Transport of Light in Classical and Quantum Models

by

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Dedication

This dissertation is dedicated to my parents, Carol and Stephen.

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Abstract

Kinetic equations have played a significant role in optics since the introduction of the radiative transport equation (RTE) by Schuster in the early 20th century. Now they are applied in many scientific fields such as atmospheric physics and biomedical imaging to understand how light interacts with a complex medium. In this thesis we investigate how these equations arise in

1. models of classical optics in an inhomogeneous disordered scattering medium
2. in fully quantum real space models with a random distribution of atoms

In certain spatial and temporal regimes it has been empirically observed that light interacting with a highly disordered scattering medium may be described by a radiative transport equation with a constant scattering coefficient. One way to derive such equations from microscopic principles is to model the medium as a random field which is statistically homogeneous. We show that in a quasi-homogeneous media, where statistical correlations vary rapidly on small length scales and slowly on large length scales, light obeys a radiative transport equation with variable scattering coefficient.

In the second part of this thesis we introduce a new model in quantum optics which allows us to treat a quantized electromagnetic field coupled to a medium of two level atoms, both in real space. The rest of the chapter focuses on the interaction of atoms with a field consisting of at most one photon. First we recover a classical result due to Wigner and Weisskopf on the rate of single atom spontaneous emission. Next we show that in a statistically homogeneous distribution of two level atoms the field and atomic amplitude satisfy linear kinetic equations. Finally we show that at sufficiently long times and large distances these amplitudes are asymptotically governed by diffusion equations which may be solved analytically.

In the final chapter we use the previous model to analyze a system in which there can be at most two photons. This allows us to study the time evolution of states that initially contain two entangled photons, a phenomenon which has no classical analogue. We show that the various probability amplitudes satisfy linear two particle kinetic equations which again may be asymptotically evaluated at large distances and long times.

Chapter 1

Introduction

1.1 Outline

This thesis is organized as follows. The remainder of this contains a brief background of some relevant material in classical optics and scattering theoring, including discussions of Maxwell's equations, the radiative transport equation (RTE), and the diffusion approximation (DA) to the RTE. This is followed by an introduction to quantum mechanics and the quantization of the free electromagnetic field in quantum field theory. Lastly, we discuss two key phenomena which may be described through the formalism of quantum optics: spontaneous and stimulated emission.

The second chapter discusses a generalization of the derivation of the RTE from the Helmholtz equation in random media. While generally the medium is assumed to be statistically homogeneous and isotropic, in which case the correlation function depends only on the distance between the two points, we consider quasihomogeneous media in which the correlation function is the product of two terms. One term varies rapidly and depends on the distance between the two points, while the other varies slowly and depends on their center of mass coordinate. We show that in such a medium one may derive a RTE with a spatially varying scattering coefficient.

In the third chapter we introduce a general model for studying the interaction of a quantized scalar field with a collection of stationary two level atoms. We use this to study the dynamics associated to a one excitation state, where an excitation is either an excited atom or a photon present in the field. First we recover a classical result due to Wigner and Weisskopf on the rate of spontaneous emission from a single atom. Next, we introduce a stochastic model of the distribution of two level atoms in real space. We show that in a high frequency limit the probability densities associated to these orthogonal states are linear combinations of solutions to transport equations. Within a suitable diffusion approximation we extract algebraic pointwise decay estimates for both the atomic and photonic probability densities.

In the fourth chapter we focus primarily on the dynamics of a general two excitation state. This state is the superposition of three possible physical cases: two photons in the field, two excited atoms, or one excited atom along with a field containing one photon. We start by calculating the rate of stimulated emission from a single atom in the presence of one photon. Then we introduce the same stochastic model of matter which was discussed in the third chapter. In order to study the affect of entangled photons on such an interaction, we assume the two photon state is initially nonfactorizable and that there are no excited atoms. We use an asymptotic approach to show that in a high frequency limit the probability densities associated to these three cases are integrated linear combinations of solutions to two particle kinetic equations. Once again we are able to show that at large times and distances far away from the origin these probability densities decay algebraically.

The final chapter is a brief discussion of several possible directions for future research. Most of these directions focus on ways to generalize or incorporate new physics into the model described in chapter 3 or the corresponding systems of equations governing the dynamics of the one and two excitation states.

1.2 Maxwell's Equations

In classical optics the electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are vector fields whose dynamics are determined by a system of four coupled partial differential equations due to James Maxwell. In SI units, in a nonmagnetic inhomogeneous medium without any external sources these equations are given

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \,, \tag{1.1a}$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \,, \tag{1.1b}$$

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \epsilon \frac{\partial \mathbf{E}}{\partial t} = 0 \,, \tag{1.1c}$$

$$\nabla \cdot \mathbf{B} = 0 \,, \tag{1.1d}$$

where μ_0, ϵ_0 and ϵ are the permeability of free space, permittivity of free space and dielectric function respectively [43]. The speed of light in vacuum c is defined as $c = 1/\sqrt{\mu_0 \epsilon_0}$. These equations, along with the Lorentz force law

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \,, \tag{1.2}$$

give a complete description of the electromagnetic field in classical optics. In the case of monochromatic light at a frequency ω, as long as the dielectric function varies slowly on the scale of the wavenumber of the light, $k_0 = \omega/c$, one can show that the field $\mathbf{E}(\mathbf{x})$ satisfies a vector Helmholtz equation of the form

$$\nabla^2 \mathbf{E} + k_0^2 \epsilon \mathbf{E} = 0 \,, \tag{1.3}$$

We note that the dielectric function ϵ determines the speed of this wave and the case $\epsilon \equiv 1$ corresponds to a homogeneous medium in which the speed is that of light in a vacuum. Although the components of the field $\mathbf{E}(\mathbf{x})$ are not independent, one frequently works with a scalar model of the electric field $U(\mathbf{x})$ which satisfies a scalar Helmholtz equation of the form

$$\Delta U + k_0^2 \epsilon U = 0 \,. \tag{1.4}$$

1.3 Scattering

Scattering occurs when the dielectric function $\epsilon(\mathbf{x})$ in Eq. (1.4) is nonconstant. Generally it has the form $\epsilon(\mathbf{x}) = 1+\eta(\mathbf{x})$ where η is a compactly supported function which corresponds to the presence of one or several scatterers. This has the affect of changing the speed of the wave in these regions in space and altering the behavior of an incident wave, even very far from the scatterers. A fundamental result in scattering theory is known as the Lippmann-Schwinger equation, which gives the field $U(\mathbf{x})$ at a point $\mathbf{x}$ in terms of its values inside the region in which the scatterers are present [16]. If we suppose that $\eta(\mathbf{x})$ is supported in a volume V, then we may decompose the field $U = U_i + U_s$ where U_i is the incident field and U_s is the scattered field which satisfy

$$\Delta U_i + k_0^2 U_i = 0\ , \tag{1.5a}$$

$$\Delta U_s + k_0^2 U_s = -k_0^2 \eta(\mathbf{x}) U(\mathbf{x})\ . \tag{1.5b}$$

Solving these equations allows us to rewrite $U(\mathbf{x})$ as

$$U(\mathbf{x}) = U_i(\mathbf{x}) + k_0^2 \int_V G_0(\mathbf{x},\mathbf{x}')\eta(\mathbf{x}')U(\mathbf{x}')d^3r'\ , \tag{1.6}$$

where G_0 is the Green's function associated to Eq. (1.4) with $\eta = 0$ and is explicitly given by the formula

$$G_0(\mathbf{x},\mathbf{x}') = \frac{e^{ik_0|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\ . \tag{1.7}$$

Eq. (1.6) is the Lippmann-Schwinger equation and may be iterated to give an infinite series expansion of $U(\mathbf{x})$ in terms of the incident field $U_i(\mathbf{x})$

$$U(\mathbf{x}) = U_i(\mathbf{x}) + k_0^2 \int_V G_0(\mathbf{x},\mathbf{x}')\eta(\mathbf{x}')U_i(\mathbf{x}')d^3x' \tag{1.8}$$

$$+ k_0^4 \int_V G_0(\mathbf{x},\mathbf{x}')\eta(\mathbf{x}')G_0(\mathbf{x}',\mathbf{x}'')\eta(\mathbf{x}'')U_i(\mathbf{x}'')d^3x'd^3x'' + \cdots\ . \tag{1.9}$$

This expansion is known as the Dyson series and when truncated to first order is referred to as the Born approximation. If we have a collection of N point scatterers at locations $\mathbf{x}_1, \cdots \mathbf{x}_N$ then the function $\eta(\mathbf{x})$ is of the form

$$\eta(\mathbf{x}) = \alpha_0 \sum_{j=1}^{N} \delta(\mathbf{x} - \mathbf{x}_j) \;, \tag{1.10}$$

with α_0 the polarizability of the scatterer which has units of volume. This leads to a system of algebraic equations for the value of the total field at the points $\mathbf{x}_k$

$$U_i(\mathbf{x}_j) = \sum_{k=1}^{N} A_{jk} U(\mathbf{x}_k) \;, \tag{1.11}$$

where

$$A_{jk} = \delta_{jk} - \alpha_0 k_0^2 G_0(\mathbf{x}_j, \mathbf{x}_k) \;. \tag{1.12}$$

These algebraic equations due to Foldy and Lax [30] are quite useful for a small number of scatterers, but are computationally costly to solve as N gets large.

1.4 Radiative Transport

The microscopic description of light given by Maxwell's equation is theoretically satisfying and valid on scales close to the wavelength of the light. However, it can be difficult to obtain useful solutions in scattering mediums where the dielectric function $\epsilon(\mathbf{x})$ is complicated. Because of this, it is reasonable to search for another description of the propagation of light waves through a scattering medium that is valid on scales larger than that of the wavelength. This scale, referred to as the mesoscale, is related to the fundamental quantity known as the scattering length ℓ_s which is the mean distance between interactions with the scatterers.

As early as 1887 physicists including von Lommel and Khvolsen wrote down integro-differential equations phenomenolically for the flow of light energy through matter on this scale [62]. These equations were influenced by the work of Bolztmann on the kinetic theory of gases and are linear analogues of the Boltzmann equation. Later work due to Schuster and Planck [62] led to the radiative transport equation (RTE) in it's modern form

$$\mathbf{k} \cdot \nabla I(\mathbf{x}, \hat{\mathbf{k}}) + (\mu_s + \mu_a) I(\mathbf{x}, \hat{\mathbf{k}}) = \mu_s \int d^2\hat{k}' A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') I(\mathbf{x}, \hat{\mathbf{k}}') \ , \tag{1.13}$$

where $I(\mathbf{x}, \hat{\mathbf{k}})$ is the specific intensity of light, a spatially and angularly resolved measurement of the light at a point $\mathbf{x}$ flowing in direction $\hat{\mathbf{k}}$. The constants μ_s and μ_a are scattering and absorption coefficients and the function $A(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$ is a phase function or scattering kernel. This equation was first derived as a conservation law for the total amount of light gained and lost through absorption and scattering at each point and direction in space. However, it is quite natural to try and derive it from microscopic principles such as the scalar wave equation in an attempt to connect the physical laws that describe the behavior of light on different scales. While in general this connection is still not fully understood, in certain cases it is possible to derive RTEs from electromagnetic theory.

1.5 Random Media

One way to arrive at at an RTE from a scalar wave equation is to model the inhomogeneous medium of scatterers as a random field with specified correlations. To this end we suppose the dielectric function $\epsilon(\mathbf{x})$ is has the form

$$\epsilon(\mathbf{x}) = 1 + \eta(\mathbf{x}) \ , \tag{1.14}$$

and that $\eta(\mathbf{x})$ is a mean zero Gaussian random field satisfying

$$\langle \eta(\mathbf{x}) \rangle = 0 \ , \tag{1.15a}$$

$$\langle \eta(\mathbf{x}) \eta(\mathbf{x}') \rangle = C(\mathbf{x} - \mathbf{x}') \ . \tag{1.15b}$$

The function $C(\mathbf{x})$ is the correlation function and $\langle \cdots \rangle$ denotes statistical averaging over realizations of the medium. Additionally, it is required that the medium is statistically homogeneous and isotropic which corresponds to imposing the restriction that the correlation function depends only on the distance between the two points $\mathbf{x}$ and $\mathbf{x}'$. The goal of introducing random media is to obtain an equation satisfied by the specific intensity $I(\mathbf{x}, \hat{\mathbf{k}})$, which may be defined in terms of average values of the correlations of the field $U(\mathbf{x})$, where the phase function $A(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$ and the scattering coefficient μ_s are determined by the correlation function $C(\mathbf{x})$. One must ask, do these solutions which involves averaging over many realizations of the medium provide useful information about a single realization of the medium? This question can be addressed in two ways: 1) If the fluctuations in the correlations between various realizations of the medium are small, then the averages give accurate answers. 2) If the randomness present is ergodic, then we may equate a time average with a spatial average to see that the results obtained are useful in a single realization after a sufficient amount of time.

1.6 Derivation of RTE

There are two main techniques that are used to derive an RTE from the scalar wave equation with a random dielectric function: diagrammatic perturbation theory and multiscale asymptotic analysis. While the two methods are quite distinct, they both require a quasiprobability distribution introduced by Eugene Wigner in 1932 known as the Wigner transform [93]. The Wigner transform $W(\mathbf{x}, \mathbf{k})$ associated to a function $U(\mathbf{x})$ is defined

$$W(\mathbf{x}, \mathbf{k}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}'} U(\mathbf{x} - \mathbf{x}'/2) U^*(\mathbf{x} + \mathbf{x}'/2) , \tag{1.16}$$

where the * denotes complex conjugation. The Wigner transform is a phase space representation of the correlations which is resolved in space and momentum. While the Wigner transform itself is not a measurable quantity, one can recover several important measurable quantities from its moments. The zeroth and first order moments are the intensity of the field and the energy current density respectively.

The diagrammatic approach, pioneered by physicists [11, 91, 31, 57], involves obtaining an expression for the field correlations through an expansion similar to the Born series given in Eq. (1.8). This infinite series solution can be expressed graphically as a collection of diagrams and the process of averaging over realization of the medium has a specific combinatorial interpretation. Additionally, the diagrams can be categorized and the expression factored which allows for only the diagrams whose contribution are the largest to be retained. This selection of the most important diagrams, known as the ladder approximation, has been placed on rigorous footing **cite**. From this simplified expression for the correlations, one can show that the Wigner transform is localized in the magnitude of the variable $\mathbf{k}$ and that it satisfies an RTE.

The other approach, involving asymptotic analysis, is due to applied mathematicians [48, 32]. In this case the goal is to study the behavior of the Wigner transform in a high frequency limit. To this end, a dimensionless parameter ϵ is introduced and the strength of the correlations are appropriately rescaled to order ϵ. Moreover, one separates the slow and fast spatial scales $\mathbf{x}$ and $\mathbf{x}/\epsilon$ and treats them as independent variables. By expanding the Wigner transform in powers of ϵ, one obtains a hierarchy of equations which may be solved given a specific closure hypothesis. Once again one finds that the Wigner transform is localized in the magnitude of $\mathbf{k}$ and that it satisfies an RTE.

The relationship between the two methods is not completely understood. In particular it is of interest to understand the analogue of the key assumption in each derivation [19]. What is the analogue of the ladder approximation in the asymptotic approach? And what is the analogue of the closure hypothesis in the diagrammatic approach?

1.7 Diffusion

Again, except in a small number of situations, exact solutions to the RTE can be quite difficult to obtain. At distances much greater than the scattering length, ℓ_s and times much greater than ℓ_s/c, the solution to the RTE can be approximated by the solution to a diffusion equation [16]. This approximation also breaks down if the system is weakly scattering or strongly absorbing. The idea is to split the intensity I into two pieces $I = I_s + I_d$ where I_s corresponds to ballistic and singularly scattered terms and I_d corresponds to all higher orders of scattering. The term I_s may be solved for exactly and the term I_d can be approximated as

$$I_d(\mathbf{x}) = \frac{c}{4\pi}\left(u(\mathbf{x}) + \ell_t \mathbf{k} \cdot \nabla u(\mathbf{x})\right) \tag{1.17}$$

where ℓ_t is the fundamental length scale associated with diffusion and u satisfies the diffusion equation

$$-D\Delta u + c\mu_a u = 0 \ . \tag{1.18}$$

The diffusion constant D and ℓ_t are determined by the scattering and absorption coefficients μ_s and μ_a as well as the phase function $A(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$.

1.8 Quantum Mechanics

In quantum mechanics, particles no longer have trajectories in phase space and instead states are given by vectors in a fixed Hilbert space, $\mathcal{H}$. The measurable quantities, such as position and momentum, which are functions in classical mechanics are replaced by self adjoint operators on $\mathcal{H}$ and the spectra of these operators correspond to the possible outcomes of measurements [75]. For any system of particles there is an associated self adjoint operator, H, called the Hamiltonian which represents the sum of potential and kenetic energies of the system. The dynamics associated to any initial state $\psi_0 \in \mathcal{H}$ is completely determined by the Schrödinger equation

$$i\hbar\partial_t\psi = H\psi \ , \tag{1.19a}$$

$$\psi|_{t=0} = \psi_0 \ , \tag{1.19b}$$

where $\hbar$ is the reduced Planck's constant. A case of particular interest is the one dimensional single particle harmonic oscillator where the potential is a quadratic function of position.

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \ . \tag{1.20}$$

Here p is the momentum operator, x is the position operator, m is the mass of the particle and ω which is the frequency of the oscillation. The position and momentum operators satisfy the canonical commutation relations

$$[x, p] = i\hbar \ . \tag{1.21}$$

In order to factor the Hamiltonian, we introduce the ladder operators $a^\dagger$ and a through the change of variables

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{i}{m\omega}p) \ , \tag{1.22a}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{i}{m\omega}p) \ . \tag{1.22b}$$

One can show that the Hamiltonian has a discrete spectrum consisting of positive eigenvalues increasing to infinity $E_0 < E_1 < E_2 < \cdots$ [75]. The operator $a^\dagger$ raises a state with energy E_n to a state of energy E_{n+1} while the operator a lowers a state with energy E_n to a state with energy E_{n-1}. Hence, these operators act as a way to move up and down the rungs of the ladder of eigenvalues associated to the Hamiltonian. Moreover, one can rewrite the Hamiltonian using these ladder operators as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \ . \tag{1.23}$$

The extra $\hbar\omega/2$ appears due to the fact that a and $a^\dagger$ do not commute but instead have a commutator equal to 1. Physically this additional term corresponds to a positive background energy present in the system and is known as the zero point energy.

1.9 Quantum Optics

Quantum optics is the study of the interaction of light and matter on the quantum scale. Often this means considering a quantum system of particles and coupling them to an electromagnetic field via an interaction term in the Hamiltonian. This formulation, in which the matter is treated quantum mechanically and the electromagnetic field is a classical field satisfying Maxwell's equations is known as semiclassical optics. While semiclassical optics is an extremely fruitful area of research and application, in a fully quantum optical model, the electromagnetic radiation must also be treated quantum mechanically. This means that the electric and magnetic fields should be elevated to fields of operators [35]. Since we work with a scalar transport equation we will describe the process of quantizing the scalar field $U(\mathbf{x})$ which satisfies a scalar wave equation

$$\Delta U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0 \; . \tag{1.24}$$

To do this we first assume we are working in a finite box of side length L with no sources present. The side length of the box L will be taken to infinity before recovering any physical result, but for the meantime we impose periodic boundary conditions on the field. The solution can be expressed as a solution of plane waves of the form

$$U(\mathbf{x}, t) = \sum_{\mathbf{k}} \left[U_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + U_{\mathbf{k}}(t)^* e^{-i\mathbf{k}\cdot\mathbf{x}} \right] \; , \tag{1.25}$$

where the sum is over a discrete set of wave vectors $\mathbf{k}$ due to the periodic boundary conditions. The coefficients $U_{\mathbf{k}}(t)$ satisfy

$$k^2 U_{\mathbf{k}} + \frac{1}{c^2}\frac{\partial^2 U_{\mathbf{k}}}{\partial t^2} = 0 \; , \tag{1.26}$$

whence this expansions is given by

$$U(\mathbf{x}, t) = \sum_{\mathbf{k}} \left[U_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t} + U_{\mathbf{k}}(t)^* e^{-(i\mathbf{k}\cdot\mathbf{x} - i\omega_k t)} \right] \; , \tag{1.27}$$

with $\omega_k = c|\mathbf{k}|$. The Hamiltonian associated to the wave equation is given by

$$H = \frac{1}{2} \int d^3x \, \left(\frac{1}{c^2} (\partial_t U)^2 + (\partial_x U)^2 \right) \; , \tag{1.28}$$

and can be expressed in terms of the modes $U_{\mathbf{k}}$ as

$$H = V \sum_{\mathbf{k}} \omega_k^2 U_{\mathbf{k}} U_{\mathbf{k}}^* \,, \tag{1.29}$$

where $V = L^3$ is the volume of the box. This is strikingly similar to the harmonic oscillator Hamiltonian in Eq. (1.23) and shows that the system is equivalent to a collection of uncoupled harmonic oscillators. This suggests rewriting the Hamiltonian in terms of the canonical position and momentum variables, elevating their status to operators and then rewriting the Hamtilonian in terms of the associated ladder operators $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$. This leads to the expression

$$H_F = \sum_{\mathbf{k}} \hbar\omega_k \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right) \,. \tag{1.30}$$

We may interpret $a_{\mathbf{k}}^\dagger$ as a creation operator which has the effect of introducing an additional photon into the system with wave vector $\mathbf{k}$, and $a_{\mathbf{k}}$ as an annihilation operator which destroys a photon of wave vector $\mathbf{k}$ if there was already one present in the system. The presence of the additional $\hbar\omega/2$ in each term is more troubling as the sum over all modes diverges. While we may resolve this issue by neglecting the vacuum energy for large frequencies, it is an extremely important term in the Hamiltonian and leads to interesting quantum phenomena such as spontaneous emission, the Lamb shift and the Casimir effect [35].

1.10 Quantized Field Coupled to a Two Level Atom

Many of the most studied phenomena in quantum optics involve the transition of an atom between various energy levels. In the absence of an an electromagnetic field, eigenstates of the Hamiltonian are by definition fixed points of the dynamics introduced by the Schrödinger equation. Once the particle is coupled to an electromagnetic field, these states are no longer eigenstates for the new Hamiltonian and can transition to higher or lower energy levels. A simple, yet effective model for studying these transitions considers a two level stationary atom. The atom can be found either in its ground state $|g\rangle$ where the energy has been shifted to 0, or its excited energy state $|e\rangle$ with an energy $\hbar\Omega$ [35]. There are raising and lowering operators $\sigma^\dagger$ and σ which when applied to one of the states outputs the other. That is

$$\sigma^\dagger|g\rangle = |e\rangle \tag{1.31a}$$

$$\sigma|e\rangle = |g\rangle \ . \tag{1.31b}$$

Since the atom has only one excited state above the ground state, these operators satisfy fermionic anticommutation relations

$$\{\sigma, \sigma^\dagger\} = 1 \ , \tag{1.32}$$

and the Hamiltonian associated to just the atom is given by

$$H_A = \hbar\Omega\sigma^\dagger\sigma \ . \tag{1.33}$$

The field $\mathbf{E}$ then interacts with the atom through the dipole moment u and leads to an interaction Hamiltonian, H_I, which is bilinear in the creation/annihilation operators of the field and raising/lowering operators associated to the atom. Specifically we have

$$H_I = \sum_{\mathbf{k},s} \hbar(\sigma + \sigma^\dagger)(g_{\mathbf{k}} a_{\mathbf{k}} + g^*_{\mathbf{k}} a^\dagger_{\mathbf{k}}) \ , \tag{1.34}$$

where $g_{\mathbf{k}}$ is the coupling of the atom to the electric field mode of wavevector $\mathbf{k}$ and polarization s. It is common to employ the rotating wave approximation (RWA) in which we neglect the quickly oscillating terms which do not conserve the number of excitations in the system, namely $\sigma^\dagger a^\dagger_{\mathbf{k}}$ and $\sigma a_{\mathbf{k}}$. It is straightforward to generalize this to a system of N two level atoms located at the points $\mathbf{x}_1, \cdots, \mathbf{x}_N$. There are now a collection of N pairs of raising and lowering operators σ_j and $\sigma_j^\dagger$, $j = 1, \cdots, N$, which satisfy anticommutation relations of the form

$$\{\sigma_j, \sigma_k^\dagger\} = \delta_{jk} \ , \tag{1.35}$$

and the atomic and interaction Hamiltonians become

$$H_A = \sum_j \hbar\Omega_j \sigma_j^\dagger \sigma_j \ , \tag{1.36a}$$

$$H_I = \sum_{j,\mathbf{k},s} \hbar(g_\mathbf{k}^* e^{-i\mathbf{k}\cdot\mathbf{x}_j} \sigma_j a_\mathbf{k}^\dagger + g_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}_j} \sigma_j^\dagger a_\mathbf{k}) \ . \tag{1.36b}$$

In order to solve for the dynamics of the system with given initial conditions, one expresses the general state of the system in a basis of simple tensors of the atomic and field states with time dependent coefficients.. These simple tensors are the eigenvectors of the sum $H_A + H_F$. By plugging these states into the Schrödinger equation and projecting back onto the basis vectors, one arrives at a system of coupled ordinary differential equations for the coefficients. However the number of these equations grows exponentially in the number of atoms present in the system and hence are costly to solve. This is analogous to the situation of the Foldy-Lax equations which arise in classical optics with N point scatterers.

1.11 Spontaneous Emission

As previously discussed, a single excited two level atom which is not coupled to an electromagnetic field will not decay to the ground state; however, in the presence of such a field the atom will transition to a lower energy state by emitting a photon, even if electromagnetic field has no photons present. This phenomenon is known as spontaneous emission. In 1916 Einstein derived a heuristic value for the rate of spontaneous emission in a cavity of two level atoms [26]. In 1930, physicists Eugene Wigner and Victor Weisskopf showed that in the case of a single two level atom initially in its excited state, the probability of finding this atom in the excited state at later times decays exponentially on short time scales [92]. This result, known as Wigner Weisskopf theory, has corrections at longer time scales which have also been investigated [64]. Here we will reproduce a calculation of Wigner Weisskopf spontaneous emission for the scalar model described in the previous section.

Recall that the Hamiltonian $H = H_A + H_F + H_I$ is given by

$$H_A = \hbar\Omega\sigma^\dagger\sigma \ , \tag{1.37a}$$
$$H_F = \sum_{\mathbf{k}} \hbar\omega_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \ , \tag{1.37b}$$
$$H_I = \hbar g \sum_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger \sigma + \sigma^\dagger a_{\mathbf{k}} \right) \ , \tag{1.37c}$$

where we have neglected the polarization of the photons and also made the Markovian approximation in which each mode of the field couples to the atom with the same strength $g_{\mathbf{k}} = g$. The general state of the system can be expressed in basis of simple tensors as

$$|\Psi(t)\rangle = c(t)|e, 0\rangle + \sum_{\mathbf{k}} c_{\mathbf{k}}(t)|g, 1_{\mathbf{k}}\rangle \ . \tag{1.38}$$

By substituting Eq. (1.38) into the Schrödinger equation

$$i\hbar\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle \ , \tag{1.39}$$

along with the definition of the Hamiltonian H given in Eq. (1.37) and projecting onto the state $|e, 0\rangle$ and $|g, 1_{\mathbf{k}}\rangle$ respectively, we arrive at a system of coupled ordinary differential equations

$$i\partial_t c = \Omega c + g \sum_{\mathbf{k}} c_{\mathbf{k}} \ , \tag{1.40a}$$
$$i\partial_t c_{\mathbf{k}} = \omega_k c_{\mathbf{k}} + gc \ , \tag{1.40b}$$

along with the initial conditions

$$c(0) = 1, \quad c_{\mathbf{k}}(0) = 0 \quad \text{for all } \mathbf{k} \ , \tag{1.40c}$$

which correspond to the atom initially in its excited state and no photons present in the electromagnetic field. Using the Laplace transform defined by

$$f(z) = \int_0^\infty e^{-zt} f(t) dt \ , \tag{1.41}$$

we can solve for $c(z)$ as

$$c(z) = \frac{1}{z + i\Omega + i\Sigma(z)} \ , \tag{1.42}$$

where the self-energy $\Sigma(z)$ is defined

$$\Sigma(z) = g^2 \sum_{\mathbf{k}} \frac{1}{\omega_k - iz + i\epsilon} \ . \tag{1.43}$$

The term $i\epsilon$ is a small regularization parameter which will be sent to 0 during the calculation. In order to compute the inverse Laplace transform, and arrive at a result for the amplitude $c(t)$ we make an approximation due to Wigner and Weisskopf in which we replace $\Sigma(z)$ by it's value of greatest contribution, $\Sigma(-i\Omega)$. This approximation is sometimes referred to as the quasiparticle pole (QPP) approximation. Upon making this approximation and inverting the Laplace transform we find that

$$c(t) = e^{-i(\Omega+\delta\Omega)t} e^{-\Gamma t/2} \ , \tag{1.44}$$

where $\delta\Omega$ and $-\Gamma/2$ are the real and imaginary parts of $\Sigma(-i\Omega)$ respectively. $\delta\Omega$ is commonly referred to as the Lamb shift and Γ is the decay rate of the atom. The decay rate can be computed directly

$$\Gamma = \frac{V g^2 \Omega^2}{\pi c^3} \ , \tag{1.45}$$

and the probability of finding the atom in its excited state is

$$|c(t)|^2 = e^{-\Gamma t} \ , \tag{1.46}$$

which is seen to decay exponentially.

1.12 Stimulated Emission

A related phenomenon is that of stimulated emission. This is the process by which the presence of a photon in an electromagnetic field coupled to an atom may interact with the atom causing it to drop to a lower energy level. We may also use the scalar model from the previous section to calculate the rate of stimulated emission for a single two level atom located at the origin. For this we assume that initially there is exactly one photon present in the field with frequency $\mathbf{k}_0$ and the atom is in its excited state, which corresponds to the initial condition

$$|\psi(0)\rangle = |1, 1_{\mathbf{k}_0}\rangle \; . \tag{1.47}$$

The general two excitation state of the system is given by

$$|\Psi(t)\rangle = \sum_{\mathbf{k}} C_{\mathbf{k}}(t)|e, 1_{\mathbf{k}}\rangle + \sum_{\mathbf{k},\mathbf{k}'} D_{\mathbf{k},\mathbf{k}'}(t)|g, 1_{\mathbf{k}} 1_{\mathbf{k}'}\rangle \; , \tag{1.48}$$

and our initial condition on the state $|\Psi\rangle$ imposes initial conditions on the $C_{\mathbf{k}}$'s and $D_{\mathbf{k},\mathbf{k}}$'s of the form

$$C_{\mathbf{k}}(0) = \delta_{\mathbf{k},\mathbf{k}_0} \; , \tag{1.49a}$$

$$D_{\mathbf{k},\mathbf{k}'}(0) = 0 \; , \quad \text{for all } \mathbf{k}, \mathbf{k}' \; . \tag{1.49b}$$

Inserting the general state into the Schrödinger equation and projecting onto the vectors $|e, 1_{\mathbf{k}}\rangle$ and $|g, 1_{\mathbf{k}} 1_{\mathbf{k}'}\rangle$ we arrive at the following system of coupled ordinary differential equations

$$i\partial_t C_{\mathbf{k}} = (\omega_0 + \omega_k)C + 2g \sum_{\mathbf{k}'} D_{\mathbf{k},\mathbf{k}'} \; , \tag{1.50a}$$

$$i\partial_t D_{\mathbf{k},\mathbf{k}'} = (\omega_k + \omega_{k'})D_{\mathbf{k},\mathbf{k}'} + \frac{g}{2}\left(C_{\mathbf{k}} + gC_{\mathbf{k}'}\right) \; . \tag{1.50b}$$

By applying the Laplace transform given in Eq. (1.41), we may eliminate the coefficient $D_{\mathbf{k},\mathbf{k}'}(z)$ and arrive at an algebraic equation for $C_{\mathbf{k}}(z)$. We apply an approximation analogous to the quasi-particle pole approximation of Wigner and Weisskopf and invert the Laplace transform to arrive at equations for the coefficients $C_{\mathbf{k}}(t)$

$$C_{\mathbf{k}}(t) = \frac{g^2 e^{-i(\omega_0-\delta\omega)t-\Gamma t/2}}{(\omega_{k_0}-\omega_0)(\omega_{k_0}-\omega_k)}\left[e^{-i\omega_k t}-e^{-i\omega_{k_0} t}\right] \ , \mathbf{k} \neq \mathbf{k}_0 \ , \tag{1.51a}$$

$$C_{\mathbf{k}_0}(t) = e^{-i(\omega_0+\omega_{k_0}-\delta\omega)t-\Gamma t/2}\left[1+\frac{ig^2 t}{\omega_{k_0}-\omega_0}\right] \ , \tag{1.51b}$$

where Γ and $\delta\omega$ are given by the same formulas as in the previous section. This shows that the coefficients $|C_{\mathbf{k}}|^2$ still decay at a rate of Γ in time.

Chapter 2

Radiative Transport in Quasi-homogeneous Random Media

2.1 Introduction

The theory of light scattering is a subject of fundamental interest and considerable applied importance. In the multiple-scattering regime, radiative transport theory is widely used to describe the propagation of light on macroscopic scales [20, 40, 18, 58, 7, 61, 6]. Nevertheless, the microscopic origins of the theory remain a topic of current research. Indeed, two conceptually different approaches to the derivation of the radiative transport equation (RTE) have been advanced. Both are based on the theory of wave propagation in random media, in one case proceeding by diagrammatic perturbation theory [11, 91, 31, 57], while in the other by multiscale asymptotic analysis [48, 32]. A comparative exposition of the two approaches has been presented in [19].

The fundamental physical quantity of radiative transport theory is the specific intensity $I(\mathbf{r}, \hat{\mathbf{s}})$, which is defined as the intensity of light at the position $\mathbf{r}$ in the direction $\hat{\mathbf{s}}$. The specific intensity obeys the RTE

$$\hat{\mathbf{s}} \cdot \nabla I + \mu_e I = \mu_s \int d\hat{\mathbf{s}}' A(\hat{\mathbf{s}}, \hat{\mathbf{s}}') I(\mathbf{r}, \hat{\mathbf{s}}') \ , \tag{2.1}$$

which is a conservation law that accounts for gains and losses of electromagnetic energy.

Here μ_e and μ_s are the extinction and scattering coefficients of the medium, and $\mu_e = \mu_s$ in the absence of absorption. It is important to note that if the random medium is *statistically homogeneous and isotropic*, then μ_e and μ_s are constant [19].

Up to now, the derivation of the RTE has been restricted to statistically homogeneous media. Indeed, the foundations of radiative transport theory for inhomogeneous media, where μ_e and μ_s are generally position-dependent, are not firmly established [58]. Since inhomogeneous media are of great interest in applications, particularly to biomedical imaging [8], the problem of justifying radiative transport theory in this setting is of some importance.

Quasi-homogeneous media are a broad class of random media with correlations that are rapidly varying on small length scales and slowly varying on large scales [78, 17]. They have been extensively studied in the weak-scattering regime, especially in connection with coherence theory and inverse problems [28, 29, 87, 95, 22]. In comparison to homogeneous media, the aforementioned separation of scales compensates for the loss of translational invariance on average.

In this paper, we investigate the theory of radiative transport in quasi-homogeneous media. By making use of diagrammatic perturbation theory, we derive the radiative transport equation that governs the propagation of light in such media. This result provides sufficient conditions under which it is justified to apply radiative transport theory to spatially-inhomogeneous media.

The remainder of the paper is organized as follows. In Section 2, we introduce the theory of scalar waves in quasi-homogeneous random media. The computation of the average field using diagrammatic perturbation theory is presented in Section 3, which is followed by a discussion of field correlations in Section 4. The RTE is then derived in Section 5. Our conclusions are formulated in Section 6.

2.2 Quasi-homogenous Media

We consider the propagation of a monochromatic scalar wave in a random medium. The field U obeys the wave equation

$$\nabla^2 U(\mathbf{r}) + k_0^2(1 + 4\pi\eta(\mathbf{r}))U(\mathbf{r}) = -4\pi S(\mathbf{r}) \ , \tag{2.2}$$

where k_0 is the wave number in vacuum and S is the source. For simplicity, the effects of polarization are not considered and the susceptibility η is taken to be purely real, so that the medium is nonabsorbing. We also assume that η is a Gaussian random field with correlations

$$\langle \eta(\mathbf{r}) \rangle = 0 \ , \tag{2.3}$$

$$\langle \eta(\mathbf{r})\eta(\mathbf{r}') \rangle = C(\mathbf{r}, \mathbf{r}') \ , \tag{2.4}$$

where C is the two-point correlation function and $\langle \cdots \rangle$ denotes statistical averaging. If C depends only upon the quantity $|\mathbf{r} - \mathbf{r}'|$, the medium is said to be statistically homogeneous and isotropic. In this situation, the specific intensity obeys the RTE with *constant* coefficients and a phase function that is invariant under rotations [19].

In a quasi-homogeneous random medium, the correlation function is taken to be of the form

$$C(\mathbf{r}, \mathbf{r}') = C_f(|\mathbf{r} - \mathbf{r}'|)C_s\left((\mathbf{r} + \mathbf{r}')/2\right) \ , \tag{2.5}$$

where C_s varies more slowly than C_f. Such a model describes a medium that is homogeneous and isotropic on small length scales, but also varies over large scales. Possible examples include biological tissue and the atmosphere. A case of particular interest has correlations of the form

$$C_f(\mathbf{r}) = C_0 e^{-r^2/l_f^2} \ , \quad C_s(\mathbf{r}) = e^{-r^2/l_s^2} \ , \tag{2.6}$$

where C_0 is constant and the correlation lengths l_s, l_f obey the condition $l_s \gg l_f$. We also require that $k_0 l_f \gg 1$, so that the average field varies slowly on the scale of the wavelength, meaning that spatial dispersion can be neglected. Models with non-Gaussian correlations can also be considered.

In an infinite medium, the solution to the wave equation (2.2) obeying the outgoing radiation condition is of the form

$$U(\mathbf{r}) = \int d^3r' G(\mathbf{r}, \mathbf{r}') S(\mathbf{r}') \ . \tag{2.7}$$

Here the Green's function G obeys the integral equation [14]

$$G(\mathbf{r}, \mathbf{r}') = G_0(\mathbf{r}, \mathbf{r}') + k_0^2 \int d^3r'' G_0(\mathbf{r}, \mathbf{r}'') \eta(\mathbf{r}'') G(\mathbf{r}'', \mathbf{r}') \ , \tag{2.8}$$

where the unperturbed Green's function G_0 is given by

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{2.9}$$

For later reference, we note that G_0 can be written as the Fourier integral

$$G_0(\mathbf{r}, \mathbf{r}') = 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2 - k_0^2 - i\epsilon} \ , \tag{2.10}$$

where the limit $\epsilon \to 0^+$ is implied, consistent with the outgoing radiation condition that G_0 obeys.

2.3 Average Field

It follows immediately from Eq. (2.7) that the average field is determined by the average Green's function, provided that the source S is deterministic. Following standard procedures [19], the average Green's function can be seen to obey the Dyson equation

$$\langle G(\mathbf{r}, \mathbf{r}') \rangle = G_0(\mathbf{r}, \mathbf{r}') + \int d^3r_1 d^3r_2 G_0(\mathbf{r}, \mathbf{r}_1) \Sigma(\mathbf{r}_1, \mathbf{r}_2) \langle G(\mathbf{r}_2, \mathbf{r}') \rangle \ , \tag{2.11}$$

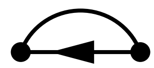
$\Sigma =$ 

Figure 2.1: The lowest order self-energy diagram.

where Σ is the self-energy. There is a convenient diagrammatic expansion for Σ. To lowest order in perturbation theory, which is known as the weak-scattering approximation, Σ is given by

$$\Sigma(\mathbf{r}_1, \mathbf{r}_2) = k_0^4 G_0(\mathbf{r}_1, \mathbf{r}_2) C(\mathbf{r}_1, \mathbf{r}_2) \ . \tag{2.12}$$

The corresponding diagram is shown in Fig. 2.1. A straight line with a left pointing arrow corresponds to a factor of the Green's function G_0 and a line connecting two vertices corresponds to a factor of $k_0^4 C$.

For a statistically homogeneous medium, Eq. (2.11) can be solved by Fourier transformation due to the translational invariance of $\langle G \rangle$ and Σ [19]. We will see that quasi-homogeneous media can also be analyzed, but the loss of translational invariance must be properly handled. To proceed, we transform to relative and center of mass coordinates according to

$$\mathbf{R} = \frac{\mathbf{r} + \mathbf{r}'}{2} \ , \quad \mathbf{R}' = \mathbf{r} - \mathbf{r}' \ . \tag{2.13}$$

Eq. (2.11) thus becomes

$$\begin{aligned} \mathscr{G}(\mathbf{R}, \mathbf{R}') = \mathscr{G}_0(\mathbf{R}') + \int d^3 r_1 d^3 r_2 \mathscr{G}_0(\mathbf{R} + \mathbf{R}'/2 - \mathbf{r}_1) \Sigma(\mathbf{r}_1, \mathbf{r}_2) \\ \times \mathscr{G}((\mathbf{r}_2 + \mathbf{R} - \mathbf{R}'/2)/2, \mathbf{r}_2 - \mathbf{R} + \mathbf{R}'/2) \ , \end{aligned} \tag{2.14}$$

where

$$\mathscr{G}(\mathbf{R}, \mathbf{R}') = \langle G(\mathbf{R} + \mathbf{R}'/2, \mathbf{R} - \mathbf{R}'/2) \rangle \ , \quad \mathscr{G}_0(\mathbf{r} - \mathbf{r}') = G_0(\mathbf{r}, \mathbf{r}') \ . \tag{2.15}$$

It will prove useful to introduce the Fourier transform of $\mathscr{G}$ with respect to $\mathbf{R}'$:

$$\tilde{\mathscr{G}}(\mathbf{R},\mathbf{k}) = \int d^3R' e^{i\mathbf{k}\cdot\mathbf{R}'} \mathscr{G}(\mathbf{R},\mathbf{R}') \ , \tag{2.16}$$

which allows the treatment of the large-scale variations of the medium in real space, and the high-frequency variations in Fourier space. After straightforward calculation, we find that

$$\widetilde{\mathscr{G}}(\mathbf{R},\mathbf{k}) = \widetilde{\mathscr{G}}_0(\mathbf{k}) + C_s(\mathbf{R})\widetilde{\mathscr{G}}_0(\mathbf{k})\Sigma_0(\mathbf{k})\widetilde{\mathscr{G}}(\mathbf{R},\mathbf{k}) \ , \tag{2.17}$$

where

$$\widetilde{\mathscr{G}}_0(\mathbf{k}) = \frac{4\pi}{k^2 - k_0^2 - i\epsilon} \ , \tag{2.18}$$

$$\Sigma_0(\mathbf{k}) = k_0^4 \int \frac{d^3k'}{(2\pi)^3} \frac{\widetilde{C}_f(\mathbf{k}-\mathbf{k}')}{k'^2 - k_0^2 - i\epsilon} \ , \tag{2.19}$$

and we have used Eqs. (2.10) and (2.12). We have also made use of the approximations

$$C_s(\mathbf{R}+\mathbf{R}') \approx C_s(\mathbf{R}) \ , \quad \widetilde{\mathscr{G}}(\mathbf{R}+\mathbf{R}',\mathbf{k}) \approx \widetilde{\mathscr{G}}(\mathbf{R},\mathbf{k}) \ , \tag{2.20}$$

consistent with the slow variation of C_s. It follows immediately from Eqs. (2.17) and (2.18) that

$$\widetilde{\mathscr{G}}(\mathbf{R},\mathbf{k}) = \frac{4\pi}{k^2 - k_0^2 - 4\pi C_s(\mathbf{R})\Sigma_0(\mathbf{k}) - i\epsilon} \ . \tag{2.21}$$

Next, we separate Σ_0 into its real and imaginary parts by making use of the identities

$$\frac{1}{k^2 - k_0^2 - i\epsilon} = P\frac{1}{k^2 - k_0^2} + i\pi\delta(k^2 - k_0^2) \ , \tag{2.22}$$

$$\delta(k^2 - k_0^2) = \frac{1}{2k_0}\left(\delta(k-k_0) + \delta(k+k_0)\right) \ , \tag{2.23}$$

where P denotes the principal value. We thus obtain

$$\mathrm{Re}\,\Sigma_0(\mathbf{k}) = k_0^4 P \int \frac{d^3k'}{(2\pi)^3} \frac{\widetilde{C}_f(\mathbf{k}-\mathbf{k}')}{k'^2 - k_0^2} , \tag{2.24}$$

$$\mathrm{Im}\,\Sigma_0(\mathbf{k}) = k_0^5 \int \frac{d\hat{\mathbf{s}}'}{4\pi} \widetilde{C}_f(\mathbf{k} - k_0\hat{\mathbf{s}}') , \tag{2.25}$$

where we have used the fact that C_f is real-valued. Now, $\mathrm{Re}\,\Sigma_0$ can be neglected by introducing a high-frequency cutoff in Eq. (2.24). Physically, such a cutoff corresponds to introducing a minimum scale for the spatial variations of C_f (the size of the smallest scatterer). Let us define the scattering length ℓ_s (on-shell) as

$$\frac{1}{\ell_s} = k_0^4 \int d\hat{\mathbf{s}}' \widetilde{C}_f(k_0|\hat{\mathbf{s}} - \hat{\mathbf{s}}'|) , \tag{2.26}$$

which is constant due to the statistical isotropy of the medium. We now find that in the weak-scattering limit $k_0\ell_s \gg C_s$, Eq. (2.21) becomes

$$\widetilde{\mathscr{G}}(\mathbf{R},\mathbf{k}) = \frac{4\pi}{k^2 - \kappa^2(\mathbf{R}) - i\epsilon} , \tag{2.27}$$

where

$$\kappa(\mathbf{R}) = k_0 \left(1 + \frac{iC_s(\mathbf{R})}{2k_0\ell_s}\right) . \tag{2.28}$$

Performing an inverse Fourier transform to obtain $\mathscr{G}(\mathbf{R},\mathbf{R}')$ and inverting the transformation to relative and center of mass coordinates, we see that the average Green's function is given by

$$\langle G(\mathbf{r},\mathbf{r}')\rangle = \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \exp\left[-C_s\left((\mathbf{r}+\mathbf{r}')/2\right)|\mathbf{r}-\mathbf{r}'|/2l_s\right] . \tag{2.29}$$

We conclude that the average field decays exponentially on the scale of the scattering length, modulated by the slowly-varying function C_s.

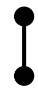
$\Gamma =$ 

Figure 2.2: The lowest order irreducible-vertex diagram.

2.4 Field Correlations

We now turn to the problem of calculating the second-order correlation function of the field. We begin by observing that

$$\langle U(\mathbf{r})U^*(\mathbf{r}')\rangle = \int d^3r_1 d^3r_2 \langle G(\mathbf{r},\mathbf{r}_1)G^*(\mathbf{r}',\mathbf{r}_2)\rangle S(\mathbf{r}_1)S^*(\mathbf{r}_2) \ , \tag{2.30}$$

which follows from Eq. (2.7) and the assumption that S is deterministic. The correlation function $\langle GG^*\rangle$ can be shown to obey the Bethe-Salpeter equation [19]

$$\begin{aligned}\langle G(\mathbf{r}_1,\mathbf{r}_2)G^*(\mathbf{r}_1',\mathbf{r}_2')\rangle &= \langle G(\mathbf{r}_1,\mathbf{r}_2)\rangle\langle G^*(\mathbf{r}_1',\mathbf{r}_2')\rangle \\ &+ \int d^3r d^3r' \langle G(\mathbf{r}_1,\mathbf{r})\rangle\langle G^*(\mathbf{r}_1',\mathbf{r}')\rangle\Gamma(\mathbf{r},\mathbf{r}') \\ &\times \langle G(\mathbf{r},\mathbf{r}_2)G^*(\mathbf{r}',\mathbf{r}_2')\rangle \ ,\end{aligned} \tag{2.31}$$

where Γ is the irreducible vertex. Here we have made the weak-scattering approximation in which Γ is of the form

$$\Gamma(\mathbf{r},\mathbf{r}') = k_0^4 C(\mathbf{r},\mathbf{r}') \ . \tag{2.32}$$

In a manner similar to the construction of the self-energy, there is an analogous diagrammatic expansion for Γ. The corresponding diagram is shown in Fig. 2.2. Note that the Bethe-Salpeter equation can be solved by iteration, leading to a sum of ladder diagrams, as shown in Fig. 2.3. The sum can be calculated analytically in a homogeneous medium with short-range correlations. At large distances, this leads to diffusive transport of light. However, for quasi-homogeneous media this approach is not applicable. Instead, it is

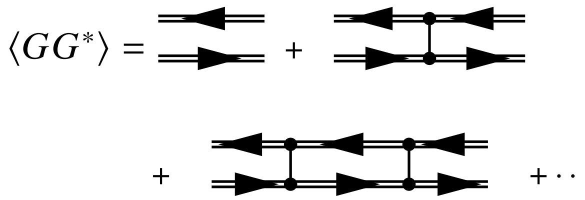


Figure 2.3: The correlation function $\langle GG^* \rangle$ in terms of ladder diagrams. A double line with a left-pointing arrow denotes a factor of $\langle G \rangle$ and a double line with a right-pointing arrow denotes a factor of $\langle G^* \rangle$.

advantageous to introduce a phase-space representation of the field correlations, which directly leads to the theory of radiative transport.

2.5 Radiative transport

In this section we present the derivation of the RTE for quasi-homogeneous media. We begin by introducing the Wigner transform of the field, which provides a phase-space representation of the field correlation function. To this end, we define the Wigner transform $W(\mathbf{r}, \mathbf{k})$ as

$$W(\mathbf{r}, \mathbf{k}) = \int \frac{d^3 r'}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}'} \langle U(\mathbf{r} - \mathbf{r}'/2) U^*(\mathbf{r} + \mathbf{r}'/2) \rangle \ . \tag{2.33}$$

The Wigner transform has several important properties. It is real-valued and related to the average intensity $I = \langle |U|^2 \rangle$ by

$$I(\mathbf{r}) = \int d^3 k W(\mathbf{r}, \mathbf{k}) \ . \tag{2.34}$$

By Fourier inversion, the Wigner transform is related to the correlation function of the field by

$$\langle U(\mathbf{r}) U^*(\mathbf{r}') \rangle = \int d^3 k e^{-\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} W\left((\mathbf{r} + \mathbf{r}')/2, \mathbf{k}\right) \ . \tag{2.35}$$

Thus the Wigner transform is a measure of the spatial coherence of the field, with randomness due to fluctuations in the medium rather than the source [94].

In order to handle the loss of translational invariance, we introduce a shift $\Delta\mathbf{r}$ in the spatial coordinate of the Wigner transform, and consider the Fourier transform of $W(\mathbf{r}+\Delta\mathbf{r},\mathbf{k})$ with respect to $\Delta\mathbf{r}$. We find, as shown in Appendix A, that $\widetilde{W}$ is given by

$$\begin{aligned}
&\widetilde{W}(\mathbf{Q},\mathbf{k}) = e^{-i\mathbf{Q}\cdot\mathbf{r}}\langle\widetilde{U}(-\mathbf{k}+\mathbf{Q}/2)\rangle\langle\widetilde{U}^*(\mathbf{k}+\mathbf{Q}/2)\rangle/(2\pi)^3 \\
&+ k_0^4 e^{-i\mathbf{Q}\cdot\mathbf{r}}\widetilde{\mathscr{G}}(\mathbf{r},-\mathbf{k}/2+\mathbf{Q})\widetilde{\mathscr{G}}^*(\mathbf{r},\mathbf{k}/2+\mathbf{Q})C_s(\mathbf{r}) \\
&\times \int \frac{d^3k'}{(2\pi)^3}\widetilde{C}_f(\mathbf{k}-\mathbf{k}')\widetilde{W}(\mathbf{Q},\mathbf{k}') \ ,
\end{aligned} \tag{2.36}$$

where the dependence of $\widetilde{W}$ on $\mathbf{r}$ is not directly indicated. Here we have made use of Eqs. (2.30), (2.31) and (2.32), and the slow variation in C_s, as reflected in the approximations (2.20). Next, we use (2.17) and the algebraic identity

$$ab = \frac{a-b}{1/b-1/a} \tag{2.37}$$

to obtain the relation

$$\widetilde{\mathscr{G}}(\mathbf{r},-\mathbf{k}/2+\mathbf{Q})\widetilde{\mathscr{G}}^*(\mathbf{r},\mathbf{k}/2+\mathbf{Q}) = \frac{\Delta\mathscr{G}(\mathbf{r},\mathbf{Q},\mathbf{k})}{2\mathbf{k}\cdot\mathbf{Q}+\Delta\Sigma(\mathbf{r},\mathbf{Q},\mathbf{k})} \ , \tag{2.38}$$

where

$$\Delta\mathscr{G}(\mathbf{r},\mathbf{Q},\mathbf{k}) = 4\pi(\widetilde{\mathscr{G}}(\mathbf{r},-\mathbf{k}+\mathbf{Q}/2) - \widetilde{\mathscr{G}}^*(\mathbf{r},\mathbf{k}+\mathbf{Q}/2)) \ , \tag{2.39}$$

$$\Delta\Sigma(\mathbf{r},\mathbf{Q},\mathbf{k}) = 4\pi C_s(\mathbf{r})\left(\Delta\Sigma_0(-\mathbf{k}+\mathbf{Q}/2) - \Delta\Sigma_0^*(\mathbf{k}+\mathbf{Q}/2)\right) \ . \tag{2.40}$$

It follows from the above that Eq. (2.36) becomes

$$\begin{aligned}&(2\mathbf{k}\cdot\mathbf{Q}+\Delta\Sigma(\mathbf{r},\mathbf{Q},\mathbf{k}))\,\widetilde{W}(\mathbf{Q},\mathbf{k}) = k_0^4\Delta\mathcal{G}(\mathbf{r},\mathbf{Q},\mathbf{k})C_s(\mathbf{r})\\&\times\int\frac{d^3k'}{(2\pi)^3}\widetilde{C}_f(\mathbf{k}-\mathbf{k}')\widetilde{W}(\mathbf{Q},\mathbf{k}')\\&+\frac{1}{(2\pi)^3}\Delta\mathcal{G}(\mathbf{r},\mathbf{Q},\mathbf{k})\widetilde{S}(-\mathbf{k}+\mathbf{Q}/2)\widetilde{S}^*(\mathbf{k}+\mathbf{Q}/2)\ .\end{aligned}\tag{2.41}$$

Finally, computing the inverse Fourier transform of $\widetilde{W}(\mathbf{Q},\mathbf{k})$, we obtain the integral equation obeyed by the Wigner transform:

$$\begin{aligned}&\mathbf{k}\cdot\nabla_{\mathbf{r}}W(\mathbf{r},\mathbf{k})+\frac{1}{2i}\int\frac{d^3q}{(2\pi)^3}\,e^{-i\mathbf{Q}\cdot\mathbf{r}}\Delta\Sigma(\mathbf{r},\mathbf{Q},\mathbf{k})\widetilde{W}(\mathbf{Q},\mathbf{k})\\&=\frac{k_0^4}{2i}\int\frac{d^3q}{(2\pi)^3}e^{-i\mathbf{Q}\cdot\mathbf{r}}\int\frac{d^3k'}{(2\pi)^3}C_s(\mathbf{r})\Delta\mathcal{G}(\mathbf{r},\mathbf{Q},\mathbf{k})\widetilde{C}_f(\mathbf{k}-\mathbf{k}')\\&\qquad\times\widetilde{W}(\mathbf{Q},\mathbf{k}')\\&+\frac{1}{2i}\int\frac{d^3q}{(2\pi)^6}e^{-i\mathbf{Q}\cdot\mathbf{r}}\Delta\mathcal{G}(\mathbf{r},\mathbf{Q},\mathbf{k})\widetilde{S}(-\mathbf{k}+\mathbf{Q}/2)\widetilde{S}^*(\mathbf{k}+\mathbf{Q}/2)\ .\end{aligned}\tag{2.42}$$

In order to obtain the RTE, we take the long wavelength limit $\mathbf{Q}\to 0$ in Eq. (2.42). That is, we assume the average Wigner transform varies slowly on the scale over which C_f varies. In performing the limit, we replace $\Delta\Sigma(\mathbf{r},\mathbf{Q},\mathbf{k})$ and $\Delta\mathcal{G}(\mathbf{r},\mathbf{Q},\mathbf{k})$ by $\Delta\Sigma(\mathbf{r},0,\mathbf{k})$ and $\Delta\mathcal{G}(\mathbf{r},0,\mathbf{k})$, respectively. In addition, we apply Eqs. (2.22) and (2.27) in the weak-scattering limit to obtain

$$\Delta\mathcal{G}(\mathbf{r},0,\mathbf{k}) = \frac{2i(2\pi)^3}{k_0}\delta(k-k_0)\ .\tag{2.43}$$

Likewise, we use Eqs. (2.25) and (2.26) to obtain

$$\Delta\Sigma(\mathbf{r},0,\mathbf{k}) = 2ikC_s(\mathbf{r})/l_s\ .\tag{2.44}$$

Using the above results along with Eq. (2.26), and defining the specific intensity I, phase function A, scattering coefficient μ_s and source I_0 by

$$\delta(k-k_0)I(\mathbf{r},\hat{\mathbf{s}}) = k_0 W(\mathbf{r},k\hat{\mathbf{s}}) \ , \tag{2.45}$$

$$A(\hat{\mathbf{s}},\hat{\mathbf{s}}') = k_0^4 l_s \widetilde{C}_f(k_0(\hat{\mathbf{s}}-\hat{\mathbf{s}}')) \ , \tag{2.46}$$

$$\mu_s(\mathbf{r}) = C_s(\mathbf{r})/l_s \ , \tag{2.47}$$

$$I_0(\mathbf{r},\hat{\mathbf{s}}) = \frac{1}{k_0^2}\int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{Q}\cdot\mathbf{r}} \widetilde{S}(-k_0\hat{\mathbf{s}}+\mathbf{Q}/2)\widetilde{S}^*(k_0\hat{\mathbf{s}}+\mathbf{Q}/2) \ , \tag{2.48}$$

we find that Eq. (2.42) becomes

$$\hat{\mathbf{s}}\cdot\nabla_{\mathbf{r}}I(\mathbf{r},\hat{\mathbf{s}}) + \mu_s(\mathbf{r})I(\mathbf{r},\hat{\mathbf{s}}) = \mu_s(\mathbf{r})\int d\hat{\mathbf{s}}' A(\hat{\mathbf{s}},\hat{\mathbf{s}}')I(\mathbf{r},\hat{\mathbf{s}}') + I_0(\mathbf{r},\hat{\mathbf{s}}) \ . \tag{2.49}$$

Eq. (2.49) is the RTE in a quasi-homogeneous random medium. Several comments on this result are necessary. First, when C_S is constant, Eq. (2.49) reduces to the RTE in a homogeneous medium. Second, we note that μ_s and A are determined by correlations in the medium. Since C_f is a function of $|\mathbf{r}-\mathbf{r}'|$, A depends only on $\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}'$, and μ_s does not depend on the direction $\hat{\mathbf{s}}$. Third, for the case of short-range correlations we have $C_f(\mathbf{r}) = C_0\delta(\mathbf{r})$, where C_0 is constant. It follows that $\mu_s(\mathbf{r}) = C_0 k_0^4 C_s(\mathbf{r})$ and $A = 1/(4\pi)$, which corresponds to spatially-modulated isotropic scattering. Fourth, our results are easily generalized to allow the susceptibility η to be complex-valued. It can then be seen that the extinction coefficient μ_e is proportional to the imaginary part of the self-energy. Fifth, the Wigner transform, although real-valued, is not necessarily nonnegative. In contrast, the specific intensity cannot take on negative values. This inconsistency can be resolved by noting that only the high-frequency behavior of the Wigner transform is of physical interest [19]. Finally, under certain conditions, the solution to the RTE can be well approximated by the solution to a diffusion equation. This so-called diffusion approximation is widely used in applications [18, 8]. For quasi-homogeneous media, it is easily seen that the diffusion approximation can be derived from Eq. (2.49), and that the corresponding diffusion equation is of the form

$$-\nabla \cdot D(\mathbf{r})\nabla u(\mathbf{r}) = Q(\mathbf{r}) \ . \tag{2.50}$$

Here the energy density u and source Q are defined by

$$u(\mathbf{r}) = \frac{1}{c}\int d\hat{\mathbf{s}} I(\mathbf{r}, \hat{\mathbf{s}}) \ , \quad Q(\mathbf{r}) = \int d\hat{\mathbf{s}} I_0(\mathbf{r}, \hat{\mathbf{s}}) \ , \tag{2.51}$$

and the diffusion coefficient is given by

$$D(\mathbf{r}) = \frac{c}{3(1-g)\mu_s(\mathbf{r})} \ , \quad g = \int \hat{\mathbf{s}} \cdot \hat{\mathbf{s}}' A(\hat{\mathbf{s}}, \hat{\mathbf{s}}') d\hat{\mathbf{s}}' \ , \tag{2.52}$$

where g is the anisotropy of scattering and c is the speed of light.

2.6 Discussion

We have derived the radiative transport equation that governs the propagation of multiply-scattered scalar waves in quasi-homogeneous random media. An alternative derivation of Eq. (2.49) may be possible using a multiscale asymptotic analysis [48, 32]. This is consistent with the equivalence, for homogeneous media, of diagrammatic perturbation theory and multiscale asymptotics [19]. We note that it would be of interest to extend our results to the radiative transport of electromagnetic and elastic waves. In the electromagnetic case, the calculation of the self-energy for quasi-homogeneous media poses a challenge, and thus the asymptotic approach to the derivation of the RTE may be worthy of exploration. The associated inverse problems is likely to have applications to polarization-sensitive imaging with diffuse light [8].

2.7 Appendix

In this appendix, we present the derivation of Eq. (2.36). We begin by noting that the Bethe-Salpeter equation (2.31) in combination with Eqs. (2.7) and (2.32) can be used to obtain the correlation function of the field. We thus obtain

$$\langle U(\mathbf{r})U(\mathbf{r}')\rangle = \langle U(\mathbf{r})\rangle\langle U(\mathbf{r}')\rangle + k_0^4 \int d^3R d^3R' \langle G(\mathbf{r},\mathbf{R})\rangle\langle G^*(\mathbf{r}',\mathbf{R}')\rangle C_f(\mathbf{R}-\mathbf{R}') \times C_s((\mathbf{R}+\mathbf{R}')/2)\langle U(\mathbf{R})U(\mathbf{R}')\rangle \ . \tag{2.53}$$

We introduce the shifted Wigner transform $W(\mathbf{r}+\Delta\mathbf{r},\mathbf{k})$, whose Fourier transform with respect to $\Delta\mathbf{r}$ is given by

$$\widetilde{W}(\mathbf{Q},\mathbf{k}) = \widetilde{W}_1(\mathbf{Q},\mathbf{k}) + \widetilde{W}_2(\mathbf{Q},\mathbf{k}) \ , \tag{2.54}$$

where

$$\widetilde{W_1}(\mathbf{Q},\mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3\Delta r d^3r' e^{i\mathbf{Q}\cdot\Delta\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}'} \langle U(\mathbf{r}+\Delta\mathbf{r}-\mathbf{r}'/2)\rangle\langle U^*(\mathbf{r}+\Delta\mathbf{r}+\mathbf{r}'/2)\rangle \tag{2.55}$$

and

$$\widetilde{W_2}(\mathbf{Q},\mathbf{k}) = \frac{k_0^4}{(2\pi)^3} \int d^3\Delta r d^3r' d^3R d^3R' e^{i\mathbf{Q}\cdot\Delta\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}'} \langle G(\mathbf{r}+\Delta\mathbf{r}-\mathbf{r}'/2,\mathbf{R})\rangle \times \langle G^*(\mathbf{r}+\Delta\mathbf{r}+\mathbf{r}'/2,\mathbf{R}')\rangle C_f(\mathbf{R}-\mathbf{R}') C_s((\mathbf{R}+\mathbf{R}')/2)\langle U(\mathbf{R})U(\mathbf{R}')\rangle \ . \tag{2.56}$$

Making a change of variables to relative and center of mass coordinates, $\widetilde{W_1}(\mathbf{Q},\mathbf{k})$ becomes

$$\widetilde{W_1}(\mathbf{Q},\mathbf{k}) = \frac{e^{-i\mathbf{Q}\cdot\mathbf{r}}}{(2\pi)^3} \langle \widetilde{U}(-\mathbf{k}+\mathbf{Q}/2)\rangle\langle \widetilde{U}(\mathbf{k}+\mathbf{Q}/2)\rangle \ , \tag{2.57}$$

where $\widetilde{U}$ is the Fourier transform of U. Recalling the definition of $\mathscr{G}$ from Eq. (2.15), Eq. (2.56) becomes

$$\widetilde{W_2}(\mathbf{Q},\mathbf{k}) = \frac{k_0^4}{(2\pi)^3}\int d^3\Delta r d^3r' d^3R d^3R' e^{i\mathbf{Q}\cdot\Delta\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}'}$$
$$\times\mathscr{G}((\mathbf{r}+\Delta\mathbf{r}-\mathbf{r}'/2+\mathbf{R})/2, \mathbf{r}+\Delta\mathbf{r}-\mathbf{r}'/2-\mathbf{R})$$
$$\times\mathscr{G}^*((\mathbf{r}+\Delta\mathbf{r}+\mathbf{r}'/2+\mathbf{R}')/2, \mathbf{r}+\Delta\mathbf{r}+\mathbf{r}'/2-\mathbf{R}')C_f(\mathbf{R}-\mathbf{R}')$$
$$\times C_s((\mathbf{R}+\mathbf{R}')/2)\langle U(\mathbf{R})U(\mathbf{R}')\rangle \ . \tag{2.58}$$

Upon making further a change of variables to relative and center of mass coordinates, invoking the approximations (similar to Eq. (2.20))

$$\mathscr{G}(\mathbf{r}+\Delta\mathbf{r},\mathbf{r}') \approx \mathscr{G}(\mathbf{r},\mathbf{r}') \ , \tag{2.59}$$

$$C_s(\mathbf{r}+\Delta\mathbf{r}) \approx C_s(\mathbf{r}) \ , \tag{2.60}$$

and defining $K(\mathbf{r},\mathbf{r}') = C_f(\mathbf{r}')\langle U(\mathbf{r}-\mathbf{r}'/2)U(\mathbf{r}+\mathbf{r}'/2)\rangle$, Eq. (2.58) becomes

$$\widetilde{W_2}(\mathbf{Q},\mathbf{k}) = \frac{k_0^4}{(2\pi)^3}\int d^3\Delta r d^3r' d^3R d^3R' \, e^{i\mathbf{Q}\cdot(\Delta\mathbf{r}+\mathbf{R})} e^{i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{R}')}\mathscr{G}(\mathbf{r},\mathbf{r}-\mathbf{r}'/2+\mathbf{R}') \tag{2.61}$$
$$\times\mathscr{G}^*(\mathbf{r},\mathbf{r}+\mathbf{r}'/2-\mathbf{R}')C_s(\mathbf{r})K(\mathbf{r}+\mathbf{R},\mathbf{R}') \ .$$

Carrying out the indicated Fourier transforms in the second arguments of $\mathscr{G}, \mathscr{G}^*$ and K, we find that

$$\widetilde{W_2}(\mathbf{Q},\mathbf{k}) = k_0^4 e^{-i\mathbf{Q}\cdot\mathbf{r}}\widetilde{\mathscr{G}}(\mathbf{r},-\mathbf{k}/2+\mathbf{Q})\widetilde{\mathscr{G}^*}(\mathbf{r},\mathbf{k}/2+\mathbf{Q})C_s(\mathbf{r})\int\frac{d^3k'}{(2\pi)^3}\widetilde{C}_f(\mathbf{k}-\mathbf{k}')\widetilde{W}(\mathbf{Q},\mathbf{k}') \ . \tag{2.62}$$

Finally, putting Eqs. (2.57) and (2.62) together, we obtain Eq. (2.36).

Chapter 3

Collective Spontaneous Emission in Random Media

3.1 Introduction

The quantum theory of light-matter interactions has historically been concerned with systems consisting of a small number of atoms [58]. To some extent, this is due to the early emphasis on such systems in atomic physics. However, the recent focus on cold atom systems [38, 33], waveguide quantum electrodynamics [96, 23], and semiconductor quantum optics [46], has served to stimulate research on quantum many-body problems. Progress in this direction can be expected to lead to significant advances in controlling quantum systems, with applications to quantum simulations, quantum information processing, and precision measurements [45, 37, 39].

Perhaps the simplest many-body problem in quantum optics arises in a system of two-level atoms interacting with a single photon. Suppose that one of the atoms is initially in its excited state and there are no photons present in the field. The atom can then decay by spontaneous emission, thereby transferring its excitation to the field. The resulting photon can then excite the remaining atoms, which likewise decay. This process, which is referred to as collective or cooperative emission, results in the transmission of light through the system. Two regimes are usually distinguished, depending on the wavelength and the size of the system: superradiance and radiation trapping. In single-photon superradiance, certain states decay much faster than the single-atom decay rate. Alternatively, there is very slow decay, and the states are said to be trapped. Moreover, in contrast to single-atom spontaneous emission, where the Lamb shift is divergent, the Lamb shift can be finite in single-photon superradiance.

The theory of collective emission has been considered from several points of view. One approach is based on a Hamiltonian describing the atoms, the optical field and their interaction. Eliminating the optical field yields an effective Hamiltonian for the atomic degrees of freedom [50, 44, 85, 9, 60]. A master equation can then be derived, and has been shown to describe quantum effects in light scattering. However, the computational cost of this procedure, which scales exponentially with the number of atoms, limits its utility to systems consisting of a small number of atoms. An alternative approach, which makes use of the eigenstates and corresponding eigenvalues of the effective Hamiltonian, can be employed to describe the dynamics of the system [69, 70, 68, 56, 2, 1, 86]. This method is especially fruitful in the setting of single-photon superradiance, where analytical expressions for the collective decay rate have been obtained for dense atomic gases.

In this paper, we consider the problem of collective emission for a random medium of two-level atoms. We emphasize that randomness is employed as a proxy for information about the medium. That is, the medium is modeled as a realization of a random process with known statistics. In this setting, we investigate the dynamics of the field and atomic probability amplitudes for a one-photon state of the system. At long times and large distances, we find that the corresponding average probability densities can be determined from the solutions to a pair of kinetic equations. There are several novel mathematical aspects of our work. We employ a real-space quantization procedure for the optical field. In contrast, quantization of the field is normally carried out in terms of Fourier modes. The advantage of the real-space approach is that it allows the field and atomic degrees of freedom to be treated on an equal footing. Moreover, the field and atomic probability amplitudes obey a system of *nonlocal* partial differential equations with random coefficients. Using this result, we show that the average Wigner transform of the amplitudes obeys a kinetic equation, whose diffusion limit is extracted. Here the average over the random medium is carried out by means of a multiscale asymptotic expansion in a suitable high-frequency limit [48, 10, 19, 16].

This paper is organized as follows. In section 3.2 we introduce the model we study, carry out the real-space quantization of the optical and atomic fields, and derive the equations obeyed by the atomic and one-photon amplitudes. These equations are studied in section 3.3 for the case of a single atom, where we recover the Wigner-Weisskopf theory of spontaneous emission, and in section 3.4 for the case of a medium of constant density. Random media are introduced in sections 3.5, where the average behavior of energy eigenstates is established. A related approach leads to the derivation of kinetic equations. The paper concludes with a discussion of our results in section 3.7. The technical details of the calculations are presented in the appendices.

3.2 Model

We consider the following model for the interaction between a quantized field and a system of two-level atoms [25, 36]. The atoms are taken to be stationary and sufficiently well separated that interatomic interactions can be neglected. The overall system is described by the Hamiltonian $H = H_F + H_A + H_I$. The Hamiltonian of the field is of the form

$$H_F = \int \frac{d^3k}{(2\pi)^3} \hbar\omega_{\mathbf{k}}\, a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \ , \tag{3.1}$$

where we have neglected the zero-point energy and for simplicity have adopted a scalar theory of the electromagnetic field. Here $\omega_{\mathbf{k}} = c|\mathbf{k}|$ is the frequency of the field mode with wave vector $\mathbf{k}$ and $a^\dagger_{\mathbf{k}}$ ($a_{\mathbf{k}}$) is the corresponding creation (annihilation) operator. The operators $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ obey the commutation relations

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}') \ , \tag{3.2}$$

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0 \ . \tag{3.3}$$

The Hamiltonian of the atoms is given by

$$H_A = \sum_j \hbar\Omega\sigma_j^\dagger\sigma_j \ , labelHA1 \tag{3.4}$$

where Ω is the resonance frequency of each atom and $\sigma_j^\dagger$ (σ_j) is the raising (lowering) operator of the jth atom. The operators σ_j and $\sigma_j^\dagger$ obey the anticommutation relations

$$\{\sigma_j, \sigma_{j'}^\dagger\} = \delta_{jj'} \ , \tag{3.5}$$

$$\{\sigma_j, \sigma_{j'}\} = 0 \ . \tag{3.6}$$

The interaction between the field and the atoms is governed by the Hamiltonian

$$H_I = \sum_j \int \frac{d^3k}{(2\pi)^3}\hbar g_{\mathbf{k}} \left(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger\right)\left(e^{i\mathbf{k}\cdot\mathbf{x}_j}\sigma_j + e^{-i\mathbf{k}\cdot\mathbf{x}_j}\sigma_j^\dagger\right) \ , \tag{3.7}$$

where $g_{\mathbf{k}}$ is the field-atom coupling and $\mathbf{x}_j$ is the position of the jth atom.

In order to treat the atoms and the field on the same footing, it is useful to introduce a real-space representation of the Hamiltonian (3.1). To this end, we define the operator $\varphi(\mathbf{x})$ as the Fourier transform of $a_{\mathbf{k}}$:

$$\varphi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} \ . \tag{3.8}$$

Making use of (3.2) we find that φ is a Bose field which obeys the commutation relations

$$[\varphi(\mathbf{x}), \varphi^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}') \ , \tag{3.9}$$

$$[\varphi(\mathbf{x}), \varphi(\mathbf{x}')] = 0 \ . \tag{3.10}$$

It follows immediately that H_F becomes

$$H_F = \hbar c \int d^3x (-\Delta)^{1/2} \varphi^\dagger(\mathbf{x})\varphi(\mathbf{x}) \ . \tag{3.11}$$

Here the operator $(-\Delta)^{1/2}$ is defined by the Fourier integral

$$(-\Delta)^{1/2} f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} |\mathbf{k}| \tilde{f}(\mathbf{k}) \ , \tag{3.12}$$

$$\tilde{f}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \ . \tag{3.13}$$

We note that $(-\Delta)^{1/2}$ has the non-local spatial representation

$$(-\Delta)^{1/2} f(\mathbf{x}) = \frac{1}{\pi^2} \int d^3y \, \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^4} \ . \tag{3.14}$$

We note that real-space quantization has proven to be a powerful tool for one-dimensional systems in the setting of waveguide quantum electrodynamics [77].

To facilitate the treatment of random media, it will prove convenient to introduce a continuum model of the atomic degrees of freedom. The atomic Hamiltonian then becomes

$$H_A = \hbar\Omega \int d^3x \rho(\mathbf{x}) \sigma^\dagger(\mathbf{x}) \sigma(\mathbf{x}) \ , \tag{3.15}$$

where ρ is the number density of atoms. In addition, the operators σ_j are replaced by a Fermi field σ which obeys the anticommutation relations

$$\{\sigma(\mathbf{x}), \sigma^\dagger(\mathbf{x})\} = \frac{1}{\rho(\mathbf{x})} \delta(\mathbf{x} - \mathbf{x}') \ , \tag{3.16}$$

$$\{\sigma(\mathbf{x}), \sigma(\mathbf{x})\} = 0 \ . \tag{3.17}$$

We find that the interaction Hamiltonian is given by

$$H_I = \hbar g \int d^3x \rho(\mathbf{x}) \left(\varphi(\mathbf{x}) + \varphi^\dagger(\mathbf{x}) \right) \left(\sigma(\mathbf{x}) + \sigma^\dagger(\mathbf{x}) \right) \ , \tag{3.18}$$

where we have made the Markovian approximation $g_{\mathbf{k}} = g$ for all $\mathbf{k}$, so that the atom-field coupling is frequency independent. We also impose the rotating wave approximation (RWA), in which we neglect the rapidly oscillating terms $\varphi^\dagger \sigma^\dagger$ and $\varphi\sigma$. The total Hamiltonian thus becomes

$$H = \hbar \int d^3x \left[c(-\Delta)^{1/2} \varphi^\dagger(\mathbf{x})\varphi(\mathbf{x}) + \Omega\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x}) \right.$$
$$\left. + g\rho(\mathbf{x}) \left(\varphi^\dagger(\mathbf{x})\sigma(\mathbf{x}) + \varphi(\mathbf{x})\sigma^\dagger(\mathbf{x}) \right) \right] , \tag{3.19}$$

which is the model we will investigate for the remainder of this paper.

We suppose that the system is in a one-photon state of the form

$$|\Psi\rangle = \int d^3x \left[\psi(\mathbf{x}, t)\varphi^\dagger(\mathbf{x}) + \rho(\mathbf{x})a(\mathbf{x}, t)\sigma^\dagger(\mathbf{x}) \right] |0\rangle , \tag{3.20}$$

where $|0\rangle$ is the combined vacuum state of the field and the ground state of the atoms. Here $a(\mathbf{x}, t)$ denotes the probability amplitude for exciting an atom at the point $\mathbf{x}$ at time t and $\psi(\mathbf{x}, t)$ is the amplitude for creating a photon. The state $|\Psi\rangle$ is the most general one-photon state that is consistent with the RWA. In addition, $|\Psi\rangle$ is normalized so that $\langle\Psi|\Psi\rangle = 1$. It follows from (4.2) and (4.7) that the amplitudes obey the normalization condition

$$\int d^3x \left(|\psi(\mathbf{x}, t)|^2 + \rho(\mathbf{x})|a(\mathbf{x}, t)|^2 \right) = 1 . \tag{3.21}$$

The dynamics of $|\Psi\rangle$ is governed by the Schrodinger equation

$$i\hbar\partial_t|\Psi\rangle = H|\Psi\rangle . \tag{3.22}$$

Projecting onto the states $\varphi^\dagger(\mathbf{x})|0\rangle$ and $\sigma^\dagger(\mathbf{x})|0\rangle$ and making use of (4.2) and (4.7), we arrive at the following system of equations obeyed by a and ψ:

$$i\partial_t\psi = c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a , \tag{3.23}$$
$$i\rho(\mathbf{x})\partial_t a = g\rho(\mathbf{x})\psi + \Omega\rho(\mathbf{x})a . \tag{3.24}$$

The details of the derivation are given in Appendix A. The overall factors of $\rho(\mathbf{x})$ in (3.24) will be cancelled as necessary.

3.3 Single Atom

In this section we consider the problem of spontaneous emission by a single atom. We assume that the atom is located at the origin and put $\rho(\mathbf{x}) = \delta(\mathbf{x})$. We also assume that the atom is initially in its excited state and that there are no photons present in the field. We thus impose the initial conditions $a(\mathbf{x}, 0) = 1$ and $\psi(\mathbf{x}, 0) = 0$. Taking the Laplace transform in t and the Fourier transform in $\mathbf{x}$ of (3.23) and (3.24), and applying the initial conditions gives

$$iz\tilde{\psi}(\mathbf{k}, z) = c|\mathbf{k}|\tilde{\psi}(\mathbf{k}, z) + ga(0, z) , \tag{3.25}$$

$$i(za(0, z) - 1) = g\psi(0, z) + \Omega a(0, z) . \tag{3.26}$$

Here we have defined the Laplace transform by

$$f(z) = \int_0^\infty dt e^{-zt} f(t) , \tag{3.27}$$

and we denote a function and its Laplace transform by the same symbol. Solving the above equations by making use of the relation

$$\psi(0, z) = \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(\mathbf{k}, z) , \tag{3.28}$$

leads to an expression for $a(0, z)$ of the form

$$a(0, z) = \frac{1}{z + i\Omega - i\Sigma(z)} , \tag{3.29}$$

where Σ is defined by

$$\Sigma(z) = g^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{c|\mathbf{k}| - iz - i\epsilon} , \tag{3.30}$$

where $\epsilon \to 0^+$. Inverting the the Laplace transform in (3.29), we obtain

$$a(0,t) = \int \frac{dz}{2\pi i} \frac{e^{zt}}{z + i\Omega - i\Sigma(z)} . \tag{3.31}$$

In order to carry out the integral (3.31), we make the pole approximation in which we evaluate Σ near resonance. That is, we replace $\Sigma(z)$ with $\Sigma(-i\Omega)$. In addition, we split $\Sigma(-i\Omega)$ into its real and imaginary parts:

$$\mathrm{Re}\,\Sigma(-i\Omega) = \delta\omega , \tag{3.32}$$

$$\mathrm{Im}\,\Sigma(-i\Omega) = \Gamma/2 , \tag{3.33}$$

which defines $\delta\omega$ and Γ. By making use of the identity

$$\frac{1}{c|\mathbf{k}| - \Omega - i\epsilon} = P\frac{1}{c|\mathbf{k}| - \Omega} + i\pi\delta(c|\mathbf{k}| - \Omega) , \tag{3.34}$$

where P denotes the principal value, we find that Γ is given by

$$\Gamma = 2g^2\pi \int \frac{d^3k}{(2\pi)^3}\, \delta(c|\mathbf{k}| - \Omega) \tag{3.35}$$

$$= \frac{g^2\Omega^2}{\pi c^3} . \tag{3.36}$$

We also obtain

$$\delta\omega = \frac{g^2}{2\pi^2} \int_0^{2\pi/\Lambda} \frac{k^2 dk}{ck - \Omega} , \tag{3.37}$$

where we have introduced a high-frequency cutoff to regularize the divergent integral. Finally, making use of (3.31), (3.32) and (3.33), we find that a is given by

$$a(0,t) = e^{-i(\Omega - \delta\omega)t} e^{-\Gamma t/2} . \tag{3.38}$$

We immediately see that the probability the atom decays is exponentially decreasing:

$$|a(0,t)|^2 = e^{-\Gamma t} . \tag{3.39}$$

We note that the decay rate Γ agrees with Wigner-Weisskopf theory formulated within scalar electrodynamics and that $\delta\omega$ is the corresponding Lamb shift.

Next we determine the behavior of the amplitude ψ. Making use of (3.25), (3.29) and inverting the Laplace transform, we find that

$$\tilde{\psi}(\mathbf{k}, t) = \int \frac{dz}{2\pi i} \frac{g e^{zt}}{(iz - c|\mathbf{k}|)\,(z + i\Omega - i\Sigma(z))} \,. \tag{3.40}$$

Carrying out the above integral in the pole approximation, we obtain

$$\tilde{\psi}(\mathbf{k}, t) = \frac{g}{c|\mathbf{k}| - (\Omega - \delta\omega) + i\Gamma/2} \left(e^{-ict|\mathbf{k}|} - e^{-\Gamma t/2} e^{-i(\Omega - \delta\omega)t} \right) \,. \tag{3.41}$$

At long times ($\Gamma t \gg 1$), we see that the one-photon probability density is given by

$$|\tilde{\psi}(\mathbf{k}, t)|^2 = \frac{|g|^2}{[c|\mathbf{k}| - (\Omega - \delta\omega)]^2 + \Gamma^2/4} \,, \tag{3.42}$$

which has the form of a Lorentzian spectral line.

3.4 Constant Density Problem

In this section we consider the problem of emission and absorption of one photon interacting with a collection of atoms with constant number density ρ_0. We will start with (3.23) and (3.24) and $\rho(\mathbf{x})$ equal to ρ_0. That is

$$i\partial_t \psi = c(-\Delta)^{1/2}\psi + g\rho_0 a \,, \tag{3.43}$$

$$i\partial_t a = g\psi + \Omega a \,, \tag{3.44}$$

where we have cancelled the density from (3.23). Defining the vector quantity $\mathbf{\Psi}(\mathbf{x}, \mathbf{t})$ as

$$\mathbf{\Psi}(\mathbf{x}, \mathbf{t}) = \begin{bmatrix} \psi(\mathbf{x}, t) \\ \sqrt{\rho_0}\, a(\mathbf{x}, t) \end{bmatrix} , \tag{3.45}$$

then the previous system becomes

$$i\partial_t \mathbf{\Psi} = \mathbf{A}\mathbf{\Psi}\,, \tag{3.46}$$

where

$$A(\mathbf{x}) = \begin{bmatrix} c(-\Delta)^{1/2} & g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & \Omega \end{bmatrix}. \tag{3.47}$$

Taking the Fourier transform of (3.46), we arrive at the system of ordinary differential equations

$$i\partial_t \hat{\mathbf{\Psi}} = \hat{A}\hat{\mathbf{\Psi}}\,, \tag{3.48}$$

where

$$\hat{\mathbf{\Psi}}(\mathbf{k}, t) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{\Psi}(\mathbf{x}, \mathbf{t})\,, \tag{3.49}$$

and

$$\hat{A}(\mathbf{k}) = \begin{bmatrix} c|\mathbf{k}| & g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & \Omega \end{bmatrix}. \tag{3.50}$$

The eigenvalues and eigenvectors of $\hat{A}$ are given by

$$\lambda_{\pm}(\mathbf{k}) = \frac{(c|\mathbf{k}| + \Omega) \pm \sqrt{(c|\mathbf{k}| - \Omega)^2 + 4g^2\rho_0}}{2}\,, \tag{3.51}$$

$$\mathbf{v}_{\pm}(\mathbf{k}) = \frac{1}{\sqrt{(\lambda_{\pm} - \Omega)^2 + g^2\rho_0}} \begin{bmatrix} \lambda_{\pm} - \Omega \\ g\sqrt{\rho_0} \end{bmatrix}. \tag{3.52}$$

The solution to (3.49) is given by

$$\hat{\mathbf{\Psi}}(\mathbf{k}, t) = C_{+}(\mathbf{k})e^{-i\lambda_{+}t}\mathbf{v}_{+}(\mathbf{k}) + C_{-}(\mathbf{k})e^{-i\lambda_{-}t}\mathbf{v}_{-}(\mathbf{k}). \tag{3.53}$$

Solving for the coefficients $C_\pm(\mathbf{k})$ we find

$$C_+(\mathbf{k}) = \frac{(\hat{\psi}_0 g - \hat{a}_0(\lambda_- - \Omega))\sqrt{(\lambda_+ - \Omega)^2 + g^2\rho_0}}{g(\lambda_+ - \lambda -)} \,, \tag{3.54}$$

$$C_-(\mathbf{k}) = \frac{(\hat{a}_0(\lambda_+ - \Omega) - g\hat{\psi}_0)\sqrt{(\lambda_- - \Omega)^2 + g^2\rho_0}}{g(\lambda_+ - \lambda -)} \,. \tag{3.55}$$

We assume that initially there is a localized region of excited atoms around the origin with width l_s. The initial amplitudes are taken to be

$$\psi(\mathbf{x}, 0) = 0 \,, \tag{3.56}$$

$$\sqrt{\rho_0}a(\mathbf{x}, 0) = \left(\frac{1}{\pi l_s^2}\right)^{3/4} e^{-|\mathbf{x}|^2/2l_s^2}. \tag{3.57}$$

Taking the Fourier transform of (3.56) and (3.57) and using (3.54) and (3.55), we see that the components of $\mathbf{\Psi}(\mathbf{k}, \mathbf{t})$ are given by

$$\hat{\psi}(\mathbf{k}, t) = \frac{g\rho_0 l_s^{3/2}}{2^{3/2}\pi^{5/2}} \frac{e^{-i\lambda_+(\mathbf{k})t} - e^{-i\lambda_-(\mathbf{k})t}}{\lambda_+(\mathbf{k}) - \lambda_-(\mathbf{k})} e^{-l_s^2|\mathbf{k}|^2/2} \,, \tag{3.58}$$

$$\sqrt{\rho_0}\hat{a}(\mathbf{k}, t) = \frac{l_s^{3/2}}{2^{3/2}\pi^{5/2}} \frac{(\lambda_+(\mathbf{k}) - \Omega)e^{-i\lambda_-(\mathbf{k})t} - (\lambda_-(\mathbf{k}) - \Omega)e^{-i\lambda_+(\mathbf{k})t}}{\lambda_+(\mathbf{k}) - \lambda_-(\mathbf{k})} e^{-l_s^2|\mathbf{k}|^2/2}. \tag{3.59}$$

Inverting the Fourier transforms, we find that

$$\psi(\mathbf{x}, t) = \frac{g\sqrt{2}\rho_0 l_s^{3/2}}{\pi^{3/2}|\mathbf{x}|} \int_0^\infty dk\, k \frac{e^{-i\lambda_+(k)t} - e^{-i\lambda_-(k)t}}{\lambda_+(k) - \lambda_-(k)} e^{-l_s^2 k^2/2} \sin(k|\mathbf{x}|) \,, \tag{3.60}$$

$$\begin{aligned}\sqrt{\rho_0}a(\mathbf{x}, t) = \frac{\sqrt{2} l_s^{3/2}}{\pi^{3/2}|\mathbf{x}|} \int_0^\infty dk\, &\frac{(\lambda_+(k) - \Omega)e^{-i\lambda_-(k)t} - (\lambda_-(k) - \Omega)e^{-i\lambda_+(k)t}}{\lambda_+(k) - \lambda_-(k)} \\ &\times e^{-l_s^2 k^2/2} k \sin(k|\mathbf{x}|) \,.\end{aligned} \tag{3.61}$$

Figure 3.1 illustrates the time-dependence of the probability densities $|\psi|^2$ and $\rho_0|a|^2$, where we have set the dimensionless quantities $\Omega/(\sqrt{\rho_0}g) = c/(l_s\sqrt{\rho_0}g) = 1$. We see that the probability densities are oscillatory and decay in time.

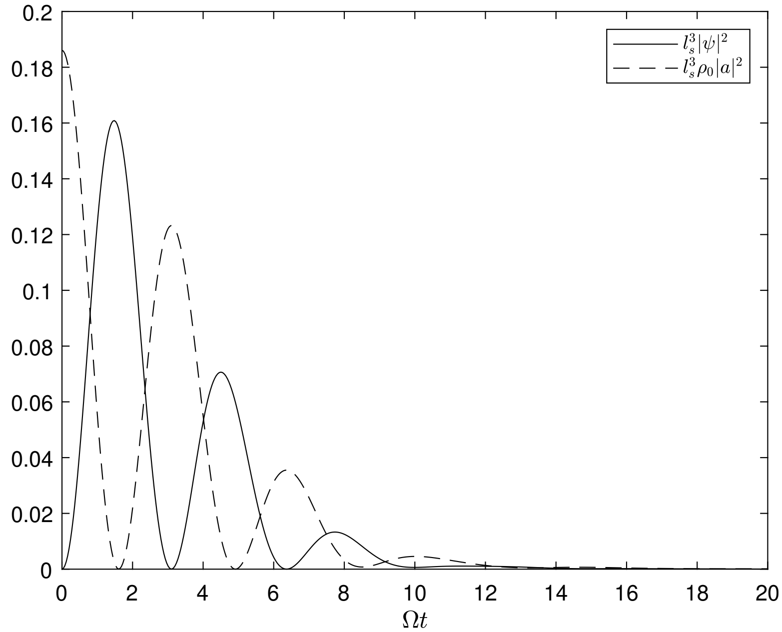


Figure 3.1: Time-dependence of atomic and field probability densities for the constant density problem with $|\mathbf{x}| = l_s$

3.5 Energy Eigenstates

3.5.1 Radiative Transport

In this section we investigate the energy eigenstates of the Hamiltonian H in a random medium. We consider the time-independent Schrodinger equation $H|\Psi\rangle = \hbar\omega|\Psi\rangle$, where $|\Psi\rangle$ is of the form (3.20) and $\hbar\omega$ is the energy. It follows that the amplitudes a and ψ, which are independent of time, obey the equations

$$c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a = \omega\psi \ , \tag{3.62}$$

$$g\psi + \Omega a = \omega a \ . \tag{3.63}$$

By eliminating a from the above system, we immediately obtain the equation obeyed by ψ, which is given by

$$(-\Delta)^{1/2}\psi + \frac{g^2\rho(\mathbf{x})}{c(\omega-\Omega)}\psi = k\psi \ , \tag{3.64}$$

where $k = \omega/c$.

For the remainder of this work, we assume that the atomic density $\rho(\mathbf{x})$ is of the form

$$\rho(\mathbf{x}) = \rho_0(1+\eta(\mathbf{x})) \ , \tag{3.65}$$

where ρ_0 is constant and $\eta(\mathbf{x})$ is a real-valued random field that accounts for statistical fluctuations in the density. We further assume that the correlations of η are given by

$$\langle \eta(\mathbf{x}) \rangle = 0 \ , \tag{3.66}$$

$$\langle \eta(\mathbf{x})\eta(\mathbf{y}) \rangle = C(\mathbf{x}-\mathbf{y}) \ , \tag{3.67}$$

where C is the two-point correlation function and $\langle \cdots \rangle$ denotes statistical averaging. If C depends only upon the quantity $|\mathbf{x}-\mathbf{y}|$, the medium is said to be statistically homogeneous and isotropic. To make further progress, we consider the relative sizes of the important physical scales. The solution to (3.64) oscillates on the scale of the wavelength $\lambda = 2\pi/k$. However, we are interested in the behavior of the solutions on the macroscopic scale $L \gg \lambda$. We thus introduce a small parameter $\epsilon = \lambda/L$ and rescale the position $\mathbf{x}$ by $\mathbf{x} \to \mathbf{x}/\epsilon$. In addition, we assume that the randomness is sufficiently weak so that the correlation function C is $O(\epsilon)$. Thus (3.64) becomes

$$\epsilon(-\Delta)^{1/2}\psi_\epsilon + k_0(1+\sqrt{\epsilon}\eta(\mathbf{x}/\epsilon))\psi_\epsilon = k\psi_\epsilon \ , \tag{3.68}$$

where the ϵ dependence of ψ is indicated explicitly and

$$k_0 = \frac{g^2\rho_0}{c(\omega-\Omega)} \ . \tag{3.69}$$

Note that we have also rescaled η to be consistent with the $O(\epsilon)$ scaling of C.

We now introduce the Wigner transform of the amplitude ψ, which provides a phase-space representation of the correlation function of ψ. The Wigner transform $W_\epsilon(\mathbf{x}, \mathbf{k})$ is defined as

$$W_\epsilon(\mathbf{x}, \mathbf{k}) = \int \frac{d^3x'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}'} \psi_\epsilon(\mathbf{x} - \epsilon\mathbf{x}'/2)\psi^*_\epsilon(\mathbf{x} + \epsilon\mathbf{x}'/2) \ . \tag{3.70}$$

The Wigner transform has several important properties. It is real-valued and related to the probability density $|\psi_\epsilon|^2$ by

$$|\psi_\epsilon(\mathbf{x})|^2 = \int d^3k W_\epsilon(\mathbf{x}, \mathbf{k}) \ . \tag{3.71}$$

Next we derive a useful relation governing the Wigner transform. Let $\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) = \psi_\epsilon(\mathbf{x}_1)\psi^*_\epsilon(\mathbf{x}_2)$. Since η is real-valued, it follows that $\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2)$ satisfies the pair of equations

$$\epsilon(-\Delta_{\mathbf{x}_1})^{1/2}\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) - k\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) + k_0(1 + \sqrt{\epsilon}\eta(\mathbf{x}_1/\epsilon))\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) = 0 \ , \tag{3.72}$$

$$\epsilon(-\Delta_{\mathbf{x}_2})^{1/2}\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) - k\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) + k_0(1 + \sqrt{\epsilon}\eta(\mathbf{x}_2/\epsilon))\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) = 0 \ . \tag{3.73}$$

Subtracting (3.72) from (3.73) yields

$$\epsilon\left[(-\Delta_{\mathbf{x}_1})^{1/2} - (-\Delta_{\mathbf{x}_2})^{1/2}\right]\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) + \sqrt{\epsilon}k_0\left[\eta(\mathbf{x}_1/\epsilon) - \eta(\mathbf{x}_2/\epsilon)\right]\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) = 0. \tag{3.74}$$

We now perform the change of variables

$$\mathbf{x}_1 = \mathbf{x} - \epsilon\mathbf{x}'/2 \ , \tag{3.75}$$

$$\mathbf{x}_2 = \mathbf{x} + \epsilon\mathbf{x}'/2 \ , \tag{3.76}$$

and Fourier transform the result with respect to $\mathbf{x}'$, thus arriving at

$$\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\left[|-\mathbf{k} + \epsilon\mathbf{q}/2| - |\mathbf{k} + \epsilon\mathbf{q}/2|\right]\tilde{W}_\epsilon(\mathbf{q}, \mathbf{k}) + \sqrt{\epsilon}LW_\epsilon(\mathbf{x}, \mathbf{k}) = 0 \ , \tag{3.77}$$

where the Fourier transform of the Wigner transform is defined by

$$\tilde{W}_\epsilon(\mathbf{q},\mathbf{k}) = \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} W_\epsilon(\mathbf{x},\mathbf{k}) \tag{3.78}$$

and

$$LW_\epsilon(\mathbf{x},\mathbf{k}) = k_0 \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\epsilon} \tilde{\eta}(\mathbf{q}) \left[W_\epsilon(\mathbf{x},\mathbf{k}+\mathbf{q}/2) - W_\epsilon(\mathbf{x},\mathbf{k}-\mathbf{q}/2)\right]. \tag{3.79}$$

The details of the calculation are given in Appendix B.

We now consider the behavior of W_ϵ in the high-frequency limit $\epsilon \to 0$, which allows for the separation of microscopic and macroscopic scales. To this end we introduce a multiscale expansion for W_ϵ of the form

$$W_\epsilon(\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k}) + \sqrt{\epsilon} W_1(\mathbf{x},\mathbf{X},\mathbf{k}) + \epsilon W_2(\mathbf{x},\mathbf{X},\mathbf{k}) + \cdots , \tag{3.80}$$

where $\mathbf{X} = \mathbf{x}/\epsilon$ is a fast variable and W_0 is taken to be deterministic. We treat $\mathbf{x}$ and $\mathbf{X}$ as independent variables and make the replacement

$$\nabla_\mathbf{x} \to \nabla_\mathbf{x} + \frac{1}{\epsilon}\nabla_\mathbf{X} . \tag{3.81}$$

Eq. (3.77) thus becomes

$$\int \frac{d^3q}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{Q}\cdot\mathbf{X}} \left[|-\mathbf{k}+\epsilon\mathbf{q}/2+\mathbf{Q}/2| - |\mathbf{k}+\epsilon\mathbf{q}/2+\mathbf{Q}/2|\right] \tilde{W}_\epsilon(\mathbf{q},\mathbf{Q},\mathbf{k}) \tag{3.82}$$

$$+\sqrt{\epsilon} LW_\epsilon(\mathbf{x},\mathbf{X},\mathbf{k}) = 0 , \tag{3.83}$$

where

$$\tilde{W}_\epsilon(\mathbf{q},\mathbf{Q},\mathbf{k}) = \int d^3x d^3X e^{-i\mathbf{q}\cdot\mathbf{x}-i\mathbf{Q}\cdot\mathbf{X}} W_\epsilon(\mathbf{x},\mathbf{X},\mathbf{k}). \tag{3.84}$$

Inserting (3.80) into (3.82) and equating terms of the same order in ϵ, we find that at $O(\sqrt{\epsilon})$

$$\int \frac{d^3Q}{(2\pi)^3} e^{i\mathbf{Q}\cdot\mathbf{X}} \left[|-\mathbf{k}+\mathbf{Q}/2| - |\mathbf{k}+\mathbf{Q}/2| \right] \tilde{W}_1(\mathbf{x},\mathbf{Q},\mathbf{k}) + \sqrt{\epsilon} L W_0(\mathbf{x},\mathbf{k}) = 0. \qquad (3.85)$$

Eq. (3.85) can be solved by Fourier transforms with the result

$$\tilde{W}_1(\mathbf{x},\mathbf{Q},\mathbf{k}) = k_0 \tilde{\eta}(\mathbf{Q}) \frac{[W_0(\mathbf{x},\mathbf{k}+\mathbf{Q}/2) - W_0(\mathbf{x},\mathbf{k}-\mathbf{Q}/2)]}{|\mathbf{k}+\mathbf{Q}/2| - |-\mathbf{k}+\mathbf{Q}/2| - i\theta}\ , \qquad (3.86)$$

where $\theta \to 0$ is a positive regularizing parameter. At order $O(\epsilon)$ we find that

$$\int \frac{d^3q}{(2\pi)^3} \frac{d^3Q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{Q}\cdot\mathbf{X}} \left[|-\mathbf{k}+\mathbf{Q}/2| - |\mathbf{k}+\mathbf{Q}/2| \right] \tilde{W}_2(\mathbf{q},\mathbf{Q},\mathbf{k}) + \hat{\mathbf{k}} \cdot \nabla W_0(\mathbf{q},\mathbf{k})$$
$$+ k_0 \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}} \tilde{\eta}(\mathbf{q}) \left[W_1(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2) - W_1(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2) \right] = 0\ . \qquad (3.87)$$

Next we average (3.87) over realizations of the random medium. To do so, we impose the condition $\langle [|-\mathbf{k}+\mathbf{Q}/2| - |\mathbf{k}+\mathbf{Q}/2|] \tilde{W}_2(\mathbf{q},\mathbf{Q},\mathbf{k}) \rangle = 0$, which closes the hierarchy relating the terms in the multiscale expansion, and corresponds to the assumption that W_2 is statistically stationary in the fast variable $\mathbf{X}$. Eq. (3.87) thus becomes

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W_0(\mathbf{x},\mathbf{k}) + k_0 \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}} \langle \tilde{\eta}(\mathbf{q}) \left[W_1(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2) - W_1(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2) \right] \rangle = 0. \qquad (3.88)$$

After substituting (3.86) into (3.88) and using the identity

$$\langle \tilde{\eta}(\mathbf{p}) \tilde{\eta}(\mathbf{q}) \rangle = (2\pi)^3 \delta(\mathbf{p}+\mathbf{q}) \tilde{C}(\mathbf{p}) \qquad (3.89)$$

we find, as shown in Appendix C, that W_0 satisfies

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W_0(\mathbf{x},\mathbf{k}) + k_0^2 \int \frac{d^3q}{(2\pi)^2} \tilde{C}(\mathbf{q}-\mathbf{k}) \delta(|\mathbf{q}| - |\mathbf{k}|) W_0(\mathbf{x},\mathbf{k})$$
$$= k_0^2 \int \frac{d^3q}{(2\pi)^2} \tilde{C}(\mathbf{q}-\mathbf{k}) \delta(|\mathbf{q}| - |\mathbf{k}|) W_0(\mathbf{x},\mathbf{q})\ . \qquad (3.90)$$

Here we define the scattering coefficient μ_s and phase function A as

$$\mu_s = \frac{k_0^2|\mathbf{k}|^2}{4\pi^2} \int d\hat{\mathbf{k}}' \, \tilde{C}\left(|\mathbf{k}|(\hat{\mathbf{k}} - \hat{\mathbf{k}}')\right) \ , \tag{3.91}$$

$$A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \frac{k_0^2|\mathbf{k}|^2}{\mu_s} \tilde{C}\left(|\mathbf{k}|(\hat{\mathbf{k}} - \hat{\mathbf{k}}')\right) \tag{3.92}$$

Making use of these definitions, (3.90) becomes

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W_0(\mathbf{x}, \mathbf{k}) + \mu_s W_0(\mathbf{x}, \mathbf{k}) = \mu_s L W_0(\mathbf{x}, \mathbf{k}) \ , \tag{3.93}$$

where the operator L is defined by

$$L W_0(\mathbf{x}, \mathbf{k}) = \int d\hat{\mathbf{k}}' A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') W_0(\mathbf{x}, \mathbf{k}'). \tag{3.94}$$

Eq. (3.93), which has the form of a time-independent radiative transport equation, is the main result of this section. We note that μ_s and A are defined in terms of the correlations of the medium. Since the density fluctuations η are statistically homogeneous and isotropic, $\tilde{C}$ depends only on the quantity $|\mathbf{k} - \mathbf{k}'|$, and hence the phase function A depends only on $\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'$ and $|\mathbf{k}|$. Similarly, μ_s only depends on the magnitude $|\mathbf{k}|$.

In the case of white noise-disorder, where $C = C_0 \delta(\mathbf{x})$ with constant C_0, the scattering coefficient and phase function are given by

$$\mu_s = 4\pi C_0 k_0^2 |\mathbf{k}|^2 \ , \tag{3.95}$$

$$A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \frac{1}{4\pi} \ , \tag{3.96}$$

which corresponds to isotropic scattering.

3.5.2 Diffusion Approximation

We now consider the diffusion limit of the radiative transport equation developed in the previous section. The diffusion approximation for a radiative transport equation of the form

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} W_0(\mathbf{x}, \mathbf{k}) + \mu_s W_0(\mathbf{x}, \mathbf{k}) = \mu_s L W_0(\mathbf{x}, \mathbf{k}) \tag{3.97}$$

is obtained by expanding W_0 in spherical harmonics [16]. To lowest order, it can be seen that

$$W_0(\mathbf{x}, \mathbf{k}) = \frac{1}{4\pi} \Big(u(\mathbf{x}, |\mathbf{k}|) - \ell^* \hat{\mathbf{k}} \cdot \nabla u(\mathbf{x}, |\mathbf{k}|) \Big) \ , \tag{3.98}$$

where the first angular moment $u(\mathbf{x}, |\mathbf{k}|)$ is defined by

$$u(\mathbf{x}, |\mathbf{k}|) = \int d\hat{\mathbf{k}} W_0(\mathbf{x}, \mathbf{k}) \ , \tag{3.99}$$

and the transport mean free path ℓ^* is given by

$$\ell^* = \frac{1}{\mu_s(1-g)}, \quad g = \int d\hat{\mathbf{k}}' \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') \ . \tag{3.100}$$

The anisotropy g takes values between -1 and 1 and vanishes for isotropic scattering. The quantity u satisfies the diffusion equation

$$\Delta u = 0 \quad \text{in} \quad \Omega \ , \tag{3.101}$$

$$u = g \quad \text{on} \quad \partial\Omega \ , \tag{3.102}$$

where we have prescribed Dirichlet boundary conditions on a bounded domain Ω and g generally depends upon k. Since $|\psi|^2$ is given by

$$|\psi(\mathbf{x})|^2 = \int d^3k \, W_0(\mathbf{x}, \mathbf{k}) = \int_0^\infty dk \, k^2 u(\mathbf{x}, k) \ , \tag{3.103}$$

it follows that $|\psi|^2$ obeys

$$\Delta|\psi|^2 = 0 \quad \text{in} \quad \Omega\,, \tag{3.104}$$

$$|\psi|^2 = \int_0^\infty dk k^2 g(\mathbf{x}, k) \quad \text{on} \quad \partial\Omega\,, \tag{3.105}$$

where the k dependence of g has been made explicit.

3.6 Collective Spontaneous Emission

3.6.1 Kinetic Equations

In this section we study the time evolution of the atomic and field amplitudes in a random medium. Our starting point is (3.23) and (3.24) (with ρ cancelled):

$$i\partial_t\psi = c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a\,, \tag{3.106}$$

$$i\partial_t a = g\psi + \Omega a\,. \tag{3.107}$$

A similar system of pseudodifferential equations with a random potential has been considered in [10]. If we define the vector quantity $\mathbf{u}(\mathbf{x}, t) = \left[\psi(\mathbf{x}, t), a(\mathbf{x}, t)\sqrt{\rho_0}\right]^T$, then $\mathbf{u}$ satisfies the equation

$$i\partial_t\mathbf{u} = A(\mathbf{x})\mathbf{u} + g\sqrt{\rho_0}\eta(\mathbf{x})K\mathbf{u}\,, \tag{3.108}$$

where

$$A(\mathbf{x}) = \begin{bmatrix} c(-\Delta_{\mathbf{x}})^{1/2} & g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & \Omega \end{bmatrix}, \tag{3.109}$$

$$K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.110}$$

This definition of $\mathbf{u}$ has the advantage that its two components have the same dimensions and that the matrix $A(\mathbf{x})$ is symmetric. We perform the same rescaling of the variables $\mathbf{x}$ and η as previously, and we also rescale the time t as $t \to t/\epsilon$. Thus (3.108) becomes

$$\epsilon i \partial_t \mathbf{u}_\epsilon = A_\epsilon(\mathbf{x})\mathbf{u}_\epsilon + \sqrt{\epsilon} g \sqrt{\rho_0} \eta(\mathbf{x}/\epsilon) K \mathbf{u}_\epsilon \ , \tag{3.111}$$

where

$$A_\epsilon(\mathbf{x}) = \begin{bmatrix} \epsilon c(-\Delta_{\mathbf{x}})^{1/2} & g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & \Omega \end{bmatrix} . \tag{3.112}$$

We now consider the Wigner transform of $\mathbf{u}$, which is matrix-valued and defined by

$$W_\epsilon(\mathbf{x}, \mathbf{k}, t) = \int \frac{d^3x'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}'} \mathbf{u}_\epsilon(\mathbf{x} - \epsilon \mathbf{x}'/2, t) \mathbf{u}_\epsilon^\dagger(\mathbf{x} + \epsilon \mathbf{x}'/2, t) \ . \tag{3.113}$$

The probability densities $|\psi_\epsilon(\mathbf{x}, t)|^2$ and $|a_\epsilon(\mathbf{x}, t)|^2$ are related to the Wigner transform by

$$|\psi_\epsilon(\mathbf{x}, t)|^2 = \int d^3k (W_\epsilon)_{11}(\mathbf{x}, \mathbf{k}, t) \ , \tag{3.114}$$

$$\rho_0 |a_\epsilon(\mathbf{x}, t)|^2 = \int d^3k (W_\epsilon)_{22}(\mathbf{x}, \mathbf{k}, t) \ . \tag{3.115}$$

If we define $\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2, t) = \mathbf{u}_\epsilon(\mathbf{x}_1, t)\mathbf{u}_\epsilon^*(\mathbf{x}_2, t)$, then Φ_ϵ satisfies the equation

$$\begin{aligned} \epsilon i \partial_t \Phi_\epsilon = {} & A_\epsilon(\mathbf{x}_1)\Phi_\epsilon + \sqrt{\epsilon} g \sqrt{\rho_0} \eta(\mathbf{x}_1/\epsilon) K \Phi_\epsilon \\ & - \Phi_\epsilon A_\epsilon(\mathbf{x}_2) + \sqrt{\epsilon} g \sqrt{\rho_0} \eta(\mathbf{x}_2/\epsilon) \Phi_\epsilon K^T . \end{aligned} \tag{3.116}$$

Next we perform the change of variables

$$\mathbf{x}_1 = \mathbf{x} - \epsilon \mathbf{x}'/2 \ , \tag{3.117}$$

$$\mathbf{x}_2 = \mathbf{x} + \epsilon \mathbf{x}'/2 \ , \tag{3.118}$$

and Fourier transform the result with respect to $\mathbf{x}'$. We thus obtain

$$
\begin{aligned}
&\epsilon i\partial_t W_\epsilon(\mathbf{x},\mathbf{k},t) \\
&= \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \left[\tilde{A}_\epsilon(\mathbf{k}/\epsilon - \mathbf{q}/2)\tilde{W}_\epsilon(\mathbf{q},\mathbf{k},t) - \tilde{W}_\epsilon(\mathbf{q},\mathbf{k},t)\tilde{A}_\epsilon(\mathbf{k}/\epsilon + \mathbf{q}/2)\right] \\
&+ \sqrt{\epsilon} g\sqrt{\rho_0} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\epsilon}\tilde{\eta}(\mathbf{q}) \left[K W_\epsilon(\mathbf{x},\mathbf{k}+\mathbf{q}/2,t) - W_\epsilon(\mathbf{x},\mathbf{k}-\mathbf{q}/2,t)K^T\right] , \quad (3.119)
\end{aligned}
$$

where

$$
\tilde{A}_\epsilon(\mathbf{k}) = \begin{bmatrix} \epsilon c|\mathbf{k}| & g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & \Omega \end{bmatrix} . \quad (3.120)
$$

The details of this calculation are given in Appendix D.

Once again we consider the behavior of W_ϵ in the high-frequency limit $\epsilon \to 0$. To this end we introduce a multiscale expansion for W_ϵ of the form

$$
W_\epsilon(\mathbf{x},\mathbf{k},t) = W_0(\mathbf{x},\mathbf{k},t) + \sqrt{\epsilon}W_1(\mathbf{x},\mathbf{X},\mathbf{k},t) + \epsilon W_2(\mathbf{x},\mathbf{X},\mathbf{k},t) + \cdots , \quad (3.121)
$$

where $\mathbf{X} = \mathbf{x}/\epsilon$ is a fast variable, and W_0 is taken to be deterministic and independent of $\mathbf{X}$. We treat $\mathbf{x}$ and $\mathbf{X}$ as independent variables and transform the derivative $\nabla_\mathbf{x}$ according to (3.81). Eq. (3.119) thus becomes

$$
\begin{aligned}
&\epsilon i\partial_t W_\epsilon(\mathbf{x},\mathbf{X},\mathbf{k},t) = \int \frac{d^3q}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{Q}\cdot\mathbf{X}} \left[\tilde{A}_\epsilon(\mathbf{k}/\epsilon - \mathbf{q}/2 - \mathbf{Q}/2\epsilon)\tilde{W}_\epsilon(\mathbf{q},\mathbf{Q},\mathbf{k},t)\right. \\
&\left.-\tilde{W}_\epsilon(\mathbf{q},\mathbf{Q},\mathbf{k},t)\tilde{A}_\epsilon(\mathbf{k}/\epsilon + \mathbf{q}/2 + \mathbf{Q}/2\epsilon)\right] + \sqrt{\epsilon} g\sqrt{\rho_0} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}} \left[K W_\epsilon(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2,t)\right. \\
&\left.-W_\epsilon(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2,t)K^T\right] . \quad (3.122)
\end{aligned}
$$

Inserting (3.121) into (3.122) and equating terms of the same order in ϵ, we find that at $O(1)$

$$
\tilde{A}_\epsilon(\mathbf{k}/\epsilon)W_0(\mathbf{x},\mathbf{k},t) - W_0(\mathbf{x},\mathbf{k},t)\tilde{A}_\epsilon(\mathbf{k}/\epsilon) = 0. \quad (3.123)
$$

Since $\tilde{A}_\epsilon(\mathbf{k}/\epsilon)$ is symmetric it can be diagonalized. Its eigenvalues are given by

$$\lambda_\pm(\mathbf{k}) = \frac{(c|\mathbf{k}| + \Omega) \pm \sqrt{(c|\mathbf{k}| - \Omega)^2 + 4g^2\rho_0}}{2} . \tag{3.124}$$

The corresponding eigenvectors are real and are of the form

$$\mathbf{b}_\pm(\mathbf{k}) = \frac{1}{\sqrt{(\lambda_\pm - \Omega)^2 + g^2\rho_0}} \begin{bmatrix} \lambda_\pm - \Omega \\ g\sqrt{\rho_0} \end{bmatrix}. \tag{3.125}$$

It follows from (3.123) that W_0 is also diagonal in the basis $\{\mathbf{b}_+(\mathbf{k}), \mathbf{b}_-(\mathbf{k})\}$ and can be expressed as

$$W_0(\mathbf{x}, \mathbf{k}, t) = a_+(\mathbf{x}, \mathbf{k}, t)\mathbf{b}_+(\mathbf{k})\mathbf{b}_+^T(\mathbf{k}) + a_-(\mathbf{x}, \mathbf{k}, t)\mathbf{b}_-(\mathbf{k})\mathbf{b}_-^T(\mathbf{k}) , \tag{3.126}$$

where $a_\pm$ are suitable coefficients.

At order $O(\sqrt{\epsilon})$ we obtain

$$\tilde{A}_\epsilon((\mathbf{k} - \mathbf{Q}/2)/\epsilon)\tilde{W}_1(\mathbf{x}, \mathbf{Q}, \mathbf{k}, t) - \tilde{W}_1(\mathbf{x}, \mathbf{Q}, \mathbf{k}, t)\tilde{A}_\epsilon((\mathbf{k} + \mathbf{Q}/2)/\epsilon) \tag{3.127}$$

$$= g\sqrt{\rho_0}\tilde{\eta}(\mathbf{q}) \left[W_0(\mathbf{x}, \mathbf{k} - \mathbf{Q}/2, t)K^T - KW_0(\mathbf{x}, \mathbf{k} + \mathbf{Q}/2, t)\right] . \tag{3.128}$$

We can then decompose $\tilde{W}_1$ as

$$\tilde{W}_1(\mathbf{x}, \mathbf{Q}, \mathbf{k}, t) = \sum_{m,n} w_{mn}(\mathbf{x}, \mathbf{Q}, \mathbf{k}, t)\mathbf{b}_m(\mathbf{k} - \mathbf{Q}/2)\mathbf{b}_n^T(\mathbf{k} + \mathbf{Q}/2) , \tag{3.129}$$

for suitable coefficients w_{mn}. Multiplying (3.127) on the left by $\mathbf{b}_m^T(\mathbf{k} - \mathbf{Q}/2)$, on the right by $\mathbf{b}_n(\mathbf{k} + \mathbf{Q}/2)$, and using the facts

$$\mathbf{b}_m^T(\mathbf{q})K\mathbf{b}_n(\mathbf{p}) = \frac{g\sqrt{\rho_0}(\lambda_m(\mathbf{q}) - \Omega)}{\sqrt{(\lambda_m(\mathbf{q}) - \Omega)^2 + g^2\rho_0}\sqrt{(\lambda_n(\mathbf{p}) - \Omega)^2 + g^2\rho_0}} , \tag{3.130}$$

$$\mathbf{b}_m^T(\mathbf{q})K^T\mathbf{b}_n(\mathbf{p}) = \frac{g\sqrt{\rho_0}(\lambda_n(\mathbf{p}) - \Omega)}{\sqrt{(\lambda_m(\mathbf{q}) - \Omega)^2 + g^2\rho_0}\sqrt{(\lambda_n(\mathbf{p}) - \Omega)^2 + g^2\rho_0}} , \tag{3.131}$$

we find that

$$w_{mn}(\mathbf{x},\mathbf{Q},\mathbf{k},t) = \frac{g^2\rho_0\tilde{\eta}(\mathbf{Q})}{(\lambda_m(\mathbf{k}-\mathbf{Q}/2)-\lambda_n(\mathbf{k}+\mathbf{Q}/2)+i\theta)}$$
$$\times \frac{((\lambda_n(\mathbf{k}+\mathbf{Q}/2)-\Omega)a_m(\mathbf{x},\mathbf{k}-\mathbf{Q}/2,t)-(\lambda_m(\mathbf{k}-\mathbf{Q}/2)-\Omega)a_n(\mathbf{x},\mathbf{k}+\mathbf{Q}/2,t))}{\sqrt{(\lambda_m(\mathbf{k}-\mathbf{Q}/2)-\Omega)^2+g^2\rho_0}\sqrt{(\lambda_n(\mathbf{k}+\mathbf{Q}/2)-\Omega)^2+g^2\rho_0}}\ , \tag{3.132}$$

where $\theta \to 0$ is a positive regularizing parameter. At order $O(\epsilon)$ we obtain

$$\begin{aligned} & i\partial_t W_0(\mathbf{x},\mathbf{k},t) \\ & = LW_2(\mathbf{x},\mathbf{X},\mathbf{k},t) + M(\mathbf{x},\mathbf{k})W_0(\mathbf{x},\mathbf{k},t) \\ & + g\sqrt{\rho_0}\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}}\tilde{\eta}(\mathbf{q})\left[KW_1(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2,t) - W_1(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2,t)K^T\right]\ , \end{aligned} \tag{3.133}$$

where

$$\begin{aligned} LW_2(\mathbf{x},\mathbf{X},\mathbf{k},t) = \int \frac{d^3q}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{Q}\cdot\mathbf{X}} & \left[\tilde{A}_\epsilon(\mathbf{k}/\epsilon-\mathbf{Q}/2\epsilon)\tilde{W}_2(\mathbf{q},\mathbf{Q},\mathbf{k},t)\right. \\ & \left. -\tilde{W}_2(\mathbf{q},\mathbf{Q},\mathbf{k},t)\tilde{A}_\epsilon(\mathbf{k}/\epsilon+\mathbf{Q}/2\epsilon)\right]\ , \end{aligned} \tag{3.134}$$

$$M(\mathbf{x},\mathbf{k}) = \begin{bmatrix} ic\,\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.135}$$

In order to obtain the equation satisfied by a_+ (a_-), we multiply (3.133) on the left by $\mathbf{b}_+^T(\mathbf{k})$ ($\mathbf{b}_-^T(\mathbf{k})$) and on the right by $\mathbf{b}_+(\mathbf{k})$ ($\mathbf{b}_-(\mathbf{k})$) and take the average. Moreover, we assume that $\langle \mathbf{b}_\pm^T LW_2\mathbf{b}_\pm\rangle = 0$, which closes the hierarchy of equations and corresponds to the assumption that W_2 is statistically stationary in the fast variable $\mathbf{X}$. This leads to the kinetic equations

$$\begin{aligned} & \frac{1}{c}\partial_t a_\pm(\mathbf{x},\mathbf{k},t) + f_\pm(\mathbf{k})\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}a_\pm(\mathbf{x},\mathbf{k},t) + \mu_\pm(\mathbf{k})a_\pm(\mathbf{x},\mathbf{k},t) \\ & \qquad = \mu_\pm(\mathbf{k})\int d\hat{\mathbf{k}}' A(\mathbf{k},\mathbf{k}')a_\pm(\mathbf{x},\mathbf{k}',t)\ , \end{aligned} \tag{3.136}$$

which is the main result of this paper. Here the scattering coefficients $\mu_\pm$, the phase function

A and transport coefficients $f_{\pm}$ are defined by

$$\mu_{\pm}(\mathbf{k}) = \frac{4\pi(g^2\rho_0)^2|\lambda_{\pm}(\mathbf{k})-\Omega|}{c^2((\lambda_{\pm}(\mathbf{k})-\Omega)^2+g^2\rho_0)^2}\sqrt{(c|\mathbf{k}|-\Omega)^2+4g^2\rho_0}|\mathbf{k}|^2 \times \int \frac{d\hat{\mathbf{k}}'}{(2\pi)^3}\tilde{C}(|\mathbf{k}|(\hat{\mathbf{k}}-\hat{\mathbf{k}}')) , \quad (3.137)$$

$$A(\mathbf{k},\mathbf{k}') = \frac{\tilde{C}(|\mathbf{k}|(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))}{\int d\hat{\mathbf{k}}'\tilde{C}(|\mathbf{k}|(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))} , \quad (3.138)$$

$$f_{\pm}(\mathbf{k}) = \frac{(\lambda_{\pm}(\mathbf{k})-\Omega)^2}{(\lambda_{\pm}(\mathbf{k})-\Omega)^2+g^2\rho_0} . \quad (3.139)$$

The details of this calculation are given in Appendix E.

Suppose that ψ and a have time dependences

$$\psi(\mathbf{x},t) = e^{-i\omega t}\psi_0(\mathbf{x}) , \quad a(\mathbf{x},t) = e^{-i\omega t}a_0(\mathbf{x}) , \quad (3.140)$$

which correspond to eigenstates of the Hamiltonian with energy $\hbar\omega$. Then using (3.136), it can be seen that the Wigner transforms of ψ_0 and a_0 satisfy the radiative transport equation (3.93). That is, the results for the time-independent problem are consistent with those of the time-dependent problem.

The Wigner transform W_0 can be obtained from the solution to the RTE (3.136) by making use of (3.126). It follows from (3.114) and (3.115) that the average probability densities $\langle|\psi|^2\rangle$ and $\langle|a|^2\rangle$ are given by

$$\begin{aligned}\langle|\psi(\mathbf{x},t)|^2\rangle &= \int d^3k (W_0)_{11}(\mathbf{x},\mathbf{k},t) \\ &= \int d^3k \left[\frac{a_+(\mathbf{x},\mathbf{k},t)(\lambda_+(\mathbf{k})-\Omega)^2}{(\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0} + \frac{a_-(\mathbf{x},\mathbf{k},t)(\lambda_-(\mathbf{k})-\Omega)^2}{(\lambda_-(\mathbf{k})-\Omega)^2+g^2\rho_0}\right] , \quad (3.141)\end{aligned}$$

$$\begin{aligned}\rho_0\langle|a(\mathbf{x},t)|^2\rangle &= \int d^3k (W_0)_{22}(\mathbf{x},\mathbf{k},t) \\ &= g^2\rho_0 \int d^3k \left[\frac{a_+(\mathbf{x},\mathbf{k},t)}{(\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0} + \frac{a_-(\mathbf{x},\mathbf{k},t)}{(\lambda_-(\mathbf{k})-\Omega)^2+g^2\rho_0}\right] . \quad (3.142)\end{aligned}$$

3.6.2 Diffusion Approximation

We now consider the diffusion approximation to the kinetic equation (3.136). The diffusion approximation for a kinetic equation of the form

$$\frac{1}{c}\partial_t I(\mathbf{x},\mathbf{k},t) + \hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} I(\mathbf{x},\mathbf{k},t) + \mu_s I(\mathbf{x},\mathbf{k},t) = \mu_s L I(\mathbf{x},\mathbf{k},t) \tag{3.143}$$

is obtained by expanding I in spherical harmonics [16]. To lowest order, it can be seen that

$$I(\mathbf{x},\mathbf{k},t) = \frac{1}{4\pi}\Big(u(\mathbf{x},|\mathbf{k}|,t) - \ell^*\hat{\mathbf{k}}\cdot\nabla u(\mathbf{x},|\mathbf{k}|,t)\Big) \ , \tag{3.144}$$

where $u(\mathbf{x},|\mathbf{k}|,t)$ is defined by

$$u(\mathbf{x},|\mathbf{k}|,t) = \int d\hat{\mathbf{k}} I(\mathbf{x},\mathbf{k},t) \ , \tag{3.145}$$

and ℓ^*, which depends on $|\mathbf{k}|$, is defined by (3.100). We then find that u satisfies the diffusion equation

$$\partial_t u = D\Delta u \ , \tag{3.146}$$

where the diffusion coefficient D is given by

$$D = \frac{1}{3}c\ell^* \ . \tag{3.147}$$

The solution to (3.146) for an infinite medium is given by

$$u(\mathbf{x},t) = \frac{1}{(4\pi Dt)^{3/2}}\int d^3x' \exp\left[-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4Dt}\right] u(\mathbf{x}',0) \ . \tag{3.148}$$

We note that the diffusion approximation is accurate at large distances and long times.

It follows from the above that the first angular moments of $a_\pm$, which are defined by

$$u_\pm(\mathbf{x},|\mathbf{k}|,t) = \int d\hat{\mathbf{k}} a_\pm(\mathbf{x},\mathbf{k},t) \ , \tag{3.149}$$

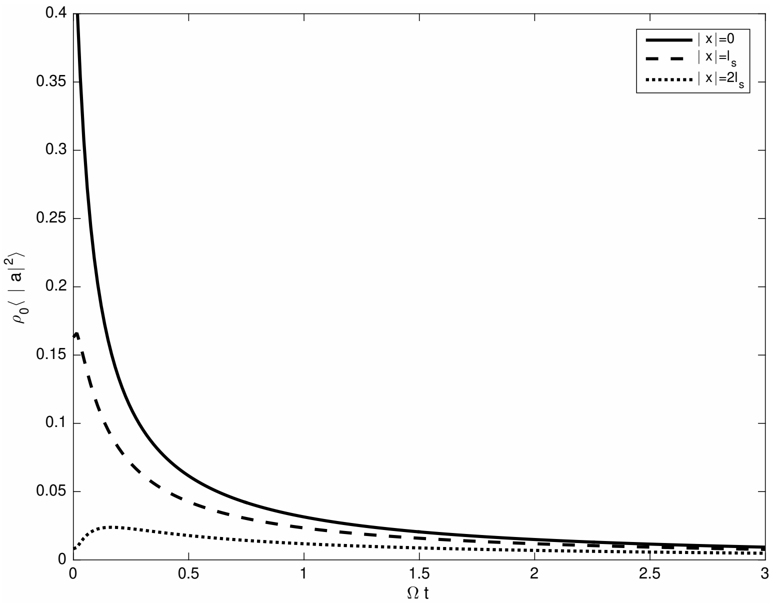


Figure 3.2: Time dependence of atomic probability density for several initial conditions with $|\mathbf{x}| = l_s$.

satisfy diffusion equations of the form

$$\partial_t u_\pm = D_\pm(k)\Delta u_\pm \ . \tag{3.150}$$

Here the diffusion coefficients are given by

$$D_\pm(k) = \frac{c f_\pm(k)^2}{3(1-g)\mu_\pm(k)} \ . \tag{3.151}$$

In order to compute $\langle|\psi(\mathbf{x},t)|^2\rangle$ and $\langle|a(\mathbf{x},t)|^2\rangle$ from (3.141) and (3.142), we must specify the initial conditions $\psi(\mathbf{x},0)$ and $a(\mathbf{x},0)$, which in turn imply initial conditions on

$a_\pm$ of the form

$$a_+(\mathbf{x}, \mathbf{k}, 0) = \frac{(W_0)_{11}(\mathbf{x}, \mathbf{k}, 0)\left[(\lambda_+(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{(\lambda_+(\mathbf{k}) - \Omega)^2 - (\lambda_-(\mathbf{k}) - \Omega)^2} - \frac{(W_0)_{22}(\mathbf{x}, \mathbf{k}, 0)(\lambda_-(\mathbf{k}) - \Omega)^2\left[(\lambda_+(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{g^2\rho_0\left[(\lambda_+(\mathbf{k}) - \Omega)^2 - (\lambda_-(\mathbf{k}) - \Omega)^2\right]}, \tag{3.152}$$

$$a_-(\mathbf{x}, \mathbf{k}, 0) = \frac{(W_0)_{11}(\mathbf{x}, \mathbf{k}, 0)\left[(\lambda_-(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{(\lambda_-(\mathbf{k}) - \Omega)^2 - (\lambda_+(\mathbf{k}) - \Omega)^2} - \frac{(W_0)_{22}(\mathbf{x}, \mathbf{k}, 0)(\lambda_+(\mathbf{k}) - \Omega)^2\left[(\lambda_-(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{g^2\rho_0\left[(\lambda_-(\mathbf{k}) - \Omega)^2 - (\lambda_+(\mathbf{k}) - \Omega)^2\right]}. \tag{3.153}$$

The corresponding initial conditions for $u_\pm(\mathbf{x}, |\mathbf{k}|, t)$ are then given by

$$u_+(\mathbf{x}, |\mathbf{k}|, 0) = \int d\hat{\mathbf{k}} \frac{(W_0)_{11}(\mathbf{x}, \mathbf{k}, 0)\left[(\lambda_+(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{(\lambda_+(\mathbf{k}) - \Omega)^2 - (\lambda_-(\mathbf{k}) - \Omega)^2} - \frac{(W_0)_{22}(\mathbf{x}, \mathbf{k}, 0)(\lambda_-(\mathbf{k}) - \Omega)^2\left[(\lambda_+(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{g^2\rho_0\left[(\lambda_+(\mathbf{k}) - \Omega)^2 - (\lambda_-(\mathbf{k}) - \Omega)^2\right]}, \tag{3.154}$$

$$u_-(\mathbf{x}, |\mathbf{k}|, 0) = \int d\hat{\mathbf{k}} \frac{(W_0)_{11}(\mathbf{x}, \mathbf{k}, 0)\left[(\lambda_-(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{(\lambda_-(\mathbf{k}) - \Omega)^2 - (\lambda_+(\mathbf{k}) - \Omega)^2} - \frac{(W_0)_{22}(\mathbf{x}, \mathbf{k}, 0)(\lambda_+(\mathbf{k}) - \Omega)^2\left[(\lambda_-(\mathbf{k}) - \Omega)^2 + g^2\rho_0\right]}{g^2\rho_0\left[(\lambda_-(\mathbf{k}) - \Omega)^2 - (\lambda_+(\mathbf{k}) - \Omega)^2\right]}. \tag{3.155}$$

We suppose that the atoms are initially excited near the origin in a volume of linear dimensions l_s and that there are no photons present in the field. We thus impose the following initial conditions on the amplitudes:

$$\sqrt{\rho_0}\, a(\mathbf{x}, 0) = \left(\frac{1}{\pi l_s^2}\right)^{3/4} e^{-|\mathbf{x}|^2/2l_s^2}, \tag{3.156}$$

$$\psi(\mathbf{x}, 0) = 0. \tag{3.157}$$

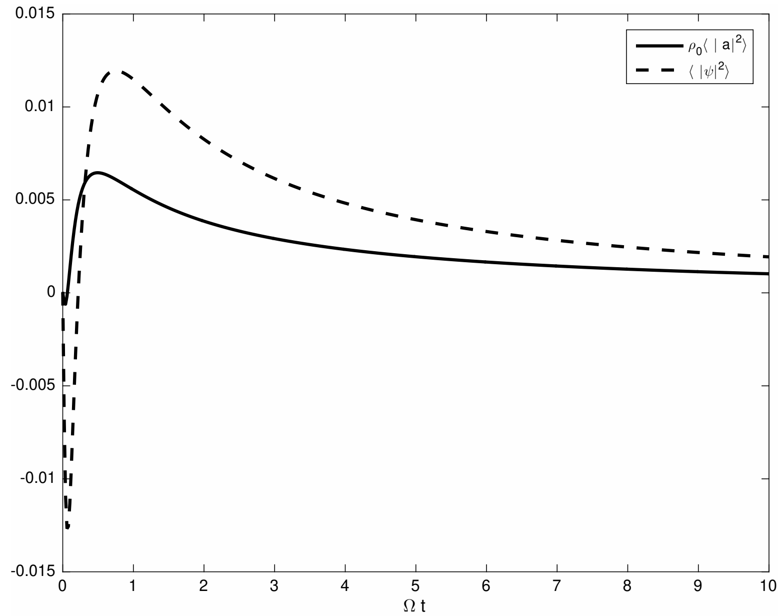


Figure 3.3: Time dependence of the field and atomic probability densities in a random medium with $|\mathbf{x}| = l_s$.

The initial conditions inherited by $u_\pm$ are then given by

$$u_+(\mathbf{x}, |\mathbf{k}|, 0) = \frac{4}{\pi^2}\frac{(\lambda_+(\mathbf{k}) - \Omega)^2 + g^2\rho_0}{(\lambda_-(\mathbf{k}) - \Omega)^2 - (\lambda_+(\mathbf{k}) - \Omega)^2}\frac{(\lambda_-(\mathbf{k}) - \Omega)^2}{g^2\rho_0}e^{-l_s^2|\mathbf{k}|^2}e^{-|\mathbf{x}|^2/l_s^2}\ , \quad (3.158)$$

$$u_-(\mathbf{x}, |\mathbf{k}|, 0) = \frac{4}{\pi^2}\frac{(\lambda_-(\mathbf{k}) - \Omega)^2 + g^2\rho_0}{(\lambda_+(\mathbf{k}) - \Omega)^2 - (\lambda_-(\mathbf{k}) - \Omega)^2}\frac{(\lambda_+(\mathbf{k}) - \Omega)^2}{g^2\rho_0}e^{-l_s^2|\mathbf{k}|^2}e^{-|\mathbf{x}|^2/l_s^2}\ . \quad (3.159)$$

Using (3.148), we find that the solutions to the diffusion equations (3.150) with initial conditions (3.158) and (3.159) are given by

$$
\begin{aligned}
u_+(\mathbf{x}, |\mathbf{k}|, t) &= \frac{4}{\pi^2} \frac{(\lambda_+(\mathbf{k}) - \Omega)^2 + g\rho_0}{(\lambda_-(\mathbf{k}) - \Omega)^2 - (\lambda_+(\mathbf{k}) - \Omega)^2} \frac{(\lambda_-(\mathbf{k}) - \Omega)^2}{g^2\rho_0} e^{-l_s^2|\mathbf{k}|^2} \\
&\quad \times \left(\frac{l_s^2}{l_s^2 + 4tD_+(|\mathbf{k}|)} \right)^{3/2} e^{-|\mathbf{x}|^2/(l_s^2 + 4tD_+(|\mathbf{k}|))} , \qquad (3.160) \\
u_-(\mathbf{x}, |\mathbf{k}|, t) &= \frac{4}{\pi^2} \frac{(\lambda_-(\mathbf{k}) - \Omega)^2 + g^2\rho_0}{(\lambda_+(\mathbf{k}) - \Omega)^2 - (\lambda_-(\mathbf{k}) - \Omega)^2} \frac{(\lambda_+(\mathbf{k}) - \Omega)^2}{g^2\rho_0} e^{-l_s^2|\mathbf{k}|^2} \\
&\quad \times \left(\frac{l_s^2}{l_s^2 + 4tD_-(|\mathbf{k}|)} \right)^{3/2} e^{-|\mathbf{x}|^2/(l_s^2 + 4tD_-(|\mathbf{k}|))} . \qquad (3.161)
\end{aligned}
$$

Using (3.141), we see that the average probability densities are given by the formulas

$$
\begin{aligned}
\langle |\psi(\mathbf{x}, t)|^2 \rangle &= \frac{4}{g^2\rho_0\pi^2} \int_0^\infty dk\, k^2 e^{-l_s^2 k^2} \\
&\quad \times \left[\frac{[(\lambda_-(k) - \Omega)(\lambda_+(k) - \Omega)]^2}{(\lambda_-(k) - \Omega)^2 - (\lambda_+(k) - \Omega)^2} \left(\frac{l_s^2}{l_s^2 + 4tD_+(k)} \right)^{3/2} e^{-|\mathbf{x}|^2/(l_s^2 + 4tD_+(k)} \right. \\
&\quad \left. + \frac{[(\lambda_+(k) - \Omega)(\lambda_-(k) - \Omega)]^2}{(\lambda_+(k) - \Omega)^2 - (\lambda_-(k) - \Omega)^2} \left(\frac{l_s^2}{l_s^2 + 4tD_-(k)} \right)^{3/2} e^{-|\mathbf{x}|^2/(l_s^2 + 4tD_-(k))} \right] ,
\end{aligned} \qquad (3.162)
$$

$$
\begin{aligned}
\rho_0 \langle |a(\mathbf{x}, t)|^2 \rangle &= \frac{4}{\pi^2} \int_0^\infty dk\, k^2 e^{-l_s^2 k^2} \\
&\quad \times \left[\frac{(\lambda_-(k) - \Omega)^2}{(\lambda_-(k) - \Omega)^2 - (\lambda_+(k) - \Omega)^2} \left(\frac{l_s^2}{l_s^2 + 4tD_+(k)} \right)^{3/2} e^{-|\mathbf{x}|^2/(l_s^2 + 4tD_+(k))} \right. \\
&\quad \left. + \frac{(\lambda_+(k) - \Omega)^2}{(\lambda_+(k) - \Omega)^2 - (\lambda_-(k) - \Omega)^2} \left(\frac{l_s^2}{l_s^2 + 4tD_-(k)} \right)^{3/2} e^{-|\mathbf{x}|^2/(l_s^2 + 4tD_-(k))} \right] .
\end{aligned} \qquad (3.163)
$$

At long times, we find that $\langle |a|^2 \rangle$ and $\langle |\psi|^2 \rangle$ decay algebraically according to

$$\rho_0 \langle |a(\mathbf{x}, t)|^2 \rangle = \frac{C_1}{t^{3/2}} - \frac{C_2}{t^{5/2}} |\mathbf{x}|^2 \,, \tag{3.164}$$

$$\langle |\psi(\mathbf{x}, t)|^2 \rangle = \frac{C_3}{t^{3/2}} - \frac{C_4}{t^{5/2}} |\mathbf{x}|^2 \tag{3.165}$$

where the C_i are given by

$$\begin{aligned} C_1 = {} & \frac{4}{\pi^2} \int_0^\infty dk \left(\frac{k^2 e^{-l_s^2 k^2}}{(\lambda_-(k) - \Omega)^2 - (\lambda_+(k) - \Omega)^2} \right) \\ & \times \left[(\lambda_-(k) - \Omega)^2 \left(\frac{l_s^2}{4D_+(k)} \right)^{3/2} - (\lambda_+(k) - \Omega)^2 \left(\frac{l_s^2}{4D_-(k)} \right)^{3/2} \right] \,, \end{aligned} \tag{3.166}$$

$$\begin{aligned} C_2 = {} & \frac{4}{l_s^2 \pi^2} \int_0^\infty dk \left(\frac{k^2 e^{-l_s^2 k^2}}{(\lambda_-(k) - \Omega)^2 - (\lambda_+(k) - \Omega)^2} \right) \\ & \times \left[(\lambda_-(k) - \Omega)^2 \left(\frac{l_s^2}{4D_+(k)} \right)^{5/2} - (\lambda_+(k) - \Omega)^2 \left(\frac{l_s^2}{4D_-(k)} \right)^{5/2} \right] \,, \end{aligned} \tag{3.167}$$

$$\begin{aligned} C_3 = {} & \frac{4}{g^2 \rho_0 \pi^2} \int_0^\infty dk \, k^2 e^{-l_s^2 k^2} \left(\frac{(\lambda_-(k) - \Omega)^2 (\lambda_+(k) - \Omega)^2}{(\lambda_-(k) - \Omega)^2 - (\lambda_+(k) - \Omega)^2} \right) \\ & \times \left[\left(\frac{l_s^2}{4D_+(k)} \right)^{3/2} - \left(\frac{l_s^2}{4D_-(k)} \right)^{3/2} \right] \,, \end{aligned} \tag{3.168}$$

$$\begin{aligned} C_4 = {} & \frac{4}{l_s^2 g^2 \rho_0 \pi^2} \int_0^\infty dk \, k^2 e^{-l_s^2 k^2} \left(\frac{(\lambda_-(k) - \Omega)^2 (\lambda_+(k) - \Omega)^2}{(\lambda_-(k) - \Omega)^2 - (\lambda_+(k) - \Omega)^2} \right) \\ & \times \left[\left(\frac{l_s^2}{4D_+(k)} \right)^{5/2} - \left(\frac{l_s^2}{4D_-(k)} \right)^{5/2} \right] . \end{aligned} \tag{3.169}$$

To illustrate the above results, we consider isotropic scattering with $A = 1/4\pi$, and put the dimensionless quantities $\Omega l_s / c = \rho_0 (g/\Omega)^2 = 1$. In Figure 3.3 we plot the time dependence of $\langle |\psi|^2 \rangle$ and $\rho_0 \langle |a|^2 \rangle$ for $|\mathbf{x}| = 3 l_s$. We note that the negative values of these quantities for small times are due to the breakdown of the diffusion approximation. Figure 3.2 shows the time dependence of $\rho_0 \langle |a(\mathbf{x}, t)|^2 \rangle$ for different values of $|\mathbf{x}|$. As may be expected, $\rho_0 \langle |a|^2 \rangle$ decays faster at larger distances away from the initial volume of

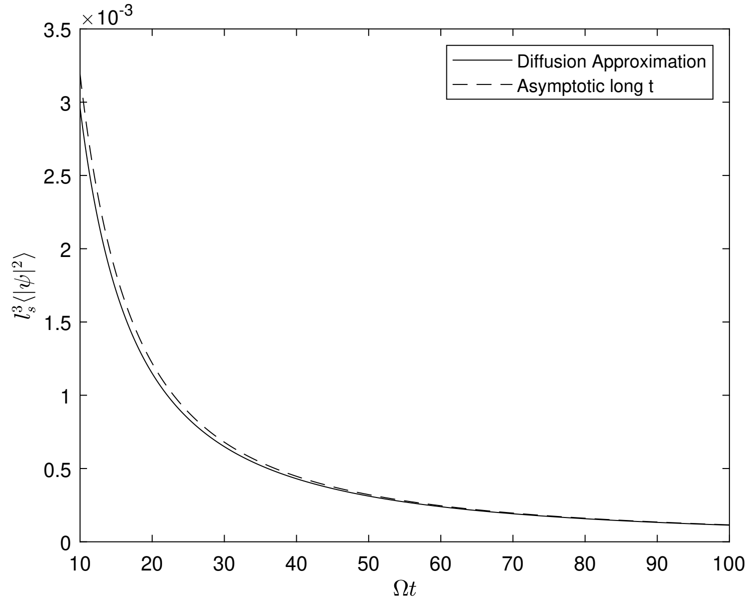


Figure 3.4: Long-time behavior of the field probability density when $|\mathbf{x}| = l_s$.

excitation. In Figures 3.4 and 3.5 we compare (3.162) and (3.163)) with the asymptotic formulas (3.164) and (3.165). There is good agreement at long times.

3.7 Discussion

We have investigated the problem of cooperative spontaneous emission in random media. Our main results are kinetic equations that govern the behavior of the one-photon and atomic probability densities. Several topics for further research are apparent. An alternative derivation of (3.136) may be possible using diagrammatic perturbation theory rather than multiscale asymptotic analysis. This is the case for the classical theory of wave propagation in random media, where a comparative exposition of the two approaches has been presented in [19]. It would also be of interest to examine the transport of two-photon states in random media. Here the evolution of the entanglement of an initially entangled state is of particular importance, especially in applications to communications and imaging. Finally, it would be of interest to extend our results to polariton transport in random media consisting of

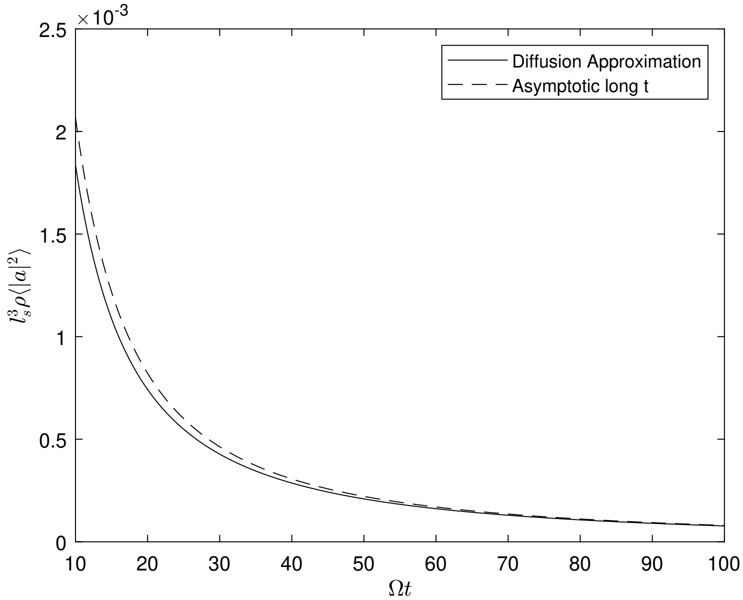


Figure 3.5: Long-time behavior of the atomic probability density when $|\mathbf{x}| = l_s$.

atoms embedded in a dielectric. In this setting, the systems of equations (3.43) and (3.44) is no longer nonlocal.

3.8 Appendix

3.8.1 Derivation of (3.23) and (3.24)

0em Here we derive the system of equations (3.23). To proceed, we compute both sides of the Schrodinger equation $i\hbar\partial_t|\Psi\rangle = H|\Psi\rangle$. Making use of

$$|\Psi\rangle = \int d^3x \left[\psi(\mathbf{x},t)\varphi^\dagger(\mathbf{x}) + \rho(\mathbf{x})a(\mathbf{x},t)\sigma^\dagger(\mathbf{x})\right]|0\rangle \,, \tag{3.170}$$

we find that the left hand side is given by

$$i\hbar\partial_t|\Psi\rangle = \int d^3x \left[i\hbar\partial_t\psi(\mathbf{x},t)\varphi^\dagger(\mathbf{x}) + i\hbar\partial_t a(\mathbf{x},t)\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\right]|0\rangle \ . \tag{3.171}$$

Next, using the definition (3.19) of the Hamiltonian H and the commutation relations (4.2) and (4.7), the right hand side becomes

$$\begin{aligned} H|\Psi\rangle = \int d^3x \big[&\hbar c(-\Delta)^{1/2}\psi(\mathbf{x},t)\varphi^\dagger(\mathbf{x}) + \hbar g\rho(\mathbf{x})\psi(\mathbf{x},t)\sigma^\dagger(\mathbf{x}) + \hbar g\rho(\mathbf{x})a(\mathbf{x},t)\varphi^\dagger(\mathbf{x}) \\ &+ \hbar\Omega\rho(\mathbf{x})a(\mathbf{x},t)\sigma^\dagger(\mathbf{x})\big]|0\rangle \ . \end{aligned} \tag{3.172}$$

It follows that

$$\langle 0|\varphi(\mathbf{x})i\hbar\partial_t|\Psi\rangle = i\hbar\partial_t\psi(\mathbf{x},t) \ , \tag{3.173}$$

$$\langle 0|\varphi(\mathbf{x})H|\Psi\rangle = \hbar c((-\Delta)^{1/2}\psi)(\mathbf{x},t) + \hbar g\rho(\mathbf{x})a(\mathbf{x},t) \ . \tag{3.174}$$

Likewise

$$\langle 0|\sigma(\mathbf{x})\rho(\mathbf{x})i\hbar\partial_t|\Psi\rangle = i\hbar\partial_t a(\mathbf{x},t) \ , \tag{3.175}$$

$$\langle 0|\sigma(\mathbf{x})\rho(\mathbf{x})H|\Psi\rangle = \hbar g\rho(\mathbf{x})\psi(\mathbf{x},t) + \hbar\Omega\rho(\mathbf{x})a(\mathbf{x},t) \ . \tag{3.176}$$

We thus obtain

$$i\partial_t\psi = c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a \ , \tag{3.177}$$

$$i\rho(\mathbf{x})\partial_t a = g\rho(\mathbf{x})\psi + \Omega\rho(\mathbf{x})a \ , \tag{3.178}$$

which are (3.23) and (3.24).

3.8.2 Derivation of Eq. (3.77)

We proceed from (3.73):

$$\epsilon c\left[(-\Delta_{\mathbf{x}_1})^{1/2}-(-\Delta_{\mathbf{x}_2})^{1/2}\right]\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_2)+\sqrt{\epsilon}\frac{\rho_0 g^2}{\omega-\Omega}\left[\eta(\mathbf{x}_1/\epsilon)-\eta(\mathbf{x}_2/\epsilon)\right]\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_2)=0 \tag{3.179}$$

and make the change of variables

$$\mathbf{x}_1=\mathbf{x}-\epsilon\mathbf{x}'/2\ , \tag{3.180}$$

$$\mathbf{x}_2=\mathbf{x}+\epsilon\mathbf{x}'/2\ . \tag{3.181}$$

We then Fourier transform the result with respect to $\mathbf{x}'$. The first term becomes

$$\begin{aligned}
&\epsilon c\int\frac{d^3x'}{(2\pi)^3}\,e^{-i\mathbf{k}\cdot\mathbf{x}'}\left[(-\Delta_{\mathbf{x}_1})^{1/2}-(-\Delta_{\mathbf{x}_2})^{1/2}\right]\Phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,\mathbf{x}+\epsilon\mathbf{x}'/2)\\
=&\epsilon c\int\frac{d^3x'}{(2\pi)^3}\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}'}e^{i\mathbf{q}_1\cdot(\mathbf{x}-\epsilon\mathbf{x}'/2)}e^{-i\mathbf{q}_2\cdot(\mathbf{x}+\epsilon\mathbf{x}'/2)}\left[|\mathbf{q}_1|-|\mathbf{q}_2|\right]\tilde{\Phi}_\epsilon(\mathbf{q}_1,\mathbf{q}_2)\\
=&c\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\left[|-\mathbf{k}+\epsilon\mathbf{q}/2|-|\mathbf{k}+\epsilon\mathbf{q}/2|\right]\tilde{W}_\epsilon(\mathbf{q},\mathbf{k})\ .
\end{aligned} \tag{3.182}$$

Continuing with the second term we have

$$\sqrt{\epsilon}\frac{\rho_0 g^2}{\omega-\Omega}\int\frac{d^3x'}{(2\pi)^3}\,e^{-i\mathbf{k}\cdot\mathbf{x}'}\,\eta(\mathbf{x}/\epsilon-\mathbf{x}'/2)\Phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,\mathbf{x}+\epsilon\mathbf{x}'/2) \tag{3.183}$$

$$=\sqrt{\epsilon}\frac{\rho_0 g^2}{\omega-\Omega}\int\frac{d^3q}{(2\pi)^3}\,e^{i\mathbf{q}\cdot\mathbf{x}/\epsilon}\,\tilde{\eta}(\mathbf{q})W_\epsilon(\mathbf{x},\mathbf{k}+\mathbf{q}/2)\ . \tag{3.184}$$

The third term follows similarly:

$$\sqrt{\epsilon}\frac{\rho_0 g^2}{\omega-\Omega}\int\frac{d^3x'}{(2\pi)^3}\,e^{-i\mathbf{k}\cdot\mathbf{x}'}\,\eta(\mathbf{x}/\epsilon+\mathbf{x}'/2)\Phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,\mathbf{x}+\epsilon\mathbf{x}'/2) \tag{3.185}$$

$$=\sqrt{\epsilon}\frac{\rho_0 g^2}{\omega-\Omega}\int\frac{d^3q}{(2\pi)^3}\,e^{i\mathbf{q}\cdot\mathbf{x}/\epsilon}\,\tilde{\eta}(\mathbf{q})W_\epsilon(\mathbf{x},\mathbf{k}-\mathbf{q}/2)\ . \tag{3.186}$$

Combining the above yields (3.77):

$$\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \left[|-\mathbf{k}+\epsilon\mathbf{q}/2| - |\mathbf{k}+\epsilon\mathbf{q}/2| \right] \tilde{W}_\epsilon(\mathbf{q},\mathbf{k}) + \sqrt{\epsilon} L W_\epsilon(\mathbf{x},\mathbf{k}) = 0 \,, \tag{3.187}$$

where

$$L W_\epsilon(\mathbf{x},\mathbf{k}) = k_0 \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\epsilon} \tilde{\eta}(\mathbf{q}) \left[W_\epsilon(\mathbf{x},\mathbf{k}+\mathbf{q}/2) - W_\epsilon(\mathbf{x},\mathbf{k}-\mathbf{q}/2) \right] \,. \tag{3.188}$$

3.8.3 Derivation of Eq. (3.90)

We evaluate the second term in (3.88) using (3.86)

$$\begin{aligned}
& k_0 \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}} \langle \tilde{\eta}(\mathbf{q}) \left[W_1(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2) - W_1(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2) \right] = \\
& = k_0^2 \int \frac{d^3q}{(2\pi)^3} \frac{d^3Q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}+i\mathbf{Q}\cdot\mathbf{X}} \langle \tilde{\eta}(\mathbf{q}) \\
& \times \left[\tilde{\eta}(\mathbf{Q}) \frac{\left[W_0(\mathbf{x},\mathbf{k}+\mathbf{q}/2+\mathbf{Q}/2) - W_0(\mathbf{x},\mathbf{k}+\mathbf{q}/2-\mathbf{Q}/2) \right]}{|\mathbf{k}+\mathbf{q}/2+\mathbf{Q}/2| - |-\mathbf{k}-\mathbf{q}/2+\mathbf{Q}/2| - i\theta} \right. \\
& \left. - \tilde{\eta}(\mathbf{Q}) \frac{\left[W_0(\mathbf{x},\mathbf{k}-\mathbf{q}/2+\mathbf{Q}/2) - W_0(\mathbf{x},\mathbf{k}-\mathbf{q}/2-\mathbf{Q}/2) \right]}{|\mathbf{k}-\mathbf{q}/2+\mathbf{Q}/2| - |-\mathbf{k}+\mathbf{q}/2+\mathbf{Q}/2| - i\theta} \right] \rangle \\
& = k_0^2 \int \frac{d^3q}{(2\pi)^3} \tilde{C}(\mathbf{q}-\mathbf{k}) \left(W_0(\mathbf{x},\mathbf{k}) - W_0(\mathbf{x},\mathbf{q}) \right) \left[\frac{1}{|\mathbf{k}| - |\mathbf{q}| - i\theta} - \frac{1}{|\mathbf{k}| - |\mathbf{q}| + i\theta} \right] \,,
\end{aligned} \tag{3.189}$$

where we have made use of the relation

$$\langle \tilde{\eta}(\mathbf{q}) \tilde{\eta}(\mathbf{Q}) \rangle = (2\pi)^3 \tilde{C}(\mathbf{q}) \delta(\mathbf{q}+\mathbf{Q}) \,. \tag{3.190}$$

Finally we put $\theta \to 0$ and use the identity

$$\lim_{\theta\to 0} \left(\frac{1}{x - i\theta} - \frac{1}{x + i\theta} \right) = 2\pi i \delta(x) \,, \tag{3.191}$$

to obtain

$$k_0^2 \int \frac{d^3q}{(2\pi)^2} \tilde{C}(\mathbf{q}-\mathbf{k}) \left(W_0(\mathbf{x},\mathbf{k}) - W_0(\mathbf{x},\mathbf{q})\right) \delta(|\mathbf{k}| - |\mathbf{q}|) \ , \tag{3.192}$$

which yields (3.90).

3.8.4 Derivation of Eq. (3.119)

We begin with the equation satisfied by $\Phi_\epsilon(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{u}_\epsilon(\mathbf{x}_1)\mathbf{u}_\epsilon^*(\mathbf{x}_2)$:

$$\begin{aligned} \epsilon i \partial_t \Phi_\epsilon &= A_\epsilon(\mathbf{x}_1)\Phi_\epsilon + \sqrt{\epsilon} g \sqrt{\rho_0} \eta(\mathbf{x}_1/\epsilon) K \Phi_\epsilon \\ &\quad - \Phi_\epsilon A_\epsilon(\mathbf{x}_2) + \sqrt{\epsilon} g \sqrt{\rho_0} \eta(\mathbf{x}_2/\epsilon) \Phi_\epsilon K^T . \end{aligned} \tag{3.193}$$

Next we make the change of variables

$$\mathbf{x}_1 = \mathbf{x} - \epsilon \mathbf{x}'/2 \ , \tag{3.194}$$

$$\mathbf{x}_2 = \mathbf{x} + \epsilon \mathbf{x}'/2 \ , \tag{3.195}$$

and Fourier transform the result with respect to $\mathbf{x}'$, which leads to

$$\begin{aligned} \epsilon i \partial_t W_\epsilon(\mathbf{x},\mathbf{k},t) &= \int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \big[A_\epsilon(\mathbf{x} - \epsilon\mathbf{x}'/2)\Phi_\epsilon(\mathbf{x} - \epsilon\mathbf{x}'/2, \mathbf{x} + \epsilon\mathbf{x}'/2) \\ &\quad - \Phi_\epsilon(\mathbf{x} - \epsilon\mathbf{x}'/2, \mathbf{x} + \epsilon\mathbf{x}'/2) A_\epsilon(\mathbf{x} + \epsilon\mathbf{x}'/2) \big] \\ &\quad + \sqrt{\epsilon} g \sqrt{\rho_0} \int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \eta(\mathbf{x}/\epsilon - \mathbf{x}'/2) K \Phi_\epsilon(\mathbf{x} - \epsilon\mathbf{x}'/2, \mathbf{x} + \epsilon\mathbf{x}'/2) \\ &\quad + \sqrt{\epsilon} g \sqrt{\rho_0} \int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \eta(\mathbf{x}/\epsilon + \mathbf{x}'/2) \Phi_\epsilon(\mathbf{x} - \epsilon\mathbf{x}'/2, \mathbf{x} + \epsilon\mathbf{x}'/2) K^T . \end{aligned} \tag{3.196}$$

The first term on the right hand side of (3.196) becomes

$$\begin{aligned}
&\int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \left[A_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2)\Phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,\mathbf{x}+\epsilon\mathbf{x}'/2)\right. \\
&\left.-\Phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,\mathbf{x}+\epsilon\mathbf{x}'/2)A_\epsilon(\mathbf{x}+\epsilon\mathbf{x}'/2)\right] \\
=&\int d^3x' \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}'+i\mathbf{q}_1\cdot(\mathbf{x}-\epsilon\mathbf{x}'/2)-i\mathbf{q}_2\cdot(\mathbf{x}+\epsilon\mathbf{x}'/2)} \\
&\times\left[\tilde{A}_\epsilon(\mathbf{q}_1)\tilde{\Phi}_\epsilon(\mathbf{q}_1,\mathbf{q}_2)-\tilde{\Phi}_\epsilon(\mathbf{q}_1,\mathbf{q}_2)\tilde{A}_\epsilon(\mathbf{q}_2)\right] \\
=&\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{x}\cdot\mathbf{q}}\left[\tilde{A}_\epsilon(\mathbf{k}/\epsilon-\mathbf{q}/2)\tilde{W}_\epsilon(\mathbf{q},\mathbf{k})-\tilde{W}_\epsilon(\mathbf{q},\mathbf{k})\tilde{A}_\epsilon(\mathbf{k}/\epsilon+\mathbf{q}/2)\right] \ .
\end{aligned} \tag{3.197}$$

The second term is seen to be

$$\begin{aligned}
&\sqrt{\epsilon}g\sqrt{\rho_0}\int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'}\eta(\mathbf{x}/\epsilon-\mathbf{x}'/2)K\Phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,\mathbf{x}+\epsilon\mathbf{x}'/2) \\
=&\sqrt{\epsilon}g\sqrt{\rho_0}\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\epsilon}\tilde{\eta}(\mathbf{q})KW_\epsilon(\mathbf{x},\mathbf{k}+\mathbf{q}/2) \ .
\end{aligned} \tag{3.198}$$

The third term is handled similarly:

$$\begin{aligned}
&\sqrt{\epsilon}g\sqrt{\rho_0}\int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'}\eta(\mathbf{x}/\epsilon+\mathbf{x}'/2)\Phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,\mathbf{x}+\epsilon\mathbf{x}'/2)K^T \\
=&\sqrt{\epsilon}g\sqrt{\rho_0}\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\epsilon}\tilde{\eta}(\mathbf{q})W_\epsilon(\mathbf{x},\mathbf{k}-\mathbf{q}/2) \ .
\end{aligned} \tag{3.199}$$

Putting the above together yields (3.119).

3.8.5 Derivation of Eq. (3.136)

The first two terms on the left hand side of (3.136) are easily obtained. The remaining terms come from considering

$$\langle \mathbf{b}_+^T(\mathbf{k})g\sqrt{\rho_0}\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}}\tilde{\eta}(\mathbf{q})\left[KW_1(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2)-W_1(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2)K^T\right]\mathbf{b}_+(\mathbf{k})\rangle \tag{3.200}$$

The first term above is given by

$$
\begin{aligned}
&\langle \mathbf{b}_+^T(\mathbf{k}) \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}} \tilde{\eta}(\mathbf{q}) K W_1(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2,t)\mathbf{b}_+(\mathbf{k})\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}+i\mathbf{Q}\cdot\mathbf{X}} \langle \mathbf{b}_+^T(\mathbf{k})\tilde{\eta}(\mathbf{q}) K \sum_{m,n} w_{mn}(\mathbf{x},\mathbf{Q},\mathbf{k}+\mathbf{q}/2,t)\mathbf{b}_m(\mathbf{k}+\mathbf{q}/2-\mathbf{Q}/2) \\
&\quad \times \mathbf{b}_n^T(\mathbf{k}+\mathbf{q}/2+\mathbf{Q}/2)\mathbf{b}_+(\mathbf{k})\rangle \\
&= \int \frac{d^3q}{(2\pi)^3} \tilde{C}(\mathbf{k}-\mathbf{q}) \\
&\quad \times \sum_m \frac{(g^2\rho_0)^{3/2}(\lambda_+(\mathbf{k})-\Omega)((\lambda_+(\mathbf{k})-\Omega)a_m(\mathbf{x},\mathbf{q},t)-(\lambda_m(\mathbf{q})-\Omega)a_+(\mathbf{x},\mathbf{k},t))}{((\lambda_m(\mathbf{q})-\Omega)^2+g^2\rho_0)((\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0)(\lambda_m(\mathbf{q})-\lambda_+(\mathbf{k})+i\theta)}\ .
\end{aligned}
\tag{3.201}
$$

The second term becomes

$$
\begin{aligned}
&\langle \mathbf{b}_+^T(\mathbf{k}) \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}} \tilde{\eta}(\mathbf{q}) W_1(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2,t) K^T \mathbf{b}_+(\mathbf{k})\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}+i\mathbf{Q}\cdot\mathbf{X}} \langle \mathbf{b}_+^T(\mathbf{k})\tilde{\eta}(\mathbf{q}) \sum_{m,n} w_{mn}(\mathbf{x},\mathbf{Q},\mathbf{k}-\mathbf{q}/2,t)\mathbf{b}_m(\mathbf{k}-\mathbf{q}/2-\mathbf{Q}/2) \\
&\quad \times \mathbf{b}_n^T(\mathbf{k}-\mathbf{q}/2+\mathbf{Q}/2)K^T\mathbf{b}_+(\mathbf{k})\rangle \\
&= \int \frac{d^3q}{(2\pi)^3} \tilde{C}(\mathbf{k}-\mathbf{q}) \\
&\quad \times \sum_n \frac{(g^2\rho_0)^{3/2}(\lambda_+(\mathbf{k})-\Omega)((\lambda_n(\mathbf{q})-\Omega)a_+(\mathbf{x},\mathbf{k},t)-(\lambda_+(\mathbf{k})-\Omega)a_n(\mathbf{x},\mathbf{q},t))}{((\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0)((\lambda_n(\mathbf{q})-\Omega)^2+g^2\rho_0)(\lambda_+(\mathbf{k})-\lambda_n(\mathbf{q})+i\theta))}\ .
\end{aligned}
\tag{3.202}
$$

Subtracting (3.201) and (3.202), letting $\theta \to 0$ and using (3.191) yields

$$
\begin{aligned}
&\langle \mathbf{b}_+^T(\mathbf{k}) g\sqrt{\rho_0} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}} \tilde{\eta}(\mathbf{q}) \left[K W_1(\mathbf{x},\mathbf{X},\mathbf{k}+\mathbf{q}/2) - W_1(\mathbf{x},\mathbf{X},\mathbf{k}-\mathbf{q}/2)K^T\right] \mathbf{b}_+(\mathbf{k})\rangle \\
&= \frac{2\pi(g^2\rho_0)^2(\lambda_+(\mathbf{k})-\Omega)^2}{(\lambda_+(\mathbf{q})-\Omega)^2+g^2\rho_0)^2} \int \frac{d^3q}{(2\pi)^3} \tilde{C}(\mathbf{k}-\mathbf{q})\delta(\lambda_+(\mathbf{q})-\lambda_+(\mathbf{k}))\left[a_+(\mathbf{x},\mathbf{k})-a_+(\mathbf{x},\mathbf{q})\right]\ ,
\end{aligned}
\tag{3.203}
$$

where only the $m = +$ contribution is included. Putting everything together we see that a_+ satisfies the equation

$$
\begin{aligned}
&\frac{1}{c}\partial_t a_+(\mathbf{x},\mathbf{k},t) + \left(\frac{(\lambda_+(\mathbf{k})-\Omega)^2}{(\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0}\right)\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} a_+(\mathbf{x},\mathbf{k},t) \\
&+\left[\frac{2\pi(g^2\rho_0)^2(\lambda_+(\mathbf{k})-\Omega)^2}{c((\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0)^2}\int\frac{d^3q}{(2\pi)^3}\,\tilde{C}(\mathbf{k}-\mathbf{q})\delta(\lambda_+(\mathbf{q})-\lambda_+(\mathbf{k}))\right]a_+(\mathbf{x},\mathbf{k},t) \\
&=\frac{2\pi(g^2\rho_0)^2(\lambda_+(\mathbf{k})-\Omega)^2}{c((\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0)^2}\int\frac{d^3q}{(2\pi)^3}\,\tilde{C}(\mathbf{k}-\mathbf{q})\delta(\lambda_+(\mathbf{q})-\lambda_+(\mathbf{k}))a_+(\mathbf{r},\mathbf{q},t)\ . \qquad (3.204)
\end{aligned}
$$

The delta function $\delta(\lambda_+(\mathbf{q})-\lambda_+(\mathbf{k}))$ can be expressed as

$$
\delta(\lambda_+(\mathbf{q})-\lambda_+(\mathbf{k})) = 2\delta(|\mathbf{q}|-|\mathbf{k}|)\frac{\sqrt{(c|\mathbf{k}|-\Omega)^2+4g^2\rho_0}}{c(\lambda_+(\mathbf{k})-\Omega)}\ . \qquad (3.205)
$$

Hence (3.204) becomes

$$
\begin{aligned}
&\frac{1}{c}\partial_t a_+(\mathbf{x},\mathbf{k},t) + \left(\frac{(\lambda_+(\mathbf{k})-\Omega)^2}{(\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0}\right)\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} a_+(\mathbf{x},\mathbf{k},t) \\
&+\mu_+(\mathbf{k})a_+(\mathbf{x},\mathbf{k},t) = \mu_+(\mathbf{k})\int\frac{d\hat{\mathbf{k}}'}{(2\pi)^3}\,A(\mathbf{k},\mathbf{k}')a_+(\mathbf{r},\mathbf{k}',t)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_+(\mathbf{k}) &= \frac{4\pi(g^2\rho_0)^2|\lambda_+(\mathbf{k})-\Omega|}{c^2((\lambda_+(\mathbf{k})-\Omega)^2+g^2\rho_0)^2}\sqrt{(c|\mathbf{k}|-\Omega)^2+4g^2\rho_0}|\mathbf{k}|^2 \\
&\quad\times\int\frac{d\hat{\mathbf{k}}'}{(2\pi)^3}\,\tilde{C}(|\mathbf{k}|(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))\ , \qquad (3.206) \\
A(\mathbf{k},\mathbf{k}') &= \frac{\tilde{C}(|\mathbf{k}|(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))}{\displaystyle\int d\hat{\mathbf{k}}'\tilde{C}(|\mathbf{k}|(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))}\ . \qquad (3.207)
\end{aligned}
$$

This is (3.136). The equation for a_- is derived in the same manner.

Chapter 4

Kinetic Equations for Two-photon Light in Random Media

4.1 Introduction

The propagation of light in random media is generally considered within the setting of classical optics [90]. However, there has been considerable recent interest in phenomena where quantum effects play a key role [54, 81, 83, 66, 67, 82, 89, 53, 51, 52, 65, 13, 88, 12]. Of particular importance is understanding the impact of multiple scattering on entangled two-photon states, with an eye towards characterizing the transfer of entanglement from the field to matter [13, 49, 66, 42, 21, 15, 55]. Progress in this direction can be expected to lead to advances in spectroscopy [80], imaging [47, 84, 3, 5, 34, 73, 72, 27, 24, 74] and communications [63, 79, 76] .

The propagation of two-photon light is generally considered either in free space or, in some cases, with account of diffraction [71, 4] or scattering [41]. In this paper, we consider the propagation of two-photon light in a random medium. A step in this direction was taken in [59], where a model in which the field is quantized and the medium is treated classically was investigated. The main drawback of that work is that it is does not allow for the transfer of entanglement between the field and the atoms or between the atoms themselves. Instead, we treat the problem from first principles, employing a model in which the field and the matter are both quantized. We show that for a medium consisting of two-level atoms, the field and atomic probability amplitudes for a two-photon state obey a system of nonlocal partial differential equations with random coefficients. Using this result, we find that at long times and large distances, the corresponding average probability densities in a random medium can be determined from the solutions to a system of kinetic equations. These equations follow from the multiscale asymptotics of the average Wigner transform of the amplitudes in a suitable high-frequency limit [48, 10, 19, 16]. This formulation of the problem builds on earlier research by the authors on collective emission of single photons in a random medium of two-level atoms. In that work, we employed a formulation of quantum electrodynamics in which the field is quantized in real space, thus allowing for the field and atomic degrees of freedom to be treated on the same footing.

This paper is organized as follows. In section II we formulate our model for the propagation of two-photon light and derive the equations governing the dynamics of the field and atomic amplitudes. Section III is concerned with the application of the real-space formalism to the problem of stimulated emission by a single atom. In section IV we study the dynamics of a two-photon state and in section V we discuss the problem of stimulated emission in a random medium and obtain the governing kinetic equations. Section VI takes up the general problem of two-photon transport in random media and presents the relevant kinetic equations. Our conclusions are formulated in section VII. The appendices contain the details of long calculations.

4.2 Model

We consider the following model for the interaction between a quantized massless scalar field and a system of two-level atoms. The atoms are taken to be identical, stationary and sufficiently well separated that interatomic interactions can be neglected. The overall system is described by the Hamiltonian $H = H_F + H_A + H_I$. In order to treat the atoms and the field on the same footing, it is useful to introduce a real-space representation of H.

The Hamiltonian of the field is of the form

$$H_F = \hbar c \int d^3x(-\Delta)^{1/2}\varphi^\dagger(\mathbf{x})\varphi(\mathbf{x}) \ , \tag{4.1}$$

where φ is a Bose field that obeys the commutation relations

$$[\varphi(\mathbf{x}), \varphi^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}') \ , \tag{4.2}$$

$$[\varphi(\mathbf{x}), \varphi(\mathbf{x}')] = 0 \ . \tag{4.3}$$

where we have neglected the zero-point energy. The nonlocal operator $(-\Delta)^{1/2}$ is defined by the Fourier integral

$$(-\Delta)^{1/2} f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} |\mathbf{k}| \tilde{f}(\mathbf{k}) \ , \tag{4.4}$$

$$\tilde{f}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \ . \tag{4.5}$$

We note that (4.1) is equivalent to the usual oscillator representation of H_F.

To facilitate the treatment of random media, it will prove convenient to introduce a continuum model of the atomic degrees of freedom. The Hamiltonian of the atoms is given by

$$H_A = \hbar\Omega \int d^3x \rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x}) \ , \tag{4.6}$$

where Ω is the atomic resonance frequency, $\rho(\mathbf{x})$ is the number density of the atoms, and σ is a Fermi field that obeys the anticommutation relations

$$\{\sigma(\mathbf{x}), \sigma^\dagger(\mathbf{x}')\} = \frac{1}{\rho(\mathbf{x})}\delta(\mathbf{x} - \mathbf{x}') \ , \tag{4.7}$$

$$\{\sigma(\mathbf{x}), \sigma(\mathbf{x}')\} = 0 \ . \tag{4.8}$$

The Hamiltonian describing the interaction between the field and the atoms is taken to be

$$H_I = \hbar g \int d^3x \rho(\mathbf{x}) \left(\varphi^\dagger(\mathbf{x})\sigma(\mathbf{x}) + \sigma^\dagger(\mathbf{x})\varphi(\mathbf{x})\right) , \tag{4.9}$$

where g is the strength of the atom-field coupling. Here we have made the Markovian approximation, in which the coupling constant is independent of the frequency of the photons or positions of the atoms, and have imposed the rotating wave approximation (RWA).

We suppose that the system is in a two-photon state of the form

$$\begin{aligned} |\Psi\rangle = \int d^3x_1 d^3x_2 \Big(&\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) + \psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\ &+ a(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2)\Big)|0\rangle , \end{aligned} \tag{4.10}$$

where $|0\rangle$ is the combined vacuum state of the field and the ground state of the atoms. Here the atomic amplitude $a(\mathbf{x}_1,\mathbf{x}_2,t)$ is the probability amplitude for exciting two atoms at the points $\mathbf{x}_1$ and $\mathbf{x}_2$ at time t, the one-photon amplitude $\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)$ is the probability amplitude for exciting an atom at $\mathbf{x}_1$ and creating a photon at $\mathbf{x}_2$, and the two-photon amplitude $\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)$ is the probability amplitude for creating photons at $\mathbf{x}_1$ and $\mathbf{x}_2$. The functions ψ_2 and a are symmetric and antisymmetric, respectively:

$$\psi_2(\mathbf{x}_2,\mathbf{x}_1,t) = \psi_2(\mathbf{x}_1,\mathbf{x}_2,t) , \quad a(\mathbf{x}_2,\mathbf{x}_1,t) = -a(\mathbf{x}_1,\mathbf{x}_2,t) , \tag{4.11}$$

consistent with the bosonic and fermionic character of the corresponding fields.

The state $|\Psi\rangle$ is the most general two-photon state within the RWA. In addition, $|\Psi\rangle$ is normalized so that $\langle\Psi|\Psi\rangle = 1$. It follows from (4.2) and (4.7) that the amplitudes obey the normalization condition

$$\int d^3x_1 d^3x_2 \left(2|\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)|^2 + \rho(\mathbf{x}_1)|\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)|^2 + 2\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)|a(\mathbf{x}_1,\mathbf{x}_2,t)|^2\right) = 1 . \tag{4.12}$$

If the amplitudes $a(\mathbf{x}_1, \mathbf{x}_2, t)$, $\psi_1(\mathbf{x}_1, \mathbf{x}_2, t)$ and $\psi_2(\mathbf{x}_1, \mathbf{x}_2, t)$ are factorizable as functions of $\mathbf{x}_1$ and $\mathbf{x}_2$, then there are no quantum correlations and the state $|\Psi\rangle$ is not entangled. Otherwise $|\Psi\rangle$ is entangled. If ψ_2 alone is not factorizable, we say that $|\Psi\rangle$ is an entangled two-photon state.

The dynamics of $|\Psi\rangle$ is governed by the Schrodinger equation

$$i\hbar\partial_t|\Psi\rangle = H|\Psi\rangle \ . \tag{4.13}$$

Projecting onto the states $\varphi^\dagger(\mathbf{x})|0\rangle$ and $\sigma^\dagger(\mathbf{x})|0\rangle$ and making use of (4.2) and (4.7), we arrive at the following system of equations obeyed by a, ψ_1 and ψ_2:

$$\begin{aligned}
&i\partial_t\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) = c(-\Delta_{\mathbf{x}_1})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) \\
&+ \frac{g}{2}(\rho(\mathbf{x}_1)\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) + \rho(\mathbf{x}_2)\psi_1(\mathbf{x}_2, \mathbf{x}_1, t)) \ , && (4.14) \\
&\rho(\mathbf{x}_1)i\partial_t\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) = 2g\rho(\mathbf{x}_1)\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + \rho(\mathbf{x}_1)\left[c(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega\right]\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) \\
&- 2g\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)a(\mathbf{x}_1, \mathbf{x}_2, t) \ , && (4.15) \\
&\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)i\partial_t a(\mathbf{x}_1, \mathbf{x}_2, t) = \rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\frac{g}{2}(\psi_1(\mathbf{x}_2, \mathbf{x}_1, t) - \psi_1(\mathbf{x}_1, \mathbf{x}_2, t)) \\
&+ 2\Omega\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\, a(\mathbf{x}_1, \mathbf{x}_2, t) \ . && (4.16)
\end{aligned}$$

The overall factors of $\rho(\mathbf{x})$ in the above will be cancelled as necessary. The details of the calculations are presented in Appendix A.

4.3 Single-Atom Stimulated Emission

In this section we consider the problem of stimulated emission by a single atom. We assume that the atom is located at the origin and put $\rho(\mathbf{x}) = \delta(\mathbf{x})$. Thus the system (4.14)–(4.16) becomes

$$i\partial_t\psi_2(\mathbf{x}_1,\mathbf{x}_2,t) = c(-\Delta_{\mathbf{x}_1})^{1/2}\psi_2(\mathbf{x}_1,\mathbf{x}_2,t) + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)$$
$$+\frac{g}{2}(\delta(\mathbf{x}_1)\psi_1(\mathbf{x}_1,\mathbf{x}_2,t) + \delta(\mathbf{x}_2)\psi_1(\mathbf{x}_2,\mathbf{x}_1,t)) , \tag{4.17}$$
$$\delta(\mathbf{x}_1)i\partial_t\psi_1(\mathbf{x}_1,\mathbf{x}_2,t) = 2g\delta(\mathbf{x}_1)\psi_2(\mathbf{x}_1,\mathbf{x}_2,t) + \delta(\mathbf{x}_1)\left[c(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega\right]\psi_1(\mathbf{x}_1,\mathbf{x}_2,t) , \tag{4.18}$$

where the term $\delta(\mathbf{x}_1)\delta(\mathbf{x}_2)a(\mathbf{x}_1,\mathbf{x}_2,t)$ does not contribute due to the antisymmetry of a. We assume that initially there is only one photon present in the system, which means that the initial conditions for the amplitudes ψ_1 and ψ_2 are given by

$$\psi_1(0,\mathbf{x},0) = e^{i\mathbf{k}_0\cdot\mathbf{x}} ,$$
$$\psi_2(\mathbf{x}_1,\mathbf{x}_2,0) = 0 ,$$

where $\mathbf{k}_0$ is the wavevector of the photon. Note that the amplitude of ψ_1 is set to unity for convenience. To proceed, we take the Laplace transform with respect to t and the Fourier transforms with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$ of (4.17) and (4.18). We thus obtain

$$iz\psi_2(\mathbf{k}_1,\mathbf{k}_2,z) = \left[c|\mathbf{k}_1| + c|\mathbf{k}_2|\right]\psi_2(\mathbf{k}_1,\mathbf{k}_2) + \frac{g}{2}\left[\psi_1(\mathbf{k}_1,z) + \psi_1(\mathbf{k}_2,z)\right] , \tag{4.19}$$

$$i\left[z\psi_1(\mathbf{k},z) - \delta(\mathbf{k}-\mathbf{k}_0)\right] = 2g\int d^3k'\psi_2(\mathbf{k}',\mathbf{k},z) + \left[c|\mathbf{k}| + \Omega\right]\psi_1(\mathbf{k},z) . \tag{4.20}$$

Here we have employed the Fourier transform convention

$$\tilde{f}(\mathbf{k}) = \int e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d^3x . \tag{4.21}$$

and have defined the Laplace transform by

$$a(z) = \int_0^\infty dt e^{-zt} a(t) , \tag{4.22}$$

where we denote a function and its Laplace transform by the same symbol. Solving (4.19) and (4.20) leads to an integral equation for $\psi_1(\mathbf{k}, z)$ of the form

$$\begin{aligned}\psi_1(\mathbf{k}, z) = &\frac{\delta(\mathbf{k}-\mathbf{k}_0)}{z + i(c|\mathbf{k}| + \Omega) - i\Sigma(\mathbf{k}, z)} \\ &+ \frac{ig^2}{z + i(c|\mathbf{k}| + \Omega) - i\Sigma(\mathbf{k}, z)} \int d^3k' \frac{\psi_1(\mathbf{k}', z)}{(c|\mathbf{k}'| + c|\mathbf{k}|) - iz} ,\end{aligned} \tag{4.23}$$

where

$$\Sigma(\mathbf{k}, z) = g^2 \int d^3k' \frac{1}{(c|\mathbf{k}'| + c|\mathbf{k}|) - iz - i\epsilon} , \tag{4.24}$$

and $\epsilon \to 0^+$. In order to evaluate the integral in (4.23), we make the pole approximation where near resonance we replace $\Sigma(\mathbf{k}, z)$ with $\Sigma(\mathbf{k}, -i(\Omega + c|\mathbf{k}|))$. We note that this quantity is independent of $\mathbf{k}$ and z, and so we will denote it by Σ. For consistency, we also replace z by $-i(\Omega + c|\mathbf{k}|)$ under the integral in (4.23). This approximation arises in the Wigner-Weisskopf theory of spontaneous emission. In addition, we split Σ into its real and imaginary parts:

$$\mathrm{Re}\,\Sigma = \delta\omega , \tag{4.25}$$

$$\mathrm{Im}\,\Sigma = \Gamma/2 . \tag{4.26}$$

We can calculate Γ and $\delta\omega$ by making use of the identity

$$\frac{1}{c|\mathbf{k}| - \Omega - i\epsilon} = P\frac{1}{c|\mathbf{k}| - \Omega} + i\pi\delta(c|\mathbf{k}| - \Omega) . \tag{4.27}$$

We find that

$$\Gamma = 2g^2\pi \int \frac{d^3k}{(2\pi)^3}\, \delta(c|\mathbf{k}| - \Omega) \tag{4.28}$$

$$= \frac{g^2\Omega^2}{\pi c^3} , \tag{4.29}$$

and

4.4 Constant Density

In this section we consider the case of a homogeneous medium and set $\rho(\mathbf{x}) = \rho_0$, where ρ_0 is constant. It is useful to define the function $\tilde{\psi}_1(\mathbf{x}_1, \mathbf{x}_2, t) = \psi_1(\mathbf{x}_2, \mathbf{x}_1, t)$ and write (4.16) as a 4×4 symmetric system. If we further define the vector

$$\boldsymbol{\Psi}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{t}) = \begin{bmatrix} \sqrt{2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) \\ \sqrt{\rho_0/2}\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) \\ \sqrt{\rho_0/2}\tilde{\psi}_1(\mathbf{x}_1, \mathbf{x}_2, t) \\ \sqrt{2}\rho_0 a(\mathbf{x}_1, \mathbf{x}_2, t) \end{bmatrix}, \tag{4.30}$$

then (4.16) can be written as

$$i\partial_t \boldsymbol{\Psi} = \mathbf{A}\boldsymbol{\Psi} , \tag{4.31}$$

where

$$A = \begin{bmatrix} c(-\Delta_{\mathbf{x}_1})^{1/2} + c(-\Delta_{\mathbf{x}_2})^{1/2} & g\sqrt{\rho_0} & g\sqrt{\rho_0} & 0 \\ g\sqrt{\rho_0} & c(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega & 0 & -g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & 0 & c(-\Delta_{\mathbf{x}_1})^{1/2} + \Omega & g\sqrt{\rho_0} \\ 0 & -g\sqrt{\rho_0} & g\sqrt{\rho_0} & 2\Omega \end{bmatrix}. \tag{4.32}$$

This definition of $\boldsymbol{\Psi}$ has the advantage that the matrix A is symmetric. The Fourier transform of (4.31) with respect to the variables $\mathbf{x}_1$ and $\mathbf{x}_2$ is given by

$$i\partial_t \hat{\boldsymbol{\Psi}} = \hat{A}\hat{\boldsymbol{\Psi}} \tag{4.33}$$

where

$$\hat{A}(\mathbf{k}_1, \mathbf{k}_2) = \begin{bmatrix} c|\mathbf{k}_1| + c|\mathbf{k}_2| & g\sqrt{\rho_0} & g\sqrt{\rho_0} & 0 \\ g\sqrt{\rho_0} & c|\mathbf{k}_2| + \Omega & 0 & -g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & 0 & c|\mathbf{k}_1| + \Omega & g\sqrt{\rho_0} \\ 0 & -g\sqrt{\rho_0} & g\sqrt{\rho_0} & 2\Omega \end{bmatrix}. \tag{4.34}$$

The matrix $\hat{A}$ has eigenvalues

$$\lambda_1 = b_1 - \frac{\sqrt{b_2 - 2\sqrt{b_3}}}{2}\,,$$
$$\lambda_2 = b_1 + \frac{\sqrt{b_2 - 2\sqrt{b_3}}}{2}\,,$$
$$\lambda_3 = b_1 - \frac{\sqrt{b_2 + 2\sqrt{b_3}}}{2}\,,$$
$$\lambda_4 = b_1 + \frac{\sqrt{b_2 + 2\sqrt{b_3}}}{2}\,,$$

where

$$b_1 = \Omega + \frac{c\,|\mathbf{k}_1|}{2} + \frac{c\,|\mathbf{k}_2|}{2}\,, \tag{4.35}$$
$$b_2 = 2\,\Omega^2 + 8\,g^2\rho_0 + c^2\,|\mathbf{k}_1|^2 + c^2\,|\mathbf{k}_2|^2 - 2\,\Omega\,c\,|\mathbf{k}_1| - 2\,\Omega\,c\,|\mathbf{k}_2|\,, \tag{4.36}$$
$$\begin{aligned} b_3 = {} & \Omega^4 - 2\,\Omega^3\,c\,|\mathbf{k}_1| - 2\,\Omega^3\,c\,|\mathbf{k}_2| + \Omega^2\,c^2\,|\mathbf{k}_1|^2 + 4\,\Omega^2\,c^2\,|\mathbf{k}_1|\,|\mathbf{k}_2| + \Omega^2\,c^2\,|\mathbf{k}_2|^2 \\ & + 8\,\Omega^2\,g^2\rho_0 - 2\,\Omega\,c^3\,|\mathbf{k}_1|^2\,|\mathbf{k}_2| - 2\,\Omega\,c^3\,|\mathbf{k}_1|\,|\mathbf{k}_2|^2 - 8\,\Omega\,c\,g^2\rho_0\,|\mathbf{k}_1| \\ & - 8\,\Omega\,c\,g^2\rho_0\,|\mathbf{k}_2| + c^4\,|\mathbf{k}_1|^2\,|\mathbf{k}_2|^2 + 4\,c^2\,g^2\rho_0\,|\mathbf{k}_1|^2 + 4\,c^2\,g^2\rho_0\,|\mathbf{k}_2|^2. \end{aligned} \tag{4.37}$$

The components of the associated eigenvectors $\mathbf{v}_i(\mathbf{k}_1, \mathbf{k}_2)$ are given by

$$\begin{aligned}
v_{i1} &= -\frac{2\,\Omega^3 - 2\,\Omega\, g^2\rho_0 + 2\,\Omega^2\, c\,|\mathbf{k}_1| + 2\,\Omega^2\, c\,|\mathbf{k}_2|}{g^2\rho_0\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} \\
&+ \frac{-c\, g^2\rho_0\,|\mathbf{k}_1| - c\, g^2\rho_0\,|\mathbf{k}_2| + 2\,\Omega\, c^2\,|\mathbf{k}_1|\,|\mathbf{k}_2|}{g^2\rho_0\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} - \frac{(4\,\Omega + c\,|\mathbf{k}_1| + c\,|\mathbf{k}_2|)\,\lambda_i^2}{g^2\rho_0\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} \\
&+ \frac{\lambda_i\left(5\,\Omega^2 - 2\,g^2\rho_0 + 3\,\Omega\, c\,|\mathbf{k}_1| + 3\,\Omega\, c\,|\mathbf{k}_2| + c^2\,|\mathbf{k}_1|\,|\mathbf{k}_2|\right)}{g^2\rho_0\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} + \frac{\lambda_i^3}{g^2\rho_0\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)}\,, \\
v_{i2} &= \frac{\lambda_i^2}{g\sqrt{\rho_0}\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} + \frac{2\,\left(\Omega^2 + c\,|\mathbf{k}_1|\,\Omega - g^2\rho_0\right)}{g\sqrt{\rho_0}\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} - \frac{(3\,\Omega + c\,|\mathbf{k}_1|)\ (\lambda_i)}{g\sqrt{\rho_0}\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)}\,,
\end{aligned} \tag{4.38}$$

$$v_{i3} = \frac{\lambda_i^2}{g\sqrt{\rho_0}\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} + \frac{2\,\left(\Omega^2 + c\,|\mathbf{k}_2|\,\Omega - g^2\rho_0\right)}{g\sqrt{\rho_0}\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)} - \frac{(3\,\Omega + c\,|\mathbf{k}_2|)\ (\lambda_i)}{g\sqrt{\rho_0}\ (c\,|\mathbf{k}_1| - c\,|\mathbf{k}_2|)}\,, \tag{4.39}$$

$$v_{i4} = 1. \tag{4.40}$$

It follows that the solution to (4.33) is

$$\mathbf{\Psi}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{t}) = \sum_{\mathbf{i=1}}^{\mathbf{4}} \int \frac{\mathbf{d^3k_1}}{(\mathbf{2}\pi)^{\mathbf{3}}} \frac{\mathbf{d^3k_2}}{(\mathbf{2}\pi)^{\mathbf{3}}} \mathbf{e}^{\mathbf{ix_1\cdot k_1 + ix_2\cdot k_2}} \mathbf{C_i}(\mathbf{k_1}, \mathbf{k_2}) \mathbf{e}^{-\mathbf{ig}\sqrt{\rho_0}\lambda_\mathbf{i}(\mathbf{k_1},\mathbf{k_2})\mathbf{t}} \mathbf{v_i}(\mathbf{k_1}, \mathbf{k_2})\,, \tag{4.41}$$

where the values C_i are solutions to the linear system

$$\hat{\mathbf{\Psi}}(\mathbf{k}_1, \mathbf{k}_2, 0) = \sum_{i=1}^{4} \mathbf{v}_i(\mathbf{k}_1, \mathbf{k}_2) C_i(\mathbf{k}_1, \mathbf{k}_2). \tag{4.42}$$

In order to study the emission of photons, we assume that initially there are two localized volumes of excited atoms of linear size l_s centered at the points $\mathbf{r}_1$ and $\mathbf{r}_2$. The initial

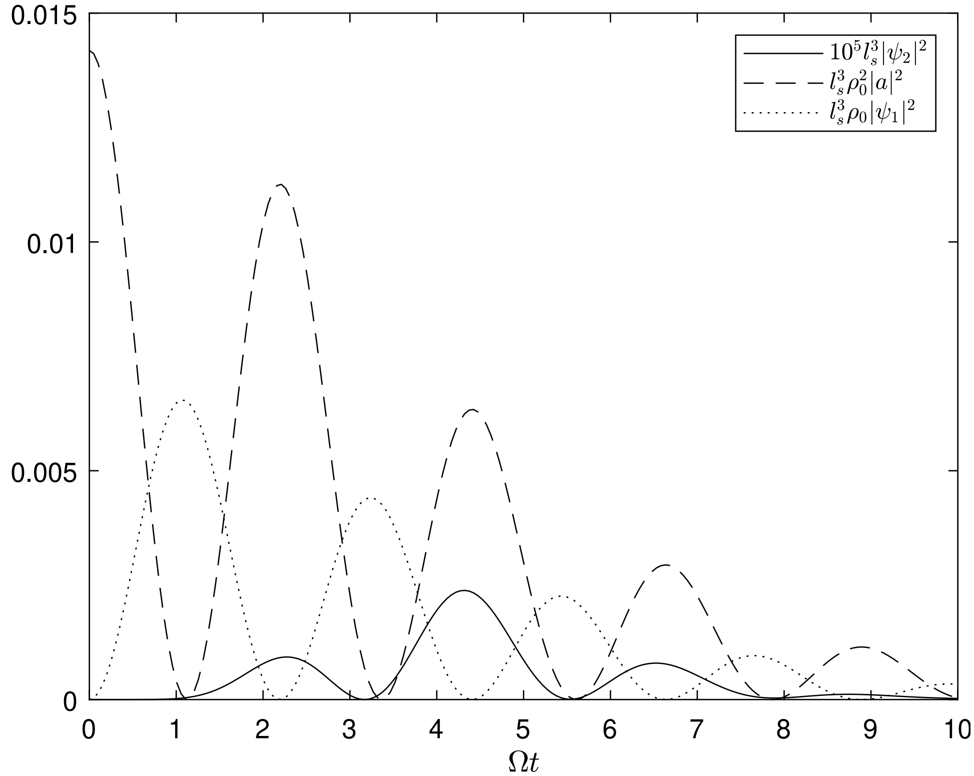


Figure 4.1: Probability densities with distances $|\mathbf{x}_1-\mathbf{r}_1| = |\mathbf{x}_2-\mathbf{r}_1| = |\mathbf{x}_1-\mathbf{r}_2| = |\mathbf{x}_2-\mathbf{r}_2| = l_s$

amplitudes are taken to be

$$\psi_2(\mathbf{x}_1, \mathbf{x}_2, 0) = 0\,, \tag{4.43}$$

$$\psi_1(\mathbf{x}_1, \mathbf{x}_2, 0) = 0\,, \tag{4.44}$$

$$a(\mathbf{x}_1, \mathbf{x}_2, 0) = \left(\frac{1}{\pi l_s^2}\right)^{3/2} \left(e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/2l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/2l_s^2} - e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/2l_s^2} e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/2l_s^2}\right). \tag{4.45}$$

Figure (4.1) illustrates the time dependence of ψ_1, ψ_2 and a, where we have set the dimensionless quantities $\Omega/\sqrt{\rho_0}g = c/l_s\sqrt{\rho_0}g = 1$. Note that the atomic amplitude is antisymmetric, consistent with (4.7) and that it corresponds to an entangled state. We observe that the amplitudes oscillate and decay in time. Moreover, the atomic and one-photon amplitudes are out of phase with one another, and the two-photon amplitude is an order of magnitude smaller.

4.5 Stimulated Emission in Random Media

4.5.1 Kinetic Equations

In this section we consider stimulated emission in a random medium. We suppose that there is at most one atomic excitation and that there are at most two photons present in the field. Thus we set the $a(\mathbf{x}_1, \mathbf{x}_2, t) = 0$ and study the dynamics of ψ_1 and ψ_2. The system (4.16) then becomes

$$\begin{aligned}
&i\partial_t\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) = c(-\Delta_{\mathbf{x}_1})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) \\
&+ \frac{g}{2}(\rho(\mathbf{x}_1)\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) + \rho(\mathbf{x}_2)\psi_1(\mathbf{x}_2, \mathbf{x}_1, t)) , \\
&i\partial_t\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) = 2g\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + \left[c(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega\right]\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) , \\
&0 = \psi_1(\mathbf{x}_2, \mathbf{x}_1, t) - \psi_1(\mathbf{x}_1, \mathbf{x}_2, t) ,
\end{aligned} \tag{4.46}$$

which can be rewritten as the pair of equations

$$\begin{aligned}
i\partial_t\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) &= c(-\Delta_{\mathbf{x}_1})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) \\
&\quad + \frac{g}{2}(\rho(\mathbf{x}_1) + \rho(\mathbf{x}_2))\psi_1(\mathbf{x}_1, \mathbf{x}_2, t)) , \\
i\partial_t\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) &= 2g\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + \left[\frac{c}{2}(-\Delta_{\mathbf{x}_1})^{1/2} + \frac{c}{2}(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega\right]\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) .
\end{aligned} \tag{4.47}$$

We will assume that the number density $\rho(\mathbf{x})$ is of the form

$$\rho(\mathbf{x}) = \rho_0(1 + \eta(\mathbf{x})) , \tag{4.48}$$

where ρ_0 is a constant and η is a real-valued random field. We assume that the correlations of η are given by

$$\langle\eta(\mathbf{x})\rangle = 0 , \tag{4.49}$$

$$\langle\eta(\mathbf{x}_1)\eta(\mathbf{x}_2)\rangle = C(\mathbf{x}_1 - \mathbf{x}_2). \tag{4.50}$$

where C is the two-point correlation function and $\langle\cdots\rangle$ denotes statistical averaging. We further assume that the medium is statistically homogeneous and isotropic, so that C depends only upon the quantity $|\mathbf{x}-\mathbf{y}|$. If we define

$$\mathbf{\Psi}(\mathbf{x_1},\mathbf{x_2},\mathbf{t}) = \begin{bmatrix} \sqrt{2}\psi_2(\mathbf{x}_1,\mathbf{x}_2,t) \\ \sqrt{\rho_0}\psi_1(\mathbf{x}_1,\mathbf{x}_2,t) \end{bmatrix} \tag{4.51}$$

then the above system of equation can be rewritten as

$$i\partial_t\mathbf{\Psi}(\mathbf{x_1},\mathbf{x_2},\mathbf{t}) = \mathbf{A}(\mathbf{x_1},\mathbf{x_2})\mathbf{\Psi}(\mathbf{x_1},\mathbf{x_2},\mathbf{t}) + \mathbf{g}\sqrt{\frac{\boldsymbol{\rho_0}}{\mathbf{2}}}(\eta(\mathbf{x_1})+\eta(\mathbf{x_2}))\mathbf{K}\mathbf{\Psi}(\mathbf{x_1},\mathbf{x_2},\mathbf{t})\,, \tag{4.52}$$

where

$$A(\mathbf{x}_1,\mathbf{x}_2) = \begin{bmatrix} c(-\Delta_{\mathbf{x}_1})^{1/2} + c(-\Delta_{\mathbf{x}_2})^{1/2} & \sqrt{2}g\sqrt{\rho_0} \\ \sqrt{2}g\sqrt{\rho_0} & \frac{c}{2}(-\Delta_{\mathbf{x}_1})^{1/2} + \frac{c}{2}(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega \end{bmatrix}, \tag{4.53}$$

$$K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{4.54}$$

To derive a kinetic equation in the high-frequency limit, we rescale the variables $t \to t/\epsilon$, $\mathbf{x}_1 \to \mathbf{x}_1/\epsilon$, and $\mathbf{x}_2 \to \mathbf{x}_2/\epsilon$. Additionally, we assume that the randomness is sufficiently weak so that the correlation function C is $O(\epsilon)$. Eq. (4.52) thus becomes

$$\begin{aligned} i\epsilon\partial_t\mathbf{\Psi}_\epsilon(\mathbf{x_1},\mathbf{x_2},\mathbf{t}) &= A_\epsilon(\mathbf{x}_1,\mathbf{x}_2)\mathbf{\Psi}_\epsilon(\mathbf{x_1},\mathbf{x_2},\mathbf{t}) \\ &\quad + \sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}(\eta(\mathbf{x}_1/\epsilon)+\eta(\mathbf{x}_2/\epsilon))K\mathbf{\Psi}_\epsilon(\mathbf{x_1},\mathbf{x_2},\mathbf{t})\,, \end{aligned} \tag{4.55}$$

where

$$A_\epsilon(\mathbf{x}_1,\mathbf{x}_2) = \begin{bmatrix} \epsilon c(-\Delta_{\mathbf{x}_1})^{1/2} + \epsilon c(-\Delta_{\mathbf{x}_2})^{1/2} & \sqrt{2}g\sqrt{\rho_0} \\ \sqrt{2}g\sqrt{\rho_0} & \epsilon\frac{c}{2}(-\Delta_{\mathbf{x}_1})^{1/2} + \epsilon\frac{c}{2}(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega \end{bmatrix}. \tag{4.56}$$

Next we introduce the scaled Wigner transform, which provides a phase space representation of the correlation functions of the various amplitudes. The Wigner transform

$W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t)$ is defined by

$$W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) = \int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3} e^{-i\mathbf{k}_1\cdot\mathbf{x}_1' - i\mathbf{k}_2\cdot\mathbf{x}_2'} \boldsymbol{\Psi}_\epsilon(\mathbf{x_1} - \epsilon\mathbf{x_1'/2}, \mathbf{x_2} - \epsilon\mathbf{x_2'/2}, \mathbf{t}) \times \boldsymbol{\Psi}^\dagger(\mathbf{x_1} + \epsilon\mathbf{x_1'/2}, \mathbf{x_2} + \epsilon\mathbf{x_2'/2}, \mathbf{t}) \,, \tag{4.57}$$

where † denotes the hermitian conjugate. The Wigner transform is real-valued and its diagonal elements are related to the probability densities $|\psi_{1\epsilon}|^2$ and $|\psi_{2\epsilon}|^2$ by

$$2|\psi_{2\epsilon}(\mathbf{x}_1, \mathbf{x}_2, t)|^2 = \int d^3k_1 d^3k_2 (W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t))_{11} \,, \tag{4.58}$$

$$\rho_0|\psi_{1\epsilon}(\mathbf{x}_1, \mathbf{x}_2, t)|^2 = \int d^3k_1 d^3k_2 (W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t))_{22}. \tag{4.59}$$

The above factor of two is due to the symmetry of the function $\psi_2(\mathbf{x}_1, \mathbf{x}_2)$. The off diagonal elements are related to correlations between the amplitudes:

$$\sqrt{2\rho_0}\psi_{2\epsilon}(\mathbf{x}_1, \mathbf{x}_2, t)\psi_{1\epsilon}^*(\mathbf{x}_1, \mathbf{x}_2, t) = \int d^3k_1 d^3k_2 (W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t))_{12}. \tag{4.60}$$

As shown in Appendix B, the Wigner transform satisfies the Liouville equation

$$\begin{aligned} &i\epsilon\partial_t W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) \\ &= \int \frac{d^3k_1'}{(2\pi)^3}\frac{d^3k_2'}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1' + i\mathbf{x}_2\cdot\mathbf{k}_2'} \hat{A}(\mathbf{k}_1 - \epsilon\mathbf{k}_1'/2, \mathbf{k}_2 - \epsilon\mathbf{k}_2'/2)\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{k}_1, \mathbf{k}_2', \mathbf{k}_2, t) \\ &- \int \frac{d^3k_1'}{(2\pi)^3}\frac{d^3k_2'}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1' + i\mathbf{x}_2\cdot\mathbf{k}_2'} \hat{W}_\epsilon(\mathbf{k}_1', \mathbf{k}_1, \mathbf{k}_2', \mathbf{k}_2, t)\hat{A}(\mathbf{k}_1 + \epsilon\mathbf{k}_1'/2, \mathbf{k}_2 + \epsilon\mathbf{k}_2'/2) \\ &+ \sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}L_1^{(0)}W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) + \sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}L_2^{(0)}W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) \,, \end{aligned} \tag{4.61}$$

where the operators $L_1^{(0)}$ and $L_2^{(0)}$ are given by

$$L_1^{(0)} W_\epsilon = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}_1/\epsilon}\, \hat{\eta}(\mathbf{q})\, [K W_\epsilon(\mathbf{x}_1, \mathbf{k}_1 + \mathbf{q}/2, \mathbf{x}_2, \mathbf{k}_2, t) \tag{4.62}$$

$$- W_\epsilon(\mathbf{x}_1, \mathbf{k}_1 - \mathbf{q}/2, \mathbf{x}_2, \mathbf{k}_2, t) K^T] \ , \tag{4.63}$$

$$L_2^{(0)} W_\epsilon = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}_2/\epsilon}\, \hat{\eta}(\mathbf{q})\, [K W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2 + \mathbf{q}/2, t) \tag{4.64}$$

$$- W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2 - \mathbf{q}/2, t) K^T] \ , \tag{4.65}$$

while $\hat{A}$ is defined

$$\hat{A}(\mathbf{k}_1, \mathbf{k}_2) = \begin{bmatrix} c|\mathbf{k}_1| + c|\mathbf{k}_2| & \sqrt{2\rho_0} g \\ \sqrt{2\rho_0} g & (c|\mathbf{k}_1| + c|\mathbf{k}_2|)/2 + \Omega \end{bmatrix} \ , \tag{4.66}$$

and the Fourier transform $\hat{W}$ is defined as

$$\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{k}_1, \mathbf{k}_2', \mathbf{k}_2, t) = \int d^3x_1 d^3x_2 e^{-i\mathbf{x}_1\cdot\mathbf{k}_1' - i\mathbf{x}_2\cdot\mathbf{k}_2'} W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t). \tag{4.67}$$

We study the behavior of W_ϵ in the high frequency limit $\epsilon \longrightarrow 0$. To this end, we introduce a multiscale expasion of the Wigner transform of the form

$$\begin{aligned} W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) &= W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) + \sqrt{\epsilon} W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\ &+ \epsilon W_2(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) + \cdots \ , \end{aligned} \tag{4.68}$$

where $\mathbf{X}_1 = \mathbf{x}_1/\epsilon$ and $\mathbf{X}_2 = \mathbf{x}_2/\epsilon$ are fast variables and W_0 is assumed to be both deterministic and independent of the fast variables. We treat the variables $\mathbf{x}_1, \mathbf{X}_1$ and $\mathbf{x}_2, \mathbf{X}_2$ as independent and make the replacements

$$\nabla_{\mathbf{x}_i} \longrightarrow \nabla_{\mathbf{x}_i} + \frac{1}{\epsilon} \nabla_{\mathbf{X}_i} \ , \quad i = 1, 2 \ . \tag{4.69}$$

Hence the Liouville equation becomes

$$
\begin{aligned}
& i\epsilon\partial_t W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\
& = \int \frac{d^3k_1'}{(2\pi)^3}\frac{d^3k_2'}{(2\pi)^3}\frac{d^3K_1}{(2\pi)^3}\frac{d^3K_2}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1' + i\mathbf{x}_2\cdot\mathbf{k}_2' + i\mathbf{X}_1\cdot\mathbf{K}_1 + i\mathbf{X}_2\cdot\mathbf{K}_2} \\
& \times \hat{A}(\mathbf{k}_1 - \epsilon\mathbf{k}_1'/2 - \mathbf{K}_1/2, \mathbf{k}_2 - \epsilon\mathbf{k}_2'/2 - \mathbf{K}_2/2)\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2, t) \\
& - \int \frac{d^3k_1'}{(2\pi)^3}\frac{d^3k_2'}{(2\pi)^3}\frac{d^3K_1}{(2\pi)^3}\frac{d^3K_2}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1' + i\mathbf{x}_2\cdot\mathbf{k}_2' + i\mathbf{X}_1\cdot\mathbf{K}_1 + i\mathbf{X}_2\cdot\mathbf{K}_2} \\
& \times \hat{W}_\epsilon(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2, t)\hat{A}(\mathbf{k}_1 - \epsilon\mathbf{k}_1'/2 + \mathbf{K}_1/2, \mathbf{k}_2 - \epsilon\mathbf{k}_2'/2 + \mathbf{K}_2/2) \\
& + \sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}L_1 W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) + \sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}L_2 W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t)\,,
\end{aligned}
\tag{4.70}
$$

where

$$
\begin{aligned}
L_1 W = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_1}\,\hat{\eta}(\mathbf{q})\,\big[& K W(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1 + \mathbf{q}/2, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\
& - W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1 - \mathbf{q}/2, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) K^T\big]\,, \qquad (4.71) \\
L_2 W = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_2}\,\hat{\eta}(\mathbf{q})\,\big[& K W(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2 + \mathbf{q}/2, t) \\
& - W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2 - \mathbf{q}/2, t) K^T\big]\,, \qquad (4.72)
\end{aligned}
$$

and the Fourier transform $\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2, t)$ is defined

$$
\begin{aligned}
& \hat{W}_\epsilon(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2, t) \\
& = \int d^3x_1 d^3x_2 d^3X_1 d^3X_2 e^{-i\mathbf{x}_1\cdot\mathbf{k}_1' - i\mathbf{X}_1\cdot\mathbf{K}_1 - i\mathbf{x}_2\cdot\mathbf{k}_2' - i\mathbf{X}_2\cdot\mathbf{K}_2} W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t)\,.
\end{aligned}
\tag{4.73}
$$

Substituting (4.68) into (4.70) and equating terms of the same order in $\sqrt{\epsilon}$ leads to a hierarchy of equations. At $O(1)$ we have

$$
\hat{A}(\mathbf{k}_1, \mathbf{k}_2)W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) - W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_1, \mathbf{k}_2, t)\hat{A}(\mathbf{k}_1, \mathbf{k}_2) = 0. \tag{4.74}
$$

Since $\hat{A}$ is symmetric it can be diagonalized by a unitary transformation. The eigenvalues and eigenvectors of $\hat{A}$ are given by

$$\lambda_{\pm}(\mathbf{k}_1, \mathbf{k}_2) = \frac{3d(\mathbf{k}_1, \mathbf{k}_2) + \Omega \pm \sqrt{(d(\mathbf{k}_1, \mathbf{k}_2) - \Omega)^2 + 8g^2\rho_0}}{2}, \tag{4.75}$$

$$\mathbf{v}_{\pm}(\mathbf{k}_1, \mathbf{k}_2) = \frac{1}{\sqrt{(\lambda_{\pm}(\mathbf{k}_1, \mathbf{k}_2) - d(\mathbf{k}_1, \mathbf{k}_2) - \Omega)^2 + 2g^2\rho_0}} \begin{bmatrix} \lambda_{\pm}(\mathbf{k}_1, \mathbf{k}_2) - d(\mathbf{k}_1, \mathbf{k}_2) - \Omega \\ g\sqrt{2\rho_0} \end{bmatrix}, \tag{4.76}$$

where

$$d(\mathbf{k}_1, \mathbf{k}_2) = \frac{c(|\mathbf{k}_1| + |\mathbf{k}_2|)}{2}. \tag{4.77}$$

Evidently W_0 is also diagonal in this basis and can be expanded as

$$\begin{aligned} W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) &= a_+(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t)\mathbf{v}_+(\mathbf{k}_1, \mathbf{k}_2)\mathbf{v}_+^T(\mathbf{k}_1, \mathbf{k}_2) \\ &\quad + a_-(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t)\mathbf{v}_-(\mathbf{k}_1, \mathbf{k}_2)\mathbf{v}_-^T(\mathbf{k}_1, \mathbf{k}_2). \end{aligned} \tag{4.78}$$

At order $O(\sqrt{\epsilon})$ we have

$$\begin{aligned} &\hat{A}(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)\hat{W}_1(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2) \\ &- \hat{W}_1(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2)\hat{A}(\mathbf{k}_1 + \mathbf{K}_2/2, \mathbf{k}_2 + \mathbf{K}_2/2) \\ &= g\sqrt{\frac{\rho_0}{2}}(2\pi)^3\left(\hat{\eta}(\mathbf{K}_1)\delta(\mathbf{K}_2) + \hat{\eta}(\mathbf{K}_2)\delta(\mathbf{K}_1)\right) \\ &\times \left[W_0(\mathbf{x}_1, \mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{x}_2, \mathbf{k}_2 - \mathbf{K}_2/2)K^T - KW_0(\mathbf{x}_1, \mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{x}_2, \mathbf{k}_2 + \mathbf{K}_2/2)\right]. \end{aligned} \tag{4.79}$$

We can then decompose $\hat{W}_1$ as

$$\begin{aligned}
&\hat{W}_1(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2) \\
&= \sum_{i,j} w_{i,j}(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2)\mathbf{v}_i(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)\mathbf{v}_j^T(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2).
\end{aligned} \tag{4.80}$$

Multiplying (4.79) on the left by $\mathbf{v}_m^T(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)$ and the right by $\mathbf{v}_n(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)$, we arrive at

$$\begin{aligned}
&(\lambda_m(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2) - \lambda_n(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2) + i\theta) \\
&\times w_{m,n}(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2) \\
=&g\sqrt{\frac{\rho_0}{2}}(2\pi)^3\left\{\eta(\mathbf{K}_1)\delta(\mathbf{K}_2) + \eta(\mathbf{K}_2)\delta(\mathbf{K}_1)\right\} \\
&\times\left[a_m(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)K_{m,n}(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2, \mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)\right] \\
-&g\sqrt{\frac{\rho_0}{2}}(2\pi)^3\left\{\eta(\mathbf{K}_1)\delta(\mathbf{K}_2) + \eta(\mathbf{K}_2)\delta(\mathbf{K}_1)\right\} \\
&\times\left[a_n(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)K_{m,n}(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2, \mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)\right],
\end{aligned} \tag{4.81}$$

where $\theta \to 0$ is a small positive regularizing parameter and

$$\begin{aligned}
&K_{m,n}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}_1, \mathbf{q}_2) \\
&= \mathbf{v}_m^T(\mathbf{k}_1, \mathbf{k}_2)K\mathbf{v}_n(\mathbf{q}_1, \mathbf{q}_2) \\
&= \frac{g\sqrt{2\rho_0}(\lambda_m(\mathbf{k}_1, \mathbf{k}_2) - d(\mathbf{k}_1, \mathbf{k}_2) - \Omega)}{\sqrt{(\lambda_m(\mathbf{k}_1, \mathbf{k}_2) - d(\mathbf{k}_1, \mathbf{k}_2) - \Omega)^2 + 2g^2\rho_0}\sqrt{(\lambda_n(\mathbf{q}_1, \mathbf{q}_2) - d(\mathbf{q}_1, \mathbf{q}_2) - \Omega)^2 + 2g^2\rho_0}}.
\end{aligned} \tag{4.82}$$

At order $O(\epsilon)$ we find that

$$
\begin{aligned}
& i\partial_t W_0(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t) = LW_2(\mathbf{x}_1,\mathbf{X}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{X}_2,\mathbf{k}_2,t) \\
& - M(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2)W_0(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t) - W_0(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t)M(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) \\
& + L_1W_1 + L_2W_1,
\end{aligned} \tag{4.83}
$$

where

$$
\begin{aligned}
& LW_2(\mathbf{x}_1,\mathbf{X}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{X}_2,\mathbf{k}_2,t) \\
& = \int \frac{d^3K_1}{(2\pi)^3}\frac{d^3K_2}{(2\pi)^3} e^{i\mathbf{X}_1\cdot\mathbf{K}_1 + i\mathbf{X}_2\cdot\mathbf{K}_2}\left[\hat{A}(\mathbf{k}_1-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)\hat{W}_2(\mathbf{x}_1,\mathbf{K}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{K}_2,\mathbf{k}_2,t)\right. \\
& \left. -\hat{W}_2(\mathbf{x}_1,\mathbf{K}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{K}_2,\mathbf{k}_2,t)\hat{A}(\mathbf{k}_1+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2)\right], \\
& M(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) = \frac{i}{2}\begin{bmatrix} c\hat{\mathbf{k}}_1\cdot\nabla_{\mathbf{x}_1} + c\hat{\mathbf{k}}_2\cdot\nabla_{\mathbf{x}_2} & 0 \\ 0 & \frac{c}{2}\hat{\mathbf{k}}_1\cdot\nabla_{\mathbf{x}_1} + \frac{c}{2}\hat{\mathbf{k}}_2\cdot\nabla_{\mathbf{x}_2} \end{bmatrix}.
\end{aligned} \tag{4.84}
$$

In order to obtain an equation satisfied by $a_\pm$, we multiply this equation on the left by $\mathbf{v}_+^T(\mathbf{k}_1,\mathbf{k}_2)$ ($\mathbf{v}_-^T(\mathbf{k}_1,\mathbf{k}_2)$) and on the right by $\mathbf{v}_+(\mathbf{k}_1,\mathbf{k}_2)$ ($\mathbf{v}_-(\mathbf{k}_1,\mathbf{k}_2)$) and take the average. Additionally, we assume that $\langle\mathbf{v}_\pm^T LW_2\mathbf{v}_\pm\rangle$ is identically zero, which corresponds to W_2 being statistically stationary with respect to the fast variables $\mathbf{X}_1$ and $\mathbf{X}_2$. This relation closes the hierarchy of equations and leads to the kinetic equation

$$
\begin{aligned}
& \frac{1}{c}\partial_t a_\pm + f_\pm(\mathbf{k}_1,\mathbf{k}_2)\left(\hat{\mathbf{k}}_1\cdot\nabla_{\mathbf{x}_1} + \hat{\mathbf{k}}_2\cdot\nabla_{\mathbf{x}_2}\right)a_\pm + \mu_{1\pm}a_\pm + \mu_{2\pm}a_\pm \\
& = \mu_{1\pm}\int d\hat{\mathbf{k}}'\,A(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}',|\mathbf{k}_1|)a_\pm(|\mathbf{k}_1|\hat{\mathbf{k}}',\mathbf{k}_2) + \mu_{2\pm}\int d\hat{\mathbf{k}}'\,A(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}',|\mathbf{k}_2|)a_\pm(\mathbf{k}_1,|\mathbf{k}_2|\hat{\mathbf{k}}'),
\end{aligned} \tag{4.85}
$$

where the scattering coefficients $\mu_{i\pm}$, the scattering kernel A, and the functions $f_\pm$ are defined as

$$\mu_{1\pm}(\mathbf{k}_1,\mathbf{k}_2) = \frac{g^2\rho_0\pi}{2}\frac{K_{\pm,\pm}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)^2}{\frac{c}{4}\left|d(\mathbf{k}_1,\mathbf{k}_2)-\Omega+\frac{3}{2}\sqrt{(d(\mathbf{k}_1,\mathbf{k}_2)-\Omega)^2+8g^2\rho_0}\right|}$$
$$\times|\mathbf{k}_1|^2\int\frac{d\hat{\mathbf{k}}'}{(2\pi)^3}\,\hat{C}(|\mathbf{k}_1|(\hat{\mathbf{k}}_1-\hat{\mathbf{k}}')), \tag{4.86}$$

$$\mu_{2\pm}(\mathbf{k}_1,\mathbf{k}_2) = \frac{g^2\rho_0\pi}{2}\frac{K_{\pm,\pm}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)^2}{\frac{c}{4}\left|d(\mathbf{k}_1,\mathbf{k}_2)-\Omega+\frac{3}{2}\sqrt{(d(\mathbf{k}_1,\mathbf{k}_2)-\Omega)^2+8g^2\rho_0}\right|}$$
$$\times|\mathbf{k}_2|^2\int\frac{d\hat{\mathbf{k}}'}{(2\pi)^3}\,\hat{C}(|\mathbf{k}_2|(\hat{\mathbf{k}}_2-\hat{\mathbf{k}}')), \tag{4.87}$$

$$A(\hat{\mathbf{k}},\hat{\mathbf{k}}',k) = \frac{\hat{C}(k(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))}{\displaystyle\int d\hat{\mathbf{k}}'\hat{C}(k(\hat{\mathbf{k}}-\hat{\mathbf{k}}'))}, \tag{4.88}$$

$$f_\pm(\mathbf{k}_1,\mathbf{k}_2) = \frac{(\lambda_\pm(\mathbf{k}_1,\mathbf{k}_2)-d(\mathbf{k}_1,\mathbf{k}_2)-\Omega)^2+g^2\rho_0}{(\lambda_\pm(\mathbf{k}_1,\mathbf{k}_2)-d(\mathbf{k}_1,\mathbf{k}_2)-\Omega)^2+2g^2\rho_0}\,. \tag{4.89}$$

Some details of the derivation of (4.85) are included in Appendix C.

4.5.2 Diffusion Approximation

We now consider the diffusion limit of the kinetic equation (4.85). The diffusion approximation (DA) to a kinetic equation of the form

$$\frac{1}{c}\partial_t I + f_1(|\mathbf{k}_1|,|\mathbf{k}_2|)\hat{\mathbf{k}}_1\cdot\nabla_{\mathbf{x}_1}I + f_2(|\mathbf{k}_1|,|\mathbf{k}_2|)\hat{\mathbf{k}}_2\cdot\nabla_{\mathbf{x}_2}I + \mu_1(|\mathbf{k}_1|,|\mathbf{k}_2|)I + \mu_2(|\mathbf{k}_1|,|\mathbf{k}_2|)I$$
$$= \mu_1(|\mathbf{k}_1|,|\mathbf{k}_2|)\int d\hat{\mathbf{k}}'\,A_1(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}')I(|\mathbf{k}_1|\hat{\mathbf{k}}',\mathbf{k}_2)$$
$$+\mu_2(|\mathbf{k}_1|,|\mathbf{k}_2|)\int d\hat{\mathbf{k}}'\,A_2(\hat{\mathbf{k}}_2,\hat{\mathbf{k}}')I(\mathbf{k}_1,|\mathbf{k}_2|\hat{\mathbf{k}}'), \tag{4.90}$$

is obtained by expanding I into spherical harmonics as

$$I = \frac{1}{16\pi^2}u + \frac{3}{16\pi^2}\mathbf{J}_1\cdot\hat{\mathbf{k}}_1 + \frac{3}{16\pi^2}\mathbf{J}_2\cdot\hat{\mathbf{k}}_2 + \cdots, \tag{4.91}$$

where

$$u(\mathbf{x}_1, \mathbf{x}_2, t) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \, I(\mathbf{x}_1, \hat{\mathbf{k}}_1, \mathbf{x}_2, \hat{\mathbf{k}}_2, t)\,, \tag{4.92}$$

$$\mathbf{J}_1(\mathbf{x}_1, \mathbf{x}_2, t) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \, \hat{\mathbf{k}}_1 I(\mathbf{x}_1, \hat{\mathbf{k}}_1, \mathbf{x}_2, \hat{\mathbf{k}}_2, t)\ , \tag{4.93}$$

$$\mathbf{J}_2(\mathbf{x}_1, \mathbf{x}_2, t) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \, \hat{\mathbf{k}}_2 I(\mathbf{x}_1, \hat{\mathbf{k}}_1, \mathbf{x}_2, \hat{\mathbf{k}}_2, t)\ . \tag{4.94}$$

Integrating (4.90) with respect to $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ we arrive at

$$\frac{1}{c}\partial_t u + f_1 \nabla_{\mathbf{x}_1} \cdot \mathbf{J}_1 + f_2 \nabla_{\mathbf{x}_2} \cdot \mathbf{J}_2 = 0. \tag{4.95}$$

If instead we multiply by $\hat{\mathbf{k}}_1$ and integrate we obtain

$$\frac{1}{c}\partial_t \mathbf{J}_1 + f_1 \nabla_{\mathbf{x}_1} \cdot \sigma_1 + f_2 \nabla_{\mathbf{x}_2} \cdot \sigma_3 + \mu_1 (1 - g_1)\mathbf{J}_1 = 0\,, \tag{4.96}$$

where

$$g_1 = \int d\hat{\mathbf{k}}_1 \, \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_1' A_1(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_1')\ , \tag{4.97}$$

and

$$\sigma_1(\mathbf{x}_1, \mathbf{x}_2, t) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \, \hat{\mathbf{k}}_1 \otimes \hat{\mathbf{k}}_1 I(\mathbf{x}_1, \hat{\mathbf{k}}_1, \mathbf{x}_2, \hat{\mathbf{k}}_2, t)\,, \tag{4.98}$$

$$\sigma_3(\mathbf{x}_1, \mathbf{x}_2, t) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \, \hat{\mathbf{k}}_1 \otimes \hat{\mathbf{k}}_2 I(\mathbf{x}_1, \hat{\mathbf{k}}_1, \mathbf{x}_2, \hat{\mathbf{k}}_2, t)\ . \tag{4.99}$$

Similarly, multiplying (4.90) by $\hat{\mathbf{k}}_2$ and carrying out the indicated integrals leads to

$$\frac{1}{c}\partial_t \mathbf{J}_2 + f_2 \nabla_{\mathbf{x}_2} \cdot \sigma_2 + f_2 \nabla_{\mathbf{x}_2} \cdot \sigma_3 + \mu_2 (1 - g_2)\mathbf{J}_2 = 0\,, \tag{4.100}$$

where

$$g_2 = \int d\hat{\mathbf{k}}_2\, \hat{\mathbf{k}}_2 \cdot \hat{\mathbf{k}}_2' A_2(\hat{\mathbf{k}}_2, \hat{\mathbf{k}}_2') \tag{4.101}$$

and

$$\sigma_2(\mathbf{x}_1, \mathbf{x}_2, t) = \int d\hat{\mathbf{k}}_2 d\hat{\mathbf{k}}_2\, \hat{\mathbf{k}}_1 \otimes \hat{\mathbf{k}}_2 I(\mathbf{x}_1, \hat{\mathbf{k}}_1, \mathbf{x}_2, \hat{\mathbf{k}}_2, t)\ . \tag{4.102}$$

Next we substitute (4.91) into (4.98), (4.99), and (4.102) and carry out the indicated integrations. We find that

$$\nabla_{\mathbf{x}_1} \cdot \sigma_1 = \frac{1}{3}\nabla_{\mathbf{x}_1} u\ , \tag{4.103}$$

$$\nabla_{\mathbf{x}_2} \cdot \sigma_2 = \frac{1}{3}\nabla_{\mathbf{x}_2} u\ , \tag{4.104}$$

$$\nabla_{\mathbf{x}_1} \cdot \sigma_3 = \nabla_{\mathbf{x}_2} \cdot \sigma_3 = 0\ . \tag{4.105}$$

Using the above results, (4.96) and (4.100) become

$$\frac{1}{c}\partial_t \mathbf{J}_1 + \mu_1(1 - g_1)\mathbf{J}_1 = -\frac{f_1}{3}\nabla_{\mathbf{x}_1} u\ , \tag{4.106}$$

$$\frac{1}{c}\partial_t \mathbf{J}_2 + \mu_2(1 - g_2)\mathbf{J}_2 = -\frac{f_2}{3}\nabla_{\mathbf{x}_2} u\ . \tag{4.107}$$

At long times ($t \gg 1/[c(1 - g_{1,2})\mu_{1,2}]$), the first terms on the right-hand sides of (4.106) and (4.107) can be neglected. Substituting the resulting expressions for $\mathbf{J}_1$ and $\mathbf{J}_2$ into (4.95), we obtain the diffusion equation obeyed by u:

$$\partial_t u - D_1 \Delta_{\mathbf{x}_1} u - D_2 \Delta_{\mathbf{x}_2} u = 0\ . \tag{4.108}$$

Here the diffusion coefficients are defined by

$$D_i = \frac{c f_i^2}{3\mu_i(1 - g_i)}\ , \quad i = 1, 2\ . \tag{4.109}$$

In the case of white noise disorder, where the correlation function $C(\mathbf{x}) = C_0\delta(\mathbf{x})$, with constant C_0, the phase functions $A_{1,2} = 1/4\pi$, which corresponds to isotropic scattering.

The solution to (4.108) for an infinite medium is given by

$$u(\mathbf{x}_1, \mathbf{x}_2, t) = \frac{1}{(4\pi D_1 t)^{3/2}(4\pi D_2 t)^{3/2}} \int d^3x_1' d^3x_2' \exp\left[-\frac{|\mathbf{x}_1 - \mathbf{x}_1'|^2}{4D_1 t} - \frac{|\mathbf{x}_2 - \mathbf{x}_2'|^2}{4D_2 t}\right] u(\mathbf{x}_1', \mathbf{x}_2', 0) \; . \tag{4.110}$$

Making use of the above results, we see that each of the modes $a_\pm$ satisfy (4.85) and thus their first angular moments, which are defined by

$$u_\pm(\mathbf{x}_1, k_1, \mathbf{x}_2, k_2, t) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 a_\pm(\mathbf{x}_1, k_1\hat{\mathbf{k}}_1, \mathbf{x}_2, k_2\hat{\mathbf{k}}_2, t) \, , \tag{4.111}$$

satisfy the equations

$$\partial_t u_\pm - D_{1,\pm}\Delta_{\mathbf{x}_1} u_\pm - D_{2,\pm}\Delta_{\mathbf{x}_2} u_\pm = 0 \, , \tag{4.112}$$

where

$$D_{i,\pm} = \frac{c f_\pm^2}{3\mu_{i\pm}} \; . \tag{4.113}$$

We assume that initially there are two photons present in the field localized around the points $\mathbf{r}_1$ and $\mathbf{r}_2$ in a volume of linear size l_s. The corresponding initial conditions are given by

$$\psi_2(\mathbf{x}_1, \mathbf{x}_2, 0) = \frac{1}{l_s^3}\left[e^{-|\mathbf{x}_1 - \mathbf{r}_1|^2/2l_s^2} e^{-|\mathbf{x}_2 - \mathbf{r}_2|^2/2l_s^2} + e^{-|\mathbf{x}_1 - \mathbf{r}_2|^2/2l_s^2} e^{-|\mathbf{x}_2 - \mathbf{r}_1|^2/2l_s^2}\right] \, , \tag{4.114}$$

$$\psi_1(\mathbf{x}_1, \mathbf{x}_2, 0) = 0 \; . \tag{4.115}$$

Note that ψ_2 corresponds to an entangled two-photon state. These initial conditions imply initial conditions for the Wigner transform W_0 from (4.57), which in turn imply initial conditions for the modes $a_\pm$ from (4.78):

$$\begin{aligned}
&a_-(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\
&= \frac{1}{\pi^3}\gamma_-(|\mathbf{k}_1|, |\mathbf{k}_2|) \Big[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/l_s^2} + e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/l_s^2} \\
&+ e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-i\mathbf{k}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)-i\mathbf{k}_2\cdot(\mathbf{r}_2-\mathbf{r}_1)} \\
&+ e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-i\mathbf{k}_1\cdot(\mathbf{r}_2-\mathbf{r}_1)-i\mathbf{k}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)} \Big] ,
\end{aligned} \tag{4.116}$$

$$\begin{aligned}
&a_+(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\
&= \frac{1}{\pi^3}\gamma_+(|\mathbf{k}_1|, |\mathbf{k}_2|) \Big[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/l_s^2} + e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/l_s^2} \\
&+ e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-i\mathbf{k}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)-i\mathbf{k}_2\cdot(\mathbf{r}_2-\mathbf{r}_1)} \\
&+ e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-i\mathbf{k}_1\cdot(\mathbf{r}_2-\mathbf{r}_1)-i\mathbf{k}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)} \Big] ,
\end{aligned} \tag{4.117}$$

where

$$\gamma_\pm(k_1, k_2) = \frac{(\lambda_\pm(k_1, k_2) - d(k_1, k_2) - \Omega)^2 + 2g^2\rho_0}{(\lambda_\pm(k_1, k_2) - d(k_1, k_2) - \Omega)^2 - (\lambda_\mp(k_1, k_2) - d(k_1, k_2) - \Omega)^2} e^{-l_s^2 k_1^2 - l_s^2 k_2^2} . \tag{4.118}$$

The initial conditions for the first angular moments $u_\pm$ are then found using (4.111):

$$\begin{aligned}
&u_-(\mathbf{x}_1, k_1, \mathbf{x}_2, k_2, 0) \\
&= \frac{16\gamma_-(k_1, k_2)}{\pi} \Big[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/l_s^2} + e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/l_s^2} \\
&+ 2e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2} \frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|} \frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|} \Big] .
\end{aligned} \tag{4.119}$$

$$
\begin{aligned}
&u_+(\mathbf{x}_1, k_1, \mathbf{x}_2, k_2, 0)\\
&= \frac{16\gamma_+(k_1,k_2)}{\pi}\left[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/l_s^2}e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/l_s^2}+e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/l_s^2}e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/l_s^2}\right.\\
&\left.+\,2e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2}e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/l_s^2}\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\right], \qquad (4.120)
\end{aligned}
$$

The above initial conditions are then inserted into (3.148) and the Gaussian integrals carried out. This results in the following formulas for the angular moments $u_\pm$:

$$
\begin{aligned}
&u_-(\mathbf{x}_1, k_1, \mathbf{x}_2, k_2, t)\\
&= \frac{16\gamma_-(k_1,k_2)}{\pi}\left(\frac{l_s^2}{l_s^2+4tD_{1,-}}\right)^{3/2}\left(\frac{l_s^2}{l_s^2+4tD_{2,-}}\right)^{3/2}\\
&\times\left[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/(l_s^2+4tD_{2,-})}+e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/(l_s^2+4tD_{2,-})}\right. \qquad (4.121)\\
&\left.+2e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{2,-})}\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\right], \qquad (4.122)
\end{aligned}
$$

$$
\begin{aligned}
&u_+(\mathbf{x}_1, k_1, \mathbf{x}_2, k_2, t)\\
&= \frac{16\gamma_+(k_1,k_2)}{\pi}\left(\frac{l_s^2}{l_s^2+4tD_{1,+}}\right)^{3/2}\left(\frac{l_s^2}{l_s^2+4tD_{2,+}}\right)^{3/2}\\
&\times\left[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/(l_s^2+4tD_{1,+})}e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/(l_s^2+4tD_{2,+})}+e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/(l_s^2+4tD_{1,+})}e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/(l_s^2+4tD_{2,+})}\right. \qquad (4.123)\\
&\left.+2e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{1,+})}e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{2,+})}\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\right]. \qquad (4.124)
\end{aligned}
$$

Finally, combining the above with (4.58), (4.78) and (4.111), we see that the average probability densities are given by

$$
\begin{aligned}
&\rho_0\langle|\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle \\
&= \frac{32g^2\rho_0}{\pi}\sum_{i=\pm}\int_0^\infty\int_0^\infty dk_1dk_2\ \eta_i(k_1,k_2)\left(\frac{l_s^2}{l_s^2+4tD_{1,\tilde{i}}}\right)^{3/2}\left(\frac{l_s^2}{l_s^2+4tD_{2,\tilde{i}}}\right)^{3/2} \\
&\times\Big[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/(l_s^2+4tD_{1,\tilde{i}})}e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/(l_s^2+4tD_{2,\tilde{i}})}+e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/(l_s^2+4tD_{2,-})} \\
&+2e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{2,-})}\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\Big]\ ,
\end{aligned}
\tag{4.125}
$$

$$
\begin{aligned}
&\langle|\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle \\
&= \frac{16}{\pi}\sum_{i=\pm}\int_0^\infty\int_0^\infty dk_1dk_2\ \zeta_i(k_1,k_2)\left(\frac{l_s^2}{l_s^2+4tD_{1,-}}\right)^{3/2}\left(\frac{l_s^2}{l_s^2+4tD_{2,-}}\right)^{3/2} \\
&\times\Big[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/(l_s^2+4tD_{2,-})}+e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/(l_s^2+4tD_{2,-})} \\
&+2e^{-|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{1,-})}e^{-|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2/(l_s^2+4tD_{2,-})}\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\Big]\ ,
\end{aligned}
$$

where we have introduced the notation $\tilde{i}=-i$ for $i=\pm$, and the coefficients $\eta_\pm$ and $\zeta_\pm$ are defined by

$$
\eta_\pm(k_1,k_2)=\frac{k_1^2k_2^2e^{-l_s^2k_1^2-l_s^2k_2^2}}{(\lambda_\pm(k_1,k_2)-d(k_1,k_2)-\Omega)^2-(\lambda_\mp(k_1,k_2)-d(k_1,k_2)-\Omega)^2}\ , \tag{4.126}
$$

$$
\zeta_\pm(k_1,k_2)=\frac{k_1^2k_2^2e^{-l_s^2k_1^2-l_s^2k_2^2}(\lambda_+(k_1,k_2)-d(k_1,k_2)-\Omega)^2}{(\lambda_+(k_1,k_2)-d(k_1,k_2)-\Omega)^2-(\lambda_-(k_1,k_2)-d(k_1,k_2)-\Omega)^2}\ . \tag{4.127}
$$

It is readily seen that long times, $\langle|\psi_1|^2\rangle$ and $\langle|\psi_2|^2\rangle$ decay algebraically according to the formulas

$$\rho_0\langle|\psi_1|^2\rangle = \frac{C_3}{t^3} - \frac{C_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2)}{t^4} \ , \tag{4.128}$$

$$\langle|\psi_2|^2\rangle = \frac{C_1}{t^3} - \frac{C_2(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2)}{t^4} \ , \tag{4.129}$$

where the constants C_i for $i = 1, 2, 3, 4$ are given by

$$C_1 = \sum_{i=\pm} \frac{64}{\pi} \int_0^\infty \int_0^\infty dk_1 dk_2 \ \zeta_i(k_1,k_2) \left(\frac{l_s^2}{4D_{1,\tilde{i}}}\right)^{3/2} \left(\frac{l_s^2}{4D_{2,\tilde{i}}}\right)^{3/2} \times \left[1 + \frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\right] \ , \tag{4.130}$$

$$\begin{aligned} C_2 = \sum_{i=\pm} \frac{16}{\pi} \int_0^\infty \int_0^\infty & dk_1 dk_2 \ \zeta_i(k_1,k_2) \left(\frac{l_s^2}{4D_{1,\tilde{i}}}\right)^{3/2} \left(\frac{l_s^2}{4D_{2,\tilde{i}}}\right)^{3/2} \\ \times \Bigg[& \frac{|\mathbf{x}_1-\mathbf{r}_1|^2}{4D_{1,i}} + \frac{|\mathbf{x}_2-\mathbf{r}_2|^2}{4D_{2,i}} + \frac{|\mathbf{x}_1-\mathbf{r}_2|^2}{4D_{1,i}} + \frac{|\mathbf{x}_2-\mathbf{r}_1|^2}{4D_{2,i}} \\ +2 & \left(\frac{|\mathbf{x}_1-(\mathbf{r}_1+\mathbf{r}_2)/2|^2}{4D_{1,i}} + \frac{|\mathbf{x}_2-(\mathbf{r}_1+\mathbf{r}_2)/2|^2}{4D_{2,i}}\right) \frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\Bigg] \ , \end{aligned} \tag{4.131}$$

$$C_3 = \sum_{i=\pm} \frac{32g^2\rho_0}{\pi} \int_0^\infty \int_0^\infty dk_1 dk_2 \ \eta_i(k_1,k_2) \left(\frac{l_s^2}{4D_{1,\tilde{i}}}\right)^{3/2} \left(\frac{l_s^2}{4D_{2,\tilde{i}}}\right)^{3/2} \times \left[2 + 2\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\right] \ , \tag{4.132}$$

$$
\begin{aligned}
C_4 = \sum_{i=\pm} \frac{32g^2\rho_0}{\pi} \int_0^\infty \int_0^\infty dk_1 dk_2 k_1^2 \, \eta_i(k_1,k_2) \left(\frac{l_s^2}{4D_{1,i}}\right)^{3/2} \left(\frac{l_s^2}{4D_{2,i}}\right)^{3/2} \\
\times \left[\frac{|\mathbf{x}_1 - \mathbf{r}_1|^2}{4D_{1,i}} + \frac{|\mathbf{x}_2 - \mathbf{r}_2|^2}{4D_{2,i}} + \frac{|\mathbf{x}_1 - \mathbf{r}_2|^2}{4D_{1,i}} + \frac{|\mathbf{x}_2 - \mathbf{r}_1|^2}{4D_{2,i}} \right. \\
\left. +2 \left(\frac{|\mathbf{x}_1 - (\mathbf{r}_1 + \mathbf{r}_2)/2|^2}{4D_{1,i}} + \frac{|\mathbf{x}_2 - (\mathbf{r}_1 + \mathbf{r}_2)/2|^2}{4D_{2,i}} \right) \frac{\sin(k_1|\mathbf{r}_1 - \mathbf{r}_2|)}{k_1|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\sin(k_2|\mathbf{r}_1 - \mathbf{r}_2|)}{k_2|\mathbf{r}_1 - \mathbf{r}_2|} \right] .
\end{aligned}
\tag{4.133}
$$

To illustrate the above results, we consider the case of isotropic scattering and set the dimensionless quantities $\Omega/\sqrt{\rho_0}g = c/l_s\sqrt{\rho_0}g = 1$. In addition, we choose $\mathbf{x}_1 = (l_s, 0, 0)$, $\mathbf{x}_2 = (-l_s, 0, 0)$, $\mathbf{r}_1 = (0, 0, l_s)$ and $\mathbf{r}_2 = (0, 0, -l_s)$, so that the distances from the points of excitation ($\mathbf{r}_1$ and $\mathbf{r}_2$) to the points of detection are equal to l_s. In Figure (4.2) we plot the time dependence of the probability densities $|\langle\psi_1\rangle|^2$ and $|\langle\psi_2\rangle|^2$. We note that $|\psi_2|^2$ is monotonically decreasing while $|\psi_1|^2$ has a peak near $\Omega t = 1$. A comparison of these results with the asymptotic formulas (4.128) and (4.129) is shown in Figures (4.4) and (4.3). There is good agreement at long times.

4.6 Two-Photon Transport in Random Media

In this section, we consider the general problem of two photons interacting with a random medium. That is, we will study the the time evolution of the amplitudes $a(\mathbf{x}_1, \mathbf{x}_2, t)$, $\psi_1(\mathbf{x}_1, \mathbf{x}_2, t)$ and $\psi_2(\mathbf{x}_1, \mathbf{x}_2, t)$. We begin with the system (4.16), where we have canceled overall factors of ρ:

$$
\begin{aligned}
& i\partial_t \psi_2(\mathbf{x}_1, \mathbf{x}_2, t) = c(-\Delta_{\mathbf{x}_1})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) \\
& + \frac{g}{2}(\rho(\mathbf{x}_1)\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) + \rho(\mathbf{x}_2)\psi_1(\mathbf{x}_2, \mathbf{x}_1, t)) , \\
& i\partial_t \psi_1(\mathbf{x}_1, \mathbf{x}_2, t) = 2g\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + \left[c(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega\right] \psi_1(\mathbf{x}_1, \mathbf{x}_2, t) \\
& - 2g\rho(\mathbf{x}_2)a(\mathbf{x}_1, \mathbf{x}_2, t) , \qquad (4.134) \\
& i\partial_t a(\mathbf{x}_1, \mathbf{x}_2, t) = \frac{g}{2}(\psi_1(\mathbf{x}_2, \mathbf{x}_1, t) - \psi_1(\mathbf{x}_1, \mathbf{x}_2, t)) + 2\Omega a(\mathbf{x}_1, \mathbf{x}_2, t) .
\end{aligned}
$$

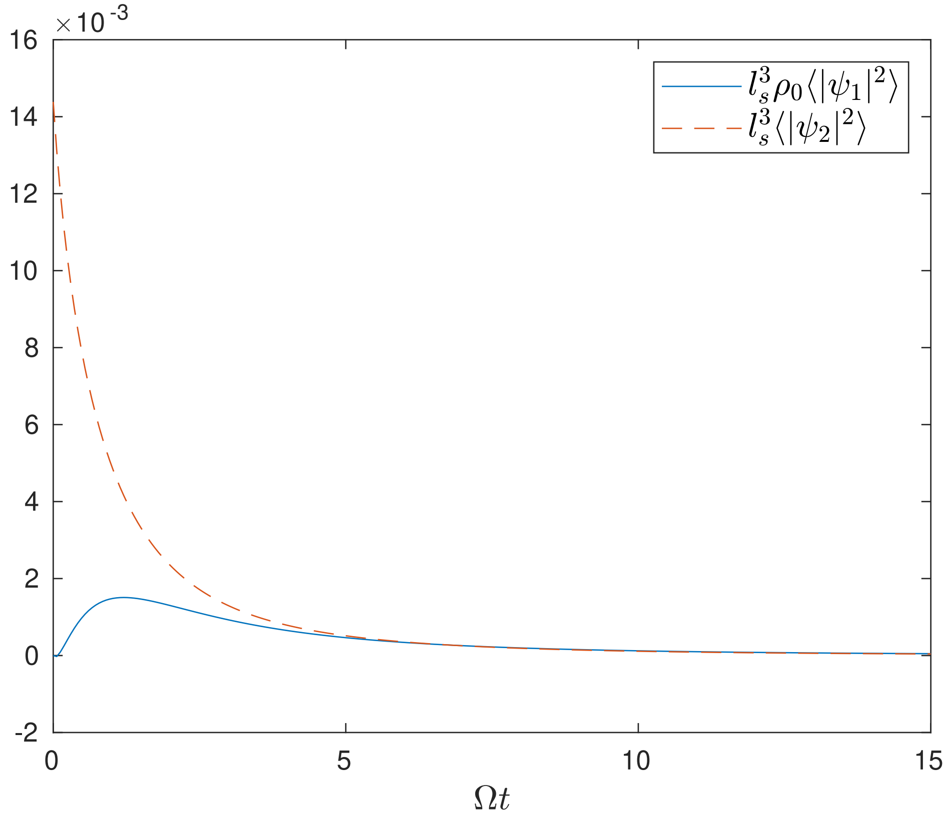


Figure 4.2: One- and two-photon probability densities for stimulated emission in a random medium.

Here the number density ρ is a random field of the form (4.48). Eq. (4.134) can now be rewritten as

$$i\partial_t \mathbf{\Psi} = \mathbf{A}(\mathbf{x_1}, \mathbf{x_2})\mathbf{\Psi} + \mathbf{g}\sqrt{\rho_0}\eta(\mathbf{x_1})\mathbf{K_1}\mathbf{\Psi} + \mathbf{g}\sqrt{\rho_0}\eta(\mathbf{x_2})\mathbf{K_2}\mathbf{\Psi}\,, \tag{4.135}$$

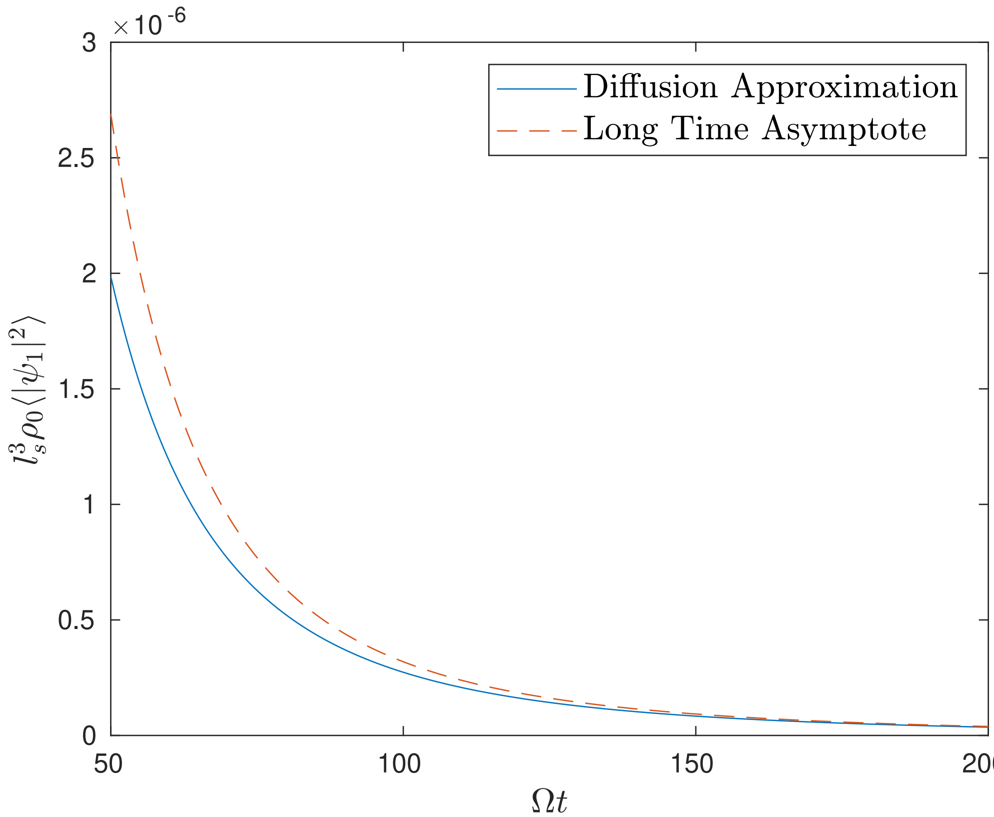


Figure 4.3: Comparison of diffusion approximation and long-time asymptote for $\langle|\psi_1|^2\rangle$.

where $\mathbf{\Psi}$ is defined by (4.30) and

$$A(\mathbf{x}_1,\mathbf{x}_2) = \begin{bmatrix} c(-\Delta_{\mathbf{x}_1})^{1/2} + c(-\Delta_{\mathbf{x}_2})^{1/2} & g\sqrt{\rho_0} & g\sqrt{\rho_0} & 0 \\ g\sqrt{\rho_0} & c(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega & 0 & -g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & 0 & c(-\Delta_{\mathbf{x}_1})^{1/2} + \Omega & g\sqrt{\rho_0} \\ 0 & -g\sqrt{\rho_0} & g\sqrt{\rho_0} & 2\Omega \end{bmatrix}, \tag{4.136}$$

$$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.137}$$

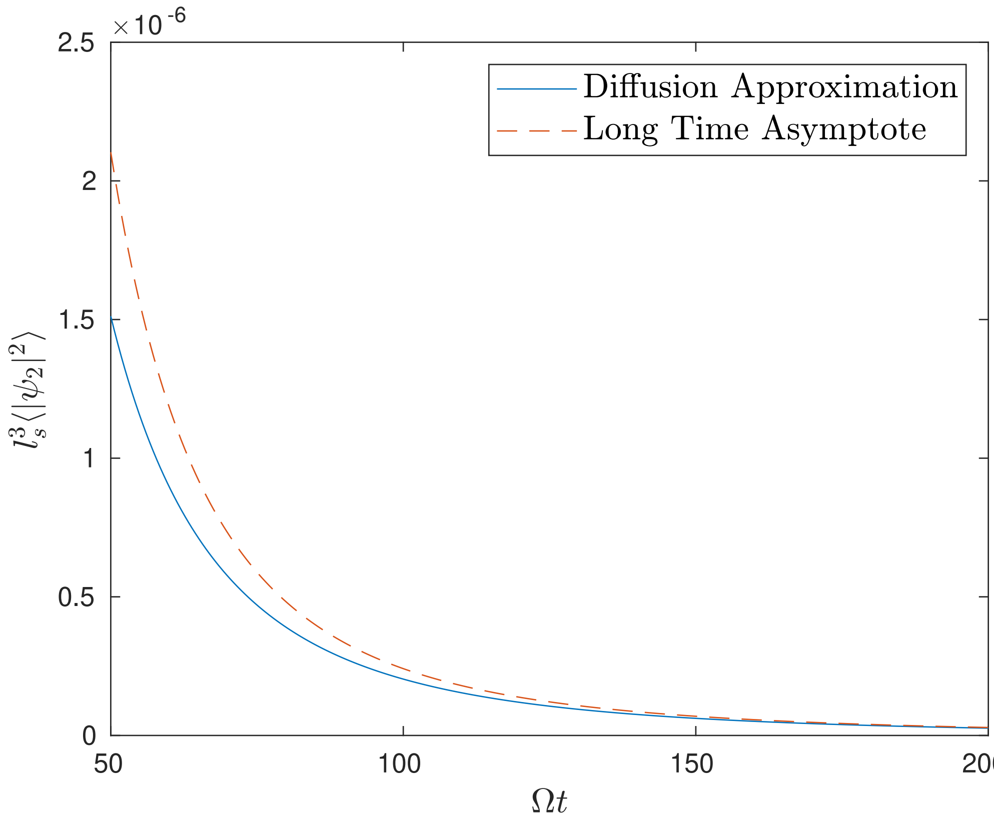


Figure 4.4: Comparison of diffusion approximation and long-time asymptote for $\langle|\psi_2|^2\rangle$.

As before. to derive a kinetic equation in the high-frequency limit, we rescale $t, \mathbf{x}_1, \mathbf{x}_2$ according to $t \to t/\epsilon$, $\mathbf{x}_1 \to \mathbf{x}_1/\epsilon$ and $\mathbf{x}_2 \to \mathbf{x}_2/\epsilon$. Additionally, we assume that the randomness is sufficiently weak so that the correlations of η are $O(\epsilon)$. Eq. (4.135) thus becomes

$$i\epsilon\partial_t \mathbf{\Psi}_\epsilon = \mathbf{A}_\epsilon(\mathbf{x_1}, \mathbf{x_2})\mathbf{\Psi}_\epsilon + \sqrt{\epsilon}\mathbf{g}\sqrt{\rho_\mathbf{0}}\eta(\mathbf{x_1}/\epsilon)\mathbf{K_1}\mathbf{\Psi}_\epsilon + \sqrt{\epsilon}\mathbf{g}\sqrt{\rho_\mathbf{0}}\eta(\mathbf{x_2}/\epsilon)\mathbf{K_2}\mathbf{\Psi}_\epsilon\,, \tag{4.138}$$

where

$$A_\epsilon(\mathbf{x}_1,\mathbf{x}_2) = \begin{bmatrix} c\epsilon(-\Delta_{\mathbf{x}_1})^{1/2}+c\epsilon(-\Delta_{\mathbf{x}_2})^{1/2} & g\sqrt{\rho_0} & g\sqrt{\rho_0} & 0 \\ g\sqrt{\rho_0} & c\epsilon(-\Delta_{\mathbf{x}_2})^{1/2}+\Omega & 0 & -g\sqrt{\rho_0} \\ g\sqrt{\rho_0} & 0 & c\epsilon(-\Delta_{\mathbf{x}_1})^{1/2}+\Omega & g\sqrt{\rho_0} \\ 0 & -g\sqrt{\rho_0} & g\sqrt{\rho_0} & 2\Omega \end{bmatrix}. \tag{4.139}$$

To proceed further, we introduce the 4×4 matrix Wigner transform W_ϵ which is defined by (4.57). The diagonal elements of W_ϵ are related to the probability densities by

$$2|\psi_{2\epsilon}(\mathbf{x}_1,\mathbf{x}_2,t)|^2 = \int d^3k_1 d^3k_2 (W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t))_{11}\ , \tag{4.140}$$

$$\frac{\rho_0}{2}|\psi_{1\epsilon}(\mathbf{x}_1,\mathbf{x}_2,t)|^2 = \int d^3k_1 d^3k_2 (W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t))_{22}\,, \tag{4.141}$$

$$2\rho_0^2|a_\epsilon(\mathbf{x}_1,\mathbf{x}_2,t)|^2 = \int d^3k_1 d^3k_2 (W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t))_{44}\,, \tag{4.142}$$

while the off diagonal elements of W_ϵ are related to correlations between the amplitudes. It can be seen by direct calculation that the Wigner transform satisfies the Liouville equation

$$\begin{aligned}
&i\epsilon\partial_t W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t)\\
&=\int \frac{d^3k_1'}{(2\pi)^3}\frac{d^3k_2'}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1'+i\mathbf{x}_2\cdot\mathbf{k}_2'}\hat{A}(\mathbf{k}_1-\epsilon\mathbf{k}_1'/2,\mathbf{k}_2-\epsilon\mathbf{k}_2'/2)\hat{W}_\epsilon(\mathbf{k}_1',\mathbf{k}_1,\mathbf{k}_2',\mathbf{k}_2,t)\\
&-\int \frac{d^3k_1'}{(2\pi)^3}\frac{d^3k_2'}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1'+i\mathbf{x}_2\cdot\mathbf{k}_2'}\hat{W}_\epsilon(\mathbf{k}_1',\mathbf{k}_1,\mathbf{k}_2',\mathbf{k}_2,t)\hat{A}(\mathbf{k}_1+\epsilon\mathbf{k}_1'/2,\mathbf{k}_2+\epsilon\mathbf{k}_2'/2)\\
&+\sqrt{\epsilon}g\sqrt{\rho_0}\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}_1/\epsilon}\,\hat{\eta}(\mathbf{q})\,[K_1W_\epsilon(\mathbf{x}_1,\mathbf{k}_1+\mathbf{q}/2,\mathbf{x}_2,\mathbf{k}_2,t)\\
&-W_\epsilon(\mathbf{x}_1,\mathbf{k}_1-\mathbf{q}/2,\mathbf{x}_2,\mathbf{k}_2,t)K_1^T]\\
&+\sqrt{\epsilon}g\sqrt{\rho_0}\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}_2/\epsilon}\,\hat{\eta}(\mathbf{q})\,[K_2W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2+\mathbf{q}/2,t)\\
&-W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2-\mathbf{q}/2,t)K_2^T]\ ,
\end{aligned} \tag{4.143}$$

where $\hat{A}$ is given by (4.34), and the Fourier transform $\hat{W}_\epsilon$ is defined by

$$\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{k}_1, \mathbf{k}_2', \mathbf{k}_2, t) = \int d^3x_1 d^3x_2 e^{-i\mathbf{x}_1\cdot\mathbf{k}_1' - i\mathbf{x}_2\cdot\mathbf{k}_2'} W_\epsilon(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t). \tag{4.144}$$

Once again we study the behavior of W_ϵ in the high-frequency limit $\epsilon \to 0$ and introduce a multiscale expansion of the Wigner transform of the form

$$\begin{aligned} W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) &= W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) + \sqrt{\epsilon} W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\ &\quad + \epsilon W_2(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) + \cdots , \end{aligned} \tag{4.145}$$

where $\mathbf{X}_1 = \mathbf{x}_1/\epsilon$ and $\mathbf{X}_2 = \mathbf{x}_2/\epsilon$ are fast variables and W_0 is both deterministic and independent of the fast variables. We will treat the slow and fast variables $\mathbf{x}_1$ and $\mathbf{X}_1$ (respectively $\mathbf{x}_2$ and $\mathbf{X}_2$) as independent and make the replacement (4.69). Hence (4.143) becomes

$$\begin{aligned} & i\epsilon\partial_t W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\ &= \int \frac{d^3k_1'}{(2\pi)^3} \frac{d^3k_2'}{(2\pi)^3} \frac{d^3K_1}{(2\pi)^3} \frac{d^3K_2}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1' + i\mathbf{x}_2\cdot\mathbf{k}_2' + i\mathbf{X}_1\cdot\mathbf{K}_1 + i\mathbf{X}_2\cdot\mathbf{K}_2} \\ &\times \left[\hat{A}(\mathbf{k}_1 - \mathbf{K}_1/2 - \epsilon\mathbf{k}_1'/2, \mathbf{k}_2 - \mathbf{K}_2/2 - \epsilon\mathbf{k}_2'/2)\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2, t)\right. \\ &- \left.\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2, t)\hat{A}(\mathbf{k}_1 + \mathbf{K}_2/2 + \epsilon\mathbf{k}_1'/2, \mathbf{k}_2 + \mathbf{K}_2/2 + \epsilon\mathbf{k}_2'/2)\right] \\ &+ \sqrt{\epsilon} g\sqrt{\rho_0} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}_1/\epsilon}\, \hat{\eta}(\mathbf{q})\, [K_1 W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1 + \mathbf{q}/2, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\ &- W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1 - \mathbf{q}/2, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) K_1^T] \\ &+ \sqrt{\epsilon} g\sqrt{\rho_0} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}_2/\epsilon}\, \hat{\eta}(\mathbf{q})\, [K_2 W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2 + \mathbf{q}/2, t) \\ &- W_\epsilon(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2 - \mathbf{q}/2, t) K_2^T] , \end{aligned} \tag{4.146}$$

where the Fourier transform $\hat{W}_\epsilon(\mathbf{k}_1', \mathbf{K}_1, \mathbf{k}_1, \mathbf{k}_2', \mathbf{K}_2, \mathbf{k}_2)$ is defined by Eq. (4.73). Next we substitute (4.145) into (4.146) and collect terms at each order of $\sqrt{\epsilon}$. At order $O(1)$ we have

$$\hat{A}(\mathbf{k}_1,\mathbf{k}_2)W_0(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) - W_0(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_1,\mathbf{k}_2)\hat{A}(\mathbf{k}_1,\mathbf{k}_2) = 0. \tag{4.147}$$

Since $\hat{A}$ is symmetric it can be diagonalized. We then define $\{\mathbf{v}_i(\mathbf{k}_1,\mathbf{k}_2), \lambda_i(\mathbf{k}_1,\mathbf{k}_2)\}$, $i = 1,2,3,4$ be the eigenvector-eigenvalue pairs given by (4.35) and (4.38). It follows from (4.147) that W_0 is also diagonal in this basis and can be expanded as

$$W_0(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t) = \sum_{i=1}^{4} a_i(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t)\mathbf{v}_i(\mathbf{k}_1,\mathbf{k}_2)\mathbf{v}_i^T(\mathbf{k}_1,\mathbf{k}_2). \tag{4.148}$$

At order $O(\sqrt{\epsilon})$ we find that

$$\begin{aligned}
&\hat{A}(\mathbf{k}_1-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)\hat{W}_1(\mathbf{x}_1,\mathbf{K}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{K}_2,\mathbf{k}_2)\\
&-\hat{W}_1(\mathbf{k}_1',\mathbf{K}_1,\mathbf{k}_1,\mathbf{k}_2',\mathbf{K}_2,\mathbf{k}_2)\hat{A}(\mathbf{k}_1+\mathbf{K}_2/2,\mathbf{k}_2+\mathbf{K}_2/2)\\
&+g\sqrt{\rho_0}(2\pi)^3\hat{\eta}(\mathbf{K}_1)\delta(\mathbf{K}_2)\left[K_1W_0(\mathbf{x}_1,\mathbf{k}_1+\mathbf{K}_1/2,\mathbf{x}_2,\mathbf{k}_2+\mathbf{K}_2)\right.\\
&\left.-W_0(\mathbf{x}_1,\mathbf{k}_1-\mathbf{K}_1/2,\mathbf{x}_2,\mathbf{k}_2-\mathbf{K}_2)K_1^T\right]\\
&+g\sqrt{\rho_0}(2\pi)^3\hat{\eta}(\mathbf{K}_2)\delta(\mathbf{K}_1)\left[K_2W_0(\mathbf{x}_1,\mathbf{k}_1+\mathbf{K}_1,\mathbf{x}_2,\mathbf{k}_2+\mathbf{K}_2/2)\right.\\
&\left.-W_0(\mathbf{x}_1,\mathbf{k}_1-\mathbf{K}_1,\mathbf{x}_2,\mathbf{k}_2-\mathbf{K}_2/2)K_2^T\right] = 0.
\end{aligned} \tag{4.149}$$

Although W_1 is not diagonal, we can still decompose its Fourier transform $\hat{W}_1$ as

$$\begin{aligned}
&\hat{W}_1(\mathbf{x}_1,\mathbf{K}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{K}_2,\mathbf{k}_2)\\
&=\sum_{i,j} w_{i,j}(\mathbf{x}_1,\mathbf{K}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{K}_2,\mathbf{k}_2)\mathbf{v}_i(\mathbf{k}_1-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)\mathbf{v}_j^T(\mathbf{k}_1+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2).
\end{aligned} \tag{4.150}$$

Multiplying (4.149) on the left by $\mathbf{v}_m^T(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)$ and the right by $\mathbf{v}_n(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)$, we obtain

$$\begin{aligned}
&(\lambda_m(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2) - \lambda_n(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2) + i\theta) \\
&\quad \times w_{m,n}(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2) \\
=& g\sqrt{\rho_0}(2\pi)^3\eta(\mathbf{K}_1)\delta(\mathbf{K}_2) \\
&\quad \times \left[a_m(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)K_{1,m,n}(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2, \mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)\right] \\
+& g\sqrt{\rho_0}(2\pi)^3\eta(\mathbf{K}_2)\delta(\mathbf{K}_1) \\
&\quad \times \left[a_m(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)K_{2,m,n}(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2, \mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)\right] \\
-& g\sqrt{\rho_0}(2\pi)^3\eta(\mathbf{K}_1)\delta(\mathbf{K}_2) \\
&\quad \times \left[a_n(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)K_{1,m,n}(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2, \mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)\right] \\
-& g\sqrt{\rho_0}(2\pi)^3\eta(\mathbf{K}_2)\delta(\mathbf{K}_1) \\
&\quad \times \left[a_n(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2)K_{2,m,n}(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2, \mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2)\right] ,
\end{aligned} \tag{4.151}$$

where

$$K_{1,m,n}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}_1, \mathbf{q}_2) = \mathbf{v}_m^T(\mathbf{k}_1, \mathbf{k}_2)K_1\mathbf{v}_n(\mathbf{q}_1, \mathbf{q}_2) , \tag{4.152}$$

$$K_{2,m,n}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}_1, \mathbf{q}_2) = \mathbf{v}_m^T(\mathbf{k}_1, \mathbf{k}_2)K_2\mathbf{v}_n(\mathbf{q}_1, \mathbf{q}_2) . \tag{4.153}$$

Here $\theta \to 0^+$ is a regularizing parameter. At order $O(\epsilon)$ we find that

$$
\begin{aligned}
& i\partial_t W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, t) = LW_2(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\
& - \frac{i}{2} M W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2) - \frac{i}{2} W_0(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2) M \\
& + g\sqrt{\rho_0} \int \frac{d^3 q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_1}\, \hat{\eta}(\mathbf{q}) \left[K_1 W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1 + \mathbf{q}/2, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2) \right. \\
& \left. - W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1 - \mathbf{q}/2, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2) K_1^T \right] \\
& + g\sqrt{\rho_0} \int \frac{d^3 q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_2}\, \hat{\eta}(\mathbf{q}) \left[K_2 W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2 + \mathbf{q}/2) \right. \\
& \left. - W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2 - \mathbf{q}/2) K_2^T \right] ,
\end{aligned}
\tag{4.154}
$$

where

$$
M = \begin{bmatrix} c\hat{\mathbf{k}}_1 \cdot \nabla_{\mathbf{x}_1} + c\hat{\mathbf{k}}_2 \cdot \nabla_{\mathbf{x}_2} & 0 & 0 & 0 \\ 0 & c\hat{\mathbf{k}}_2 \cdot \nabla_{\mathbf{x}_2} & 0 & 0 \\ 0 & 0 & c\hat{\mathbf{k}}_1 \cdot \nabla_{\mathbf{x}_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.155}
$$

and

$$
\begin{aligned}
& LW_2(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2, t) \\
& = \int \frac{d^3 K_1}{(2\pi)^3} \frac{d^3 K_2}{(2\pi)^3} e^{i\mathbf{K}_1\cdot\mathbf{X}_1 + i\mathbf{K}_2\cdot\mathbf{X}_2} \hat{A}(\mathbf{k}_1 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2) \hat{W}_2(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2, t) \\
& - \hat{W}_2(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2, t) \hat{A}(\mathbf{k}_1 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2).
\end{aligned}
\tag{4.156}
$$

In order to obtain the equations satisfied by the a_i, we multiply (4.154) on the left by $\mathbf{v}_i^T(\mathbf{k}_1, \mathbf{k}_2)$ and the right by $\mathbf{v}_i(\mathbf{k}_1, \mathbf{k}_2)$, and take the ensemble average. In order to close the hierarchy of equations, we assume that $\langle \mathbf{v}_i^T LW_2 \mathbf{v}_i \rangle = 0$, which corresponds to the assumption that W_2 is statistically stationary with respect to the fast variables $\mathbf{X}_1$ and $\mathbf{X}_2$.

Following procedures similar to those in Appendix C, we find that

$$\begin{aligned}
&\frac{1}{c}\partial_t a_i + f_{1i}(\mathbf{k}_1,\mathbf{k}_2)\hat{\mathbf{k}}_1\cdot\nabla_{\mathbf{x}_1}a_i + f_{2i}(\mathbf{k}_1,\mathbf{k}_2)\hat{\mathbf{k}}_2\cdot\nabla_{\mathbf{x}_2}a_i \\
&\qquad + \mu_{1i}(\mathbf{k}_1,\mathbf{k}_2)a_i(\mathbf{k}_1,\mathbf{k}_2) + \mu_{2i}(\mathbf{k}_1,\mathbf{k}_2)a_i(\mathbf{k}_1,\mathbf{k}_2) \\
&\qquad = \mu_{1i}(\mathbf{k}_1,\mathbf{k}_2)\int d^2\hat{K}A(\hat{\mathbf{k}_1},\hat{\mathbf{K}},|\mathbf{k}_1|)a_i(|\mathbf{k}_1|\hat{\mathbf{K}},\mathbf{k}_2) \\
&\qquad + \mu_{2i}(\mathbf{k}_1,\mathbf{k}_2)\int d^2\hat{K}A(\hat{\mathbf{k}_2},\hat{\mathbf{K}},|\mathbf{k}_2|)a_i(\mathbf{k}_1,|\mathbf{k}_2|\hat{\mathbf{K}})\ ,
\end{aligned} \tag{4.157}$$

where

$$f_{1i}(\mathbf{k}_1,\mathbf{k}_2) = \mathbf{v}_{i1}(\mathbf{k}_1,\mathbf{k}_2)^2 + \mathbf{v}_{i3}(\mathbf{k}_1,\mathbf{k}_2)^2\,, \tag{4.158}$$

$$f_{2i}(\mathbf{k}_1,\mathbf{k}_2) = \mathbf{v}_{i1}(\mathbf{k}_1,\mathbf{k}_2)^2 + \mathbf{v}_{i2}(\mathbf{k}_1,\mathbf{k}_2)^2\,, \tag{4.159}$$

$$\mu_{1i}(\mathbf{k}_1,\mathbf{k}_2) = \frac{g^2\rho_0\pi K_{1,i,i}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)^2|\mathbf{k}_1|^2}{|\partial_{\mathbf{k}_1}\lambda_i(\mathbf{k}_1,\mathbf{k}_2)|}\,, \tag{4.160}$$

$$\mu_{2i}(\mathbf{k}_1,\mathbf{k}_2) = \frac{g^2\rho_0\pi K_{2,i,i}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)^2|\mathbf{k}_2|^2}{|\partial_{\mathbf{k}_2}\lambda_i(\mathbf{k}_1,\mathbf{k}_2)|}\,, \tag{4.161}$$

$$A(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2,k) = \frac{\tilde{C}(k(\hat{\mathbf{k}}_1-\hat{\mathbf{k}}_2))}{\displaystyle\int d^2\hat{k}'\tilde{C}(k(\hat{\mathbf{k}}_1-\hat{\mathbf{k}}'))}. \tag{4.162}$$

4.6.1 Diffusion Approximation

In this section we consider the diffusion limit of the kinetic equation (4.157). We again specialize to the case of white-noise correlations, which leads to the phase function $A = 1/4\pi$, corresponding to isotropic scattering. Making use of the diffusion approximation developed in section 5.2, we see that the first angular moments

$$u_i(\mathbf{x}_1,|\mathbf{k}_1|,\mathbf{x}_2,|\mathbf{k}_2|,t) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 a_i(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t) \tag{4.163}$$

satisfy the equations

$$\partial_t u_i - D_{1i}(|\mathbf{k}_1|, |\mathbf{k}_2|)\Delta_{\mathbf{x}_1} u_i - D_{2i}(|\mathbf{k}_1|, |\mathbf{k}_2|)\Delta_{\mathbf{x}_2} u_i = 0 \,, \tag{4.164}$$

where

$$D_{1i} = \frac{c f_{1i}^2}{3\mu_{1i}} \,, \tag{4.165}$$

$$D_{2i} = \frac{c f_{2i}^2}{3\mu_{2i}} \,. \tag{4.166}$$

Suppose that initially there are two photons in the field localized around the points $\mathbf{r}_1$ and $\mathbf{r}_2$ in a volume of linear size l_s. The corresponding initial conditions are given by

$$\sqrt{2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, 0) = \frac{1}{C}\left[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/2l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/2l_s^2} + e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/2l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/2l_s^2}\right] , \tag{4.167}$$

$$\psi_1(\mathbf{x}_1, \mathbf{x}_2, 0) = a(\mathbf{x}_1, \mathbf{x}_2, 0) = 0 \,, \tag{4.168}$$

where

$$C = \| e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/2l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/2l_s^2} + e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/2l_s^2} e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/2l_s^2} \|_{L^2_{\mathbf{x}_1,\mathbf{x}_2}} \,. \tag{4.169}$$

Note that this corresponds to an entangled two-photon state.

The above initial conditions imply initial conditions for the Wigner transform W_0. The initial conditions for the modes a_i are determined by solving the linear system

$$\begin{bmatrix} (W_0)_{11}(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\ (W_0)_{22}(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\ (W_0)_{33}(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\ (W_0)_{44}(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \end{bmatrix} = V(\mathbf{k}_1, \mathbf{k}_2) \begin{bmatrix} a_1(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\ a_2(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\ a_3(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \\ a_4(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0) \end{bmatrix} , \tag{4.170}$$

where

$$(V(\mathbf{k}_1, \mathbf{k}_2))_{ij} = \mathbf{v}_{ji}(\mathbf{k}_1, \mathbf{k}_2)^2. \tag{4.171}$$

It follows that the initial conditions for the first angular moments are given by

$$u_i(\mathbf{x}_1, |\mathbf{k}_1|, \mathbf{x}_2, |\mathbf{k}_2|, 0) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 a_i(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2, 0). \tag{4.172}$$

The diffusion equations (4.164) can then be solved using (3.148). Combining this result with (4.140)–(4.142), (4.148) and (4.163) we see that the average probability densities are given by

$$\begin{aligned}
&\langle|\psi_2(\mathbf{x}_1, \mathbf{x}_2, t)|^2\rangle \\
&= \frac{1}{2C^2}\sum_{i=1}^{4}\int_0^\infty\int_0^\infty dk_1 dk_2 k_1^2 k_2^2 e^{-l_s^2 k_1^2 - l_s^2 k_2^2}\mathbf{v}_{1i}^2(k_1, k_2)(V^{-1})_{i1}(k_1, k_2)\left(\frac{l_s^2}{l_s^2 + 4tD_{1i}}\right)^{3/2} \\
&\times\left(\frac{l_s^2}{l_s^2 + 4tD_{2i}}\right)^{3/2}\Big[e^{-|\mathbf{x}_1 - \mathbf{r}_1|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2 - \mathbf{r}_2|^2/(l_s^2+4tD_{2i})} \\
&+ e^{-|\mathbf{x}_1 - \mathbf{r}_2|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2 - \mathbf{r}_1|^2/(l_s^2+4tD_{2i})} \\
&+2e^{-|\mathbf{x}_1 - \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2 - \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2/(l_s^2+4tD_{2i})}\frac{\sin(k_1|\mathbf{r}_1 - \mathbf{r}_2|)}{k_1|\mathbf{r}_1 - \mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1 - \mathbf{r}_2|)}{k_2|\mathbf{r}_1 - \mathbf{r}_2|}\Big],
\end{aligned} \tag{4.173}$$

$$\begin{aligned}
&\langle|\psi_1(\mathbf{x}_2, \mathbf{x}_1, t)|^2\rangle \\
&= \frac{2}{\rho_0 C^2}\sum_{i=1}^{4}\int_0^\infty\int_0^\infty dk_1 dk_2 k_1^2 k_2^2 e^{-l_s^2 k_1^2 - l_s^2 k_2^2}\mathbf{v}_{2i}^2(k_1, k_2)(V^{-1})_{i1}(k_1, k_2)\left(\frac{l_s^2}{l_s^2 + 4tD_{1i}}\right)^{3/2} \\
&\times\left(\frac{l_s^2}{l_s^2 + 4tD_{2i}}\right)^{3/2}\Big[e^{-|\mathbf{x}_1 - \mathbf{r}_1|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2 - \mathbf{r}_2|^2/(l_s^2+4tD_{2i})} \\
&+ e^{-|\mathbf{x}_1 - \mathbf{r}_2|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2 - \mathbf{r}_1|^2/(l_s^2+4tD_{2i})} \\
&+2e^{-|\mathbf{x}_1 - \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2 - \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2/(l_s^2+4tD_{2i})}\frac{\sin(k_1|\mathbf{r}_1 - \mathbf{r}_2|)}{k_1|\mathbf{r}_1 - \mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1 - \mathbf{r}_2|)}{k_2|\mathbf{r}_1 - \mathbf{r}_2|}\Big],
\end{aligned} \tag{4.174}$$

$$
\begin{aligned}
&\langle |a(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle \\
&= \frac{1}{2\rho_0^2 C^2}\sum_{i=1}^{4}\int_0^\infty\int_0^\infty dk_1 dk_2 k_1^2 k_2^2 e^{-l_s^2 k_1^2 - l_s^2 k_2^2}\mathbf{v}_{4i}^2(k_1,k_2)(V^{-1})_{i1}(k_1,k_2)\left(\frac{l_s^2}{l_s^2+4tD_{1i}}\right)^{3/2} \\
&\times\left(\frac{l_s^2}{l_s^2+4tD_{2i}}\right)^{3/2}\Big[e^{-|\mathbf{x}_1-\mathbf{r}_1|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2-\mathbf{r}_2|^2/(l_s^2+4tD_{2i})} \\
&+e^{-|\mathbf{x}_1-\mathbf{r}_2|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2-\mathbf{r}_1|^2/(l_s^2+4tD_{2i})} \\
&+2e^{-|\mathbf{x}_1-\frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2/(l_s^2+4tD_{1i})}e^{-|\mathbf{x}_2-\frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2/(l_s^2+4tD_{2i})}\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_2|)}{k_1|\mathbf{r}_1-\mathbf{r}_2|}\frac{\sin(k_2|\mathbf{r}_1-\mathbf{r}_2|)}{k_2|\mathbf{r}_1-\mathbf{r}_2|}\Big]\,.
\end{aligned}
\tag{4.175}
$$

We note that at long times, the average probability densities decay algebraically according to

$$\langle|\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle = \frac{B_{11}}{t^3} - \frac{B_{21}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2)}{t^4}\,, \tag{4.176}$$

$$\langle|\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle = \frac{4B_{12}}{\rho_0 t^3} - \frac{4B_{22}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2)}{\rho_0 t^4}\,, \tag{4.177}$$

$$\langle|a(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle = \frac{B_{14}}{\rho_0^2 t^3} - \frac{B_{24}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2)}{\rho_0^2 t^4}\,, \tag{4.178}$$

where the constants B_{1j} and B_{2j}, $j = 1, 2, 4$ are given by

$$B_{1j} = \frac{1}{C^2} \sum_{i=1}^{4} \int_0^\infty \int_0^\infty dk_1 dk_2 k_1^2 k_2^2 e^{-l_s^2 k_1^2 - l_s^2 k_2^2} \mathbf{v}_{ji}^2(k_1, k_2)(V^{-1})_{i1}(k_1, k_2) \left(\frac{l_s^2}{4D_{1i}}\right)^{3/2} \tag{4.179}$$

$$\times \left(\frac{l_s^2}{4D_{2i}}\right)^{3/2} \left[1 + 1 \frac{\sin(k_1|\mathbf{r}_1 - \mathbf{r}_2|)}{k_1|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\sin(k_2|\mathbf{r}_1 - \mathbf{r}_2|)}{k_2|\mathbf{r}_1 - \mathbf{r}_2|}\right],$$

$$B_{2j} = \frac{1}{2C^2} \sum_{i=1}^{4} \int_0^\infty \int_0^\infty dk_1 dk_2 k_1^2 k_2^2 e^{-l_s^2 k_1^2 - l_s^2 k_2^2} \mathbf{v}_{ji}^2(k_1, k_2)(V^{-1})_{i1}(k_1, k_2) \left(\frac{l_s^2}{4D_{1i}}\right)^{3/2} \tag{4.180}$$

$$\times \left(\frac{l_s^2}{4D_{2i}}\right)^{3/2} \left[\frac{|\mathbf{x}_1 - \mathbf{r}_1|^2}{4D_{1i}} + \frac{|\mathbf{x}_2 - \mathbf{r}_2|^2}{4D_{2i}} + \frac{|\mathbf{x}_1 - \mathbf{r}_2|^2}{4D_{1i}} + \frac{|\mathbf{x}_2 - \mathbf{r}_1|^2}{4D_{2i}}\right.$$
$$\left. +2\left(\frac{|\mathbf{x}_1 - \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2}{4D_{1i}} + \frac{|\mathbf{x}_2 - \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}|^2}{4D_{2i}}\right) \frac{\sin(k_1|\mathbf{r}_1 - \mathbf{r}_2|)}{k_1|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\sin(k_2|\mathbf{r}_1 - \mathbf{r}_2|)}{k_2|\mathbf{r}_1 - \mathbf{r}_2|}\right]. \tag{4.181}$$

In order to illustrate the above results, we consider the case of isotropic scattering and set the dimensionless quantities $\Omega/\sqrt{\rho_0}g = c/l_s\sqrt{\rho_0}g = 1$. In addition, we choose $\mathbf{x}_1 = (l_s, 0, 0)$, $\mathbf{x}_2 = (-l_s, 0, 0)$, $\mathbf{r}_1 = (0, 0, l_s)$ and $\mathbf{r}_2 = (0, 0, -l_s)$, so that the distances from the points of excitation ($\mathbf{r}_1$ and $\mathbf{r}_2$) to the points of detection are equal to l_s. In Figure (4.5) we plot the time dependence of the probability densities a, $|\langle\psi_1\rangle|^2$ and $|\langle\psi_2\rangle|^2$. We note that the negative values of these quantities are due to the breakdown of the diffusion approximation at short times. We observe that the two-photon probability density increases before eventually decaying. A comparison of these results with the asymptotic formulas (4.176), (4.177) and (4.178) is shown in Figures (4.6), (4.7) and (4.8). There is good agreement at long times.

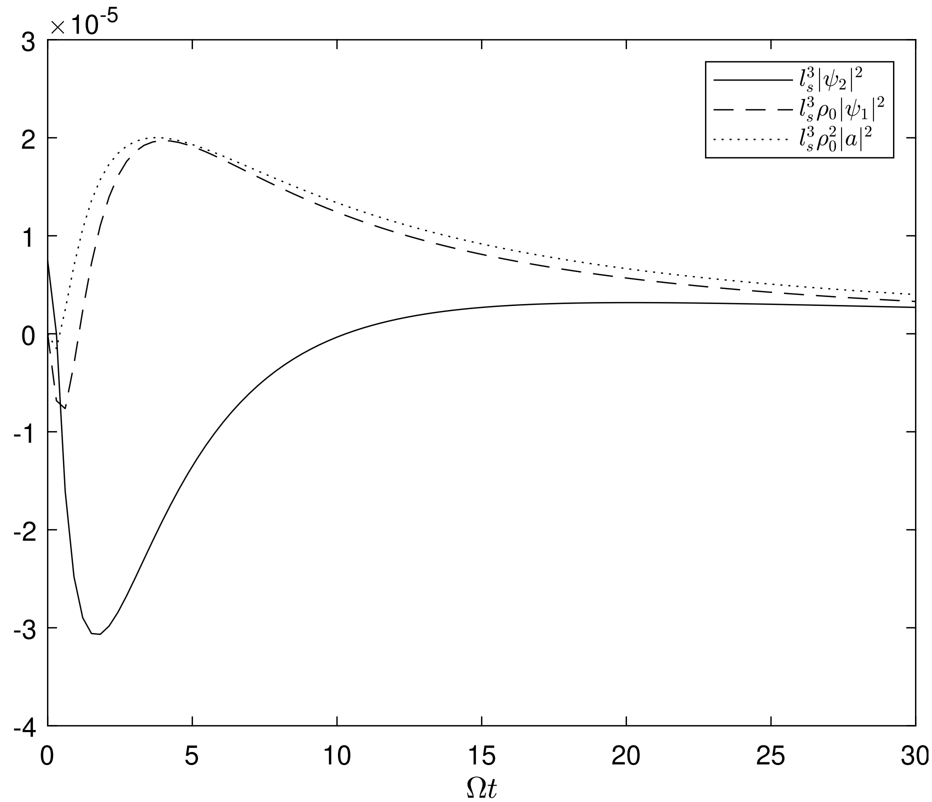


Figure 4.5: Atomic, one-photon, and two-photon probability densities in a random medium.

4.7 Appendix

4.7.1 Derivation of the System (4.16)

Here we derive the system (4.16). We begin by computing both sides of the time-dependent Schrodinger equation (3.22), employing the Hamiltonian H and the state (3.170). The left-hand side is equal to

$$\begin{aligned} i\hbar\partial_t|\Psi\rangle = \int d^3x_1 d^3x_2 \Big(& i\hbar\partial_t\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\ & + i\hbar\partial_t\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\ & + i\hbar\partial_t a(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2)\Big)|0\rangle\,, \end{aligned} \quad (4.182)$$

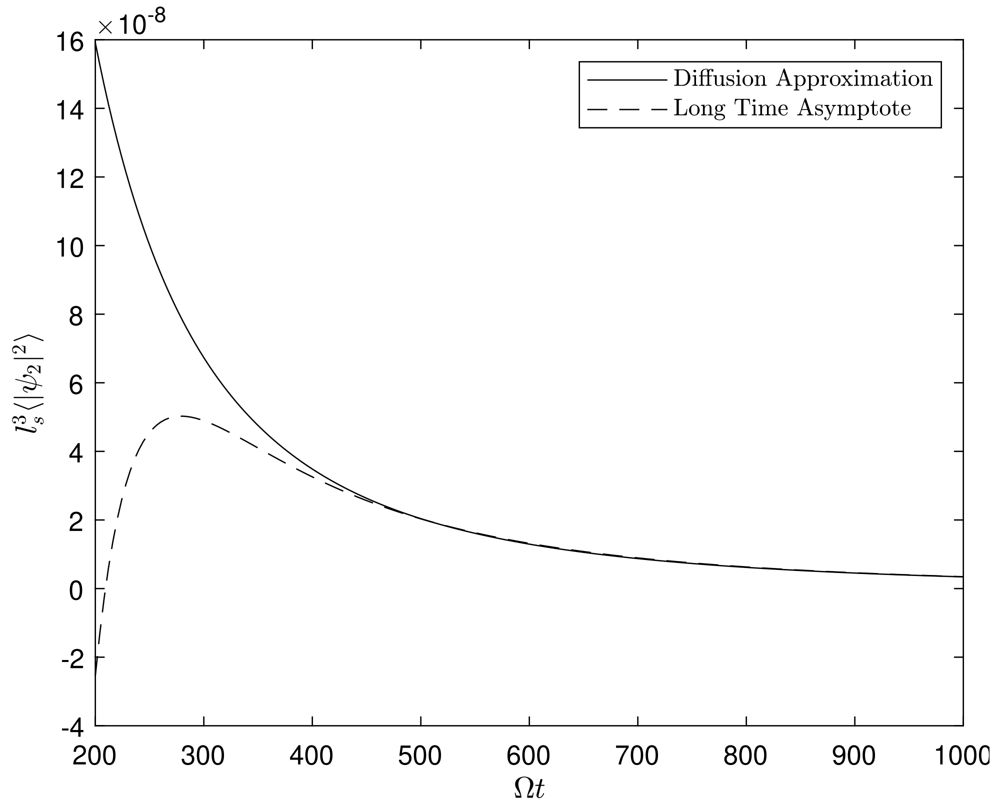


Figure 4.6: Comparison of diffusion approximation and long-time asymptote for $\langle|\psi_2|^2\rangle$.

while the right-hand side is given by

$$H|\Psi\rangle = H_F|\Psi\rangle + H_A|\Psi\rangle + H_I|\Psi\rangle \ . \tag{4.183}$$

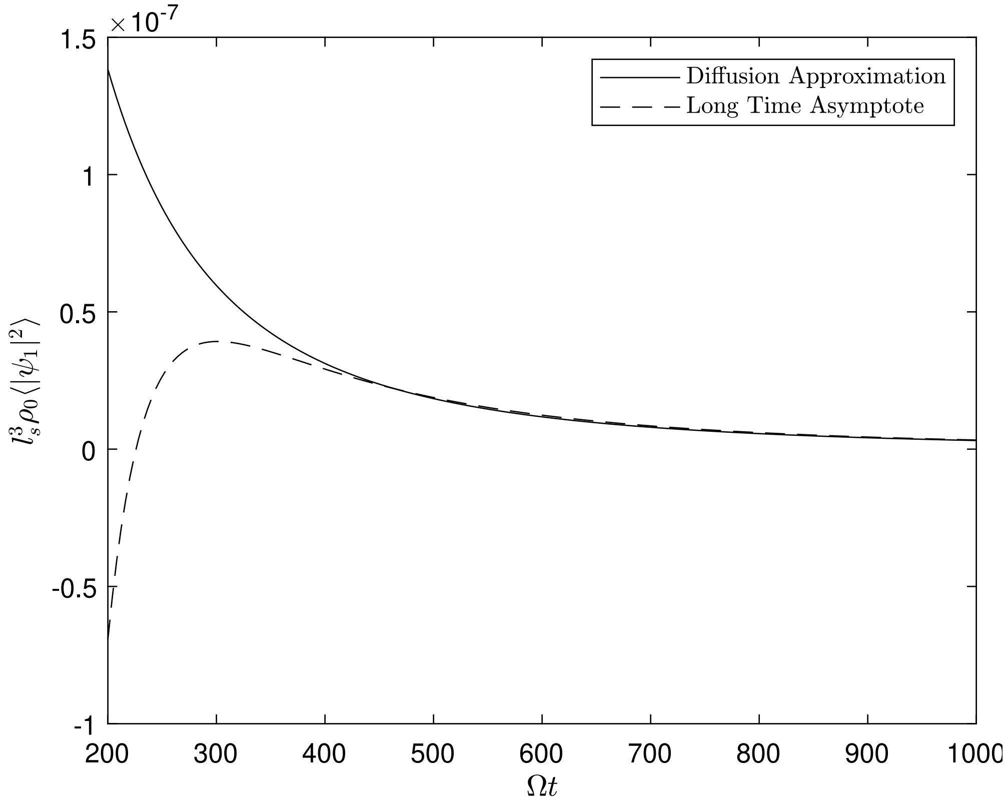


Figure 4.7: Comparison of diffusion approximation and long-time asymptote for $\langle|\psi_1|^2\rangle$.

We compute each term separately.

$$
\begin{aligned}
H_F|\Psi\rangle &= \hbar \int d^3x d^3x_1 d^3x_2 \left\{c(-\Delta)^{1/2}\varphi^\dagger(\mathbf{x})\varphi(\mathbf{x})\right\} \\
&\times \{\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) + \psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
&+ a(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2)\}|0\rangle \\
&= \hbar \int d^3x d^3x_1 d^3x_2 \{\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)c(-\Delta)^{1/2}\varphi^\dagger(\mathbf{x})\varphi(\mathbf{x})\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
&+ \psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)c(-\Delta)^{1/2}\varphi^\dagger(\mathbf{x})\varphi(\mathbf{x})\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2)\}|0\rangle \ . \qquad (4.184)
\end{aligned}
$$

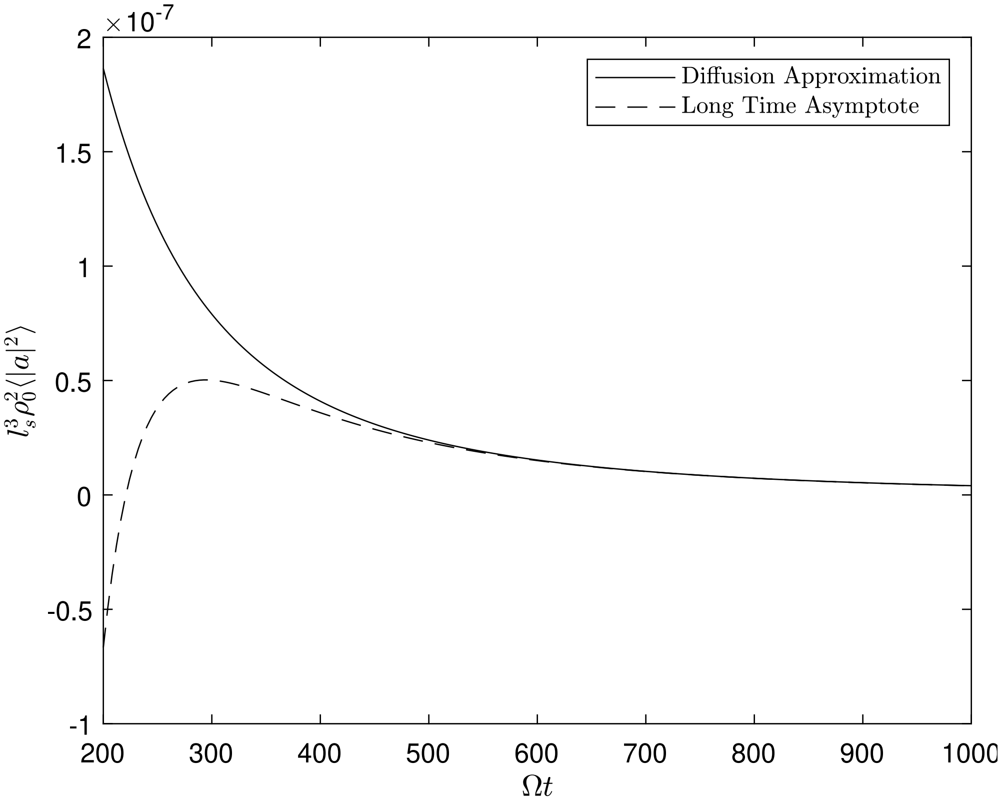


Figure 4.8: Comparison of diffusion approximation and long-time asymptote for $\langle|a|^2\rangle$.

$$
\begin{aligned}
H_A|\Psi\rangle &= \hbar \int d^3x d^3x_1 d^3x_2 \left\{\Omega\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x})\right\} \\
&\times \{\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
&+ \psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
&+ a(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2)\}|0\rangle \\
&= \hbar \int d^3x d^3x_1 d^3x_2 \{\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\Omega\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x})\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
&+ a(\mathbf{x}_1,\mathbf{x}_2,t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\Omega\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x})\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2)\ .
\end{aligned}
\tag{4.185}
$$

$$
\begin{aligned}
H_I|\Psi\rangle = \hbar \int & d^3x d^3x_1 d^3x_2 \left\{g\rho(\mathbf{x})\left(\varphi^\dagger(\mathbf{x})\sigma(\mathbf{x}) + \varphi(\mathbf{x})\sigma^\dagger(\mathbf{x})\right)\right\} \\
& \times \{\psi_2(\mathbf{x}_1, \mathbf{x}_2, t)\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) + \psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
& + a(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2)\}|0\rangle \\
& = \hbar \int d^3x d^3x_1 d^3x_2 \{\psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1) g\rho(\mathbf{x})\varphi^\dagger(\mathbf{x})\sigma(\mathbf{x})\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
& + a(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2) g\rho(\mathbf{x})\varphi^\dagger(\mathbf{x})\sigma(\mathbf{x})\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2) \\
& + \psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1) g\rho(\mathbf{x})\varphi(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
& + \psi_2(\mathbf{x}_1, \mathbf{x}_2, t) g\rho(\mathbf{x})\varphi(\mathbf{x})\sigma^\dagger(\mathbf{x})\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2)\}|0\rangle \ . \qquad (4.186)
\end{aligned}
$$

Combining these results and using the commutation and anticommutation relations (4.2) and (4.7) we arrive at

$$
\begin{aligned}
& H|\Psi\rangle \\
& = \hbar \int d^3x_1 d^3x_2 \{(c(-\Delta_{\mathbf{x}_1})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_2(\mathbf{x}_1, \mathbf{x}_2, t))\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
& + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) + \psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\Omega\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
& + 2a(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\Omega\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2) + \psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1) g\varphi^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2) \\
& - 2ga(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\varphi^\dagger(\mathbf{x}_2)\sigma^\dagger(\mathbf{x}_1) + g\psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2) \\
& + 2g\psi_2(\mathbf{x}_1, \mathbf{x}_2, t)\rho(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_1)\varphi^\dagger(\mathbf{x}_2)\}|0\rangle. \qquad (4.187)
\end{aligned}
$$

Computing the inner products

$$
\langle 0|\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2) i\hbar\partial_t|\Psi\rangle = 2i\hbar\partial_t\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) \ , \qquad (4.188)
$$

$$
\begin{aligned}
\langle 0|\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2) H|\Psi\rangle = & \ 2(c(-\Delta_{\mathbf{x}_1})^{1/2} + c(-\Delta_{\mathbf{x}_2})^{1/2})\psi_2(\mathbf{x}_1, \mathbf{x}_2, t) \\
& + g\rho(\mathbf{x}_1)\psi_1(\mathbf{x}_1, \mathbf{x}_2, t) + g\rho(\mathbf{x}_2)\psi_1(\mathbf{x}_2, \mathbf{x}_1, t) \ , \qquad (4.189)
\end{aligned}
$$

yields (4.14). The inner products

$$\langle 0|\varphi(\mathbf{x}_2)\sigma(\mathbf{x}_1)\rho(\mathbf{x}_1)i\hbar\partial_t|\Psi\rangle = \rho(\mathbf{x}_1)i\hbar\partial_t\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\ , \tag{4.190}$$

$$\begin{aligned}\langle 0|\varphi(\mathbf{x}_2)\sigma(\mathbf{x}_1)\rho(\mathbf{x}_1)H|\Psi\rangle &= (c(-\Delta_{\mathbf{x}_2})^{1/2}+\Omega)\rho(\mathbf{x}_1)\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\\ &\quad + 2g\rho(\mathbf{x}_1)\psi_2(\mathbf{x}_1,\mathbf{x}_2,t) - 2g\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)a(\mathbf{x}_1,\mathbf{x}_2,t)\ ,\end{aligned} \tag{4.191}$$

give (4.15), while

$$\langle 0|\sigma(\mathbf{x}_1)\sigma(\mathbf{x}_2)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)i\hbar\partial_t|\Psi\rangle = 2\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)i\hbar\partial_t a(\mathbf{x}_1,\mathbf{x}_2,t)\ , \tag{4.192}$$

$$\begin{aligned}\langle 0|\sigma(\mathbf{x}_1)\sigma(\mathbf{x}_2)\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)H|\Psi\rangle &= g\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\\ &\quad - g\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\psi_1(\mathbf{x}_2,\mathbf{x}_1,t)\\ &\quad + 4\Omega\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)a(\mathbf{x}_1,\mathbf{x}_2,t)\ ,\end{aligned} \tag{4.193}$$

gives (4.16).

4.7.2 Derivation of the Liouville Equation (4.61)

Here we derive (4.61). We first define

$$\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t) = \mathbf{\Psi}_\epsilon(\mathbf{x_1}-\epsilon\mathbf{x_1'}/\mathbf{2},\mathbf{x_2}-\epsilon\mathbf{x_2'}/\mathbf{2},\mathbf{t})\mathbf{\Psi}_\epsilon^\dagger(\mathbf{x_1}+\epsilon\mathbf{x_1'}/\mathbf{2},\mathbf{x_2}+\epsilon\mathbf{x_2'}/\mathbf{2},\mathbf{t})\ . \tag{4.194}$$

The Wigner transform is defined by

$$W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2,t) = \int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\ . \tag{4.195}$$

We begin by computing $i\epsilon\partial_t W_\epsilon$:

$$
\begin{aligned}
i\epsilon\partial_t W_\epsilon &= \int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}i\epsilon\partial_t\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\\
&= \int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\Big[\Big\{A_\epsilon(\mathbf{x}_1-\epsilon\mathbf{x}_1'/2,\mathbf{x}_2-\epsilon\mathbf{x}_2'/2)\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\\
&+\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}(\eta(\mathbf{x}_1/\epsilon-\mathbf{x}_1'/2)+\eta(\mathbf{x}_2/\epsilon-\mathbf{x}_2'/2))K\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\Big\}\\
&-\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)A_\epsilon(\mathbf{x}_1+\epsilon\mathbf{x}_1'/2,\mathbf{x}_2+\epsilon\mathbf{x}_2'/2)\\
&+\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}(\eta(\mathbf{x}_1/\epsilon+\mathbf{x}_1'/2)+\eta(\mathbf{x}_2/\epsilon+\mathbf{x}_2'/2))\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)K^T\Big\}\Big]\\
&= \int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\Big\{A_\epsilon(\mathbf{x}_1-\epsilon\mathbf{x}_1'/2,\mathbf{x}_2-\epsilon\mathbf{x}_2'/2)\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\\
&-\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)A_\epsilon(\mathbf{x}_1+\epsilon\mathbf{x}_1'/2,\mathbf{x}_2+\epsilon\mathbf{x}_2'/2)\Big\}\\
&+\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\Big\{\eta(\mathbf{x}_1/\epsilon-\mathbf{x}_1'/2)K\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\\
&-\eta(\mathbf{x}_1/\epsilon+\mathbf{x}_1'/2)\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)K^T\Big\}\\
&+\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\Big\{\eta(\mathbf{x}_2/\epsilon-\mathbf{x}_2'/2)K\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\\
&-\eta(\mathbf{x}_2/\epsilon+\mathbf{x}_2'/2)\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)K^T\Big\}\,. \qquad (4.196)
\end{aligned}
$$

The first term becomes

$$
\begin{aligned}
&\int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\Big\{A_\epsilon(\mathbf{x}_1-\epsilon\mathbf{x}_1'/2,\mathbf{x}_2-\epsilon\mathbf{x}_2'/2)\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\\
&-\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)A_\epsilon(\mathbf{x}_1+\epsilon\mathbf{x}_1'/2,\mathbf{x}_2+\epsilon\mathbf{x}_2'/2)\Big\}\\
&= \int \frac{d^3k_1'}{(2\pi)^3}\frac{d^3k_2'}{(2\pi)^3}e^{i\mathbf{x}_1\cdot\mathbf{k}_1'+i\mathbf{x}_2\cdot\mathbf{k}_2'}\Big\{\hat{A}(\mathbf{k}_1-\epsilon\mathbf{k}_1'/2,\mathbf{k}_2-\epsilon\mathbf{k}_2'/2)\hat{W}_\epsilon(\mathbf{k}_1',\mathbf{k}_1,\mathbf{k}_2',\mathbf{k}_2)\\
&-\hat{W}_\epsilon(\mathbf{k}_1',\mathbf{k}_1,\mathbf{k}_2',\mathbf{k}_2)\hat{A}(\mathbf{k}_1+\epsilon\mathbf{k}_1'/2,\mathbf{k}_2+\epsilon\mathbf{k}_2'/2)\Big\}\;, \qquad (4.197)
\end{aligned}
$$

where

$$(A_\epsilon f)(\mathbf{x}_1,\mathbf{x}_2) = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} e^{i\mathbf{x}_1\cdot\mathbf{k}_1+i\mathbf{x}_2\cdot\mathbf{k}_2}\hat{A}(\epsilon\mathbf{k}_1,\epsilon\mathbf{k}_2)\hat{f}(\mathbf{k}_1,\mathbf{k}_2)\ . \tag{4.198}$$

Likewise, the second term becomes

$$\begin{aligned}
&\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\left\{\eta(\mathbf{x}_1/\epsilon-\mathbf{x}_1'/2)K\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\right.\\
&\left.-\eta(\mathbf{x}_1/\epsilon+\mathbf{x}_1'/2)\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)K^T\right\}\\
&=\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'+i\mathbf{q}\cdot(\mathbf{x}_1/\epsilon-\mathbf{x}_1'/2)}\hat{\eta}(\mathbf{q})K\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\\
&-\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'+i\mathbf{q}\cdot(\mathbf{x}_1/\epsilon+\mathbf{x}_1'/2)}\hat{\eta}(\mathbf{q})\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)K^T\\
&=\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}_1/\epsilon}\hat{\eta}(\mathbf{q})KW_\epsilon(\mathbf{x}_1,\mathbf{k}_1+\mathbf{q}/2,\mathbf{x}_2,\mathbf{k}_2,t)\\
&-\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}_1/\epsilon}\hat{\eta}(\mathbf{q})W_\epsilon(\mathbf{x}_1,\mathbf{k}_1-\mathbf{q}/2,\mathbf{x}_2,\mathbf{k}_2,t)K^T\\
&=\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}_1/\epsilon}\hat{\eta}(\mathbf{q})\\
&\times\left[KW_\epsilon(\mathbf{x}_1,\mathbf{k}_1+\mathbf{q}/2,\mathbf{x}_2,\mathbf{k}_2,t)-W_\epsilon(\mathbf{x}_1,\mathbf{k}_1-\mathbf{q}/2,\mathbf{x}_2,\mathbf{k}_2,t)K^T\right].
\end{aligned}\tag{4.199}$$

Finally, the third term becomes

$$\begin{aligned}
&\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3x_1'}{(2\pi)^3}\frac{d^3x_2'}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1'-i\mathbf{k}_2\cdot\mathbf{x}_2'}\left\{\eta(\mathbf{x}_2/\epsilon-\mathbf{x}_2'/2)K\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)\right.\\
&\left.-\eta(\mathbf{x}_2/\epsilon+\mathbf{x}_2'/2)\Phi_\epsilon(\mathbf{x}_1,\mathbf{x}_1',\mathbf{x}_2,\mathbf{x}_2',t)K^T\right\}\\
&=\sqrt{\epsilon}g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}_1/\epsilon}\hat{\eta}(\mathbf{q})\\
&\times\left[KW_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2+\mathbf{q}/2,t)-W_\epsilon(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2-\mathbf{q}/2,t)K^T\right].
\end{aligned}\tag{4.200}$$

4.7.3 Derivation of the Kinetic Equation (4.85)

Here we derive the kinetic equation (4.85) which is satisfied by the quantity a_+. The first two terms are elementary and so we must compute

$$\langle \mathbf{v}_+^T L_1 W_1 \mathbf{v}_+ \rangle \, , \quad \langle \mathbf{v}_+^T L_2 W_1 \mathbf{v}_+ \rangle \, . \tag{4.201}$$

We compute each of the above in two steps. The first term is

$$\begin{aligned}
&\langle \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_1} \, \hat{\eta}(\mathbf{q}) \left[\mathbf{v}_+^T(\mathbf{k}_1, \mathbf{k}_2) K W_1(\mathbf{x}_1, \mathbf{X}_1, \mathbf{k}_1 + \mathbf{q}/2, \mathbf{x}_2, \mathbf{X}_2, \mathbf{k}_2) \mathbf{v}_+(\mathbf{k}_1, \mathbf{k}_2) \right] \rangle \\
=&\langle \int \frac{d^3q}{(2\pi)^3} \frac{d^3K_1}{(2\pi)^3} \frac{d^3K_2}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_1 + i\mathbf{K}_1\cdot\mathbf{X}_1 + i\mathbf{K}_2\cdot\mathbf{X}_2} \, \hat{\eta}(\mathbf{q}) \\
&\times \left[\mathbf{v}_+^T(\mathbf{k}_1, \mathbf{k}_2) K \hat{W}_1(\mathbf{x}_1, \mathbf{K}_1, \mathbf{k}_1 + \mathbf{q}/2, \mathbf{x}_2, \mathbf{K}_2, \mathbf{k}_2) \mathbf{v}_+(\mathbf{k}_1, \mathbf{k}_2) \right] \rangle \\
=&\langle \int \frac{d^3q}{(2\pi)^3} \frac{d^3K_1}{(2\pi)^3} \frac{d^3K_2}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_1 + i\mathbf{K}_1\cdot\mathbf{X}_1 + i\mathbf{K}_2\cdot\mathbf{X}_2} \, \hat{\eta}(\mathbf{q}) \mathbf{v}_+^T(\mathbf{k}_1, \mathbf{k}_2) K \\
&\times \Bigg\{ \sum_{m,n=\pm} w_{m,n}(\mathbf{K}_1, \mathbf{k}_1 + \mathbf{q}/2, \mathbf{K}_2, \mathbf{k}_2) \mathbf{v}_m(\mathbf{k}_1 + \mathbf{q}/2 - \mathbf{K}_1/2, \mathbf{k}_2 - \mathbf{K}_2/2) \\
&\times \mathbf{v}_n^T(\mathbf{k}_1 + \mathbf{q}/2 + \mathbf{K}_1/2, \mathbf{k}_2 + \mathbf{K}_2/2) \mathbf{v}_+(\mathbf{k}_1, \mathbf{k}_2) \Bigg\} \rangle .
\end{aligned} \tag{4.202}$$

Next we substitute the expression $w_{m,n}$ using equation (4.81) to arrive at

$$\Big\langle \int \frac{d^3q}{(2\pi)^3}\frac{d^3K_1}{(2\pi)^3}\frac{d^3K_2}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{X}_1+i\mathbf{K}_1\cdot\mathbf{X}_1+i\mathbf{K}_2\cdot\mathbf{X}_2}\,\hat{\eta}(\mathbf{q})\mathbf{v}_+^T(\mathbf{k}_1,\mathbf{k}_2)K$$
$$\times\left\{\sum_{m,n=\pm} g\sqrt{\frac{\rho_0}{2}}(2\pi)^3\left\{\eta(\mathbf{K}_1)\delta(\mathbf{K}_2)+\eta(\mathbf{K}_2)\delta(\mathbf{K}_1)\right\}\right\}$$
$$\times\Bigg(\frac{a_m(\mathbf{k}_1+\mathbf{q}/2-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)}{\lambda_m(\mathbf{k}_1+\mathbf{q}/2-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)-\lambda_n(\mathbf{k}_1+\mathbf{q}/2+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2)+i\theta}$$
$$\times K_{m,n}(\mathbf{k}_1+\mathbf{q}/2-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2,\mathbf{k}_1+\mathbf{q}/2+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2)$$
$$-\frac{a_n(\mathbf{k}_1+\mathbf{q}/2+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2)}{\lambda_m(\mathbf{k}_1+\mathbf{q}/2-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)-\lambda_n(\mathbf{k}_1+\mathbf{q}/2+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2)+i\theta}$$
$$\times K_{m,n}(\mathbf{k}_1+\mathbf{q}/2+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2,\mathbf{k}_1+\mathbf{q}/2-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)\Big)$$
$$\times\,\mathbf{v}_m(\mathbf{k}_1+\mathbf{q}/2-\mathbf{K}_1/2,\mathbf{k}_2-\mathbf{K}_2/2)\mathbf{v}_n^T(\mathbf{k}_1+\mathbf{q}/2+\mathbf{K}_1/2,\mathbf{k}_2+\mathbf{K}_2/2)\mathbf{v}_+(\mathbf{k}_1,\mathbf{k}_2)\Big\rangle.$$

We separate the above into two terms and use the orthogonality of the basis $\{\mathbf{v}_i\}$ to see that $n=+$ in the first term. We thus obtain

$$=g\sqrt{\frac{\rho_0}{2}}\int\frac{d^3K_1}{(2\pi)^3}\,\hat{C}(\mathbf{K}_1)\sum_{m=\pm}\Bigg(\frac{\left[a_m(\mathbf{k}_1-\mathbf{K}_1,\mathbf{k}_2)K_{m,+}(\mathbf{k}_1-\mathbf{K}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)\right]}{\lambda_m(\mathbf{k}_1-\mathbf{K}_1,\mathbf{k}_2)-\lambda_+(\mathbf{k}_1,\mathbf{k}_2)+i\theta}$$
$$-\frac{\left[a_+(\mathbf{k}_1,\mathbf{k}_2)K_{m,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{K}_1,\mathbf{k}_2)\right]}{\lambda_m(\mathbf{k}_1-\mathbf{K}_1,\mathbf{k}_2)-\lambda_+(\mathbf{k}_1,\mathbf{k}_2)+i\theta}\Bigg)K_{+,m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{K}_1,\mathbf{k}_2)$$
$$+g\sqrt{\frac{\rho_0}{2}}\int\frac{d^3K_2}{(2\pi)^3}\,e^{i\mathbf{K}_2\cdot(\mathbf{X}_2-\mathbf{X}_1)}\hat{C}(\mathbf{K}_2)\mathbf{v}_+^T(\mathbf{k}_1,\mathbf{k}_2)K\sum_{m,n=\pm}$$
$$\times\Bigg(\frac{\left[a_m(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2-\mathbf{K}_2/2)K_{m,n}(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2-\mathbf{K}_2/2,\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2+\mathbf{K}_2/2)\right]}{\lambda_m(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2-\mathbf{K}_2/2)-\lambda_n(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2+\mathbf{K}_2/2)+i\theta}$$
$$-\frac{\left[a_n(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2+\mathbf{K}_2/2)K_{m,n}(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2+\mathbf{K}_2/2,\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2-\mathbf{K}_2/2)\right]}{\lambda_m(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2-\mathbf{K}_2/2)-\lambda_n(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2+\mathbf{K}_2/2)+i\theta}\Bigg)$$
$$\times\,\mathbf{v}_m(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2-\mathbf{K}_2/2)\mathbf{v}_n^T(\mathbf{k}_1-\mathbf{K}_2/2,\mathbf{k}_2+\mathbf{K}_2/2)\mathbf{v}_+(\mathbf{k}_1,\mathbf{k}_2). \tag{4.203}$$

As $\epsilon \to 0$ the above converges to

$$g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_1-\mathbf{K})\sum_{m=\pm}\left(\frac{\left[a_m(\mathbf{K},\mathbf{k}_2)K_{m,+}(\mathbf{K},\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)\right]}{\lambda_m(\mathbf{K},\mathbf{k}_2)-\lambda_+(\mathbf{k}_1,\mathbf{k}_2)+i\theta}\right. \tag{4.204}$$

$$\left.-\ \frac{\left[a_+(\mathbf{k}_1,\mathbf{k}_2)K_{m,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)\right]}{\lambda_m(\mathbf{K},\mathbf{k}_2)-\lambda_+(\mathbf{k}_1,\mathbf{k}_2)+i\theta}\right)K_{+,m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)\ , \tag{4.205}$$

which follows from the Riemann-Lebesgue Lemma and the fact that W_0 is independent of the fast variables $\mathbf{X}_1$ and $\mathbf{X}_2$. Similarly the other three terms become

$$\begin{aligned}
&-g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_1-\mathbf{K})\sum_{n=\pm}\left(\frac{\left[a_+(\mathbf{k}_1,\mathbf{k}_2)K_{+,n}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)\right]}{\lambda_+(\mathbf{k}_1,\mathbf{k}_2)-\lambda_n(\mathbf{K},\mathbf{k}_2)+i\theta}\right.\\
&\left.-\ \frac{\left[a_n(\mathbf{K},\mathbf{k}_2)K_{+,n}(\mathbf{K},\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)\right]}{\lambda_+(\mathbf{k}_1,\mathbf{k}_2)-\lambda_n(\mathbf{K},\mathbf{k}_2)+i\theta}\right)K_{+,n}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)\\
&+g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_2-\mathbf{K})\sum_{m=\pm}\left(\frac{\left[a_m(\mathbf{k}_1,\mathbf{K})K_{m,+}(\mathbf{k}_1,\mathbf{K},\mathbf{k}_1,\mathbf{k}_2)\right]}{\lambda_m(\mathbf{k}_1,\mathbf{K})-\lambda_+(\mathbf{k}_1,\mathbf{k}_2)+i\theta}\right.\\
&\left.-\ \frac{\left[a_+(\mathbf{k}_1,\mathbf{k}_2)K_{m,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K})\right]}{\lambda_m(\mathbf{k}_1,\mathbf{K})-\lambda_+(\mathbf{k}_1,\mathbf{k}_2)+i\theta}\right)K_{+,m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K})\\
&-g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_2-\mathbf{K})\sum_{+,n=\pm}\left(\frac{\left[a_+(\mathbf{k}_1,\mathbf{k}_2)K_{+,n}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K})\right]}{\lambda_+(\mathbf{k}_1,\mathbf{k}_2)-\lambda_n(\mathbf{k}_1,\mathbf{K})+i\theta}\right.\\
&\left.-\ \frac{\left[a_n(\mathbf{k}_1,\mathbf{K})K_{+,n}(\mathbf{k}_1,\mathbf{K},\mathbf{k}_1,\mathbf{k}_2)\right]}{\lambda_+(\mathbf{k}_1,\mathbf{k}_2)-\lambda_n(\mathbf{k}_1,\mathbf{K})+i\theta}\right)K_{+,n}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K}).
\end{aligned} \tag{4.206}$$

Making use of the identity

$$\lim_{\theta\to 0}\left(\frac{1}{x-i\theta}-\frac{1}{x+i\theta}\right)=2\pi i\delta(x)\ , \tag{4.207}$$

we see that we must have $m, n = +$, to ensure that the support of the delta function is nonempty since λ_+ and λ_- are never equal. Thus the above equation simplifies as

$$
\begin{aligned}
&-i\pi g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_1-\mathbf{K})\delta(\lambda_+(\mathbf{K},\mathbf{k}_2)-\lambda_+(\mathbf{k}_1,\mathbf{k}_2))\\
&\times\Big(a_+(\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)^2-a_+(\mathbf{K},\mathbf{k}_2)K_{+,+}(\mathbf{K},\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)\Big)\\
&-i\pi g\sqrt{\frac{\rho_0}{2}}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_2-\mathbf{K})\delta(\lambda_+(\mathbf{k}_1,\mathbf{K})-\lambda_+(\mathbf{k}_1,\mathbf{k}_2))\\
&\times\Big(a_+(\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K})^2-a_+(\mathbf{k}_1,\mathbf{K})K_{+,+}(\mathbf{k}_1,\mathbf{K},\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K})\Big)\,.
\end{aligned}
\tag{4.208}
$$

Putting everything together, we see that a_+ satisfies the equation

$$
\begin{aligned}
&\frac{1}{c}\partial_t a_+ + \left[\frac{(\lambda_+(\mathbf{k}_1,\mathbf{k}_2)-d(\mathbf{k}_1,\mathbf{k}_2)-\Omega)^2+g^2\rho_0}{(\lambda_+(\mathbf{k}_1,\mathbf{k}_2)-d(\mathbf{k}_1,\mathbf{k}_2)-\Omega)^2+2g^2\rho_0}\right]\Big(\hat{\mathbf{k}}_1\cdot\nabla_{\mathbf{x}_1}+\hat{\mathbf{k}}_2\cdot\nabla_{\mathbf{x}_2}\Big)\,a_+\\
=&-\pi g\frac{\rho_0}{2}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_1-\mathbf{K})\delta(\lambda_+(\mathbf{K},\mathbf{k}_2)-\lambda_+(\mathbf{k}_1,\mathbf{k}_2)) \qquad (4.209)\\
&\times\Big(a_+(\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)^2-a_+(\mathbf{K},\mathbf{k}_2)K_{+,+}(\mathbf{K},\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{K},\mathbf{k}_2)\Big)\\
&-\pi g^2\frac{\rho_0}{2}\int \frac{d^3K}{(2\pi)^3}\,\hat{C}(\mathbf{k}_2-\mathbf{K})\delta(\lambda_+(\mathbf{k}_1,\mathbf{K})-\lambda_+(\mathbf{k}_1,\mathbf{k}_2))\\
&\times\Big(a_+(\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K})^2-a_+(\mathbf{k}_1,\mathbf{K})K_{+,+}(\mathbf{k}_1,\mathbf{K},\mathbf{k}_1,\mathbf{k}_2)K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{K})\Big)\,.
\end{aligned}
\tag{4.210}
$$

The delta function can be simplified using the identity

$$
\delta(g(x)) = \frac{\delta(x-x_0)}{|g'(x_0)|} \tag{4.211}
$$

where g has a single real root at $x = x_0$. Thus

$$\begin{aligned}
&\delta(\lambda_+(\mathbf{K},\mathbf{k}_2) - \lambda_+(\mathbf{k}_1,\mathbf{k}_2)) \\
&= \frac{\delta(|\mathbf{K}| - |\mathbf{k}_1|)}{\frac{c}{4}\left|c(|\mathbf{k}| + |\mathbf{k}_2|)/2 - \Omega + \frac{3}{2}\sqrt{(c(|\mathbf{k}| + |\mathbf{k}_2|)/2 - \Omega)^2 + 8g^2\rho_0}\right|} \,. \qquad (4.212)
\end{aligned}$$

Hence (4.209) becomes

$$\begin{aligned}
&\frac{1}{c}\partial_t a_+ + \left[\frac{(\lambda_+(\mathbf{k}_1,\mathbf{k}_2) - d(\mathbf{k}_1,\mathbf{k}_2) - \Omega)^2 + g^2\rho_0}{(\lambda_+(\mathbf{k}_1,\mathbf{k}_2) - d(\mathbf{k}_1,\mathbf{k}_2) - \Omega)^2 + 2g^2\rho_0}\right]\left(\hat{\mathbf{k}}_1\cdot\nabla_{\mathbf{x}_1} + \hat{\mathbf{k}}_2\cdot\nabla_{\mathbf{x}_2}\right)a_+ \\
&= -\frac{2g^2\rho_0\pi}{c}\frac{K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)^2}{\left|d(\mathbf{k}_1,\mathbf{k}_2) - \Omega + \frac{3}{2}\sqrt{(d(\mathbf{k}_1,\mathbf{k}_2) - \Omega)^2 + 8g^2\rho_0}\right|}|\mathbf{k}_1|^2 \\
&\quad\times\int\frac{d\hat{\mathbf{k}}'}{(2\pi)^3}\,\hat{C}(|\mathbf{k}_1|(\hat{\mathbf{k}}_1 - \hat{\mathbf{k}}'))\left(a_+(\mathbf{k}_1,\mathbf{k}_2) - a_+(|\mathbf{k}_1|\hat{\mathbf{k}}',\mathbf{k}_2)\right) \\
&\quad-\frac{2g^2\rho_0\pi}{c}\frac{K_{+,+}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)^2}{\left|d(\mathbf{k}_1,\mathbf{k}_2) - \Omega + \frac{3}{2}\sqrt{(d(\mathbf{k}_1,\mathbf{k}_2) - \Omega)^2 + 8g^2\rho_0}\right|}|\mathbf{k}_2|^2 \\
&\quad\times\int\frac{d\hat{\mathbf{k}}'}{(2\pi)^3}\,\hat{C}(|\mathbf{k}_2|(\hat{\mathbf{k}}_2 - \hat{\mathbf{k}}'))\left(a_+(\mathbf{k}_1,\mathbf{k}_2) - a_+(\mathbf{k}_1,|\mathbf{k}_2|\hat{\mathbf{k}}')\right)\,, \qquad (4.213)
\end{aligned}$$

as desired.

Chapter 5

Future Work

The model developed in the beginning of Chapter 3 is may be applied to many physical scenarios and thus contains many possible directions for future research. Below we briefly describe several of these potential projects.

5.1 Reintroduce Polarization

While the model introduced in Chapters 3 considers a scalar model of an electromagnetic field, we could reintroduce the polarization of the field back into the model in order to study the effects on the dynamics. This would allow us to have the coupling constant g depend on the polarization while still working within the Markovian approximation. One other aspect of such a model is that the differential operator $(-\Delta)^{1/2}$ which is nonlocal in our scalar model could be traded out for a local operator.

5.2 More General Coupling

Additionally, in the model presented in this paper we have made the Markovian approximation, which replaces a wave number dependent coupling $g_{\mathbf{k}}$ for a constant strength of coupling between the matter and all modes of the electromagnetic field. There are several ways that we could consider a more general version of coupling between the field and the collection of atoms.

- Reintroduce the $\mathbf{k}$ dependence on g before transferring the model to real space.
- Introduce a spatially dependent coupling constant $g(\mathbf{x})$.
- Introduce a time dependent coupling $g(t)$. One such time dependent coupling would be modeled as a pulse train where a compactly supported function $g_0(t)$ is repeated at regular or stochastic time intervals.

5.3 Other Spatial Distributions of Matter

While chapters 3 and 4 consider a medium consisting of two level atoms fluctuating randomly about a density ρ_0, there are many other distributions $\rho(\mathbf{x})$ to consider. These can be broken down into two main categories:

1. Deterministic $\rho(\mathbf{x})$
2. Stochastic $\rho(\mathbf{x})$

One extremely interesting example of a deterministic density is the case of periodic $\rho(\mathbf{x})$ which is supported on a Bravais lattice. In this setting one may investigate the corresponding band structure of the Hamiltonian. A key example of such a lattice is the hexagonal honeycomb lattice.

A related phenomenon occurs in media that vary on small scales, one can derive effective equations for the behavior of waves in the medium on larger scales. This process of replacing a rapidly oscillating coefficient by an effective overall coefficient is known as homogenization. Such a constant coefficient system of equations could be derived for the systems studied in chapters 3 and 4 given that the density $\rho(\mathbf{x})$ oscillates rapidly.

Many other stochastic models may be considered as well. One such generalization would be to consider a medium which is not isotropic and statistically homogeneous. In terms of the correlation function, this would mean removing the assumption that $C(\mathbf{x}, \mathbf{y})$ depends only on the quantity $|\mathbf{x} - \mathbf{y}|$. Additionally, while the work presented in this thesis primarily deals with densities which fluctuate about an average value ρ_0, one could study many other models of random densities. Examples include perturbations of periodic densities where the location of the sites are given by random variables.

5.4 Time Dependent Distribution of Matter

In a similar vein to the previous subsection, it would be quite interesting to alter the model in such a way that allowed for the density to depend on space and time $\rho(\mathbf{x}, t)$. This would make it possible to study the effect of the motion of the atoms on the transport or diffusion of the excitations in the electromagnetic field. Additionally, one may be able to derive a transport equation rigorously in this case as long as there is a sufficient mixing hypothesis placed on the time dependence of the density.

In order to introduce such a density one would need to go back and reconfigure the Hamiltonian instead of altering the equations of motion directly. It seems that the right approach would be to make the operators $\varphi(\mathbf{x})$ and $\sigma(\mathbf{x})$ time dependent and work in the Heisenberg picture of quantum mechanics.

5.5 Nonlinear Hamiltonians

Incorporating terms which are nonlinear in the field operators would allow for the consideration of particle particle interactions in the model. This would likely lead to systems of coupled nonlinear pseudodifferential equations akin to the focusing or defocusing nonlinear Schrödinger equations (NLS). There has been some work on the derivation of transport and diffusion equations from the NLS and analogous questions for the model presented here are quite interesting.

5.6 Coherent States

Another direction to go in is to consider the dynamics associated to different initial states of the electromagnetic field. One important set of states are coherent states which are most similar to classical states of light. Comparing the dynamics of states which are initially coherent to that of states which are initially entangled could provide insight into the effect of entanglement on the transport of light.

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