal to d. (One way to see this is that the corresponding ring map $\mathbb{C}[x, x^{-1}] \to \mathbb{C}[x]$ must send x to a unit and hence to something in k.) So from the formula for ϕ_U, the composition $\det_{\mathbb{P}} \circ \tilde{F}$ is also constant with value $[d : 1]$. In particular $\tilde{F}$ factors through $GL_n \subset \mathbb{A}^{n^2}$. But G is a closed subgroup of GL_n, so $\tilde{F}$ factors through G.

By (3.40), the morphisms ϕ_U and ϕ_V define an automorphism of $\mathscr{P}_{\tilde{\alpha}}$, and by (3.39) its inverse sends σ to a section ρ with $\rho_V = \sigma_V(0)$ a constant function.

□

Our next application of Lemma 3.3.4 is to characterize the fixed locus $F_{\tilde{\alpha}}(X /\!\!/ T)$. A portion of this lemma is stated without proof in [CK10, 29].

Lemma 3.3.6. *There is a subspace $X_{\tilde{\alpha}} \subset X$, invariant under $P_{\tilde{\alpha}}$, such that $ev_\bullet : F_{\tilde{\alpha}}(X /\!\!/ T) \to X /\!\!/ T$ is a closed embedding with image $X_{\tilde{\alpha}} /\!\!/ T$. The universal family on*

$X_{\tilde\alpha}/\!\!/T$ defines a vector bundle $\mathcal{X}$ on $\mathbb{P}^1_{X_{\tilde\alpha}/\!\!/T}$ and section $\mathcal{S}$:

$$\mathcal{X} = \frac{X^s_{\tilde\alpha}(T) \times \mathbb{C}^2 \times X}{(x,u,y) \sim (tx, su, \tau_{\tilde\alpha}(s)ty)} \qquad \mathcal{S}(x,u) = (x, u, \tau_{\tilde\alpha}(u_1)x) \tag{3.41}$$

where $X^s_{\tilde\alpha}(T) = X_{\tilde\alpha} \cap X^s(T)$ and we have $(x,u,y) \in X^s_{\tilde\alpha}(T) \times \mathbb{C}^2 \times X$ with $u = (u_1, u_2)$ and $(t,s) \in T \times \mathbb{C}^*$.

Proof. Define $X_{\tilde\alpha}$ to be the subspace of X spanned by those elements of $\mathcal{B}$ whose corresponding weights ξ_i satisfy $\tilde\alpha(\xi_i) \geq 0$ (note the subspace $X_{\tilde\alpha}$ is independent of the choice of basis $\mathcal{B}$). To see that $X_{\tilde\alpha}$ is invariant under $P_{\tilde\alpha}$, let $g \in P_{\tilde\alpha}$ and let $x \in X_{\tilde\alpha}$ have weight ζ with $\tilde\alpha(\zeta) \geq 0$. From the definition (3.35) of $P_{\tilde\alpha}$ we know

$$\lim_{u\to 0} \tau_{\tilde\alpha}(u)^{-1} g \tau_{\tilde\alpha}(u) x \qquad \text{exists in } X$$

for $u \in \mathbb{C}^*$. But $\tau_{\tilde\alpha}(u)x = u^{-\tilde\alpha(\zeta)}x$ (using (3.34)). Write $x = (c_i)_{i=1}^r$ in the coordinates of the basis $\mathcal{B}$. Since G acts linearly on X, we have

$$\lim_{u\to 0} u^{-\tilde\alpha(\zeta)} u^{\tilde\alpha(\xi_i)} c_i \qquad \text{exists in } \mathbb{C}$$

for every $i = 1, \ldots, r$. If $c_i \neq 0$ this implies $\tilde\alpha(\xi_i) \geq \tilde\alpha(\zeta)$, so in particular $X_{\tilde\alpha}$ is invariant.

To show that $ev_\bullet$ is a closed embedding with image $X_{\tilde\alpha}/\!\!/T$, note that $X_{\tilde\alpha}/\!\!/T$ is a smooth closed subvariety of $X/\!\!/T$, and $ev_\bullet$ factors through $X_{\tilde\alpha}/\!\!/T$. We will show that

$$ev_\bullet : F_{\tilde\alpha}(X/\!\!/T) \to X_{\tilde\alpha}/\!\!/T$$

induces a bijection of $\mathbb{C}$-points; by Lemma 3.3.2 and Lemma 3.6.2 the result follows.

By Lemma 3.3.4 a quasimap in $F_{\tilde\alpha}(X/\!\!/T)$ has the form $(\mathbb{P}^1, \mathscr{T}_{\tilde\alpha}, c_i u^{\tilde\alpha(\xi_i)}, id)$ for some complex numbers $(c_i)_{i=1}^r$. Suppose $(d_i)_{i=1}^r$ is another vector defining a quasimap with the same image under $ev_\bullet$. Then for some $t \in T$ we have

$$(c_i)_{i=1}^r = t \cdot (d_i)_{i=1}^r.$$

Hence if we define $\phi_V(v) = \phi_U(u) = t$ for every $u \in U$ and $v \in V$, the equations (3.13) are satisfied, and the quasimaps are isomorphic.

On the other hand, if $(c_i)_{i=1}^r$ are the $\mathcal{B}$-coordinates of a point in $X_{\tilde\alpha}$, it is the image of a quasimap $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, \sigma := (c_i u^{\tilde{\alpha}(\xi_i)}), id)$ which is in $F_{\tilde{\alpha}}(X/\!\!/T)$ by Lemma 3.3.4.

Turning to the claimed universal family (3.41), note that the quantity $\tau_{\tilde{\alpha}}(u_1)x$ in the definition of $\mathcal{S}$ is well-defined for $u_1 \in \mathbb{C}$ because x is in $X_{\tilde{\alpha}}$: when $u_1 = 0$ we have $\tau_{\tilde{\alpha}}(u_1)x = 0$. Using that T is abelian, one checks that $\mathcal{S}$ is in fact a section of $\mathcal{X}$. Moreover, (3.41) defines a family of fixed quasimaps because its geometric fibers are fixed (this suffices since we are working with smooth schemes). From the definitions, one sees that the map from $X_{\tilde{\alpha}}/\!\!/T$ to $F_{\tilde{\alpha}}(X/\!\!/T)$ induced by (3.41) is a section of $ev_\bullet$, hence an isomorphism. □

Lemma 3.3.7. *Proposition 3.3.1 holds when X is a vector space and G acts linearly.*

Proof. Let F be the pullback of the T-torsor $X^s(G) \to X^s(G)/T$ to $F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G)$ along $ev_\bullet$. By Lemmas 3.6.1 and 3.3.6 and the universal property of categorical quotients, we have a diagram of schemes

$$\begin{array}{ccccc} F/P_{\tilde{\alpha}} & \longleftarrow & F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G) & \longleftarrow & F \\ \big\downarrow & & \big\downarrow{\scriptstyle ev_\bullet} & & \big\downarrow \\ X^s(G)/P_{\tilde{\alpha}} & \longleftarrow & X^s(G)/T & \longleftarrow & X^s(G) \end{array}$$

where the outer square is a fiber diagram of $P_{\tilde{\alpha}}$ torsors and the right square is a fiber diagram of T-torsors. By the universal property of the categorical quotient $F/T = F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G)$, the factorization of $F \to F/P_{\tilde{\alpha}}$ through such a quotient is unique, and hence the left square is also fibered (see [Bri11, Thm 3.3]).

Let $\tilde{\psi}_{\tilde{\alpha}} : F \to F_\alpha(X/\!\!/G)$ denote the composition

$$F \to F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G) \xrightarrow{\psi_{\tilde{\alpha}}} F_\alpha(X/\!\!/G)$$

and let $F_{\tilde{\alpha}}(X/\!\!/G)$ denote its image (also the image of $\psi_{\tilde{\alpha}}$), i.e., the closed substack defined by

$$\ker(\mathscr{O}_{QG_\alpha} \to (\psi_{\tilde{\alpha}})_* \mathscr{O}_{F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G)}).$$

This kernel-sheaf is quasicoherent because $\psi_{\tilde{\alpha}}$ is qcqs (its domain is a Noetherian scheme).

We show that the map $\tilde{\psi}_{\tilde{\alpha}}$ is invariant under the action of $P_{\tilde{\alpha}}$ on F. Because $F_\alpha(X/\!\!/G)$ is a separated algebraic space and F is a reduced scheme, it suffices to check invariance of $\mathbb{C}$-points. Let $(c_i)_{i=1}^r$ be a point of $X^s_{\tilde{\alpha}}(G)$ with image $(c_i')_{i=1}^r$ under $g \in P_{\tilde{\alpha}}$ (note that $X^s_{\tilde{\alpha}}(G) := X_{\tilde{\alpha}} \cap X^s(G)$ is identified with F via $ev_\bullet$). The corresponding quasimaps to $X/\!\!/G$ are $(\mathbb{P}^1, \mathscr{P}_{\tilde{\alpha}}, (c_i u^{\tilde{\alpha}(\xi_i)}), id)$ and $(\mathbb{P}^1, \mathscr{P}_{\tilde{\alpha}}, ((c_i')u^{\tilde{\alpha}(\xi_i)}), id)$, but these are isomorphic via the automorphism of $\mathscr{T}_{\tilde{\alpha}} \times_T G$ defined by $g \in P_{\tilde{\alpha}}$ (see (3.35)).

Since $\tilde{\psi}_{\tilde{\alpha}}$ is invariant, we get an induced map

$$A : F/P_{\tilde{\alpha}} \to F_{\tilde{\alpha}}(X/\!\!/G)$$

factoring $\tilde{\psi}_{\tilde{\alpha}}$. We show that A is an isomorphism using Lemma 3.6.2.

By Lemma 3.3.5, we have $F_\alpha(X/\!\!/G) = \bigcup_{\tilde{\alpha}\to\alpha} F_{\tilde{\alpha}}(X/\!\!/G)$. In fact this union is disjoint and finite by Remark 3.2.13, so since each $F_{\tilde{\alpha}}(X/\!\!/G)$ is closed, we see that $F_{\tilde{\alpha}}(X/\!\!/G)$ is an open subset of $F_\alpha(X/\!\!/G)$. Thus $F_{\tilde{\alpha}}(X/\!\!/G)$ is smooth by Lemma 3.3.2. Moreover $F/P_{\tilde{\alpha}}$ is smooth by Luna's étale slice theorem, and it is proper (it is a closed subscheme of the flag bundle $X^s(G)/P_{\tilde{\alpha}}$ on the projective variety $X/\!\!/G$), so A is proper.

It remains to show that A induces a bijection of $\mathbb{C}$-points and their automorphism groups. Since A is surjective by construction and both $F/P_{\tilde{\alpha}}$ and $F_{\tilde{\alpha}}(X/\!\!/G)$ are algebraic spaces (using Lemma 3.3.2), we only need to show A is injective on $\mathbb{C}$-points. Let $(c_i)_{i=1}^r$ and $(d_i)_{i=1}^r$ be two elements of $X_{\tilde{\alpha}} \cap X^s(G)$ (this space is identified with F via $ev_\bullet$). If their images under $\tilde{\psi}_{\tilde{\alpha}}$ are equal, then the quasimaps $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, (c_i u^{\tilde{\alpha}(\xi_i)}), id)$ and $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, (d_i u^{\tilde{\alpha}(\xi_i)}), id)$ are isomorphic as quasimaps to $X/\!\!/G$. That is, there are maps $\phi_U : U \to X$ and $\phi_V : V \to X$ (defining an element of $\mathrm{Aut}(\mathscr{T}_{\tilde{\alpha}} \times_T G)$) that satisfy

$$\phi_V(v)(c_i)_{i=1}^r = (d_i)_{i=1}^r \qquad \text{for each } v \in V$$
$$\phi_U(u) = \tau_{\tilde{\alpha}}^{-1}(u)\phi_V(u^{-1})\tau_{\tilde{\alpha}}(u) \qquad \text{for each } u \in U \setminus 0$$

(see (3.13)). The first equation implies that ϕ_V is constant and hence may be identified with an element $g \in G$. The second equation implies that $\lim_{t\to 0} \tau_{\tilde{\alpha}}(t)^{-1} g \tau_{\tilde{\alpha}}(t)$ exists in G (and equals $\phi_U(0)$), so we have $g \in P_{\tilde{\alpha}}$.

□

3.3.4 Proof of Proposition 3.3.1 for general Z

Now let Z be an l.c.i. affine variety with $Z^s(G)$ smooth, and let X be a G-representation with a closed equivariant embedding $Z \hookrightarrow X$ inducing an identification $Z^s_\theta(G) = X^s_\theta(G) \cap Z$. This exists by our assumption in Section 3.1.1. Then the natural map

$$\iota : F_\beta(Z/\!\!/G) \to F_{\ell(\beta)}(X/\!\!/G) \tag{3.42}$$

induced by the inclusion $Z \subset X$ is a closed embedding (because it is a proper monomorphism, see for example Lemma 2.3.4). When G is the torus T and $\tilde{\alpha} = \ell_T(\tilde{\beta})$ we have a fiber diagram

$$\begin{array}{ccccc} Z_{\tilde{\beta}} & \hookrightarrow & X_{\tilde{\alpha}} \cap X^s(G) & \hookrightarrow & X^s(G) \\ \downarrow & & \downarrow & & \downarrow \\ F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G) & \xhookrightarrow{\iota} & F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G) & \xhookrightarrow{ev_\bullet} & X^s(G)/T \end{array} \tag{3.43}$$

where the scheme $Z_{\tilde{\beta}}$ is defined to be the fiber product, a closed subscheme of $X^s(G)$.

Lemma 3.3.8. *The subscheme $Z_{\tilde{\beta}} \subset X$ is invariant under the action of $P_{\tilde{\alpha}}$.*

Proof. Let $f : S \to Z_{\tilde{\beta}}$ be a map from a scheme S. The composition

$$S \xrightarrow{f} Z_{\tilde{\beta}} \xrightarrow{\iota} X_{\tilde{\alpha}} \cap X^s(G) \to F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G)$$

defines a family of fixed quasimaps to $X/\!\!/T$ of degree $\tilde{\alpha}$ whose section factors through $Z^s(G) \subset X$. Using the universal family (3.41) we can describe this family as follows. The underlying principal bundle is $\mathbb{P}^1_S \times \mathscr{T}_{\tilde{\alpha}}$, which is trivial on $U \times S$ and $V \times S$ and has transition function $U \times S \xrightarrow{pr_1} U \xrightarrow{\tau_{\tilde{\alpha}}} T$. The section σ_f is defined by setting $\sigma_{f,V\times S}$ equal to the composition

$$V \times S \xrightarrow{pr_2} S \xrightarrow{f} Z_{\tilde{\beta}} \subset X.$$

If $g \in P_{\tilde{\alpha}}$, then $gf : S \to X_{\tilde{\alpha}}$ also defines a family of quasimaps to $X/\!\!/T$ of degree $\tilde{\alpha}$ that factors through $Z^s(G) \subset X$ (since $Z^s(G)$ is G-invariant). This family is given by the same principal bundle $\mathbb{P}^1_S \times \mathscr{T}_{\tilde{\alpha}}$ but with section σ_{gf} defined by setting $\sigma_{gf,V\times S}$ equal to the composition

$$V \times S \xrightarrow{pr_2} S \xrightarrow{gf} X_{\tilde{\alpha}} \cap Z^s(G) \subset X.$$

In particular this is still a family of quasimaps to $Z^s(G)/T \subset Z/\!\!/T$; to show that the induced map to $F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G)$ factors through $F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G)$, we need to show it has degree $\tilde{\beta}$. This may be checked at geometric points of S.

Let $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, \sigma, id)$ be a geometric fiber of the family defined by f on S, a quasimap to $Z/\!\!/T$ of degree $\tilde{\beta}$. The corresponding fiber of the family defined by gf is $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, \wp\circ\sigma, id)$ where $\wp \in \mathrm{Aut}(\mathscr{T}_{\tilde{\alpha}} \times_T Z)$ is the automorphism defined by $g \in P_{\tilde{\alpha}}$ (see (3.35)). Then as a quasimap to $Z/\!\!/T$, the degree of $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, \wp \circ \sigma, id)$ is the homomorphism that sends $\mathscr{L} \in \mathrm{Pic}^T(Z)$ to

$$\deg_{\mathbb{P}^1}((\wp \circ \sigma)^*(\mathscr{T} \times_T \mathscr{L})).$$

Because $P_{\tilde{\alpha}}$ is a connected subgroup of $\mathrm{Aut}(\mathscr{T}_{\tilde{\alpha}} \times_T Z)$, there is a (piecewise linear) homotopy from the automorphism $\wp$ to the identity on $\mathscr{T}_{\tilde{\alpha}} \times_T Z$. In particular the images of σ and $\wp \circ \sigma$ are rationally equivalent, hence the degree of $\mathscr{T}_{\tilde{\alpha}} \times_T \mathscr{L}$ along these two rational curves is the same. □

Proof of Proposition 3.3.1. Let $\beta \in \mathrm{Hom}(\mathrm{Pic}^G(Z), \mathbb{Z})$ be effective and choose $\tilde{\beta} \in r_{\mathrm{Pic}}^{-1}(\beta)$. Set $\alpha = \ell(\beta)$ and $\tilde{\alpha} = \ell_T(\tilde{\beta})$. By Lemma 3.3.8 and Lemma 3.3.7, the map $F_{\tilde{\beta}}(Z/\!\!/T) \to X^s(T)/T$ is a closed embedding, but the image is clearly in $Z^s(T)/T$. So we get the top right fibered square in (3.2).

Recall that $F_{\tilde{\beta}}(Z/\!\!/G)$ is the image of $\psi_{\tilde{\beta}}$ in $F_\beta(Z/\!\!/G)$, and similarly $F_{\tilde{\alpha}}(X/\!\!/G)$ is the image of $\psi_{\tilde{\alpha}}$ in $F_\alpha(X/\!\!/G)$. We have a (a priori solid) commuting diagram

$$\begin{array}{ccccccc} F_\beta(Z/\!\!/G) & \hookleftarrow & F_{\tilde{\beta}}(Z/\!\!/G) & \xleftarrow{\psi_{\tilde{\beta}}} & F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G) & \longleftarrow & Z_{\tilde{\beta}} \\ \downarrow \iota & & \downarrow j & & \downarrow \iota & & \downarrow \\ F_\alpha(X/\!\!/G) & \hookleftarrow & F_{\tilde{\alpha}}(X/\!\!/G) & \xleftarrow{\psi_{\tilde{\alpha}}} & F_{\tilde{\alpha}}(X/\!\!/T) \cap X^s(G) & \longleftarrow & X_{\tilde{\alpha}} \cap X^s(G) \end{array} \tag{3.44}$$

where the rightmost square is the left square of (3.43) and j is induced by the universal property of the image of a morphism. Note that $F_{\tilde{\beta}}(Z/\!\!/G)$ is also the image of $Z_{\tilde{\beta}}$ under the composition

$$Z_{\tilde{\beta}} \to X_{\tilde{\alpha}} \cap X^s(G) \to F_{\tilde{\alpha}}(X/\!\!/T) \xrightarrow{\psi_{\tilde{\alpha}}} F_{\tilde{\alpha}}(X/\!\!/G).$$

But we saw in the proof of Lemma 3.3.7 that $X_{\tilde{\alpha}} \cap X^s(G) \to F_{\tilde{\alpha}}(X/\!\!/G)$ is a principal $P_{\tilde{\alpha}}$-bundle. Hence by Lemmas 3.3.8 and 3.6.1, the arrow $Z_{\tilde{\beta}} \to F_{\tilde{\beta}}(Z/\!\!/G)$ is also a principal $P_{\tilde{\alpha}}$-bundle, and one may argue as in the proof of Lemma 3.3.7 to show that the middle square in (3.44) is fibered. Since the images of ι and j land in $F_{\tilde{\alpha}}(Z/\!\!/T)$ and $F_{\tilde{\alpha}}(Z/\!\!/G)$, respectively, this proves the proposition. □

3.4 Abelianization for quasimap I-functions

In this section we prove Theorem 3.1.1.

3.4.1 Weyl group action

It is now our goal to show that the images $F_{\tilde\beta}(Z/\!\!/G)$ are always disjoint or equal, and to write $F_\beta(Z/\!\!/G)$ as a specific disjoint union of these images. Define

$$F_\beta(Z/\!\!/T) = \sqcup_{\tilde\beta\to\beta} F_{\tilde\beta}(Z/\!\!/T)\cap X^s(G)$$

and let $\psi : F_\beta(Z/\!\!/T) \to F_\beta(Z/\!\!/G)$ be defined to equal $\psi_{\tilde\beta}$ on $F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G)$. Similarly, let $ev_\bullet : F_\beta(Z/\!\!/T) \to Z^s(G)/T$ be defined to equal $ev_\bullet$ on each component. Notice that $F_\beta(Z/\!\!/T)$ is a stack of maps to $[Z/T]$ and hence carries a W-action as in (3.28). Under this action, $ev_\bullet$ is equivariant and ψ is invariant.

For $\tilde\alpha \in \mathrm{Hom}(\chi(T),\mathbb{Z})$, let $W_{\tilde\alpha} = N_{L_{\tilde\alpha}}(T)/T$ be the Weyl group of $L_{\tilde\alpha}$, the unique Levi subgroup of $P_{\tilde\alpha}$ containing T. Recall that W acts on $\mathrm{Hom}(\mathrm{Pic}^T(Z),\mathbb{Z})$ as in Section 3.2.4.

Lemma 3.4.1. *The action of W on $F_\beta(Z/\!\!/T)$ has the following properties.*

1. *If $(C,\mathscr{T},\sigma,\mathbf{x})$ is a quasimap of degree $\tilde\beta$, then $w\cdot(C,\mathscr{T},\sigma,\mathbf{x})$ has degree $w\cdot\tilde\beta$. In particular the action of W permutes the components $F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G)$ of $\tilde F_\beta(Z/\!\!/T)$.*

2. *If $\tilde\alpha = \ell_T(\tilde\beta)$, then the stabilizer of a quasimap of degree $\tilde\beta$ is $W_{\tilde\alpha}$..*

Consequently, if $F_{\tilde\beta_1}(Z/\!\!/T)$ and $F_{\tilde\beta_2}(Z/\!\!/T)$ are W-related, then they have the same image in $F_\beta(Z/\!\!/G)$. Conversely,

3. *If $F_{\tilde\beta_1}(Z/\!\!/T)$ and $F_{\tilde\beta_2}(Z/\!\!/T)$ are not W-related, then their images in $F_\beta(Z/\!\!/G)$ are disjoint.*

Proof. To prove (1), let $(\mathbb{P}^1,\mathscr{T},\sigma,id)$ be a $\mathbb{C}$-quasimap in $F_{\tilde\beta}(Z/\!\!/T)$ and choose an equivariant line bundle $\mathscr{L}\in\mathrm{Pic}^T(Z)$. Then from (3.31) we have a fiber square

$$\begin{array}{ccc} \mathscr{T}\times w^*\mathscr{L} & \longrightarrow & w\mathscr{T}\times\mathscr{L} \\ \downarrow & & \downarrow \\ \mathscr{T}\times X & \xrightarrow{(id,w\cdot)} & w\mathscr{T}\times X \end{array}$$

where the horizontal maps are twisted-equivariant isomorphisms and vertical maps are T-equivariant. Hence we can quotient the square by T to obtain a fiber square over ϖ defined in (3.29). From this it follows that

$$\deg_{\mathbb{P}^1}(\varpi \circ \sigma)^*(w\mathscr{T} \times_T \mathscr{L}) = \deg_{\mathbb{P}^1} \sigma^*(\mathscr{T} \times_T w^*\mathscr{L}) = \deg_{\mathbb{P}^1} \sigma^*(\mathscr{T} \times_T (w^{-1} \cdot \mathscr{L})),$$

or in other words, the degree of the quasimap $(\mathbb{P}^1, w\mathscr{T}, \varpi \circ \sigma, id)$ applied to $\mathscr{L}$ is $(w \cdot \tilde{\beta})(\mathscr{L})$.

For (2), first note that by definition, $L_{\tilde{\alpha}}$ is the G-stabilizer of the cocharacter $\tau_{\tilde{\alpha}}$.Since the identification (3.34) of $\tilde{\alpha}$ and $\tau_{\tilde{\alpha}}$ is W-equivariant, we see that $W_{\tilde{\alpha}}$ is the stabilizer of $\tilde{\alpha}$. So the stabilizer of $\tilde{\beta}$ is a subgroup of $W_{\tilde{\alpha}}$. Conversely, if $w \in N_{L_{\tilde{\alpha}}}(T)$ we want to show $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, \sigma, id)$ and $(\mathbb{P}^1, \mathscr{T}_{w\tilde{\alpha}}, \varpi \circ \sigma, id)$ have the same degree. Because w is in the stabilizer of $\tilde{\alpha}$, the bundles $\mathscr{T}_{\tilde{\alpha}}$ and $\mathscr{T}_{w\tilde{\alpha}}$ are identically the same. In fact the morphism

$$\varpi : \mathscr{T}_{\tilde{\alpha}} \times_T G \to \mathscr{T}_{w\tilde{\alpha}} \times_T G$$

defined in (3.29) is the same as the automorphism $\wp \in \mathrm{Aut}(\mathscr{T}_{\tilde{\alpha}} \times_T G)$ determined by $p = w$ as an element of $P_{\tilde{\alpha}}$. We have

$$\deg_{\mathbb{P}^1}(\varpi \circ \sigma)^*(\mathscr{T}_{w\tilde{\alpha}} \times_T \mathscr{L}) = \deg_{\mathbb{P}^1} \sigma^*(\varpi^*(\mathscr{T}_{\tilde{\alpha}} \times_T \mathscr{L})) = \deg_{\mathbb{P}^1} \sigma^*(\mathscr{T}_{\tilde{\alpha}} \times_T \mathscr{L}),$$

where in the first equality we have replaced $\mathscr{T}_{w\tilde{\alpha}}$ with $\mathscr{T}_{\tilde{\alpha}}$, and the second equality can be argued as in Lemma 3.3.8.

To prove (3), let $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}_1}, \sigma_1, id)$ and $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}_2}, \sigma_2, id)$ be two $\mathbb{C}$-quasimaps with the same image in $F_\beta(z/\!\!/G)$. Then in particular the associated G-bundles $\mathscr{P}_{\tilde{\alpha}_1}$ and $\mathscr{P}_{\tilde{\alpha}_2}$ are isomorphic, so by Theorem 3.2.20 there is some $w \in N_G(T)$ such that $w\tilde{\alpha}_1 = \tilde{\alpha}_2$. Then the injectivity argument in the proof of Lemma 3.3.7 shows that there exists $p \in P_\tau$ such that $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}_2}, w\sigma_1, id) = (\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}_2}, \wp \circ \sigma_2, id)$ (in particular, this argument did not require the two quasimaps to have the same degree, just the same bundle type). Finally the argument of Lemma 3.3.8 shows that $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}_2}, \wp \circ \sigma_2, id)$ and $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}}, \sigma_2, id)$ have the same degree. So the degree of $w \cdot (\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}_1}, \sigma_1, id)$ equals the degree of $(\mathbb{P}^1, \mathscr{T}_{\tilde{\alpha}_2}, \sigma_2, id)$. □

Lemma 3.4.1 shows that the images $F_{\tilde{\beta}}(Z/\!\!/G)$ in $F_\beta(Z/\!\!/G)$ are either disjoint or identical. Moreover, by Lemma 3.3.5 we know every element of $F_\beta(Z/\!\!/G) \subset F_\alpha(X/\!\!/G)$ is the image of some element of $F_\alpha(X/\!\!/T)$ under ψ, but this element must be a quasimap to $Z/\!\!/T$ (because its image is). So the images $F_{\tilde{\beta}}(Z/\!\!/G)$ cover $F_\beta(Z/\!\!/G)$.

Let $\tilde{\beta}_i$ be elements of $\mathrm{Hom}(\mathrm{Pic}^T(Z), \mathbb{Z})$ such that

$$F_\beta(Z/\!\!/G) = \sqcup_i F_{\tilde{\beta}_i}(Z/\!\!/G). \tag{3.45}$$

This is a decomposition of $F_\beta(Z/\!\!/G)$ as a disjoint union of closed subschemes.

3.4.2 Relate the perfect obstruction theories

The main goal of this section is to relate the perfect obstruction theory of $F_{\tilde{\beta}}(Z/\!\!/T)$ to the pullback of the perfect obstruction theory of $F_{\tilde{\beta}}(Z/\!\!/G)$ under $\psi_{\tilde{\beta}}$. Let

$$E_G^\bullet := E^\bullet_{QG_\beta(Z/\!\!/G)} \qquad E_T^\bullet := E^\bullet_{QG_{\tilde{\beta}}(Z/\!\!/T)}$$

denote the absolute perfect obstruction theories defined in (3.18). In what follows, if $A^\bullet$ (resp. $B^\bullet$) is a complex on $QG_\beta(Z/\!\!/G)$ (resp. $QG_{\tilde{\beta}}(Z/\!\!/T)$), we will use the notation

$$A^\bullet|_F := A^\bullet|_{F_{\tilde{\beta}}(Z/\!\!/G)} \qquad (\text{resp. } B^\bullet|_F := B^\bullet|_{F_{\tilde{\beta}}(Z/\!\!/T)\cap Z^s(G)})$$

for the restricted complex whenever the intended degree $\tilde{\beta}$ is clear. In particular, we have

$$E_G^\bullet|_F := E_G^\bullet|_{F_{\tilde{\beta}}(Z/\!\!/G)} \qquad E_T^\bullet|_F := E_T^\bullet|_{F_{\tilde{\beta}}(Z/\!\!/T)\cap Z^s(G)}.$$

Lemma 3.4.2. *In the derived category of $F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G)$, there is a morphism of distinguished triangles*

$$\begin{array}{ccccccc}
\psi_{\tilde{\beta}}^*(E_G^\bullet|_F) & \longrightarrow & E_T^\bullet|_F & \longrightarrow & (R^\bullet\pi_* u_F^* \mathbb{T}_\psi)^\vee & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\psi_{\tilde{\beta}}^*(\mathbb{L}_{QG_\beta(Z/\!\!/G)}|_F) & \longrightarrow & \mathbb{L}_{QG_{\tilde{\beta}}(Z/\!\!/T)}|_F & \longrightarrow & \mathbb{L}_{\psi^\circ}|_F & \longrightarrow
\end{array} \tag{3.46}$$

where ψ is the canonical map $[Z/T] \to [Z/G]$ and u is the restriction to $F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G)$ of the map defined in Section 3.2.3.

The map ψ° will be defined in the proof below.

Proof. We have a tower of morphisms $C_{\mathfrak{M}(\mathbb{P}^1)} \times [X/T] \xrightarrow{\psi} C_{\mathfrak{M}(\mathbb{P}^1)} \times [X/G] \to C_{\mathfrak{M}(\mathbb{P}^1)} \to \mathfrak{M}(\mathbb{P}^1)$ as in (3.16), which leads to a morphism of moduli of sections

$$\psi : \underline{\mathrm{Sec}}(C_{\mathfrak{M}(\mathbb{P}^1)} \times [Z/T]/C_{\mathfrak{M}(\mathbb{P}^1)}) \to \underline{\mathrm{Sec}}(C_{\mathfrak{M}(\mathbb{P}^1)} \times [Z/G]/C_{\mathfrak{M}(\mathbb{P}^1)}).$$

Define $QG_\beta(Z/\!\!/T)^\circ$ to be the fiber product

$$\begin{array}{ccc} QG_\beta(Z/\!\!/T)^\circ & \longrightarrow & \underline{\mathrm{Sec}}(C_{\mathfrak{M}(\mathbb{P}^1)} \times [Z/T]/C_{\mathfrak{M}(\mathbb{P}^1)}) \\ \Big\downarrow \psi^\circ & & \Big\downarrow \psi \\ QG_\beta(Z/\!\!/G) & \longrightarrow & \underline{\mathrm{Sec}}(C_{\mathfrak{M}(\mathbb{P}^1)} \times [Z/G]/C_{\mathfrak{M}(\mathbb{P}^1)}) \end{array}$$

Notice that $QG_\beta(Z/\!\!/T)^\circ$ is an open substack of the (finite) disjoint union of moduli spaces $QG_{\tilde{\beta}}(Z/\!\!/T)$ with $\tilde{\beta}$ mapping to β. On $QG_\beta(Z/\!\!/T)^\circ$ we have the following morphism of distinguished triangles, where the left vertical arrows are the absolute perfect obstruction theories of (3.18) (see Lemma 2.3.6).

$$\begin{array}{ccccccc} (\psi^\circ)^* E^\bullet_G & \longrightarrow & E^\bullet_T & \longrightarrow & (R^\bullet \pi_* u^* \mathbb{T}_\psi)^\vee & \longrightarrow & \\ \Big\downarrow (\psi^\circ)^*\phi_G & & \Big\downarrow (\psi^\circ)^*\phi_T & & \Big\downarrow & & \\ (\psi^\circ)^* \mathbb{L}_{QG_{\tilde{\beta}}(Z/\!\!/G)} & \longrightarrow & \mathbb{L}_{QG_{\tilde{\beta}}(Z/\!\!/T)} & \longrightarrow & \mathbb{L}_{\psi_\circ} & \longrightarrow & \end{array} \tag{3.47}$$

Now restrict this diagram to $F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G)$ and use base change as in (2.18).

□

We use Lemma 3.4.2 to relate the virtual and euler classes appearing in the definition (3.23) of the I-function. We recall the definitions of these classes. According to [CKL17, Sec 3], the composition

$$E^\bullet_G|^{\mathrm{fix}}_F \xrightarrow{\phi|^{\mathrm{fix}}_F} \mathbb{L}_{QG_\beta(Z/\!\!/G)}|^{\mathrm{fix}}_F \to \mathbb{L}_{F_{\tilde{\beta}}(Z/\!\!/G)}$$

is a perfect obstruction theory on $F_{\tilde{\beta}}(Z/\!\!/G)$. The virtual class $[F_{\tilde{\beta}}(Z/\!\!/G)]^{\mathrm{vir}}$ in (3.23) is the one defined by this perfect obstruction theory. By definition we have

$$N^{\mathrm{vir}}_{F_{\tilde{\beta}}(Z/\!\!/G)} := (E^\bullet_G|^{\mathrm{mov}}_F)^\vee. \tag{3.48}$$

We note that the complex (3.48) has a global resolution by vector bundles. (This is because it is a perfect complex on a projective variety, see [Stacks, Tag 0F87].) Thus we may define its euler class as in [Stacks, Tag 0F9E].

Corollary 3.4.3. *We have the following relationships on* $F_{\tilde\beta}(X/\!\!/T)$*:*

$$\psi_{\tilde\beta}^*[F_{\tilde\beta}(Z/\!\!/G)]^{\mathrm{vir}} = [F_{\tilde\beta}(Z/\!\!/T)]^{\mathrm{vir}} \tag{3.49}$$

$$\psi_{\tilde\beta}^* e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{\tilde\beta(Z/\!\!/G)}}) = e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{\tilde\beta(Z/\!\!/T)}}) \frac{\prod_{\tilde\beta(\rho)<0}\prod_{k=\tilde\beta(\rho)+1}^{-1}(c_1(\mathscr{L}_\rho)+kz)}{\prod_{\tilde\beta(\rho)\geq 0}\prod_{k=1}^{\tilde\beta(\rho)}(c_1(\mathscr{L}_\rho)+kz)}. \tag{3.50}$$

Here, ρ *ranges over characters of* T*.*

Proof. For the equality of virtual classes, modify (3.46) by applying the "fix" functor, and then use the commuting square

$$\begin{array}{ccc} F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G) & \longrightarrow & QG_\beta(Z/\!\!/T)^\circ \\ \downarrow & & \downarrow \\ F_{\tilde\beta}(Z/\!\!/G) & \longrightarrow & QG_\beta(Z/\!\!/G) \end{array}$$

and Lemma 2.2.12 part 2 to map the bottom row to the canonical distinguished triangle for $\psi : F_{\tilde\beta}(X/\!\!/T)\cap X^s(G) \to F_{\tilde\beta}(X/\!\!/G)$. The resulting morphism of distinguished triangles

$$\begin{array}{ccccccc} \psi_{\tilde\beta}^*(E_G^\bullet|_F^{\mathrm{fix}}) & \longrightarrow & E_T^\bullet|_F^{\mathrm{fix}} & \longrightarrow & ((R^\bullet\pi_* u_F^*\mathbb{T}_\psi)^{\mathrm{fix}})^\vee & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ \psi_{\tilde\beta}^*(\mathbb{L}_{F_{\tilde\beta}(Z/\!\!/G)}) & \longrightarrow & \mathbb{L}_{F_{\tilde\beta}(Z/\!\!/T)} & \longrightarrow & \mathbb{L}_{\psi_{\tilde\beta}} & \longrightarrow \end{array}$$

is exactly the diagram for checking that we can define virtual pullback along $\psi_{\tilde\beta}$ as in [Man12]. Because the complex $\mathbb{T}_\psi$ is a vector bundle in degree 0, Lemma 3.3.3 shows that $(R\pi_* u^*\mathbb{T}_\psi)|_F^{\mathrm{fix}}$ is also a vector bundle in degree 0. So virtual pullback is defined and agrees with the usual flat pullback [Man12, Rmk 3.10]. By [Man12, Cor 4.9], we get (3.49).

To compute the euler class, pull back the universal family (3.41) along the closed embedding (3.42). We see that we may write the universal curve $[F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G)]\times\mathbb{P}^1$ as the quotient

$$(Z_{\tilde\beta}\times\mathbb{C}^2)/(T\times\mathbb{C}^*)$$

and that with this presentation, the vector bundle $u_F^*\mathbb{T}_\psi$ on $[F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G)]\times\mathbb{P}^1$ is induced from a topologically trivial bundle on $Z_{\tilde\beta}\times\mathbb{C}^2$. This trivial bundle has fiber equal to the subspace of the lie algebra $\mathfrak{g}$ of G with nontrivial weights, viewed as a $T \times \mathbb{C}^*$ representation, where T acts via the adjoint representation and $\mathbb{C}^*$ acts trivially. In particular, $u_F^*\mathbb{T}_\psi$ splits into a sum of line bundles corresponding to the weights ρ of the T-action on $\mathfrak{g}$.

Now we apply [Stacks, Tag 0F9F] to the top row of (3.46), recalling the definition (3.48). Since $R^i\pi_*u_F^*\mathbb{T}_\psi$ is locally free for $i = 0, 1$ (its fibers all have the same rank, see [Har77, Exercise II.5.8]), we get

$$\psi_{\tilde{\beta}}^* e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}(X/\!\!/G)}}) = e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}(X/\!\!/T)}}) \frac{e_{\mathbb{C}^*}((R^1\pi_*u_F^*\mathbb{T}_\psi)^{\mathrm{mov}})}{e_{\mathbb{C}^*}((R^0\pi_*u_F^*\mathbb{T}_\psi)^{\mathrm{mov}})} (-1)^{rk(R^1\pi_*u_F^*\mathbb{T}_\psi)-rk(R^0\pi_*u_F^*\mathbb{T}_\psi)}. \tag{3.51}$$

To compute the sign in (3.51), recall that for the reductive group G the weights ρ come in pairs $(\rho, -\rho)$. Since the euler characteristic of $\mathscr{O}_{\mathbb{P}^1}(d)$ and $\mathscr{O}_{\mathbb{P}^1}(-d)$ have the same parity, we see that the sign is always equal to $+1$.

If $\tilde{\beta}(\rho)$ is nonnegative, then $R^0\pi_*u_F^*\mathbb{T}_\psi$ is nonzero on a closed fiber of π, and a basis is given by the monomials $u^{\tilde{\beta}(\rho)}, u^{\tilde{\beta}(\rho)-1}v, u^{\tilde{\beta}(\rho)-2}v^2, \ldots, v^{\tilde{\beta}(\rho)}$ which have $\mathbb{C}^*$-weights $0, 1, 2, \ldots, \tilde{\beta}(\rho)$, respectively. Hence the euler class of the moving part of this bundle is $\prod_{k=1}^{\tilde{\beta}(\rho)}(c_1(\mathscr{L}_\rho) + kz)$.

If $\tilde{\beta}(\rho)$ is less than -1, then $R^1\pi_*u_F^*\mathbb{T}_\psi$ is nonzero on a closed fiber of π, and a basis is given by monomials $uv^{\tilde{\beta}(\rho)+1}, u^2v^{\tilde{\beta}(\rho)+2}, \ldots, u^{\tilde{\beta}(\rho)+1}v$. Hence the euler class of the moving part of this bundle is $\prod_{k=\tilde{\beta}(\rho)+1}^{-1}(c_1(\mathscr{L}_\rho) + kz)$. □

3.4.3 Proof of the main theorem

The following lemma, a restatement of [Bri96, Prop 2.1], lets us navigate around the bottom left triangle of (3.33).

Lemma 3.4.4. *For any $\delta \in A_*(Z^s(G)/P_{\tilde{\alpha}})$, we have*

$$g^*f_*\delta = \sum_{\omega \in W/W_{\tilde{\alpha}}} \omega^* \left[\frac{p^*\delta}{\prod_{\rho \in R^+_{\tilde{\alpha}}} c_1(\mathscr{L}_\rho)} \right] \tag{3.52}$$

where $R^+_{\tilde{\alpha}}$ is the set of roots of G whose inner product with the dual character $\tilde{\alpha}$ is positive.

Proof. We reduce this statement to the one in [Bri96, Prop 2.1]. Using the dynamic method, one may obtain a Borel subgroup of G, contained in $P_{\tilde{\alpha}}$, equal to P_μ for some cocharacter μ that is positive on any root where $\tau_{\tilde{\alpha}}$ is positive (see e.g. [CF, 45]). So the opposite roots of this Borel, minus the roots of $L_{\tilde{\alpha}}$, are precisely those roots where $\tau_{\tilde{\alpha}}$ is negative. Recalling the relationship (3.34), we see that this is the set $R^+_{\tilde{\alpha}}$.

Using the Leray-Hirsch theorem for $Z^s(G)/P_{\tilde{\alpha}} \to Z^s(G)/G$ and the fact that the characteristic homomorphism $(Sym(\chi(T))^{W_{\tilde{\alpha}}} \to A_*(G/P_{\tilde{\alpha}})$ is surjective (this follows from [Dem74, Sec 1.5] and [RWY11, (3.1)]), it is enough to show that (3.52) holds when δ is in the image of $c^P : (Sym(\chi(T)))^{W_{\tilde{\alpha}}} \to A_*(Z^s(G)/P_{\tilde{\alpha}})$.

For such $\delta = c^P(\xi)$, the result [Bri96, Prop 2.1] tells us

$$g^* f_* \delta = p^* f^* f_* c^P(\xi) = p^* c^P \left(\sum_{w \in W/W_{\tilde{\alpha}}} w \cdot (\xi / \prod_{\rho \in R^+_{\tilde{\alpha}}} \rho) \right).$$

Note that $f_* c^P$ is a restriction of the characteristic homomorphism $c^T : Sym(\chi(T)) \to A^*(Z^s(G)/T)$, by functoriality of c_1. The map c^T is W-equivariant (3.30), so (3.52) follows.

□

Let $\tilde{\alpha}_i = \ell_T(\tilde{\beta}_i)$. Turning to formula (3.23) for $I^{Z/\!\!/G}_\beta(z)$, we first write it as a sum of pushforwards from $F_{\tilde{\beta}_i}(Z/\!\!/G)$ using (3.45). We use Proposition 3.3.1 to identify the evaluation map on each component, and then apply Lemma 3.4.4, obtaining

$$g^* I^{Z/\!\!/G}_\beta = \sum_{\tilde{\beta}_i} \sum_{w \in W/W_{\ell_T(\tilde{\beta}_i)}} w^* \left[\frac{p^* i_*([F_{\tilde{\beta}_i}(Z/\!\!/G)]^{vir} e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}_i}(Z/\!\!/G)})^{-1})}{\prod_{\rho \in R^+_{\tilde{\alpha}_i}} c_1(\mathscr{L}_\rho)} \right]. \qquad (3.53)$$

Let us simplify the numerator of a summand of (3.53). From Lemma 3.4.1 and the equivariance of $ev_\bullet$, we have a commuting diagram

$$\begin{array}{ccccc}
F_{w^{-1}\tilde{\beta}_i}(Z/\!\!/T) \cap Z^s(G) & \xrightarrow{w} & F_{\tilde{\beta}_i}(Z/\!\!/T) \cap Z^s(G) & \xrightarrow{\psi_{\tilde{\beta}_i}} & F_{\tilde{\beta}_i}(Z/\!\!/G) \\
\downarrow{\scriptstyle ev_\bullet} & & \downarrow{\scriptstyle ev_\bullet} & & \downarrow{\scriptstyle i} \\
Z^s(G)/T & \xrightarrow{w} & Z^s(G)/T & \xrightarrow{p_{\tilde{\alpha}_i}} & Z^s(G)/P_{\tilde{\alpha}_i}
\end{array}$$

The square on the left is fibered because w is an isomorphism and the square on the right is fibered by Proposition 3.3.1, so the outer square is fibered. Because w and $p_{\tilde{\alpha}_i}$ are flat, by [Ful98, Prop 1.7] we have $w^* p^*_{\tilde{\alpha}_i} i_* = (ev_\bullet)_* w^* \psi^*_{\tilde{\beta}_i}$, so that the numerator of a summand in (3.53) is

$$(ev_\bullet)_*\psi^*_{w^{-1}\tilde{\beta}_i}\big([F_{\tilde{\beta}_i}(Z/\!\!/G)]^{vir}e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}_i}(Z/\!\!/G)})^{-1}\big) \tag{3.54}$$

where we have also used that ψ (defined on $F_\beta(Z/\!\!/T)$) is equivariant.

Now let us compute the denominator of a summand of (3.53). We get

$$w^*\prod_{\rho\in R^+_{\tilde{\alpha}_i}} c_1(\mathscr{L}_\rho) = \prod_{\rho\in R^+_{\tilde{\alpha}_i}} c_1(\mathscr{L}_{w^{-1}\cdot\rho}) = \prod_{\rho\in R^+_{w^{-1}\cdot\tilde{\alpha}_i}} c_1(\mathscr{L}_\rho). \tag{3.55}$$

The first equality uses (3.30) and the second follows from the fact that the natural pairing between $\chi(T)$ and $\mathrm{Hom}(\chi(T),\mathbb{Z})$ is invariant.

Finally we apply equations (3.54) and (3.55) and use Lemma 3.4.1 to combine the double sum in (3.53) into a single sum, obtaining

$$g^*I^{Z/\!\!/G}_\beta = \sum_{\tilde{\beta}\mapsto\beta}\frac{(ev_\bullet)_*\psi^*_{\tilde{\beta}}([F_{\tilde{\beta}}(Z/\!\!/G)]^{vir}e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}}(Z/\!\!/G)})^{-1})}{\prod_{\rho\in R^+_{\ell_T(\tilde{\beta})}} c_1(\mathscr{L}_\rho)}. \tag{3.56}$$

We can compute the pullbacks in the numerator with Corollary 3.4.3. Finally, applying the projection formula and recalling that $R^+_{\ell_T(\tilde{\beta})}$ is just the set of roots with $\ell_T(\tilde{\beta})(\rho) = \tilde{\beta}(\rho) > 0$, we recover Theorem 3.1.1.

3.5 Extensions and applications

3.5.1 Equivariant I-functions

Let S be a torus and suppose that we have an action of $S\times G$ on Z extending the action of $G = \{1\}\times G$ on Z. In other words, S acts on Z and this action commutes with the action of G. Then S acts on $[Z/G]$ and $[Z/T]$ (see [Rom05b, Rmk 2.4]) and this defines actions on $QG_\beta(X/\!\!/G)$ and $QG_{\tilde{\beta}}(X/\!\!/T)$ and their universal families by Lemma 2.3.9, viewing them as substacks of the moduli of sections as in Section 3.2.3. Moreover the perfect obstruction theory $E^\bullet$ in (3.18) is canonically S-equivariant as in Lemma 2.3.10.

Because the actions of S and $\mathbb{C}^*$ on $\mathbb{P}^1\times[Z/G]$ commute, the $\mathbb{C}^*$-fixed locus $F_\beta(Z/\!\!/G)$ is invariant under the action of S and the $\mathbb{C}^*$-fixed and moving parts of the perfect obstruction theory $E^\bullet$ are also S equivariant. Finally the map $ev_\bullet$ is S-equivariant since the universal family on $QG_\beta(X/\!\!/G)$ is. These statements also hold for T in place of G. Since the spaces $F_\beta(Z/\!\!/G)$ are schemes, we can use the equivariant intersection theory of [EG98] to define $[F_\beta(Z/\!\!/G)]^{S,\text{vir}}$ in $A^S_*(F)$. The class $e_{S\times\mathbb{C}^*}(N^{\text{vir}}_{F_\beta})$ is defined as in [Stacks, Tag 0F9E] but with the euler classes replaced by their $S\times\mathbb{C}^*$-equivariant counterparts. Hence we can define the S-equivariant I-function via the same formulas (3.23), but with all objects replaced by their S-equivariant counterparts.

For $\rho\in\chi(T)$, let L_ρ be the S-equivariant line bundle on $X^s(T)/T$ given by

$$L_\rho = X^s(T)\times_T \mathbb{C}_\rho \tag{3.57}$$

where $\mathbb{C}_\rho$ is the $S\times T$-equivariant representation where S acts trivially and T acts with character ρ.

Corollary 3.5.1. *The S-equivariant I-functions of $Z/\!\!/G$ and $Z/\!\!/T$ satisfy the equation* (3.5)*, with $I^{S,Z/\!\!/G}_\beta(z)$ and $I^{S,Z/\!\!/T}_\beta(z)$ in place of $I^{Z/\!\!/G}_\beta(z)$ and $I^{Z/\!\!/T}_\beta(z)$.*

Proof. First note that Proposition 3.2.21 and Lemma 3.4.4 also hold S-equivariantly (in Lemma 3.4.4, the line bundles $c_1(\mathscr{L}_\rho)$ are S-equivariant as in (3.57) and we take the S-equivariant first chern class). The same proofs work after replacing $Z^s(G)$ with $Z^s(G)\times_S U$, where $U\to U/S$ is an appropriate approximation of the universal S-bundle (definition as in [EG98, Sec 2.2]).

Now the computation in Section 3.4 proceeds as follows. The hardest part is showing that the whole diagram (3.46) is equivariant; i.e., it is isomorphic to the pullback of a morphism of distinguished triangles on $[(F_{\tilde\beta}(Z/\!\!/T)\cap Z^s(G))/S]$ (compare with the proof of Lemma 2.3.11). It suffices to show that the diagram (3.47) is equivariant. In fact, it is the pullback of the analogous diagram on $[QG_\beta(Z/\!\!/T)^\circ/T]$. This uses Lemma 2.3.10, as well as Corollary 2.3.7 to recognize the map

$$(R^\bullet\pi_*u^*\mathbb{T}_\psi)^\vee\to\mathbb{L}_{\psi^\circ}$$

and its analog on $[QG_\beta(Z/\!\!/T)^\circ/T]$ as canonical obstruction theories so that Lemma 2.3.10 applies to this morphism as well. To get commutativity of the remaining squares, use Lemma 2.2.12 part 2 to get two morphisms of distinguished triangles, to one of which apply the functor $R^\bullet\pi_*u^*$.

To compute the equivariant euler class in Corollary 3.4.3, note that since S commutes with G its action on the lie algebra $\mathfrak{g}$ is trivial. The remainder of the proof is the same as in the non-equivariant case. □

3.5.2 Twisted I-functions

Let S be a torus and suppose we have an action of $S \times G$ on Z as in Section 3.5.1. Furthermore, let $R = \mathbb{C}^*$ act trivially on Z with equivariant parameter μ; note this induces the trivial action on F_β as a moduli space of maps. Let E be a $S \times G$-equivariant vector bundle on Z, and let $\mathbb{C}_\mu$ be the R-equivariant vector bundle on Z that is topologically trivial and has its R-action given by scaling fibers. Let E_G denote the $S \times R$-equivariant vector bundle on $[Z/G]$ corresponding to $E \otimes \mathbb{C}_\mu$. Let $\pi : F_\beta \times \mathbb{P}^1 \to F_\beta$ be the universal curve and $u : F_\beta \times \mathbb{P}^1 \to [Z/G]$ the universal map. Recall that we have an additional $\mathbb{C}^*$-action on $F_\beta \times \mathbb{P}^1$ that is trivial on F_β and acts on $\mathbb{P}^1$ via (3.19), and that u is invariant with respect to this action. So $u^* E_G$ is naturally $S \times R \times \mathbb{C}^*$-equivariant. We assume that the complex $R\pi_* u^* E_G$ has a $S \times R \times \mathbb{C}^*$-equivariant global resolution by vector bundles; i.e., it is an element of the rational Grothendieck group

$$K^\circ_{S\times R\times\mathbb{C}^*}(F_\beta) = K^\circ_{S\times\mathbb{C}^*}(F_\beta) \otimes \mathbb{Q}[\mu, \mu^{-1}]$$

of $S \times R \times \mathbb{C}^*$-equivariant vector bundles on F_β. This assumption holds, for example, if $R^1\pi_* u^* E_G$ is zero and $R^0\pi_* u^* E_G$ is a vector bundle (see also [CKM14, Sec 6.2]).

Fix an invertible multiplicative characteristic class $\mathbf{c}$ defining a group homomorphism

$$\mathbf{c} : K^\circ_{S\times R\times\mathbb{C}^*}(F_\beta) \to (H^*_{S\times R\times\mathbb{C}^*}(F_\beta, \mathbb{Q}))^\times$$

to the group of units in $H^*_{S\times R\times\mathbb{C}^*}(F_\beta, \mathbb{Q})$. A priori, $\mathbf{c}$ may be defined only for vector bundles; its invertibility means its definition extends to elements of K-theory. Let $\underline{E}_G$ denote the $S \times R$-equivariant vector bundle on $Z/\!\!/G$ induced by $E \otimes \mathbb{C}_\mu$. Now we define the S-equivariant, $\mathbf{c}(E)$-twisted I-function to be

$$I^{Z/\!\!/G,\, S,\, \mathbf{c}(E)}(z) = 1 + \sum_{\beta\neq 0} q^\beta I_\beta^{Z/\!\!/G,\, S,\, \mathbf{c}(E)}(z)$$

where

$$I_\beta^{Z/\!\!/G,\, S,\, \mathbf{c}(E)}(z) = \mathbf{c}(\underline{E}_G)^{-1}(ev_\bullet)_* \left(\frac{[F_\beta]^{S\times R,\mathrm{vir}} \cap \mathbf{c}(R\pi_* u^* E_G)}{e_{S\times R\times\mathbb{C}^*}(N^{vir}_{F_\beta})} \right). \tag{3.58}$$

(see [CK14b, (7.2.3)]). Note that the torus R is omitted from the superscripts in the

I-function notation.

For the abelianization theorem, observe that E is naturally a T-equivariant vector bundle on Z, so we can also define the $\mathbf{c}(E)$-twisted I-function of $Z/\!\!/T$.

Corollary 3.5.2. *If the class* $\mathbf{c}$ *is functorial with respect to pullback, then Theorem* (3.1.1) *holds with* $I_\beta^{Z/\!\!/G,\,S,\,\mathbf{c}(E)}(z)$ *and* $I_\beta^{Z/\!\!/T,\,S,\,\mathbf{c}(E)}(z)$ *in place of* $I_\beta^{Z/\!\!/G}(z)$ *and* $I_\beta^{Z/\!\!/T}(z)$.

Proof. To complete the computation in Section 3.4.3, first note that

$$g^*\mathbf{c}(\underline{E}_G)^{-1} = \mathbf{c}(g^*\underline{E}_G)^{-1} = \mathbf{c}(\underline{E}_T)^{-1}.$$

The remainder of the computation is the same until the last line when we replace the numerator in the right-hand side of (3.56) with

$$(ev_\bullet)_*\psi_{\tilde{\beta}}^*([F_{\tilde{\beta}}(Z/\!\!/G)]^{vir} \cap e_{S\times R\times\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}}(Z/\!\!/G)})^{-1} \cap \mathbf{c}(R\pi_*u^*E_G)).$$

By functoriality of $\mathbf{c}$, the term $\psi_{\tilde{\beta}}^*(\mathbf{c}(R\pi_*u^*E_G))$ is equal to

$$\mathbf{c}(\psi_{\tilde{\beta}}^*R\pi_*u^*E_G) = \mathbf{c}(R\pi_*u^*\psi^*E_G)$$

where ψ is the natural map from $[X/T]$ to $[X/G]$. The bundle ψ^*E_G is just E_T. $\square$

Remark 3.5.3. The standard application of twisted invariants is to choose E that satisfies $R^1\pi_*u^*E_G = 0$ and set $\mathbf{c}$ to be the R-equivariant euler class e_S. Then the non-equivariant limit of (3.58) exists—i.e., one can set $\mu = 0$. This non-equivariant limit is the definition of the twisted I-function in [CK14b, Sec 7.2]. Taking the non-equivariant limit of Corollary 3.5.2, we see that abelianization holds for these twisted I-functions as well. We will denote these non-equivariant, euler-twisted I-functions by $I^{Z/\!\!/G,E}$.

3.5.3 Big I-functions

The I-function we have been discussing in this paper is often called the *small* I-function because it is related to Gromov-Witten invariants with insertions all in $H^2(Z/\!\!/G,\mathbb{Q})$. The *big I-function* is expected to encode Gromov-Witten invariants with arbitrary insertions, and it is defined in [CK16] to be the generating series

$$\mathbb{I}^{Z/\!\!/G}(z) = 1 + \sum_{\beta \neq 0} q^\beta \mathbb{I}^{Z/\!\!/G}_\beta(z) \quad \text{where} \quad \mathbb{I}^{Z/\!\!/G}_\beta(z) = (ev_\bullet)_* \left(\exp(\hat{ev}^*_\beta(\mathbf{t})/z) \frac{[F_\beta]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{F_\beta})} \right) \tag{3.59}$$

F_β and $ev_\bullet$ are defined as in (3.23), and the sum is over all θ-effective classes β of (Z, G) (but we have yet to define the notation $\exp(\hat{ev}^*_\beta(\mathbf{t})/z)$). The goal of this section is to prove an abelianization formula for $\mathbb{I}^{Z/\!\!/G}(z)$ and use it to derive a closed formula for $\mathbb{I}^{Z/\!\!/G}(z)$ when Z is a vector space.

Let S be a torus acting on Z as in Section 3.5.1—we will define (3.59) S-equivariantly. If $Z^s \subset Z$ is any locus of stable points, recall the Kirwan map

$$\kappa_G : H^*_{S\times G}(Z, \mathbb{Q}) \to H^*_S(Z^s/G, \mathbb{Q}).$$

We will write out the definition when S is trivial. Let EG be the universal principal G-bundle. Then we have maps

$$EG \times_G Z \xleftarrow{a} EG \times_G Z^s \xrightarrow{b} Z^s(G)/G$$

where a is an open embedding and b is projection to the second factor. Then b^* induces an isomorphism on cohomology, and the Kirwan map is defined by $\kappa_G = (b^*)^{-1} \circ a^*$. This map is surjective by [Kir84].

In similar spirit we define

$$\hat{ev}^*_\beta : H^*_{G\times S}(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z] \to H^*_{S\times\mathbb{C}^*}(F_\beta, \mathbb{Q}).$$

Let $\mathscr{P} \to F_\beta \times \mathbb{P}^1$ be the universal principal bundle and let $\sigma : F_\beta \times \mathbb{P}^1 \to \mathscr{P} \times_G Z$ be the universal section. When S is trivial, the map $\hat{ev}^*_\beta$ is simply the pullback in cohomology along the composition of maps

$$F_\beta \xrightarrow{(id,0)} F_\beta \times \mathbb{P}^1 \xrightarrow{\sigma} \mathscr{P} \times_G Z \to EG \times_G Z.$$

Now we can define the notation in (3.59). Fix a homogeneous basis γ_i of $H^*_S(Z/\!\!/G, \mathbb{Q})$. Let $\tilde{\gamma}_i \in H^*_{S\times G}(Z, \mathbb{Q})$ be classes such that $\kappa_G(\tilde{\gamma}_i) = \gamma_i$, and set

$$\mathbf{t} = \sum_i \tilde{\gamma}_i t_i$$

for t_i some formal variables. The term $\exp(\hat{ev}_\beta^*(\mathbf{t})/z)$ is interpereted as a polynomial in the t_i with coefficients in $H^*_{S\times\mathbb{C}^*}(F_\beta,\mathbb{Q})$ via the power series expansion of the exponential.

When Z is a vector space, we can explicitly compute (3.59) as follows. By Proposition 3.2.21, the classes γ_i are uniquely determined by their pullbacks $g^*\gamma_i \in H^*_S(Z^s(G)/T,\mathbb{Q})$, and these pullbacks may be expressed as W-invariant polynomials in the classes $c_1(\mathscr{L}_{\xi_j})$, where ξ_j are the characters of the T-action on Z. Write

$$g^*\gamma_i = q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}}))$$

for these polynomials, where $q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}}))$ is shorthand for $q_i(c_1(\mathscr{L}_{\xi_1}),\dots,c_1(\mathscr{L}_{\xi_r}))$.

Corollary 3.5.4. *If the GIT chamber of θ has dimension equal to the rank of the group of rational characters $\chi(G)\otimes\mathbb{Q}$, then the coefficient $g^*\mathbb{I}_\beta^{Z/\!\!/G}(z)$ of the big I-function of $Z/\!\!/_\theta G$ equals*

$$j^*\left[\sum_{\tilde{\beta}\to\beta}\exp\left(\sum_i t_i q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}})+\tilde{\beta}(\boldsymbol{\xi})z)/z\right)\left(\prod_\alpha\frac{\prod_{k=-\infty}^{\tilde{\beta}(\alpha)}(c_1(\mathscr{L}_\alpha)+kz)}{\prod_{k=-\infty}^{0}(c_1(\mathscr{L}_\alpha)+kz)}\right)I_{\tilde{\beta}}^{Z/\!\!/T}(z)\right], \tag{3.60}$$

where

$$q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}})+\tilde{\beta}(\boldsymbol{\xi})z) := q_i(c_1(\mathscr{L}_{\xi_1})+\tilde{\beta}(\xi_1)z,\dots,c_1(\mathscr{L}_{\xi_r})+\tilde{\beta}(\xi_r)z)$$

and the sum is over all $\tilde{\beta}$ mapping to β under the natural map $\mathrm{Hom}(\mathrm{Pic}^T(X),\mathbb{Z})\to\mathrm{Hom}(\mathrm{Pic}^G(X),\mathbb{Z})$ *and the product is over all roots α of G.*

This corollary extends the procedure in [CK16, Sec 5.3].

Proof. The first step is to carefully choose the lifts $\tilde{\gamma}_i$. We have a commuting diagram of topological spaces

$$\begin{array}{ccccc}
ET\times_T Z & \xleftarrow{a} & ET\times_Z Z^s(G) & \xrightarrow[\sim]{b} & Z^s(G)/T \\
\big\downarrow \psi & & \big\downarrow & & \big\downarrow g \\
EG\times_G Z & \xleftarrow{a} & EG\times_G Z^s(G) & \xrightarrow[\sim]{b} & Z^s(G)/G
\end{array}$$

which leads to a commuting diagram of cohomology maps

$$\begin{array}{ccc} H^*_{S\times T}(Z,\mathbb{Q})^W & \xrightarrow{\kappa_T} & H^*_S(Z^s(G)/T,\mathbb{Q})^W \\ \psi^*\big\uparrow\sim & & \sim\big\uparrow g^* \\ H^*_{S\times G}(Z,\mathbb{Q}) & \xrightarrow{\kappa_G} & H^*_S(Z/\!\!/G,\mathbb{Q}) \end{array} \tag{3.61}$$

The right vertical arrow is an isomorphism by Proposition 3.2.21, and the left vertical arrow is an isomorphism by [Bri98, Prop 1]. Let

$$\tilde{\delta}_i = q_i(c_1(L_{\boldsymbol{\xi}})) \in H^*_{S\times T}(Z,\mathbb{Q})^W,$$

so $\kappa_T(\tilde{\delta}_i) = g^*\gamma_i$. Then set $\tilde{\gamma}_i = \psi^*\tilde{\delta}_i$. Commutativity of (3.61) implies that $\kappa_G(\tilde{\gamma}_i) = \gamma_i$ as desired.

Now we apply the computation in Section 3.4.3 to $\mathbb{I}^{Z/\!\!/G}_\beta$. In place of (3.56) we arrive at the formula

$$g^*\mathbb{I}^{Z/\!\!/G}_\beta = \sum_{\tilde{\beta}\mapsto\beta} \frac{(ev_\bullet)_*\psi^*_{\tilde{\beta}}(\exp(\hat{ev}^*_{\tilde{\beta}}(\mathbf{t})/z)[F_{\tilde{\beta}}(Z/\!\!/G)]^{vir}e_{\mathbb{C}^*}(N^{vir}_{F_{\tilde{\beta}}(Z/\!\!/G)})^{-1}))}{\prod_{\rho\in R^+_{\ell_T(\tilde{\beta})}} c_1(\mathscr{L}_\rho)}$$

where $\hat{ev}^*_{\tilde{\beta}}$ denotes the composition

$$H^*_{G\times S}(Z,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{Q}[z] \xrightarrow{\hat{ev}^*_\beta} H^*_{S\times\mathbb{C}^*}(F_\beta(Z/\!\!/G),\mathbb{Q}) \to H^*_{S\times\mathbb{C}^*}(F_{\tilde{\beta}}(Z/\!\!/G),\mathbb{Q})$$

where the second map is restriction to an open and closed subspace of $F_\beta(Z/\!\!/G)$. To compute $\psi^*_{\tilde{\beta}}\hat{ev}^*_{\tilde{\beta}}$ we use the commuting diagram

$$\begin{array}{ccc} H^*_{S\times T}(Z,\mathbb{Q}) & \xrightarrow{\hat{ev}^*_{\tilde{\beta}}} & H^*_S(F_{\tilde{\beta}}(Z/\!\!/T)\cap Z^s(G),\mathbb{Q}) \\ \psi^*\big\uparrow & & \psi^*_{\tilde{\beta}}\big\uparrow \\ H^*_{S\times G}(Z,\mathbb{Q}) & \xrightarrow{\hat{ev}^*_{\tilde{\beta}}} & H^*_S(F_{\tilde{\beta}}(Z/\!\!/G),\mathbb{Q}) \end{array}$$

which follows from the commuting diagram of topological spaces (in the case when S

is trivial)

$$\begin{array}{ccccccc}
ET \times_T Z & \longleftarrow & \mathscr{T} \times_T Z & \xleftarrow[\sigma]{} & F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G) \times \mathbb{P}^1 & \xleftarrow[(id,0)]{} & F_{\tilde{\beta}}(Z/\!\!/T) \cap Z^s(G) \\
\downarrow \psi & & \downarrow & & \downarrow & & \downarrow \psi_{\tilde{\beta}} \\
EG \times_G Z & \longleftarrow & \mathscr{P} \times_G Z & \xleftarrow[\sigma]{} & F_{\tilde{\beta}}(Z/\!\!/G) \times \mathbb{P}^1 & \xleftarrow[(id,0)]{} & F_{\tilde{\beta}}(Z/\!\!/G)
\end{array}$$

We see that

$$\psi_{\tilde{\beta}}^* \hat{ev}_{\tilde{\beta}}^*(\tilde{\gamma}_i) = \hat{ev}_{\tilde{\beta}}^* \psi^*(\tilde{\gamma}_i) = \hat{ev}_{\tilde{\beta}}^*(\tilde{\delta}_i)$$

by the definition of $\tilde{\gamma}_i$. But by [CK16, Lem 5.2, Rmk 5.3], we have

$$\hat{ev}_{\tilde{\beta}}^*(\tilde{\delta}_i) = ev_\bullet^* q_i(c_1(\mathscr{L}_{\boldsymbol{\xi}}) + \tilde{\beta}(\boldsymbol{\xi})z)$$

(this is where we use the assumption that the GIT chamber of θ has full dimension). The rest of the computation proceeds via the projection formula as in Section 3.4.3 and [CK16, Sec 5.1]. □

3.5.4 Applications to Gromov-Witten theory

With the wall-crossing theorems of [CK14b], Theorem 3.1.1 can be used to compute certain small J-functions. In order to have a clear statement to use in our applications, we summarize [CK14b, Cor 7.3.2] here. If E is a vector space with a linear G-action, we say that the resulting vector bundle on $Z/\!\!/G$ is *convex* if $(\mathbb{P}^1, \mathscr{P}, \sigma, \mathbf{x})$ is a quasimap to $Z/\!\!/G$, then $H^1(\mathbb{P}^1, \mathscr{P} \times_G E) = 0$. (See [CKM14, Prop 6.2.3] for some sufficient conditions for E to be convex.) The following theorem applies to the twisted theory described in Remark 3.5.3.

Theorem 3.5.5 ([CK14b, Cor 7.3.2])**.** *Assume $Z/\!\!/_\theta G$ is Fano of index at least 2, and has an S-action with isolated fixed points. Let E be a convex representation satisfying*

$$\beta(\det(T_Z)) - \beta(\det(Z \times E)) \geq 0$$

for all θ-effective classes β, where T_Z is the (G-equivariant) tangent bundle of Z. Then $J^{X/\!\!/G,E} = I^{X/\!\!/G,E}$, both S-equivariantly and nonequivariantly.

We remark that the big I-func in (3.59) is also known to lie on the Lagrangian cone in good circumstances (see [CK16, Thm 3.3]).

One application of [CK14b, Cor 7.3.2] is as follows. In [CKS08, Conj 3.7.1] the authors conjecture a relationship between the Frobenius manifolds defined by the Gromov-Witten theories of $Z/\!\!/T$ and $Z/\!\!/G$. Theorem 3.1.1, together with several substantial results in the literature, implies their conjecture in certain cases. Let S be a torus acting as in Section 3.5.1.

Corollary 3.5.6. *If $Z/\!\!/G$ is Fano of index at least 2, and if the S-action has isolated fixed points, then the conjecture [CKS08, Conj 3.7.1] holds.*

Proof. We may apply the reconstruction theorem in [CKS08, Thm 4.3.6] because the localized equivariant cohomology ring

$$H_S^*(Z/\!\!/G, \mathbb{Q}) \otimes \mathrm{Frac}(H_S^*(pt, \mathbb{Q}))$$

is generated by divisors (this follows from the torus localization theorem). By the reconstruction theorem, it suffices to relate the S-equivariant small J-functions of $Z/\!\!/G$ and $Z/\!\!/T$ (by the same factor as in Theorem 3.1.1 for I-functions). By the mirror theorem of [CK14b, Cor 7.3.2] (restated in Theorem 3.5.5 above), this equality follows from the equality in Theorem 3.1.1 of small I-functions. □

3.5.5 Example: A Grassmann bundle on a Grassmannian variety

In this section we will apply Theorem 3.1.1 to a family of Calabi-Yau hypersurfaces in Fano GIT quotients, proving the following.

Theorem 3.5.7. *Let $Y := Gr_{Gr(k,n)}(\ell, U^{\oplus m})$ be the Grassmann bundle of ℓ-planes in m copies of the tautological bundle U on $Gr(k,n)$, and assume $n > \ell m$ and $\gcd(n-\ell m, km) > 1$. Let $\omega^\vee$ be the anticanonical bundle of Y with trivial linearization. Let $S = (\mathbb{C}^*)^{n+m}$ act on Y as defined in Section 3.5.5. Then the $\omega^\vee$-twisted equivariant* *small J-function of Y equals* $1 + \sum_{d,e>0} q_1^d q_2^e J_{(d,e)}(z)$*, where* $J_{(d,e)}(z)$ *equals*

$$\sum_{\substack{d_1+\ldots+d_k=d \\ e_1+\ldots+e_\ell=e}} \prod_{\substack{i,j=1 \\ i\neq j}}^{k} \left(\frac{\prod_{h=-\infty}^{d_i-d_j}(x_i - x_j + hz)}{\prod_{h=-\infty}^{0}(x_i - x_j + hz)} \right) \prod_{\substack{i,j=1 \\ i\neq j}}^{\ell} \left(\frac{\prod_{h=-\infty}^{e_i-e_j}(y_i - y_j + hz)}{\prod_{h=-\infty}^{0}(y_i - y_j + hz)} \right)$$
$$\cdot \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left(\frac{\prod_{h=-\infty}^{0}(x_i + \lambda_\alpha^1 + hz)}{\prod_{h=-\infty}^{d_i}(x_i + \lambda_\alpha^1 + hz)} \right) \prod_{i=1}^{k} \prod_{j=1}^{\ell} \prod_{\beta=1}^{m} \left(\frac{\prod_{h=-\infty}^{0}(y_j - x_i + \lambda_\beta^2 + hz)}{\prod_{h=-\infty}^{e_j-d_i}(y_j - x_i + \lambda_\beta^2 + hz)} \right)$$
$$\cdot \left(\frac{\prod_{h=-\infty}^{(n-\ell m)d+kme}((n-\ell m)\sum_{i=1}^{k} x_i + km\sum_{j=1}^{\ell} y_j + hz)}{\prod_{h=-\infty}^{0}((n-\ell m)\sum_{i=1}^{k} x_i + km\sum_{j=1}^{\ell} y_j + hz)} \right), \tag{3.62}$$

where the x_i *are the Chern roots of* U*, the* y_j *are the Chern roots of the tautological bundle for the Grassmann bundle* Y*, and the* λ*'s are the equivariant parameters.*

In the formula (3.62), the first line is the factor coming from the roots of G, the second line is the J-function of the abelian quotient, and the last line is the $\omega^\vee$-twisting factor.

Defining the target

To define the GIT target, choose integers k, n, ℓ, and m with $k < n$ and $\ell < km$. Let $M_{k\times n}$ denote the space of $k \times n$ matrices with complex entries, and set

- the vector space $X = M_{k\times n} \times M_{\ell\times km}$
- the group $G = GL_k \times GL_\ell$
- the action $(g, h) \cdot (V, W) = (gV, hW\mathrm{diag}(g^{-1}))$ for $(g, h) \in G$ and $(U, W) \in X$ where $\mathrm{diag}(g^{-1})$ is the block diagonal $km \times km$ matrix with g^{-1} repeated m times
- the character $\theta(g, h) = \det(g)\det(h)$

We can check stability of points using the numerical criterion [Kin94, Prop 2.5]. It is straightforward to compute that

$$X_\theta^{ss}(G) = X_\theta^{s}(G) = (M_{k\times n} \setminus \Delta) \times (M_{\ell\times km} \setminus \Delta),$$

where Δ denotes matrices of less than full rank. Thus,

$$X /\!\!/_\theta G = Gr_{Gr(k,n)}(\ell, U^{\oplus m}) =: Y$$

is the Grassmann bundle of ℓ-planes in m copies of the tautological bundle U on $Gr(k,n)$.

On $X/\!\!/_{\theta}G$ we have the euler sequence

$$0 \to X^s(G) \times_G \mathfrak{g} \to X^s(G) \times_G X \to T_{X/\!\!/G} \to 0,$$

which is just the dual of the short exact sequence of cotangent complexes for the smooth morphisms $X^s(G) \to X/\!\!/G \to pt$. From this sequence we compute that for any ϑ the anticanonical bundle of $V/\!\!/_{\vartheta}G$ is the line bundle corresponding to the character

$$(g,h) \mapsto \det(g)^{n-\ell m} \det(h)^{km}.$$

One can also check that if $\vartheta(g,h) = \det(g)^a \det(g)^b$ with $a, b > 0$ then $X^{ss}_{\vartheta}(G) = X^s_{\vartheta}(G) = X^s_{\theta}(G)$ so also $X/\!\!/_{\vartheta}G = X/\!\!/_{\theta}G$. Since $X^s(G) \times_G \mathbb{C}_{\vartheta}$ is always ample on $X/\!\!/_{\vartheta}G$, this implies that when $n > \ell m$, the target Y is Fano. If $gcd(n-\ell m, km) > 1$ then we also know it has index at least 2. A generic section of the anticanonical bundle defines a Calabi-Yau subariety of Y.

Quasimaps and I-function

Let $\omega^{\vee}$ denote the anticanonical bundle of Y. To check that it is convex, so that we can write down the I-function a corresponding hypersurface, we need to briefly investigate quasimaps to Y. We will also describe the fixed locus $F_{\tilde{\alpha}}(X/\!\!/G)$ in this case, for $\tilde{\alpha} \in \mathrm{Hom}(\chi(T), \mathbb{Z})$. A stable quasimap to Y is equivalent to the following data:

- a rank-k vector bundle $\oplus_{i=1}^k \mathscr{O}_{\mathbb{P}^1}(d_i)$ and a rank-ℓ vector bundle $\oplus_{j=1}^{\ell} \mathscr{O}_{\mathbb{P}^1}(e_j)$
- a section σ of $\left[\oplus_{i=1}^k \mathscr{O}_{\mathbb{P}^1}(d_i)^{\oplus n}\right] \oplus \left[\oplus_{j=1}^{\ell} \oplus_{i=1}^k \mathscr{O}_{\mathbb{P}^1}(e_j - d_i)^{\oplus m}\right]$, written as a $k \times n$ and $\ell \times mk$ matrix of polynomials, such that all but finitely many points $\mathbf{x} \in \mathbb{P}^1$ satisfy $\sigma(\mathbf{x}) \in V^s$.

In fact, this data defines a quasimap to $X/\!\!/_{\theta}T$ of degree $\tilde{\alpha} = (d_1, \ldots, d_k, e_1, \ldots, e_{\ell})$; the degree as a quasimap to $V/\!\!/G$ is $\alpha = (\sum d_i, \sum e_j)$. In order to have finitely many basepoints, a stable quasimap must have $d_i \geq 0$, hence also $e_j \geq 0$, so that if $n > \ell m$ then $\omega^{\vee}$ is indeed convex.

We can now read off the twisted I-function using Theorem 3.1.1, obtaining formula (3.62) with the equivariant parameters λ set to zero. In the next section we will derive the full equivariant formula.

Finally we identify $F_{\tilde{\alpha}}(X/\!\!/G)$. For simplicity assume that the sequences $d_1, \ldots, d_k$ and $e_1, \ldots, e_\ell$ are ordered from smallest to largest. The subspace $X_{\tilde{\alpha}} \subset X$ is $M_{k\times n} \times X'_{\tilde{\alpha}}$, where $X'_{\tilde{\alpha}}$ is the subspace of $M_{\ell\times km}$ consisting of matrices (m_{ij}) where

$$m_{ij} = 0 \qquad \text{if} \qquad e_i - d_{(j \bmod m)+1} < 0.$$

Such a matrix looks something like the picture in Figure 3.3.

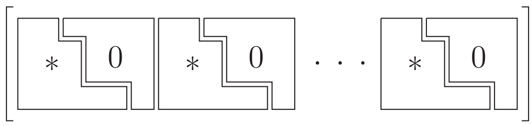


Figure 3.3: An element of $X'_{\tilde{\alpha}}$
This diagram represents a matrix. The entries labeled "0" are required to be zero and the entries labeled "*" are not. The same $\ell \times k$ pattern of *'s and 0's is repeated m times.

The group $P_{\tilde{\alpha}} \subset G$ is $P_1 \times P_2$ where P_1 is the parabolic subgroup of GL_k equal to block lower triangular matrices with blocks determined by the multiplicities of the d_i, and $P_2 \subset GL_\ell$ is similarly defined by the e_j. Hence from Lemma 3.3.6 and Proposition 3.3.1 we have a series of maps

$$F_{\tilde{\alpha}}(X/\!\!/G) = X^s_{\tilde{\alpha}}(G)/P_{\tilde{\alpha}} \hookrightarrow X^s(G)/P_{\tilde{\alpha}} \to X^s(G)/G = X/\!\!/G$$

whose composition is $ev_\bullet$. The first arrow is a closed embedding and the second is a flag bundle.

A good torus action

The target Y has a torus action with isolated fixed points. Let $S = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^m$ act on X as follows: if s_1 is an $n \times n$ diagonal matrix and s_2 is a $km \times km$ diagonal matrix with m constant $k \times k$ diagonal blocks, and both s_1 and s_2 are filled with numbers from $\mathbb{C}^*$ then

$$(s_1, s_2) \cdot (V, W) = (W s_1, V s_2).$$

This action commutes with the action of G. We can extend it to a linearization of $\omega^\vee$ as follows. The total space of $\omega^\vee$ is $X \times_G \mathbb{C}_\kappa$, where $\kappa(g,h) = \det(g)^{n-\ell m}\det(h)^{km}$. Define $(s_1, s_2)\cdot(V, W, z) = (Vs_1, Ws_2, z)$ for $(V, W) \in X$ and $z \in \mathbb{C}$.

The S-action on Y has isolated fixed points as follows. For $I \subset \{1, \ldots, n\}$ let D_I denote the $k \times n$ matrix which has the identity matrix in the I-columns and zeros elsewhere. Similarly, for $J \subset \{1, \ldots, km\}$, let D_J denote the $\ell \times km$ matrix which has the identity matrix in the I-columns and zeros elsewhere. Then the fixed points of the S-action are (D_I, D_J) for all possible combinations of I and J. Now we can apply the mirror theorem in [CK14b, Cor 7.3.2] (see also Theorem 3.5.5), noting that the quantity $\beta(\det(T_X)) - \beta(X \times E)$ in the hypotheses is 0 since $E = \det(T_X)$, and conclude that (3.62) holds.

3.6 Two lemmas

Here we prove two geometric lemmas that are probably well-known, but we could not find a reference. Compare the following statement and its proof with [Con, Cor 1.2].

Lemma 3.6.1. *Let X and Y be finite type schemes over $\mathbb{C}$ and let G be an algebraic group, also of finite type over $\mathbb{C}$. If $\pi : X \to Y$ is a principal G-bundle and $V \subset X$ is a closed invariant subscheme, then there is a fibered square*

$$\begin{array}{ccc} V & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \pi(V) & \longrightarrow & Y \end{array}$$

where $\pi(V)$ is the scheme-theoretic image of V.

Proof. We claim that $V = \pi^{-1}(Z)$ for some closed subscheme $Z \subset Y$. Granting this claim, we must have $\pi(V) \subset Z$ from the universal property of images, and hence

$$V \subset \pi^{-1}(\pi(V)) \subset \pi^{-1}(Z) = V.$$

This implies $V = \pi^{-1}(\pi(V))$.

Now we prove the claim. By fppf descent for closed subschemes, it suffices to show that $p_1^{-1}(V) = p_2^{-1}(V)$ for the projection morphisms $p_i : X \times_Y X \to X$. Clearly $p_1^{-1}(V) = V \times_Y X$ and $p_2^{-1}(V) = X \times_Y V$. On the other hand we have a morphism

$\phi : G \times X \to X \times_Y X$ given by $\phi = (\alpha, p_2)$, where $\alpha : G \times X \to X$ is action and p_2 is projection to the second factor. We have

$$X \times_Y V \subset \phi(G \times V) \subset V \times_Y V \subset V \times_Y X$$

where the second inclusion uses the G-invariance of V. An analogous argument shows that the opposite containment $V \times_Y X \subset X \times_Y V$ also holds. □

The following result is probably known, but due to a lack of reference we provide a proof. Compare for example with [Stacks, Tag 0DUD].

Lemma 3.6.2. *Let X and Y be smooth algebraic stacks over $\mathbb{C}$, locally of finite type with diagonals represented by separated algebraic spaces of finite type, with Y quasicompact. If $\pi : X \to Y$ is a proper morphism inducing bijections of $\mathbb{C}$-points and of automorphism groups of $\mathbb{C}$-points, then π is an isomorphism.*

The heuristic for this lemma is that every locally closed subset of a stack contains a point of finite type [Stacks, Tag 06G2], and for stacks locally of finite type over $\mathbb{C}$, the points of finite type are precisely the $\mathbb{C}$-points. Hence to check that an open or closed property is true for an algebraic stack, it suffices to check it for $\mathbb{C}$-points of the stack.

Proof. We first show that π is universally injective by checking that the diagonal $\Delta_\pi : X \to X \times_Y X$ is surjective. Because π is separated, Δ_π is closed, so the complement of the image $|\Delta_\pi|^C$ is an open subset of $|X \times_Y X|$ and contains a $\mathbb{C}$-point if it is nonempty. So it suffices to show that Δ_π is surjective on $\mathbb{C}$-points.

To this end, let (x_1, x_2, β) be an element of $(X \times_Y X)(\mathbb{C})$, so we have $x_i \in X(\mathbb{C})$ and $\beta : \pi(x_1) \to \pi(x_2)$ a morphism lying over the identity. We find an arrow from (x_1, x_1, id) to (x_1, x_2, id). It suffices to find an arrow α from x_1 to x_2 in X such that $\pi(\alpha) = \beta$, yielding a commuting square

$$\begin{array}{ccc} \pi(x_1) & \xrightarrow{id} & \pi(x_1) \\ \downarrow & & \downarrow{\scriptstyle \pi(\alpha)} \\ \pi(x_1) & \xrightarrow{\beta} & \pi(x_2) \end{array}$$

Because π is injective on $\mathbb{C}$-points, and the $\mathbb{C}$-points $\pi(x_1)$ and $\pi(x_2)$ are isomorphic, the points x_1 and x_2 are also isomorphic via some arrow γ in X. Then $\pi(\gamma)^{-1} \circ \beta$ is in $Aut(\pi(x_1))$. Because we've assumed π induces bijections of isotropy groups of

$\mathbb{C}$-points, there is some $\delta \in Aut(x_1)$ with $\pi(\delta) = \pi(\gamma)^{-1} \circ \beta$. Then $\beta = \pi(\gamma \circ \delta)$, as desired.

Next we show that π is representable by algebraic spaces. It suffices to show that the inertia stack $I_{X/Y} \to X$ is trivial ([Stacks, Tag 04YY]). By assumption and the exact sequence of automorphism groups in [Stacks, Tag 0CPK], we see that the relative automorphism group of every $\mathbb{C}$-point of X is trivial. We can check triviality of $I_{X/Y} \to X$ on a smooth cover by a scheme $U \to X$. Let $G = U \times_X I_{X/Y}$ denote the fiber product; then $G \to U$ is a finite type group algebraic space over a locally finite type scheme with trivial fibers at $\mathbb{C}$-points. From here we may argue as in [Con07, Thm 2.2.5], replacing geometric points with $\mathbb{C}$-points. For example, [Stacks, Tag 04NW] implies that $G \to U$ is locally quasi-finite.

In fact, our assumptions imply that π is separated and locally quasi-finite, so that by [Stacks, Tag 03XX] π is representable by schemes. Hence we have reduced to the case where X and Y are schemes.

To prove the lemma in this case, observe that π is injective: the rank of $\pi_*\mathscr{O}_X$ is 1 at every $\mathbb{C}$-point, hence its rank is 1 everywhere (dimension can only go up on closed sets). Likewise π is surjective. Now restrict π to a connected component U of Y; its preimage must be a component V of X. Then $\pi : V \to U$ is a quasifinite, proper (hence finite) morphism of integral schemes; since it is injective, it induces an isomorphism of function fields. Hence π is a birational morphism of smooth integral finite-type schemes, and therefore an isomorphism. □

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