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Eigenvalues of the Thurston operator

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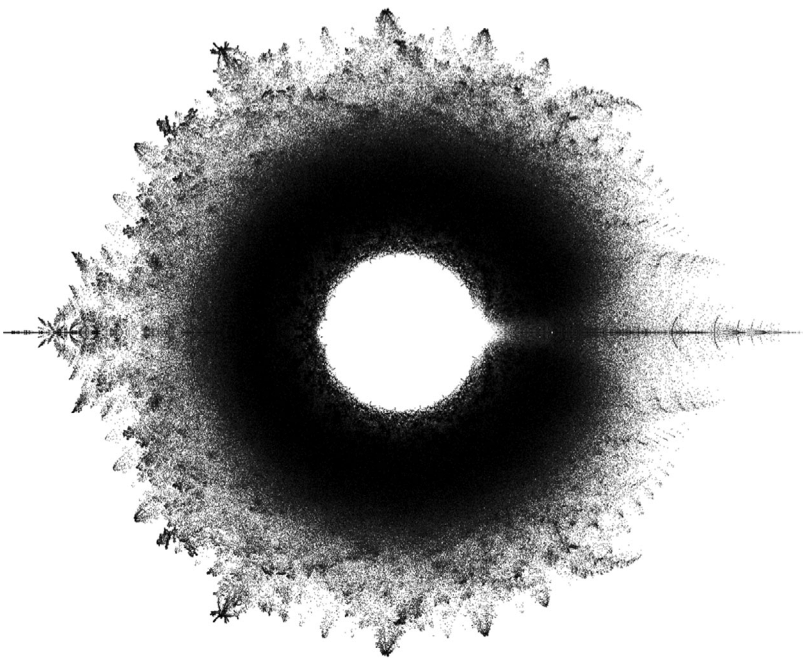
Dedicated to Bill

Abstract

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite rational map, and let $\mathcal{Q}(\widehat{\mathbb{C}})$ be the space of meromorphic quadratic differentials on $\widehat{\mathbb{C}}$ with simple poles. We study the set of eigenvalues of the pushforward operator $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$. In particular, we show that when $f : \mathbb{C} \to \mathbb{C}$ is a unicritical polynomial of degree D with periodic critical point, the eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ are contained in the annulus $\left\{\frac{1}{4D} < |\lambda| < 1\right\}$ and belong to $\frac{1}{D}\mathbb{U}$ where $\mathbb{U}$ is the group of algebraic units.



1. *Introduction*

Throughout this article, $D \geqslant 2$ and Rat_D is the space of rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere of degree D. We denote the set of critical points of f by $\mathcal{C}_f$ and the set of

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critical values by $\mathcal{V}_f$. The postcritical set $\mathcal{P}_f$ is the smallest forward invariant subset of $\widehat{\mathbb{C}}$ which contains $\mathcal{V}_f$:

$$\mathcal{P}_f := \bigcup_{n\geqslant 1} f^{\circ n}(\mathcal{C}_f).$$

We study *postcritically finite rational maps*; that is, rational maps $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ for which $\mathcal{P}_f$ is finite. It follows from work of Thurston that with the exception of flexible Lattès maps (see § 2.3 for the definition), postcritically finite rational maps are rigid: if two postcritically finite rational maps are topologically conjugate, then either they are flexible Lattès maps, or they are conjugate by a Möbius transformation [4].

In fact, Thurston worked with the Teichmüller space $\mathcal{T}_f$ of the Riemann sphere marked with the postcritical set $\mathcal{P}_f$ and he considered the holomorphic self-map $\sigma_f:\mathcal{T}_f\to\mathcal{T}_f$ induced by pulling back complex structures on $\widehat{\mathbb{C}}\smallsetminus\mathcal{P}_f$. The fact that the map $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is holomorphic is equivalent to the fact that the point $\tau_f\in\mathcal{T}_f$ represented by the standard complex structure on $\widehat{\mathbb{C}}\smallsetminus\mathcal{P}_f$ is fixed by σ_f. Thurston proved that the pullback map σ_f is contracting for the Teichmüller metric on $\mathcal{T}_f$, so that it has a unique fixed point; rigidity follows.

A more elementary result that does not appeal to Teichmüller spaces concerns infinitesimal rigidity: if $t\mapsto f_t$ is an analytic family of postcritically finite rational maps, then either the maps are flexible Lattès maps, or there is an analytic family of Möbius transformations $t\mapsto M_t$ such that $M_0=\mathrm{id}$ and $f_t\circ M_t = M_t\circ f_0$. The proof of this result relies on the following lemma, in which $\mathcal{Q}(\widehat{\mathbb{C}})$ is the space of meromorphic quadratic differentials on $\widehat{\mathbb{C}}$ with simple poles and $f_*:\mathcal{Q}(\widehat{\mathbb{C}})\to\mathcal{Q}(\widehat{\mathbb{C}})$ is the Thurston pushforward operator (see § 3 for the definition).

Lemma 1.1 (Thurston). *Assume $f\in\mathrm{Rat}_D$ is postcritically finite. If λ is an eigenvalue of $f_*:\mathcal{Q}(\widehat{\mathbb{C}})\to\mathcal{Q}(\widehat{\mathbb{C}})$, then $|\lambda|\leqslant 1$. In addition, $\lambda=1$ is an eigenvalue if and only if f is a flexible Lattès map.*

We derive infinitesimal rigidity in § 5. We then study the eigenvalues of $f_*:\mathcal{Q}(\widehat{\mathbb{C}})\to\mathcal{Q}(\widehat{\mathbb{C}})$. The subspace $\mathcal{Q}_f\subset\mathcal{Q}(\widehat{\mathbb{C}})$ of quadratic differentials with poles contained in $\mathcal{P}_f$ has finite dimension $\mathrm{card}(\mathcal{P}_f)-3$ and is invariant by f_*. Let Σ_f be the set of eigenvalues of $f_*:\mathcal{Q}_f\to\mathcal{Q}_f$, and let Λ_f be the set of eigenvalues of the induced operator $f_*:\mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f\to\mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$.

Remark. The space $\mathcal{Q}_f$ canonically identifies with the cotangent space to the Teichmüller space $\mathcal{T}_f$ at the base point τ_f and the coderivative of σ_f at τ_f identifies with $f_*:\mathcal{Q}_f\to\mathcal{Q}_f$. Thus, Σ_f coincides with the spectrum of the derivative of the Thurston pullback map $\sigma_f:\mathcal{T}_f\to\mathcal{T}_f$ at its unique fixed point.

In § 6, we study Λ_f and in § 7, we study Σ_f, establishing the following results.

Theorem 1.2. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a postcritically finite rational map. The set Λ_f consists of 0 and the complex numbers $\lambda\in\mathbb{C}\smallsetminus\{0\}$ such that $1/\lambda^m$ is the multiplier of a cycle of f of period m which is not contained in $\mathcal{P}_f$. If $\lambda\in\Lambda_f\smallsetminus\{0\}$, then λ is an algebraic number but not an algebraic integer.*

Theorem 1.3. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a postcritically finite rational map. If $\lambda\in\Sigma_f$, then λ is an algebraic number. If λ is an algebraic integer, then either $\lambda=0$, or f is a Lattès map and*

$$\lambda\in\left\{\pm 1,\ \pm \mathrm{i},\ \frac{1}{2}\pm\mathrm{i}\frac{\sqrt{3}}{2},\ -\frac{1}{2}\pm\mathrm{i}\frac{\sqrt{3}}{2}\right\}.$$

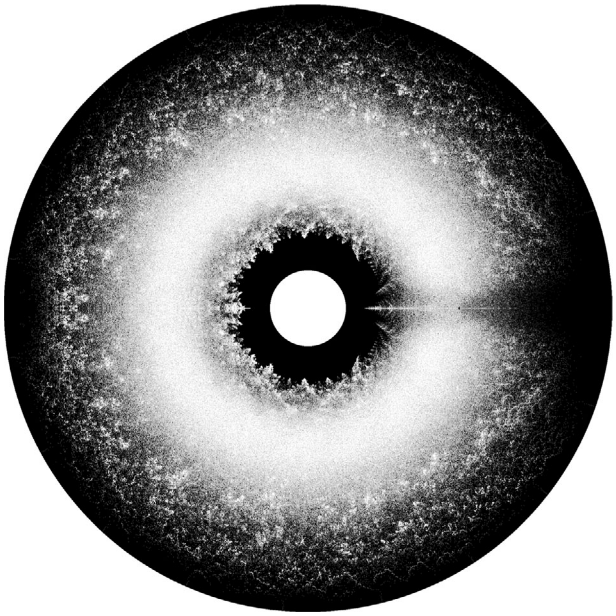


FIGURE 1. *The set of eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ for unicritical polynomials f of degree 2 with periodic critical point, up to period 19. The set of eigenvalues (white) is contained in the annulus $1/8 < |\lambda| < 1$ (black). The picture on the first page of this article shows the reciprocal of the eigenvalues (black).*

In § 8, we describe a way to compute the characteristic polynomial of the operator $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$. We then apply this in § 9 to the case where f is a unicritical polynomial with periodic critical point. We establish estimates on the size of the coefficients of the characteristic polynomial, which enable us to derive the following result.

THEOREM 1.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a unicritical polynomial of degree D with periodic critical point. Then*

- *if $\lambda \in \Lambda_f$, we have that $\frac{1}{2D} < |\lambda| < 1$ and*
- *if $\lambda \in \Sigma_f$, we have that $\frac{1}{4D} < |\lambda| < 1$.*

In both cases, $D\lambda$ is an algebraic unit.

The case of eigenvalues in Λ_f is covered by Proposition 9.6 and the case of eigenvalues in Σ_f is covered by Theorem 9.9 and Proposition 9.12.

Given $D \geqslant 2$, set

$$\Sigma(D) := \bigcup_f \Sigma_f,$$

where the union is taken over all unicritical polynomials $f : \mathbb{C} \to \mathbb{C}$ of degree D with periodic critical point.

THEOREM 1.5. *The closure of $\Sigma(D)$ contains the annulus $r_D \leqslant |\lambda| \leqslant 1$ where r_D is defined by*

$$\frac{1}{r_D} = \begin{cases} 2D & \text{if } D \text{ is even} \\ 2D\cos\left(\frac{\pi}{2D}\right) & \text{if } D \text{ is odd.} \end{cases}$$

The proof of this theorem relies on estimating the modulus of the multipliers of unicritical postcritically finite polynomials of degree D at their fixed points. We have an optimal estimate in the case when D is even, and we do not in the case when D is odd. The proof also relies on the following equidistribution result which might be of independent interest.

Lemma 1.6. *Let*

$$P_n = 1 + \cdots + c_n z^{d_n} \in \mathbb{C}[z] \quad \textit{and} \quad Q_n = \frac{P_n}{c_n z^{d_n}} = 1 + \cdots + \frac{1}{c_n z^{d_n}} \in \mathbb{C}[1/z].$$

If

- *the sequence* (d_n) *tends to* ∞ *as* $n \to \infty$,
- *the sequence* (P_n) *is normal in the unit disk* $\mathbb{D}$, *and*
- *the sequence* (Q_n) *is normal in* $\mathbb{C} \setminus \overline{\mathbb{D}}$,

then as $n \to \infty$, *the roots of* P_n *equidistribute on the unit circle* S^1.

The proof of Theorem 1.4 is given in § 9.5 and § 9.7, and the proof of Theorem 1.5 is given in § 9.8. Finally, in § 10, we pose some questions for further study.

2. *Postcritically finite rational maps*

Fix $f \in \mathrm{Rat}_D$. It follows from the Riemann–Hurwitz formula that f has $2D - 2$ critical points counted with multiplicity, and that f has at least two distinct critical values. As a consequence, $\mathrm{card}(\mathcal{P}_f) \geqslant 2$. If $\mathrm{card}(\mathcal{P}_f) = 2$, then $\mathcal{V}_f = \mathcal{P}_f$, and f is conjugate to $z \mapsto z^{\pm D}$.

2.1. *Examples*

In the following two examples, $\mathrm{card}(\mathcal{P}_f) = 3$.

- The polynomial $f : z \mapsto 1 - z^D$ has critical set $\mathcal{C}_f = \{0, \infty\}$, postcritical set $\mathcal{P}_f = \{0, 1, \infty\}$, and postcritical dynamics:

$$\begin{tikzcd} \infty \arrow[loop right, "D"] & 0 \arrow[r, bend left, "D"] & 1 . \arrow[l, bend left] \end{tikzcd}$$

- The rational map $f : z \mapsto 1 - 1/z^D$ has critical set $\mathcal{C}_f = \{0, \infty\}$, postcritical set $\mathcal{P}_f = \{0, 1, \infty\}$, and postcritical dynamics:

$$\begin{tikzcd} \infty \arrow[r, bend left, "D"] & 1 \arrow[r, bend left] & 0 . \arrow[ll, bend left, "D"] \end{tikzcd}$$

Unicritical polynomials. For much of this article, we will focus on polynomials $\mathbb{C} \to \mathbb{C}$ of degree D which have a unique critical point; these polynomials are called *unicritical*. Every unicritical polynomial is affine conjugate to a polynomial of the form $f_c(z) = z^D + c$, where $z_0 = 0$ is the unique critical point with critical value $f_c(0) = c$. Fix an integer $m \geqslant 1$. In parameter space, the roots of the polynomial $G_m(c) := f_c^{\circ m}(0)$ correspond to polynomials f_c for which 0 is periodic of period dividing m. These maps f_c are necessarily postcritically finite with postcritical set equal to

$$\{\infty\} \ \cup \bigcup_{1 \leqslant k \leqslant m} f_c^{\circ k}(0).$$

An argument due to Gleason shows that G_m has simple roots (see Lemma 9.4), so there are lots of postcritically finite polynomials. In fact, for $D = 2$, the boundary of the Mandelbrot set is contained in the closure of the set $\bigcup_{m\geqslant 1}\{\text{roots of } G_m\}$.

2.2. *Cycles are superattracting or repelling*

Recall that the multiplier of a periodic m-cycle $\{x, f(x), \ldots, f^{\circ(m-1)}(x)\}$ is the eigenvalue λ of the linear map $\mathrm{D}_x f^{\circ m} : \mathrm{T}_x\widehat{\mathbb{C}} \to \mathrm{T}_x\widehat{\mathbb{C}}$ (this eigenvalue does not depend on the point in the cycle). The periodic cycles of a postcritically finite rational map are either *superattracting*; that is, $\lambda = 0$, or *repelling*; that is, $|\lambda| > 1$.

2.3. *Lattès maps*

The rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *Lattès map* if there is:

- a complex torus $\mathcal{T} := \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice of rank 2,
- an affine map $L : \mathcal{T} \to \mathcal{T}$, and
- a finite branched cover $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$

so that the following diagram commutes.

$$\begin{tikzcd} \mathcal{T} \arrow[r, "L"] \arrow[d, "\Theta"'] & \mathcal{T} \arrow[d, "\Theta"] \\ \widehat{\mathbb{C}} \arrow[r, "f"'] & \widehat{\mathbb{C}} \end{tikzcd}$$

The rational map f is necessarily postcritically finite, and $\mathcal{P}_f$ is the set of critical values of $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$. In addition, $\mathrm{card}(\mathcal{P}_f) \in \{3, 4\}$. We shall use the following characterization of Lattès maps with four postcritical points (see [**5**, §4]).

Proposition 2.1. *A postcritically finite rational map with* $\mathrm{card}(\mathcal{P}_f) = 4$ *is a Lattès map if and only if every critical point if simple* (*with local degree* 2) *and no critical point is postcritical.*

Lattès maps are either *flexible* or *rigid*. The map f is flexible if

- for L of the form $L : w \mapsto \alpha w + \beta$, we have $\alpha \in \mathbb{Z}$, and
- the map Θ has degree 2.

Equivalently, f is flexible if it can be *deformed*, that is, if it is part of a one-parameter isospectral family that is nontrivial [**6**]. The Lattès map f is rigid if it is not flexible. Flexible Lattès maps have four postcritical points (this follows from the fact that Θ has degree 2). Rigid Lattès maps may have three or four postcritical points.

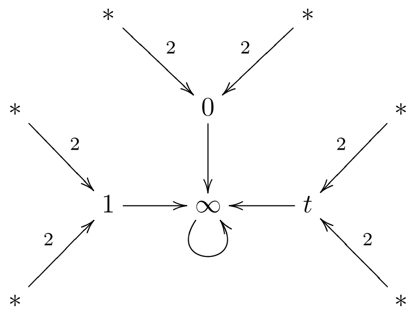
Example. The map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $f : z \mapsto (1 - 2/z)^2$ is a rigid Lattès map with $\mathcal{P}_f = \{0, 1, \infty\}$ and has the following postcritical dynamics.

$$2 \xrightarrow{2} 0 \xrightarrow{2} \infty \longrightarrow 1 \circlearrowleft .$$

Example. The family

$$\{f_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}\}_{t\in\mathbb{C}\smallsetminus\{0,1\}} \quad \text{given by} \quad f_t : z \mapsto \frac{(z^2 - t)^2}{4z(z-1)(z-t)}$$

consists entirely of flexible Lattès maps. The postcritical set of f_t is $\{0, 1, t, \infty\}$, and f_t has the following postcritical dynamics.



3. *Quadratic differentials*

Let U be a Riemann surface. A *quadratic differential* on U is a section of the square of the cotangent bundle $\mathrm{T}^*U \otimes \mathrm{T}^*U$. We shall usually think of a quadratic differential $\boldsymbol{q}$ as a field of quadratic forms. In particular, if $\boldsymbol{\theta}$ is a vector field on U and ϕ is a function on U, then $\boldsymbol{q}(\boldsymbol{\theta})$ is a function on U and $\boldsymbol{q}(\phi\boldsymbol{\theta}) = \phi^2\boldsymbol{q}(\boldsymbol{\theta})$.

If $\zeta : U \to \mathbb{C}$ is a coordinate, we shall use the notation $(\mathrm{d}\zeta)^2 = \mathrm{d}\zeta \otimes \mathrm{d}\zeta$ (not to be confused with the 1-form $\mathrm{d}(\zeta^2)$). On U (whose complex dimension is 1), the ratio of two quadratic differentials is a function. In other words, any quadratic differential $\boldsymbol{q}$ on U may be written as

$$\boldsymbol{q} = q\ (\mathrm{d}\zeta)^2 \quad \text{for some function } q.$$

3.1. *Meromorphic quadratic differentials*

A quadratic differential $\boldsymbol{q}$ on $\widehat{\mathbb{C}}$ is meromorphic if $\boldsymbol{q} = q\ (\mathrm{d}z)^2$ for some meromorphic function $q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The quadratic differential $(\mathrm{d}z)^2$ has no zero and has a pole of order 4 at ∞. Since the number of zeros of the function q equals the number of poles of q, counting multiplicities, the number of poles minus the number of zeros of $\boldsymbol{q}$ is equal to four. In particular, $\boldsymbol{q}$ has at least four poles (counting multiplicities).

Let $\mathcal{Q}(\widehat{\mathbb{C}})$ be the set of meromorphic quadratic differentials with only simple poles. For $X \subset \widehat{\mathbb{C}}$, let $\mathcal{Q}(\widehat{\mathbb{C}}; X) \subset \mathcal{Q}(\widehat{\mathbb{C}})$ be the subset of quadratic differentials whose poles are contained in X. For $k \geqslant 0$, let $\mathcal{Q}_k(\mathbb{C})$ be the set of meromorphic quadratic differentials whose poles in $\mathbb{C}$ are all simple and which have at worst a pole of order k at ∞.

Example. The quadratic differential $(\mathrm{d}z)^2$ belongs to $\mathcal{Q}_4(\mathbb{C})$, and for any $x \in \mathbb{C}$, the quadratic differential $\frac{(\mathrm{d}z)^2}{z-x}$ belongs to $\mathcal{Q}_3(\mathbb{C}) \subset \mathcal{Q}_4(\mathbb{C})$.

3.2. *Pullback*

The derivative $\mathrm{D}f : \mathrm{T}U \to \mathrm{T}V$ of a holomorphic map $f : U \to V$ naturally induces a pullback map f^* from quadratic differentials on V to quadratic differentials on U:

$$f^*\boldsymbol{q} := \boldsymbol{q} \circ \mathrm{D}f.$$

If $f : (U, x) \to (V, y)$ is holomorphic at x, and q is meromorphic at $y = f(x)$, then

$$2 + \mathrm{ord}_x(f^*\boldsymbol{q}) = \deg_x f \cdot (2 + \mathrm{ord}_y\boldsymbol{q}).$$

3.3. *The Thurston pushforward operator*

If $f : U \to V$ is a covering map and $\boldsymbol{q}$ is a quadratic differential on U, then we can define a quadratic differential $f_*\boldsymbol{q}$ on V by

$$f_*\boldsymbol{q} := \sum_g g^*\boldsymbol{q}$$

where the sum is taken over all inverse branches g of f. If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a nonconstant rational map and $\boldsymbol{q} = q\,(\mathrm{d}z)^2$ is a meromorphic quadratic differential on $\widehat{\mathbb{C}}$, then the quadratic differential $f_*\boldsymbol{q}$, which is a priori defined on $\widehat{\mathbb{C}}\smallsetminus\mathcal{V}_f$, is globally meromorphic on $\widehat{\mathbb{C}}$, and

$$f_*\boldsymbol{q} := r\,(\mathrm{d}z)^2 \quad \text{with} \quad r(y) := \sum_{x\in f^{-1}(y)} \frac{q(x)}{f'(x)^2}.$$

If $\boldsymbol{q}$ is a meromorphic quadratic differential on $\widehat{\mathbb{C}}$, and if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map, then or all $y \in \widehat{\mathbb{C}}$, we have

$$2 + \mathrm{ord}_y(f_*\boldsymbol{q}) \geqslant \min_{x\in f^{-1}(y)} \frac{2+\mathrm{ord}_x\boldsymbol{q}}{\deg_x f}.$$

As a consequence,

- if X is the set of poles of $\boldsymbol{q}$, then the set of poles of $f_*\boldsymbol{q}$ is contained in $f(X)\cup\mathcal{V}_f$,
- if $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$, then $f_*\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$ and
- if f fixes ∞ and if $\boldsymbol{q} \in \mathcal{Q}_k(\mathbb{C})$ for some $k \geqslant 0$, then $f_*\boldsymbol{q} \in \mathcal{Q}_k(\mathbb{C})$.

3.4. *Transposition*

If $\boldsymbol{q}$ is a quadratic differential on U and $\boldsymbol{\theta}$ is a vector field on U, we may consider the 1-form $\boldsymbol{q}\otimes\boldsymbol{\theta}$ defined on U by its action on vector fields $\boldsymbol{\xi}$:

$$\boldsymbol{q}\otimes\boldsymbol{\theta}(\boldsymbol{\xi}) = \frac{1}{4}\big(\boldsymbol{q}(\boldsymbol{\theta}+\boldsymbol{\xi}) - \boldsymbol{q}(\boldsymbol{\theta}-\boldsymbol{\xi})\big).$$

If

$$\boldsymbol{q} = q\,(\mathrm{d}z)^2 \quad \text{and} \quad \boldsymbol{\theta} = \theta\,\frac{\mathrm{d}}{\mathrm{d}z}, \quad \text{then} \quad \boldsymbol{q}\otimes\boldsymbol{\theta} = q\theta\,\mathrm{d}z.$$

We shall use the following lemma which, in some sense, asserts that the transpose of pushing forward a quadratic differential is pulling back a vector field.

Lemma 3.1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map, let $\boldsymbol{\theta}$ be a meromorphic vector field on $\widehat{\mathbb{C}}$, and let $\boldsymbol{q}$ be a meromorphic quadratic differential on $\widehat{\mathbb{C}}$. Then*

$$\mathrm{residue}\big((f_*\boldsymbol{q})\otimes\boldsymbol{\theta}, y\big) \;=\; \sum_{x\in f^{-1}(y)} \mathrm{residue}\big(\boldsymbol{q}\otimes f^*\boldsymbol{\theta}, x\big).$$

Proof. Let γ be a small loop around y with basepoint a. Then

$$\int_\gamma (f_*\boldsymbol{q})\otimes\boldsymbol{\theta} = \sum_g \int_{\gamma\smallsetminus\{a\}} (g^*\boldsymbol{q})\otimes\boldsymbol{\theta} = \sum_g \int_{g(\gamma\smallsetminus\{a\})} \boldsymbol{q}\otimes f^*\boldsymbol{\theta} = \int_{f^{-1}(\gamma)} \boldsymbol{q}\otimes f^*\boldsymbol{\theta},$$

where the sum ranges over the inverse branches g of f defined on $\gamma\smallsetminus\{a\}$. □

4. *The contraction principle*

If $\boldsymbol{q}$ is a quadratic differential on U, we denote by $|\boldsymbol{q}|$ the positive (1,1)-form on U defined by

$$|\boldsymbol{q}|(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) := \frac{1}{2}\big|\boldsymbol{q}(\boldsymbol{\theta}_1 - \mathrm{i}\boldsymbol{\theta}_2)\big| - \frac{1}{2}\big|\boldsymbol{q}(\boldsymbol{\theta}_1 + \mathrm{i}\boldsymbol{\theta}_2)\big|.$$

If $\boldsymbol{q} = q\ (\mathrm{d}\zeta)^2$, then

$$|\boldsymbol{q}| = |q|\cdot\frac{\mathrm{i}}{2}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}.$$

We shall say that $\boldsymbol{q}$ is *integrable* on U if

$$\|\boldsymbol{q}\|_{L^1(U)} := \int_U |\boldsymbol{q}| < \infty.$$

Note that $\boldsymbol{q}$ is integrable in a neighborhood of a pole if and only if the pole is simple.

The following results due to Thurston will be crucial for our purposes. The proof, based on the triangle inequality (see [4], for example), is transcendental.

LEMMA 4.1 (Contraction principle). *Let $f: U\to V$ be a covering map and let $\boldsymbol{q}$ be an integrable quadratic differential on U. Then,*

$$\|f_*\boldsymbol{q}\|_{L^1(V)} \leqslant \|\boldsymbol{q}\|_{L^1(U)}$$

and equality holds if and only if $f^*(f_*\boldsymbol{q}) = \phi\ \boldsymbol{q}$ with $\phi: U\to[0,+\infty)$ a real and positive function.

COROLLARY 4.2. *If $f\in\mathrm{Rat}_D$ is postcritically finite, then $f_*:\mathcal{Q}(\widehat{\mathbb{C}})\to\mathcal{Q}(\widehat{\mathbb{C}})$ is (weakly) contracting. In particular, the eigenvalues of f_* have modulus at most 1.*

COROLLARY 4.3. *Let $f:\mathbb{C}\to\mathbb{C}$ be a polynomial of degree D. Then for all $k\geqslant 0$, the eigenvalues of $f_*:\mathcal{Q}_k(\mathbb{C})\to\mathcal{Q}_k(\mathbb{C})$ have modulus less than 1.*

Proof. Suppose that λ is an eigenvalue and $\boldsymbol{q}\in\mathcal{Q}_k(\mathbb{C})$ is an associated eigenvector; that is, $\boldsymbol{q}\neq 0$ and $f_*\boldsymbol{q} = \lambda\boldsymbol{q}$. Let V be a sufficiently large disk so that $U := f^{-1}(V)$ is compactly contained in V. Set $V' := V\smallsetminus\mathcal{V}_f$ and $U' := f^{-1}(V')$. Then

$$|\lambda|\cdot\|\boldsymbol{q}\|_{L^1(V)} = \|\lambda\boldsymbol{q}\|_{L^1(V')} = \|f_*\boldsymbol{q}\|_{L^1(V')} \leqslant \|\boldsymbol{q}\|_{L^1(U')} = \|\boldsymbol{q}\|_{L^1(U)} < \|\boldsymbol{q}\|_{L^1(V)}.$$

The first inequality is an application of the contraction principle. The last inequality is strict since U is compactly contained in V and $\boldsymbol{q}\neq 0$. In addition, $\|\boldsymbol{q}\|_{L^1(V)} > 0$, so $|\lambda| < 1$. □

The proof of the following result is given in [4].

PROPOSITION 4.4 (Thurston). *Let $f\in\mathrm{Rat}_D$, and suppose that λ is an eigenvalue of $f_*:\mathcal{Q}(\widehat{\mathbb{C}})\to\mathcal{Q}(\widehat{\mathbb{C}})$. If $|\lambda| = 1$, then f is a Lattès map with four postcritical points. If $\lambda = 1$, then f is a flexible Lattès map.*

5. *Infinitesimal rigidity*

We now present a proof that with the exception of flexible Lattès maps, postcritically finite rational maps are infinitesimally rigid. This proof only relies on the fact that when f is not a flexible Lattès map, $1\notin\Sigma_f$.

Here and henceforth, we consider holomorphic families $t \mapsto \gamma_t$ defined near $t = 0$ in $\mathbb{C}$. We shall employ the notation

$$\gamma := \gamma_0 \quad \text{and} \quad \dot\gamma := \left.\frac{\mathrm{d}\gamma_t}{\mathrm{d}t}\right|_{t=0}.$$

If $t \mapsto f_t$ is a family of rational maps of degree D, then $\boldsymbol{\xi} := \dot f \in \mathrm{T}_f\mathrm{Rat}_D$ is a section of the pullback bundle $f^\star\mathrm{T}\widehat{\mathbb{C}}$: for each $z \in \widehat{\mathbb{C}}$, $\boldsymbol{\xi}(z) \in T_{f(z)}\widehat{\mathbb{C}}$. Setting

$$\boldsymbol{\tau}(x) := (\mathrm{D}_x f)^{-1}\big(\boldsymbol{\xi}(x)\big) \quad \text{if} \quad x \notin \mathcal{C}_f,$$

we define a meromorphic vector field $\boldsymbol{\tau}$ on $\widehat{\mathbb{C}}$, holomorphic outside $\mathcal{C}_f$, with poles of order at most the multiplicity of x as a critical point of f when $x \in \mathcal{C}_f$. This vector field satisfies

$$\boldsymbol{\xi} = \mathrm{D}f \circ \boldsymbol{\tau}.$$

THEOREM 5.1 (Thurston). *Let $t \mapsto f_t$ be a holomorphic family of postcritically finite rational maps of degree D, parameterized by a neighborhood of 0 in $\mathbb{C}$. Then either the maps are flexible Lattés maps, or there is an analytic family of Möbius transformations $t \mapsto M_t$ such that $M_0 = \mathrm{id}$ and $f_t \circ M_t = M_t \circ f_0$.*

Proof. Without loss of generality, we may assume that f is not a flexible Lattès map. The fixed points of f_t are superattracting or repelling and depend holomorphically on t. There are $D + 1 \geqslant 3$ such fixed points. Conjugating the family $t \mapsto f_t$ with a holomorphic family $t \mapsto M_t$ of Möbius transformations, we may assume that f_t fixes 0, 1, and ∞. We will show that in this case, the holomorphic family $t \mapsto f_t$ is constant. It is enough to show that $\frac{\mathrm{d}f_t}{\mathrm{d}t}$ identically vanishes, and since $t = 0$ plays no particular role, it is enough to prove that $\dot f \equiv 0$.

Set $\boldsymbol{\xi} := \dot f \in \mathrm{T}_f\mathrm{Rat}_D$ and let $\boldsymbol{\tau}$ be the globally meromorphic vector field on $\widehat{\mathbb{C}}$ such that $\boldsymbol{\xi} = \mathrm{D}f \circ \boldsymbol{\tau}$.

As t varies, the set $Y_t := \mathcal{P}_{f_t} \cup \{0, 1, \infty\}$ moves holomorphically and $f_t(Y_t) = Y_t$. For each $y \in Y$, let $t \mapsto y_t$ be the holomorphic curve satisfying $y_0 = y$ and $y_t \in Y_t$. Set

$$\boldsymbol{\vartheta}(y) := \left.\frac{\mathrm{d}y_t}{\mathrm{d}t}\right|_{t=0} \in \mathrm{T}_y\widehat{\mathbb{C}}.$$

If $y \in Y$ and $z := f(y) \in Y$, then $z_t = f_t(y_t)$, so that

$$\boldsymbol{\vartheta} \circ f = \boldsymbol{\xi} + \mathrm{D}f \circ \boldsymbol{\vartheta} \quad \text{on } Y \quad \text{and} \quad \boldsymbol{\vartheta} \circ f = \boldsymbol{\xi} \quad \text{on } \mathcal{C}_f.$$

Let $\boldsymbol{\theta}$ be a vector field, defined and holomorphic near Y, with $\boldsymbol{\theta}|_Y = \boldsymbol{\vartheta}$. Then, $f^*\boldsymbol{\theta} - \boldsymbol{\tau}$ is holomorphic near $f^{-1}(Y)$ and coincides with $\boldsymbol{\theta}$ on Y. Also note that since f_t fixes 0, 1 and ∞, $\boldsymbol{\theta}$ vanishes at 0, 1, and ∞.

Let $\nabla_f := \mathrm{id} - f_* : \mathcal{Q}(\widehat{\mathbb{C}}; Y) \to \mathcal{Q}(\widehat{\mathbb{C}}; Y)$. Observe that for $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}}; Y)$,

$$\begin{aligned}
\sum_{y \in Y} \mathrm{residue}\big((\nabla_f \boldsymbol{q}) \otimes \boldsymbol{\theta}, y\big) &\underset{(1)}{=} \sum_{y \in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) - \sum_{y \in Y} \mathrm{residue}\big((f_*\boldsymbol{q}) \otimes \boldsymbol{\theta}, y\big) \\
&\underset{(2)}{=} \sum_{y \in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) - \sum_{x \in f^{-1}(Y)} \mathrm{residue}(\boldsymbol{q} \otimes f^*\boldsymbol{\theta}, x) \\
&\underset{(3)}{=} \sum_{y \in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) - \sum_{x \in f^{-1}(Y)} \mathrm{residue}(\boldsymbol{q} \otimes (f^*\boldsymbol{\theta} - \boldsymbol{\tau}), x) \\
&\underset{(4)}{=} 0.
\end{aligned}$$

Equality (1) holds by definition of ∇_f; Equality (2) follows from Lemma 3.1; Equality (3) follows from the fact that $\boldsymbol{q} \otimes \boldsymbol{\tau}$ is globally meromorphic on $\widehat{\mathbb{C}}$ with poles contained in

$Y \cup \mathcal{C}_f \subseteq f^{-1}(Y)$, so that the sum of its residues on $f^{-1}(Y)$ is 0; Equality (4) follows from the fact that $f^*\boldsymbol{\theta} - \boldsymbol{\tau}$ is holomorphic near $f^{-1}(Y)$ and coincides with $\boldsymbol{\theta}$ on Y which contains the set of poles of $\boldsymbol{q}$.

According to Proposition 4.4, since f is not a flexible Lattès map, $\lambda = 1$ is not an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$. The operator $\nabla_f : \mathcal{Q}(\widehat{\mathbb{C}}; Y) \to \mathcal{Q}(\widehat{\mathbb{C}}; Y)$ is therefore injective, thus surjective. It follows that for any $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}}; Y)$,

$$\sum_{y \in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) = 0.$$

Equivalently, $\boldsymbol{\vartheta}$ is the restriction to Y of a globally holomorphic vector field $\boldsymbol{\theta}$. Since $\boldsymbol{\theta}$ vanishes at 0, 1, and ∞, we have $\boldsymbol{\theta} = 0$. The vector field $-\boldsymbol{\tau} = f^*\boldsymbol{\theta} - \boldsymbol{\tau}$ is globally holomorphic and coincides with $\boldsymbol{\theta} = 0$ on Y. So $\boldsymbol{\tau} = 0$ and $\boldsymbol{\xi} = \mathrm{D}f \circ \boldsymbol{\theta} = 0$ as required. □

The following corollary of infinitesimal rigidity is part of the folklore, but we are not aware of a written proof. Our presentation provides a systematic way of defining algebraic equations for the set of postcritically finite rational maps with prescribed dynamics on the postcritical set.

Proposition 5.2. *If $f \in \mathrm{Rat}_D$ is postcritically finite but not a flexible Lattès map, then the Möbius conjugacy class of f contains a representative with algebraic coefficients.*

Proof. As in the previous proof, conjugating f with a Möbius transformation if necessary, we may assume that f fixes 0, 1, and ∞ and set $Y := \mathcal{P}_f \cup \{0, 1, \infty\}$. In addition, set $X := f^{-1}(Y)$ and let $\delta : X \to \mathbb{N}$ be defined by

$$\delta(x) := \deg_x f.$$

Let us identify $\widehat{\mathbb{C}}$ and $\mathbb{P}^1(\mathbb{C})$ via the usual map $\mathbb{P}^1(\mathbb{C}) \ni [u : v] \mapsto u/v \in \widehat{\mathbb{C}}$. For $N \geqslant 1$, let us denote by $\mathcal{H}_N$ the vector space of homogeneous polynomials of degree N from $\mathbb{C}^2$ to $\mathbb{C}$. There is a canonical isomorphism between $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}(\mathcal{H}_1)$: a point of $\mathbb{P}^1(\mathbb{C})$, a 1-dimensional linear subspace of $\mathbb{C}^2$, is identified with the space of forms on $\mathbb{C}^2$ vanishing on this linear subspace; that is, a 1-dimensional linear subspace of $\mathcal{H}_1$. This subsequently yields an identification of $\widehat{\mathbb{C}}$ with $\mathbb{P}(\mathcal{H}_1)$.

Note that Rat_D may be identified with the open subset of $\mathbb{P}(\mathcal{H}_D \times \mathcal{H}_D)$ corresponding to pairs of coprime homogeneous polynomials of degree D. Such a pair of polynomials defines a nondegenerate homogeneous polynomial map $\mathbb{C}^2 \to \mathbb{C}^2$ of degree D which induces an endomorphism $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree D.

We shall denote as $\widehat{\mathbb{C}}_D$ the D-fold symmetric product of the Riemann sphere; that is, the quotient of $\widehat{\mathbb{C}}^D$ by the group of permutation of the coordinates. The map

$$\mathcal{H}_1^D \ni (P_1, \ldots, P_D) \mapsto P_1 \times \cdots \times P_D \in \mathcal{H}_D$$

induces an identification between $\widehat{\mathbb{C}}_D$ and $\mathbb{P}(\mathcal{H}_D)$.

Set

$$\mathcal{X} := \mathrm{Rat}_D \times \widehat{\mathbb{C}}^X \times \widehat{\mathbb{C}}^Y \quad \text{and} \quad \mathcal{Y} := (\widehat{\mathbb{C}}_D \times \widehat{\mathbb{C}})^Y.$$

A point $(g, \alpha, \beta) \in \mathcal{X}$ may be represented by a triple $\big(G, (A_x)_{x \in X}, (B_y)_{y \in Y}\big)$ where $G := (G_1, G_2) \in \mathcal{H}_D \times \mathcal{H}_D$ is a nondegenerate homogeneous polynomial map $\mathbb{C}^2 \to \mathbb{C}^2$ of degree D, and where $A_x \in \mathcal{H}_1$ and $B_y \in \mathcal{H}_1$ are linear forms $\mathbb{C}^2 \to \mathbb{C}$. Recall that $Y \subset X$ and consider the algebraic map $\Phi : \mathcal{X} \to \mathcal{Y}$ induced by

$$\big(G, (A_x)_{x\in X}, (B_y)_{y\in Y}\big) \mapsto \left(\prod_{x\in f^{-1}(y)} A_x^{\delta(x)}, A_y\right)_{y\in Y}$$

and the algebraic map $\Psi : \mathcal{X} \to \mathcal{Y}$ induced by

$$\big(G, (A_x)_{x\in X}, (B_y)_{y\in Y}\big) \mapsto \big(B_y \circ G, B_y\big)_{y\in Y}.$$

Let us consider the algebraic set $\mathcal{Z} \subset \mathcal{X}$ defined by the equation $\Phi = \Psi$.

We claim that the triple (g, α, β) belongs to $\mathcal{Z}$ if and only if we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & \widehat{\mathbb{C}} \arrow[d, "g"] \\ Y \arrow[r, "\beta"'] & \widehat{\mathbb{C}} \end{tikzcd} \quad \text{with} \quad \deg_{\alpha(x)} g = \deg_x f, \quad \text{and} \quad \alpha|_Y = \beta|_Y.$$

Indeed,

$$B_y \circ G = \prod_{x\in f^{-1}(y)} A_x^{\delta(x)}$$

if and only if for each $x \in f^{-1}(y)$, the point $\alpha(x) \in \widehat{\mathbb{C}}$ is a preimage of $\beta(y) \in \widehat{\mathbb{C}}$ by g taken with multiplicity $\delta(x) = \deg_x f$. In addition, $A_y = B_y$ if and only if $\alpha(y) = \beta(y)$.

As a consequence, if $(g, \alpha, \beta) \in \mathcal{Z}$, then $\mathcal{C}_g = \alpha(\mathcal{C}_f) \subseteq \alpha(X)$. In that case, $\mathcal{P}_g = \beta(\mathcal{P}_f) \subseteq \beta(Y)$, and g is postcritically finite.

Set

$$\mathcal{Z}_0 := \big\{(g, \alpha, \beta) \in \mathcal{Z} \mid \alpha(0) = 0,\ \alpha(1) = 1,\ \text{and}\ \alpha(\infty) = \infty\big\}.$$

Note that the triple $(f, \mathrm{id}, \mathrm{id})$ belongs to $\mathcal{Z}_0$. According to Theorem 5.1, if f is not a flexible Lattès map, then the algebraic set $\mathcal{Z}_0$ has dimension 0 at f. This implies that f has algebraic coefficients. □

6. *The eigenvalues of* $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$

Proposition 6.1. *If* $f \in \mathrm{Rat}_D$*, then* 0 *is an eigenvalue of* $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$.

Proof. Let Y be a subset of $\widehat{\mathbb{C}} \smallsetminus \mathcal{V}_f$ with $\mathrm{card}(Y) = 3$. Set $X := f^{-1}(Y)$. Note that X is disjoint from $\mathcal{C}_f$ and

$$\mathrm{card}(X) - \mathrm{card}(\mathcal{C}_f) \geqslant 3D - (2D - 2) \geqslant D + 2 \geqslant 4.$$

So there is a nonzero quadratic differential $\boldsymbol{q}$ which vanishes on $\mathcal{C}_f$ and whose poles are simple and contained in X. The quadratic differential $f_*\boldsymbol{q}$ is holomorphic on $\widehat{\mathbb{C}} \smallsetminus Y$ and has at most simple poles along Y. Thus, it has at most three poles counting multiplicities, which forces $f_*\boldsymbol{q} = 0$. □

Let us now assume that $f \in \mathrm{Rat}_D$ is postcritically finite and set

$$\mathcal{Q}_f := \mathcal{Q}(\widehat{\mathbb{C}}; \mathcal{P}_f).$$

Note that $f_*(\mathcal{Q}_f) \subseteq \mathcal{Q}_f$. Indeed, if $\boldsymbol{q} \in \mathcal{Q}_f$, then the poles of $f_*\boldsymbol{q}$ are contained in $f(\mathcal{P}_f) \cup \mathcal{V}_f = \mathcal{P}_f$. So, the set of eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ may be written as the union $\Sigma_f \cup \Lambda_f$ with

$$\Sigma_f := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } f_* : \mathcal{Q}_f \to \mathcal{Q}_f\},$$

and

$$\Lambda_f := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f\}.$$

We postpone the study of Σ_f and focus now on Λ_f.

Proposition 6.2. *The elements of $\Lambda_f \setminus \{0\}$ are the complex numbers λ such that $1/\lambda^m$ is the multiplier of a cycle of f of period m which is not contained in $\mathcal{P}_f$.*

Proof. To begin with, let us describe the space $\mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$. First, observe that if $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$, then the residue of $\boldsymbol{q}$ at a point $x \in \widehat{\mathbb{C}}$ is naturally a form on $\mathrm{T}_x\widehat{\mathbb{C}}$; that is, an element of $\mathrm{T}^*_x\widehat{\mathbb{C}}$. It may be defined as follows: if $\theta \in \mathrm{T}_x\widehat{\mathbb{C}}$ and $\boldsymbol{\theta}$ is a vector field defined and holomorphic near x with $\boldsymbol{\theta}(x) = \theta$, then

$$\mathrm{residue}(\boldsymbol{q}, x)(\theta) := \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, x).$$

The result does not depend on the extension $\boldsymbol{\theta}$ since if $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are two holomorphic vector fields which coincide at x, then $\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2$ vanishes at x, so that $\boldsymbol{q} \otimes (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)$ is holomorphic near x and $\mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}_1, x) = \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}_2, x)$.

Second, set

$$B := \begin{cases} \mathcal{P}_f & \text{if } \mathrm{card}(\mathcal{P}_f) \geqslant 3, \\ \mathcal{P}_f \cup \{\alpha\} \text{ with } \alpha \text{ a repelling fixed point of } f & \text{if } \mathrm{card}(\mathcal{P}_f) = 2. \end{cases}$$

So $\mathrm{card}(B) \geqslant 3$, $f(B) = B$, and $\mathcal{Q}_f = \mathcal{Q}(\widehat{\mathbb{C}}; B)$ (the equality holds even when $\mathrm{card}(\mathcal{P}_f) = 2$ since in that case, both spaces are reduced to $\{0\}$). Set

$$\Omega_f := \bigoplus_{x \in \widehat{\mathbb{C}} \setminus B} \mathrm{T}^*_x\widehat{\mathbb{C}}.$$

Note that Ω_f is the space of 1-forms on $\widehat{\mathbb{C}} \setminus B$ which vanish outside a finite set. Consider the map $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ defined by

$$\mathrm{Res}\big([\boldsymbol{q}]\big)(x) := \mathrm{residue}(\boldsymbol{q}, x).$$

This map is well defined since if $\boldsymbol{q}_1 \in \mathcal{Q}(\widehat{\mathbb{C}})$ and $\boldsymbol{q}_2 \in \mathcal{Q}(\widehat{\mathbb{C}})$ satisfy $\boldsymbol{q}_1 - \boldsymbol{q}_2 \in \mathcal{Q}_f$, then $\boldsymbol{q}_1 - \boldsymbol{q}_2$ is holomorphic on $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$, so that $\mathrm{residue}(\boldsymbol{q}_1, x) = \mathrm{residue}(\boldsymbol{q}_2, x)$ for all $x \in \widehat{\mathbb{C}} \setminus B$.

Lemma 6.3. *The map* $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ *is an isomorphism of vector spaces.*

Proof. First, if $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$ and $\mathrm{residue}(\boldsymbol{q}, x) = 0$ for all $x \in \widehat{\mathbb{C}} \setminus B$, then $\boldsymbol{q}$ is holomorphic outside B, so that $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}}; B) = \mathcal{Q}_f$. It follows that $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ is injective.

Second, given two distinct points x_1 and x_2 in $\widehat{\mathbb{C}}$, let $\boldsymbol{\omega}_{x_1,x_2}$ be the meromorphic 1-form on $\widehat{\mathbb{C}}$ which is holomorphic outside $\{x_1, x_2\}$, has residue 1 at x_1, and residue -1 at x_2:

$$\boldsymbol{\omega}_{x_1,x_2} = \begin{cases} \mathrm{d}z/(z - x_1) - \mathrm{d}z/(z - x_2) & \text{if } x_1 \neq \infty \text{ and } x_2 \neq \infty, \\ \mathrm{d}z/(z - x_1) & \text{if } x_2 = \infty, \\ -\mathrm{d}z/(z - x_2) & \text{if } x_1 = \infty. \end{cases}$$

Note that $\boldsymbol{\omega}_{x_1,x_2}$ does not vanish.

Third, choose three distinct points x_1, x_2, and x_3 in B. Given $\omega \in \Omega_f$, we may define a function $\phi : \widehat{\mathbb{C}}\smallsetminus B \to \mathbb{C}$ by

$$\omega(x) = \phi(x) \cdot \boldsymbol{\omega}_{x_2,x_3}(x).$$

Since ω vanishes outside a finite set, ϕ also vanishes outside a finite set. Set

$$\boldsymbol{q} := \sum_{x \in \widehat{\mathbb{C}}\smallsetminus B} \phi(x) \cdot \boldsymbol{\omega}_{x,x_1} \otimes \boldsymbol{\omega}_{x_2,x_3}.$$

Since x, x_1, x_2, and x_3 are distinct, $\boldsymbol{\omega}_{x,x_1} \otimes \boldsymbol{\omega}_{x_2,x_3} \in \mathcal{Q}(\widehat{\mathbb{C}})$. Since ϕ vanishes outside a finite set, the sum is finite and $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$. By construction, for $x \in \widehat{\mathbb{C}}\smallsetminus B$,

$$\operatorname{residue}(\boldsymbol{q}, x) = \phi(x) \cdot \boldsymbol{\omega}_{x_2,x_3}(x) = \omega(x),$$

so that $\operatorname{Res}\big([\boldsymbol{q}]\big) = \omega$. It follows that $\operatorname{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ is surjective. □

Observe that if $y \in \widehat{\mathbb{C}}\smallsetminus B \subseteq \widehat{\mathbb{C}}\smallsetminus \mathcal{V}_f$, then $\mathrm{D}_x f : \mathrm{T}_x\widehat{\mathbb{C}} \to \mathrm{T}_y\widehat{\mathbb{C}}$ is invertible for any $x \in f^{-1}(y)$, and in that case, the isomorphism $\operatorname{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ conjugates $f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$ to the linear map $f_* : \Omega_f \to \Omega_f$ defined by

$$f_*\omega(y) := \sum_{x \in f^{-1}(y)} \omega \circ (\mathrm{D}_x f)^{-1}.$$

Next, let $X \subset \widehat{\mathbb{C}}\smallsetminus B$ be a cycle of f of period m and multiplier μ. Note that the space

$$E_X := \bigoplus_{x \in X} \mathrm{T}^*_x\widehat{\mathbb{C}} \subset \Omega_f$$

has dimension m and is invariant by $f_* : \Omega_f \to \Omega_f$.

LEMMA 6.4. *The endomorphism $f_* : E_X \to E_X$ is diagonalizable. Its eigenvalues are the mth roots of $1/\mu$.*

Proof. Suppose $\lambda^m = 1/\mu$, and let $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{m-1} \mapsto x_m = x_0$ be the points of X. Let $\omega_0 \in \mathrm{T}^*_{x_0}\widehat{\mathbb{C}}$ be any nonzero form. For $1 \leqslant j \leqslant m$, define recursively

$$\omega_j := \lambda\omega_{j-1} \circ (\mathrm{D}_{x_{j-1}} f)^{-1} \in \mathrm{T}^*_{x_j}\widehat{\mathbb{C}}.$$

Then

$$\omega_m = \lambda^m \omega_0 \circ (\mathrm{D}_{x_0} f^{\circ m})^{-1} = \frac{\lambda^m}{\mu}\omega_0 = \omega_0$$

since $\mathrm{D}_{x_0} f^{\circ m} : \mathrm{T}_{x_0}\widehat{\mathbb{C}} \to \mathrm{T}_{x_0}\widehat{\mathbb{C}}$ is multiplication by μ and $\lambda^m = 1/\mu$. It follows that the 1-form $\omega \in \Omega_f$ defined by

$$\omega(x) = \begin{cases} 0 & \text{if } x \notin X, \\ \omega_j & \text{if } x = x_j \in X \end{cases}$$

satisfies $f_*\omega = \lambda\omega$ and $\omega \neq 0$ since $\omega_0 \neq 0$. □

Finally, assume $\lambda \neq 0$ is an eigenvalue of $f_* : \Omega_f \to \Omega_f$ and let $\omega \in \Omega_f$ be an eigenvector associated to λ. Set $X := \{x \in \widehat{\mathbb{C}} \mid \omega(x) \neq 0\}$. If $\omega(y) \neq 0$, then there exists $x \in f^{-1}(y)$ such that $\omega(x) \neq 0$. Thus, $X \subseteq f(X)$ and since the cardinality of $f(X)$ is always less than or equal to the cardinality of X, we necessarily have $X = f(X)$. So $X \subset \widehat{\mathbb{C}}\smallsetminus B$ is a union of cycles of f. It follows from Lemma 6.4 that the eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$ other than $\lambda = 0$

are the complex numbers λ such that $1/\lambda^m$ is the multiplier of a cycle of f of period m which is not contained in B. This completes the proof of Proposition 6.2 when $B = \mathcal{P}_f$; that is, when $\operatorname{card}(\mathcal{P}_f) \geqslant 3$.

To complete the proof of Proposition 6.2 when $\operatorname{card}(\mathcal{P}_f) = 2$, observe that

- either f is conjugate to $z \mapsto 1/z^D$ in which case there are $D+1$ repelling fixed points, each with multiplier $\mu = -D$, so that the multipliers of the fixed points which are not contained in B are the multipliers of the fixed points which are not contained in $\mathcal{P}_f$;
- or f is conjugate to $z \mapsto z^D$; if $D > 2$, the proof is similar; if $D = 2$, there is a single repelling fixed point at $z = 1$, its multiplier is 2 and we must show that $1/2$ is an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$; in that case, the multiplier of the cycle of period $m = 2$ is $\mu = 4$, so that $1/2$ is a mth root of $1/\mu$.

This completes the proof of Proposition 6.2. □

COROLLARY 6.5. *Let $f \in \mathrm{Rat}_D$ be postcritically finite. Then, $\Lambda_f \subset \mathbb{D}$.*

Proof. A cycle of f not contained in $\mathcal{P}_f$ is repelling. □

EXAMPLE. If $f(z) = z^{\pm D}$, then a cycle of period m not contained in $\mathcal{P}_f$ is a repelling cycle of multiplier $(\pm D)^m$ and there is at least one such cycle for each period $m \geqslant 2$. It follows that

$$\Lambda_f = \{0\} \cup \left\{ \frac{\mathrm{e}^{2\pi \mathrm{i} p/q}}{D},\ p/q \in \mathbb{Q}/\mathbb{Z} \right\}.$$

Note that $\operatorname{card}(\mathcal{P}_f) = 2$, so that $\mathcal{Q}_f = \{0\}$ and $\Sigma_f = \emptyset$.

EXAMPLE. If $f \in \mathrm{Rat}_D$ is a flexible Lattès map, then a cycle of period m not contained in $\mathcal{P}_f$ is a repelling cycle of multiplier $\sqrt{D}^m$ and there is at least one such cycle for each period $m \geqslant 1$. It follows that

$$\Lambda_f = \{0\} \cup \left\{ \frac{\mathrm{e}^{2\pi \mathrm{i} p/q}}{\sqrt{D}},\ p/q \in \mathbb{Q}/\mathbb{Z} \right\}.$$

PROPOSITION 6.6. *Let $f \in \mathrm{Rat}_D$ be postcritically finite. If $\lambda \in \Lambda_f \smallsetminus \{0\}$, then λ is an algebraic number but not an algebraic integer.*

Proof. As discussed in the previous example, the proposition holds for flexible Lattès maps, so we may assume that f is not a flexible Lattès map in this proof.

If $f \in \mathrm{Rat}_D$ and $g \in \mathrm{Rat}_D$ are conjugate by a Möbius transformation M; that is, $M \circ f = g \circ M$, then the linear map $M_* : \mathcal{Q}_f \to \mathcal{Q}_g$ conjugates $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ to $g_* : \mathcal{Q}_g \to \mathcal{Q}_g$:

$$g_*(M_* \boldsymbol{q}) = M_*(f_* \boldsymbol{q}) = M_*(\lambda \boldsymbol{q}) = \lambda M_* \boldsymbol{q}.$$

Thus, $\Lambda_f = \Lambda_g$.

According to Proposition 5.2, the conjugacy class of f contains a representative with algebraic coefficients. Without loss of generality, we may therefore assume that this is the case for f and consider f as a rational map $f : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$. Working over the algebraically closed field $\overline{\mathbb{Q}}$, we deduce that the multipliers of cycles of f are algebraic numbers, so Λ_f consists of algebraic numbers.

If $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ is a Galois automorphism, then $g := \sigma \circ f \circ \sigma^{-1} : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$ is postcritically finite. In addition, since $g_*(\sigma_* \boldsymbol{q}) = \sigma_*(f_* \boldsymbol{q})$ and $\sigma(\lambda)\ \sigma_* \boldsymbol{q} = \sigma_*(\lambda \boldsymbol{q})$, we have that

$$f_* \boldsymbol{q} = \lambda \boldsymbol{q} \quad \Longleftrightarrow \quad g_*(\sigma_* \boldsymbol{q}) = \sigma(\lambda)\ \sigma_* \boldsymbol{q}.$$

Thus, $\lambda \in \Lambda_f$ if and only if $\sigma(\lambda) \in \Lambda_g$. According to Corollary 6.5, if $\lambda \in \Lambda_f$, then $\left|\sigma(\lambda)\right| < 1$ for any Galois automorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$. Thus, if λ is an algebraic integer, the product of λ with its Galois conjugates is an integer of modulus less than 1, which forces $\lambda = 0$. □

7. *The eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$*

From now on, we assume that $\mathrm{card}(\mathcal{P}_f) \geqslant 4$ so that $\mathcal{Q}_f$ is not reduced to $\{0\}$. In that case, the dimension of $\mathcal{Q}_f$ is $\mathrm{card}(\mathcal{P}_f) - 3$, so $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ has at most $\mathrm{card}(\mathcal{P}_f) - 3$ eigenvalues.

7.1. *Lattès maps with four postcritical points*

According to Proposition 4.4, if Σ_f contains an eigenvalue of modulus 1, then f is a Lattès map with $\mathrm{card}(\mathcal{P}_f) = 4$. The converse is also true.

Proposition 7.1. *Suppose that $f \in \mathrm{Rat}_D$ is a Lattès map with $\mathrm{card}(\mathcal{P}_f) = 4$. Then $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ is multiplication by λ with $|\lambda| = 1$. In addition, $\lambda = \pm 1$, or λ belongs to an imaginary quadratic number field. Any imaginary quadratic number of modulus 1 may arise for some Lattès map. If λ is an algebraic integer, then λ is a root of unity of order 1, 2, 3, 4, or 6.*

Proof. Since $\mathrm{card}(\mathcal{P}_f) = 4$, the dimension of $\mathcal{Q}_f$ is 1, so $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ has a unique eigenvalue, and f_* is multiplication by this eigenvalue. By assumption, there is a complex torus $\mathcal{T}$, a ramified cover $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$ ramifying at each point above $\mathcal{P}_f$ with local degree 2, and an endomorphism $L : \mathcal{T} \to \mathcal{T}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "L"] \arrow[d, "\Theta"'] & \mathcal{T} \arrow[d, "\Theta"] \\ \widehat{\mathbb{C}} \arrow[r, "f"'] & \widehat{\mathbb{C}}. \end{tikzcd}$$

As mentioned in Proposition 4.4, if $\boldsymbol{q} \in \mathcal{Q}_f$, then $\Theta^*\boldsymbol{q}$ is a multiple of $(\mathrm{d}z)^2$ and if λ is the eigenvalue of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$, then $L_*(\Theta^*\boldsymbol{q}) = \frac{1}{D\lambda}\Theta^*\boldsymbol{q}$. Thus, if $L(w) = \alpha w + \beta$, then $\alpha^2 = D\lambda$.

According to [**5**, § 5], we have $|\alpha|^2 = D$, so that $|\lambda| = 1$. In addition, either

- $\alpha \in \mathbb{Z}$ in which case $\lambda = 1$, and f is a flexible Lattès map, or
- α is an imaginary quadratic integer; that is, $\alpha^2 - C\alpha + D = 0$ with $C \in \mathbb{Z}$ and $C^2 < 4D$.

In the latter case, $\lambda = \alpha^2/D \in \mathbb{Q}[\alpha]$ is either -1 or an imaginary quadratic number of modulus 1.

Conversely, suppose $\lambda = -1$ or λ is an imaginary quadratic number of modulus 1. Let $k \geqslant 2$ be a sufficiently large integer so that $\alpha := k\sqrt{\lambda}$ is an imaginary quadratic integer and set $D := k^2$. According to [**5**, § 5], there exists a Lattès map $f \in \mathrm{Rat}_D$ with $L(w) = \alpha w$. According to the previous discussion, the eigenvalue of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ is λ.

Finally, if λ is a quadratic integer, then it is a unit since $|\lambda| = 1$. Thus, it is a root of unity of order 1, 2, 3, 4, or 6. □

7.2. *Non-Lattès maps*

We now assume that f is not a Lattès map. In that case, according to Corollary 4.2 and Proposition 4.4, the eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ are contained in the unit disk.

Proposition 7.2. *Let $f \in \mathrm{Rat}_D$ be postcritically finite with $\mathrm{card}(\mathcal{P}_f) \geqslant 4$, and suppose that f is not a Lattès map. If $\lambda \in \Sigma_f \setminus \{0\}$, then λ is an algebraic number but not an algebraic integer.*

Proof. We proceed as in the proof of Proposition 6.6. Conjugating f with a Möbius transformation if necessary, we may assume that f is a rational map $f : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$. Working over $\overline{\mathbb{Q}}$, we deduce that the eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ are algebraic numbers.

Let $\lambda \in \Sigma_f$, and let $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ be a Galois automorphism. Then, $\sigma(\lambda) \in \Sigma_g$ with $g := \sigma \circ f \circ \sigma^{-1} : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$, so $\left|\sigma(\lambda)\right| < 1$. Thus, if λ is an algebraic integer, then $\lambda = 0$. □

7.3. *An example where $\Sigma_f = \{0\}$*

Proposition 6.1 establishes that $0 \in \Lambda_f$. However, 0 does not necessarily belong to Σ_f. For example, if $f : \mathbb{C} \to \mathbb{C}$ is a polynomial with periodic critical points, then $0 \notin \Sigma_f$ (see [2]).

We now present an example of a postcritically finite rational map f for which $0 \in \Sigma_f$; this example appears in [2].

PROPOSITION 7.3. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rational map given by $f(z) = \frac{3z^2}{2z^3+1}$. Then $\Sigma_f = \{0\}$.*

Proof. The critical set of f is $\mathcal{C}_f = \{0, 1, \omega, \bar{\omega}\}$, where

$$\omega := -1/2 + \mathrm{i}\sqrt{3}/2 \quad \text{and} \quad \bar{\omega} := -1/2 - \mathrm{i}\sqrt{3}/2$$

are cube roots of unity. The postcritical set of f is $\mathcal{P}_f = \{0, 1, \omega, \bar{\omega}\}$, and f has the following postcritical dynamics.

$$0 \circlearrowright 2 \qquad 1 \circlearrowright 2 \qquad \omega \underset{2}{\overset{2}{\rightleftarrows}} \bar{\omega}\ .$$

Since $\operatorname{card}(\mathcal{P}_f) = 4$, the space $\mathcal{Q}_f$ is 1-dimensional, and there is a single eigenvalue λ. Consider

$$\boldsymbol{q} := \frac{(\mathrm{d}z)^2}{z(z^3 - 1)} \in \mathcal{P}_f, \quad \text{so that} \quad f_*\boldsymbol{q} = \lambda \boldsymbol{q}.$$

Let $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rotation $z \mapsto \omega z$. Then,

$$f \circ g(z) = f(\omega z) = \omega^2 f(z) = g^{\circ 2} \circ f(z).$$

Setting $u = g(z) = \omega z$, we have that

$$g_*\boldsymbol{q} = g_*\left(\frac{(\mathrm{d}z)^2}{z(z^3 - 1)}\right) = \frac{(\mathrm{d}u)^2/\omega^2}{u/\omega \cdot (u^3 - 1)} = \frac{\boldsymbol{q}}{\omega}.$$

As a consequence,

$$f_*(g_*\boldsymbol{q}) = f_*\left(\frac{\boldsymbol{q}}{\omega}\right) = \frac{f_*\boldsymbol{q}}{\omega} \quad \text{and} \quad g_*^{\circ 2}(f_*\boldsymbol{q}) = \frac{f_*\boldsymbol{q}}{\omega^2}.$$

It follows that

$$\frac{f_*\boldsymbol{q}}{\omega} = \frac{f_*\boldsymbol{q}}{\omega^2}$$

and since $\omega \neq \omega^2$, we necessarily have $f_*\boldsymbol{q} = 0$. □

8. *Characteristic polynomials*

In this section, the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is postcritically finite with postcritical set $\mathcal{P}_f$ and $f(\infty) = \infty$. Let d_∞ be the local degree of f at ∞, and let μ_∞ be the multiplier of f at ∞. Note that $d_\infty \geqslant 2$ if and only if $\mu_\infty = 0$.

Our goal is to compute the characteristic polynomial χ_f of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$:

$$\chi_f(\lambda) := \det(\lambda \cdot \mathrm{id} - f_*).$$

Set $X := \mathcal{P}_f \setminus \{\infty\}$ and consider the square matrix A_f whose coefficients $a_{y,x}$, indexed by $X \times X$, are defined by:

$$a_{y,x} := \sum_{w \in f^{-1}(y) \cap (\mathcal{C}_f \cup \{x\})} \mathrm{residue}\left(\frac{\mathrm{d}z}{(z-x)f'(z)}, w\right).$$

PROPOSITION 8.1. *We have that* $\det(\lambda \cdot \mathrm{I} - A_f) = \xi_f(\lambda) \cdot \chi_f(\lambda)$ *with*

$$\xi_f(\lambda) := \begin{cases} (\lambda - \mu_\infty)(\lambda - 1/d_\infty) & \text{if } \infty \in \mathcal{P}_f, \\ (\lambda - \mu_\infty)(\lambda - 1/d_\infty)(\lambda - 1/\mu_\infty) & \text{if } \infty \notin \mathcal{P}_f. \end{cases}$$

The remainder of § 8 is devoted to the proof of this proposition. We first outline a sketch of the proof.

Step 1. Instead of working in $\mathcal{Q}_f$, we introduce the following vector spaces of meromorphic quadratic differentials:

- $\mathcal{Q}_f^1$ for those with at worst simple poles at the points in $\mathcal{P}_f \cup \{\infty\}$,
- $\mathcal{Q}_f^2$ for those with at worst simple poles at points in $\mathcal{P}_f$, and at worst a double pole at ∞, and
- $\mathcal{Q}_f^3$ for those with at worst simple poles at points in $\mathcal{P}_f$, and at worst a triple pole at ∞.

We will show that each of these spaces is invariant under f_*. As subspaces,

$$\mathcal{Q}_f \subseteq \mathcal{Q}_f^1 \subset \mathcal{Q}_f^2 \subset \mathcal{Q}_f^3.$$

If $\infty \in \mathcal{P}_f$, then $\mathcal{Q}_f = \mathcal{Q}_f^1$. Otherwise $\dim \mathcal{Q}_f^1/\mathcal{Q}_f = 1$. In all cases,

$$\dim \mathcal{Q}_f^3/\mathcal{Q}_f^2 = \dim \mathcal{Q}_f^2/\mathcal{Q}_f^1 = 1.$$

Step 2. We will show that the eigenvalues of the induced endomorphisms

$$\mathcal{Q}_f^3/\mathcal{Q}_f^2 \to \mathcal{Q}_f^3/\mathcal{Q}_f^2, \quad \mathcal{Q}_f^2/\mathcal{Q}_f^1 \to \mathcal{Q}_f^2/\mathcal{Q}_f^1 \quad \text{and} \quad \mathcal{Q}_f^1/\mathcal{Q}_f \to \mathcal{Q}_f^1/\mathcal{Q}_f$$

are given in Table 1.

Step 3. We will then compute the eigenvalues of $f_* : \mathcal{Q}_f^3 \to \mathcal{Q}_f^3$ as follows. The quadratic differentials

$$\left\{ \boldsymbol{q}_x := \frac{\mathrm{d}z^2}{z-x} \right\}_{x \in \mathcal{P}_f \setminus \{\infty\}}$$

TABLE 1. *The eigenvalues of the quotient maps induced by f_* where μ_∞ is the multiplier of f at ∞, and d_∞ is the local degree of f at ∞.*

	$\mathcal{Q}_f^3/\mathcal{Q}_f^2$	$\mathcal{Q}_f^2/\mathcal{Q}_f^1$	$\mathcal{Q}_f^1/\mathcal{Q}_f$
$\infty \notin \mathcal{P}_f$, $d_\infty = 1$	μ_∞	1	$1/\mu_\infty$
$\infty \in \mathcal{P}_f$, $d_\infty = 1$	μ_∞	1	None
$\infty \in \mathcal{P}_f$, $d_\infty \geqslant 2$	0	$1/d_\infty$	None

form a basis of $\mathcal{Q}_f^3$. According to Lemma 8.5 below, the matrix of $f_*:\mathcal{Q}_f^3\to\mathcal{Q}_f^3$ in the basis $\{\boldsymbol{q}_x\}_{x\in\mathcal{P}_f\smallsetminus\{\infty\}}$ is the matrix A_f.

Step 4. For $k\in\{1,2,3\}$, let $\chi_{f,k}$ be the characteristic polynomial of $f_*:\mathcal{Q}_f^k\to\mathcal{Q}_f^k$, and let $\xi_{f,k}$ be the characteristic polynomial of $f_*:\mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}\to\mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}$, with the convention that $\mathcal{Q}_f^0:=\mathcal{Q}_f$ and $\xi_{f,1}=1$ if $\mathcal{Q}_f^1=\mathcal{Q}_f$; that is, if $\infty\notin\mathcal{P}_f$.

Since

$$\mathcal{Q}_f\subseteq\mathcal{Q}_f^1\subset\mathcal{Q}_f^2\subset\mathcal{Q}_f^3$$

are invariant by f_*, we have

$$\chi_{f,3}=\xi_{f,3}\cdot\chi_{f,2}=\xi_{f,3}\cdot\xi_{f,2}\cdot\chi_{f,1}=\xi_{f,3}\cdot\xi_{f,2}\cdot\xi_{f,1}\cdot\chi_f.$$

According to Step 2,

$$\xi_{f,3}(\lambda)=\lambda-\mu_\infty,\quad \xi_{f,2}(\lambda)=\lambda-\frac{1}{d_\infty},\quad \text{and when } \infty\notin\mathcal{P}_f,\quad \xi_{f,1}(\lambda)=\lambda-\frac{1}{\mu_\infty}.$$

Therefore, $\xi_{f,3}\cdot\xi_{f,2}\cdot\xi_{f,1}=\xi_f$ and $\chi_{f,3}=\xi_f\cdot\chi_f$. Proposition 8.1 follows from Step 3:

$$\det(\lambda\cdot\mathrm{I}-A_f)=\chi_{f,3}(\lambda)=\xi_f(\lambda)\cdot\chi_f(\lambda).$$

We now proceed with the proof working step by step.

8.1. *Invariant subspaces*

LEMMA 8.2. *The vector spaces $\mathcal{Q}_f^1$, $\mathcal{Q}_f^2$, and $\mathcal{Q}_f^3$ are invariant by f_*.*

Proof. Suppose $\boldsymbol{q}\in\mathcal{Q}_f^k$ with $k\in\{1,2,3\}$. The poles of $f_*\boldsymbol{q}$ are contained in $f\big(\mathcal{P}_f\cup\{\infty\}\big)\cup\mathcal{V}_f=\mathcal{P}_f\cup\{\infty\}$.

Assume $k=1$. Then $\mathcal{Q}_f^1\subset\mathcal{Q}(\widehat{\mathbb{C}})$ and $f_*\boldsymbol{q}\in\mathcal{Q}(\widehat{\mathbb{C}})$, so that the poles of $f_*\boldsymbol{q}$ are simple. Thus, $f_*\boldsymbol{q}\in\mathcal{Q}_f^1$. In other words, $f_*(\mathcal{Q}_f^1)\subseteq\mathcal{Q}_f^1$.

Assume $k\in\{2,3\}$ and $y\in\mathcal{P}_f$. On the one hand, if $x\in f^{-1}(y)\smallsetminus\{\infty\}$, then $2+\mathrm{ord}_x\boldsymbol{q}\geqslant 1$ and

$$\frac{2+\mathrm{ord}_x\boldsymbol{q}}{\deg_x f}-2\geqslant\frac{1}{\deg_x f}-2>-2.$$

In particular, if $y\neq\infty$, then $f_*\boldsymbol{q}$ has at worst a simple pole at y. On the other hand, if $x=\infty$, then $2+\mathrm{ord}_\infty\boldsymbol{q}\geqslant 2-k$ and since $2-k\leqslant 0$,

$$\frac{2+\mathrm{ord}_\infty\boldsymbol{q}}{\deg_\infty f}-2\geqslant 2-k-2=-k.$$

It follows that $f_*\boldsymbol{q}$ has at worst a pole of order k at ∞. □

8.2. *Extra eigenvalues*

Here, we identify the eigenvalues arising from the induced operators $f_*:\mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}\to\mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}$ for $k\in\{1,2,3\}$ (see Table 1). The case of $\mathcal{Q}_f^1/\mathcal{Q}_f$ is covered by § 6, more precisely by Lemma 6.4: if ∞ is a fixed point with multiplier μ_∞ not contained in $\mathcal{P}_f$, then $1/\mu_\infty$ is an eigenvalue of $f_*:\mathcal{Q}_f^1/\mathcal{Q}_f\to\mathcal{Q}_f^1/\mathcal{Q}_f$. We therefore only need to deal with $\mathcal{Q}_f^2/\mathcal{Q}_f^1$ and $\mathcal{Q}_f^3/\mathcal{Q}_f^2$.

Fix two vector fields $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$, where $\boldsymbol{\theta}_k$ is holomorphic near ∞ and vanishes to order $k-1$ at ∞. Let $\alpha_k:\mathcal{Q}_f^k\to\mathbb{C}$ be the form defined by

$$\alpha_k(\boldsymbol{q}):=\mathrm{residue}(\boldsymbol{q}\otimes\boldsymbol{\theta}_k,\infty).$$

This form is in the annihilator of $\mathcal{Q}_f^{k-1}$, and as such, α_k may be canonically identified with an element in the dual of the quotient $\mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}$. Therefore, if λ_k is the eigenvalue of the induced operator $f_*:\mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}\to\mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}$, then

$$\alpha_k(f_*\boldsymbol{q})=\lambda_k\ \alpha_k(\boldsymbol{q}).$$

The data presented in Table 1 are a consequence of the following lemma.

LEMMA 8.3. *If $\boldsymbol{q}\in\mathcal{Q}_f^3$, then*

$$\alpha_3(f_*\boldsymbol{q})=\mu_\infty\ \alpha_3(\boldsymbol{q}).$$

If $\boldsymbol{q}\in\mathcal{Q}_f^2$, then

$$\alpha_2(f_*\boldsymbol{q})=\frac{1}{d_\infty}\ \alpha_2(\boldsymbol{q}).$$

Proof. Assume $\boldsymbol{q}\in\mathcal{Q}_f^k$, where $k\in\{2,3\}$. Observe that for $x\in f^{-1}(\infty)\smallsetminus\{\infty\}$, the 1-form $\boldsymbol{q}\otimes f^*\boldsymbol{\theta}_k$ is holomorphic at x. Indeed, $\boldsymbol{\theta}_k$ vanishes at ∞, $f^*\boldsymbol{\theta}_k$ vanishes at x, and $\boldsymbol{q}$ has at worst a simple pole at x. Therefore,

$$\begin{aligned}\alpha_k(f_*\boldsymbol{q})\quad&:=\quad\operatorname{residue}\big((f_*\boldsymbol{q})\otimes\boldsymbol{\theta}_k,\infty\big)\\&\underset{\text{Lemma 3.1}}{=}\sum_{x\in f^{-1}(\infty)}\operatorname{residue}(\boldsymbol{q}\otimes f^*\boldsymbol{\theta}_k,x)=\operatorname{residue}(\boldsymbol{q}\otimes f^*\boldsymbol{\theta}_k,\infty).\end{aligned}$$

Case 1. If $k=3$, then $\boldsymbol{\theta}_3$ vanishes to order 2 at ∞. It follows from Lemma 8.4 below that $f^*\boldsymbol{\theta}_3-\mu_\infty\boldsymbol{\theta}_3$ vanishes to order 3 at ∞. Since $\boldsymbol{q}$ has at worst a triple pole at ∞, we have

$$\alpha_3(f_*\boldsymbol{q})=\operatorname{residue}(\boldsymbol{q}\otimes f^*\boldsymbol{\theta}_3,\infty)=\operatorname{residue}(\boldsymbol{q}\otimes\mu_\infty\boldsymbol{\theta}_3,\infty)=\mu_\infty\alpha_3(\boldsymbol{q}).$$

Case 2. If $k=2$, then $\boldsymbol{\theta}_2$ vanishes to order 1 at ∞. It follows from Lemma 8.4 below that $f^*\boldsymbol{\theta}_2-\frac{1}{d_\infty}\boldsymbol{\theta}_2$ vanishes to order 2 at ∞. Since $\boldsymbol{q}$ has at worst a double pole at ∞, we have

$$\alpha_2(f_*\boldsymbol{q})=\operatorname{residue}(\boldsymbol{q}\otimes f^*\boldsymbol{\theta}_2,\infty)=\operatorname{residue}\left(\boldsymbol{q}\otimes\frac{1}{d_\infty}\boldsymbol{\theta}_2,\infty\right)=\frac{1}{d_\infty}\alpha_2(\boldsymbol{q}).$$

□

LEMMA 8.4. *Let f be a germ of a holomorphic map fixing a point x with multiplier μ and local degree d. Let $\boldsymbol{\theta}$ be a germ of a holomorphic vector field vanishing at x with order m.*

- *If $d=1$, then $f^*\boldsymbol{\theta}-\mu^{m-1}\boldsymbol{\theta}$ vanishes to order $m+1$ at x.*
- *If $m=1$, then $f^*\boldsymbol{\theta}-\frac{1}{d}\boldsymbol{\theta}$ vanishes to order $m+1$ at x.*
- *If $d\geqslant 2$ and $m\geqslant 2$, then $f^*\boldsymbol{\theta}$ vanishes to order $m+1$ at x.*

Proof. Let ζ be a local coordinate vanishing at x. We may write

$$\zeta\circ f=a\zeta^d\cdot\big(1+\mathcal{O}(\zeta)\big)\quad\text{and}\quad\boldsymbol{\theta}=b\zeta^m\frac{\mathrm{d}}{\mathrm{d}\zeta}\cdot\big(1+\mathcal{O}(\zeta)\big)$$

with $a\neq 0$ and $b\neq 0$. In addition, if $d=1$, then $a=\mu$. Then,

$$f^*\boldsymbol{\theta}=\frac{ba^m\zeta^{dm}}{da\zeta^{d-1}}\frac{\mathrm{d}}{\mathrm{d}\zeta}\cdot\big(1+\mathcal{O}(\zeta)\big)=\frac{a^{m-1}}{d}\zeta^{(d-1)(m-1)}\boldsymbol{\theta}\cdot\big(1+\mathcal{O}(\zeta)\big).$$

□

8.3. *The matrix A_f*

Lemma 8.5. *The matrix of $f_* : \mathcal{Q}^3_f \to \mathcal{Q}^3_f$ in the basis $\{\boldsymbol{q}_x\}_{x\in\mathcal{P}_f\smallsetminus\{\infty\}}$ is A_f.*

Proof. Since $\{\boldsymbol{q}_y\}_{y\in\mathcal{P}_f\smallsetminus\{\infty\}}$ forms a basis of $\mathcal{Q}^3_f$, we may write

$$f_*\boldsymbol{q}_x = \sum_{y\in\mathcal{P}_f\smallsetminus\{\infty\}} f_{y,x}\cdot\boldsymbol{q}_y.$$

We need to show that $f_{y,x} = a_{y,x}$ for all $x, y \in \mathcal{P}_f\smallsetminus\{\infty\}$. We shall apply Lemma 3.1 with

$$\boldsymbol{q} := \boldsymbol{q}_x \quad\text{and}\quad \boldsymbol{\theta} := \frac{\mathrm{d}}{\mathrm{d}z}.$$

Fix $y_0 \in \mathcal{P}_f\smallsetminus\{\infty\}$. Note that $\boldsymbol{\theta}$ is holomorphic at y_0, and for $y \neq y_0$, $\boldsymbol{q}_y$ is holomorphic at y_0. In addition, $\boldsymbol{q}_{y_0}\otimes\boldsymbol{\theta} = \frac{\mathrm{d}z}{z-y_0}$. Therefore,

$$\begin{aligned} f_{y_0,x} &= \operatorname{residue}\big((f_*\boldsymbol{q}_x)\otimes\boldsymbol{\theta}, y_0\big) \\ &\underset{\text{Lemma 3.1}}{=} \sum_{w\in f^{-1}(y_0)} \operatorname{residue}(\boldsymbol{q}_x\otimes f^*\boldsymbol{\theta}, w) \\ &= \sum_{w\in f^{-1}(y_0)\cap(\mathcal{C}_f\cup\{x\})} \operatorname{residue}\left(\frac{\mathrm{d}z}{(z-x)f'(z)}, w\right) = a_{y_0,x}. \end{aligned}$$

For the third equality, we use that

$$\boldsymbol{q}_x\otimes f^*\boldsymbol{\theta} = \frac{\mathrm{d}z}{(z-x)f'(z)}$$

is holomorphic outside of $\mathcal{C}_f\cup\{x\}$. □

9. *Periodic unicritical polynomials*

From now on, we shall restrict our study to the case of unicritical polynomials. Any such polynomial is conjugate by an affine map to a polynomial of the form

$$f_c(z) := z^D + c \quad\text{with}\quad c\in\mathbb{C}.$$

The map f_c has a critical point at $z_0 = 0$ and a critical value at $f_c(0) = c$.

Such a polynomial is postcritically finite if and only if the critical point 0 is either periodic or preperiodic. We will restrict our study to the periodic case, and abusing terminology, we shall say that f_c *is periodic of period* m if 0 is periodic of period m for f_c.

9.1. *Families of unicritical polynomials*

Before studying the corresponding sets Σ_{f_c}, we introduce some subsets of parameter space. The Multibrot set $\mathcal{M}_D$ is defined as

$$\mathcal{M}_D := \Big\{c\in\mathbb{C} \mid \text{the sequence } \big(f_c^{\circ n}(0)\big)_{n\geqslant 1} \text{ is bounded}\Big\}.$$

The set $\mathcal{M}_2$ is called the *Mandelbrot set.*

Note that the set of parameters $c\in\mathbb{C}$ such that f_c has a periodic critical point is contained in the interior of $\mathcal{M}_D$. In addition, each component U of the interior of $\mathcal{M}_D$ contains at most one parameter c such that f_c is periodic; in that case, c is called the center of U.

The boundary of $\mathcal{M}_D$ is contained in the closure of the set of centers. Indeed, for any $c_0\in\partial\mathcal{M}_D$ and any small neighborhood U of c_0, the family $(h_n : U\to\mathbb{C})_{n\geqslant 3}$ defined by

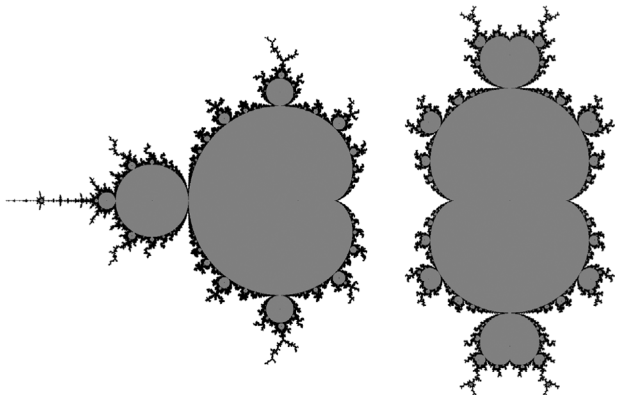


FIGURE 2. *Left: the Mandelbrot set. Right: the Multibrot set $\mathcal{M}_3$.*

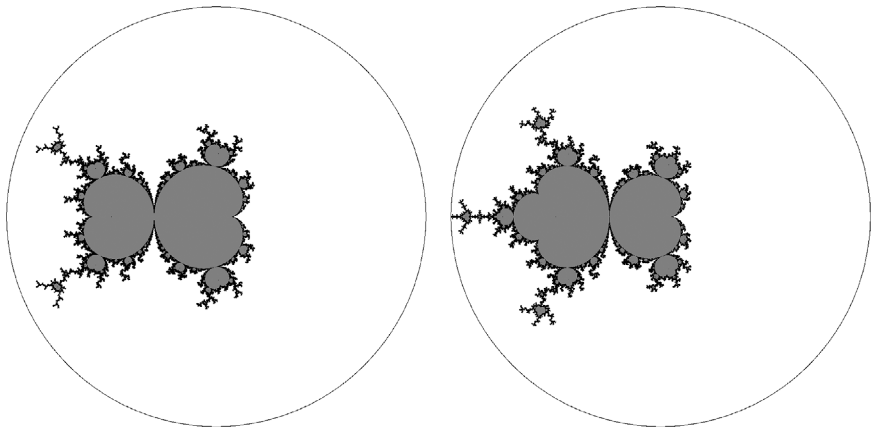


FIGURE 3. *Left: the set $\mathcal{N}_3$. Right: the set $\mathcal{N}_4$. Both are contained in the closed disk $|b| \leqslant 2$.*

$h_n(c) := f_c^{\circ n}(0)/c$ is uniformly bounded on $\mathcal{M}_D$ and converges to infinity outside $\mathcal{M}_D$. So it is not normal and cannot avoid 0 and 1 by Montel's theorem. When $h_n(c) = 0$, then $f_c^{\circ n}(0) = 0$; when $h_n(c) = 1$, then $f_c^{\circ(n-1)}(0) = 0$.

Observe that $\mathcal{M}_D$ has a symmetry of order $D-1$. Indeed, for $c \neq 0$, the linear map $z \mapsto w := z/c$ conjugates the polynomial f_c to the polynomial

$$g_b : w \mapsto bw^D + 1 \quad \text{with} \quad b := c^{D-1}.$$

The polynomial g_b has a critical point at 0 and a critical value at $g_b(0) = 1$. We set

$$\mathcal{N}_D := \Big\{ b \in \mathbb{C} \mid \text{the sequence } \big(g_b^{\circ n}(0)\big)_{n \geqslant 1} \text{ is bounded} \Big\}.$$

Then, $c \in \mathcal{M}_D$ if and only if $c^{D-1} \in \mathcal{N}_D$.

If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, the filled-in Julia set K_f is defined as

$$K_f := \Big\{ z \in \mathbb{C} \mid \text{the sequence } \big(f^{\circ n}(z)\big)_{n \geqslant 1} \text{ is bounded} \Big\}.$$

The following estimates are easily established.

LEMMA 9.1. *If $b \in \mathcal{N}_D$, then $|b| < 2$.*

COROLLARY 9.2. *If $c \in \mathcal{M}_D$, then $|c|^{D-1} \leqslant 2$.*

LEMMA 9.3. *If $c \in \mathcal{M}_D$ and $z \in K_{f_c}$, then $|z|^{D-1} \leqslant 2$.*

9.2. *Gleason polynomials*

For $n \geqslant 0$, let $G_n \in \mathbb{Z}[c]$ be defined by

$$G_n(c) := f_c^{\circ n}(0).$$

Alternatively, the polynomials $G_n \in \mathbb{Z}[c]$ may be defined recursively by:

$$G_0 := 0 \quad \text{and} \quad G_n(c) := G_{n-1}^D(c) + c.$$

Then, $G_n(c) = 0$ if and only if c is a center of period p dividing n.

EXAMPLE. For $D = 2$, we have

$$G_1(c) = c, \quad G_2(c) = c^2 + c, \quad G_3(c) = c^4 + 2c^3 + c^2 + c.$$

Note that G_n is a monic polynomial. The following result is proven in [3] for $D = 2$. The proof easily generalizes to arbitrary degrees.

LEMMA 9.4 (Gleason). *For each $n \geqslant 0$, the polynomial G_n has simple roots.*

Proof. For each $n \geqslant 1$,

$$G_n' = DG_{n-1}^{D-1}G_{n-1}' + 1 \equiv 1 \mod D.$$

In addition, G_n is monic, and so, the resultant of G_n and G_n' is equal to $1 \mod D$. Indeed, this resultant is equal to the determinant of a matrix that modulo D is triangular with coefficients equal to 1 on the diagonal. In particular, it does not vanish. □

As a corollary, we deduce that for $m \geqslant 1$, there exists a (unique) monic polynomial $H_m \in \mathbb{Z}[c]$, such that for $n \geqslant 1$

$$G_n = \prod_{m|n} H_m.$$

The roots of H_m are exactly the centers of period m. The monomial of least degree of $G_n(c)$ is c with coefficient 1. Since $H_1(c) = c$, we see that for $m \geqslant 2$ the constant coefficient of H_m is 1. In particular, with the exception of $c = 0$, centers are algebraic units.

EXAMPLE. For $D = 2$, we have $G_4 = H_1H_2H_4$ with

$$H_1(c) = c, \quad H_2(c) = c + 1 \quad \text{and} \quad H_4(c) = c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1.$$

We shall also use the following result due to Bjorn Poonen.

LEMMA 9.5 (Poonen). *For $m \neq n$,* $\mathrm{resultant}(H_m, H_n) = \pm 1$.

Proof. Assume $n > m$. It is not hard to see by induction on $k \geqslant 1$, that

$$G_{m+k} \equiv G_k \mod G_m^D.$$

This implies that, $G_{mn} \equiv G_m \mod G_m^D$. Since $m < n$, G_mH_n divides G_{mn}. So, there are polynomials $A \in \mathbb{Z}[c]$ and $B \in \mathbb{Z}[c]$ such that

$$AG_mH_n = G_{mn} = G_m + BG_m^D.$$

Dividing by G_m yields $AH_n - BG_m^{D-1} = 1$. It follows that H_n and H_m are relatively prime in $\mathbb{Z}[c]$ and $\mathrm{resultant}(H_m, H_n) = \pm 1$. □

REMARK. For $m \geqslant 2$, we have $H_m(c) = J_m(c^{D-1})$ for some polynomial $J_m \in \mathbb{Z}[b]$. It might be tempting to conjecture that for all $D \geqslant 2$ and all $m \geqslant 2$, the polynomial J_m is irreducible. However, for $D = 7$, J_3 is reducible:

$$J_3(b) = (b^2 + b + 1)(b^6 + 6b^5 + 14b^4 + 15b^3 + 6b^2 + 1).$$

This is further discussed in [**1**].

9.3. *Periodic points*

PROPOSITION 9.6. *Let f be a periodic unicritical polynomial of degree D. If $\lambda \in \Lambda_f \setminus \{0\}$, then $D\lambda$ is an algebraic unit and $\frac{1}{2D} \leqslant |\lambda| < 1$.*

Proof. According to Proposition 6.2, $\lambda \in \Lambda_f \setminus \{0\}$ if and only if $1/\lambda^n$ is the multiplier of a cycle of period n not contained in $\mathcal{P}_f$, that is, a repelling cycle of f_c. In particular, $|\lambda| < 1$.

Assume that the critical point of f is periodic of period m. Conjugating f with an affine map, we may assume that

$$f(z) = f_c(z) := z^D + c \quad \text{with} \quad H_m(c) = 0.$$

Now, let $z_1 \mapsto z_2 \mapsto \cdots \mapsto z_n \mapsto z_1$ be a cycle of period n. The multiplier of the cycle is

$$\mu = D^n (z_1 z_2 \cdots z_n)^{D-1}.$$

According to Lemma 9.3, $|z_j|^{D-1} \leqslant 2$, so that $|\mu| \leqslant (2D)^n$. Thus, it suffices to prove that the points z_j are algebraic units.

Let us first assume that m does not divide n, so that $f_c^{\circ n}(0) \neq 0$. The points z_j are roots of the polynomial $f_c^{\circ n}(z) - z \in \mathbb{Z}[c, z]$. Denote by R_z the polynomial $f_c^{\circ n}(z) - z$ considered as a polynomial of the variable c with coefficients in $\mathbb{Z}[z]$ and set

$$S(z) := \mathrm{resultant}\big(H_m, R_z\big) = \prod_{H_m(c)=0} \big(f_c^{\circ n}(z) - z\big).$$

Note that, as a product of monic polynomials, S is a monic polynomial. In addition, $f_c^{\circ n}(0) = G_n(c)$ and the constant coefficient of S is

$$S(0) = \prod_{H_m(c)=0} \big(f_c^{\circ n}(0) - 0\big) = \prod_{H_m(c)=0} G_n(c) = \mathrm{resultant}(H_m, G_n).$$

According to Lemma 9.5, since m does not divide n, this resultant is equal to ± 1. This shows that the points z_j are algebraic units.

Let us now assume that m divides n. Then, the constant coefficient of $f_c^{\circ n}(z) - z$ vanishes and

$$g_c(z) := \frac{f_c^{\circ n}(z) - z}{z}$$

is a monic polynomial with constant coefficient $g_c(0) = -1$. Denote by R_z the polynomial $g_c(z)$ considered as a polynomial of the variable c with coefficients in $\mathbb{Z}[z]$ and set

$$S(z) := \mathrm{resultant}\big(H_m, R_z\big) = \prod_{H_m(c)=0} g_c(z).$$

Again, S is a monic polynomial and its constant coefficient is

$$S(0) = \prod_{H_m(c)=0} g_c(0) = \pm 1.$$

Thus, the points z_j are algebraic units. □

9.4. *Characteristic polynomials*

We assume that c is a center of period $m \geqslant 3$ and $f := f_c$, so that $\dim(\mathcal{Q}_f) = m-2 \geqslant 1$.

Note that $f(\infty) = \infty \in \mathcal{P}_f$. We denote by χ_f the characteristic polynomial of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ and for $k \in \{2,3\}$, we denote by $\chi_{f,k}$ the characteristic polynomial of $f_* : \mathcal{Q}_f^k \to \mathcal{Q}_f^k$. The local degree of f at ∞ is D and the multiplier of f at ∞ is 0. According to §8, we have

$$\chi_{f,3}(\lambda) = \lambda \cdot \chi_{f,2}(\lambda) = \lambda\left(\lambda - \frac{1}{D}\right)\chi_f(\lambda).$$

For $n \in \mathbb{Z}/m\mathbb{Z}$, set

$$\zeta_n := f^{\circ n}(0), \quad \delta_n := f'(\zeta_n) = D\zeta_n^{D-1}.$$

Set

$$\Delta_1 := \delta_1, \quad \Delta_2 := \delta_1\delta_2, \quad \ldots, \quad \Delta_{m-1} := \delta_1\delta_2\cdots\delta_{m-1}.$$

PROPOSITION 9.7. *We have that*

$$\chi_{f,2}(\lambda) = \lambda^{m-1} + \frac{1}{\Delta_1}\lambda^{m-2} + \cdots + \frac{1}{\Delta_{m-2}}\lambda + \frac{1}{\Delta_{m-1}}.$$

Proof. For $n \in \mathbb{Z}/m\mathbb{Z}$, set

$$\boldsymbol{q}_n := \frac{(\mathrm{d}z)^2}{z-\zeta_n}.$$

The matrix A_f of $f_* : \mathcal{Q}_f^3 \to \mathcal{Q}_f^3$ in the basis $\{\boldsymbol{q}_n\}_{n\in\mathbb{Z}/m\mathbb{Z}}$ is provided by § 8. We have

$$f_*\boldsymbol{q}_n = \begin{cases} 0 & \text{if } n = 0 \text{ and} \\ \dfrac{1}{\delta_n}\boldsymbol{q}_{n+1} - \dfrac{1}{\delta_n}\boldsymbol{q}_1 & \text{if } n \neq 0. \end{cases}$$

and so

$$A_f := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_{m-1} \\ 0 & -a_1 & -a_2 & \cdots & -a_{m-2} & -a_{m-1} \\ 0 & a_1 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & a_{m-2} & 0 \end{bmatrix} \quad \text{with} \quad a_n := \frac{1}{\delta_n}.$$

The characteristic polynomial of A_f is $\chi_{f,3}(\lambda) = \lambda\chi_{f,2}(\lambda)$, so that

$$\begin{aligned}\chi_{f,2}(\lambda) &= (\lambda + a_1)\lambda^{m-2} + a_1a_2\big(\lambda^{m-3} + a_3(\lambda^{m-4} + \cdots)\big) \\ &= \lambda^{m-1} + \frac{1}{\Delta_1}\lambda^{m-2} + \cdots + \frac{1}{\Delta_{m-2}}\lambda + \frac{1}{\Delta_{m-1}}.\end{aligned}$$

□

It will be convenient to work with the polynomial $\chi_{f,2}/\chi_{f,2}(0)$ which has the same zeros as $\chi_{f,2}$. Setting

$$\Delta_0 := 1, \quad \Delta_{-1} := \delta_{m-1}, \quad \Delta_{-2} := \delta_{m-1}\delta_{m-2}, \quad \ldots, \quad \Delta_{-(m-1)} := \delta_{m-1}\delta_{m-2}\cdots\delta_1,$$

we get that for $n \in [\![1, m-1]\!]$,

$$\Delta_n \cdot \Delta_{-(m-1-n)} = \Delta_{m-1} = \frac{1}{\chi_{f,2}(0)},$$

so that

$$\frac{\chi_{f,2}(\lambda)}{\chi_{f,2}(0)} = 1 + \Delta_{-1}\lambda + \Delta_{-2}\lambda^2 + \cdots + \Delta_{-(m-1)}\lambda^{m-1}.$$

The polynomials $G_n \in \mathbb{Z}[c]$ defined by $G_n(c) = f_c^{\circ n}(0)$ have simple roots (see Lemma 9.4). This is related to the fact that for a postcritically finite rational map f which is not a flexible Lattès map, $1 \notin \Sigma_f$ (see Proposition 4.4).

Proposition 9.8. *If c is a center of period $m \geqslant 3$ and $f := f_c$, then*

$$G'_m(c) = (1-D)\frac{\chi_f(1)}{\chi_f(0)}.$$

Proof. We have $G_1(c) = c$ and $G_{n+1} = G_n^D(c) + c$, so that

$$G'_m = 1 + D\, G_{m-1}^{D-1} G'_{m-1} = 1 + \delta_{m-1} G'_{m-1}.$$

Since $G'_1 = 1$, we have

$$\begin{aligned} G'_m(c) &= 1 + \delta_{m-1} \cdot \big(1 + \delta_{m-2} \cdot (1 + \cdots (1 + \delta_1))\big) \\ &= 1 + \delta_{m-1} + \delta_{m-1}\delta_{m-2} + \cdots + \delta_{m-1}\delta_{m-2}\cdots\delta_1 \\ &= 1 + \Delta_{-1} + \Delta_{-2} + \cdots + \Delta_{-(m-1)} = \chi_{f,2}(1)/\chi_{f,2}(0). \end{aligned}$$

Now,

$$\chi_{f,2}(0) = -\frac{1}{D}\chi_f(0) \quad \text{and} \quad \chi_{f,2}(1) = \left(1 - \frac{1}{D}\right)\chi_f(1)$$

so that

$$\frac{\chi_{f,2}(1)}{\chi_{f,2}(0)} = (1-D)\frac{\chi_f(1)}{\chi_f(0)}.$$

□

9.5. *Algebraic units*

Theorem 9.9. *Let f be a periodic unicritical polynomial of degree D. If λ is an eigenvalue of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$, then $D\lambda$ is an algebraic unit.*

Proof. The polynomial f is conjugate to some polynomial f_c where c is a center of period m; that is, $H_m(c) = 0$. The eigenvalues of f_* are those of $(f_c)_*$.

For $n \in [\![0, m-1]\!]$, let $\Gamma_n \in \mathbb{Z}[c]$ be the polynomial defined by

$$\Gamma_n := \prod_{k=1}^{n} G_{m-k}^{D-1},$$

where as usual, an empty product is equal to 1. Then

$$\Delta_{-n} = D^n \prod_{k=1}^{n} \zeta_{m-k}^{D-1} = D^n \prod_{k=1}^{n} \big(f_c^{\circ(m-k)}(0)\big)^{D-1} = D^n \Gamma_n(c).$$

As a consequence,

$$\frac{\chi_{f_c,2}(\lambda)}{\chi_{f_c,2}(0)} = \sum_{n=0}^{m-1} D^n \Gamma_n(c)\lambda^n = R(c, D\lambda),$$

where $R \in \mathbb{Z}[c,\nu]$ is defined by

$$R(c,\nu) := \sum_{n=0}^{m-1} \Gamma_n(c)\nu^n.$$

We shall denote by R_ν the polynomial $R(c,\nu)$ considered as a polynomial in the variable c with coefficients in $\mathbb{Z}[\nu]$. Let $S_m \in \mathbb{Z}[\nu]$ be defined by

$$S_m(\nu) := \text{resultant}(H_m, R_\nu) = \prod_{H_m(c)=0} R(c,\nu) = \prod_{H_m(c)=0} \frac{\chi_{f_c,2}(\nu/D)}{\chi_{f_c,2}(0)}. \tag{1}$$

If $\lambda \in \Sigma_{f_c}$ with $H_m(c) = 0$, then $\nu := D\lambda$ is a root of S_m.

On the one hand, the constant coefficient of S_m is $S_m(0) = 1$. On the other hand, the leading monomial of $R_m(c,\nu)$ considered as a polynomial of ν is $\Gamma_{m-1}(c)\nu^{m-1}$, so that the leading coefficient of S_m is:

$$\prod_{H_m(c)=0} \Gamma_{m-1}(c) = \text{resultant}(H_m, \Gamma_{m-1}) = \text{resultant}\left(H_m, \prod_{k=1}^{m-1} G_{m-k}^{D-1}\right).$$

By Lemma 9.5, this resultant is equal to ± 1. It follows that the roots of S_m are algebraic units. $\square$

We know that $\chi^2_{f_c}(\lambda) = (\lambda - 1/D)\chi_{f_c}(\lambda)$, so that the factor $(\nu - 1)$ appears $\deg(H_m)$ times in the product (1) defining S_m. In addition, if $c_1^{D-1} = c_2^{D-1}$, then the polynomials f_{c_1} and f_{c_2} are conjugate. So, each eigenvalue is counted $D-1$ times in the product. Thus, there is a (unique) polynomial $\Upsilon_m \in \mathbb{Z}[\nu]$ with constant coefficient 1 such that

$$S_m(\nu) = (1-\nu)^{\deg(H_m)} \cdot \Upsilon_m^{D-1}(\nu).$$

It would be interesting to prove that the roots of Υ_m are simple. For example, when this is the case, $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ is diagonalizable.

9.6. *The case $m = 3$*

We will prove that the roots of Υ_3 are simple.

LEMMA 9.10. *For all $D \geqslant 2$,*

$$\Upsilon_3(\nu) = (\nu+1)^{D+1} - \nu^D.$$

Proof. We have that $G_3(c) = (c^D + c)^D + c$, so that

$$H_3(c) = A(c^{D-1}) \quad \text{with} \quad A(b) = b(b+1)^D + 1.$$

In addition, an elementary computation shows that

$$R(c,\nu) = (1-\nu)B_\nu(c^{D-1}) \quad \text{with} \quad B_\nu(b) = 1 - \nu b^2(b+1)^{D-1}.$$

Then,

$$\Upsilon_3(\nu) = \text{resultant}(A, B_\nu).$$

Observe that

$$\nu b A(b) + (b+1)B_\nu(b) = (\nu+1)b + 1.$$

Note that A is monic of degree $D+1$. We therefore have

$$\begin{aligned}(\nu+1)^{D+1}\cdot \operatorname{resultant}\left(A(b), b+\frac{1}{\nu+1}\right) &= \operatorname{resultant}\big(A(b), (\nu+1)b+1\big)\\ &= \operatorname{resultant}\big(A(b), (b+1)B_\nu(b)\big)\\ &= \operatorname{resultant}\big(A(b), b+1\big)\cdot \Upsilon_3(\nu).\end{aligned}$$

As a consequence,

$$\Upsilon_3(\nu) = (\nu+1)^{D+1}\cdot \frac{A\big(-1/(\nu+1)\big)}{A(-1)} = (\nu+1)^{D+1} - \nu^D.$$

□

Lemma 9.11. *For all $D \geqslant 2$, the roots of Υ_3 are simple.*

Proof. Note that $\nu\Upsilon_3'(\nu) - D\Upsilon_3(\nu) = (1+\nu)^D(\nu - D)$. So, if Υ_3 and Υ_3' had a common root, this would be either -1 or D. None of those are roots of Υ_3. □

9.7. *Spectral gap*

For $m \geqslant 3$, let $\Sigma(D,m)$ be the union of the sets of eigenvalues in Σ_f for all unicritical polynomials f of degree D which are periodic of period m. Set

$$\Sigma(D) := \bigcup_{m\geqslant 3} \Sigma(D,m).$$

The following proposition completes the proof of Theorem 1.4.

Proposition 9.12 (Spectral gap). *If $\lambda \in \Sigma(D)$, then*

$$\frac{1}{4D} < |\lambda| < 1.$$

Proof. Assume that c is a center of period m and $f = f_c$. Assume $\lambda \in \Sigma_f$. According to Corollary 4.3, we have that $|\lambda| < 1$. We must show that $|\lambda| > 1/(4D)$.

Let us recall that

$$0 = \frac{\chi_{f,2}(\lambda)}{\chi_{f,2}(0)} = 1 + \Delta_{-1}\lambda + \Delta_{-2}\lambda^2 + \cdots + \Delta_{-(m-1)}\lambda^{m-1}$$

with

$$\Delta_0 := 1, \quad \Delta_{-1} := \delta_{m-1}, \quad \Delta_{-2} := \delta_{m-1}\delta_{m-2}, \quad \ldots, \quad \Delta_{-(m-1)} := \delta_{m-1}\delta_{m-2}\cdots\delta_1.$$

Lemma 9.13. *For all $n \in \mathbb{Z}/m\mathbb{Z}$, we have that $|\delta_n| \leqslant 2D$.*

Proof. Set $\zeta_n := f_c^{\circ n}(0)$. Note that $c \in \mathcal{M}_D$ and $\zeta_n \in K_{f_c}$. According to Lemma 9.3, we have $|\zeta_n|^{D-1} \leqslant 2$ for all $n \in \mathbb{Z}/m\mathbb{Z}$, so

$$|\delta_n| = D|\zeta_n|^{D-1} \leqslant 2D.$$

□

If $|z| \leqslant \frac{1}{4D}$, then for all $k \in [\![1, m-1]\!]$,

$$|\Delta_{-k} z^k| < \frac{(2D)^k}{(4D)^k} = \frac{1}{2^k},$$

so

$$\left|\frac{\chi_{f,2}(z)}{\chi_{f,2}(0)}\right| \geqslant 1 - \sum_{k=1}^{m-1} \frac{1}{2^k} > 0.$$

Since $\chi_{f,2}(\lambda) = 0$, we necessarily have $|\lambda| > \frac{1}{4D}$. □

9.8. *Equidistribution*

This section is devoted to the proof of Theorem 1.5.

In order to prove this result, we will show that when $r \in [r_D, 1]$, there exists a sequence of centers c_n such that the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $|z| = r$ as $n \to \infty$. This means the following.

Given a polynomial $P \in \mathbb{C}[z]$, we denote by $\mathfrak{m}_P$ the probability measure

$$\mathfrak{m}_P := \frac{1}{\deg(P)} \sum_{x \in \mathbb{C}} \operatorname{ord}_x(P),$$

where $\operatorname{ord}_x(P)$ is the order of vanishing of P at x.

Assume that $(P_n \in \mathbb{C}[z])_{n \geqslant 0}$ is a sequence of polynomials. We say that as $n \to \infty$, the roots of P_n equidistribute according to a probability measure $\mathfrak{m}$ if the sequence of probability measures $(\mathfrak{m}_{P_n})_{n \geqslant 0}$ converges weakly to $\mathfrak{m}$.

We say that as $n \to \infty$, the roots of P_n equidistribute on a Euclidean circle if the roots of P_n equidistribute according to the normalized 1-dimensional Lebesgue measure on this circle.

Example. As $n \to \infty$, the roots of the polynomials $z^n - 1$ equidistribute on the unit circle $S^1 := \{|z| = 1\}$.

Proposition 9.14. *Suppose that the critical point of $f_{c_0}(z) = z^D + c_0$ is preperiodic to a repelling fixed point β_0 of multiplier μ. Then, there exists a sequence of centers $c_n \in \mathcal{M}_D$ converging to c_0 such that as $n \to \infty$, the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $\big\{|\lambda| = 1/|\mu|\big\}$.*

Remark. A similar argument establishes that when the critical point of $f_c : z \mapsto z^D + c$ is preperiodic to a repelling cycle of multiplier μ and period m, then there exists a sequence of centers $c_n \in \mathcal{M}_D$ converging to c such that as $n \to \infty$, the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $\big\{|\lambda| = 1/\sqrt[m]{|\mu|}\big\}$. We will not use this fact.

Proof. Fix $r_1 < r_0 < |\beta_0|$ and $\epsilon > 0$ so that f_c has an inverse branch

$$g_c : D(\beta_0, r_0) \to D(\beta_0, r_1)$$

for every $c \in D(c_0, \epsilon)$. For $c \in D(c_0, \epsilon)$, let $\beta(c)$ be the unique (repelling) fixed point of f_c in $D(\beta_0, r_0)$. The map β is holomorphic on $D(c_0, \epsilon)$ and $\beta(c_0) = \beta_0$.

Choose a point $z_{-n_0} \in D(\beta_0, r_0)$ in the backward orbit of 0 of f_{c_0}; that is, $f_{c_0}^{\circ n_0}(z_{-n_0}) = 0$ with $n_0 > 0$. Since 0 is not periodic (it is preperiodic to β_0), $(f_{c_0}^{\circ n_0})'(z_{-n_0}) \neq 0$. Thus, taking $\epsilon > 0$ closer to 0 if necessary, we may assume that there is a holomorphic function $\zeta_{-n_0} : D(c_0, \epsilon) \to D(\beta_0, r_0)$ defined implicitly by $f_c^{\circ n_0}(\zeta_{-n_0}(c)) = 0$. Set

$$\zeta_j(c) := f_c^{\circ(n_0+j)}(\zeta_{-n_0}(c)) \text{ for } j \geqslant -n_0$$

and

$$\zeta_j(c) := g_c^{\circ(-n_0-j)}(\zeta_{-n_0}(c)) \text{ for } j \leqslant -n_0.$$

We have specified a distinguished orbit for f_c

$$\cdots \mapsto \zeta_{-2}(c) \mapsto \zeta_{-1}(c) \mapsto \zeta_0(c) = 0 \mapsto \zeta_1(c) = c \mapsto \zeta_2(c) \mapsto \cdots$$

which converges to $\beta(c)$ in backward time.

Let k_0 be the preperiod of 0 to β_0. As $n \to \infty$, the sequence $(\zeta_{k_0} - \zeta_{-n})$ converges uniformly on $D(c_0, \epsilon)$, to the function $\zeta_{k_0} - \beta$. This function vanishes at c_0, but it is not identically 0. In particular, we can find a sequence (c_n) such that $c_n \to c_0$ as $n \to \infty$, and $\zeta_{k_0}(c_n) = \zeta_{-n}(c_n)$. Then, c_n is a center of period $m_n := n + k_0$.

Lemma 9.15. *There exists a constant K such that for all sufficiently large n and all $1 \leqslant j \leqslant m_n - 1$*

$$\frac{1}{K} < \left|\frac{\Delta_j(c_n)}{\mu^j}\right| \leqslant K \quad \text{and} \quad \frac{1}{K} < \left|\frac{\Delta_{-j}(c_n)}{\mu^j}\right| \leqslant K.$$

Proof. For $j \in \mathbb{Z}$, let $\delta_j(c)$ be the derivative of f_c at $\zeta_j(c)$. Taking ϵ closer to 0 if necessary, we may assume that there is a constant $K_0 > 0$ so that

$$\frac{1}{K_0} < \left|\frac{\delta_j(c)}{\mu}\right| < K_0 \tag{2}$$

for $c \in D(c_0, \epsilon)$ and $j \in \{-n_0, \ldots, -1\} \cup \{1, \ldots, k_0\}$.

Let d be the order of $\zeta_{k_0} - \beta$ at c_0 (it is true that $d = 1$ [**3**], but we will not require this fact). Now there is a constant K_1 so that

$$K_1|c - c_0|^d < |\zeta_{k_0}(c) - \beta(c)|. \tag{3}$$

Since g_c maps $D(\beta_0, r_0)$ to $D(\beta_0, r_1)$, by the Schwarz lemma, there exists $\kappa < 1$ such that for all $z \in D(\beta_0, r_0)$ and all $c \in D(c_0, \epsilon)$,

$$|g_c(z) - \beta(c)| < \kappa|z - \beta(c)|.$$

Then, since $\zeta_{-n_0}(c) \in D(\beta_0, r_0)$ and $\zeta_{-n}(c) = g_c^{\circ(n-n_0)}\big(\zeta_{-n_0}(c)\big)$, we have

$$|\zeta_{-n}(c) - \beta(c)| < r_0\kappa^{n-n_0}. \tag{4}$$

Set

$$r_n := \left(\frac{r_0\kappa^n}{K_1\kappa^{n_0}}\right)^{1/d} = K_2\kappa^{n/d} \quad \text{with} \quad K_2 := \left(\frac{r_0}{K_1\kappa^{n_0}}\right)^{1/d}. \tag{5}$$

For $c \in D(c_0, \epsilon) \smallsetminus D(c_0, r_n)$, so that $r_n \leqslant |c - c_0|$, we have

$$|\zeta_{-n}(c) - \beta(c)| \underset{(4)}{<} r_0\frac{\kappa^n}{\kappa^{n_0}} \underset{(5)}{=} K_1 r_n^d \leqslant K_1|c - c_0|^d \underset{(3)}{<} |\zeta_{k_0}(c) - \beta(c)|,$$

so that the leftmost quantity cannot be equal to the rightmost quantity. As a consequence, for n large enough,

$$|c_n - c_0| < r_n.$$

For $j \leqslant -n_0$ and $c \in D(c_0, \epsilon)$, the point $\zeta_j(c)$ belongs to $D(\beta_0, r_0)$. Since $\delta_j(c) = D\zeta_j^{D-1}(c)$, the branch of

$$c \mapsto \log\frac{\delta_j(c)}{\delta_j(c_0)},$$

which vanishes at c_0 is bounded by some constant K_3. According to the Schwarz lemma,

$$\left|\log\frac{\delta_j(c_n)}{\delta_j(c_0)}\right| < \frac{K_3 r_n}{\epsilon} \underset{(5)}{=} K_4\kappa^{n/d} \quad \text{with} \quad K_4 = \frac{K_2K_3}{\epsilon}.$$

Then, for $-n \leqslant j_1 < j_2 \leqslant -n_0$, we have

$$\left|\log\frac{\delta_{j_1}(c_n)\cdots\delta_{j_2-1}(c_n)}{\delta_{j_1}(c_0)\cdots\delta_{j_2-1}(c_0)}\right| < nK_4\kappa^{n/d} \underset{n\to\infty}{\longrightarrow} 0.$$

In addition, we have

$$\delta_{j_1}(c_0)\cdots\delta_{j_2-1}(c_0) = \mu^{j_2-j_1}\frac{\phi'\big(\zeta_{j_1}(c_0)\big)}{\phi'\big(\zeta_{j_2}(c_0)\big)},$$

where $\phi : D(\beta_0, r_0) \to \mathbb{C}$ is the linearizing map conjugating f_{c_0} to multiplication by μ. So, there is a constant K_5 such that for n large enough and $-n \leqslant j_1 < j_2 \leqslant -n_0$,

$$\frac{1}{K_5} < \left|\frac{\delta_{j_1}(c_n)\cdots\delta_{j_2-1}(c_n)}{\mu^{j_2-j_1}}\right| < K_5.$$

Using inequality (2) and $\delta_j(c_n) = \delta_{m_n+j}(c_n)$ with $m_n = n + k_0$, we deduce that for n large enough and $1 \leqslant j_1 < j_2 \leqslant m_n - 1$

$$\frac{1}{K} < \left|\frac{\delta_{j_1}(c_n)\cdots\delta_{j_2-1}(c_n)}{\mu^{j_2-j_1}}\right| < K \quad \text{with} \quad K := K_5K_0^{k_0+n_0-1}.$$

The lemma follows since for $1 \leqslant j \leqslant m_n - 1$,

$$\Delta_j = \delta_1(c_n)\cdots\delta_j(c_n) \quad \text{and} \quad \Delta_{-j} = \delta_{m_n-j}(c_n)\cdots\delta_{m_n-1}(c_n).$$

□

According to Lemma 9.15, the coefficients of

$$P_n(z) := \frac{\chi_{f_{c_n},2}(z/\mu)}{\chi_{f_{c_n},2}(0)} = 1 + \frac{\Delta_{-1}}{\mu}z + \cdots + \frac{\Delta_{-(n-1)}}{\mu^{n-1}}z^{n-1}$$

and

$$Q_n(z) := \frac{\mu^{n-1}P_n(z)}{\Delta_{-(n-1)}z^{n-1}} = 1 + \frac{\mu}{\Delta_1}\frac{1}{z} + \cdots + \frac{\mu^{n-1}}{\Delta_{n-1}}\frac{1}{z^{n-1}}$$

are uniformly bounded. In particular, the sequence (P_n) is normal in $\mathbb{D}$, and the sequence (Q_n) is normal outside $\overline{\mathbb{D}}$. In Lemma 9.16 below, we prove that as $n \to \infty$, the roots of P_n equidistribute on the unit circle, so the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $\big\{|\lambda| = 1/|\mu|\big\}$. □

LEMMA 9.16. *Let*

$$P_n = 1 + \cdots + c_nz^{d_n} \in \mathbb{C}[z] \quad \textit{and} \quad Q_n = \frac{P_n}{c_nz^{d_n}} = 1 + \cdots + \frac{1}{c_nz^{d_n}} \in \mathbb{C}[1/z].$$

If

- *the sequence (d_n) tends to ∞ as $n \to \infty$,*
- *the sequence (P_n) is normal in the unit disk $\mathbb{D}$, and*
- *the sequence (Q_n) is normal in $\mathbb{C}\smallsetminus\overline{\mathbb{D}}$,*

then as $n \to \infty$, the roots of P_n equidistribute on the unit circle S^1.

Proof. Extracting a subsequence if necessary, we may assume that the sequence $\mathfrak{m}_{P_n}$ converges to a probability measure $\mathfrak{m}$ on $\widehat{\mathbb{C}}$. It is enough to show that $\mathfrak{m}$ coincides with the normalized Lebesgue measure on S^1. We first show that the support of $\mathfrak{m}$ is contained in S^1. We then show that its Fourier coefficients all vanish except the constant coefficient.

Extracting a further subsequence if necessary, we may assume that

- the sequence (P_n) converges locally uniformly in $\mathbb{D}$ to a holomorphic map ϕ, and
- the sequence (Q_n) converges locally uniformly in $\widehat{\mathbb{C}}\smallsetminus\overline{\mathbb{D}}$ to a holomorphic map ψ.

Since $P_n(0) = Q_n(\infty) = 1$, the limits satisfy $\phi(0) = \psi(\infty) = 1$. As a consequence, the zeros of P_n stay bounded away from 0 and ∞. In addition, ϕ and ψ do not identically vanish, so their zeros are isolated; and within any compact subset of $\mathbb{C}\smallsetminus S^1$, the number of zeros of P_n (counting multiplicities) is uniformly bounded. This shows that the support of $\mathfrak{m}$ is contained in S^1.

Now choose $r < 1 < R$ so that all the roots of P_n remain in the annulus $A := \{r < |z| < R\}$. For $k \in \mathbb{Z}$, let m_k be the Fourier coefficient

$$m_k := \int_{S^1} z^k \,\mathrm{d}\mathfrak{m} = \int_A z^k \,\mathrm{d}\mathfrak{m} = \lim_{n\to\infty} \int_A z^k \,\mathrm{d}\mathfrak{m}_{P_n}.$$

By the residue theorem, if $k > 0$, then

$$\begin{aligned}\frac{d_n}{2\pi \mathrm{i}} \cdot \int_A z^k \,\mathrm{d}\mathfrak{m}_{P_n} = \int_{|z|=R} z^k \frac{P_n'(z)}{P_n(z)} \,\mathrm{d}z &= \int_{|z|=R} z^k \cdot \left(\frac{d_n}{z} + \frac{Q_n'(z)}{Q_n(z)}\right) \mathrm{d}z \\ &= \int_{|z|=R} z^k \frac{Q_n'(z)}{Q_n(z)} \,\mathrm{d}z \\ &\underset{n\to\infty}{\longrightarrow} \int_{|z|=R} z^k \frac{\psi'(z)}{\psi(z)} \,\mathrm{d}z.\end{aligned}$$

Similarly, if $k < 0$, then

$$\frac{d_n}{2\pi \mathrm{i}} \cdot \int_A z^k \mathrm{d}\mathfrak{m}_{P_n} = \int_{|z|=r} z^k \frac{P_n'(z)}{P_n(z)} \,\mathrm{d}z \underset{n\to\infty}{\longrightarrow} \int_{|z|=r} z^k \frac{\phi'(z)}{\phi(z)} \,\mathrm{d}z.$$

In both cases, the limit is finite and since $d_n \to \infty$, we deduce that $m_k = 0$. □

To complete the proof of Theorem 1.5, it is enough to show that the set of t such that there exists a $c_0 \in \mathbb{C}$ for which the critical point of f_{c_0} is preperiodic to a fixed point of multiplier of modulus t is dense in $[1, 1/r_D]$. This follows from Lemmas 9.17 and 9.18 below.

LEMMA 9.17. *For each $t \in [1, 1/r_D]$, there is a parameter $c \in \partial\mathcal{M}_D$ such that f_c has a fixed point with multiplier of modulus t.*

Proof. The boundary of $\mathcal{M}_D$ is connected. As c varies in the boundary of $\mathcal{M}_D$, the multipliers of fixed points vary continuously. Thus, it suffices to show that $\partial\mathcal{M}_D$ contains a parameter c_0 for which f_{c_0} has a fixed point with multiplier of modulus 1, and a parameter c_1 for which f_{c_1} has a fixed point of modulus $1/r_D$.

Note that $f_c(\beta) = \beta$ and $f_c'(\beta) = \mu$ if and only if

$$c^{D-1} = \frac{\mu}{D}\left(1 - \frac{\mu}{D}\right)^{D-1} \quad \text{and} \quad \beta = \frac{c}{1 - \mu/D}. \tag{6}$$

First, when $c^{D-1} = \frac{1}{D}(1-\frac{1}{D})^{D-1}$, then f_c has a fixed point of multiplier 1. The corresponding parameters c belong to the boundary of $\mathcal{M}_D$. Second, let ω be a Dth root of unity closest to -1. If D is even, we have $\omega = -1$. If D is odd, we have $\omega = \exp(\pm\pi\mathrm{i}\frac{D-1}{D})$. Note that

$$|1-\omega|^2 = \left(1+\cos\frac{\pi}{D}\right)^2 + \sin^2\frac{\pi}{D} = 2+2\cos\frac{\pi}{D} = 4\cos^2\frac{\pi}{2D}.$$

In both cases,

$$|1-\omega| = \frac{1}{Dr_D}.$$

Set $\mu := D(1-\omega)$, choose c and define β so that equation (6) holds. Then, f_c has a fixed point at β with multiplier μ of modulus $1/r_D$. In addition,

$$f_c(0) = c = \left(1-\frac{\mu}{D}\right)\beta = \omega\beta, \quad \text{so that} \quad f_c^{\circ 2}(0) = \beta.$$

Thus, c is a *Misiurewicz parameter*; that is, f_c is postcritically finite with $c \in \partial\mathcal{M}_D$, as required. □

LEMMA 9.18. *Let $c_0 \in \partial\mathcal{M}_D$, and let β_0 be a repelling fixed point of f_{c_0} of multiplier μ_0. Then, there exists a sequence of parameters c_n converging to c_0 such that f_{c_n} has a fixed point $\beta_n \in \mathcal{P}_{f_{c_n}}$ converging to β_0.*

Proof. Since β_0 is repelling, there is a function β defined and holomorphic near c_0, such that $f_c \circ \beta_c = \beta_c$. Let $\omega \neq 1$ be a Dth root of unity. Since $c_0 \in \partial\mathcal{M}_D$, the sequence of functions $\zeta_k := c \mapsto f_c^{\circ k}(0)$ is not normal at c_0. It follows from Montel's theorem that in any neighborhood of c_0, the sequence (ζ_k) cannot avoid both β and $\omega\beta$. When $\zeta_k = \omega\beta$, then $\zeta_{k+1} = \beta$. So there is a sequence of complex numbers (c_n) converging to c_0 and a sequence of integers (k_n) tending to ∞ such that $\zeta_{k_n}(c_n) = \beta(c_n) =: \beta_n$. □

10. *Questions for further study*

We conclude with some remaining questions.

10.1. *Periodic unicritical polynomials*

Let f be a periodic unicritical polynomial of degree D. An eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ satisfies $\frac{1}{4D} < |\lambda| < 1$ and for D even $\Sigma(D)$ contains the annulus $\left\{\frac{1}{2D} \leqslant |\lambda| \leqslant 1\right\}$. The estimate obtained for D odd is not as good.

QUESTION 1. For D odd, does $\Sigma(D)$ contain the annulus $\left\{\frac{1}{2D} \leqslant |\lambda| \leqslant 1\right\}$?

We shall say that an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ is a *small eigenvalue* if $|\lambda| < \frac{1}{2D}$. According to Proposition 9.6, a small eigenvalue belongs to Σ_f. In particular, if the critical point is periodic of period m, there are at most $m-2$ small eigenvalues.

QUESTION 2. How many small eigenvalues can a periodic unicritical polynomial have?

10.2. *Spectral gap*

Let f be a postcritically finite rational map. We saw that for periodic unicritical polynomials of degree D, the eigenvalues in $\Sigma_f \smallsetminus \{0\}$ and $\Lambda_f \smallsetminus \{0\}$ remain uniformly bounded away from 0.

At the same time, the set of periodic unicritical polynomials of degree D is bounded in moduli space of degree D polynomials.

Conversely, let $\mathcal{M}$ be a compact subset of moduli space of degree D rational maps. Conjugacy classes in $\mathcal{M}$ may be represented by rational maps whose derivatives (for the spherical metric on $\widehat{\mathbb{C}}$) are uniformly bounded by some constant K. It follows that if μ is the multiplier of a cycle of period m of such a rational map f, then $|\mu|^{1/m} \leqslant K$. This implies that if $\lambda \in \Lambda_f \smallsetminus \{0\}$, then $|\lambda| \geqslant 1/K$. We may ask whether a similar result holds for Σ_f.

Question 3. If the conjugacy class of f remains in a compact subset of moduli space of degree D rational maps, is the set $\Sigma_f \smallsetminus \{0\}$ bounded away from 0?

We saw that the eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ are related to the multipliers of cycles of f. These, in turn, are related to the Lyapunov exponent $L(f)$ of f with respect to the equilibrium measure (which remains bounded on compact subsets of moduli space since it is continuous).

Question 4. Is there a relation between $\exp\bigl(-L(f)\bigr)$ and $\inf\bigl\{|\lambda| : \lambda \in \Sigma_f \smallsetminus \{0\}\bigr\}$ or $\inf\bigl\{|\lambda| : \lambda \in \Lambda_f \smallsetminus \{0\}\bigr\}$?

10.3. *Diagonalizability*

Question 5. Let f be postcritically finite. Is $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ diagonalizable? Is $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ diagonalizable?

Question 6. Let f be a periodic unicritical polynomial. Are the roots of the characteristic polynomial χ_f simple?

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