EIGENVALUES OF THE THURSTON OPERATOR

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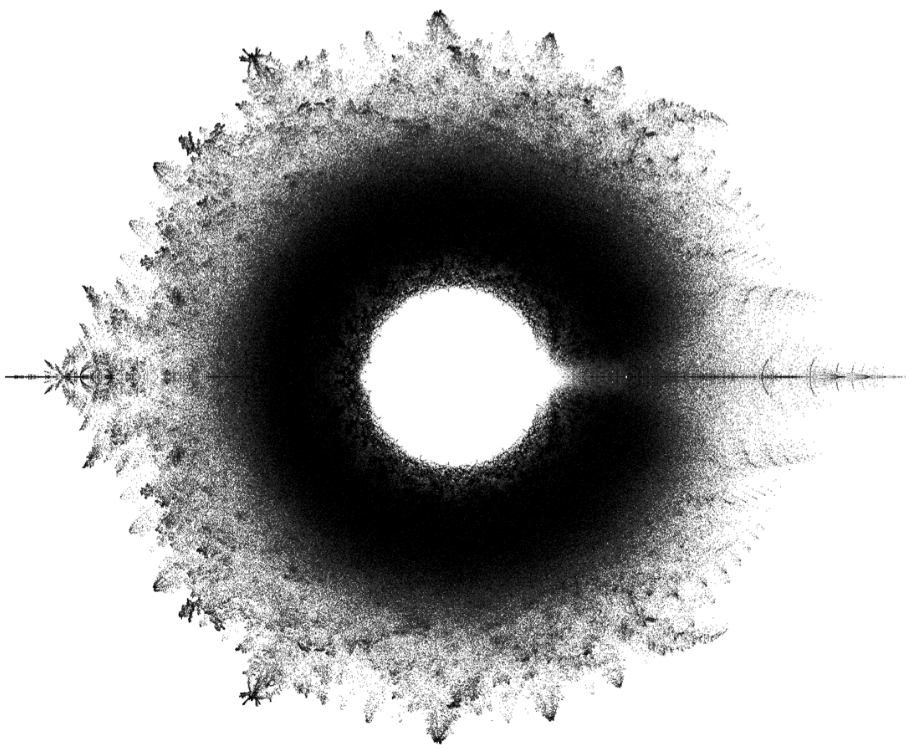
for Bill

Abstract. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite rational map, and let $\mathcal{Q}(\widehat{\mathbb{C}})$ be the space of meromorphic quadratic differentials on $\widehat{\mathbb{C}}$ with simple poles. We study the set of eigenvalues of the pushforward operator $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$. In particular, we show that when $f : \mathbb{C} \to \mathbb{C}$ is a unicritical polynomial of degree D with periodic critical point, the eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ are contained in the annulus $\left\{\frac{1}{4D} < |\lambda| < 1\right\}$ and belong to $\frac{1}{D}\mathbb{U}$ where $\mathbb{U}$ is the group of algebraic units.



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1. Introduction

Throughout this article, $D \geq 2$ and Rat_D is the space of rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere of degree D. We denote the set of critical points of f by $\mathcal{C}_f$ and the set of critical values by $\mathcal{V}_f$. The postcritical set $\mathcal{P}_f$ is the smallest forward invariant subset of $\widehat{\mathbb{C}}$ which contains $\mathcal{V}_f$:

$$\mathcal{P}_f := \bigcup_{n \geq 1} f^{\circ n}(\mathcal{C}_f).$$

We study *postcritically finite rational maps*; that is, rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which $\mathcal{P}_f$ is finite. It follows from work of Thurston that with the exception of flexible Lattès maps (see §2.3 for the definition), postcritically finite rational maps are rigid: if two postcritically finite rational maps are topologically conjugate, then either they are flexible Lattès maps, or they are conjugate by a Möbius transformation [DH2].

In fact, Thurston worked with the Teichmüller space $\mathcal{T}_f$ of the Riemann sphere marked with the postcritical set $\mathcal{P}_f$ and he considered the holomorphic self-map $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ induced by pulling back complex structures on $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$. The fact that the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic is equivalent to the fact that the point $\tau_f \in \mathcal{T}_f$ represented by the standard complex structure on $\widehat{\mathbb{C}} \setminus \mathcal{P}_f$ is fixed by σ_f. Thurston proved that the pullback map σ_f is contracting for the Teichmüller metric on $\mathcal{T}_f$, so that it has a unique fixed point; rigidity follows.

A more elementary result that does not appeal to Teichmüller spaces, concerns infinitesimal rigidity: if $t \mapsto f_t$ is an analytic family of postcritically finite rational maps, then either the maps are flexible Lattès maps, or there is an analytic family of Möbius transformations $t \mapsto M_t$ such that $M_0 = \mathrm{id}$ and $f_t \circ M_t = M_t \circ f_0$. The proof of this result relies on the following lemma, in which $\mathcal{Q}(\widehat{\mathbb{C}})$ is the space of meromorphic quadratic differentials on $\widehat{\mathbb{C}}$ with simple poles and $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ is the Thurston pushforward operator (see §3 for the definition).

Lemma 1.1 (Thurston)**.** *Assume $f \in \mathrm{Rat}_D$ is postcritically finite. If λ is an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$, then $|\lambda| \leq 1$. In addition, $\lambda = 1$ is an eigenvalue if and only if f is a flexible Lattès map.*

We derive infinitesimal rigidity in §5. We then study the eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$. The subspace $\mathcal{Q}_f \subset \mathcal{Q}(\widehat{\mathbb{C}})$ of quadratic differentials with poles contained in $\mathcal{P}_f$ has finite dimension $\mathrm{card}(\mathcal{P}_f) - 3$ and is invariant by f_*. Let Σ_f be the set of eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$, and let Λ_f be the set of eigenvalues of the induced operator $f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$.

Remark. The space $\mathcal{Q}_f$ canonically identifies with the cotangent space to the Teichmüller space $\mathcal{T}_f$ at the base point τ_f and the coderivative of σ_f at τ_f identifies with $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$. Thus, Σ_f coincides with the spectrum of the derivative of the Thurston pullback map $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ at its unique fixed point.

In §6, we study Λ_f and in §7, we study Σ_f, establishing the following results.

Theorem 1.2. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite rational map. The set Λ_f consists of 0 and the complex numbers $\lambda \in \mathbb{C} \setminus \{0\}$ such that $1/\lambda^m$ is the multiplier of a cycle of f of period m which is not contained in $\mathcal{P}_f$. If $\lambda \in \Lambda_f \setminus \{0\}$, then λ is an algebraic number but not an algebraic integer.*

Theorem 1.3. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite rational map. If $\lambda \in \Sigma_f$, then λ is an algebraic number. If λ is an algebraic integer, then either $\lambda = 0$, or f is a Lattès map and*

$$\lambda \in \left\{ \pm 1,\ \pm \mathrm{i},\ \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2},\ -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2} \right\}.$$

In §8, we describe a way to compute the characteristic polynomial of the operator $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$. We then apply this in §9 to the case where f is a unicritical polynomial with periodic critical point. We establish estimates on the size of the coefficients of the characteristic polynomial, which enable us to derive the following result.

Theorem 1.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a unicritical polynomial of degree D with periodic critical point. Then*

- *if $\lambda \in \Lambda_f$, we have that $\dfrac{1}{2D} < |\lambda| < 1$ and*
- *if $\lambda \in \Sigma_f$, we have that $\dfrac{1}{4D} < |\lambda| < 1$.*

In both cases, $D\lambda$ is an algebraic unit.

The case of eigenvalues in Λ_f is covered by Proposition 9.6 and the case of eigenvalues in Σ_f is covered by Theorem 9.9 and Proposition 9.12.

Given $D \geq 2$, set

$$\Sigma(D) := \bigcup_f \Sigma_f,$$

where the union is taken over all unicritical polynomials $f : \mathbb{C} \to \mathbb{C}$ of degree D with periodic critical point.

Theorem 1.5. *The closure of $\Sigma(D)$ contains the annulus $r_D \leq |\lambda| \leq 1$ where r_D is defined by*

$$\frac{1}{r_D} = \begin{cases} 2D & \text{if } D \text{ is even} \\ 2D\cos\left(\dfrac{\pi}{2D}\right) & \text{if } D \text{ is odd.} \end{cases}$$

The proof of this theorem relies on estimating the modulus of the multipliers of unicritical postcritically finite polynomials of degree D at their fixed points. We have an optimal estimate in the case when D is even, and we do not in the case when D is odd. The proof also relies on the following equidistribution result which might be of independent interest.

Lemma 1.6. *Let*

$$P_n = 1 + \cdots + c_n z^{d_n} \in \mathbb{C}[z] \quad \text{and} \quad Q_n = \frac{P_n}{c_n z^{d_n}} = 1 + \cdots + \frac{1}{c_n z^{d_n}} \in \mathbb{C}[1/z].$$

If

- *the sequence (d_n) tends to ∞ as $n \to \infty$,*
- *the sequence (P_n) is normal in the unit disk $\mathbb{D}$, and*
- *the sequence (Q_n) is normal in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$,*

then as $n \to \infty$, the roots of P_n equidistribute on the unit circle S^1.

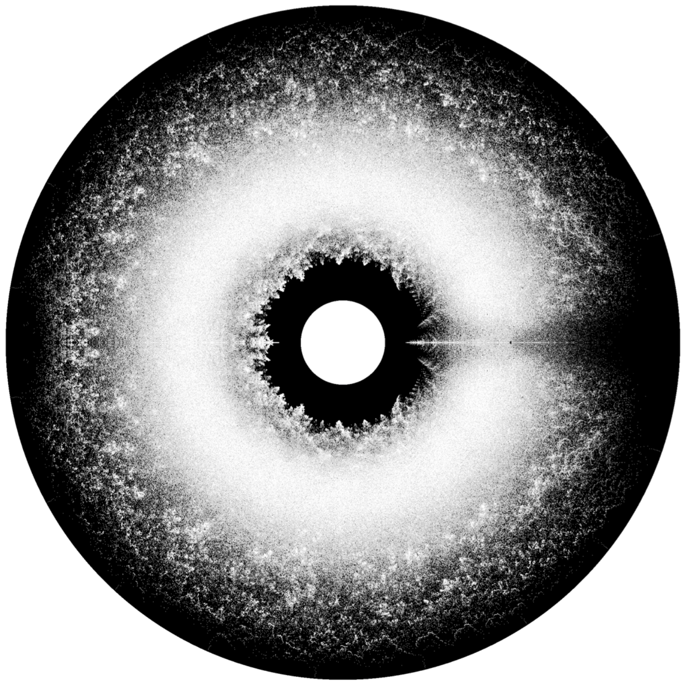

FIGURE 1. The set of eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ for unicritical polynomials f of degree 2 with periodic critical point, up to period 19. The set of eigenvalues (white) is contained in the annulus $1/8 < |\lambda| < 1$ (black). The picture on the first page of this article shows the reciprocal of the eigenvalues (black).

The proof of Theorem 1.4 is given in §9.5 and §9.7, and the proof of Theorem 1.5 is given in §9.8. Finally, in §10, we pose some questions for further study.

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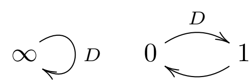
2. Postcritically finite rational maps

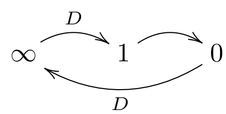
Fix $f \in \mathrm{Rat}_D$. It follows from the Riemann-Hurwitz formula that f has $2D-2$ critical points counted with multiplicity, and that f has at least two distinct critical values. As a consequence, $\mathrm{card}(\mathcal{P}_f) \geq 2$. If $\mathrm{card}(\mathcal{P}_f) = 2$, then $\mathcal{V}_f = \mathcal{P}_f$, and f is conjugate to $z \mapsto z^{\pm D}$.

2.1. **Examples.** In the following two examples, $\mathrm{card}(\mathcal{P}_f) = 3$.

- The polynomial $f : z \mapsto 1 - z^D$ has critical set $\mathcal{C}_f = \{0, \infty\}$, postcritical set $\mathcal{P}_f = \{0, 1, \infty\}$, and postcritical dynamics:



- The rational map $f : z \mapsto 1 - 1/z^D$ has critical set $\mathcal{C}_f = \{0, \infty\}$, postcritical set $\mathcal{P}_f = \{0, 1, \infty\}$, and postcritical dynamics:



Unicritical polynomials. For much of this article, we will focus on polynomials $\mathbb{C} \to \mathbb{C}$ of degree D which have a unique critical point; these polynomials are called *unicritical*. Every unicritical polynomial is affine conjugate to a polynomial of the form $f_c(z) = z^D + c$, where $z_0 = 0$ is the unique critical point with critical value $f_c(0) = c$. Fix an integer $m \geq 1$. In parameter space, the roots of the polynomial $G_m(c) := f_c^{\circ m}(0)$ correspond to polynomials f_c for which 0 is periodic of period dividing m. These maps f_c are necessarily postcritically finite with postcritical set equal to

$$\{\infty\} \;\cup \bigcup_{1 \leq k \leq m} f_c^{\circ k}(0).$$

An argument due to Gleason shows that G_m has simple roots (see Lemma 9.4), so there are lots of postcritically finite polynomials. In fact, for $D = 2$ the boundary of the Mandelbrot set is contained in the closure of the set $\bigcup_{m \geq 1} \{\text{roots of } G_m\}$.

2.2. **Cycles are superattracting or repelling.** Recall that the multiplier of a periodic m-cycle $\{x, f(x), \ldots, f^{\circ(m-1)}(x)\}$ is the eigenvalue λ of the linear map $\mathrm{D}_x f^{\circ m} : \mathrm{T}_x\widehat{\mathbb{C}} \to \mathrm{T}_x\widehat{\mathbb{C}}$ (this eigenvalue does not depend on the point in the cycle). The periodic cycles of a postcritically finite rational map are either *superattracting*; that is, $\lambda = 0$, or *repelling*; that is, $|\lambda| > 1$.

2.3. **Lattès maps.** The rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *Lattès map* if there is:

- a complex torus $\mathcal{T} := \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice of rank 2,
- an affine map $L : \mathcal{T} \to \mathcal{T}$, and
- a finite branched cover $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$

so that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{L} & \mathcal{T} \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ \widehat{\mathbb{C}} & \xrightarrow[f]{} & \widehat{\mathbb{C}} \end{array}$$

The rational map f is necessarily postcritically finite, and $\mathcal{P}_f$ is the set of critical values of $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$. In addition, $\mathrm{card}(\mathcal{P}_f) \in \{3, 4\}$. We shall use the following characterization of Lattès maps with four postcritical points (see [M, §4]).

Proposition 2.1. *A postcritically finite rational map with* $\mathrm{card}(\mathcal{P}_f) = 4$ *is a Lattès map if and only if every critical point if simple (with local degree 2) and no critical point is postcritical.*

Lattès maps are either *flexible* or *rigid*. The map f is flexible if

- for L of the form $L : w \mapsto \alpha w + \beta$, we have $\alpha \in \mathbb{Z}$, and
- the map Θ has degree 2.

Equivalently, f is flexible if it can be *deformed*, that is, if it is part of a one-parameter isospectral family that is nontrivial [McM]. The Lattès map f is rigid if it is not flexible. Flexible Lattès maps have four postcritical points (this follows from the fact that Θ has degree 2). Rigid Lattès maps may have three or four postcritical points.

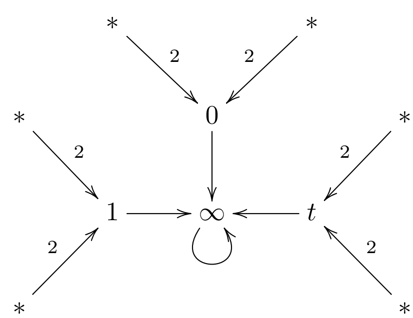
Example. The map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $f : z \mapsto (1 - 2/z)^2$ is a rigid Lattès map with $\mathcal{P}_f = \{0, 1, \infty\}$ and has the following postcritical dynamics.

$$2 \xrightarrow{2} 0 \xrightarrow{2} \infty \longrightarrow 1 \circlearrowleft$$

Example. The family

$$\{f_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}\}_{t \in \mathbb{C} \smallsetminus \{0,1\}} \quad \text{given by} \quad f_t : z \mapsto \frac{(z^2 - t)^2}{4z(z-1)(z-t)}$$

consists entirely of flexible Lattès maps. The postcritical set of f_t is $\{0, 1, t, \infty\}$, and f_t has the following postcritical dynamics.



3. Quadratic differentials

Let U be a Riemann surface. A *quadratic differential* on U is a section of the square of the cotangent bundle $\mathrm{T}^*U \otimes \mathrm{T}^*U$. We shall usually think of a quadratic differential $\boldsymbol{q}$ as a field of quadratic forms. In particular, if $\boldsymbol{\theta}$ is a vector field on U and ϕ is a function on U, then $\boldsymbol{q}(\boldsymbol{\theta})$ is a function on U and $\boldsymbol{q}(\phi\boldsymbol{\theta}) = \phi^2 \boldsymbol{q}(\boldsymbol{\theta})$.

If $\zeta : U \to \mathbb{C}$ is a coordinate, we shall use the notation $(\mathrm{d}\zeta)^2 = \mathrm{d}\zeta \otimes \mathrm{d}\zeta$ (not to be confused with the 1-form $\mathrm{d}(\zeta^2)$). On U (whose complex dimension is 1), the ratio of two quadratic differentials is a function. In other words, any quadratic differential $\boldsymbol{q}$ on U may be written as

$$\boldsymbol{q} = q\ (\mathrm{d}\zeta)^2 \quad \text{for some function } q.$$

3.1. **Meromorphic quadratic differentials.** A quadratic differential $\boldsymbol{q}$ on $\widehat{\mathbb{C}}$ is meromorphic if $\boldsymbol{q} = q\ (\mathrm{d}z)^2$ for some meromorphic function $q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The quadratic differential $(\mathrm{d}z)^2$ has no zero and has a pole of order 4 at ∞. Since the number of zeros of the function q equals the number of poles of q, counting multiplicities, the number of poles minus the number of zeros of $\boldsymbol{q}$ is equal to four. In particular, $\boldsymbol{q}$ has at least four poles (counting multiplicities).

Let $\mathcal{Q}(\widehat{\mathbb{C}})$ be the set of meromorphic quadratic differentials with only simple poles. For $X \subset \widehat{\mathbb{C}}$, let $\mathcal{Q}(\widehat{\mathbb{C}}; X) \subset \mathcal{Q}(\widehat{\mathbb{C}})$ be the subset of quadratic differentials whose poles are contained in X. For $k \geq 0$, let $\mathcal{Q}_k(\mathbb{C})$ be the set of meromorphic quadratic differentials whose poles in $\mathbb{C}$ are all simple and which have at worst a pole of order k at ∞.

Example. The quadratic differential $(\mathrm{d}z)^2$ belongs to $\mathcal{Q}_4(\mathbb{C})$ and for any $x \in \mathbb{C}$, the quadratic differential $\dfrac{(\mathrm{d}z)^2}{z-x}$ belongs to $\mathcal{Q}_3(\mathbb{C}) \subset \mathcal{Q}_4(\mathbb{C})$.

3.2. **Pullback.** The derivative $\mathrm{D}f : \mathrm{T}U \to \mathrm{T}V$ of a holomorphic map $f : U \to V$ naturally induces a pullback map f^* from quadratic differentials on V to quadratic differentials on U:

$$f^*\boldsymbol{q} := \boldsymbol{q} \circ \mathrm{D}f.$$

If $f : (U, x) \to (V, y)$ is holomorphic at x, and q is meromorphic at $y = f(x)$, then

$$2 + \mathrm{ord}_x(f^*\boldsymbol{q}) = \deg_x f \cdot (2 + \mathrm{ord}_y \boldsymbol{q}).$$

3.3. **The Thurston pushforward operator.** If $f : U \to V$ is a covering map and $\boldsymbol{q}$ is a quadratic differential on U, then we can define a quadratic differential $f_*\boldsymbol{q}$ on V by

$$f_*\boldsymbol{q} := \sum_g g^*\boldsymbol{q}$$

where the sum is taken over all inverse branches g of f. If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a nonconstant rational map and $\boldsymbol{q} = q\ (\mathrm{d}z)^2$ is a meromorphic quadratic differential on $\widehat{\mathbb{C}}$, then the quadratic differential $f_*\boldsymbol{q}$, which is a priori defined on $\widehat{\mathbb{C}} \smallsetminus \mathcal{V}_f$, is globally meromorphic on $\widehat{\mathbb{C}}$, and

$$f_*\boldsymbol{q} := r\ (\mathrm{d}z)^2 \quad \text{with} \quad r(y) := \sum_{x \in f^{-1}(y)} \frac{q(x)}{f'(x)^2}.$$

If $\boldsymbol{q}$ is a meromorphic quadratic differential on $\widehat{\mathbb{C}}$, and if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map, then or all $y \in \widehat{\mathbb{C}}$, we have

$$2 + \mathrm{ord}_y(f_*\boldsymbol{q}) \geq \min_{x \in f^{-1}(y)} \frac{2 + \mathrm{ord}_x \boldsymbol{q}}{\deg_x f}.$$

As a consequence,

- if X is the set of poles of $\boldsymbol{q}$, then the set of poles of $f_*\boldsymbol{q}$ is contained in $f(X) \cup \mathcal{V}_f$,
- if $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$, then $f_*\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$ and
- if f fixes ∞ and if $\boldsymbol{q} \in \mathcal{Q}_k(\mathbb{C})$ for some $k \geq 0$, then $f_*\boldsymbol{q} \in \mathcal{Q}_k(\mathbb{C})$.

3.4. **Transposition.** If $\boldsymbol{q}$ is a quadratic differential on U and $\boldsymbol{\theta}$ is a vector field on U, we may consider the 1-form $\boldsymbol{q}\otimes\boldsymbol{\theta}$ defined on U by its action on vector fields $\boldsymbol{\xi}$:

$$\boldsymbol{q}\otimes\boldsymbol{\theta}(\boldsymbol{\xi}) = \frac{1}{4}\big(\boldsymbol{q}(\boldsymbol{\theta}+\boldsymbol{\xi}) - \boldsymbol{q}(\boldsymbol{\theta}-\boldsymbol{\xi})\big).$$

If

$$\boldsymbol{q} = q\ (\mathrm{d}z)^2 \quad \text{and} \quad \boldsymbol{\theta} = \theta\,\frac{\mathrm{d}}{\mathrm{d}z}, \quad \text{then} \quad \boldsymbol{q}\otimes\boldsymbol{\theta} = q\theta\ \mathrm{d}z.$$

We shall use the following lemma which, in some sense, asserts that the transpose of pushing forward a quadratic differential is pulling back a vector field.

Lemma 3.1. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational map, let $\boldsymbol{\theta}$ be a meromorphic vector field on $\widehat{\mathbb{C}}$, and let $\boldsymbol{q}$ be a meromorphic quadratic differential on $\widehat{\mathbb{C}}$. Then*

$$\mathrm{residue}\big((f_*\boldsymbol{q})\otimes\boldsymbol{\theta}, y\big) \ = \sum_{x\in f^{-1}(y)} \mathrm{residue}\big(\boldsymbol{q}\otimes f^*\boldsymbol{\theta}, x\big).$$

Proof. Let γ be a small loop around y with basepoint a. Then

$$\int_\gamma (f_*\boldsymbol{q})\otimes\boldsymbol{\theta} = \sum_g \int_{\gamma\smallsetminus\{a\}} (g^*\boldsymbol{q})\otimes\boldsymbol{\theta} = \sum_g \int_{g(\gamma\smallsetminus\{a\})} \boldsymbol{q}\otimes f^*\boldsymbol{\theta} = \int_{f^{-1}(\gamma)} \boldsymbol{q}\otimes f^*\boldsymbol{\theta},$$

where the sum ranges over the inverse branches g of f defined on $\gamma\smallsetminus\{a\}$. □

4. The contraction principle

If $\boldsymbol{q}$ is a quadratic differential on U, we denote by $|\boldsymbol{q}|$ the positive $(1,1)$-form on U defined by

$$|\boldsymbol{q}|(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) := \frac{1}{2}\big|\boldsymbol{q}(\boldsymbol{\theta}_1 - \mathrm{i}\boldsymbol{\theta}_2)\big| - \frac{1}{2}\big|\boldsymbol{q}(\boldsymbol{\theta}_1 + \mathrm{i}\boldsymbol{\theta}_2)\big|.$$

If $\boldsymbol{q} = q\ (\mathrm{d}\zeta)^2$, then

$$|\boldsymbol{q}| = |q|\cdot\frac{\mathrm{i}}{2}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}.$$

We shall say that $\boldsymbol{q}$ is *integrable* on U if

$$\|\boldsymbol{q}\|_{L^1(U)} := \int_U |\boldsymbol{q}| < \infty.$$

Note that $\boldsymbol{q}$ is integrable in a neighborhood of a pole if and only if the pole is simple.

The following results due to Thurston will be crucial for our purposes. The proof, based on the triangle inequality (see [DH2] for example), is transcendental.

Lemma 4.1 (Contraction Principle)**.** *Let $f:U\to V$ be a covering map and let $\boldsymbol{q}$ be an integrable quadratic differential on U. Then,*

$$\|f_*\boldsymbol{q}\|_{L^1(V)} \le \|\boldsymbol{q}\|_{L^1(U)}$$

and equality holds if and only if $f^*(f_*\boldsymbol{q}) = \phi\ \boldsymbol{q}$ with $\phi: U\to[0,+\infty)$ a real and positive function.

Corollary 4.2. *If $f\in\mathrm{Rat}_D$ is postcritically finite, then $f_*:\mathcal{Q}(\widehat{\mathbb{C}})\to\mathcal{Q}(\widehat{\mathbb{C}})$ is (weakly) contracting. In particular, the eigenvalues of f_* have modulus at most 1.*

Corollary 4.3. *Let $f:\mathbb{C}\to\mathbb{C}$ be a polynomial of degree D. Then for all $k\ge 0$, the eigenvalues of $f_*:\mathcal{Q}_k(\mathbb{C})\to\mathcal{Q}_k(\mathbb{C})$ have modulus less than 1.*

Proof. Suppose λ is an eigenvalue and $\boldsymbol{q} \in \mathcal{Q}_k(\mathbb{C})$ is an associated eigenvector; that is, $\boldsymbol{q} \neq 0$ and $f_*\boldsymbol{q} = \lambda\boldsymbol{q}$. Let V be a sufficiently large disk so that $U := f^{-1}(V)$ is compactly contained in V. Set $V' := V \smallsetminus \mathcal{V}_f$ and $U' := f^{-1}(V')$. Then

$$|\lambda| \cdot \|\boldsymbol{q}\|_{L^1(V)} = \|\lambda\boldsymbol{q}\|_{L^1(V')} = \|f_*\boldsymbol{q}\|_{L^1(V')} \leq \|\boldsymbol{q}\|_{L^1(U')} = \|\boldsymbol{q}\|_{L^1(U)} < \|\boldsymbol{q}\|_{L^1(V)}.$$

The first inequality is an application of the contraction principle. The last inequality is strict since U is compactly contained in V and $\boldsymbol{q} \neq 0$. In addition, $\|\boldsymbol{q}\|_{L^1(V)} > 0$, so $|\lambda| < 1$. □

The proof of the following result is given in [DH2].

Proposition 4.4 (Thurston)**.** *Let $f \in \mathrm{Rat}_D$, and suppose λ is an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$. If $|\lambda| = 1$, then f is a Lattès map with four postcritical points. If $\lambda = 1$, then f is a flexible Lattès map.*

5. Infinitesimal rigidity

We now present a proof that with the exception of flexible Lattès maps, postcritically finite rational maps are infinitesimally rigid. This proof only relies on the fact that when f is not a flexible Lattès map, $1 \notin \Sigma_f$.

Here and henceforth, we consider holomorphic families $t \mapsto \gamma_t$ defined near $t = 0$ in $\mathbb{C}$. We shall employ the notation

$$\gamma := \gamma_0 \quad \text{and} \quad \dot{\gamma} := \left.\frac{\mathrm{d}\gamma_t}{\mathrm{d}t}\right|_{t=0}.$$

If $t \mapsto f_t$ is a family of rational maps of degree D, then $\boldsymbol{\xi} := \dot{f} \in \mathrm{T}_f\mathrm{Rat}_D$ is a section of the pullback bundle $f^\star\mathrm{T}\widehat{\mathbb{C}}$: for each $z \in \widehat{\mathbb{C}}$, $\boldsymbol{\xi}(z) \in T_{f(z)}\widehat{\mathbb{C}}$. Setting

$$\boldsymbol{\tau}(x) := (\mathrm{D}_x f)^{-1}\big(\boldsymbol{\xi}(x)\big) \quad \text{if} \quad x \notin \mathcal{C}_f,$$

we define a meromorphic vector field $\boldsymbol{\tau}$ on $\widehat{\mathbb{C}}$, holomorphic outside $\mathcal{C}_f$, with poles of order at most the multiplicity of x as a critical point of f when $x \in \mathcal{C}_f$. This vector field satisfies

$$\boldsymbol{\xi} = \mathrm{D}f \circ \boldsymbol{\tau}.$$

Theorem 5.1 (Thurston)**.** *Let $t \mapsto f_t$ be a holomorphic family of postcritically finite rational maps of degree D, parameterized by a neighborhood of 0 in $\mathbb{C}$. Then either the maps are flexible Lattés maps, or there is an analytic family of Möbius transformations $t \mapsto M_t$ such that $M_0 = \mathrm{id}$ and $f_t \circ M_t = M_t \circ f_0$.*

Proof. Without loss of generality, we may assume that f is not a flexible Lattès map. The fixed points of f_t are superattracting or repelling and depend holomorphically on t. There are $D + 1 \geq 3$ such fixed points. Conjugating the family $t \mapsto f_t$ with a holomorphic family $t \mapsto M_t$ of Möbius transformations, we may assume that f_t fixes 0, 1 and ∞. We will show that in this case, the holomorphic family $t \mapsto f_t$ is constant. It is enough to show that $\frac{\mathrm{d}f_t}{\mathrm{d}t}$ identically vanishes, and since $t = 0$ plays no particular role, it is enough to prove that $\dot{f} \equiv 0$.

Set $\boldsymbol{\xi} := \dot{f} \in \mathrm{T}_f\mathrm{Rat}_D$ and let $\boldsymbol{\tau}$ be the globally meromorphic vector field on $\widehat{\mathbb{C}}$ such that $\boldsymbol{\xi} = \mathrm{D}f \circ \boldsymbol{\tau}$.

As t varies, the set $Y_t := \mathcal{P}_{f_t} \cup \{0,1,\infty\}$ moves holomorphically and $f_t(Y_t) = Y_t$. For each $y \in Y$, let $t \mapsto y_t$ be the holomorphic curve satisfying $y_0 = y$ and $y_t \in Y_t$. Set

$$\boldsymbol{\vartheta}(y) := \frac{\mathrm{d}y_t}{\mathrm{d}t}\Big|_{t=0} \in \mathrm{T}_y\widehat{\mathbb{C}}.$$

If $y \in Y$ and $z := f(y) \in Y$, then $z_t = f_t(y_t)$, so that

$$\boldsymbol{\vartheta} \circ f = \boldsymbol{\xi} + \mathrm{D}f \circ \boldsymbol{\vartheta} \quad \text{on } Y \quad \text{and} \quad \boldsymbol{\vartheta} \circ f = \boldsymbol{\xi} \quad \text{on } \mathcal{C}_f.$$

Let $\boldsymbol{\theta}$ be a vector field, defined and holomorphic near Y, with $\boldsymbol{\theta}|_Y = \boldsymbol{\vartheta}$. Then, $f^*\boldsymbol{\theta} - \boldsymbol{\tau}$ is holomorphic near $f^{-1}(Y)$ and coincides with $\boldsymbol{\theta}$ on Y. Also note that since f_t fixes 0, 1 and ∞, $\boldsymbol{\theta}$ vanishes at 0, 1 and ∞.

Let $\nabla_f := \mathrm{id} - f_* : \mathcal{Q}(\widehat{\mathbb{C}};Y) \to \mathcal{Q}(\widehat{\mathbb{C}};Y)$. Observe that for $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}};Y)$,

$$\begin{aligned}
\sum_{y\in Y} \mathrm{residue}\big((\nabla_f \boldsymbol{q}) \otimes \boldsymbol{\theta}, y\big) &\underset{(1)}{=} \sum_{y\in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) - \sum_{y\in Y} \mathrm{residue}\big((f_*\boldsymbol{q}) \otimes \boldsymbol{\theta}, y\big) \\
&\underset{(2)}{=} \sum_{y\in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) - \sum_{x\in f^{-1}(Y)} \mathrm{residue}(\boldsymbol{q} \otimes f^*\boldsymbol{\theta}, x) \\
&\underset{(3)}{=} \sum_{y\in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) - \sum_{x\in f^{-1}(Y)} \mathrm{residue}(\boldsymbol{q} \otimes (f^*\boldsymbol{\theta} - \boldsymbol{\tau}), x) \\
&\underset{(4)}{=} 0.
\end{aligned}$$

Equality (1) holds by definition of ∇_f; Equality (2) follows from Lemma 3.1; Equality (3) follows from the fact that $\boldsymbol{q} \otimes \boldsymbol{\tau}$ is globally meromorphic on $\widehat{\mathbb{C}}$ with poles contained in $Y \cup \mathcal{C}_f \subseteq f^{-1}(Y)$, so that the sum of its residues on $f^{-1}(Y)$ is 0; Equality (4) follows from the fact that $f^*\boldsymbol{\theta} - \boldsymbol{\tau}$ is holomorphic near $f^{-1}(Y)$ and coincides with $\boldsymbol{\theta}$ on Y which contains the set of poles of $\boldsymbol{q}$.

According to Proposition 4.4, since f is not a flexible Lattès map, $\lambda = 1$ is not an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$. The operator $\nabla_f : \mathcal{Q}(\widehat{\mathbb{C}};Y) \to \mathcal{Q}(\widehat{\mathbb{C}};Y)$ is therefore injective, thus surjective. It follows that for any $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}};Y)$,

$$\sum_{y\in Y} \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, y) = 0.$$

Equivalently, $\boldsymbol{\vartheta}$ is the restriction to Y of a globally holomorphic vector field $\boldsymbol{\theta}$. Since $\boldsymbol{\theta}$ vanishes at 0, 1 and ∞, we have $\boldsymbol{\theta} = 0$. The vector field $-\boldsymbol{\tau} = f^*\boldsymbol{\theta} - \boldsymbol{\tau}$ is globally holomorphic and coincides with $\boldsymbol{\theta} = 0$ on Y. So $\boldsymbol{\tau} = 0$ and $\boldsymbol{\xi} = \mathrm{D}f \circ \boldsymbol{\theta} = 0$ as required. □

The following corollary of infinitesimal rigidity is part of the folklore, but we are not aware of a written proof. Our presentation provides a systematic way of defining algebraic equations for the set of postcritically finite rational maps with prescribed dynamics on the postcritical set.

Proposition 5.2. *If $f \in \mathrm{Rat}_D$ is postcritically finite but not a flexible Lattès map, then the Möbius conjugacy class of f contains a representative with algebraic coefficients.*

Proof. As in the previous proof, conjugating f with a Möbius transformation if necessary, we may assume that f fixes 0, 1 and ∞ and set $Y := \mathcal{P}_f \cup \{0,1,\infty\}$. In addition, set $X := f^{-1}(Y)$ and let $\delta : X \to \mathbb{N}$ be defined by

$$\delta(x) := \deg_x f.$$

Let us identify $\widehat{\mathbb{C}}$ and $\mathbb{P}^1(\mathbb{C})$ via the usual map $\mathbb{P}^1(\mathbb{C}) \ni [u:v] \mapsto u/v \in \widehat{\mathbb{C}}$. For $N \geq 1$, let us denote by $\mathcal{H}_N$ the vector space of homogeneous polynomials of degree N from $\mathbb{C}^2$ to $\mathbb{C}$. There is a canonical isomorphism between $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}(\mathcal{H}_1)$: a point of $\mathbb{P}^1(\mathbb{C})$, a 1-dimensional linear subspace of $\mathbb{C}^2$, is identified with the space of forms on $\mathbb{C}^2$ vanishing on this linear subspace; that is, a 1-dimensional linear subspace of $\mathcal{H}_1$. This subsequently yields an identification of $\widehat{\mathbb{C}}$ with $\mathbb{P}(\mathcal{H}_1)$.

Note that Rat_D may be identified with the open subset of $\mathbb{P}(\mathcal{H}_D \times \mathcal{H}_D)$ corresponding to pairs of coprime homogeneous polynomials of degree D. Such a pair of polynomials defines a nondegenerate homogeneous polynomial map $\mathbb{C}^2 \to \mathbb{C}^2$ of degree D which induces an endomorphism $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree D.

We shall denote as $\widehat{\mathbb{C}}_D$ the D-fold symmetric product of the Riemann sphere; that is, the quotient of $\widehat{\mathbb{C}}^D$ by the group of permutation of the coordinates. The map

$$\mathcal{H}_1^D \ni (P_1, \ldots, P_D) \mapsto P_1 \times \cdots \times P_D \in \mathcal{H}_D$$

induces an identification between $\widehat{\mathbb{C}}_D$ and $\mathbb{P}(\mathcal{H}_D)$.

Set

$$\mathcal{X} := \mathrm{Rat}_D \times \widehat{\mathbb{C}}^X \times \widehat{\mathbb{C}}^Y \quad \text{and} \quad \mathcal{Y} := (\widehat{\mathbb{C}}_D \times \widehat{\mathbb{C}})^Y.$$

A point $(g, \alpha, \beta) \in \mathcal{X}$ may be represented by a triple $\big(G, (A_x)_{x\in X}, (B_y)_{y\in Y}\big)$ where $G := (G_1, G_2) \in \mathcal{H}_D \times \mathcal{H}_D$ is a nondegenerate homogeneous polynomial map $\mathbb{C}^2 \to \mathbb{C}^2$ of degree D, and where $A_x \in \mathcal{H}_1$ and $B_y \in \mathcal{H}_1$ are linear forms $\mathbb{C}^2 \to \mathbb{C}$. Recall that $Y \subset X$ and consider the algebraic map $\Phi : \mathcal{X} \to \mathcal{Y}$ induced by

$$\big(G, (A_x)_{x\in X}, (B_y)_{y\in Y}\big) \mapsto \left(\prod_{x \in f^{-1}(y)} A_x^{\delta(x)}, A_y \right)_{y \in Y}$$

and the algebraic map $\Psi : \mathcal{X} \to \mathcal{Y}$ induced by

$$\big(G, (A_x)_{x\in X}, (B_y)_{y\in Y}\big) \mapsto \left(B_y \circ G, B_y\right)_{y\in Y}.$$

Let us consider the algebraic set $\mathcal{Z} \subset \mathcal{X}$ defined by the equation $\Phi = \Psi$.

We claim that the triple (g, α, β) belongs to $\mathcal{Z}$ if and only if we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \widehat{\mathbb{C}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\beta]{} & \widehat{\mathbb{C}} \end{array} \quad \text{with} \quad \deg_{\alpha(x)} g = \deg_x f, \quad \text{and} \quad \alpha|_Y = \beta|_Y.$$

Indeed,

$$B_y \circ G = \prod_{x \in f^{-1}(y)} A_x^{\delta(x)}$$

if and only if for each $x \in f^{-1}(y)$, the point $\alpha(x) \in \widehat{\mathbb{C}}$ is a preimage of $\beta(y) \in \widehat{\mathbb{C}}$ by g taken with multiplicity $\delta(x) = \deg_x f$. In addition, $A_y = B_y$ if and only if $\alpha(y) = \beta(y)$.

As a consequence, if $(g, \alpha, \beta) \in \mathcal{Z}$, then $\mathcal{C}_g = \alpha(\mathcal{C}_f) \subseteq \alpha(X)$. In that case, $\mathcal{P}_g = \beta(\mathcal{P}_f) \subseteq \beta(Y)$, and g is postcritically finite.

Set

$$\mathcal{Z}_0 := \big\{(g, \alpha, \beta) \in \mathcal{Z} \mid \alpha(0) = 0,\ \alpha(1) = 1,\ \text{and } \alpha(\infty) = \infty\big\}.$$

Note that the triple $(f, \mathrm{id}, \mathrm{id})$ belongs to $\mathcal{Z}_0$. According to Theorem 5.1, if f is not a flexible Lattès map, then the algebraic set $\mathcal{Z}_0$ has dimension 0 at f. This implies that f has algebraic coefficients. □

6. The eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$

Proposition 6.1. *If $f \in \mathrm{Rat}_D$, then 0 is an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$.*

Proof. Let Y be a subset of $\widehat{\mathbb{C}} \smallsetminus \mathcal{V}_f$ with $\mathrm{card}(Y) = 3$. Set $X := f^{-1}(Y)$. Note that X is disjoint from $\mathcal{C}_f$ and

$$\mathrm{card}(X) - \mathrm{card}(\mathcal{C}_f) \geq 3D - (2D-2) \geq D+2 \geq 4.$$

So there is a nonzero quadratic differential $\boldsymbol{q}$ which vanishes on $\mathcal{C}_f$ and whose poles are simple and contained in X. The quadratic differential $f_*\boldsymbol{q}$ is holomorphic on $\widehat{\mathbb{C}} \smallsetminus Y$ and has at most simple poles along Y. Thus, it has at most 3 poles counting multiplicities, which forces $f_*\boldsymbol{q} = 0$. □

Let us now assume $f \in \mathrm{Rat}_D$ is postcritically finite and set

$$\mathcal{Q}_f := \mathcal{Q}(\widehat{\mathbb{C}}; \mathcal{P}_f).$$

Note that $f_*(\mathcal{Q}_f) \subseteq \mathcal{Q}_f$. Indeed, if $\boldsymbol{q} \in \mathcal{Q}_f$, then the poles of $f_*\boldsymbol{q}$ are contained in $f(\mathcal{P}_f) \cup \mathcal{V}_f = \mathcal{P}_f$. So, the set of eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ may be written as the union $\Sigma_f \cup \Lambda_f$ with

$$\Sigma_f := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } f_* : \mathcal{Q}_f \to \mathcal{Q}_f\},$$

and

$$\Lambda_f := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f\}.$$

We postpone the study of Σ_f and focus now on Λ_f.

Proposition 6.2. *The elements of $\Lambda_f \smallsetminus \{0\}$ are the complex numbers λ such that $1/\lambda^m$ is the multiplier of a cycle of f of period m which is not contained in $\mathcal{P}_f$.*

Proof. To begin with, let us describe the space $\mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$. First, observe that if $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$, then the residue of $\boldsymbol{q}$ at a point $x \in \widehat{\mathbb{C}}$ is naturally a form on $\mathrm{T}_x\widehat{\mathbb{C}}$; that is, an element of $\mathrm{T}^*_x\widehat{\mathbb{C}}$. It may be defined as follows: if $\theta \in \mathrm{T}_x\widehat{\mathbb{C}}$ and $\boldsymbol{\theta}$ is a vector field defined and holomorphic near x with $\boldsymbol{\theta}(x) = \theta$, then

$$\mathrm{residue}(\boldsymbol{q}, x)(\theta) := \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}, x).$$

The result does not depend on the extension $\boldsymbol{\theta}$ since if $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are two holomorphic vector fields which coincide at x, then $\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2$ vanishes at x, so that $\boldsymbol{q} \otimes (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)$ is holomorphic near x and $\mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}_1, x) = \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}_2, x)$.

Second, set

$$B := \begin{cases} \mathcal{P}_f & \text{if } \mathrm{card}(\mathcal{P}_f) \geq 3, \\ \mathcal{P}_f \cup \{\alpha\} \text{ with } \alpha \text{ a repelling fixed point of } f & \text{if } \mathrm{card}(\mathcal{P}_f) = 2. \end{cases}$$

So $\mathrm{card}(B) \geq 3$, $f(B) = B$, and $\mathcal{Q}_f = \mathcal{Q}(\widehat{\mathbb{C}}; B)$ (the equality holds even when $\mathrm{card}(\mathcal{P}_f) = 2$ since in that case, both spaces are reduced to $\{0\}$) . Set

$$\Omega_f := \bigoplus_{x \in \widehat{\mathbb{C}} \smallsetminus B} \mathrm{T}^*_x\widehat{\mathbb{C}}.$$

Note that Ω_f is the space of 1-forms on $\widehat{\mathbb{C}}\smallsetminus B$ which vanish outside a finite set. Consider the map $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ defined by

$$\mathrm{Res}\big([\boldsymbol{q}]\big)(x) := \mathrm{residue}(\boldsymbol{q}, x).$$

This map is well defined since if $\boldsymbol{q}_1 \in \mathcal{Q}(\widehat{\mathbb{C}})$ and $\boldsymbol{q}_2 \in \mathcal{Q}(\widehat{\mathbb{C}})$ satisfy $\boldsymbol{q}_1 - \boldsymbol{q}_2 \in \mathcal{Q}_f$, then $\boldsymbol{q}_1 - \boldsymbol{q}_2$ is holomorphic on $\widehat{\mathbb{C}}\smallsetminus\mathcal{P}_f$, so that $\mathrm{residue}(\boldsymbol{q}_1, x) = \mathrm{residue}(\boldsymbol{q}_2, x)$ for all $x \in \widehat{\mathbb{C}}\smallsetminus B$.

Lemma 6.3. *The map* $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ *is an isomorphism of vector spaces.*

Proof. First, if $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$ and $\mathrm{residue}(\boldsymbol{q}, x) = 0$ for all $x \in \widehat{\mathbb{C}}\smallsetminus B$, then $\boldsymbol{q}$ is holomorphic outside B, so that $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}}; B) = \mathcal{Q}_f$. It follows that $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ is injective.

Second, given two distinct points x_1 and x_2 in $\widehat{\mathbb{C}}$, let $\boldsymbol{\omega}_{x_1,x_2}$ be the meromorphic 1-form on $\widehat{\mathbb{C}}$ which is holomorphic outside $\{x_1, x_2\}$, has residue 1 at x_1 and residue -1 at x_2:

$$\boldsymbol{\omega}_{x_1,x_2} = \begin{cases} \mathrm{d}z/(z - x_1) - \mathrm{d}z/(z - x_2) & \text{if } x_1 \neq \infty \text{ and } x_2 \neq \infty, \\ \mathrm{d}z/(z - x_1) & \text{if } x_2 = \infty, \\ -\mathrm{d}z/(z - x_2) & \text{if } x_1 = \infty. \end{cases}$$

Note that $\boldsymbol{\omega}_{x_1,x_2}$ does not vanish.

Third, choose three distinct points x_1, x_2 and x_3 in B. Given $\omega \in \Omega_f$, we may define a function $\phi : \widehat{\mathbb{C}}\smallsetminus B \to \mathbb{C}$ by

$$\omega(x) = \phi(x) \cdot \boldsymbol{\omega}_{x_2,x_3}(x).$$

Since ω vanishes outside a finite set, ϕ also vanishes outside a finite set. Set

$$\boldsymbol{q} := \sum_{x \in \widehat{\mathbb{C}}\smallsetminus B} \phi(x) \cdot \boldsymbol{\omega}_{x,x_1} \otimes \boldsymbol{\omega}_{x_2,x_3}.$$

Since x, x_1, x_2, and x_3 are distinct, $\boldsymbol{\omega}_{x,x_1} \otimes \boldsymbol{\omega}_{x_2,x_3} \in \mathcal{Q}(\widehat{\mathbb{C}})$. Since ϕ vanishes outside a finite set, the sum is finite and $\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$. By construction, for $x \in \widehat{\mathbb{C}}\smallsetminus B$,

$$\mathrm{residue}(\boldsymbol{q}, x) = \phi(x) \cdot \boldsymbol{\omega}_{x_2,x_3}(x) = \omega(x),$$

so that $\mathrm{Res}\big([\boldsymbol{q}]\big) = \omega$. It follows that $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ is surjective. □

Observe that if $y \in \widehat{\mathbb{C}}\smallsetminus B \subseteq \widehat{\mathbb{C}}\smallsetminus\mathcal{V}_f$, then $\mathrm{D}_x f : \mathrm{T}_x\widehat{\mathbb{C}} \to \mathrm{T}_y\widehat{\mathbb{C}}$ is invertible for any $x \in f^{-1}(y)$, and in that case, the isomorphism $\mathrm{Res} : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \Omega_f$ conjugates $f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$ to the linear map $f_* : \Omega_f \to \Omega_f$ defined by

$$f_*\omega(y) := \sum_{x \in f^{-1}(y)} \omega \circ (\mathrm{D}_x f)^{-1}.$$

Next, let $X \subset \widehat{\mathbb{C}}\smallsetminus B$ be a cycle of f of period m and multiplier μ. Note that the space

$$E_X := \bigoplus_{x \in X} \mathrm{T}^*_x\widehat{\mathbb{C}} \subset \Omega_f$$

has dimension m and is invariant by $f_* : \Omega_f \to \Omega_f$.

Lemma 6.4. *The endomorphism* $f_* : E_X \to E_X$ *is diagonalizable. Its eigenvalues are the* m*-th roots of* $1/\mu$.

Proof. Suppose $\lambda^m = 1/\mu$, and let $x_0 \mapsto x_1 \mapsto \ldots \mapsto x_{m-1} \mapsto x_m = x_0$ be the points of X. Let $\omega_0 \in \mathrm{T}^*_{x_0}\widehat{\mathbb{C}}$ be any nonzero form. For $1 \le j \le m$, define recursively

$$\omega_j := \lambda\omega_{j-1} \circ (\mathrm{D}_{x_{j-1}} f)^{-1} \in \mathrm{T}^*_{x_j}\widehat{\mathbb{C}}.$$

Then

$$\omega_m = \lambda^m \omega_0 \circ (\mathrm{D}_{x_0} f^{\circ m})^{-1} = \frac{\lambda^m}{\mu}\omega_0 = \omega_0$$

since $\mathrm{D}_{x_0} f^{\circ m} : \mathrm{T}_{x_0}\widehat{\mathbb{C}} \to \mathrm{T}_{x_0}\widehat{\mathbb{C}}$ is multiplication by μ and $\lambda^m = 1/\mu$. It follows that the 1-form $\omega \in \Omega_f$ defined by

$$\omega(x) = \begin{cases} 0 & \text{if } x \notin X, \\ \omega_j & \text{if } x = x_j \in X \end{cases}$$

satisfies $f_*\omega = \lambda\omega$ and $\omega \neq 0$ since $\omega_0 \neq 0$. . □

Finally, assume $\lambda \neq 0$ is an eigenvalue of $f_* : \Omega_f \to \Omega_f$ and let $\omega \in \Omega_f$ be an eigenvector associated to λ. Set $X := \{x \in \widehat{\mathbb{C}} \mid \omega(x) \neq 0\}$. If $\omega(y) \neq 0$, then there exists $x \in f^{-1}(y)$ such that $\omega(x) \neq 0$. Thus, $X \subseteq f(X)$ and since the cardinality of $f(X)$ is always less than or equal to the cardinality of X, we necessarily have $X = f(X)$. So $X \subset \widehat{\mathbb{C}} \smallsetminus B$ is a union of cycles of f. It follows from Lemma 6.4 that the eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$ other than $\lambda = 0$ are the complex numbers λ such that $1/\lambda^m$ is the multiplier of a cycle of f of period m which is not contained in B. This completes the proof of Proposition 6.2 when $B = \mathcal{P}_f$; that is, when $\mathrm{card}(\mathcal{P}_f) \ge 3$.

To complete the proof of Proposition 6.2 when $\mathrm{card}(\mathcal{P}_f) = 2$, observe that

- either f is conjugate to $z \mapsto 1/z^D$ in which case there are $D+1$ repelling fixed points, each with multiplier $\mu = -D$, so that the multipliers of the fixed points which are not contained in B are the multipliers of the fixed points which are not contained in $\mathcal{P}_f$;
- or f is conjugate to $z \mapsto z^D$; if $D > 2$, the proof is similar; if $D = 2$, there is a single repelling fixed point at $z = 1$, its multiplier is 2 and we must show that $1/2$ is an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f \to \mathcal{Q}(\widehat{\mathbb{C}})/\mathcal{Q}_f$; in that case the multiplier of the cycle of period $m = 2$ is $\mu = 4$, so that $1/2$ is a m-th root of $1/\mu$.

This completes the proof of Proposition 6.2. □

Corollary 6.5. *Let $f \in \mathrm{Rat}_D$ be postcritically finite. Then, $\Lambda_f \subset \mathbb{D}$.*

Proof. A cycle of f not contained in $\mathcal{P}_f$ is repelling. □

Example. If $f(z) = z^{\pm D}$, then a cycle of period m not contained in $\mathcal{P}_f$ is a repelling cycle of multiplier $(\pm D)^m$ and there is at least one such cycle for each period $m \ge 2$. It follows that

$$\Lambda_f = \{0\} \cup \left\{ \frac{\mathrm{e}^{2\pi \mathrm{i} p/q}}{D},\ p/q \in \mathbb{Q}/\mathbb{Z} \right\}.$$

Note that $\mathrm{card}(\mathcal{P}_f) = 2$, so that $\mathcal{Q}_f = \{0\}$ and $\Sigma_f = \emptyset$.

Example. If $f \in \mathrm{Rat}_D$ is a flexible Lattès map, then a cycle of period m not contained in $\mathcal{P}_f$ is a repelling cycle of multiplier $\sqrt{D}^m$ and there is at least one such cycle for each period $m \geq 1$. It follows that

$$\Lambda_f = \{0\} \cup \left\{ \frac{\mathrm{e}^{2\pi \mathrm{i} p/q}}{\sqrt{D}},\ p/q \in \mathbb{Q}/\mathbb{Z} \right\}.$$

Proposition 6.6. *Let $f \in \mathrm{Rat}_D$ be postcritically finite. If $\lambda \in \Lambda_f \smallsetminus \{0\}$, then λ is an algebraic number but not an algebraic integer.*

Proof. As discussed in the previous example, the proposition holds for flexible Lattès maps, so we may assume f is not a flexible Lattès map in this proof.

If $f \in \mathrm{Rat}_D$ and $g \in \mathrm{Rat}_D$ are conjugate by a Möbius transformation M; that is, $M \circ f = g \circ M$, then the linear map $M_* : \mathcal{Q}_f \to \mathcal{Q}_g$ conjugates $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ to $g_* : \mathcal{Q}_g \to \mathcal{Q}_g$:

$$g_*(M_* \boldsymbol{q}) = M_*(f_* \boldsymbol{q}) = M_*(\lambda \boldsymbol{q}) = \lambda M_* \boldsymbol{q}.$$

Thus, $\Lambda_f = \Lambda_g$.

According to Proposition 5.2, the conjugacy class of f contains a representative with algebraic coefficients. Without loss of generality, we may therefore assume that this is the case for f and consider f as a rational map $f : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$. Working over the algebraically closed field $\overline{\mathbb{Q}}$, we deduce that the multipliers of cycles of f are algebraic numbers, so Λ_f consists of algebraic numbers.

If $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ is a Galois automorphism, then $g := \sigma \circ f \circ \sigma^{-1} : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$ is postcritically finite. In addition, since $g_*(\sigma_* \boldsymbol{q}) = \sigma_*(f_* \boldsymbol{q})$ and $\sigma(\lambda)\ \sigma_* \boldsymbol{q} = \sigma_*(\lambda \boldsymbol{q})$, we have that

$$f_* \boldsymbol{q} = \lambda \boldsymbol{q} \quad \Longleftrightarrow \quad g_*(\sigma_* \boldsymbol{q}) = \sigma(\lambda)\ \sigma_* \boldsymbol{q}.$$

Thus, $\lambda \in \Lambda_f$ if and only if $\sigma(\lambda) \in \Lambda_g$. According to Corollary 6.5, if $\lambda \in \Lambda_f$, then $\left|\sigma(\lambda)\right| < 1$ for any Galois automorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$. Thus, if λ is an algebraic integer, the product of λ with its Galois conjugates is an integer of modulus less than 1, which forces $\lambda = 0$. □

7. The eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$

From now on, we assume that $\mathrm{card}(\mathcal{P}_f) \geq 4$ so that $\mathcal{Q}_f$ is not reduced to $\{0\}$. In that case, the dimension of $\mathcal{Q}_f$ is $\mathrm{card}(\mathcal{P}_f) - 3$, so $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ has at most $\mathrm{card}(\mathcal{P}_f) - 3$ eigenvalues.

7.1. Lattès maps with four postcritical points.

According to Proposition 4.4, if Σ_f contains an eigenvalue of modulus 1, then f is a Lattès map with $\mathrm{card}(\mathcal{P}_f) = 4$. The converse is also true.

Proposition 7.1. *Suppose $f \in \mathrm{Rat}_D$ is a Lattès map with $\mathrm{card}(\mathcal{P}_f) = 4$. Then $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ is multiplication by λ with $|\lambda| = 1$. In addition, $\lambda = \pm 1$, or λ belongs to an imaginary quadratic number field. Any imaginary quadratic number of modulus 1 may arise for some Lattès map. If λ is an algebraic integer, then λ is a root of unity of order 1, 2, 3, 4, or 6.*

Proof. Since $\mathrm{card}(\mathcal{P}_f) = 4$, the dimension of $\mathcal{Q}_f$ is 1, so $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ has a unique eigenvalue, and f_* is multiplication by this eigenvalue. By assumption, there is a complex torus $\mathcal{T}$, a ramified cover $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$ ramifying at each point above $\mathcal{P}_f$

with local degree 2, and an endomorphism $L : \mathcal{T} \to \mathcal{T}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{L} & \mathcal{T} \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ \widehat{\mathbb{C}} & \xrightarrow[f]{} & \widehat{\mathbb{C}}. \end{array}$$

As mentioned in Proposition 4.4, if $\boldsymbol{q} \in \mathcal{Q}_f$, then $\Theta^*\boldsymbol{q}$ is a multiple of $(\mathrm{d}z)^2$ and if λ is the eigenvalue of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$, then $L_*(\Theta^*\boldsymbol{q}) = \dfrac{1}{D\lambda}\Theta^*\boldsymbol{q}$. Thus, if $L(w) = \alpha w + \beta$, then $\alpha^2 = D\lambda$.

According to [M, §5], we have $|\alpha|^2 = D$, so that $|\lambda| = 1$. In addition, either

- $\alpha \in \mathbb{Z}$ in which case $\lambda = 1$, and f is a flexible Lattès map, or
- α is an imaginary quadratic integer; that is, $\alpha^2 - C\alpha + D = 0$ with $C \in \mathbb{Z}$ and $C^2 < 4D$.

In the latter case, $\lambda = \alpha^2/D \in \mathbb{Q}[\alpha]$ is either -1 or an imaginary quadratic number of modulus 1.

Conversely, suppose $\lambda = -1$ or λ is an imaginary quadratic number of modulus 1. Let $k \geq 2$ be a sufficiently large integer so that $\alpha := k\sqrt{\lambda}$ is an imaginary quadratic integer and set $D := k^2$. According to [M, §5], there exists a Lattès map $f \in \mathrm{Rat}_D$ with $L(w) = \alpha w$. According to the previous discussion, the eigenvalue of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ is λ.

Finally, if λ is a quadratic integer, then it is a unit since $|\lambda| = 1$. Thus, it is a root of unity of order 1, 2, 3, 4, or 6. □

7.2. **Non Lattès maps.** We now assume that f is not a Lattès map. In that case, according to Corollary 4.2 and Proposition 4.4, the eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ are contained in the unit disk.

Proposition 7.2. *Let $f \in \mathrm{Rat}_D$ be postcritically finite with $\mathrm{card}(\mathcal{P}_f) \geq 4$, and suppose that f is not a Lattès map. If $\lambda \in \Sigma_f \setminus \{0\}$, then λ is an algebraic number but not an algebraic integer.*

Proof. We proceed as in the proof of Proposition 6.6. Conjugating f with a Möbius transformation if necessary, we may assume that f is a rational map $f : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$. Working over $\overline{\mathbb{Q}}$, we deduce that the eigenvalues of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ are algebraic numbers.

Let $\lambda \in \Sigma_f$, and let $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ be a Galois automorphism. Then, $\sigma(\lambda) \in \Sigma_g$ with $g := \sigma \circ f \circ \sigma^{-1} : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$, so $\bigl|\sigma(\lambda)\bigr| < 1$. Thus, if λ is an algebraic integer, then $\lambda = 0$. □

7.3. **An example where $\Sigma_f = \{0\}$.** Proposition 6.1 establishes that $0 \in \Lambda_f$. However, 0 does not necessarily belong to Σ_f. For example, if $f : \mathbb{C} \to \mathbb{C}$ is a polynomial with periodic critical points, then $0 \notin \Sigma_f$ (see [BEKP]).

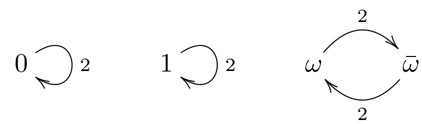
We now present an example of a postcritically finite rational map f for which $0 \in \Sigma_f$; this example appears in [BEKP].

Proposition 7.3. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rational map given by $f(z) = \dfrac{3z^2}{2z^3 + 1}$. Then $\Sigma_f = \{0\}$.*

Proof. The critical set of f is $\mathcal{C}_f = \{0, 1, \omega, \bar{\omega}\}$, where

$$\omega := -1/2 + \mathrm{i}\sqrt{3}/2 \quad \text{and} \quad \bar{\omega} := -1/2 - \mathrm{i}\sqrt{3}/2$$

are cube roots of unity. The postcritical set of f is $\mathcal{P}_f = \{0, 1, \omega, \bar{\omega}\}$, and f has the following postcritical dynamics.



Since $\mathrm{card}(\mathcal{P}_f) = 4$, the space $\mathcal{Q}_f$ is 1-dimensional, and there is a single eigenvalue λ. Consider

$$\boldsymbol{q} := \frac{(\mathrm{d}z)^2}{z(z^3-1)} \in \mathcal{P}_f, \quad \text{so that} \quad f_*\boldsymbol{q} = \lambda \boldsymbol{q}.$$

Let $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rotation $z \mapsto \omega z$. Then,

$$f \circ g(z) = f(\omega z) = \omega^2 f(z) = g^{\circ 2} \circ f(z).$$

Setting $u = g(z) = \omega z$, we have that

$$g_*\boldsymbol{q} = g_*\left(\frac{(\mathrm{d}z)^2}{z(z^3-1)}\right) = \frac{(\mathrm{d}u)^2/\omega^2}{u/\omega \cdot (u^3-1)} = \frac{\boldsymbol{q}}{\omega}.$$

As a consequence,

$$f_*(g_*\boldsymbol{q}) = f_*\left(\frac{\boldsymbol{q}}{\omega}\right) = \frac{f_*\boldsymbol{q}}{\omega} \quad \text{and} \quad g_*^{\circ 2}(f_*\boldsymbol{q}) = \frac{f_*\boldsymbol{q}}{\omega^2}.$$

It follows that

$$\frac{f_*\boldsymbol{q}}{\omega} = \frac{f_*\boldsymbol{q}}{\omega^2}$$

and since $\omega \neq \omega^2$, we necessarily have $f_*\boldsymbol{q} = 0$. □

8. Characteristic polynomials

In this section, the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is postcritically finite with postcritical set $\mathcal{P}_f$ and $f(\infty) = \infty$. Let d_∞ be the local degree of f at ∞, and let μ_∞ be the multiplier of f at ∞. Note that $d_\infty \geq 2$ if and only if $\mu_\infty = 0$.

Our goal is to compute the characteristic polynomial χ_f of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$:

$$\chi_f(\lambda) := \det(\lambda \cdot \mathrm{id} - f_*).$$

Set $X := \mathcal{P}_f \smallsetminus \{\infty\}$ and consider the square matrix A_f whose coefficients $a_{y,x}$, indexed by $X \times X$, are defined by:

$$a_{y,x} := \sum_{w \in f^{-1}(y) \cap (\mathcal{C}_f \cup \{x\})} \mathrm{residue}\left(\frac{\mathrm{d}z}{(z-x)f'(z)}, w\right).$$

Proposition 8.1. *We have that* $\det(\lambda \cdot \mathrm{I} - A_f) = \xi_f(\lambda) \cdot \chi_f(\lambda)$ *with*

$$\xi_f(\lambda) := \begin{cases} (\lambda - \mu_\infty)(\lambda - 1/d_\infty) & \text{if } \infty \in \mathcal{P}_f, \\ (\lambda - \mu_\infty)(\lambda - 1/d_\infty)(\lambda - 1/\mu_\infty) & \text{if } \infty \notin \mathcal{P}_f. \end{cases}$$

The remainder of §8 is devoted to the proof of this proposition. We first outline a sketch of the proof.

Step 1. Instead of working in $\mathcal{Q}_f$, we introduce the following vector spaces of meromorphic quadratic differentials:

- $\mathcal{Q}_f^1$ for those with at worst simple poles at the points in $\mathcal{P}_f \cup \{\infty\}$,
- $\mathcal{Q}_f^2$ for those with at worst simple poles at points in $\mathcal{P}_f$, and at worst a double pole at ∞, and
- $\mathcal{Q}_f^3$ for those with at worst simple poles at points in $\mathcal{P}_f$, and at worst a triple pole at ∞.

We will show that each of these spaces is invariant under f_*. As subspaces,

$$\mathcal{Q}_f \subseteq \mathcal{Q}_f^1 \subset \mathcal{Q}_f^2 \subset \mathcal{Q}_f^3.$$

If $\infty \in \mathcal{P}_f$, then $\mathcal{Q}_f = \mathcal{Q}_f^1$. Otherwise $\dim \mathcal{Q}_f^1/\mathcal{Q}_f = 1$. In all cases

$$\dim \mathcal{Q}_f^3/\mathcal{Q}_f^2 = \dim \mathcal{Q}_f^2/\mathcal{Q}_f^1 = 1.$$

Step 2. We will show that the eigenvalues of the induced endomorphisms

$$\mathcal{Q}_f^3/\mathcal{Q}_f^2 \to \mathcal{Q}_f^3/\mathcal{Q}_f^2, \quad \mathcal{Q}_f^2/\mathcal{Q}_f^1 \to \mathcal{Q}_f^2/\mathcal{Q}_f^1 \quad \text{and} \quad \mathcal{Q}_f^1/\mathcal{Q}_f \to \mathcal{Q}_f^1/\mathcal{Q}_f$$

are given in Table 1.

	$\mathcal{Q}_f^3/\mathcal{Q}_f^2$	$\mathcal{Q}_f^2/\mathcal{Q}_f^1$	$\mathcal{Q}_f^1/\mathcal{Q}_f$
$\infty \notin \mathcal{P}_f$, $d_\infty = 1$	μ_∞	1	$1/\mu_\infty$
$\infty \in \mathcal{P}_f$, $d_\infty = 1$	μ_∞	1	none
$\infty \in \mathcal{P}_f$, $d_\infty \geq 2$	0	$1/d_\infty$	none

Table 1. The eigenvalues of the quotient maps induced by f_* where μ_∞ is the multiplier of f at ∞, and d_∞ is the local degree of f at ∞.

Step 3. We will then compute the eigenvalues of $f_* : \mathcal{Q}_f^3 \to \mathcal{Q}_f^3$ as follows. The quadratic differentials

$$\left\{ \boldsymbol{q}_x := \frac{\mathrm{d}z^2}{z-x} \right\}_{x \in \mathcal{P}_f \smallsetminus \{\infty\}}$$

form a basis of $\mathcal{Q}_f^3$. According to Lemma 8.5 below, the matrix of $f_* : \mathcal{Q}_f^3 \to \mathcal{Q}_f^3$ in the basis $\{\boldsymbol{q}_x\}_{x \in \mathcal{P}_f \smallsetminus \{\infty\}}$ is the matrix A_f.

Step 4. For $k \in \{1,2,3\}$, let $\chi_{f,k}$ be the characteristic polynomial of $f_* : \mathcal{Q}_f^k \to \mathcal{Q}_f^k$, and let $\xi_{f,k}$ be the characteristic polynomial of $f_* : \mathcal{Q}_f^k/\mathcal{Q}_f^{k-1} \to \mathcal{Q}_f^k/\mathcal{Q}_f^{k-1}$, with the convention that $\mathcal{Q}_f^0 := \mathcal{Q}_f$ and $\xi_{f,1} = 1$ if $\mathcal{Q}_f^1 = \mathcal{Q}_f$; that is, if $\infty \notin \mathcal{P}_f$.

Since

$$\mathcal{Q}_f \subseteq \mathcal{Q}_f^1 \subset \mathcal{Q}_f^2 \subset \mathcal{Q}_f^3$$

are invariant by f_*, we have

$$\chi_{f,3} = \xi_{f,3} \cdot \chi_{f,2} = \xi_{f,3} \cdot \xi_{f,2} \cdot \chi_{f,1} = \xi_{f,3} \cdot \xi_{f,2} \cdot \xi_{f,1} \cdot \chi_f.$$

According to Step 2,

$$\xi_{f,3}(\lambda) = \lambda - \mu_\infty, \quad \xi_{f,2}(\lambda) = \lambda - \frac{1}{d_\infty}, \quad \text{and when } \infty \notin \mathcal{P}_f, \quad \xi_{f,1}(\lambda) = \lambda - \frac{1}{\mu_\infty}.$$

Therefore, $\xi_{f,3} \cdot \xi_{f,2} \cdot \xi_{f,1} = \xi_f$ and $\chi_{f,3} = \xi_f \cdot \chi_f$. Proposition 8.1 follows from Step 3:

$$\det(\lambda \cdot \mathrm{I} - A_f) = \chi_{f,3}(\lambda) = \xi_f(\lambda) \cdot \chi_f(\lambda).$$

We now proceed with the proof working step by step.

8.1. **Invariant subspaces.**

Lemma 8.2. *The vector spaces* $\mathcal{Q}^1_f, \mathcal{Q}^2_f$, *and* $\mathcal{Q}^3_f$ *are invariant by* f_*.

Proof. Suppose $\boldsymbol{q} \in \mathcal{Q}^k_f$ with $k \in \{1,2,3\}$. The poles of $f_*\boldsymbol{q}$ are contained in $f\big(\mathcal{P}_f \cup \{\infty\}\big) \cup \mathcal{V}_f = \mathcal{P}_f \cup \{\infty\}$.

Assume $k = 1$. Then $\mathcal{Q}^1_f \subset \mathcal{Q}(\widehat{\mathbb{C}})$ and $f_*\boldsymbol{q} \in \mathcal{Q}(\widehat{\mathbb{C}})$, so that the poles of $f_*\boldsymbol{q}$ are simple. Thus, $f_*\boldsymbol{q} \in \mathcal{Q}^1_f$. In other words, $f_*(\mathcal{Q}^1_f) \subseteq \mathcal{Q}^1_f$.

Assume $k \in \{2,3\}$ and $y \in \mathcal{P}_f$. On the one hand, if $x \in f^{-1}(y) \setminus \{\infty\}$, then $2 + \mathrm{ord}_x\boldsymbol{q} \geq 1$ and

$$\frac{2 + \mathrm{ord}_x\boldsymbol{q}}{\deg_x f} - 2 \geq \frac{1}{\deg_x f} - 2 > -2.$$

In particular, if $y \neq \infty$, then $f_*\boldsymbol{q}$ has at worst a simple pole at y. On the other hand, if $x = \infty$, then $2 + \mathrm{ord}_\infty\boldsymbol{q} \geq 2 - k$ and since $2 - k \leq 0$,

$$\frac{2 + \mathrm{ord}_\infty\boldsymbol{q}}{\deg_\infty f} - 2 \geq 2 - k - 2 = -k.$$

It follows that $f_*\boldsymbol{q}$ has at worst a pole of order k at ∞. □

8.2. **Extra eigenvalues.** Here, we identify the eigenvalues arising from the induced operators $f_* : \mathcal{Q}^k_f/\mathcal{Q}^{k-1}_f \to \mathcal{Q}^k_f/\mathcal{Q}^{k-1}_f$ for $k \in \{1,2,3\}$ (see Table 1). The case of $\mathcal{Q}^1_f/\mathcal{Q}_f$ is covered by §6, more precisely by Lemma 6.4: if ∞ is a fixed point with multiplier μ_∞ not contained in $\mathcal{P}_f$ then $1/\mu_\infty$ is an eigenvalue of $f_* : \mathcal{Q}^1_f/\mathcal{Q}_f \to \mathcal{Q}^1_f/\mathcal{Q}_f$. We therefore only need to deal with $\mathcal{Q}^2_f/\mathcal{Q}^1_f$ and $\mathcal{Q}^3_f/\mathcal{Q}^2_f$.

Fix two vector fields $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$, where $\boldsymbol{\theta}_k$ is holomorphic near ∞ and vanishes to order $k-1$ at ∞. Let $\alpha_k : \mathcal{Q}^k_f \to \mathbb{C}$ be the form defined by

$$\alpha_k(\boldsymbol{q}) := \mathrm{residue}(\boldsymbol{q} \otimes \boldsymbol{\theta}_k, \infty).$$

This form is in the annihilator of $\mathcal{Q}^{k-1}_f$, and as such, α_k may be canonically identified with an element in the dual of the quotient $\mathcal{Q}^k_f/\mathcal{Q}^{k-1}_f$. Therefore, if λ_k is the eigenvalue of the induced operator $f_* : \mathcal{Q}^k_f/\mathcal{Q}^{k-1}_f \to \mathcal{Q}^k_f/\mathcal{Q}^{k-1}_f$, then

$$\alpha_k(f_*\boldsymbol{q}) = \lambda_k \ \alpha_k(\boldsymbol{q}).$$

The data presented in Table 1 is a consequence of the following lemma.

Lemma 8.3. *If* $\boldsymbol{q} \in \mathcal{Q}^3_f$, *then*

$$\alpha_3(f_*\boldsymbol{q}) = \mu_\infty \ \alpha_3(\boldsymbol{q}).$$

If $\boldsymbol{q} \in \mathcal{Q}^2_f$, *then*

$$\alpha_2(f_*\boldsymbol{q}) = \frac{1}{d_\infty} \ \alpha_2(\boldsymbol{q}).$$

Proof. Assume $\boldsymbol{q} \in \mathcal{Q}^k_f$, where $k \in \{2,3\}$. Observe that for $x \in f^{-1}(\infty) \smallsetminus \{\infty\}$, the 1-form $\boldsymbol{q} \otimes f^*\boldsymbol{\theta}_k$ is holomorphic at x. Indeed, $\boldsymbol{\theta}_k$ vanishes at ∞, $f^*\boldsymbol{\theta}_k$ vanishes at x, and $\boldsymbol{q}$ has at worst a simple pole at x. Therefore,

$$\begin{aligned}\alpha_k(f_*\boldsymbol{q}) &:= \mathrm{residue}\big((f_*\boldsymbol{q}) \otimes \boldsymbol{\theta}_k, \infty\big) \\ &\underset{\text{Lemma 3.1}}{=} \sum_{x \in f^{-1}(\infty)} \mathrm{residue}(\boldsymbol{q} \otimes f^*\boldsymbol{\theta}_k, x) = \mathrm{residue}(\boldsymbol{q} \otimes f^*\boldsymbol{\theta}_k, \infty).\end{aligned}$$

Case 1. If $k = 3$, then $\boldsymbol{\theta}_3$ vanishes to order 2 at ∞. It follows from Lemma 8.4 below that $f^*\boldsymbol{\theta}_3 - \mu_\infty\boldsymbol{\theta}_3$ vanishes to order 3 at ∞. Since $\boldsymbol{q}$ has at worst a triple pole at ∞, we have

$$\alpha_3(f_*\boldsymbol{q}) = \text{residue}(\boldsymbol{q} \otimes f^*\boldsymbol{\theta}_3, \infty) = \text{residue}(\boldsymbol{q} \otimes \mu_\infty\boldsymbol{\theta}_3, \infty) = \mu_\infty\alpha_3(\boldsymbol{q}).$$

Case 2. If $k = 2$, then $\boldsymbol{\theta}_2$ vanishes to order 1 at ∞. It follows from Lemma 8.4 below that $f^*\boldsymbol{\theta}_2 - \dfrac{1}{d_\infty}\boldsymbol{\theta}_2$ vanishes to order 2 at ∞. Since $\boldsymbol{q}$ has at worst a double pole at ∞, we have

$$\alpha_2(f_*\boldsymbol{q}) = \text{residue}(\boldsymbol{q} \otimes f^*\boldsymbol{\theta}_2, \infty) = \text{residue}\left(\boldsymbol{q} \otimes \frac{1}{d_\infty}\boldsymbol{\theta}_2, \infty\right) = \frac{1}{d_\infty}\alpha_2(\boldsymbol{q}). \qquad \square$$

Lemma 8.4. *Let f be a germ of a holomorphic map fixing a point x with multiplier μ and local degree d. Let $\boldsymbol{\theta}$ be a germ of a holomorphic vector field vanishing at x with order m.*

- *If $d = 1$, then $f^*\boldsymbol{\theta} - \mu^{m-1}\boldsymbol{\theta}$ vanishes to order $m + 1$ at x.*
- *If $m = 1$, then $f^*\boldsymbol{\theta} - \frac{1}{d}\boldsymbol{\theta}$ vanishes to order $m + 1$ at x.*
- *If $d \geq 2$ and $m \geq 2$, then $f^*\boldsymbol{\theta}$ vanishes to order $m + 1$ at x.*

Proof. Let ζ be a local coordinate vanishing at x. We may write

$$\zeta \circ f = a\zeta^d \cdot \big(1 + \mathcal{O}(\zeta)\big) \quad \text{and} \quad \boldsymbol{\theta} = b\zeta^m \frac{\mathrm{d}}{\mathrm{d}\zeta} \cdot \big(1 + \mathcal{O}(\zeta)\big)$$

with $a \neq 0$ and $b \neq 0$. In addition, if $d = 1$, then $a = \mu$. Then,

$$f^*\boldsymbol{\theta} = \frac{ba^m\zeta^{dm}}{da\zeta^{d-1}} \frac{\mathrm{d}}{\mathrm{d}\zeta} \cdot \big(1 + \mathcal{O}(\zeta)\big) = \frac{a^{m-1}}{d}\zeta^{(d-1)(m-1)}\boldsymbol{\theta} \cdot \big(1 + \mathcal{O}(\zeta)\big). \qquad \square$$

8.3. **The matrix A_f.**

Lemma 8.5. *The matrix of $f_* : \mathcal{Q}^3_f \to \mathcal{Q}^3_f$ in the basis $\{\boldsymbol{q}_x\}_{x \in \mathcal{P}_f \smallsetminus \{\infty\}}$ is A_f.*

Proof. Since $\{\boldsymbol{q}_y\}_{y \in \mathcal{P}_f \smallsetminus \{\infty\}}$ forms a basis of $\mathcal{Q}^3_f$, we may write

$$f_*\boldsymbol{q}_x = \sum_{y \in \mathcal{P}_f \smallsetminus \{\infty\}} f_{y,x} \cdot \boldsymbol{q}_y.$$

We need to show that $f_{y,x} = a_{y,x}$ for all $x, y \in \mathcal{P}_f \smallsetminus \{\infty\}$. We shall apply Lemma 3.1 with

$$\boldsymbol{q} := \boldsymbol{q}_x \quad \text{and} \quad \boldsymbol{\theta} := \frac{\mathrm{d}}{\mathrm{d}z}.$$

Fix $y_0 \in \mathcal{P}_f \smallsetminus \{\infty\}$. Note that $\boldsymbol{\theta}$ is holomorphic at y_0, and for $y \neq y_0$, $\boldsymbol{q}_y$ is holomorphic at y_0. In addition, $\boldsymbol{q}_{y_0} \otimes \boldsymbol{\theta} = \dfrac{\mathrm{d}z}{z - y_0}$. Therefore

$$\begin{aligned} f_{y_0,x} &= \text{residue}\big((f_*\boldsymbol{q}_x) \otimes \boldsymbol{\theta}, y_0\big) \\ &\underset{\text{Lemma 3.1}}{=} \sum_{w \in f^{-1}(y_0)} \text{residue}(\boldsymbol{q}_x \otimes f^*\boldsymbol{\theta}, w) \\ &= \sum_{w \in f^{-1}(y_0) \cap (\mathcal{C}_f \cup \{x\})} \text{residue}\left(\frac{\mathrm{d}z}{(z - x)f'(z)}, w\right) = a_{y_0,x}. \end{aligned}$$

For the third equality, we use that

$$\boldsymbol{q}_x \otimes f^*\boldsymbol{\theta} = \frac{\mathrm{d}z}{(z - x)f'(z)}$$

is holomorphic outside of $\mathcal{C}_f \cup \{x\}$. □

9. Periodic unicritical polynomials

From now on, we shall restrict our study to the case of unicritical polynomials. Any such polynomial is conjugate by an affine map to a polynomial of the form

$$f_c(z) := z^D + c \quad \text{with} \quad c \in \mathbb{C}.$$

The map f_c has a critical point at $z_0 = 0$ and a critical value at $f_c(0) = c$.

Such a polynomial is postcritically finite if and only if the critical point 0 is either periodic or preperiodic. We will restrict our study to the periodic case, and abusing terminology, we shall say that f_c *is periodic of period* m if 0 is periodic of period m for f_c.

9.1. **Families of unicritical polynomials.** Before studying the corresponding sets Σ_{f_c}, we introduce some subsets of parameter space. The Multibrot set $\mathcal{M}_D$ is defined as

$$\mathcal{M}_D := \left\{c \in \mathbb{C} \mid \text{the sequence } \left(f_c^{\circ n}(0)\right)_{n \geq 1} \text{ is bounded}\right\}.$$

The set $\mathcal{M}_2$ the set is called the *Mandelbrot set*.

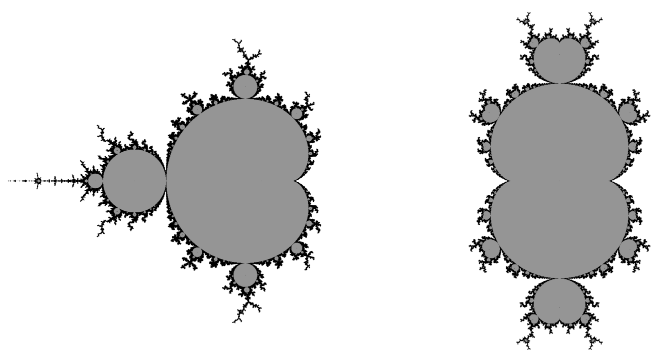


Figure 2. Left: the Mandelbrot set. Right: the Multibrot set $\mathcal{M}_3$.

Note that the set of parameters $c \in \mathbb{C}$ such that f_c has a periodic critical point is contained in the interior of $\mathcal{M}_D$. In addition, each component U of the interior of $\mathcal{M}_D$ contains at most one parameter c such that f_c is periodic; in that case, c is called the center of U.

The boundary of $\mathcal{M}_D$ is contained in the closure of the set of centers. Indeed, for any $c_0 \in \partial\mathcal{M}_D$ and any small neighborhood U of c_0, the family $(h_n : U \to \mathbb{C})_{n\geq 3}$ defined by $h_n(c) := f_c^{\circ n}(0)/c$ is uniformly bounded on $\mathcal{M}_D$ and converges to infinity outside $\mathcal{M}_D$. So it is not normal and cannot avoid 0 and 1 by Montel's theorem. When $h_n(c) = 0$, then $f_c^{\circ n}(0) = 0$; when $h_n(c) = 1$, then $f_c^{\circ(n-1)}(0) = 0$.

Observe that $\mathcal{M}_D$ has a symmetry of order $D - 1$. Indeed, for $c \neq 0$, the linear map $z \mapsto w := z/c$ conjugates the polynomial f_c to the polynomial

$$g_b : w \mapsto bw^D + 1 \quad \text{with} \quad b := c^{D-1}.$$

The polynomial g_b has a critical point at 0 and a critical value at $g_b(0) = 1$. We set

$$\mathcal{N}_D := \{b \in \mathbb{C} \mid \text{the sequence } \big(g_b^{\circ n}(0)\big)_{n\geq 1} \text{ is bounded}\big\}.$$

Then, $c \in \mathcal{M}_D$ if and only if $c^{D-1} \in \mathcal{N}_D$.

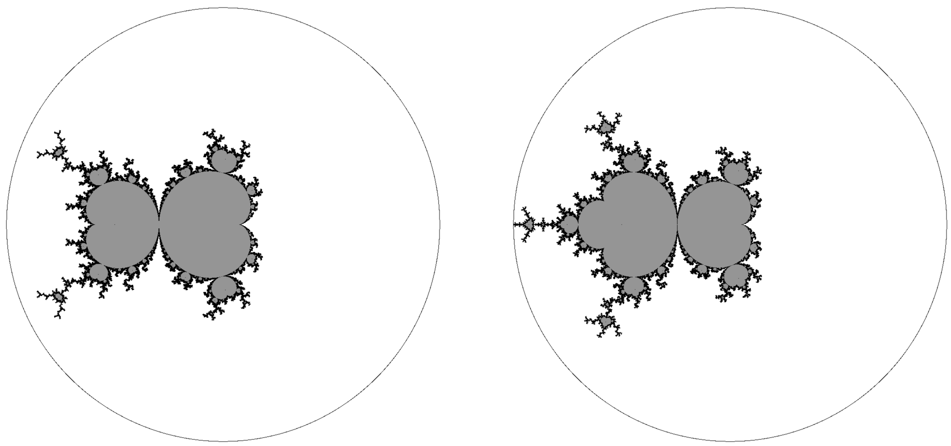


FIGURE 3. Left: the set $\mathcal{N}_3$. Right: the set $\mathcal{N}_4$. Both are contained in the closed disk $|b| \leq 2$.

If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, the filled-in Julia set K_f is defined as

$$K_f := \big\{z \in \mathbb{C} \mid \text{the sequence } \big(f^{\circ n}(z)\big)_{n\geq 1} \text{ is bounded}\big\}.$$

The following estimates are easily established.

Lemma 9.1. *If $b \in \mathcal{N}_D$, then $|b| < 2$.*

Corollary 9.2. *If $c \in \mathcal{M}_D$, then $|c|^{D-1} \leq 2$.*

Lemma 9.3. *If $c \in \mathcal{M}_D$ and $z \in K_{f_c}$, then $|z|^{D-1} \leq 2$.*

9.2. **Gleason polynomials.** For $n \geq 0$, let $G_n \in \mathbb{Z}[c]$ be defined by

$$G_n(c) := f_c^{\circ n}(0).$$

Alternatively, the polynomials $G_n \in \mathbb{Z}[c]$ may be defined recursively by:

$$G_0 := 0 \quad \text{and} \quad G_n(c) := G_{n-1}^D(c) + c.$$

Then, $G_n(c) = 0$ if and only if c is a center of period p dividing n.

Example. For $D = 2$, we have

$$G_1(c) = c, \quad G_2(c) = c^2 + c, \quad G_3(c) = c^4 + 2c^3 + c^2 + c.$$

Note that G_n is a monic polynomial. The following result is proven in [DH1] for $D = 2$. The proof easily generalizes to arbitrary degrees.

Lemma 9.4 (Gleason). *For each $n \geq 0$, the polynomial G_n has simple roots.*

Proof. For each $n \geq 1$,

$$G'_n = DG_{n-1}^{D-1}G'_{n-1} + 1 \equiv 1 \mod D.$$

In addition, G_n is monic, and so, the resultant of G_n and G'_n is equal to $1 \mod D$. Indeed, this resultant is equal to the determinant of a matrix that modulo D, is triangular with coefficients equal to 1 on the diagonal. In particular, it does not vanish. □

As a corollary, we deduce that for $m \geq 1$, there exists a (unique) monic polynomial $H_m \in \mathbb{Z}[c]$, such that for $n \geq 1$

$$G_n = \prod_{m|n} H_m.$$

The roots of H_m are exactly the centers of period m. The monomial of least degree of $G_n(c)$ is c with coefficient 1. Since $H_1(c) = c$, we see that for $m \geq 2$ the constant coefficient of H_m is 1. In particular, with the exception of $c = 0$, centers are algebraic units.

Example. For $D = 2$, we have $G_4 = H_1H_2H_4$ with

$$H_1(c) = c, \quad H_2(c) = c + 1 \quad \text{and} \quad H_4(c) = c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1.$$

We shall also use the following result due to Bjorn Poonen.

Lemma 9.5 (Poonen)**.** *For $m \neq n$,* resultant$(H_m, H_n) = \pm 1$.

Proof. Assume $n > m$. It is not hard to see by induction on $k \geq 1$, that

$$G_{m+k} \equiv G_k \mod G_m^D.$$

This implies that, $G_{mn} \equiv G_m \mod G_m^D$. Since $m < n$, G_mH_n divides G_{mn}. So, there are polynomials $A \in \mathbb{Z}[c]$ and $B \in \mathbb{Z}[c]$ such that

$$AG_mH_n = G_{mn} = G_m + BG_m^D.$$

Dividing by G_m yields $AH_n - BG_m^{D-1} = 1$. It follows that H_n and H_m are relatively prime in $\mathbb{Z}[c]$ and resultant$(H_m, H_n) = \pm 1$. □

Remark. For $m \geq 2$, we have $H_m(c) = J_m(c^{D-1})$ for some polynomial $J_m \in \mathbb{Z}[b]$. It might be tempting to conjecture that for all $D \geq 2$ and all $m \geq 2$, the polynomial J_m is irreducible. However for $D = 7$, J_3 is reducible:

$$J_3(b) = (b^2 + b + 1)(b^6 + 6b^5 + 14b^4 + 15b^3 + 6b^2 + 1).$$

This is further discussed in [B].

9.3. **Periodic points.**

Proposition 9.6. *Let f be a periodic unicritical polynomial of degree D. If $\lambda \in \Lambda_f \setminus \{0\}$, then $D\lambda$ is an algebraic unit and $\frac{1}{2D} \leq |\lambda| < 1$.*

Proof. According to Proposition 6.2, $\lambda \in \Lambda_f \setminus \{0\}$ if and only if $1/\lambda^n$ is the multiplier of a cycle of period n not contained in $\mathcal{P}_f$, i.e. a repelling cycle of f_c. In particular, $|\lambda| < 1$.

Assume the critical point of f is periodic of period m. Conjugating f with an affine map, we may assume that

$$f(z) = f_c(z) := z^D + c \quad \text{with} \quad H_m(c) = 0.$$

Now, let $z_1 \mapsto z_2 \mapsto \cdots \mapsto z_n \mapsto z_1$ be a cycle of period n. The multiplier of the cycle is

$$\mu = D^n (z_1 z_2 \cdots z_n)^{D-1}.$$

According to Lemma 9.3, $|z_j|^{D-1} \leq 2$, so that $|\mu| \leq (2D)^n$. Thus, it suffices to prove that the points z_j are algebraic units.

Let us first assume that m does not divide n, so that $f_c^{\circ n}(0) \neq 0$. The points z_j are roots of the polynomial $f_c^{\circ n}(z) - z \in \mathbb{Z}[c, z]$. Denote by R_z the polynomial $f_c^{\circ n}(z) - z$ considered as a polynomial of the variable c with coefficients in $\mathbb{Z}[z]$ and set

$$S(z) := \operatorname{resultant}\big(H_m, R_z\big) = \prod_{H_m(c)=0} \big(f_c^{\circ n}(z) - z\big).$$

Note that, as a product of monic polynomials, S is a monic polynomial. In addition, $f_c^{\circ n}(0) = G_n(c)$ and the constant coefficient of S is

$$S(0) = \prod_{H_m(c)=0} \big(f_c^{\circ n}(0) - 0\big) = \prod_{H_m(c)=0} G_n(c) = \operatorname{resultant}(H_m, G_n).$$

According to Lemma 9.5, since m does not divide n, this resultant is equal to ± 1. This shows that the points z_j are algebraic units.

Let us now assume that m divides n. Then, the constant coefficient of $f_c^{\circ n}(z) - z$ vanishes and

$$g_c(z) := \frac{f_c^{\circ n}(z) - z}{z}$$

is a monic polynomial with constant coefficient $g_c(0) = -1$. Denote by R_z the polynomial $g_c(z)$ considered as a polynomial of the variable c with coefficients in $\mathbb{Z}[z]$ and set

$$S(z) := \operatorname{resultant}\big(H_m, R_z\big) = \prod_{H_m(c)=0} g_c(z).$$

Again, S is a monic polynomial and its constant coefficient is

$$S(0) = \prod_{H_m(c)=0} g_c(0) = \pm 1.$$

Thus, the points z_j are algebraic units. □

9.4. **Characteristic polynomials.** We assume that c is a center of period $m \geq 3$ and $f := f_c$, so that $\dim(\mathcal{Q}_f) = m - 2 \geq 1$.

Note that $f(\infty) = \infty \in \mathcal{P}_f$. We denote by χ_f the characteristic polynomial of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ and for $k \in \{2, 3\}$, we denote by $\chi_{f,k}$ the characteristic polynomial of $f_* : \mathcal{Q}_f^k \to \mathcal{Q}_f^k$. The local degree of f at ∞ is D and the multiplier of f at ∞ is 0. According to §8, we have

$$\chi_{f,3}(\lambda) = \lambda \cdot \chi_{f,2}(\lambda) = \lambda \left(\lambda - \frac{1}{D}\right) \chi_f(\lambda).$$

For $n \in \mathbb{Z}/m\mathbb{Z}$, set

$$\zeta_n := f^{\circ n}(0), \quad \delta_n := f'(\zeta_n) = D\zeta_n^{D-1}.$$

Set

$$\Delta_1 := \delta_1, \quad \Delta_2 := \delta_1 \delta_2, \quad \ldots, \quad \Delta_{m-1} := \delta_1 \delta_2 \cdots \delta_{m-1}.$$

Proposition 9.7. *We have that*

$$\chi_{f,2}(\lambda) = \lambda^{m-1} + \frac{1}{\Delta_1}\lambda^{m-2} + \cdots + \frac{1}{\Delta_{m-2}}\lambda + \frac{1}{\Delta_{m-1}}.$$

Proof. For $n \in \mathbb{Z}/m\mathbb{Z}$, set

$$\boldsymbol{q}_n := \frac{(\mathrm{d}z)^2}{z-\zeta_n}.$$

The matrix A_f of $f_* : \mathcal{Q}_f^3 \to \mathcal{Q}_f^3$ in the basis $\{\boldsymbol{q}_n\}_{n\in\mathbb{Z}/m\mathbb{Z}}$ is provided by §8. We have

$$f_*\boldsymbol{q}_n = \begin{cases} 0 & \text{if } n = 0 \text{ and} \\ \dfrac{1}{\delta_n}\boldsymbol{q}_{n+1} - \dfrac{1}{\delta_n}\boldsymbol{q}_1 & \text{if } n \neq 0. \end{cases}$$

and so

$$A_f := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_{m-1} \\ 0 & -a_1 & -a_2 & \cdots & -a_{m-2} & -a_{m-1} \\ 0 & a_1 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & a_{m-2} & 0 \end{bmatrix} \quad \text{with} \quad a_n := \frac{1}{\delta_n}.$$

The characteristic polynomial of A_f is $\chi_{f,3}(\lambda) = \lambda\chi_{f,2}(\lambda)$, so that

$$\begin{aligned}\chi_{f,2}(\lambda) &= (\lambda + a_1)\lambda^{m-2} + a_1a_2\big(\lambda^{m-3} + a_3(\lambda^{m-4} + \cdots)\big) \\ &= \lambda^{m-1} + \frac{1}{\Delta_1}\lambda^{m-2} + \cdots + \frac{1}{\Delta_{m-2}}\lambda + \frac{1}{\Delta_{m-1}}.\end{aligned}$$ □

It shall be convenient to work with the polynomial $\chi_{f,2}/\chi_{f,2}(0)$ which has the same zeros as $\chi_{f,2}$. Setting

$$\Delta_0 := 1, \quad \Delta_{-1} := \delta_{m-1}, \quad \Delta_{-2} := \delta_{m-1}\delta_{m-2}, \quad \ldots, \quad \Delta_{-(m-1)} := \delta_{m-1}\delta_{m-2}\cdots\delta_1,$$

we get that for $n \in [\![1, m-1]\!]$,

$$\Delta_n \cdot \Delta_{-(m-1-n)} = \Delta_{m-1} = \frac{1}{\chi_{f,2}(0)},$$

so that

$$\frac{\chi_{f,2}(\lambda)}{\chi_{f,2}(0)} = 1 + \Delta_{-1}\lambda + \Delta_{-2}\lambda^2 + \cdots + \Delta_{-(m-1)}\lambda^{m-1}.$$

The polynomials $G_n \in \mathbb{Z}[c]$ defined by $G_n(c) = f_c^{\circ n}(0)$ have simple roots (see Lemma 9.4). This is related to the fact that for a postcritically finite rational map f which is not a flexible Lattès map, $1 \notin \Sigma_f$ (see Proposition 4.4).

Proposition 9.8. *If c is a center of period $m \geq 3$ and $f := f_c$, then*

$$G_m'(c) = (1-D)\frac{\chi_f(1)}{\chi_f(0)}.$$

Proof. We have $G_1(c) = c$ and $G_{n+1} = G_n^D(c) + c$, so that

$$G_m' = 1 + D\, G_{m-1}^{D-1}G_{m-1}' = 1 + \delta_{m-1}G_{m-1}'.$$

Since $G'_1 = 1$, we have

$$G'_m(c) = 1 + \delta_{m-1} \cdot \big(1 + \delta_{m-2} \cdot (1 + \cdots (1 + \delta_1))\big)$$
$$= 1 + \delta_{m-1} + \delta_{m-1}\delta_{m-2} + \cdots + \delta_{m-1}\delta_{m-2}\cdots\delta_1$$
$$= 1 + \Delta_{-1} + \Delta_{-2} + \cdots + \Delta_{-(m-1)} = \chi_{f,2}(1)/\chi_{f,2}(0).$$

Now,

$$\chi_{f,2}(0) = -\frac{1}{D}\chi_f(0) \quad \text{and} \quad \chi_{f,2}(1) = \left(1 - \frac{1}{D}\right)\chi_f(1)$$

so that

$$\frac{\chi_{f,2}(1)}{\chi_{f,2}(0)} = (1 - D)\frac{\chi_f(1)}{\chi_f(0)}. \qquad \square$$

9.5. **Algebraic units.**

Theorem 9.9. *Let f be a periodic unicritical polynomial of degree D. If λ is an eigenvalue of $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$, then $D\lambda$ is an algebraic unit.*

Proof. The polynomial f is conjugate to some polynomial f_c where c is a center of period m; that is, $H_m(c) = 0$. The eigenvalues of f_* are those of $(f_c)_*$.

For $n \in [\![0, m-1]\!]$, let $\Gamma_n \in \mathbb{Z}[c]$ be the polynomial defined by

$$\Gamma_n := \prod_{k=1}^{n} G_{m-k}^{D-1},$$

where as usual, an empty product is equal to 1. Then

$$\Delta_{-n} = D^n \prod_{k=1}^{n} \zeta_{m-k}^{D-1} = D^n \prod_{k=1}^{n} \big(f_c^{\circ(m-k)}(0)\big)^{D-1} = D^n \Gamma_n(c).$$

As a consequence,

$$\frac{\chi_{f_c,2}(\lambda)}{\chi_{f_c,2}(0)} = \sum_{n=0}^{m-1} D^n \Gamma_n(c)\lambda^n = R(c, D\lambda),$$

where $R \in \mathbb{Z}[c, \nu]$ is defined by

$$R(c, \nu) := \sum_{n=0}^{m-1} \Gamma_n(c)\nu^n.$$

We shall denote by R_ν the polynomial $R(c, \nu)$ considered as a polynomial in the variable c with coefficients in $\mathbb{Z}[\nu]$. Let $S_m \in \mathbb{Z}[\nu]$ be defined by

$$\text{(1)} \qquad S_m(\nu) := \text{resultant}(H_m, R_\nu) = \prod_{H_m(c)=0} R(c, \nu) = \prod_{H_m(c)=0} \frac{\chi_{f_c,2}(\nu/D)}{\chi_{f_c,2}(0)}.$$

If $\lambda \in \Sigma_{f_c}$ with $H_m(c) = 0$, then $\nu := D\lambda$ is a root of S_m.

On the one hand, the constant coefficient of S_m is $S_m(0) = 1$. On the other hand, the leading monomial of $R_m(c, \nu)$ considered as a polynomial of ν is $\Gamma_{m-1}(c)\nu^{m-1}$, so that the leading coefficient of S_m is:

$$\prod_{H_m(c)=0} \Gamma_{m-1}(c) = \text{resultant}(H_m, \Gamma_{m-1}) = \text{resultant}\left(H_m, \prod_{k=1}^{m-1} G_{m-k}^{D-1}\right).$$

By Lemma 9.5, this resultant is equal to ± 1. It follows that the roots of S_m are algebraic units. $\square$

We know that $\chi^2_{f_c}(\lambda) = (\lambda - 1/D)\chi_{f_c}(\lambda)$, so that the factor $(\nu - 1)$ appears $\deg(H_m)$ times in the product (1) defining S_m. In addition, if $c_1^{D-1} = c_2^{D-1}$, then the polynomials f_{c_1} and f_{c_2} are conjugate. So, each eigenvalue is counted $D-1$ times in the product. Thus, there is a (unique) polynomial $\Upsilon_m \in \mathbb{Z}[\nu]$ with constant coefficient 1 such that

$$S_m(\nu) = (1-\nu)^{\deg(H_m)} \cdot \Upsilon_m^{D-1}(\nu).$$

It would be interesting to prove that the roots of Υ_m are simple. For example, when this is the case, $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ is diagonalizable.

9.6. **The case $m = 3$.** We will prove that the roots of Υ_3 are simple.

Lemma 9.10. *For all $D \geq 2$,*

$$\Upsilon_3(\nu) = (\nu+1)^{D+1} - \nu^D.$$

Proof. We have that $G_3(c) = (c^D + c)^D + c$, so that

$$H_3(c) = A(c^{D-1}) \quad \text{with} \quad A(b) = b(b+1)^D + 1.$$

In addition, an elementary computation shows that

$$R(c,\nu) = (1-\nu)B_\nu(c^{D-1}) \quad \text{with} \quad B_\nu(b) = 1 - \nu b^2(b+1)^{D-1}.$$

Then,

$$\Upsilon_3(\nu) = \text{resultant}(A, B_\nu)$$

Observe that

$$\nu b A(b) + (b+1)B_\nu(b) = (\nu+1)b + 1.$$

Note that A is monic of degree $D+1$. We therefore have

$$\begin{aligned}(\nu+1)^{D+1} \cdot \text{resultant}\left(A(b), b + \frac{1}{\nu+1}\right) &= \text{resultant}\big(A(b), (\nu+1)b+1\big) \\ &= \text{resultant}\big(A(b), (b+1)B_\nu(b)\big) \\ &= \text{resultant}\big(A(b), b+1\big) \cdot \Upsilon_3(\nu)\end{aligned}$$

As a consequence,

$$\Upsilon_3(\nu) = (\nu+1)^{D+1} \cdot \frac{A\big(-1/(\nu+1)\big)}{A(-1)} = (\nu+1)^{D+1} - \nu^D. \qquad \square$$

Lemma 9.11. *For all $D \geq 2$, the roots of Υ_3 are simple.*

Proof. Note that $\nu\Upsilon_3'(\nu) - D\Upsilon_3(\nu) = (1+\nu)^D(\nu - D)$. So, if Υ_3 and Υ_3' had a common root, this would be either -1 or D. None of those are roots of Υ_3. $\square$

9.7. **Spectral gap.** For $m \geq 3$, let $\Sigma(D,m)$ be the union of the sets of eigenvalues in Σ_f for all unicritical polynomials f of degree D which are periodic of period m. Set

$$\Sigma(D) := \bigcup_{m\geq 3} \Sigma(D,m).$$

The following proposition completes the proof of Theorem 1.4.

Proposition 9.12 (Spectral gap)**.** *If $\lambda \in \Sigma(D)$, then*

$$\frac{1}{4D} < |\lambda| < 1.$$

Proof. Assume c is a center of period m and $f = f_c$. Assume $\lambda \in \Sigma_f$. According to Corollary 4.3, we have that $|\lambda| < 1$. We must show that $|\lambda| > 1/(4D)$.

Let us recall that

$$0 = \frac{\chi_{f,2}(\lambda)}{\chi_{f,2}(0)} = 1 + \Delta_{-1}\lambda + \Delta_{-2}\lambda^2 + \cdots + \Delta_{-(m-1)}\lambda^{m-1}.$$

with

$$\Delta_0 := 1, \quad \Delta_{-1} := \delta_{m-1}, \quad \Delta_{-2} := \delta_{m-1}\delta_{m-2}, \quad \ldots, \quad \Delta_{-(m-1)} := \delta_{m-1}\delta_{m-2}\cdots\delta_1.$$

Lemma 9.13. *For all $n \in \mathbb{Z}/m\mathbb{Z}$, we have that $|\delta_n| \le 2D$.*

Proof. Set $\zeta_n := f_c^{\circ n}(0)$. Note that $c \in \mathcal{M}_D$ and $\zeta_n \in K_{f_c}$. According to Lemma 9.3, we have $|\zeta_n|^{D-1} \le 2$ for all $n \in \mathbb{Z}/m\mathbb{Z}$, so

$$|\delta_n| = D|\zeta_n|^{D-1} \le 2D. \qquad \square$$

If $|z| \le \frac{1}{4D}$, then for all $k \in [\![1, m-1]\!]$,

$$|\Delta_{-k} z^k| < \frac{(2D)^k}{(4D)^k} = \frac{1}{2^k},$$

so

$$\left|\frac{\chi_{f,2}(z)}{\chi_{f,2}(0)}\right| \ge 1 - \sum_{k=1}^{m-1} \frac{1}{2^k} > 0.$$

Since $\chi_{f,2}(\lambda) = 0$, we necessarily have $|\lambda| > \frac{1}{4D}$. $\square$

9.8. **Equidistribution.** This section is devoted to the proof of Theorem 1.5.

In order to prove this result, we will show that when $r \in [r_D, 1]$, there exists a sequence of centers c_n such that the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $|z| = r$ as $n \to \infty$. This means the following.

Given a polynomial $P \in \mathbb{C}[z]$, we denote by $\mathfrak{m}_P$ the probability measure

$$\mathfrak{m}_P := \frac{1}{\deg(P)} \sum_{x \in \mathbb{C}} \mathrm{ord}_x(P)$$

where $\mathrm{ord}_x(P)$ is the order of vanishing of P at x.

Assume $(P_n \in \mathbb{C}[z])_{n\ge 0}$ is a sequence of polynomials. We say that as $n \to \infty$, the roots of P_n equidistribute according to a probability measure $\mathfrak{m}$ if the sequence of probability measures $(\mathfrak{m}_{P_n})_{n\ge 0}$ converges weakly to $\mathfrak{m}$.

We say that as $n \to \infty$, the roots of P_n equidistribute on a Euclidean circle if the roots of P_n equidistribute according to the normalized 1-dimensional Lebesgue measure on this circle.

Example. As $n \to \infty$, the roots of the polynomials $z^n - 1$ equidistribute on the unit circle $S^1 := \{|z| = 1\}$.

Proposition 9.14. *Suppose that the critical point of $f_{c_0}(z) = z^D + c_0$ is preperiodic to a repelling fixed point β_0 of multiplier μ. Then, there exists a sequence of centers $c_n \in \mathcal{M}_D$ converging to c_0 such that as $n \to \infty$, the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $\{|\lambda| = 1/|\mu|\}$.*

Remark. A similar argument establishes that when the critical point of $f_c : z \mapsto z^D + c$ is preperiodic to a repelling cycle of multiplier μ and period m, then there exists a sequence of centers $c_n \in \mathcal{M}_D$ converging to c such that as $n \to \infty$, the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $\{|\lambda| = 1/\sqrt[m]{|\mu|}\}$. We will not use this fact.

Proof. Fix $r_1 < r_0 < |\beta_0|$ and $\epsilon > 0$ so that f_c has an inverse branch

$$g_c : D(\beta_0, r_0) \to D(\beta_0, r_1)$$

for every $c \in D(c_0, \epsilon)$. For $c \in D(c_0, \epsilon)$, let $\beta(c)$ be the unique (repelling) fixed point of f_c in $D(\beta_0, r_0)$. The map β is holomorphic on $D(c_0, \epsilon)$ and $\beta(c_0) = \beta_0$.

Choose a point $z_{-n_0} \in D(\beta_0, r_0)$ in the backward orbit of 0 of f_{c_0}; that is, $f_{c_0}^{\circ n_0}(z_{-n_0}) = 0$ with $n_0 > 0$. Since 0 is not periodic (it is preperiodic to β_0), $(f_{c_0}^{\circ n_0})'(z_{-n_0}) \neq 0$. Thus, taking $\epsilon > 0$ closer to 0 if necessary, we may assume that there is a holomorphic function $\zeta_{-n_0} : D(c_0, \epsilon) \to D(\beta_0, r_0)$ defined implicitly by $f_c^{\circ n_0}(\zeta_{-n_0}(c)) = 0$. Set

$$\zeta_j(c) := f_c^{\circ(n_0+j)}(\zeta_{-n_0}(c)) \text{ for } j \geq -n_0$$

and

$$\zeta_j(c) := g_c^{\circ(-n_0-j)}(\zeta_{-n_0}(c)) \text{ for } j \leq -n_0.$$

We have specified a distinguished orbit for f_c

$$\ldots \mapsto \zeta_{-2}(c) \mapsto \zeta_{-1}(c) \mapsto \zeta_0(c) = 0 \mapsto \zeta_1(c) = c \mapsto \zeta_2(c) \mapsto \ldots$$

which converges to $\beta(c)$ in backward time.

Let k_0 be the preperiod of 0 to β_0. As $n \to \infty$, the sequence $(\zeta_{k_0} - \zeta_{-n})$ converges uniformly on $D(c_0, \epsilon)$, to the function $\zeta_{k_0} - \beta$. This function vanishes at c_0, but it is not identically 0. In particular, we can find a sequence (c_n) such that $c_n \to c_0$ as $n \to \infty$, and $\zeta_{k_0}(c_n) = \zeta_{-n}(c_n)$. Then, c_n is a center of period $m_n := n + k_0$.

Lemma 9.15. *There exists a constant K such that for all sufficiently large n and all $1 \leq j \leq m_n - 1$*

$$\frac{1}{K} < \left|\frac{\Delta_j(c_n)}{\mu^j}\right| \leq K \quad \textit{and} \quad \frac{1}{K} < \left|\frac{\Delta_{-j}(c_n)}{\mu^j}\right| \leq K$$

Proof. For $j \in \mathbb{Z}$, let $\delta_j(c)$ be the derivative of f_c at $\zeta_j(c)$. Taking ϵ closer to 0 if necessary, we may assume that there is a constant $K_0 > 0$ so that

$$\frac{1}{K_0} < \left|\frac{\delta_j(c)}{\mu}\right| < K_0 \tag{2}$$

for $c \in D(c_0, \epsilon)$ and $j \in \{-n_0, \ldots, -1\} \cup \{1, \ldots, k_0\}$.

Let d be the order of $\zeta_{k_0} - \beta$ at c_0 (it is true that $d = 1$ [DH1], but we will not require this fact). Now there is a constant K_1 so that

$$K_1|c - c_0|^d < |\zeta_{k_0}(c) - \beta(c)|. \tag{3}$$

Since g_c maps $D(\beta_0, r_0)$ to $D(\beta_0, r_1)$, by the Schwarz Lemma, there exists $\kappa < 1$ such that for all $z \in D(\beta_0, r_0)$ and all $c \in D(c_0, \epsilon)$,

$$|g_c(z) - \beta(c)| < \kappa|z - \beta(c)|.$$

Then, since $\zeta_{-n_0}(c) \in D(\beta_0, r_0)$ and $\zeta_{-n}(c) = g_c^{\circ(n-n_0)}\big(\zeta_{-n_0}(c)\big)$, we have

$$|\zeta_{-n}(c) - \beta(c)| < r_0\kappa^{n-n_0} \tag{4}$$

Set

$$r_n := \left(\frac{r_0\kappa^n}{K_1\kappa^{n_0}}\right)^{1/d} = K_2\kappa^{n/d} \quad \text{with} \quad K_2 := \left(\frac{r_0}{K_1\kappa^{n_0}}\right)^{1/d}. \tag{5}$$

For $c \in D(c_0, \epsilon) \smallsetminus D(c_0, r_n)$, so that $r_n \leq |c - c_0|$, we have

$$|\zeta_{-n}(c) - \beta(c)| \underset{(4)}{<} r_0 \frac{\kappa^n}{\kappa^{n_0}} \underset{(5)}{=} K_1 r_n^d \leq K_1 |c - c_0|^d \underset{(3)}{<} |\zeta_{k_0}(c) - \beta(c)|,$$

so that the leftmost quantity cannot be equal to the rightmost quantity. As a consequence, for n large enough,

$$|c_n - c_0| < r_n.$$

For $j \leq -n_0$ and $c \in D(c_0, \epsilon)$, the point $\zeta_j(c)$ belongs to $D(\beta_0, r_0)$. Since $\delta_j(c) = D\zeta_j^{D-1}(c)$, the branch of

$$c \mapsto \log \frac{\delta_j(c)}{\delta_j(c_0)}$$

which vanishes at c_0 is bounded by some constant K_3. According to the Schwarz Lemma,

$$\left| \log \frac{\delta_j(c_n)}{\delta_j(c_0)} \right| < \frac{K_3 r_n}{\epsilon} \underset{(5)}{=} K_4 \kappa^{n/d} \quad \text{with} \quad K_4 = \frac{K_2 K_3}{\epsilon}.$$

Then, for $-n \leq j_1 < j_2 \leq -n_0$ we have

$$\left| \log \frac{\delta_{j_1}(c_n) \cdots \delta_{j_2-1}(c_n)}{\delta_{j_1}(c_0) \cdots \delta_{j_2-1}(c_0)} \right| < n K_4 \kappa^{n/d} \underset{n\to\infty}{\longrightarrow} 0.$$

In addition, we have

$$\delta_{j_1}(c_0) \cdots \delta_{j_2-1}(c_0) = \mu^{j_2 - j_1} \frac{\phi'\big(\zeta_{j_1}(c_0)\big)}{\phi'\big(\zeta_{j_2}(c_0)\big)}$$

where $\phi : D(\beta_0, r_0) \to \mathbb{C}$ is the linearizing map conjugating f_{c_0} to multiplication by μ. So, there is a constant K_5 such that for n large enough and $-n \leq j_1 < j_2 \leq -n_0$,

$$\frac{1}{K_5} < \left| \frac{\delta_{j_1}(c_n) \cdots \delta_{j_2-1}(c_n)}{\mu^{j_2 - j_1}} \right| < K_5.$$

Using Inequality (2) and $\delta_j(c_n) = \delta_{m_n + j}(c_n)$ with $m_n = n + k_0$, we deduce that for n large enough and $1 \leq j_1 < j_2 \leq m_n - 1$

$$\frac{1}{K} < \left| \frac{\delta_{j_1}(c_n) \cdots \delta_{j_2-1}(c_n)}{\mu^{j_2 - j_1}} \right| < K \quad \text{with} \quad K := K_5 K_0^{k_0 + n_0 - 1}.$$

The lemma follows since for $1 \leq j \leq m_n - 1$,

$$\Delta_j = \delta_1(c_n) \cdots \delta_j(c_n) \quad \text{and} \quad \Delta_{-j} = \delta_{m_n - j}(c_n) \cdots \delta_{m_n - 1}(c_n). \qquad \square$$

According to Lemma 9.15, the coefficients of

$$P_n(z) := \frac{\chi_{f_{c_n},2}(z/\mu)}{\chi_{f_{c_n},2}(0)} = 1 + \frac{\Delta_{-1}}{\mu} z + \cdots + \frac{\Delta_{-(n-1)}}{\mu^{n-1}} z^{n-1}$$

and

$$Q_n(z) := \frac{\mu^{n-1} P_n(z)}{\Delta_{-(n-1)} z^{n-1}} = 1 + \frac{\mu}{\Delta_1} \frac{1}{z} + \cdots + \frac{\mu^{n-1}}{\Delta_{n-1}} \frac{1}{z^{n-1}}$$

are uniformly bounded. In particular, the sequence (P_n) is normal in $\mathbb{D}$, and the sequence (Q_n) is normal outside $\overline{\mathbb{D}}$. In Lemma 9.16 below, we prove that as $n \to \infty$, the roots of P_n equidistribute on the unit circle, so the roots of $\chi_{f_{c_n}}$ equidistribute on the circle $\big\{|\lambda| = 1/|\mu|\big\}$. $\square$

Lemma 9.16. *Let*

$$P_n = 1 + \cdots + c_n z^{d_n} \in \mathbb{C}[z] \quad \textit{and} \quad Q_n = \frac{P_n}{c_n z^{d_n}} = 1 + \cdots + \frac{1}{c_n z^{d_n}} \in \mathbb{C}[1/z].$$

If

- *the sequence* (d_n) *tends to* ∞ *as* $n \to \infty$,
- *the sequence* (P_n) *is normal in the unit disk* $\mathbb{D}$, *and*
- *the sequence* (Q_n) *is normal in* $\mathbb{C} \setminus \overline{\mathbb{D}}$,

then as $n \to \infty$, *the roots of* P_n *equidistribute on the unit circle* S^1.

Proof. Extracting a subsequence if necessary, we may assume that the sequence $\mathfrak{m}_{P_n}$ converges to a probability measure $\mathfrak{m}$ on $\widehat{\mathbb{C}}$. It is enough to show that $\mathfrak{m}$ coincides with the normalized Lebesgue measure on S^1. We first show that the support of $\mathfrak{m}$ is contained in S^1. We then show that its Fourier coefficients all vanish except the constant coefficient.

Extracting a further subsequence if necessary, we may assume that

- the sequence (P_n) converges locally uniformly in $\mathbb{D}$ to a holomorphic map ϕ, and
- the sequence (Q_n) converges locally uniformly in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to a holomorphic map ψ.

Since $P_n(0) = Q_n(\infty) = 1$, the limits satisfy $\phi(0) = \psi(\infty) = 1$. As a consequence, the zeros of P_n stay bounded away from 0 and ∞. In addition, ϕ and ψ do not identically vanish, so their zeros are isolated, and within any compact subset of $\mathbb{C} \setminus S^1$, the number of zeros of P_n (counting multiplicities) is uniformly bounded. This shows that the support of $\mathfrak{m}$ is contained in S^1.

Now choose $r < 1 < R$ so that all the roots of P_n remain in the annulus $A := \{r < |z| < R\}$. For $k \in \mathbb{Z}$, let m_k be the Fourier coefficient

$$m_k := \int_{S^1} z^k \, \mathrm{d}\mathfrak{m} = \int_A z^k \, \mathrm{d}\mathfrak{m} = \lim_{n\to\infty} \int_A z^k \, \mathrm{d}\mathfrak{m}_{P_n}$$

By the Residue Theorem, if $k > 0$, then

$$\begin{aligned} \frac{d_n}{2\pi \mathrm{i}} \cdot \int_A z^k \, \mathrm{d}\mathfrak{m}_{P_n} = \int_{|z|=R} z^k \frac{P_n'(z)}{P_n(z)} \, \mathrm{d}z &= \int_{|z|=R} z^k \cdot \left(\frac{d_n}{z} + \frac{Q_n'(z)}{Q_n(z)} \right) \, \mathrm{d}z \\ &= \int_{|z|=R} z^k \frac{Q_n'(z)}{Q_n(z)} \, \mathrm{d}z \\ &\underset{n\to\infty}{\longrightarrow} \int_{|z|=R} z^k \frac{\psi'(z)}{\psi(z)} \, \mathrm{d}z. \end{aligned}$$

Similarly, if $k < 0$, then

$$\frac{d_n}{2\pi \mathrm{i}} \cdot \int_A z^k \mathrm{d}\mathfrak{m}_{P_n} = \int_{|z|=r} z^k \frac{P_n'(z)}{P_n(z)} \, \mathrm{d}z \underset{n\to\infty}{\longrightarrow} \int_{|z|=r} z^k \frac{\phi'(z)}{\phi(z)} \, \mathrm{d}z.$$

In both cases, the limit is finite and since $d_n \to \infty$, we deduce that $m_k = 0$. □

To complete the proof of Theorem 1.5, it is enough to show that the set of t such that there exists a $c_0 \in \mathbb{C}$ for which the critical point of f_{c_0} is preperiodic to a fixed point of multiplier of modulus t, is dense in $[1, 1/r_D]$. This follows from Lemmas 9.17 and 9.18 below.

Lemma 9.17. *For each $t \in [1, 1/r_D]$, there is a parameter $c \in \partial\mathcal{M}_D$ such that f_c has a fixed point with multiplier of modulus t.*

Proof. The boundary of $\mathcal{M}_D$ is connected. As c varies in the boundary of $\mathcal{M}_D$, the multipliers of fixed points vary continuously. Thus, it suffices to show that $\partial\mathcal{M}_D$ contains a parameter c_0 for which f_{c_0} has a fixed point with multiplier of modulus 1, and a parameter c_1 for which f_{c_1} has a fixed point of modulus $1/r_D$.

Note that $f_c(\beta) = \beta$ and $f'_c(\beta) = \mu$ if and only if

$$c^{D-1} = \frac{\mu}{D}\left(1 - \frac{\mu}{D}\right)^{D-1} \quad \text{and} \quad \beta = \frac{c}{1 - \mu/D}. \tag{6}$$

First, when $c^{D-1} = \frac{1}{D}\left(1 - \frac{1}{D}\right)^{D-1}$, then f_c has a fixed point of multiplier 1. The corresponding parameters c belong to the boundary of $\mathcal{M}_D$. Second, let ω be a D-th root of unity closest to -1. If D is even, we have $\omega = -1$. If D is odd, we have $\omega = \exp\left(\pm\pi \mathrm{i}\frac{D-1}{D}\right)$. Note that

$$|1-\omega|^2 = \left(1 + \cos\frac{\pi}{D}\right)^2 + \sin^2\frac{\pi}{D} = 2 + 2\cos\frac{\pi}{D} = 4\cos^2\frac{\pi}{2D}.$$

In both cases,

$$|1-\omega| = \frac{1}{Dr_D}.$$

Set $\mu := D(1-\omega)$, choose c and define β so that Equation (6) holds. Then, f_c has a fixed point at β with multiplier μ of modulus $1/r_D$. In addition,

$$f_c(0) = c = \left(1 - \frac{\mu}{D}\right)\beta = \omega\beta, \quad \text{so that} \quad f_c^{\circ 2}(0) = \beta.$$

Thus, c is a *Misiurewicz parameter*; that is, f_c is postcritically finite with $c \in \partial\mathcal{M}_D$, as required. □

Lemma 9.18. *Let $c_0 \in \partial\mathcal{M}_D$, and let β_0 be a repelling fixed point of f_{c_0} of multiplier μ_0. Then, there exists a sequence of parameters c_n converging to c_0 such that f_{c_n} has a fixed point $\beta_n \in \mathcal{P}_{f_{c_n}}$ converging to β_0.*

Proof. Since β_0 is repelling, there is a function β defined and holomorphic near c_0, such that $f_c \circ \beta_c = \beta_c$. Let $\omega \neq 1$ be a D-th root of unity. Since $c_0 \in \partial\mathcal{M}_D$, the sequence of functions $\zeta_k := c \mapsto f_c^{\circ k}(0)$ is not normal at c_0. It follows from Montel's Theorem that in any neighborhood of c_0, the sequence (ζ_k) cannot avoid both β and $\omega\beta$. When $\zeta_k = \omega\beta$, then $\zeta_{k+1} = \beta$. So there is a sequence of complex numbers (c_n) converging to c_0 and a sequence of integers (k_n) tending to ∞ such that $\zeta_{k_n}(c_n) = \beta(c_n) =: \beta_n$. □

10. Questions for further study

We conclude with some remaining questions.

10.1. **Periodic unicritical polynomials.** Let f be a periodic unicritical polynomial of degree D. An eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ satisfies $\frac{1}{4D} < |\lambda| < 1$ and for D even $\Sigma(D)$ contains the annulus $\left\{\frac{1}{2D} \le |\lambda| \le 1\right\}$. The estimate obtained for D odd is not as good.

Question 1. For D odd, does $\Sigma(D)$ contain the annulus $\left\{\frac{1}{2D} \le |\lambda| \le 1\right\}$?

We shall say that an eigenvalue of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ is a *small eigenvalue* if $|\lambda| < \frac{1}{2D}$. According to Proposition 9.6, a small eigenvalue belongs to Σ_f. In particular, if the critical point is periodic of period m, there are at most $m-2$ small eigenvalues.

Question 2. How many small eigenvalues can a periodic unicritical polynomial have?

10.2. **Spectral gap.** Let f be a postcritically finite rational map. We saw that for periodic unicritical polynomials of degree D, the eigenvalues in $\Sigma_f \smallsetminus \{0\}$ and $\Lambda_f \smallsetminus \{0\}$ remain uniformly bounded away from 0. At the same time, the set of periodic unicritical polynomials of degree D is bounded in moduli space of degree D polynomials.

Conversely, let $\mathcal{M}$ be a compact subset of moduli space of degree D rational maps. Conjugacy classes in $\mathcal{M}$ may be represented by rational maps whose derivatives (for the spherical metric on $\widehat{\mathbb{C}}$) are uniformly bounded by some constant K. It follows that if μ is the multiplier of a cycle of period m of such a rational map f, then $|\mu|^{1/m} \le K$. This implies that if $\lambda \in \Lambda_f \smallsetminus \{0\}$, then $|\lambda| \ge 1/K$. We may ask whether a similar result holds for Σ_f.

Question 3. If the conjugacy class of f remains in a compact subset of moduli space of degree D rational maps, is the set $\Sigma_f \smallsetminus \{0\}$ bounded away from 0?

We saw that the eigenvalues of $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ are related to the multipliers of cycles of f. These in turn are related to the Lyapunov exponent $L(f)$ of f with respect to the equilibrium measure (which remains bounded on compact subsets of moduli space since it is continuous).

Question 4. Is there a relation between $\exp\bigl(-L(f)\bigr)$ and $\inf\bigl\{|\lambda| \ : \ \lambda \in \Sigma_f \smallsetminus \{0\}\bigr\}$ or $\inf\bigl\{|\lambda| \ : \ \lambda \in \Lambda_f \smallsetminus \{0\}\bigr\}$?

10.3. **Diagonalizability.**

Question 5. Let f be postcritically finite. Is $f_* : \mathcal{Q}_f \to \mathcal{Q}_f$ diagonalizable? Is $f_* : \mathcal{Q}(\widehat{\mathbb{C}}) \to \mathcal{Q}(\widehat{\mathbb{C}})$ diagonalizable?

Question 6. Let f be a periodic unicritical polynomial. Are the roots of the characteristic polynomial χ_f simple?

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