Moduli of Curves at Infinite Level

by

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To my parents

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ABSTRACT

We present some new results for perfectoid rings and spaces and use them to study moduli of the following classes of complex algebraic curves: smooth, compact type, and stable. Full level-n structures on such curves are trivializations of the n-torsion points of their Jacobians. We give an algebraic proof that the étale cohomology groups of all three moduli spaces vanish in high degrees at "infinite level." For smooth curves, this yields a new perspective on a result of Harer who showed such vanishing already at finite level using topological methods. The statements for stable curves and curves of compact type are not covered by Harer's methods. Two of the main ingredients in the proofs are a vanishing statement for certain constructible sheaves on perfectoid spaces and a comparison of the étale cohomology groups of different towers of Deligne–Mumford stacks in the presence of ramification.

CHAPTER 1

Introduction

In 2011, P. Scholze introduced the theory of perfectoid spaces in order to relate questions about number systems in mixed and positive characteristics [Sch12]. Although his initial motivation arose from problems in arithmetic geometry, perfectoid spaces have since seen applications to a wide scope of mathematical areas, such as algebraic geometry [CS19], commutative algebra [And18b, Bha18b], number theory and the Langlands program [Sch15], p-adic Hodge theory [Sch13a, BMS18, BS19], algebraic topology [BMS19], and algebraic K-theory [BCM20]. The purpose of this thesis is twofold:

1. to explain some new results for perfectoid rings and spaces;
2. to apply some of these results to an object of classical algebraic geometry, the moduli space of algebraic curves, also known as closed Riemann surfaces.

We begin with a brief overview of the motivation behind these applications.

1.1 Moduli of curves and their cohomology

Riemann surfaces lie at the crossroads of many different mathematical disciplines; their study extends far across the boundaries of the subject, reaching applications in fields such as physics or cryptography. Informally, one can think of closed Riemann surfaces as two-dimensional objects in the shape of a donut with several "holes," or "handles," together with a way to measure lengths and angles between curves on them. This so-called conformal structure arises from an additional structure over the complex numbers.

A mathematically precise characterization is the following.

Definition 1.1.1. A closed Riemann surface is a one-dimensional, compact manifold over the complex numbers $\mathbf{C}$ without boundary.

Note that since complex numbers are parametrized by two different real parameters—the real and the imaginary part—, a "real-world surface" must be a space of complex dimension

1. By the uniformization theorem, all closed Riemann surfaces are zero sets of polynomials in several complex variables. Thus, they fall under the umbrella of algebraic geometry, where they are called complex algebraic curves and have been studied extensively.

Moduli of curves provide a way to classify all algebraic curves. A first attempt at such a classification might use the genus, an integer which counts the number of holes. For a more detailed analysis, we fix an integer $g \geq 2$ and classify all algebraic curves of fixed genus g. This can be done via a space (more precisely, a complex orbifold) $\mathscr{M}_g(\mathbf{C})$ of complex dimension $3g-3$. The various points of $\mathscr{M}_g(\mathbf{C})$ correspond to all different curves of genus g. Similarly, lines inside $\mathscr{M}_g(\mathbf{C})$ correspond to families of curves varying over a line, and so on.

This thesis studies an invariant of $\mathscr{M}_g(\mathbf{C})$ called its cohomology. A strong interest in this invariant dates back as far as the 1980s. Its computation could, for example, help to understand the enumerative geometry of $\mathscr{M}_g(\mathbf{C})$ (cf. e.g. [Mum83]) or predict the global structure of $\mathscr{M}_g(\mathbf{C})$ (e.g., Looijenga's conjecture that the coarse space $M_g(\mathbf{C})$ can be covered by $g-1$ open affine subschemes). Unfortunately, only little is known about this invariant for general g. In low degrees, genus-independent computations of the rational singular cohomology $\mathrm{H}^i(\mathscr{M}_g(\mathbf{C}), \mathbf{Q})$ are due to Mumford [Mum67, Thm. 1] ($i = 1$) and Harer [Har83] ($i = 2$) [Har91] ($i = 2, 3$). On the other extreme, Harer showed the following.

Theorem 1.1.2 ([Har86, Cor. 4.3])**.** *For $i > 4g-5$, we have* $\mathrm{H}^i(\mathscr{M}_g(\mathbf{C}), \mathbf{Q}) = 0$.

A common thread in all these computations is the pervasive use of tools from geometric topology and Teichmüller theory, i.e., using the construction of $\mathscr{M}_g(\mathbf{C})$ as the quotient of contractible Teichmüller space $\mathscr{T}_g$ by the mapping class group Γ_g. There has been some interest in finding proofs that rely on more algebro-geometric methods; see e.g. [Cor98, p. 251]. For example, Arbarello–Cornalba realized that the cohomology computations in low degrees follow via Hodge theory from Theorem 1.1.2 [AC98] [AC09, Thm. 10]. In the next section, we describe a new, essentially algebro-geometric perspective on Theorem 1.1.2.

1.2 Algebraic approach

In [Har86], Harer showed in fact that the virtual cohomological dimension of Γ_g is $4g-5$. This has the following consequence: Let $n \in \mathbf{N}$ be a natural number such that $p^n \geq 3$. Then $\mathrm{H}^i(\mathscr{M}_g[p^n](\mathbf{C}), \Lambda) = 0$ for all $i > 4g-5$ and all coefficient systems Λ. Here, $\mathscr{M}_g[p^n]$ is the moduli space of smooth, complex curves of genus $g \geq 2$ with full level-p^n structure; it parametrizes smooth, complex curves C of genus $g \geq 2$ together with an isomorphism $\mathrm{H}^1(C, \mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\sim} (\mathbf{Z}/p^n\mathbf{Z})^{2g}$. The natural map $\mathscr{M}_g[p^n] \to \mathscr{M}_g$ of moduli spaces, which corresponds on the level of classification problems to "forgetting the level structure," is a finite étale cover. It thus corresponds to a finite-index subgroup of Γ_g, which turns out to be torsion-free.

We will be interested in the groups $\mathrm{H}^i(\mathscr{M}_g[p^n](\mathbf{C}), \mathbf{F}_p)$ with coefficients in the finite field $\mathbf{F}_p$, where p is a fixed prime number. Since $\dim_{\mathbf{Q}} \mathrm{H}^i(\mathscr{M}_g[p^n](\mathbf{C}), \mathbf{Q}) \leq \dim_{\mathbf{F}_p} \mathrm{H}^i(\mathscr{M}_g[p^n](\mathbf{C}), \mathbf{F}_p)$, the Cartan–Leray spectral sequence combined with the vanishing of $\mathrm{H}^i(\mathscr{M}_g[p^n](\mathbf{C}), \mathbf{F}_p)$ in degrees $i > 4g-5$ implies Theorem 1.1.2. Our main result is a vanishing statement "at infinite level" in this direction, proved without geometric topology, for $\mathscr{M}_g[p^n]$ and some (partial) compactifications thereof, which we recall in detail in § 3.2.

Theorem 1.2.1. *Let $g \geq 2$ and p be a prime. Let $\mathscr{M}[p^n]$ be one of the following:*

(i) *the moduli space $\mathscr{M}_g[p^n]$ of smooth curves of genus g over $\mathbf{C}$ with full level-p^n structure,*

(ii) *the moduli space $\mathscr{M}_g^{\mathrm{c}}[p^n]$ of curves of compact type of genus g over $\mathbf{C}$ with full level-p^n structure, or*

(iii) *the moduli space $\overline{\mathscr{M}_g}[p^n]$ of pre-level-p^n curves of genus g over $\mathbf{C}$ with full level-p^n structure.*

Then we have

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}[p^n], \mathbf{F}_p) = 0$$

for all $i > 4g-5$ in case (i) *and for all $i > \left\lfloor \frac{7g}{2} \right\rfloor - 4$ in cases* (ii) *and* (iii).

The stack $\overline{\mathscr{M}_g}[p^n]$ is a smooth, modular compactification of $\mathscr{M}_g[p^n]$ that was first introduced in [ACV03]; when $n = 0$, a pre-level-p^n curve is simply a stable curve. A curve of compact type is a stable curve whose dual graph is a tree.

Remark 1.2.2. While our bound in case (i) is the same as the one in Theorem 1.1.2, the only known upper bound on the virtual cohomological dimension (and hence vanishing at finite level) for $\mathscr{M}_g^{\mathrm{c}}$ is $5g-6$ [Mon08, Cor. 0.4] and thus significantly higher than our bound $\left\lfloor \frac{7g}{2} \right\rfloor - 4$ for vanishing at infinite level. It may be worthwile to further investigate this discrepancy.

Remark 1.2.3. The groups $\operatorname{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g[p^n], \mathbf{F}_p)$ are the mod p analogs of the completed cohomology groups for the tower of spaces

$$\cdots \to \mathscr{M}_g[p^n] \to \cdots \to \mathscr{M}_g[p] \to \mathscr{M}_g;$$

cf. [Eme06, CE12]. A dévissage argument shows that $\operatorname{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g[p^n], \mathbf{Z}/p^s\mathbf{Z}) = 0$ in degrees $i > 4g-5$ for all $s \in \mathbf{N}$ and thus the vanishing of the integral completed cohomology groups

$$\lim_s \operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g[p^n], \mathbf{Z}/p^s\mathbf{Z})$$

in this range of degrees.

The beginning of Chapter 3 outlines the strategy of the proof of Theorem 1.2.1. To conclude this introduction, we would like to highlight its connection with the theory of perfectoid spaces.

1.3 Relation to perfectoid geometry

The significance of perfectoid spaces in regard to cohomological vanishing statements was first observed by Scholze in his construction of Galois representations attached to torsion classes in the cohomology of locally symmetric spaces [Sch15]. For many moduli problems parametrized by Shimura varieties, level structures give rise to towers of moduli spaces similar to the ones from § 1.2. Scholze realized that the inverse limits of such towers are often similar to a perfectoid space "at infinite level"; in particular, the $\mathbf{F}_p$-cohomology of this perfectoid space is simply the direct limit of the $\mathbf{F}_p$-cohomology at finite levels. He then discovered a vanishing statement for perfectoid spaces arising from towers of proper varieties by establishing surprising relationships between

1. their $\mathbf{F}_p$-étale and coherent étale cohomologies, and
2. their coherent étale and coherent analytic cohomologies.

In Chapter 2, we give a new version of this vanishing result for more general coefficients. We only state a special case here and refer to Theorem 2.6.3 for the most general version.

Theorem 1.3.1. *Let $\cdots \to X_{n+1} \to X_n \to \cdots \to X_0$ be a tower of proper varieties with finite transition maps over a complete, algebraically closed extension C of $\mathbf{Q}_p$. Assume there exists a perfectoid space X over C such that $X^\diamondsuit \simeq \lim_n X_n^\diamondsuit$ as diamonds. Then for all constructible sheaves of $\mathbf{F}_p$-modules F_0 on X_0 with pullbacks F_n to X_n, we have*

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(X_n, F_n) = 0$$

for all $i > \dim \operatorname{supp}(F_0)$.

Note that [Sch15, § 4.2] treats sheaves of the form $F_0 = j_{0,!}\mathbf{F}_p$ for some dense, open $j_0 \colon U_0 \hookrightarrow X_0$. Our proof relies on the recent notions of diamonds [Sch17] and mixed characteristic perfectoidizations [BS19], which are introduced in Chapter 2. At present, it is not clear how to apply Theorem 1.3.1 to the tower from § 1.2 directly: while the diamond $\lim_n \mathscr{M}_g[p^n]^\diamondsuit$ is represented by a perfectoid space, we do not know what can be said about the compactifications $\overline{\mathscr{M}_g}[p^n]$, which are not even varieties, but genuine Deligne–Mumford stacks. Instead, we will extract cohomological information about moduli of curves via the Torelli morphism from the cohomology of moduli of principally polarized abelian varieties, which are perfectoid at infinite level by [Sch15]. In this approach, we need the full strength of Theorem 1.3.1 even if we are only interested in cohomology with $\mathbf{F}_p$-coefficients.

1.4 Notation and conventions

Throughout this paper, we fix a prime $p \in \mathbf{N}$. For any Deligne–Mumford stack $\mathscr{X}$, we denote the bounded derived category of constructible sheaves of $\mathbf{F}_p$-modules on the small étale site of $\mathscr{X}$ by $\mathrm{D}(\mathscr{X})$; for the algebraic stacks appearing in Appendix A, we use the lisse-étale site as developed in [Ols07b] or [LZ17] instead of the small étale site in the definition of $\mathrm{D}(\mathscr{X})$.

CHAPTER 2

Perfectoid geometry

The basic building block of perfectoid geometry is the class of perfectoid rings, a mixed characteristic version of the class of perfect rings from positive characteristic. In § 2.1, we motivate the definition of perfectoid rings from the viewpoint of Bhatt–Scholze's theory of prisms [BS19] in a way that we hope illuminates the analogy with perfect rings. The general structure of perfectoid rings is described in Corollary 2.1.17, which shows that perfectoid rings can be decomposed into perfect rings of characteristic p and perfectoid Tate rings in mixed characteristic.

To globalize this notion, Scholze used Huber's theory of adic spaces from nonarchimedean geometry, which we do not attempt to explain here; instead, we refer the reader to [Sch12, § 2] for a short overview and to [Con15] for a comprehensive introduction. In § 2.2, we then define perfectoid spaces, following Scholze, as adic spaces which are locally given by the adic spectrum of a perfectoid Huber pair. The highlight of this section is Scholze's tilting equivalence from [Sch12], which allowed Scholze to relate questions in positive and mixed characteristic and forms one of the keystones of the theory.

An important method later will be the perfectoidization of a p-adically complete $\mathbf{Z}_p$-algebra from [BS19], i.e., a universal morphism to a perfectoid ring, which we discuss in § 2.3. Although perfectoidizations tend to be quite unwieldy in general, we can use the structure theorem for perfectoid rings to explicitly describe the perfectoidization of $\mathbf{Z}_p^{\mathrm{cycl}}\langle x^{1/p^\infty}\rangle/(x-1)$ in Corollary 2.3.7. In § 2.3.2, we also explain a hands-on approach to perfectoidizations via the perfectoid Riemann extension theorem, which applies in certain nonarchimedean situations.

Again, the notion of perfectoidizations can be globalized; to do so, we have chosen to adopt the perspective of diamonds from [Sch17]. We present some of the salient features of this theory in § 2.4. A concrete connection to perfectoidizations appears in Proposition 2.4.20.

In § 2.5, we describe how derived categories behave in inverse systems of perfectoid rings that appear naturally in the perfectoid Riemann extension theorem. Although this section is somewhat removed from the remainder of the thesis, we hope that it may prove useful in the future with regard to questions related to the approach to perfectoidizations from § 2.3.2.

While we were not able to locate references in the literature to several of the examples and statements from the aforementioned sections, they are likely known to experts in the field. The main new results of this chapter are contained in § 2.6. There, we build on the concepts introduced so far and prove the more general version of Theorem 1.3.1 mentioned in the introduction, which will form one of the ingredients in our study of the moduli of curves.

2.1 Perfectoid rings

Geometry in positive characteristic comes with a priceless tool: the Frobenius morphism. Its ubiquity sparks interest in the following class of rings.

Definition 2.1.1. A ring R is *perfect* if its Frobenius morphism $\varphi_R\colon R \to R$ is an isomorphism.

Moreover, there is a canonical operation to pass to perfect rings.

Definition 2.1.2. The *(colimit) perfection* of a ring R of positive characteristic is given by

$$R_{\mathrm{perf}} := \operatorname{colim}(R \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} \cdots).$$

Example 2.1.3. Let $R := \mathbf{F}_p[t]$. Then $R_{\mathrm{perf}} \simeq \mathbf{F}_p[t^{1/p^\infty}]$.

Since the Frobenius morphism does not change topological or étale information, perfections can lead to elegant proofs of statements about such data even for nonperfect rings. As an example, we just mention Bhatt–Scholze's new proof of Kunz's theorem that rings of positive characteristic with flat Frobenius are regular [BS17, Cor. 11.35] (the converse is also true and straightforward). Essentially, it suffices to show that such rings have finite Tor dimension, which can be seen after passage to perfections and is easy for perfect rings.

Next, we consider an analog of perfect rings in mixed characteristic. First, we need to single out a class of rings for which it still makes sense to talk about Frobenius morphisms. Recall that we have fixed a prime $p \in \mathbf{N}$ throughout this paper.

Definition 2.1.4. Let A be a ring over $\mathbf{Z}_{(p)}$. A *Frobenius morphism* on A is a morphism $\varphi\colon A \to A$ whose reduction $\overline{\varphi}\colon A/p \to A/p$ modulo p is the Frobenius of A/p.

It is technically advantageous (e.g., for the existence of equalizers in the desired category) to record the "reason" Frobenius lifts to a mixed characteristic ring instead of simply the lift itself.

Definition 2.1.5 ([Joy85, Def. 1])**.** A *δ-ring* is a ring A together with a map of sets $\delta\colon A \to A$ such that

- $\delta(1) = \delta(0) = 0$,
- $\delta(ab) = a^p\delta(b) + \delta(a)b^p + p\delta(a)\delta(b)$, and
- $\delta(a+b) = \delta(a) + \delta(b) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a^i b^{p-i}$.

The map $\varphi\colon A \to A$ given by $\varphi(a) = a^p + p\delta(a)$ is the *Frobenius* of the δ-ring, and is a morphism thanks to the previous conditions.

When A is p-torsionfree, the δ-ring structure can be completely recovered from φ. The category of δ-rings enjoys many felicitous properties; e.g., it is closed under limits and colimits, many localizations and completions, etc. We do not state them here rigorously and instead refer to [BS19, § 2].

Definition 2.1.6. Let A be a δ-ring. We call A *perfect* if its associated Frobenius morphism φ is an isomorphism. The *(colimit) perfection* of R is given by

$$A_{\mathrm{perf}} := \operatorname{colim}(A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots).$$

The class of p-adically complete, perfect δ-rings can be related back to perfect $\mathbf{F}_p$-algebras.

Proposition 2.1.7 ([BS19, Cor. 2.31])**.** *Reduction modulo p induces an equivalence of categories*

$$\{p\text{-complete, perfect } \delta\text{-rings}\} \xrightarrow{\sim} \{\text{perfect } \mathbf{F}_p\text{-algebras}\}$$

whose quasi-inverse is given by the Witt vector construction.

Such an equivalence continues to hold if one reduces modulo other, more general elements than p.

Definition 2.1.8 ([BS19, Def. 3.2])**.** A *perfect prism* is a pair $(A, (d))$ consisting of a perfect δ-ring A and a principal ideal $(d) \subset A$ such that A is (p, d)-adically complete and $\delta(d)$ is a unit of A.

The last condition is equivalent to $p \in (d, \varphi(d))$ [BS19, Lem. 3.8.(1)] and therefore encodes that the ideals (d) and $(\varphi(d))$ only meet in characteristic p. Perfect prisms still satisfy an analog of Proposition 2.1.7, in which the full subcategory of perfectoid rings replaces the category of perfect rings.

Definition 2.1.9. A *perfectoid ring* is a ring of the form A/d for some perfect prism $(A,(d))$.

As promised, this notion is justified by the following.

Theorem 2.1.10 ([BS19, Thm.3.10])**.** *Reduction modulo d induces an equivalence of categories*

$$\{\text{perfect prisms}\} \xrightarrow{\sim} \{\text{perfectoid rings}\}.$$

Perfectoid rings bear similarities to perfect $\mathbf{F}_p$-algebras in the following sense.

Lemma 2.1.11 ([BMS18, § 3], [BS19, § 2])**.** *Let R be a perfectoid ring. Then there exists $\pi \in R$ such that*

(i) *R is p-adically complete,*

(ii) *$\pi^p = p \cdot u$ for some unit $u \in R$, and*

(iii) *the Frobenius map $\varphi\colon R/\pi \to R/\pi^p$ (a morphism because $\operatorname{char} R/\pi = \operatorname{char} R/\pi^p = p$) is an isomorphism.*

Proof. Write $R = A/d$ for some perfect prism $(A,(d))$. Since A is p-adically complete and taking cokernels preserves derived completeness, R is derived p-adically complete. A calculation in [BS19, Lem. 2.34] shows that $d \in A$ is a nonzerodivisor and R has bounded p^∞-torsion (i.e., there exists $c \in \mathbf{N}$ such that $R[p^{c'}] = R[p^c]$ for all $c' \geq c$). Therefore, R is also (classically) p-adically complete; cf. [SP20, Lem. 0923].

By Proposition 2.1.7, $A \simeq W(A/p)$ for the perfect ring A/p. Let π be the image of $[\bar{a}_0^{1/p}]$ in R, where $[\bar{a}_0]$ is the zeroth coefficient of the Teichmüller expansion $d = \sum_{i=0}^\infty [\bar{a}_i]p^i$. Since

$$\delta(d) = \frac{1}{p}(\varphi(d) - d^p) = \frac{1}{p}\Big(\sum_{i=0}^\infty [\bar{a}_i^p]p^i - \Big(\sum_{i=0}^\infty [\bar{a}_i]p^i\Big)^p\Big) \equiv \bar{a}_1^p \mod p,$$

$\delta(d)$ is a unit of A if and only if $\bar{a}_1$ is a unit of A/p. Thus, π is the desired element for (ii) because $\pi^p = [\bar{a}_0] \equiv -p \cdot [\bar{a}_1] \mod p^2$ in R and R is p-adically complete.

Lastly, with the identification

$$\begin{array}{ccc} R/\pi & \xrightarrow{\overline{\varphi}} & R/\pi^p \\ \wr\| & & \wr\| \\ (A/p)/\bar{a}_0^{1/p} & \xrightarrow{\overline{\varphi}} & (A/p)/\bar{a}_0, \end{array}$$

(iii) follows from the perfectness of A/p. ∎

2.1.1 Classification

Finally, we provide a structure theorem which roughly states that perfectoid rings are made up of perfect rings in characteristic p and certain rings in characteristic 0 which arise in nonarchimedean geometry and are thus amenable to more geometric methods. The latter class of rings is described as follows.

Definition 2.1.12. A *perfectoid Huber pair* is a pair (R, R^+) consisting of a complete topological ring R and a subring $R^+ \subset R$ such that

(i) R^+ is an integrally closed, open, bounded subring of R,

(ii) the induced topology on R^+ is ϖ-adic for some $\varpi \in R^+$ which becomes a unit in R, and

(iii) R^+ is perfectoid.

Remark 2.1.13. In the language of analytic geometry, the ring R is Tate. Whether a Huber pair (R, R^+) for some Tate ring R and an integrally closed, open ring of integral elements $R^+ \subset R$ is perfectoid, turns out to be independent of the choice of R^+ [BMS18, Lem. 3.20]. This explains the antecedent definitions of perfectoid rings by Scholze [Sch12] and Fontaine [Fon13, § 1.1] as uniform Tate rings R with a topologically nilpotent unit $\varpi \in R^\circ$ such that $\varpi^p \mid p$ in R° and the Frobenius map $R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism (cf. Lemma 2.1.11).

Lemma 2.1.14 ([BMS18, Lem. 3.21])**.** *Let R' be a perfectoid ring. Assume there exists a nonzerodivisor $\varpi \in R'$ for which $\varpi^p \mid p$ and R' is ϖ-adically complete. Let R^+ be the integral closure of R' in $R'[\frac{1}{\varpi}]$. Then $(R'[\frac{1}{\varpi}], R^+)$ is a perfectoid Huber pair, where $R'[\frac{1}{\varpi}]$ is given the topology in which the image of $R' \to R'[\frac{1}{\varpi}]$ is an open and bounded subring.*

When $\operatorname{char} R' = 0$, the element $\pi \in R'$ from Lemma 2.1.11 is a possible choice for ϖ. The next statement, which is a special case of [BS19, Cor. 8.11], provides the recipe to decompose a perfectoid ring into a perfect and a "nonarchimedean" part.

Proposition 2.1.15. *Let R' be a perfectoid ring. Let R^+ be the integral closure of the image of R' in $R'[\frac{1}{p}]$. Set $\overline{R'} := (R'/p)_{\mathrm{perf}}$ and $\overline{R^+} := (R^+/p)_{\mathrm{perf}}$. Then R^+ is perfectoid and*

$$\begin{array}{ccc} R' & \longrightarrow & R^+ \\ \downarrow & & \downarrow \\ \overline{R'} & \longrightarrow & \overline{R^+} \end{array} \tag{2.1}$$

is a homotopy fiber square of abelian groups. Moreover, the maps $R' \to \overline{R'}$ and $R^+ \to \overline{R^+}$ are surjective and $\overline{R'} \to \overline{R^+}$ is integral.

Recall that a square of abelian groups is a homotopy fiber square if the induced map between the cones of the two horizontal (or equivalently, the two vertical) arrows is an isomorphism. In particular, (2.1) is a fiber square of commutative rings because the forgetful functor from commutative rings to abelian groups preserves limits and homotopy fiber squares of abelian groups are fiber squares.

Remark 2.1.16. Note that $(R'/p)_{\mathrm{perf}} \simeq R'/\sqrt{pR'}$ for any perfectoid ring R'. Indeed, since perfect rings are reduced, any map from R'/p to a perfect ring must factor through $R'/\sqrt{pR'}$. On the other hand, $R'/\sqrt{pR'}$ is perfect: the injectivity of the Frobenius morphism $\varphi\colon R'/\sqrt{pR'} \to R'/\sqrt{pR'}$ follows from the fact $\sqrt{pR'}$ is a radical ideal and its surjectivity from the observation that the Frobenius on R'/p is surjective (Lemma 2.1.11.(iii)) and the commutativity of

$$\begin{tikzcd}
R'/p \arrow[r, two heads, "\varphi_{R'/p}"] \arrow[d, two heads] & R'/p \arrow[d, two heads] \\
R'/\sqrt{pR'} \arrow[r, "\varphi_{R'/\sqrt{pR'}}"] & R'/\sqrt{pR'}.
\end{tikzcd}$$

Proof of Proposition 2.1.15. Remark 2.1.16 shows that $R \to \overline{R}$ and $R^+ \to \overline{R^+}$ are surjective, and thus also the integrality of $\overline{R} \to \overline{R^+}$. The diagram (2.1) is a homotopy fiber square by [BS19, Cor. 8.11]: to apply this criterion, we need to check that any map $\theta'\colon R' \to V$ to a p-adically complete valuation ring V of rank 1 whose kernel does not contain p extends uniquely to a map $\theta^+\colon R^+ \to V$. Since $R'[\frac{1}{p}] \to R^+[\frac{1}{p}]$ is an isomorphism and $p \notin \ker(\theta)$, the induced map $R' \to \mathrm{Frac}(V)$ extends uniquely to $R^+ \to \mathrm{Frac}(V)$. The morphism $R' \to R^+$ is by definition integral and hence a directed colimit of finite morphisms. Therefore, the valuative criterion for properness guarantees the existence of the desired unique extension θ^+. □

Alternatively, Proposition 2.1.15 can be proved with more elementary methods along the lines of [Bha18a, § IV.3], which also contains a detailed argument why $R'/R'[\sqrt{pR'}]$ (and thus R^+ by Lemma 2.1.14) is perfectoid. We leave the details to the reader. Next, we give the promised classification result.

Corollary 2.1.17. *The natural functor*

$$\Delta\colon \{\text{perfectoid rings}\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{pairs } ((R^+[\tfrac{1}{p}], R^+), \overline{R'} \to (R^+/p)_{\mathrm{perf}}) \text{ where} \\ \bullet\ \ (R^+[\tfrac{1}{p}], R^+) \text{ is a perfectoid Huber pair over } (\mathbf{Q}_p, \mathbf{Z}_p) \text{ and} \\ \bullet\ \ \overline{R'} \to (R^+/p)_{\mathrm{perf}} \text{ is an integral map of perfect } \mathbf{F}_p\text{-algebras} \end{array} \right\}$$

(where the morphisms of the target category are given by compatible morphisms of perfectoid Huber pairs and commutative diagrams of perfect $\mathbf{F}_p$-algebras) induced by Proposition 2.1.15 is an equivalence of categories.

Proof. We construct a quasi-inverse functor Γ. Pick $(R^+[\frac{1}{p}], R^+)$ and $\overline{R'} \to \overline{R^+} := (R^+/p)_{\mathrm{perf}}$ as above. Let $(A^+, (d^+))$ be the perfect prism corresponding to R^+; the perfect prisms for $\overline{R'}$ and $\overline{R^+}$ are given by the Witt vector construction. Let $A := A^+ \times_{W(\overline{R^+})} W(\overline{R'})$ in the category of δ-rings. Since the Frobenius of A is the limit of the Frobenii of the factors and thus an isomorphism, A is perfect.

By the proof of Lemma 2.1.11.(ii), $d^+ = [\bar{a}_0] + p \cdot u$ for some $\bar{a}_0 \in A^+/p$ and some unit $u \in A^+$. Hence,

$$A^+/\Big(\big[\bar{a}_0^{1/p^\infty}\big], p\Big) \simeq R^+/\Big(\big[\bar{a}_0^{1/p^\infty}\big], p\Big) \simeq R^+/\sqrt{pR^+} \simeq \overline{R^+},$$

and $A^+/\Big(\big[\bar{a}_0^{1/p^\infty}\big]\Big) \simeq W(\overline{R^+})$ by Proposition 2.1.7. In particular, after replacing d^+ by $d^+ \cdot u^{-1}$, we may assume that the projection $A^+ \to W(\overline{R^+})$ maps d^+ to p.

Set $d := (d^+, p) \in A$; this is well-defined by the assumption of the previous paragraph. Then $\delta(d) = (\delta(d^+), \delta(p))$ is a unit. Furthermore,

$$\begin{aligned}\lim_n A/(p,d)^n &\simeq \lim_n A/(p^n, d^n) \simeq \lim_n (A^+/p^n \times_{W(\overline{R^+})/p^n} W(\overline{R})/p^n)/d^n \\ &\simeq \lim_n A^+/(p^n, d^n) \times_{W(\overline{R^+})/p^n} W(\overline{R})/p^n \simeq A^+ \times_{W(\overline{R^+})} W(\overline{R}) \simeq A\end{aligned}$$

by the (p, d^+)-adic completeness of A^+ and the p-adic completeness of $W(\overline{R'})$ and $W(\overline{R^+})$. Thus, A is (p,d)-adically complete.

We claim that $R' := A/d$ is the desired perfectoid ring for the quasi-inverse functor Γ (with values on morphisms induced by the fiber products). Indeed, since the vertical maps in the diagram

$$\begin{array}{ccc} A & \longrightarrow & A^+ \\ \downarrow & & \downarrow \\ W(\overline{R'}) & \longrightarrow & W(\overline{R^+}) \end{array}$$

are surjective, it is a homotopy fiber square. As d is a nonzerodivisor on all rings involved, derived reduction modulo d gives a homotopy fiber square

$$\begin{array}{ccc} R' & \longrightarrow & R^+ \\ \downarrow & & \downarrow \\ \overline{R'} & \longrightarrow & \overline{R^+}. \end{array}$$

This shows $\Gamma \circ \Delta \simeq \mathrm{id}$.

Conversely, if $(R^+[\frac{1}{p}], R^+)$ and $\overline{R'} \to \overline{R^+} := (R^+/p)_{\mathrm{perf}}$ are as before and R' is their image under Γ, then $R'[\frac{1}{p}] \simeq R^+[\frac{1}{p}]$ and the projection $R' \to R^+$ is integral by Lemma 2.1.18. Since $R^+ \subset R^+[\frac{1}{p}]$ is integrally closed, it must be the integral closure of the image of R' in $R'[\frac{1}{p}]$. Furthermore,

$$(R'/p)_{\mathrm{perf}} \simeq ((R^+ \times_{\overline{R^+}} \overline{R'})/p)_{\mathrm{perf}} \simeq ((R^+/p) \times_{\overline{R^+}} \overline{R'})_{\mathrm{perf}} \simeq \overline{R^+} \times_{\overline{R^+}} \overline{R'} \simeq \overline{R'}.$$

Thus, $\Delta \circ \Gamma \simeq \mathrm{id}$, concluding the proof. □

Lemma 2.1.18. *Let $g\colon R \to S$ be an integral morphism of commutative rings and*

$$\begin{tikzcd} R' \arrow[r, "g'"] \arrow[d, two heads, "f'"] & S' \arrow[d, two heads, "f"] \\ R \arrow[r, "g"] & S \end{tikzcd}$$

be a fiber square in which the vertical morphisms are surjective. Then $R' \to S'$ is integral.

Proof. Let $s' \in S'$. Since g is integral, there is a monic polynomial $P \in R[x]$ such that $P^g(f(s')) = 0$. By the surjectivity of f', we can pick a lift $\widetilde{P} \in R'[x]$ of P. Then $f(\widetilde{P}^{g'}(s')) = P^g(f(s')) = 0$, giving $\widetilde{P}^{g'}(s') \in \ker(f)$. However, the vertical morphisms are both surjective and the diagram is a fiber square, so the natural map $\ker(f') \to \ker(f)$ is an isomorphism. Therefore, s' is a root of the monic polynomial $Q := \widetilde{P} - \widetilde{P}^{g'}(s') \in R'[x]$. □

2.2 Globalization: perfectoid spaces

In this section, we globalize the notion of perfectoid rings in the context of nonarchimedean geometry. In order to use adic spaces, we will restrict to the class of perfectoid Huber pairs from Definition 2.1.12. All definitions and statements in this section are due to [KL15] and [Sch12].

Definition 2.2.1. An affinoid perfectoid space is an adic space of the form $\mathrm{Spa}(R, R^+)$ for some perfectoid Huber pair (R, R^+).

In the setting of perfectoid Huber pairs, we can translate questions between mixed and positive characteristics while retaining all topological and étale cohomological data. The translation process relies on the tilting procedure.

Definition 2.2.2. Let (R, R^+) be a perfectoid Huber pair. Let $\pi \in R^+$ be as in Lemma 2.1.11. The *tilt* of (R, R^+) is the Huber pair $(R^\flat, R^{\flat+})$, where $R^{\flat+} := \lim_\varphi R^+/\pi$ is the limit perfection of R^+/π and $R^\flat := R^{\flat+}[\frac{1}{\pi}]$.

An explicit computation with p-adic limits yields the following direct connection between R and $R^\flat$.

Lemma 2.2.3 ([Sch12, Prop. 5.17]). *The projection $R^{\flat+} \to R^+/\varpi$ onto the first factor factors through a continuous morphism of multiplicative monoids $\sharp\colon R^{\flat+} \to R^+$.*

Abusing notation, we sometimes denote the induced map $\sharp\colon R^\flat \to R$ by the same symbol.

Definition 2.2.4. A morphism $(R, R^+) \to (S, S^+)$ of perfectoid Huber pairs is *finite étale* if $R \to S$ is a finite étale ring homomorphism and S^+ is the integral closure of R^+ in S.

One can check that any complete Huber pair which is finite étale over a perfectoid Huber pair is itself perfectoid.

Theorem 2.2.5 (Tilting equivalence [KL15, Sch12]). *Let (R, R^+) be a perfectoid Huber pair. Then tilting induces*

(i) *a homeomorphism $|\mathrm{Spa}(R, R^+)| \to |\mathrm{Spa}(R^\flat, R^{\flat+})|$ given by precomposition with $\sharp$, which is compatible with localizations at rational subsets,*

(ii) *an equivalence of categories between perfectoid Huber pairs over (R, R^+) and over $(R^\flat, R^{\flat+})$, and*

(iii) *an equivalence of categories between finite étale perfectoid Huber pairs over (R, R^+) and over $(R^\flat, R^{\flat+})$.*

We now globalize the aforementioned notions.

Definition 2.2.6. A *perfectoid space* is an adic space that can be covered by affinoid perfectoid spaces.

Definition 2.2.7. A morphism $f\colon X \to Y$ of perfectoid spaces is

(i) *finite étale* if for every affinoid perfectoid open $\mathrm{Spa}(R, R^+) \subset Y$ the preimage $X \times_Y \mathrm{Spa}(R, R^+) \simeq \mathrm{Spa}(S, S^+)$ is affinoid perfectoid and the induced morphism $(R, R^+) \to (S, S^+)$ is a finite étale morphism of perfectoid Huber pairs,

(ii) *étale* if for every $x \in X$ there exist open neighborhoods $x \in U$ and $f(U) \subseteq V$ such that the induced morphism $U \to V$ is finite étale, and

(iii) *pro-étale* if for every $x \in X$ there exist affinoid perfectoid open neighborhoods $x \in U = \mathrm{Spa}(S, S^+)$ and $f(U) \subseteq V = \mathrm{Spa}(R, R^+)$ such that the induced morphism $(R, R^+) \to (S, S^+)$ is the completed filtered colimit of étale morphisms.

Corollary 2.2.8. *There is a tilting functor $X \mapsto X^\flat$ which is given affinoid locally by the functor of Definition 2.2.2. Moreover, for a fixed perfectoid space X, tilting induces*

(i) *a homeomorphism $|X| \to |X^\flat|$ compatible with localizations at rational subsets,*

(ii) *an equivalence of categories between perfectoid spaces over X and over $X^\flat$, and*

(iii) *an equivalence of categories between perfectoid spaces that are (finite) étale (thus in particular perfectoid) over X and over $X^\flat$.*

2.3 Perfectoidizations

The perfection of an $\mathbf{F}_p$-algebra (resp. δ-ring) R from § 2.1 is the universal perfect $\mathbf{F}_p$-algebra (resp. δ-ring) that receives a map from R. In [BS19], Bhatt–Scholze discovered that there is a similar notion for perfectoid rings.

Theorem 2.3.1 ([BS19, Thm. 1.16.(1)])**.** *Let R be a perfectoid ring and $R \to S$ be an integral morphism. Then S admits a* perfectoidization $S \to S_{\mathrm{perfd}}$*, i.e., a morphism to a perfectoid ring S_{perfd} which is universal with that property.*

Combining the previous result with Theorem 2.1.10, the category of perfect prisms (A, d) with a morphism $S \to A/d$ admits an initial object $(\Delta^{\mathrm{init}}_{S,\mathrm{perf}}, d^{\mathrm{init}})$. The original description of S_{perfd} in [BS19, Def. 8.2] is as $\Delta^{\mathrm{init}}_{S,\mathrm{perf}}/d^{\mathrm{init}}$; this definition lends itself to a vast generalization for any derived p-complete simplicial R-algebra S and leads to the introduction of prismatic cohomology. For this generality, however, S_{perfd} is merely a cosimplicial ring, even when the original ring S is discrete. Concretely, $S_{\mathrm{perfd}} \simeq \mathrm{Rlim}_{S \to T} T$ is the homotopy limit over all perfectoid S-algebras T [BS19, Prop. 8.5].

When S is perfectoid, $S_{\mathrm{perfd}} \simeq S$. Otherwise, perfectoidizations tend to be quite hard to compute explicitly. We demonstrate this at the following example, in which we assume more advanced knowledge of perfectoid spaces.

2.3.1 Example: the perfectoidization of $\mathrm{L}^1(\mathbf{Q}_p/\mathbf{Z}_p, K^\circ)$

Let $K := \mathbf{Q}_p^{\mathrm{cycl}}$ with the p-adic valuation. This is a perfectoid field with subring of powerbounded elements $K^\circ = \mathbf{Z}_p^{\mathrm{cycl}}$. In this subsection, we compute the perfectoidization of the K°-algebra $S^+ := \mathrm{L}^1(\mathbf{Q}_p/\mathbf{Z}_p, K^\circ)$ of functions $f\colon \mathbf{Q}_p/\mathbf{Z}_p \to K^\circ$ such that for all $\varepsilon > 0$, all but finitely many $y \in \mathbf{Q}_p/\mathbf{Z}_p$ satisfy $|f(y)| < \varepsilon$; here, the product of two functions $f, g \in \mathrm{L}^1(\mathbf{Q}_p/\mathbf{Z}_p, K^\circ)$ is given by the convolution

$$(f * g)(y) = \sum_{z \in \mathbf{Q}_p/\mathbf{Z}_p} f(z)g(y-z).$$

First, we explain why S^+ admits a finite morphism from the perfectoid K°-algebra $R^+ := K^\circ\langle x^{1/p^\infty}\rangle$.

Lemma 2.3.2. *The identification of the algebra of functions with finite support on a group with its group algebra induces an isomorphism* $\mathrm{L}^1(\mathbf{Q}_p/\mathbf{Z}_p, K^\circ) \simeq R^+/(x-1)$.

Proof. By [KL15, Lem. 2.8.8], the ideal $(x-1) \subset R^+$ is closed and the quotient $R^+/(x-1)$ thus p-adically separated. Since derived p-adic completeness is always preserved under taking quotients, S^+ must be p-adically complete, so

$$S^+ \simeq \lim_n \frac{K^\circ/p^n[x^{1/p^\infty}]}{(x-1)} \simeq \lim_n \operatorname*{colim}_m \frac{K^\circ/p^n[x^{1/p^m}]}{(x-1)}.$$

The maps $x^{1/p^m} \mapsto \left[\frac{1}{p^m}\right]$ determine isomorphisms of $\frac{K^\circ/p^n[x^{1/p^m}]}{(x-1)}$ with the group algebras $K^\circ/p^n\left[\frac{1}{p^m}\mathbf{Z}/\mathbf{Z}\right]$. Therefore,

$$\lim_n \operatorname*{colim}_m \frac{K^\circ/p^n[x^{1/p^m}]}{(x-1)} \simeq \lim_n \operatorname*{colim}_m K^\circ/p^n\left[\tfrac{1}{p^m}\mathbf{Z}/\mathbf{Z}\right] \simeq \lim_n K^\circ\left[\mathbf{Q}_p/\mathbf{Z}_p\right]/p^n.$$

Identifying $K^\circ\left[\mathbf{Q}_p/\mathbf{Z}_p\right]$ with the algebra of K°-valued functions on $\mathbf{Q}_p/\mathbf{Z}_p$ with finite support, we conclude that $\lim_n K^\circ\left[\mathbf{Q}_p/\mathbf{Z}_p\right]/p^n \simeq \mathrm{L}^1(\mathbf{Q}_p/\mathbf{Z}_p, K^\circ)$, with the product given by convolution. $\square$

Next, we compute $S^+_{\mathrm{perfd}}[\frac{1}{p}]$ explicitly. To do so, we need the following general statement.

Lemma 2.3.3. *Let (R, R^+) be a perfectoid Huber pair with* $\operatorname{char} R = 0$. *Let $\varpi \in R^+$ be a pseudouniformizer and $X := \mathrm{Spa}(R, R^+)$. Let $g \in R^+$ and $S^+ := R^+/g$. Then the perfectoid Huber pair $(\widetilde{S}, \widetilde{S}^+)$, where $\widetilde{S} := S^+_{\mathrm{perfd}}[\frac{1}{\varpi}]$ and $\widetilde{S}^+$ is the integral closure of S^+_{perfd} in $\widetilde{S}$, is computed as the colimit* $\operatorname{colim}_{n\in\mathbf{N}}\Big(\mathscr{O}_X(X(\frac{g}{\varpi^n})), \mathscr{O}_X^+(X(\frac{g}{\varpi^n}))\Big)$ *in the category of perfectoid Huber pairs.*

Proof. Since $\operatorname{char} R = 0$, $\widetilde{S}$ is p-adically (or equivalently, ϖ-adically) complete. Thus, the map $(R, R^+) \to (\widetilde{S}, \widetilde{S}^+)$ is the universal morphism of perfectoid Huber pairs whose kernel contains g. However,

$$(R, R^+) \to (\widetilde{S}', \widetilde{S}'^+) := \operatorname*{colim}_{n\in\mathbf{N}}\Big(\mathscr{O}_X(X(\tfrac{g}{\varpi^n})), \mathscr{O}_X^+(X(\tfrac{g}{\varpi^n}))\Big)$$

satisfies the same universal property. Namely, let $f\colon (R, R^+) \to (T, T^+)$ be a morphism of perfectoid Huber pairs with $f(g) = 0$. Since $\varpi^n \mid f(g) = 0$ in T^+, the map f factors uniquely through $(\widetilde{S}', \widetilde{S}'^+)$. Lastly, g maps to 0 in the $(\widetilde{S}', \widetilde{S}'^+)$ because colimits in the category of perfectoid Huber pairs over (R, R^+) are ϖ-completed. $\square$

Lemma 2.3.4. *The K-algebra $\widetilde{S} := S^{+}_{\mathrm{perfd}}[\frac{1}{p}]$ is isomorphic to $\mathrm{C}^0(\mathbf{Z}_p(1)(K), K)$. Under this isomorphism, the quotient map $q\colon R \to \widetilde{S}$ from Lemma 2.3.2 is identified with the "evaluation map" given on x^{1/p^m} by sending $(\zeta_r) \in \lim_r \mu_{p^r}(K) \simeq \mathbf{Z}_p(1)(K)$ to ζ_m.*

Proof. By Lemma 2.3.3, the natural morphism $\operatorname{colim}_n \mathscr{O}^{+}_{X}(X(\frac{x-1}{p^n})) \to \widetilde{S}^{+}$ is an isomorphism after p-adic completion. Consider the inverse system of rigid spaces

$$\cdots \to X_m := \operatorname{Spa}(K\langle x^{1/p^m}\rangle, K^{\circ}\langle x^{1/p^m}\rangle) \to X_{m-1} := \operatorname{Spa}(K\langle x^{1/p^{m-1}}\rangle, K^{\circ}\langle x^{1/p^{m-1}}\rangle) \to \cdots$$

with transition maps induced by the natural inclusions. Each $\mathscr{O}^{+}_{X}(X(\frac{x-1}{p^n}))$ in the colimit is itself the p-adic completion of $\operatorname{colim}_m \mathscr{O}^{+}_{X_m}(X_m(\frac{x-1}{p^n}))$. Since the $\mathscr{O}^{+}_{X_m}(X_m(\frac{x-1}{p^n}))$ are p-torsion free, $\widetilde{S}^{+}$ can simply be computed as the p-adic completion of $\operatorname{colim}_{n,m} \mathscr{O}^{+}_{X_m}(X_m(\frac{x-1}{p^n}))$.

By [Bha17, Cor. 7.5.8.5], the p-adic completion of $\operatorname{colim}_n \mathscr{O}^{+}_{X_m}(X_m(\frac{x-1}{p^n}))$ is given by the integral closure of $K^{\circ}\langle x^{1/p^m}\rangle$ in $K\langle x^{1/p^m}\rangle/(x-1)$. Under the isomorphism $K\langle x^{1/p^m}\rangle/(x-1) \xrightarrow{\sim} \prod_{\mu_{p^m}(K)} K$ sending x^{1/p^m} to ζ in the factor corresponding to $\zeta \in \mu_{p^m}(K)$, the image of $K^{\circ}\langle x^{1/p^m}\rangle$ is contained in the integrally closed subring $\prod K^{\circ}$. On the other hand, the composition

$$K^{\circ} \hookrightarrow K^{\circ}\langle x^{1/p^m}\rangle \to \prod K^{\circ}$$

is the diagonal map; since all factors are spanned by idempotents, it is integral. In particular, the integral closure of $K^{\circ}\langle x^{1/p^m}\rangle$ in $K\langle x^{1/p^m}\rangle/(x-1)$ can be identified with $\prod K^{\circ}$.

Using the additional identification $\prod_{\mu_{p^m}(K)} K^{\circ} \simeq \mathrm{C}^0(\mu_{p^m}(K), K^{\circ})$, we see that $\widetilde{S}^{+}$ is the p-adic completion of $\operatorname{colim}_m \mathrm{C}^0(\mu_{p^m}(K), K^{\circ})$. The first statement then follows from Lemma 2.3.5. For the second statement, note that it suffices to verify the formula for the map $K^{\circ}\langle x^{1/p^m}\rangle \to \mathrm{C}^0(\mu_{p^m}(K), K^{\circ})$ at finite level, where it is a consequence of the previous identifications. $\square$

Lemma 2.3.5. *Every continuous map $f\colon G \to X$ from a profinite group G to a discrete space X factors through a finite quotient.*

Proof. Since the continuous image of a compact space is compact and X is discrete, the image of f must be finite. Let $G_1, \dots, G_\rho$ be the finite number of nonempty preimages. Each G_i is open and thus contains open subspaces of the form $g_i H_i$ for some $g_i \in G_i$ and open, normal subgroups H_i of G. Therefore, f factors through the finite quotient of G by the open, normal subgroup $\bigcap H_i$. $\square$

Under the identifications from Lemma 2.3.4, the "rational perfectoidization" $S := S^{+}[\frac{1}{p}] \to \widetilde{S} := S^{+}_{\mathrm{perfd}}[\frac{1}{p}]$ becomes the Fourier transform $\mathrm{L}^1(\mathbf{Q}_p/\mathbf{Z}_p, K) \to \mathrm{C}^0(\mathbf{Z}_p(1)(K), K)$, under which f maps to the function $\widehat{f}(\zeta) = \sum_{x \in \mathbf{Q}_p/\mathbf{Z}_p} f(x)\zeta^x$. This Fourier transform had been studied in [FdM78] in a different language.

Finally, in order to obtain S^+_{perfd} from $S^+_{\mathrm{perfd}}[\frac{1}{p}]$, we record the following consequence of the structure theorems of § 2.1.1.

Corollary 2.3.6. *Let S' be integral over a perfectoid ring. Then S'_{perfd} can be described by the fiber square*

$$\begin{tikzcd} S'_{\mathrm{perfd}} \arrow[r] \arrow[d] & S^+_{\mathrm{perfd}} \arrow[d] \\ (S'/p)_{\mathrm{perf}} \arrow[r] & (S^+_{\mathrm{perfd}}/p)_{\mathrm{perf}}, \end{tikzcd}$$

where S^+_{perfd} is the integral closure of S'_{perfd} in $S'_{\mathrm{perfd}}[\frac{1}{p}]$.

Proof. Thanks to Proposition 2.1.15 applied to $R' = S'_{\mathrm{perfd}}$, it only remains to show that $(S'_{\mathrm{perfd}}/p)_{\mathrm{perf}} \simeq (S'/p)_{\mathrm{perf}}$. We check that every perfect S'/p-algebra $S'/p \to T$ factors uniquely through a morphism $(S'_{\mathrm{perfd}}/p)_{\mathrm{perf}} \to T$. First, as T is perfectoid of characteristic p, there is a unique factorization $S'/p \to S'_{\mathrm{perfd}}/p \to T$ by the universal properties of perfectoidizations and quotients. Then, perfectness of T gives the desired unique factorization of $S'_{\mathrm{perfd}}/p \to T$ through $(S'_{\mathrm{perfd}}/p)_{\mathrm{perf}}$. □

Corollary 2.3.7. *The (integral) perfectoidization of $S^+ := \mathrm{L}^1(\mathbf{Q}_p/\mathbf{Z}_p, K^\circ)$ is given by*

$$S^+_{\mathrm{perfd}} \simeq \{f \in \mathrm{C}^0(\mathbf{Z}_p(1)(K), K^\circ) \mid f \textit{ is constant} \mod (p^{1/p^\infty})\}.$$

Proof. We have $\widetilde{S}^+ \simeq \mathrm{C}^0(\mathbf{Z}_p(1)(K), K^\circ)$ by Lemma 2.3.4 and

$$(S^+/p)_{\mathrm{perf}} \simeq \Big((\mathbf{F}_p[t^{1/p^\infty}]/(t)\Big)[x^{1/p^\infty}]/(x-1)_{\mathrm{perf}} \simeq \mathbf{F}_p$$

by Lemma 2.3.2. The fiber square of Corollary 2.3.6 therefore becomes

$$\begin{tikzcd} S^+_{\mathrm{perfd}} \arrow[r] \arrow[d, two heads] & \mathrm{C}^0(\mathbf{Z}_p(1)(K), K^\circ) \arrow[d, two heads] \\ \mathbf{F}_p \arrow[r] & \mathrm{C}^0(\mathbf{Z}_p(1)(K), \mathbf{F}_p), \end{tikzcd}$$

where the bottom map is the inclusion of the constant functions. The statement follows. □

2.3.2 Perfectoidizations of Huber pairs

In this subsection, we describe the perfectoidizations from Theorem 2.3.1 more concretely in case the integral morphism arises from a finite morphism of Huber pairs. Besides its geometric significance in the context of (global) perfectoid spaces, which we will explore in § 2.4, this setting can also be useful as a first step toward determining integral perfectoidizations, as seen in § 2.3.1. Subsequently, we will discuss a related question of André.

First, recall the definition of a finite morphism of complete Huber pairs.

Definition 2.3.8 ([Hub96, (1.4.2)]). A morphism $(R, R^+) \to (S, S^+)$ of complete Huber pairs is *finite* if S is a finite R-algebra with the induced topology and S^+ is the integral closure of R^+ in S.

Theorem 2.3.1 directly implies the following statement.

Corollary 2.3.9. *Let (R, R^+) be a perfectoid Huber pair and $(R, R^+) \to (S, S^+)$ be a finite morphism of complete Huber pairs. Then (S, S^+) admits a* perfectoidization $(S, S^+) \to (\widetilde{S}, \widetilde{S}^+)$*, i.e., a morphism to a perfectoid Huber pair $(\widetilde{S}, \widetilde{S}^+)$ which is universal with that property.*

Proof. Choose a pseudouniformizer $\varpi \in R^+$. Since $R^+ \to S^+$ is integral, we can set $\widetilde{S} := (S^+_{\mathrm{perfd}})^\wedge[\frac{1}{\varpi}]$, where the completion is ϖ-adic. Let $\widetilde{S}^+$ be the integral closure of $(S^+_{\mathrm{perfd}})^\wedge$ in $\widetilde{S}$. The universal morphism $S^+ \to S^+_{\mathrm{perfd}}$ induces a map $(S, S^+) \to (\widetilde{S}, \widetilde{S}^+)$. To check that it is universal for morphisms to perfectoid Huber pairs, let $(S, S^+) \to (T, T^+)$ be such a morphism. Then the universal properties of $S^+ \to S^+_{\mathrm{perfd}}$ and of completions yield a unique morphism $(S^+_{\mathrm{perfd}})^\wedge \to T^+$ and thus $\widetilde{S} \to T$. We obtain $\widetilde{S}^+ \to T^+$ from the integral closedness of T^+. □

Since a finite morphism is étale away from the (closed) discriminant locus, we can Zariski locally choose $g \in R^+$ such that $R[\frac{1}{g}] \to S[\frac{1}{g}]$ is étale. Under additional hypotheses on g, the perfectoidization of (S, S^+) can almost be identified with a (nonderived) limit in the category of Huber pairs.

Proposition 2.3.10. *Fix a perfectoid Huber pair (R, R^+). Choose a pseudouniformizer $\varpi^\flat \in R^{\flat,+}$ and set $\varpi := \sharp(\varpi^\flat)$. Let $(R, R^+) \to (S, S^+)$ be a finite morphism of complete Huber pairs and set $X := \mathrm{Spa}(S, S^+)$. Assume there exists $g^\flat \in R^{\flat,+}$ such that $R[\frac{1}{g}] \to S[\frac{1}{g}]$ is finite étale for $g := \sharp(g^\flat) \in R^+$. Then the natural map*

$$\widetilde{S}^+ \to \lim_{n\in\mathbf{N}} \mathscr{O}^+_X(X(\tfrac{\varpi^n}{g}))$$

is an almost isomorphism with respect to the ideal $(\varpi g)^{1/p^\infty} := \bigcup_{\ell\in\mathbf{N}}\Big(\sharp\Big((\varpi^\flat g^\flat)^{1/p^\ell}\Big)\Big)$. In particular, the map $\widetilde{S} \to \lim_n \mathscr{O}_X(X(\frac{\varpi^n}{g}))$ is a (g^{1/p^∞})-almost isomorphism.

To simplify notation, we will from here on write $S\langle\frac{\varpi^n}{g}\rangle := \mathscr{O}_X(X(\frac{\varpi^n}{g}))$ and $S\langle\frac{\varpi^n}{g}\rangle^+ := \mathscr{O}^+_X(X(\frac{\varpi^n}{g}))$ for the rings of functions on the rational localizations of the adic spectrum of a complete Huber pair (S, S^+). Note that although in mixed characteristic limits in the category of perfectoid Huber pairs need not coincide with limits in the category of complete Huber pairs (see [And18a, § 3.8.2]), they are almost isomorphic in the setting of Proposition 2.3.10 by [And18a, § 3.8.4, Thm. 4.4.2, Prop. 3.5.4]. In the proof of Proposition 2.3.10 and in later sections, we will need a perfectoid version of the Riemann extension theorem, due to Scholze [Sch15, § 2.3], André [And18a, Thm. 4.4.2], and Bhatt.

Theorem 2.3.11 ([Bha18b, Thm. 4.2])**.** *Let (R, R^+) be a perfectoid Huber pair with pseudouniformizer $\varpi \in R^+$. Choose $g^\flat \in R^{\flat,+}$ and set $g := \sharp(g^\flat)$. For all $m \in \mathbf{N}$, the pro-system of morphisms*

$$\left\{ f_n \colon R^+/\varpi^m \to R\left\langle \tfrac{\varpi^n}{g} \right\rangle^+ /\varpi^m \right\}_{n \in \mathbf{N}}$$

is an almost-pro-isomorphism with respect to $(\varpi g)^{1/p^\infty}$. Moreover, the morphisms f_n are (ϖ^{1/p^∞})-almost injective.

We refer to § 2.5.1 for the definition of almost-pro-isomorphisms. For now, we simply note that the almost-pro-isomorphism of the inverse systems from Theorem 2.3.11 yields a $(\varpi g)^{1/p^\infty}$-almost isomorphism

$$R^+ \xrightarrow{\sim} \lim_{n \in \mathbf{N}} R\langle \tfrac{\varpi^n}{g} \rangle^+ \tag{2.2}$$

upon taking inverse limits over m.

Proof of Proposition 2.3.10. Equation (2.2) applied to $\widetilde{S}^+$ gives a $(\varpi g)^{1/p^\infty}$-almost isomorphism

$$\widetilde{S}^+ \xrightarrow{\sim} \lim_{n \in \mathbf{N}} \widetilde{S} \left\langle \frac{\varpi^n}{g} \right\rangle^+.$$

Thus, it suffices to show that $\widetilde{S}\langle \frac{\varpi^n}{g} \rangle^+$ and $S\langle \frac{\varpi^n}{g} \rangle^+$ are almost isomorphic.

Fix $n \in \mathbf{N}$. Let $\lambda_n \colon S \to S(\frac{\varpi^n}{g})$ be the initial topological S-algebra such that $\lambda_n(g)$ is invertible and $\frac{\lambda_n(\varpi^n)}{\lambda_n(g)}$ is bounded; the underlying ring of $S(\frac{\varpi^n}{g})$ is $S[\frac{1}{g}]$ and its completion is $S\langle \frac{\varpi^n}{g} \rangle$ [Hub94, § 1]. By [BS19, Thm. 10.9], $S[\frac{1}{g}] = S^+[\frac{1}{\varpi g}] \to \widetilde{S}[\frac{1}{g}] = S^+_{\mathrm{perfd}}[\frac{1}{\varpi g}]$ is an isomorphism. Hence, so is the completion $S\langle \frac{\varpi^n}{g} \rangle \to \widetilde{S}\langle \frac{\varpi^n}{g} \rangle$, and the morphism $S\langle \frac{\varpi^n}{g} \rangle^+ \to \widetilde{S}\langle \frac{\varpi^n}{g} \rangle^+$ of integral elements must be a (ϖ^{1/p^∞})-almost isomorphism. $\square$

Example 2.3.12. Let K be a perfectoid field and $K^\flat$ be its tilt. Choose a pseudouniformizer $\varpi^\flat \in K^\flat$ and set $\varpi := \sharp(\varpi^\flat)$. Assume that $\operatorname{char} K^\flat \neq 2$.

Let $(R, R^+) := (K\langle x^{1/p^\infty} \rangle, K^\circ\langle x^{1/p^\infty} \rangle)$ be the perfectoid Huber pair corresponding to the perfectoid closed unit disk. We consider the finite morphism of Huber pairs $(R, R^+) \to (S, S^+) := (R[x^{1/2}], R[x^{1/2}]^+)$, where $R[x^{1/2}]^+$ is the integral closure of R^+ in $R[x^{1/2}]$. Then $R[\frac{1}{x}] \to S[\frac{1}{x}]$ is finite étale. Since x lies in the image of $\sharp \colon R^{\flat,+} \to R^+$, we can compute the perfectoidization of S^+ via Proposition 2.3.10.

For any $n \in \mathbf{N}$, the natural inclusion of perfectoid K-algebras

$$S\langle \tfrac{\varpi^n}{x} \rangle \hookrightarrow K\Big\langle x^{1/(2p^\infty)}, (\tfrac{\varpi^n}{x})^{1/(2p^\infty)} \Big\rangle := \Big(\bigcup_{\ell \in \mathbf{N}} K^\circ \Big[x^{1/(2p^\ell)}, (\tfrac{\varpi^n}{x})^{1/(2p^\ell)} \Big] \Big)^\wedge \Big[\tfrac{1}{\varpi} \Big]$$

is an isomorphism because $x^{1/(2p^\ell)} = \frac{1}{\varpi^{n(p^\ell-1)/(2p^\ell)}} \cdot x^{1/2} \cdot \Big((\frac{\varpi^n}{x})^{1/p^\ell} \Big)^{(p^\ell-1)/2}$ and $(\frac{\varpi^n}{x})^{1/(2p^\ell)} = \frac{1}{\varpi^{n/(2p^\ell)}} \cdot x^{1/(2p^\ell)} \cdot (\frac{\varpi^n}{x})^{1/p^\ell}$ are contained in $S\langle \frac{\varpi^n}{x} \rangle$ and powerbounded. Thus, $S\langle \frac{\varpi^n}{x} \rangle^+ \to K\langle x^{1/(2p^\infty)}, (\frac{\varpi^n}{x})^{1/(2p^\infty)} \rangle^+$ must be a (ϖ^{1/p^∞})-almost isomorphism. Since $K^\circ\langle x^{1/(2p^\infty)} \rangle$ is perfectoid, another application of the perfectoid Riemann extension theorem as in the proof of Proposition 2.3.10 shows that the natural map $K^\circ\langle x^{1/(2p^\infty)} \rangle \xrightarrow{\sim} \lim_n K^\circ\langle x^{1/(2p^\infty)}, (\frac{\varpi^n}{x})^{1/(2p^\infty)} \rangle$ is an $(\varpi x)^{1/p^\infty}$-almost isomorphism. All in all, Proposition 2.3.10 then shows that $\widetilde{S}^+$ can $(\varpi x)^{1/p^\infty}$-almost be identified with $K^\circ\langle x^{1/(2p^\infty)} \rangle$.

One of the ingredients in the proof of Proposition 2.3.10 can be phrased by saying that under the given hypotheses, $\widetilde{S}^+$ is almost isomorphic to its image under the counit of an adjunction

$$\begin{tikzcd} R^{+,a}\text{-Perfd} \arrow[r, bend left, "\Lambda"] & 2\text{-}\lim_n R\langle \tfrac{\varpi^n}{g} \rangle^{+,a}\text{-Perfd}. \arrow[l, bend left, "\lim^\natural"] \end{tikzcd}$$

Here, $R^{+,a}$-Perfd denotes the category of ϖ-complete, ϖ-torsionfree perfectoid R^+-algebras, up to $(\varpi g)^{1/p^\infty}$-almost isomorphism, and $R\langle \frac{\varpi^n}{g} \rangle^{+,a}$-Perfd denotes the category of ϖ-complete, ϖ-torsionfree perfectoid $R\langle \frac{\varpi^n}{g} \rangle^+$-algebras, up to (ϖ^{1/p^∞})-almost isomorphism. The top, left adjoint functor Λ is given by the localizations $S^+ \mapsto \{S\langle \frac{\varpi^n}{g} \rangle^+\}_{n \in \mathbf{N}}$, the bottom, right adjoint functor $\lim^\natural$ by the perfectoid inverse limit $\{S_n^+\}_{n \in \mathbf{N}} \mapsto (\lim_{n \in \mathbf{N}} S_n^{+,\flat})^\sharp$.[1] By [And18a, § 4.4.2], the counit of this adjunction is always an almost isomorphism, or equivalently, Λ is fully faithful. This naturally leads to the following question.

Question 2.3.13 ([And18a, Qn. 4.4.3])**.** *Is Λ also essentially surjective (and thus an equivalence)?*

As the following example due to Lütkebohmert and Lütkebohmert–Schmechta shows, the analog of this question in positive characteristic rigid geometry has in general a negative answer.

Example 2.3.14 ([Lüt93, Ex. 2.10],[LS05, Ex. 5.3])**.** Consider the Tate algebra $R = K\langle x \rangle$ over a (complete) nonarchimedean field K of characteristic p with uniformizer ϖ. Inside the

[1] We describe the inverse limit via the tilting equivalence (i.e., the equivalence of Theorem 2.2.5.(ii)) so that its image lies in the correct category: limits in perfectoid Huber pairs coincide with limits in complete Huber pairs in positive characteristic, but not necessarily in mixed characteristic (cf. also [And18a, § 3.8.4]).

rigid closed unit disk $\mathbf{D} := \mathrm{Spa}(R, R^\circ)$ over K, we have annuli $\mathbf{D}(\frac{t^n}{x})$ with associated rings of functions $R_n := R\langle\frac{\varpi^n}{g}\rangle = \mathrm{H}^0(\mathbf{D}(\frac{\varpi^n}{x}), \mathscr{O}_\mathbf{D})$ and the punctured disk $\mathbf{D}^* := \mathbf{D} \smallsetminus \{0\} = \bigcup_n \mathbf{D}(\frac{\varpi^n}{x})$ with ring of functions $\widetilde{R} = \mathrm{H}^0(\mathbf{D}^*, \mathscr{O}_\mathbf{D})$. Let $h := \sum_{i=1}^\infty \varpi^{i^2} x^{1-pi} \in \widetilde{R}$; it has an essential singularity at the origin. For each $n \in \mathbf{N}$, we can rewrite $h = \sum_{i=1}^\infty \varpi^{i^2+n(1-pi)}(\frac{\varpi^n}{x})^{pi-1}$ as an element of R_n.

Let $R_n \to S_n := R_n[y]/(y^p - y - h)$ be the Artin–Schreier covers associated with h. Then the S_n form a compatible inverse system of finite étale extensions of R_n that do not extend to R.

Here, we show that this example does not carry over to the perfectoid setting, leaving open Question 2.3.13. Let us take up some of the notation of Example 2.3.12 and Example 2.3.14. Fix a perfectoid field K *of characteristic* $p > 0$ with pseudouniformizer ϖ. Let $R := K\langle x^{1/p^\infty}\rangle$ be the perfectoid Tate algebra and $\mathbf{D} := \mathrm{Spa}(R, R^\circ)$.

We can represent elements of $R_n := R\langle\frac{\varpi^n}{x}\rangle \simeq K^\circ[x^{1/p^\infty}, (\frac{\varpi^n}{x})^{1/p^\infty}]^\wedge[\frac{1}{\varpi}]$ (where the completion is ϖ-adic) by formal power series $f = \sum_{i\in\mathbf{Z}[p^{-1}]} a_i x^i$ with the constraint that for all $\varepsilon > 0$, we have $|a_i| < \varepsilon$ and $|a_{-i}\varpi^{-ni}| < \varepsilon$ for all but finitely many $i \in \mathbf{Z}[p^{-1}]_{>0}$. The transition morphisms $R_{n+1} \to R_n$ are injective and given by the natural inclusions.

Consider the limit $\widetilde{R} := \lim_n R_n \simeq \bigcap_n R_n$ in the category of topological rings. If $\mathbf{D}^* := \mathbf{D} \smallsetminus \{0\}$ denotes the punctured perfectoid closed unit disk, we have

$$\widetilde{R} = \mathrm{H}^0(\mathbf{D}^*, \mathscr{O}_\mathbf{D}) = \left\{ f = \sum_{i\in\mathbf{Z}[p^{-1}]} a_i x^i \;\middle|\; \begin{array}{l} \text{for all } \varepsilon > 0,\ |a_i| < \varepsilon \text{ and } |a_{-i}|^{1/(i+1)} < \varepsilon \\ \text{for all but finitely many } i \in \mathbf{Z}[p^{-1}]_{>0} \end{array} \right\}. \tag{2.3}$$

In this concrete description, we used that all the R_n admit compatible inclusions into the K-module $\mathrm{Map}(\mathbf{Z}[p^{-1}], K)$ of formal expressions $f = \sum_{i\in\mathbf{Z}[p^{-1}]} a_i x^i$; their filtered limits and filtered colimits can thus be computed as intersections and unions inside $\mathrm{Map}(\mathbf{Z}[p^{-1}], K)$. Note that although the Cauchy product does not equip $\mathrm{Map}(\mathbf{Z}[p^{-1}], K)$ with a ring structure due to the appearance of infinite sums, the sums occuring in the Cauchy products of series coming from the various R_n do converge and recover the product structure.

Fix any $h := \sum_{i\in\mathbf{Z}[p^{-1}]} h_i x^i \in \widetilde{R}$ and consider the associated Artin–Schreier extension $\widetilde{R} \to \widetilde{S}$. In the rigid setting of Example 2.3.14, a suitable choice of h_i yielded an extension which does not arise via base change from a map $R \to S$. We now exhibit the contrasting behavior in the perfectoid setting.

Lemma 2.3.15. *The étale extension $\widetilde{R} \to \widetilde{S}$ is an Artin–Schreier extension associated with some $m \in R[\frac{1}{x}]$.*

Proof. By Artin–Schreier theory, we need to find $m \in R[\frac{1}{x}]$ and $e \in \widetilde{R}$ such that $h = m + e^p - e$. Since we are free to add the degree ≥ -1 part of h to m, we may assume without loss of generality that $h_i = 0$ for all $i \geq -1$. Further, we can only have $|h_i| \geq 1$ for finitely many i, so we may also assume that $|h_i| < 1$ for all i. Denote by log the logarithm with base p. Set

$$m := \sum_{i \in \mathbf{Z}[p^{-1}]_{<-1}} h_i^{1/p^{\lceil \log(-i) \rceil}} x^{i/p^{\lceil \log(-i) \rceil}} \quad \text{and} \quad e := \sum_{i \in \mathbf{Z}[p^{-1}]_{<-1}} \sum_{j=1}^{\lceil \log(-i) \rceil} h_i^{1/p^j} x^{i/p^j}.$$

First, we check that $e \in \widetilde{R}$ and $m \in R[\frac{1}{x}]$ using the concrete descriptions of these rings from above. Under the standing assumption $|h_i| < 1$, we have for all $i \in \mathbf{Z}[p^{-1}]_{<-1}$ and all $1 \leq j \leq \lceil \log(-i) \rceil$,

$$|h_i^{1/p^j}|^{\frac{1}{1-i/p^j}} = |h_i|^{\frac{1}{p^j - i}} \leq |h_i|^{\frac{1}{-(p+1)i}} \leq \left(|h_i|^{\frac{1}{1-i}}\right)^{1/(p+1)}.$$

For all $\varepsilon > 0$, there exist only finitely many $i \in \mathbf{Z}[p^{-1}]_{<-1}$ such that $|h_i|^{\frac{1}{1-i}} \geq \varepsilon^{p+1}$ because $h \in \widetilde{R}$. Hence, the coefficients of e meet the criterion of (2.3). Likewise, we have

$$|h_i^{1/p^{\lceil \log(-i) \rceil}}| \leq |h_i|^{1/-pi} \leq \left(|h_i|^{\frac{1}{1-i}}\right)^p < \varepsilon$$

for all but finitely many $i \in \mathbf{Z}[p^{-1}]_{<-1}$, so that $x \cdot m \in R$ and thus $m \in R[\frac{1}{x}]$.

These convergence properties ensure that products and sums of m and e can be calculated via the usual formal Cauchy product and formal sum formulas for formal series. Therefore, a telescoping sum argument shows

$$\begin{aligned} m + e^p - e &= \sum_i h_i^{1/p^{\lceil \log(-i) \rceil}} x^{i/p^{\lceil \log(-i) \rceil}} + \sum_i \sum_{j=1}^{\lceil \log(-i) \rceil} h_i^{1/p^{j-1}} x^{i/p^{j-1}} - \sum_i \sum_{j=1}^{\lceil \log(-i) \rceil} h_i^{1/p^j} x^{i/p^j} \\ &= \sum_i \left(h_i^{1/p^{\lceil \log(-i) \rceil}} x^{i/p^{\lceil \log(-i) \rceil}} + h_i^{1/p^0} x^{i/p^0} - h_i^{1/p^{\lceil \log(-i) \rceil}} x^{i/p^{\lceil \log(-i) \rceil}} \right) = \sum_i h_i x^i = h. \quad \square \end{aligned}$$

Corollary 2.3.16. *The system of integral rings corresponding to the inverse system $\{R_n \to S_n\}_{n \in \mathbf{N}}$ of Artin–Schreier extensions associated with h lies in the essential image of the localization functor Λ.*

Proof. Follows from [And18a, Prop. 4.4.4]. □

In § 2.5, we will see that Question 2.3.13 has a positive answer if one considers derived categories instead of categories of perfectoid algebras.

2.4 Globalization: diamonds

In this section, we describe a procedure to perfectoidify general analytic adic spaces over a perfectoid base. To do so, we rely on the notion of diamonds, which has been developed by Scholze in [Sch17]; although we give all necessary definitions, we refer to [SW20] for a more comprehensive introduction. We begin by recasting the perfection functor for rings from Definition 2.1.2 in more abstract terms. Fix a perfect ring R. Denote the opposite category of R-algebras by $\mathrm{Alg}_R^{\mathrm{op}}$, and the opposite category of perfect R-algebras by $\mathrm{Alg}_R^{\mathrm{perf,op}}$, both with the structure of a site via the étale topology.

Lemma 2.4.1. *The perfection functor* $\mathrm{Alg}_R^{\mathrm{op}} \to \mathrm{Alg}_R^{\mathrm{perf,op}}$ *defines an embedding of topoi* $\iota\colon \mathrm{Shv}(\mathrm{Alg}_R^{\mathrm{perf,op}}) \hookrightarrow \mathrm{Shv}(\mathrm{Alg}_R^{\mathrm{op}})$ *given by* $(\iota_* F)(S) = F(S_{\mathrm{perf}})$ *and* $(\iota^{-1}G)(S) = G(S)$*. This embedding admits a retraction* $\rho\colon \mathrm{Shv}(\mathrm{Alg}_R^{\mathrm{op}}) \to \mathrm{Shv}(\mathrm{Alg}_R^{\mathrm{perf,op}})$ *with* $(\rho_* G)(S) = G(S)$*, induced by the inclusion* $\mathrm{Alg}_R^{\mathrm{perf,op}} \hookrightarrow \mathrm{Alg}_R^{\mathrm{op}}$ *as full subcategory. Moreover,* ι_* *and* ρ_* *both preserve representable sheaves. In particular, the perfection of any* R*-algebra* S *can be identified with* $\rho_* \mathrm{Hom}_{\mathrm{Alg}_R^{\mathrm{op}}}(-, S)$*.*

Proof. The statement follows from general results in topos theory; see e.g. [SP20, Lem. 00XU, Lem. 00XY]. We need to use that $\mathrm{Alg}_R^{\mathrm{perf,op}}$ admits fiber products (tensor products of perfect R-algebras are perfect), the full embedding $\mathrm{Alg}_R^{\mathrm{perf,op}} \hookrightarrow \mathrm{Alg}_R^{\mathrm{op}}$ is continuous (preserves covers and fiber products), cocontinuous (an étale cover of a perfect R-algebra is perfect because étale morphisms are relatively perfect) and has the perfection functor as continuous right adjoint, and that R is perfect. $\square$

Remark 2.4.2. Since the *limit* perfection functor $R \mapsto R^{\mathrm{perf}} := \lim(\cdots \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} R)$ is left adjoint to the inclusion $\mathrm{Alg}_R^{\mathrm{perf,op}} \hookrightarrow \mathrm{Alg}_R^{\mathrm{op}}$, the inverse image $\rho^{-1}F$ is the sheafification of the presheaf $S \mapsto F(S^{\mathrm{perf}})$ [SP20, Lemma 00XU, Lemma 08NH].

In the global setting of § 2.2, we can mimic this description via sheaves to define the promised perfectoidizations. Fix a perfectoid space T; in later applications, we will take $T = \mathrm{Spa}(K, K^\circ)$ for a perfectoid field K. Denote by An_T the category of analytic adic spaces over T and by Perfd_T the big pro-étale site of T in which the objects are all perfectoid spaces $X \to T$ and covers are families of morphisms $\{f_i\colon X_i \to X\}_{i\in I}$ such that

- each f_i is pro-étale,
- $\bigcup_{i\in I} f_i(X_i) = X$, and
- for each quasi-compact open subspace $U \subseteq X$, there exists a finite subset $J \subseteq I$ and quasi-compact open subspaces $U_j \subseteq X_j$ for all $j \in J$ such that $U = \bigcup_{j\in J} f_j(U_j)$.

Note that by the tilting equivalence (Corollary 2.2.8), $\mathrm{Perfd}_T \simeq \mathrm{Perfd}_{T^\flat}$ can be manifestly described using only positive characteristic. As in Lemma 2.4.1, we then obtain a functor

$$\mathrm{An}_T \to \mathrm{PreShv}(\mathrm{Perfd}_T), \quad Z \mapsto Z^\diamondsuit \colon (X \mapsto \mathrm{Hom}(X, Z)) \tag{2.4}$$

Scholze realized that the presheaf $Z^\diamondsuit$ has good geometric properties, analogous to those of algebraic spaces in algebraic geometry.

Definition 2.4.3 ([Sch17, Def. 11.1])**.** (i) A pro-étale equivalence relation on a perfectoid space $X \to T$ is a monomorphism of perfectoid spaces $R \hookrightarrow X \times_T X$ such that the two projections $s, t \colon R \hookrightarrow X \times_T X \to X$ are pro-étale and $R(Y)$ induces an equivalence relation on $X(Y) \times_{T(Y)} X(Y)$ for all perfectoid spaces Y over T.

(ii) A sheaf Δ on Perfd_T is called a *diamond over* T if it fits into a coequalizer diagram

$$R \underset{t}{\overset{s}{\rightrightarrows}} X \longrightarrow \Delta$$

for some perfectoid space $X \to T$ and a pro-étale equivalence relation $R \subset X \times_T X$.

Despite their abstract definition, diamonds still carry topological information.

Definition 2.4.4 ([Sch17, Def. 11.14, Def. 11.17])**.** Let Δ be a diamond over X.

1. The *underlying topological space* of Δ is $|\Delta| := |X|/|R|$ for a presentation $\Delta \simeq X/R$ as in Definition 2.4.3; this is independent of the choice of presentation by [Sch17, Prop. 11.13].

2. Δ is *spatial* if it is quasi-compact and quasi-separated (see [SGA4, § VI.I], [Sch17, § 8]) and $|U| \subset |X|$ gives a basis of the topology, where U ranges over all quasi-compact open subsheaves of X (and thus again diamonds).

The underlying topological space of a spatial diamond is a spectral space by [Sch17, Prop. 11.18.(i)].

Lemma 2.4.5 ([Sch17, Lem. 15.6])**.** *Let Z be a quasi-compact and quasi-separated analytic adic space over T. Then the presheaf $Z^\diamondsuit$ defined above is a spatial diamond and $|Z^\diamondsuit| \simeq |Z|$.*

We call $Z^\diamondsuit$ the *diamondification* of Z and think of it as an abstract perfectoidization.

Remark 2.4.6. For any Tate–Huber pair (A, A^+), [Sch17, Lem. 15.1] defines a sheaf

$$\mathrm{Spd}(A, A^+) \colon X \mapsto \mathrm{Hom}\Big((A, A^+), (\mathscr{O}_X(X), \mathscr{O}_X^+(X))\Big)$$

in analogy with (2.4). When $Z = \mathrm{Spa}(A, A^+)$, we have $Z^\lozenge \simeq \mathrm{Spd}(A, A^+)$; however, even when (A, A^+) is not sheafy, $\mathrm{Spd}(A, A^+)$ can be defined and is a spatial diamond by [Sch17, Prop. 15.4].

Remark 2.4.7. In [Sch17], Scholze studies a decidedly more flexible, absolute notion of diamonds, which does not require a perfectoid base space. He also introduces diamondifications $Z^\lozenge$ of analytic adic spaces Z over $\mathbf{Z}_p$ in that generality. However, in this absolute case one cannot simply think of diamondifications as perfectoidizations anymore: passing to $Z^\lozenge$ also "forgets" the structure morphism to $\mathbf{Z}_p$. Here, we content ourselves with the simpler relative theory over a perfectoid base, which will be sufficient for our purposes.

Besides the topological information, diamondifications also preserve étale cohomological data. For simplicity, we will focus on a well-behaved class of adic spaces that was introduced by Huber in [Hub96, Cond. 1.1.1] and has a good notion of étale site [Hub96, § 2.1].

Definition 2.4.8. An analytic adic space is *locally noetherian* if it is covered by affinoid adic spaces $\mathrm{Spa}(R, R^+)$ for which R is a strongly noetherian Tate ring.

A morphism of diamonds is *étale* if it is locally separated, representable in perfectoid spaces, and the pullback to every perfectoid space is étale in the sense of Definition 2.2.7.(ii) [Sch17, Def. 10.1.(ii)]. One can then define the small étale site $\Delta_{\text{ét}}$ of a diamond Δ as usual.

Lemma 2.4.9 ([Sch17, Lem. 15.6])**.** *For any locally noetherian or perfectoid space Z over T, the diamondification functor induces an equivalence of sites $Z^\lozenge_{\text{ét}} \xrightarrow{\sim} Z_{\text{ét}}$.*

Remark 2.4.10. The statement remains true for general analytic adic spaces over T if one defines the étale site $Z_{\text{ét}}$ using preadic spaces; the complication arises because it is not clear whether finite étale extensions of sheafy Huber pairs are sheafy.

2.4.1 Relation to the perfectoidizations from § 2.3

In Definition 2.3.8, we reviewed the notion of a finite morphism of complete Huber pairs. In the locally noetherian setup, this definition can immediately be globalized.

Definition 2.4.11 ([Hub96, (1.4.4)])**.** A morphism $f\colon Y \to Z$ of locally noetherian adic spaces is finite if the following equivalent conditions hold:

- every $z \in Z$ admits an open affinoid neighborhood V such that $U := f^{-1}(V)$ is affinoid and the induced morphism $(\mathscr{O}_Z(V), \mathscr{O}_Z^+(V)) \to (\mathscr{O}_Y(U), \mathscr{O}_Y^+(U))$ is finite;
- for every open affinoid $V \subseteq Z$, the preimage $U := f^{-1}(V)$ is affinoid and the induced morphism $(\mathscr{O}_Z(V), \mathscr{O}_Z^+(V)) \to (\mathscr{O}_Y(U), \mathscr{O}_Y^+(U))$ is finite.

Unfortunately, as of now it seems unclear whether there is a good notion of finite morphisms between general analytic adic spaces that behaves nicely with respect to localizations even in the analytic topology (e.g., so that the conditions of Definition 2.4.11 continue to be equivalent). We can therefore only make sense of the phrase "finite morphism of perfectoid spaces" if it is pulled back from locally noetherian spaces. The next result, which is a direct consequence of Theorem 2.3.1, states roughly that the question whether "diamond perfectoidizations" are indeed perfectoid spaces behaves well under finite morphisms.

Proposition 2.4.12. *Let $X_0 = \mathrm{Spa}(A_0, A_0^+)$ be a locally noetherian, affinoid analytic adic space over T. Let $X = \mathrm{Spa}(A, A^+) \to X_0$ be an affinoid perfectoid space over X_0 and $Y_0 = \mathrm{Spa}(B_0, B_0^+) \to X_0$ be a finite morphism. Then the diamond $Y_0^\Diamond \times_{X_0^\Diamond} X^\Diamond$ is the diamondification of an affinoid perfectoid space Y.*

Proof. Let ϖ be a pseudouniformizer for A_0. Since the diamondification functor from (2.4) commutes with limits, $Y_0^\Diamond \times_{X_0^\Diamond} X^\Diamond \simeq \mathrm{Spd}(B, B^+)$, where (B, B^+) is the finite (A, A^+)-algebra with $B = B_0 \otimes_{A_0} A$ and B^+ the integral closure of A^+ in B (cf. [Hub96, Lem. 1.4.5.i)]). Let $(B, B^+) \to (\widetilde{B}, \widetilde{B}^+)$ be the perfectoidization morphism from Corollary 2.3.9. Then $Y := \mathrm{Spa}(\widetilde{B}, \widetilde{B}^+)$ is perfectoid and the morphisms $Y \to Y_0$ and $Y \to X$ coming from the morphisms of affinoid Tate rings $(B_0, B_0^+) \to (\widetilde{B}, \widetilde{B}^+)$ and $(A, A^+) \to (\widetilde{B}, \widetilde{B}^+)$ induce the desired isomorphism of diamonds $Y^\Diamond \xrightarrow{\sim} Y_0^\Diamond \times_{X_0^\Diamond} X^\Diamond$. □

The affinoid statement of Proposition 2.4.12 globalizes by a standard gluing argument.

Proposition 2.4.13. *Let X_0 be a locally noetherian analytic adic space over T. Let $X \to X_0$ be a perfectoid space over X_0 and $f_0 \colon Y_0 \to X_0$ be a finite morphism. Then the diamond $Y_0^\Diamond \times_{X_0^\Diamond} X^\Diamond$ is the diamondification of a perfectoid space Y.*

Proof. Choose an open cover $X = \bigcup_{\alpha \in J} \widetilde{U}_\alpha$ by affinoid perfectoid spaces $\widetilde{U}_\alpha$ for which the compositions $\widetilde{U}_\alpha \to X_0$ factor through affinoid open subspaces $U_\alpha \subseteq X_0$. By Proposition 2.4.12, there are perfectoid spaces $f_\alpha \colon \widetilde{V}_\alpha \to \widetilde{U}_\alpha$ together with isomorphisms $\xi_\alpha \colon \widetilde{V}_\alpha^\Diamond \xrightarrow{\sim} \left(f_0^{-1}(U_\alpha)\right)^\Diamond \times_{U_\alpha^\Diamond} \widetilde{U}_\alpha^\Diamond \simeq Y_0^\Diamond \times_{X_0^\Diamond} \widetilde{U}_\alpha^\Diamond$; as we work over a fixed perfectoid base, this simply means that $\widetilde{V}_\alpha^\flat$ represents the latter sheaf.

For all $\alpha, \beta \in J$, set $\widetilde{U}_{\alpha\beta} := \widetilde{U}_\alpha \cap \widetilde{U}_\beta \subseteq X$, and similarly for triple intersections $\widetilde{U}_{\alpha\beta\gamma}$. Let $\widetilde{V}_{\alpha\beta} := f_\alpha^{-1}\left(\widetilde{U}_{\alpha\beta}\right) \subseteq \widetilde{V}_\alpha$. Since $\widetilde{V}_{\alpha\beta} \simeq \widetilde{V}_\alpha \times_{\widetilde{U}_\alpha} \widetilde{U}_{\alpha\beta}$, the tilt $\widetilde{V}_{\alpha\beta}^\flat$ represents

$$\widetilde{V}_\alpha^\Diamond \times_{\widetilde{U}_\alpha^\Diamond} \widetilde{U}_{\alpha\beta}^\Diamond \xrightarrow[\xi_\alpha \restriction \widetilde{V}_{\alpha\beta}^\flat]{\sim} Y_0^\Diamond \times_{X_0^\Diamond} \widetilde{U}_\alpha^\Diamond \times_{\widetilde{U}_\alpha^\Diamond} \widetilde{U}_{\alpha\beta}^\Diamond \simeq Y_0^\Diamond \times_{X_0^\Diamond} \widetilde{U}_{\alpha\beta}^\Diamond.$$

The corresponding strings of isomorphisms for the perfectoid spaces $\widetilde{V}_{\beta\alpha}^\flat \subseteq \widetilde{V}_\beta^\flat$, $\widetilde{V}_{\alpha\beta}^\flat \cap \widetilde{V}_{\alpha\gamma}^\flat \subseteq \widetilde{V}_\alpha^\flat$, and $\widetilde{V}^\flat_{\beta\alpha} \cap \widetilde{V}^\flat_{\beta\gamma} \subseteq \widetilde{V}^\flat_\beta$ yield the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Mor}\big(-, \widetilde{V}^\flat_{\alpha\beta} \cap \widetilde{V}^\flat_{\alpha\gamma}\big) \arrow[r, "\sim"] \arrow[d] & Y_0^\diamondsuit \times_{X_0^\diamondsuit} \widetilde{U}^\diamondsuit_{\alpha\beta\gamma} \arrow[d] & \mathrm{Mor}\big(-, \widetilde{V}^\flat_{\beta\alpha} \cap \widetilde{V}^\flat_{\beta\gamma}\big) \arrow[l, "\sim"'] \arrow[d] \\
\mathrm{Mor}\big(-, \widetilde{V}^\flat_{\alpha\beta}\big) \arrow[r, "\xi_\alpha \restriction \widetilde{V}^\flat_{\alpha\beta}", "\sim"'] & Y_0^\diamondsuit \times_{X_0^\diamondsuit} \widetilde{U}^\diamondsuit_{\alpha\beta} & \mathrm{Mor}\big(-, \widetilde{V}^\flat_{\beta\alpha}\big). \arrow[l, "\xi_\beta \restriction \widetilde{V}^\flat_{\beta\alpha}"', "\sim"]
\end{tikzcd}$$

By the full faithfulness of the Yoneda functor, the bottom lines are induced by unique isomorphisms $\varphi_{\alpha\beta} \colon \widetilde{V}^\flat_{\alpha\beta} \xrightarrow{\sim} \widetilde{V}^\flat_{\beta\alpha}$ such that $\varphi^{-1}_{\alpha\beta}\big(\widetilde{V}^\flat_{\beta\alpha} \cap \widetilde{V}^\flat_{\beta\gamma}\big) = \widetilde{V}^\flat_{\alpha\beta} \cap \widetilde{V}^\flat_{\alpha\gamma}$ and the cocycle condition

$$\varphi_{\beta\gamma}\Big|_{\widetilde{V}^\flat_{\beta\alpha} \cap \widetilde{V}^\flat_{\beta\gamma}} \circ \varphi_{\alpha\beta}\Big|_{\widetilde{V}^\flat_{\alpha\beta} \cap \widetilde{V}^\flat_{\alpha\gamma}} = \varphi_{\alpha\gamma}\Big|_{\widetilde{V}^\flat_{\alpha\beta} \cap \widetilde{V}^\flat_{\alpha\gamma}}$$

is satisfied. Using these isomorphisms and the tilting equivalence, we can glue the various $\widetilde{V}_\alpha$ to a perfectoid space Y.

Moreover, the ξ_α glue to a morphism $\xi \colon Y^\diamondsuit \to Y_0^\diamondsuit \times_{X_0^\diamondsuit} X^\diamondsuit$ because $\xi_\beta \circ \varphi^\diamondsuit_{\alpha\beta} = \xi_\alpha\Big|_{\widetilde{V}^\flat_{\alpha\beta}}$ essentially by definition. We can check that ξ is an isomorphism after base change along the open cover $Y_0^\diamondsuit \times_{X_0^\diamondsuit} \widetilde{U}^\diamondsuit_\alpha$. In that case, this follows from the fact that the ξ_α are isomorphisms. $\square$

Example 2.4.14. When $Y_0 \to X_0$ is a Zariski-closed immersion with corresponding coherent ideal sheaf $\mathscr{J}_0 \subset \mathscr{O}_{X_0}$, the perfectoid space Y is given in the affinoid setting of Proposition 2.4.12 by the Zariski-closed subspace corresponding to the ideal $\mathscr{J}_0(\mathrm{Spa}(A_0, A_0^+)) \cdot A \subseteq A$ with its perfectoid structure from [Sch15, Lem. 2.2.2]; cf. also [BS19, Rmk. 7.5]. In the general setting of Proposition 2.4.13, these spaces glue again to $Y \subset X$ by universality.

2.4.2 Structure sheaves

Away from the setting of § 2.4.1, the diamond $Z^\diamondsuit$ attached to an analytic adic space Z over T is usually not representable by a perfectoid space anymore, corresponding to the fact that the perfectoidization of R needs to be thought of as a cosimplicial ring. To make statements in that generality, we need to use structure sheaves on diamonds, following e.g. [CGH+20, § 2.3]. While they can in principle be defined similarly to those of algebraic spaces, we have to iron out an additional wrinkle in the theory: whereas algebraic spaces can be equivalently described as étale sheaf quotiens of schemes by étale equivalence relations or as étale sheaves with an étale surjection from a scheme (and representable diagonal), it is no longer true that every diamond admits a pro-étale surjection from a perfectoid space. Thus, one cannot obtain a pro-étale structure sheaf on diamonds by simply bootstrapping from perfectoid spaces via pro-étale covers. Instead, one needs to work with a slightly more general class of covers, which is pro-étale local on the target.

Definition 2.4.15 ([Sch17, Def. 10.1.(i)])**.** A map $f\colon F \to G$ of pro-étale sheaves on Perfd_T is *quasi-pro-étale* if it is locally on F separated and for every strictly totally disconnected perfectoid space Y and every morphism $Y \to G$, the product $F \times_G Y$ is a perfectoid space and the induced morphism $F \times_G Y \to Y$ is pro-étale.

This condition can be checked "pointwise": if f is quasi-compact and separated, then f is quasi-pro-étale if and only if the pullback $F \times_G \mathrm{Spa}(C, C^\circ) \to \mathrm{Spa}(C, C^\circ)$ to every geometric point $\mathrm{Spa}(C, C^\circ) \to G$ of rank 1 is affinoid pro-étale (i.e., a profinite set) [Sch17, Lem. 7.19].

Proposition 2.4.16 ([Sch17, Prop. 11.5])**.** *A pro-étale sheaf Δ on Perfd_T is a diamond if and only if it admits a quasi-pro-etale surjection $X \twoheadrightarrow \Delta$ from a perfectoid space X.*

Definition 2.4.17 ([Sch17, Def. 14.1])**.** Let Δ be a diamond. The *(small) quasi-pro-étale site* $\Delta_{\text{qproét}}$ has the underlying category of all quasi-pro-étale maps of diamonds $\Gamma \to \Delta$ and covers given by families of morphisms $\{f_i\colon \Gamma_i \to \Gamma\}_{i \in I}$ such that

- each f_i is quasi-pro-étale,
- the induced morphism $\bigsqcup_{i \in I} f_i\colon \bigsqcup_{i\in I} \Gamma_i \to \Gamma$ is surjective, and
- for each quasi-compact open $\Gamma' \subseteq \Gamma$, there exist a finite subset $J \subseteq I$ and quasi-compact open subdiamonds $\Gamma'_j \subseteq \Gamma_j$ for all $j \in J$ such that $\bigsqcup_{j\in J} f_j\big|_{\Gamma'_j}\colon \bigsqcup_{j \in J} \Gamma'_J \to \Gamma'$ is surjective.

Lastly, we will need the following lemma.

Lemma 2.4.18. *Let Δ be a diamond. Let $\Delta_{\text{ét}}$ be its étale site, $\Delta_{\text{qproét}}$ its quasi-pro-étale site, and $\Delta^{\text{perfd}}_{\text{qproét}} \subset \Delta_{\text{qproét}}$ the restriction of the quasi-pro-étale site to all perfectoid spaces which are quasi-pro-étale over Δ. Then*

(i) *The natural inclusion of categories $\Delta^{\text{perfd}}_{\text{qproét}} \to \Delta_{\text{qproét}}$ induces an equivalence of topoi $\mathrm{Shv}(\Delta^{\text{perfd}}_{\text{qproét}}) \xrightarrow{\sim} \mathrm{Shv}(\Delta_{\text{qproét}})$;*

(ii) *If Δ is locally spatial, the natural morphism of sites $\nu_\Delta\colon \Delta_{\text{qproét}} \to \Delta_{\text{ét}}$ induces a fully faithful functor $\nu^*_\Delta\colon \mathrm{Shv}(\Delta_{\text{ét}}) \to \mathrm{Shv}(\Delta_{\text{qproét}})$.*

Proof. Since every diamond admits a quasi-pro-étale cover by a perfectoid space, (i) follows from general topos theory; cf. e.g. [SP20, Lem. 03A0]. Part (ii) is [Sch17, Prop. 14.8]. ☐

Definition 2.4.19. Let Δ be a diamond. The *integral structure sheaf* $\mathscr{O}^+_\Delta$ on $\Delta_{\text{qproét}}$ is given under the equivalence of Lemma 2.4.18.(i) by the assignment

$$(Y \in \Delta^{\text{perfd}}_{\text{qproét}}) \mapsto \mathscr{O}^+_Y(Y).$$

By means of the integral structure sheaf, we can now give a description of perfectoidizations in mixed characteristic without the integrality constraints of Theorem 2.3.1, at least up to almost isomorphism. Note that although this fails in positive characteristic due to completion issues, perfectoidizations are in that case simply given by the perfections from Definition 2.1.2. The following statement is essentially [BS19, Prop. 8.5], bar some translations from the language of diamonds to that of integral perfectoid rings. For the proof, we assume that the reader is familiar with the more general notion of perfectoidizations from [BS19].

Proposition 2.4.20. *Let (R, R^+) be a perfectoid Huber pair with* $\operatorname{char} R = 0$*. Fix a pseudouniformizer $\varpi^\flat \in R^{\flat,+}$ and set $\varpi := \sharp(\varpi^\flat)$. Let (S, S^+) be a p-adically complete (R, R^+)-algebra. Then there is a (ϖ^{1/p^∞})-almost isomorphism*

$$S^+_{\mathrm{perfd}} \overset{a}{\simeq} \mathrm{R}\Gamma(\mathrm{Spd}(S, S^+)_{\mathrm{qpro\acute{e}t}}, \mathscr{O}^+).$$

Proof. Set $\Delta := \mathrm{Spd}(S, S^+)$. Consider the big quasi-pro-étale site $(\mathrm{Dmd}/\Delta)_{\mathrm{qpro\acute{e}t}}$ and the big affinoid perfectoid quasi-pro-étale site $(\mathrm{Aff}/\Delta)_{\mathrm{qpro\acute{e}t}}$; their underlying categories are the category of all diamonds over Δ and the full subcategory of affinoid perfectoid spaces over Δ, respectively, and their coverings those of Definition 2.4.17.[2] As in Lemma 2.4.18, the inclusion functor $(\mathrm{Aff}/\Delta)_{\mathrm{qpro\acute{e}t}} \to (\mathrm{Dmd}/\Delta)_{\mathrm{qpro\acute{e}t}}$ induces an equivalence of topoi $\mathrm{Shv}((\mathrm{Aff}/\Delta)_{\mathrm{qpro\acute{e}t}}) \xrightarrow{\sim} \mathrm{Shv}((\mathrm{Dmd}/\Delta)_{\mathrm{qpro\acute{e}t}})$ because every diamond admits a quasi-pro-étale cover by a collection of affinoid perfectoid spaces [SP20, Lem. 03A0]. In particular, we can still define an integral structure sheaf $\mathscr{O}^+$ on $(\mathrm{Dmd}/\Delta)_{\mathrm{qpro\acute{e}t}}$ following the procedure of Definition 2.4.19.

On the other hand, since the inclusion functor $\Delta_{\mathrm{qpro\acute{e}t}} \to (\mathrm{Dmd}/\Delta)_{\mathrm{qpro\acute{e}t}}$ is continuous, cocontinuous, commutes with fiber products, and preserves the final object Δ, it induces a morphism of topoi $\iota\colon \mathrm{Shv}(\Delta_{\mathrm{qpro\acute{e}t}}) \to \mathrm{Shv}((\mathrm{Dmd}/\Delta)_{\mathrm{qpro\acute{e}t}})$; moreover, the inverse image ι^{-1} is given by precomposition with this inclusion, i.e., by restriction to the small site [SP20, Lem. 00XU]. By [SP20, Lem. 03YU], we conclude that

$$\mathrm{R}\Gamma(\Delta_{\mathrm{qpro\acute{e}t}}, \mathscr{O}^+_\Delta) \simeq \mathrm{R}\Gamma(\Delta_{\mathrm{qpro\acute{e}t}}, \iota^{-1}\mathscr{O}^+) \simeq \mathrm{R}\Gamma((\mathrm{Dmd}/\Delta)_{\mathrm{qpro\acute{e}t}}, \mathscr{O}^+) \simeq \mathrm{R}\Gamma((\mathrm{Aff}/\Delta)_{\mathrm{qpro\acute{e}t}}, \mathscr{O}^+).$$

Next, we analyze the morphisms of sites

$$(\mathrm{Aff}/\Delta)_{\mathrm{qpro\acute{e}t}} \xrightarrow{\mu} (\mathrm{Aff}/\Delta)_{\mathrm{pro\acute{e}t}} \xrightarrow{\nu} (\mathrm{Aff}/\Delta)$$

of affinoid perfectoid spaces over Δ with the quasi-pro-étale, the pro-étale, and the indiscrete topology, respectively. By [Sch17, Lem. 7.18], every affinoid perfectoid space U admits an affinoid pro-étale cover $\widetilde{U} \to U$ by a strictly totally disconnected affinoid perfectoid

[2] We ignore all set-theoretic issues here, which have been treated at length in [Sch17].

space $\widetilde{U}$. The natural morphism of sites $\widetilde{U}_{\mathrm{qpro\acute{e}t}} \to \widetilde{U}_{\mathrm{pro\acute{e}t}}$ is an isomorphism essentially by definition, so that $\mathrm{H}^i(U_{\mathrm{qpro\acute{e}t}}, \mathscr{O}^+) \simeq \mathrm{H}^i(U_{\mathrm{pro\acute{e}t}}, \mathscr{O}^+)$ is almost zero for all $i > 0$ thanks to [Sch17, Prop. 8.5]. Since the higher direct images $\mathrm{R}^i\mu_*\mathscr{O}^+$ are the sheafifications of the presheaf $(U \to \Delta) \mapsto \mathrm{H}^i(U_{\mathrm{qpro\acute{e}t}}, \mathscr{O}^+)$ [SP20, Lem. 072W], we can conclude that they are almost zero for $i > 0$. Another application of [Sch17, Prop. 8.5] and [SP20, Lem. 072W] shows similarly that the $\mathrm{R}^i\nu_*\mathscr{O}^+$ are almost zero for all $i > 0$. The Leray spectral sequence therefore yields an almost isomorphism

$$\mathrm{R}\Gamma((\mathrm{Aff}/\Delta)_{\mathrm{qpro\acute{e}t}}, \mathscr{O}^+) \overset{a}{\simeq} \mathrm{R}\Gamma((\mathrm{Aff}/\Delta), (\nu \circ \mu)_*\mathscr{O}^+) \simeq \mathrm{R}\Gamma((\mathrm{Aff}/\Delta), \mathscr{O}^+).$$

However, the derived global sections of the presheaf topos for the catgory (Aff/Δ) are given by the homotopy limit

$$\mathrm{R}\Gamma((\mathrm{Aff}/\Delta), \mathscr{O}^+) \simeq \operatorname*{Rlim}_{U \in (\mathrm{Aff}/\Delta)} \mathscr{O}^+(U) \simeq \operatorname*{Rlim}_{S^+ \to T^+} T^+, \tag{2.5}$$

over all perfectoid S^+ algebras T^+ which are ϖ-torsionfree, ϖ-adically complete, and integrally closed in $T'[\frac{1}{\varpi}]$. For any perfectoid algebra $S^+ \to T'$, the integral closure T^+ of the image of T' in $T'[\frac{1}{\varpi}]$ is of this form by Lemma 2.1.14; since $\operatorname{char} R = 0$, the ϖ-completeness of T' and T^+ is automatic. Moreover, Proposition 2.1.15 provides a homotopy fiber square

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & T^+ \arrow[d] \\ \overline{T'} \arrow[r] & \overline{T^+}, \end{tikzcd}$$

where $\overline{T'} := (T'/p)_{\mathrm{perf}} \simeq T'/\sqrt{\varpi T'}$ (cf. Remark 2.1.16), and similarly for $\overline{T^+}$. In particular, the (isomorphic) cones of the two horizontal maps are (ϖ^{1/p^∞})-torsion so that $T' \to T^+$ is a (ϖ^{1/p^∞})-almost isomorphism. Therefore, the homotopy limit in (2.5) is almost isomorphic to the homotopy limit over all perfectoid S^+-algebras T' and thus S^+_{perfd} by [BS19, Prop. 8.5]. $\square$

We note for later use that if we work over a perfectoid field (K, K°) with pseudouniformizer ϖ, [CGH+20, § 2.3] checks that the quotient $\mathscr{O}^+/\varpi$ is well-behaved with respect to the étale topology and the diamondification functor.

Lemma 2.4.21 ([CGH+20, Lem. 2.3.2, Lem. 2.3.3])**.** (i) *If Δ is a locally spatial diamond over* $\mathrm{Spd}(K, K^\circ)$*, then $\mathscr{O}^+_\Delta/\varpi$ is an étale sheaf, i.e., in the essential image of the fully faithful functor* $\nu^*_\Delta\colon \mathrm{Shv}(\Delta_{\mathrm{\acute{e}t}}) \to \mathrm{Shv}(\Delta_{\mathrm{qpro\acute{e}t}})$ *from Lemma 2.4.18.*(ii)*.*

(ii) *If Z is a rigid space over (K, K°), then the pullback functor for the equivalence of sites*

$Z^{\lozenge}_{\text{ét}} \simeq Z_{\text{ét}}$ *of Lemma 2.4.9 identifies the sheaf of almost* $\mathscr{O}_K$*-modules* $(\mathscr{O}^+_Z/\varpi)^a$ *with* $(\mathscr{O}^+_{Z^\lozenge}/\varpi)^a$.

2.5 Derived categories for inverse systems of perfectoid rings

In this section, we explore the behavior of geometric objects such as derived categories in certain inverse systems of rings. This is inspired by the discussion in § 2.3.2, in particular André's Question 2.3.13 on the equivalence of certain categories of almost perfectoid algebras with inverse systems of their localizations. We answer a derived category analog of this question positively.

Proposition 2.5.1. *Let* (R, R^+) *be a perfectoid Huber pair with pseudouniformizer* $\varpi \in R^+$. *Choose* $g^\flat \in R^{\flat,+}$ *and set* $g := \sharp(g^\flat)$. *Then the pair of adjoint functors*

$$\mathrm{D}_{\mathrm{comp}}(R^{+,a}) \to \lim_n \mathrm{D}_{\mathrm{comp}}\Big(R\Big\langle \tfrac{\varpi^n}{g}\Big\rangle^{+,a}\Big), \quad M \mapsto \Big\{\Big(M \otimes^{\mathrm{L}}_{R^{+,a}} R\Big\langle \tfrac{\varpi^n}{g}\Big\rangle^{+,a}\Big)^\wedge\Big\}_{n\in\mathbf{N}}$$

and

$$\lim_n \mathrm{D}_{\mathrm{comp}}\Big(R\Big\langle \tfrac{\varpi^n}{g}\Big\rangle^{+,a}\Big) \to \mathrm{D}_{\mathrm{comp}}(R^{+,a}), \quad \{M_n\}_{n\in\mathbf{N}} \mapsto \mathrm{R}\lim_n M_n.$$

gives an adjoint equivalence of categories, where almost mathematics is with respect to $(\varpi g)^{1/p^\infty}$.

Here, $(-)^\wedge$ denotes the derived ϖ-completion functor and $\mathrm{D}_{\mathrm{comp}}(-) \subseteq \mathrm{D}(-)$ is the full subcategory of derived ϖ-complete objects; see Definition 2.5.15. The starting point of our discussion is Bhatt's quantitative form of the perfectoid Riemann extension theorem from Theorem 2.3.11. In § 2.5.1, we show that the conclusion of that theorem yields a comparison of the derived categories of the reductions modulo ϖ^m in a more general setting. This comparison is then bootstrapped to one for the integral perfectoid rings in § 2.5.2.

In order to properly work with the inverse limits of the derived categories in Proposition 2.5.1, it will be necessary to use their ∞-categorical enhancements. In this section, we will therefore consider the derived categories of all Grothendieck abelian categories as stable ∞-categories instead of their associated homotopy categories.

2.5.1 Derived categories on almost-pro-isomorphic systems

Throughout this section, fix a basic setup for almost mathematics: a ring A together with an ideal $\mathfrak{m} \subset A$ such that $\mathfrak{m} = \mathfrak{m}^2$. For simplicity, we assume that $(A, \mathfrak{m})$ satisfies hypothesis (**B**) of [GR03] that for all $n > 1$, the n-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$.

In [Bha18b], Bhatt introduces a notion of almost mathematics for pro-systems of A-modules and bounded complexes. We provide a slightly modified definition for the unbounded derived category.

Definition 2.5.2. Let $\mathfrak{m}_0 \subseteq A$ be an ideal. An object M of $\mathrm{D}(A^a)$ is $\mathfrak{m}_0$-torsion if for all $\varepsilon \in \mathfrak{m}_0$, the morphism

$$\varepsilon \cdot \mathrm{id} \colon M \to M$$

induced by the A-linear structure of $\mathrm{D}(A^a)$ is homotopic to 0.

If $M \in \mathrm{D}(A^a)$ is $\mathfrak{m}_0$-torsion, then $\mathrm{H}^i(M)$ is an $\mathfrak{m}_0$-torsion A^a-module for all $i \in \mathbf{Z}$. Beware, however, that the converse may not hold.

Definition 2.5.3. A pro-object $\{M_n\}_{n\in\mathbf{N}}$ of $\mathrm{D}(A^a)$ is *almost-pro-zero* if for all $\varepsilon \in \mathfrak{m}$ and all $n \in \mathbf{N}$, there exists some $m \geq n$ such that the diagram

$$\begin{tikzcd} M_m \arrow[r, "\varepsilon\cdot\mathrm{id}"] \arrow[d] \arrow[dr, "0"] & M_m \arrow[d] \\ M_n \arrow[r, "\varepsilon\cdot\mathrm{id}"] & M_n \end{tikzcd}$$

commutes up to homotopy. A map $\{f_n\}_{n\in\mathbf{N}}$ of pro-objects in $\mathrm{D}(A^a)$ is an *almost-pro-isomorphism* if $\{\mathrm{fib}(f_n)\}_{n\in\mathbf{N}}$ is almost-pro-zero.

The significance of this definition to us lies in the following versions of [Bha18b, Lem. 3.5, Lem. 3.6].

Lemma 2.5.4. *Let $\{f_n \colon M_n \to N_n\}_{n\in\mathbf{N}}$ be an almost-pro-isomorphism in $\mathrm{D}(A^a)$. Then the induced morphism $\mathrm{Rlim}\, M_n \to \mathrm{Rlim}\, N_n$ is an almost isomorphism.*

Proof. We need to show that the inverse limit of the almost-pro-zero object $\{\mathrm{fib}(f_n)\}_{n\in\mathbf{N}}$ is almost zero. For every $i \in \mathbf{Z}$, we have a short exact sequence

$$0 \to \mathrm{R}^1 \lim \mathrm{H}^{i-1}(\mathrm{fib}(f_n)) \to \mathrm{H}^i(\mathrm{Rlim}\, \mathrm{fib}(f_n)) \to \lim \mathrm{H}^i(\mathrm{fib}(f_n)) \to 0$$

Since the outer terms are almost zero by [Bha18b, Lem. 3.4], so is the middle term and thus $\mathrm{Rlim}\, \mathrm{fib}(f_n)$. $\square$

Lemma 2.5.5. *Let $\{M_n\}_{n\in\mathbf{N}}$ be an almost-pro-zero object of $\mathrm{D}(A^a)$ and $F \colon \mathrm{D}(A^a) \to \mathrm{D}(A^a)$ be an A-linear functor. Then $\{F(M_n)\}_{n\in\mathbf{N}}$ is almost-pro-zero.*

Proof. Let $\varepsilon \in \mathfrak{m}$ and $m \geq n$. If

$$\begin{tikzcd} M_m \arrow[r, "\varepsilon\cdot\mathrm{id}"] \arrow[d] \arrow[dr, "0"] & M_m \arrow[d] \\ M_n \arrow[r, "\varepsilon\cdot\mathrm{id}"] & M_n \end{tikzcd}$$

commutes up to homotopy, then so does

$$\begin{tikzcd} F(M_m) \arrow[r, "\varepsilon\cdot\mathrm{id}"] \arrow[d] \arrow[dr, "0"] & F(M_m) \arrow[d] \\ F(M_n) \arrow[r, "\varepsilon\cdot\mathrm{id}"] & F(M_n) \end{tikzcd}$$

by the A-linearity of F. $\square$

Let now $\{A_n\}_{n\in\mathbf{N}}$ be a pro-system of A-algebras such that the map of pro-systems $\{A \to A_n\}_{n\in\mathbf{N}}$ is an almost-pro-isomorphism. For simplicity, assume that the structure map $A^a \to A_0^a$ (and hence every $A^a \to A_n^a$) is a monomorphism of almost algebras. This is the case in Theorem 2.3.11 if we take $A = R^+/\varpi^m$ and $A_n = R\langle\frac{\varpi^n}{g}\rangle^+/\varpi^m$.

We wish to compare compatible systems of almost complexes over the A_n with almost complexes over A. The former are captured by the category $\lim \mathrm{D}(A_n^a)$ whose objects are collections of complexes $M_n \in \mathrm{D}(A_n^a)$ together with compatible (up to higher homotopy) isomorphisms $M_m \otimes^{\mathrm{L}}_{A_m^a} A_n^a \xrightarrow{\sim} M_n$ for all $m \geq n$.

Remark 2.5.6. If an object $\{M_n\}_{n\in\mathbf{N}}$ of $\lim \mathrm{D}(A_n^a)$ is almost-pro-zero, all individual M_n are already almost zero. To see this, fix $n \in \mathbf{N}$ and $\varepsilon \in \mathfrak{m}$. Choose $m \geq n$ as in Definition 2.5.3. Then $M_m \to M_n \xrightarrow{\cdot\varepsilon} M_n$ is homotopic to 0, and hence by extending scalars to A_n also $M_m \otimes^{\mathrm{L}}_{A_m^a} A_n^a \xrightarrow{\sim} M_n \xrightarrow{\cdot\varepsilon} M_n$.

Proposition 2.5.7. *The pair of adjoint functors*

$$\mathrm{D}(A^a) \to \lim \mathrm{D}(A_n^a), \quad M \mapsto \{M \otimes^{\mathrm{L}}_{A^a} A_n^a\}_{n\in\mathbf{N}}$$

and

$$\lim \mathrm{D}(A_n^a) \to \mathrm{D}(A^a), \quad \{M_n\}_{n\in\mathbf{N}} \mapsto \mathrm{R}\lim_n M_n.$$

gives an adjoint equivalence of categories.

Before we begin the proof, we take on some preparatory work. First, two facts about stable ∞-categories which we will need.

Lemma 2.5.8. *Let $\mathscr{D}$ be an A-linear stable ∞-category and $K \to L \to M$ an exact triangle of $\mathscr{D}$. Assume that for some $\varepsilon \in A$, two out of K, L, and M are ε-torsion. Then the third object of the exact triangle is ε^2-torsion.*

Proof. The statement can be checked in the homotopy category $D := h\mathscr{D}$. Rotating the triangle, we may assume that K and M are ε-torsion. By the Yoneda lemma, it suffices to show that for any object N of D the map

$$\mathrm{Hom}_D(N, L) \xrightarrow{\cdot\varepsilon^2} \mathrm{Hom}_D(N, L)$$

induced by multiplication with ε^2 on the second factor is 0. This follows from a diagram chase in

$$\begin{tikzcd}
\mathrm{Hom}_D(N,K) \arrow[r] \arrow[d, "\cdot\varepsilon"', "=0"] & \mathrm{Hom}_D(N,L) \arrow[r] \arrow[d, "\cdot\varepsilon"'] & \mathrm{Hom}_D(N,M) \arrow[d, "\cdot\varepsilon"', "=0"] \\
\mathrm{Hom}_D(N,K) \arrow[r] \arrow[d, "\cdot\varepsilon"', "=0"] & \mathrm{Hom}_D(N,L) \arrow[r] \arrow[d, "\cdot\varepsilon"'] & \mathrm{Hom}_D(N,M) \arrow[d, "\cdot\varepsilon"', "=0"] \\
\mathrm{Hom}_D(N,K) \arrow[r] & \mathrm{Hom}_D(N,L) \arrow[r] & \mathrm{Hom}_D(N,M).
\end{tikzcd}$$

□

Lemma 2.5.9. *Let $\mathscr{D}$ be an A-linear stable ∞-category. Let $f\colon K \to L$ and $g\colon L \to M$ be morphisms in $\mathscr{D}$ such that two out of the three fibers $\mathrm{fib}(f)$, $\mathrm{fib}(g)$, and $\mathrm{fib}(g \circ f)$ are ε-torsion for some $\varepsilon \in A$. Then the remaining third fiber is ε^2-torsion.*

Proof. Consider the diagram of pushout squares for the octahedral axiom

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d] & L \arrow[r, "g"] \arrow[d] & M \arrow[d] \\
0 \arrow[r] & \mathrm{cof}(f) \arrow[r] \arrow[d] & \mathrm{cof}(g \circ f) \arrow[d] \\
 & 0 \arrow[r] & \mathrm{cof}(g).
\end{tikzcd}$$

Since $\mathrm{cof}(f)$ and $\mathrm{cof}(g)$ are ε-torsion by assumption, we can apply Lemma 2.5.8 to the exact triangle that arises from the bottom square. □

Now fix $\varepsilon \in \mathfrak{m}$. For every $n \in \mathbf{N}$, let $B_{n,\varepsilon} := A + \varepsilon A_n \subseteq A_n$. Every transition map $A_m \to A_n$ induces a natural morphism $\rho_{mn,\varepsilon}\colon B_{m,\varepsilon} \to B_{n,\varepsilon}$. First, we show a technical lemma about ε-torsion properties of several morphisms.

Lemma 2.5.10. *Let $m \geq n$. There exists $J \in \mathbf{N}$ (independent of m, n, and ε) such that the fibers of the following morphisms in $\mathrm{D}(A)$ are ε^J-torsion:*

(i) *the base change* $A_m \otimes^{\mathrm{L}}_{B_{m,\varepsilon}} B_{n,\varepsilon} \to A_m \otimes^{\mathrm{L}}_{B_{m,\varepsilon}} A_n$ *of the inclusion* $B_{n,\varepsilon} \hookrightarrow A_n$.

(ii) *the universal morphism*

$$\mu\colon A_m \otimes^{\mathrm{L}}_{B_{m,\varepsilon}} A_n \to A_m \otimes_{B_{m,\varepsilon}} A_n \to A_n \otimes_{B_{n,\varepsilon}} A_n \to A_n$$

induced by left truncation, the transition maps of $\{A_n\}_{n\in\mathbf{N}}$, *and multiplication.*

(iii) *the universal morphism* $A_m \otimes^{\mathrm{L}}_{B_{m,\varepsilon}} B_{n,\varepsilon} \to A_n$ *given by the composition of the morphisms from* (i) *and* (ii).

Proof. (i). Follows from $\varepsilon \cdot (A_n/B_{n,\varepsilon}) = 0$ and the A-linearity of the derived tensor product.

(ii). We compare both source and target to $\varepsilon A_m \otimes^{\mathrm{L}}_{B_{m,\varepsilon}} A_n$ via the morphism $\iota\colon \varepsilon A_m \otimes^{\mathrm{L}}_{B_{m,\varepsilon}} A_n \to A_m \otimes^{\mathrm{L}}_{B_{m,\varepsilon}} A_n$ induced by the inclusion $\varepsilon A_m \hookrightarrow A_m$. Similarly to (i), $\mathrm{cof}(\iota)$ is ε-torsion because $\varepsilon \cdot (A_m/\varepsilon A_m) = 0$ and the derived tensor product is A-linear. Moreover, $\mu \circ \iota$ can alternatively be obtained by taking the derived tensor product of $\varepsilon A_m \hookrightarrow B_{m,\varepsilon}$ with A_n so that $\mathrm{cof}(\mu \circ \iota)$ is ε-torsion by the same argument. Lemma 2.5.9 now yields the statement.

(iii). Apply (i) and (ii) together with Lemma 2.5.9. □

Since $A \to \{A_n\}_{n\in\mathbf{N}}$ is an almost-pro-isomorphism and $A^a \to A^a_0$ is a monomorphism, we can find $\ell \in \mathbf{N}$ such that $\rho^a_{n0,\varepsilon}$ factors through a unique A^a-algebra morphism $\sigma_{n,\varepsilon}\colon B^a_{n,\varepsilon} \to A^a$ for all $n \geq \ell$; this morphism is a section of $A^a \to B^a_{n,\varepsilon}$. The uniqueness guarantees that $\sigma_{m,\varepsilon} = \sigma_{n,\varepsilon} \circ \rho^a_{mn,\varepsilon}$.

Lemma 2.5.11. *The map of pro-*A^a*-modules* $\{\sigma_{n,\varepsilon}\colon B^a_{n,\varepsilon} \to A^a\}_{n\geq\ell}$ *is a pro-isomorphism (in the almost category).*

Proof. For all $n \geq \ell$, set $K_{n,\varepsilon} := \ker \sigma_{n,\varepsilon}$. As each $\sigma_{n,\varepsilon}$ is a section and hence surjective, it suffices to show that the induced pro-system of kernels $\{K_{n,\varepsilon}\}_{n\geq\ell}$ is pro-zero. Given $n \geq \ell$, choose $m \geq n$ such that $\rho^a_{mn,\varepsilon}$ factors through some $B^a_{m,\varepsilon} \to A^a$. This morphism is $\sigma_{m,\varepsilon}$, once more by uniqueness. Thus, $\rho^a_{mn,\varepsilon}(K_{m,\varepsilon}) = 0$ as desired. □

Lemma 2.5.12. *Let* $\{M_n\}_{n\geq\ell}$ *be a pro-system of objects of* $\mathrm{D}(A^a_n)$. *Then the natural base change map*

$$\{M_n \to M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} A^a\}_{n\geq\ell}$$

of pro-objects in $\mathrm{D}(A^a)$ *is an pro-isomorphism.*

Proof. The fiber of $M_n \to M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} A^a$ is given by $M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} K^a_{n,\varepsilon}$, where again $K_{n,\varepsilon} := \ker \sigma_{n,\varepsilon}$. Since $\{K_{n,\varepsilon}\}_{n\geq\ell}$ is pro-zero and the derived tensor product is a linear functor, $\{M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} K^a_{n,\varepsilon}\}_{n\geq\ell}$ is pro-zero as well. The assertion follows. □

Lemma 2.5.13. *Let $\{M_n\}_{n\geq \ell}$ be an object of* $\lim \mathrm{D}(A_n^a)$. *Then the fiber of the projection*

$$\operatorname*{Rlim}_n (M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} A) \to M_\ell \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a$$

is ε^J-torsion for some $J \in \mathbf{N}$ (independent of m, n, and ε).

Proof. By Lemma 2.5.10.(iii), there is $J \in \mathbf{N}$ such that for each $n \geq \ell$, the fiber of

$$M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} B^a_{\ell,\varepsilon} \simeq M_n \otimes^{\mathrm{L}}_{A^a_n} A^a_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} B^a_{\ell,\varepsilon} \to M_n \otimes^{\mathrm{L}}_{A^a_n} A^a_\ell \xrightarrow{\sim} M_\ell$$

is ε^J-torsion. Hence, so is

$$\operatorname{fib}\left(\operatorname*{Rlim}_n (M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} A^a) \simeq \operatorname*{Rlim}_n (M_n \otimes^{\mathrm{L}}_{B^a_{n,\varepsilon}} B^a_{\ell,\varepsilon} \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a) \to M_\ell \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a\right)$$

by the A^a-linearity of the functors involved. □

Corollary 2.5.14. *Let $\{M_n\}_{n\geq \ell}$ be an object of* $\lim \mathrm{D}(A_n^a)$. *Then the fiber of the natural map*

$$\operatorname*{Rlim}_n M_n \to M_\ell \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a$$

is ε^J-torsion for some $J \in \mathbf{N}$ (independent of m, n, and ε).

Proof. Combine Lemma 2.5.12 and Lemma 2.5.13. □

Proof of Proposition 2.5.7. We show that unit and counit are natural isomorphisms.

For the unit map, let M be an object of $\mathrm{D}(A^a)$. Since $\{A \to A_n\}_{n\in\mathbf{N}}$ is an almost-pro-isomorphism, so is $\{M \to M \otimes^{\mathrm{L}}_{A^a} A^a_n\}_{n\in\mathbf{N}}$; cf. Lemma 2.5.5. In particular,

$$M \to \operatorname*{Rlim}_n (M \otimes^{\mathrm{L}} A^a_n)$$

is an almost isomorphism by Lemma 2.5.4.

Let now $\{M_n\}_{n\in\mathbf{N}}$ be an object of $\lim \mathrm{D}(A^a_n)$. Let $n \in \mathbf{N}$ and $\varepsilon \in \mathfrak{m}$. In the above discussion, we can pick ℓ such that $\ell \geq n$ and $\rho^a_{\ell n,\varepsilon}$ factors through $\sigma_{\ell,\varepsilon}$. Then the counit map

$$\left\{\left(\operatorname*{Rlim}_m M_m\right) \otimes^{\mathrm{L}}_{A^a} A^a_n\right\}_{n\in\mathbf{N}} \to \{M_n\}_{n\in\mathbf{N}}$$

can be decomposed in the n-th spot as

$$\left(\operatorname*{Rlim}_m M_m\right) \otimes^{\mathrm{L}}_{A^a} A^a_n \to M_\ell \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a \otimes^{\mathrm{L}}_{A^a} A^a_n \to M_n.$$

The fiber of the first map is ε^J-torsion by Corollary 2.5.14. Since

$$M_\ell \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a \otimes^{\mathrm{L}}_{A^a} A^a_n = M_\ell \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a_n = M_\ell \otimes^{\mathrm{L}}_{A^a_\ell} A^a_\ell \otimes^{\mathrm{L}}_{B^a_{\ell,\varepsilon}} A^a_n,$$

the fiber of the second map is ε^J-torsion by Lemma 2.5.10.(ii). Thus, the fiber of the counit map is ε^{2J}-torsion. Since the $2J$-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$ by hypothesis (**B**), the counit map is an almost isomorphism as well. $\square$

2.5.2 Consequences for perfectoid rings

As mentioned, Proposition 2.5.7 applies to $A = R^+/\varpi^m$ and $A_n = R\langle\frac{\varpi^n}{g}\rangle^+/\varpi^m$ from Theorem 2.3.11. Next, we would like to bootstrap this to a result for R^+ and $R\langle\frac{\varpi^n}{g}\rangle^+$. For now, let us only assume that R^+ is a ring with ideal $\mathfrak{m} = \mathfrak{m}^2 \subseteq R^+$ so that $(R^+, \mathfrak{m})$ is a basic setup as before. Assume further that R^+ has bounded ϖ^∞-torsion for some $\varpi \in R^+$.

Definition 2.5.15. The *derived ϖ-completion* of an object M of $\mathrm{D}(R^{+,a})$ is given by

$$\widehat{M} := \mathrm{Rlim}(M \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n).$$

If the natural map $M \to \widehat{M}$ is an isomorphism, M is called *derived ϖ-complete*. Denote by $\mathrm{D}_{\mathrm{comp}}(R^{+,a}) \subseteq \mathrm{D}(R^{+,a})$ the full stable ∞-subcategory of all derived ϖ-complete objects.

We need the following lemma, in which the use of ∞-categories is essential.

Lemma 2.5.16. *Let R^+ be a ring and $\varpi \in R^+$ be a nonzerodivisor. Then the pair of adjoint functors*

$$\mathrm{D}_{\mathrm{comp}}(R^{+,a}) \to \lim \mathrm{D}(R^{+,a}/\varpi^n), \quad M \mapsto \{M \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n\}_{n\in\mathbf{N}}$$

and

$$\lim \mathrm{D}(R^{+,a}/\varpi^n) \to \mathrm{D}_{\mathrm{comp}}(R^{+,a}), \quad \{M_n\}_{n\in\mathbf{N}} \mapsto \mathrm{R}\lim_n M_n.$$

gives an adjoint equivalence of categories.

Proof. The unit morphism $M \to \mathrm{Rlim}(M \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n)$ is an equivalence for every object M of $\mathrm{D}_{\mathrm{comp}}(R^{+,a})$ by the definition of derived ϖ-completeness. Next, since R^+ is ϖ-torsionfree, R^+/ϖ^n is perfect for all $n \in N$. Hence, the natural morphisms

$$(\mathrm{R}\lim_m M_m) \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n \to \mathrm{R}\lim_m(M_m \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n) \simeq \mathrm{R}\lim_m(M_n) \simeq M_n$$

are equivalences. Concretely, as $R^{+,a}/\varpi^n \simeq (R^{+,a} \xrightarrow{\cdot\varpi} R^{+,a})$, the stupid truncation gives rise to an exact triangle $R^{+,a}[0] \to R^{+,a}/\varpi^n \to R^{+,a}[1]$ and thus a commutative diagram

$$\begin{tikzcd}
(\mathrm{Rlim}_m M_m) \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a} \arrow[r] \arrow[d, "\wr"] & (\mathrm{Rlim}_m M_m) \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n \arrow[r] \arrow[d] & (\mathrm{Rlim}_m M_m) \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}[1] \arrow[d, "\wr"] \\
\mathrm{Rlim}_m(M_m \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}) \arrow[r] & \mathrm{Rlim}_m(M_m \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n) \arrow[r] & \mathrm{Rlim}_m(M_m \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}[1])
\end{tikzcd}$$

in which the two horizontal lines are exact triangles. Since the two outer vertical maps are equivalences, so is the middle one. Thus, we see at once that $\mathrm{Rlim}_n M_n$ is derived ϖ-complete and the second functor hence well-defined, and that the counit map

$$\left\{\left(\operatorname*{Rlim}_m M_m\right) \otimes^{\mathrm{L}}_{R^{+,a}} R^{+,a}/\varpi^n\right\}_{n\in\mathbf{N}} \to \{M_n\}_{n\in\mathbf{N}}$$

is an equivalence. □

Proof of Proposition 2.5.1. This is now a combination of Proposition 2.5.7, Lemma 2.5.16, and Theorem 2.3.11. □

2.6 Perfectoid Artin vanishing

The results of this section are inspired by the following statement due to Artin and Grothendieck.

Theorem 2.6.1 ([SGA4, Cor. XIV.3.2])**.** *Suppose X is an affine algebraic variety over a separably closed field. Let F be a torsion abelian étale sheaf on X. Then $\mathrm{H}^n_{\text{ét}}(X, F) = 0$ for all $n > \dim X$.*

When X is proper, Theorem 2.6.1 fails; nonetheless, we can prove a similar version for certain inverse systems of varieties. From here on, fix a complete, algebraically closed extension C of $\mathbf{Q}_p$. Let $\mathscr{O}_C$ be its ring of integers and ϖ be a pseudouniformizer. We will work in the following setting.

Situation 2.6.2. Let I be a cofiltered category with final object $0 \in I$. Let X_i, $i \in I$, be a cofiltered inverse system of quasi-compact and quasi-separated rigid spaces over C of dimension d with finite transition maps $\pi_{ji}\colon X_j \to X_i$. Assume that the diamond $\lim_i X_i^\diamondsuit$ is representable by a perfectoid space X.

In order to have a good theory of étale cohomology, we will always identify rigid spaces with their associated adic spaces [Hub94, Prop. 4.3]. Later in this thesis, we will mostly be interested in the special case where the X_i are algebraic varieties and there exists a perfectoid space X such that $X \sim \lim_i X_i$ in the sense of [SW13, Def. 2.4.1].

Note that by Proposition 2.4.13, Situation 2.6.2 is preserved under base change along any finite morphism $Y_0 \to X_0$, a fact that will turn out to be crucial later in this section: the isomorphism $Y^\lozenge \simeq Y_0^\lozenge \times_{X_0^\lozenge} X^\lozenge$ from Proposition 2.4.13 yields, via projection to the two factors and the definition of the diamonds $Y_0^\lozenge$ and $X^\lozenge$, morphisms of adic spaces $Y \to Y_0$ and $Y \to X$. Setting $Y_i := Y_0 \times_{X_0} X_i$ for all $i \in I$, we obtain the desired inverse system of maps $Y \to Y_i$. On the other hand, it is not even clear if the fiber product $Y_0 \times_{X_0} X$ exists as an adic space—we do not know whether its structure presheaf is a sheaf. Therefore, it is really advantageous to formulate Situation 2.6.2 in the category of diamonds.

Theorem 2.6.3 (Perfectoid Artin vanishing)**.** *In Situation 2.6.2, let F_0 be a Zariski-constructible sheaf of $\mathbf{F}_p$-modules on X_0 and $F_i := \pi_{i0}^* F_0$ for all $i \in I$. Assume that X_0 is proper. Then for all $n > d$*

$$\operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(X_i, F_i) = 0.$$

An étale sheaf F_0 of $\mathbf{F}_p$-modules on X_0 is Zariski-constructible if there is a finite stratification $X_0 = \bigsqcup X_{0,\alpha}$ such that each stratum is locally closed in the Zariski topology of X_0 and the restrictions $F_0\big|_{X_{0,\alpha}}$ are finite locally constant. For example, every constructible sheaf of $\mathbf{F}_p$-modules on a scheme of finite type over C (in the usual algebro-geometric sense) gives rise to a Zariski-constructible sheaf on its analytification. We refer to [Han20] for a detailed account in the general rigid-analytic setting. When $F_0 = j_{0,!}\mathbf{F}_p$ for some dense, Zariski-open $j_0 \colon U_0 \hookrightarrow X_0$, Theorem 2.6.3 was proven in [Sch15, § 4.2].

Let us briefly elucidate the connection between the cohomologies of the rigid spaces at finite level and the inverse limit diamond.

Lemma 2.6.4. *In Situation 2.6.2, let F_0 be an étale sheaf on X_0 with pullbacks F_i to Y_i and F to Y. Then the natural map $\operatorname{colim}_i \mathrm{H}^n(X_i, F_i) \to \mathrm{H}^n(X, F)$ is an isomorphism.*

Proof. By Lemma 2.4.5 and Lemma 2.4.9, the diamonds $X_i^\lozenge$ attached to X_i are spatial and the diamondification functor induces a natural equivalence of sites $X^\lozenge_{i,\text{ét}} \xrightarrow{\sim} X_{i,\text{ét}}$. Denoting the pullbacks of the F_i and F under these equivalences by $F_i^\lozenge$ and $F^\lozenge$, respectively, it therefore suffices to show that the natural map

$$\operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(X_i^\lozenge, F_i^\lozenge) \to \mathrm{H}^n_{\text{ét}}(X^\lozenge, F^\lozenge)$$

is an isomorphism. This is [Sch17, Prop. 14.9]. □

Example 2.6.5. Lemma 2.4.21 applies to the setting of Lemma 2.6.4 as follows: When

$F_0 = (\mathscr{O}_{X_0}^+/\varpi)^a$, chasing through the diagram

$$\begin{tikzcd} X^\lozenge_{i,\mathrm{qproét}} \arrow[r] \arrow[d] & X^\lozenge_{i,\mathrm{ét}} \arrow[r, "\sim"] \arrow[d] & X_{i,\mathrm{ét}} \arrow[d, "\rho_i"] \\ X^\lozenge_{0,\mathrm{qproét}} \arrow[r] & X^\lozenge_{0,\mathrm{ét}} \arrow[r, "\sim"] & X_{0,\mathrm{ét}} \end{tikzcd}$$

shows that the pullback $F_i := \rho_i^* F_0$ is given by $(\mathscr{O}_{X_i}^+/\varpi)^a$ because the diamondification $\rho_i^\lozenge\colon X_i^\lozenge \to X_0^\lozenge$ of the finite morphism $X_i \to X_0$ is quasi-pro-étale [CGH+20, Prop. 2.3.4]; virtually the same argument applies to the pullback to $X_{\mathrm{ét}}$. Therefore, the natural map $\operatorname{colim}_i \mathrm{H}^n(X_i, \mathscr{O}_{X_i}^+/\varpi) \to \mathrm{H}^n(X, \mathscr{O}_X^+/\varpi)$ is an almost isomorphism by Lemma 2.6.4.

2.6.1 Examples and counterexamples

Before delving into the proof of Theorem 2.6.3, we give some examples and applications.

Example 2.6.6. When $X_0 \subseteq \mathbf{P}^N_C$ is a projective variety, we obtain a natural inverse system with the required properties by putting $X_i := X_0 \times_{\mathbf{P}^N_C, \Phi^i} \mathbf{P}^N_C$, $i \in \mathbf{Z}_{\geq 0}$, where Φ is the "mock Frobenius"

$$\Phi\colon \mathbf{P}^N_C \to \mathbf{P}^N_C, \quad [x_0 : \cdots : x_N] \mapsto [x_0^p : \cdots : x_N^p].$$

In this setting, Esnault [Esn18, Thm. 5.1] gave a proof of Theorem 2.6.3 that does not require perfectoid techniques. In fact, her argument establishes this special case over any algebraically closed field of characteristic not equal to p. It seems interesting to investigate whether Theorem 2.6.3 holds true over more general algebraically closed, nonarchimedean fields as well.

Example 2.6.7. Let A be an abelian variety, or more generally an abeloid variety, over C. Consider the cofiltered inverse system

$$\cdots \xrightarrow{[p]} A \xrightarrow{[p]} A \xrightarrow{[p]} A$$

in which the transition maps are multiplication by p. In [BGH+18], the authors construct a perfectoid space A_∞ with $A_\infty \sim \lim_{[p]} A$ and thus $A_\infty^\lozenge \simeq \lim_{[p]} A^\lozenge$. In particular, this produces interesting nonalgebraic examples.

Example 2.6.8. Fix, once and for all, a smooth, $\mathrm{GL}_g(\mathbf{Z})$-admissible polyhedral decomposition of the cone of positive semi-definite quadratic forms on $\mathbf{R}^g$ whose null space is defined over $\mathbf{Q}$; see [FC90, Def. IV.2.2, IV.2.3]. Let $\overline{\mathscr{A}_g}[m]$ be the toroidal compactification of the moduli space $\mathscr{A}_g[m]$ of principally polarized abelian varieties of dimension g over C with full level-m

structure which is determined by this decomposition. By [FC90, Thm. IV.6.7], $\overline{\mathscr{A}_g}[m]$ is a smooth and proper algebraic stack in which $\overline{\mathscr{A}_g}[m] \smallsetminus \mathscr{A}_g[m]$ is a normal crossings divisor (see Definition 3.1.1). If $m \geq 3$, it is an algebraic space [FC90, Cor. IV.6.9], and even a projective variety under certain convexity conditions on the decomposition [FC90, § V.5].

We obtain a cofiltered inverse system $X_i := \overline{\mathscr{A}_g}[p^i]$, $i \in \mathbf{Z}_{\geq 0}$, of projective varieties with finite transition maps "forgetting level structure." Work of Scholze [Sch15] and Pilloni–Stroh [PS16, Thm. 0.4] constructs a perfectoid space $X \sim \lim_i X_i^{\mathrm{ad}}$. Thus, Theorem 2.6.3 applies to this inverse system.

For later purposes, it will be beneficial to formulate Theorem 2.6.3 for arbitrary systems of semiperverse sheaves. We only consider the case where the X_i are (the analytifications of) proper schemes over C, in which we can use the notation set up in Appendix A.

Corollary 2.6.9. *Let $m \in \mathbf{Z}$ and $K_i \in {}^p\mathrm{D}(X_i)^{\leq m}$ for all $i \in I$. Assume that for every morphism $i \to j$ of I, there exists $\varphi_{ij} \colon \pi_{ij}^* K_j \to K_i$ such that $\varphi_{ik} = \varphi_{ij} \circ \pi_{ij}^* \varphi_{jk}$ for all $i \to j \to k$. Then*

$$\operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(X_i, K_i) = 0$$

for all $n > m$.

Proof. By Lemma 2.6.10 below applied to the $\mathrm{Arr}(I^{\mathrm{op}})$-shaped diagram $H_{j\to i} := \mathrm{H}^n_{\text{ét}}(X_i, \pi_{ij}^* K_j)$ with transition maps induced by the natural pullback morphisms and the φ_{ij}, we may assume that $K_i \simeq \pi_{i0}^* K_0$ for all $i \in I$. Consider the hypercohomology spectral sequences

$$\mathrm{H}^r_{\text{ét}}(X_i, \mathscr{H}^s(K_i)) \Longrightarrow \mathrm{H}^{r+s}_{\text{ét}}(X_i, K_i),$$

which are compatible in $i \in I$. By the "colimit lemma" (cf. e.g. [Mit97, Prop. 3.3]), it suffices to check that

$$\operatorname*{colim}_i \mathrm{H}^r_{\text{ét}}(X_i, \mathscr{H}^s(K_i)) = 0$$

whenever $r + s > m$. By Theorem 2.6.3 and Proposition 2.4.13 below, the colimit is 0 when $r > \dim \operatorname{supp} \mathscr{H}^s(K_0)$, so the statement follows from the semiperversity condition $\dim \operatorname{supp} \mathscr{H}^s(K_0) \leq m - s$. □

It remains to prove Lemma 2.6.10. Recall that the arrow category of a category J is the category $\mathrm{Fun}(\{0 \to 1\}, J)$ of functors from the interval category $\{0 \to 1\}$ to J whose objects are the morphisms of J. In § 3.3, we will take $J = \mathbf{Z}_{\geq 0}$; in that case, $\mathrm{Arr}(J)$ can be identified with the "staircase" diagram $\{(j,k) \in (\mathbf{Z}_{\geq 0})^2 \mid j \leq k\}$, considered as a partially ordered set via the product order.

Lemma 2.6.10. *Let $\mathscr{C}$ be a category that admits all filtered colimits. Let J be a filtered category with associated arrow category* $\mathrm{Arr}(J)$ *and* $H\colon \mathrm{Arr}(J) \to \mathscr{C}$ *be an* $\mathrm{Arr}(J)$*-shaped diagram of* $\mathscr{C}$*. Then*

$$\operatorname*{colim}_{j\in J} H_{(\mathrm{id}\colon j\to j)} \simeq \operatorname*{colim}_{j\in J} \operatorname*{colim}_{(j\to k)\in \mathrm{Arr}(J)} H_{(j\to k)}.$$

Proof. Let $F\colon \mathrm{Arr}(J) \to J$, $(j \to k) \mapsto j$ be the projection onto the source and $G\colon J \to *$ be the functor to the terminal category. Since the fully faithful "diagonal" subcategory $J \subset \mathrm{Arr}(J)$ consisting of all identity morphisms $(\mathrm{id}\colon j \to j)$ is cofinal, we have $\operatorname{colim}_{j\in J} H_{(\mathrm{id}\colon j\to j)} \simeq \operatorname{colim}_{\mathrm{Arr}(J)} H$. However, we can compute $\operatorname{colim}_{\mathrm{Arr}(J)} H$ another way: it is the left Kan extension $\mathrm{Lan}_{G\circ F} H$ of H along $G \circ F$, which is naturally isomorphic to $\mathrm{Lan}_G \mathrm{Lan}_F H$. By the universal property of left Kan extensions, the functor $\mathrm{Lan}_F H$ is the J-shaped diagram $(\operatorname{colim}_{(j\to k)\in\mathrm{Arr}(J)} H_{(j\to k)})_{j\in J}$ of colimits with fixed source, hence

$$\mathrm{Lan}_G \mathrm{Lan}_F H \simeq \mathrm{Lan}_G \Big(\operatorname*{colim}_{(j\to k)\in \mathrm{Arr}(J)} H_{(j\to k)} \Big)_{j\in J} \simeq \operatorname*{colim}_{j\in J} \operatorname*{colim}_{(j\to k)\in \mathrm{Arr}(J)} H_{(j\to k)}. \qquad \square$$

We conclude with two further examples to indicate the limits of our methods.

Example 2.6.11. We work in Situation 2.6.2 and assume that X_0 is proper. If $f_0\colon Y_0 \to X_0$ is a finite morphism and $Y_i := Y_0 \times_{X_0} X_i$, Theorem 2.6.3 applied to $f_{0,*}\mathbf{F}_p$ on X_0 (which is Zariski-constructible by [Han20, Prop. 2.3]) shows that $\operatorname{colim}_i \mathrm{H}^n_{\text{ét}}(Y_i, \mathbf{F}_p) = 0$ for all $n > d$. In fact, this will be established independently in Lemma 2.6.13 as a key step toward the proof of Theorem 2.6.3. When f_0 is only required to be generically finite, the statement is already false for the blowup of $\mathbf{P}^d_C$ in a point.

More precisely, let $X_i := \mathbf{P}^d_C$, $i \in \mathbf{Z}_{\geq 0}$, be the inverse system whose transition maps are the mock Frobenii from Example 2.6.6. Let $Q := [1 : 1 : \ldots : 1]$ and $Y_0 := \mathrm{Bl}_Q \mathbf{P}^d_C \xrightarrow{f_0} \mathbf{P}^d_C$. Since blowing up commutes with flat base change [SP20, Lem. 0805], Y_i is the blowup of $\mathbf{P}^d_C$ in the points $Q_{i1}, \ldots, Q_{ip^{id}}$ whose coordinates are all p^i-th roots of unity. Let E be the exceptional divisor in Y_0 and $E_{i1}, \ldots, E_{ip^{id}}$ the exceptional divisors in Y_i.

On cohomology, we have

$$\mathrm{H}^{2d-2}_{\text{ét}}(Y_i, \mathbf{F}_p) = \mathrm{H}^{2d-2}_{\text{ét}}(\mathbf{P}^d_C, \mathbf{F}_p) \oplus \bigoplus_j \mathrm{H}^{2d-4}_{\text{ét}}(E_{ij}, \mathbf{F}_p),$$

where the summands coming from the E_{ij} are generated by the pushforwards of $\mathrm{c}_1(\mathscr{O}_{E_{ij}}(1))^{d-2}$ along the inclusions $E_{ij} \hookrightarrow X_i$. Moreover, the pullback of $\mathrm{c}_1(\mathscr{O}_E(1))$ along $Y_i \to Y_0$ is $\sum_j \mathrm{c}_1(\mathscr{O}_{E_{ij}}(1))$ and different Chern classes in the sum intersect to 0, so $\mathrm{c}_1(\mathscr{O}_E(1))^{d-2}$ pulls back to $\sum_j \mathrm{c}_1(\mathscr{O}_{E_{ij}}(1))^{d-2}$. In other words, the transition map

$$\mathrm{H}^{2d-2}_{\text{ét}}(\mathbf{P}^d_C, \mathbf{F}_p) \oplus \mathrm{H}^{2d-4}_{\text{ét}}(E, \mathbf{F}_p) \to \mathrm{H}^{2d-2}_{\text{ét}}(\mathbf{P}^d_C, \mathbf{F}_p) \oplus \bigoplus_j \mathrm{H}^{2d-4}_{\text{ét}}(E_{ij}, \mathbf{F}_p)$$

is multiplication by p^i on the summand $H^{2d-2}_{\text{ét}}(\mathbf{P}^d_C, \mathbf{F}_p)$, but the diagonal map on $\mathrm{H}^{2d-4}_{\text{ét}}(E, \mathbf{F}_p)$. In particular, $\operatorname{colim}_i \mathrm{H}^{2d-2}_{\text{ét}}(Y_i, \mathbf{F}_p) \neq 0$.

The next example shows why Theorem 2.6.3 does not hold for more general classes of constructible sheaves which can be found in [Hub96].

Example 2.6.12. Let $X_i := \mathbf{P}^1_{\mathbf{C}_p}$, $i \in \mathbf{Z}_{\geq 0}$, with the transition maps from Example 2.6.6. Let $j_i := U_i \hookrightarrow X_i$ be the closed unit disc; in particular, $U_i \simeq U_0 \times_{X_0} X_i$. We claim that for $F_0 := j_{0,!}\mathbf{F}_p$, we have $\operatorname{colim}_i \mathrm{H}^2_{\text{ét}}(X_i, F_i) \neq 0$.

Let $X = \mathbf{P}^{1,\text{perf}}_{\mathbf{C}_p}$ and $j\colon U \hookrightarrow X$ be the perfectoid inverse limit of the system X_i and U_i, respectively. Likewise, let $Y_i := \mathbf{P}^1_{\mathbf{C}_p^\flat}$, $i \in \mathbf{Z}_{\geq 0}$, be the inverse system of projective lines over $\mathbf{C}_p^\flat$ with relative Frobenius as transition maps, $k_i\colon V_i \hookrightarrow Y_i$ the closed unit discs, and $Y \simeq \mathbf{P}^{1,\text{perf}}_{\mathbf{C}_p^\flat}$ and $k\colon V \hookrightarrow Y$ their perfectoid inverse limits. The tilting equivalence [Sch12, Thm. 7.12] [KL15, Cor. 8.3.6] and [Sch12, Thm. 7.17, Cor. 7.19] yield a commutative diagram of topoi

$$\begin{tikzcd}
V^\sim_{0,\text{ét}} \arrow[d, hook, "k_0"] & V^\sim_{\text{ét}} \arrow[l, "\sim"'] \arrow[d, hook, "k"] \arrow[r, phantom, "\simeq"] & U^\sim_{\text{ét}} \arrow[d, hook, "j"] \arrow[r, phantom, "\simeq"] & \lim_i U^\sim_{i,\text{ét}} \arrow[r] & U^\sim_{0,\text{ét}} \arrow[d, hook, "j_0"] \\
Y^\sim_{0,\text{ét}} & Y^\sim_{\text{ét}} \arrow[l, "\sim"'] \arrow[r, phantom, "\simeq"] & X^\sim_{\text{ét}} \arrow[r, phantom, "\simeq"] & \lim_i X^\sim_{i,\text{ét}} \arrow[r] & X^\sim_{0,\text{ét}}
\end{tikzcd}$$

and thus via base change [SGA4, Lem. XVII.5.1.2] the identification

$$\operatorname*{colim}_i \mathrm{H}^2_{\text{ét}}(X_i, F_i) \simeq \mathrm{H}^2_{\text{ét}}(X, j_!\mathbf{F}_p) \simeq \mathrm{H}^2_{\text{ét}}(Y, k_!\mathbf{F}_p) \simeq \mathrm{H}^2_{\text{ét}}(Y_0, k_{0,!}\mathbf{F}_p).$$

Now we can proceed similarly to [Hub96, § 0.2]. We have a short exact sequence of étale sheaves on $Y_0 = \mathbf{P}^1_{\mathbf{C}_p^\flat}$

$$0 \to k_{0,!}\mathbf{F}_p \to \mathbf{F}_p \to \iota_*\mathbf{F}_p \to 0, \tag{2.6}$$

where $\iota\colon W^\sim_{\text{ét},Y_0} \hookrightarrow Y^\sim_{0,\text{ét}}$ is the inclusion of the closed subtopos complementary to $V^\sim_{0,\text{ét}}$ [SGA4, § IV.9.3], which corresponds to $W := \{x \in Y_0 \mid |x| > 1\}$ (in the language of [Hub96], the étale topos of the pseudo-adic space (Y_0, W)). By GAGA for rigid étale cohomology in the proper case (cf. [BM18, Cor. 6.18] or [Hub96, Thm. 3.2.10]), the Artin–Schreier sequence, and [SGA4, Cor. X.5.2], $\mathrm{H}^1_{\text{ét}}(Y_0, \mathbf{F}_p) \simeq \mathrm{H}^2_{\text{ét}}(Y_0, \mathbf{F}_p) \simeq 0$. Thus, the long exact sequence in cohomology for (2.6) gives an isomorphism

$$\mathrm{H}^1(W_{\text{ét},Y_0}, \mathbf{F}_p) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(Y_0, k_{0,!}\mathbf{F}_p).$$

Let $V := \mathscr{O}_{\mathbf{C}_p^\flat}$ and $V\{T\}$ be the henselization of $V[T]_{(\mathfrak{m}_V,T)}$. Choose a pseudouniformizer $\varpi^\flat$ of V. As in [Hub96, Ex. 0.2.5], we can conclude from [Hub96, Thm. 0.2.4/Thm. 3.2.1] that

$$\mathrm{H}^1(W_{\text{ét},Y_0}, \mathbf{F}_p) \simeq \mathrm{H}^1_{\text{ét}}\Big(\operatorname{Spec} V\{T\}\Big[\tfrac{1}{\varpi^\flat}\Big], \mathbf{F}_p\Big).$$

Next, we show that the étale $\mathbf{F}_p$-torsor $V\{T\}\left[\frac{1}{\varpi^\flat}\right] \to \left(V\{T\}\left[\frac{1}{\varpi^\flat}\right]\right)[S]/\left(S^p - S - \frac{T}{\varpi^\flat}\right)$ is nontrivial; this implies $\mathrm{H}^1_{\text{ét}}\left(\operatorname{Spec} V\{T\}\left[\frac{1}{\varpi^\flat}\right], \mathbf{F}_p\right) \neq 0$ and hence finishes the proof of the claim.

Note that the map of pairs

$$(V[T]_{(\mathfrak{m}_V,T)}, (\varpi^\flat, T)) \to (V[T]_{(\mathfrak{m}_V,T)}, (\mathfrak{m}_V, T))$$

induces an isomorphism on henselizations by [SP20, Lem. 0F0L]. Therefore, $V[T]_{(\mathfrak{m}_V,T)}$ and $V\{T\}$ have isomorphic $(\varpi^\flat, T)$-adic completions [SP20, Lem. 0AGU]. On the other hand, the $(\varpi^\flat, T)$-adic completion of $V[T]_{(\mathfrak{m}_V,T)}$ is identified via the string of isomorphisms

$$\begin{aligned}\lim_n V[T]_{(\mathfrak{m}_V,T)}/(\varpi^\flat, T)^n &\simeq \lim_{a,b} V[T]_{(\mathfrak{m}_V,T)}/(\varpi^{\flat,a}, T^b) \simeq \lim_{a,b} (V[T]/(\varpi^{\flat,a}, T^b))_{(\mathfrak{m}_V,T)} \\ &\simeq \lim_{a,b} V[T]/(\varpi^{\flat,a}, T^b) \simeq \lim_b \Big(\lim_a V/(\varpi^{\flat,a})\Big)[T]/(T^b) \\ &\simeq \lim_b V[T]/(T^b) \simeq V[\![T]\!]\end{aligned}$$

because the chain of ideals $(\varpi^{\flat,n}, T^n)$ is cofinal with $(\varpi^\flat, T)^n$, localization is exact, $(\mathfrak{m}_V, T)$ is maximal and $\operatorname{rad}(\varpi^{\flat,a}, T^b) = (\mathfrak{m}_V, T)$, completion commutes with finite sums, and V is $\varpi^\flat$-adically complete, respectively. We obtain an induced map $\theta\colon \left(V\{T\}\left[\frac{1}{\varpi^\flat}\right]\right)[S] \to \left(V[\![T]\!]\left[\frac{1}{\varpi^\flat}\right]\right)[S] \subset (\mathbf{C}_p^\flat[\![T]\!])[S]$.

Assume that $S^p - S - \frac{T}{\varpi^\flat} = g \cdot h$ in $\left(V\{T\}\left[\frac{1}{\varpi^\flat}\right]\right)[S]$. Set $d := \deg(g)$; this is also the degree of $\theta(g)$ because the degree of $S^p - S - \frac{T}{\varpi^\flat}$ does not decrease under θ. Since the roots of $S^p - S - \frac{T}{\varpi^\flat}$ in $(\mathbf{C}_p^\flat[\![T]\!])[S]$ are given by $\alpha + \sum_{\nu=0}^\infty \left(\frac{T}{\varpi^\flat}\right)^{p^\nu}$, where α ranges over all elements of $\mathbf{F}_p$, the $(d-1)$-st coefficient of g is $-\sum_{\mu=1}^d \alpha_\mu - d \cdot \sum_{\nu=0}^\infty \left(\frac{T}{\varpi^\flat}\right)^{p^\nu}$ for some (pairwise different) $\alpha_1, \ldots, \alpha_d \in \mathbf{F}_p$. As the image of θ is contained in $\left(V[\![T]\!]\left[\frac{1}{\varpi^\flat}\right]\right)[S]$, where the coefficients in all appearing power series have bounded denominators, $d \equiv 0 \mod p$ and therefore $\deg(g) = 0$ or $\deg(h) = 0$. However, $S^p - S - \frac{T}{\varpi^\flat}$ is not divisible by any nonunit of $V\{T\}\left[\frac{1}{\varpi^\flat}\right]$, so either g or h must be a unit. In other words, $S^p - S - \frac{T}{\varpi^\flat}$ is irreducible and the torsor $V\{T\}\left[\frac{1}{\varpi^\flat}\right] \to \left(V\{T\}\left[\frac{1}{\varpi^\flat}\right]\right)[S]/\left(S^p - S - \frac{T}{\varpi^\flat}\right)$ is nontrivial.

2.6.2 Proof of Theorem 2.6.3

We begin with the following key assertion, whose proof is inspired by [Sch15, § 4.2].

Lemma 2.6.13. *In Situation 2.6.2, let $Y_0 \to X_0$ be a finite morphism and $Y_i := Y_0 \times_{X_0} X_i$ for all $i \in I$. Assume that X_0 is proper. Then for all $n > d$*

$$\operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(Y_i, \mathbf{F}_p) = 0.$$

Proof. It suffices to show that the free $\mathscr{O}_C/p$-module

$$\operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(Y_i, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathscr{O}_C/p$$

is almost zero for all $n > d$. By Scholze's primitive comparison theorem [Sch13b, Thm. 3.17] (here properness of the X_i is crucial!), the stability of Situation 2.6.2 under base change along finite morphisms, and Example 2.6.5, there is an almost isomorphism

$$\operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(Y_i, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathscr{O}_C/p \xrightarrow{\sim} \operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(Y_i, \mathscr{O}^+_{Y_i}/p) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(Y, \mathscr{O}^+_Y/p),$$

where Y is the perfectoid space from Proposition 2.4.13. The Čech-to-derived functor spectral sequence for an affinoid perfectoid cover of Y and [Sch12, Prop. 6.14, Prop. 7.13] or [KL15, Prop. 8.3.2.(c)] show that there is an almost isomorphism of cohomology groups $\mathrm{H}^n_{\text{ét}}(Y, \mathscr{O}^+_Y/p)^a \simeq \mathrm{H}^n(Y, \mathscr{O}^+_Y/p)^a$ for the étale and analytic topology. However, since $Y^\Diamond \simeq Y_0^\Diamond \times_{X_0^\Diamond} X^\Diamond \simeq \lim_i Y_i^\Diamond$ and thus $|Y| \simeq \lim_i |Y_i|$ [Sch17, Lem. 11.22, Lem. 15.6] and since the cohomological dimension of the $|Y_i|$ is at most $\dim Y_i = d$ [Sch92, Cor. 4.6], the statement follows; cf. the proof of [Sch15, Cor. 4.2.2]. □

Remark 2.6.14. In the proof of Lemma 2.6.13, we could have proceeded along the lines of [CGH+20] and used Proposition 2.4.12 directly instead of referring to the global statement of Proposition 2.4.13. Namely, we can simply define the spatial diamond $Y^\Diamond := \lim_i Y_i^\Diamond$; the natural map

$$\operatorname*{colim}_i \mathrm{H}^n_{\text{ét}}(Y_i^\Diamond, \mathscr{O}^+_{Y_i^\Diamond}/p) \to \mathrm{H}^n_{\text{ét}}(Y^\Diamond, \mathscr{O}^+_{Y^\Diamond}/p)$$

is still an isomorphism by [Sch17, Prop. 14.9]. The morphism $\pi \colon Y^\Diamond_{\text{ét}} \to |Y_0^\Diamond| = |Y_0|$ from the étale site of $Y^\Diamond$ to the analytic site of Y_0 induces an isomorphism

$$\mathrm{R}\Gamma_{\text{ét}}(Y^\Diamond, \mathscr{O}^+_{Y^\Diamond}/p) \simeq \mathrm{R}\Gamma(|Y_0|, \mathrm{R}\pi_*(\mathscr{O}^+_{Y^\Diamond}/p)).$$

Since the cohomological dimension of $|Y_0|$ is at most $\dim Y_0 = d$ [Sch92, Cor. 4.6], we are left to prove that $\mathrm{R}^i\pi_*(\mathscr{O}^+_{Y^\Diamond}/p)^a = 0$ for all $i > 0$. This statement can be checked locally on $|Y_0|$ [Sch17, Cor. 16.10], so we may assume that we are in the affinoid situation of Proposition 2.4.12 and conclude with [Sch12, Prop. 7.13] or [KL15, Prop. 8.3.2.(c)].

At last, we can finish the proof of Theorem 2.6.3 by reducing the assertion to Lemma 2.6.13 via a standard dévissage argument.

Proof of Theorem 2.6.3. It suffices to prove the following statement for all $d \geq 0$ and all $n > d$ via ascending induction on d and descending induction on n:

$$\operatorname*{colim}_k \mathrm{H}^n_{\text{ét}}(X_k, F_k) = 0. \tag{$V_{d,n}$}$$

The base case $n > 2d$ is [Hub96, Cor. 2.8.3]. For the inductive step, we fix $d > 0$ and $n > d$ and assume $(V_{d',n'})$ for all d', n' such that either $d' < d$ and $n' > d'$, or $d' = d$ and $n' > n$.

First, by the topological invariance of the étale site and Proposition 2.4.13, we may, after possibly replacing X_k with $X_{0,\mathrm{red}} \times_{X_0} X_k$, assume that X_0 is reduced. Choose a dense, Zariski-open subspace $j_0 \colon U_0 \hookrightarrow X_0$ for which $j_0^* F_0$ is locally constant. By shrinking U_0 if necessary, we can ensure that U_0 is normal; cf. [Con99, Thm. 2.1.2]. Let $i_0 \colon Z_0 \hookrightarrow X_0$ be the inclusion of the complement of U_0 with its reduced closed subspace structure. For all $k \in I$, define $U_k := U_0 \times_{X_0} X_k$, $Z_k := Z_0 \times_{X_0} X_k$, and let $j_k \colon U_k \hookrightarrow X_k$ and $i_k \colon Z_k \hookrightarrow X_k$ be the projections to the second factor.

By Proposition 2.4.13, there is a perfectoid space Z representing $\lim_k Z_k^\diamondsuit$. Thus, by the induction hypothesis, $\operatorname{colim}_k \mathrm{H}^{n'}_{\text{ét}}(X_k, i_{k,*} i_k^* F_k) = \operatorname{colim}_k \mathrm{H}^{n'}_{\text{ét}}(Z_k, \pi_{k0}^* i_0^* F_0) = 0$ for $n' > d - 1$. Taking cohomology of the direct system of short exact sequences

$$0 \to j_{k,!} j_k^* F_k \to F_k \to i_{k,*} i_k^* F_k \to 0,$$

we see that the resulting map

$$\operatorname*{colim}_k \mathrm{H}^n_{\text{ét}}(X_k, j_{k,!} j_k^* F_k) \to \operatorname*{colim}_k \mathrm{H}^n_{\text{ét}}(X_k, F_k)$$

is an isomorphism. Since $j_{k,!} j_k^* F_k \simeq \pi_{k0}^* j_{0,!} j_0^* F_0$ by proper base change, we may assume that $F_0 = j_{0,!} L$ for some local system L on U_0.

In this case, we can choose a finite étale cover $f_0 \colon V_0 \to U_0$ for which $f_0^* L \simeq \mathbf{F}_p^{\oplus r}$. Let $\nu_0 \colon \widetilde{X}_0 \to X_0$ be the normalization of X_0; then ν_0 is an isomorphism over U_0 (see e.g. [Con99, § 2] for basic facts about normalizations of rigid spaces). By [Han20, Thm. 1.6], f_0 can be extended to a finite cover $\bar{f}_0 \colon Y_0 \to \widetilde{X}_0$. Denote by $\bar{\jmath}_0 \colon V_0 \hookrightarrow Y_0$ the induced open immersion. The trace map yields a surjective morphism

$$\nu_{0,*} \bar{f}_{0,*} \bar{\jmath}_{0,!} \mathbf{F}_p^{\oplus r} \simeq j_{0,!} f_{0,*} \mathbf{F}_p^{\oplus r} \simeq j_{0,!} f_{0,*} f_0^* L \twoheadrightarrow j_{0,!} L. \tag{2.7}$$

Let K denote the kernel of this map. For all $k \in I$, set $\widetilde{X}_k := \widetilde{X}_0 \times_{X_0} X_k$, $V_k := V_0 \times_{X_0} X_k$, and $Y_k := Y_0 \times_{X_0} X_k$. The projections $\nu_k \colon \widetilde{X}_k \to X_k$ are still isomorphisms over U_k. We obtain finite étale covers $f_k \colon V_k \to U_k$ extending to finite covers $\bar{f}_k \colon Y_k \to \widetilde{X}_k$ and open immersions $\bar{\jmath}_k \colon V_k \hookrightarrow Y_k$ as base changes from the respective morphisms over X_0.

Since ν_0, f_0, and $\bar{f}_0$ are finite, proper base change shows that pulling back (2.7) along π_{k0} yields a direct system of short exact sequences

$$0 \to \pi_{k0}^* K \to \nu_{k,*} \bar{f}_{k,*} \bar{\jmath}_{k,!} \mathbf{F}_p^{\oplus r} \to \pi_{k0}^* j_{0,!} L \to 0,$$

From the induction hypothesis $(V_{d,n+1})$, we know that $\operatorname{colim}_k \mathrm{H}^{n+1}_{\text{ét}}(X_k, \pi_{k0}^* K) = 0$. It remains to show that $\operatorname{colim}_k \mathrm{H}^n_{\text{ét}}(X_k, \nu_{k,*} \bar{f}_{k,*} \bar{\jmath}_{k,!} \mathbf{F}_p^{\oplus r}) = \operatorname{colim}_k \mathrm{H}^n_{\text{ét}}(Y_k, \bar{\jmath}_{k,!} \mathbf{F}_p^{\oplus r}) = 0$ as well.

Let $A_k := Z_k \times_{X_k} Y_k$ and $\bar{\imath}_k \colon A_k \hookrightarrow Y_k$ be the projection to the second factor. Consider the short exact sequences

$$0 \to \bar{\jmath}_{k,!} \mathbf{F}_p^{\oplus r} \to \mathbf{F}_p^{\oplus r} \to \bar{\imath}_{k,*} \mathbf{F}_p^{\oplus r} \to 0$$

on Y_k. Lemma 2.6.13 applied to the systems of finite morphisms $Y_k \to X_k$ and $A_k \to Z_k$ shows that $\operatorname{colim}_k \mathrm{H}^{n-1}(Y_k, \bar{\imath}_{k,*} \mathbf{F}_p^{\oplus r}) = (\operatorname{colim}_k \mathrm{H}^{n-1}(A_k, \mathbf{F}_p))^{\oplus r} = 0$ and $\operatorname{colim}_k \mathrm{H}^n_{\text{ét}}(Y_k, \mathbf{F}_p^{\oplus r}) = 0$, respectively, whence the claim. $\square$

CHAPTER 3

Moduli of curves

In this section, we apply the perfectoid methods introduced in Chapter 2 to the study of moduli spaces of curves. Recall from Chapter 1 that we want to prove the following statement.

Theorem 1.2.1. *Let $g \geq 2$ and p be a prime. Let $\mathscr{M}[p^n]$ be one of the following:*

(i) *the moduli space $\mathscr{M}_g[p^n]$ of smooth curves of genus g over $\mathbf{C}$ with full level-p^n structure,*

(ii) *the moduli space $\mathscr{M}_g^{\mathrm{c}}[p^n]$ of curves of compact type of genus g over $\mathbf{C}$ with full level-p^n structure, or*

(iii) *the moduli space $\overline{\mathscr{M}_g}[p^n]$ of pre-level-p^n curves of genus g over $\mathbf{C}$ with full level-p^n structure.*

Then we have

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}[p^n], \mathbf{F}_p) = 0$$

for all $i > 4g-5$ in case (i) *and for all $i > \left\lfloor \frac{7g}{2} \right\rfloor - 4$ in cases* (ii) *and* (iii).

Before we describe the content of the individual sections of this chapter, let us sketch the strategy of the proof of Theorem 1.2.1. We reduce (i) and (iii) to statements about $\mathscr{M}_g^{\mathrm{c}}$. To do so, first observe that since the boundaries $\mathscr{M}_g^{\mathrm{c}}[p^n] \smallsetminus \mathscr{M}_g[p^n]$ are Cartier divisors, the inclusions $j_n \colon \mathscr{M}_g[p^n] \hookrightarrow \mathscr{M}_g^{\mathrm{c}}[p^n]$ are affine morphisms. In particular, the functors $\mathrm{R}j_{n,*}$ are right t-exact for the perverse t-structure; see Appendix A for a review of the necessary prerequisites concerning perverse t-structures. Parts (i) and (ii) of Theorem 1.2.1 thus follow from parts (i) and (ii), respectively, of the following, more general statement.

Theorem 3.0.1. *Let $g \geq 2$ and p be a prime. Let $\mathscr{M}_g^{\mathrm{c}}[p^n]$ be the moduli space of curves of compact type of genus g over $\mathbf{C}$ with full level-p^n structure. Let $\pi_n \colon \mathscr{M}_g^{\mathrm{c}}[p^n] \to \mathscr{M}_g^{\mathrm{c}}$ be the maps "forgetting the level structure."*

(i) *If* $K \in {}^p\mathrm{D}(\mathscr{M}_g^{\mathrm{c}}, \mathbf{F}_p)^{\leq 0}$*, we have* $\mathrm{colim}_n\, \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g^{\mathrm{c}}[p^n], \pi_n^* K) = 0$ *for all* $i > g-2$*.*

(ii) *If* F *is a constructible sheaf of* $\mathbf{F}_p$*-modules on* $\mathscr{M}_g^{\mathrm{c}}$*, we have* $\mathrm{colim}_n\, \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g^{\mathrm{c}}[p^n], \pi_n^* F) = 0$ *for all* $i > \left\lfloor \frac{7g}{2} \right\rfloor - 4$*.*

Here, $\mathrm{D}(\mathscr{M}_g^{\mathrm{c}}, \mathbf{F}_p)$ denotes the bounded derived category of constructible sheaves of $\mathbf{F}_p$-modules on $\mathscr{M}_g^{\mathrm{c}}$. Theorem 3.0.1 is shown in three steps.

1. *Analysis of the Torelli morphism.* Let $\mathscr{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g over $\mathbf{C}$. The Torelli morphism $t_g \colon \mathscr{M}_g^{\mathrm{c}} \to \mathscr{A}_g$ sends a curve of compact type to the product of the Jacobians of its irreducible components. Analyzing the fiber dimensions of t_g, we produce bounds on its cohomological amplitude: in case (i), $\mathrm{R}t_{g,*}K \in {}^p\mathrm{D}(\mathscr{A}_g, \mathbf{F}_p)^{\leq g-2}$ and in case (ii), $\mathrm{R}t_{g,*}(F[3g-3]) \in {}^p\mathrm{D}(\mathscr{A}_g, \mathbf{F}_p)^{\leq \lfloor \frac{g}{2} \rfloor - 1}$. The same bounds hold in the presence of level structures. See § 3.3.1 for details.

2. *Passage to toroidal compactifications.* Let $\rho_n \colon \mathscr{A}_g[p^n] \to \mathscr{A}_g$ be the natural covering maps. By Step 1, it remains to prove that the functor

$$\mathrm{D}(\mathscr{A}_g, \mathbf{F}_p) \to \mathrm{D}(\mathbf{F}_p), \quad K \mapsto \operatorname*{colim}_n \mathrm{R}\Gamma(\mathscr{A}_g[p^n], \rho_n^* K)$$

takes ${}^p\mathrm{D}(\mathscr{A}_g, \mathbf{F}_p)^{\leq m}$ to $\mathrm{D}(\mathbf{F}_p)^{\leq m}$ for all $m \in \mathbf{Z}$. After fixing some additional auxiliary data, there are compatible toroidal compactifications $\mathscr{A}_g[p^n] \subset \overline{\mathscr{A}_g}[p^n]$ with forgetful maps $\bar{\rho}_{mn} \colon \overline{\mathscr{A}_g}[p^m] \to \overline{\mathscr{A}_g}[p^n]$. As in the case of $\mathscr{M}_g[p^n]$, the boundaries are Cartier divisors, and we can reduce the statement to showing that

$$\mathrm{D}(\overline{\mathscr{A}_g}[p^n], \mathbf{F}_p) \to \mathrm{D}(\mathbf{F}_p), \quad L \mapsto \operatorname*{colim}_m \mathrm{R}\Gamma(\overline{\mathscr{A}_g}[p^m], \bar{\rho}_{mn}^* L)$$

takes ${}^p\mathrm{D}(\overline{\mathscr{A}_g}[p^n], \mathbf{F}_p)^{\leq \ell}$ to $\mathrm{D}(\mathbf{F}_p)^{\leq \ell}$ for all $n \in \mathbf{Z}_{\geq 0}$ and $\ell \in \mathbf{Z}$.

3. *Use of perfectoid covers.* By work of Scholze and Pilloni–Stroh, the inverse limit of the projective system $\overline{\mathscr{A}_g}[p^n]$, $n \in \mathbf{Z}_{\geq 0}$, is similar (in the sense of [SW13, Def. 2.4.1]) to a perfectoid space; see Example 2.6.8. Using his "primitive comparison theorem," Scholze [Sch15, § 4.2] deduced from this that $\mathrm{colim}_n\, \mathrm{H}^i(\overline{\mathscr{A}_g}[p^n], \mathbf{F}_p) = 0$ for $i > \dim \mathscr{A}_g$. In § 2.6, we proved a more general statement for Zariski-constructible sheaves on cofiltered inverse systems of proper rigid spaces with finite transition maps whose inverse limit is similar to a perfectoid space; this yields the claim about perverse t-structures from Step 2.

In order to reduce Theorem 1.2.1.(iii) to a statement about $\mathscr{M}_g^{\mathrm{c}}$, we show the following assertion, which holds over any algebraically closed field k of characteristic not equal to p.

Theorem 3.0.2. *Let Λ_0 be an étale $\mathbf{F}_p$-local system on $\overline{\mathscr{M}_g}$, with pullbacks Λ_n to $\overline{\mathscr{M}_g}[p^n]$. Then for all $i \geq 0$, the natural map*

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\overline{\mathscr{M}_g}[p^n], \Lambda_n) \to \operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}^{\mathrm{c}}_g[p^n], \Lambda_n)$$

is an isomorphism.

The key observation in the proof is that the transition maps $\overline{\mathscr{M}_g}[p^{n+1}] \to \overline{\mathscr{M}_g}[p^n]$ are highly ramified over the boundary $\overline{\mathscr{M}_g}[p^n] \smallsetminus \mathscr{M}^{\mathrm{c}}_g[p^n]$: on complete local rings, they are given by

$$k[\![t_1, \dots, t_{3g-3}]\!] \to k[\![t_1, \dots, t_{3g-3}]\!], \quad t_i \mapsto \begin{cases} t_i^p & \text{if } 1 \leq i \leq \gamma, \\ t_i & \text{if } \gamma < i \leq 3g-3 \end{cases}$$

after a suitable choice of coordinates t_i for which $\overline{\mathscr{M}_g}[p^n] \smallsetminus \mathscr{M}^{\mathrm{c}}_g[p^n]$ corresponds to $\{t_1 \cdots t_\gamma = 0\}$; see Lemma 3.2.11. The claim then follows from a careful analysis of the maps between the excision triangles for the inclusions $\mathscr{M}^{\mathrm{c}}_g[p^n] \subset \overline{\mathscr{M}_g}[p^n]$. We refer to § 3.1 for details and a more general statement about projective systems of smooth Deligne–Mumford stacks with similar ramification over a system of normal crossings divisors.

In the argument, it really seems necessary to use the compactifictions via pre-level-p^n curves from [ACV03]. A "naïve" compactification of $M_g[p^n]$ as the normalization of $\overline{M_g}$ in the function field of $M_g[p^n]$, which turns out to be the coarse moduli space of $\overline{\mathscr{M}_g}[p^n]$, does not exhibit the same ramification behavior; see Example 3.3.12. For the toroidal compactifications $\mathscr{A}_g[p^n] \subset \overline{\mathscr{A}_g}[p^n]$, such complications do not arise: the $\overline{\mathscr{A}_g}[p^n]$ are algebraic spaces for $p^n \geq 3$ and the maps $\overline{\mathscr{A}_g}[p^{n+1}] \to \overline{\mathscr{A}_g}[p^n]$ are sufficiently ramified over the boundary $\overline{\mathscr{A}_g}[p^n] \smallsetminus \mathscr{A}_g[p^n]$ so that

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\overline{\mathscr{A}_g}[p^n], \mathbf{F}_p) \to \operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{A}_g[p^n], \mathbf{F}_p)$$

is again an isomorphism. Thus, the aforementioned results of Scholze [Sch15] and Pilloni–Stroh [PS16, Thm. 0.4] yield the analogous statement $\operatorname{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{A}_g[p^n], \mathbf{F}_p) = 0$ for $i > \dim \mathscr{A}_g = \frac{g(g+1)}{2}$, as had been observed earlier by Scholze.

Now for the content of the individual sections of this chapter. In § 3.1, we prove a general result about the étale cohomology of different towers of Deligne-Mumford stacks in the presence of ramification (Theorem 3.1.5) which, applied to moduli of curves, yields Theorem 3.0.2 from above. In § 3.2, we set up the necessary general theory for moduli of curves with level structure and their compactifications from [ACV03], which may not be well-known to a broader audience. We conclude in § 3.3 with the the proofs of Theorem 3.0.1 and Theorem 1.2.1 as sketched above.

3.1 Interlude: vanishing from ramification of the boundary divisor

Throughout this section, we work over a fixed algebraically closed field k.

Definition 3.1.1. Let $\mathscr{Y}$ be a smooth Deligne–Mumford stack over k. A *normal crossings divisor* on $\mathscr{Y}$ is an effective Cartier divisor $\mathscr{D} \subset \mathscr{Y}$ such that for any $y \in \mathscr{Y}(k)$, there is an integer $0 \le \gamma \le d$ and an isomorphism

$$\mathscr{O}^{\wedge}_{\mathscr{Y},y} \simeq k[\![t_1, \dots, t_d]\!]$$

under which the restriction of $\mathscr{D}$ to $\operatorname{Spec} \mathscr{O}^{\wedge}_{\mathscr{Y},y}$ is identified with $\{t_1 \cdots t_\gamma = 0\}$.

Note that if $\mathscr{Y}$ is a scheme, Definition 3.1.1 agrees with the usual one by [SP20, Lem. 0CBS] and Artin approximation. We will use the following notion from [AVA18, Def. 4.2].

Definition 3.1.2. Let $\pi\colon \mathscr{Y}' \to \mathscr{Y}$ be a proper and quasi-finite morphism of smooth Deligne–Mumford stacks over k. Let $\mathscr{D} \subset \mathscr{Y}$ be a normal crossings divisor. Then π is called *m-ramified over $\mathscr{D}$* if for all $y \in \mathscr{Y}'(k)$ the induced map $\pi^*\colon \mathscr{O}^{\wedge}_{\mathscr{Y},\pi(y)} \to \mathscr{O}^{\wedge}_{\mathscr{Y}',y}$ can be identified with a morphism

$$k[\![t_1, \dots, t_d]\!] \to k[\![t_1, \dots, t_d]\!]$$

such that the restriction of $\mathscr{D}$ to $\operatorname{Spec} \mathscr{O}^{\wedge}_{\mathscr{Y},\pi(y)}$ is given by $\{t_1 \cdots t_\gamma = 0\}$ and there exist units $\nu_i \in k[\![t_1, \dots, t_d]\!]$ with $\pi^*(t_i) = \nu_i t_i^m$ for all $1 \le i \le \gamma$.

Example 3.1.3. If $\mathscr{Y}' = \mathscr{Y} = \mathbf{P}^N_k$, the mock Frobenius

$$\pi\colon \mathbf{P}^N_k \to \mathbf{P}^N_k, \quad [x_0 : \cdots : x_N] \mapsto [x_0^p : \cdots : x_N^p]$$

from Example 2.6.6 is p-ramified over the divisor of coordinate hyperplanes $\mathscr{D} := \{x_0 \cdots x_N = 0\}$.

From now on assume $\operatorname{char} k \neq p$. The importance of Definition 3.1.2 lies in the following calculation.

Lemma 3.1.4. *Let $Y := \operatorname{Spec} k[\![t_1, \dots, t_d]\!]$ and D be the divisor $\{t_1 \cdots t_\gamma = 0\}$ for some $0 \le \gamma \le d$. Set $U := Y \smallsetminus D$. Let $\pi\colon Y \to Y$ be a morphism such that for all $1 \le r \le \gamma$ there exist units $\nu_r \in k[\![t_1, \dots, t_d]\!]$ with $\pi^*(t_r) = \nu_r t_r^p$. Then the induced map*

$$\pi_U^*\colon \mathrm{H}^i(U, \mathbf{F}_p) \to \mathrm{H}^i(U, \mathbf{F}_p)$$

is 0 for all $i > 0$.

Proof. For all $1 \leq r \leq \gamma$, let $U_r := Y \smallsetminus \{t_r = 0\}$. By cohomological purity (and the assumption on $\operatorname{char} k$), $\mathrm{H}^1(U_r, \mathbf{F}_p) \simeq \mathbf{F}_p$ and the natural map

$$\bigwedge^{i}\Big(\oplus_{r=1}^{\gamma} \mathrm{H}^1(U_r, \mathbf{F}_p)\Big) \to \mathrm{H}^i(U, \mathbf{F}_p)$$

given by the cup product is an isomorphism; see e.g. [SGA4, Thm. XIX.1.2] or [ILO14, Cor. XVI.3.1.4] for the most general version. Thus, we may assume $\gamma = 1$ and $i = 1$.

Since $z^p - t_1$ is irreducible in $(k[\![t_1, \dots, t_d]\!]_{t_1})[z]$, the group $\mathrm{H}^1(U, \mathbf{F}_p)$ is (under the non-canonical isomorphism $\mathbf{F}_p \simeq \mu_p$) generated by the class of the μ_p-torsor

$$\operatorname{Spec} \tfrac{k[\![t_1,\dots,t_d]\!]_{t_1}[z]}{(z^p - t_1)} \to \operatorname{Spec} k[\![t_1, \dots, t_d]\!]_{t_1}.$$

Its image under π_U^* is the class of μ_p-torsor

$$\operatorname{Spec} \tfrac{k[\![t_1,\dots,t_d]\!]_{t_1}[z]}{(z^p - \nu_1 t_1^p)} \to \operatorname{Spec} k[\![t_1, \dots, t_d]\!]_{t_1}.$$

However, this torsor is trivial because the unit ν_1 has a p-th root $\lambda_1 \in k[\![t_1, \dots, t_d]\!]$ by Hensel's lemma and thus

$$z^p - \nu_1 t_1^p = \prod_{\zeta \in \mu_p(k)} (z - \zeta \lambda_1 t_1). \qquad \square$$

Theorem 3.1.5. *Let $\mathscr{Y}_n$, $n \in \mathbf{Z}_{\geq 0}$, be a projective system of smooth, tame Deligne–Mumford stacks of finite type over k with proper and quasi-finite transition maps $\pi_{mn} \colon \mathscr{Y}_m \to \mathscr{Y}_n$ for all $m \geq n$. Let Λ_0 be an étale $\mathbf{F}_p$-local system on $\mathscr{Y}_0$ and $\Lambda_n := \pi_{n0}^* \Lambda_0$. Assume there exist normal crossings divisors $\mathscr{D}_n \subset \mathscr{Y}_n$ such that $\mathscr{D}_m = (\pi_{mn}^* \mathscr{D}_n)_{\mathrm{red}}$ and $\pi_{n+1,n}$ is p-ramified over $\mathscr{D}_n$ for all n. Set $\mathscr{U}_n := \mathscr{Y}_n \smallsetminus \mathscr{D}_n$. Then the natural map*

$$\operatorname*{colim}_n \mathrm{H}^i(\mathscr{Y}_n, \Lambda_n) \to \operatorname*{colim}_n \mathrm{H}^i(\mathscr{U}_n, \Lambda_n)$$

is an isomorphism for all $i \geq 0$.

Throughout the proof, we denote by $j_n \colon \mathscr{U}_n \hookrightarrow \mathscr{Y}_n$ and $i_n \colon \mathscr{D}_n \hookrightarrow \mathscr{Y}_n$ the canonical inclusions. To simplify notation, we moreover set $\pi_n := \pi_{n0}$. The strategy of the proof is as follows. By the Gysin sequence, the assertion is implied by $\operatorname{hocolim}_n \mathrm{R}\pi_{n,*} i_{n,*} \mathrm{R} i_n^! \Lambda_n \simeq 0$, which amounts to the vanishing of $\operatorname{colim}_n \mathscr{H}^\ell(\mathrm{R}\pi_{n,*} i_{n,*} \mathrm{R} i_n^! \Lambda_n)_{y_0}$ for all ℓ and all $y_0 \in \mathscr{Y}(k)$. In fact, we show that any composition of ℓ transition maps in the latter directed systems is 0, which can be checked on completions by a standard reduction.

Thus, we first treat the complete, local picture. Let $y_0 \in \mathscr{Y}_0(k)$. Choose a system of points $y_m \in \mathscr{Y}_m(k)$ with $\pi_{m+1,m}(y_{m+1}) = y_m$ for all $m \geq 0$. Set $V_m := \operatorname{Spec} \mathscr{O}^\wedge_{\mathscr{Y}_m, y_m}$. Let $Z_m \subset V_m$

be the restriction of $\mathscr{D}_m$ to V_m and $U_m := V_m \smallsetminus Z_m \subseteq V_m$ be its complement.

Let $H_m := \mathrm{Aut}(y_m)$ be the automorphism group scheme of y_m. It acts naturally on V_m. Since $\mathscr{D}_m \subset \mathscr{Y}_m$ is a substack, U_m and Z_m are invariant under this action. We obtain diagrams

$$\begin{tikzcd} {[U_m/H_m]} \arrow[r, hook, "\hat{\jmath}_m"] \arrow[d] & {[V_m/H_m]} \arrow[d, "\hat{\pi}_m"] & {[Z_m/H_m]} \arrow[l, hook', "\hat{\imath}_m"'] \arrow[d] \\ {[U_0/H_0]} \arrow[r, hook, "\hat{\jmath}_0"] & {[V_0/H_0]} & {[Z_0/H_0]\,,} \arrow[l, hook', "\hat{\imath}_0"'] \end{tikzcd}$$

where $\hat{\jmath}_m$ is an open and $\hat{\imath}_m$ a closed immersion. Denote the pullback of Λ_m to $[V_m/H_m]$ by $\widehat{\Lambda}_m$; we have $\widehat{\Lambda}_m = \hat{\pi}_m^* \widehat{\Lambda}_0$ for all $m \geq 0$.

Lemma 3.1.6. *For any $\ell, n \geq 0$, the pullback morphism*

$$\mathscr{H}^\ell\Big((\mathrm{R}\hat{\pi}_{n,*}\hat{\imath}_{n,*}\mathrm{R}\hat{\imath}_n^!\widehat{\Lambda}_n)_{y_0}\Big) \to \mathscr{H}^\ell\Big((\mathrm{R}\hat{\pi}_{n+\ell,*}\hat{\imath}_{n+\ell,*}\mathrm{R}\hat{\imath}_{n+\ell}^!\widehat{\Lambda}_{n+\ell})_{y_0}\Big)$$

of $\mathbf{F}_p$-vector spaces is 0.

Here, the stalk at y_0 is understood to be the derived pullback under $y_0\colon \mathrm{Spec}\, k \to \mathscr{Y}_0$.

Proof. For any $m \geq 0$, the exceptional inverse image fits into the distinguished triangle

$$(\mathrm{R}\hat{\pi}_{m,*}\widehat{\Lambda}_m)_{y_0} \to (\mathrm{R}\hat{\pi}_{m,*}\mathrm{R}\hat{\jmath}_{m,*}\hat{\jmath}_m^*\widehat{\Lambda}_m)_{y_0} \to (\mathrm{R}\hat{\pi}_{m,*}\hat{\imath}_{m,*}\mathrm{R}\hat{\imath}_m^!\widehat{\Lambda}_m)_{y_0}[1] \to . \qquad (3.1)$$

Set $K_m := \ker(H_m \to H_0)$. Via base change (cf. e.g. [ACV03, Thm. A.0.2]) across the diagram of fiber squares

$$\begin{tikzcd} & {[U_m/K_m]} \arrow[r, "\alpha'"] \arrow[d, hook, "\jmath'"] & {[U_m/H_m]} \arrow[d, hook, "\hat{\jmath}_m"] \\ {[\mathrm{Spec}\, k/K_m]} \arrow[r, "\tilde{y}_m"] \arrow[d, "\hat{\rho}"] & {[V_m/K_m]} \arrow[r, "\alpha_m"] \arrow[d, "\hat{\pi}'"] & {[V_m/H_m]} \arrow[d, "\hat{\pi}_m"] \\ \mathrm{Spec}\, k \arrow[r, "\tilde{y}_0"] & V_0 \arrow[r, "\alpha_0"] & {[V_0/H_0]} \end{tikzcd}$$

(in which $\tilde{y}_0$ is the lift of y_0 and α_0 is the canonical étale atlas), (3.1) is identified with

$$\mathrm{R}\hat{\rho}_*\tilde{y}_m^*\hat{\pi}'^{,*}\alpha_0^*\widehat{\Lambda}_0 \to \mathrm{R}\hat{\rho}_*\tilde{y}_m^*\mathrm{R}\jmath'_*\jmath'^{,*}\hat{\pi}'^{,*}\alpha_0^*\widehat{\Lambda}_0 \to (\mathrm{R}\hat{\pi}_{m,*}\hat{\imath}_{m,*}\mathrm{R}\hat{\imath}_m^!\widehat{\Lambda}_m)_{y_0}[1] \to .$$

Since V_0 is strictly henselian, $\alpha_0^*\widehat{\Lambda}_0 \simeq \mathbf{F}_p^{\oplus r}$, where r is the rank of Λ_0. Thus, $\tilde{y}_m^*\hat{\pi}'^{,*}\alpha_0^*\widehat{\Lambda}_0 \simeq \mathbf{F}_p^{\oplus r}$ with the trivial K_m-action and $\tilde{y}_m^*\mathrm{R}\jmath'_*\jmath'^{,*}\hat{\pi}'^{,*}\alpha_0^*\widehat{\Lambda}_0 \simeq \mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r})$ with K_m-action induced by the K_m-action on U_m. Moreover, $\tilde{y}_m^*\hat{\pi}'^{,*}\alpha_0^*\widehat{\Lambda}_0 \to \tilde{y}_m^*\mathrm{R}\jmath'_*\jmath'^{,*}\hat{\pi}'^{,*}\alpha_0^*\widehat{\Lambda}_0$ is the natural morphism

$\tau^{\leq 0}(\mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r})) \to \mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r})$, where τ denotes the truncation with respect to the natural t-structure on the derived category of $\mathbf{F}_p[K_m]$-modules [BBD82, Prop. 1.3.3]. This proves

$$(\mathrm{R}\hat{\pi}_{m,*}\hat{\imath}_{m,*}\mathrm{R}\hat{\imath}_m^!\widehat{\Lambda}_m)_{y_0} \simeq \mathrm{R}\hat{\rho}_*\Big(\tau^{>0}(\mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r}))\Big)[-1] \simeq \mathrm{R}\Gamma\Big(K_m, \tau^{>0}(\mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r}))\Big)[-1]$$

and hence $\mathscr{H}^\ell\Big((\mathrm{R}\hat{\pi}_{m,*}\hat{\imath}_{m,*}\mathrm{R}\hat{\imath}_m^!\widehat{\Lambda}_m)_{y_0}\Big) \simeq \mathrm{H}^{\ell-1}\Big(K_m, \tau^{>0}(\mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r}))\Big)$.

It remains to show that the pullback morphism

$$\hat{\pi}^*_{n+\ell,n} \colon \mathrm{H}^{\ell-1}\Big(K_n, \tau^{>0}(\mathrm{R}\Gamma(U_n, \mathbf{F}_p^{\oplus r}))\Big) \to \mathrm{H}^{\ell-1}\Big(K_{n+\ell}, \tau^{>0}(\mathrm{R}\Gamma(U_{n+\ell}, \mathbf{F}_p^{\oplus r}))\Big)$$

is 0 for all ℓ. For $n \leq m \leq n+\ell-1$, $\hat{\pi}^*_{m+1,m}$ is induced by the natural morphism of hypercohomology spectral sequences

$$\begin{tikzcd}
\mathrm{H}^p\Big(K_m, \mathscr{H}^q\Big(\tau^{>0}(\mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r}))\Big)\Big) \arrow[r, Rightarrow] \arrow[d, "\mathscr{H}^q(\hat{\pi}^*_{m+1,m})"] & \mathrm{H}^{\ell-1}\Big(K_m, \tau^{>0}(\mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r}))\Big) \arrow[d, "\hat{\pi}^*_{m+1,m}"] \\
\mathrm{H}^p\Big(K_{m+1}, \mathscr{H}^q\Big(\tau^{>0}(\mathrm{R}\Gamma(U_{m+1}, \mathbf{F}_p^{\oplus r}))\Big)\Big) \arrow[r, Rightarrow] & \mathrm{H}^{\ell-1}(K_{m+1}, \tau^{>0}(\mathrm{R}\Gamma(U_{m+1}, \mathbf{F}_p^{\oplus r}))\Big).
\end{tikzcd}$$

This follows from the functoriality properties of the Cartan–Eilenberg resolution, which is used in the construction of the hypercohomology spectral sequence, and of the injective resolutions from which the pullback maps are computed.

The morphisms

$$\mathscr{H}^q(\mathrm{R}\Gamma(U_m, \mathbf{F}_p^{\oplus r})) \to \mathscr{H}^q(\mathrm{R}\Gamma(U_{m+1}, \mathbf{F}_p^{\oplus r}))$$

of $\mathbf{F}_p[K_{m+1}]$-modules are 0 for all $q > 0$: this can be checked on the étale cover $\operatorname{Spec} k \to [\operatorname{Spec} k/K_{m+1}]$, where it follows from Definition 3.1.2, Lemma 3.1.4, and the additivity of the cohomology functors. Therefore, the morphisms between the second pages are 0, and so are the morphisms between the graded rings associated with the induced ℓ-step filtrations of the abutments. Consequently, $\hat{\pi}^*_{n+\ell,n} = \hat{\pi}^*_{n+\ell,n+\ell-1} \circ \cdots \circ \hat{\pi}^*_{n+1,n} = 0$. □

We return to the setting of Theorem 3.1.5.

Lemma 3.1.7. *For all $y_0 \in \mathscr{Y}_0(k)$ and all $\ell, n \geq 0$, the pullback morphism*

$$\mathscr{H}^\ell(\mathrm{R}\pi_{n,*}i_{n,*}\mathrm{R}i_n^!\Lambda_n)_{y_0} \to \mathscr{H}^\ell(\mathrm{R}\pi_{n+\ell,*}i_{n+\ell,*}\mathrm{R}i_{n+\ell}^!\Lambda_{n+\ell})_{y_0}$$

of $\mathbf{F}_p$-vector spaces is 0.

Proof. For all $m \geq 0$, let Y_m be the coarse space of $\mathscr{Y}_m$ and $\bar{\pi}_m \colon Y_m \to Y_0$ be the map induced by π_m. For any $y_m \in \mathscr{Y}_m(k)$, the corresponding point in $Y_m(k)$ will be denoted by $\bar{y}_m$. Let $\mathscr{Y}^\wedge_{m,y_m} := \mathscr{Y}_m \times_{Y_m} \operatorname{Spec} \mathscr{O}^\wedge_{Y_m,\bar{y}_m}$ be the completion of $\mathscr{Y}_m$ at y_m. In the previous notation, we have $\mathscr{Y}^\wedge_{m,y_m} = [V_m/H_m]$, and similarly $\mathscr{U}_m \times_{\mathscr{Y}_m} \mathscr{Y}^\wedge_{m,y_m} = [U_m/H_m]$ and $\mathscr{D}_m \times_{\mathscr{Y}_m} \mathscr{Y}^\wedge_{m,y_m} = [Z_m/H_m]$ (cf. the proof of [Ols16, Thm. 11.3.1]).

The maps $\bar{\pi}_m \colon Y_m \to Y_0$ are still quasi-finite and proper, hence finite. Thus, by the theorem on formal functions

$$\mathrm{Y}^\wedge_{0,\bar{y}_0} \times_{\mathrm{Y}_0} Y_m \simeq \operatorname{Spec}(\bar{\pi}_* \mathscr{O}_{\mathrm{Y}_m})^\wedge_{\bar{y}_m} \simeq \coprod_{\bar{\pi}_m(\bar{y}_m)=\bar{y}_0} \mathrm{Y}^\wedge_{m,\bar{y}_m}.$$

As $\mathscr{Y}^\wedge_{0,y_0} \times_{\mathscr{Y}_0} \mathscr{Y}_m \simeq (\mathrm{Y}^\wedge_{0,\bar{y}_0} \times_{\mathrm{Y}_0} Y_m) \times_{Y_m} \mathscr{Y}_m$, the diagram

$$\begin{tikzcd}
\coprod_{\pi_m(y_m)=y_0} \mathscr{Y}^\wedge_{m,y_m} \arrow[r] \arrow[d] & \mathscr{Y}_m \arrow[d] \\
\mathscr{Y}^\wedge_{0,y_0} \arrow[r] & \mathscr{Y}_0
\end{tikzcd}$$

is cartesian.

Since $y_0 \colon \operatorname{Spec} k \to \mathscr{Y}_0$ factors through $\mathscr{Y}^\wedge_{0,y_0}$, Lemma 3.1.8 shows that the vector spaces $\mathscr{H}^\ell(\mathrm{R}\pi_{m,*} i_{m,*} \mathrm{R}i^!_m \Lambda_m)_{y_0}$ as well as the morphisms between them may be computed on completions. The statement therefore follows from Lemma 3.1.6 and exactness of the stalk functor. $\square$

Lemma 3.1.8. *Let $f \colon \mathscr{W} \to \mathscr{Y}$ be a morphism of finite type between noetherian Deligne–Mumford stacks over k. Assume the coarse moduli space Y of $\mathscr{Y}$ is excellent. Let $y \in \mathscr{Y}(k)$, giving rise to the cartesian square*

$$\begin{tikzcd}
\mathscr{W} \times_{\mathscr{Y}} \mathscr{Y}^\wedge_y \arrow[r, "h'"] \arrow[d, "\hat{f}"] & \mathscr{W} \arrow[d, "f"] \\
\mathscr{Y}^\wedge_y \arrow[r, "h"] & \mathscr{Y}.
\end{tikzcd}$$

Let F be an étale abelian torsion sheaf on $\mathscr{W}$ and G be an étale abelian torsion sheaf on $\mathscr{Y}$ whose torsion orders are invertible in k. Then

(i) $h^* \mathrm{R} f_* F \simeq \mathrm{R}\hat{f}_* h'^{,*} F$

(ii) *If f is a closed immersion, $h^* f_* \mathrm{R} f^! G \simeq \hat{f}_* \mathrm{R} \hat{f}^! h^* G$.*

Proof. (i). By Néron–Popescu desingularization [Pop86] and excellence of Y, the natural map $Y^\wedge_y \to Y$ is a cofiltered limit of smooth morphisms. Taking fiber products with $\mathscr{Y}$, we see that $h\colon \mathscr{Y}_y^\wedge \to \mathscr{Y}$ is a cofiltered limit of smooth morphisms $h_\nu\colon \mathscr{Y}_\nu \to \mathscr{Y}$. For each such morphism, we have the following fiber square:

$$\begin{tikzcd} \mathscr{W} \times_{\mathscr{Y}} \mathscr{Y}_\nu \arrow[r, "h'_\nu"] \arrow[d, "f_\nu"] & \mathscr{W} \arrow[d, "f"] \\ \mathscr{Y}_\nu \arrow[r, "h_\nu"] & \mathscr{Y}. \end{tikzcd}$$

Let further $q_\nu\colon \mathscr{Y}_y^\wedge \to \mathscr{Y}_\nu$ denote the canonical morphisms. By smooth base change,

$$h^*\mathrm{R}f_*F \simeq \operatorname{hocolim} q_\nu^* h_\nu^* \mathrm{R}f_*F \simeq \operatorname{hocolim} q_\nu^* \mathrm{R}f_{\nu,*}h_\nu^{\prime,*}F \simeq \mathrm{R}\hat{f}_* h^{\prime,*}F.$$

(ii). Let $j\colon \mathscr{U} \hookrightarrow \mathscr{Y}$ be the complement of $\mathscr{W}$ with its canonical open substack structure. By (i), $h^*\mathrm{R}j_*j^*G \simeq \mathrm{R}\hat{\jmath}_*\hat{\jmath}^*h^*G$. Using the distinguished triangles

$$h^*f_*\mathrm{R}f^!G \to h^*G \to h^*\mathrm{R}j_*j^*G \to \quad \text{and} \quad \hat{f}_*\mathrm{R}\hat{f}^!h^*G \to h^*G \to \mathrm{R}\hat{\jmath}_*\hat{\jmath}^*h^*G \to,$$

we see that likewise $h^*f_*\mathrm{R}f^!G \simeq \hat{f}_*\mathrm{R}\hat{f}^!h^*G$. □

Proof of Theorem 3.1.5. The projective system $\mathscr{Y}_n$ gives rise to the distinguished triangle

$$\operatorname*{hocolim}_n \mathrm{R}\pi_{n,*}i_{n,*}\mathrm{R}i_n^!\Lambda_n \to \operatorname*{hocolim}_n \mathrm{R}\pi_{n,*}\Lambda_n \to \operatorname*{hocolim}_n \mathrm{R}\pi_{n,*}\mathrm{R}j_{n,*}j_n^*\Lambda_n \to .$$

It suffices to show that $\mathscr{H}^\ell(\operatorname{hocolim}_n \mathrm{R}\pi_{n,*}i_{n,*}\mathrm{R}i_n^!\Lambda_n) = 0$ for all ℓ. This can be checked on the stalks at all finite type points $y_0 \in \mathscr{Y}_0(k)$. By [SP20, Lem. 0CRK] and the cocontinuity of pullback functors, $\mathscr{H}^\ell(\operatorname{hocolim}_n \mathrm{R}\pi_{n,*}i_{n,*}\mathrm{R}i_n^!\Lambda_n)_{y_0} \simeq \operatorname{colim}_n \mathscr{H}^\ell(\mathrm{R}\pi_{n,*}i_{n,*}\mathrm{R}i_n^!\Lambda_n)_{y_0}$, so that the assertion follows from Lemma 3.1.7. □

Inspired by Theorem 2.6.3, one might try to generalize Theorem 3.1.5 and ask whether for an arbitrary constructible sheaf F_0 on $\mathscr{Y}_0$ with pullbacks F_n to $\mathscr{Y}_n$, the natural map

$$\operatorname*{colim}_n \mathrm{H}^i(\mathscr{Y}_n, F_n) \to \operatorname*{colim}_n \mathrm{H}^i(\mathscr{U}_n, F_n)$$

is still an isomorphism for all $i \geq 0$. Any such hope is quickly shattered by the next example.

Example 3.1.9. Let $\mathscr{Y}_n := \mathbf{P}_k^1$ for all $n \in \mathbf{Z}_{\geq 0}$, with the transition maps given by the mock Frobenii from Example 2.6.6 and hence p-ramified as in Example 3.1.3. Let $i_0\colon \operatorname{Spec} k \hookrightarrow \mathscr{Y}_0$ be the inclusion of the closed point $[0 : 1] \in \mathbf{P}_k^1$. The pullback of i_0 to $\mathscr{Y}_n$ is the closed immersion $i_n\colon \operatorname{Spec} k[x]/(x^{p^n}) \hookrightarrow \mathbf{P}_k^1$ corresponding to the p^n-th infinitesimal thickening of

$[0:1]$ in $\mathbf{P}^1_k$. For $F_0 := i_{0,*}\mathbf{F}_p$, we have $F_n = i_{n,*}\mathbf{F}_p$. Let $\mathscr{U}_n$ be the complement of $[0:1]$ in $\mathscr{Y}_n$. Then by the topological invariance of the étale site,

$$\operatorname*{colim}_n \mathrm{H}^0(\mathscr{Y}_n, F_n) \simeq \operatorname*{colim}_n \mathrm{H}^0\Big(\operatorname{Spec} k[x]/(x^{p^n}), \mathbf{F}_p\Big) \simeq \operatorname*{colim}_n \mathrm{H}^0(\operatorname{Spec} k, \mathbf{F}_p) = \mathbf{F}_p,$$

whereas $F_n\big|_{\mathscr{U}_n} \simeq 0$ and thus $\operatorname{colim}_n \mathrm{H}^0(\mathscr{U}_n, F_n) \simeq 0$. Hence, the map from Theorem 3.1.5 is not an isomorphism in this case.

3.2 Moduli spaces of curves

3.2.1 The moduli problems

We review different moduli spaces of curves from the literature, in part to fix some notation. Let k be an algebraically closed field. Fix an integer $g \geq 2$. We begin with a stack-theoretic version of stable curves, which in this generality is taken from [AOV11, Def. 2.1, Prop. 2.3].

Definition 3.2.1. A *twisted curve* is a flat, proper, tame algebraic stack $\mathscr{C} \to S$ such that each geometric fiber $\mathscr{C}_{\bar{s}}$ satisfies the following conditions:

(i) $\mathscr{C}_{\bar{s}}$ is purely 1-dimensional and connected;

(ii) The coarse moduli space $C_{\bar{s}}$ is a nodal curve;

(iii) The natural map $\mathscr{C}_{\bar{s}} \to C_{\bar{s}}$ is an isomorphism over the smooth locus of $C_{\bar{s}}$;

(iv) For the strictly henselian local ring $\mathscr{O}^{\mathrm{sh}}_{C_{\bar{s}},x}$ at a node $x \in C_{\bar{s}}$, there is $m \in \mathbf{Z}_{>0}$ such that

$$\mathscr{C}^{\mathrm{sh}}_{\bar{s},x} := \mathscr{C}_{\bar{s}} \times_{C_{\bar{s}}} \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{C_{\bar{s}},x} \simeq \Big[\operatorname{Spec}(k[z,w]/(zw))^{\mathrm{sh}}/\mu_m\Big],$$

where $\zeta \in \mu_m$ acts on $\operatorname{Spec}(k[z,w]/(zw))^{\mathrm{sh}}$ by $z \mapsto \zeta z$ and $w \mapsto \zeta^{-1} w$.

Definition 3.2.2 ([ACV03, Def. 6.1.1])**.** Let $m \in \mathbf{Z}_{>0}$. A *pre-level-m curve* is a twisted curve $\mathscr{C} \to S$ whose coarse moduli space is a stable curve and whose geometric fibers have trivial stabilizer at each separating node and stabilizer μ_m at each nonseparating node.

Pre-level-m curves without nontrivial stabilizers are exactly the curves of compact type.

Definition 3.2.3. A stable curve $C \to S$ is *of compact type* if each geometric fiber $C_{\bar{s}}$ satisfies one of the following equivalent conditions:

(i) The Jacobian $J(C_{\bar{s}})$ is compact;

(ii) All nodes of $C_{\bar{s}}$ are separating;

(iii) The dual graph of $C_{\bar{s}}$ is a tree.

From now on, we will assume that $m \in \mathbf{Z}_{>0}$ is invertible in k. In that case, we can endow pre-level-m curves with level structures.

Definition 3.2.4 ([ACV03, § 6])**.** Let $f\colon \mathscr{C} \to S$ be a pre-level-m curve of genus g. A *full level-m structure* on $\mathscr{C}$ is the choice of a symplectic isomorphism of local systems

$$\mathrm{R}^1 f_*(\mathbf{Z}/m\mathbf{Z}) \xrightarrow{\sim} \underline{(\mathbf{Z}/m\mathbf{Z})^{2g}}.$$

We consider the following smooth Deligne–Mumford stacks:

$\mathscr{M}_g$	moduli stack of smooth curves of genus g over k
$\mathscr{M}_g^{\mathrm{c}}$	moduli stack of curves of compact type of genus g over k
$\overline{\mathscr{M}_g}$	moduli stack of stable curves of genus g over k
$\mathscr{M}_g[m]$	moduli stack of smooth curves of genus g over k with full level-m structure
$\mathscr{M}_g^{\mathrm{c}}[m]$	moduli stack of curves of compact type of genus g over k with full level-m structure
$\overline{\mathscr{M}_g}[m]$	moduli stack of pre-level-m curves of genus g over k with full level-m structure.

We use Roman letters (i.e., M_g, M_g^{c}, etc.) for the corresponding coarse spaces. By covering space theory, $\overline{\mathscr{M}_g}[m]$ can be identified with the stack of connected $(\mathbf{Z}/m\mathbf{Z})^{2g}$-torsors over twisted curves with stable coarse moduli space, rigidified along $(\mathbf{Z}/m\mathbf{Z})^{2g}$ [ACV03, Thm. 6.2.4]. There are natural open embeddings $\mathscr{M}_g \subset \mathscr{M}_g^{\mathrm{c}} \subset \overline{\mathscr{M}_g}$. The natural covers $\mathscr{M}_g[m] \to \mathscr{M}_g$ and $\mathscr{M}_g^{\mathrm{c}}[m] \to \mathscr{M}_g^{\mathrm{c}}$ are finite étale. Although the cover $\overline{\mathscr{M}_g}[m] \to \overline{\mathscr{M}_g}$ is flat, proper, and quasi-finite [ACV03, Cor. 3.0.5], it is not representable [ACV03, Rem. 5.2.4.(b)] and highly ramified over $\overline{\mathscr{M}_g} \smallsetminus \mathscr{M}_g^{\mathrm{c}}$ (see Lemma 3.2.11).

3.2.2 Local structure of $\overline{\mathscr{M}_g}[m]$

Next, we summarize the description of the complete, local picture from § 3.1 for a given point $y \in \overline{\mathscr{M}_g}[m](k)$ in a sequence of well-known lemmas. Most of what follows can be found in [ACV03].

Fix $m \in \mathbf{Z}_{>0}$ invertible in k. Set $G := (\mathbf{Z}/m\mathbf{Z})^{2g}$. The datum parametrized by y is a pre-level-m curve $\mathscr{C}$ over k of genus g together with a full level-m structure $\mathrm{H}^1(\mathscr{C}, \mathbf{Z}/m\mathbf{Z}) \xrightarrow{\sim} G$, or equivalently a connected G-torsor $P \to \mathscr{C}$. Let C be the coarse space of $\mathscr{C}$. Assume that C has nonseparating nodes $x_1, \ldots, x_\gamma$ and separating nodes $x_{\gamma+1}, \ldots, x_\delta$.

Since $\overline{\mathscr{M}_g}[m]$ is Deligne–Mumford and $P \to \mathscr{C}$ is étale, $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$ is the universal deformation ring of $\mathscr{C}$. Let $\mathfrak{C} \to \operatorname{Spec} \mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$ be the universal curve.

Lemma 3.2.5. (i) *The completed local ring $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$ is regular and of dimension $3g-3$.*

(ii) *There is an isomorphism $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y} \simeq k[\![t_1, \dots, t_{3g-3}]\!]$ such that the locus over which the node x_i persists in $\mathfrak{C}$ is $\{t_i = 0\}$.*

Proof. (i). Since $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$ is the universal deformation ring of $\mathscr{C}$, both properties follow from the usual analysis of the deformation theory of $\mathscr{C}$; see [ACV03, § 3].

(ii). The local-to-global Ext spectral sequence decomposes the tangent space $\operatorname{Ext}^1(\Omega^1_{\mathscr{C}}, \mathscr{O}_{\mathscr{C}})$ of $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$ via the short exact sequence

$$0 \to \mathrm{H}^1(\mathscr{C}, \mathscr{T}_{\mathscr{C}}) \to \operatorname{Ext}^1(\Omega^1_{\mathscr{C}}, \mathscr{O}_{\mathscr{C}}) \to \prod_{i=1}^{\delta} \operatorname{Ext}^1\left(\Omega^{1,\wedge}_{\mathscr{C},x_i}, \mathscr{O}^{\wedge}_{\mathscr{C},x_i}\right) \to 0 \tag{3.2}$$

into a "global contribution" from the left and a "local contribution" from the right term; cf. [DM69, Prop. 1.5]. To analyze the local part, note that for each node x_i, the completed local ring $\mathscr{O}^{\wedge}_{\mathscr{C},x_i} \simeq k[\![z,w]\!]/(zw)$ of $\mathscr{C}$ at x_i admits the universal formal deformation $k[\![z,w,t_i]\!]/(zw - t_i)$. Thus, the universal deformation ring of the node is given by $k[\![t_i]\!]$ and its tangent space $\operatorname{Ext}^1(\Omega^{1,\wedge}_{\mathscr{C},x_i}, \mathscr{O}^{\wedge}_{\mathscr{C},x_i})$ is of dimension 1.

Since the $\mathscr{O}^{\wedge}_{\mathfrak{C},x_i}$ are deformations of $\mathscr{O}^{\wedge}_{\mathscr{C},x_i}$, there are natural morphisms $\varphi_i \colon k[\![t_i]\!] \to \mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$. Their completed tensor product

$$\varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_\delta \colon k[\![t_1, \dots, t_\delta]\!] \to \mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$$

is formally smooth because its tangent map from (3.2) is surjective. Therefore, we can choose an isomorphism $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y} \simeq \operatorname{Spec} k[\![t_1, \dots, t_{3g-3}]\!]$ so that φ_i is identified with the inclusion of the i-th coordinate for all $1 \le i \le \delta$. In particular, the node x_i persists over $\{t_i = 0\}$ in $\mathfrak{C}$. ∎

Corollary 3.2.6. *The locus $\mathscr{D}_m := \overline{\mathscr{M}_g}[m] \smallsetminus \mathscr{M}^{\mathrm{c}}_g[m]$ of curves not of compact type with its reduced closed substack structure is a normal crossings divisor in $\overline{\mathscr{M}_g}[m]$ whose pullback to $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[m],y}$ is given by $\{t_1 \cdots t_\gamma = 0\}$ under the isomorphism from Lemma 3.2.5.*(ii).

We refer to Definition 3.1.1 for the notion of a normal crossings divisor on a smooth Deligne–Mumford stack.

Lemma 3.2.7. *When $m \ge 3$, the automorphism group $\operatorname{Aut}^G(P \to \mathscr{C})$ of the G-torsor $P \to \mathscr{C}$ is $G \times H_y$, where H_y is a subgroup of $\operatorname{Aut}_C(\mathscr{C})$.*

Proof. As $m \geq 3$, a lemma of Serre shows that $\mathrm{Aut}^G(P \to \mathscr{C})$ is contained in the group $\mathrm{Aut}_C(P \to \mathscr{C})$ of automorphism of $P \to \mathscr{C}$ which act trivially on the underlying coarse space C [ACV03, Lem. 7.2.1]. By [ACV03, Lem. 7.3.3], the sequence

$$1 \to \mathrm{Aut}_{\mathscr{C}}(P) \to \mathrm{Aut}_C(P \to \mathscr{C}) \to \mathrm{Aut}_C(\mathscr{C}) \to 1$$

is split exact. Since $\mathrm{Aut}^G(P \to \mathscr{C})$ is the centralizer of $\mathrm{Aut}_{\mathscr{C}}(P) \simeq G$ in $\mathrm{Aut}_C(P \to \mathscr{C})$ and G is abelian, the statement follows with $H_y := \ker(\mathrm{Aut}_C(\mathscr{C}) \to \mathrm{Aut}(G))$. $\square$

Corollary 3.2.8. *The completion of $\overline{\mathscr{M}_g}[m]$ at y for $m \geq 3$ is given by*

$$\overline{\mathscr{M}_g}[m]^\wedge_y := \overline{\mathscr{M}_g}[m] \times_{\overline{M_g[m]}} \operatorname{Spec} \mathscr{O}^\wedge_{\overline{M_g[m]},y} \simeq [\operatorname{Spec} k[\![t_1, \ldots, t_{3g-3}]\!]/H_y] \,,$$

with the open substacks $\mathscr{M}^{\mathrm{c}}_g[m] \times_{\overline{\mathscr{M}_g[m]}} \overline{\mathscr{M}_g}[m]^\wedge_y$ and $\mathscr{M}_g[m] \times_{\overline{\mathscr{M}_g[m]}} \overline{\mathscr{M}_g}[m]^\wedge_y$ corresponding to the loci $[(\operatorname{Spec} k[\![t_1, \ldots, t_{3g-3}]\!]_{t_1 \cdots t_\gamma})/H_y]$ and $[(\operatorname{Spec} k[\![t_1, \ldots, t_{3g-3}]\!]_{t_1 \cdots t_\delta})/H_y]$, respectively.

Proof. As we already saw in Lemma 3.1.7, the proof of [Ols16, Thm. 11.3.1] shows that $\overline{\mathscr{M}_g}[m]^\wedge_y \simeq \left[\operatorname{Spec} \mathscr{O}^\wedge_{\overline{\mathscr{M}_g}[m],y} / \mathrm{Aut}(y)\right]$. Since $\overline{\mathscr{M}_g}[m]$ is rigidified along G, we have $\mathrm{Aut}(y) \simeq H_y$ by Lemma 3.2.7. Now use Lemma 3.2.5. $\square$

We will need a more concrete description of $\mathrm{Aut}_C(\mathscr{C})$ in Example 3.3.12.

Lemma 3.2.9 ([ACV03, § 7.1])**.** *There is an isomorphism*

$$\mathrm{Aut}_C(\mathscr{C}) \simeq \prod_{i=1}^{\gamma} \mu_m;$$

if $U := \operatorname{Spec}(k[z,w]/(zw))^{\mathrm{sh}}$ is an étale atlas for the strict henselization $\mathscr{C}^{\mathrm{sh}}_{x_i}$ at a nonseparating node x_i as in Definition 3.2.1, the automorphisms from the i-th factor act trivially on $\mathscr{C} \smallsetminus \{x_i\}$, and on $\mathscr{C}^{\mathrm{sh}}_{x_i}$ as

$$\mathrm{Aut}_{C^{\mathrm{sh}}_{x_i}} \mathscr{C}^{\mathrm{sh}}_{x_i} \simeq (\mathrm{Aut}_{U/\mu_m} U)/(\mathrm{Aut}_{[U/\mu_m]} U) \simeq \mu_m^2/\mu_m \simeq \mu_m.$$

Example 3.2.10. Lemma 3.2.9 leads to a hands-on description of H_y when $k = \mathbf{C}$. Let $\mathscr{C}_\Delta \to \Delta$ be a deformation of $\mathscr{C}$ over a small polydisc Δ with smooth general fiber C_η. For $1 \leq i \leq \gamma$, let c_i be the vanishing cycle of C_η corresponding to the node x_i of C. A full level-m structure on $\mathscr{C}$ induces an isomorphism $G \simeq \mathrm{H}_1(C_\eta, \mathbf{Z}/m\mathbf{Z})$. With this identification, we can pick $(\zeta_1, \ldots, \zeta_\gamma) \in \prod_{i=1}^{\gamma} \mu_m \simeq \mathrm{Aut}_C(\mathscr{C})$ such that ζ_i acts on G via the Dehn twist

$$\mathrm{H}_1(C_\eta, \mathbf{Z}/m\mathbf{Z}) \to \mathrm{H}_1(C_\eta, \mathbf{Z}/m\mathbf{Z}), \quad \alpha \mapsto \alpha + (\alpha \cdot c_i)c_i$$

[ACV03, Lem. 7.3.3]. Since the intersection pairing on $\mathrm{H}_1(C_\eta, \mathbf{Z})$ is unimodular and the vanishing cycles form part of a basis, $H_y \simeq \ker\Big(\bigoplus_i \mathrm{H}_1(c_i, \mathbf{Z}/m\mathbf{Z}) \to \mathrm{H}_1(C_\eta, \mathbf{Z}/m\mathbf{Z})\Big)$.

From the long exact sequence in homology for the pair $(C, \bigcup_i c_i)$, we see further that $H_y \simeq \mathrm{im}\Big(\mathrm{H}_2(C_\eta, \bigcup_i c_i; \mathbf{Z}/m\mathbf{Z}) \to \bigoplus_i \mathrm{H}_1(c_i, \mathbf{Z}/m\mathbf{Z})\Big)$. Let ν be the number of irreducible components of C. Then $C_\eta \smallsetminus (\bigcup_i c_i)$ has $\nu - (\delta - \gamma)$ connected components $C_1, \ldots, C_{\nu-\delta+\gamma}$. In particular, $\mathrm{H}_2(C_\eta, \bigcup_i c_i; \mathbf{Z}/m\mathbf{Z}) \simeq \bigoplus_j \mathbf{Z}/m\mathbf{Z} \cdot [C_j]$, where $[C_j]$ denotes the fundamental class of the closure of C_j in C_η.

Let Γ be the dual graph of C. Let Γ' be the graph obtained from Γ by contracting all bridges. Its vertices correspond to the connected components C_j and its edges to the nonseparating nodes x_i. After fixing compatible orientations for the c_i and Γ', we can identify the complex $\mathrm{H}_2(C_\eta, \bigcup_i c_i; \mathbf{Z}/m\mathbf{Z}) \to \bigoplus_i \mathrm{H}_1(c_i, \mathbf{Z}/m\mathbf{Z})$ with the cellular cochain complex $\mathrm{C}^0(\Gamma', \mathbf{Z}/m\mathbf{Z}) \to \mathrm{C}^1(\Gamma', \mathbf{Z}/m\mathbf{Z})$. In particular, H_y is a free $\mathbf{Z}/m\mathbf{Z}$-module of rank $\gamma - b_1(\Gamma') = \gamma - b_1(\Gamma) = \gamma - \delta + \nu - 1$, as can also be seen directly from the long exact homology sequence.

3.2.3 Towers of moduli spaces

Assume $\operatorname{char} k \neq p$. For each $n \in \mathbf{Z}_{\geq 0}$, set $G_n := (\mathbf{Z}/p^n\mathbf{Z})^{2g}$. We have a system of natural flat, proper, quasi-finite maps

$$\cdots \to \overline{\mathscr{M}_g}[p^{n+1}] \to \overline{\mathscr{M}_g}[p^n] \to \cdots,$$

which are given on S-valued points by

$$(P_{n+1} \to \mathscr{C}_{n+1} = [P_{n+1}/G_{n+1}] \to S) \mapsto (P_n := P_{n+1}/(p^n\mathbf{Z}/p^{n+1}\mathbf{Z})^{2g} \to \mathscr{C}_n := [P_n/G_n] \to S).$$

The next lemma, which uses the notion of p-ramified morphisms from Definition 3.1.2, will allow us to apply Theorem 3.1.5 to the moduli spaces of curves from above.

Lemma 3.2.11 ([BR11, Thm. 5.1.5] or [Ols07a, Rmk. 1.11])**.** *The maps $\overline{\mathscr{M}_g}[p^{n+1}] \to \overline{\mathscr{M}_g}[p^n]$ are p-ramified over the normal crossings divisor $\mathscr{D}_{p^n}$ from Corollary 3.2.6.*

The proof relies on the description of the completed local rings from Lemma 3.2.5. The "local contributions" of the nodes to a map $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[p^n], y_n} \to \mathscr{O}^{\wedge}_{\overline{\mathscr{M}_g}[p^{n+1}], y_{n+1}}$ can be computed using the universal formal deformations of the nodes described in the proof of Lemma 3.2.5.(ii); this yields the desired ramification at every node with nontrivial stabilizer.

3.3 Vanishing for moduli spaces of curves

In this section, we discuss how to apply the vanshing theorems from Appendix A, § 2.6, and § 3.1 to the moduli spaces of curves reviewed in § 3.2. We retain the conventions and notation from § 3.2.

3.3.1 The Torelli morphism

The Torelli morphism is the functor

$$t_g \colon \mathscr{M}_g^{\mathrm{c}} \to \mathscr{A}_g$$

which sends a curve of compact type to its (principally polarized) Jacobian. A detailed account of the construction of t_g is, for example, given in [Lan19]. For $m \in \mathbf{Z}_{>0}$ invertible in k, let $\mathscr{A}_g[m]$ be the moduli space of principally polarized abelian varieties of dimension g over k with full level-m structure. Since full level-m structures on curves of compact type correspond to full level-m structures on their Jacobians, the stack $\mathscr{M}_g^{\mathrm{c}}[m]$ from § 3.2 fits into a pullback square

$$\begin{tikzcd} \mathscr{M}_g^{\mathrm{c}}[m] \arrow[r, "t_g^m"] \arrow[d, "\pi_m"] & \mathscr{A}_g[m] \arrow[d, "\rho_m"] \\ \mathscr{M}_g^{\mathrm{c}} \arrow[r, "t_g"] & \mathscr{A}_g \end{tikzcd}$$

and we obtain a Torelli map $t_g^m \colon \mathscr{M}_g^{\mathrm{c}}[m] \to \mathscr{A}_g[m]$ at level m. The scheme-theoretic image of t_g^m ([SP20, § 0CMH]) is called the Torelli locus $\mathscr{T}_g[m] \subseteq \mathscr{A}_g[m]$. Set $\mathscr{T}_g := \mathscr{T}_g[0]$ and $\mathscr{T}_1 := \mathscr{A}_1$.

Since t_g^m is only generically finite, we cannot expect that $\operatorname{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g^{\mathrm{c}}[p^n], \mathbf{F}_p) = 0$ for all $i > \dim \mathscr{M}_g^{\mathrm{c}} = 3g-3$, even though $\overline{\mathscr{A}_g}[p^n]$ is perfectoid at infinite level; cf. Example 2.6.8 and Example 2.6.11. In this subsection, we instead apply the results from Appendix A to t_g^m. First, we recall two well-known statements.

Lemma 3.3.1. *The morphism t_g^m is representable and proper.*

Proof. Representable and proper morphisms are stable under base change, so it suffices to show both properties for $m = 0$. To prove that t_g is representable, we only have to see that for every curve C of compact type over an algebraically closed field, the induced group homomorphism $\mathrm{Aut}(C) \to \mathrm{Aut}(J(C))$ is injective; cf. e.g. [AV02, Lem. 4.4.3]. This is [DM69, Thm. 1.13].

Since $\mathscr{M}_g^{\mathrm{c}}$ is of finite type and separated and $\mathscr{A}_g$ is locally noetherian with separated diagonal, properness follows from the existence part of the valuative criterion for all discrete valuation rings V with fraction field K. Let $\mathfrak{A} \in \mathscr{A}_g(V)$ (suppressing principal polarizations from the notation) and $C \in \mathscr{M}_g^{\mathrm{c}}(K)$ such that $\mathfrak{A}_K \simeq J(C)$. The stable reduction theorem for curves produces an extension $K \subseteq K'$ and a stable curve $\mathfrak{C}$ over a valuation ring $V' \subset K'$ dominating V with residue field κ whose generic fiber is isomorphic to $C_{K'}$. By Weil's extension theorem for rational maps into group schemes, $\mathfrak{A}_{V'}$ is the Néron model of $\mathfrak{A}_{K'}$. In particular, a theorem of Raynaud [Ray70, Thm. 8.2.1] (cf. also [DM69, Thm. 2.5]) shows that $J(\mathfrak{C}_\kappa) \simeq \mathfrak{A}_\kappa$, so $\mathfrak{C}_\kappa$ is of compact type and $\mathfrak{C} \in \mathscr{M}_g^{\mathrm{c}}(V')$ is the desired lift of $\mathfrak{A}_{V'}$. □

Lemma 3.3.2 (Torelli for compact type curves). *Let C and D be two curves of compact type of genus g over k. Let $C_1, \ldots, C_\delta$ and $D_1, \ldots, D_\varepsilon$ be their irreducible components of positive genus. If $t_g(C) \simeq t_g(D)$, then $\delta = \varepsilon$ and $C_i \simeq D_{\sigma(i)}$ for some permutation $\sigma \in S_\delta$.*

Proof. We have $\prod_{i=1}^{\delta} J(C_i) \simeq J(C) \simeq J(D) \simeq \prod_{j=1}^{\varepsilon} J(D_j)$ as principally polarized abelian varieties. Since a principally polarized abelian variety over an algebraically closed field uniquely decomposes into an unordered product of indecomposable principally polarized abelian varieties ([CG72, Lem. 3.20], [Deb96, Cor. 2]) and the polarization for each C_i and D_j is induced by its irreducible theta divisor, we must have $\delta = \varepsilon$ and $J(C_i) \simeq J(D_{\sigma(i)})$ for some $\sigma \in S_\delta$. The statement now follows from the usual Torelli theorem for smooth, projective curves. □

In other words, $t_g(C)$ determines the irreducible components of C of positive genus, but does not depend on the position of the nodes or the number of rational irreducible components. A dimension analysis (keeping in mind the dimensions of the automorphism groups of curves of genus 0 and 1) then shows the following statement.

Lemma 3.3.3 ([CV11, Prop. 5.2.1]). *Let C be a curve of compact type of genus g over k. Let δ and δ_1 be the number of irreducible components of C of positive genus and genus 1, respectively. Then*

$$\dim\Big((\mathscr{M}_g^{\mathrm{c}})_{t_g(C)}\Big) = 2\delta - \delta_1 - 2.$$

Although the statement is a special case of [CV11, Prop. 5.2.1], we provide the proof for the convenience of the reader.

Proof. It suffices to show that the preimage of $J(C)$ under the Torelli map $M_g^{\mathrm{c}} \to A_g$ of coarse spaces has dimension $2\delta - \delta_1 - 2$. Let δ_0 be the number of irreducible components of C of genus 0 and Γ be the dual graph of C. Let $D_\Gamma \subset M_g^{\mathrm{c}}$ be the locally closed subvariety corresponding to curves with dual graph Γ. If we denote the irreducible components of C by $C_1, \ldots, C_n$, then $J(C) = J(C_1) \times \cdots \times J(C_n)$. In particular, $J(C)$ does not depend on the position of the nodes so that the preimage of $J(C)$ in D_Γ has dimension

$$2 \cdot \#E(\Gamma) - 3\delta_0 - \delta_1$$

(keeping in mind the dimensions of the automorphisms groups of curves of genus 0, 1, and at least 2). Therefore, if Γ' is a graph that is obtained from Γ by succesively contracting edges with a vertex of weight 0, then the preimage of $J(C)$ in $D_{\Gamma'}$ has maximal dimension. Since Γ' is a tree and thus $\#E(\Gamma') = \#V(\Gamma') - 1 = \delta - 1$, we obtain the desired formula. $\square$

Next, we describe a stratification on $|\mathscr{T}_g|$ that is suitable to determine the defect of semi-smallness of t_g (see Definition A.0.5).

Lemma 3.3.4. *Let $\lambda = (\lambda_1, \dots, \lambda_\delta)$ be a partition of g, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\delta$ and $\sum_{i=1}^{\delta} \lambda_i = g$. Then the product morphism $\xi_\lambda \colon \prod_{i=1}^{\delta} \mathscr{T}_{\lambda_i} \to \mathscr{T}_g$ is finite.*

Proof. We show that ξ_λ is representable, proper, and locally quasi-finite. Representability follows from the faithfulness of the product functor [SP20, Lem. 04Y5]. Since the domain is separated and of finite type and the target has separated diagonal, ξ_λ is separated and of finite type.

For universal closedness, we can again verify the existence part of the valuative criterion for all discrete valuation rings V with fraction field K because $\mathscr{T}_g$ is locally noetherian. Let $\mathfrak{A} \in \mathscr{T}_g(V)$ and $A_i \in \mathscr{T}_{\lambda_i}(K)$ such that $\mathfrak{A}_K \simeq \prod A_i$. The Néron–Ogg–Shafarevich criterion [ST68, Thm. 1] and the compatibility of Tate modules with products show that the A_i have good reduction. Furthermore, their principal polarizations extend to the integral models; cf. e.g. the argument in [FC90, p. 6]. Since $\mathscr{T}_{\lambda_i} \subseteq \mathscr{A}_{\lambda_i}$ is closed, we thus obtain $\mathfrak{A}_i \in \mathscr{T}_{\lambda_i}(V)$ with generic fiber isomorphic to A_i. Separatedness of $\mathscr{A}_g$ gives $\mathfrak{A} \simeq \prod \mathfrak{A}_i$ as wanted.

Finally, to prove that ξ_λ is locally quasi-finite, we can check that for every morphism $\operatorname{Spec}(K) \to \mathscr{T}_g$ from a field K, the space $|\operatorname{Spec} K \times_{\mathscr{T}_g} \prod \mathscr{T}_{\lambda_i}|$ is discrete [SP20, Lem. 06UA]. Since the base change morphism $|\operatorname{Spec} \overline{K} \times_{\mathscr{T}_g} \prod \mathscr{T}_{\lambda_i}| \to |\operatorname{Spec} K \times_{\mathscr{T}_g} \prod \mathscr{T}_{\lambda_i}|$ is surjective and integral (hence closed), it suffices to show that $|\operatorname{Spec} \overline{K} \times_{\mathscr{T}_g} \prod \mathscr{T}_{\lambda_i}|$ is discrete. This follows from the uniqueness of the indecomposable factors of principally polarized abelian varieties used in the proof of Lemma 3.3.2. $\square$

Below, we will use the partial order on the set of integer partitions given by refinement, as introduced in [Bir67, Ex. I.8.10].

Definition 3.3.5. Let $\lambda = (\lambda_1, \dots, \lambda_\delta)$ and $\mu = (\mu_1, \dots, \mu_\varepsilon)$ be two partitions of g. Then μ *refines* λ, or $\mu \leq \lambda$, if there exists a set partition $\{1, \dots, \varepsilon\} = I_1 \cup \cdots \cup I_\delta$ such that $\lambda_j = \sum_{i \in I_j} \mu_i$ for all $1 \leq j \leq \delta$.

Definition 3.3.6. For each partition λ of g, let $\mathscr{T}_\lambda$ be the scheme-theoretic image of ξ_λ. Set

$$S_\lambda := |\mathscr{T}_\lambda| \smallsetminus \bigcup_{\mu<\lambda} |\mathscr{T}_\mu|.$$

Lemma 3.3.7. (i) *The subspaces $S_\lambda \subset |\mathscr{T}_g|$ associated with the partitions λ of g are locally closed and parametrize the Jacobians of curves of compact type whose geometric fibers have exactly δ irreducible components of positive genus λ_i, giving rise to a finite stratification $|\mathscr{T}_g| = \bigsqcup_\lambda S_\lambda$.*

(ii) *For each integer partition λ of g into δ parts, we have $\dim S_\lambda = 3g - 3\delta + \delta_1$, where $\delta_1 := \#\{i \mid \lambda_i = 1\}$.*

Proof. (i). Since ξ_λ is closed by Lemma 3.3.4, it surjects onto its scheme-theoretic image. Thus, the closed substack $\mathscr{T}_\lambda \subseteq \mathscr{T}_g$ parametrizes those families whose geometric fibers are products of δ Jacobians of compact type curves of genus $\lambda_1, \dots, \lambda_\delta$. By Lemma 3.3.2, the geometric fibers are the Jacobians of exactly those curves of compact type that have δ (not necessarily irreducible) components of genus $\lambda_1, \dots, \lambda_\delta$. This yields the first statement as $|\mathscr{T}_\mu| \subseteq |\mathscr{T}_\lambda|$ if $\mu \leq \lambda$ and $|\mathscr{T}_\mu| \cap |\mathscr{T}_\lambda| = \varnothing$ if not.

(ii). Since ξ_λ is finite by Lemma 3.3.4, t_{λ_i} is generically finite, and $\mathscr{T}_\lambda$ is Deligne–Mumford,

$$\dim|\mathscr{T}_\lambda| = \dim \mathscr{T}_\lambda = \sum_{i=1}^{\delta} \mathscr{T}_{\lambda_i} = \sum_{\lambda_i \geq 2} (3\lambda_i - 3) + \sum_{\lambda_i = 1} 1 = 3g - 3\delta + \delta_1;$$

cf. [SP20, Rem. 0DRK]. The number of summands of a partition increases with refinement, hence $\dim S_\lambda = \dim|\mathscr{T}_\lambda| = 3g - 3\delta + \delta_1$. □

Remark 3.3.8. The preceding arguments also show the existence of a finite stratification of $\mathscr{A}_g$ by number and dimension of indecomposable principally polarized factors.

Proposition 3.3.9. *The Torelli morphism $t_g \colon \mathscr{M}_g^{\mathrm{c}} \to \mathscr{A}_g$ has maximal fiber dimension $g - 2$ and defect of semi-smallness $r(t_g) = \left\lfloor \frac{g}{2} \right\rfloor - 1$.*

Proof. The claim about the fiber dimension follows from Lemma 3.3.3 and Lemma 3.3.10.(i) below. For the defect of semi-smallness, we use the stratification from Lemma 3.3.7.(i). Let $\lambda = (\lambda_1, \dots, \lambda_\delta)$ be an integer partition of g. As before, set $\delta_1 := \#\{i \mid \lambda_i = 1\}$. The dimension of S_λ is $3g - 3\delta + \delta_1$ by Lemma 3.3.7.(ii) and the relative dimension of t_g over S_λ is $2\delta - \delta_1 - 2$ by Lemma 3.3.3. The statement about $r(t_g)$ is therefore a consequence of Lemma 3.3.10.(ii) and the equality

$$2 \cdot (2\delta - \delta_1 - 2) + (3g - 3\delta + \delta_1) - (3g - 3) = \delta - \delta_1 - 1. \qquad \square$$

Lemma 3.3.10. *Let C be a curve of compact type of genus g over k. Let δ and δ_1 be the number of irreducible components of C of positive genus and genus 1, respectively. Then*

(i) $2\delta - \delta_1 \leq g$

(ii) $\delta - \delta_1 \leq \left\lfloor \frac{g}{2} \right\rfloor$

and both bounds are sharp.

Proof. Since both $2\delta - \delta_1$ and $\delta - \delta_1$ do not change under contraction of rational components, we may assume that C does not have any rational components. Moreover, since both expressions do not decrease when a component of genus $g' \geq 3$ specializes to a union of a component of genus $g' - 2$ and 2, we may further assume that all irreducible components of C have genus 1 or 2. In that case, $2\delta - \delta_1 = g$ and $\delta - \delta_1$ is maximal when C has $\left\lfloor \frac{g}{2} \right\rfloor$ irreducible components of genus 2, with one component of genus 1 if g is odd, yielding the inequalities. □

Corollary 3.3.11. *Let $K \in \mathrm{D}(\mathscr{M}_g^{\mathrm{c}}[m])$. Then*

(i) $\mathrm{R}t^m_{g,*}K \in {}^p\mathrm{D}(\mathscr{A}_g[m])^{\leq n+g-2}$ *if* $K \in {}^p\mathrm{D}(\mathscr{M}_g^{\mathrm{c}}[m])^{\leq n}$ *and*

(ii) $\mathrm{R}t^m_{g,*}K \in {}^p\mathrm{D}(\mathscr{A}_g[m])^{\leq \left\lfloor \frac{g}{2} \right\rfloor - 1}$ *if* $K = F[3g-3]$ *for some constructible sheaf F of $\mathbf{F}_p$-modules on $\mathscr{M}_g^{\mathrm{c}}[m]$.*

Proof. Since the forgetful maps $\rho_m \colon \mathscr{A}_g[m] \to \mathscr{A}_g$ are finite, the direct image functors $\rho_{m,*}$ are t-exact for the perverse t-structure ([BBD82, Cor. 2.2.6.(i)]) and conservative. Consequently, $\mathrm{R}t^m_{g,*}K \in {}^p\mathrm{D}(\mathscr{A}_g[m])^{\leq \ell}$ if and only if $\rho_{m,*}\mathrm{R}t^m_{g,*}K \in {}^p\mathrm{D}(\mathscr{A}_g)^{\leq \ell}$, and it suffices to show the statement when $m = 0$. In that case, it follows from Lemma 3.3.1, Proposition A.0.4, Lemma A.0.7, and Proposition 3.3.9. □

3.3.2 Proof of the main results

From here on, we additionally assume that $k = \mathbf{C}$. By fixing a (noncanonical) isomorphism $\mathbf{C} \simeq \mathbf{C}_p$, we then have all results of § 2.6 at our disposal. We are now ready to prove Theorem 3.0.1 and Theorem 3.0.2, which we restate for the convenience of the reader.

Theorem 3.0.1. *Let $g \geq 2$ and p be a prime. Let $\mathscr{M}_g^{\mathrm{c}}[p^n]$ be the moduli space of curves of compact type of genus g over $\mathbf{C}$ with full level-p^n structure. Let $\pi_n \colon \mathscr{M}_g^{\mathrm{c}}[p^n] \to \mathscr{M}_g^{\mathrm{c}}$ be the maps "forgetting the level structure."*

(i) *If $K \in {}^p\mathrm{D}(\mathscr{M}_g^{\mathrm{c}}, \mathbf{F}_p)^{\leq 0}$, we have* $\mathrm{colim}_n\, \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g^{\mathrm{c}}[p^n], \pi_n^* K) = 0$ *for all $i > g - 2$.*

(ii) *If F is a constructible sheaf of $\mathbf{F}_p$-modules on $\mathscr{M}_g^{\mathrm{c}}$, we have* $\operatorname{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g^{\mathrm{c}}[p^n], \pi_n^* F) = 0$ *for all* $i > \left\lfloor \frac{7g}{2} \right\rfloor - 4$.

Proof. Let $\rho_n \colon \mathscr{A}_g[p^n] \to \mathscr{A}_g$ be the natural maps forgetting the level structure. Since the canonical maps $\rho_n^* \mathrm{R}t_{g,*}K \to \mathrm{R}t_{g,*}^{p^n} \pi_n^* K$ are isomorphisms for all $n \geq 0$ and all $K \in \mathrm{D}(\mathscr{M}_g^{\mathrm{c}})$ by proper base change, Corollary 3.3.11 reduces the assertion to showing that $\operatorname{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{A}_g[p^n], \rho_n^* L) = 0$ for all $L \in {}^p\mathrm{D}(\mathscr{A}_g)^{\leq \ell}$ and all $i > \ell$.

As in Example 2.6.8, we denote by $\overline{\mathscr{A}_g}[p^n]$ the toroidal compactifications of $\mathscr{A}_g[p^n]$ determined by a fixed smooth, projective $\mathrm{GL}_g(\mathbf{Z})$-admissible polyhedral decomposition of the cone of positive semi-definite quadratic forms on $\mathbf{R}^g$ whose null space is defined over $\mathbf{Q}$. Let $\overline{\rho}_{mn} \colon \overline{\mathscr{A}_g}[p^m] \to \overline{\mathscr{A}_g}[p^n]$ be the natural transition maps. The inclusions $\iota_n \colon \mathscr{A}_g[p^n] \hookrightarrow \overline{\mathscr{A}_g}[p^n]$ cut out the complement of a Cartier divisor and are thus affine morphisms. Therefore, $\mathrm{R}\iota_{n,*}\rho_n^* L \in {}^p\mathrm{D}(\overline{\mathscr{A}_g}[p^n])^{\leq \ell}$ (Theorem A.0.3) and

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{A}_g[p^n], \rho_n^* L) \simeq \operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\overline{\mathscr{A}_g}[p^n], \mathrm{R}\iota_{n,*}\rho_n^* L) = 0$$

for all $i > \ell$ by Example 2.6.8 and Corollary 2.6.9 applied to $K_n = \mathrm{R}\iota_{n,*}\rho_n^* L$ and the natural base change maps $\varphi_{mn}^* \colon \overline{\rho}_{mn}^* \mathrm{R}\iota_{n,*}\rho_n^* L \to \mathrm{R}\iota_{m,*}\rho_m^* L$. □

Theorem 3.0.2. *Let Λ_0 be an étale $\mathbf{F}_p$-local system on $\overline{\mathscr{M}_g}$, with pullbacks Λ_n to $\overline{\mathscr{M}_g}[p^n]$. Then for all $i \geq 0$, the natural map*

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\overline{\mathscr{M}_g}[p^n], \Lambda_n) \to \operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}_g^{\mathrm{c}}[p^n], \Lambda_n)$$

is an isomorphism.

Proof. By Corollary 3.2.6, the locus $\mathscr{D}_{p^n} \subset \overline{\mathscr{M}_g}[p^n]$ of curves not of compact type is a normal crossings divisor. Lemma 3.2.11 shows that the transition maps $\overline{\mathscr{M}_g}[p^{n+1}] \to \overline{\mathscr{M}_g}[p^n]$ are p-ramified over $\mathscr{D}_{p^n}$. Thus, the assertion follows from Theorem 3.1.5. □

Theorem 1.2.1 is a consequence of these two results.

Theorem 1.2.1. *Let $g \geq 2$ and p be a prime. Let $\mathscr{M}[p^n]$ be one of the following:*

(i) *the moduli space $\mathscr{M}_g[p^n]$ of smooth curves of genus g over $\mathbf{C}$ with full level-p^n structure,*

(ii) *the moduli space $\mathscr{M}_g^{\mathrm{c}}[p^n]$ of curves of compact type of genus g over $\mathbf{C}$ with full level-p^n structure, or*

(iii) *the moduli space $\overline{\mathscr{M}_g}[p^n]$ of pre-level-p^n curves of genus g over $\mathbf{C}$ with full level-p^n structure.*

Then we have

$$\operatorname*{colim}_n \mathrm{H}^i_{\text{ét}}(\mathscr{M}[p^n], \mathbf{F}_p) = 0$$

for all $i > 4g-5$ *in case* (i) *and for all* $i > \left\lfloor \frac{7g}{2} \right\rfloor - 4$ *in cases* (ii) *and* (iii).

Proof. Cases (ii) and (iii) are immediate from Theorem 3.0.1.(ii) and Theorem 3.0.2. For (i), note that $\mathscr{M}_g[p^n] \subset \mathscr{M}^{\mathrm{c}}_g[p^n]$ is the inclusion of the complement of a Cartier divisor and the derived direct image of $\mathbf{F}_p[3g-3]$ is therefore semiperverse by Theorem A.0.3. We conclude from Theorem 3.0.1.(i) and smooth base change. □

Example 3.3.12. When $p^n \geq 3$, the stack $\mathscr{M}^{\mathrm{c}}_g[p^n]$ is isomorphic to its coarse space $M^{\mathrm{c}}_g[p^n]$. Another compactification $\overline{M_g}[p^n]$ of $M^{\mathrm{c}}_g[p^n]$ due to Mumford is given by the normalization of $\overline{M_g}$ inside the function field of $M^{\mathrm{c}}_g[p^n]$. This is the coarse space of $\overline{\mathscr{M}_g}[p^n]$.

Since $\overline{M_g}[p^n]$ is in general singular at the boundary $D_{p^n} := \overline{M_g}[p^n] \smallsetminus M^{\mathrm{c}}_g[p^n]$ (as follows for example from (3.3) below), D_{p^n} with its induced reduced subscheme structure cannot be a normal crossings divisor and it does not make sense to ask if the transition maps of the projective system $\overline{M_g}[p^n]$ are p-ramified over D_{p^n}. However, one might wonder if methods akin to those of § 3.1 can still be used to prove a version of Theorem 3.0.2. We show now that this is in fact not the case.

Let $y \in \overline{M_3}(\mathbf{C})$ be a closed point corresponding to a "wheel" of two smooth genus 1 curves attached at two points. Let $y_n \in \overline{M_3}[p^n](\mathbf{C})$ be a compatible system of lifts of y. For any n, there is an isomorphism $\theta_n \colon \mathscr{O}^{\wedge}_{\overline{\mathscr{M}_3}[p^n],y_n} \simeq \mathbf{C}[\![t_1, \ldots, t_6]\!]$, with the singular curves parametrized by the locus $\{t_1 t_2 = 0\}$. Since the vanishing cycles corresponding to the two nodes are homologous, the description of H_{y_n} for $k = \mathbf{C}$ via Dehn twists from Example 3.2.10 shows that H_{y_n} is given by the antidiagonal in $\mathrm{Aut}_C(\mathscr{C}) \simeq \mu_{p^n} \times \mu_{p^n}$. By [Ols07a, Lem. 5.3] (or a direct tangent space calculation), θ_n can be chosen such that the action of $(\zeta, \zeta^{-1}) \in H_{y_n}$ on $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_3}[p^n],y_n}$ from Lemma 3.2.7 is identified with

$$t_i \mapsto \begin{cases} \zeta^{3-2i} t_i & \text{if } 1 \leq i \leq 2, \\ t_i & \text{if } 3 \leq i \leq 6, \end{cases}$$

and the transition maps become

$$t_i \mapsto \begin{cases} t_i^p & \text{if } 1 \leq i \leq 2, \\ t_i & \text{if } 3 \leq i \leq 6. \end{cases}$$

The completed local ring of the coarse moduli space at y_n is computed as the H_{y_n}-invariants of $\mathscr{O}^{\wedge}_{\overline{\mathscr{M}_3}[p^n],y_n}$, and thus in these coordinates as the $\mathbf{C}$-subalgebra of $\mathbf{C}[\![t_1, \ldots, t_6]\!]$ generated by

$t_1^{p^n}$, t_1t_2, $t_2^{p^n}$, and t_i for $i \geq 3$. In other words,

$$\mathscr{O}^{\wedge}_{\overline{M_3[p^n]},y_n} \simeq \mathbf{C}[\![t_1^{p^n}, t_2^{p^n}, t_3, \ldots, t_6, z]\!]/(t_1^{p^n} t_2^{p^n} - z^{p^n}), \tag{3.3}$$

with the transition maps given as before and by $z \mapsto z^p$. Further, the complement of D_{p^n} in $\operatorname{Spec} \mathscr{O}^{\wedge}_{\overline{M_3[p^n]},y_n}$ is $\operatorname{Spec} k[\![t, t_3, \ldots, t_6, z]\!]_{tz}$ via the isomorphism taking $t_1^{p^n}$ to t and $t_2^{p^n}$ to $\frac{z^{p^n}}{t}$. Here, the transition maps are $(t, t_3, \ldots, t_6, z) \mapsto (t, t_3, \ldots, t_6, z^p)$. Thus,

$$\operatorname{hocolim}(\hat{\imath}_{n,*}\mathrm{R}\hat{\imath}_n^!\widehat{\Lambda}_n)_{y_n} \simeq \operatorname{hocolim} \mathrm{R}\widetilde{\Gamma}(S^1 \times S^1, \mathbf{F}_p^{\oplus r})[-1]$$

does not vanish.

APPENDIX A

Vanishing from perverse sheaves

In this appendix, we discuss some vanishing results for perverse sheaves on algebraic stacks, which are without doubt known to the experts, but for which we could not track down a suitably general reference in the literature. Throughout, we fix a separably closed field k of characteristic not equal to p. Recall the following definition from [LO09, § 4].

Definition A.0.1. Let $\mathscr{X}$ be an algebraic stack of finite type over k. Let $\pi\colon X \to \mathscr{X}$ be a smooth surjection from a scheme X of finite type over k. Denote the *relative dimension* of π at a point $x \in X$ by $d_\pi(x) := \dim_x(X_{\pi(x)})$ and set $\dim(x) := \mathrm{trdeg}(\kappa(x)/k)$. For any $n \in \mathbf{Z}$, the category of *complexes bounded above n for the perverse t-structure* is the full subcategory

$$^p\mathrm{D}(\mathscr{X})^{\leq n} := \{K \in \mathrm{D}(\mathscr{X}) \mid \mathscr{H}^j((\pi^*K)_x) = 0 \quad \forall x \in X,\, j > n + d_\pi(x) - \dim(x)\} \subset \mathrm{D}(\mathscr{X}),$$

where $(\pi^*K)_x$ is the derived pullback of π^*K under the inclusion of x into X,

By [LO09, Lem. 4.1], Definition A.0.1 is independent of the choice of $\pi\colon X \to \mathscr{X}$. Moreover, since the restriction functor induces an equivalence between the categories of constructible sheaves on $X_{\text{lis-ét}}$ and $X_{\text{ét}}$, we identify π^*K with a sheaf on the small étale site of X.

In this section, we collect some known foundational results about the derived direct image of $^p\mathrm{D}(\mathscr{X})^{\leq n}$ under certain morphisms for later reference. Although not strictly necessary, it will be convenient later on to work in the generality of stacks. We use the dimension theory for stacks developed in [SP20, § 0DRE].

Remark A.0.2. The relative dimension of any smooth morphism is locally constant on the domain [SP20, Lem. 0DRQ]. In order to show that a given $K \in \mathrm{D}(\mathscr{X})$ is contained in $^p\mathrm{D}(\mathscr{X})^{\leq n}$, it therefore suffices to check that $\pi^*K \in {}^p\mathrm{D}(X)^{\leq n+d_\pi}$ for every smooth (not necessarily surjective) morphism $\pi\colon X \to \mathscr{X}$ from a connected scheme X of finite type with constant relative dimension d_π at every $x \in X$.

Theorem A.0.3. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be an affine morphism between algebraic stacks of finite type over k. Then* $\mathrm{R}f_*({}^p\mathrm{D}(\mathscr{X})^{\leq n}) \subseteq {}^p\mathrm{D}(\mathscr{Y})^{\leq n}$ *for all* $n \in \mathbf{Z}$.

Proof. Let $K \in {}^p\mathrm{D}(\mathscr{X})^{\leq n}$. By Remark A.0.2, it suffices to check that $\pi^*\mathrm{R}f_*K \in {}^p\mathrm{D}(Y)^{\leq n+d_\pi}$ for every smooth morphism $\pi\colon Y \to \mathscr{Y}$ from a connected scheme Y of finite type over k of relative dimension d_π. Since f is affine, $X := \mathscr{X} \times_{\mathscr{Y}} Y$ is a scheme as well. Denote the projection to the first and second factor by π' and f', respectively. By [SP20, Lem. 0DRN], $d_{\pi'}$ is constant and equals d_π. In particular, $\pi'^{,*}K \in {}^p\mathrm{D}(X)^{\leq n+d_\pi}$. By [Ols07b, Prop. 9.8.(i)], $\pi^*\mathrm{R}f_*K \simeq \mathrm{R}f'_*\pi'^{,*}K$ as complexes on $Y_{\text{ét}}$, so the assertion follows from [SGA4, Thm. XIV.3.1]. □

We will mainly use Theorem A.0.3 when f is the inclusion of the complement of a Cartier divisor.

Proposition A.0.4. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a proper morphism between algebraic stacks of finite type over k which is representable by algebraic spaces. Assume that for any $y \in |\mathscr{Y}|$, the fiber dimension* $\dim(\mathscr{X}_y)$ *is at most d. Then* $\mathrm{R}f_*({}^p\mathrm{D}(\mathscr{X})^{\leq n}) \subseteq {}^p\mathrm{D}(\mathscr{Y})^{\leq n+d}$ *for all* $n \in \mathbf{Z}$.

Proof. As in the proof of Theorem A.0.3, choose a smooth morphism $\pi\colon Y \to \mathscr{Y}$ from a connected scheme of finite type over k with relative dimension d_π at all $y \in Y$. The fiber product $X := \mathscr{X} \times_{\mathscr{Y}} Y$ is an algebraic space, again of constant relative dimension d_π over $\mathscr{X}$. By [Ols07b, Prop. 9.8.(i)], it therefore suffices to prove the statement in case $\mathscr{Y}$ is a scheme and $\mathscr{X}$ an algebraic space.

We proceed by induction on $\dim \mathscr{X}$, the case $\dim \mathscr{X} = 0$ being trivial. For the inductive step, since the assertion is local on $\mathscr{Y}$, we may assume that $\mathscr{Y}$, and hence $\mathscr{X}$, is separated. In particular, $\mathscr{X}$ has dense, open schematic locus; we can choose a dense, open subscheme $U \subseteq \mathscr{Y}$ such that $f^{-1}(U) \subset \mathscr{X}$ is represented by a scheme because f is closed.

Denote by $j\colon U \hookrightarrow \mathscr{Y}$ and $j'\colon f^{-1}(U) \hookrightarrow \mathscr{X}$ the open immersions and by i and i' the closed immersions of their complements with their reduced subspace structures, respectively. For any $K \in {}^p\mathrm{D}(\mathscr{X})^{\leq n}$, the induction hypothesis ensures $i_*i^*\mathrm{R}f_*K \simeq i_*\mathrm{R}f_*i'^{,*}K \in {}^p\mathrm{D}^{\leq n+d}(\mathscr{Y})$. On the other hand, $j_!j^*\mathrm{R}f_*K \simeq j_!\mathrm{R}f_*j'^{,*}K \in {}^p\mathrm{D}^{\leq n+d}(\mathscr{Y})$ by [BBD82, § 4.2.4]. The assertion then follows from the long exact excision sequence for $\mathrm{R}f_*K$. □

Under stricter assumptions on the allowable loci with a given fiber dimension, more can be said about the direct image of $\mathbf{F}_p[\dim \mathscr{X}]$.

Definition A.0.5. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a proper morphism between algebraic stacks of finite type over k which is representable by algebraic spaces. Let $|\mathscr{Y}| = \bigsqcup_{i \in I} S_i$ be a finite stratification of $|\mathscr{Y}|$ into locally closed subspaces S_i. Let $\mathscr{Y}_i$ be the corresponding locally closed substacks with their reduced induced substack structure (cf. [SP20, Remark 06FK]). Assume that f has constant relative dimension d_i over $\mathscr{Y}_i$. Then the *defect of semi-smallness of f* is

$$r(f) := \max_{i\in I}\{2d_i + \dim \mathscr{Y}_i - \dim \mathscr{X}\}.$$

Example A.0.6. If f is the blowup of a closed subvariety of codimension $c \geq 2$ in a variety Y, we have $r(f) = c - 2$.

Lemma A.0.7. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ and $|\mathscr{Y}| = \coprod_{i\in I} S_i$ be as in Definition A.0.5. Let F be a constructible sheaf of $\mathbf{F}_p$-modules on $\mathscr{X}$. Then $\mathrm{R}f_*(F[\dim \mathscr{X}]) \in {}^p\mathrm{D}(\mathscr{Y})^{\leq r(f)}$.*

Proof. Choose a smooth surjection $Y \to \mathscr{Y}$ from a scheme Y of finite type over k, giving rise to a fiber square

$$\begin{array}{ccc} X := \mathscr{X} \times_{\mathscr{Y}} Y & \xrightarrow{f'} & Y \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ \mathscr{X} & \xrightarrow{f} & \mathscr{Y}. \end{array}$$

Let $y \in Y$. Then

$$(\pi^*\mathrm{R}f_*(F[\dim \mathscr{X}]))_y \simeq (\mathrm{R}f'_*\pi'^{,*}(F[\dim \mathscr{X}]))_y \simeq \mathrm{R}\Gamma(X_y, \pi'^{,*}F[\dim \mathscr{X}])$$

by proper base change. If $\pi(y) \in S_i$, we have $\dim X_y = d_i$ and hence

$$\mathscr{H}^j\Big((\pi^*\mathrm{R}f_*(F[\dim \mathscr{X}]))_y\Big) \simeq \mathrm{H}^{j+\dim \mathscr{X}}(X_y, \pi'^{,*}F) = 0$$

for all $j > 2d_i - \dim \mathscr{X}$. The assertion thus follows from the inequality

$$2d_i - \dim \mathscr{X} \leq r(f) - \dim \mathscr{Y}_i \leq r(f) + d_\pi(y) - \dim(y). \qquad \square$$

Remark A.0.8. In fact, a combination of Lemma A.0.7 and Proposition A.0.4 shows that $\mathrm{R}f_*(F[\dim \mathscr{X}]) \in {}^p\mathrm{D}(\mathscr{Y})^{\leq r}$, where $r := \min\{r(f), 2\dim \operatorname{supp} F - \dim \mathscr{X}\}$. On the other hand, the bound of Lemma A.0.7 is sharp in general. For example, choose $i \in I$ such that $\dim \mathscr{Y}_i = r(f) + \dim \mathscr{X} - 2d_i$. Let $y \in \pi^{-1}(S_i)$ be the generic point of an irreducible component such that $\dim \mathscr{Y}_i = \dim_{\pi(y)} \mathscr{Y}_i$. Then equality holds at every step of the last inequality in the proof and thus $\mathrm{R}f_*(\mathbf{F}_p[\dim \mathscr{X}]) \notin {}^p\mathrm{D}(\mathscr{Y})^{\leq r(f)-1}$.

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