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NUMERICAL INVESTIGATION OF SEVERAL PROBLEMS IN NUMBER
THEORY

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**NUMERICAL INVESTIGATION OF SEVERAL
PROBLEMS IN NUMBER THEORY**

by
William Reynolds Monach

**A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
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Doctoral Committee:

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NOTATION

The letters p and q denote prime numbers. We use χ to denote a Dirichlet character mod q and χ_0 to denote the principal character. We have $e = \exp(1)$ and $e(\theta) = \exp(2\pi i \theta)$. Also we have

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(n) = \#\{m, 1 \leq m \leq n: m \text{ and } n \text{ are relatively prime}\},$$

$$\pi(u) = \sum_{p \leq u} 1,$$

$$\theta(u) = \sum_{p \leq u} \log p,$$

$$\psi(u) = \sum_{n \leq u} \Lambda(n),$$

$$\psi(u, \chi) = \sum_{n \leq u} \Lambda(n) \chi(n).$$

and

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt, \quad |\arg z| < \pi.$$

We let $N(T) = \#\{\gamma: 0 \leq \gamma \leq T\}$, where $\rho = \beta + i\gamma$ is a nontrivial zero of $\zeta(s)$, the Riemann zeta function. We use the Bessel functions

$$J_j(z) = \left(\frac{1}{2} z\right)^j \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4} z^2\right)^k}{k! \Gamma(j+k+1)}$$

and

$$I_j(z) = e^{-\frac{1}{2} j \pi i} J_j\left(ze^{\frac{1}{2} \pi i}\right) \quad (-\pi \leq \arg z \leq \frac{1}{2} \pi).$$

When we use Euler-Maclaurin summation, we make frequent use of the Bernoulli polynomials. The n th Bernoulli polynomial is by definition the unique polynomial of degree n with the property that $\int_x^{x+1} B_n(t) dt = x^n$. We make use of the fact that $B'_n(x) = nB_{n-1}(x)$. The Bernoulli polynomials which we use are

$$B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x,$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

and

$$B_5(x) = x^5 - \frac{5}{2} x^4 + \frac{5}{3} x^3 - \frac{x}{6}.$$

We use the standard notation

$$\hat{B}_n(x) = B_n(\{x\}),$$

where $\{x\}$ is the fractional part of x .

CHAPTER I

THE DISTRIBUTION OF $\arg L(1, \chi)$

1. Statement of results. Dirichlet L-functions and other sums involving Dirichlet characters have always attracted a great deal of attention. Here we discuss the distribution of $\arg L(1, \chi)$, where

$$L(1, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-1}$$

and χ is a Dirichlet character. More precisely, we examine

$$D(x) = \lim_{\substack{q \rightarrow \infty \\ q \text{ prime}}} D_q(x),$$

where

$$(1) \quad D_q(x) = \frac{1}{q-2} \# \{ \chi \pmod q, \chi \neq \chi_0 : \arg L(1, \chi) \leq x \} .$$

In section 2 we show that $D(x)$ is the distribution function of a sum of independent random variables. In sections 3 and 4, we show how to calculate the values of $D(x)$. The following table contains typical values for $D(x)$:

x	D(x)	Maximum possible error
$-(.5)\pi$	0.0000001894	0.000000002
$-(.4)\pi$	0.0007492671	0.000000002
$-(.3)\pi$	0.0208192351	0.000000002
$-(.2)\pi$	0.1080262718	0.000000002
$-(.1)\pi$	0.2767402886	0.000000002
0	0.5	0.0
$(.1)\pi$	0.7232597114	0.000000002
$(.2)\pi$	0.8919737282	0.000000002
$(.3)\pi$	0.9791807649	0.000000002
$(.4)\pi$	0.9992507329	0.000000002
$(.5)\pi$	0.9999998106	0.000000002

A graph of $D(x)$ for $0 \leq x \leq (.5)\pi$ appears as Figure 1. We also show that

$$\lim_{\substack{q \rightarrow \infty \\ q \text{ prime}}} \frac{1}{q-2} \# \{ \chi \bmod q, \chi \neq \chi_0 : \operatorname{Re} L(1, \chi) < 0 \} = 0.378 \cdot 10^{-6} + \epsilon,$$

where $|\epsilon| \leq 4 \cdot 10^{-9}$. We obtain these results using computer programs (Appendix A) based on Theorem 1.16.

P. D. T. A. Elliott [8], [9], [10] has studied the distribution of $L(1, \chi)$ and $\arg L(1, \chi)$, but has not obtained precise numerical results. Our computational approach is similar to that used by Rice [21].

It is interesting to observe that although $L(1, \chi)$ has negative real part for about one in every three million characters, no such character is known. We have computed $L(1, \chi)$ for all characters with prime modulus ≤ 1300 using the formulas in section 5. The smallest real part which we found was 0.1886 and the greatest argument which we found was $1.3865 = (0.4413)\pi$.

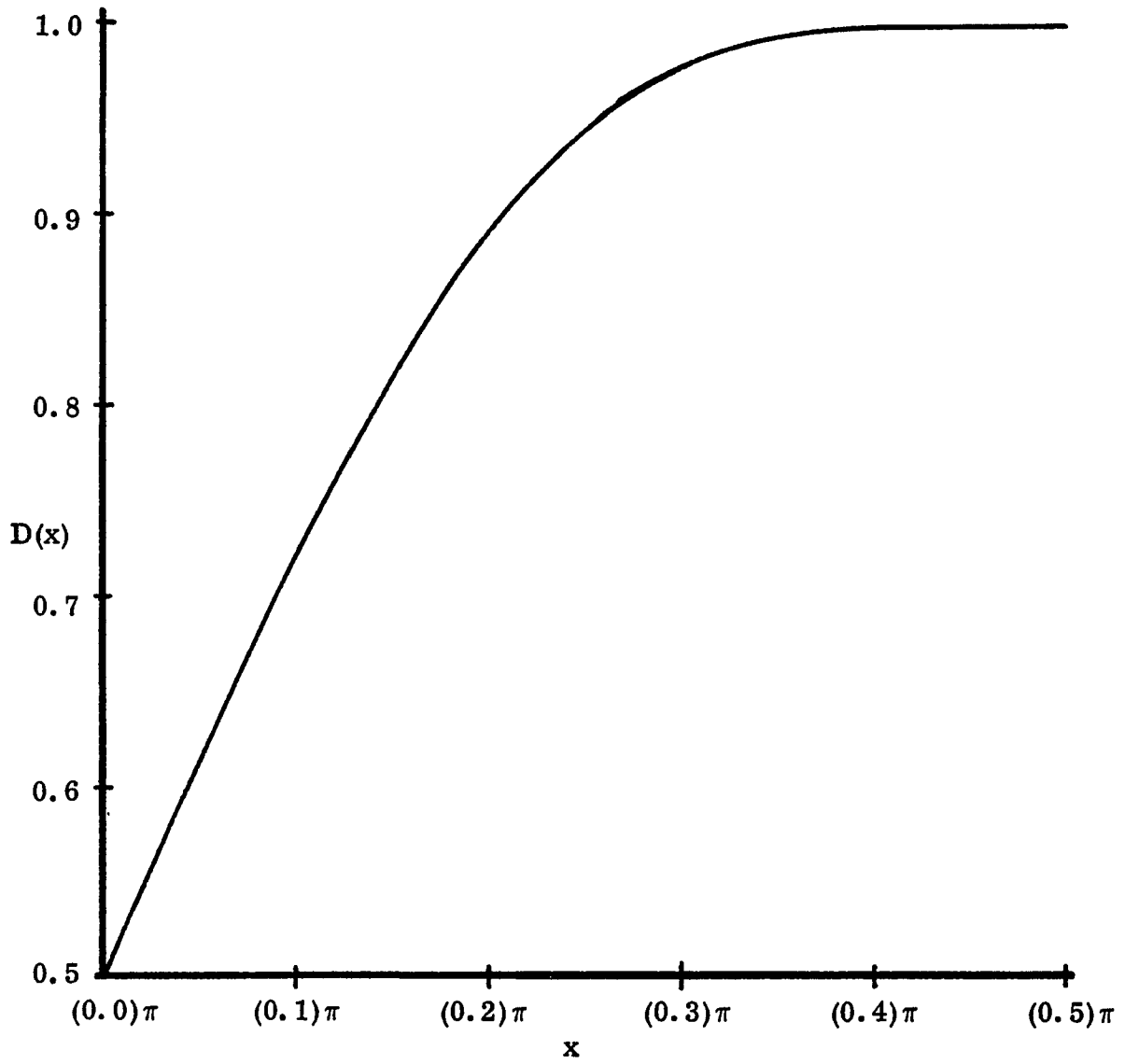


Fig. 1 - A Graph of $D(x)$ for $0 \leq x \leq (0.5)\pi$

2. Reduction to random variables. We wish to reduce the problem of finding $D(x)$ to the problem of determining the distribution of a random variable.

Since $L(1, \chi) = \prod_p (1 - \chi(p)/p)^{-1}$ for $\chi \neq \chi_0$, we have $\arg L(1, \chi) = \sum_p \arg (1 - \chi(p)/p)^{-1}$. As $\chi(2), \chi(3), \chi(5), \dots$ tend to be independently uniformly distributed on $|z| = 1$, we consider the possibility that the random variable

$\mathbb{Y} = \sum_p \arg(1 - e(\theta_p)/p)^{-1}$, where the θ_p are independent and uniformly distributed in $[0, 1)$, has a distribution similar to that of $\arg L(1, \chi)$. Using a standard result from probability theory (see Petrov [20, p.266]), we have $P(\mathbb{Y} \text{ converges}) = 1$, since $E(\arg(1 - e(\theta_p)/p)^{-1}) = 0$ and

$$\sum_{n=1}^{\infty} E(\arg(1 - e(\theta_p)/p)^{-1})^2 \ll \sum_{n=1}^{\infty} \frac{1}{p} \ll 1.$$

The relation between these two distributions is given in

Theorem 1.1. Let \mathbb{Y} be as above. Then $D(x)$ is the distribution function of \mathbb{Y} .

In addition, $D(x)$ is differentiable, $D(-x) + D(x) = 1$ and for $x \geq \frac{\pi}{2}$

$$e^{-e^{c_1} e^x} \leq 1 - D(x) \leq e^{-e^{c_2} e^x}$$

Here c_1 and c_2 are positive absolute constants.

Since the summands $\arg(1 - e(\theta_p)/p)^{-1}$ are symmetrically distributed, we have $F(-x) + F(x) = 1$, where $F(x)$ is the distribution function of \mathbb{Y} .

From Montgomery [16, Theorem 6.2], we have

Lemma 1.2. Let $s(\chi) = \sum_{n=M+1}^{M+N} a_n \chi(n)$, where χ is a character modulo q . Then

$$\sum_{\chi} |s(\chi)|^2 \leq \varphi(q) \left(1 + \left[\frac{N-1}{q} \right] \right) \sum_{\substack{M+1 \\ (n, q)=1}}^{M+N} |a_n|^2.$$

If $N \leq q$, then this holds with equality.

We now prove

Lemma 1.3. For all sufficiently large prime q ,

$$\sum_{\chi \neq \chi_0} \left| \log L(1, \chi) - \sum_{p \leq A} \log(1 - \chi(p)/p)^{-1} \right| < \frac{2(q-2)}{(A \log A)^{\frac{1}{2}}}$$

uniformly for $e^{631} \leq A \leq q(\log q)^{-7}$.

Proof. Since $L(1, \chi) = \prod_p (1 - \chi(p)/p)^{-1}$ for $\chi \neq \chi_0$, we have

$$(2) \left| \log L(1, \chi) - \sum_{p \leq A} \log(1 - \chi(p)/p)^{-1} \right| = \left| \sum_{p > A} \log(1 - \chi(p)/p)^{-1} \right|.$$

As $\log(1-z)^{-1} = \sum_{k=1}^{\infty} z^k/k$ for $|z| < 1$, the above is

$$\left| \sum_{p > A} \sum_{k=1}^{\infty} \chi(p)^k / kp^k \right| =$$

$$\left| \sum_{n > A} \Lambda(n) \chi(n) / n \log n - \sum_{p \leq A} \sum_{k > \log A / \log p} \chi(p)^k / kp^k \right|$$

$$\leq \left| \sum_{A < n \leq q} \Lambda(n) \chi(n) / n \log n \right| + \left| \sum_{q < n < U} \Lambda(n) \chi(n) / n \log n \right|$$

$$+ \left| \sum_{U \leq n} \Lambda(n) \chi(n) / n \log n \right| + \left| \sum_{p \leq A} \sum_{k > \log A / \log p} \chi(p)^k / kp^k \right|$$

$$= T_1(\chi) + T_2(\chi) + T_3(\chi) + T_4(\chi).$$

Using the Lemma 1.2, we obtain

$$(3) \quad \sum_{\chi \neq \chi_0} |T_1(\chi)| \leq \left(\sum_{\chi \neq \chi_0} 1 \right)^{\frac{1}{2}} \left(\sum_{\chi \neq \chi_0} \left| \sum_{A < n \leq q} \Lambda(n) \chi(n) / n \log n \right|^2 \right)^{\frac{1}{2}}$$

$$\leq (q-2)^{\frac{1}{2}} (q-1)^{\frac{1}{2}} \left(\sum_{A < n < q} \left| \Lambda(n) / n \log n \right|^2 \right)^{\frac{1}{2}}.$$

If $A > 1$, then

$$(4) \quad \sum_{A < n < q} \Lambda^2(n) / n^2 \log^2 n < \frac{1}{\log A} \sum_{n > A} \frac{\Lambda(n)}{n^2} < \frac{2}{\log A} \int_A^\infty \frac{\psi(u) du}{u^3}.$$

For $17 \leq u$, we have $|\psi(u) - u| < u \sqrt{8/17\pi} Y^{\frac{1}{2}} \exp(-Y)$, where

$Y = \sqrt{\log u / 9.645908801}$, as shown by Schoenfeld [24, Theorem 11]. Since $Y^{\frac{1}{2}} \exp(-Y)$ attains its maximum at $Y = \frac{1}{2}$, we have $\psi(u) < (1.166)u$ for $17 \leq u$.

As $\psi(u) < u$ for $u < 19$, we obtain $\psi(u) < (1.166)u$ for $0 < u$. Using this

inequality in (4), we obtain

$$(5) \quad \sum_{A < n < q} \frac{\Lambda^2(n)}{n^2 \log^2 n} < \frac{2.332}{A \log A}.$$

Combining (3) and (5), we have

$$(6) \quad \sum_{\chi \neq \chi_0} |T_1(\chi)| < \frac{(1.55)q}{(A \log A)^{\frac{1}{2}}}.$$

We observe that

$$|T_2(\chi)| = \left| \int_q^U \frac{d\psi(u, \chi)}{u \log u} \right| << \frac{|\psi(q, \chi)|}{q \log q} + \frac{|\psi(U, \chi)|}{U \log U} + \int_q^U \frac{|\psi(u, \chi)|}{u^2 \log u} du.$$

Vaughn [28, Theorem 2] has shown that if $q \geq 1$ and $u \geq 2$, then

$$\sum_{\chi} \max_{y \leq u} |\psi(y, \chi)| \ll u l^3 + u^{3/4} q^{5/8} l^{23/8} + u^{1/2} q l^{7/2},$$

where $l = \log(uq)$. Hence,

$$\begin{aligned} \sum_{\chi \neq \chi_0} |T_2(\chi)| &\ll q^{1/2} (\log q)^{5/2} + (1 + q^{5/8} U^{-1/4} + q U^{-1/2}) (\log U)^{5/2} \\ &\quad + (\log U)^3 + q^{5/8} U^{-1/4} (\log U)^{23/8} + q U^{-1/2} (\log U)^{7/2}. \end{aligned}$$

We now take $U = \exp(q^{1/7})$, so that

$$(7) \quad \sum_{\chi \neq \chi_0} |T_2(\chi)| \ll q^{1/2} (\log q)^{5/2}.$$

It is well known (see Davenport [6, pp. 135-136]) that $\psi(u, \chi) \ll u \exp(-c(\log u)^{1/2})$ for $\chi \neq \chi_0 \pmod{q}$, provided that $q \leq (\log u)^7$. Hence,

$$\begin{aligned} |T_3(\chi)| &= \left| \int_U^\infty \frac{d\psi(u, \chi)}{u \log u} \right| \ll \frac{|\psi(U, \chi)|}{U \log U} + \int_U^\infty \frac{|\psi(u, \chi)|}{u^2 \log u} du \\ &\ll \exp\left(-\frac{1}{2} c (\log U)^{1/2}\right) \ll q^{-1}, \end{aligned}$$

so that

$$(8) \quad \sum_{\chi \neq \chi_0} |T_3(\chi)| \ll 1.$$

As for $T_4(\chi)$, we note that, for $p \leq A$,

$$\left| \sum_{k > \log A / \log p} \frac{\chi(p)^k}{kp^k} \right| \leq \sum_{k > \log A / \log p} \frac{1}{p^k} \leq \min(p^{-2}, A^{-1}) \frac{1}{1 - \frac{1}{p}} \\ \leq 2 \min(p^{-2}, A^{-1}).$$

Hence,

$$|T_4(\chi)| \leq 2A^{-1} \sum_{p \leq A^{1/2}} 1 + 2 \sum_{A > p > A^{1/2}} p^{-2}.$$

From Rosser and Schoenfeld's work [23, Corollary 2], we have

$\pi(x) < (1.25506)x/\log x$ for $1 < x$. This gives us

$$\begin{aligned} |T_4(\chi)| &\leq 2A^{-1} \pi(A^{1/2}) + 2 \int_{A^{1/2}}^{\infty} \frac{d\pi(u)}{u^2} \\ &\leq 2A^{-1} \pi(A^{1/2}) + 2 \left(\frac{-\pi(A^{1/2})}{A} + 2 \int_{A^{1/2}}^{\infty} \frac{\pi(u) du}{u^3} \right) \\ &\leq 5.02024 \int_{A^{1/2}}^{\infty} \frac{du}{u^2 \log u} \leq \frac{5.02024}{\log A^{1/2}} \frac{1}{A^{1/2}} \\ &\leq \frac{10.04048}{A^{1/2} \log A}, \end{aligned}$$

so that

$$(9) \quad \sum_{\chi \neq \chi_0} |T_4(\chi)| \leq \frac{(10.04048)g}{A^{1/2} \log A}.$$

This completes the proof of Lemma 1.3, since by combining (2), (6), (7), (8), and (9), we obtain our desired result.

Our next step is to relate this lemma to the distribution of $\arg L(1, \chi)$.

Lemma 1.4. Let

$$D_q(x) = \frac{1}{q-2} \# \{ \chi \neq \chi_0 : \arg L(1, \chi) \leq x \}$$

and

$$D_q(x, A) = \frac{1}{q-2} \# \{ \chi \neq \chi_0 : \sum_{p \leq A} \arg(1 - \chi(p)/p)^{-1} \leq x \}$$

for prime q . If q is sufficiently large and $e^{631} \leq A \leq q(\log q)^{-7}$, then

$$D_q(x - \delta, A) - \frac{2}{\delta (A \log A)^{1/2}} < D_q(x) < D_q(x + \delta, A) + \frac{2}{\delta (A \log A)^{1/2}}$$

for all $\delta > 0$.

Proof. Let N denote the number of $\chi \neq \chi_0 \pmod{q}$ such that

$$\sum_{p \leq A} \arg(1 - \chi(p)/p)^{-1} \leq x - \delta \text{ but } \arg L(1, \chi) > x. \text{ Then } D_q(x - \delta, A) \leq D_q(x) + \frac{N}{q-2}.$$

But by Lemma 1.3,

$$N \leq \frac{2(q-2)}{\delta (A \log A)^{1/2}}.$$

This gives the lower bound for $D_q(x)$, and the upper bound is proved similarly.

Finally, we relate $D_q(x, A)$ to the distribution of a random variable. We first need

Lemma 1.5 (Weyl's criterion). Let μ_1, μ_2, \dots be probability measures on $\mathbb{I}^{\mathbb{R}}$.

Then the following are equivalent:

- (a) the μ_k tend weakly to Lebesgue measure on $\mathbb{I}^{\mathbb{R}}$;
 (b) for any Riemann-integrable function f defined on $\mathbb{I}^{\mathbb{R}}$,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{I}^{\mathbb{R}}} f(\underline{\theta}) d\mu_k(\underline{\theta}) = \int_{\mathbb{I}^{\mathbb{R}}} f(\underline{\theta}) d\underline{\theta} ;$$

- (c) for any $\underline{h} \in \mathbb{Z}^{\mathbb{R}}$, $\underline{h} \neq 0$,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{I}^{\mathbb{R}}} e(\underline{h} \cdot \underline{\theta}) d\mu_k(\underline{\theta}) = 0.$$

Proof. This lemma is a special case of the continuity theorem for R-dimensional characteristic functions. Billingsley [3, p. 329-335], proves that for probability measures μ_k and μ on $\mathbb{R}^{\mathbb{R}}$, the following are equivalent:

- (i) μ_k converges weakly to μ ;
 (ii) $\lim_k \int_{\mathbb{R}} q d\mu_k = \int_{\mathbb{R}} q d\mu$ for bounded continuous q ;
 (iii) $\lim_k \mu_k(A) = \mu(A)$ for all Borel sets A with $\mu(\partial A) = 0$.

Since the measures in the lemma are zero outside the unit cube, we see immediately that (a) \rightarrow (ii) \rightarrow (c) and (b) \rightarrow (ii) \rightarrow (a). By the definition of Riemann integration, we have (a) \rightarrow (iii) \rightarrow (b). Finally, we have (c) \rightarrow (ii) \rightarrow (a) using the Weierstrass approximation theorem. Using this lemma, we prove

Lemma 1.6. Let $F(x, A)$ be the distribution function of the random variable $\mathbb{Y}_A = \sum_{p \leq A} \arg(1 - e(\theta_p)/p)^{-1}$, where the θ_p are independent and uniformly distributed in $[0, 1)$. Given any $\epsilon > 0$, there is a $q_0(\epsilon)$ such that if $q > q_0(\epsilon)$, then

$$|D_q(x, A) - F(x, A)| < \epsilon$$

for all x .

Proof. The function $F(x, A)$ is continuous; thus, by compactness and monotonicity, it suffices to establish the inequality for any fixed x . We use Lemma 1.5 with $R = \pi(A)$; the primes $p \leq A$ index the coordinates of our vectors. Let

$$f(\underline{\theta}) = \begin{cases} 1 & \text{if } \sum_{p \leq A} \arg(1 - e(\theta_p)/p)^{-1} \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$F(x, A) = \int_{\mathbb{I}^R} f(\underline{\theta}) \, d\underline{\theta}$$

and

$$D_q(x, A) = \int_{\mathbb{I}^R} f(\underline{\theta}) \, d\mu_q(\underline{\theta}),$$

where μ_q is the probability measure which has point masses of weight $\frac{1}{q-2}$ at each of the points $\left(\frac{\arg \chi(p)}{2\pi}\right)$ for $\chi \neq \chi_0 \pmod{q}$. Thus, the desired result follows from (b). We establish (c), which is to say

$$\lim_{\substack{q \rightarrow \infty \\ q \text{ prime}}} \int_{\mathbb{I}^R} e(\underline{h} \cdot \underline{\theta}) \, d\mu_q(\underline{\theta}) = 0$$

for $\underline{h} \in \mathbb{Z}^R$, $\underline{h} \neq \underline{0}$. If $\underline{\theta} = \left(\frac{\arg \chi(p)}{2\pi}\right)$, then

$$\underline{h} \cdot \underline{\theta} = \sum_{p \leq A} h_p \frac{\arg \chi(p)}{2\pi} = \frac{1}{2\pi} \arg \prod_{p \leq A} \chi(p)^{h_p}.$$

Put $a/b = \prod_{p \leq A} p^{\underline{h}_p}$ in lowest terms; here a and b depend only on A and \underline{h} .
 Then $e(\underline{h} \cdot \underline{\theta}) = \chi(a) \bar{\chi}(b)$, and the integral above is

$$\frac{1}{q-2} \sum_{\chi \neq \chi_0} \chi(a) \bar{\chi}(b) = \begin{cases} 1 & \text{if } a \equiv b \pmod{q}, \\ -\frac{1}{q-2} & \text{otherwise.} \end{cases}$$

If q is large and $a \equiv b \pmod{q}$, then $a = b$. But $(a, b) = 1$, so that $a = b = 1$.

But then $\underline{h} = \underline{0}$, contrary to supposition. Hence,

$$\int_{\mathbb{R}} e(\underline{h} \cdot \underline{\theta}) d\mu_q(\underline{\theta}) = -\frac{1}{q-2}$$

for all large primes q and (c) is established. This completes the proof of Lemma 1.6.

Combining Lemma 1.2, Lemma 1.4, and Lemma 1.6, we obtain

Lemma 1.7. If $A \geq e^{631}$, then

(10)

$$F(x-\delta, A) - \frac{2}{\delta (A \log A)^{1/2}} < D(x) < F(x+\delta, A) + \frac{2}{\delta (A \log A)^{1/2}}$$

for any x and any $\delta > 0$.

We can now complete the proof of the theorem. Taking $\delta = A^{-1/4}$ and letting $A \rightarrow \infty$ in (10), we obtain $D(x) = \lim_{A \rightarrow \infty} F(x, A) = F(x)$, where $F(x)$ is the distribution function of \mathbb{Y} , at every point x where the limit exists and $F(x)$ is continuous. We have shown that \mathbb{Y} converges almost surely.

Thus, $F(x)$ exists for all x . In order to show that $F(x)$ is continuous, we examine the characteristic function

$$\hat{F}(\lambda) = \int_{-\infty}^{\infty} e(-\lambda x) dF(x) = \int_{-\infty}^{\infty} e(-\lambda \mathbf{Y}) d\underline{\theta} = \prod_p G(p, 2\pi\lambda),$$

where

$$G(p, \mu) = \int_0^1 e^{i\mu \arg(1-e(\theta)/p)} d\theta.$$

In order to bound $\hat{F}(\lambda)$, we need

Lemma 1.8. Let

$$B(p, \mu_0) = \sqrt{\frac{\mu_0}{p} \left(6 - \sqrt{36 - \frac{24\pi p}{\mu_0} - \frac{12\pi^2}{\mu_0^2}} \right)}.$$

If $\mu \geq \mu_0 \geq \pi$ and $\frac{2}{\pi} B(p, \mu_0) \sqrt{\frac{p}{\mu_0}} \leq 1$, then

$$|G(p, \mu)| \leq \frac{2}{\pi} B(p, \mu_0) \sqrt{\frac{p}{\mu}}.$$

Proof. We first show that if $-1 \leq p \sin \frac{\pi}{2\mu} \leq 1$ and $-1 \leq p \sin \left(\sin^{-1} \frac{1}{p} - \frac{\pi}{\mu} \right) \leq 1$, then

$$|G(p, \mu)| \leq \frac{2}{\pi} \left(\frac{\pi}{2} - \sin^{-1} \left(p \sin \left(\sin^{-1} \frac{1}{p} - \frac{\pi}{\mu} \right) \right) - \sin^{-1} p \sin \frac{\pi}{2\mu} \right).$$

We have

$$G(p, \mu) = \int_0^1 e^{i\mu \arg(1-e(\theta)/p)} d\theta = \int_{-\infty}^{\infty} e^{i\mu t} h_p(t) dt,$$

where $h_p(t)$ is the density function of $\arg(1-e(\theta)/p)$. If we let $H_p(t)$ be the distribution function of $\arg(1-e(\theta)/p)$, then

(11)

$$H_p(t) = \begin{cases} 1 & \text{if } t \geq \tan^{-1} \frac{1}{\sqrt{p^2-1}}, \\ \frac{1}{2} + \frac{\sin^{-1} p \sin t}{\pi} & \text{if } -\tan^{-1} \frac{1}{\sqrt{p^2-1}} \leq t \leq \tan^{-1} \frac{1}{\sqrt{p^2-1}}, \\ 0 & \text{if } t \leq -\tan^{-1} \frac{1}{\sqrt{p^2-1}}. \end{cases}$$

Hence,

$$h_p(t) = \begin{cases} \frac{p \cos t}{\pi \sqrt{1-p^2} \sin^2 t} & \text{if } -\tan^{-1} \frac{1}{\sqrt{p^2-1}} \leq t \leq \tan^{-1} \frac{1}{\sqrt{p^2-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Using this result, we obtain

$$\begin{aligned} G(p, \mu) &= \int_{-\infty}^{\infty} e^{i\mu t} h_p(t) dt = -\int_{-\infty}^{\infty} e^{i\mu t} h_p\left(t + \frac{\pi}{\mu}\right) dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{i\mu t} (h_p(t) - h_p\left(t + \frac{\pi}{\mu}\right)) dt. \end{aligned}$$

This gives us

(12)

$$|G(p, \mu)| \leq \frac{2}{\pi} \left(\frac{\pi}{2} - \sin^{-1} \left(p \sin \left(\sin^{-1} \frac{1}{p} - \frac{\pi}{\mu} \right) \right) - \sin^{-1} p \sin \frac{\pi}{2\mu} \right).$$

We next show that for all $\mu \geq \mu_0 \geq \pi$

$$(13) \quad \sin^{-1} p \sin \left(\sin^{-1} \frac{1}{p} - \frac{\pi}{\mu} \right) \geq \frac{\pi}{2} - B(p, \mu_0) \sqrt{\frac{p}{\mu}}.$$

It is sufficient to demonstrate that

$$\cos \frac{\pi}{\mu} - p \sin \frac{\pi}{\mu} - \cos B(p, \mu) \sqrt{\frac{p}{\mu}} \geq 0.$$

This inequality follows from the fact that

$$\begin{aligned} \cos \frac{\pi}{\mu} &\geq 1 - \frac{\left(\frac{\pi}{\mu}\right)^2}{2!} + \frac{\left(\frac{\pi}{\mu}\right)^4}{4!} - \frac{\left(\frac{\pi}{\mu}\right)^6}{6!}, \\ p \sin \frac{\pi}{\mu} &\leq p \left(\frac{\pi}{\mu} - \frac{\left(\frac{\pi}{\mu}\right)^3}{3!} + \frac{\left(\frac{\pi}{\mu}\right)^5}{5!} \right) \end{aligned}$$

and

$$\cos B(p, \mu) \sqrt{\frac{p}{\mu}} \leq 1 - \frac{B^2(p, \mu)p}{2! \mu} + \frac{B^4(p, \mu)p^2}{4! \mu^2} - \frac{B^6(p, \mu)p^3}{6! \mu^3} + \frac{B^8(p, \mu)p^4}{8! \mu^4}.$$

Combining (12) and (13), we complete the proof of the lemma.

To show that $F(x)$ has a continuous derivative, it is sufficient to show that

$$\int_{-\infty}^{\infty} |\hat{F}(\lambda)| d\lambda < \infty \text{ (see Billingsley [3, p.301]).}$$

This follows from the fact that $|G(p, \mu)| \leq 1$ and $|G(p, \mu)| \ll \mu^{-\frac{1}{2}}$ for $\mu \geq \mu_0 \geq \pi$. Thus, $F(x)$ is continuous and hence, $D(x) = F(x)$ for all x (remark: this also has an elementary proof). We now determine upper and lower bounds for $1-D(x) = P(\mathbb{Y} \geq x)$. The upper bound is standard. Let $\lambda > 0$.

Since $e^{\lambda \mathbb{Y}} \geq e^{\lambda x}$ for $\mathbb{Y} \geq x$, we have $P(\mathbb{Y} \geq x) \leq e^{-\lambda x} E(e^{\lambda \mathbb{Y}})$.

We write

$$\begin{aligned} E(e^{\lambda \mathbb{Y}}) &= \prod_p \int_0^1 e^{\lambda \arg(1-e(\theta)/p)^{-1}} d\theta \\ &= \prod_p \int_0^1 e^{\lambda \tan^{-1} \frac{\sin 2\pi \theta}{p - \cos 2\pi \theta}} d\theta = \prod_p E(p, \lambda). \end{aligned}$$

Since $\tan^{-1} \frac{\sin 2\pi \theta}{p - \cos 2\pi \theta} \leq \sin^{-1} \frac{1}{p}$, we have

$$E(p, \lambda) \leq e^{\lambda \sin^{-1} \frac{1}{p}}.$$

In addition, we have

$$\begin{aligned} E(p, \lambda) &= \int_0^1 \left(1 + \lambda \tan^{-1} \frac{\sin 2\pi \theta}{p - \cos 2\pi \theta} + \frac{(\lambda \tan^{-1} \frac{\sin 2\pi \theta}{p - \cos 2\pi \theta})^2}{2!} + \dots \right) d\theta \\ &\leq e^{\frac{\lambda^2}{2} (\sin^{-1} \frac{1}{p})^2}. \end{aligned}$$

Combining the above results, we obtain

$$E(e^{\lambda \mathbb{Y}}) \leq e^{\lambda \sum_{p \leq \lambda} \sin^{-1} \frac{1}{p}} e^{\frac{\lambda^2}{2} \sum_{p > \lambda} (\sin^{-1} \frac{1}{p})^2}.$$

We have

$$\sum_{p \leq \lambda} \sin^{-1} \frac{1}{p} = \sum_{p \leq \lambda} \frac{1}{p} + O(1) = \log \log \lambda + O(1)$$

and

$$\sum_{p > \lambda} (\sin^{-1} \frac{1}{p})^2 = O\left(\sum_{p > \lambda} \frac{1}{p^2}\right) = O\left(\frac{1}{\lambda \log \lambda}\right).$$

Hence,

$$P(\mathbb{Y} \geq x) \leq e^{-\lambda x + \lambda \log \log \lambda + O(\lambda)}.$$

Thus, we can choose $\lambda = e^{c_4 e^x}$ so that $P(\mathbb{Y} \geq x) \leq e^{-e^{c_2 e^x}}$.

We now obtain the lower bound. From (11), we have $\arg(1-e(\theta)/p)^{-1} \geq \sin^{-1} \frac{1}{p} - t_p$ with probability

$$\frac{1}{2} - \frac{\sin^{-1} \frac{1}{p} \sin(\sin^{-1} \frac{1}{p} - t_p)}{\pi}.$$

From (13), we have

$$\sin^{-1} \frac{1}{p} \sin(\sin^{-1} \frac{1}{p} - t_p) \geq \frac{\pi}{2} - B(p, \pi p) \sqrt{\frac{pt}{p}}$$

for $t_p \leq \frac{1}{p}$. Since $B(p, \pi p) \geq \sqrt{3}$, we see that $\arg(1-e(\theta)/p)^{-1} \geq \sin^{-1} \frac{1}{p} - t_p$ with probability $\geq \sqrt{\frac{3pt}{\pi}}$. Write

$$\mathbb{Y} = \sum_{p \leq T} \arg(1-e(\theta)/p)^{-1} + \sum_{p > T} \arg(1-e(\theta)/p)^{-1} = \mathbb{Y}_1 + \mathbb{Y}_2.$$

Then $P(\mathbb{Y} \geq x) \geq P(\mathbb{Y}_1 \geq x) P(\mathbb{Y}_2 \geq 0) \geq \frac{1}{2} P(\mathbb{Y}_1 \geq x)$. But we can choose $T = e^{c_3 e^x}$

so that

$$\mathbb{Y}_1 \geq \sum_{p \leq T} \sin^{-1} \frac{1}{p} - \frac{1}{T} = \log \log T + O(1) \geq x$$

with probability

$$\geq \prod_{p \leq T} \sqrt{\frac{3p}{\pi T}} = \left(\frac{3}{\pi T}\right)^{\frac{\pi(T)}{2}} e^{\frac{1}{2}\theta(T)} \geq 2e^{-e^{c_1 e^x}}.$$

Thus,

$$P(\mathbb{Y} \geq x) \geq e^{-e^{c_1 e^x}},$$

which completes the proof of the theorem.

3. Approximation of $F(x, A)$ by an infinite sum. In the previous section,

we reduced the problem of determining $D(x)$ to the problem of determining

the values of $F(x+\delta, A)$. In this section, we establish the following

Theorem 1.9. Let $G(p, \mu) = \sum_{k=0}^{\infty} \prod_{j=0}^{k-1} (j^2 - \frac{\mu^2}{4}) / (k!)^2 p^{2k}$, where $\prod_{j=0}^{-1} (j^2 - \frac{\mu^2}{4}) = 1$. If $0 \leq x \leq \frac{3\pi}{2}$, $0 < h \leq \frac{1}{4\pi}$, $A = e^{647.5336}$ and $R = \pi(A)$, then

$$F(x, A) = \frac{1}{2} + h \sum_{n=-\infty}^{\infty} \left(\frac{e(nhx)-1}{2\pi i nh} \prod_{p \leq A} G(p, 2\pi nh) \right).$$

To calculate $F(x, A)$, we write

$$F(x, A) = \frac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda x)-1}{2\pi i \lambda} \hat{F}(\lambda, A) d\lambda,$$

where

$$\hat{F}(\lambda, A) = \int_{-\infty}^{\infty} e(-\lambda x) dF(x, A);$$

see Kawata [13, p.128] with a change of variables. Here

$$\begin{aligned} \hat{F}(\lambda, A) &= \int_{\Gamma_R} e(-\lambda \mathbb{Y}_A) d\underline{\theta} = \prod_{p \leq A} \int_0^1 e(-\lambda \arg(1-e(\theta)/p)^{-1}) d\theta \\ &= \prod_{p \leq A} G(p, 2\pi \lambda), \end{aligned}$$

where

$$G(p, \mu) = \int_0^1 e^{i \mu \arg(1-e(\theta)/p)} d\theta.$$

But $e^{i \arg z} = \frac{z^{\frac{1}{2}}}{|z|^{\frac{1}{2}}}$, so that

$$e^{i \mu \arg(1-e(\theta)/p)} = \left(\frac{1-e(\theta)/p}{|1-e(-\theta)/p|} \right)^{\mu/2}.$$

Put

$$(1-e(\theta)/p)^{\mu/2} = \sum_{k=0}^{\infty} a_k e(k\theta),$$

where

$$a_k = (-1)^k \binom{\frac{\mu}{2}}{k} / p^k$$

and

$$(1-e(-\theta)/p)^{-\mu/2} = \sum_{l=0}^{\infty} b_l e(-l\theta),$$

where

$$b_l = (-1)^l \binom{-\frac{\mu}{2}}{l} / p^l.$$

Then the integral is

$$\int_0^1 \left(\sum_k a_k e(k\theta) \right) \left(\sum_l b_l e(-l\theta) \right) d\theta = \sum_{k=0}^{\infty} a_k b_k.$$

But

$$\binom{\frac{\mu}{2}}{k} \binom{-\frac{\mu}{2}}{k} = \frac{1}{(k!)^2} \prod_{j=0}^{k-1} \left(j^2 - \frac{\mu^2}{4} \right),$$

so that

$$G(p, \mu) = \sum_{k=0}^{\infty} \prod_{j=0}^{k-1} \left(j^2 - \frac{\mu^2}{4} \right) / (k!)^2 p^{2k}.$$

In order to write $F(x, A)$ as a sum, we use a method introduced by

Rice [22]. From Weiss and Stein [31, Corollary 2.6], we take

Lemma 1.10 (Poisson summation formula). Let $\hat{f}(t) = \int_{-\infty}^{\infty} e(-t\lambda) f(\lambda) d\lambda$. If $f(\lambda) = \int_{-\infty}^{\infty} \hat{f}(t) e(t\lambda) dt$, $|f(\lambda)| \leq A(1+|\lambda|)^{-1-\delta}$ and $|\hat{f}(t)| \leq A(1+|t|)^{-1-\delta}$ for some $\delta > 0$, then

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \hat{f}(m).$$

If $f(\lambda)$ is integrable and of bounded variation over every finite interval, and $f(\lambda) = \frac{1}{2}(f(\lambda+0) + f(\lambda-0))$ for all λ , then $f(\lambda) = \int_{-\infty}^{\infty} \hat{f}(t) e(t\lambda) dt$ (see Kawata [13, Theorem 4.34]). Thus, we have

Corollary 1.11. If f satisfies the above conditions, and $\hat{f}(t) = 0$ for $|t| \geq \epsilon > 0$, then

$$\int_{-\infty}^{\infty} f(\lambda) d\lambda = h \sum_{n=-\infty}^{\infty} f(nh)$$

for $0 < h \leq 1/\epsilon$.

We need to show that if $0 \leq x \leq \frac{3\pi}{2}$, $|t| \geq 4\pi$ and $f(\lambda) = \frac{e(\lambda x) - 1}{2\pi i \lambda} \hat{F}(\lambda, A)$, then $\hat{f}(t) = 0$. In demonstrating this, we make use of the following result (see Whittaker and Watson [32, p. 115]).

Lemma 1.12 (Jordan). Let Γ be the semicircle of radius r and center at the origin which lies above the real axis. If $Q(z) \rightarrow 0$ uniformly with respect to $\arg z$ as $|z| \rightarrow \infty$ for $0 \leq \arg z \leq \pi$ and $Q(z)$ is analytic when $|z| > c$ and

$0 \leq \arg z \leq \pi$, then $\lim_{r \rightarrow \infty} \int_{\Gamma} e^{imz} Q(z) dz = 0$ for all positive integers m .

Using these two preliminary lemmas, we prove

Lemma 1.13. Let $f(z) = \frac{e(zx)-1}{2\pi iz} \hat{F}(z, A)$. If $0 \leq x \leq \frac{3\pi}{2}$ and

$$0 < h \leq \frac{1}{\sum_{p \leq A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x},$$

then $\int_{-\infty}^{\infty} f(\lambda) d\lambda = h \sum_{-\infty}^{\infty} f(nh)$.

Proof. By Corollary 1.11, it is sufficient to show that $\hat{f}(t) = 0$ for

$|t| \geq \sum_{p \leq A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x$, since $|G(p, \mu)| \leq 1$ and $|G(p, \mu)| \ll \mu^{\frac{1}{2}}$ for $\mu \geq \mu_0 \geq \pi$. We have

$$\begin{aligned} \hat{f}(t) &= \int_{-\infty}^{\infty} e(-tz) f(z) dz \\ &= \lim_{r \rightarrow \infty} \int_{\Gamma} e(-tz) f(z) dz. \end{aligned}$$

From Lemma 1.12, we know that the above limit equals zero if

$e(-tz - \frac{z}{2\pi}) f(z) \rightarrow 0$ uniformly with respect to $\arg z$ as $|z| \rightarrow \infty$ for

$0 \leq \arg z \leq \pi$. For $t \leq 0$, $z = r \cos \theta + ir \sin \theta$ and $0 \leq \theta \leq \pi$, we have

(14)

$$\begin{aligned} \left| e(-tz - \frac{z}{2\pi}) f(z) \right| &= \left| \exp(-2\pi izt - iz) \right| \left| \frac{e(zx)-1}{r} \right| \prod_{p \leq A} |G(p, 2\pi z)| \\ &\leq \left| \exp((2\pi t+1)r \sin \theta) \right| \left| \exp((-2\pi t-1)ir \cos \theta) \right| \\ &\quad \frac{\left| \exp(2\pi ixr \cos \theta) \right| \left| \exp(2\pi xr \sin \theta) \right| + 1}{r} \prod_{p \leq A} |G(p, 2\pi z)| \\ &\leq \frac{2 \exp(r \sin \theta (1+2\pi x+2\pi t))}{r} \prod_{p \leq A} |G(p, 2\pi z)|. \end{aligned}$$

In addition,

$$(15) \quad \begin{aligned} G(p, 2\pi z) &= \int_0^1 e^{2\pi iz \arg\left(1 - \frac{e(u)}{p}\right)} du \\ &= \int_0^1 e^{2\pi iz \tan^{-1} \frac{\sin 2\pi u}{p - \cos 2\pi u}} du \end{aligned}$$

and for $0 \leq u \leq 1$

$$(16) \quad \left| \tan^{-1} \frac{\sin 2\pi u}{p - \cos 2\pi u} \right| \leq \tan^{-1} \frac{1}{\sqrt{p^2 - 1}} \leq \frac{1}{\sqrt{p^2 - 1}}.$$

Combining (15) and (16), we obtain

$$(17) \quad \begin{aligned} |G(p, 2\pi z)| &\leq \max_{0 \leq t \leq 1} \left| e^{2\pi iz \tan^{-1} \frac{\sin 2\pi t}{p - \cos 2\pi t}} \right| \\ &\leq \exp\left(\frac{2\pi r \sin \theta}{\sqrt{p^2 - 1}}\right). \end{aligned}$$

Using (17) in (14), we obtain

$$e(-tz - \frac{z}{2\pi})f(z) \leq \frac{2 \exp\left(2\pi r \sin \theta \left(\sum_{p < A} \frac{1}{\sqrt{p^2 - 1}} + \frac{1}{2\pi} + x + t\right)\right)}{r}.$$

Thus, $\left| e(-tz - \frac{z}{2\pi})f(z) \right| \rightarrow 0$ uniformly with respect to $\arg z$ as $|z| \rightarrow \infty$ for $0 \leq \arg z \leq \pi$ if $t \leq -\left(\sum_{p < A} \frac{1}{\sqrt{p^2 - 1}} + \frac{1}{2\pi} + x\right)$, which shows that $\hat{f}(t) = 0$ for these t . For $t \geq 0$, we have

$$\int_{-\infty}^{\infty} e(-tz)f(z)dz = -\int_{-\infty}^{\infty} e(tz) \frac{e(-zx)-1}{2\pi iz} \prod_{p \leq A} G(p, 2\pi z) dz,$$

and proceeding as before, we obtain $\hat{f}(t) = 0$ for $t \geq \sum_{p \leq A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x$.

This completes the proof of Lemma 1.13. Finally, we need

Lemma 1.14. If $0 \leq x \leq \frac{3\pi}{2}$ and $A = e^{647.5336}$, then

$$\frac{1}{4\pi} \leq \frac{1}{\sum_{p \leq A} \frac{1}{\sqrt{p^2-1}} + \frac{1}{2\pi} + x}.$$

Proof. Put

$$\sum_{p \leq A} \frac{1}{\sqrt{p^2-1}} = \sum_{p \leq 1010} \frac{1}{\sqrt{p^2-1}} + \sum_{1010 < p \leq A} \frac{1}{\sqrt{p^2-1}} = S_1 + S_2.$$

We obtain $S_1 \leq 2.30316$ by direct calculation. We have

$$S_2 \leq \frac{1010}{\sqrt{(1010)^2-1}} \sum_{1010 < p \leq A} \frac{1}{p} \leq 1.0000005 \sum_{1010 < p \leq A} \frac{1}{p}.$$

Using Lebesgue-Stieltjes integration, we write the last quantity as

$$1.0000005 \left(\frac{\theta(A)}{A \log A} - \frac{\theta(1010)}{1010 \log 1010} + \int_{1010}^{A^+} \frac{(\log u+1) \theta(u) du}{u \log^2 u} \right).$$

Schoenfeld [24, p.360] has shown that $\theta(u) \leq (1.000081)u$ for $u \geq 2$ and we

have calculated that $\theta(1010) \geq 963.149658$. Thus, the above expression is less

than 4.55. Hence,

$$\sum_{p \leq A} \frac{1}{\sqrt{p^2-1}} \leq 6.8532$$

and

$$\frac{1}{4\pi} < \frac{1}{\sum_{p \leq A} \frac{1}{p^2 - 1} + \frac{1}{2\pi} + x}$$

for $0 \leq x \leq \frac{3\pi}{2}$, which completes the proof of the lemma.

The theorem follows directly from Lemma 1. 13 and Lemma 1. 14.

4. The calculation of $F(x, A)$. In Theorem 1. 9 we reduced the problem of evaluating $F(x, A)$ for $0 \leq x \leq \frac{3\pi}{2}$ to that of evaluating

$$\frac{1}{2} + h \sum_{n=-\infty}^{\infty} \frac{e(nhx) - 1}{2\pi inh} \prod_{p \leq A} G(p, 2\pi nh)$$

when $0 < h \leq \frac{1}{4\pi}$ and $A = e^{647.5336}$. For $n = 0$, we have

$$\lim_{n \rightarrow 0} h \frac{e(nhx) - 1}{2\pi inh} \prod_{p \leq A} G(p, 2\pi nh) = hx.$$

For $n \geq 0$, we have

$$\begin{aligned} & \frac{e(nhx) - 1}{2\pi inh} \prod_{p \leq A} G(p, 2\pi nh) + \frac{e(-nhx) - 1}{-2\pi inh} \prod_{p \leq A} G(p, -2\pi nh) \\ &= \frac{2 \sin 2\pi nhx}{2\pi nh} \prod_{p \leq A} G(p, 2\pi nh). \end{aligned}$$

Using the above equalities, we obtain

$$F(x, A) = \frac{1}{2} + hx + 2h \sum_{n=1}^{\infty} \frac{\sin 2\pi nhx}{2\pi nh} \prod_{p \leq A} G(p, 2\pi nh).$$

We now take $h = \frac{1}{4\pi}$. We first truncate the above sum. By taking $\mu_0 = 800$ in Lemma 1.8, we obtain

$$\begin{aligned} \left| \frac{1}{2\pi} \sum_{n=1601}^{\infty} \frac{\sin nx/2}{n/2} \prod_{p \leq A} G(p, n/2) \right| &\leq \frac{1}{2\pi} \sum_{n=1601}^{\infty} \frac{\prod_{p \leq 241} \frac{2}{\pi} B(p, 800) \sqrt{\frac{p}{n/2}}}{n/2} \\ &\leq \frac{(800)^{53/2}}{2\pi} \int_{800}^{\infty} \frac{\prod_{p \leq 241} \frac{2}{\pi} \sqrt{6 - \sqrt{36 - \frac{24\pi p}{800} - \frac{12\pi^2}{800^2}}} d\mu}{\mu^{55/2}} \\ &\leq 0.5 \cdot 10^{-16}. \end{aligned}$$

Using this, we obtain

$$F(x, A) = \frac{1}{2} + \frac{x}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{1600} \frac{\sin nx/2}{n/2} \prod_{p \leq A} G(p, n/2) + \epsilon,$$

with $|\epsilon| \leq 0.5 \cdot 10^{-16}$. To further truncate our sum, we need to compute

$G(p, \mu)$. The method we use is justified by

Lemma 1.15. Let $L = \left\lceil \frac{\mu}{\sqrt{2}} \right\rceil$ and $f(p, \mu, N) = \frac{\left(\frac{\mu}{2}\right)^{2(N+1)} e^{2(N+1)}}{2\pi (N+1)^{2N+3} p^{2(N+1)}}.$

If we choose

$$N = \begin{cases} L & \text{if } f(p, \mu, L) = 10^{-16}, \\ \left[\frac{8 \log 10 + \left(\left\lceil \frac{\mu}{\sqrt{2}} \right\rceil + 1\right) \log \left(\frac{\mu e}{2 \left(\left\lceil \frac{\mu}{\sqrt{2}} \right\rceil + 1\right)} \right) - \frac{1}{2} \log 2\pi \left(\left\lceil \frac{\mu}{\sqrt{2}} \right\rceil + 1\right)}{\log p} \right] & \text{if } f(p, \mu, L) > 10^{-16}, \\ \text{greatest integer } k \text{ such that } \left(\frac{\mu e}{2^{(k+1)} p}\right)^{2(k+1)} \frac{1}{2\pi^{(k+1)}} < 10^{-16} & \\ & \text{if } f(p, \mu, L) < 10^{-16}, \end{cases}$$

then

$$G(p, \mu) = \sum_{k=0}^N \frac{\prod_{j=0}^{k-1} \left(j^2 - \frac{\mu^2}{4} \right)}{(k!)^2 p^{2k}} + R_N,$$

where $|R_N| \leq \frac{4}{3} \cdot 10^{-16}$.

Proof. Put

$$u_k = \frac{\prod_{j=0}^{k-1} \left(j^2 - \frac{\mu^2}{4} \right)}{(k!)^2 p^{2k}} = \frac{-1}{k^2} \left(\frac{\mu^2}{4} \right) \left(1 - \frac{\mu^2}{4} \right) \left(1 - \frac{\mu^2}{4(2)^2} \right) \dots \left(1 - \frac{\mu^2}{4(k-1)^2} \right) \frac{1}{p^{2k}}$$

and

$$T_N = \max \left(\frac{\mu^2}{4(N+1)^2 p^2} \left(\frac{1}{1 - \frac{\mu^2}{4(N+1)^2 p^2}} \right), \frac{1}{p^2} \left(\frac{1}{1 - \frac{1}{p^2}} \right) \right).$$

We first show that

$$(18) \quad u_{N+1} (1 - \operatorname{sgn}(u_{N+1}) T_N) \leq R_N \leq u_{N+1} (1 + \operatorname{sgn}(u_{N+1}) T_N).$$

We have

$$R_N = u_{N+1} \left(1 + \frac{(N+1)^2}{(N+2)^2} \left(1 - \frac{\mu^2}{4(N+1)^2} \right) \frac{1}{p^2} + \dots \right).$$

If we put

$$S_N = \frac{(N+1)^2}{(N+2)^2} \left| 1 - \frac{\mu^2}{4(N+1)^2} \right| \frac{1}{p^2} + \frac{(N+1)^2}{(N+3)^2} \left| 1 - \frac{\mu^2}{4(N+1)^2} \right| \left| 1 - \frac{\mu^2}{4(N+2)^2} \right| \frac{1}{p^4} + \dots$$

then

$$u_{N+1}(1-S_N) \leq R_N \leq u_{N+1}(1+S_N) \quad (u_{N+1} \geq 0)$$

and

$$u_{N+1}(1+S_N) \leq R_N \leq u_{N+1}(1-S_N) \quad (u_{N+1} < 0).$$

Obviously,

$$S_N \leq \left| 1 - \frac{\mu^2}{4(N+1)^2} \right| \frac{1}{p^2} + \left| 1 - \frac{\mu^2}{4(N+2)^2} \right| \frac{1}{p^4} + \dots$$

Less obviously, we have, for all $M \geq N$,

$$\left| 1 - \frac{\mu^2}{4(M+1)^2} \right| \leq \max\left(\frac{\mu^2}{4(M+1)^2}, 1\right) \leq \begin{cases} \frac{\mu^2}{4(N+1)^2} & \text{if } N+1 \leq \frac{\mu}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

This is certainly true if $M \leq \frac{\mu}{2}$, since

$$\left| 1 - \frac{\mu^2}{4M^2} \right| = \frac{\mu^2}{4M^2} - 1 \leq \frac{\mu^2}{4M^2}.$$

When $\frac{\mu}{2} \leq M \leq \frac{\mu}{\sqrt{2}}$, we have

$$\left| 1 - \frac{\mu^2}{4M^2} \right| \leq 1 - \frac{\mu^2}{4(\mu/\sqrt{2})^2} \leq \frac{\mu^2}{4(\mu/\sqrt{2})^2} \leq \frac{\mu^2}{4M^2}.$$

Finally, if $M \geq \frac{\mu}{2\sqrt{2}}$, then

$$\left| 1 - \frac{\mu^2}{4M^2} \right| \leq 1.$$

From the above, we have

$$S_N \leq \begin{cases} \frac{\mu^2}{4(N+1)^2 p^2} + \left(\frac{\mu^2}{4(N+1)^2 p^2} \right)^2 + \dots & \text{if } N+1 \leq \frac{\mu}{2}, \\ \frac{1}{p^2} + \frac{1}{p^4} + \dots & \text{otherwise,} \end{cases}$$

so that (18) is true. We have

$$|u_{N+1}| \leq \begin{cases} \frac{1}{(N+1)^2} \left(\frac{\mu^2}{4} \right) \left(\frac{\mu^2}{4} \right) \left(\frac{\mu^2}{4(2)^2} \right) \cdots \left(\frac{\mu^2}{4N^2} \right) \frac{1}{p^{2(N+1)}} \\ \leq \frac{\left(\frac{\mu}{2} \right)^{2(N+1)}}{((N+1)!)^2 p^{2(N+1)}} & \text{if } N \leq \frac{\mu}{\sqrt{2}}, \\ \frac{1}{(N+1)^2} \left(\frac{\mu^2}{4} \right) \left(\frac{\mu^2}{4} \right) \left(\frac{\mu^2}{4(2)^2} \right) \cdots \left(\frac{\mu^2}{4(\lceil \frac{\mu}{2} \rceil)^2} \right) \frac{1}{p^{2(N+1)}} \\ \leq \frac{\left(\frac{\mu}{2} \right)^{2(\lceil \frac{\mu}{2} \rceil + 1)}}{((\lceil \frac{\mu}{2} \rceil + 1)!)^2 p^{2(N+1)}} & \text{if } N > \frac{\mu}{\sqrt{2}}. \end{cases}$$

Since $(M-1)! = M^{M-\frac{1}{2}} e^{-M} (2\pi)^{\frac{1}{2}} e^{s/12M}$ for $0 \leq s \leq 1$ (see Whittaker and Watson [32, p.253]), we have $M! \geq M^{M+\frac{1}{2}} e^{-M} (2\pi)^{\frac{1}{2}}$ and thus,

$$|u_{N+1}| \leq \begin{cases} \frac{\left(\frac{\mu}{2}\right)^{2(N+1)} e^{2(N+1)}}{2\pi (N+1)^{2N+3} p^{2(N+1)}} & \text{if } N \leq \frac{\mu}{\sqrt{2}} \\ \frac{\left(\frac{\mu}{2}\right)^{2\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)} e^{2\left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)}}{2\pi \left(\left[\frac{\mu}{\sqrt{2}}\right]+1\right)^{2\left[\frac{\mu}{\sqrt{2}}\right]+3} p^{2(N+1)}} & \text{if } N > \frac{\mu}{\sqrt{2}}. \end{cases}$$

If we choose N according to our algorithm, then

$$N \geq \left\lceil \frac{\mu e}{2p} \right\rceil, \quad N+1 \geq \frac{\mu e}{2p} \quad \text{and} \quad T_N \leq \frac{1}{3}.$$

Thus, our algorithm insures that $|R_N| \leq \frac{4}{3} \cdot 10^{-16}$.

Using (12) to bound $|G(p, \mu)|$ for $p \leq 101$, and Lemma 1.15 to compute $|G(p, \mu)|$ for $101 < p \leq 1871$, we obtain

$$\left| \frac{1}{2\pi} \sum_{n=401}^{1600} \frac{\sin nx/2}{n/2} \prod_{p \leq A} G(p, n/2) \right| \leq 0.44249 \cdot 10^{-16}.$$

From this, we obtain

$$F(x, A) = \frac{1}{2} + \frac{x}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{400} \frac{\sin xn/2}{n/2} \prod_{p \leq A} G(p, n/2) + \epsilon,$$

where $|\epsilon| \leq 10^{-16}$.

Since we cannot compute $G(p, \mu)$ for all $p \leq A$, we need

Theorem 1.16. Let

$$G(p, \mu) = \sum_{k=0}^{\infty} u_k, \quad S_L(p, \mu) = \sum_{k=1}^5 u_k + u_6(1 - \operatorname{sgn}(u_6)(2 \cdot 1 \cdot 10^{-10}))$$

and

$$S_H(p, \mu) = \sum_{k=1}^5 u_k + u_6(1 + \operatorname{sgn}(u_6)(2 \cdot 1 \cdot 10^{-10})).$$

Let

$$\begin{aligned} B_L(p, \mu) &= \sum_{m=1}^{36} \underline{b}(m, \mu) p^{-2m} \\ &= S_L(p, \mu) - \frac{S_L(p, \mu)^2}{2} + \frac{S_L(p, \mu)^3}{3} - \frac{S_L(p, \mu)^4}{4} \\ &\quad + \frac{S_L(p, \mu)^5}{5} - \frac{1.00000001497 S_L(p, \mu)^6}{6} \end{aligned}$$

and

$$\begin{aligned} B_H(p, \mu) &= \sum_{m=1}^{36} \bar{b}(m, \mu) p^{-2m} \\ &= S_H(p, \mu) - \frac{S_H(p, \mu)^2}{2} + \frac{S_H(p, \mu)^3}{3} - \frac{S_H(p, \mu)^4}{4} \\ &\quad + \frac{S_H(p, \mu)^5}{5} - \frac{S_H(p, \mu)^6}{6}. \end{aligned}$$

Let $V = 1155901$,

$$I_L(n) = \frac{0.998697 - \frac{\theta(V)}{V}}{V^{2n-1} \log V} - \frac{0.998697 - \frac{\theta(A)}{A}}{A^{2n-1} \log A} \\ + 0.998697(E_1((2n-1)\log V) - E_1((2n-1)\log A))$$

and

$$I_H(n) = \frac{1 - \frac{\theta(V)}{V}}{V^{2n-1} \log V} + \frac{0.000081}{(10^{11})^{2n-1} \log 10^{11}} + E_1((2n-1)\log V) \\ + 0.000081 E_1((2n-1)\log 10^{11}).$$

Let

$$J_L(n, \mu) = \begin{cases} I_L(n) & \text{if } \underline{b}(n, \mu) \geq 0, \\ I_H(n) & \text{if } \underline{b}(n, \mu) < 0, \end{cases}$$

$$J_H(n, \mu) = \begin{cases} I_H(n) & \text{if } \bar{b}(n, \mu) \geq 0, \\ I_L(n) & \text{if } \bar{b}(n, \mu) < 0, \end{cases}$$

$$Q_L(\mu) = \sum_{n=1}^{36} \underline{b}(n, \mu) J_L(n, \mu)$$

and

$$Q_H(\mu) = \sum_{n=1}^{36} \bar{b}(n, \mu) J_H(n, \mu).$$

Let

$$R_L(\mu) = \begin{cases} \exp(Q_L(\mu)) & \text{if } \frac{\sin x\mu}{\mu} \prod_{2 \leq p \leq V} G(p, \mu) \geq 0, \\ \exp(Q_H(\mu)) & \text{otherwise} \end{cases}$$

and

$$R_H(\mu) = \begin{cases} \exp(Q_H(\mu)) & \text{if } \frac{\sin x\mu}{\mu} \prod_{2 \leq p \leq V} G(p, \mu) \geq 0, \\ \exp(Q_L(\mu)) & \text{otherwise.} \end{cases}$$

If $0 \leq x \leq \frac{3\pi}{2}$, then

$$\begin{aligned} & \frac{1}{2} + \frac{x}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{400} \frac{\sin nx/2}{n/2} \prod_{2 \leq p \leq V} G(p, n/2) R_L(n/2) - \epsilon \leq F(x, A) \\ & \leq \frac{1}{2} + \frac{x}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{400} \frac{\sin nx/2}{n/2} \prod_{2 \leq p \leq V} G(p, n/2) R_H(n/2) + \epsilon, \end{aligned}$$

where $0 \leq \epsilon \leq 10^{-16}$. To prove Theorem 1.16, we need the following auxiliary results.

Lemma 1.17. If $0 \leq y < 1$ and $M \in \mathbb{Z}^+$, then

$$-\sum_{m=1}^M \frac{y^m}{m} - \frac{y^{M+1}}{M+1} \left(\frac{1}{1-y} \right) \leq \log(1-y) \leq -\sum_{m=1}^M \frac{y^m}{m} - \frac{y^{M+1}}{M+1} \left(\frac{1}{1-(y/2)} \right).$$

Proof. For $0 \leq y < 1$, we have

$$\log(1-y) = -\sum_{m=1}^M \frac{y^m}{m} - \sum_{m=M+1}^{\infty} \frac{y^m}{m}.$$

The lemma follows from the fact that

$$\sum_{m=M+1}^{\infty} \frac{y^m}{m} \leq \frac{y^{M+1}}{M+1} \left(\frac{1}{1-y} \right)$$

and

$$\sum_{m=M+1}^{\infty} \frac{y^m}{m} \geq \sum_{m=M+1}^{\infty} \frac{y^m}{(M+1)2^{(M+1)-m}} \geq \frac{y^{M+1}}{M+1} \left(\frac{1}{1-(y/2)} \right).$$

Lemma 1.18. If $0 \leq \mu \leq 200$, then

$$\begin{aligned} \exp \left(\sum_{n=1}^{36} \bar{b}(n, \mu) \int_{V^+}^A \frac{d\theta(u)}{u^{2n} \log u} \right) &\leq \prod_{V < p < A} G(p, \mu) \\ &\leq \exp \left(\sum_{n=1}^{36} \bar{b}(n, \mu) \int_{V^+}^A \frac{d\theta(u)}{u^{2n} \log u} \right). \end{aligned}$$

Proof. From (16), we obtain

$$1 + \sum_{k=1}^5 u_k + u_6 (1 - \operatorname{sgn}(u_6) T_5) \leq G(p, \mu) \leq 1 + \sum_{k=1}^5 u_k + u_6 (1 + \operatorname{sgn}(u_6) T_5).$$

Since $T_5 \leq 2.1 \cdot 10^{-10}$, we have

$$1 + S_L(p, \mu) \leq G(p, \mu) \leq 1 + S_H(p, \mu).$$

Since $0 \leq -S_H(p, \mu) \leq -S_L(p, \mu) \leq 7.485 \cdot 10^{-9}$, we can use the above lemma to obtain

$$B_L(p, \mu) \leq \log G(p, \mu) \leq B_H(p, \mu).$$

To complete the proof of Lemma 1. 18, we note that

$$\sum_{V < p < A} B_L(p, \mu) = \sum_{n=1}^{36} \bar{b}(n, \mu) \sum_{V < p < A} p^{-2n} = \sum_{n=1}^{36} \bar{b}(n, \mu) \int_{V^+}^A \frac{d\theta(u)}{u^{2n} \log u}$$

and

$$\sum_{V < p < A} B_H(p, \mu) = \sum_{n=1}^{36} \bar{b}(n, \mu) \int_{V^+}^A \frac{d\theta(u)}{u^{2n} \log u}.$$

To bound the integrals in the above lemma, we have

Lemma 1. 19. If $n \geq 1$, then

$$I_L(n) \leq \int_{V^+}^A \frac{d\theta(u)}{u^{2n} \log u} \text{ and } \int_{V^+}^A \frac{d\theta(u)}{u^{2n} \log u} \leq I_H(n).$$

Proof. Schoenfeld [24, p360] has shown that $\theta(u) < (1.000081)u$ for all

$u > 0$ and $\theta(u) < u$ for $0 < u \leq 10^{11}$. Using this result, we obtain

$$\begin{aligned} \int_{V^+}^A \frac{d\theta(u)}{u^{2n} \log u} &= \int_{V^+}^{10^{11}} \frac{d\theta(u)}{u^{2n} \log u} + \int_{10^{11}}^A \frac{d\theta(u)}{u^{2n} \log u} \\ &= \frac{\theta(10^{11})}{(10^{11})^{2n} \log 10^{11}} - \frac{\theta(V)}{V^{2n} \log V} + \int_{V^+}^{10^{11}} \frac{2n \theta(u) du}{u^{2n+1} \log^2 u} \\ &\quad + \int_{V^+}^{10^{11}} \frac{\theta(u) du}{u^{2n+1} \log^2 u} \end{aligned}$$

$$\begin{aligned}
& + \frac{\theta(A)}{A^{2n} \log 10^{11}} - \frac{\theta(10^{11})}{(10^{11})^{2n} \log 10^{11}} + \int_{10^{11}}^A \frac{2n\theta(u)du}{u^{2n+1} \log u} \\
& + \int_{10^{11}}^A \frac{\theta(u)du}{u^{2n+1} \log^2 u} \\
& \leq I_H(n).
\end{aligned}$$

The lower bound is obtained similarly.

Using Lemma 1.18 and Lemma 1.19, we have

$$\begin{aligned}
\exp(Q_L(\mu)) &= \exp \sum_{n=1}^{36} \underline{b}(n, \mu) J_L(n, \mu) \leq \prod_{V < p < A} G(p, \mu) \\
&\leq \exp \sum_{n=1}^{36} \bar{b}(n, \mu) J_H(n, \mu) = \exp(Q_H(\mu)),
\end{aligned}$$

which completes the proof of Theorem 1.16.

In computing $F(x, A)$, we use the fact that $\theta(1155901) = 1150824.716$.

In addition, we use the following result from Abramowitz and Stegun [1, p.231]:

Lemma 1.20. Let

$$a_1 = 8.5733287401,$$

$$a_2 = 18.0590169730,$$

$$a_3 = 8.6347608925,$$

$$a_4 = 0.2677727343,$$

$$b_1 = 9.5733223454,$$

$$b_2 = 25.6329561486,$$

$$b_3 = 21.0996530827 \text{ and}$$

$$b_4 = 3.9584969228.$$

Then

$$E_1(v) = \int_v^\infty \frac{e^{-y} dy}{y} = \frac{1}{ve^v} \left(\frac{v^4 + a_1 v^3 + a_2 v^2 + a_3 v + a_4}{v^4 + b_1 v^3 + b_2 v^2 + b_3 v + b_4} + \epsilon(v) \right),$$

where $|\epsilon(v)| \leq 2 \cdot 10^{-8}$.

Since we compute $G(p, \mu)$ with an error of $\leq \frac{4}{3} \cdot 10^{-16}$, the relative error in computing $\prod_{2 \leq p \leq V} G(p, \mu)$ is less than 10^{-10} . Thus, we can compute the bounds in Theorem 1.16 with an absolute error $< 1.9 \cdot 10^{-9}$. Taking $\delta = e^{-40}$ in Lemma 1.7, we see that we can compute $D(x)$ with an absolute error $< 2 \cdot 10^{-9}$.

5. Calculation of $L(1, \chi)$. It is well known (see Davenport [6, pp. 67-69])

that

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{m=1}^q \bar{\chi}(m) e(mn/q),$$

where q is a prime and

$$\tau(\chi) = \sum_{m=1}^q \chi(m) e(m/q).$$

Hence

$$\begin{aligned} L(1, \chi) &= \frac{1}{\tau(\bar{\chi})} \sum_{m=1}^{q-1} \bar{\chi}(m) \sum_{n=1}^{\infty} e(mn/q)/n \\ &= \frac{-1}{\tau(\bar{\chi})} \sum_{m=1}^{q-1} \bar{\chi}(m) \log(1-e(m/q)). \end{aligned}$$

Suppose that $\chi(-1) = -1$. We have

$$\bar{\chi}(m) \log(1-e(m/q)) + \bar{\chi}(q-m) \log(1-e((-m)/q)) = \bar{\chi}(m) \log \left(\frac{1-e(m/q)}{1-e(-m/q)} \right).$$

But $(1-e(m/q))/(1-e(-m/q)) = e^{\pi i(-1 + \frac{2m}{q})}$, so the above is

$$\bar{\chi}(m) \pi i \left(-1 + \frac{2m}{q} \right).$$

Thus,

$$\begin{aligned} L(1, \chi) &= \frac{-1}{2\tau(\bar{\chi})} \sum_{m=1}^{q-1} \bar{\chi}(m) \pi i \left(-1 + \frac{2m}{q} \right) \\ &= -\frac{\pi i}{\tau(\bar{\chi})q} \sum_{m=1}^{q-1} m \bar{\chi}(m). \end{aligned}$$

Suppose that $\chi(-1) = 1$. We have

$$\bar{\chi}(m) \log(1-e(m/q)) + \bar{\chi}(q-m) \log(1-e((-m)/q)) = 2\bar{\chi}(m) \log |1-e(m/q)|.$$

But $|1-e(m/q)| = 2 \sin \frac{\pi m}{q}$, so the above is

$$2\bar{\chi}(m) \left(\log 2 + \log \sin \frac{\pi m}{q} \right).$$

Hence

$$L(1, \chi) = \begin{cases} \frac{-1}{\tau(\bar{\chi})} \sum_{m=1}^{q-1} \bar{\chi}(m) \log \sin \frac{\pi m}{q} & \text{if } \bar{\chi}(-1) = 1, \\ \frac{-\pi i}{\tau(\bar{\chi})q} \sum_{m=1}^{q-1} m \bar{\chi}(m) & \text{if } \bar{\chi}(-1) = -1. \end{cases}$$

Given a primitive root $g \pmod{q}$, we compute

$$(19) \tau(\bar{\chi}_k) = \sum_{r=1}^{q-1} e\left(\frac{-kr}{q-1}\right) e((g^r \pmod{q})/q),$$

$$(20) \quad L(1, \chi_k) = \begin{cases} \frac{-1}{\tau(\bar{\chi}_k)} \sum_{r=1}^{q-1} e\left(\frac{-kr}{q-1}\right) \log \sin \frac{\pi (g^r \pmod{q})}{q} & \text{if } \bar{\chi}(-1) = 1, \\ \frac{-\pi i}{\tau(\bar{\chi}_k)q} \sum_{r=1}^{q-1} e\left(\frac{-kr}{q-1}\right) (g^r \pmod{q}) & \text{if } \bar{\chi}(-1) = -1, \end{cases}$$

for $1 \leq k \leq \frac{q-1}{2}$.

CHAPTER II

THE DISTRIBUTION OF THE ERROR TERM IN THE PRIME NUMBER THEOREM

1. Statement of results. When we examine the distribution of prime numbers, $\psi(u)$ is often a more natural function to use than $\pi(u)$. If we define

$$\psi_0(u) = \begin{cases} \psi(u) & \text{if } u \neq p^m, \\ \psi(u) - \frac{1}{2}\Lambda(u) & \text{otherwise,} \end{cases}$$

then we have the von Mangoldt formula

$$\psi_0(u) = u - \sum_{\rho} \frac{u^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1-u^{-2})$$

for $u > 1$, where the sum is taken over all non-trivial zeros of $\zeta(s)$. We will examine

$$H(u) = \frac{\psi_0(u) - u + \log 2\pi + \frac{1}{2} \log(1-u^{-2})}{u^{\frac{1}{2}}} = - \sum_{\rho} \frac{u^{\rho - \frac{1}{2}}}{\rho}$$

assuming the Riemann hypothesis and the rational linear independence of the positive imaginary parts of the non-trivial zeros of $\zeta(s)$.

Given the Riemann hypothesis, we write the non-trivial zeros of $\zeta(s)$ in the form $\rho = \frac{1}{2} + i\gamma$, and obtain

$$H(u) = - \sum_{\gamma} \frac{u^{i\gamma}}{\frac{1}{2} + i\gamma}.$$

Setting $u = e^v$, we write

$$\begin{aligned} (1) \quad H(u) = h(v) &= - \sum_{\gamma > 0} \left(\frac{\exp(i\gamma v)}{\frac{1}{2} + i\gamma} + \frac{\exp(-i\gamma v)}{\frac{1}{2} - i\gamma} \right) \\ &= - \sum_{\gamma > 0} \frac{\exp(i(\gamma v - \arg \rho))}{|\rho|} - \sum_{\gamma > 0} \frac{\exp(-i(\gamma v - \arg \rho))}{|\rho|} = h_1(v) + h_2(v). \end{aligned}$$

This function is related to the probability distribution of a random variable in Theorem 2.1. Let $\mathbb{Y} = \sum_{\gamma > 0} \frac{2 \sin 2\pi \theta_{\gamma}}{|\rho|}$, where the θ_{γ} are independent and uniformly distributed in $[0, 1)$ (almost sure convergence follows by arguments analogous to those used in Theorem 1.1). Let $G(x)$ be the distribution function of \mathbb{Y} . If the $\gamma > 0$ are linearly independent over the rationals, then for every real x ,

$$\lim_{V \rightarrow \infty} \frac{1}{V} \text{meas} \{v : 0 \leq v \leq V, h(v) \leq x\} = G(x).$$

We need several preliminary results. From Wintner and Jessen [33, Theorem 29] we have Lemma 2.2. Let ϕ_j be the circular equidistribution on $S_j = \{z : |z| = r_j\}$, where for any Borel set E , $\phi_j(E) = \mu(S_j \cap E) / \mu(S_j)$ (μ is one-dimensional measure). Let r_1, r_2, \dots, r_n be positive, $\lambda_1, \lambda_2, \dots, \lambda_n$ be linearly independent and $\delta_1, \delta_2, \dots, \delta_n$ be real. Then the asymptotic distribution of

$$s_n(t) = r_1 \exp(i(\lambda_1 t + \delta_1)) + r_2 \exp(i(\lambda_2 t + \delta_2)) + \dots + r_n \exp(i(\lambda_n t + \delta_n))$$

is the distribution $\psi_n = \phi_1 * \phi_2 * \dots * \phi_n$.

From Wintner and Jessen [33, Theorem 7] we have

Lemma 2.3. The convergence of the series $r_1^2 + r_2^2 + \dots$ is necessary and sufficient both for the convergence and the absolute convergence of the infinite convolution $\phi_1 * \phi_2 * \dots$.

Since $N(T) \ll T \log T$, we have

$$\sum_{\gamma > 0} \frac{1}{|\rho|^2} \leq \sum_{\gamma > 0} \frac{1}{\gamma^2} \ll \sum_{n=1}^{\infty} \frac{\log n}{n^2} < \infty.$$

Using the previous two lemmas we see that the asymptotic distribution of

$h_1(v)$ is $\phi_1 * \phi_2 * \dots$ and the asymptotic distribution of $h_2(v)$ is $\bar{\phi}_1 * \bar{\phi}_2 * \dots$,

where ϕ_j is the distribution of $\frac{e(\theta \gamma_j)}{|\rho_j|}$. Hence, $h(v)$ has asymptotic distribution $\phi_1 * \bar{\phi}_1 * \phi_2 * \bar{\phi}_2 * \dots$. But the distribution $\phi_j * \bar{\phi}_j$ is the distribution of $\frac{2 \sin 2\pi \theta \gamma_j}{|\rho_j|}$. Thus, the asymptotic distribution of $h(v)$ is the distribution of \mathbb{Y} , which completes the proof of the theorem.

The following table contains typical values for $G(x)$:

x	G(x)	Maximum possible error
-1.0	0.000000263	0.000000194
-0.8	0.000047207	0.0000002
-0.6	0.00213582	0.0000002
-0.4	0.0309492	0.0000002
-0.2	0.178533	0.000001
0.0	0.5	0.0
0.2	0.821467	0.000001
0.4	0.9690508	0.0000002
0.6	0.99786418	0.0000002
0.8	0.999952793	0.0000002
1.0	0.999999737	0.000000194

We obtain these results using computer programs (Appendix B) based on

Theorem 2.12. A graph of $G(x)$ for $0 \leq x \leq 0.8$ appears in Figure 2.

Montgomery [18] has shown that $G(x)$ is continuous,

$$G(-x) + G(x) = 1 \text{ and for } x \geq 1$$

$$\exp\left(-c_1\sqrt{x} e^{\sqrt{2\pi x}}\right) \leq 1 - G(x) \leq \exp\left(-c_2\sqrt{x} e^{\sqrt{2\pi x}}\right).$$

Here c_1 and c_2 are positive absolute constants.

For large values of u , we show in section 4 that

$$G'(u) = \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}} \left(1 + O\left(\frac{1}{\sqrt{T \log T}}\right)\right),$$

where

$$f(z) = \sum_{\gamma > 0} \log I_0\left(\frac{z}{|\rho|}\right) - zu/2$$

and T is chosen so that $f'(T) = 0$.

Using this formula, we can find constants a_1, a_2, a_3 and a_4 such that

$$G'(u) = \exp\left(-\left(\sqrt{a_1 u + a_2} - a_3\right) e^{\sqrt{2\pi u + a_4}} + O(\sqrt{u})\right).$$

2. Approximation of $G(x)$ by an infinite sum. In this section, we first establish

Theorem 2.4. Let $G(x, B)$ be the distribution function of $\mathbb{Y}_B = \sum_{0 < \gamma \leq B} \frac{2\sin 2\pi \theta}{|\rho|} \gamma$.
If $0 \leq x \leq 1$, $B \geq e^{58.223085}$ and $R = N(B)$, then

$$|G(x) - G(x, B)| < 10^{-12.24}.$$

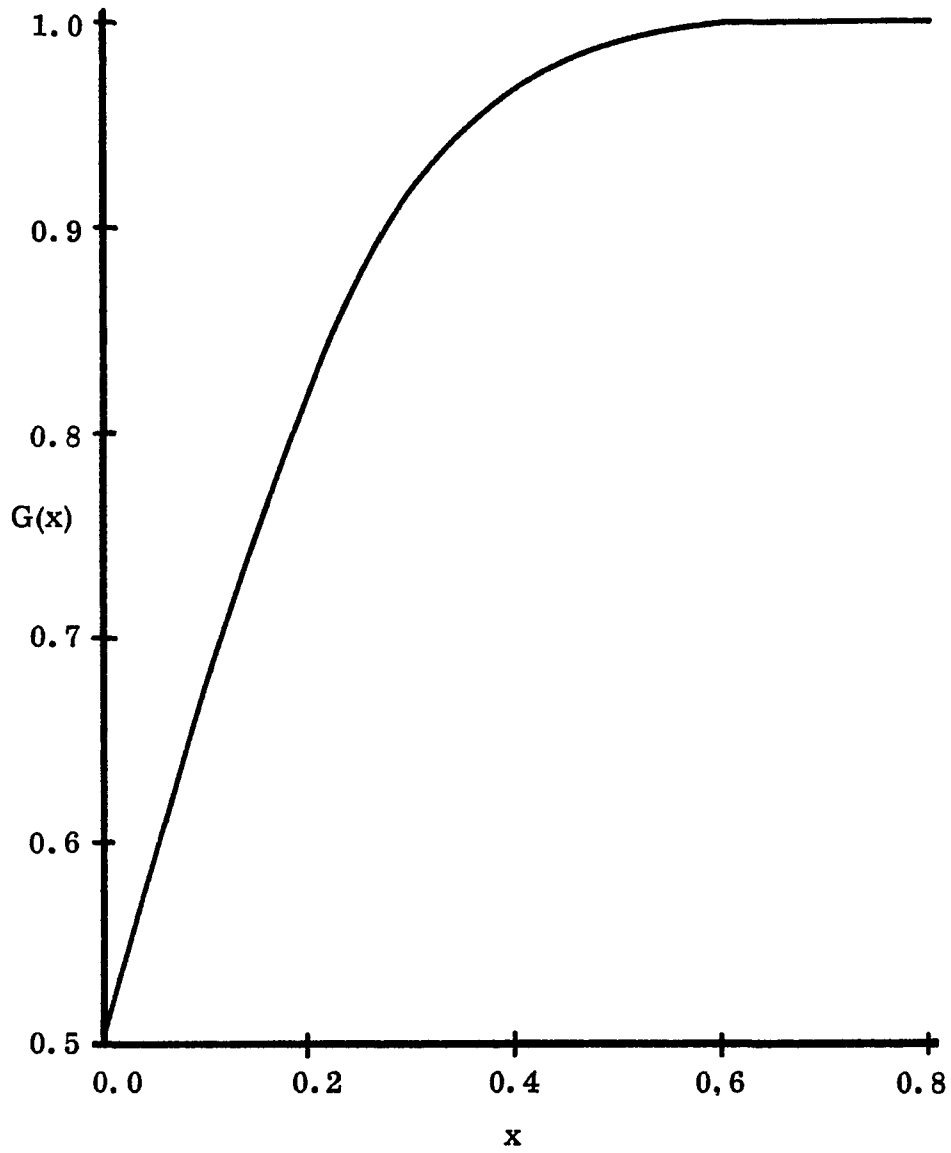


Fig. 2 - A Graph of $G(x)$ for $0 \leq x \leq 0.8$

We need several preliminary results. From Edwards [7, p. 45] we have

Lemma 2.5. Let $f(z)$ be an analytic function on the disk $|z| \leq r$, let $f(0) = 0$, and let M be the maximum value of $\operatorname{Re} f(z)$ on the circle $|z| = r$. If $r_1 < r$, then $|f(z)| \leq 2r_1 M / (r - r_1)$ on the disk $|z| \leq r_1$.

We use this result in proving

Lemma 2.6. Let $S_1(T) = \int_0^T S(t) dt = \frac{1}{\pi} \int_0^T \arg \zeta(\frac{1}{2} + it) dt$. Let

$$A_L = -1.7412588,$$

$$B_L = -2.864789,$$

$$C_L = -0.3936986,$$

$$D_L = -3.8661494,$$

$$A_H = 0.0198943,$$

$$B_H = 0.4774648 \text{ and}$$

$$D_H = -0.666871.$$

If the Riemann hypothesis is true and $T \geq 1005$, then

$$\begin{aligned} A_L \log T + B_L \log \log T + C_L \log \log \left(T + \sqrt{\frac{7}{4}} \right) + D_L \\ \leq S_1(t) \leq A_H \log T + B_H \log \log T + D_H \end{aligned}$$

Proof. From Titchmarsh [26, p. 188] we have

$$S_1(T) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \log |\zeta(\sigma + iT)| d\sigma - \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \log |\zeta(\sigma)| d\sigma.$$

From Edwards [7, p. 192] we obtain

$$\left| \int_2^{\infty} \log |\zeta(\sigma + iT)| d\sigma \right| \leq \int_2^{\infty} 3 \cdot 2^{-\sigma} d\sigma = \frac{3}{4 \log 2}$$

and

$$0 \leq \int_2^\infty \log |\zeta(\sigma)| d\sigma \leq \frac{3}{4 \log 2}.$$

Hence,

(1)

$$S_1(T) = \frac{1}{\pi} \int_{\frac{1}{2}}^2 \log |\zeta(\sigma + iT)| d\sigma - \frac{1}{\pi} \int_{\frac{1}{2}}^2 \log |\zeta(\sigma)| d\sigma + R(T) = K_1 - K_2 + R(T),$$

where

$$\frac{-3}{\pi 2 \log 2} \leq R(T) \leq 0.$$

We first bound K_1 . For a fixed T we let $z_0 = 2 + iT$ and let C be the circle centered at z_0 with radius $R = 2 + \epsilon$ such that no zeros of $\zeta(s)$ lie on C . Since ϵ can be made arbitrarily small, we can assume that $R = 2$ in our computations. Assuming the Riemann hypothesis, we let $z_j = \frac{1}{2} + it_j$, for $1 \leq j \leq n$, be the n zeros of $\zeta(s)$ inside C . Finally we let $F(z) = \zeta(z) \prod_{j=1}^n a_j(z)$, where $a_j(z) = \frac{R^2 - (\bar{z}_j - \bar{z}_t)(z - z_0)}{R(z - z_j)}$.

We apply Lemma 2.5 to the disk $|z - z_0| \leq R$ with $f(z) = \operatorname{Re} \log(F(z)/F(z_0))$.

Since $f(z)$ is analytic in the disk and $f(z_0) = 0$, $f(z)$ satisfies the conditions of the lemma. Thus to use the lemma we need only to bound

$$(2) \quad \operatorname{Re} \log(F(z)/F(z_0)) = \operatorname{Re} \log F(z) - \operatorname{Re} \log F(z_0).$$

On $|z - z_0| = R$, we have

$$\operatorname{Re} \log F(z) = \log |F(z)| = \log |\zeta(z)| + \sum_{j=1}^n \log |a_j(z)|.$$

From Backland [2, pp. 362-367], we obtain, for $z = \sigma + it$ and $50 \leq t$,

$$|\zeta(z)| \leq \begin{cases} \log t - .048 & \text{if } 1 < \sigma \\ \left(\frac{t}{2\pi}\right)^{\frac{1-\sigma}{2}} \log t \left(1 + \frac{4}{t^2}\right)^{-1} \left(1 + \frac{5}{t^2}\right)^{-1} & \text{if } \frac{1}{2} \leq \sigma \leq 1 \\ \left(\frac{t}{2\pi}\right)^{\frac{1-\sigma}{2}} \log t \left(\frac{t^2}{t^2-4}\right) & \text{if } 0 \leq \sigma \leq \frac{1}{2}. \end{cases}$$

Since $|a_j(z)| = 1$ on $|z - z_0| = R$ we have

$$(3) \quad \operatorname{Re} \log F(z) \leq \operatorname{Re} \log \zeta(z)$$

$$\leq \begin{cases} \log(\log t - .048) & \text{if } 1 < \sigma \\ \frac{1-\sigma}{2} \log \frac{t}{2\pi} + \log \log t - \log \left(1 + \frac{4}{t^2}\right) \left(1 + \frac{5}{t^2}\right) & \text{if } \frac{1}{2} \leq \sigma \leq 1 \\ \frac{1-\sigma}{2} \log \frac{t}{2\pi} + \log \log t + \log \left(\frac{t^2}{t^2-4}\right) & \text{if } 0 \leq \sigma \leq \frac{1}{2} \end{cases}$$

for $|z - z_0| = R$.

We next obtain a lower bound for

$$(4) \quad \operatorname{Re} \log F(z_0) = \log |\zeta(z_0)| + \sum_{j=1}^n \log |a_j(z_0)|.$$

From Edwards [7, p.190] we have

$$(5) \quad \log |\zeta(z_0)| \geq -\log \zeta(2) = -\log \frac{\pi^2}{6}.$$

In addition,

$$|a_j(z_0)| = \left| \frac{R}{(z_0 - z_j)} \right| \geq 1$$

and thus

$$(6) \quad \sum_{j=1}^n \log |a_j(z_0)| \geq 0.$$

Combining (2), (3), (4), (5), and (6) we obtain

$$f(z) = \operatorname{Re} \log(F(z)/F(z_0)) \leq \frac{1}{2} \log T + \log \log T - 0.4212345 = M$$

for $1005 \leq t$ and $|z - z_0| = R$. Applying Lemma 2.5 to $f(z)$ we have

$$(7) \quad |f(z)| \leq 6M$$

in $|z - z_0| \leq \frac{3}{2}$. From this it follows directly that

$$|\operatorname{Re} \log F(z) - \operatorname{Re} \log F(z_0)| \leq 6M.$$

and combining this with (4), (5) and (6) we obtain

$$(8) \quad \operatorname{Re} \log F(z) \geq -6M - \log \frac{\pi^2}{6}.$$

We can now bound K_1 . Using (3), we obtain

$$(9) \quad \begin{aligned} K_1 &= \frac{1}{\pi} \int_{\frac{1}{2}}^1 \operatorname{Re} \log \zeta(\sigma + iT) d\sigma + \frac{1}{\pi} \int_1^2 \operatorname{Re} \log \zeta(\sigma + iT) d\sigma \\ &\leq \frac{1}{\pi} \int_{\frac{1}{2}}^1 \left(\frac{1-\sigma}{2} \log \frac{T}{2\pi} + \log \log T \right) d\sigma + \frac{1}{\pi} \int_1^2 \log \log T d\sigma \\ &\leq 0.0198943 \log T + 0.4774648 \log \log T - 0.0365633. \end{aligned}$$

Using (8), we obtain

$$\begin{aligned}
 (10) \quad K_1 &= \frac{1}{\pi} \int_{\frac{1}{2}}^2 \operatorname{Re} \log \zeta(\sigma + iT) d\sigma \\
 &= \frac{1}{\pi} \int_{\frac{1}{2}}^2 \operatorname{Re} \log F(\sigma + iT) d\sigma - \frac{1}{\pi} \sum_{j=1}^n \int_{\frac{1}{2}}^2 \operatorname{Re} \log a_j(\sigma + iT) d\sigma \\
 &> \frac{1}{\pi} \left(-6M - \log \frac{\pi^2}{6} \right) - \frac{1}{\pi} \sum_{j=1}^n \int_{\frac{1}{2}}^2 \operatorname{Re} \log a_j(\sigma + iT) d\sigma.
 \end{aligned}$$

Thus we need to find an upper bound for

$$(11) \quad \sum_{j=1}^n \int_{\frac{1}{2}}^2 \operatorname{Re} \log a_j(\sigma + iT) d\sigma \leq n \max_j \int_{\frac{1}{2}}^2 \log |a_j(\sigma + iT)|.$$

From Backland [2, p. 355], we have

$$(12) \quad \left| N(T) - \left(\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} \right) \right| < 0.137 \log T + 0.443 \log \log T + 4.350$$

for $T > 2$. Since we are assuming the Riemann hypothesis, the zeros z_j inside $|z - z_0| = R$ have real part equal to $\frac{1}{2}$ and for $T > 1005$

$$\begin{aligned}
 (13) \quad n &\leq N\left(T + \sqrt{\frac{7}{4}}\right) - N\left(T - \sqrt{\frac{7}{4}}\right) \\
 &\leq \frac{T}{2\pi} \left(\log \frac{T + \sqrt{\frac{7}{4}}}{2\pi} - \log \frac{T - \sqrt{\frac{7}{4}}}{2\pi} \right) + \frac{1}{2\pi} \sqrt{\frac{7}{4}} \left(\log \frac{T + \sqrt{\frac{7}{4}}}{2\pi} + \log \frac{T - \sqrt{\frac{7}{4}}}{2\pi} \right) - \frac{1}{\pi} \sqrt{\frac{7}{4}} \\
 &\quad + 0.137 \left(\log \left(T + \sqrt{\frac{7}{4}} \right) + \log \left(T - \sqrt{\frac{7}{4}} \right) \right) + 0.443 \left(\log \log \left(T + \sqrt{\frac{7}{4}} \right) + \log \log \left(T - \sqrt{\frac{7}{4}} \right) \right) + 8.7
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{T}{2\pi} \left(\frac{2\sqrt{\frac{7}{4}}}{T-\sqrt{\frac{7}{4}}} \right) + \frac{1}{2\pi} \sqrt{\frac{7}{4}} \left(2 \log \frac{T}{2\pi} \right) - \frac{1}{\pi} \sqrt{\frac{7}{4}} \\
&\quad + 0.137 (2 \log T) + 0.886 \log \log \left(T + \sqrt{\frac{7}{4}} \right) + 8.7 \\
&\leq 0.6950843 \log T + 0.886 \log \log \left(T + \sqrt{\frac{7}{4}} \right) + 7.9266538.
\end{aligned}$$

From the definition of $a_j(\sigma+iT)$, we write

$$\begin{aligned}
&\int_{\frac{1}{2}}^2 \log |a_j(\sigma+iT)| d\sigma = \int_{\frac{1}{2}}^2 \log \left| \frac{R^2 - (\bar{z}_j - \bar{z}_0)(z - z_0)}{R(z - z_j)} \right| d\sigma \\
&= \int_{\frac{1}{2}}^2 \log \left| \frac{4 - ((\frac{1}{2} - it_j)(2 - iT))((\sigma+iT) - (2+iT))}{2((\sigma+iT) - (\frac{1}{2} + it_j))} \right| d\sigma \\
&= \int_{\frac{1}{2}}^2 \log \left| \frac{1 + \frac{3}{2}\sigma - ib_j(\sigma-2)}{2\sigma-1 + i2b_j} \right| d\sigma,
\end{aligned}$$

where $b_j = T - t_j$. Since we are assuming the Riemann hypothesis, $|b_j| \leq \sqrt{\frac{7}{4}}$, and in this range

$$\begin{aligned}
0 \leq \log \left| \frac{1 + \frac{3}{2}\sigma - ib_j(\sigma-2)}{2\sigma-1 + i2b_j} \right| &= \frac{1}{2} \log \left(\frac{(1 + \frac{3}{2}\sigma)^2 + (\sigma-2)^2 b_j^2}{(2\sigma-1)^2 + 4b_j^2} \right) \\
&< \frac{1}{2} \log \left(\frac{(1 + \frac{3}{2}\sigma)^2}{(2\sigma-1)^2} \right).
\end{aligned}$$

Combining the above two results we obtain

$$(14) \quad 0 \leq \int_{\frac{1}{2}}^2 \log |a_j(\sigma+iT)| d\sigma \leq \frac{1}{2} \int_{\frac{1}{2}}^2 \log \left(\frac{(1 + \frac{3}{2}\sigma)^2}{(2\sigma-1)^2} \right) d\sigma \leq 1.3959830.$$

Combining (10), (11), (13) and (14) we obtain

(15)

$$\begin{aligned} \frac{1}{\pi} \int_{\frac{1}{2}}^2 \log |\zeta(\sigma + iT)| d\sigma \geq \frac{1}{\pi} \left(-6M - \log \frac{\pi^2}{6} \right) - \frac{1}{\pi} (1.3959830) (0.6950843 \log T \\ + 0.886 \log \log \left(T + \sqrt{\frac{7}{4}} \right) + 7.9266538), \end{aligned}$$

which gives us a lower bound for K_1 .

To bound K_2 we use the formula (see Edwards [7, p.114-115])

$$\zeta(\sigma) = \frac{1}{\sigma-1} + \frac{1}{2} - \sigma \int_1^\infty \frac{\hat{B}_1(u) du}{u^{\sigma+1}} = \frac{1}{\sigma-1} + \frac{1}{2} + R_0,$$

where $0 \leq R_0 \leq \frac{1}{12}$. This gives us

$$\frac{1}{2} + \frac{1}{\sigma-1} \leq \zeta(\sigma) \leq \frac{1}{2} + \frac{1}{\sigma-1} + \frac{\sigma}{12}.$$

Thus,

$$(16) \quad \log \left| \frac{1}{\sigma-1} + \frac{7}{12} \right| \leq \log |\zeta(\sigma)| \leq \log \left| \frac{1}{\sigma-1} + \frac{1}{2} \right| \quad \left(\frac{1}{2} \leq \sigma < 1 \right)$$

and

$$(17) \quad \log \left| \frac{1}{\sigma-1} + \frac{1}{2} \right| \leq \log |\zeta(\sigma)| \leq \log \left| \frac{1}{\sigma-1} + \frac{2}{3} \right| \quad (1 < \sigma \leq 2).$$

Since

$$\int_{\frac{1}{2}}^1 \log \left| \frac{1}{\sigma-1} + c \right| = \frac{1}{2} \log |2-c| - \frac{1}{c} \log \left| 1 - \frac{c}{2} \right|$$

for $\frac{1}{2} \leq c \leq \frac{7}{12}$, we use (16) to obtain

$$(18) \quad \frac{1}{2} \log \frac{17}{12} - \frac{12}{7} \log \frac{17}{24} \leq \int_{\frac{1}{2}}^1 \log |\zeta(\sigma)| d\sigma \leq \frac{1}{2} \log \frac{3}{2} - 2 \log \frac{3}{4}.$$

Since

$$\int_1^2 \log \left| \frac{1}{\sigma-1} + c \right| = (1 + \frac{1}{c}) \log |1 + c|$$

for $\frac{1}{2} \leq c \leq \frac{2}{3}$, we use (17) to obtain

$$(19) \quad 3 \log \frac{3}{2} \leq \int_1^2 \log |\zeta(\sigma)| d\sigma \leq \frac{5}{2} \log \frac{5}{3}.$$

Combining (18) and (19), we have

$$(20) \quad 0.6303077 \leq K_2 \leq 0.6541774.$$

Combining (1), (9), (15) and (20) we complete the proof of the lemma.

We next prove

Lemma 2.7. If $m \geq 1$, then

$$(i) \quad \int_a^b \frac{\log y dy}{y^{m+2}} = \frac{1}{(m+1)a^{m+1}} \left(\log a + \frac{1}{m+1} \right) - \frac{1}{(m+1)b^{m+1}} \left(\log b + \frac{1}{m+1} \right)$$

$$\int_a^b \frac{\log \log y dy}{y^{m+2}} \leq \frac{1}{(m+1)a^{m+1}} \left(\log \log a + \frac{1}{(m+1) \log a} \right)$$

(ii)

$$- \frac{1}{(m+1)b^{m+1}} \left(\log \log b + \frac{1}{(m+1) \log a} \right)$$

$$(iii) \quad \int_a^b \frac{\log \log \left(y + \sqrt{\frac{7}{4}} \right) dy}{y^{m+2}} \leq \frac{1}{(m+1)a^{m+1}} \left(\log \log \left(a + \sqrt{\frac{7}{4}} \right) + \frac{1}{(m+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \\ - \frac{1}{(m+1)b^{m+1}} \left(\log \log \left(b + \sqrt{\frac{7}{4}} \right) + \frac{1}{(m+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right)$$

and

$$(iv) \quad \int_a^b \frac{dy}{y^{m+2}} = \frac{1}{m+1} \left(\frac{1}{a^{m+1}} + \frac{1}{b^{m+1}} \right).$$

Proof. We have

$$\begin{aligned} \int_a^b \frac{\log y \, dy}{y^{m+2}} &= \frac{\log a}{(m+1)a^{m+1}} - \frac{\log b}{(m+1)b^{m+1}} + \frac{1}{m+1} \int_a^b \frac{dy}{y^{m+2}} \\ &= \frac{1}{(m+1)a^{m+1}} \left(\log a + \frac{1}{(m+1)} \right) - \frac{1}{(m+1)b^{m+1}} \left(\log b + \frac{1}{m+1} \right), \end{aligned}$$

$$\begin{aligned} \int_a^b \frac{\log \log y \, dy}{y^{m+2}} &= \frac{\log \log a}{(m+1)a^{m+1}} - \frac{\log \log b}{(m+1)b^{m+1}} + \frac{1}{(m+1)} \int_a^b \frac{dy}{y^{m+2} \log y} \\ &\leq \frac{\log \log a}{(m+1)a^{m+1}} - \frac{\log \log b}{(m+1)b^{m+1}} + \frac{1}{(m+1) \log a} \int_a^b \frac{dy}{y^{m+2}} \\ &= \frac{1}{(m+1)a^{m+1}} \left(\log \log a + \frac{1}{(m+1) \log a} \right) - \frac{1}{(m+1)b^{m+1}} \left(\log \log b + \frac{1}{(m+1) \log a} \right), \end{aligned}$$

$$\begin{aligned} \int_a^b \frac{\log \log(y + \sqrt{\frac{7}{4}}) \, dy}{y^{m+2}} &\leq \frac{1}{(m+1)a^{m+1}} \left(\log \log \left(a + \sqrt{\frac{7}{4}} \right) + \frac{1}{(m+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \\ &= \frac{1}{(m+1)b^{m+1}} \left(\log \log \left(b + \sqrt{\frac{7}{4}} \right) + \frac{1}{(m+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \end{aligned}$$

and

$$\int_a^b \frac{1}{y^{m+2}} = \frac{1}{m+1} \left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}} \right).$$

We use this lemma in proving

Lemma 2.8. Let $n \in \mathbb{Z}^+$. Let $A_L, B_L, C_L, D_L, A_H, B_H$ and D_H be as in

Lemma 2.6. Let $N(y) = \frac{y}{2\pi} \log \frac{y}{2\pi} - \frac{y}{2\pi} + \frac{7}{8} + S(y) + G(y)$, where

$S(y) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iy)$. If $a \geq 1005$, then

$$\int_a^b \frac{dN(y)}{|\frac{1}{2} + iy|} \leq \frac{1}{4\pi} \left(\left(\log \frac{b}{2\pi} \right)^2 - \left(\log \frac{a}{2\pi} \right)^2 \right) + \frac{S(b) + G(b)}{b} - \frac{S(a) + G(a)}{a}$$

$$+ \frac{1}{2} \left(\frac{25}{96\pi} + (A_H - A_L) \log a + (B_H - B_L) \log \log a - C_L \log \log \left(a + \sqrt{\frac{7}{4}} \right) + (D_H - D_L) \right.$$

$$\left. + \frac{A_H}{2} + \frac{B_H}{2 \log a} \right) - \frac{1}{2b} \left(\frac{25}{48\pi} + A_H + \frac{B_H}{\log a} \right),$$

$$\int_a^b \frac{dN(y)}{|\frac{1}{2} + iy|^{2n}} \leq \frac{1}{2\pi(2n-1)a^{2n-1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n-1} \right) - \frac{1}{2\pi(2n-1)b^{2n-1}} \left(\log \frac{b}{2\pi} + \frac{1}{2n-1} \right)$$

$$+ \frac{S(b) + G(b)}{b^{2n}} - \frac{S(a) + G(a)}{a^{2n}} + \frac{2n}{a^{2n+1}} \left(\frac{25}{48\pi(2n+1)} + (A_H - A_L) \log a + (B_H - B_L) \log \log a \right.$$

$$\left. - C_L \log \log \left(a + \sqrt{\frac{7}{4}} \right) + (D_H - D_L) + \frac{A_H}{2n+1} + \frac{B_H}{(2n+1) \log a} \right)$$

$$- \frac{2n}{(2n+1)b^{2n+1}} \left(\frac{25}{48\pi} + A_H + \frac{B_H}{\log a} \right)$$

and

$$\int_a^b \frac{dN(y)}{|\frac{1}{2} + iy|^{2n}} \geq \frac{1}{2\pi(2n-1)a^{2n-1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n-1} \right) - \frac{1}{2\pi(2n-1)b^{2n-1}} \left(\log \frac{b}{2\pi} + \frac{1}{2n-1} \right)$$

$$+ \frac{S(b) + G(b)}{b^{2n}} - \frac{S(a) + G(a)}{a^{2n}} + \frac{2n}{a^{2n+1}} \left(\frac{-25}{48\pi(2n+1)} + (A_L - A_H) \log a + (B_L - B_H) \log \log a \right.$$

$$\begin{aligned}
& + C_L \log \log \left(a + \sqrt{\frac{7}{4}} \right) + (D_L - D_H) + \frac{A_L}{2n+1} + \frac{B_L}{(2n+1) \log a} + \frac{C_L}{(2n+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \\
& \quad - \frac{2n}{(2n+1)b} \left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \\
& - \frac{n}{4} \left(\frac{1}{2\pi(2n+1)a} \left(\log \frac{a}{2\pi} + \frac{1}{2n+1} \right) - \frac{1}{2\pi(2n+1)b} \left(\log \frac{b}{2\pi} + \frac{1}{2n+1} \right) \right. \\
& + \frac{S(b)+G(b)}{b^{2n+2}} - \frac{S(a)+G(a)}{a^{2n+2}} + \frac{2n+2}{a^{2n+3}} \left(\frac{-25}{48\pi(2n+3)} + (A_L - A_H) \log a + (B_L - B_H) \log \log a \right. \\
& \quad \left. \left. + C_L \log \log \left(a + \sqrt{\frac{7}{4}} \right) + (D_L - D_H) + \frac{A_L}{2n+3} + \frac{B_L}{(2n+3) \log a} + \frac{C_L}{(2n+3) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \right. \\
& \quad \left. - \frac{2n+2}{(2n+3)b} \left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \right).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
(21) \quad & \int_a^b \frac{dN(y)}{\left| \frac{1}{2} + iy \right|} \leq \int_a^b \frac{dN(y)}{y} \\
& = \int_a^b \frac{d \left(\frac{y}{2\pi} \log \frac{y}{2\pi} - \frac{y}{2\pi} + \frac{7}{8} \right)}{y} + \int_a^b \frac{d(S(y) + G(y))}{y} \\
& = \frac{1}{4\pi} \left(\left(\log \frac{b}{2\pi} \right)^2 - \left(\log \frac{a}{2\pi} \right)^2 \right) + \int_a^b \frac{d(S(y) + G(y))}{y}
\end{aligned}$$

and similarly

$$(22) \quad \int_a^b \frac{dN(y)}{\left| \frac{1}{2} + iy \right|^{2n}} \leq \int_a^b \frac{dN(y)}{y^{2n}}$$

$$= \frac{1}{2\pi(2n-1)a^{2n-1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n-1} \right) - \frac{1}{2\pi(2n-1)b^{2n-1}} \left(\log \frac{b}{2\pi} + \frac{1}{2n-1} \right) \\ + \int_a^b \frac{d(S(y) + G(y))}{y^{2n}}.$$

Since

$$\left(y^2 + \frac{1}{4}\right)^n - y^{2n} \leq \frac{n}{4} \left(y^{2n-2} + \binom{n-1}{1} y^{2n-4} \left(\frac{1}{4}\right) + \binom{n-1}{2} y^{2n-6} \left(\frac{1}{4}\right)^2 + \dots + \left(\frac{1}{4}\right)^{n-1} \right) \\ = \frac{n}{4} \left(y^2 + \frac{1}{4}\right)^{n-1}$$

we have

$$\int_a^b \frac{dN(y)}{|\frac{1}{2} + iy|^{2n}} \geq \int_a^b \frac{dN(y)}{y^{2n}} - \frac{n}{4} \int_a^b \frac{dN(y)}{y^{2n+2}} \\ = \frac{1}{2\pi(2n-1)a^{2n-1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n-1} \right) - \frac{1}{2\pi(2n-1)b^{2n-1}} \left(\log \frac{b}{2\pi} + \frac{1}{2n-1} \right) \\ - \frac{n}{4} \left(\frac{1}{2\pi(2n+1)a^{2n+1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n+1} \right) - \frac{1}{2\pi(2n+1)b^{2n+1}} \left(\log \frac{b}{2\pi} + \frac{1}{2n+1} \right) \right) \\ + \int_a^b \frac{d(S(y) + G(y))}{y^{2n}} - \frac{n}{4} \int_a^b \frac{d(S(y) + G(y))}{y^{2n}}.$$

For $m \geq 1$, we integrate by parts to see that

$$(24) \int_a^b \frac{d(S(y) + G(y))}{y^m} = \frac{S(b) + G(b)}{b^m} - \frac{S(a) + G(a)}{a^m} + m \int_a^b \frac{S(y) dy}{y^{m+1}} + m \int_a^b \frac{G(y) dy}{y^{m+1}} \\ = \frac{S(b) + G(b)}{b^m} - \frac{S(a) + G(a)}{a^m} + m \left(\frac{S_1(b)}{b^{m+1}} - \frac{S_1(a)}{a^{m+1}} + (m+1) \int_a^b \frac{S_1(y) dy}{y^{m+2}} + \int_a^b \frac{G(y) dy}{y^{m+1}} \right).$$

Using Lemma 2.6 and Lemma 2.7, we obtain

$$\begin{aligned}
 (26) \quad \int_a^b \frac{S_1(y)dy}{y^{m+2}} &\leq \frac{A_H}{m+1} \left(\frac{1}{a^{m+1}} \left(\log a + \frac{1}{m+1} \right) - \frac{1}{b^{m+1}} \left(\log b + \frac{1}{m+1} \right) \right) \\
 &+ \frac{B_H}{m+1} \left(\frac{1}{a^{m+1}} \left(\log \log a + \frac{1}{(m+1) \log a} \right) - \frac{1}{b^{m+1}} \left(\log \log b + \frac{1}{(m+1) \log a} \right) \right) \\
 &+ \frac{D_H}{m+1} \left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 (27) \quad \int_a^b \frac{S_1(y)dy}{y^{m+2}} &\geq \frac{A_L}{m+1} \left(\frac{1}{a^{m+1}} \left(\log a + \frac{1}{m+1} \right) - \frac{1}{b^{m+1}} \left(\log b + \frac{1}{m+1} \right) \right) \\
 &+ \frac{B_L}{m+1} \left(\frac{1}{a^{m+1}} \left(\log \log a + \frac{1}{(m+1) \log a} \right) - \frac{1}{b^{m+1}} \left(\log \log b + \frac{1}{(m+1) \log a} \right) \right) \\
 &+ \frac{C_L}{m+1} \left(\frac{1}{a^{m+1}} \left(\log \log \left(a + \sqrt{\frac{7}{4}} \right) + \frac{1}{(m+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right) - \frac{1}{b^{m+1}} \left(\log \log \left(b + \sqrt{\frac{7}{4}} \right) \right. \right. \\
 &\quad \left. \left. + \frac{1}{(m+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \right) \\
 &+ \frac{D_L}{m+1} \left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}} \right).
 \end{aligned}$$

Backland [2, p. 374] showed that

$$|G(y)| \leq \frac{25}{48\pi y}$$

for $y \geq 50$. Hence for $m \geq 1$

$$\frac{-25}{48\pi(m+1)}\left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}}\right) \leq \int_a^b \frac{G(y)dy}{y^{m+1}} \leq \frac{25}{48\pi(m+1)}\left(\frac{1}{a^{m+1}} - \frac{1}{b^{m+1}}\right).$$

The upper bound for

$$\int_a^b \frac{dN(y)}{\left|\frac{1}{2}+iy\right|}$$

follows from (21), (24), (26), (28) and Lemma 2.6. The upper bound for

$$\int_a^b \frac{dN(y)}{\left|\frac{1}{2}+iy\right|^{2n}}$$

follows from (22), (24), (26), (28) and Lemma 2.6. The lower bound for

$$\int_a^b \frac{dN(y)}{\left|\frac{1}{2}+iy\right|^{2n}}$$

follows from (23), (24), (27), (28) and Lemma 2.6.

We now have the tools which we need to prove the theorem. To calculate $G(x, B)$, we write

$$G(x, B) = \frac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda \cdot x) - 1}{2\pi i \lambda} \hat{G}(\lambda, B) d\lambda,$$

where

$$\hat{G}(\lambda, B) = \int_{-\infty}^{\infty} e(-\lambda x) dG(x, B)$$

as in Theorem 1.9. Here

$$\hat{G}(\lambda, B) = \int_{\mathbb{I}^{\mathbb{R}}} e(-\lambda \mathbf{Y}_B) d\underline{\theta} = \prod_{0 < \gamma \leq B} \int_0^1 e\left(\frac{-2\lambda}{|\rho|} \sin 2\pi \theta\right) d\theta.$$

$$\begin{aligned}
&= \prod_{0 < \gamma \leq B} \frac{1}{\pi} \int_0^\pi e^{i \frac{4\pi\lambda}{|\rho|} \cos \theta} d\theta \\
&= \prod_{0 < \gamma \leq B} J_0 \left(\frac{4\pi\lambda}{|\rho|} \right),
\end{aligned}$$

since $J_0(z) = \frac{1}{\pi} \int_0^\pi e^{i z \cos \theta} d\theta$ (see Abramowitz and Stegun [1, p. 360]). We note here that $|J_0(z)| \leq \exp(|\operatorname{Im}(z)|)$. Hence

$$G(x, B) = \frac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda x) - 1}{2\pi i \lambda} \prod_{0 < \gamma \leq B} J_0 \left(\frac{4\pi\lambda}{|\rho|} \right)$$

and similarly

$$G(x) = \frac{1}{2} + \int_{-\infty}^{\infty} \frac{e(\lambda x) - 1}{2\pi i \lambda} \prod_{\gamma > 0} J_0 \left(\frac{4\pi\lambda}{|\rho|} \right).$$

Thus for any $c \leq \frac{B}{4\pi}$ we have

(29)

$$\begin{aligned}
|G(x) - G(x, B)| &\leq \left| \int_0^c \frac{2 \sin 2\pi\lambda x}{2\pi\lambda} \prod_{0 < \gamma \leq B} J_0 \left(\frac{4\pi\lambda}{|\rho|} \right) \left(1 - \prod_{\gamma > B} J_0 \left(\frac{4\pi\lambda}{|\rho|} \right) \right) d\lambda \right| \\
&\leq 2 \int_0^c \left| 1 - \prod_{\gamma > B} J_0 \left(\frac{4\pi\lambda}{|\rho|} \right) \right| d\lambda + \int_c^\infty \frac{1}{\lambda} \prod_{0 < \gamma \leq B} \left| J_0 \left(\frac{4\pi\lambda}{|\rho|} \right) \right| d\lambda = L_1 + L_2.
\end{aligned}$$

To bound the first integral, we observe that for $|z| \leq 6$,

$$J_0(z) \geq 1 - \frac{z^2}{4}.$$

Thus

$$L_1 \leq 2 \int_0^c \left| 1 - \exp \left(\sum_{\gamma > B} \log \left(1 - \frac{(4\pi\lambda)^2}{4|\rho|^2} \right) \right) \right| d\lambda.$$

Applying Lemma 1.17, we see that the above integral

$$\begin{aligned} &\leq 2 \int_0^c \left| 1 - \exp \left(- \sum_{\gamma > B} \left(\frac{(4\pi\lambda)^2}{4|\rho|^2} + \frac{(4\pi\lambda)^4}{24|\rho|^4} \right) \right) \right| d\lambda. \\ &= 2 \int_0^c \left| 1 - \exp \left(- \frac{(4\pi\lambda)^2}{4} \int_{B^+}^{\infty} \frac{dN(y)}{|\frac{1}{2} + iy|^2} - \frac{(4\pi\lambda)^4}{24} \int_B^{\infty} \frac{dN(y)}{|\frac{1}{2} + iy|^4} \right) \right| d\lambda. \end{aligned}$$

Next we apply Lemma 2.8 and obtain

$$\begin{aligned} (30) \quad L_1 &\leq 2 \int_0^c \left| 1 - \exp \left(\frac{-(4\pi\lambda)^2 \log B}{8\pi B} - \frac{(4\pi\lambda)^4 \log B}{144\pi B^3} \right) \right| d\lambda \\ &\leq 2 \left(\frac{(4\pi)^2 c^3 \log B}{24\pi B} + \frac{(4\pi)^4 c^5 \log B}{720\pi B^3} \right). \end{aligned}$$

To bound the second integral, we use the well-known result (see Watson [30, pp.206-208]) that

$$J_0(t) = \left(\frac{2}{\pi t} \right)^{\frac{1}{2}} \left(\cos(t - \frac{\pi}{4}) P(t, 0) - \sin(t - \frac{\pi}{4}) Q(t, 0) \right)$$

where $0 \leq t$, $0 \leq P(t, 0) \leq 1$ and $\frac{-1}{8t} \leq Q(t, 0) \leq 0$. Thus,

$$(31) \quad |J_0(t)| \leq \min \left(\left(\frac{2}{\pi t} \right)^{\frac{1}{2}} \left(1 + \frac{1}{8t} \right), 1 \right).$$

Using the above result, we obtain

$$(32) \quad L_2 \leq \frac{2}{653} \prod_{0 < \gamma < 1005} \left(\left(\frac{|\rho|}{2\pi^2 c} \right)^{\frac{1}{2}} \left(1 + \frac{|\rho|}{32\pi c} \right) \right).$$

We choose $c = e^{10}/4\pi$. Using (30) we obtain $L_1 \leq 10^{-12.25}$. Using (32) we obtain $L_2 \leq 10^{-78}$. Combining these results with (29), we complete the proof of the theorem.

Next we prove

Theorem 2.9. If $0 \leq x \leq 1$, $h \leq \frac{1}{512}$ and $B = e^{58.2230805}$, then

$$\left| G(x) - \left(\frac{1}{2} + h \sum_{n=-\infty}^{\infty} \left(\frac{e(nhx)-1}{2\pi inh} \prod_{0 < \gamma \leq B} J_0 \left(\frac{4\pi nh}{|\rho|} \right) \right) \right) \right| \leq 10^{-12}.$$

We need several preliminary results.

Lemma 2.10. Let $f(z) = \frac{e(zx)-1}{2\pi iz} \hat{G}(z, B)$. If $0 \leq x \leq 1$ and

$$0 < h \leq \frac{1}{2 \left(\sum_{0 < \gamma \leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} \right)},$$

then $\int_{-\infty}^{\infty} f(\lambda) d\lambda = h \sum_{-\infty}^{\infty} f(nh)$.

Proof. By Corollary 1.11 and (31), it is sufficient to show that

$$\hat{f}(t) = 0 \text{ for } |t| \geq 2 \left(\sum_{0 < \gamma \leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} \right).$$

We have

$$\begin{aligned} \hat{f}(t) &= \int_{-\infty}^{\infty} e(-tz) f(z) dz \\ &= \lim_{r \rightarrow \infty} \int_{\Gamma} e(-tz) f(z) dz. \end{aligned}$$

From Lemma 1.12, we know that the above limit equals zero if

$e(-tz - \frac{z}{2\pi}) f(z) \rightarrow 0$ uniformly with respect to $\arg z$ as $|z| \rightarrow \infty$ for $0 \leq \arg z \leq \pi$.

For $t \leq 0$, $z = r \cos \theta + i r \sin \theta$ and $0 \leq \theta \leq \pi$, we have

$$\left| e\left(-tz - \frac{z}{2\pi}\right) f(z) \right| \leq \frac{2 \exp(r \sin \theta (1 + 2\pi x + 2\pi t))}{r} \prod_{0 < \gamma \leq B} J_0 \left(\frac{4\pi z}{|\rho|} \right).$$

In addition

$$\prod_{0 < \gamma \leq B} \left| J_0 \left(\frac{4\pi z}{|\rho|} \right) \right| \leq \exp \left(r \sin \theta \sum_{0 < \gamma \leq B} \frac{4\pi z}{|\rho|} \right)$$

so that

$$\left| e \left(-tz - \frac{z}{2\pi} \right) f(z) \right| \leq \frac{2}{r} \exp \left(4\pi r \sin \theta \left(\sum_{0 < \gamma \leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} + \frac{t}{2} \right) \right).$$

Thus $\left| e \left(-tz - \frac{z}{2\pi} \right) f(z) \right| \rightarrow 0$ uniformly with respect to $\arg z$ as $|z| \rightarrow \infty$ for $0 \leq \arg z \leq \pi$ if $t \leq -2 \left(\sum_{0 < \gamma \leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} \right)$, which shows that $\hat{f}(t) = 0$

for these t . For $t \geq 0$, we have

$$\int_{-\infty}^{\infty} e(-tz) f(z) dz = - \int_{-\infty}^{\infty} e(tz) \frac{e(-zx) - 1}{2\pi iz} \prod_{0 < \lambda \leq B} J_0 \left(\frac{4\pi z}{|\rho|} \right),$$

and proceeding as before, we obtain $\hat{f}(t) = 0$ for

$t \geq 2 \left(\sum_{0 < \gamma \leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} \right)$. This completes the proof of the

lemma.

We next prove

Lemma 2.11. If $0 \leq x \leq 1$ and $B \geq e^{58.223085}$, then

$$\frac{1}{512} \leq \frac{1}{2 \left(\sum_{0 < \gamma \leq B} \frac{1}{|\rho|} + \frac{1}{4\pi} + \frac{x}{2} \right)}.$$

Proof. Put

$$\sum_{0 < \gamma \leq B} \frac{1}{|\rho|} = \sum_{0 < \gamma < 1005} \frac{1}{|\rho|} + \sum_{1005 < \gamma \leq B} \frac{1}{|\rho|} = T_1 + T_2.$$

We obtain $T_1 = 2.0325543$ by direct calculation. From Lemma 2.8, we have

$$\begin{aligned} T_2 &\leq \int_{1005}^{B^+} \frac{dN(\gamma)}{|\frac{1}{2} + iy|} \leq \frac{1}{4\pi} \left(\left(\log \frac{B}{2\pi} \right)^2 - \left(\log \frac{1005}{2\pi} \right)^2 \right) \\ &+ \frac{S(B^+) + G(B^+)}{B} - \frac{S(1005) + G(1005)}{1005} + \frac{1}{(1005)^2} \left(\frac{25}{96\pi} + (A_H - A_L) \log 1005 \right) \\ &+ (B_H - B_L) \log \log 1005 - C_L \log \log \left(1005 + \sqrt{\frac{7}{4}} \right) + D_H - D_L + \frac{A_H}{2} + \frac{B_H}{2 \log 1005} \\ &\quad - \frac{1}{B^2} \left(\frac{25}{96\pi} + \frac{A_H}{2} + \frac{B_H}{2 \log 1005} \right). \end{aligned}$$

Since $|S(B^+) + G(B^+)| \leq 0.137 \log B + 0.443 \log \log B + 4.35$ and $S(1005) + G(1005) \geq 0.34741$ we have

$$T_2 \leq 251.$$

This completes the proof of the lemma.

The theorem follows directly from Theorem 2.9, Lemma 2.10 and Lemma 2.11.

3. The calculation of $G(x)$. In Theorem 2.9, we reduced the problem of evaluating $G(x)$ for $0 \leq x \leq 1$ to that of evaluating

$$\frac{1}{2} + h \sum_{n=-\infty}^{\infty} \left(\frac{e(nhx) - 1}{2\pi inh} \prod_{0 < \nu \leq B} J_0 \left(\frac{4\pi n h}{|\rho|} \right) \right)$$

when $0 < h \leq \frac{1}{512}$ and $B = e^{58.2230805}$.

For $n = 0$, we have

$$\lim_{n \rightarrow 0} h \frac{e(nhx) - 1}{2\pi inh} \prod_{0 < \nu \leq B} J_0 \left(\frac{4\pi n h}{|\rho|} \right) = hx.$$

For $n > 0$, we have

$$\begin{aligned} & \frac{e(nhx) - 1}{2\pi inh} \prod_{0 < \nu \leq B} J_0 \left(\frac{4\pi n h}{|\rho|} \right) + \frac{e(-nhx) - 1}{-2\pi inh} \prod_{0 < \nu \leq B} J_0 \left(\frac{-4\pi n h}{|\rho|} \right) \\ &= \frac{2 \sin 2\pi nhx}{2\pi nh} \prod_{0 < \nu \leq B} J_0 \left(\frac{4\pi n h}{|\rho|} \right). \end{aligned}$$

Using the above equalities, we obtain

$$G(x) = \frac{1}{2} + hx + 2h \sum_{n=1}^{\infty} \frac{\sin 2\pi nhx}{2\pi nh} \prod_{0 < \nu \leq B} J_0 \left(\frac{4\pi n h}{|\rho|} \right) + \epsilon$$

where $|\epsilon| \leq 10^{-12.24}$.

We now take $h = \frac{1}{512}$. We first truncate the above sum.

Using (31), we obtain

$$\begin{aligned}
& \left| \frac{1}{256\pi} \sum_{n=8065}^{\infty} \frac{\sin \pi n x / 256}{n/256} \prod_{0 < \gamma \leq B} J_0 \left(\frac{\pi n / 128}{|\rho|} \right) \right| \\
& \leq \frac{1}{256\pi} \sum_{n=8065}^{\infty} \frac{1}{n/256} \prod_{0 < \gamma \leq B} \min \left(\left(\frac{256 |\rho|}{\pi^2 n} \right)^{\frac{1}{2}}, \left(1 + \frac{16 |\rho|}{\pi n} \right), 1 \right) \\
& \leq \frac{1}{\pi} \int_{31.5}^{\infty} \frac{1}{t} \prod_{0 < \gamma \leq 235} \left(\frac{|\rho|}{\pi^2 t} \right)^{\frac{1}{2}} \left(1 + \frac{|\rho|}{16\pi t} \right) dt \\
& \leq 0.5 \cdot 10^{-18}.
\end{aligned}$$

Using this, we obtain

$$G(x) = \frac{1}{2} + \frac{x}{512} + \frac{1}{256\pi} \sum_{n=1}^{8064} \frac{\sin \pi n x / 256}{n/256} \prod_{0 < \gamma \leq B} J_0 \left(\frac{\pi n / 128}{|\rho|} \right) + \epsilon,$$

$$\text{with } |\epsilon| \leq 10^{-12.23}.$$

Through calculation we show that

$$\left| \frac{1}{256\pi} \sum_{n=5377}^{8064} \frac{\sin \pi n x / 256}{n/256} \prod_{0 < \gamma \leq B} J_0 \left(\frac{\pi n / 128}{|\rho|} \right) \right| \leq 3.36 \cdot 10^{-24}.$$

Hence

$$G(x) = \frac{1}{2} + \frac{x}{512} + \frac{1}{256\pi} \sum_{n=1}^{5376} \frac{\sin \pi nx/256}{n/256} \prod_{0 < \gamma \leq B} J_0 \left(\frac{\pi n/128}{|\rho|} \right) + \epsilon$$

with $|\epsilon| \leq 10^{-12.22}$.

Since we cannot compute $J_0 \left(\frac{t}{|\rho|} \right)$ for all $0 < \gamma \leq B$, we need

Theorem 2.12. Let $J_0 \left(\frac{t}{|\rho|} \right) = \sum_{k=0}^{\infty} u_k$,

$$T_L \left(\frac{t}{|\rho|} \right) = \sum_{k=1}^5 u_k + 0.9999121 u_6$$

and

$$T_H \left(\frac{t}{|\rho|} \right) = \sum_{k=1}^6 u_k.$$

Let

$$\begin{aligned} C_L (|\rho|, t) &= \sum_{m=1}^{36} \underline{c} (m, t) |\rho|^{-2m} \\ &= T_L \left(\frac{t}{|\rho|} \right) - \frac{T_L \left(\frac{t}{|\rho|} \right)^2}{2} + \frac{T_L \left(\frac{t}{|\rho|} \right)^3}{3} - \frac{T_L \left(\frac{t}{|\rho|} \right)^4}{4} + \frac{T_L \left(\frac{t}{|\rho|} \right)^5}{5} \\ &\quad - \frac{1.0043233 T_L \left(\frac{t}{|\rho|} \right)^6}{6} \end{aligned}$$

and

$$C_H (|\rho|, t) = \sum_{m=1}^{36} \bar{c} (m, t) |\rho|^{-2m}$$

$$= T_H\left(\frac{t}{|\rho|}\right) - \frac{T_H\left(\frac{t}{|\rho|}\right)^2}{2} + \frac{T_H\left(\frac{t}{|\rho|}\right)^3}{3} - \frac{T_H\left(\frac{t}{|\rho|}\right)^4}{4} + \frac{T_H\left(\frac{t}{|\rho|}\right)^5}{5} - \frac{T_H\left(\frac{t}{|\rho|}\right)^6}{6}.$$

Put $a = 1005$. Put $b = B$. Let

$$\begin{aligned} K_L(n) &= \frac{1}{2\pi(2n-1)a^{2n-1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n-1} \right) - \frac{1}{2\pi(2n-1)b^{2n-1}} \left(\log \frac{b}{2\pi} + \frac{1}{2n-1} \right) \\ &+ \frac{S(b)+G(b)}{b^{2n}} - \frac{0.34741}{a^{2n}} + \frac{2n}{a^{2n+1}} \left(\frac{-25}{48\pi(2n+1)} + (A_L - A_H) \log a \right. \\ &+ (B_L - B_H) \log \log a + C_L \log \log \left(a + \sqrt{\frac{7}{4}} \right) + (D_L - D_H) + \frac{A_L}{2n+1} + \frac{B_L}{(2n+1)\log a} \\ &+ \left. \frac{C_L}{(2n+1) \log \left(a + \sqrt{\frac{7}{4}} \right)} \right) - \frac{2n}{(2n+1)b^{2n+1}} \left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log \left(a + \sqrt{\frac{7}{4}} \right)} \right) \\ &- \frac{n}{4} \left(\frac{1}{2\pi(2n+1)a^{2n+1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n+1} \right) - \frac{1}{2\pi(2n+1)b^{2n+1}} \left(\log \frac{b}{2\pi} \right. \right. \\ &+ \left. \left. \frac{1}{2n+1} \right) + \frac{S(b)+G(b)}{b^{2n+2}} - \frac{0.34741}{a^{2n+2}} + \frac{2n+2}{a^{2n+3}} \left(\frac{-25}{48\pi(2n+3)} + (A_L - A_H) \log a \right. \right. \\ &+ (B_L - B_H) \log \log a + C_L \log \log \left(a + \sqrt{\frac{7}{4}} \right) + (D_L - D_H) + \frac{A_L}{2n+3} + \frac{B_L}{(2n+3)\log a} \\ &+ \left. \left. \frac{C_L}{(2n+3) \log a + \sqrt{\frac{7}{4}}} \right) - \frac{2n+2}{(2n+3)b^{2n+3}} \left(\frac{-25}{48\pi} + A_L + \frac{B_L}{\log a} + \frac{C_L}{\log a + \sqrt{\frac{7}{4}}} \right) \right) \end{aligned}$$

and

$$\begin{aligned}
K_H(n) &= \frac{1}{2\pi(2n-1)a^{2n-1}} \left(\log \frac{a}{2\pi} + \frac{1}{2n-1} \right) - \frac{1}{2\pi(2n-1)b^{2n-1}} \left(\log \frac{b}{2\pi} \right. \\
&+ \left. \frac{1}{2n-1} \right) + \frac{S(b) + G(b)}{b^{2n}} - \frac{0.34741}{a^{2n}} + \frac{2n}{a^{2n+1}} \left(\frac{25}{48\pi(2n+1)} + (A_H - A_L) \log a \right. \\
&+ (B_H - B_L) \log \log a - C_L \log \log \left(a + \sqrt{\frac{7}{4}} \right) + (D_H - D_L) + \frac{A_H}{2n+1} \\
&\left. + \frac{B_H}{(2n+1) \log a} \right) - \frac{2n}{(2n+1)b^{2n+1}} \left(\frac{25}{48\pi} + A_H + \frac{B_H}{\log a} \right).
\end{aligned}$$

Let

$$L_L(n, t) = \begin{cases} K_L(n) & \text{if } \underline{c}(n, t) \geq 0, \\ K_H(n) & \text{if } \underline{c}(n, t) < 0, \end{cases}$$

$$L_H(n, t) = \begin{cases} K_H(n) & \text{if } \bar{c}(n, t) \geq 0, \\ K_L(n) & \text{if } \bar{c}(n, t) < 0, \end{cases}$$

$$M_L(t) = \sum_{n=1}^{36} \underline{c}(n, t) L_L(n, t)$$

and

$$M_H(t) = \sum_{n=1}^{36} \bar{c}(n, t) L_H(n, t).$$

Let

$$N_L(t) = \begin{cases} \exp(M_L(t)) & \text{if } \frac{\sin xt/2}{t} \underset{0 < \gamma < 1005}{\overset{\Pi}{J_0}} \left(\frac{t}{|\rho|} \right) \geq 0, \\ \exp(M_H(t)) & \text{otherwise} \end{cases}$$

and

$$N_H(t) = \begin{cases} \exp(M_H(t)) & \text{if } \frac{\sin xt/2}{t} \underset{0 < \gamma < 1005}{\overset{\Pi}{J_0}} \left(\frac{t}{|\rho|} \right) \geq 0, \\ \exp(M_L(t)) & \text{otherwise.} \end{cases}$$

If $0 \leq x \leq 1$, then

$$\frac{1}{2} + \frac{x}{512} + \frac{1}{256\pi} \sum_{n=1}^{5376} \frac{\sin \pi nx/256}{n/256} \underset{0 < \gamma < 1005}{\overset{\Pi}{J_0}} \left(\frac{\pi n/128}{|\rho|} \right) N_L \left(\frac{\pi n}{128} \right)$$

$$- \epsilon \leq G(x) \leq \frac{1}{2} + \frac{x}{512}$$

$$+ \frac{1}{256\pi} \sum_{n=1}^{5376} \frac{\sin \pi nx/256}{n/256} \underset{0 < \gamma < 1005}{\overset{\Pi}{J_0}} \left(\frac{\pi n/128}{|\rho|} \right) N_H \left(\frac{\pi n}{128} \right) + \epsilon,$$

where $|\epsilon| \leq 10^{-12}$.

To prove Theorem 2.12, we need the following auxilliary results.

Lemma 2.13. Let $J_0(z) = \sum_{n=0}^N \frac{(-1)^n (z/2)^{2n}}{(n!)^2} + R_N$. If N is odd and

$z \leq 2(N+3)$, then

$$\frac{\left(\frac{z^2}{4}\right)^{N+1}}{((N+1)!)^2} \left(1 - \frac{\left(\frac{z^2}{4}\right)}{(N+2)^2} + \frac{\left(\frac{z^2}{4}\right)^2}{(N+2)^2(N+3)^2} - \frac{\left(\frac{z^2}{4}\right)^3}{(N+2)^2(N+3)^2(N+4)^2} \right)$$

$$\leq R_N \leq \frac{\left(\frac{z^2}{4}\right)^{N+1}}{((N+1)!)^2} .$$

Proof. We have

$$R_N = \frac{\left(\frac{z^2}{4}\right)^{N+1}}{((N+1)!)^2} - \frac{\left(\frac{z^2}{4}\right)^{N+2}}{((N+2)!)^2} + \dots .$$

If $M \geq N$ and M is odd, then

$$- \frac{\left(\frac{z^2}{4}\right)^{M+2}}{((M+2)!)^2} + \frac{\left(\frac{z^2}{4}\right)^{M+3}}{((M+3)!)^2}$$

$$= \frac{\left(\frac{z^2}{4}\right)^{M+2}}{((M+2)!)^2} \left(\frac{\frac{z^2}{4}}{(M+3)^2} - 1 \right) \leq 0.$$

The lemma follows directly.

Lemma 2.14. If $0 \leq t \leq 42\pi$, then

$$\exp\left(\sum_{n=1}^{36} \underline{c}(n, t) \int_{1005}^{B^+} \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}}\right) \leq_{1005 < \gamma \leq B} \Pi J_0\left(\frac{t}{|\rho|}\right) \leq$$

$$\exp\left(\sum_{n=1}^{36} \bar{c}(n, t) \int_{1005}^{B^+} \frac{dN(y)}{|\frac{1}{2}+iy|^{2n}}\right).$$

Proof. From the previous lemma we have

$$1 + \sum_{k=1}^5 u_k + u_6 \left(1 - \frac{\left(\frac{t}{2|\rho|}\right)^2}{7^2 8^2} + \frac{\left(\frac{t}{2|\rho|}\right)^4}{7^2 8^2} - \frac{\left(\frac{t}{2|\rho|}\right)^6}{7^2 8^2 9^2}\right) \leq J_0\left(\frac{t}{|\rho|}\right)$$

$$\leq 1 + \sum_{k=1}^6 u_k.$$

Since $\frac{t}{|\rho|} \leq 0.1312904$, we have

$$1 + T_L\left(\frac{t}{|\rho|}\right) \leq J_0\left(\frac{t}{|\rho|}\right) \leq 1 + T_H\left(\frac{t}{|\rho|}\right).$$

Since $0 \leq -T_H\left(\frac{t}{|\rho|}\right) \leq -T_L\left(\frac{t}{|\rho|}\right) \leq 0.0043047$, we can use Lemma 1.17

to obtain

$$C_L(|\rho|, t) \leq \log J_0\left(\frac{t}{|\rho|}\right) \leq C_H(|\rho|, t).$$

To complete the proof of Lemma 2.14, we note that

$$\begin{aligned} \sum_{1005 < \gamma \leq B} c_L(|\rho|, t) &= \sum_{1005 < \gamma \leq B} \sum_{n=1}^{36} \underline{c}(n, t) |\rho|^{-2n} \\ &= \sum_{n=1}^{36} \underline{c}(n, t) \int_{1005}^{B^+} \frac{dN(y)}{|\frac{1}{2} + iy|^{2n}} \end{aligned}$$

and

$$\sum_{1005 < \gamma \leq B} c_H(|\rho|, t) = \sum_{n=1}^{36} \bar{c}(n, t) \int_{1005}^{B^+} \frac{dN(y)}{|\frac{1}{2} + iy|^{2n}} .$$

Using Lemma 28, we obtain

$$K_L(n) \leq \int_{1005}^{B^+} \frac{dN(y)}{|\frac{1}{2} + iy|^{2n}} \leq K_H(n).$$

Thus

$$\begin{aligned} \exp(M_L(t)) &= \exp\left(\sum_{n=1}^{36} \underline{c}(|\rho|, t) L_N(n, t)\right) \leq \prod_{1005 < \gamma \leq B} J_0\left(\frac{t}{|\rho|}\right) \\ &\leq \exp\left(\sum_{n=1}^{36} \bar{c}(|\rho|, t) L_H(n, t)\right) = \exp(M_H(t)), \end{aligned}$$

which completes the proof of the theorem.

In computing $G(x)$, we use the tables in Haselgrove [12, pp. 58-64] to approximate the value of γ with an error of $\leq 2 \cdot 10^{-6}$. Denoting the values from the table by $\underline{\gamma}$, we have

$$2.032554186 \leq \sum_{0 < \underline{\gamma} < 1005} \frac{1}{|\frac{1}{2} + i\underline{\gamma}|} \leq 2.032554275.$$

Thus, if $H(x)$ is the number we calculate

$$H(x - 10^{-7}) \leq G(x) \leq H(x + 10^{-7}).$$

Our calculations show that this causes a relative error in $G(x)$ of at most 10^{-7} .

The relative error in calculating $J_0\left(\frac{t}{|\underline{\rho}|}\right)$, where $\underline{\rho} = \frac{1}{2} + i\underline{\gamma}$, is

$$\leq \begin{cases} 0.30 \cdot 10^{-15} & \text{if } 0 \leq \frac{t}{|\underline{\rho}|} \leq 4, \\ 0.60 \cdot 10^{-15} & \text{if } 4 \leq \frac{t}{|\underline{\rho}|} \leq 8, \\ 0.20 \cdot 10^{-11} & \text{if } 8 \leq \frac{t}{|\underline{\rho}|} \leq 10. \end{cases}$$

Thus for $0 \leq t \leq 21\pi$ we have a relative error in calculating $\sum_{0 < \underline{\gamma} < 1005}^{\Pi} J_0\left(\frac{t}{|\underline{\rho}|}\right)$ of $\leq 2.0 \cdot 10^{-13}$. For $21\pi \leq t \leq 42\pi$ we have a relative error in calculating $\sum_{0 < \underline{\gamma} < 1005}^{\Pi} J_0\left(\frac{t}{|\underline{\rho}|}\right)$ of $\leq 0.30 \cdot 10^{-11}$. Therefore, our absolute error in calculating $H(x)$ is $\leq 5 \cdot 10^{-10}$.

4. Asymptotic results for $G'(u)$. In this section we prove

Theorem 2.15. Let $f(z) = \sum_{\gamma > 0} \log I_0 \left(\frac{z}{|\rho|} \right) - zu/2$. If $T > 50$ is chosen so that $f'(T) = 0$, then

$$G'(u) = \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}} \left(1 + O\left(\frac{1}{\sqrt{T \log T}} \right) \right).$$

We need several preliminary results.

Lemma 2.16. Let $G'(u)$ be the density function of the random variable

$$Y = \sum_{\gamma > 0} \frac{2 \sin 2\pi \theta \gamma}{|\rho|}. \text{ Then}$$

$$G'(u) = \frac{1}{4\pi i} \int_{T-i\infty}^{T+i\infty} \exp(f(z)) dz.$$

Proof. As in Theorem 2.4 we have

$$G'(u) = \int_{-\infty}^{\infty} e(\lambda u) \hat{G}(\lambda) d\lambda$$

where

$$G(\lambda) = \int_{-\infty}^{\infty} e(-\lambda x) dG(x) = \sum_{\gamma > 0} J_0 \left(\frac{4\pi\lambda}{|\rho|} \right).$$

Since $I_0(z) = J_0(-iz)$ we have

$$\begin{aligned} G'(u) &= -i \int_{-i\infty}^{i\infty} \exp(-2\pi zu) \prod_{\gamma > 0} I_0 \left(\frac{4\pi z}{|\rho|} \right) dz \\ &= \frac{1}{4\pi i} \int_{T-i\infty}^{T+i\infty} \exp(-zu/2) \prod_{\gamma > 0} I_0 \left(\frac{z}{|\rho|} \right) dz, \end{aligned}$$

which completes the proof of the lemma.

Lemma 2.17. Let

$$U_1 = \frac{1}{4\pi i} \int_{T-iT}^{T-iT^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(f(z)) dz + \frac{1}{4\pi i} \int_{T+iT^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T+iT} \exp(f(z)) dz$$

and

$$U_2 = \frac{1}{4\pi i} \int_{T-i\infty}^{T-iT} \exp(f(z)) dz + \frac{1}{4\pi i} \int_{T+iT}^{T+i\infty} \exp(f(z)) dz.$$

If $T > 50$, then

$$G'(u) = \frac{1}{4\pi i} \int_{T-iT^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T+iT^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(f(z)) dz + U_1 + U_2,$$

where

$$|U_1| \leq \exp(f(T)) \left(\frac{\exp(-0.0207121 T^{1/3}) T^{1/3}}{0.0414242 \pi \sqrt{\log \frac{T}{2\pi}} \left(1 + \sqrt{1 + \frac{4}{0.0207121 \pi T^{1/3}}}\right)} \right)$$

and

$$|U_2| < \exp(f(T)) \frac{\exp(-0.0429599T)}{0.0859198\pi}.$$

Proof. From the previous lemma we have

$$G'(u) = \frac{1}{4\pi i} \int_{T-iT^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}}^{T+iT^{2/3}/\left(\log \frac{T}{2\pi}\right)^{\frac{1}{2}}} \exp(f(z)) dz + U_1 + U_2.$$

In order to estimate U_1 and U_2 we first examine, for $z = T + it$,

$$\operatorname{Re} f(z) - f(T) = \operatorname{Re} f(T+it) - f(T)$$

$$= \log_{\gamma > 0} \prod_0 \left| I_0 \left(\frac{z}{|\rho|} \right) \right| - \log_{0 < \gamma} \prod_0 \left| I_0 \left(\frac{T}{|\rho|} \right) \right|$$

$$= \log \prod_{\gamma > 0} \left| \frac{I_0\left(\frac{T+it}{|\rho|}\right)}{I_0\left(\frac{T}{|\rho|}\right)} \right|.$$

We have

$$\begin{aligned} \left| I_0\left(\frac{T+it}{|\rho|}\right) \right| &\leq \frac{1}{\pi} \int_0^\pi \left| e^{\frac{T}{|\rho|} \cos \theta} \right| \left| e^{\frac{it}{|\rho|} \cos \theta} \right| d\theta \\ &\leq \frac{1}{\pi} \int_0^\pi e^{\frac{T}{|\rho|} \cos \theta} d\theta = I_0\left(\frac{T}{|\rho|}\right). \end{aligned}$$

Combining the two above equations we obtain

$$(33) \quad \operatorname{Re} f(z) - f(t) \leq \sum_{\gamma > 0} \log \left| \frac{I_0\left(\frac{T+it}{|\rho|}\right)}{I_0\left(\frac{T}{|\rho|}\right)} \right| = \frac{1}{2} \sum_{\gamma > 0} \log \left| \frac{I_0\left(\frac{T+it}{|\rho|}\right)}{I_0\left(\frac{T}{|\rho|}\right)} \right|^2$$

$$\frac{T}{|\rho|} \leq 1, \quad \frac{|t|}{|\rho|} \leq 1, \quad T, |t| \leq |\rho|$$

$$= \frac{1}{2} \sum_{\gamma > 0} \log \left(1 + I_0\left(\frac{T}{|\rho|}\right)^{-2} g(z, T, |\rho|) \right)$$

$$T, |t| < |\rho|$$

where

$$g(z, T, |\rho|) = 2 \operatorname{Re} \left(I_0\left(\frac{z}{|\rho|}\right) - I_0\left(\frac{T}{|\rho|}\right) \right) I_0\left(\frac{T}{|\rho|}\right) + \operatorname{Re}^2 \left(I_0\left(\frac{z}{|\rho|}\right) - I_0\left(\frac{T}{|\rho|}\right) \right) + \operatorname{Im}^2 \left(I_0\left(\frac{z}{|\rho|}\right) - I_0\left(\frac{T}{|\rho|}\right) \right)$$

We have, for $|x|, |y| \leq 1$, $\operatorname{Re}(I_0(x+iy)) - I_0(x) \leq 0$, and hence

(34)

$$\begin{aligned} &2 \operatorname{Re}(I_0(x+iy) - I_0(x)) I_0(x) + \operatorname{Re}^2(I_0(x+iy) - I_0(x)) + \operatorname{Im}^2(I_0(x+iy) - I_0(x)) \\ &\leq 2 \operatorname{Re}(I_0(x+iy) - I_0(x)) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^6} + \frac{x^6}{2^8 3^2} + \cdots + \frac{x^{18}}{2^{32} 3^8 5^2 7^2} \right) \end{aligned}$$

$$+ \operatorname{Re}^2(I_0(x+iy) - I_0(x)) + \operatorname{Im}^2 I_0(x+iy).$$

For $s = x+iy$ we write

$$I_0(s) = 1 + \frac{s^2}{2^2} + \frac{s^4}{2^6} + \frac{s^6}{2^8 3^2} + \frac{s^8}{2^{14} 3^2} + \frac{s^{10}}{2^{16} 3^2 5^2} + \frac{s^{12}}{2^{20} 3^4 5^2} E(s),$$

where $E(s) = 1 + \frac{s^2}{4 \cdot 7^2} + \frac{s^4}{4^2 7^2 8^2} + \dots$. Using this in (34), we obtain

(35)

$$\begin{aligned} & 2 \operatorname{Re}(I_0(x+iy) - I_0(x)) I_0(x) + \operatorname{Re}^2(I_0(x+iy) - I_0(x)) + \operatorname{Im}^2(I_0(x+iy) - I_0(x)) \\ & \leq f_0 y^2 + f_2 y^2 x^2 + f_4 y^2 x^4 + f_6 y^2 x^6 + f_8 y^2 x^8 + f_{10} x^{10} + g(x, y), \end{aligned}$$

where

$$\begin{aligned} (36) \quad g(x, y) = & f_{12} x^{12} + f_{14} x^{14} + f_{16} x^{16} + f_{18} x^{18} + f_{20} x^{20} + f_{22} x^{22} \\ & + f_{24} x^{24} + f_{26} x^{26} + f_{28} x^{28} + f_{30} x^{30} \end{aligned}$$

and

(38)

$$\begin{aligned} f_0 = & \frac{-1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13} 3^2} - \frac{7y^8}{2^{14} 5^2} + \frac{461y^{10}}{2^{19} 3^4 5^2} - \frac{17y^{12}}{2^{20} 3^4 5^2} + \frac{19y^{14}}{2^{28} 3^3 5^2} \\ & - \frac{y^{16}}{2^{29} 3^4 5^2} + \frac{y^{18}}{2^{32} 3^4 5^4} + \left(\frac{y^{10}}{2^{14} 3^4 5^2} - \frac{y^{12}}{2^{21} 3^4 5^2} + \frac{y^{14}}{2^{25} 3^4 5^2} - \frac{y^{16}}{2^{27} 3^6 5^2} \right. \\ & \left. + \frac{y^{18}}{2^{33} 3^6 5^2} - \frac{y^{20}}{2^{35} 3^6 5^4} \right) \operatorname{Re} E(s) + \frac{|E(s)|^2 y^{22}}{2^{40} 3^8 5^4} + x \left(\frac{y^9}{2^{17} 3^3 5^2} - \frac{y^{11}}{2^{20} 3^4 5} + \frac{y^{13}}{2^{22} 3^4 5^2} \right) \end{aligned}$$

$$-\frac{y^{15}}{2^{26}3^55^2} + \frac{y^{17}}{2^{31}3^65^2} - \frac{y^{19}}{2^{34}3^65^4} \Big) \text{Im } E(s),$$

$$(38) \quad f_2 = \frac{-1}{2^4} + \frac{y^2}{2^63} - \frac{7y^4}{2^{12}3^2} + \frac{7y^6}{2^{14}3^25} + \frac{19y^8}{2^{18}3^35^2} - \frac{13y^{10}}{2^{20}3^45^2} + \frac{y^{12}}{2^{25}3^4} + \frac{53y^{14}}{2^{29}3^45^2}$$

$$+ \frac{y^{16}}{2^{31}3^45^3} + \left(\frac{-11y^8}{2^{18}3^35^2} + \frac{43y^{10}}{2^{21}3^45^2} - \frac{y^{12}}{2^{22}3^35^2} + \frac{y^{14}}{2^{27}3^45^2} + \frac{31y^{16}}{2^{32}3^75^2} - \frac{y^{18}}{2^{32}3^55^3} \right)$$

$$\cdot \text{Re } E(s) - \frac{41 |E(s)|^2 y^{20}}{2^{40}3^75^4} + x \left(\frac{-11y^7}{2^{17}3^45} + \frac{y^9}{2^{20}3^45} - \frac{y^{11}}{2^{22}3^35^2} - \frac{y^{13}}{2^{23}3^65^2} + \frac{7y^{15}}{2^{31}3^45^2} \right.$$

$$\left. + \frac{53y^{17}}{2^{33}3^65^4} \right) \text{Im } E(s),$$

$$(39) \quad f_4 = \frac{-5}{2^73} + \frac{y^2}{2^{12}3} - \frac{11y^4}{2^{13}3^5} - \frac{19y^6}{2^{19}3^5} + \frac{7y^8}{2^{20}3^25^2} + \frac{y^{10}}{2^{26}3^3} - \frac{29y^{12}}{2^{26}3^45} + \frac{2569y^{14}}{2^{32}3^45^3}$$

$$+ \left(\frac{11y^6}{2^{19}3^25} - \frac{121y^8}{2^{21}3^45^2} - \frac{y^{10}}{2^{23}3^25^2} + \frac{y^{12}}{2^{25}3^55} - \frac{139y^{14}}{2^{33}3^45^2} - \frac{893y^{16}}{2^{35}3^55^4} \right) \text{Re } E(s)$$

$$+ \frac{209 |E(s)|^2 y^{18}}{2^{37}3^75^4} + x \left(\frac{11y^5}{2^{16}3^25^2} - \frac{11y^7}{2^{20}3^45^2} - \frac{y^9}{2^{22}3^35} + \frac{y^{11}}{2^{26}3^55^2} - \frac{151y^{13}}{2^{29}3^75} \right.$$

$$\left. - \frac{15719y^{15}}{2^{34}3^65^4} \right) \text{Im } E(s),$$

$$(40) \quad f_6 = \frac{-7}{2^{12}3^2} + \frac{29y^2}{2^{12}3^25} - \frac{3599y^4}{2^{17}3^45^2} - \frac{41y^6}{2^{19}3^35^2} + \frac{41y^8}{2^{25}3^25^2} + \frac{217y^{10}}{2^{24}3^45^2} - \frac{71y^{12}}{2^{29}3^55^3}$$

$$+ \left(\frac{-77y^4}{2^{17}3^35^2} - \frac{11y^6}{2^{21}3^35} + \frac{11y^8}{2^{22}3^45^2} + \frac{29y^{10}}{2^{25}3^65^2} + \frac{571y^{12}}{2^{29}3^65^2} + \frac{8729y^{14}}{2^{35}3^65^3} \right) \text{Re } E(s)$$

$$- \frac{12595 |E(s)|^2 y^{16}}{2^{40} 3^8 5^3} + x \left(\frac{-11y^3}{2^{16} 3^2 5^2} - \frac{11y^5}{2^{18} 3^3 5^2} - \frac{11y^7}{2^{22} 3^4 5^2} + \frac{y^9}{2^{23} 3^4 5^2} + \frac{931y^{11}}{2^{29} 3^6 5^2} \right. \\ \left. + \frac{11803y^{13}}{2^{32} 3^6 5^4} \right) \text{Im } E(s),$$

$$(41) \quad f_8 = \frac{-67}{2^{14} 3^2 5} + \frac{701y^2}{2^{19} 3^3 5} + \frac{41y^4}{2^{19} 3^3 5} + \frac{1099y^6}{2^{27} 3^4 5} - \frac{8209y^8}{2^{28} 3^4 5} + \frac{6407y^{10}}{2^{31} 3^3 5} \\ + \left(\frac{11y^2}{2^{14} 3^2 5} + \frac{11y^4}{2^{21} 3^3 5} + \frac{11y^6}{2^{24} 3^2 5^2} + \frac{11y^8}{2^{26} 3^5 5^2} - \frac{34811y^{10}}{2^{33} 3^6 5^2} - \frac{36086y^{12}}{2^{34} 3^6 5^4} \right) \text{Re } E(s) \\ + \frac{7667 |E(s)|^2 y^{14}}{2^{40} 3^6 5^3} + x \left(\frac{11y}{2^{17} 3^4 5} - \frac{11y^3}{2^{20} 3^4 5^2} + \frac{11y^5}{2^{22} 3^4 5^2} + \frac{11y^7}{2^{25} 3^6 5} - \frac{1841y^9}{2^{30} 3^6 5^2} \right. \\ \left. - \frac{47757y^{11}}{2^{33} 3^6 5^4} \right) \text{Im } E(s),$$

$$(42) \quad f_{10} = \frac{-31y^2}{2^{18} 3^3 5^2} - \frac{7y^4}{2^{20} 3^2 5^2} - \frac{41y^6}{2^{25} 3^2 5^2} + \frac{6169y^8}{2^{28} 3^4 5^2} - \frac{63937y^{10}}{2^{30} 3^4 5^4} + \left(\frac{-11y^2}{2^{18} 3^3 5^2} \right. \\ \left. - \frac{121y^4}{2^{21} 3^4 5^2} + \frac{11y^6}{2^{22} 3^4 5^2} - \frac{11y^8}{2^{26} 3^5 5^2} + \frac{2233y^{10}}{2^{31} 3^6 5^2} + \frac{97571y^{12}}{2^{35} 3^6 5^4} \right) \text{Re } E(s) \\ - \frac{17017 |E(s)|^2 y^{14}}{2^{39} 3^5 5^4} + x \left(\frac{-y}{2^{17} 3^3 5^2} + \frac{y^3}{2^{18} 3^4} + \frac{y^5}{2^{22} 3^5} + \frac{y^7}{2^{23} 3^4 5^2} + \frac{1519y^9}{2^{30} 3^7 5^2} \right. \\ \left. + \frac{3812y^{11}}{2^{31} 3^5 5^4} \right) \text{Im } E(s),$$

$$\begin{aligned}
(43) \quad f_{12} &= \frac{y^2}{2^{21} 3^4} + \frac{17y^4}{2^{26} 3^4 5^2} - \frac{3143y^6}{2^{27} 3^6 5} + \frac{691961y^8}{2^{33} 3^6 5^3} \\
&+ \frac{(\operatorname{Re} E(s) - \operatorname{Re} E(s) E(x) + 634032 |E(s)|^2)y^{12}}{2^{34} 3^8 5^4} + \left(\frac{1}{2^{19} 3^4 5^2} - \frac{43y^2}{2^{21} 3^4 5^2} - \frac{y^4}{2^{23} 3^2 5^2} \right. \\
&- \left. \frac{29y^6}{2^{25} 3^6 5^2} - \frac{49y^8}{2^{32} 3^5 5} - \frac{24011y^{10}}{2^{34} 3^5 5^4} \right) \operatorname{Re} E(s) + \left(\frac{-1}{2^{19} 3^4 5^2} + \frac{y^2}{2^{24} 3^3 5^2} - \frac{y^4}{2^{25} 3^4 5^2} \right. \\
&+ \left. \frac{y^6}{2^{27} 3^6 5^2} + \frac{y^8}{2^{33} 3^6 5^2} + \frac{y^{10}}{2^{35} 3^6 5^4} \right) E(x) + x \left(\frac{-y}{2^{19} 3^3 5^2} + \frac{y^3}{2^{22} 3^3 5^2} + \frac{y^5}{2^{24} 3^5 5^2} \right. \\
&- \left. \frac{7y^7}{2^{24} 3^5 5^2} - \frac{30817y^9}{2^{33} 3^6 5^4} + \frac{(1 - E(x))y^{11}}{2^{37} 3^7 5^4} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$\begin{aligned}
(44) \quad f_{14} &= \frac{103y^2}{2^{25} 3^2 5^2 7^2} + \frac{21439y^4}{2^{26} 3^5 5^2 7^2} - \frac{101293y^6}{2^{29} 3^6 5 \cdot 7^2} + \frac{7y^8}{2^{34} 3^6 5^3} \\
&+ \frac{(-22 \operatorname{Re} E(s) + 22 \operatorname{Re} E(s) E(x) - 153153 |E(s)|^2)y^{10}}{2^{39} 3^7 5^4} + \left(\frac{1}{2^{21} 3^4 5^2} - \frac{y^2}{2^{22} 3^3 5^2} \right. \\
&- \left. \frac{y^4}{2^{25} 3^5 5} - \frac{y^6}{2^{25} 3^7 5^2} + \frac{1577y^8}{2^{34} 3^6 5^2} + \frac{y^{12}}{2^{41} 3^8 5^4 7^2} \right) \operatorname{Re} E(s) + \left(\frac{-1}{2^{21} 3^4 5^2} + \frac{y^2}{2^{24} 3^3 5^2} \right. \\
&- \left. \frac{y^4}{2^{27} 3^5 5} + \frac{7y^6}{2^{31} 3^7 5^2} + \frac{y^8}{2^{35} 3^4 5^3} \right) E(x) + x \left(\frac{-y}{2^{22} 3^4 5^2} - \frac{y^3}{2^{33} 3^6 5^2} - \frac{y^5}{2^{29} 3^5 5^2} \right. \\
&+ \left. \frac{5069y^7}{2^{32} 3^6 5^4} + \frac{(-11 + 11 E(x))y^9}{2^{37} 3^8 5^3} + \frac{y^{11}}{2^{39} 3^7 5^4 7^2} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$\begin{aligned}
(45) \quad f_{16} &= \frac{-7747y^2}{2^{31} 3^5 5 \cdot 7^2} + \frac{9238237y^4}{2^{35} 3^6 5^3 7^2} - \frac{8797y^6}{2^{37} 3^6 5^3 7^2} + \frac{83y^8}{2^{43} 3^6 5^3 7} \\
&+ \frac{(22 \operatorname{Re} E(s) - 22 \operatorname{Re} E(s) E(x) + 7667 |E(s)|^2)y^8}{2^{40} 3^6 5^3} + \left(\frac{1}{2^{25} 3^4 5^2} - \frac{y^2}{2^{27} 3^4 5^2} \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{19y^4}{2^{33}3^65^2} - \frac{3157y^6}{2^{35}3^65^3} - \frac{11y^{10}}{2^{40}3^75^47^2} + \frac{y^{12}}{2^{41}3^85^47^2} \Big) \operatorname{Re} E(s) + \left(\frac{-1}{2^{25}3^45^2} \right. \\
& + \frac{y^2}{2^{27}3^55} + \frac{7y^4}{2^{32}3^65} + \frac{7y^6}{2^{34}3^65^3} \Big) \operatorname{Re} E(x) + x \left(\frac{-y}{2^{26}3^55^2} - \frac{7y^3}{2^{31}3^65^2} - \frac{2237y^5}{2^{34}3^65^4} \right. \\
& + \left. \frac{11(1-E(x))y^7}{2^{36}3^65^4} - \frac{11y^9}{2^{39}3^85^37^2} + \frac{y^{11}}{2^{47}3^75^47^2} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$\begin{aligned}
(46) \quad f_{18} &= \frac{181799y^2}{2^{34}3^85^37^2} + \frac{87368y^4}{2^{37}3^85^37^2} - \frac{2729y^6}{2^{41}3^{10}5^37^2} + \frac{367y^8}{2^{44}3^{10}5^37^2} \\
& + \frac{(1848(1-E(x)) \operatorname{Re} E(s) - 62975 |E(s)|^2)y^6}{2^{40}3^85^4} + \left(\frac{1}{2^{27}3^65^2} + \frac{y^2}{2^{32}3^65^2} + \frac{803y^4}{2^{35}3^65^4} \right. \\
& + \frac{11y^8}{2^{41}3^65^37^2} - \frac{11y^{10}}{2^{48}3^75^47^2} + \frac{y^{12}}{2^{51}3^{12}5^47^4} \Big) \operatorname{Re} E(s) + \left(\frac{-1}{2^{27}3^65^2} + \frac{7y^2}{2^{31}3^65^2} \right. \\
& - \left. \frac{7y^4}{2^{34}3^55^3} \right) E(x) + x \left(\frac{-y}{2^{31}3^65^2} - \frac{y^3}{2^{33}3^45^3} + \frac{11(E(x)-1)y^5}{2^{36}3^65^4} + \frac{11y^7}{2^{38}3^65^47^2} \right. \\
& - \left. \frac{11y^9}{2^{47}3^85^37^2} + \frac{y^{11}}{2^{49}3^{11}5^47^4} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$\begin{aligned}
(47) \quad f_{20} &= \frac{-7207y^2}{2^{37}3^75^37^2} + \frac{3919y^4}{2^{42}3^65^37^2} - \frac{1771y^6}{2^{44}3^{10}5^37^2} + \frac{y^8}{2^{47}3^85^37^2} \\
& + \frac{(165(1-E(x)) \operatorname{Re} E(s) + 836 |E(s)|^2)y^4}{2^{39}3^75^4} + \left(\frac{1}{2^{33}3^65^2} + \frac{y^2}{2^{35}3^45^4} - \frac{11y^6}{2^{39}3^75^47} \right. \\
& + \frac{11y^8}{2^{49}3^65^37^2} - \frac{11y^{10}}{2^{50}3^{11}5^47^4} \Big) \operatorname{Re} E(s) + \left(\frac{-1}{2^{33}3^65^4} + \frac{y^2}{2^{35}3^45^3} \right) E(x) + x \left(\frac{-y}{2^{34}3^65^4} \right. \\
& + \left. \frac{11(1-E(x))y^3}{2^{37}3^85^4} - \frac{11y^5}{2^{38}3^65^47^2} + \frac{11y^7}{2^{46}3^65^47^2} - \frac{11y^9}{2^{49}3^{12}5^37^4} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$\begin{aligned}
(48) \quad f_{22} &= \frac{-12709y^2}{2^{41} 3^9 5^3 7^2} + \frac{377y^4}{2^{43} 3^{10} 5^3 7} - \frac{y^6}{2^{46} 3^9 5^3 7} \\
&+ \frac{(44(E(x)-1) \operatorname{Re} E(s) - 41 |E(s)|^2)y^2}{2^{40} 3^7 5^4} + \frac{\operatorname{Re} E(s) - E(x)}{2^{35} 3^6 5^4} + \left(\frac{11y^4}{2^{41} 3^6 5^3 7^2} \right. \\
&- \left. \frac{11y^6}{2^{47} 3^7 5^4 7} + \frac{11y^8}{2^{51} 3^{10} 5^3 7^4} \right) \operatorname{Re} E(s) + x \left(\frac{(E(x)-1)y}{2^{37} 3^7 5^4} + \frac{11y^3}{2^{39} 3^8 5^3 7^2} - \frac{11y^5}{2^{46} 3^6 5^4 7^2} \right. \\
&\left. + \frac{11y^7}{2^{48} 3^{10} 5^4 7^4} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$\begin{aligned}
(49) \quad f_{24} &= \frac{-869y^2}{2^{45} 3^{10} 5^3 7^2} + \frac{y^4}{2^{46} 3^9 5^3 7} + \frac{2(\operatorname{Re} E(s) - \operatorname{Re} E(s)E(x) - E(x)) + E(x)^2 + |E(s)|^2}{2^{40} 3^8 5^4} \\
&+ \left(\frac{-11y^2}{2^{40} 3^5 5^7 7^2} + \frac{11y^4}{2^{49} 3^6 5^3 7^2} - \frac{11y^6}{2^{44} 3^{11} 5^4 7^3} \right) \operatorname{Re} E(s) + x \left(\frac{-y}{2^{39} 3^7 5^4 7^2} + \frac{11y^3}{2^{47} 3^8 5^3 7^2} \right. \\
&\left. - \frac{11y^5}{2^{48} 3^{10} 5^4 7^4} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$\begin{aligned}
(50) \quad f_{26} &= \frac{-1}{2^{47} 3^8 5^3 7^2} + \frac{\operatorname{Re} E(s) - E(x)}{2^{41} 3^8 5^4 7^2} + \left(\frac{-11y^2}{2^{48} 3^7 5^4 7^2} + \frac{11y^4}{2^{51} 3^{10} 5^3 7^4} \right) \operatorname{Re} E(s) \\
&+ x \left(\frac{-y}{2^{47} 3^7 5^4 7^2} + \frac{11y^3}{2^{49} 3^{12} 5^3 7^4} \right) \operatorname{Im} E(s),
\end{aligned}$$

$$(51) \quad f_{28} = \frac{\operatorname{Re} E(s) - E(x)}{2^{49} 3^8 5^4 7^2} - \frac{11y^2 \operatorname{Re} E(s)}{2^{50} 3^{11} 5^4 7^4} + \frac{-xy \operatorname{Im} E(s)}{2^{49} 3^{11} 5^4 7^4}$$

and

$$(52) \quad f_{30} = \frac{\operatorname{Re} E(s) - E(x)}{2^{51} 3^{12} 5^4 7^4}.$$

We now assume that $0 \leq x \leq 1$ and $-1 \leq y \leq 0$. In order to bound the terms in (35), we need to examine $E(x+iy)$ more closely. We have, for $s=x+iy$

$$\begin{aligned} E(s) &= 1 + \frac{s^2}{4 \cdot 7^2} + \frac{s^4}{4^2 \cdot 7^2 \cdot 8^2} + \frac{s^6}{4^3 \cdot 7^2 \cdot 8^2 \cdot 9^2} + \dots \\ &= 1 + \frac{x^2 + \binom{2}{1}x(iy) + (iy)^2}{4 \cdot 7^2} + \frac{x^4 + \binom{4}{1}x^3(iy) + \binom{4}{2}x^2(iy)^2 + \binom{4}{3}x(iy)^3 + (iy)^4}{4^2 \cdot 7^2 \cdot 8^2} + \dots \end{aligned}$$

Thus

$$(53) \quad \operatorname{Re} E(s) = 1 + \frac{x^2 - y^2}{4 \cdot 7^2} + \frac{x^4 - \binom{4}{2}x^2y^2 + y^4}{4^2 \cdot 7^2 \cdot 8^2} + \dots$$

and

$$(54) \quad \operatorname{Im} E(s) = \frac{\binom{2}{1}xy}{4 \cdot 7^2} + \frac{\binom{4}{1}x^3y - \binom{4}{3}xy^3}{4^2 \cdot 7^2 \cdot 8^2} + \dots$$

In addition, we write

$$E(x) = 1 + \frac{x^2}{4 \cdot 7^2} + \frac{x^4}{4^2 \cdot 7^2 \cdot 8^2} + \dots$$

Since $\sum_{i=0}^n \binom{n}{i} = 2^n$, the sum of the coefficients of the degree n polynomials in (53) and (54) is $< 2^n$. Thus

$$(55) \quad \left| \operatorname{Re} E(s) - 1 \right| \leq \frac{1}{4 \cdot 7^2} + \frac{1}{4^2 \cdot 7^2 \cdot 8} + \frac{1}{7^2 \cdot 8^2 \cdot 9^2} + \frac{1}{7^2 \cdot 8^2 \cdot 9^2 \cdot 10^2} + \dots$$

$$\leq 0.0052653,$$

$$\frac{xy}{2 \cdot 7^2} + 0.0001633xy \leq \operatorname{Im} E(s) \leq \frac{xy}{2 \cdot 7^2} - 0.0001633xy$$

and

$$\frac{-y^2}{4 \cdot 7^2} - 0.0001633y^2 \leq \operatorname{Re} E(s) - E(x) \leq \frac{-y^2}{4 \cdot 7^2} + 0.0001633y^2.$$

From the above two equations we obtain

$$(56) \quad 0.0103673xy \leq \operatorname{Im} E(s) \leq 0.0100407xy$$

and

$$(57) \quad -0.0052653y^2 \leq \operatorname{Re} E(s) - E(x) \leq -0.0049387y^2.$$

Using (55), (56), and (57) in (36) we obtain

$$g(x, y) < \frac{y^2 x^{12}}{2^{20} 3^4} + \frac{2(\operatorname{Re} E(s) - \operatorname{Re} E(s)E(x) - E(x)) + E(x)^2 + |E(s)|^2}{2^{40} 3^8 5^4} - \frac{(0.0103673)^2 y^2}{2^{40} 3^8 5^4}.$$

Since

$$\begin{aligned} & 2(\operatorname{Re} E(s) - \operatorname{Re} E(s)E(x) - E(x)) + E(x)^2 + |E(s)|^2 \\ &= (\operatorname{Re} E(s) - E(x)) (\operatorname{Re} E(s) - E(x) + 2) + \operatorname{Im} E(s)^2 \\ &\leq (0.0103673)^2 y^2, \end{aligned}$$

we have

$$(58) \quad g(x, y) < \frac{x^{12} y^2}{2^{20} 3^4}.$$

We next examine the leading terms in (35). We have, from (37),

$$\begin{aligned}
(59) \quad f_0 &= \frac{-1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13} 3^2} - \frac{y^8}{2^{14} 5} + \frac{y^8}{2^{14} 5^2} \left(-2 + \frac{xy \operatorname{Im} E(s)}{2^3 3^3} \right) \\
&+ \frac{y^2 (461 + \operatorname{Re} E(s))}{2^5 3^4} - \frac{5xy^3 \operatorname{Im} E(s)}{2^6 3^4} \Big) + \frac{y^{12}}{2^{21} 3^4 5^2} \left(-34 - \operatorname{Re} E(s) + 2xy \operatorname{Im} E(s) \right) \\
&+ \frac{y^2 (57 + 128 \operatorname{Re} E(s))}{2^7} - \frac{xy^3 \operatorname{Im} E(s)}{2^3 3} \Big) + \frac{y^{16}}{2^{29} 3^6 5^2} \left(-9 - 4 \operatorname{Re} E(s) + \frac{xy \operatorname{Im} E(s)}{2^2} \right) \\
&+ \frac{y^2 (18 + 25 \operatorname{Re} E(s))}{2^5 5^2} - \frac{xy^3 \operatorname{Im} E(s)}{2^5 5^2} \Big) + \frac{y^{20}}{2^{35} 3^6 5^4} \left(-\operatorname{Re} E(s) + \frac{y^2 |E(s)|^2}{2^5 3^2} \right) \\
&\leq \frac{-1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13} 3^2} - \frac{y^8}{2^{14} 5} \leq -0.3994805y^2,
\end{aligned}$$

since each of the terms in parentheses is negative and $-\frac{1}{2} + \frac{3y^2}{2^5} - \frac{5y^4}{2^6 3^2} + \frac{35y^6}{2^{13} 3^2} - \frac{y^8}{2^{14} 5}$ is an increasing function of y^2 for $y^2 \leq 1$. From (38) we obtain

$$\begin{aligned}
(60) \quad f_2 &= \frac{-1}{2^4} + \frac{y^2}{2^6 3} - \frac{7y^4}{2^{13} 3^2} + \frac{y^4}{2^{13} 3^2} \left(-7 + \frac{7y^2}{2 \cdot 5} - \frac{11xy^3 \operatorname{Im} E(s)}{2^4 3^2 5} + \frac{19y^4}{2^5 \cdot 3 \cdot 5^2} \right) \\
&+ \frac{y^5 \operatorname{Im} E(s)}{2^7 3^2 5} \Big) + \frac{y^8}{2^{18} 3^3 5^2} \left(-11 \operatorname{Re} E(s) - \frac{13y^2}{2^2 \cdot 3} + \frac{43y^2 \operatorname{Re} E(s)}{2^3 \cdot 3} - \frac{y^4 \operatorname{Re} E(s)}{2^4} \right) \\
&+ \frac{y^4}{2^7 \cdot 3 \cdot 5^2} + \frac{y^6 (53 + 4 \operatorname{Re} E(s))}{2^{11} 3} + \frac{y^8 (54 + 155 \operatorname{Re} E(s))}{2^{14} 3^4 5} - \frac{y^2 \operatorname{Re} E(s)}{2^{14} 3^2 5} \\
&- \frac{41y^2 |E(s)|^2}{2^{22} 3^4 5^2} - \frac{xy^3 \operatorname{Im} E(s)}{2^4} - \frac{xy^5 \operatorname{Im} E(s)}{2^5 3} \Big) \leq \frac{-1}{2^4} + \frac{y^2}{2^6 3} - \frac{7y^4}{2^{13} 3^2} \leq -0.0573866,
\end{aligned}$$

since each of the terms in parentheses is negative and $\frac{-1}{2^4} + \frac{y^2}{2^6 3} - \frac{7y^4}{2^{13} 3^2}$ is an increasing function of y^2 for $y^2 \leq 1$. From (39) we have

$$\begin{aligned}
(61) \quad f_4 &= \frac{-5}{2^7 3} + \frac{y^2}{2^{12} 3} - \frac{10y^4}{2^{13} \cdot 3 \cdot 5} + \frac{y^4}{2^{13} 3 \cdot 5} \left(-1 + \frac{11xy \operatorname{Im} E(s)}{2^3 3 \cdot 5} \right. \\
&+ \frac{y^2(-19 + 33 \operatorname{Re} E(s))}{2^6 3^2} - \frac{11xy^3 \operatorname{Im} E(s)}{2^7 3^3 5} + \frac{y^4(126 - 121 \operatorname{Re} E(s))}{2^8 3^3 5} - \frac{xy^5 \operatorname{Im} E(s)}{2^9 3^2} \\
&+ \frac{y^6(5 - 24 \operatorname{Re} E(s))}{2^{13} 3^2 5} + \frac{xy^7 \operatorname{Im} E(s)}{2^{13} 3^4 5} \left. \right) + \frac{y^{12}}{2^{26} 3^4 5} \left(-29 + \frac{2 \operatorname{Re} E(s)}{3} - \frac{151xy \operatorname{Im} E(s)}{2^3 3^3} \right. \\
&+ \frac{2569y^2}{2^6 5^2} - \frac{139y^2 \operatorname{Re} E(s)}{2^7 5} - \frac{15719xy^3 \operatorname{Im} E(s)}{2^8 3^2 5^3} - \frac{893y^4 \operatorname{Re} E(s)}{2^9 3 \cdot 5^3} + \frac{209y^6 |E(s)|^2}{2^{11} 3^3 5^3} \left. \right) \\
&\leq \frac{-5}{2^7 3} + \frac{y^2}{2^{12} 3} - \frac{10y^4}{2^{13} 3 \cdot 5} \leq -.01298,
\end{aligned}$$

since each of the terms in parentheses is negative and $\frac{-5}{2^7 3} + \frac{y^2}{2^{12} 3} - \frac{10y^4}{2^{13} 3 \cdot 5}$ has its maximum at $y^2 = \frac{1}{2}$. From (40) we obtain

$$\begin{aligned}
(62) \quad f_6 &= \frac{-7}{2^{12} 3^2} + \frac{29y^2}{2^{12} 3^2 5} - \frac{11xy^3 \operatorname{Im} E(s)}{2^{16} 3^2 5^2} - \frac{3599y^4}{2^{17} 3^4 5^2} + \frac{y^4}{2^{17} 3^4 5^2} \left(-231 \operatorname{Re} E(s) \right. \\
&- \frac{33xy \operatorname{Im} E(s)}{2} - \frac{3y^2(164 + 55 \operatorname{Re} E(s))}{2^4} - \frac{11xy^3 \operatorname{Im} E(s)}{2^5} + \frac{y^4(369 + 88 \operatorname{Re} E(s))}{2^8} \\
&+ \frac{xy^5 \operatorname{Im} E(s)}{2^6} + \frac{y^6(3906 + 29 \operatorname{Re} E(s))}{2^8 3^2} + \frac{931xy^7 \operatorname{Im} E(s)}{2^{12} 3^2} + \frac{y^8(-17253 + 2855 \operatorname{Re} E(s))}{2^{12} 3^2 5} \\
&+ \frac{11803xy^9 \operatorname{Im} E(s)}{2^{15} 3^2 5^2} + \frac{8729y^{10} \operatorname{Re} E(s)}{2^{18} 3^2 5^2} - \frac{12595y^{12} |E(s)|^2}{2^{23} 3^4 5} \left. \right) \leq \frac{1}{2^{12} 3^2} \left(-7 + \frac{29y^2}{5} \right. \\
&- \left. \frac{11xy^3 \operatorname{Im} E(s)}{2^4 5} - \frac{3599y^4}{2^{17} 3^4 5^2} \right) \leq \frac{-3.4627824}{2^{12} 3^2},
\end{aligned}$$

since each of the terms in parentheses is negative and $-7 + \frac{29y^2}{5}$
 $-\frac{11xy^3 \text{Im } E(s)}{2^4 3^2 5} - \frac{3599y^4}{2^{17} 3^4 5^2}$ is a decreasing function of y for $-1 \leq y \leq 0$.

From (41) we have

$$\begin{aligned}
(63) \quad f_8 &= \frac{-67}{2^{14} 3^2 5} + \frac{11xy \text{Im } E(s)}{2^{17} 3^4 5} + \frac{y^2(701+33 \text{Re } E(s))}{2^{19} 3^3 5} - \frac{11xy^3 \text{Im } E(s)}{2^{20} 3^4 5^2} \\
&+ \frac{y^4(164+55 \text{Re } E(s))}{2^{21} 3^3 5^2} + \frac{11xy^5 \text{Im } E(s)}{2^{22} 3^4 5^2} + \frac{y^6(1099+792 \text{Re } E(s))}{2^{27} 3^4 5^2} + \frac{11xy^7 \text{Im } E(s)}{2^{25} 3^6 5} \\
&+ \frac{y^8}{2^{28} 3^4 5^2} \left(-8209 + \frac{44 \text{Re } E(s)}{3} - \frac{1841xy \text{Im } E(s)}{2^2 3^2} + \frac{y^2(-691956-174055 \text{Re } E(s))}{2^5 3^2 5} \right. \\
&\left. - \frac{47757xy^3 \text{Im } E(s)}{2^5 3^2 5^2} - \frac{36086y^4 \text{Re } E(s)}{2^6 3^2 5^2} + \frac{7667y^6 |E(s)|^2}{2^{12} 3^2 5} \right) \\
&\leq \frac{1}{2^{14} 3^2 5} \left(-67 + \frac{11xy \text{Im } E(s)}{2^3 3^2} + \frac{y^2(701+33 \text{Re } E(s))}{2^5 3} - \frac{11xy^3 \text{Im } E(s)}{2^6 3^2 5} \right. \\
&\left. + \frac{y^4(164+55 \text{Re } E(s))}{2^7 3 \cdot 5} + \frac{11xy^5 \text{Im } E(s)}{2^8 3^2 5} + \frac{y^6(1099+792 \text{Re } E(s))}{2^{13} 3^2 5} + \frac{11xy^7 \text{Im } E(s)}{2^{11} 3^4} \right) \\
&\leq \frac{-59.23165}{2^{14} 3^2 5},
\end{aligned}$$

since each of the terms in parentheses is negative and

$$\begin{aligned}
&-67 + \frac{11xy \text{Im } E(s)}{2^3 3^2} + \frac{y^2(701+33 \text{Re } E(s))}{2^5 3} - \frac{11xy^3 \text{Im } E(s)}{2^6 3^2 5} + \frac{y^4(164+55 \text{Re } E(s))}{2^7 3 \cdot 5} \\
&+ \frac{11xy^5 \text{Im } E(s)}{2^8 3^2 5} + \frac{y^6(1099+792 \text{Re } E(s))}{2^{13} 3^2 5} + \frac{11xy^7 \text{Im } E(s)}{2^{11} 3^4} \text{ is a decreasing function}
\end{aligned}$$

of y for $-1 \leq y \leq 0$. Finally, it is clear from (42) that

$$(64) \quad f_{10} \leq 0 \text{ for } -1 \leq y \leq 0.$$

Combining (35), (58), (59), (60), (61), (62), (63) and (64) we obtain

$$(65) \quad 2 \operatorname{Re}(I_0(x+iy) - I_0(x))I_0(x) + \operatorname{Re}^2(I_0(x+iy) - I_0(x)) + \operatorname{Im}^2(I_0(x+iy) - I_0(x)) \\ \leq -0.3994805y^2.$$

Using the fact that $I_0(x) \leq 0.46576e$ and (65) in (33), we have, for $t \leq 0$,

$$(66) \quad \operatorname{Re}f(T+it) - f(T) \leq \frac{1}{2} \sum_{\substack{\gamma > 0 \\ T \leq |\rho| \\ -|\rho| \leq t \leq 0}} \log \left(1 - \frac{0.3994805 \frac{t^2}{|\rho|^2}}{(0.46576)^2 e^2} \right)$$

$$\leq \frac{1}{2} \sum_{\substack{\gamma > 0 \\ T \leq |\rho| \\ -|\rho| \leq t \leq 0}} \log \exp \left(-0.260276 \frac{t^2}{|\rho|^2} \right)$$

$$= -0.130138 \sum_{\substack{\gamma > 0 \\ T \leq |\rho| \\ -|\rho| \leq t \leq 0}} \frac{t^2}{|\rho|^2}$$

$$\leq \begin{cases} -0.130138t^2 \int_{t-}^{\infty} \frac{dN(y)}{|\frac{1}{2}+iy|^2} & \text{if } T \leq |t|, \\ -0.130138t^2 \int_{T-}^{\infty} \frac{dN(y)}{|\frac{1}{2}+iy|^2} & \text{if } |t| \leq T. \end{cases}$$

Using Lemma 2.8 we obtain

$$\int_c^\infty \frac{dN(\gamma)}{|\mathfrak{w}|^2} \geq \frac{\log \frac{c}{2\pi}}{2\pi c} - \frac{1}{c^2}$$

for $c \geq 50$. Applying this to (66), we obtain

$$(67) \quad \operatorname{Re} f(T+it) - f(T) \leq \begin{cases} -0.0207121 t \log \frac{t}{2\pi} & \text{if } T \leq |t|, \\ -0.0207121 \frac{t^2}{T} \log \frac{T}{2\pi} & \text{if } |t| \leq T. \end{cases}$$

We can now bound U_1 and U_2 . Since $f(T-it) = \overline{f(T+it)}$, we have

$$U_1 = \frac{1}{2\pi} \int_{-T}^{-T^{2/3}/(\log \frac{T}{2\pi})^{1/2}} \operatorname{Re} \exp(f(T+it)) dt.$$

Using (67) we obtain

$$\begin{aligned} |U_1| &< \frac{1}{2\pi} \int_{-T}^{-T^{2/3}/(\log \frac{T}{2\pi})^{1/2}} \exp(\operatorname{Re} f(T+it)) dt \\ &\leq \frac{1}{2\pi} \exp(f(T)) \int_{-T}^{-T^{2/3}/(\log \frac{T}{2\pi})^{1/2}} \exp\left(-0.0207121 \frac{t^2}{T} \log \frac{T}{2\pi}\right) dt \\ &= \frac{1}{2\pi} \exp(f(T)) \sqrt{\frac{T}{0.0207121 \log \frac{T}{2\pi}}} \int \sqrt{\frac{0.0207121 T \log \frac{T}{2\pi}}{0.0207121 T^{1/3}}} \exp(-u^2) du. \end{aligned}$$

Using a result from Abromowitz and Stegun [1, p.298], that

$$\frac{\exp(-x^2)}{x + \sqrt{x^2 + 2}} < \int_x^\infty \exp(-t^2) dt \leq \frac{\exp(-x^2)}{x + \sqrt{x^2 + \frac{4}{\pi}}} \quad (x \geq 0),$$

we have

$$(69) \quad |U_1| \leq \exp(f(T)) \exp(-0.0207121 T^{1/3})$$

$$\cdot \left(\frac{T^{1/3}}{0.0414242 \pi \left(1 + \sqrt{1 + \frac{4}{0.0207121 \pi T^{1/3}}} \right) \sqrt{\log \frac{T}{2\pi}}} \right)$$

Similarly

$$U_2 = \frac{1}{2\pi} \int_{-\infty}^{-T} \operatorname{Re} \exp(f(T+it)) dt$$

and for $50 \leq T$,

$$(70) \quad |U_2| \leq \exp(f(T)) \frac{\exp(-0.0429599T)}{0.0859198 \pi}.$$

This completes the proof of Lemma 2.17.

Finally, we prove

Lemma 2.18. If $f'(T) = 0$, then

$$\frac{1}{4\pi i} \int_{T-iT^{2/3}/(\log \frac{T}{2\pi})^{1/2}}^{T+iT^{2/3}/(\log \frac{T}{2\pi})^{1/2}} \exp(f(z)) dz = \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}} \left(1 + O\left(\frac{1}{\sqrt{T \log T}}\right) \right).$$

Proof. We write

$$(71) \quad \frac{1}{4\pi i} \int_{T-iT^{2/3}/(\log \frac{T}{2\pi})^{1/2}}^{T+iT^{2/3}/(\log \frac{T}{2\pi})^{1/2}} \exp(f(z)) dz = \frac{1}{4\pi} \int_{-T^{2/3}/(\log \frac{T}{2\pi})^{1/2}}^{T^{2/3}/(\log \frac{T}{2\pi})^{1/2}} \exp(f(T)) \\ + \frac{df(T+iv)}{dv} \Big|_{v=0} t + \frac{d^2 f(T+iv)}{dv^2} \Big|_{v=0} \frac{t^2}{2} + R_2(t) dt,$$

where $|R_2(t)| \leq \max_{0 \leq v \leq t} \left| \frac{d^3(f(T+iv))}{dv^3} \right| \frac{|t|^3}{6}$. We compute the derivatives using the formula $\frac{d^n f(T+iv)}{dv^n} = (i)^n \frac{d^n f(z)}{dz^n} \Big|_{T+iv}$. From Abramowitz and Stegun [1, p.376], we have

$$\frac{dI_j(z)}{dz} = I_{j-1}(z) - \frac{j}{z} I_j(z) = I_{j+1}(z) + \frac{j}{z} I_j(z).$$

Since $f(z) = \sum_{\gamma > 0} \log I_0\left(\frac{z}{|\rho|}\right) - zu/2$, we obtain

$$f'(z) = \sum_{\gamma > 0} \frac{1}{|\rho|} \frac{I_1\left(\frac{z}{|\rho|}\right)}{I_0\left(\frac{z}{|\rho|}\right)} - u/2,$$

$$f''(z) = \sum_{\gamma > 0} \frac{1}{|\rho|^2} \frac{I_0\left(\frac{z}{|\rho|}\right)^2 - I_1\left(\frac{z}{|\rho|}\right)^2 - \frac{|\rho|}{z} I_0\left(\frac{z}{|\rho|}\right) I_1\left(\frac{z}{|\rho|}\right)}{I_0\left(\frac{z}{|\rho|}\right)^2}$$

and

$$f'''(z) = \sum_{\gamma > 0} \frac{h(z, |\rho|)}{|\rho|^3 I_0\left(\frac{z}{|\rho|}\right)^3},$$

where

$$\begin{aligned} h(z, |\rho|) = & \frac{|\rho|}{z} I_0\left(\frac{z}{|\rho|}\right) \left(I_1\left(\frac{z}{|\rho|}\right)^2 - I_0\left(\frac{z}{|\rho|}\right)^2 \right) + 2 I_1\left(\frac{z}{|\rho|}\right) \left(I_1\left(\frac{z}{|\rho|}\right)^2 - I_0\left(\frac{z}{|\rho|}\right)^2 \right) \\ & + \frac{|\rho|}{z} I_0\left(\frac{z}{|\rho|}\right) I_1\left(\frac{z}{|\rho|}\right) + 2 \frac{|\rho|^2}{z^2} I_0\left(\frac{z}{|\rho|}\right) I_1\left(\frac{z}{|\rho|}\right)^2. \end{aligned}$$

From Olver [19, p 269] we have, for $\text{Re } s \geq 1$

$$I_0(s) = \frac{\exp(s)}{\sqrt{2\pi s}} (A_0(s) - i \exp(-2s) B_0(s))$$

and

$$I_1(s) = \frac{\exp(s)}{\sqrt{2\pi s}} (A_1(s) + i \exp(-2s) B_1(s)),$$

where

$$A_0(s) = 1 + \frac{1}{8s} + \frac{9}{128s^2} + \frac{75}{1024s^3} + C_0(s),$$

$$B_0(s) = 1 - \frac{1}{8s} + \frac{9}{128s^2} - \frac{75}{1024s^3} + D_0(s),$$

$$A_1(s) = 1 - \frac{3}{8s} - \frac{15}{128s^2} - \frac{105}{1024s^3} + C_1(s),$$

$$B_1(s) = 1 + \frac{3}{8s} - \frac{15}{128s^2} + \frac{105}{1024s^3} + D_1(s),$$

$$|C_0(s)|, |D_0(s)| \leq \frac{2.637}{(\operatorname{Re} s)^4}$$

and

$$|C_1(s)|, |D_1(s)| \leq \frac{16.31}{(\operatorname{Re} s)^4}.$$

Hence,

(73)

$$\begin{aligned} & \frac{1}{s} I_0(s)(I_1(s)^2 - I_0(s)^2) + 2I_1(s)(I_1(s)^2 - I_0(s)^2) + \frac{1}{s} I_0(s)I_1(s) + \frac{2}{s^2} I_0(s)I_1(s)^2 \\ &= \frac{\exp(3s)}{(2\pi s)^{3/2}} \left(\frac{A_0(s)(A_1(s)^2 - A_0(s)^2)}{s} + 2A_1(s) \left(A_1(s)^2 - A_0(s)^2 + \frac{A_0(s)A_1(s)}{s} \right) \right. \\ &+ \frac{2A_0(s)A_1(s)^2}{s^2} + \exp(-4s) \frac{A_0(s)(B_1(s)^2 - B_0(s)^2)}{s} + \frac{2B_0(s)(A_1(s)B_1(s) + A_0(s)B_0(s))}{s} \\ &+ 2A_1(s) \left(B_1(s)^2 - B_0(s)^2 - \frac{B_0(s)B_1(s)}{s} \right) + 2B_1(s) \left(2A_1(s)A_0(s) + 2A_0(s)A_1(s) \right. \\ &\left. + \frac{A_0(s)B_1(s) - B_0(s)A_1(s)}{s} \right) + \frac{2A_0(s)B_1(s)^2}{s^2} + \frac{4A_1(s)B_1(s)B_0(s)}{s^2} \left. \right) + 1 \left(\exp(-2s) \right) \end{aligned}$$

$$\begin{aligned}
& \left(\frac{A_0(s)(A_0(s)B_0(s)+A_1(s)B_1(s))}{s} - \frac{B_0(s)(A_1(s)^2-B_0(s)^2)}{s} \right. \\
& + 2B_1(s) \left(A_1(s)^2 - A_0(s)^2 + \frac{A_0(s)A_1(s)}{s} \right) \\
& + 2A_1(s) \left(2A_1(s)B_1(s) + 2A_0(s)B_0(s) + \frac{A_0(s)B_1(s) - B_0(s)A_1(s)}{s} \right) \\
& + \left. \frac{4A_0(s)A_1(s)B_1(s)}{s^2} - \frac{2B_0(s)A_1(s)^2}{s^2} \right) \\
& + \exp(-6s) \left(\frac{B_0(s)(B_1(s)^2 - A_0(s)^2)}{s} - 2B_1(s) \left(B_1(s)^2 - B_0(s)^2 - \frac{B_0(s)B_1(s)}{s} \right. \right. \\
& \left. \left. + \frac{2B_0(s)B_1(s)^2}{s^2} \right) \right) .
\end{aligned}$$

Using our definitions of $A_0(s)$ and $A_1(s)$, we obtain

$$\begin{aligned}
(74) \quad & \frac{A_0(s)(A_1(s)^2 - A_0(s)^2)}{s} + A_1(s) \left(A_1(s)^2 - A_0(s)^2 + \frac{A_0(s)A_1(s)}{s} \right) + \frac{2A_0(s)A_1(s)}{s^2} \\
& = \frac{-1}{s^2} - \frac{3}{8s^3} - \frac{49}{128s^4} - \left(\frac{9183}{16384} + \frac{9}{8 \cdot 32} + \frac{9}{4 \cdot 128} + \frac{75}{8 \cdot 128} \right) \frac{1}{s^4} - \left(\frac{9183}{8 \cdot 16384} + \frac{81}{32 \cdot 128} \right. \\
& + \left. \frac{75}{32 \cdot 128} - \frac{225}{16384} \right) \frac{1}{s^6} - \left(\frac{9 \cdot 9183}{128 \cdot 16384} + \frac{9 \cdot 75}{8 \cdot 128 \cdot 32} - \frac{225}{8 \cdot 16384} \right) \frac{1}{s^7} \\
& - \left(\frac{75 \cdot 9183}{8 \cdot 128 \cdot 16384} - \frac{9 \cdot 225}{128 \cdot 16384} \right) \frac{1}{s^8} + \frac{75 \cdot 225}{8 \cdot 128 \cdot 16384} \frac{1}{s^9} + C_0(s) \left(\frac{-1}{s^2} - \frac{1}{4s^3} - \frac{9}{32s^4} \right. \\
& - \left. \frac{9183}{16384s^5} + \frac{225}{16384s^6} \right) + \left(2 + \frac{1}{4s} + \frac{9}{64s^2} + \frac{75}{4 \cdot 128s^3} + C_0(s) \right) \left(\frac{-C_0(s)}{s} - \frac{1}{s} - \frac{1}{8s^2} \right. \\
& - \left. \frac{9}{128s^3} - \frac{75}{8 \cdot 128s^4} \right) C_0(s) + \left(2 + \frac{3}{4s} + \frac{15}{64s^2} + \frac{105}{4 \cdot 128s^3} - C_1(s) \right) \left(\frac{-C_0(s)}{s} - \frac{1}{s} - \frac{1}{8s^2} \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{9}{128s^3} - \frac{75}{8 \cdot 128s^4} \Big) C_1(s) + \frac{2}{s^2} - \frac{5}{4s^3} - \frac{15}{64s^4} - \left(\frac{15}{64} + \frac{3}{8 \cdot 16} + \frac{27}{2 \cdot 128} - \frac{150}{8 \cdot 128} \right) \\
& \cdot \frac{1}{s^5} - \left(\frac{15}{8 \cdot 64} + \frac{27}{16 \cdot 128} + \frac{225}{16 \cdot 128} - \frac{1485}{16 \cdot 128} \right) \frac{1}{s^6} - \left(\frac{15 \cdot 9}{64 \cdot 128} + \frac{225}{8 \cdot 16 \cdot 128} \right. \\
& - \left. \frac{1575}{16 \cdot 128} - \frac{1485}{8 \cdot 64 \cdot 128} \right) \frac{1}{s^7} + \left(\frac{(105)^2}{32(128)^2} + \frac{1575}{16(128)^2} + \frac{9 \cdot 1485}{64(128)^2} - \frac{15 \cdot 75}{8 \cdot 64 \cdot 128} \right) \frac{1}{s^8} \\
& + \left(\frac{(105)^2}{4 \cdot 8^2(128)^2} + \frac{9 \cdot 1575}{2(128)^3} + \frac{75 \cdot 1485}{8 \cdot 64(128)^2} \right) \frac{1}{s^9} + \left(\frac{9(105)^2}{32(128)^3} + \frac{75 \cdot 1575}{16(128)^3} \right) \frac{1}{s^{10}} \\
& + \frac{75(105)^2}{8 \cdot 32(128)^3} \frac{1}{s^{11}} - C_1(s) \left(\frac{2}{s^2} + \frac{3}{4s^3} + \frac{15}{64s^4} + \frac{105}{4 \cdot 128s^5} - \frac{C_1(s)}{s^2} \right) (2) \left(1 + \frac{1}{8s} \right. \\
& + \left. \frac{9}{128s^2} + \frac{75}{8 \cdot 128s^3} + C_0(s) \right) + \frac{2C_0(s)}{s^2} \left(1 - \frac{3}{4s} - \frac{3}{32s^2} - \frac{15}{128s^3} + \frac{1485}{(128)^2 s^4} \right. \\
& + \left. \frac{1575}{4(128)^2 s^5} + \frac{(105)^2}{8^2(128)^2 s^6} \right) - \frac{1}{s^2} - \frac{3}{8s^3} + \left(\frac{-2 \cdot 10335}{16384} + \frac{9}{32} + \frac{15}{128} \right) \frac{1}{s^4} + \left(\frac{-2 \cdot 36480}{8^2(128)^2} \right. \\
& + \left. \frac{6 \cdot 10335}{8 \cdot 16384} + \frac{45}{4 \cdot 128} + \frac{105}{8 \cdot 128} \right) \frac{1}{s^5} + \left(\frac{-2 \cdot 2070}{8(128)^2} + \frac{6 \cdot 36480}{8^3(128)^2} + \frac{30 \cdot 10335}{128 \cdot 16384} + \frac{315}{8 \cdot 4 \cdot 128} \right) \\
& \cdot \frac{1}{s^6} + \left(\frac{-150 \cdot 105}{(8 \cdot 128)^2} + \frac{6 \cdot 2070}{(8 \cdot 128)^2} + \frac{30 \cdot 36480}{8^2(128)^3} + \frac{2 \cdot 105 \cdot 10335}{8 \cdot 128 \cdot 16384} \right) \frac{1}{s^7} + \left(\frac{450 \cdot 105}{8^3(128)^2} \right. \\
& + \left. \frac{30 \cdot 2070}{8(128)^3} + \frac{210 \cdot 36480}{8^3(128)^3} \right) \frac{1}{s^8} + \left(\frac{15 \cdot 150 \cdot 105}{8^2(128)^3} + \frac{210 \cdot 2070}{8^2(128)^3} \right) \frac{1}{s^9} + \frac{(105)^2 150}{(8 \cdot 128)^3} \frac{1}{s^{10}} \\
& + 2C_1(s) \left(\frac{-1}{2s^2} - \frac{3}{8s^3} - \frac{10335}{16384s^4} - \frac{36480}{8^2(128)^2 s^5} - \frac{2070}{8(128)^2 s^6} - \frac{75 \cdot 105}{(8 \cdot 128)^2 s^7} \right) \\
& + 2 \left(1 - \frac{3}{8s} - \frac{15}{128s^2} - \frac{105}{8 \cdot 128s^3} + C_1(s) \right) \left(C_1(s) \left(\frac{1}{s} + \frac{1}{8s} + \frac{9}{128s^3} + \frac{75}{8 \cdot 128s^4} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& + C_0(s) \left(\frac{1}{s} - \frac{3}{8s^2} - \frac{15}{128s^3} - \frac{105}{8 \cdot 128s^4} + \frac{C_1(s)}{s} \right) - \frac{C_1(s)}{s} \left(2 + \frac{3}{4s} + \frac{15}{64s^2} + \frac{105}{4 \cdot 128s^3} \right. \\
& \left. - C_1(s) - \frac{C_0(s)}{s} \left(2 + \frac{1}{4s} + \frac{9}{64s^2} + \frac{75}{4 \cdot 128s^3} + C_0(s) \right) \right) \\
& = \frac{-2}{s^3} + O\left(\frac{1}{(\operatorname{Re} s)^4}\right) = O\left(\frac{1}{(\operatorname{Re} s)^3}\right).
\end{aligned}$$

Combining (73) and (74), we obtain, for $\operatorname{Re} s \geq 1$.

$$\begin{aligned}
(75) \quad & \frac{1}{s} I_0(s) (I_1(s)^2 - I_0(s)^2) + 2 I_1(s) \left(I_1(s)^2 - I_0(s)^2 + \frac{I_0(s) I_1(s)}{s} \right) \\
& + \frac{2 I_0(s) I_1(s)^2}{s^2} = O\left(\frac{\exp(3s)}{s^{3/2} (\operatorname{Re} s)^3}\right).
\end{aligned}$$

We have

$$I_0(s) = \sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{(k!)^2}$$

and

$$I_1(s) = \frac{s}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{k! (k+1)!}.$$

Thus

$$I_1(s) = I_0(s) \frac{I_1(s)}{I_0(s)} = \frac{s}{2} I_0(s) Q(s),$$

where

$$Q(s) = \frac{\sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{(k!)(k+1)!}}{\sum_{k=0}^{\infty} \frac{\left(\frac{s^2}{4}\right)^k}{(k!)^2}} = \frac{1 + \frac{s^2}{8} + \frac{s^4}{192} + \dots}{1 + \frac{s^2}{4} + \frac{s^4}{64} + \dots}.$$

For $\text{Re } s \leq 1$ and $|\text{Im } s| \leq \text{Re } s$, we have

$$|Q(s)| \leq 1 + \frac{2}{8} + \frac{2^2}{4^2 \cdot 2 \cdot 6} + \frac{2^3}{4^3 \cdot 6 \cdot 24} + \dots \leq 1.28.$$

Hence for $\text{Re } s \leq 1$ and $|\text{Im } s| \leq \text{Re } s$ we have

(76)

$$\begin{aligned} & \frac{1}{s} I_0(s) (I_1(s)^2 - I_0(s)^2) + 2 I_1(s) \left(I_1(s)^2 - I_0(s)^2 + \frac{I_0(s) I_1(s)}{s} \right) + \frac{2 I_0(s) I_1(s)^2}{s^2} \\ &= \frac{1}{s} I_0(s) \left(\frac{s^2}{4} I_0(s)^2 Q(s)^2 - I_0(s)^2 \right) + s I_0(s) Q(s) \left(\frac{s^2}{4} I_0(s)^2 Q(s)^2 - I_0(s)^2 \right. \\ & \quad \left. + \frac{I_0(s)^2 Q(s)}{2} \right) + \frac{I_0(s)^3 Q(s)^2}{2} \\ &= I_0(s)^3 \left(\frac{s^3 Q(s)^3}{4} + s \left(\frac{3Q(s)^2}{4} - Q(s) \right) + \frac{Q(s)^2}{2} - \frac{1}{s} \right). \end{aligned}$$

Using (72), (75), (76) and Lemma 2.8, we obtain

(77)

$$\begin{aligned} |f'''(z)| &<< \sum_{\substack{0 < \gamma \\ |\rho| \leq T}} \frac{1}{|\rho|^3 \text{Re} \left(\frac{z}{|\rho|} \right)^3} + \sum_{|\rho| \geq T} \frac{1}{|\rho|^3} \left(\frac{|\rho|}{z} + 1 + \frac{z}{|\rho|} + \frac{z^3}{|\rho|^3} \right) \\ &<< \frac{\log T}{T^2}. \end{aligned}$$

We next estimate

$$f''(\Gamma) = \sum_{\gamma > 0} \frac{I_0\left(\frac{\Gamma}{|\rho|}\right)^2 - I_1\left(\frac{\Gamma}{|\rho|}\right)^2 - \frac{I_0\left(\frac{\Gamma}{|\rho|}\right)I_1\left(\frac{\Gamma}{|\rho|}\right)}{(\Gamma/|\rho|)}}{|\rho|^2 I_0\left(\frac{\Gamma}{|\rho|}\right)^2}.$$

From Abromowitz and Stegun [1, 9.1.14, 9.63], we have, for real x

$$I_0(x)^2 = J_0(ix)^2 = \sum_{k=0}^{\infty} \frac{(2k)! \left(\frac{x^2}{4}\right)^k}{(k!)^4},$$

$$I_1(x)^2 = -J_1(ix)^2 = \left(\frac{x}{2}\right)^2 \sum_{k=0}^{\infty} \frac{(2k+2)! \left(\frac{x^2}{4}\right)^k}{k! \cdot ((k+1)!)^2 (k+2)!}$$

and

$$I_0(x)I_1(x) = -i J_0(ix)J_1(ix) = \frac{x}{2} \sum_{k=0}^{\infty} \frac{(2k+1)! \left(\frac{x^2}{4}\right)^k}{(k!)^2 ((k+1)!)^2}.$$

Hence

$$\begin{aligned} I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x} &= \sum_{k=0}^{\infty} \frac{(2k)! \left(\frac{x^2}{4}\right)^k}{(k!)^4} - \sum_{k=0}^{\infty} \frac{(2k+2)! \left(\frac{x^2}{4}\right)^{k+1}}{k! \cdot ((k+1)!)^2 (k+2)!} \\ &\quad - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(2k+1)! \left(\frac{x^2}{4}\right)^k}{(k!)^2 ((k+1)!)^2} \\ &= \frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{(2k)! \left(\frac{x^2}{4}\right)^k}{(k!)^2 ((k+1)!)^2} \right) \leq \frac{1}{2} I_0(x)^2. \end{aligned}$$

Thus

$$(78) \quad 0 \leq \frac{I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x}}{I_0(x)^2} \leq \frac{1}{2}.$$

As before, we write $I_0(x)$ and $I_1(x)$ in terms of $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ for $x \geq 1$. We have

$$\begin{aligned} I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x} &= \frac{\exp(2x)}{2\pi x} \left(A_0(x)^2 - A_1(x)^2 - \frac{A_0(x)A_1(x)}{x} \right. \\ &+ \exp(-4x) \left(-B_0(x)^2 + B_1(x)^2 - \frac{B_0(x)B_1(x)}{x} \right) + i \exp(-2x) (-2A_0(x)B_0(x) \\ &\quad \left. + 2A_1(x)B_1(x) + A_0(x)B_1(x) - A_1(x)B_0(x)) \right) \end{aligned}$$

and

$$\begin{aligned} A_0(x)^2 - A_1(x)^2 - \frac{A_0(x)A_1(x)}{x} &= \frac{1}{2x^2} + \frac{3}{8x^3} + \frac{3}{1024x^4} + \frac{285}{8192x^5} \\ &\quad + \frac{1395}{131072x^6} + \frac{7875}{(1024)^2 x^7} \\ &+ C_0(x) \left(2 - \frac{3}{4x} + \frac{66}{128x^2} + \frac{270}{1024x^3} + \frac{105}{1024x^4} + C_0(x) - C_1(x) \right) \\ &- C_1(x) \left(2 + \frac{1}{4x} - \frac{14}{128x^2} - \frac{138}{1024x^3} + \frac{75}{1024x^4} + C_1(x) \right). \end{aligned}$$

Since $|C_0(x)| \leq \frac{2.637}{x}$ and $|C_1(x)| \leq \frac{16.31}{x}$, we have, for $x \geq 8$

$$\frac{0.0324768}{x^2} \leq A_0(x)^2 - A_1(x)^2 - \frac{A_0(x)A_1(x)}{x} \leq \frac{1.1422185}{x^2}.$$

Thus there are constants $d_1, d_2, d_3 > 0$ such that if $x \geq d_1$,

$$(79) \quad \frac{d_2}{x^2} \leq \frac{I_0(x)^2 - I_1(x)^2 - \frac{I_0(x)I_1(x)}{x}}{I_0(x)^2} \leq \frac{d_3}{x}.$$

Combining (78) and (79), we have

$$\sum_{\substack{\gamma > 0 \\ |\rho| \leq T/d_1}} \frac{d_2}{T^2} \leq f''(T) \leq \sum_{\substack{\gamma > 0 \\ |\rho| \leq T/d_1}} \frac{d_3}{T^2} + \frac{1}{2} \sum_{|\rho| > T/d_1} \frac{1}{|\rho|^2}.$$

Applying Lemma 2.8 to the above equation, we obtain

$$(80) \quad \frac{\log T}{T} \ll f''(T) \ll \frac{\log T}{T}.$$

Combining (71) and (77), we obtain

$$G'(u) - U_1 - U_2 = \frac{1}{4\pi} \int_{-T^{2/3}/(\log \frac{T}{2\pi})^{1/2}}^{T^{2/3}/(\log \frac{T}{2\pi})^{1/2}} \exp(f(T) - \frac{f''(T)t^2}{2} - R_2(t)) dt,$$

where $|R_2(t)| \ll \frac{\log T |t|^3}{T^2}$. Letting $v = \sqrt{\frac{f''(T)}{2}} t$, we write

$$(81) \quad G'(u) - U_1 - U_2 = \frac{\exp(f(T))}{4\pi \sqrt{\frac{f''(T)}{2}}} \int_{-T^{2/3}/\sqrt{\frac{f''(T)}{2}}/(\log \frac{T}{2\pi})^{1/2}}^{T^{2/3}/\sqrt{\frac{f''(T)}{2}}/(\log \frac{T}{2\pi})^{1/2}} \exp\left(-v^2 + R_2\left(\frac{v}{\sqrt{\frac{f''(T)}{2}}}\right)\right) dv$$

$$= \frac{\exp(f(T))}{\pi \sqrt{8f''(T)}} \int_{-T^{2/3}/\sqrt{\frac{f''(T)}{2}}/(\log \frac{T}{2\pi})^{1/2}}^{T^{2/3}/\sqrt{\frac{f''(T)}{2}}/(\log \frac{T}{2\pi})^{1/2}} \exp(-v^2) \left(1 + O\left(\frac{\log T |v|^3}{T^2 (f''(T))^{3/2}}\right)\right) dv$$

$$= \frac{\exp(f(T))}{\sqrt{8\pi f''(T)}} \left(1 + O\left(\frac{1}{\sqrt{T \log T}}\right) \right),$$

which completes the proof of Lemma 2.18.

We obtain Theorem 2.15 by combining Lemma 2.16, Lemma 2.17, and Lemma 2.18.

We now examine $f(x)$ more closely. We have

$$(82) \quad \begin{aligned} f(x) &= \sum_{\gamma > 0} \log I_0 \left(\frac{x}{|\rho|} \right) \\ &= \int_1^\infty \log I_0 \left(\frac{x}{|\frac{1}{2} + iy|} \right) dN(y). \end{aligned}$$

Olver [19, p.93] has shown that

$$I_0(v) = e^v \left\{ \sum_{s=0}^{n-1} \frac{\Gamma(s+\frac{1}{2}) a_{2s}}{v^{s+\frac{1}{2}}} - \epsilon_{2n,1}(v) + \epsilon_{2n,2}(v) \right\} + \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} e^{v \cos t} dt,$$

where for $v > \frac{1}{2}$,

$$a_0 = \frac{1}{\pi 2^{\frac{1}{2}}},$$

$$|\epsilon_{2,1}(v)| < \frac{e^{-v}}{\pi v}$$

and

$$|\epsilon_{2,2}(v)| < \frac{\Gamma(3/2)}{(v-\frac{1}{2})^{3/2} \pi 2^{5/2}}$$

Thus for $v \geq 1$, we have

$$(83) \quad I_0(v) = \frac{\exp(v)}{\sqrt{2\pi v}} E(v),$$

where

$$E(v) = 1 + O\left(\frac{1}{v}\right).$$

Since

$$I_0(v) = \sum_{k=0}^{\infty} \frac{\left(\frac{v}{2}\right)^{2k}}{(k!)^2} = 1 + \frac{v^2}{4} + \dots,$$

we have, for $0 < v < 1$,

$$(84) \quad I_0(v) = 1 + \frac{v^2}{4} + O(v^4).$$

Since

$$0 \leq \frac{1}{y} - \frac{1}{\left(\frac{1}{2} + y^2\right)^{\frac{1}{2}}} = \frac{\left(1 + \frac{1}{2y^2}\right)^{\frac{1}{2}} - 1}{\left(\frac{1}{2} + y^2\right)^{\frac{1}{2}}} \leq \frac{1}{2y^3},$$

we can write

$$\frac{1}{\left(\frac{1}{2} + y^2\right)^{\frac{1}{2}}} = \frac{1}{y} (1 + F(y))$$

where

$$\frac{-1}{2y^2} \leq F(y) < 0.$$

We write $N(y) = \frac{y}{2\pi} \log \frac{y}{2\pi} - \frac{y}{2\pi} + \frac{7}{8} + S(y) + G(y)$. Then

$$f(x) = \frac{1}{2\pi} \int_1^x \log I_0\left(\frac{x}{y} (1 + F(y))\right) \log \frac{y}{2\pi} dy + \int_x^\infty \log I_0\left(\frac{x}{y} (1 + F(y))\right) \log \frac{y}{2\pi} dy$$

$$\begin{aligned}
& + \int_1^x \log I_0\left(\frac{x}{y} (1 + F(y))\right) d(S(y) + G(y)) + \int_x^\infty \log I_0\left(\frac{x}{y} (1 + F(y))\right) d(S(y) + G(y)) \\
& = K_1 + K_2 + K_3 + K_4 .
\end{aligned}$$

Using (83) and letting $t = \frac{x}{y}$, we obtain

$$\begin{aligned}
K_1 & = \frac{x}{2\pi} \int_1^x \left(t - t F\left(\frac{x}{t}\right) - \frac{1}{2} \log 2\pi t - \frac{1}{2} \log \left(1 + F\left(\frac{x}{t}\right)\right) \right. \\
& \quad \left. + \log E\left(t\left(1 + F\left(\frac{x}{t}\right)\right)\right) \right) \log \frac{x}{2\pi t} \frac{dt}{t^2} \\
& = \frac{x}{2\pi} \int_1^x \left(t - \frac{1}{2} \log 2\pi t + \log E(t) \right) \log \frac{x}{2\pi t} \frac{dt}{t^2} \\
& + \frac{x}{2\pi} \int_1^u \left(t F\left(\frac{x}{t}\right) - \frac{1}{2} \log \left(1 + F\left(\frac{x}{t}\right)\right) + \left(\log E\left(t\left(1 + F\left(\frac{x}{t}\right)\right)\right) - \log E(t) \right) \right) \log \frac{x}{2\pi t} \frac{dt}{t^2} \\
& = L_1 + L_2 .
\end{aligned}$$

Since

$$\begin{aligned}
\frac{x}{2\pi} \int_1^x \log E(t) \log \frac{x}{2\pi t} \frac{dt}{t^2} & = \frac{x}{2\pi} \log \frac{x}{2\pi} \left(\int_1^\infty \frac{\log E(t) dt}{t^2} + O\left(\frac{1}{x^2}\right) \right) \\
& - \frac{x}{2\pi} \left(\int_1^\infty \frac{\log E(t) \log t dt}{t^2} + O\left(\frac{\log x}{x^2}\right) \right) ,
\end{aligned}$$

we have

(85)

$$L_1 = \frac{x}{4\pi} \log^2 x + C_{1,1} x \log x + C_{1,2} x + C_{1,3} \log x + C_{1,4} + O\left(\frac{\log x}{x}\right).$$

To evaluate L_2 we let $y = \frac{x}{t}$, so that

(86)

$$\begin{aligned}
L_2 &= \frac{1}{2\pi} \int_1^x \left(\frac{x}{y} F(y) - \frac{1}{2} \log(1+F(y)) + \log E\left(\frac{x}{y}(1+F(y))\right) - \log E\left(\frac{x}{y}\right) \right) \log \frac{y}{2\pi} dy \\
&= \frac{1}{2\pi} \int_1^\infty \left(\frac{x}{y} F(y) - \frac{1}{2} \log(1+F(y)) \right) \log \frac{y}{2\pi} dy + O\left(\frac{\log x}{x}\right) \\
&= C_{1,2} x + C_{1,4} + O\left(\frac{\log x}{x}\right).
\end{aligned}$$

Using (84) and letting $t = \frac{x}{y}$, we obtain

$$\begin{aligned}
K_2 &= \frac{x}{2\pi} \int_0^1 \log I_0\left(t\left(1+F\left(\frac{x}{t}\right)\right)\right) \log \frac{x}{2\pi t} \frac{dt}{t^2} \\
&= \frac{x}{2\pi} \int_0^1 \log\left(1 + \frac{t^2\left(1+F\left(\frac{x}{t}\right)\right)^2}{4} + \dots\right) \log \frac{x}{2\pi t} \frac{dt}{t^2} \\
&= C_{2,1} x \log x + C_{2,2} x + O\left(x \int_0^1 \frac{t^4}{x} \log \frac{x}{t} \frac{dt}{t^2}\right) \\
&= C_{2,1} x \log x + C_{2,2} x + O\left(\frac{\log x}{x}\right).
\end{aligned}$$

Recall that $S(y) + G(y) = O(\log y)$. Using (83), we have

(88)

$$\begin{aligned}
K_3 &= \int_1^x \left(\frac{x}{y} (1+F(y)) - \frac{1}{2} \log \frac{2\pi x(1+F(y))}{y} + \log E\left(\frac{x}{y}(1+F(y))\right) \right) d(S(y) + G(y)) \\
&= (S(x) + G(x)) (1+F(x)) - \frac{1}{2} \log 2\pi(1+F(x)) + \log E(1+F(x)) \\
&\quad - (S(1) + G(1)) (x(1+F(1))) - \frac{1}{2}(\log 2\pi x)(1+F(1)) + \log E(x(1+F(1)))
\end{aligned}$$

$$\begin{aligned}
& + \int_1^x (S(y) + G(y)) \left(\frac{-1}{y} (1 + F(y)) + \frac{F'(y)}{y} \right) - \frac{1}{2} \frac{F'(y)}{1 + F(y)} \\
& + \frac{1}{2} \frac{1}{y} + \frac{E' \left(\frac{x}{y} (1 + F(y)) \right) \left(\frac{-x}{y} (1 + F(y)) + \frac{x}{y} F'(y) \right)}{E \left(\frac{x}{y} (1 + F(y)) \right)} \\
& = C_{3,3} x + O(\log x).
\end{aligned}$$

Similarly, using (84), we have

$$(89) \quad K_4 \ll \int_x^\infty \frac{x^2}{y} d(S(y) + G(y)) \ll \log x.$$

Combining (85), (86), (87), (88), and (89), we obtain

$$\begin{aligned}
(90) \quad f(x) &= \frac{x}{4\pi} \log x + C_{5,1} x \log x + C_{5,2} x - xu/2 + O(\log x) \\
&= g(x) + O(\log x),
\end{aligned}$$

say. We have

$$g'(x) = \frac{1}{4\pi} \log^2 x + \left(\frac{1}{2\pi} + C_{5,1} \right) \log x + (C_{5,1} + C_{5,2}^{-u/2}),$$

so the minimum of $g(x)$ occurs at

$$X_0 = e^{-1 - 2\pi C_{5,1}} + \sqrt{2\pi u + 1 + 4\pi^2 C_{5,1}^2 - 4\pi C_{5,2}}.$$

Since there are constants $C_{6,1}$ and $C_{6,2}$ such that

$$g(x) + C_{6,1} \log x \leq f(x) \leq g(x) + C_{6,2} \log x,$$

we have

$$\lim_{x \rightarrow \infty} g'(x) = f'(x)$$

and

$$\lim_{x \rightarrow \infty} X_0 = T,$$

where $f'(T) = 0$.

Taking $T = X_0$, we have

$$f(T) = -\frac{1}{2\pi} \left(\sqrt{2\pi u + 4\pi^2 C_{5,1}^2 + 1 - 4\pi C_{5,2}^{-1}} \right) T + O(\log T).$$

Since $\frac{\log T}{T} \ll f''(T) \ll \frac{\log T}{T}$, there are constants a_1, a_2, a_3 and a_4

such that

$$G'(u) = \exp \left(- \left(\sqrt{a_1 u + a_2} - a_3 \right) e^{\sqrt{2\pi u + a_4}} + O(\sqrt{u}) \right).$$

CHAPTER III

THE ZEROS OF $\zeta_N(s)$

1. Introduction. Let $\zeta_N(s) = \sum_{n=1}^N n^{-s}$. We show that $\zeta_N(s)$ has a zero $s = \sigma + it$ with $\sigma > 1$ for all $N > 30$. The question is of particular interest because Turán [27] showed that the Riemann hypothesis is true if there exists an N_0 such that for all $N > N_0$, $\zeta_N(s)$ has no zero with $\sigma > 1$. The question has been considered by many others, including Haselgrove [11], Spira [25] and Voronin [29]. Montgomery [17] has shown that if $0 < c < \frac{4}{\pi} - 1$, then for all $N > N_0(c)$, $\zeta_N(s)$ has a zero with $\sigma > 1 + c(\log \log N)/\log N$.

The following table summarizes the current knowledge concerning zeros of $\zeta_N(s)$ with $\sigma > 1$:

N	zero with $\sigma > 1$?
1	No (Turán)
2	No (Turán)
3	No (Turán)
4	No (Turán)
5	No (Turán)
6	No (Spira)
7	No (Spira)
8	No (Spira)
9	No (Spira)
10	?
11	?
12	?
13	?
14	?

N	zero with $\sigma > 1$?
15	?
16	?
17	?
18	?
19	Yes (Spira)
20	?
21	?
22	Yes (Spira)
23	Yes (Spira)
24	Yes (Spira)
25	Yes (Spira)
26	Yes (Spira)
27	Yes (Spira)
28	?
29-50	Yes (Spira)
51- N_0	Yes (Monach)
$>N_0$	Yes (Montgomery)

Van de Lune and Te Riele [15] have explicitly computed zeros of $\zeta_N(s)$ with $\sigma > 1$ for small values of N .

By the classical work of Bohr [4], $\zeta_N(s)$ takes the same values in the half-plane $\text{Re } s > \sigma_0$ as $f_N(s) = \sum_{n=1}^N a(n) n^{-s}$, where $|a(n)| = 1$ for all n and $a(n)$ is totally multiplicative. Hence, $f_N(s)$ has a zero with $\sigma > 1$ if and only if $\zeta_N(s)$ has such a zero. We write

$$f_N(s) = \sum_{n=1}^N a(n)n^{-s} + \sum_{N^{\frac{1}{2}} < p \leq N^{19/20}} a(p)p^{-s} \sum_{m \leq N/p} a(m)m^{-s}$$

where $\sum_{n=1}^N$ is the sum over all n such that $p|n \rightarrow p \leq N^{\frac{1}{2}}$ or $p > N^{19/20}$. We choose $a(p) = p^{i\tau}$ for $p \leq N^{1/2}$ and $p > N^{19/20}$, leaving the $a(p)$ for $N^{1/2} < p \leq N^{19/20}$ to be chosen later. Thus, $f_N(\sigma + i2\tau)$ has a zero with $\sigma > 1$ if for some τ

$$(1) \quad G(1+i\tau) > |F(1+i\tau)|$$

and there is no prime q , $N^{1/2} < q \leq N^{19/20}$, for which

(2)

$$q^{-1} \left| \sum_{m \leq N/q} m^{-1-i\tau} \right| > N^{1/2} \sum_{\substack{p \leq N^{19/20} \\ p \neq q}} p^{-1} \left| \sum_{m \leq N/p} m^{-1-i\tau} \right|.$$

Here,

$$G(s) = N^{1/2} \sum_{p \leq N^{19/20}} p^{-1} \left| \sum_{m \leq N/p} m^{-s} \right|$$

and

$$F(s) = \sum_{n=1}^N n^{-s}.$$

We will show that (1) and (2) are true for $\tau = \frac{2\pi}{\log N}$ for every $N > 549798$. The choice of τ is inspired by Levinson's work [14] on the zeros of $\zeta_N(s)$ near 1.

2. An upper bound for $|F(1+i\tau)|$. In this section, we prove

Theorem 3.1. Let $\tau = \frac{2\pi}{\log N}$. Let $a = N^{1/2}$ and $b = N^{19/20}$. Let

$$\begin{aligned} F_{r,0} = & \frac{1}{\tau} (-0.4257265 + K + L) - 0.05177 \epsilon(a) - I + \left(\frac{1}{2N} - \frac{1}{12N^2} + \frac{1-\tau^2}{120N^4} \right) \\ & + \left(0.575 + \frac{\tau^2}{120} \right) \left(1.1466715 - \frac{1.0011126\epsilon(b)}{\log N} - \frac{2\epsilon(a)}{\log N} - I - J + \tau K \right) \\ & + \left(\frac{49\tau + \tau^3}{720} \right) \left(0.4257265 + \frac{0.325281\epsilon(b)}{\log N} - K - L + \tau I \right) \end{aligned}$$

$$- \frac{1}{2N} \left(\int_a^b \frac{du}{\log u} + \frac{T(b)}{\log b} - \frac{T(a)}{\log a} + \int_{a^+}^{b^+} \frac{\epsilon(u) du}{\log^2 u} \right)$$

and

$$\begin{aligned} F_{1,0} = & \frac{1}{\tau} \left(-0.7885255 + I + J - \int_{a^+}^{b^+} \frac{\epsilon(u) \left(1 + \frac{1}{\log u}\right) du}{u \log u} \right) + 0.6366197 \epsilon(a) \\ & + K - 0.0081994 \epsilon(b) + \left(0.575 + \frac{\tau^2}{120}\right) \left(-0.4257265 - \frac{0.325281 \epsilon(b)}{\log N}\right) \\ & + K + L - \tau I + \left(\frac{49\tau + \tau^3}{720}\right) \left(1.1466715 - \frac{1.0011126 \epsilon(b)}{\log N}\right) \\ & - \frac{2\epsilon(a)}{\log N} - I - J - \tau K \left) - \frac{\tau}{12N^2} + \frac{11\tau - \tau^3}{720N^4} \right. \\ & \left. - \frac{\tau}{2N^2} \left(\int_a^b \frac{u du}{\log u} + \frac{bT(b)}{\log b} - \frac{aT(a)}{\log a} - \int_{a^+}^{b^+} \frac{T(u) \left(1 - \frac{1}{\log u}\right) du}{\log u} \right), \right. \end{aligned}$$

where $T(u) = \theta(u) - u$, $\epsilon(u) = \frac{T(u)}{u}$,

$$I = \int_{a^+}^{b^+} \frac{\epsilon(u) \cos \tau \log u du}{u \log u},$$

$$J = \int_{a^+}^{b^+} \frac{\epsilon(u) \cos \tau \log u du}{u \log^2 u},$$

$$K = \int_{a^+}^{b^+} \frac{\epsilon(u) \sin \tau \log u du}{u \log u},$$

and

$$L = \int_{a^+}^{b^+} \frac{\epsilon(u) \sin \tau \log u du}{u \log^2 u}.$$

Let

$$E(u, s) = \frac{-s(s+1)(s+2)(s+3)(s+4)}{120} \int_u^\infty \hat{B}_5(x) x^{-s-5} dx$$

and $r(p) = N/p$. Let θ represent any number such that $|\theta| \leq 1$. If $N \geq 549798$, then

$$\begin{aligned} |F(1+i\tau)| &\leq (F_{r,0}^2 + F_{i,0}^2)^{\frac{1}{2}} + |E(1, 1+i\tau) - E(N, 1+i\tau)| \\ &\quad + \left| N^{1/2} \sum_{p < N}^{19/20} p^{-1-i\tau} (E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)) \right| \\ &\quad + 5 \cdot 10^{-7} (\log N) \theta. \end{aligned}$$

We need several preliminary results. From Edwards [7, pp.114-115],

we have

Lemma 3.2. If $\sigma \geq 1$, then

$$\begin{aligned} \zeta(s) &= \zeta_N(s) + \frac{N^{1-s}}{s-1} - \frac{1}{2} N^{-s} + \frac{s}{12} N^{-s-1} - \frac{s(s+1)(s+2)}{720} N^{-s-3} \\ &\quad + E(N, s). \end{aligned}$$

Using this lemma, we prove

Lemma 3.3. If N is an integer > 1 , then

$$\begin{aligned} \zeta_N(1+i\tau) &= \left(0.575 + \frac{\tau^2}{120} \right) + \left(\frac{1}{2N} - \frac{1}{12N^2} + \frac{(1-\tau^2)}{120N^4} \right) \\ &\quad + i \left(\frac{49\tau + \tau^3}{720} - \frac{\tau}{12N^2} + \frac{11\tau - \tau^3}{720N^4} \right) + E(1, 1+i\tau) - E(N, 1+i\tau). \end{aligned}$$

Proof. Using the previous lemma, we write

$$\begin{aligned} \zeta_N(1+i\tau) &= \zeta(1+i\tau) - \frac{1}{i\tau} N^{-i\tau} + \frac{1}{2} N^{-1-i\tau} - \frac{(1+i\tau)}{12} N^{-2-i\tau} \\ &\quad + \frac{(1+i\tau)(2+i\tau)(3+i\tau)}{720} N^{-4-i\tau} - E(N, 1+i\tau) \end{aligned}$$

and

$$\zeta(1+i\tau) = 1 + \frac{1}{i\tau} - \frac{1}{2} + \frac{1+i\tau}{12} - \frac{(1+i\tau)(2+i\tau)(3+i\tau)}{720} + E(1, 1, i\tau).$$

Combining these equalities with the fact that $N^{-i\tau} = 1$, we complete the proof of the lemma.

Next we have

Lemma 3.4. Let $h(p, s) = \sum_{m \leq N/p} m^{-s}$,

$$\begin{aligned} C(p) &= -\hat{B}_1(r(p)) \frac{p}{N} - \hat{B}_2(r(p)) \frac{p^2}{2N^2} - \hat{B}_3(r(p)) \frac{p^3(2-\tau^2)}{6N^3} \\ &\quad - \hat{B}_4(r(p)) \frac{p^4(1-\tau^2)}{4N^4} - \hat{B}_5(r(p)) \frac{p^5(24-35\tau^2+\tau^4)}{120N^4} \end{aligned}$$

and

$$\begin{aligned} D(p) &= -\hat{B}_2(r(p)) \frac{p^2\tau}{2N^2} - \hat{B}_3(r(p)) \frac{p^3\tau}{2N^3} - \hat{B}_4(r(p)) \frac{p^4(11\tau-\tau^3)}{24N^4} \\ &\quad - \hat{B}_5(r(p)) \frac{p^5(5\tau-\tau^3)}{12N^5}. \end{aligned}$$

If $N \geq 549798$, then

$$\begin{aligned} h(p, 1+i\tau) &= \left(0.575 + \frac{\tau^2}{120}\right) - \frac{1}{\tau} \sin \tau \log p + C(p) \cos \tau \log p \\ &\quad - D(p) \sin \tau \log p + i \left(\frac{1}{\tau} (\cos \tau \log p - 1) + \left(\frac{49\tau + \tau^3}{720} \right) \right. \\ &\quad \left. + D(p) \cos \tau \log p + C(p) \sin \tau \log p \right) + E(1, 1+i\tau) - E(r(p)^+, 1+i\tau). \end{aligned}$$

Proof. For $\sigma > 1$, we have

$$\begin{aligned} h(p, s) &= \zeta(s) - \sum_{n > N/p} n^{-s} \\ &= \zeta(s) - \frac{\left(\frac{N}{p}\right)^{1-s}}{s-1} - \hat{B}_1(r(p)) \left(\frac{N}{p}\right)^{-s} + s \int_{r(p)^+}^{\infty} \hat{B}_1(x) x^{-s-1} dx. \end{aligned}$$

Integrating by parts, we see that this

$$\begin{aligned} &= \zeta(s) - \frac{\left(\frac{N}{p}\right)^{1-s}}{s-1} - \hat{B}_1(r(p)) \left(\frac{N}{p}\right)^{-s} - \frac{s}{2} \hat{B}_2(r(p)) \left(\frac{N}{p}\right)^{-s-1} \\ &\quad - \frac{s(s+1)}{6} \hat{B}_3(r(p)) \left(\frac{N}{p}\right)^{-s-2} - \frac{s(s+1)(s+2)}{24} \hat{B}_4(r(p)) \left(\frac{N}{p}\right)^{-s-3} \\ &\quad - \frac{s(s+1)(s+2)(s+3)}{120} \hat{B}_5(r(p)) \left(\frac{N}{p}\right)^{-s-4} \\ &\quad + \frac{s(s+1)(s+2)(s+3)(s+4)}{120} \int_{r(p)^+}^{\infty} \hat{B}_5(x) x^{-s-5} dx. \end{aligned}$$

Since this expression remains valid for $\sigma = 1$, we set $s = 1 + 17\tau$ and obtain the desired result.

To bound the $C(p)$ and $D(p)$ in the previous lemma, we need

Lemma 3.5. If $N \geq 549798$ and $p \leq N^{19/20}$, then

$$\frac{p}{N} \left(-\frac{1}{2} - \frac{p}{12N} + \frac{p^3(1-\tau^2)}{120N^3} \right) \leq C(p) \leq \frac{p}{N} \left(\frac{1}{2} - \frac{p}{12N} + \frac{p^3(1-\tau^2)}{120N^3} \right)$$

and

$$-\frac{p^2\tau}{N^2} \left(\frac{1}{12} + \frac{p}{4N} \right) \leq D(p) \leq \frac{p^2\tau}{2N^2}.$$

Proof. We have $t \leq 0.4753755$ and $\frac{p}{N} \leq 0.5164054$. Replace $\{r(p)\}$ by x in the definition of $C(p)$. We have

$$\begin{aligned} \frac{\partial C(p)}{\partial x} &= \frac{p}{N} \left(\frac{-p^4(24-35\tau^2+\tau^4)}{24N^4} x^4 + \left(\frac{p^4(24-35\tau^2+\tau^4)}{12N^4} - \frac{p^3(1-\tau^2)}{N^3} \right) x^3 \right. \\ &\quad + \left(\frac{-p^4(24-35\tau^2+\tau^4)}{24N^4} + \frac{3p^3(1-\tau^2)}{2N^3} - \frac{p^2(2-\tau^2)}{2N^2} \right) x^2 \\ &\quad + \left(\frac{-p^3(1-\tau^2)}{2N^3} + \frac{p^2(2-\tau^2)}{2N^2} - \frac{p}{N} \right) x + \left(\frac{p^4(24-35\tau^2+\tau^4)}{720N^4} - \frac{p^2(2-\tau^2)}{12N^2} \right. \\ &\quad \left. \left. + \frac{p}{2N} - 1 \right) \right). \end{aligned}$$

Since this is < 0 for $0 \leq x < 1$, we have our desired result for $C(p)$. In

the case of $D(p)$, we replace $\{r(p)\}$ by x and observe that our desired result

is true since $x^5 \leq x^4 \leq x^3 \leq x^2 \leq x$ for $0 \leq x \leq 1$.

We now prove the theorem. Write

$$F(1+i\tau) = \zeta_N(1+i\tau) - N^{1/2} \sum_{p \leq N}^{19/20} p^{-1-i\tau} h(p, 1+i\tau).$$

Using Lemma 3.3 and Lemma 3.4, we obtain

$$\begin{aligned} (3) \quad F(1+i\tau) &= \left(.575 + \frac{\tau^2}{120} \right) + \left(\frac{1}{2N} - \frac{1}{12N^2} + \frac{(1-\tau^2)}{120N^4} \right) + i \left(\frac{49\tau + \tau^3}{720} - \frac{\tau}{12N^2} + \frac{11\tau - \tau^3}{720N^4} \right) \\ &+ E(1, 1+i\tau) - E(N, 1+i\tau) - N^{1/2} \sum_{p \leq N}^{19/20} p^{-1-i\tau} \left(.575 + \frac{\tau^2}{120} \right) \\ &+ i \left(\frac{-1}{\tau} + \frac{49\tau + \tau^3}{720} \right) + E(1, 1+i\tau) + \left(\frac{-1}{i\tau} + C(p) + iD(p) \right) p^{i\tau} - E(r(p)^+, 1+i\tau). \end{aligned}$$

We need to evaluate several summations over primes in this expression.

Using Lebesgue-Stieltjes integration, we obtain

$$\begin{aligned} (4) \quad N^{1/2} \sum_{p \leq N}^{19/20} p^{-1-i\tau} &= \int_a^{b^+} \frac{u^{-1-i\tau} d\theta(u)}{\log u} \\ &= -0.1466715 + 0.4257265i \\ &+ \frac{1.0011126\epsilon(b)}{\log N} + \frac{0.325281\epsilon(b)i}{\log N} \\ &+ \frac{2\epsilon(a)}{\log N} + I+J+ K-iK-iL+i\tau I+2 \cdot 10^{-7} \theta. \end{aligned}$$

Similarly, we have

(5)

$$N^{1/2} \sum_{19/20 < p \leq N} p^{-1} = 0.641854 + \frac{\epsilon(b)}{\log b} - \frac{\epsilon(a)}{\log a} + \int_a^{b^+} \frac{\epsilon(u)(1 + \frac{1}{\log u}) du}{u \log u} + 10^{-6} \theta,$$

(6)

$$N^{1/2} \sum_{19/20 < p \leq N} 1 = \int_a^b \frac{du}{\log u} + \frac{T(b)}{\log b} - \frac{T(a)}{\log a} + \int_a^{b^+} \frac{\epsilon(u) du}{\log^2 u}$$

and

(7)

$$N^{1/2} \sum_{19/20 < p \leq N} p = \int_a^b \frac{u du}{\log u} + \frac{bT(b)}{\log b} - \frac{aT(a)}{\log a} + \int_a^{b^+} \frac{T(u)(\frac{1}{\log u} - 1) du}{\log u}.$$

Using Lemma 3.5, (6) and (7), we obtain

(8)

$$N^{1/2} \sum_{19/20 < p \leq N} p^{-1} C(p) \leq \frac{1}{2N} \left(\int_a^b \frac{du}{\log u} + \frac{T(b)}{\log b} - \frac{T(a)}{\log a} + \int_a^{b^+} \frac{\epsilon(u) du}{\log^2 u} \right)$$

and

(9)

$$N^{1/2} \sum_{19/20 < p \leq N} p^{-1} D(p) \leq \frac{\tau}{2N^2} \left(\int_a^b \frac{u du}{\log u} + \frac{bT(b)}{\log b} - \frac{aT(a)}{\log a} - \int_a^{b^+} \frac{T(u)(1 - \frac{1}{\log u}) du}{\log u} \right).$$

We note that $\operatorname{Re} F(1+i\tau)$ and $\operatorname{Im} F(1+i\tau)$ are both negative, so that

$|F(1+i\tau)|$ is maximized when $\sum_{N^{1/2} < p \leq N^{19/20}} p^{-1} C(p)$ and $\sum_{N^{1/2} < p \leq N^{19/20}} p^{-1} D(p)$ are maximized. Combining (3), (4), (5), (8), and (9), we complete the proof of the theorem.

3. A lower bound for $G(1+i\tau)$. In this section, we prove

Theorem 3.6. Let

$$P = \int_{a^+}^{b^+} \frac{\epsilon(u) \cos \frac{\tau}{2} \log u \, du}{u \log u},$$

$$Q = \int_{a^+}^{b^+} \frac{\epsilon(u) \sin \frac{\tau}{2} \log u \, du}{u \log u} \quad \text{and}$$

$$R = \int_{a^+}^{b^+} \frac{\epsilon(u) \sin \frac{\tau}{2} \log u \, du}{u \log^2 u}.$$

If $N \geq 549798$, then

$$G(1+i\tau) \geq 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a)$$

$$-P + \frac{\log N}{\pi} Q + \frac{\log N}{\pi} R - \sum_{N^{1/2} < p \leq N^{19/20}} p^{-1} |E(1, 1+i\tau)$$

$$-E(r(p), 1+i\tau)| + 5 \cdot 10^{-7} (\log N)\theta.$$

We need a lower bound for $|h(p, 1+i\tau)|$. Using Lemma 3.4, we obtain

(10)

$$\begin{aligned}
|h(1, 1+i\tau)| &\geq \left(\frac{2}{\tau} (1-\cos \tau \log p) + \left(0.575 + \frac{\tau^2}{120} \right)^2 \right. \\
&\quad + \left(\frac{49\tau + \tau^3}{720} \right)^2 - \frac{49 + \tau^2}{360} + C(p)^2 + D(p)^2 + \frac{2D(p)}{\tau} \\
&\quad + \sin \tau \log p \left(\frac{-1}{\tau} \left(1.15 + \frac{\tau^2}{60} \right) + C(p) \left(\frac{49\tau + \tau^3}{360} \right) \right. \\
&\quad - D(p) \left(1.15 + \frac{\tau^2}{60} \right) - \frac{2C(p)}{\tau} \left. \right) + \cos \tau \log p \left(\frac{49 + \tau^2}{360} \right. \\
&\quad \left. \left. + C(p) \left(1.15 + \frac{\tau^2}{60} \right) + D(p) \left(\frac{49\tau + \tau^3}{360} \right) - \frac{2D(p)}{\tau} \right) \right)^{1/2} \\
&\quad - |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)|.
\end{aligned}$$

We simplify this result in

Lemma 3.7. If $N \geq 549798$ and $p \leq N^{19/20}$, then

$$|h(p, 1+i\tau)| \geq \left(\frac{2}{\tau} (1-\cos \tau \log p) \right)^{1/2} - |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)|.$$

Proof. We first assume that $N^{1/2} < p \leq N^{3/4}$. We have $\cos \tau \log p \leq 0$, $\sin \tau \log p \leq 0$, $\frac{p}{N} \leq 0.0367239$, $\frac{p^2}{N^2} \leq 0.001349$ and $\tau \leq 0.4753755$.

From Lemma 3.5, we have

$$-\frac{p}{N} \left(\frac{1}{2} + \frac{p}{12N} \right) \leq C(p) \leq \frac{p}{2N}$$

and

$$-\frac{p^2 \tau}{N^2} \left(\frac{1}{12} + \frac{p}{4N} \right) \leq D(p) \leq \frac{p^2 \tau}{2N^2}.$$

Thus, the coefficient of $\sin \tau \log p$ in (10) is negative and the right-hand side of (10) is minimized when this coefficient is replaced by 0 and we choose $C(p) = \frac{-p}{N} \left(\frac{1}{2} + \frac{p}{12N} \right)$ and $D(p) = \frac{-p^2 \tau}{N^2} \left(\frac{1}{12} + \frac{p}{4N} \right)$. The right-hand side of (10) is obviously still

$$\geq \left(\frac{2}{\tau} (1 - \cos \tau \log p) \right)^{1/2} - |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)|,$$

which proves the lemma for $N^{1/2} < p \leq N^{3/4}$.

We now assume that $N^{3/4} < p \leq N^{19/20}$. We have $0 \leq \cos \tau \log p \leq 0.951057$, $-0.309017 \leq \sin \tau \log p \leq 0$, $\frac{p}{N} \leq 0.5164043$, $\frac{p^2}{N^2} \leq 0.266674$, and $\tau \leq 0.4753755$. As before, the coefficient of $\sin \tau \log p$ in (10) is negative. Thus, the right-hand side of (10) is minimized when this coefficient is replaced by 0 and we choose $\cos \tau \log p = 0.951057$, $C(p) = \frac{-p}{N} \left(\frac{1}{2} + \frac{p}{12N} \right)$ and $D(p) = \frac{-p^2 \tau}{N^2} \left(\frac{1}{12} + \frac{p}{4N} \right)$. The right-hand side of (10) is obviously still

$$\geq \left(\frac{2}{\tau} (1 - \cos \tau \log p) \right)^{1/2} - |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)|,$$

which completes the proof of the lemma.

Using this lemma, we have

$$\begin{aligned}
G(1+i\tau) &= N^{1/2} \sum_{p \leq N} 19/20 p^{-1} |h(p, 1+i\tau)| \\
&\geq N^{1/2} \sum_{p \leq N} 19/20 p^{-1} \left(\frac{2}{\tau} (1 - \cos \tau \log p) \right)^{1/2} \\
&\quad - N^{1/2} \sum_{p \leq N} 19/20 p^{-1} |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)|.
\end{aligned}$$

Using Lebesgue-Stieltjes integration, we obtain

$$\begin{aligned}
N^{1/2} \sum_{p \leq N} 19/20 p^{-1} \left(\frac{2}{\tau} (1 - \cos \tau \log p) \right)^{1/2} &= \frac{2}{\tau} N^{1/2} \sum_{p \leq N} 19/20 p^{-1} \sin \frac{\tau}{2} \log p \\
&= \frac{2}{\tau} \int_a^b \frac{\sin \frac{\tau}{2} \log u \, d\theta(u)}{u \log u} \\
&= 0.1518588 \log N + 0.0524155\epsilon(b) \\
&\quad - 0.6366197\epsilon(a) \\
&\quad - P + \frac{\log N}{\pi} Q + \frac{\log N}{\pi} R \\
&\quad + 5 \cdot 10^{-7} (\log N)\theta,
\end{aligned}$$

which completes the proof of the theorem.

4. The zeros of $\zeta_N(s)$. In this section we prove

Theorem 3.8. For every $N \geq 30$, $\zeta_N(s)$ has a zero with $\sigma > 1$.

We first establish

Lemma 3.9. If $N \geq 549798$, then

$$|E(1, 1+i\tau) - E(N, 1+i\tau)| \leq 0.0040729$$

and

$$N^{1/2} \sum_{N^{19/20} < p \leq N} p^{-1} |E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)| \leq 0.002627.$$

Proof. We have

$$|E(1, 1+i\tau) - E(N, 1+i\tau)| \leq \left| \frac{(1+i\tau)(2+i\tau)(3+i\tau)(4+i\tau)(5+i\tau)}{120} \right| \int_1^N |\hat{B}_5(x)| x^{-6} dx.$$

Since $\tau \leq 0.4753755$, the first factor above is ≤ 1.1656289 . In addition, the minimum of $\hat{B}_5(x)$ occurs at 0.2403353 ± 10^{-7} and the maximum of $\hat{B}_5(x)$ occurs at 0.7596647 ± 10^{-7} . Thus, $|\hat{B}_5(x)| \leq 0.0244588$. Using these results, we obtain

$$|E(1, 1+i\tau) - E(N, 1+i\tau)| \leq 0.0040729.$$

Similarly, we have

$$|E(1, 1+i\tau) - E(r(p)^+, 1+i\tau)| \leq 0.0040729.$$

Using the formula for $N^{1/2} \sum_{N^{19/20} < p \leq N} p^{-1}$ in (8), we obtain

$$N^{1/2} \sum_{p \leq N} p^{-1} \left| E(1, 1+i\tau) - E(r(p)^+, 1+i\tau) \right| \leq 0.002627.$$

This completes the proof of the lemma.

Combining the results of Theorem 3.1, Theorem 3.6, and Lemma 3.8, we see that $G(1+i\tau) > |F(1+i\tau)|$ for a given $N > 549798$ if

$$D_0 = G_0 + \frac{\log N}{\pi} Q - (F_{r,0}^2 + F_{i,0}^2)^{1/2} > 0.0093269 + 10^{-6}(\log N),$$

where

$$G_0 = 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a) \\ - P + \frac{\log N}{\pi} R.$$

We first examine the case when $N \geq 10^{11}$.

Lemma 3.10. If $N \geq 10^{11}$, then

$$G(1+i\tau) > |F(1+i\tau)|.$$

Proof. We have $N^{1/2} \geq 316226$, $\log N \geq 25.32835$ and $t \leq 0.2480684$.

Schoenfeld [24, p.360] has shown that $\theta(u) < 1.000081$, for all $u > 0$.

Hence, $\epsilon(u) < 0.000081$ for all $u > 0$. Schoenfeld [24, Corollary 2*] has determined values of c and d such that if $d \leq u$, then

$$u - u/(c \log u) < \theta(u).$$

Thus,

(11)

$$-\frac{1}{c \log u} < \epsilon(u).$$

In particular, (11) is true for $d = 315437$ and $c = 29$. Hence, for $a \leq u \leq b$

$$-0.00273 < \epsilon(u).$$

From Schoenfeld [24, 5.66*], we have, for $u > e^{22}$,

$$|\theta(u) - u| < 0.0077629 u / \log u.$$

Hence, for $u > e^{22}$

$$-0.0077629 / \log u < \epsilon(u).$$

Therefore,

$$-0.000030649 < \epsilon(b).$$

Combining the above results, we obtain

$$-0.00273 < \epsilon(u) < 0.000081 \quad (a \leq u \leq b)$$

and

$$-0.000030649 < \epsilon(b) < 0.000081.$$

Thus,

$$0.0524155\epsilon(b) > -0.000017$$

and

$$-0.6366197\epsilon(a) > -0.000052.$$

In addition, we have

$$-P = -\int_{\left(\frac{\pi}{2}\right)^+}^{\left(\frac{19\pi}{20}\right)^+} \frac{\epsilon(e^{2v/\tau}) \cos v \, dv}{v} > -0.00273 \int_{\left(\frac{\pi}{2}\right)^+}^{\left(\frac{19\pi}{20}\right)^+} \frac{\cos v \, dv}{v} > -0.001726.$$

In a similar manner, we obtain

$$\frac{\log N}{\pi} Q \geq -0.0007593 \log N$$

and

$$\frac{\log N}{\pi} R \geq -0.0023854.$$

Combining these results, we have

$$G_0 + \frac{\log N}{\pi} Q \geq 0.1510995 \log N - 0.0041804.$$

To bound $F_{r,0}$ and $F_{i,0}$, we need bounds on I, J, K, and L. Using the same method as we used to bound P, we obtain

$$|I| \leq 0.0019875,$$

$$|J| \leq 0.0004931,$$

$$-0.0000345 \leq K \leq 0.0021287$$

and

$$-0.0000085 \leq L \leq 0.0005217.$$

Combining these results with those previously obtained, we have

$$F_{r,0} \geq -0.0677634 \log N + 0.6450713$$

and

$$F_{i,0} \geq -0.125909 \log N - .2483715.$$

Hence,

$$\begin{aligned} D_0 &\geq 0.0091103 \log N - 0.0041804 \\ &\geq .2012154 + 10^{-6} \log N. \end{aligned}$$

Since $D_0 > .0093269 + 10^{-6}(\log N)$, we have $G(1+ir) > |F(1+ir)|$ for every $N \geq 10^{11}$, which completes the proof of the lemma.

Next, we examine the case when $549798 \leq N \leq 10^{11}$. In this range, we use Schoenfeld's result [24, p. 360] that $\theta(u) < u$ for $0 < u < 10^{11}$. This result implies that $\epsilon(u) < 0$ for $0 < u \leq 10^{11}$.

Treat

$$H_0 = (F_{r,0}^2 + F_{i,0}^2)^{1/2}$$

as a function of K and L. Since $F_{r,0}$ and $F_{i,0}$ are negative for $N \geq 549798$, it is obvious that H_0 is maximized when K and L are minimized. But $K \geq 0$ and $L \geq 0$, so $H_0 \leq H_1$, where

$$H_1 = (F_{r,1}^2 + F_{i,1}^2)^{1/2}.$$

and $F_{r,1}$ and $F_{i,1}$ are defined by replacing K and L by 0 in the definitions of $F_{r,0}$ and $F_{i,0}$.

Treat H_1 as a function of I. Let

$$H_2 = (F_{r,2}^2 + F_{i,2}^2)^{1/2},$$

where $F_{r,2}$ and $F_{i,2}$ are defined by replacing I by 0 in the definitions of $F_{r,1}$ and $F_{i,1}$. We have

$$\begin{aligned} \frac{\partial H_1}{\partial I} &= (F_{r,1}^2 + F_{i,1}^2)^{-\frac{1}{2}} \left(F_{r,1} \left(-1 - \left(0.575 + \frac{\tau^2}{120} \right) + \tau \left(\frac{49\tau + \tau^3}{720} \right) \right) \right. \\ &\quad \left. + F_{i,1} \left(\frac{1}{\tau} - \tau \left(0.575 + \frac{\tau^2}{120} \right) - \left(\frac{49\tau + \tau^3}{720} \right) \right) \right) \end{aligned}$$

$$\leq 0.$$

Thus, H_1 is a decreasing function of I . Hence, if $I > 0$, then $H_1 \leq H_2$.

If $I \leq 0$, then

$$d(I) = H_2 - \frac{1}{\tau} I - H_1$$

is a decreasing function of I which is 0 when $I = 0$. Thus, $H_1 \leq H_2 - \frac{1}{\tau} I$

if $I \leq 0$. We now examine D_0 . If $I \geq 0$, then

$$(12) \quad D_0 \geq G_0 + \frac{\log N}{\pi} Q - H_2.$$

If $I \leq 0$, then

$$D_0 \geq G_0 + \frac{\log N}{\pi} Q - H_2 + \frac{1}{\tau} I.$$

But

$$\frac{\log N}{2\pi} Q + \frac{1}{\tau} I = \frac{1}{\tau} \int_a^+ b^+ \frac{\epsilon(u) (\sin \frac{\tau}{2} \log u + \cos \tau \log u) du}{u \log u}$$

and

$$\sin \frac{\tau}{2} \log u + \cos \tau \log u \geq 0$$

for $a \leq u \leq b$. Thus, if we let

$$m(j) = \min_{x_{j-1} \leq u \leq x_j} \epsilon(u),$$

$$Q_1 = \sum_{j=1}^n m(j) \int_{x_{j-1}}^{x_j} \frac{\sin \frac{\tau}{2} \log u}{u \log u} du$$

and

$$I_1 = \sum_{j=1}^n m(j) \int_{x_{j-1}}^{x_j} \frac{\cos \tau \log u \, du}{u \log u}$$

for $a = x_0 < x_1 < x_2 < \dots < x_n = b$, then

$$(13) \quad D_0 \geq G_0 + \frac{\log N}{2\pi} Q - H_2 + \frac{1}{\tau} (Q_1 - I_1).$$

But $m(j)$ is an increasing function of j , so $I_1 \geq 0$. Hence, for

$549798 \leq N \leq 10^{11}$, we have

$$D_0 \geq G_0 + \frac{\log N}{2\pi} Q - H_2 + \frac{1}{\tau} Q_1.$$

Treat H_2 as a function of J , and let

$$H_3 = (F_{r,3}^2 + F_{i,3}^2)^{1/2},$$

where $F_{r,3}$ and $F_{i,3}$ are defined by replacing J by 0 in the definitions of $F_{r,2}$ and $F_{i,2}$. Using the same methods we used above, we obtain, for $549798 \leq N \leq 10^{11}$,

$$D_0 \geq G_0 - H_3 + \frac{\log N}{\pi} Q_1.$$

Let

$$P_1 = \sum_{j=1}^n m(j) \int_{x_{j-1}}^{x_j} \frac{\cos \frac{\tau}{2} \log u \, du}{u \log u}$$

and

$$R_1 = \sum_{j=1}^n m(j) \int_{x_{j-1}}^{x_j} \frac{\sin \frac{\tau}{2} \log u \, du}{u \log^2 u}.$$

Then

$$(14) \quad D_0 \geq D_1 = G_1 - H_3,$$

where

$$G_1 = 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a) \\ - P_1 + \frac{\log N}{\pi} Q_1 + \frac{\log N}{\pi} R_1.$$

Let D_3 be defined by replacing $\epsilon(a)$ by ϵ^* in D_1 , where

$$0 \geq \epsilon^* \geq N_1^{1/2} \max_{N_1 \leq u \leq N_2} \epsilon(u).$$

Let D_2 be defined by replacing $\epsilon(a)$ by v in D_1 . We have

$$\frac{\partial D_2}{\partial v} = -0.636619 - (F_{r,3}^2 + F_{i,3}^2)^{-1/2} \left(F_{r,3} \left(-0.05177 - \left(0.575 + \frac{\tau^2}{120} \right) \left(\frac{2}{\log N} \right) \right. \right. \\ \left. \left. - \frac{a}{2N \log a} \right) + F_{i,3} \left(0.6366197 - \left(\frac{49\tau + \tau^3}{720} \right) \left(\frac{2}{\log N} \right) + \frac{\tau a^2}{2n} \right) \right) \leq 0.$$

Thus, $D_1 \geq D_3$ for $549798 \leq N_1 \leq N \leq N_2 \leq 10^{11}$, where

$$D_3 = 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197 \epsilon^* \\ - P_1 + \frac{\log N}{\pi} Q_1 + \frac{\log N}{\pi} R_1 - \left(-0.0677564 \log N - 0.05177 \epsilon^* \right) \\ + \left(0.575 + \frac{\tau^2}{120} \right) \left(1.1466715 - \left(\frac{1.0011126\epsilon(b)}{\log N} - \frac{2\epsilon^*}{\log N} \right) \right)$$

$$\begin{aligned}
& + \left(\frac{49\tau + \tau^3}{720} \right) \left(0.4257265 + \frac{0.325281\epsilon(b)}{\log N} \right) - \frac{1}{2N} \left(\int_a^b \frac{du}{\log u} + \frac{b\epsilon(b)}{\log b} - \frac{a\epsilon^*}{\log a} \right)^2 \\
& + \left(-0.1254977 \log N + 0.6366197 \epsilon^* - 0.0081994 \epsilon(b) \right. \\
& + \left. \left(0.575 + \frac{\tau^2}{120} \right) \left(-0.4257265 - \frac{0.325281\epsilon(b)}{\log N} \right) \right. \\
& + \left. \left(\frac{49\tau + \tau^3}{720} \right) \left(1.1466715 - \frac{1.0011126\epsilon(b)}{\log N} - \frac{2\epsilon^*}{\log N} \right) \right. \\
& \left. - \frac{\tau}{2N^2} \left(\int_a^b \frac{udu}{\log u} + \frac{b^2 \epsilon(b)}{\log b} - \frac{a^2 \epsilon^*}{\log a} - \int_{a^+}^{b^+} \frac{\epsilon(u)u(1 - \frac{1}{\log u})du}{\log u} \right) \right)^2 \Big)^{1/2}.
\end{aligned}$$

Replacing τ in D_3 by W , we have $\frac{\partial D_3}{\partial W} \geq 0$. Thus, $D_0 \geq D_4$ for $549798 \leq N_1 \leq N \leq N_2 \leq 10^{11}$, where D_4 is defined by replacing τ by τ_* in D_3 , with $0 \leq \tau_* \leq \frac{2\pi}{\log N_2}$.

Hence, for $549798 \leq N_1 \leq N < N_2 \leq 10^{11}$, we have

$$(15) \quad D_0 > D_4 = G_4 - (F_{r,4}^2 + F_{i,4}^2)^{1/2}$$

where

$$\begin{aligned}
G_4 &= 0.1518588 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon^* - P_1 \\
&+ \frac{\log N}{\pi} Q_1 + \frac{\log N}{\pi} R_1, \\
F_{r,4} &= -0.0677564 \log N - 0.05177\epsilon^* + \left(0.575 + \frac{\tau_*^2}{120} \right) \\
&\left(1.1466715 - \frac{1.0011126\epsilon(b)}{\log N} - \frac{2\epsilon^*}{\log N} + \left(\frac{49\tau_* + \tau_*^3}{720} \right) \left(0.4257265 \right. \right. \\
&\left. \left. + \frac{0.325281\epsilon(b)}{\log N} \right) - \frac{1}{2N} \left(\int_a^b \frac{du}{\log u} + \frac{b\epsilon(b)}{\log b} - \frac{a\epsilon^*}{\log a} \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
F_{i,4} = & -0.1254977 \log N + 0.6366197\epsilon^* - 0.0081994\epsilon(b) \\
& + \left(0.575 + \frac{\tau_*^2}{120}\right) \left(-0.4257265 - \frac{0.325281\epsilon(b)}{\log N}\right) \\
& + \left(\frac{49\tau_* + \tau_*^3}{720}\right) \left(1.466715 - \frac{1.0011126\epsilon(b)}{\log N} - \frac{2\epsilon^*}{\log N}\right) \\
& - \frac{\tau_*}{2N^2} \left(\int_a^b \frac{u du}{\log u} + \frac{b^2 \epsilon(b)}{\log b} - \frac{a^2 \epsilon^*}{\log a} - \int_a^{b^+} \frac{\epsilon(u)u(1 - \frac{1}{\log u}) du}{\log u} \right).
\end{aligned}$$

We use the following upper bounds when calculating D_4 :

$$\begin{aligned}
\frac{1}{2N} \int_a^b \frac{du}{\log u} \leq & \frac{1}{2 \log N} \left(\frac{1.1111112}{N^{1/20}} + \frac{0.1388889}{N^{1/10}} + \frac{0.1785714}{N^{1/5}} \right. \\
& \left. + \frac{0.2380952}{N^{3/10}} + \frac{0.3333334}{N^{2/5}} - \frac{2}{N^{1/2}} \right) = g(N)
\end{aligned}$$

and

$$\frac{1}{2N^2} \int_a^b \frac{u du}{\log u} \leq \frac{1}{N^{1/20}} g(N).$$

The x_i used in computing P_1 , Q_1 , and R_1 are

$$\begin{aligned}
x_0 &= a = N^{1/2}, x_1 = N^{1.58/\pi}, x_2 = N^{1.59/\pi}, x_3 = N^{1.60/\pi} \\
x_4 &= N^{1.65/\pi}, x_5 = N^{1.70/\pi}, x_6 = N^{1.75/\pi}, x_7 = N^{1.8/\pi} \\
x_8 &= N^{1.85/\pi}, x_9 = N^{1.90/\pi}, x_{10} = N^{1.95/\pi}, x_{11} = N^{2/\pi} \\
x_{12} &= N^{2.1/\pi}, x_{13} = N^{2.2/\pi}, x_{14} = N^{2.3/\pi}, x_{15} = N^{2.4/\pi} \\
x_{16} &= N^{2.5/\pi}, x_{17} = b = N^{19/20}.
\end{aligned}$$

We now prove

Lemma 3.11. If $2193361 \leq N \leq 10^{11}$, then

$$G(1+i\tau) > |F(1+i\tau)|.$$

Proof. We have $-P_1 \geq -0.0058212$, $\frac{\log N}{\pi} Q_1 \geq -0.0034335 \log N$ and $\frac{\log N}{\pi} R_1 \geq -0.006867$. We choose $\epsilon^* = 0$. From Schoenfeld [24, 5.2*], we have $0.998697 u < \theta(u)$ for $u \geq 1155901$, so that $-0.001303 \leq \epsilon(b) \leq 0$. Hence, $G_1 \geq 0.1484253 \log N - 0.0127565$.

In evaluating the rest of D_4 , we choose $\tau_* = 0$. Since $g(N) \leq 0.0164211$, we have $D_4 \geq 0.0058052 \log N - 0.0362244 \geq 0.0485223 + 10^{-6} \log N$, which completes the proof of the lemma.

Continuing, we have

Lemma 3.12. If $549798 \leq N \leq 2193361$, then

$$G(1+i\tau) > |F(1+i\tau)|.$$

Proof. Schoenfeld [24, Corollary 2*] has shown that (11) holds with $d = 8623$ and $c = 6$ as well as with $d = 486377$ and $c = 37$. Using (11), we obtain

$$-0.0178032 \leq \epsilon(u) \quad (u \geq 11633)$$

and

$$-0.0020448 \leq \epsilon(u) \quad (u \geq 549798).$$

Supplementing these results with our own calculations of minimum values of $\epsilon(u)$ for $1 \leq u \leq 11633$, we obtain

$$\epsilon(u) \leq -0.01872009 \quad (549798)^{1/2} \leq u \leq (2193361)^{1/2}$$

and

$$-0.0020448 \leq \epsilon(u) \leq 0 \quad 549798 \leq u \leq 2193361.$$

Hence, we choose $\epsilon^* = -0.01872009$ and we have $-0.0020448 \leq \epsilon(b) \leq 0$.

We have $-P_1 \geq -0.0068286$, $\frac{\log N}{\pi} Q_1 \geq -0.004764 \log N$ and $\frac{\log N}{\pi} R_1 \geq -0.009528$, so that $G_4 \geq 0.1470948 \log N - 0.0052253$.

In evaluating the rest of D_4 , we choose $\tau_* = 0.4303274$. Since $g(N) \leq 0.0237707$, we have

$$D_4 \geq 0.0044747 \log N - 0.0221691 \geq 0.0369611 + 10^{-6} \log N,$$

which completes the proof of Lemma 3.12.

We have now shown that $G(1+i\tau) > |F(1+i\tau)|$ for $N \geq 549798$. To show that $\zeta_N(s)$ has a zero with $\sigma > 1$ for $N \geq 549798$, we need

Lemma 3.13. If $N \geq 549798$, then there is no prime q , $N^{1/2} < q \leq N^{19/20}$, such that

$$q^{-1} \left| \sum_{m \leq N/q} m^{-1-i\tau} \right| > N^{1/2} \sum_{\substack{p \leq N^{19/20} \\ p \neq q}}^{-1} \left| \sum_{m \leq N/p} m^{-1-i\tau} \right|.$$

Proof. It is sufficient to show that

$$q^{-1} |h(q, 1+i\tau)| \leq \frac{1}{2} G(1+i\tau)$$

for all primes q , $N^{1/2} < q \leq N^{19/20}$. Using Theorem 3.6 and Lemma 3.9, we obtain

$$\begin{aligned} G(1+i\tau) &\geq 0.1518578 \log N + 0.0524155\epsilon(b) - 0.6366197\epsilon(a) \\ &\quad - P + \frac{\log N}{\pi} Q + \frac{\log N}{\pi} R - 0.002627 \\ &\geq 0.1470948 \log N - 0.0198213. \end{aligned}$$

Thus, it is sufficient to show that for all primes q , $N^{1/2} < q \leq N^{19/20}$,

$$q^{-1} |h(q, 1+i\tau)| \leq 0.0735474 \log N - 0.0099106.$$

Using Lemma 3.4, Lemma 3.5, and Lemma 3.9, we obtain

$$\begin{aligned}
|h(q, 1+i\tau)| &\leq \left(0.575 + \frac{\tau^2}{120}\right) + \frac{\log N}{2\pi} + \frac{p}{N} \left(\frac{1}{2} + \frac{p}{12N}\right) + \frac{p^2 \tau}{2N^2} \\
&\quad + \frac{\log N}{\pi} + \frac{49\tau + \tau^3}{720} + \frac{p^2 \tau}{2N^2} + \frac{p}{N} \left(\frac{1}{2} + \frac{p}{12N}\right) + 0.002627 + 10^{-6} \log N \\
&\leq 0.4774657 \log N + 3.
\end{aligned}$$

Hence,

$$q^{-1} |h(q, 1+i\tau)| \leq 0.07354741 \log N - 0.0099106,$$

which completes the proof of the lemma.

Combining the results of Lemma 3.10, Lemma 3.11, Lemma 3.12, and Lemma 3.13, we see that

$\zeta_N(s)$ has a zero with $\sigma > 1$ for every $N \geq 549798$.

Finally we have

Lemma 3.14. If $30 < N < 549798$, then

$\zeta_N(s)$ has a zero with $\sigma > 1$.

Proof. Using Bohr's result as before, we have a zero of $\zeta_N(s)$ with $\sigma > 1$ if for some τ

$$(16) \quad N^{1/2} \sum_{\substack{p \leq N \\ p \leq N}}^{-1} \left| \sum_{m \leq N/p} m^{-1-i\tau} \right| > \left| \sum_{\substack{n=1 \\ p/n \rightarrow p \leq N^{1/2}}}^N n^{-1-i\tau} \right|$$

and there is no prime q , $N^{1/2} < q \leq N$, such that

$$(17) \quad q^{-1} \left| \sum_{m \leq N/q} m^{-1-i\tau} \right| > \sum_{\substack{N^{1/2} < p \leq N \\ p \neq q}} p^{-1} \left| \sum_{m \leq N/p} m^{-1-i\tau} \right|.$$

Using a computer program (Appendix C), we showed that the above inequalities are true for $30 < N < 549798$. Thus, $\zeta_N(s)$ has a zero with $\sigma > 1$ for $30 < N < 549798$.

This completes the proof of the theorem.

APPENDICES

APPENDIX A

The following programs are used to obtain the results in Chapter 1. Our polynomials are stored as vectors of coefficients, with the first element equal to the position of the highest degree non-zero term, the second element equal to the value of the constant term, and the remaining elements equal to the values of the other coefficients. The program listings follow:

A program to add two polynomials

```

SUBROUTINE ADD (P, Q, R)
  IMPLICIT REAL*8 (A-H, O-Z)
  DIMENSION P (38)
  DIMENSION Q (38)
  DIMENSION R (38)
  MDEGP = P(1)
  MDEGQ = Q(1)
  MDEGR = MDEGP
  IF (MDEGQ .GT. MDEGP) MDEGR = MDEGQ
  R(1) = MDEGR
  DO 10 I = 2, 38
  R(I) = P(I) + Q(I)
10 CONTINUE
  RETURN
  END

```

A program to multiply a polynomial by a scalar

```

SUBROUTINE SMULT (P, S, R)
  IMPLICIT REAL*8 (A-H, O-Z)
  DIMENSION P(38)
  DIMENSION R(38)
  MDEG = P(1)
  DO 10 I = 2, 38
  R(I) = P(I) * S
10 CONTINUE
  R(1) = MDEG
  RETURN
  END

```

A program to multiply two polynomials

```

SUBROUTINE MULT (P, Q, R)
  IMPLICIT REAL * 8 (A-H, O-Z)
  DIMENSION P(38)
  DIMENSION Q(38)
  DIMENSION R(38)
  MDEGP = P(1)
  MDEGQ = Q(1)
  R(1) = MDEGP + MDEGQ - 2
  DO 5 K = 2, 38
    R(K) = 0.0D0
5  CONTINUE
  DO 20 I = 2, MDEGP
    DO 10 J = 2, MDEGQ
      IF (P(I) .EQ. 0.0D0) GO TO 10
      IF (Q(J) .EQ. 0.0D0) GO TO 10
      R(I + J - 2) = R(I + J - 2) + (P(I) * Q(J))
10  CONTINUE
20  CONTINUE
  RETURN
  END

```

A program to evaluate polynomials

```

SUBROUTINE EVAL (P, X, Y)
  IMPLICIT REAL * 8 (A-H, O-Z)
  DIMENSION P(300)
  MDEGP = P(1)
  Y = P(MDEGP)
  MSTOP = MDEGP - 2
  IF (MSTOP .LT. 1) GO TO 10
  DO 10 I = 1, MSTOP
    INDEX = MDEGP - I
    Y = P (INDEX) + (X * Y)
10  CONTINUE
  RETURN
  END

```

$$\text{A program to bound } \prod_{p \leq p_0} \frac{2}{\pi} \sqrt{6 - \sqrt{36 - \frac{24\pi p}{B} - \frac{12\pi^2}{B^2}}}$$

```

IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION MP(13)
FPI = 3.14159265358973D0
CONSQ = 2.0D0/FPI
TOTAL = 1.0D0
N = 0
READ (4, 50) T
50  FORMAT (F13.6)
10  READ (5, 100) (MP(I) , I = 1, 13)
100 FORMAT (13I6)
    I = 1
15  IF (MP(I) .EQ. 0) GO TO 20
    P = MP(I)
    BOUNDP = (2.0D0 * T)/FPI
    IF (P .GT. BOUNDP) GO TO 20
    BOUND1 = 36.0D0 - (((24.0D0*FPI*P)/T) + ((12.0D0*FPI*FPI)/(T*T)))
    BOUND2 = 6.0D0 - DSQRT (BOUND1)
    BOUND = CONSQ * DSQRT (BOUND2)
    IF (BOUND .GT. 1.0D0) GO TO 20
    TOTAL = TOTAL * BOUND
    N = N + 1
    WRITE (6, 200) N, TOTAL
    I = I + 1
    IF (I .EQ. 14) GO TO 10
    GO TO 15
20  WRITE (6, 200) N, TOTAL
200 FORMAT (I5, E15.6)
RETURN
END

```

A program to bound $\sum_{p \leq 1010} \frac{1}{\sqrt{p^2-1}}$

```

IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION IP(169)
READ (5,100) (IP(I) , I = 1, 169)
100 FORMAT (13I6)
TOTAL = 0.0
DO 10 I = 1, 169
P = IP(I)
TOTAL = TOTAL + (1.0D0/DSQRT ((P * P) - 1.0D0))
WRITE (6, 200) IP(I), TOTAL
200 FORMAT (I6, E20.6)
10 CONTINUE
RETURN
END

```

A program to compute $\prod_{p \leq 101} \frac{2}{\pi} \left(\frac{\pi}{2} - \sin^{-1} p \sin \left(\sin^{-1} \frac{1}{p} - \frac{\pi}{nh} \right) - \sin^{-1} p \sin \frac{\pi}{2nh} \right)$

```

IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION MP(26)
FPI = 3.14159265358973D0
CONSQ = 2.0D0/FPI
READ (5,100) (MP(I) , I = 1, 26)
100 FORMAT (13I6)
DO 40 IT = 401, 410
T = IT
T = T*.5D0
TOTAL = 1.0D0
DO 30 I = 1, 26
P = MP(I)
BOUNDP = (2.0D0 * T)/FPI
IF (P .GT. BOUNDP) GO TO 20
BOUND1 = DARSIN (P * DSIN(DARSIN(1.0D0/P) - (FPI/T)))
BOUND2 = DARSIN (P * DSIN(FPI/(2.0D0 * T)))
BOUND = 1.0D0 - (BOUND1 + BOUND2) * CONSQ
IF (BOUND .GT. 1.0D0) GO TO 20
TOTAL = TOTAL * BOUND

```

```

30 CONTINUE
20 WRITE (6, 200) TOTAL
200 FORMAT (D24.16)
40 CONTINUE
RETURN
END

```

A program to compute $\prod_{p_0 \leq p \leq p_1} G(p, nh)$ using Lemma 1.15

```

IMPLICIT REAL *8 (A-H, O-Z)
DIMENSION MP(13)
DIMENSION MPRIME (200)
DIMENSION P(160)
DIMENSION TOTAL (400)
N = 0
10 READ (5, 100) (MP(I) , I = 1, 13)
100 FORMAT (13I6)
I = 1
15 IF (MP(I) .EQ. 0) GO TO 20
N = N + 1
MPRIME (N) = MP(I)
I = I + 1
IF (I .EQ. 14) GO TO 10
GO TO 15
20 CONTINUE
DO 40 I = 1, 400
TOTAL(I) = 1.0D0
FI = I
FI = FI *.5D0
CALL COMFPT (43, FI, M)
CALL PREPOL (FI, M, P)
DO 30 J = 1, N
CALL COMFPT (MPRIME(J), FI, IDEG)
P(1) = IDEG + 2
FP = MPRIME (J)
X = 1.0D0/(FP * FP)
CALL PREEVA (P, X, Y)
TOTAL (I) = TOTAL (I) * Y
30 CONTINUE
WRITE (6, 150) TOTAL (I)
150 FORMAT (D24.16)
40 CONTINUE
RETURN
END

```


A program to compute the number of terms of $\sum_{k=0}^N \frac{\prod_{j=0}^{k-1} \left(j^2 - \frac{(nh)^2}{4} \right)}{(k!)^2 p^{2k}}$

necessary to compute G(p, nh) using Lemma 1.15

```

SUBROUTINE COMFPT (IP, T, N)
IMPLICIT REAL*8 (A-H, O-Z)
C0 = -8.0D0 * DLOG (10.0D0)
C1 = DLOG (2.0D0 * 3.14159265358973D0) / 2.0D0
C2 = DEXP (1.0D0)
C3 = T / 2.0D0
K = T / DSQRT (2.0D0)
FP = IP
FK = K
FK1 = FK + 1.0D0
REMAIN = (FK1 * DLOG ((T * C2) / (2.0D0 * FP * FK1))) - (C1 + (DLOG
1(FK1) / 2.0D0))
N = K
IF (REMAIN .EQ. C0) GO TO 90
IF (REMAIN .LT. C0) GO TO 10
N = ((FK1 + (FK1 * DLOG(C3))) - (C0 + C1 + ((FK1 + .5D0) * DLOG
1(FK1)))) / DLOG(FP)
GO TO 90
10 NLOW = (C2 * C3) / FP
FNLOW1 = NLOW + 1
REMAIN = (FNLOW1 * DLOG ((T * C2) / (2.0D0 * FP * FNLOW1))) -
1(C1 + (DLOG(FNLOW1) / 2.0D0))
IF (REMAIN .GT. C0) GO TO 15
N = NLOW
GO TO 90
15 NHIGH = N
DO 40 IBISEC = 1, 40
NDIFF = NHIGH - NLOW
IF (NDIFF .LT. 2) GO TO 50
NMID = (NHIGH + NLOW) / 2
FNMID1 = NMID + 1
REMAIN = (FNMID1 * DLOG ((T * C2) / (2.0D0 * FP * FNMID1))) -
1(C1 + DLOG(FNMID1) / 2.0D0))
IF (REMAIN .GT. C0) GO TO 30
IF (REMAIN .EQ. C0) GO TO 20
NHIGH = NMID
GO TO 40

```

```

20 N = NMID
   GO TO 90
30 NLOW = NMID
40 CONTINUE
50 N = NHIGH
90 RETURN
   END

```

A program to compute $\sum_{k=0}^N \frac{\prod_{j=0}^{k-1} \left(j^2 - \frac{(nh)^2}{4} \right)}{(k!)^2 p^{2k}}$ as a

polynomial in p for given n

```

SUBROUTINE PREPOL (T, N, P)
  IMPLICIT REAL * 8 (A-H, O-Z)
  DIMENSION P(160)
  NPLUS2 = N + 2
  P(1) = NPLUS2
  P(2) = 1.0D0
  P(3) = -(T * T) / (4.0D0)
  DO 10 I = 4, NPLUS2
    FI = I
    P(I) = P(I - 1) * (FI - 3.0D0) * (FI - 3.0D0) / (FI - 2.0D0) *
1(FI - 2.0D0)
    P(I) = P(I) * ((1.0D0) - ((T * T) / (4.0D0 * (FI - 3.0D0) *
1(FI - 3.0D0))))
10 CONTINUE
  NPLUS3 = N + 3
  DO 20 I = NPLUS3, 160
    P(I) = 0.0D0
20 CONTINUE
  RETURN
  END

```

A program to evaluate polynomials

```

SUBROUTINE PREEVA (P, X, Y)
IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION P(160)
DIMENSION PPOS(160)
DIMENSION PNEG(160)
MDEGP = P(1)
PPOS (1) = MDEGP
PNEG(1) = MDEGP
DO 20 I = 2, =DEGP
PPOS(I) = 0.0D0
PNEG(I) = 0.0D0
IF (P(I) .GT. 0.0D0) GO TO 10
PNEG (I) = P(I)
GO TO 20
10 PPOS (I) = P(I)
20 CONTINUE
CALL EVAL (PPOS, X, YPOS)
CALL EVAL (PNEG, X, YNEG)
Y = YPOS + YNEG
RETURN
END

```

A program to compute $\prod_{p_0 \leq p \leq p_1} G(p, nh)$ in multiple precision

using Lemma 1.15

```

IMPLICIT REAL * 8(A-H, O-Z)
INTEGER B, T, R
DIMENSION IP(32, 160)
DIMENSION R(258)
DIMENSION JP(32)
DIMENSION KP(32)
DIMENSION LP(32)
DIMENSION MPRIME(13)
COMMON B, T, M, LUN, MXR, R
MXR = 258
LUN = 8
B = 10000
T = 30
M = 800

```

```

      READ (5, 100) (MPRIME(I) , I = 1, 13)
100  FORMAT (13I6)
      DO 40 I = 1, 400
      TOTAL = 1.0D0
      FI = I
      FI = FI*.5D0
      CALL COMFPT (2, FI, MDEG2)
      CALL PREPMP (FI, MDEG2, IP)
      DO 30 J = 1, 13
      CALL COMFPT(MPRIME (J), FI, IDEG)
      IDEG2 = IDEG + 2
      CALL MPCIM(IDEG2, IP (1, 1))
      MPS = MPRIME(J) * MPRIME(J)
      CALL EVALMP (IP, MPS, Y)
      TOTAL = TOTAL * Y
30   CONTINUE
      WRITE (6, 200) TOTAL
200  FORMAT (D24.16)
40   CONTINUE
      RETURN
      END

```

The MP routines are due to Brent and are available in the Michigan computer library.

A program to compute
$$\sum_{k=0}^N \frac{\prod_{j=0}^{k-1} \left(j^2 - \frac{(nh)^2}{4} \right)}{(k!)^2 p^{2k}}$$
 as a

polynomial in p for given n in multiple precision

```

SUBROUTINE PREPMP(X, N, IP)
IMPLICIT REAL * 8 (A-H, O-Z)
INTEGER B, T, R
DIMENSION IP (32, 160)
DIMENSION R(258)
DIMENSION JP(32)
DIMENSION KP(32)
DIMENSION LP(32)
COMMON B, T, M, LUN, MXR, R
MXR = 258
LUN = 8

```

```

B = 10000
T = 30
M = 800
Y = (X*X)/4.0D0
I1 = 1
NPLUS2 = N + 2
CALL MPCIM (NPLUS2, IP (1, 1))
CALL MPCIM (I1, IP (1, 2))
OPPY = -Y
CALL MPCDM (OPPY, IP(1, 3))
DO 10 I = 4, NPLUS2
IL2SQ = (I - 2) * (I - 2)
IL3SQ = (I - 3) * (I - 3)
CALL MPMULQ (IP(1, I-1), IL3SQ, IL2SQ, JP)
CALL MPDIVI (IP (1, 3), IL3SQ, KP)
CALL MPADDI (KP, I1, LP)
CALL MPMUL (JP, LP, IP (1, I))
10 CONTINUE
NPLUS3 = N + 3
DO 20 I = NPLUS3, 160
IP (1, I) = 0
20 CONTINUE
RETURN
END

```

A program to evaluate polynomials in multiple precision

```

SUBROUTINE EVALMP (IP, MP, Y)
IMPLICIT REAL * 8 (A-H, O-Z)
INTEGER B, T, R
DIMENSION IP(32, 160)
DIMENSION R(258)
DIMENSION JP(32)
DIMENSION KP(32)
DIMENSION LP(32)
COMMON B, T, M, LUN, MXR, R
MXR = 258
LUN = 8
B = 10000
T = 30
M = 800
CALL MPCMI(IP (1, 1), MDEGP)
CALL MPSTR(IP (1, MDEGP), JP)
MSTOP = MDEG - 2
IF (MSTOP .LT. 1) GO TO 10

```

```

DO 10 I = 1, MSTOP
INDEX = MDEGP - I
CALL MPDIVI(JP, MP, JP)
CALL MPADD(JP, IP(1, INDEX), JP)
10 CONTINUE
CALL MPCMD(JP, Y)
RETURN
END

```

A program to compute $R_L^{(nh)}$ and $R_H^{(nh)}$ using Theorem 1.16

```

IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION S1(38)
DIMENSION S2(38)
DIMENSION S3(38)
DIMENSION S4(38)
DIMENSION S5(38)
DIMENSION S6(38)
DIMENSION BL(38)
DIMENSION BH(38)
DIMENSION H(38)
DIMENSION HL(38)
DIMENSION HH(38)
U = 1155901.0D0
THETAU = 1150824.716334041D0/U
DLOGU = DLOG(U)
DLOG11 = DLOG(1155901.0D0)
DO 60 IT = 1, 400
T = IT
T = T*.5D0
T5 = 1.0D0/(((144.0D0 * U * U)/(T * T)) - 1.0D0)
IF (T .LT. 12.0D0) T5 = 1.0D0/((U * U) - 1.0D0)
S1(1) = 8.0D0
S1(2) = 0.0D0
S1(3) = -(T * T)/(4.0D0 * U * U)
DO 3 I = 4, 8
FI = I
S1(I) = S1(I - 1) * (FI - 3.0D0) * (FI - 3.0D0)/((FI - 2.0D0) *
1(FI - 2.0D0))
S1(I) = S1(I) * ((1.0D0) - ((T * T)/(4.0D0 * (FI - 3.0D0) * (FI - 3.0D0)))
1)/(U * U)
3 CONTINUE
SAVS18 = S1(8)
S1(8) = SAVS18 - (DABS(SAVS18) * T5)

```

```

DO 5 K = 9, 38
S1(K) = 0.0D0
5 CONTINUE
CALL MULT (S1, S1, S2)
CALL MULT (S1, S2, S3)
CALL MULT (S1, S3, S4)
CALL MULT (S1, S4, S5)
CALL MULT (S1, S5, S6)
S1(8) = SAVS18 + (DABS(SAVS18) * T5)
CALL MULT (S1, S1, S2)
CALL MULT (S1, S2, S3)
CALL MULT (S1, S3, S4)
CALL MULT (S1, S4, S5)
CALL MULT (S1, S5, S6)
DO 10 K = 1, 38
BL(K) = S1(K)
10 CONTINUE
D2 = -.5D0
CALL SMULT(S2, D2, H)
CALL ADD(BL, H, BL)
D3 = 1.0D0/3.0D0
CALL SMULT(S3, D3, H)
CALL ADD(BL, H, BL)
D4 = -1.0D0/4.0D0
CALL SMULT(S4, D4, H)
CALL ADD(BL, H, BL)
D5 = 1.0D0/5.0D0
CALL SMULT(S5, D5, H)
CALL ADD(BL, H, BL)
D6 = -1.0098775D0/6.0D0
CALL SMULT(S6, D6, H)
CALL ADD(BL, H, BL)
DO 20 K = 1, 38
BH(K) = S1(K)
20 CONTINUE
D2 = -.5D0
CALL SMULT(S2, D2, H)
CALL ADD(BH, H, BH)
D3 = 1.0D0/3.0D0
CALL SMULT(S3, D3, H)
CALL ADD(BH, H, BH)
D4 = -1.0D0/4.0D0
CALL SMULT(S4, D4, H)
CALL ADD(BH, H, BH)

```

```

D5 = 1.0D0/5.0D0
CALL SMULT(S5, D5, H)
CALL ADD(BH, H, BH)
D6 = -1.0D0/6.0D0
CALL SMULT(S6, D6, H)
CALL ADD(BH, H, BH)
I = 41 - K
IF (BL(I) .NE. 0.0D0) GO TO 22
IF (BR(I) .NE. 0.0D0) GO TO 22
GO TO 30
22 I2L1 = (I*2) - 5
   FI2L1 = I2L1
   FI = I - 2
   P1 = ((1.0D0 - (THETAU)) + (EI(FI2L1 * DLOGU)/FI2L1))/DLOGU
   P2 = .00000003D0/(FI2L1 * DLOGU)
   P3 = .001303D0 * (1.0D0 + ((EI(FI2L1 * DLOG11) + .0000000300)
1/FI2L1)) * ((U/115901.0D0)**I2L1)/DLOG11
   HL(I) = BL(I) * U * (P1 - (P2 + P3))
   IF (BL(I) .LT. 0.0D0) HL(I) = BL(I) * U * (P1 + P2)
   HH(I) = BH(I) * U * (P1 + P2)
   IF (BH(I) .LT. 0.0D0) HH(I) = BH(I) * U * (P1 - (P2 + P3))
   TOTHL = TOTHL + HL(I)
   TOTHH = TOTHH + HH(I)
30 CONTINUE
   BOUNDL = DEXP(TOTHL)
   BOUNDH = DEXP(TOTHH)
   WRITE (6, 400) BOUNDL
   WRITE (7, 400) BOUNDH
400 FORMAT (D24.16)
60 CONTINUE
   RETURN
   END

```

A program to compute exponential integrals using Lemma 1.20

```

REAL FUNCTION EI*8(V)
IMPLICIT REAL*8(A-H, O-Z)
DIMENSION Q1(6)
DIMENSION Q2(6)
Q1(1) = 6.0D0
Q2(1) = 6.0D0
Q1(2) = .2677737343D0
Q2(2) = 3.9584969228D0
Q1(3) = 8.6347608925D0
Q2(3) = 21.0996530827D0

```



```

Q1(4) = 18.0590159730D0
Q2(4) = 25.6329561486D0
Q1(5) = 8.5733287401D0
Q2(5) = 9.5733223454D0
Q1(6) = 1.0D0
Q2(6) = 1.0D0
CALL EVAL(Q1, V, Y1)
CALL EVAL(Q2, V, Y2)
EI = Y1/Y2
RETURN
END

```

A program to compute $\epsilon(u)$

```

IMPLICIT REAL * 8(A-H, O-Z)
DIMENSION IP(9592)
READ (5,100) (IP(I) , I = 1, 9592)
100 FORMAT (13I6)
TOTAL = 0.0D0
DO 10 I = 1, 9592
P = IP(I)
TOTAL = TOTAL + DLOG (P)
10 CONTINUE
WRITE (6,200) TOTAL
200 FORMAT (D30.16)
RETURN
END

```

A program to compute upper and lower bounds for

$F(x, A)$ using Theorem 1.16

```

IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION FUNVAL(400)
DIMENSION BOUNDL(400)
DIMENSION BOUNDH(400)
DIMENSION TOTLOW(601)
DIMENSION TOTHIGH(601)
DIMENSION TOTLOD(601)
DIMENSION TOTHD(601)
DIMENSION XDIST(601)
NPOINT = 400
READ (5,100) (FUNVAL(I), I = 1, NPOINT)

```

```

100  FORMAT (3D24.16)
      READ (2,200) (BOUNDI(I), I = 1, NPOINT)
      READ (3,200) (BOUNDH(I), I = 1, NPOINT)
200  FORMAT (D24.16)
      FPI = 3.14159265358973D0
      NDIST = 601
      DC 2 J = 1, DIST
      FJ = J - 1
      XDIST(J) = FJ*.001D0 * FPI
  2  CONTINUE
      H = .5D0
  5  READ (4,50) INCR
50  FORMAT (I6)
      IF (INCR .EQ. 0) GO TO 40
      NUSED = NPOINT/INCR
      DO 10 K = 1, NDIST
      TOTLOW(K) = 0.0D0
      TOTHIG(K) = 0.0D0
      TOTLOD(K) = 0.0D0
      TOTHID(K) = 0.0D0
10  CONTINUE
      DO 30 J = 1, NUSED
      I = (NUSED + 1) - J
      INDEX = I*INCR
      FINDEX = INDEX
      X = FINDEX*H
      DO 20 K = 1, NDIST
      FUNC = DSIN(X*XDIST(K))/X
      F = FUNC * FUNVAL(INDEX)
      TRALOW = F*BOUNDL(INDEX)
      IF (F .LT. 0.0D0) TRALOW = F*BOUNDH(INDEX)
      TRAHIG = F*BOUNDH(INDEX)
      IF (F .LT. 0.0D0) TRAHIG = F*BOUNDL(INDEX)
      TOTLOW(K) = TOTLOW(K) + TRALOW
      TOTHIG(K) = TOTHIG(K) + TRAHIG
      FUNC = DCOS(X*XDIST(K))
      F = FUNC * FUNVAL(INDEX)
      TRALOD = F*BOUNDL(INDEX)
      IF (F .LT. 0.0D0) TRALOD = F*BOUNDH(INDEX)
      TRAHID = F*BOUNDH(INDEX)
      IF (F .LT. 0.0D0) TRAHID = F*BOUNDL(INDEX)
      TOTLOD(K) = TOTLOD(K) + TRALOD
      TOTHID(K) = TOTHID(K) + TRAHID
20  CONTINUE
30  CONTINUE

```

```

FINCR = INCR
DO 35 K = 1, NDIST
TOTLOW(K) = TOTLOW(K) + (.5D0 * XDIST(K))
TOTHIG(K) = TOTHIG(K) + (.5D0 * XDIST(K))
TOTLOD(K) = TOTLOD(K) + .5D0
TOTHID(K) = TOTHID(K) + .5D0
TOTLOW(K) = TOTLOW(K) * H * FINCR / FPI
TOTHIG(K) = TOTHIG(K) * H * FINCR / FPI
TOTLOD(K) = TOTLOD(K) * H * FINCR / FPI
TOTHID(K) = TOTHID(K) * H * FINCR / FPI
PRINT1 = TOTLOW(K) + .5D0
PRINT2 = TOTHIG(K) + .5D0
PRINT3 = .5D0 - TOTHIG(K)
PRINT4 = .5D0 - TOTLOW(K)
FK1 = K - 1
PRINT0 = FK1 * .001D0
WRITE (6,250) PRINT0, PRINT1, PRINT2, PRINT3, PRINT4
250 FORMAT (F13.3, 'PI', 4D24.16)
WRITE (6,260) TOTLOD(K), TOTHID(K)
260 FORMAT (2D24.16)
35 CONTINUE
GO TO 5
40 RETURN
END

```

A program to compute $L(1, X)$ using (19) and (20)

```

IMPLICIT REAL * 8(A-H, O-Z)
DIMENSION MPRIME(400)
DIMENSION MPRIMR(400)
DIMENSION FG(2000)
DIMENSION FLSIN(2000)
DIMENSION FRETAU(2000)
DIMENSION FIMTAU(2000)
PI = 3.14159265358979323D0
TWOPI = PI * 2.0D0
READ (4,100) M, N
100 FORMAT (13I6)
READ (4,100) (MPRIME(I), I = 1, N)
READ (5,100) (MPRIMR(I), I = 1, N)
DO 40 I = M, N
    IPRIME = MPRIME(I)
    IPRIMR = MPRIMR(I)

```

```

FPRIME = IPRIME
IPHI = IPRIME - 1
FIPHI = IPHI
KSTOP = IPHI/2
FM = TWOPI/FIPHI
IPROD = 1
DO 10 IR = 1, IPHI
    IPROD = IPROD * IPRIMR
    IF (IPROD .GT. IPRIME) IPROD = IPROD -
1      (IPRIME * (IPROD/IPRIME))
    FIPROD = IPROD
    FG(IR) = FIPROD
    FLSIN(IR) = DLOG(DSIN(PI * FIPROD/FPRIME))
    FRETAU(IR) = DCOS(TWOPI * FIPROD/FPRIME)
    FIMTAU(IR) = DSIN(TWOPI * FIPROD/FPRIME)
10  CONTINUE
    FREMIN = 1000.0D0
    ARGMAX = 0.0D0
    DO 30 K = 1, KSTOP
        FK = K
        FRE1 = DCOS(TWOPI * FK/FIPHI)
        FIM1 = -DSIN(TWOPI * FK/FIPHI)
        FRE = 1.0D0
        FIM = 0.0D0
        TOTRE = 0.0D0
        TOTIM = 0.0D0
        TAURE = 0.0D0
        TAUIM = 0.0D0
        KEY = -1
        QUO = K/2
        IF ((FK/2.0D0) .EQ. QUO) KEY = 1
        DO 20 IR = 1, IPHI
            HRE = (FRE * FRE1) - (FIM * FIM1)
            HIM = (FRE * FIM1) + (FIM * FRE1)
            FRE = HRE
            FIM = HIM
            FMULT = FLSIN(IR)
            IF (KEY .EQ. -1) FMULT = FG(IR)
            TOTRE = TOTRE + (FRE * FMULT)
            TOTIM = TOTIM + (FIM * FMULT)
            TAURE = TAURE + (FRE * FRETAU(IR)) - (FIM *
1      FIMTAU(IR))
            TAUIM = TAUIM + (FRE * FIMTAU(IR)) + (FIM *
1      FRETAU(IR))

```

```
20      CONTINUE
      HRE = (TOTRE * TAURE) + (TOTIM * TAUM)
      HIM = (TOTIM * TAURE) - (TOTRE * TAUM)
      FMOD = (TAURE **2) + (TAUM **2)
      FRE = -HRE/FMOD
      FIM = -HIM/FMOD
      IF (KEY .EQ. 1) GO TO 25
      HRE = -FIM * PI/FPRIME
      HIM = FRE * PI/FPRIME
      FRE = HRE
      FIM = HIM
25      ARG = DATAN(FIM/FRE)
      ARG = DABS(ARG)
      IF (FRE .LT. FREMIN) FREMIN = FRE
      IF (ARG .GT. ARGMAX) ARGMAX = ARG
30      CONTINUE
      WRITE (6,300) IPRIME, FREMIN, ARGMAX
300     FORMAT (' ', I4, 2F16.8)
40     CONTINUE
      STOP
      END
```

APPENDIX B

The following programs are used to obtain the results in Chapter II. The polynomial manipulation routines are the same as those in Appendix A. The program listings follow.

A program to bound $\prod_{0 \leq y \leq 1005} \left(\frac{2|\rho|}{\pi c} \right)^{\frac{1}{2}} \left(1 + \frac{|\rho|}{8c} \right)$ using (31)

```

DIMENSION GAMMA(6)
CONSQ = 2.0/3.141593
CONS = SQRT(CONSQ)
TOTAL = 1.0
N = 0
T = EXP(10.0)
10 READ (5,100) (GAMMA(I), I = 1, 6)
100 FORMAT (6F13.6)
I = 1
15 IF (GAMMA(I) .EQ. 0.0) GO TO 20
N = N+1
RHO = SQRT(.25 + (GAMMA(I) * GAMMA(I)))
XRECIP = RHO/T
TOTAL = TOTAL * CONS * (1.0 + (XRECIP/8.0)) * SQRT(XRECIP)
WRITE (6,200) N, TOTAL
I = I + 1
IF (I .EQ. 7) GO TO 10
GO TO 15
20 WRITE (6,200) N, TOTAL
200 FORMAT (I5, E15.6)
RETURN
END

```

A program to compute $\sum_{0 < \gamma \leq 1005} \frac{1}{|\rho|}$ and related sums

```

IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION GAMMA (653)
READ (5,100) (GAMMA(I), I = 1, 653)
100 FORMAT (6F13.6)
TOTALL = 0.0D0
TOTAL = 0.0D0
TOTALH = 0.0D0
DO 10 I = 1, 653
  RHOL = DSQRT(.25D0 + ((GAMMA(I) + .000002D0) * (GAMMA(I) +
1.000002D0)))
  RHO = DSQRT(.25D0 + ((GAMMA(I)) * (GAMMA(I))))
  RHOH = DSQRT(.25D0 + ((GAMMA(I) - .000002D0) * (GAMMA(I) -
1.000002D0)))
  TOTALL = TOTALL + (1.0D0/RHOL)
  TOTAL = TOTAL + (1.0D0/RHO)
  TOTALH = TOTALH + (1.0D0/RHOH)
WRITE = (6,200) TOTALL, TOTAL, TOTALH
200 FORMAT (3D24.16)
10 CONTINUE
RETURN
END

```

A program to calculate $\prod_{0 < \gamma \leq 1005} J_0 \left(\frac{n\hbar}{|\rho|} \right)$

```

IMPLICIT REAL * 8(A-H, O-Z)
DIMENSION GAMMA(6)
DIMENSION RHO(653)
DIMENSION TOTAL(2688)
N = 0
10 READ (5,100) (GAMMA(I), I = 1, 6)
100 FORMAT (6F13.6)
I = 1
15 IF (GAMMA(I) .EQ. 0.0D0) GO TO 20
N = N + 1
RHO(N) = DSQRT(.25D0 + GAMMA(I) * GAMMA(I))
I = I + 1
IF (I .EQ. 7) GO TO 10
GO TO 15
20 CONTINUE

```

```

FPI = 3.14159265358973D0
H = FPI/128.0D0
DO 40 I = 1, 2688
TOTAL(I) = 1.0D0
FI = I
FI = FI * H
FI = FI + (21.0D0 * FPI)
DO 30 J = 1, N
X = FI/RHO(J)
F = BESJ0(X)
TOTAL(I) = TOTAL(I) * F
30 CONTINUE
40 CONTINUE
WRITE (6,200) (TOTAL(I), I = 1, 2688)
200 FORMAT (3D26.16)
RETURN
END

```

The program BESJ0 is in the Michigan computer library.

A program to compute $C_L(|\rho|, t)$ and $C_H(|\rho|, t)$ using Theorem 2.12

```

IMPLICIT REAL * 8 (A-H, O-Z)
DIMENSION S1(38)
DIMENSION S2(38)
DIMENSION S3(38)
DIMENSION S4(38)
DIEMNSION S5(38)
DIMENSION S6(38)
DIMENSION BL(38)
DIMENSION BH(38)
DIMENSION H(38)
R = (1.0D0/(4.0D0 * 36.0D0 * 14745600.0D0))
C1 = .9999121D0
C2 = 1.0D0
S1(1) = 8.0D0
S1(2) = 0.0D0
S1(3) = -(1.0D0/4.0D0)
S1(4) = (1.0D0/64.0D0)
S1(5) = -(1.0D0/2304.0D0)
S1(6) = (1.0D0/147456.0D0)
S1(7) = -(1.0D0/14745600.0D0)
S1(8) = C1 * R

```



```

DO 5 K = 9, 38
S1(K) = 0.0D0
5 CONTINUE
CALL MULT(S1, S1, S2)
CALL MULT(S1, S2, S3)
CALL MULT(S1, S3, S4)
CALL MULT(S1, S4, S5)
CALL MULT(S1, S5, S6)
WRITE (5, 200) (S1(I), I = 1, 38)
WRITE (5, 200) (S2(I), I = 1, 38)
WRITE (5, 200) (S3(I), I = 1, 38)
WRITE (5, 200) (S4(I), I = 1, 38)
WRITE (5, 200) (S5(I), I = 1, 38)
WRITE (5, 200) (S6(I), I = 1, 38)
DO 10 K = 1, 38
BL(K) = S1(K)
10 CONTINUE
D2 = -.5D0
CALL SMULT(S2, D2, H)
CALL ADD(BL, H, BL)
D3 = 1.0D0/3.0D0
CALL SMULT(S3, D3, H)
CALL ADD(BL, H, BL)
D4 = -1.0D0/4.0D0
CALL SMULT(S4, D4, H)
CALL ADD(BL, H, BL)
D5 = 1.0D0/5.0D0
CALL SMULT(S5, D5, H)
CALL ADD(BL, H, BL)
D6 = -1.0043233D0/6.0D0
CALL SMULT(S6, D6, H)
CALL ADD(BL, H, BL)
WRITE (6, 200) (BL(I), I = 1, 38)
200 FORMAT (3D26.16)
S1(8) = C2 * R
CALL MULT(S1, S1, S2)
CALL MULT(S1, S2, S3)
CALL MULT(S1, S3, S4)
CALL MULT(S1, S4, S5)
CALL MULT(S1, S5, S6)
WRITE (5, 200) (S1(I), I = 1, 38)
WRITE (5, 200) (S2(I), I = 1, 38)
WRITE (5, 200) (S3(I), I = 1, 38)
WRITE (5, 200) (S4(I), I = 1, 38)
WRITE (5, 200) (S5(I), I = 1, 38)
WRITE (5, 200) (S6(I), I = 1, 38)

```

```

DO 20 K = 1, 38
BH(K) = S1(K)
20 CONTINUE
D2 = -.5D0
CALL SMULT(S2, D2, H)
CALL ADD(BH, H, BH)
D3 = 1.0D0/3.0D0
CALL SMULT(S3, D3, H)
CALL ADD(BH, H, BH)
D4 = -1.0D0/4.0D0
CALL SMULT(S4, D4, H)
CALL ADD(BH, H, BH)
D5 = 1.0D0/5.0D0
CALL SMULT(S5, D5, H)
CALL ADD(BH, H, BH)
D6 = -1.0D0/6.0D0
CALL SMULT(S6, D6, H)
CALL ADD(BH, H, BH)
WRITE (7,200) (BH(I), I = 1, 38)
RETURN
END

```

A program to calculate $M_L(t)$ and $M_H(t)$ using Theorem 2.12

```

IMPLICIT REAL * 8(A-H, O-Z)
DIMENSION BLVEC(38)
DIMENSION BHVEC(38)
DIMENSION QL(38)
DIMENSION QH(38)
T = 1005.0D0
E = .34741D0
READ (2,200) (BLVEC(I), I = 1, 38)
READ (3,200) (BHVEC(I), I = 1, 38)
200 FORMAT (3D26.16)
A = 3.0767998D0
B = 1.2046751D0
C = (DLOG((T * T)/((T * T) - 4.0D0)) * 2.864789D0) + 7.2371007D0
D = .1657863D0
C0 = DLOG(T + 1.322875D0)
C1 = (A * C0) + (B * DLOG(C0)) + C
C2 = A + (B * (DLOG(C0)/C0))
C3 = 2.0D0 * 3.14159265358973D0
C4 = DLOG(T/C3)

```

```

AL = -1.7412588D0
BL = -2.864789D0
CL = -.3936986
DL = -3.8661494
AH = .0198943
BH = .4774648
DH = -.666871
IBOUND = BLVEC(1)
QL(1) = IBOUND
QH(1) = IBOUND
DO 5 K = 2, 38
QL(K) = 0.0D0
QH(K) = 0.0D0
5 CONTINUE
DO 10 I = 3, IBOUND
FI2 = 2 * (I - 2)
FI2P1 = FI2 + 1.0D0
P1 = T * (C4 + (1.0D0/(FI2 - 1.0D0)))/((C3 * (FI2 - 1.0D0)))
P1 = P1 - E
PH2 = D/FI2P1
PH2 = PH2 + ((AH-AL) * DLOG(T)) + (AH/FI2P1)
PH2 = PH2 + ((BH-BL) * DLOG(DLOG(T))) + (BH/FI2P1 * DLOG(T))
PH2 = PH2 + ((-CL) * DLOG(DLOG(T) + 1.322875D0))
PH2 = PH2 + (DH-DL)
PH2 = (PH2 * FI2)/T
PL2 = D/FI2P1
PL2 = PL2 + ((AH-AL) * DLOG(T)) - (AL/FI2P1)
PL2 = PL2 + ((BH-BL) * DLOG(DLOG(T))) - (BL/(FI2P1 * DLOG(T)))
PL2 = PL2 + ((-CL) * DLOG(DLOG(T) + 1.322875D0)) - (CL/(FI2P1 *
1DLOG(T + 1.322875D0)))
PL2 = PL2 + (DH-DL)
PL2 = -(PL2 * FI2)/T
FI2 = FI2 + 2.0D0
FI2P1 = FI2 + 1.0D0
PL3 = T * (C4 + (1.0D0/(FI2 - 1.0D0)))/((C3 * (FI2 - 1.0D0)))
PL3 = PL3 - E
PL3 = PL3 * ((FI2 - 2.0D0)/(T * T * 8.0D0))
PL4 = D/FI2P1
PL4 = PL4 + ((AH-AL) * DLOG(T)) - (AL/FI2P1)
PL4 = PL4 + ((BH-BL) * DLOG(DLOG(T))) - (BL/(FI2P1 * DLOG(T)))
PL4 = PL4 + ((-CL) * DLOG(DLOG(T) + 1.322875D0)) - (CL/(FI2P1 *
1DLOG(T + 1.322875D0)))
PL4 = PL4 + (DH-DL)
PL4 = -PL4 * (((FI2 - 2.0D0) * FI2)/(T * T * T * 8.0D0))
QL(I) = BLVEC(I) * (P1 + PL2 - (PL3 + PL4))
IF (BLVEC(I) .LT. 0.0D0) QL(I) = BLVEC(I) * (P1 + PH2)

```

```

      QH(I) = BHVEC(I) * (P1 + PH2)
      IF (BHVEC(I) .LT. 0.0D0) QH(I) = BHVEC(I) * (P1 + PL2 - (PL3 +
10  1 PL4))
      CONTINUE
      WRITE (6,200) (QL(I), I = 1, 38)
      WRITE (7,200) (QH(I), I = 1, 38)
      RETURN
      END

```

A program to compute upper and lower bounds for
G(X) using Theorem 2.12

```

      IMPLICIT REAL * 8(A-H, O-Z)
      DIMENSION FUNVAL(5376)
      DIMENSION QL(38)
      DIMENSION QH(38)
      DIMENSION TOTLOW(40)
      DIMENSION TOTHIG(40)
      DIMENSION TOTLOD(40)
      DIMENSION TOTHD(40)
      DIMENSION XDIST(40)
      READ (2,200) (QL(I), I = 1, 38)
      READ (3,200) (QH(I), I = 1, 38)
200  FORMAT (3D26.16)
      NPOINT = 5376
      READ (5,200) (FUNVAL(I), I = 1, NPOINT)
100  FORMAT (2D30.16)
      FPI = 3.14159265358973D0
      READ (1,150) NDIST
150  FORMAT (I6)
      READ (1,160) (XDIST(I), I = 1, NDIST)
160  FORMAT (6F13.6)
      T = 1005.0D0
      H = FPI/128.0D0
      5  READ (4,50) INCR
      50  FORMAT (I6)
      IF (INCR .EQ. 0) GO TO 40
      NUSED = NPOINT/INCR
      DO 10 K = 1, NDIST
      TOTLOW(K) = 0.0D0
      TOTHIG(K) = 0.0D0
      TOTLOD(K) = 0.0D0
      TOTHD(K) = 0.0D0
10  CONTINUE

```

```

DO 30 J = 1, NUSED
I = (NUSED + 1) - J
INDEX = I*INCR
FINDEX = INDEX
X = FINDEX*H
X2 = (X*X)/(T*T)
CALL EVAL(QL, X2, Y1)
BOUNDL = DEXP(Y1)
CALL EVAL(QH, X2, Y2)
BOUNDH = DEXP(Y2)
DO 20 K = 1, NDIST
FUNC = DSIN(X*XDIST(K))/X
F = FUNC * FUNVAL(INDEX)
TRALOW = F*BOUNDL
IF (F .LT. 0.0D0) TRALOW = F*BOUNDH
TRAHIG = F*BOUNDH
IF (F .LT. 0.0D0) TRAHIG = F*BOUNDL
TOTLOW(K) = TOTLOW(K) + TRALOW
TOTHIG(K) = TOTHIG(K) + TRAHIG
FUNC = DCOS(X*XDIST(K))
F = FUNC * FUNVAL(INDEX)
TRALOD = F*BOUNDL
IF (F .LT. 0.0D0) TRALOD = F*BOUNDH
TRAHID = F*BOUNDH
IF (F .LT. 0.0D0) TRAHID = F*BOUNDL
TOTLOD(K) = TOTLOD(K) + TRALOD
TOTHID(K) = TOTHID(K) + TRAHID
20 CONTINUE
30 CONTINUE
FINCR = INCR
DO 35 K = 1, NDIST
TOTLOW(K) = TOTLOW(K) + (.5D0 * XDIST(K))
TOTHIG(K) = TOTHIG(K) + (.5D0 * XDIST(K))
TOTLOD(K) = TOTLOD(K) + .5D0
TOTHID(K) = TOTHID(K) + .5D0
TOTLOW(K) = TOTLOW(K) * H * FINCR / FPI
TOTHIG(K) = TOTHIG(K) * H * FINCR / FPI
TOTLOD(K) = TOTLOD(K) * H * FINCR / FPI
TOTHID(K) = TOTHID(K) * H * FINCR / FPI
PRINT1 = TOTLOW(K) + .5D0
PRINT2 = TOTHIG(K) + .5D0
PRINT3 = .5D0 - TOTHIG(K)
PRINT4 = .5D0 - TOTLOW(K)
WRITE (6, 250) XDIST(K), PRINT1, PRINT2, PRINT3, PRINT4

```

```
250  FORMAT (F13.6, 4D24.16)
      WRITE (6,260) TOTLOD(K), TOTHD(K)
260  FORMAT (2D24.16)
35   CONTINUE
      GO TO 5
40   RETURN
      END
```

APPENDIX C

The following program is used to obtain the results in Chapter 3. The program listing follows.

A program to verify (16) and (17) for $N = \text{NBEGIN}, \text{NEND}$

```

      IMPLICIT REAL * 8 (A-H, O-Z)
      DIMENSION MPRIME(50000)
      DIMENSION SLHSRE(50000)
      DIMENSION SLHSIM(50000)
      DIMENSION AM(1000)
      READ (4, 51) MPRIME
51  FORMAT (6I8)
      READ (5, 100, END = 6000) NBEGIN, NEND
100 FORMAT (2I8)
      N = NBEGIN
10  IF (N .GT. NEND) GO TO 5000
      FLOATN = N
      TEE = 6.2831852/(DLOG(FLOATN))
      MBOUND = DSQRT(FLOATN)
      DO 50 I = 1, MBOUND
      AI = I
      AM(I) = TEE * DLOG(AI)
50  CONTINUE
      DO 120 I = 1, N
      MPRIM = MPRIME(I)
      IF (MPRIM .GT. N) GO TO 121
      IF (MPRIM .GT. MBOUND) GO TO 120
      NLASTP = 1
120 CONTINUE
121 MSTART = I - 1
      TOTLHS = 0.0
      NSTART = 1
      TLHSRE = 0.0
      TLHSIM = 0.0
      TOTLRE = 0.0
      TOTLIM = 0.0
      DO 250 K = 1, MSTART
      I = (MSTART + 1) - K

```

```

MPRIM = MPRIME(I)
IF (.NOT. (MPRIM .GT. MBOUND)) GO TO 251
NSTOP = N/MPRIM
DO 225 J = NSTART, NSTOP
IF (NSTART .GT. NSTOP) GO TO 225
A = AM(J)
AJ = J
TLHSRE = TLHSRE + ((DCOS(A))/AJ)
TLHSIM = TLHSIM - ((DSIN(A))/AJ)
225 CONTINUE
FMPRIM = MPRIM
TOTHOL = ((1.0/FMPRIM))
TOTHOL = TOTHOL * DSQRT(TLHSRE ** 2 + TLHSIM ** 2)
TOTLHS = TOTLHS + TOTHOL
A = TEE * (DLOG(FMPRIM))
PRE = (DCOS(A))/FMPRIM
PIM = -(DSIN(A))/FMPRIM
TOTLRE = TOTLRE + ((PRE * TLHSRE) - (PIM * TLHSIM))
TOTLIM = TOTLIM + ((PRE * TLHSIM) + (PIM * TLHSRE))
235 SLHSRE(I) = TLHSRE
SLHSIM(I) = TLHSIM
NSTART = NSTOP + 1
250 CONTINUE
251 TRHSRE = TLHSRE
TRHSIM = TLHSIM
DO 200 I = NSTART, N
AI = I
A = TEE * DLOG(AI)
TRHSRE = TRHSRE + ((DCOS(A))/AI)
TRHSIM = TRHSIM - ((DSIN(A))/AI)
200 CONTINUE
TRHSRE = TRHSRE - TOTLRE
TRHSIM = TRHSIM - TOTLIM
TOTRHS = DSQRT((TRHSRE * TRHSRE) + (TRHSIM * TRHSIM))
DIFF = TOTLHS - TOTRHS
WRITE (6, 252) N, TOTLHS, TOTRHS, DIFF
252 FORMAT (I8, 3F16.8)
WRITE (6, 255) TRHSRE, TRHSIM
255 FORMAT (2F16.8)
IC = 0
NPLUS1 = N + 1
NEXTP = NLASTP + 1
DO 500 L = NPLUS1, NEND
IC = IC + 1
FLOATL = L
DSQRTL = DSQRT(FLOATL)

```



```

IF (MPRIME(NEXTP) .GT. DSQRTL) GO TO 305
FMPRIM = MPRIME(NEXTP)
HOLDTO = (DSQRT(SLHSRE(NEXTP)**2 + SLHSIM(NEXTP)**2))/
1FMPRIM
TOTLHS = TOTLHS - HOLDTO
NLASTP = NEXTP
NEXTP = NLASTP + 1
MPRIM = MPRIME(NLASTP)
DO 302 M = 1, MPRIM
K = M * MPRIM
FK = K
A = TEE * DLOG(FK)
IF (M .EQ. 1) AM(K) = A
TRHSRE = TRHSRE + ((DCOS(A))/FK)
TRHSIM = TRHSIM - ((DSIN(A))/FK)
302 CONTINUE
GO TO 430
305 HOLDL = FLOATL
DO 407 I = 1, NLASTP
FMPRIM = MPRIME(I)
310 SAVE = HOLDL/FMPRIM
ISAVE = SAVE
IF (.NOT. (ISAVE .EQ. SAVE)) GO TO 405
HOLDL = SAVE
GO TO 310
405 IF (HOLDL .EQ. 1.0) GO TO 410
407 CONTINUE
410 IF (.NOT. (HOLDL .EQ. 1.0)) GO TO 413
A = TEE * DLOG(FLOATL)
TRHSRE = TRHSRE + ((DCOS(A))/FLOATL)
TRHSIM = TRHSIM - ((DSIN(A))/FLOATL)
GO TO 430
413 ITEST = MSTART + 1
IF (MPRIME(ITEST) .GT. L) GO TO 900
HOLD = MPRIME(ITEST)
TOTLHS = TOTLHS + (1.0/HOLD)
MSTART = ITEST
SLHSRE(ITEST) = 1.0
SLHSIM(ITEST) = 0.0
GO TO 430
900 IHOLDL = HOLDL
IBEGIN = NEXTP
IEND = MSTART
DO 1100 IBISEC = 1, 20
ISEAR = (IBEGIN + IEND)/2
MPRIM = MPRIME(ISEAR)

```

```

      IF (MPRIM .LT. IHOLDL) GO TO 1050
      IF (MPRIM .EQ. IHOLDL) TO TO 1040
      IEND = ISEAR - 1
      GO TO 1100
1040  J = ISEAR
      GO TO 418
1050  IBEGIN = ISEAR + 1
1100  CONTINUE
      415  DO 417 J = IBEGIN, IEND
          FMPRIM = MPRIME(J)
          IF (FMPRIM .EQ. HOLDL) GO TO 418
      417  CONTINUE
          WRITE (6,255) FMPRIM, HOLDL
          STOP
      418  NSTOP = L/IHOLDL
          A = AM(NSTOP)
          AJ = NSTOP
          HOLDTO = DSQRT(SLHSRE(J)**2 + SLHSIM(J)**2)/HOLDL
          SLHSRE(J) = SLHSRE(J) + ((DCOS(A))/AJ)
          SLHSIM(J) = SLHSIM(J) - ((DSIN(A))/AJ)
          TOTHOL = DSQRT(SLHSRE(J)**2 + SLHSIM(J)**2)/HOLDL
          TOTLHS = (TOTLHS + TOTHOL) - HOLDTO
      430  TOTRHS = DSQRT(TRHSRE**2 + TRHSIM**2)
          DIFF = TOTLHS - TOTRHS
          IF (TOTLHS .LT. 1.0) GO TO 999
          IF (IC .LT. 1000) GO TO 450
          IC = 0
      999  WRITE (6,252) L, TOTLHS, TOTRHS, DIFF
          WRITE (6,255) TRHSRE, TRHSIM
      450  IF (DIFF .GT. .1) GO TO 500
          N = L
          GO TO 10
      500  CONTINUE
      5000 CONTINUE
      6000 CALL EXIT
          END

```

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