Non-Supersymmetric Black Holes: A Step Toward Quantum Gravity

by

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To my grandpa and grandma.

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ABSTRACT

One of the greatest challenges of modern theoretical physics is to find a quantum theory of gravity that could unify general relativity and quantum mechanics. The microscopic understanding of black holes plays an essential role in pursuing this goal for its unique connection between gravity and quantum effects.

Despite the remarkable progress that has been made towards this direction, most developments rely heavily on supersymmetry and elude more realistic non-supersymmetric (nonBPS) black holes. This thesis is to improve this situation and develop both the macroscopic and microscopic sides of nonBPS black holes.

In the first part of this thesis, we study a special class of nonBPS black hole solutions in supergravity theories and compute logarithmic corrections to black hole entropy. We find that the correction is universal and independent of black hole parameters only when the number of supercharges $\mathcal{N} \geq 6$.

In the second part, we compute the spectrum of extremal nonBPS black holes introduced in the previous part by studying supergravity on their near-horizon geometry. We find that the spectrum exhibits significant simplifications even though supersymmetry is completely broken and we interpret our results in the framework of the AdS/CFT correspondence.

In the final part, we analyze AdS_5 black holes that are nearly supersymmetric. They depart from the BPS limit either by having nonzero temperature or by violating a constraint on potentials. We study the thermodynamics of these deformations and their interplay. We discuss microscopic computations of BPS black hole entropy in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and generalize the arguments to compute entropy of nearBPS black holes, which agrees with results from the gravity side.

CHAPTER I

Introduction

For more than half a century, quantum gravity has been one of the greatest challenges of modern theoretical physics. A complete understanding of it requires a consistent unification of the two backbones of modern physics: general relativity and quantum mechanics. To find this unified theory, it is necessary to understand subjects where both gravitational and quantum effects are significant. Black holes, a region of spacetime where the gravitational field is so strong that not even light can escape from it, provide such a setting.

A quantum theory of gravity has to provide microscopic quantum descriptions of black holes that are consistent with their macroscopic gravitational descriptions. The existing microscopic understandings of black holes heavily rely on the existence of supersymmetry, while little is known for more general non-supersymmetric (nonBPS) black holes.

The goal of this thesis is to improve this situation and quantitatively explore the entropy, spectrum and other properties of non-supersymmetric black holes, both macroscopically and microscopically.

In this chapter, we present a brief overview of the developments on black hole physics, as well as the motivations that lead us to the research in later chapters. In the end we overview and summarize key features of our results.

1.1 Black Holes

A black hole is a spacetime region where the gravitational field is so strong that nothing can escape from it. It was first discovered by Karl Schwarzschild in 1915 as the first exact solution to the Einstein field equations of general relativity. In astrophysics, a black hole is one of the possible outcomes of gravitational collapse of a massive object, such as a star, at the end of its life cycle. If the object is massive enough, its internal pressure will eventually not be able to prevent it from collapse due to its own gravity. When the objects collapses within its Schwarzschild radius, a black hole forms. The boundary of it is referred to as the event horizon, within which everything is forced to move inward and no escape is possible. Once the black hole achieves a stable state, its geometry is uniquely determined by its mass, angular momenta and charges, irrespective of the original collapsing object and how it collapses. This uniqueness statement is called the no-hair theorem [1].

1.1.1 Black Hole Thermodynamics

At first glance, the no-hair theorem seems to imply that a black hole can only have one internal configuration, and we might be prone to get to the conclusion that the entropy of any black hole would be $S = \ln 1 = 0$. This, however, can not be correct because it violates the second law of thermodynamics, which states that the entropy of a closed system can not decrease. To illustrate the contradiction, one can imagine a closed system consisting of a black hole and some other matter with $S > 0$ falling into it. Assuming our conclusion is correct, after all the matter falls into the black hole, the closed system is left with a final black hole, and hence has vanishing entropy, which violates the second law. Therefore our previous conclusion is false, despite the implication from the no-hair theorem.

In the 1970's, the puzzle mentioned above was partially understood thanks to the development of black hole thermodynamics [2], which relates black hole parameters to thermodynamics quantities. It was identified by Bekenstein and Hawking [3] that:

$$S_{\mathrm{BH}} = \frac{A_H}{4\hbar G} \ , \tag{1.1}$$

where A_H is the area of the event horizon. Meanwhile, Hawking [4] showed that by considering particle-antiparticle pair production in the region near the event horizon, a black hole emits thermal radiation (known as Hawking radiation) with temperature

$$T_{\mathrm{BH}} = \frac{\hbar\kappa}{2\pi} \ , \tag{1.2}$$

where κ is the surafce gravity of the black hole.

Here we would like to emphasize the quantum nature of black hole thermodynamics. The presence of $\hbar$ in the entropy and temperature of black holes clearly shows that black hole thermodynamics is originated from quantum effects. If we take the classical limit $\hbar \to 0$, we would see vanishing black hole temperature, no Hawking radiation and breakdown of the entropy formula (1.1). To fully understand black hole thermodynamics, we need both general relativity and quantum mechanics, which is exactly why black hole thermodynamics is an ideal candidate for studying quantum gravity.

1.1.2 Macroscopic and Microscopic Descriptions of Black Holes

The Bekenstein-Hawking area law (1.1) tells us how to compute entropy of a black hole from its geometry. However, it does not fully solve the puzzle implied by the no-hair theorem. From statistical mechanics, we know that the entropy can be derived by counting the degeneracy Ω of all possible microscopic states given fixed macroscopic thermodynamic quantities, i.e.

$$S = \ln \Omega \; . \tag{1.3}$$

Therefore the statistical meaning of having a nonzero entropy is that a black hole with fixed charges has different possible microstates. However, the no-hair theorem which fixes the entire geometry does not leave us any room for different states.

The contradiction above is originated from a fundamental problem of modern physics: the incompatibility of general relativity which describes the dynamics of spacetime, and quantum mechanics which dominates microscopic quantum effects. The presence of $\hbar$ in the black hole entropy formula (1.1) and the statistical interpretation of entropy (1.3) together indicate that to understand a black hole as a collection of microscopic states, we need a quantum theory of gravity.

One of the most promising quantum gravity theories is string theory. In string theory, the fundamental objects are one-dimensional strings, and ordinary point-like particles are nothing but different modes of string excitations. It also proposes high-dimensional objects called D-branes, which can carry electric or magnetic charges. An important fact of string theory is that the excitation modes of closed strings contain a massless spin-2 state, which can be identified as graviton. Therefore string theory is naturally a quantum theory of gravity. Meanwhile, string theory is an ultraviolet (UV) complete quantum theory, which means it is well-defined at arbitrarily high energy scales.

With the development of string theory, the first example of black hole microstates counting was given by Strominger and Vafa [5] in 1996. They studied a class of five-dimensional black holes that has explicit string realizations and reproduced the Bekenstein-Hawking area law $S_{\rm BH} = \frac{A_H}{4\hbar G}$ by counting the leading degeneracies of microstates of strings and D-branes carrying the same charges of the black holes in the large charge limit. Their success confirmed the consistency between macroscopic and microscopic descriptions of black holes and hence indicated a new direction of studying quantum gravity. We should be able to understand black holes in either its macroscopic or microscopic description, and these two approaches should be consistent with each other. Therefore, even the information on the macroscopic side can be valuable in understanding quantum gravity on the microscopic side.

More precisely, to study black holes in the macroscopic approach, we can start with a black hole background as a classical solution of some low-energy effective theory, then quantize the gravitational field, as well as coupled matter fields, and study the quantum fluctuations of these fields. The information contained in these quantum fluctuations can be used to put strong constraints on any quantum gravity theory. For example, the Bekenstein-Hawking area law (1.1) is a semi-classical result and only the leading term of the full formula in the large area expansion,

$$S = \frac{A_H}{4G} + C \ln A_H + \ldots \ , \tag{1.4}$$

for some coefficient C. Generally, the overall quantum corrections to black hole entropy are sensitive to the details of the UV-complete quantum gravity. However, the subleading logarithmic term in (1.4) is the one-loop quantum correction that is governed by quantum fluctuations in the low-energy effective theory. Therefore it can be computed without any knowledge of the UV-complete quantum gravity, whereas any quantum gravity theory has to give microscopic counting consistent with it in order to be possibly correct. In other words, the logarithmic correction to black hole entropy can serve as a criterion to the correctness of any quantum gravity theory.

The logarithmic corrections have been computed for various classes of black holes in the macroscopic side. Some of them [6–9] have also been done by microstates counting in string theory and matched with macroscopic results. In chapter II, we compute the logarithmic corrections for a special class of black holes, for which how to do microstates counting is still a mystery. In order to learn more about these black holes, we compute the mass spectrum of their quantum fluctuations in chapter III, which provides us with more detailed information about their microstates. These computations put strong constraints on any theory of quantum gravity that attempts to understand these black holes microscopically.

1.2 Supersymmetry and Supergravity

Supersymmetry (SUSY) is the only possible nontrivial extension of the Poincaré group for spacetime symmetry of a realistic quantum field theory by the Haag-Łopuszański-Sohnius theorem [10]. It is a conjectured symmetry between particles with integer spin (bosons) and those with half-integer spin (fermions). A supersymmetry transformation turns a boson into a fermion, and vice versa. The fermionic operators Q_i that generate the transformations are anti-commuting spinors called supercharges, with

$$Q_i \left|\text{Boson}\right\rangle = \left|\text{Fermion}\right\rangle \ , \qquad Q_i \left|\text{Fermion}\right\rangle = \left|\text{Boson}\right\rangle \ . \tag{1.5}$$

Particles in a supersymmetric theory fall into irreducible representations of the supersymmetry algebra. These representations are called supermultiplets. Each supermultiplet contains an equal number of bosonic and fermionic degrees of freedom, and particles in the same supermultiplet have the same mass.

Supersymmetry is a spacetime symmetry because the anti-commutation relation of supercharges $\{Q_a, Q_b^\dagger\} = (\sigma^\mu)_{ab}P_\mu$ involves the momentum operator P_μ. If we require supersymmetric transforamtions to be local, i.e. the transformation parameters are arbitrary functions of spacetime coordinates, the supersymmetry algebra will involves momentum P_μ with spacetime-dependent coefficients, which must be regarded as diffeomorphisms. Therefore gravity is naturally required in local supersymmetry, and the result is the theory of supergravity (SUGRA). A supergravity theory is a nonlinear field theory with supersymmetry that contains the gravity multiplet. The gravity multiplet consists of the spin-2 graviton and $\mathcal{N}$ spin-$\frac{3}{2}$ gravitinos (superpartners of graviton), as well as additional fields with lower spins depending on the number of supercharges $\mathcal{N}$. Supergravity theory combines supersymmetry and general relativity, and more importantly, is a low energy limit of string theory, which means black holes in supergravity should have microscopic interpretations in string theory.

1.2.1 BPS and nonBPS Black Holes

Supersymmetry plays a cruicial role in the existing microscopic descriptions of black holes. Examples of microstates counting in string theory are mostly based on black holes that preserve at least some of the supersymmetries in the supergravity theory. These black holes saturate the Bogomol'nyi-Prasad-Sommerfield (BPS) bound and therefore are referred to as BPS black holes.

The microstates counting is difficult for general nonBPS black holes because a classical black hole has a large radius of curvature near the horizon compared to the length scale of strings, which makes it into the strong coupling regime of string theory, while our current techniques and understanding of strings only allow us to do computations in the weak coupling regime. The advantage of having a BPS black hole is that the preserved supersymmetry can protect some physical quantities from changing when we continuously move from weak to strong coupling, so that our results in the weak coupling regime are still valid when applied to black holes in the strong coupling regime.

In reality, supersymmetry is broken, if it ever exists. This is basically because supersymmetry reuqires identical mass for particles in the same supermultiplet, which is not observed in laboratories for particles in the standard model. This means all realistic black holes in our universe are necessarily nonBPS, and therefore the microscopic understanding of nonBPS black holes is important and necessary. However, we have mentioned that the microstates counting technique for BPS black holes does not work for general cases, so how far could we go with nonBPS black holes? We could first consider nearBPS black holes that are infinitesimally close to BPS ones. For example, because BPS black holes are necessarily extremal, i.e. they have zero temperature, by increasing an infinitesimal amount of temperature, we get near-extremal nonBPS black holes that are nearBPS. There are results [11, 12] indicating that microstates counting is possible for certain near-extremal black holes by perturbative computations. In chapter IV, we develop both microscopic and macroscopic sides of nonBPS black holes in AdS_5 space with general infinitesimal deviations from BPS considered. If we go beyond nearBPS, it becomes much more difficult. Several results [13–15] imply that for limited cases in string theory, microscopic computations in the weak coupling regime can also reproduce the entropy of corresponding black holes arbitrarily far from extremality. However, these identifications and matches are far from complete proofs, and generally only the scaling of entropy with area can be reproduced [16].

Though we lose control over the microstates counting when adding finite temperature to BPS black holes, some of these black holes surprisingly share certain universalities independent of continuous black hole parameters, such as universal logarithmic corrections to entropy [17]. This motivates us to study the extent of these universalities and their related properties for more general cases. The class of nonBPS black holes we consider in chapters II and III is referred to as the nonBPS branch in the sense that all of them including the extremal ones are far from BPS. We find that the non-trivial universality of logarithmic corrections is rarer in this case, while the mass spectrum of these black holes exhibits significant simplifications that can be interpreted in the framework of the AdS/CFT correspondence. These results and interpretations might serve as targets and inspirations for future microscopic research.

1.3 The AdS/CFT Correspondence

The AdS/CFT correspondence, also known as gauge/gravity duality, is a conjectured holographic duality between theories of quantum gravity on $n+1$ dimensional Anti-de-Sitter (AdS) space and n-dimensional conformal field theories (CFT). It was first proposed by Juan Maldacena [18] in 1997. Despite the lack of rigorous proof, the majority of the community is now convinced that this duality holds, largely because attempts to find a clear counter-example of it all failed and these failures in return deepened our understanding of how and why it works.

The most famous example of the AdS/CFT correspondence states that type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory on the four-dimensional boundary. It also specifies the identifications between dimensionless parameters on two sides. These parameters are, on the AdS side, the string coupling g_s and the AdS radius in string length unit ℓ/ℓ_s, and on the CFT side, the Yang-Mills coupling $g_{\rm YM}$ and the rank of the gauge group N. Their correspondences are given by

$$4\pi g_s = g_{\rm YM}^2 \ , \qquad \left(\frac{\ell}{\ell_s}\right)^4 = 4\pi g_s N \ . \tag{1.6}$$

It was shown by 't Hooft [19] that the gauge theory becomes effectively classical when taking the limit $N \to \infty$ with $\lambda = g_{\rm YM}^2 N$ fixed. The 't Hooft limit corresponds to taking $g_s \to 0$ in the type IIB string theory where loop contributions from strings are suppressed. If we additionally take $\lambda \to \infty$ after the 't Hooft limit, which means the AdS radius is much larger than the string length scale ($\ell \gg \ell_5$), higher-derivative corrections also become negligible. Therefore, the limit $N \gg \lambda \gg 1$ corresponds to our classical general relativity. This shows one of the key features of the AdS/CFT correspondence: it is a strong/weak coupling duality. It is when the gauge coupling is strong ($\lambda \gg 1$) that the field theory is dual to the weakly-coupled gravity theory.

The AdS/CFT correspondence offers a particularly useful tool for studying microscopic descriptions of asymptotically-AdS black holes, for it rephrases black hole microstates into corresponding states in the dual CFT. There are many successes in studying black holes using quantum field theory, mostly based on AdS$_3$/CFT$_2$. However, despite the fact that black hole solutions in AdS$_5$ have been known for a long time [20–23], it remains a long-standing challenge how to do microstates counting for them. It was not until recent years that the entropy for BPS ones were reproduced microscopically [24–26]. In chapter IV, we work on both gravitational and field theory sides of nearBPS AdS$_5$ black holes. Our results generalize the results on BPS cases and develop the emerging microscopics of nearBPS AdS$_5$ black holes.

1.4 Overview of Results

The remainder of this thesis is organized as follows. In chapter II, we compute logarithmic corrections to the entropy of black holes on the nonBPS branch. We start with a discussion about Kaluza-Klein theory and its black hole solutions. Then we embed Kaluza-Klein black holes into $\mathcal{N} = 8$ supergravity and derive quadratic fluctuations around the background. The background solutions do not preserve supersymmetry even in their extremal limit, and neither do their quadratic fluctuations. They break the original $SU(8)$ symmetry group but still respect the subgroup $USp(8)$. The preserved $USp(8)$ symmetry decouples the quadratic fluctuations into blocks corresponding to different irreducible representations. The symmetry structure also facilitates consistent truncations of $\mathcal{N} = 8$ results into $\mathcal{N} = 6, 4, 2$ supergravity. For $\mathcal{N} = 2$ supergravity, we also discuss embeddings with more general prepotentials. Based on the quadratic fluctuations, we compute the resulting logarithmic corrections in various embeddings using heat kernel expansion. Our analysis shows that only when $\mathcal{N} \geq 6$, the logarithmic correction on the nonBPS branch is universal and independent of the parameters of black holes, in contrast to the BPS branch where it is universal for all $\mathcal{N} \geq 2$. This chapter is based on [27], written in collaboration with Alejandra Castro, Victor Godet and Finn Larsen.

To better understand black holes on the nonBPS branch, chapter III focuses on the spectrum of $\mathcal{N} = 8$ (or $\mathcal{N} = 4$) extremal nonBPS black holes introduced in chapter II by studying their $\mathrm{AdS}_2 \times S^2$ near horizon geometry. As we previously mentioned, the quadratic fluctuations respect $USp(8)$ (or $USp(4) \times SO(n_V - 1)$) symmetry and decouple into blocks corresponding to different irreducible representations. We compute the mass spectrum of these blocks and use the result to reproduce logarithmic corrections determined in chapter II as a consistency check. The mass spectrum of these nonBPS black holes exhibits surprising simplifications that are usually expected only for BPS ones. To explain this, we study the dimensional reduction from $\mathrm{AdS}_3 \times S^2$ to $\mathrm{AdS}_2 \times S^2$ and show it reproduces either the nonBPS spectrum or the BPS spectrum on $\mathrm{AdS}_2 \times S^2$ depending on a choice of chirality. We end the chapter with a discussion of broken supersymmetry on both BPS and nonBPS branches to understand these simplifications in another approach. This chapter is based on [28], written in collaboration with Finn Larsen.

In chapter IV, we turn to study thermodynamics of AdS_5 black holes that are nearly supersymmetric. First we develop the gravitational thermodynamics of nearBPS AdS_5 black holes. There are two distinct ways to depart from the BPS limit: temperature takes them above extremality and a potential maintains extremality but violates a certain constraint. We study the thermodynamics of these deformations and their interplay in detail, and find an unexpected identity between the heat capacity and the capacitance. Then on the microscopic side, we review the partition function of $\mathcal{N} = 4$ SYM in the free field limit and derive the free energy from its leading contribution in the large-N and low-temperature limit. We generalize the arguments to the nearBPS regime and compute the resulting nearBPS AdS_5 black hole entropy by relaxing the potential constraint imposed by supersymmetry. Our methods recover gravitational results from microscopic theory also for nearBPS black holes. This chapter is based on [29], written in collaboration with Finn Larsen and Jun Nian.

CHAPTER II

Logarithmic Corrections to Black Hole Entropy: the NonBPS Branch

2.1 Introduction and Summary

A remarkable feature of the Bekenstein-Hawking entropy formula is its universality: the leading contribution to the black hole entropy is controlled by the area of the event horizon, regardless of the details of the solutions or the matter content of the theory. It is therefore interesting to investigate if there is any notion of universality and/or robustness in the quantum corrections to the entropy of a black hole.

Generically there is no expectation that the quantum corrections to the Bekenstein-Hawking area law are universal: according to effective quantum field theory they are sensitive to the details of the UV completion of the low energy theory in consideration. However, there is a special class of quantum corrections that are entirely determined by the low energy theory [30–38]: the leading *logarithmic* correction is governed by the one-loop effective action of the low energy modes in the gravitational theory. These corrections, therefore, provide a powerful *infrared window into the microstates.*

The claim that logarithmic corrections computed from the IR theory agree with results for the UV completion has been successfully tested in many cases where string theory provides a microscopic counting formula for black hole microstates. We refer to [39, 40] for a broad overview and [41–44] for more recent developments in AdS_4/CFT_3. Logarithmic corrections have also been evaluated for a plethora of other black holes [17, 45] where a microscopic account still awaits.[1]

The coefficients multiplying these logarithms follow some interesting patterns. The black

[1] In certain cases the logarithm can be accounted for very simply by using thermodynamics [45, 46]: the measure that controls the change from, for example, the microcanonical to the canonical ensemble correctly reproduces the gravitational result without leading to new insight in the microscopic theory.

hole entropy has the schematic structure

$$S_{\rm BH} = \frac{A_H}{4G} + \frac{1}{2}(C_{\rm local} + C_{\rm zm}) \log \frac{A_H}{G} + \cdots \ , \tag{2.1}$$

where we highlight the two terms (area law+logarithm) controlled by low energy gravity and use dots to denote subleading corrections that generally depend on the UV completion. $C_{\rm zm}$ is an integer that accounts for zero modes in the path integral. $C_{\rm local}$ refers to the constant term in the heat kernel that captures the non-zero eigenvalues of the one-loop determinant [47]. It is expressed as a density [48, 49]

$$C_{\rm local} = \int d^4x \sqrt{g}\, a_4(x) \ , \tag{2.2}$$

where the integrand takes the form

$$a_4(x) = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 \ , \tag{2.3}$$

for the backgrounds we will consider. In this expression, E_4 is the Gauss-Bonnet term and $W_{\mu\nu\rho\sigma}$ is the Weyl tensor. The constants c and a are familiar from related computations of the trace anomaly of the stress tensor. Their values depend on the content of matter fields and their couplings to the background black hole solution.[2]

$C_{\rm local}$ contains non-trivial information about the background so this function generally depends greatly on the matter content of the theory and the parameters of the black hole [45]. However, under certain conditions $C_{\rm local}$ has a universal structure [17, 50]: for Kerr-Newman black holes embedded in $\mathcal{N} \geq 2$ supergravity, the c-anomaly vanishes. This leads to a remarkable simplification since then the integral in (2.2) is just a topological invariant. The logarithmic correction is therefore universal in the sense that its does not depend on details of the black hole background; it is determined entirely by the content of massless fields.

The class of backgrounds considered in [17] was constructed such that, in the extremal limit, they continuously connect to BPS solutions. For this reason we denote this class as the *BPS branch*. The black holes on the BPS branch are not generally supersymmetric, but their couplings to matter are arranged such that supersymmetry is attained in the limit. One of the motivations for the present chapter is to study universality of logarithmic corrections outside of the BPS branch in $D = 4$ supergravity.

Supergravity (with $\mathcal{N} \geq 2$) also allows for black holes that do not approach BPS solutions

[2]It is important to note that the couplings are not necessarily minimal, so the values of c and a may be nonstandard functions of the matter content.

in the extremal limit. We refer to such solutions as the *nonBPS branch*. In their minimal incarnation, they correspond to solutions of the $D = 4$ theory obtained by a Kaluza-Klein reduction of five dimensional Einstein gravity [51]. In a string theory setup it is natural to identify the compact Kaluza-Klein dimension with the M-theory circle, and then these solutions are charged with respect to electric $D0$-brane charge and magnetic $D6$-brane charge. Such configurations break supersymmetry even in the extremal limit. Therefore, they offer an interesting arena for studying logarithmic corrections and their possible universality.

The minimal Kaluza-Klein theory needed to describe the nonBPS branch is a four dimensional Einstein-Maxwell-dilaton theory where the couplings are dictated by the reduction from five dimensions. We will refer to the black hole solutions of this theory as "Kaluza-Klein black holes." These solutions can be embedded in supergravity, as we will discuss in detail. In particular, we will consider the embedding of the Kaluza-Klein theory in $\mathcal{N} = 4, 6, 8$ supergravity and for $\mathcal{N} = 2$ we consider $ST(n)$ models [3], which include the well-known STU-model as a special case.

Our technical goal is to evaluate the Seeley-DeWitt coefficient $a_4(x)$ for the Kaluza-Klein black hole when it is embedded in one of the supergravities. This involves the study of quadratic fluctuations around the background, potentially a formidable task since there are many fields and generally they have non-minimal couplings to the background and to each other. Fortunately we find that, in the cases we consider, global symmetries of supergravity organize the quadratic fluctuations into manageable groups of fields that are decoupled from one another. We refer to such groups of fields as "blocks". There are only five distinct types of blocks, summarized in Table 2.1.

Multiplet	Block content
KK block	1 graviton, 1 vector, 1 scalar
Vector block	1 vector and 1 (pseudo)scalar
Scalar block	1 real scalar
Gravitino block	2 gravitini and 2 gaugini
Gaugino block	2 gaugini

Table 2.1: Decomposition of quadratic fluctuations.

The KK block comprises the quadratic fluctuations in the seed theory, *i.e.* the Kaluza Klein theory with no additional matter fields. The scalar block is a single minimally coupled spectator scalar field. The remaining matter blocks have unfamiliar field content and their

[3] We work out the bosonic fluctuations for $\mathcal{N} = 2$ with any prepotential. It is only for fermionic fluctuations that we restrict our attention to the $ST(n)$ models.

couplings to the background are non-standard. The great simplification is that the spectrum of quadratic fluctuations of each supergravity theory we consider can be characterized by the number of times each type of block appears. We record those degeneracies in Tables 2.4 and 2.8.

Once the relevant quadratic fluctuations are identified it is a straightforward (albeit cumbersome) task to evaluate the Seeley-DeWitt coefficient $a_4(x)$. We do this for every block listed above and so determine their contribution to C_{local} in (2.1). Having already computed the degeneracies of the blocks, it is elementary algebra to find the values of c and a for each supergravity theory. Our results for individual blocks are given in Table 2.7 and those for theories are given in Table 2.8.

One of our main motivation is to identify theories where $c = 0$ since for those the coefficient of the logarithm is universal. We find that the non-trivial cancellations on the BPS branch reported in [17] are much rarer on the nonBPS branch. For example, on the nonBPS branch the c coefficient does not vanish for any $\mathcal{N} = 2, 4$ supergravity we consider, whatever their matter content. Therefore, as we discuss in section 2.7, this implies that the logarithmic correction to the entropy depends on black hole parameters in a combination different from the horizon area.

In contrast, for $\mathcal{N} = 6, 8$ we find that $c = 0$. The vanishing of c on the nonBPS branch is rather surprising, since it is apparently due to a different balance among the field content and couplings than the analogous cancellation on the BPS-branch. It would be very interesting to understand the origin of this cancellation from a more fundamental principle. In our closing remarks we discuss some directions to pursue.

The outline of this chapter is as follows. In section 2.2, we discuss Kaluza-Klein theory and its Kaluza-Klein black hole solution. This gives the "seed solution", the minimal incarnation of the nonBPS branch. In section 2.3, we embed this theory into $\mathcal{N} = 8$ supergravity, and in section 2.4, we derive the quadratic fluctuations around the black hole in the $\mathcal{N} = 8$ environment. In section 2.5, we discuss the embedding of the Kaluza-Klein black hole into theories with less supersymmetry by truncating our previous results for $\mathcal{N} = 8$ and then exploiting global symmetries of supergravity. In section 2.6, we discuss the embedding of the nonBPS branch directly into $\mathcal{N} = 2$ supergravity, without making reference to $\mathcal{N} = 8$. This generalizes some $\mathcal{N} = 2$ results to a general prepotential. In section 2.7, we evaluate the c and a coefficients for the Kaluza-Klein black hole in its various embeddings and discuss the resulting quantum corrections to the black hole entropy. Finally, section 2.8 summarizes our results and discusses future directions. Appendix A contains the technical details behind the Seeley-DeWitt coefficients presented in section 2.7.

2.2 The Kaluza-Klein Black Hole

Our starting point is a black hole solution to Kaluza-Klein theory. It is sufficient for our purposes to consider the original version of Kaluza-Klein theory: the compactification to four spacetime dimensions of Einstein gravity in five dimensions. In this section, we briefly present the theory and its black hole solutions. In the following sections we embed the theory and its solutions into supergravity and study perturbations around the Kaluza-Klein black holes in the framework of supergravity.

The Lagrangian of Kaluza-Klein theory is given by[4]

$$e^{-1}\mathcal{L}_{\rm KK} = \frac{1}{16\pi G}\left(R - 2D_\mu\Phi D^\mu\Phi - \frac{1}{4}e^{-2\sqrt{3}\Phi}F_{\mu\nu}F^{\mu\nu}\right) \ . \tag{2.4}$$

The scalar field Φ parametrizes the size of the compact fifth dimension and the field strength $F_{\mu\nu}$ is the 4D remnant of the metric with one index along the fifth dimension. The Lagrangian (2.4) gives the equations of motion

$$D^2\Phi + \frac{\sqrt{3}}{8}e^{-2\sqrt{3}\Phi}F_{\mu\nu}F^{\mu\nu} = 0 \ , \tag{2.5}$$

$$D_\mu\left(e^{-2\sqrt{3}\Phi}F^{\mu\nu}\right) = 0 \ , \tag{2.6}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = (2D_\mu\Phi D_\nu\Phi - g_{\mu\nu}D^\rho\Phi D_\rho\Phi) + \frac{1}{2}e^{-2\sqrt{3}\Phi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \ . \tag{2.7}$$

Some of our considerations will apply to any solution of the Kaluza-Klein theory (2.4) but our primary interest is in asymptotically flat black holes. We therefore focus on the general Kaluza-Klein black hole [51–53]. It is characterized by the black hole mass M and angular momentum J, along with the electric/magnetic charges (Q, P) of the Maxwell field. Its 4D metric is given by

$$ds_4^2 = g_{\mu\nu}^{(\rm KK)}dx^\mu dx^\nu = -\frac{H_3}{\sqrt{H_1H_2}}(dt - B)^2 + \sqrt{H_1H_2}\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3}\sin^2\theta\, d\phi^2\right) \ , \tag{2.8}$$

where

$$\begin{aligned} H_1 &= r^2 + a^2\cos^2\theta + r(p-2m) + \frac{p}{p+q}\frac{(p-2m)(q-2m)}{2} \\ &\quad - \frac{p}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}a\cos\theta \ , \end{aligned} \tag{2.9}$$

[4]We use e and $\sqrt{-g}$ interchangeably, to denote the square root of the determinant of the metric.

$$
\begin{aligned}
H_2 &= r^2 + a^2\cos^2\theta + r(q-2m) + \frac{q}{p+q}\frac{(p-2m)(q-2m)}{2} \\
&\quad + \frac{q}{2m(p+q)}\sqrt{(q^2-4m^2)(p^2-4m^2)}a\cos\theta\ , && (2.10)\\
H_3 &= r^2 - 2mr + a^2\cos^2\theta\ , && (2.11)\\
\Delta &= r^2 - 2mr + a^2\ , && (2.12)
\end{aligned}
$$

and the 1-form B is given by

$$
B = \sqrt{pq}\frac{(pq+4m^2)r - m(p-2m)(q-2m)}{2m(p+q)H_3}a\sin^2\theta\, d\phi\ . \tag{2.13}
$$

The matter fields are the gauge field

$$
\begin{aligned}
A^{(\mathrm{KK})} &= -\left[2Q\left(r + \frac{p-2m}{2}\right) + \sqrt{\frac{q^3(p^2-4m^2)}{4m^2(p+q)}}a\cos\theta\right] H_2^{-1}dt \\
&\quad - \left[2P\left(H_2 + a^2\sin^2\theta\right)\cos\theta + \sqrt{\frac{p(q^2-4m^2)}{4m^2(p+q)^3}}\right. \\
&\quad \left.\times\left[(p+q)(pr - m(p-2m)) + q(p^2-4m^2)\right]a\sin^2\theta\right] H_2^{-1}d\phi\ , && (2.14)
\end{aligned}
$$

and the dilaton

$$
e^{-4\Phi^{(\mathrm{KK})}/\sqrt{3}} = \sqrt{\frac{H_2}{H_1}}\ . \tag{2.15}
$$

The superscript "KK" on $g_{\mu\nu}^{(\mathrm{KK})}$, $A^{(\mathrm{KK})}$, and $\Phi^{(\mathrm{KK})}$ refers to the Kaluza-Klein black hole. These background fields should be distinguished from the exact fields in (2.4-2.7) which generally include fluctuations around the background.

The four parameters m, a, p, q appearing in the solution determine the four physical parameters M, J, Q, P as

$$
\begin{aligned}
2GM &= \frac{p+q}{2}\ , && (2.16)\\
GJ &= \frac{\sqrt{pq}(pq+4m^2)}{4(p+q)}\frac{a}{m}\ , && (2.17)\\
Q^2 &= \frac{q(q^2-4m^2)}{4(p+q)}\ , && (2.18)\\
P^2 &= \frac{p(p^2-4m^2)}{4(p+q)}\ . && (2.19)
\end{aligned}
$$

Note that $q, p \geq 2m$, with equality corresponding to the absence of electric or magnetic charge, respectively.

The spectrum of quadratic fluctuations around the general black hole solution to Kaluza-Klein theory is complicated. In section 2.6 we start with a general solution to the equations of motion (2.5-2.7) such as the Kaluza-Klein black hole $g_{\mu\nu}^{(\mathrm{KK})}$, $A_\mu^{(\mathrm{KK})}$, and $\Phi^{(\mathrm{KK})}$ presented above. We construct an embedding into $\mathcal{N}=2$ SUGRA with arbitrary cubic prepotential and study fluctuations around the background. Although we make some progress in this general setting it proves notable that the analysis simplifies greatly when the background dilaton is constant $\Phi^{(\mathrm{KK})}=0$.

In the predominant part of the chapter we therefore focus on the simpler case from the outset and assume $\Phi^{(\mathrm{KK})}=0$. We arrange this by considering the non-rotating black hole $J=0$ with $P^2=Q^2$. In this special case the metric $g_{\mu\nu}^{(\mathrm{KK})}$ is (2.8) with

$$
\begin{aligned}
H_1 = H_2 &= \left(r + \frac{q-2m}{2}\right)^2 , \\
H_3 = \Delta &= r^2 - 2mr , \qquad (2.20)
\end{aligned}
$$

and the gauge field (2.14) becomes

$$
A^{(\mathrm{KK})} = -2Q\left(r+\frac{q-2m}{2}\right)^{-1} dt - 2P\cos\theta d\phi \ . \qquad (2.21)
$$

In the simplified setting it is easy to eliminate the parameters m, q in favor of the physical mass $2GM = q$ and charges $P^2 = Q^2 = \frac{1}{8}(q^2-4m^2)$ but we do not need to do so.

When $\Phi^{(\mathrm{KK})}=0$ the geometry of the Kaluza-Klein black hole is in fact the same as the Reissner-Nordström black hole. Indeed, they both satisfy the standard Einstein-Maxwell equations

$$
R_{\mu\nu}^{(\mathrm{KK})} = \frac{1}{2}\left(F_{\mu\rho}^{(\mathrm{KK})}F_{\nu}^{(\mathrm{KK})\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}^{(\mathrm{KK})}F^{(\mathrm{KK})\rho\sigma}\right) , \qquad (2.22)
$$

$$
D_\mu F^{(\mathrm{KK})\mu\nu} = 0 \ . \qquad (2.23)
$$

However, whereas the Reissner-Nordström solution can be supported by any combination of electric and magnetic charges (Q, P) with the appropriate value of $Q_{\mathrm{eff}} = \sqrt{P^2+Q^2}$, for the Kaluza-Klein black hole we must set $P^2 = Q^2$ so

$$
F_{\mu\nu}^{(\mathrm{KK})}F^{(\mathrm{KK})\mu\nu} = 0 \ , \qquad (2.24)
$$

or else the dilaton equation of motion (2.5) is inconsistent with a constant dilaton $\Phi^{(\mathrm{KK})}$. This difference between the two cases is closely related to the fact that, after embedding in supergravity, the Kaluza-Klein black hole does not preserve supersymmetry in the extremal limit.

2.3 The KK Black Hole in $\mathcal{N}=8$ SUGRA

In this section, we review $\mathcal{N}=8$ SUGRA and show how to embed a solution of $D=4$ Kaluza-Klein theory with constant dilaton into $\mathcal{N}=8$ SUGRA.

2.3.1 $\mathcal{N}=8$ Supergravity in Four Dimensions

The matter content of $\mathcal{N}=8$ SUGRA is a spin-2 graviton $g_{\mu\nu}$, 8 spin-3/2 gravitini $\psi_{A\mu}$ (with $A=1,...,8$), 28 spin-1 vectors B_μ^{MN} (antisymmetric in $M,N=1,...,8$), 56 spin-1/2 gaugini λ_{ABC} (antisymmetric in $A,B,C=1,...,8$), and 70 spin-0 scalars. The Lagrangian can be presented as [54][5]

$$\begin{aligned}
e^{-1}\mathcal{L}^{(\mathcal{N}=8)} = {} & \frac{1}{4}R-\frac{1}{2}\bar{\psi}_{A\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{A\rho}-\frac{i}{8}G_{\mu\nu}^{MN}\widetilde{H}_{MN}^{(\mathrm{F})\mu\nu}-\frac{1}{12}\bar{\lambda}_{ABC}\gamma^\mu D_\mu\lambda_{ABC} \\
& -\frac{1}{24}P_{\mu ABCD}\bar{P}^{\mu ABCD}-\frac{1}{6\sqrt{2}}\bar{\psi}_{A\mu}\gamma^\nu\gamma^\mu\left(\bar{P}_\nu^{ABCD}+\hat{\bar{P}}_\nu^{ABCD}\right)\lambda_{BCD} \\
& +\frac{1}{8\sqrt{2}}\left(\bar{\psi}_{A\mu}\gamma^\nu\hat{\mathcal{F}}_{AB}\gamma^\mu\psi_{\nu B}-\frac{1}{\sqrt{2}}\bar{\psi}_{C\mu}\hat{\mathcal{F}}_{AB}\gamma^\mu\lambda_{ABC}+\frac{1}{72}\epsilon^{ABCDEFGH}\bar{\lambda}_{ABC}\hat{\mathcal{F}}_{DE}\lambda_{FGH}\right) ,
\end{aligned} \tag{2.25}$$

in conventions where all fermions are in Majorana form, the metric is "mostly plus", and Hodge duality is defined by

$$\widetilde{H}_{MN}^{(\mathrm{F})\mu\nu}=-\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}H_{MN\rho\sigma}^{(\mathrm{F})} ,\ \ \epsilon_{0123}=e\ . \tag{2.26}$$

Below we also use (R/L) superscripts on fermions, to denote their right- and left-handed components.

We include all the glorious details of $\mathcal{N}=8$ SUGRA to facilitate comparison with other references. The symmetry structure is the most important aspect for our applications so we focus on that in the following. The starting point is the 56-bein

$$\mathcal{V}=\begin{pmatrix} {U_{AB}}^{MN} & V_{ABMN} \\ \bar{V}^{ABMN} & {\bar{U}^{AB}}_{MN}\end{pmatrix} , \tag{2.27}$$

[5] To match with the conventions of many authors, when discussing $\mathcal{N}=8$ supergravity, we set Newton constant to $\kappa^2=8\pi G=2$. In section 2.6, we will restore the explicit κ dependence.

that is acted on from the left by a local $SU(8)$ symmetry (with indices $A, B, \ldots$) and from the right by a global $E_{7(7)}$ duality symmetry (with indices M, N). The connection

$$\partial_\mu \mathcal{V}\mathcal{V}^{-1} = \begin{pmatrix} 2Q_{\mu[A}{}^{[C}\delta_{B]}{}^{D]} & P_{\mu ABCD} \\ \bar{P}_\mu^{ABCD} & 2\bar{Q}_\mu{}^{[A}{}_{[C}\delta^{B]}{}_{D]} \end{pmatrix} , \tag{2.28}$$

defines an $SU(8)$ gauge field $Q_{\mu A}{}^B$ that renders the $SU(8)$ redundant. We therefore interpret $P_{\mu ABCD}$ as covariant derivatives of scalar fields that belong to the coset $E_{7(7)}/SU(8)$ with dimension $133 - 63 = 70$. The term in (2.25) that is quadratic in $P_{\mu ABCD}$ is therefore a standard kinetic term for the physical scalars. The terms linear in $P_{\mu ABCD}$, including

$$\hat{P}_{\mu ABCD} = P_{\mu ABCD} + 2\sqrt{2}\left(\bar{\psi}^{(L)}_{\mu[A}\lambda^{(R)}_{BCD]} + \frac{1}{24}\epsilon_{ABCDEFGH}\bar{\psi}^{(R)E}_\mu\lambda^{(L)FGH}\right) , \tag{2.29}$$

do not contribute to quadratic fluctuations around a background with constant scalars. The covariant derivatives D_μ that act on fermions are $SU(8)$ covariant so at this point the Lagrangian is manifestly invariant under the local $SU(8)$.

The gauge fields and their duals are

$$G^{MN}_{\mu\nu} = \partial_\mu B^{MN}_\nu - \partial_\nu B^{MN}_\mu , \tag{2.30}$$

$$\widetilde{H}^{(\mathrm{F})\mu\nu}_{MN} = \frac{4i}{e}\frac{\partial\mathcal{L}}{\partial G^{MN}_{\mu\nu}} . \tag{2.31}$$

They enter the Lagrangian (2.25) explicitly. Their Pauli couplings are written in terms of

$$\hat{\mathcal{F}}_{AB} = \gamma^{\mu\nu}\mathcal{F}_{AB\mu\nu} , \tag{2.32}$$

where

$$\mathcal{F}_{AB\mu\nu} = \mathcal{F}^{(\mathrm{F})}_{AB\mu\nu} + \sqrt{2}\left(\bar{\psi}^{(R)}_{[A[\mu}\psi^{(L)}_{[B[\nu} - \frac{1}{\sqrt{2}}\bar{\psi}^{(L)C}_{[\mu}\gamma_{\nu]}\lambda^{(L)}_{ABC} - \frac{1}{288}\epsilon_{ABCDEFGH}\bar{\lambda}^{CDE}_{(L)}\gamma_{\mu\nu}\lambda^{FGH}_{(R)}\right) ,$$

with

$$\begin{pmatrix} \mathcal{F}^{(\mathrm{F})}_{AB\mu\nu} \\ \bar{\mathcal{F}}^{(\mathrm{F})AB}_{\mu\nu} \end{pmatrix} = \frac{1}{\sqrt{2}}\mathcal{V}\begin{pmatrix} G^{MN}_{\mu\nu} + iH^{(\mathrm{F})}_{MN\mu\nu} \\ G^{MN}_{\mu\nu} - iH^{(\mathrm{F})}_{MN\mu\nu} \end{pmatrix} . \tag{2.33}$$

These relatives of the gauge fields encode couplings and $E_{7(7)}$ duality symmetries. They satisfy the self-duality constraint

$$\mathcal{F}_{\mu\nu AB} = \widetilde{\mathcal{F}}_{\mu\nu AB} . \tag{2.34}$$

This self-duality constraint is a complex equation that relates the real fields $G^{MN}_{\mu\nu}$, $H^{(\mathrm{F})}_{MN\mu\nu}$ and their duals linearly, with coefficients that depend nonlinearly on scalar fields. It has a solution of the form

$$\widetilde{H}^{(\mathrm{F})}_{MN\mu\nu} = -\, i \left(\mathcal{N}_{MNPQ} G^{-PQ}_{\mu\nu} + \text{h.c.} \right) + (\text{terms quadratic in fermions}) \ , \tag{2.35}$$

where the self-dual (anti-self-dual) parts of the field strengths are defined as

$$G^{\pm MN}_{\mu\nu} = \frac{1}{2} \left(G^{MN}_{\mu\nu} \pm \widetilde{G}^{MN}_{\mu\nu} \right) \ , \tag{2.36}$$

and the gauge coupling function is

$$\mathcal{N}_{MNPQ} = \left(U_{AB}{}^{MN} - V_{ABMN} \right)^{-1} \left(U_{AB}{}^{MN} + V_{ABPQ} \right) \ . \tag{2.37}$$

Using (2.35) for $\widetilde{H}^{(\mathrm{F})}_{MN\mu\nu}$ and (2.32–2.33) for $\hat{\mathcal{F}}_{AB}$ we can eliminate these fields from the Lagrangian (2.25) in favor of the dynamical gauge field $G^{MN}_{\mu\nu}$, embellished by scalar fields and fermion bilinears.

The relatively complicated classical dynamics of $\mathcal{N} = 8$ SUGRA is due to the interplay between fermion bilinears, duality, and the scalar coset. These disparate features are all important in our considerations but they largely decouple. For example, although we need the Pauli couplings of fermions, we need them only for trivial scalars.

In our explicit computations it is convenient to remove the $SU(8)$ gauge redundancy by writing the 56-bein (2.27) in a symmetric gauge

$$\mathcal{V} = \exp \begin{pmatrix} 0 & W_{ABCD} \\ \bar{W}^{ABCD} & 0 \end{pmatrix} \ , \tag{2.38}$$

where the 70 complex scalars W_{ABCD} are subject to the constraint

$$\bar{W}^{ABCD} = \frac{1}{24} \epsilon^{ABCDEFGH} W_{EFGH} \ . \tag{2.39}$$

After fixing the local $SU(8)$ symmetry, the theory still enjoys a global $SU(8)$ symmetry. Moreover, it is linearly realized when compensated by $SU(8) \subset E_{7(7)}$. We identify this residual global $SU(8)$ as the R-symmetry $SU(8)_R$. This identification proves useful repeatedly. For example, it is according to this residual symmetry that W_{ABCD} transforms as an antisymmetric four-tensor.

2.3.2 The Embedding into $\mathcal{N} = 8$ SUGRA

The embedding of the Kaluza-Klein black hole (2.8, 2.20, 2.21) in $\mathcal{N} = 8$ SUGRA is implemented by

$$\begin{aligned}
&\mathring{g}_{\mu\nu}^{(\mathrm{SUGRA})} = g_{\mu\nu}^{(\mathrm{KK})} \ , \\
&\mathring{G}_{\mu\nu}^{MN} = \frac{1}{4}\Omega^{MN} F_{\mu\nu}^{(\mathrm{KK})} \ , \\
&\mathring{W}_{ABCD} = 0 \ , \\
&(\text{All background fermionic fields}) = 0 \ ,
\end{aligned} \tag{2.40}$$

where

$$\Omega^{MN} = \mathrm{diag}(\epsilon, \epsilon, \epsilon, \epsilon) \ , \ \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{2.41}$$

In this section (and beyond) we shall often declutter formulae by omitting the superscript "KK" when referring to fields of the seed solution.

To establish the consistency of our embedding, in the following we explicitly check that the $\mathcal{N} = 8$ SUGRA equations of motion are satisfied by the background (2.40). Vanishing fermions satisfy trivially their equations of motion, because they appear at least quadratically in the action. The equations of motion for the scalars W_{ABCD} take the form

$$\begin{aligned}
&(\text{Terms at least linear in } \mathring{W}_{ABCD} \text{ or quadratic in fermions}) \\
&= \ 3\,\mathring{G}_{\mu\nu}^{+[AB}\mathring{G}^{+CD]\mu\nu} + \frac{1}{8}\epsilon_{ABCDEFGH}\mathring{G}_{\mu\nu}^{-EF}\mathring{G}^{-GH\mu\nu} \ .
\end{aligned} \tag{2.42}$$

The scalars $\mathring{W}_{ABCD}$ and the fermions vanish so the right-hand side of the equation must also vanish. Inserting $\mathring{G}_{\mu\nu}^{MN}$ from our embedding (2.40), we find the condition $F_{\mu\nu}^{(\mathrm{KK})}F^{(\mathrm{KK})\mu\nu} = 0$. This condition is satisfied by the seed solution (2.24) because the electric and magnetic charges are equal $P = Q$. Therefore it is consistent to take all scalars $\mathring{W}_{ABCD} = 0$ in $\mathcal{N} = 8$ SUGRA.

The $\mathcal{N} = 8$ Einstein equation is given by

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \ &= \ \frac{1}{6}P_{ABCD\{\mu}\bar{P}_{\nu\}}^{ABCD} - \frac{1}{12}g_{\mu\nu}P_{\rho ABCD}\bar{P}^{\rho ABCD} \\
&\quad + \mathrm{Re}(\mathcal{N}_{MNPQ})\left(G_{\mu\rho}^{MN}G_{\nu}{}^{\rho PQ} - \frac{1}{4}g_{\mu\nu}G_{\rho\sigma}^{MN}G^{\rho\sigma PQ}\right) \ .
\end{aligned} \tag{2.43}$$

The vanishing of the scalars $\mathring{W}_{ABCD} = 0$ implies

$$\mathring{\mathcal{V}} = \begin{pmatrix} \delta_{[A}^{\ [M}\delta_{B]}^{\ N]} & 0 \\ 0 & \delta^{[A}_{\ [M}\delta^{B]}_{\ N]} \end{pmatrix} , \qquad \mathring{\mathcal{N}}_{MNPQ} = \mathbf{1}_{MNPQ} , \tag{2.44}$$

so the Einstein equation simplifies to

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\mathring{R} = \mathring{G}^{MN}_{\mu\rho}\mathring{G}_{\nu MN}^{\ \rho} - \frac{1}{4}\mathring{g}_{\mu\nu}\mathring{G}^{MN}_{\rho\sigma}\mathring{G}^{\rho\sigma}_{MN} . \tag{2.45}$$

The embedding (2.40) reduces the right-hand side so that these equations coincide with the Einstein equation (2.22) satisfied by the seed solution.

Finally, the equations of motion for the vector fields in $\mathcal{N} = 8$ SUGRA are

$$D_\mu \left(\mathcal{N}_{MNPQ}G^{-\mu\nu PQ} + \bar{\mathcal{N}}_{MNPQ}G^{+\mu\nu PQ}\right) = 0 . \tag{2.46}$$

The embedding (2.40) and the simplifications (2.44) reduce these equations to the Maxwell equation $D_\mu F^{(\mathrm{KK})\mu\nu} = 0$, consistent with the seed equation of motion (2.23).

In summary, the equations of motion in $\mathcal{N} = 8$ SUGRA are satisfied by the embedding (2.40). Therefore, for any seed solution that satisfies (2.22-2.24), the embedding (2.40) gives a solution to $\mathcal{N} = 8$ SUGRA. Our primary example is the Kaluza-Klein black hole with dilaton $\Phi^{(\mathrm{KK})} = 0$.

2.4 Quadratic Fluctuations in $\mathcal{N} = 8$ SUGRA

In this section we expand the Lagrangian (2.25) for $\mathcal{N} = 8$ SUGRA to quadratic order around the background (2.40). We reparametrize the fluctuation fields so that they all transform in representations of the global $USp(8)$ symmetry group preserved by the background. We then partially decouple the quadratic fluctuations into different blocks corresponding to different representations of $USp(8)$.

2.4.1 Global Symmetry of Fluctuations

The $\mathcal{N} = 8$ SUGRA theory has a global $SU(8)$ symmetry, as discussed at the end of section 2.3. The graviton, gravitini, vectors, gaugini, and scalars transform in the representations **1**, **8**, **28**, **56** and **70** of this $SU(8)$ group. The **28**, **56**, and **70**, are realized as antisymmetric combinations of the fundamental representation **8**.

A generic background solution does not respect all the symmetries of the theory, so the global $SU(8)$ symmetry is not generally helpful for analyzing fluctuations around the background. Our embedding (2.40) into $\mathcal{N}=8$ SUGRA indeed breaks the $SU(8)$ symmetry since $\mathring{G}^{MN}_{\mu\nu}=\frac{1}{4}\Omega^{MN}F^{(\mathrm{KK})}_{\mu\nu}$ is not invariant under the $SU(8)$ group. However, the matrix Ω^{MN} (2.41) can be interpreted as a canonical symplectic form so our embedding respects most of the global $SU(8)$, it preserves a $USp(8)$ subgroup. Therefore, different $USp(8)$ representations cannot couple at quadratic order and it greatly simplifies the analysis to organize fluctuations around the background as representations of $USp(8)$. In the following we analyze one $USp(8)$ representation at a time.

- *Graviton*

  The graviton $h_{\mu\nu}=\delta g_{\mu\nu}=g_{\mu\nu}-\mathring{g}_{\mu\nu}$ is a singlet of $SU(8)$ and remains a singlet of $USp(8)$.

- *Vectors*

  The fluctuations of the gauge fields $\delta G^{MN}_{\mu\nu}=G^{MN}_{\mu\nu}-\mathring{G}^{MN}_{\mu\nu}$ transform in the **28** of $SU(8)$ which has the branching rule to $USp(8)$ $\mathbf{28}\to\mathbf{1}\oplus\mathbf{27}$. We realize this decomposition directly on the fluctuations by defining

$$f_{\mu\nu}=\Omega_{MN}\delta G^{MN}_{\mu\nu}\ ,\qquad f^{MN}_{\mu\nu}=\delta G^{MN}_{\mu\nu}-\frac{1}{8}\Omega^{MN}f_{\mu\nu}\ . \tag{2.47}$$

  The $f^{MN}_{\mu\nu}$ are Ω-traceless $f^{MN}_{\mu\nu}\Omega_{MN}=0$ by construction so they have only $2\times(28-1)$ degrees of freedom which transform in the **27** of $USp(8)$. The remaining 2 degrees of freedom are in $f_{\mu\nu}$, which transforms in the **1** of $USp(8)$. This decomposition under the global symmetry shows that the graviton can only mix with the "overall" gauge field $f_{\mu\nu}$ and not with $f^{MN}_{\mu\nu}$.

- *Scalars*

  The scalars transform in **70** of $SU(8)$ and the branching rule to $USp(8)$ is $\mathbf{70}\to\mathbf{1}\oplus\mathbf{27}\oplus\mathbf{42}$. We realize this decomposition by defining

$$\begin{aligned}&W'=W_{ABCD}\Omega^{AB}\Omega^{CD}\ ,\quad W'_{AB}=W_{ABCD}\Omega^{CD}-\frac{1}{8}W'\Omega_{AB}\ ,\\&W'_{ABCD}=W_{ABCD}-\frac{3}{2}W'_{[AB}\Omega_{CD]}-\frac{1}{16}W'\Omega_{[AB}\Omega_{CD]}\ .\end{aligned} \tag{2.48}$$

  W'_{ABCD} is antisymmetric in all indices and Ω-traceless on any pair or pairs, so it is in the **42** of $USp(8)$. W'_{AB} is antisymmetric, Ω-traceless, and hence in the **27** of $USp(8)$. The remainder W' has no index and is in the **1** of $USp(8)$. The obvious construction of an antisymmetric four-tensor representation of $SU(8)$ has 70 *complex* degrees of freedom, but the scalars W_{ABCD} in $\mathcal{N}=8$ SUGRA have 70 *real* degrees of freedom that realize an irreducible representation, as implemented by the reality constraint (2.39). The decomposition of this reality constraint under $SU(8) \to USp(8)$ shows that the scalar W' that couples to gravity is real $\overline{W}' = W'$, as expected from Kaluza-Klein theory. It also implies the reality condition on the four-tensor

$$\overline{W}'^{ABCD} = \frac{1}{24}\epsilon^{ABCDEFGH} W'_{EFGH} \ , \tag{2.49}$$

and an analogous condition on the two-tensor W'^{AB}. An interesting aspect of these reality conditions is that, just like the KK block must couple to a scalar (as opposed to a pseudoscalar), the condition on the $USp(8)$ four-tensor demonstrates that the scalar moduli must comprise exactly 22 scalars and 20 pseudoscalars. The vector multiplet couples vectors and scalars/pseudoscalars precisely so that it restores the overall balance between scalars and pseudoscalars required by $\mathcal{N} = 8$ SUGRA, with 12 scalars and 15 pseudoscalars.

The distinctions between scalars and pseudoscalars are interesting because these details must be reproduced by viable microscopic models of black holes. Extrapolations far off extremality of phenomenological models that are motivated by the BPS limit lead to entropy formulae [55–57] with moduli dependence that is very similar but not identical to the result found here. It would be interesting to construct a model for non-extremal black holes that combines the features of the BPS and the nonBPS branch.

- *Gravitini*

  The gravitini $\psi_{A\mu}$ transform in the fundamental **8** of $SU(8)$. The gravitini only carry one $SU(8)$ index which cannot be contracted with the symplectic form Ω^{AB}. Therefore, the gravitini also transform in the **8** of $USp(8)$.

- *Gaugini*

  The gaugini λ_{ABC} of $\mathcal{N} = 8$ SUGRA transform in the **56** of the global $SU(8)$. The branching rule to $USp(8)$ is $\mathbf{56} \to \mathbf{8} \oplus \mathbf{48}$. We can realize this decomposition by introducing

$$\lambda'_A = \frac{1}{\sqrt{12}}\lambda_{ABC}\Omega^{BC} \ , \tag{2.50}$$

and

$$\lambda'_{ABC} = \lambda_{ABC} - \frac{1}{8}(\lambda_{ADE}\Omega^{DE})\Omega_{BC} \ . \tag{2.51}$$

The gaugini λ'_A transform in the **8** of $USp(8)$. We will find that these gaugini are coupled to the gravitini. This is allowed because they have the same quantum numbers under the global $USp(8)$. The normalization $1/\sqrt{12}$ introduced in (2.50) ensures that the gaugini retain a canonical kinetic term after the field redefinition.

The gaugini λ'_{ABC} introduced in (2.51) satisfy the constraint $\lambda'_{ABC}\Omega^{BC} = 0$. This ensures that they transform in the **48** of $USp(8)$. No other fields transform in the same way under the global symmetry so these gaugini decouple from other fields. They can of course mix among themselves and we will find that they do in fact have nontrivial Pauli couplings. However, the normalization of the fields is inconsequential and we have retained the normalization inherited from the full $\mathcal{N} = 8$ SUGRA.

Table 2.2 summarizes the decomposition of quadratic fluctuations according to their representations under the global $USp(8)$ that is preserved by the background.

Representations	Fields
1	$h_{\mu\nu},\ f_{\mu\nu},\ W'$
8	$\psi_{A\mu},\ \lambda'_A$
27	$f^{\mu\nu}_{AB},\ W'_{AB}$
42	W'_{ABCD}
48	λ'_{ABC}

Table 2.2: The $USp(8)$ representation content of the quadratic fluctuations.

2.4.2 The Decoupled Fluctuations

The quadratic fluctuations around any bosonic background decouple into a bosonic part $\delta^2\mathcal{L}_{\text{bosons}}$ and a fermionic part $\delta^2\mathcal{L}_{\text{fermions}}$ because fermions always appear quadratically in the Lagrangian. As we expand the Lagrangian (2.25) around the background (2.40) to quadratic order, these parts further decouple into representations of the preserved $USp(8)$ global symmetry.

The bosonic fluctuations therefore decouple into three blocks

$$\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{bosons}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}} + \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{vector}} + \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{scalar}} \ . \tag{2.52}$$

- *KK block*

The first block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\rm KK}$, which we call the "KK block", consists of all fields that are singlets of $USp(8)$: the graviton $h_{\mu\nu}$, 1 vector with field strength $f_{\mu\nu}$, and 1 scalar W'. The Lagrangian for this block is given by

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\rm KK} &= \bar{h}^{\mu\nu}\Box\bar{h}_{\mu\nu} - \frac{1}{4}h\Box h + 2\bar{h}^{\mu\nu}\bar{h}^{\rho\sigma}R_{\mu\rho\nu\sigma} - 2\bar{h}^{\mu\nu}\bar{h}_{\mu\rho}R^{\rho}{}_{\nu} - h\bar{h}^{\mu\nu}R_{\mu\nu} \\ &\quad -F_{\mu\nu}F_{\rho\sigma}\bar{h}^{\mu\rho}\bar{h}^{\nu\sigma} + a^{\mu}\left(\Box g_{\mu\nu} - R_{\mu\nu}\right)a^{\nu} + 2\sqrt{2}F_{\nu}{}^{\rho}f_{\mu\rho}\bar{h}^{\mu\nu} \\ &\quad -4\partial_{\mu}\phi\partial^{\mu}\phi + 2\sqrt{3}F^{\mu\nu}f_{\mu\nu}\phi - 4\sqrt{6}R_{\mu\nu}\bar{h}^{\mu\nu}\phi \ , \end{aligned} \tag{2.53}$$

after the fields were redefined as $h_{\mu\nu} \to \sqrt{2}h_{\mu\nu}$, $f_{\mu\nu} \to 4f_{\mu\nu}$, and $\phi = -\frac{1}{8\sqrt{3}}W'$. We also decomposed the graviton into its trace $h = g^{\rho\sigma}h_{\rho\sigma}$ and its traceless part $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}h_{\rho\sigma}$, and further included the gauge-fixing term

$$e^{-1}\mathcal{L}_{\rm g.f.} = -\left(D^{\mu}\bar{h}_{\mu\rho} - \frac{1}{2}D_{\rho}h\right)\left(D^{\nu}\bar{h}_{\nu}{}^{\rho} - \frac{1}{2}D^{\rho}h\right) - \left(D^{\mu}a_{\mu}\right)^2 \ . \tag{2.54}$$

The rather complicated Lagrangian (2.53) represents the theory of fluctuations around any solution of Kaluza-Klein theory (2.4) with constant dilaton. The fields $f_{\mu\nu}$ and ϕ correspond to the fluctuations of the field strength and the dilaton. The gauge-fixed theory (2.53) must be completed with additional ghost terms. We discuss those in Appendix A.

- *Vector blocks*

The second block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\rm vector}$ consists of all fields that transform in the **27** of $USp(8)$: $f^{\mu\nu}_{AB}$ and W'_{AB}. We use $f^{\mu\nu}_{a}$ and W'_{a} to denote the 27 independent vectors and scalars respectively. It includes two slightly different parts. One part has 12 copies of a vector coupled to a scalar $W'^{(\rm R)}_{a}$ with the Lagrangian

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)(\rm R)}_{\rm vector} = -\frac{1}{2}\partial^{\mu}W'^{(\rm R)}_{a}\partial_{\mu}W'^{(\rm R)}_{a} - f^{\mu\nu}_{a}f_{a\mu\nu} - W'^{(\rm R)}_{a}f_{a\mu\nu}F^{\mu\nu} \ , \ a = 1, ..., 12 \ , \tag{2.55}$$

and the other has 15 copies of a vector coupled to a pseudoscalar $W'^{(\rm P)}_{a}$ given by

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)(\rm P)}_{\rm vector} = -\frac{1}{2}\partial^{\mu}W'^{(\rm P)}_{a}\partial_{\mu}W'^{(\rm P)}_{a} - f^{\mu\nu}_{a}f_{a\mu\nu} - iW'^{(\rm P)}_{a}f_{a\mu\nu}\widetilde{F}^{\mu\nu} \ , \ a = 13, ..., 27. \tag{2.56}$$

Although these two Lagrangians are distinct, they give equations of motion that are equivalent under a duality transformation. This is consistent with the fact that $SU(8)$

duality symmetry is the diagonal combination of local $SU(8)$ and global $E_{7(7)}$ duality symmetry, where the latter is not realized at the level of the Lagrangian.

- *Scalar blocks*

  The last bosonic block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{scalar}}$ consists of the remaining 42 scalars, transforming in the **42** of $USp(8)$. There are no other bosonic fields with the same quantum numbers so, these fields can only couple to themselves. The explicit expansion around the background (2.22-2.24) shows that all these scalars are in fact minimally coupled

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{scalar}} = -\frac{1}{24}\partial^\mu W'_{ABCD}\partial_\mu \overline{W}'^{ABCD} \ . \tag{2.57}$$

We now turn to the quadratic fluctuations for the fermions. Since they appear at least quadratically in the Lagrangian the bosonic fields can be fixed to their background values. In this case, the $\mathcal{N}=8$ SUGRA Lagrangian (2.25) simplifies to

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{fermions}} \ &= \ -\frac{1}{2}\bar{\psi}_{A\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{A\rho} - \frac{1}{12}\bar{\lambda}_{ABC}\gamma^\mu D_\mu\lambda_{ABC} + \frac{1}{4\sqrt{2}}\bar{\psi}_{A\mu}\gamma^\nu\mathring{\mathcal{F}}_{AB}\gamma^\mu\psi_{\nu B} \\ &\quad -\frac{1}{8}\bar{\psi}_{C\mu}\mathring{\mathcal{F}}_{AB}\gamma^\mu\lambda_{ABC} + \frac{1}{288\sqrt{2}}\epsilon^{ABCDEFGH}\bar{\lambda}_{ABC}\mathring{\mathcal{F}}_{DE}\lambda_{FGH} \ , \end{aligned} \tag{2.58}$$

where all fermions are in Majorana form and

$$\mathring{\mathcal{F}}_{AB} \ = \ \frac{1}{\sqrt{2}}\left(\mathring{G}_{AB\mu\nu} + \gamma_5\mathring{\tilde{G}}_{AB\mu\nu}\right)\gamma^{\mu\nu} = \frac{1}{2\sqrt{2}}\Omega_{AB}F_{\mu\nu}\gamma^{\mu\nu} \ . \tag{2.59}$$

The field redefinitions introduced in section 2.4.1 decouple this Lagrangian as

$$\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{fermions}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}} + \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}} \ . \tag{2.60}$$

- *Gravitino blocks*

  The first block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}}$ consists of the 8 gravitini $\psi_{A\mu}$ and the 8 gaugini λ'_A singled out by the projection (2.50). The gravitini and the gaugini both transform in **8** of $USp(8)$ and couple through the Lagrangian

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}} \ &= \ -\bar{\psi}_{A\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{A\rho} - \bar{\lambda}'_A\gamma^\mu D_\mu\lambda'_A + \frac{1}{4}\Omega^{AB}\bar{\psi}_{A\mu}\left(F^{\mu\nu} + \gamma_5\widetilde{F}^{\mu\nu}\right)\psi_{B\nu} \\ &\quad -\frac{\sqrt{6}}{8}\bar{\psi}_{A\mu}F_{\rho\sigma}\gamma^{\rho\sigma}\gamma^\mu\lambda'_A + \frac{1}{4}\Omega^{AB}\bar{\lambda}'_A F_{\rho\sigma}\gamma^{\rho\sigma}\lambda'_B \ . \end{aligned} \tag{2.61}$$

  The indices take values $A, B = 1, \ldots 8$. However, this block actually decouples into

4 identical pairs, with a single pair comprising two gravitini and two gaugini. The canonical pair is identified by restricting the indices to $A, B = 1, 2$ and so $\Omega_{AB} \to \epsilon_{AB}$. The other pairs correspond to $A, B = 3, 4,\ A, B = 5, 6$, and $A, B = 7, 8$.

- *Gaugino blocks*

  The second block $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}}$ consists of the 48 gaugini (2.51) that transform in the **48** of $USp(8)$. These 48 gaugini decompose into 24 identical groups that decouple from one another. Each group has 2 gaugini and a Lagrangian given by

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}} = -\,\bar{\lambda}_a\gamma^\mu D_\mu\lambda_a - \frac{1}{8}\epsilon^{ab}\bar{\lambda}_a F_{\mu\nu}\gamma^{\mu\nu}\lambda_b\ , \tag{2.62}$$

  where $a, b = 1, 2$ denote the 2 different gaugini in one group. It is interesting that *no* fermions in the theory are minimally coupled. Moreover, the numerical strength of the Pauli couplings to black holes on the nonBPS branch are different from the corresponding Pauli couplings for fermions on the BPS branch [17].

2.4.3 Summary of Quadratic Fluctuations

In the previous sections we defined a seed solution (2.22-2.24) of Kaluza-Klein theory with vanishing dilaton and embedded it into $\mathcal{N} = 8$ SUGRA through (2.40). In this section, we have studied fluctuations around the background by expanding the $\mathcal{N} = 8$ SUGRA Lagrangian (2.25) to quadratic order. In section 2.4.1, we decomposed the fluctuations in representations of the $USp(8)$ symmetry preserved by the background. In section 2.4.2, we have decoupled the quadratic fluctuations into blocks corresponding to distinct representations of $USp(8)$. They are summarized in Table 2.3.

Degeneracy	Multiplet	Block content	$USp(8)$	Lagrangian
1	KK block	1 graviton, 1 vector, 1 scalar	**1**	(2.53)
27	Vector block	1 vector and 1 (pseudo)scalar	**27**	(2.55)
42	Scalar block	1 real scalar	**42**	(2.57)
4	Gravitino block	2 gravitini and 2 gaugini	**8**	(2.61)
24	Gaugino block	2 gaugini	**48**	(2.62)

Table 2.3: Decoupled quadratic fluctuations in $\mathcal{N} = 8$ supergravity around the KK black hole.

2.5 Consistent Truncations of $\mathcal{N} = 8$ SUGRA

In this section we present consistent truncations from $\mathcal{N} = 8$ SUGRA to $\mathcal{N} = 6$, $\mathcal{N} = 4$, $\mathcal{N} = 2$ and $\mathcal{N} = 0$. These truncations are well adapted to the KK black hole in that all its nontrivial fields are retained. In other words, the truncations amount to removal of fields that are trivial in the background solution.

It is easy to analyse the spectrum of quadratic fluctuations around the KK black hole in the truncated theories. In each case some of the fluctuating fields are removed, but always consistently so that blocks of fields that couple to each other are either all retained or all removed. Therefore, the fluctuation spectrum in all these theories can be described in terms of the same simple blocks that appear in $\mathcal{N} = 8$ supergravity. For these truncations the entire dependence on the theory is encoded in the degeneracy of each type of block. They are summarized in Table 2.4.

Multiplet \ Theory	$\mathcal{N} = 8$	$\mathcal{N} = 6$	$\mathcal{N} = 4$	$\mathcal{N} = 2$	$\mathcal{N} = 0$
KK block	1	1	1	1	1
Gravitino block	4	3	2	1	0
Vector block	27	15	$n + 5$	n_V	0
Gaugino block	24	10	$2n$	$n_V - 1$	0
Scalar block	42	14	$5n - 4$	$n_V - 1$	0

Table 2.4: The degeneracy of multiplets in the spectrum of quadratic fluctuations around the KK black hole embedded in various theories. For $\mathcal{N} = 4$, the integer n is the number of $\mathcal{N} = 4$ matter multiplets. For $\mathcal{N} = 2$, the integer n_V refers to the $ST(n_V - 1)$ model.

All the truncations in this section heavily utilize the $SU(8)_R$ global symmetry of $\mathcal{N} = 8$ supergravity. We therefore recall from the outset that the gravitons, gravitini, vectors, gaugini, and scalars transform in the irreducible representations **1**, **8**, **28**, **56**, **70** of $SU(8)_R$.

2.5.1 The $\mathcal{N} = 6$ Truncation

The $\mathcal{N} = 6$ truncation restricts $\mathcal{N} = 8$ SUGRA to fields that are *even* under the $SU(8)_R$ element $\mathrm{diag}(I_6, -I_2)$. This projection preserves $\mathcal{N} = 6$ local supersymmetry since the 8 gravitini of $\mathcal{N} = 8$ SUGRA are in the fundamental **8** of $SU(8)_R$ and so exactly two gravitini are odd under $\mathrm{diag}(I_6, -I_2)$ and projected out. The branching rules of the matter multiplets under $SU(8)_R \to SU(6)_R \times SU(2)_{\rm matter}$ are

$$\begin{aligned} \mathbf{70} &\to (\mathbf{15},\mathbf{1}) \oplus (\overline{\mathbf{15}},\mathbf{1}) \oplus (\mathbf{20},\mathbf{2}) \ , \\ \mathbf{56} &\to (\mathbf{20},\mathbf{1}) \oplus (\mathbf{15},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1}) \ , \\ \mathbf{28} &\to (\mathbf{15},\mathbf{1}) \oplus (\mathbf{6},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1}) \ . \end{aligned} \tag{2.63}$$

These branching rules follow from decomposition of the $SU(8)_R$ four-tensor T_{ABCD} (**70**), the three-tensor T_{ABC} (**56**), and the two-tensor T_{AB} (**28**), by splitting the $SU(8)_R$ indices as $A, B, ... \to (\alpha, a), (\beta, b), ...$ where the lower case indices refer to $SU(2)_{\rm matter}$ (greek) and $SU(6)_R$ (latin). The truncation to $\mathcal{N} = 6$ SUGRA retains only the fields that are invariant under $SU(2)_{\rm matter}$ so fields in the **2** are removed. Therefore the truncated theory has 30 scalar fields, 26 gaugini, and 16 vector fields. Taking the 6 gravitini and the graviton into account as well, the total field content comprises 64 bosonic and 64 fermionic degrees of freedom.

The claim that the truncation is *consistent* means that the equations of motion *of the retained fields* are sufficient to guarantee that *all* equations of motion are satisfied, as long as the removed fields vanish. In general, the primary obstacle to truncation is that the equations of motion for the omitted fields may fail. This is addressed here because the equations of motion for fields in the **2** of $SU(2)_{\rm matter}$ only involve terms in the **2**. Therefore their equations of motion are satisfied when all fields in the **2** vanish.

Our interest in the consistent truncation of $\mathcal{N} = 8$ SUGRA to $\mathcal{N} = 6$ SUGRA is the application to the KK black hole. The embedding (2.40) of the Kaluza-Klein black hole into $\mathcal{N} = 8$ SUGRA turns on the four field strengths on the skew-diagonal of the **28** (which is realized by an antisymmetric 8×8 matrix of field strengths F_{AB}). The entries on the skew diagonal are all contained in the $SU(6)_R \times SU(2)_{\rm matter}$ subgroup of $SU(8)_R$, because the antisymmetric representation of $SU(2)$ is trivial. The embedding of the KK black hole in $\mathcal{N} = 8$ SUGRA therefore defines an embedding in $\mathcal{N} = 6$ SUGRA as well. In other words, the truncation and the embedding are *compatible*.

We can find the spectrum of quadratic fluctuations in $\mathcal{N} = 6$ SUGRA either by truncating the spectrum determined in the $\mathcal{N} = 8$ SUGRA context, or by directly analyzing the spectrum of fluctuations around the $\mathcal{N} = 6$ solution. Consistency demands that these procedures agree.

We begin from the $SU(6)$ content of $\mathcal{N} = 6$ SUGRA: **1** graviton, **6** gravitini, $\mathbf{15} \oplus \mathbf{1}$ vectors, $\mathbf{20} \oplus \mathbf{6}$ gaugini, and 2(**15**) scalars. The KK black hole in $\mathcal{N} = 6$ SUGRA breaks the global symmetry $SU(6) \to USp(6)$. Therefore, the quadratic fluctuations around the background need not respect the $SU(6)$ symmetry, but they must respect the $USp(6)$. Their

$USp(6)$ content is: $\mathbf{1}$ graviton, $\mathbf{6}$ gravitini, $\mathbf{14} \oplus 2(\mathbf{1})$ vectors, $\mathbf{14} \oplus 2(\mathbf{6})$ gaugini, $2(\mathbf{14} \oplus \mathbf{1})$ scalars. The black hole background breaks Lorentz invariance so the equations of motion for fluctuations generally mix Lorentz representations, as we have seen explicitly in section 2.4, but they always preserve global symmetries. In the present context the mixing combines the fields into $\mathbf{1}$ KK block (gravity + 1 vector + 1 scalar), 3 gravitino blocks (1 gravitino + 1 gaugino) (transforming in the $\mathbf{6}$), $\mathbf{14} \oplus \mathbf{1}$ vector blocks (1 vector + 1 scalar), 10 gaugino blocks (transforming in the $\mathbf{14} \oplus \mathbf{6}$), and $\mathbf{14}$ (minimally coupled) scalars.

To verify these claims and find the specific couplings for each block, we could analyze the equations of motion for $\mathcal{N} = 6$ SUGRA using the methods of section 2.4. However, no new computations are needed because it is clear that the fields in the truncated theory are a subset of those in $\mathcal{N} = 8$ SUGRA. In that context we established that the fluctuations decompose into $\mathbf{1}$ (KK block), $\mathbf{8}$ (gravitini mixing with gaugini), $\mathbf{27}$ (vectors mixing with scalars), $\mathbf{24}$ (gaugini with Pauli couplings to the background), and $\mathbf{42}$ (minimal scalars) of the $USp(8)$ that is preserved by the background. The consistent truncation to $\mathcal{N} = 6$ SUGRA removes some of these fluctuations as it projects the global symmetry $USp(8) \to USp(6)$. This rule not only establishes the mixing claimed in the preceding paragraph but also shows that all couplings must be the same in the $\mathcal{N} = 8$ and $\mathcal{N} = 6$ theories. It is only the degeneracy of each type of block that is reduced by the truncation.

2.5.2 The $\mathcal{N} = 4$ Truncation

The $\mathcal{N} = 4$ truncation restricts $\mathcal{N} = 8$ SUGRA to fields that are even under the $SU(8)_R$ element $\mathrm{diag}(I_4, -I_4)$. This projection breaks the global symmetry $SU(8)_R \to SU(4)_R \times SU(4)_{\mathrm{matter}}$. It preserves $\mathcal{N} = 4$ local supersymmetry since the 8 gravitini of $\mathcal{N} = 8$ SUGRA are in the $\mathbf{4}$ of $SU(4)_R$. The branching rules of the matter multiplets under the symmetry breaking are

$$\begin{aligned} \mathbf{70} &\to 2(\mathbf{1},\mathbf{1}) \oplus (\mathbf{6},\mathbf{6}) \oplus (\mathbf{4},\bar{\mathbf{4}}) \oplus (\bar{\mathbf{4}},\mathbf{4}) \ , \\ \mathbf{56} &\to (\bar{\mathbf{4}},\mathbf{1}) \oplus (\mathbf{6},\mathbf{4}) \oplus (\mathbf{4},\mathbf{6}) \oplus (\mathbf{1},\bar{\mathbf{4}}) \ , \\ \mathbf{28} &\to (\mathbf{1},\mathbf{6}) \oplus (\mathbf{6},\mathbf{1}) \oplus (\mathbf{4},\mathbf{4}) \ . \end{aligned} \tag{2.64}$$

The consistent truncation preserving $\mathcal{N} = 4$ supersymmetry is defined by omission of all fields in the $\mathbf{4}$ (or $\bar{\mathbf{4}}$) of $SU(4)_{\mathrm{matter}}$.

There is a unique supergravity with n $\mathcal{N} = 4$ matter multiplets. It has a global $SU(4)_R$ symmetry that acts on its supercharges and also a global $SO(n)_{\mathrm{matter}}$ that reflects the equivalence of all matter multiplets. The consistent truncation of $\mathcal{N} = 8$ by the element $\mathrm{diag}(I_4, -I_4)$ retains a $SU(4)_R \times SU(4)_{\mathrm{matter}}$ symmetry so, recalling that $SO(6)$ and $SU(4)$

are equivalent as Lie algebras, the truncated theory must be $\mathcal{N} = 4$ SUGRA with $n = 6$ matter multiplets.

Several important features of $\mathcal{N} = 4$ SUGRA are succinctly summarized by the scalar coset

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6) \times SO(n)} \ . \tag{2.65}$$

It has dimension $6n + 2$ with scalars transforming in $2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{n})$ under $SU(4)_R \times SO(n)_{\text{matter}}$. It also encodes the $SU(1,1) \simeq SL(2)$ electromagnetic duality of the $6 + n$ vector fields in the fundamental of $SO(6,n)$. The representation content obtained by removal of $\mathbf{4}$ (and $\bar{\mathbf{4}}$) from the branchings (2.64) is consistent with these expectations when $n = 6$.

The $\mathcal{N} = 4$ truncation has a natural interpretation in perturbative Type II string theory. There is a simple duality frame where the diagonal element $\text{diag}(I_4, -I_4)$ changes the sign on the RR sector and interchanges the RNS and NSR sectors; so the consistent truncation projects on to the common sector of Type IIA and Type IIB supergravity. The complete string theory orbifold includes twisted sectors as well. It is conveniently implemented by a flip of the GSO projection and is equivalent to T-duality between Type IIA and Type IIB string theory.

The embedding of the KK black hole into $\mathcal{N} = 8$ SUGRA is compatible with the truncation to $\mathcal{N} = 4$ SUGRA: the four field strengths on the skew-diagonal of the $\mathbf{28}$ are all contained in the $SU(4)_R \times SU(4)_{\text{matter}}$ subgroup of $SU(8)_R$ and therefore retained in the truncation to $\mathcal{N} = 4$ SUGRA. The embedding of the KK black hole in $\mathcal{N} = 8$ SUGRA therefore defines an embedding in $\mathcal{N} = 4$ SUGRA as well. The consistent truncation just removes fields that are not excited by the KK black hole in $\mathcal{N} = 8$ SUGRA.

The quadratic fluctuations around the KK black hole in $\mathcal{N} = 8$ SUGRA similarly project on to the $\mathcal{N} = 4$ setting. As discussed in section 2.4, the KK black hole in $\mathcal{N} = 8$ SUGRA breaks the global symmetry $SU(8)_R \to USp(8)$ and this symmetry breaking pattern greatly constrains the spectrum of fluctuations around the black hole. Moreover, the symmetry breaking pattern is largely preserved by the consistent truncation: the analogous breaking pattern in $\mathcal{N} = 4$ SUGRA is $SU(4)_R \times SU(4)_{\text{matter}} \to USp(4)_R \times USp(4)_{\text{matter}}$. For example, the entire KK block (with a graviton, a vector, and a scalar), identified as the $\mathbf{1}$ of $USp(8)$, is unchanged by the consistent truncation.

The 27 vector blocks (2.55-2.56), each with a vector coupled to a scalar, are perturbations of the 8×8 matrix of field strengths F_{AB} after its symplectic trace is removed. The branching (2.64) of the $\mathbf{28}$ under $SU(4)_R \times SU(4)_{\text{matter}}$ shows that 16 vector blocks are projected out by the truncation. None of these are affected by the symplectic trace so $27 - 16 = 11$ vector blocks remain in $\mathcal{N}=4$ SUGRA. Among the 38 scalars from the coset (2.65) with $n=6$ there is 1 coupled to gravity and 11 that couple to the vectors, so 26 minimally coupled scalars remain. They parametrize the coset

$$SU(1,1)\times\frac{SO(5,5)}{USp(4)\times USp(4)}\ . \tag{2.66}$$

The fermionic sector is simpler because the truncation removes exactly one half of the fermions. The retained fermions are essentially identical to those that are projected away, they differ at most in their chirality and the KK black holes is insensitive to this distinction. The quadratic fluctuations for the fermions in $\mathcal{N}=8$ SUGRA are 4 gravitino pairs (with each pair including two gravitini coupled to two Weyl fermions, a total of 32 degrees of freedom) and 24 gaugino pairs with Pauli couplings to the background field strength. In $\mathcal{N}=4$ SUGRA with 6 matter multiplets there are 4 gravitino pairs and 12 gaugino pairs.

There is a simple extension of these results to the case of $\mathcal{N}=4$ SUGRA with $n\neq 6$ matter multiplets. For this generalization, we recast the symmetry breaking by the field strengths that have been designated $\mathcal{N}=4$ matter as $SO(6)_{\rm matter}\to SO(5)_{\rm matter}$ using the equivalences $SU(4)=SO(6)$ and $USp(4)=SO(5)$ as Lie algebras. In this form the symmetry breaking just amounts to picking the direction of a vector on an S^5. We can equally consider any number n of matter fields and break the symmetry $SO(n)_{\rm matter}\to SO(n-1)_{\rm matter}$ by picking a vector on S^{n-1}. The only restriction is $n\geq 1$ in order to ensure that there is a direction to pick in the first place. This more general construction gives the scalar manifold

$$SU(1,1)\times\frac{SO(5,n-1)}{SO(5)\times SO(n-1)}\ . \tag{2.67}$$

In particular, it has $5n-4$ dimensions, each corresponding to a minimally coupled scalar field. The duality group read off from the numerator correctly indicates $n+5$ vector fields, not counting the one coupling to gravity. Each of these vector fields couples to a scalar field, as in (2.55-2.56).

The black hole attractor mechanism offers a perspective on the scalar coset (2.67). The attractor mechanism is usually formulated in the context of extremal black holes in $\mathcal{N}\geq 2$ supergravity where it determines the value of some of the scalars at the horizon in terms of black hole charges. Importantly, the attractor mechanism generally leaves other scalars undetermined. Such undetermined scalars can take any value, so they are moduli. The hyper-scalars in $\mathcal{N}=2$ BPS black hole backgrounds are well-known examples of black hole moduli.

In the case of extremal (but non-supersymmetric) black holes in $\mathcal{N}\geq 2$ supergravity the moduli space is determined by the centralizer remaining after extremization of the black hole potential over the full moduli space of the theory. The result for nonBPS black holes in $\mathcal{N}=4$ supergravity was obtained in [58] and agrees with (2.67). Our considerations generalize this result to a moduli space of non-extremal KK black holes. The exact masslessness of moduli is protected by the breaking of global symmetries so supersymmetry is not needed.

2.5.3 The $\mathcal{N}=2$ Truncation

Starting from $\mathcal{N}=4$ SUGRA with n $\mathcal{N}=4$ matter multiplets, there is a consistent truncation to $\mathcal{N}=2$ SUGRA with $n+1$ $\mathcal{N}=2$ vector multiplets that respects the KK black hole background. It is defined by keeping only fields that are even under the $SU(4)_R$ element $\mathrm{diag}(I_2,-I_2)$.

All fermions, both gravitini and gaugini are in the fundamental $\mathbf{4}$ of $SU(4)_R$ so the consistent truncation retains exactly 1/2 of them. In particular, the SUSY is reduced from $\mathcal{N}=4$ to $\mathcal{N}=2$. The bosons are either invariant under $SU(4)_R$ or they transform as an antisymmetric tensor $\mathbf{6}$. The branching rule $\mathbf{6}\to 2(1,1)\oplus(2,2)$ under $SU(4)_R\to SU(2)^2$ determines that its truncation retains only the 2 fields on the skew-diagonal of the antisymmetric 4×4 tensor.

The truncated theory has $2(2n+4)$ fermionic degrees of freedom and the same number of bosonic ones. We can implement the truncation directly on the $\mathcal{N}=4$ coset (2.65) and find that scalars of the truncated theory parametrize

$$\frac{SU(1,1)}{U(1)}\times\frac{SO(2,n)}{SO(2)\times SO(n)}\ . \tag{2.68}$$

This theory is known as the $ST(n)$ model. In the special case $n=2$ the $ST(2)$ model is the well-known STU model. This model has enhanced symmetry ensuring that its 3 complex scalar fields are equivalent and similarly that its 4 field strengths are equivalent. The STU model often appears as a subsector of more general $\mathcal{N}=2$ SUGRA theories, such as those defined by a cubic prepotential. These in turn arise as the low energy limit of string theory compactified on a Calabi-Yau manifold, so the STU model may capture some generic features of such theories.

The consistent truncation to the $ST(n)$ model in $\mathcal{N}=2$ SUGRA is compatible with the embedding of the KK black hole in $\mathcal{N}=8$ SUGRA. The embedding (2.40) in $\mathcal{N}=8$ excites precisely the field strengths on the skew-diagonal, breaking $SU(8)_R\to USp(8)$. As discussed in (2.5.2), they were retained by the truncation to $\mathcal{N}=4$ SUGRA. The further truncation of the antisymmetric representation to $\mathcal{N}=2$ SUGRA projects $\mathbf{6}\to 2(\mathbf{1},\mathbf{1})$ and so it specifically retains field strengths on the skew diagonal. Moreover, the gauge fields that are projected out are in the **2** of an $SU(2)$ so they are not coupled to other fields at quadratic order.

It can be shown that the $\mathcal{N} = 4$ embedding identifies the "dilaton" of the KK black hole with the scalar (as opposed to the pseudoscalar) in the coset $SU(1,1)/U(1)$. This part of the scalar coset is untouched by the truncation to $\mathcal{N} = 2$ SUGRA. Therefore, the truncation to $\mathcal{N} = 2$ does not remove any of the fields that are turned on in the background, nor any of those that couple to them at quadratic order. This shows that the consistent truncation to $\mathcal{N} = 2$ SUGRA, like other truncations considered in this section, removes only entire blocks of fluctuations: the fields that remain have the same couplings as they do in the $\mathcal{N} = 8$ context.

The breaking pattern determines the moduli space of scalars for the black hole background as

$$SU(1,1) \times \frac{SO(1,n-1)}{SO(n-1)} \ . \tag{2.69}$$

In particular this confirms that, among the $2n+2$ scalars of the $ST(n)$ model, exactly n are moduli and so are minimally coupled massless scalars.

2.5.4 More Comments on Consistent Truncations

The natural endpoint of the consistent truncations is $\mathcal{N} = 0$ SUGRA, *i.e.* the pure Kaluza-Klein theory (2.4). We constructed our embedding (2.40) into $\mathcal{N} = 8$ SUGRA so that the Kaluza-Klein black hole would remain a solution also to the full $\mathcal{N} = 8$ SUGRA. Thus we arranged that all the additional fields required by $\mathcal{N} = 8$ supersymmetry would be "unimportant", in the sense that they can be taken to vanish on the Kaluza-Klein black hole. It is therefore consistent to remove them again, and that is the content of the "truncation to $\mathcal{N} = 0$ SUGRA".

From this perspective, the truncations considered in this section are intermediate stages between $\mathcal{N} = 8$ and $\mathcal{N} = 0$ in that only some of the "unimportant" fields are included. For each value of $\mathcal{N} = 6, 4, 2$, the requirement that the Kaluza-Klein black hole is a solution largely determines the truncation. The resulting embedding of the STU model into $\mathcal{N} = 8$ SUGRA is very simple, and possibly simpler than others that appear in the literature, in that symmetries between fields in the STU model are manifest even without performing any electromagnetic duality.

Having analyzed the spectrum of fluctuations around Kaluza-Klein black holes in the context of SUGRA with $\mathcal{N} = 8, 6, 4, 2$ (and even $\mathcal{N} = 0$), it is natural to inquire about the situation for SUGRA with odd $\mathcal{N}$. Our embeddings in $\mathcal{N} = 6, 4, 2$ rely on the skew-diagonal nature of the embedding in $\mathcal{N} = 8$ so they do not have any generalizations to odd $\mathcal{N}$. This fact is vacuous for $\mathcal{N}=7$ SUGRA which automatically implies $\mathcal{N}=8$. Moreover, it is interesting that $\mathcal{N}=3,5$ SUGRA do not have any nonBPS branch at all: all extremal black holes in these theories must be BPS (they preserve supersymmetry) [58]. This may indicate that our examples exhaust a large class of nonBPS embeddings.

2.6 The General KK Black Hole in $\mathcal{N}=2$ SUGRA

In this section, we start afresh with an *arbitrary* solution to the $D=4$ Kaluza-Klein theory (2.4), such as the general Kaluza-Klein black hole (2.5-2.7). We embed this solution into $\mathcal{N}=2$ SUGRA with a *general cubic prepotential* and analyze the quadratic fluctuations around the background in this setting. Along the way we make additional assumptions that further decouple the fluctuations, and ultimately specialize to a constant background dilaton and $ST(n)$ prepotential. In this case the final results of the direct computations will be consistent with those found in section 2.5.3, by truncation from $\mathcal{N}=8$ SUGRA, and summarized in section 2.4.3.

The setup in this section complements our discussion of the Kaluza-Klein black hole in $\mathcal{N}=8$ SUGRA and its truncations to $\mathcal{N}<8$ SUGRA. Here we do not assume vanishing background dilaton $\Phi^{(\mathrm{KK})}=0$ from the outset and we consider more general theories.

2.6.1 $\mathcal{N}=2$ SUGRA with Cubic Prepotential

We first introduce $\mathcal{N}=2$ SUGRA. We allow for matter in the form of n_V $\mathcal{N}=2$ vector multiplets with couplings encoded in a cubic prepotential

$$F=\frac{1}{\kappa^2}\frac{d_{ijk}X^iX^jX^k}{X^0}\ , \tag{2.70}$$

where d_{ijk} is totally symmetric. We also include n_H $\mathcal{N}=2$ hypermultiplets. The theory is described by the $\mathcal{N}=2$ SUGRA Lagrangian

$$\begin{aligned}
e^{-1}\mathcal{L}^{(\mathcal{N}=2)} \;=\;& \kappa^{-2}\left(\frac{R}{2}-\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho\right)-g_{\alpha\bar{\beta}}\partial^\mu z^\alpha\partial_\mu z^{\bar{\beta}}-\frac{1}{2}h_{uv}\partial_\mu q^u\partial^\mu q^v\\
&+\left(-\frac{1}{4}i\mathcal{N}_{IJ}F^{+I}_{\mu\nu}F^{+\mu\nu J}+F^{-I}_{\mu\nu}\,\mathrm{Im}\,\mathcal{N}_{IJ}Q^{\mu\nu-J}\right.\\
&\left.-\frac{1}{4}g_{\alpha\bar{\beta}}\bar{\chi}^\alpha_i\not{D}\chi^{i\bar{\beta}}-\bar{\zeta}_A\not{D}\zeta^A+\frac{1}{2}g_{\alpha\bar{\beta}}\bar{\psi}_{i\mu}\not{\partial}z^\alpha\gamma^\mu\chi^{i\bar{\beta}}+\text{h.c.}\right)\ ,
\end{aligned} \tag{2.71}$$

where

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu}\right), \text{ with } \widetilde{F}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \ , \tag{2.72}$$

$$\begin{aligned} Q^{\mu\nu-J} \equiv\ & \overline{\nabla}_{\bar{\alpha}}\bar{X}^J\left(\frac{1}{8}g^{\beta\bar{\alpha}}C_{\beta\gamma\delta}\bar{\chi}^{\gamma}_i\gamma^{\mu\nu}\chi^{\delta}_j\epsilon^{ij} + \bar{\chi}^{\bar{\alpha}i}\gamma^{\mu}\psi^{\nu j}\epsilon^{ij}\right) \\ & + X^J\left(\bar{\psi}^{\mu}_i\psi^{\nu}_j\epsilon^{ij} + \frac{1}{2}\kappa^2\bar{\zeta}^A\gamma^{\mu\nu}\zeta^B C_{AB}\right) \ . \end{aligned} \tag{2.73}$$

We follow the notations and conventions from [59]. In particular, the $\chi^{\alpha}_i = P_L\chi^{\alpha}_i, \alpha = 1, \ldots, n_V$ denote the physical gaugini and $\zeta^A = P_L\zeta^A, A = 1, \ldots, 2\,n_H$ denote the hyperfermions. The Kähler covariant derivatives are

$$\nabla_{\alpha}X^I = \left(\partial_{\alpha} + \frac{1}{2}\kappa^2\partial_{\alpha}\mathcal{K}\right)X^I \ , \tag{2.74}$$

$$\overline{\nabla}_{\bar{\alpha}}X^I = \left(\partial_{\bar{\alpha}} - \frac{1}{2}\kappa^2\partial_{\bar{\alpha}}\mathcal{K}\right)X^I \ , \tag{2.75}$$

where the Kähler potential $\mathcal{K}$

$$e^{-\kappa^2\mathcal{K}} = -i(X^I\bar{F}_I - F_I\bar{X}^I) \ , \tag{2.76}$$

with $F_I = \partial_I F = \frac{\partial F}{\partial X^I}$.

The projective coordinates X^I (with $I = 0, \ldots, n_V$) are related to physical coordinates as $z^i = X^i/X^0$ (with $i = 1, \ldots, n_V$). We split the complex scalars z^i into real and imaginary parts

$$z^i = x^i - iy^i \ . \tag{2.77}$$

With cubic prepotential (2.70) we have

$$g_{i\bar{j}} = \partial_I\partial_{\bar{J}}\mathcal{K} = \kappa^{-2}\left(-\frac{3d_{ij}}{2d} + \frac{9d_id_j}{4d^2}\right) \ , \tag{2.78}$$

where we define

$$d_{ij} \equiv d_{ijk}y^k \ , \qquad d_i \equiv d_{ijk}y^jy^k \ , \qquad d \equiv d_{ijk}y^iy^jy^k \ . \tag{2.79}$$

Finally, the scalar-vector coupling are encoded in

$$\mathcal{N}_{IJ} = \mu_{IJ} + i\nu_{IJ} \ , \tag{2.80}$$

with

$$\mu_{IJ} = \kappa^{-2} \begin{pmatrix} 2d_{ijk}x^i x^j x^k & -3d_{ijk}x^j x^k \\ -3d_{ijk}x^j x^k & 6d_{ijk}x^k \end{pmatrix} , \tag{2.81}$$

and

$$\nu_{IJ} = \kappa^{-2} \begin{pmatrix} -d + 6d_{\ell m}x^\ell x^m - \frac{9}{d}(d_\ell x^\ell)^2 & \frac{9}{d}(d_\ell x^\ell)d_i - 6d_{i\ell}x^\ell \\ \frac{9}{d}(d_\ell x^\ell)d_i - 6d_{i\ell}x^\ell & 6d_{ij} - \frac{9}{d}(d_i d_j) \end{pmatrix} . \tag{2.82}$$

2.6.2 The Embedding into $\mathcal{N} = 2$ SUGRA

We want to embed our seed solution into $\mathcal{N} = 2$ SUGRA. The starting point is a solution to the equations of motion (2.5, 2.6, 2.7) of the Kaluza-Klein theory. We denote the corresponding fields $g_{\mu\nu}^{(\mathrm{KK})}$, $F_{\mu\nu}^{(\mathrm{KK})}$ and $\Phi^{(\mathrm{KK})}$. The fields of $\mathcal{N} = 2$ SUGRA are then defined to be

$$\begin{aligned}
&g_{\mu\nu}^{(\mathrm{SUGRA})} = g_{\mu\nu}^{(\mathrm{KK})} , \\
&F_{\mu\nu}^0 = \frac{1}{\sqrt{2}} F_{\mu\nu}^{(\mathrm{KK})}, \quad F_{\mu\nu}^i = 0, \quad \text{for } 1 \le i \le n_V \\
&x^i = 0, \quad \text{for } 1 \le i \le n_V , \\
&y^i = c^i y_0, \text{ with } y_0 = \frac{\exp\left(-2\Phi^{(\mathrm{KK})}/\sqrt{3}\right)}{(d_{ijk}c^i c^j c^k)^{1/3}} , \\
&(\text{All other bosonic fields in } \mathcal{N} = 2 \text{ SUGRA}) = 0 , \\
&(\text{All fermionic fields in } \mathcal{N} = 2 \text{ SUGRA}) = 0 .
\end{aligned} \tag{2.83}$$

This field configuration solves the equations of motion of $\mathcal{N} = 2$ SUGRA for any seed solution to the Kaluza-Klein theory. In the following, we will often declutter formulae by omitting the superscript "KK" when referring to fields in the seed solution.

The embedding (2.83) is really a family of embeddings parameterized by the n_V constants c^i (with $i = 1, \ldots, n_V$). They are projective coordinates on the moduli space parametrized by the n_V scalar fields y_i with the constraint

$$d = d_{ijk}y^i y^j y^k = \exp\left(-2\sqrt{3}\Phi^{(\mathrm{KK})}\right) . \tag{2.84}$$

In the special case of the non-rotating Kaluza-Klein black hole with $P = Q$, we have $\Phi^{(\mathrm{KK})} =$

0 and so the constraint is $d = 1$. More generally, d is the composite field defined through the constraints (2.79) and related to the Kaluza-Klein dilaton by (2.84).

2.6.3 Decoupled Fluctuations: General Case

The Lagrangian for quadratic fluctuations around a bosonic background always decouples into a bosonic sector and fermionic sector,

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{bosons}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{fermions}} \; . \tag{2.85}$$

With the above embedding into $\mathcal{N} = 2$, each sector further decouples into several blocks.

The bosonic sector decomposes as the sum of three blocks

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{bosons}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vectors}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{scalars}} \; . \tag{2.86}$$

The "gravity block" $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}}$ consists of the graviton $\delta g_{\mu\nu}$, the gauge field δA^0_μ, and the n_V real scalars δy^i:

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}} = \frac{1}{\sqrt{-g}}\delta^2\left[\sqrt{-g}\left(\frac{R}{2\kappa^2} - g_{ij}\partial_\mu y^i\partial^\mu y^j + \frac{d}{4\kappa^2}F^0_{\mu\nu}F^{\mu\nu 0}\right)\right] \; . \tag{2.87}$$

Generically, the fields $\delta g_{\mu\nu}, \delta A^0_\mu$ and δy^i all mix together. This block can nonetheless be further decoupled with simplifying assumptions, as we will discuss later.

The block $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vectors}}$ consists of the n_V vector fields δA^i_μ and the n_V real pseudoscalars δx^i:

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vectors}} = g_{ij}\left(-\partial_\mu\delta x^i\partial^\mu\delta x^j - \frac{1}{2}dF_{\mu\nu}F^{\mu\nu}\delta x^i\delta x^j + \sqrt{2}dF_{\mu\nu}\delta x^i\delta F^{\mu\nu j} - d\delta F^i_{\mu\nu}\delta F^{\mu\nu j}\right) \; .$$

The Kähler metric g_{ij} can be diagonalized and we obtain n_V identical decoupled copies, that we call "vector block", each consisting in one vector field and one real scalar. Denoting the fluctuating field $f_{\mu\nu}$, one such copy has the Lagrangian

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vector}} = -\frac{1}{2}\partial_\mu x\partial^\mu x - \frac{d}{4}F_{\mu\nu}F^{\mu\nu}x^2 + \frac{d}{2}F_{\mu\nu}f^{\mu\nu}x - \frac{d}{4}f_{\mu\nu}f^{\mu\nu} \; , \tag{2.88}$$

using conventional normalizations for the scalar fields.

The last bosonic block contains the hyperbosons:

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{scalars}} = -\frac{1}{2}h_{uv}\partial_\mu\delta q^u\partial^\mu\delta q^v \; . \tag{2.89}$$

The quaternionic Kähler metric h_{uv} is trivial on the background. Hence, this block decouples at quadratic order into $4\,n_H$ independent minimally coupled massless scalars.

We next turn to the fermions. The Lagrangian (2.71) is the sum of the decoupled Lagrangians

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{fermions}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermions}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino-gaugino}} \,. \tag{2.90}$$

The hyperfermions consist of n_H identical copies, that we call "hyperfermion block", each containing two hyperfermions. For any two such fermions we can take $C_{AB} = \epsilon_{AB}$ with $A, B = 1, 2$. The resulting Lagrangian is

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermion}} \;=\; -2\bar{\zeta}_A \not{D}\zeta^A + \left(\frac{\kappa^2}{2}F^{-I}_{\mu\nu}\nu_{IJ}X^J\bar{\zeta}^A\gamma^{\mu\nu}\zeta^B\epsilon_{AB} + \text{h.c.}\right) \,. \tag{2.91}$$

In our background, we use (2.83, 2.82) to find

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermion}} \;=\; -2\bar{\zeta}_A \not{D}\zeta^A - \left(\frac{d^{\frac{1}{2}}}{8}F^{-}_{\mu\nu}\bar{\zeta}^A\gamma^{\mu\nu}\zeta^B\epsilon_{AB} + \text{h.c.}\right) \,. \tag{2.92}$$

We used the T-gauge [59] to fix the projective coordinates X^I resulting in $X^0 = (8d)^{-1/2}$.

The "gravitino-gaugino block" contains two gravitini and n_V gaugini and has Lagrangian

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino-gaugino}} \;=\; & -\frac{1}{\kappa^2}\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho + \Bigg(-\frac{d^{\frac{1}{2}}}{4\kappa^2}F^{-}_{\mu\nu}\bar{\psi}^\mu_i\psi^\nu_j\epsilon^{ij} \\ & + \frac{9}{256\kappa^2 d^{\frac{3}{2}}}F^{-}_{\mu\nu}d_{\bar{\alpha}}g^{\beta\bar{\alpha}}d_{\beta\gamma\delta}\bar{\chi}^\gamma_i\gamma^{\mu\nu}\chi^\delta_j\epsilon^{ij} - \frac{3i}{8\kappa^2 d^{\frac{1}{2}}}F^{-}_{\mu\nu}d_{\bar{\alpha}}\bar{\chi}^{\bar{\alpha}i}\gamma^\mu\psi^{\nu j}\epsilon^{ij} \\ & - \frac{1}{4}g_{\alpha\bar{\beta}}\bar{\chi}^\alpha_i\not{D}\chi^{i\bar{\beta}} + \frac{1}{2}g_{\alpha\bar{\beta}}\bar{\psi}_{ia}\not{\partial}z^\alpha\gamma^a\chi^{i\bar{\beta}} + \text{h.c.}\Bigg) \,. \end{aligned} \tag{2.93}$$

Generally, all the gravitini and gaugini couple nontrivially but they can be further decoupled in simpler cases, as we will discuss later.

Summarizing so far: given any Kaluza-Klein solution, the embedding (2.83) provides solutions of $\mathcal{N} = 2$ SUGRA. We have expanded the $\mathcal{N} = 2$ Lagrangian around this background to quadratic order and observed that the fluctuations can be decoupled as shown in Table 2.5.

Degeneracy	Multiplet	Block content	Lagrangian
1	Gravity block	1 graviton, 1 vector, n_V scalars	(2.87)
n_V	Vector block	1 vector and 1 (pseudo)scalar	(2.88)
$4n_H$	Scalar block	1 real scalar	(2.89)
1	Gravitino-gaugino block	2 gravitini and $2\,n_V$ gaugini	(2.93)
n_H	Hyperfermion block	2 hyperfermions	(2.92)

Table 2.5: Decoupled quadratic fluctuations in $\mathcal{N}=2$ SUGRA around a general KK black hole.

These results are reminiscent of the analogous structure for $\mathcal{N}=8$ SUGRA, summarized in (2.40). However, with the more general assumptions made here, there are more scalars in the $\mathcal{N}=2$ gravity block than in the analogous $\mathcal{N}=8$ KK block and these additional scalars do not generally decouple from gravity. Similarly, the $\mathcal{N}=2$ gravitino-gaugino block here includes more gaugini than the analogous $\mathcal{N}=8$ gravitino block.

2.6.4 Decoupled Fluctuations: Constant Dilaton

So far, we have been completely general about the underlying Kaluza-Klein solution. In this section, we further decouple the quadratic fluctuations by assuming that the scalar fields of $\mathcal{N}=2$ SUGRA are constant

$$y^i = \text{constant},\ \ i=1,...,n_V\ . \tag{2.94}$$

From the embedding (2.83), this is equivalent to taking the Kaluza-Klein dilaton to vanish

$$\Phi^{(\mathrm{KK})} = 0\ , \tag{2.95}$$

since we can always rescale the field strengths to arrange for $d = d_{ijk}y^iy^jy^k = 1$. As noted previously, this is satisfied by the non-rotating Kaluza-Klein black hole with $P=Q$. This is the simplified background that we already studied in $\mathcal{N}=8$ SUGRA, but it is embedded here in $\mathcal{N}=2$ SUGRA with arbitrary prepotential. As in the $\mathcal{N}=8$ case, we will use that the background satisfies

$$R = 0\ , \qquad F_{\mu\nu}F^{\mu\nu} = 0 \tag{2.96}$$

to decouple further the quadratic fluctuations.

- *Gravity*

The gravity block decouples as

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravity}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}} \ , \tag{2.97}$$

where $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}}$ is the "KK block", consisting of the graviton $\delta g_{\mu\nu}$, the graviphoton δA^0_μ and the center-of-mass scalar $\delta y'^1$. $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}}$ denotes $n_V - 1$ free massless scalars $\delta y'^i, \ i = 2, \ldots n_V$. This decoupling is obtained by center-of-mass diagonalization: the $\delta y'^i$ are linear combinations of δy^i such that $\delta y'^1$ is precisely the combination that couples to the graviton and graviphoton at quadratic order. Then, the "relative scalars" $\delta y'^i, \ i = 2, \ldots, n_V$ are minimally coupled to the background

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}} \ = \ -\frac{2}{\kappa^2}\partial_\mu \delta y'^i \partial^\mu \delta y'^i \quad (\text{for } i = 2, \ldots, n_V) \ , \tag{2.98}$$

The center-of-mass Lagrangian turns out to be exactly the same as the $\mathcal{N} = 8$ KK block (2.53)

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}} \ , \tag{2.99}$$

with the identifications

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{2}}\left(\delta g_{\mu\nu} - \frac{1}{4} g_{\mu\nu} g^{\rho\sigma}\delta g_{\rho\sigma}\right) \ , \quad h = \frac{1}{\sqrt{2}} g^{\rho\sigma}\delta g_{\rho\sigma} \ , \tag{2.100}$$

$$a_\mu = \sqrt{2}\delta A^0_\mu \ , \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \ , \tag{2.101}$$

$$\phi = \delta y'^1 = -\frac{\sqrt{3} d_i}{2d}\delta y^i = \delta\Phi \ . \tag{2.102}$$

The equality between $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{KK}}$ and $\delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{KK}}$ is expected because the KK block is the same for any $\mathcal{N} = 2$ SUGRA and in particular for the $\mathcal{N} = 2$ truncations of $\mathcal{N} = 8$ SUGRA.

The $n_V - 1$ minimally coupled massless scalars $\delta y'^i, i = 2, \ldots, n_V$ parameterize flat directions in the moduli space, at least at quadratic order. In important situations with higher symmetry, including homogeneous spaces constructed as coset manifolds, it can be shown that these $n_V - 1$ directions are exactly flat at all orders. This implies that, in particular, these models are stable [60, 61]. In such situations the "relative" coordinates $\delta y'^i$ are Goldstone bosons parameterizing symmetries of the theories.

- *Vector block*

Using the fact that $F_{\mu\nu}F^{\mu\nu} = 0$, the vector block becomes

$$e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vector}} = -\frac{1}{2}\partial_\mu x\partial^\mu x + \frac{1}{2}F_{\mu\nu}f^{\mu\nu}x - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} \ . \tag{2.103}$$

Again, we find that $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{vector}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{vector}}$ after proper normalization of the field strength.

- *Scalar block*

  The Lagrangian for hyperbosons $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{scalars}}$ consists of $4n_H$ minimally coupled scalars. In addition, the center-of-mass diagonalization has brought $n_V - 1$ minimally coupled "relative" scalars $\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{relative}}$. This gives a total of $n_V + 4n_H - 1$ minimally coupled scalars.

We now turn to fermions. The interactions between gravitini and gaugini simplify greatly when scalars are constant. However, they still depend on the prepotential through the structure constants $d_{\alpha\beta\gamma}$. The fermionic fluctuations in $\mathcal{N} = 2$ SUGRA are therefore qualitatively different from the bosonic fluctuations which, as we just saw, reduce to the form found in $\mathcal{N} = 8$ SUGRA.

For fermions we need to further specialize and study the $ST(n)$ model. This model already appeared in section 2.5.3, as a truncation of $\mathcal{N} = 8$ SUGRA to $\mathcal{N} = 2$. Presently, we introduce it as the model with $n_V = n + 1$ vector multiplets and prepotential

$$F = \frac{1}{\kappa^2}\frac{X^1(X^2X^2 - X^\alpha X^\alpha)}{2X^0} \qquad (\alpha = 3, \ldots, n_V) \ . \tag{2.104}$$

We take the background scalars

$$y^1 = 1, \quad y^2 = \sqrt{2}, \quad y^\alpha = 0 \quad (\alpha = 3, \ldots, n_V) \ , \tag{2.105}$$

such that the normalization is $d = 1$ and therefore $\Phi^{(\text{KK})} = 0$. As mentioned already in section 2.5.3, this model generalizes the STU model which is equivalent to $ST(2)$.

- *Gravitino-gaugino block*

  The Lagrangian for the gravitino-gaugino block decouples as

$$\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino-gaugino}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino}} + \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} \ , \tag{2.106}$$

  after using center-of-mass diagonalization. We call χ'^{i1} the center-of-mass gaugini, *i.e.*

the gaugini that couples to the gravitini. More precisely, we define

$$
\begin{aligned}
&\chi'^{i1} = \frac{1}{4}\left(\frac{\sqrt{3}}{3}\chi^{i1} + \frac{\sqrt{6}}{3}\chi^{i2}\right) , \quad \chi'^{i2} = \frac{1}{4}\left(\frac{\sqrt{6}}{3}\chi^{i1} - \frac{\sqrt{3}}{3}\chi^{i2}\right) , \\
&\chi'^{i\alpha} = \frac{1}{4}\chi^{i\alpha} \quad \text{for } \alpha = 3, \ldots, n_V . \qquad (2.107)
\end{aligned}
$$

We find a center-of-mass multiplet that we call "gravitino block"

$$
\begin{aligned}
e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino}} \;=\;& -\frac{1}{\kappa^2}\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho + \frac{1}{\kappa^2}\Big(-\bar{\chi}'^1_i \not{D}\chi'^{i1} - \frac{1}{4}\bar{\psi}^\mu_i F^-_{\mu\nu}\psi^\nu_j\epsilon^{ij} \\
&+\frac{1}{4}\bar{\chi}'^1_i F^-_{\mu\nu}\gamma^{\mu\nu}\chi'^1_j\epsilon^{ij} - \frac{\sqrt{3}i}{2}\bar{\chi}'^{i1}\gamma^\mu F^-_{\mu\nu}\psi^{\nu j}\epsilon^{ij} + \text{h.c.}\Big) , \qquad (2.108)
\end{aligned}
$$

This Lagrangian couples the two gravitini to two center-of-mass gaugini. The "relative" multiplets are $n_V - 1$ identical copies of a "gaugino block"

$$
e^{-1}\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} \;=\; -\frac{2}{\kappa^2}\bar{\chi}'^\alpha_i \not{D}\chi'^i_\alpha - \left(\frac{1}{8\kappa^2}\bar{\chi}'^\alpha_i F^-_{\mu\nu}\gamma^{\mu\nu}\chi'_{j\alpha}\epsilon^{ij} + \text{h.c.}\right) , \qquad (2.109)
$$

where $\alpha = 2, \ldots, n_V$.

- *Hyperfermion block*

  The hyperfermion Lagrangian is given in (2.92). We notice that

$$
\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{hyperfermion}} = \delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} , \qquad (2.110)
$$

  The fluctuations of "relative" gaugini are therefore the same as the fluctuations of hyperfermions. Therefore, we call both of them "gaugino block".

The Lagrangians (2.108) and (2.109) are written in terms of Weyl fermions. If we rewrite them with Majorana fermions, we find that

$$
\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gravitino}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gravitino}} , \qquad (2.111)
$$

$$
\delta^2\mathcal{L}^{(\mathcal{N}=2)}_{\text{gaugino}} = \delta^2\mathcal{L}^{(\mathcal{N}=8)}_{\text{gaugino}} , \qquad (2.112)
$$

where the right-hand sides were defined in (2.61) and (2.62). The agreement between our explicit computations of the fermionic blocks for the $ST(n)$ model in $\mathcal{N} = 2$ SUGRA and the analogous results in $\mathcal{N} = 8$ SUGRA is an important consistency check on the truncations discussed in section 2.5.3. This also explains the agreement (2.110) between fermionic fluctuations that are in different $\mathcal{N} = 2$ multiplets. $\mathcal{N} = 2$ gaugini and hyperfermions becomes equivalent when embedded into some larger structure, ultimately furnished by $\mathcal{N} = 8$ SUGRA.

In summary, taking the dilaton to be constant has further decoupled the fluctuations in $\mathcal{N} = 2$ SUGRA around the KK background, as shown in Table 2.6. For bosons, we recover the results of $\mathcal{N} = 8$ SUGRA as expected, although we are more general here since we allow for an arbitrary prepotential. For fermions, we have to specialize to the $ST(n)$ model to be able to further decouple the fluctuations. The resulting fermionic fluctuations also reproduce the fluctuations of $\mathcal{N} = 8$ SUGRA.

Degeneracy	Multiplet	Block content	Lagrangian
1	KK block	1 graviton, 1 vector, 1 scalar	(2.99)
n_V	Vector block	1 vector and 1 (pseudo)scalar	(2.103)
$n_V + 4n_H - 1$	Scalar block	1 real scalar	(2.89, 2.98)
1	Gravitino block	2 gravitini and 2 gaugini	(2.108)
$n_V + n_H - 1$	Gaugino block	2 spin 1/2 fermions	(2.92, 2.109)

Table 2.6: Decoupled fluctuations in $\mathcal{N} = 2$ SUGRA around the KK black hole with constant dilaton. The decoupling in the bosonic sector holds for an arbitrary prepotential. The fermionic sector has been further decoupled by specializing to the $ST(n)$ model.

2.7 Logarithmic Corrections to Black Hole Entropy

The logarithmic correction controlled by the size of the horizon in Planck units is computed by the functional determinant of the quadratic fluctuations of light fields around the background solution. The arguments establishing this claim for non-extremal black holes are made carefully in [45]. In this section we give a brief summary of the steps needed to extract the logarithm using the heat kernel approach. It follows the discussion in [17] and we refer to [49] for background literature on technical aspects.

Naturally, we apply the procedure to the Kaluza-Klein black holes on the nonBPS branch. This gives our final results for the coefficients of the logarithmic corrections, summarized in Table 2.8.

2.7.1 General Framework: Heat Kernel Expansion

In Euclidean signature, the effective action W for the quadratic fluctuations takes the schematic form

$$e^{-W} = \int \mathcal{D}\phi \exp\left(-\int d^4x\sqrt{g}\,\phi_n \Lambda^n_m \phi^m\right) = \det{}^{\mp 1/2}\Lambda \ , \tag{2.113}$$

where Λ is a second order differential operator that characterizes the background solution, and ϕ_n embodies the entire field content of the theory. The sign $\mp$ is $-$ for bosons and $+$ for fermions. The formal determinant of Λ diverges and a canonical way to regulate it is by introducing a heat kernel: if $\{\lambda_i\}$ is the set of eigenvalues of Λ, then the heat kernel $D(s)$ is defined by

$$D(s) = \mathrm{Tr}\, e^{-s\Lambda} = \sum_i e^{-s\lambda_i} \ , \tag{2.114}$$

and the effective action becomes

$$W = \mp\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s} D(s) \ . \tag{2.115}$$

Here ϵ is an ultraviolet cutoff, which is typically controlled by the Planck length, *i.e.* $\epsilon \sim \ell_P^2 \sim G$.

In our setting it is sufficient to focus on the contribution of massless fields in the two derivative theory. For this part of the spectrum, the scale of the eigenvalues λ_i is set by the background size which in our case is identified with the size of the black hole horizon, denoted by A_H. The integral (2.115) is therefore dominated by the integration range $\epsilon \ll s \ll A_H$, and there is a logarithmic contribution

$$\int_\epsilon^\infty \frac{ds}{s} D(s) = \cdots + C_{\text{local}} \log(A_H/G) + \cdots \ . \tag{2.116}$$

with coefficient denoted by C_{local}. This term comes from the constant term in the Laurent expansion of the heat kernel $D(s)$. Introducing the heat kernel density $K(x,x;s)$ which satisfies

$$D(s) = \int d^4x\sqrt{g}\, K(x,x;s) \ , \tag{2.117}$$

it is customary to cast the perturbative expansion in s as

$$K(x,x;s) = \sum_{n=0}^\infty s^{n-2} a_{2n}(x) \ , \tag{2.118}$$

and we identify

$$C_{\text{local}} = \int d^4x \sqrt{g}\, a_4(x) \ . \tag{2.119}$$

The functions $\{a_{2n}(x)\}$ are known as the Seeley-DeWitt coefficients. The logarithmic term that we need is controlled by $a_4(x)$. The omitted terms denoted by ellipses in (2.116) are captured by the other Seeley-DeWitt coefficients. For example, the term $a_0(x)$ induces a cosmological constant at one-loop and the term $a_2(x)$ renormalizes Newton's constant.

There is a systematic way to evaluate the Seeley-DeWitt coefficients in terms of the background fields and covariant derivatives appearing in the operator Λ [49]. The procedure assumes that the quadratic fluctuations can be cast in the form

$$-\Lambda^n_m = (\Box) I^n_m + 2(\omega^\mu D_\mu)^n_m + P^n_m \ . \tag{2.120}$$

Here, I^n_m is the identity matrix in the space of fields, ω^μ and P are matrices constructed from the background fields, and $\Box = D_\mu D^\mu$. From this data, the Seeley-DeWitt coefficient $a_4(x)$ is given by the expression

$$(4\pi)^2 a_4(x) = \mathrm{Tr}\left[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{360}(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu})\right] \ , \tag{2.121}$$

where

$$E = P - \omega^\mu\omega_\mu - (D^\mu\omega_\mu) \ , \qquad \Omega_{\mu\nu} = [D_\mu + \omega_\mu, D_\nu + \omega_\nu] \ . \tag{2.122}$$

This is the advantage of the heat kernel approach: after explicitly expanding the action around the background to second order, we have a straightforward formula to compute the Seeley-DeWitt coefficients from Λ (2.120).

The preceding discussion is based on the operator Λ (2.120) that is second order in derivatives. For fermions, the quadratic fluctuations are described by a first order operator H so the discussion must be modified slightly. We express the quadratic Lagrangian as

$$\delta^2\mathcal{L} = \bar{\Psi} H \Psi \ . \tag{2.123}$$

Following the conventions in [17], we always cast the quadratic fluctuations for the fermions in terms of Majorana spinors. The one-loop action is obtained by applying heat kernel techniques to the operator $H^\dagger H$ and using

$$\log\det H = \frac{1}{2}\log\det H^\dagger H \ . \tag{2.124}$$

Fermi-Dirac statistics also gives an additional minus sign. Thus, the fermionic contribution is obtained by multiplying (2.121) with an additional factor of $-1/2$.

2.7.2 Local Contributions

It is conceptually straightforward to compute $a_4(x)$ via (2.121). However, it can be cumbersome to decompose the differential operators, write them in the form (2.120) and compute their traces. The main complication is that our matter content is not always minimally coupled, as emphasized in sections 2.4 and 2.6.

To overcome these technical challenges we automated the computations using Mathematica with the symbolic tensor manipulation package `xAct`[6]. In particular, we used the subpackage `xPert` [62] to expand the bosonic Lagrangian to second order. We created our own package for treatment of Euclidean spinors. The computation proceeds as follows:

1. Expand the Lagrangian to second order.
2. Gauge-fix and identify the appropriate ghosts.
3. Reorganize the fluctuation operator Λ_m^n and extract the operators ω_μ and P from (2.120).
4. Compute the Seeley-DeWitt coefficient $a_4(x)$ using formula (2.121).
5. Simplify $a_4(x)$ using the background equations of motion, tensor and gamma matrix identities.

The results of the expansion to second order with `xPert` match with the bosonic Lagrangians summarized in Table 2.3. In Appendix A we elaborate on the intermediate steps and record the traces of E and $\Omega_{\mu\nu}$ for each of the blocks encountered in our discussion.

A priori, the Seeley-DeWitt coefficient $a_4(x)$ is a functional of both the geometry and the matter fields. The fact that the dilaton $\Phi^{(\mathrm{KK})}$ is constant on our background simplifies the situation greatly. By using the equations of motion, $a_4(x)$ can be recast as a functional of the geometry alone. We list the equations that we use to simplify $a_4(x)$ explicitly in Appendix A.

As a result, for our background, the Seeley-DeWitt coefficient at four derivative order can be arranged in the canonical form

$$a_4(x) = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 \ , \tag{2.125}$$

[6] www.xact.es

where a and c are constants governed by the couplings and field content of the theory and the curvature invariants are defined in (A.3) and (A.4). The values of c and a are summarized in Tables 2.7 and 2.8.

Multiplet \ Properties	Content	d.o.f.	c	a	$c-a$
Minimal boson	1 real scalar	1	$\frac{1}{120}$	$\frac{1}{360}$	$\frac{1}{180}$
Gaugino block	2 gaugini	4	$\frac{13}{960}$	$-\frac{17}{2880}$	$\frac{7}{360}$
Vector block	1 vector and 1 (pseudo)scalar	3	$\frac{1}{40}$	$\frac{11}{120}$	$-\frac{1}{15}$
Gravitino block	2 gravitini and 2 gaugini	8	$-\frac{347}{480}$	$-\frac{137}{1440}$	$-\frac{113}{180}$
KK block	1 graviton, 1 vector, 1 scalar	5	$\frac{37}{24}$	$\frac{31}{72}$	$\frac{10}{9}$

Table 2.7: Contributions to $a_4(x)$ decomposed in the multiplets that are natural to the KK black hole.

Multiplet / Theory	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=4$	$\mathcal{N}=2$	$\mathcal{N}=0$
KK block	1	1	1	1	1
Gravitino block	4	3	2	1	0
Vector block	27	15	$n+5$	n_V	0
Gaugino block	24	10	$2n$	n_V+n_H-1	0
Scalar block	42	14	$5n-4$	n_V+4n_H-1	0
a	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{32}(22+3n)$	$\frac{1}{192}(65+17n_V+n_H)$	$\frac{31}{72}$
c	0	0	$\frac{3}{32}(2+n)$	$\frac{3}{64}(17+n_V+n_H)$	$\frac{37}{24}$

Table 2.8: The degeneracy of multiplets in the spectrum of quadratic fluctuations around the KK black hole embedded in to various theories, and their respective values of the c and a coefficients defined in (2.125). For $\mathcal{N}=4$, the integer n is the number of $\mathcal{N}=4$ matter multiplets. For $\mathcal{N}=2$, the recorded values of c and a for the gravitino and the gaugino blocks were only established for $ST(n_V-1)$ models.

It is worth making a few remarks.

1. The value of $c-a$ in each case is independent of the couplings of the theory. In other words, $c-a$ can be reproduced by an equal number of minimally coupled fields on the same black hole background. This property is due to the fact that none of the non-minimal couplings appearing in our blocks involve the Riemann tensor $R_{\mu\nu\rho\sigma}$. Therefore, the coefficient of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is insensitive to the non-trivial couplings.

2. The values of c for blocks recorded in Table 2.7 do not have any obvious regularity, they are not suggestive of any cancellations. The vanishing of the c-anomaly for the $\mathcal{N}=6$ and $\mathcal{N}=8$ theories, exhibited in Table 2.8, seems therefore rather miraculous. Somehow these embeddings with large supersymmetry have special properties that are not shared by those with lower supersymmetry.

2.7.3 Quantum Corrections to Black Hole Entropy

The logarithmic terms in the one-loop effective action of the massless modes correct the entropy of the black hole as

$$\delta S_{\mathrm{BH}} = \frac{1}{2}(C_{\mathrm{local}} + C_{\mathrm{zm}}) \log \frac{A_H}{G} \ . \tag{2.126}$$

In this subsection we gather our results and evaluate the quantum contribution for the Kaluza-Klein black hole.

The local contribution is given by the integrated form of the Seeley-DeWitt coefficient $a_4(x)$:

$$C_{\mathrm{local}} = \frac{c}{16\pi^2}\int \sqrt{g}\, d^4x\, W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2}\int \sqrt{g}\, d^4x\, E_4 \ . \tag{2.127}$$

The second term is essentially the Euler characteristic

$$\chi = \frac{1}{32\pi^2}\int d^4x \sqrt{g}\, E_4 = 2 \ , \tag{2.128}$$

for any non-extremal black hole. It is a topological invariant so it does not depend on black hole parameters. In contrast, the first integral in (2.127) depends sensitively on the details of the black hole background. Using the KK black hole presented in section 2.2 with $J=0$ and $P=Q$ we find

$$\frac{1}{16\pi^2}\int d^4x \sqrt{g}\, W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = 4 + \frac{8}{5\,\xi(1+\xi)} \ , \tag{2.129}$$

where $\xi \geq 0$ is a dimensionless parameter related to the black hole parameters as

$$\frac{Q}{GM} = \frac{P}{GM} = \frac{\sqrt{2(1+\xi)}}{2+\xi} \ . \tag{2.130}$$

In this parametrization the extremal (zero temperature) limit corresponds to $\xi \to 0$ and the Schwarzschild (no charge) limit corresponds to $\xi \to \infty$.

We also need to review the computation of C_{zm}, the integer that captures corrections to the effective action due to zero modes. In our schematic notation zero modes $\lambda_i = 0$ are included in the heat kernel (2.114) and therefore contribute to the local term $C_{\rm local}$. However, the zero mode contribution to the effective action is not computed correctly by the Gaussian path integral implied in (2.113) and should instead be replaced by an overall volume of the symmetry group responsible for the zero mode. It is the combination of removing the zero-mode from the heat kernel and adding it back in again as a volume factor that gives the correction $C_{\rm zm}$.

Additionally, the effective action defined by the Euclidean path integral with thermal boundary conditions is identified with the free energy in the canonical ensemble whereas the entropy is computed in the microcanonical ensemble where mass and charges are fixed. The Legendre transform relating these ensembles gives a logarithmic contribution to the entropy that we have absorbed into $C_{\rm zm}$, for brevity.

The various contributions to $C_{\rm zm}$ are not new, they were analyzed in [45]. The result can be consolidated in the formula [17]

$$C_{\rm zm} = -(3+K) + 2N_{\rm SUSY} + 3\,\delta_{\text{non-ext}}\ . \tag{2.131}$$

Here K is the number of rotational isometries of the black hole, $N_{\rm SUSY}$ is the number of preserved real supercharges. $\delta_{\text{non-ext}}$ is 0 if the black hole is extremal and 1 otherwise. The non-extremal KK black hole with $J = 0$ is spherically symmetric and has $K = 3$, $N_{\rm SUSY} = 0$ and $\delta_{\text{non-ext}} = 1$. Therefore, $C_{\rm zm} = -3$ for all the non-extremal black holes we consider in this chapter but $C_{\rm zm} = -6$ in the extreme limit.

Combining all contributions, our final result for the coefficient of the logarithmic correction to the non-extreme black hole entropy is

$$\frac{1}{2}(C_{\rm local} + C_{\rm zm}) = 2(c-a) - \frac{3}{2} + \frac{4}{5\,\xi(1+\xi)}\,c\ , \tag{2.132}$$

where the values of c and a for the theories discussed in this chapter are given in Table 2.8. The expression manifestly shows that when $c \neq 0$, which is the case for $\mathcal{N} = 0, 2, 4$, the quantum correction to the entropy depends on black hole parameters through ξ or, by the relation (2.130), through the physical ratio Q/GM. The cases with very high supersymmetry are special since $c = 0$ when $\mathcal{N} \geq 6$ and then the coefficient of the logarithm is purely numerical. For example, we find the quantum corrections

$$\delta S^{(\mathcal{N}=6)}_{\text{non-ext}} = -\frac{9}{2}\log\frac{A_H}{G}\ , \qquad \delta S^{(\mathcal{N}=8)}_{\text{non-ext}} = -\frac{13}{2}\log\frac{A_H}{G}\ , \tag{2.133}$$

to the non-extremal black holes on the nonBPS branch.

As we have stressed, the KK black hole on the nonBPS branch is not intrinsically exceptional. In the non-rotating case with $P = Q$ that is our primary focus, the geometry is the standard Reissner-Nordström black hole. However, Kaluza-Klein theory includes a scalar field, the dilaton, and this dilaton couples non-minimally to gravity and to the gauge field. According to Table 2.8 we find $c = \frac{37}{24}$ for the KK black hole that is, after all, motivated by a higher dimensional origin.

An appropriate benchmark for this result is the minimally coupled Einstein-Maxwell theory, which has Reissner-Nordström as a solution, with an additional minimally coupled scalar field. The KK theory and the minimal theory both have $c - a = \frac{10}{9}$, because these theories have the same field content, and the zero-mode content of the black holes in the two theories is also identical, because the geometries are the same. However, $c = \frac{55}{24}$ for the minimally coupled black hole, a departure from the KK black holes. Thus, as one would expect, the quantum corrections to the black hole entropy depend not only on the field content but also on the couplings to low energy matter.

Although the focus in this chapter has been on the non-extreme case, and specifically whether the logarithmic corrections to the black hole entropy depend on the departure from extremality, it is worth highlighting the extremal limit since in this special case a detailed microscopic model is the most realistic. In the extremal case we find the quantum correction on the nonBPS branch

$$\delta S_{\text{ext}} = -\mathcal{N} \log \frac{A_H}{G} \ , \tag{2.134}$$

for $\mathcal{N} = 6, 8$. The surprising simplicity of this result is inspiring.

2.8 Discussion

In summary, we have shown that the spectrum of quadratic fluctuations around static Kaluza-Klein black holes in four dimensional supergravity partially diagonalizes into blocks of fields. Tables 2.7 and 2.8 give the c and a coefficients that control the Seeley-DeWitt coefficient $a_4(x)$ for each block and, taking into account appropriate degeneracies, for each supergravity theory. These coefficients directly yield the logarithmic correction to the black hole entropy via (2.126-2.127).

The detailed computations are quite delicate since any improper sign or normalization can dramatically change our conclusions. We therefore proceeded with extreme care, devoting several sections to explain the embedding of the Kaluza-Klein black hole into a range of supergravities and carefully record the action for quadratic fluctuations of the fields around the background. Moreover, we allowed for considerable redundancy, with indirect symmetry arguments supporting explicit computations and also performing many computations both analytically and using Mathematica. These steps increase our confidence in the results we report.

The prospect that interesting patterns in these corrections could lead to novel insights into black hole microstates is our main motivation for computing these quantum corrections in supergravity theories. Our discovery that $c = 0$ for $\mathcal{N} = 6, 8$ on the nonBPS branch is therefore gratifying. Recall that when c vanishes, the quantum correction is universal, it depends on the matter content of the theory but not on the parameters of the black hole. This property therefore holds out promise for a detailed microscopic description of these corrections. Such progress would be welcome since our current understanding of, for example, the $D0 - D6$ system leaves much to be desired [63–66] for the nonBPS branch.

Conversely, our analysis shows that on the nonBPS branch $c \neq 0$ for $\mathcal{N} \leq 4$. On the BPS-branch not only has it been found that $c = 0$ for all $\mathcal{N} \geq 2$ but this fact has also been shown to be a consequence of $\mathcal{N} = 2$ supersymmetry [50]. It would be interesting to similarly understand why $c = 0$ requires $\mathcal{N} \geq 6$ on the nonBPS branch.

To date, there is no known microstate counting formula that, when compared to the black hole entropy, accounts for terms that involve $c \neq 0$. For example, in *all* cases considered in [39, 40, 67], the object of interest is an index, or a closely related avatar, and the resulting logarithmic terms nicely accommodate quantum corrections when C_{local} is controlled by a alone. The challenge of reproducing the logarithmic correction when c is non-vanishing comes from the intricate dependence on the black hole parameters that the Weyl tensor gives to C_{local}. It would be interesting to understand which properties a partition function must possess in order that the logarithmic correction to the thermodynamic limit leads to $c \neq 0$.

An interesting concrete generalization of the present work would be to increase the scope of theories considered. In section 2.6 our main obstacle to covering all $\mathcal{N} = 2$ theories is the complicated structure of fermion couplings for a generic prepotential, and hence we restrict the discussion in section 2.6.4 to the $ST(n)$ models. Nevertheless, we suspect that for a generic prepotential our conclusions would not be significantly different. In particular, we predict that $c \neq 0$ on the nonBPS branch for any $\mathcal{N} = 2$ supergravity. It would of course be desirable to confirm this explicitly.

A more ambitious generalization would be to consider more general black hole solutions, specifically those where the dilaton $\Phi^{(\text{KK})}$ is not constant. Our assumption that $\Phi^{(\text{KK})} = 0$ simplified our computations greatly by sorting quadratic fluctuations into blocks that are decoupled from one another. By addressing the technical complications due to relaxation of this assumption and so computing $a_4(x)$ for black holes with non-trivial dilaton we could, in particular, access solutions with non-zero angular momentum $J \neq 0$. The rotating black holes on the nonBPS branch are novel since they never have constant dilaton, even in the extremal limit [68]. Therefore, they offer an interesting contrast to the Kerr-Newman black hole, their counterparts on the BPS branch [17]. Rotation is quite sensitive to microscopic details so any differences or similarities between the quantum corrections to rotating black holes on the BPS and nonBPS branches may well provide valuable clues towards a comprehensive microscopic model. A nonconstant dilaton is also the linchpin to connections with the new developments in AdS_2 holography for rotating black holes such as in [69, 70].

CHAPTER III

Black Hole Spectroscopy and AdS_2 Holography

3.1 Introduction and Summary

An important step towards a detailed understanding of quantum black holes is the determination of their spectrum [71]. However, with the exception of BPS black holes, it has generally proven quite difficult to compute the black hole spectrum precisely. In this chapter we find the spectrum of extremal nonrotating black holes on the nonBPS branch of $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravity.

The black holes we consider are solutions to theories with extended supersymmetry and have $\mathrm{AdS}_2 \times S^2$ near horizon geometry, just like BPS black holes; but they are supported by fluxes that are inconsistent with supersymmetry. In this situation it is not expected that the spectrum is organized by supersymmetry and our explicit computations confirm this generic expectation. However, we find that nonetheless the *black hole spectrum exhibits significant simplifications* that are *reminiscent of the familiar ones that are due to supersymmetry*. This finding does not conform with textbook BPS-ology but we will explain how it fits nicely with other expectations.

The spectrum of the black holes we consider is described by the quantum numbers of the $SL(2) \times SU(2)$ isometries of $\mathrm{AdS}_2 \times S^2$, i.e. the conformal weight h and the partial wave number j. The conformal weight is equivalent to the mass m of the perturbations in units of the AdS_2 radius ℓ through

$$h = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2\ell^2} \; , \tag{3.1}$$

for scalar fields. For BPS black holes the supersymmetry algebra guarantees that the supergravity mass spectrum corresponds to conformal weights h that are integers for bosons and half-integers for fermions. For nonBPS black holes the masses of fluctuations in supergravity are not constrained a priori but our explicit computations establish that, in fact, the values of m^2 for scalar fields are all such that the conformal weights (3.1) are integers. This is part of our claim that the spectrum is reminiscent of supersymmetry. In particular, the result suggests that the supergravity spectrum on the nonBPS branch is protected against quantum corrections and, if so, it should offer detailed guidance towards construction of the UV complete string theory describing extreme nonBPS black holes, despite the absence of supersymmetry.

The technical aspects of our explicit computations follow the strategy that is very well known from similar problems addressed in the past, such as spherical reduction of type IIB supergravity in ten dimensions on $\mathrm{AdS}_5 \times S^5$ [72, 73]. Accordingly, we first find the equations of motion of 4D supergravity and then linearize them around our $\mathrm{AdS}_2 \times S^2$ background solution. We then expand all fluctuating fields in their partial wave components and impose gauge conditions. It is no surprise that the 2D equations that result from these steps are messy, but fortunately they are sufficiently block diagonal that they can be disentangled and solved, despite the absence of supersymmetry. The final mass matrices therefore straightforwardly give eigenvalues for the masses of each partial wave that we can insert in (3.1) and so identify the conformal weights in AdS_2.

The only subtlety that is special to two dimensions is the spin of the fields [8, 74, 75]. In AdS_2 we can generally represent vectors and tensors as scalar fields and similarly recast gravitinos as Majorana-Weyl fermions. However, the dualization of fields with with spin require special considerations for *harmonic modes* because those are generated by gauge symmetries that are "large" in the sense that they are non-normalizable on AdS_2. Therefore, such transformations are not true symmetries, they generate field configurations that are physical and interpreted as excitations that are localized on the boundary. They can be identified with the modes that are described by a Schwarzian action (and its generalizations) in the Jackiw-Teitelboim model (and its relatives) [76–81]. We refer to these modes as *boundary modes* following the terminology previously used in the context of logarithmic corrections to black hole entropy in four dimensions. Thus the spectrum of extremal black holes on the nonBPS branch is characterized by

- *Bulk modes* that, from the AdS_2 point of view, are organized in infinite towers of Kaluza-Klein modes (partial waves).

- *Boundary modes* that, from the AdS_2 point of view, are field configurations that are physical even though they can be represented as "pure gauge" locally. These modes are closely related to *harmonic modes*.

Our result for the quantum numbers of supergravity on the nonBPS branch of $\mathrm{AdS}_2 \times S^2$ are reported in table 3.2. As a test of this spectrum we have computed the quantum contributions due to these modes by explicitly summing over all physical states. We find agreement with logarithmic corrections to the black hole entropy previously found using local methods [27]. This gives great confidence in the black hole spectrum we find.

We have already mentioned that on the nonBPS branch all fields in AdS_2 have integral conformal weight h and table 3.2 shows that we mean this quite literally: the *conformal weight is integral even for fermions.* This assignment is unusual but not inconsistent because the familiar relation between spin and statistics does not apply in two dimensions, at least in its standard form. Indeed, we will confirm our finding that fermions have integral weight on the nonBPS branch by recovering this assignment in settings where the AdS_2 geometry descends from an AdS_3 factor.

The standard simplification due to supersymmetry is that, when certain conditions are satisfied, the spectrum is organized into short multiplets that enjoy some protection against quantum corrections. However, there is also a less frequently exploited simplification that is due to broken supersymmetry. On the BPS branch both simplifications are relevant but on the nonBPS branch it is only the latter one that applies. It can be interpreted as a global supersymmetry that is implemented directly on the black hole spectrum. We discuss this symmetry in detail in section 3.6.

Before getting to details of our computations we must carefully consider the *meaning* of the spectrum of quadratic fluctuations around $\mathrm{AdS}_2 \times S^2$. Indeed, several well-known results prompt the question of whether such a spectrum makes any sense at all. For example, [1]

- Finite energy excitations in AdS_2 are incompatible with asymptotically AdS_2 boundary conditions: they elicit strong gravitational backreaction that modifies the asymptotic structure of spacetime [84]. Therefore, quadratic fluctuations are not intrinsic to AdS_2.

- In constructions where AdS_2 arises from AdS_3 through reduction along a null direction it was argued that the excitations with the lowest energy depend on the compact null coordinate but not on the AdS_2 that is retained by the compactification [85]. Therefore, the perturbations varying over AdS_2 that we consider do not dominate in the infrared limit.

In view of such results it is, for example, not obvious that the AdS_2 conformal weight h is a useful quantum number in AdS_2 quantum gravity. However, the recent development of $\mathrm{nAdS}_2/\mathrm{nCFT}_1$ correspondence [86] addresses these obstacles:

- The strict AdS_2 theory is interpreted as an inert IR fixed point of a dual CFT_1.

  An interesting holographic theory is obtained only by perturbing away from the fixed point by irrelevant operators. These operators dominate the far UV, corresponding

[1] There are closely related results for the near horizon Kerr geometry and our discussion below should apply to that case as well [82, 83].

to the asymptotic AdS_2 boundary breaking down. However, their description of the approach to the IR is controlled.

The spectrum we compute classifies the irrelevant operators in the IR fixed point theory that may serve as appropriate deformations. When these operators are added to the Lagrangian they deform the theory such that conformal symmetry is broken and new length scales are introduced. The most important scales appearing in this manner are associated with $h = 2$ operators and were discussed in [87].

- In constructions where AdS_2 arises from AdS_3 through a null reduction the dependence on the null direction indeed dominates in the strict infrared limit. However, the irrelevant operators controlling the *near* infrared regime are transverse to the direction of dimensional reduction and such excitations depend on position in the AdS_2 geometry. We identify our spectrum with such operators.

In short, the spectrum given in table 3.2 does not describe the ground state of AdS_2 quantum gravity but rather the low lying excitations above the ground state. In terms of a CFT_2, the ground state has huge degeneracy and is referred to as left moving in our conventions. The nAdS_2 theory with the spectrum we compute characterizes the leading excitations which, for kinematic reasons, are entirely right moving and only weakly coupled to the left moving ground state. The discussion in section 3.5 elaborates on this interpretation and related conceptual challenges.

The simplifications we observe by explicit computations are, as mentioned, reminiscent of those that are due to supersymmetry. In section 3.6 we develop this point of view and identify fermionic operators that generate the black hole spectra. It would be interesting to recover the same generators from *ab initio* considerations. Progress in this direction could yield clues to the microscopic description of these black holes.

This chapter is organized as follows. In section 3.2 we describe the extremal nonBPS black hole backgrounds we consider as solutions to $\mathcal{N} = 8$ (or $\mathcal{N} = 4$) supergravity in $D = 4$. They all have $AdS_2 \times S^2$ near horizon geometry and in these contexts they respect $USp(8)$ (or $USp(4) \times SO(n_V - 1)$) global symmetry. This symmetry structure partially diagonalizes the quadratic fluctuations around the backgrounds by organizing them into manageable blocks that are decoupled from one another. In section 3.3 we compute the mass spectrum of these blocks and obtain the conformal weights h of the corresponding fields. In section 3.4 we compute the logarithmic correction to the black hole entropy due the one loop contributions of all these states and find agreement with the results recently found using very different methods [27]. In section 3.5, we study the dimensional reduction from $\mathrm{AdS}_3 \times S^2$ to $\mathrm{AdS}_2 \times S^2$ and show how, depending on a choice of chirality, we reproduce either the nonBPS spectrum or the BPS spectrum on $\mathrm{AdS}_2 \times S^2$. This not only yields yet another consistency check on our computations but, as we discuss, it also enlightens the relation between the $\mathrm{nAdS}_2/\mathrm{nCFT}_1$ correspondence and black holes in string theory. We finish in section 3.6 with a discussion of broken supersymmetry.

3.2 Black Holes and Their Fluctuations

In this section we introduce the nonBPS black holes in $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravity. We exploit symmetries to establish the partial decoupling of quadratic fluctuations around these backgrounds into blocks.

3.2.1 The $\mathrm{AdS}_2 \times S^2$ Backgrounds in $\mathcal{N} = 8$ Supergravity

$\mathcal{N} = 8$ supergravity in $D = 4$ spacetime dimensions consists of one graviton, 8 gravitini $\Psi_{\hat{\mu}A}$, 28 $U(1)$ vector fields $A_{\hat{\mu}}^{AB}$, 56 Majorana spinors Λ_{ABC}, and 70 scalars W_{ABCD}. The hatted greek indices $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3$ denote 4D Lorentz indices and capital latin letters $A = 1, ..., 8$ refer to the global $SU(8)_R$ symmetry of $\mathcal{N} = 8$ SUGRA. The $SU(8)_R$ indices are fully antisymmetrized so the graviton, gravitini, vectors, gaugini, and scalars transform in representations **1**, **8**, **28**, **56** and **70** of the $SU(8)_R$ group.

The black hole backgrounds we consider all have an $\mathrm{AdS}_2 \times S^2$ near horizon geometry,

$$R_{\mu\nu\lambda\rho} = -\frac{1}{\ell^2}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \ , \tag{3.2}$$

$$R_{\alpha\beta\gamma\delta} = +\frac{1}{\ell^2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \ , \tag{3.3}$$

where unhatted indices $\mu, \nu = 0, 1$ and $\alpha, \beta = 2, 3$ refer to AdS_2 and S^2, respectively. ℓ is the radius of curvature of both 2D spaces.

The scalar fields are all constant on $\mathrm{AdS}_2 \times S^2$ and the fermions vanish. Thus the only matter supporting the geometry is the 28 field strengths $G_{\hat{\mu}\hat{\nu}}^{AB} = 2\partial_{[\hat{\mu}}A_{\hat{\nu}]}^{AB}$. The 28 electric charges (field components on AdS_2) and 28 magnetic charges (field components on S^2) characterizing the field strengths can famously be organized into a fundamental representation **56** of $E_{7(7)}$ duality symmetry [54]. However, it is convenient to focus on the $SU(8)_R$ symmetry that is the maximal compact subgroup of $E_{7(7)}$ and express the charges by the complex antisymmetric central charge matrix Z_{AB}. After block diagonalization by an

$SU(8)_R$ transformation we can present it as

$$Z_{AB} = \mathrm{diag}(\lambda_1\epsilon, \lambda_2\epsilon, \lambda_3\epsilon, \lambda_4\epsilon) \ , \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{3.4}$$

The canonical example of a charge configuration that corresponds to a BPS solution is $\lambda_1 = \ell^{-1}$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$. These skew-eigenvalues preserve a $SU(2)_R \times SU(6)$ subgroup of $SU(8)_R$. The symmetry breaking pattern $SU(8)_R \to SU(2)_R \times SU(6)$ constitutes a more general characterization of the charges corresponding to BPS black holes with finite area.

A charge configuration that corresponds to the nonBPS black holes we focus on is [88, 89]

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{e^{i\frac{\pi}{4}}}{2\ell} \ . \tag{3.5}$$

It realizes the symmetry breaking pattern $SU(8)_R \to USp(8)$ that is characteristic of the nonBPS branch. To see this, note that the central charge matrix (3.4) with the skew-eigenvalues (3.5) is proportional to the symplectic matrix

$$\Omega_8 = \mathrm{diag}(\epsilon, \epsilon, \epsilon, \epsilon) \ . \tag{3.6}$$

The antisymmetric tensor representation of $USp(8)$ is inherited from that of $SU(8)$ by imposing tracelessness upon contraction with Ω_8 so the symmetry breaking $SU(8)_R \to USp(8)$ is manifest.

The phase appearing in (3.5) ensures that the central charge matrix Z_{AB} has determinant $+1$, as it must to be an element of $SU(8)_R$. Physically, the phase shows that the nonBPS branch has equal electric and magnetic charges, in contrast to the BPS solutions that can be chosen to have only electric charge. The factor $\frac{1}{2}$ on the right hand side of (3.5) is such that the quadratic invariant $Z_{AB}Z^{AB}$ has the same magnitude for BPS and nonBPS black holes. This means the energy momentum tensor will be the same on the two branches which show that they share the same geometry.

In contrast, fermions enjoy Pauli couplings that depend linearly on the field strengths so supersymmetry acts differently on the two branches. Supersymmetry is preserved when the fermion transformations

$$\delta\lambda^{ABC} = -\frac{3}{\sqrt{2}}\hat{G}^{[AB}\epsilon^{C]} \ , \tag{3.7}$$

$$\delta\psi^A_{\hat{\mu}} = \left(\delta^{AB} D_{\hat{\mu}} + \frac{1}{2}\hat{G}^{AB}\Gamma_{\hat{\mu}}\right)\epsilon_B \ , \tag{3.8}$$

vanish, where the field strengths $\hat{G}^{AB} \equiv \frac{1}{2}\Gamma^{\hat{\mu}\hat{\nu}}G^{AB}_{\hat{\mu}\hat{\nu}}$. We can assume without loss of generality that $\hat{G}^{AB}$ are block diagonal in the (AB) indices, as for the central charge in (3.4). Thus the 4 sectors $(12),(34),(56),(78)$ do not couple to each other. On the BPS branch only $\hat{G}^{12}$ is nonvanishing. In this case there are no solutions for ϵ_B in the $(34),(56),(78)$ sectors but, in the (12) sector, there is a solution with nontrivial $\epsilon_{1,2}$ and so the BPS solutions preserve $\mathcal{N}=2$ supersymmetry. On the nonBPS branch the $(12),(34),(56),(78)$ sectors give equivalent conditions but, because of the factor $\frac{1}{2}$ in (3.5) that was discussed in the preceding paragraph, there is a mismatch between the magnitude of the field strength and the AdS_2 with scale ℓ. Therefore, there are no solutions for ϵ_B on the nonBPS branch.

3.2.2 Adaptation to $\mathcal{N}=4$ Supergravity

We also want to discuss the spectrum of nonBPS black holes in $\mathcal{N}=4$ supergravity. It will ultimately follow automatically from the results in $\mathcal{N}=8$ supergravity, after a few modest reinterpretations.

In order to show this we first truncate $\mathcal{N}=8$ supergravity to $\mathcal{N}=4$ supergravity. This truncation breaks the global symmetry $SU(8)_R \to SU(4)_R \times SU(4)_{\text{matter}}$. The branching rules of this symmetry breaking are

$$\begin{aligned}
\mathbf{70} &\to 2(\mathbf{1},\mathbf{1}) \oplus (\mathbf{6},\mathbf{6}) \oplus (\mathbf{4},\bar{\mathbf{4}}) \oplus (\bar{\mathbf{4}},\mathbf{4}) \ , \\
\mathbf{56} &\to (\bar{\mathbf{4}},\mathbf{1}) \oplus (\mathbf{6},\mathbf{4}) \oplus (\mathbf{4},\mathbf{6}) \oplus (\mathbf{1},\bar{\mathbf{4}}) \ , \\
\mathbf{28} &\to (\mathbf{1},\mathbf{6}) \oplus (\mathbf{6},\mathbf{1}) \oplus (\mathbf{4},\mathbf{4}) \ , \\
\mathbf{8} &\to (\mathbf{1},\mathbf{4}) \oplus (\mathbf{4},\mathbf{1}) \ , \\
\mathbf{1} &\to (\mathbf{1},\mathbf{1}) \ .
\end{aligned} \tag{3.9}$$

It is a consistent truncation that preserves $\mathcal{N}=4$ supersymmetry to omit all fields in the $\mathbf{4}$ (or $\bar{\mathbf{4}}$) of $SU(4)_{\text{matter}}$. The truncated theory obtained this way comprises an $\mathcal{N}=4$ supergravity multiplet (in the $\mathbf{1}$ of $SU(4)_{\text{matter}}$) and $n_V=6$ matter multiplets (in the $\mathbf{6}$ of $SU(4)_{\text{matter}}$).

The matter supporting $\mathrm{AdS}_2 \times S^2$ solutions in $\mathcal{N}=8$ supergravity is encoded in the spacetime central charges (3.4). The nontrivial fields can be chosen without loss of generality as the four skew-diagonal ones and these are all retained in the truncation of $SU(8)_R$ to its $SU(4)_R \times SU(4)_{\text{matter}}$ subgroup. Therefore these background configurations are also solutions to the truncated theory with $\mathcal{N}=4$ supersymmetry. We focus on the nonBPS branch with skew-eigenvalues (3.5) and the symmetry breaking pattern $SU(8)_R \to USp(8)$ in $\mathcal{N}=8$ SUGRA. This case descends to a nonBPS branch of $\mathcal{N}=4$ SUGRA with the symmetry breaking pattern $SU(4)_R \times SU(4)_{\text{matter}} \to USp(4) \times USp(4)_{\text{matter}}$.

There is a simple generalization of this result to $\mathcal{N} = 4$ SUGRA with a general number $n_V \geq 1$ of matter multiplets [27]. Since $SU(4) = SO(6)$ and $USp(4) = SO(5)$ as Lie algebras, the symmetry breaking pattern of the nonBPS branch found in the preceding paragraph for $n_V = 6$ matter multiplets is equivalent to $SO(n_V)_{\rm matter} \to SO(n_V - 1)_{\rm matter}$. This is the pattern that characterizes the nonBPS solutions of theories with any $n_V \geq 1$.

3.2.3 Structure of Fluctuations

As we have stressed, our background solution breaks the global $SU(8)_R$ of $\mathcal{N} = 8$ SUGRA theory to a $USp(8)$ subgroup. This greatly simplifies the analysis of fluctuations around the background because it shows that different $USp(8)$ representations cannot couple at quadratic order. We can therefore organize the spectrum as representations of $USp(8)$.

The branchings of $SU(8)_R \to USp(8)$ for the matter representations in $\mathcal{N} = 8$ SUGRA can be realized explicitly by removing contractions with the symplectic invariant (3.6) from $SU(8)_R$ representations. This gives

$$\begin{aligned} \mathbf{70} &\to \mathbf{42} \oplus \mathbf{27} \oplus \mathbf{1} \ , \\ \mathbf{56} &\to \mathbf{48} \oplus \mathbf{8} \ , \\ \mathbf{28} &\to \mathbf{27} \oplus \mathbf{1} \ , \\ \mathbf{8} &\to \mathbf{8} \ , \\ \mathbf{1} &\to \mathbf{1} \ . \end{aligned} \tag{3.10}$$

Collecting all singlets we find that on the nonBPS branch gravity can mix with one linear combination of the vector fields and similarly with one scalar. This is the field content of minimal Kaluza-Klein gravity in 4D, obtained by dimensional reduction of Einstein gravity in 5D. Truncation of $\mathcal{N} = 8$ SUGRA to this sector is consistent and identifies the black holes on the nonBPS branch with the black holes in Kaluza-Klein theory [51, 52]. Moreover, the quadratic fluctuations of these fields is identical whether we consider the nonBPS branch of $\mathcal{N} = 8$ SUGRA or minimal Kaluza-Klein theory. We therefore refer to the singlet sector as the “Kaluza-Klein block”.

The other $USp(8)$ representations similarly present “blocks” that do not mix with each other. We summarize these decoupled sectors in table 3.1. The partial diagonalization of quadratic fluctuations into blocks was previously established away from extremality [27].

The spectrum of the KK black hole in $\mathcal{N} = 4$ SUGRA can be computed directly, or by truncating the fluctuations analyzed for $\mathcal{N} = 8$ SUGRA. The blocks of decoupled quadratic fluctuations are unchanged, it is only their degeneracy that is modified. Table 3.1 lists the multiplicity of block in $\mathcal{N} = 4$ SUGRA with $n_V \geq 1$ matter multiplets and their representations under the global $USp(4) \times SO(n_V - 1)_{\text{matter}}$ symmetry.

Multiplet	Block content	d.o.f.	$\mathcal{N} = 8$		$\mathcal{N} = 4$ with n_V matter multiplets	
			$USp(8)$	#	$USp(4) \times SO(n_V - 1)_{\text{matter}}$	#
KK block	1 graviton, 1 vector, 1 scalar	5	$\mathbf{1}$	1	$(\mathbf{1},\mathbf{1})$	1
Gravitino block	2 gravitini and 2 gaugini	8	$\mathbf{8}$	4	$(\mathbf{4},\mathbf{1})$	2
Vector block	1 vector and 1 (pseudo)scalar	3	$\mathbf{27}$	27	$(\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},n_V - 1) \oplus (\mathbf{1},\mathbf{1})$	$n_V + 5$
Gaugino block	2 gaugini	4	$\mathbf{48}$	24	$(\mathbf{4},n_V - 1) \oplus (\mathbf{4},\mathbf{1})$	$2n_V$
Scalar block	1 real scalar	1	$\mathbf{42}$	42	$(\mathbf{5},n_V - 1) \oplus (\mathbf{1},\mathbf{1})$	$5n_V - 4$

Table 3.1: Decoupled quadratic fluctuations around the KK black hole in $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravity. The columns # denote the multiplicity of the blocks.

3.3 Mass Spectrum

In this section we compute the mass spectrum of fields on $\text{AdS}_2 \times S^2$. Global symmetries partially decouple the fluctuations so we can consider one block at a time, as discussed in section 3.2 and summarized in table 3.1. For each block we start from the linearized equations of motion in 4D and expand the perturbations in spherical harmonics on S^2, before diagonalizing the resulting 2D equations of motion explicitly. Bulk modes are analyzed in section 3.3.2 through 3.3.6 and boundary modes are considered in section 3.3.7. From now on, we set the AdS_2 radius ℓ to 1 for simplicity.

This section is long and relatively technical. Readers who are not interested in the detailed computations can jump directly to section 3.3.8 where the results are summarized.

3.3.1 Partial Wave Expansion on S^2 and Dualization of AdS_2 Vectors

The standard basis elements for the partial wave expansion of a scalar field on S^2 are the spherical harmonics $Y_{(lm)}$, i.e. the eigenfunctions of the 2D Laplacian ∇_S^2 on S^2 with eigenvalues $-l(l+1)$. The analogous spherical harmonics for vector (or tensor) fields on S^2 are easily formed by taking one (or two) derivatives of $Y_{(lm)}$ along the S^2. Thus we can expand a 4D scalar w, a 4D vector $a_{\hat{\mu}}$, and 4D gravity $h_{\hat{\mu}\hat{\nu}}$ as

$$
\begin{aligned}
w &= \sum_{lm} \varphi^{(lm)} Y_{(lm)} \ , && (3.11)\\
a_\mu &= \sum_{lm} b_\mu^{(lm)} Y_{(lm)} \ , && (3.12)\\
a_\alpha &= \sum_{lm} \left(b_1^{(lm)} \nabla_\alpha Y_{(lm)} + b_2^{(lm)} \epsilon_{\alpha\beta} \nabla^\beta Y_{(lm)} \right) \ , && (3.13)\\
h_{\mu\nu} &= \sum_{lm} H_{\mu\nu}^{(lm)} Y_{(lm)} \ , && (3.14)\\
h_{\mu\alpha} &= \sum_{lm} \left(B_{1\mu}^{(lm)} \nabla_\alpha Y_{(lm)} + B_{2\mu}^{(lm)} \epsilon_{\alpha\beta} \nabla^\beta Y_{(lm)} \right) \ , && (3.15)\\
h_{\alpha\beta} &= \sum_{lm} \left(\phi_1^{(lm)} \nabla_{\{\alpha} \nabla_{\beta\}} Y_{(lm)} + \phi_2^{(lm)} \epsilon_{\{\alpha}{}^{\gamma} \nabla_{\beta\}} \nabla_\gamma Y_{(lm)} + \phi_3^{(lm)} g_{\alpha\beta} Y_{(lm)} \right) \ . && (3.16)
\end{aligned}
$$

Curly brackets indicate traceless symmetrization of indices such as $\nabla_{\{\alpha}\nabla_{\beta\}} = \frac{1}{2}(\nabla_\alpha\nabla_\beta + \nabla_\beta\nabla_\alpha - g_{\alpha\beta}\nabla^2)$. The coefficient functions $H_{\mu\nu}^{(lm)}, B_{1\mu}^{(lm)}, \ldots$ are fields on the AdS_2 base with AdS_2 tensor structure given by the indices $\mu, \nu, \ldots$ and degeneracy enumerated by the angular momentum quantum numbers (lm).

Fermion fields can similarly be expanded on a basis of spinor spherical harmonics $\eta_{(\sigma lm)}$ satisfying $\gamma^\alpha D_\alpha \eta_{(\sigma lm)} = i(l+1)\eta_{(\sigma lm)}$ where γ^α denotes gamma matrices on S^2. We will use γ^μ for gamma matrices on AdS_2 and $\Gamma^{\hat{\mu}}$ for 4D gamma matrices. The partial wave expansion of a gaugino Λ and a gravitino $\Psi_{\hat{\mu}}$ are

$$
\begin{aligned}
\Lambda &= \lambda_+^{(\sigma lm)} \otimes \eta_{(\sigma lm)} + \lambda_-^{(\sigma lm)} \otimes \gamma_S \eta_{(\sigma lm)} \ , && (3.17)\\
\Psi_\mu &= \psi_{\mu+}^{(\sigma lm)} \otimes \eta_{(\sigma lm)} + \psi_{\mu-}^{(\sigma lm)} \otimes \gamma_S \eta_{(\sigma lm)} \ , && (3.18)\\
\Psi_\alpha &= \psi_+^{(\sigma lm)} \otimes D_{(\alpha)} \eta_{(\sigma lm)} + \psi_-^{(\sigma lm)} \otimes D_{(\alpha)} \gamma_S \eta_{(\sigma lm)} && \\
&\quad + \chi_+^{(\sigma lm)} \otimes \gamma_\alpha \eta_{(\sigma lm)} + \chi_-^{(\sigma lm)} \otimes \gamma_\alpha \gamma_S \eta_{(\sigma lm)} \ , && (3.19)
\end{aligned}
$$

where the summation symbol is suppressed for brevity. The chirality operator γ_S is the S^2 analogue of Γ_5 in 4D and the symbol $D_{(\alpha)} = D_\alpha - \frac{1}{2}\gamma_\alpha\gamma^\beta D_\beta$. The indices $\pm$ on the fields on AdS_2 thus refer to chirality and the four terms in (3.17) correspond to projection on to the four helicities, $\pm\frac{3}{2}, \pm\frac{1}{2}$. There is a detailed discussion of spinors on S^2 in [90].

It will be sufficient to discuss bulk modes on-shell. Therefore, we can impose gauge conditions from the outset. The Lorentz-deDonder (LdD) gauge

$$
\nabla^\alpha h_{\alpha\mu} = \nabla^\alpha h_{\{\alpha\beta\}} = 0 \ , \qquad \nabla^\alpha a_\alpha = 0 \ , \qquad \gamma^\alpha \Psi_\alpha = 0 \ , \tag{3.20}
$$

amounts to the conditions on AdS_2 fields

$$\phi_1^{(lm)} = \phi_2^{(lm)} = 0 \ , \quad B_{1\mu}^{(lm)} = 0 \ , \tag{3.21}$$

$$b_1^{(lm)} = 0 \ , \tag{3.22}$$

$$\chi_+^{(\sigma lm)} = \chi_-^{(\sigma lm)} = 0 \ . \tag{3.23}$$

This simplifies the expansions (3.13, 3.15, 3.16, 3.19). Importantly, the LdD gauge (3.20) is complete only for partial waves with $l \geq 2$. For $l = 0, 1$ some of the LdD gauge conditions are vacuous so additional gauge fixing is needed. We will discuss this on a case by case basis.

A vector field in AdS_2 can be dualized to two scalars as

$$b_\mu^{(lm)} = \epsilon_{\mu\nu}\nabla^\nu a_\perp^{(lm)} + \nabla_\mu a_\parallel^{(lm)} \ . \tag{3.24}$$

This decomposition into transverse and longitudinal modes is unique when there are no normalizable harmonic scalars, as in Euclidean AdS_2. In Lorentzian signature there are nontrivial harmonic modes but they are not physical as they can be presented in longitudinal form where they manifestly decouple from physical processes. The determination of boundary modes in section 3.3.7 will further refine these statements by considering nonnormalizable harmonic modes.

3.3.2 Bulk Modes of the Scalar Block

The scalar block consists of just one 4D scalar that is minimally coupled. Upon expansion in partial waves following (3.11), the 4D Klein-Gordon equation becomes

$$\left(\nabla_A^2 - l(l+1)\right)\varphi^{(lm)} = 0 \ , \quad l \geq 0 \ . \tag{3.25}$$

The effective 2D mass is therefore $m^2 = l(l+1) = j(j+1)$ after identification of the orbital angular momentum l with the total angular momentum j, as usual for scalar fields. Therefore (3.1) gives the conformal weight

$$h = j + 1 \ . \tag{3.26}$$

This result applies for all integral $j \geq 0$.

3.3.3 Bulk Modes of the Vector Block

The 4D vector block couples a scalar field x and a gauge field through the Lagrangian [27]

$$e^{-1}\mathcal{L}_{\text{vector}} = -\frac{1}{2}\nabla_{\hat{\mu}}x\nabla^{\hat{\mu}}x - \frac{1}{4}f_{\hat{\mu}\hat{\nu}}f^{\hat{\mu}\hat{\nu}} + xf_{\hat{\mu}\hat{\nu}}G^{\hat{\mu}\hat{\nu}} \ , \tag{3.27}$$

where the background gauge field $G^{\hat{\mu}\hat{\nu}}$ has AdS_2 and S^2 components $G^{\mu\nu} = \frac{1}{\sqrt{2}}\epsilon^{\mu\nu}$ and $G^{\alpha\beta} = \frac{1}{\sqrt{2}}\epsilon^{\alpha\beta}$. The resulting 4D equations of motion for the scalar and the vector are

$$\nabla^2 x + f_{\hat{\mu}\hat{\nu}}G^{\hat{\mu}\hat{\nu}} = 0 \ , \tag{3.28}$$

$$\nabla^{\hat{\mu}}\left(f_{\hat{\mu}\hat{\nu}} - 2xG^{\hat{\mu}\hat{\nu}}\right) = 0 \ . \tag{3.29}$$

Applying partial wave expansions of the form (3.12, 3.13), the gauge condition (3.22), and dualization (3.24) we find

$$\left[\left(\nabla_A^2 - l(l+1)\right)x + \sqrt{2}l(l+1)b_2 + \sqrt{2}\nabla_A^2 a_\perp\right]Y = 0 \ , \tag{3.30}$$

$$\epsilon_{\nu\mu}\nabla^{\mu}\left[\left(\nabla_A^2 - l(l+1)\right)a_\perp + \sqrt{2}x\right]Y + \nabla_\nu\left[\left(\nabla_A^2 - l(l+1)\right)a_\parallel\right]Y = 0 \ , \tag{3.31}$$

$$\left[-\nabla_A^2 a_\parallel\right]\nabla_\alpha Y + \left[\left(\nabla_A^2 - l(l+1)\right)b_2 + \sqrt{2}x\right]\epsilon_{\alpha\beta}\nabla^\beta Y = 0 \ . \tag{3.32}$$

The partial wave numbers (lm) on the 2D fields $x, b_2, a_\perp, a_\parallel$ and on the spherical harmonics Y are suppressed for brevity. Since $Y_{(00)}$ is a constant on S^2 (3.32) has no content for $l = 0$. For the same reason, the expansion (3.13) in vector harmonics on S^2 leaves the component $b_2^{(00)}$ undefined. Importantly, the combination $l(l+1)b_2^{(00)}$ unambiguously vanishes for $l = 0$, so (3.30) is meaningful for all $l \geq 0$.

The 4D equations of motion (3.30, 3.31, 3.32) are equivalent to the vanishing of each expression in square bracket by itself, due to orthogonality of the spherical harmonics. For (3.31) we also appeal to uniqueness of dualization in order to remove the gradients on AdS_2. In the following we diagonalize these 2D equations of motion. We first discuss modes with $l \geq 1$ and then address the special case $l = 0$.

Vector block: $l \geq 1$ ***modes***

For $l \geq 1$ we can apply (3.32). In particular, the first equation shows that $a_\parallel = 0$, due to the absence of propagating harmonic modes. Then (3.30, 3.31, 3.32) give

$$\left(\nabla_A^2 - l(l+1)\right) x + \sqrt{2} l(l+1) b_2 + \sqrt{2}\nabla_A^2 a_\perp = 0 \ , \tag{3.33}$$
$$\left(\nabla_A^2 - l(l+1)\right) a_\perp + \sqrt{2} x = 0 \ , \tag{3.34}$$
$$\left(\nabla_A^2 - l(l+1)\right) b_2 + \sqrt{2} x = 0 \ , \tag{3.35}$$

which can be reordered into the diagonal form

$$\left(\nabla_A^2 - (l-1)l\right)\left(\sqrt{2}x + (l+1)(a_\perp + b_2)\right) = 0 \ , \tag{3.36}$$
$$\left(\nabla_A^2 - l(l+1)\right)(a_\perp - b_2) = 0 \ , \tag{3.37}$$
$$\left(\nabla_A^2 - (l+1)(l+2)\right)\left(\sqrt{2}x - l(a_\perp + b_2)\right) = 0 \ , \tag{3.38}$$

The eigenvalues of the AdS$_2$ Laplacian ∇_A^2 thus give the scalar masses

$$m^2 = (l-1)l \ , \ l(l+1) \ , \ (l+1)(l+2) \ , \tag{3.39}$$

and so the conformal weights (3.1) become

$$h = j \ , \ j+1 \ , \ j+2 \ , \tag{3.40}$$

for all integral $j \geq 1$. We identified the angular quantum number $j = l$ by noting that each value of the conformal weight has degeneracy $(2l+1)$, the dimension of the irreducible representation of $SU(2)$ with $j = l$.

Vector block: $l = 0$ ***modes***

In the $l = 0$ sector The 4D gauge field $b_{\hat{\mu}}$ has no components on the S^2 so the only non-vanishing field components are $a_\perp$, $a_\parallel$, and x. Since $Y_{(00)} = 1$ the LdD gauge condition (3.20) is empty for $l = 0$. On the other hand, the standard 4D gauge transformation $b_{\hat{\mu}} \to b_{\hat{\mu}} + \partial_{\hat{\mu}}\Lambda$ reduces to a 2D symmetry acting on the AdS$_2$ components b_μ because for $l = 0$ it does not act on the (non-existent) components b_α of the vector field on S^2. We can exploit this gauge symmetry to set the longitudinal component $a_\parallel = 0$. The equations of motion (3.30, 3.31) then give

$$\begin{cases} \nabla_A^2 x + \sqrt{2}\nabla_A^2 a_\perp = 0 \\ \nabla_A^2 a_\perp + \sqrt{2}x = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \left(\nabla_A^2 - 2\right) x = 0 \\ \nabla_A^2\left(\sqrt{2}a_\perp + x\right) = 0 \end{cases} \ . \tag{3.41}$$

The lower equation becomes a constraint $\sqrt{2}a_\perp + x = 0$ up to a harmonic solution for

$a_\perp$ which is equivalent to $a_\parallel$ that vanishes due to the gauge condition. The $l = 0$ sector therefore reduces to one degree of freedom which we can identify as the scalar field x. This is the expected result because the vector block consists of a scalar and a vector but 2D vector fields have no degrees of freedom.

The upper equation in (3.41) identifies the eigenvalue of the scalar as $m^2 = 2$ which corresponds to conformal weight $h = 2$. We can present this in terms of the result (3.40) for $j \geq 1$: the tower with $h = j + 2$ is completed so it includes an entry for $j = 0$ while the other two towers have no $j = 0$ mode.

3.3.4 Bulk Modes of the KK Block

Expansion of the Kaluza-Klein Lagrangian to quadratic order around the $\text{AdS}_2 \times S^2$ background supported by nonBPS fluxes yields a Lagrangian for the quadratic fluctuations (given explicitly in [27]). This in turn gives the equations of motion for the KK block, summarized in the following.

KK block: Einstein equation

The 4D Einstein equation is given by

$$\begin{aligned}
&\nabla^2 h_{\hat\mu\hat\nu} + \nabla_{\hat\mu}\nabla_{\hat\nu} h - 2\nabla_{(\hat\mu}\nabla^{\hat\alpha} h_{\hat\nu)\hat\alpha} - 2R^{\hat\alpha}{}_{(\hat\mu} h_{\hat\nu)\hat\alpha} - 2R_{\hat\alpha\hat\mu\hat\nu\hat\beta} h^{\hat\alpha\hat\beta} + h_{\hat\mu\hat\nu} R \\
&+ g_{\hat\mu\hat\nu}(-\nabla^2 h + \nabla_{\hat\alpha}\nabla_{\hat\beta} h^{\hat\alpha\hat\beta} - h^{\hat\alpha\hat\beta} R_{\hat\alpha\hat\beta}) = -8G_{(\hat\mu}{}^{\hat\alpha} f_{\hat\nu)\hat\alpha} + 4G_{\hat\mu\hat\alpha} G_{\hat\nu\hat\beta} h^{\hat\alpha\hat\beta} \\
&+ 2g_{\hat\mu\hat\nu}(G^{\hat\alpha\hat\beta} f_{\hat\alpha\hat\beta} - G_{\hat\alpha}{}^{\hat\nu} G_{\hat\beta\hat\nu} h^{\hat\alpha\hat\beta}) + h_{\hat\mu\hat\nu} G^{\hat\alpha\hat\beta} G_{\hat\alpha\hat\beta} + 8\sqrt{3}\varphi G_{\hat\mu}{}^{\hat\alpha} G_{\hat\nu\hat\alpha} \ .
\end{aligned} \tag{3.42}$$

The background is described by the 4D metric $g_{\hat\mu\hat\nu}$ with Riemann curvature $R^{\hat\alpha}{}_{\hat\beta\hat\gamma\hat\delta}$ as well as the gauge fields $G_{\mu\nu} = \frac{1}{\sqrt{2}}\epsilon_{\mu\nu}$ along AdS_2 and $G_{\alpha\beta} = \frac{1}{\sqrt{2}}\epsilon_{\alpha\beta}$ through S^2. The fluctuations are the metric $h_{\hat\mu\hat\nu}$, the field strength $f_{\hat\mu\hat\nu}$, and the scalar field φ.

The partial wave expansions of the 4D fields take the form (3.11-3.16). Considering first the equations where $\hat\mu\hat\nu = \mu\nu$ so both indices are within AdS_2 we find

$$\Big[\left(l(l+1)+2\right)H - 2\left(\nabla_A^2 - l(l+1)\right)\phi_3 + 4\sqrt{2}\nabla_A^2 a_\perp - 4\sqrt{2}l(l+1)b_2 + 8\sqrt{3}\varphi\Big]\, Y = 0 \ , \tag{3.43}$$

$$\left[-l(l+1)H_{\{\mu\nu\}} + 2\nabla_{\{\mu}\nabla_{\nu\}}\phi_3\right] Y = 0 \ , \tag{3.44}$$

for the scalar and symmetric traceless components of the AdS_2 indices $\mu\nu$. We suppress the partial wave indices (lm) on the 2D fields to avoid clutter. The analogous equations for $\hat\mu\hat\nu = \alpha\beta$ so both indices of the Einstein equation (3.42) are on the S^2 give

$$\Big[\nabla_\rho\nabla_\sigma H^{\rho\sigma} - \left(\nabla_A^2 - \frac{1}{2}l(l+1)\right)H - \left(\nabla_A^2 + 2\right)\phi_3 - 2\sqrt{2}\nabla_A^2 a_\perp + 2\sqrt{2}l(l+1)b_2$$

$$-4\sqrt{3}\varphi\Big]\, g_{\alpha\beta}Y + [H]\ \nabla_{\{\alpha}\nabla_{\beta\}}Y - \left[2\nabla_A^2 B_{2\parallel}\right]\ \epsilon_{\{\alpha}{}^{\gamma}\nabla_{\beta\}}\nabla_{\gamma}Y = 0\ . \tag{3.45}$$

Finally, the partial wave expansion of the Einstein equation with mixed indices $\hat{\mu}\hat{\nu} = \mu\alpha$ becomes

$$\begin{aligned}&\left(\epsilon_{\mu\nu}\nabla^{\nu}\left[(\nabla_A^2 - l(l+1))B_{2\perp} - 2\sqrt{2}b_2 - 2\sqrt{2}a_{\perp}\right] - \nabla_{\mu}\left[l(l+1)B_{2\parallel} + 2\sqrt{2}a_{\parallel}\right]\right)\epsilon_{\alpha\beta}\nabla^{\beta}Y \\ &+ \left[\nabla_{\mu}H - \nabla^{\nu}H_{\mu\nu} + \nabla_{\mu}\phi_3 + 2\sqrt{2}\epsilon_{\mu\nu}b^{\nu} - 2\sqrt{2}\nabla_{\mu}b_2 + 2\epsilon_{\mu\nu}B_2^{\nu}\right]\nabla_{\alpha}Y = 0\ .\end{aligned} \tag{3.46}$$

KK block: vector equation

The equation of motion for the vector field in KK theory is

$$\nabla^{\hat{\mu}}\left(f_{\hat{\mu}\hat{\nu}} - h_{\hat{\mu}\hat{\rho}}G^{\hat{\rho}}{}_{\hat{\nu}} + h_{\hat{\nu}\hat{\rho}}G^{\hat{\rho}}{}_{\hat{\mu}} + \frac{1}{2}hG_{\hat{\mu}\hat{\nu}} - 2\sqrt{3}\varphi G_{\hat{\mu}\hat{\nu}}\right) = 0\ , \tag{3.47}$$

after linearizing around our background. For $\hat{\nu} = \nu$ the 4D index is along AdS_2 and the partial wave expansions (3.11-3.16) give

$$\begin{aligned}&\Bigg(\epsilon_{\nu\mu}\nabla^{\mu}\left[\left(\nabla_A^2 - l(l+1)\right)a_{\perp} - \frac{1}{\sqrt{2}}l(l+1)B_{2\perp} + \frac{1}{2\sqrt{2}}H - \frac{1}{\sqrt{2}}\phi_3 + \sqrt{6}\varphi\right] \\ &\quad -\nabla_{\nu}\left[\nabla_A^2 a_{\parallel} + \frac{1}{\sqrt{2}}l(l+1)B_{2\parallel}\right]\Bigg)Y = 0\ .\end{aligned} \tag{3.48}$$

We used the identity $\nabla_{\mu}H^{\{\mu\rho\}}\epsilon_{\rho\nu} = \nabla_{\mu}H_{\{\nu\rho\}}\epsilon^{\rho\mu}$. The partial wave expansion of the 4D field equation (3.47) for $\hat{\nu} = \alpha$ similarly gives

$$\begin{aligned}&\left[\left(\nabla_A^2 - l(l+1)\right)b_2 + \frac{1}{\sqrt{2}}\phi_3 - \frac{1}{\sqrt{2}}\nabla_A^2 B_{2\perp} - \frac{1}{2\sqrt{2}}H + \sqrt{6}\varphi\right]\epsilon_{\alpha\beta}\nabla^{\beta}Y \\ &-\nabla_A^2\left[a_{\parallel} + \frac{1}{\sqrt{2}}B_{2\parallel}\right]\nabla_{\alpha}Y = 0\ .\end{aligned} \tag{3.49}$$

KK block: scalar equation

The last equation of motion for KK theory is the one for the KK scalar:

$$8\nabla^2\varphi + 8\sqrt{3}G^{\hat{\mu}\hat{\nu}}f_{\hat{\mu}\hat{\nu}} - 4\sqrt{3}R^{\hat{\mu}\hat{\nu}}h_{\hat{\mu}\hat{\nu}} = 0\ . \tag{3.50}$$

The partial wave expansion gives

$$\left[\left(\nabla_A^2 - l(l+1)\right)\varphi + \sqrt{6}\nabla_A^2 a_{\perp} + \sqrt{6}l(l+1)b_2 + \frac{\sqrt{3}}{2}H - \sqrt{3}\phi_3\right]Y = 0\ . \tag{3.51}$$

At this point we must solve all these equations. Orthogonality of spherical harmonics show that all terms in square brackets vanish. However, we must take into account that gradients $\nabla_\alpha Y$ of the spherical harmonics vanish for $l = 0$ and traceless combinations of the double gradients $\nabla_\alpha \nabla_\beta Y$ vanish also for $l = 1$. Therefore we first discuss the equations for $l \geq 2$ and then address $l = 1, 0$.

KK block: $l \geq 2$ ***modes***

From (3.45) and (3.49) we find

$$\nabla_A^2 B_{2\|} = \nabla_A^2 a_\| = 0 \ , \tag{3.52}$$

$$H = 0 \ . \tag{3.53}$$

The uniqueness of AdS_2 dualization (up to modes that decouple) means we can take all these fields to vanish $B_{2\|} = a_\| = H = 0$. Additionally (3.44) shows that the graviton perturbations $H_{\{\mu\nu\}}$ can be expressed in terms of ϕ_3 so they do not represent independent degrees of freedom.

Taking these simplification into account, we gather the equations of motions (3.43, 3.46, 3.48, 3.49, 3.51) and find

$$\left(\nabla_A^2 - l(l+1)\right)\phi_3 = 2\sqrt{2}\nabla_A^2 a_\perp - 2\sqrt{2}l(l+1)b_2 + 4\sqrt{3}\varphi \ , \tag{3.54}$$

$$\left(\nabla_A^2 - l(l+1)\right)B_{2\perp} = 2\sqrt{2}b_2 + 2\sqrt{2}a_\perp \ , \tag{3.55}$$

$$\left(\nabla_A^2 - l(l+1)\right)a_\perp = \frac{1}{\sqrt{2}}l(l+1)B_{2\perp} + \frac{1}{\sqrt{2}}\phi_3 - \sqrt{6}\varphi \ , \tag{3.56}$$

$$\left(\nabla_A^2 - l(l+1)\right)b_2 = -\frac{1}{\sqrt{2}}\phi_3 + \frac{1}{\sqrt{2}}\nabla_A^2 B_{2\perp} - \sqrt{6}\varphi \ , \tag{3.57}$$

$$\left(\nabla_A^2 - l(l+1)\right)\varphi = -\sqrt{6}\nabla_A^2 a_\perp - \sqrt{6}l(l+1)b_2 + \sqrt{3}\phi_3 \ . \tag{3.58}$$

We can reorganize these equations as

$$\left(\nabla_A^2 - (l+2)(l+3)\right)\left[2\sqrt{3}\varphi - l(l+1)B_{2\perp} - 2\sqrt{2}la_\perp - 2\sqrt{2}lb_2\right] = 0 \ , \tag{3.59}$$

$$\left(\nabla_A^2 - (l+1)(l+2)\right)\left[-\phi_3 - lB_{2\perp} - \sqrt{2}la_\perp + \sqrt{2}lb_2\right] = 0 \ , \tag{3.60}$$

$$\left(\nabla_A^2 - l(l+1)\right)\left[2\varphi + \sqrt{3}(l^2+l-1)B_{2\perp} + \sqrt{6}a_\perp + \sqrt{6}b_2\right] = 0 \ , \tag{3.61}$$

$$\left(\nabla_A^2 - (l-1)l\right)\left[\phi_3 - (l+1)B_{2\perp} - \sqrt{2}(l+1)a_\perp + \sqrt{2}(l+1)b_2\right] = 0 \ , \tag{3.62}$$

$$\left(\nabla_A^2 - (l-2)(l-1)\right)\left[2\sqrt{3}\varphi - l(l+1)B_{2\perp} + 2\sqrt{2}(l+1)\left(a_\perp + b_2\right)\right] = 0 \ . \tag{3.63}$$

The scalar masses read off from the eigenvalues of ∇_A^2 are

$$m^2 \;=\; (l-2)(l-1)\ ,\ (l-1)l\ ,\ l(l+1)\ ,\ (l+1)(l+2)\ ,\ (l+2)(l+3)\ . \qquad (3.64)$$

Each of the AdS$_2$ scalars have degeneracy $(2l+1)$ so we identify $j = l$, where j is the angular quantum number labeling the irreducible representation of $SU(2)$. The conformal weights (3.1) of the 1D conformal fields dual to the five partial wave towers of the KK block become

$$h = \ j-1\ ,\ j\ ,\ j+1\ ,\ j+2\ ,\ j+3\ . \qquad (3.65)$$

This result is valid for $j \geq 2$.

KK block: $l = 1$ ***modes***

The $l = 1$ sector is special because $\epsilon_{\alpha\beta}\nabla^\beta Y_{(1m)}$ / $\nabla_\alpha Y_{(1m)}$ are Killing Vectors (KVs) / Conformal Killing Vectors (CKVs) on S^2. Therefore $\epsilon_{\{\alpha}{}^\gamma\nabla_{\beta\}}\nabla_\gamma Y_{(1m)} = \nabla_{\{\alpha}\nabla_{\beta\}}Y_{(1m)} = 0$ and so the partial wave expansion (3.16) does not include the coefficient functions $\phi_1^{(1m)}$ and $\phi_2^{(1m)}$. Moreover, the gauge conditions $\nabla^\alpha h_{\{\alpha\beta\}} = 0$ are automatic, they fail to constrain diffeomorphisms ξ_α on the S^2.

We gauge fix the diffeomorphisms along the KVs by setting $B_{2\|}^{(1m)} = 0$ and those along the CKVs by taking $\phi_3^{(1m)} = 0$. With these conditions (3.44) becomes a constraint that sets $H_{\{\mu\nu\}}^{(1m)} = 0$ and the vanishing of the second square bracket in (3.46) demands that also $a_\|^{(1m)} = 0$.

After gauge fixing the 15 partial wave components in the generic KK-block have been reduced to only 5. We gather the remaining terms in (3.43, 3.46, 3.48, 3.49, 3.51) for $l = 1$ and get the equations of motion for these 5 fields in AdS$_2$:

$$H^{(1m)} = -\sqrt{2}\nabla_A^2 a_\perp^{(1m)} + 2\sqrt{2}b_2^{(1m)} - 2\sqrt{3}\varphi^{(1m)}\ , \qquad (3.66)$$

$$\left(\nabla_A^2 - 2\right)B_{2\perp}^{(1m)} = 2\sqrt{2}b_2^{(1m)} + 2\sqrt{2}a_\perp^{(1m)}\ , \qquad (3.67)$$

$$\left(\nabla_A^2 - 2\right)a_\perp^{(1m)} = -\frac{1}{2\sqrt{2}}H^{(1m)} + \sqrt{2}B_{2\perp}^{(1m)} - \sqrt{6}\varphi^{(1m)}\ , \qquad (3.68)$$

$$\left(\nabla_A^2 - 2\right)b_2^{(1m)} = \frac{1}{\sqrt{2}}\nabla_A^2 B_{2\perp}^{(1m)} + \frac{1}{2\sqrt{2}}H^{(1m)} - \sqrt{6}\varphi^{(1m)}\ , \qquad (3.69)$$

$$\left(\nabla_A^2 - 2\right)\varphi^{(1m)} = -\sqrt{6}\nabla_A^2 a_\perp^{(1m)} - 2\sqrt{6}b_2^{(1m)} - \frac{\sqrt{3}}{2}H^{(1m)}\ . \qquad (3.70)$$

Simplifying the first of these equations using the others we find

$$H^{(1m)} = -4\sqrt{2}a_\perp^{(1m)} + 4\sqrt{2}b_2^{(1m)} - 4B_{2\perp}^{(1m)}\ . \qquad (3.71)$$

Therefore $H^{(1m)}$ is not an independent field. We diagonalize the remaining equations as

$$(\nabla_A^2 - 12)\left(-B_{2\perp}^{(1)} - \sqrt{2}a_\perp^{(1)} - \sqrt{2}b_2^{(1)} + \sqrt{3}\varphi^{(1)}\right) = 0 \ , \tag{3.72}$$

$$(\nabla_A^2 - 6)\left(-B_{2\perp}^{(1)} - \sqrt{2}a_\perp^{(1)} + \sqrt{2}b_2^{(1)}\right) = 0 \ , \tag{3.73}$$

$$(\nabla_A^2 - 2)\left(\sqrt{3}B_{2\perp}^{(1)} + \sqrt{6}a_\perp^{(1)} + \sqrt{6}b_2^{(1)} + 2\varphi^{(1)}\right) = 0 \ , \tag{3.74}$$

$$\nabla_A^2\left(-B_{2\perp}^{(1)} + 2\sqrt{2}a_\perp^{(1)} + 2\sqrt{2}b_2^{(1)} + \sqrt{3}\varphi^{(1)}\right) = 0 \ . \tag{3.75}$$

The final equation amounts to the constraint

$$-B_{2\perp}^{(1)} + 2\sqrt{2}a_\perp^{(1)} + 2\sqrt{2}b_2^{(1)} + \sqrt{3}\varphi^{(1)} = 0 \ . \tag{3.76}$$

up to a harmonic function that can be fixed by residual symmetry.

The three eigenvectors that remain represent propagating modes. This is the expected net number of physical fields from a gauge field and a scalar, the field content in the $l = 1$ sector of the KK block. The source of all the complications addressed here is the mixing of these degrees of freedom with gravity and with each other.

Therefore, for $j = l = 1$, the eigenvalues of ∇_A^2 are $m^2 = 12, 6, 2$, corresponding to the conformal weights $h = 4, 3, 2$ respectively. Among the five towers in (3.65), we thus find that those with $h = j+1, j+2, j+3$ are extended to $j = 1$ while the towers with $h = j-1, j$ do not include modes $j = 1$. Indeed, the three eigenvectors (3.72, 3.73, 3.74) with eigenvalues $12, 6, 2$ found for $l = 1$ extend those identified in (3.59, 3.60, 3.61) for $l \geq 2$.

KK block: $l = 0$ ***modes***

The spherical harmonic $Y_{(00)} = 1$ is constant, so for $l = 0$ the only non-vanishing terms defined by the partial wave expansions (3.11-3.16) are the 2D metric $H_{\mu\nu}^{(00)}$, the 2D gauge field $b_\mu^{(00)}$, the KK scalar $\varphi^{(00)}$ and the S^2 volume $\phi_3^{(00)}$. This is a total of 7 non-vanishing 2D field components for $l = 0$. In the $l = 0$ sector the LdD gauge conditions (3.20) place no restrictions on the fields. The 2D diffeomorphism symmetry generated by an AdS$_2$ vector ξ_μ is therefore unfixed, as is the 2D gauge symmetry. We fix these three symmetries by imposing

$$\nabla^\mu\nabla^\nu H_{\{\mu\nu\}}^{(00)} = 0 \ , \qquad a_\parallel^{(00)} = 0 \ . \tag{3.77}$$

Notice that there are still residual diffeomorphisms that satisfy $\nabla^\mu\nabla^\nu\nabla_{\{\mu}\xi_{\nu\}}^{(00)} = 0$, which we will take advantage of later.

The 2D equations of motion (3.43, 3.44, 3.45, 3.48, 3.51) of the remaining 4 field components $a_\perp^{(00)}$, $\phi_3^{(00)}$, $H^{(00)}$, and $\varphi^{(00)}$ can be organized as

$$\nabla_A^2\left(\sqrt{6}a_\perp^{(00)}+\varphi^{(00)}+\frac{\sqrt{3}}{4}H^{(00)}+2\sqrt{3}\phi_3^{(00)}\right)=0\ , \tag{3.78}$$

$$\nabla_{\{\mu}\nabla_{\nu\}}\phi_3^{(00)}=0\ , \tag{3.79}$$

$$(\nabla_A^2-2)\phi_3^{(00)}=0\ , \tag{3.80}$$

$$(\nabla_A^2-2)H^{(00)}=-12\phi_3^{(00)}\ , \tag{3.81}$$

$$\left(\nabla_A^2-6\right)\varphi^{(00)}=0\ . \tag{3.82}$$

Now (3.78) amounts to a constraint that expresses $a_\perp^{(00)}$ in terms of other fields, up to a harmonic mode that is inconsequential for the physical spectrum. Similarly, (3.79) define Conformal Killing Vectors (CKVs) $\nabla_\mu\phi_3^{(00)}$ but, since there are no normalizable CKVs on (Euclidean) AdS$_2$, we must have $\phi_3^{(00)}=0$. Then (3.81) becomes

$$(\nabla_A^2-2)H^{(00)}=0\ . \tag{3.83}$$

However, the gauge conditions (3.77) permit residual diffeomorphisms ξ_μ satisfying

$$\nabla^\mu\nabla^\nu\nabla_{\{\mu}\xi_{\nu\}}^{(00)}=0 \quad\Leftrightarrow\quad (\nabla_A^2-2)\delta H^{(00)}=0\ . \tag{3.84}$$

Such ξ_μ are CKVs that are necessarily nonnormalizable, but they correspond to normalizable $\delta H_{\mu\nu}^{(00)}$. Comparison of (3.83) and (3.84) shows that $H^{(00)}$ is pure gauge; it can be set to be zero by residual diffeomorphisms $\xi_\mu^{(00)}$.

In summary, in the $l=0$ sector of the KK-block there is only one physical degree of freedom which can be identified as $\varphi^{(00)}$. This mode generalizes the partial wave tower (3.59) to $l=0$. It is an eigenfunction of ∇_A^2 with eigenvalue $m^2=6$, corresponding to $h=3$. Thus it extends the final entry $h=j+3$ in (3.65) to the value $j=0$.

3.3.5 Bulk Modes of the Gaugino Block

The gaugino block has the 4D Lagrangian [27]

$$e^{-1}\delta^2\mathcal{L}_{\text{gaugino}}=-\bar{\Lambda}_A\Gamma^{\hat{\mu}}D_{\hat{\mu}}\Lambda_A-\frac{1}{2}\epsilon_{AB}\bar{\Lambda}_A\hat{G}\Lambda_B\ , \tag{3.85}$$

where $\hat{G} \equiv \frac{1}{2}\Gamma^{\hat{\mu}\hat{\nu}}G_{\hat{\mu}\hat{\nu}}$, summation over the indices $A, B = 1, 2$ is implied, and ϵ_{AB} is antisymmetric with $\epsilon_{12} = +1$. It gives the 4D equation of motion

$$\Gamma^{\hat{\mu}} D_{\hat{\mu}} \Lambda_A + \frac{1}{2} \hat{G} \epsilon_{AB} \Lambda_B = 0 \ . \tag{3.86}$$

Applying the partial wave expansion (3.17) to the two Majorana gaugini Λ_A we find

$$\Lambda_A^{(L/R)} = \frac{1}{2}(1 \pm \Gamma_5)\Lambda_A = \frac{1}{2}\left(\lambda_{A+} \pm \gamma_A \lambda_{A-}\right) \otimes \eta + \frac{1}{2}\left(\lambda_{A-} \pm \gamma_A \lambda_{A+}\right) \otimes \gamma_S \eta \ . \tag{3.87}$$

for their left- and right-handed components. The indices $(\sigma l m)$ on the 2D fields $\lambda_{A\pm}$ and the spinor harmonics η are suppressed for brevity. Inserting the expansion in spinor partial waves (3.87) into the 4D equations of motion (3.86) projected on to the right helicity by the operator $\frac{1}{2}(1 - \Gamma_5)$ we get

$$\begin{aligned} 0 = & \left[\gamma^\mu D_\mu \left(\lambda_{A-} + \gamma_A \lambda_{A+}\right) + i(l+1)\left(\lambda_{A+} + \gamma_A \lambda_{A-}\right) - \frac{1}{2} e^{i\frac{\pi}{4}} \epsilon_{AB} \left(\lambda_{B-} - \gamma_A \lambda_{B+}\right)\right] \otimes \eta \\ & + \left[\gamma^\mu D_\mu \left(\lambda_{A+} + \gamma_A \lambda_{A-}\right) - i(l+1)\left(\lambda_{A-} + \gamma_A \lambda_{A+}\right) - \frac{1}{2} e^{i\frac{\pi}{4}} \epsilon_{AB} \left(\lambda_{B+} - \gamma_A \lambda_{B-}\right)\right] \otimes \gamma_S \eta \ . \end{aligned}$$

Orthogonality of spinor harmonics then give us the 2D equation of motion

$$\gamma^\mu D_\mu \hat{\lambda}_A^{(L)} - (l+1)\hat{\lambda}_A^{(L)} - \frac{1}{2} e^{i\frac{\pi}{4}} \epsilon_{AB} \hat{\lambda}_B^{(R)} = 0 \ , \tag{3.88}$$
$$\gamma^\mu D_\mu \widetilde{\lambda}_A^{(L)} + (l+1)\widetilde{\lambda}_A^{(L)} - \frac{1}{2} e^{i\frac{\pi}{4}} \epsilon_{AB} \widetilde{\lambda}_B^{(R)} = 0 \ , \tag{3.89}$$

for every spinor harmonic index $(\sigma l m)$. Here $\hat{\lambda}_A^{(L/R)}$ and $\widetilde{\lambda}_A^{(L/R)}$ are defined by

$$\hat{\lambda}_A^{(L/R)} \equiv \left(\lambda_{A+} \pm \gamma_A \lambda_{A-}\right) + i\left(\lambda_{A-} \pm \gamma_A \lambda_{A+}\right) \ , \tag{3.90}$$
$$\widetilde{\lambda}_A^{(L/R)} \equiv \left(\lambda_{A+} \pm \gamma_A \lambda_{A-}\right) - i\left(\lambda_{A-} \pm \gamma_A \lambda_{A+}\right) \ . \tag{3.91}$$

Similarly acting with the left projection operator $\frac{1}{2}(1 + \Gamma_5)$ on the 4D equations of motion (3.86), we find the 2D wave equations that are conjugate of (3.88, 3.89):

$$\gamma^\mu D_\mu \widetilde{\lambda}_B^{(R)} + (l+1)\widetilde{\lambda}_B^{(R)} + \frac{1}{2} e^{-i\frac{\pi}{4}} \epsilon_{BA} \widetilde{\lambda}_A^{(L)} = 0 \ , \tag{3.92}$$
$$\gamma^\mu D_\mu \hat{\lambda}_B^{(R)} - (l+1)\hat{\lambda}_B^{(R)} + \frac{1}{2} e^{-i\frac{\pi}{4}} \epsilon_{BA} \hat{\lambda}_A^{(L)} = 0 \ . \tag{3.93}$$

Combining (3.88) and (3.93), as well as (3.89) and (3.92) with $A = 1, B = 2$, we get

$$(\gamma^\mu D_\mu - (l+1)) \begin{pmatrix} \hat{\lambda}_1^{(L)} \\ \hat{\lambda}_2^{(R)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & e^{i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & 0 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_1^{(L)} \\ \hat{\lambda}_2^{(R)} \end{pmatrix} . \tag{3.94}$$

$$(\gamma^\mu D_\mu + (l+1)) \begin{pmatrix} \widetilde{\lambda}_1^{(L)} \\ \widetilde{\lambda}_2^{(R)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & e^{i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\lambda}_1^{(L)} \\ \widetilde{\lambda}_2^{(R)} \end{pmatrix} . \tag{3.95}$$

We are giving these results in full gory detail because the phases $e^{\pm i\frac{\pi}{4}}$ in the final result are physical consequences of the interplay between electric and magnetic fields which can be technically challenging to account for.

The matrices on the right hand side of (3.94) have eigenvalues $\pm\frac{1}{2}$. Therefore, the eigenvalues of the Dirac operator $\gamma^\mu D_\mu$ give the four AdS_2 spinor masses

$$m = \pm(l + \frac{1}{2}) \ , \ \pm(l + \frac{3}{2}) \ . \tag{3.96}$$

The sign of the fermion mass is formal and has no physical meaning. The conformal weight of the 1D conformal operator dual to an AdS_2 spinor is given by the relation $h_{\text{spinor}} = |m| + \frac{1}{2}$, so we find $h = l+1, l+2$, each with multiplicity 2.

The harmonic expansion for spinor fields has degeneracy $2(l+1)$, while the irreducible representation of $SU(2)$ labeled by the angular quantum number j has $(2j+1)$ states. We therefore identify $j = l + \frac{1}{2}$ for spinors. This gives our final result for the spectrum of the gaugino block

$$h = 2 \times (j + \frac{1}{2}) \ , \ 2 \times (j + \frac{3}{2}) \ , \tag{3.97}$$

where "2×" denotes multiplicity 2, not the normal multiplication. This result is valid for all $j \geq \frac{1}{2}$.

3.3.6 Bulk Modes of Gravitino Block

The gravitino block has the 4D Lagrangian [27]

$$\begin{aligned} e^{-1}\delta^2\mathcal{L}_{\text{gravitino}} &= -\bar{\Psi}_{A\hat{\mu}}\Gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}D_{\hat{\nu}}\Psi_{A\hat{\rho}} - 2\bar{\Lambda}_A\Gamma^{\hat{\mu}}D_{\hat{\mu}}\Lambda_A - \frac{1}{2}\epsilon_{AB}\bar{\Psi}_{A\hat{\mu}}\left(G^{\hat{\mu}\hat{\nu}} + \Gamma_5\widetilde{G}^{\hat{\mu}\hat{\nu}}\right)\Psi_{B\hat{\nu}} \\ &\quad -\frac{\sqrt{3}}{2}\left(\bar{\Psi}_{A\hat{\mu}}\hat{G}\Gamma^{\hat{\mu}}\Lambda_A + \bar{\Lambda}_A\Gamma^{\hat{\mu}}\hat{G}\Psi_{A\hat{\mu}}\right) + 2\epsilon_{AB}\bar{\Lambda}_A\hat{G}\Lambda_B \ , \end{aligned} \tag{3.98}$$

where $\widetilde{G}^{\hat{\mu}\hat{\nu}} \equiv -\frac{i}{2}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}G_{\hat{\rho}\hat{\sigma}}$. It gives the 4D equations of motion

$$\Gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}D_{\hat{\nu}}\Psi_{A\hat{\rho}} + \frac{1}{2}\left(G^{\hat{\mu}\hat{\nu}} + \Gamma_5\widetilde{G}^{\hat{\mu}\hat{\nu}}\right)\epsilon_{AB}\Psi_{B\hat{\nu}} + \frac{\sqrt{3}}{2}\hat{G}\Gamma^{\hat{\mu}}\Lambda_A = 0 \ , \tag{3.99}$$

$$2\Gamma^{\hat{\mu}}D_{\hat{\mu}}\Lambda_A - 2\epsilon_{AB}\hat{G}\Lambda_B + \frac{\sqrt{3}}{2}\Gamma^{\hat{\mu}}\hat{G}\Psi_{A\hat{\mu}} = 0 \ . \tag{3.100}$$

In the following we work out the corresponding 2D equations of motion.

Gravitino block: gravitino equation

We first act with the right projection operator $\frac{1}{2}(1-\Gamma_5)$ on the 4D equations of motion for gravitini (3.99) and then insert partial wave expansions in spinor harmonics (3.17-3.18). The S^2 components $\hat{\mu}=\alpha$ of the equations become

$$\begin{aligned}
0 = &\left[\frac{1}{2}i(l+1)\gamma_\mu\left(\psi^\mu_{A-}+\gamma_A\psi^\mu_{A+}\right) + \gamma_{\mu\nu}D^\mu\left(\psi^\nu_{A+}+\gamma_A\psi^\nu_{A-}\right) + \frac{\sqrt{3}}{2}ie^{-i\frac{\pi}{4}}\left(\lambda_{A-}+\gamma_A\lambda_{A+}\right)\right]\otimes\gamma^\alpha\eta \\
+ &\left[-\frac{1}{2}i(l+1)\gamma_\mu\left(\psi^\mu_{A+}+\gamma_A\psi^\mu_{A-}\right) + \gamma_{\mu\nu}D^\mu\left(\psi^\nu_{A-}+\gamma_A\psi^\nu_{A+}\right) + \frac{\sqrt{3}}{2}ie^{-i\frac{\pi}{4}}\left(\lambda_{A+}+\gamma_A\lambda_{A-}\right)\right]\otimes\gamma^\alpha\gamma_S\eta \\
+ &\left[-\gamma_\mu\left(\psi^\mu_{A-}+\gamma_A\psi^\mu_{A+}\right) + \gamma^\mu D_\mu\left(\psi_{A-}+\gamma_A\psi_{A+}\right) - \frac{1}{2}ie^{i\frac{\pi}{4}}\epsilon_{AB}\left(\psi_{B-}-\gamma_A\psi_{B+}\right)\right]\otimes D^{(\alpha)}\eta \\
+ &\left[-\gamma_\mu\left(\psi^\mu_{A+}+\gamma_A\psi^\mu_{A-}\right) + \gamma^\mu D_\mu\left(\psi_{A+}+\gamma_A\psi_{A-}\right) - \frac{1}{2}ie^{i\frac{\pi}{4}}\epsilon_{AB}\left(\psi_{B+}-\gamma_A\psi_{B-}\right)\right]\otimes D^{(\alpha)}\gamma_S\eta
\end{aligned} \tag{3.101}$$

The AdS_2 components $\hat{\mu}=\mu$ of the equations similarly give

$$\begin{aligned}
0 = &\Big[-i(l+1)\gamma_{\mu\nu}\left(\psi^\nu_{A+}+\gamma_A\psi^\nu_{A-}\right) + \frac{1}{2}((l+1)^2-1)\gamma_\mu\left(\psi_{A-}+\gamma_A\psi_{A+}\right) \\
&-\frac{\sqrt{3}}{2}ie^{i\frac{\pi}{4}}\gamma_\mu\left(\lambda_{A+}+\gamma_A\lambda_{A-}\right) + \frac{1}{2}e^{-i\frac{\pi}{4}}\gamma_{\mu\nu}\epsilon_{AB}\left(\psi^\nu_{B-}-\gamma_A\psi^\nu_{B+}\right)\Big]\otimes\eta \\
+ &\Big[i(l+1)\gamma_{\mu\nu}\left(\psi^\nu_{A-}+\gamma_A\psi^\nu_{A+}\right) + \frac{1}{2}((l+1)^2-1)\gamma_\mu\left(\psi_{A+}+\gamma_A\psi_{A-}\right) \\
&-\frac{\sqrt{3}}{2}ie^{i\frac{\pi}{4}}\gamma_\mu\left(\lambda_{A-}+\gamma_A\lambda_{A+}\right) + \frac{1}{2}e^{-i\frac{\pi}{4}}\gamma_{\mu\nu}\epsilon_{AB}\left(\psi^\nu_{B+}-\gamma_A\psi^\nu_{B-}\right)\Big]\otimes\gamma_S\eta \ .
\end{aligned} \tag{3.102}$$

Again, we suppress the indices (σlm) of the spinor harmonics. Orthogonality of the spinor harmonics mean each square bracket vanishes by itself. This gives four towers of equations from the S^2 but only two from the AdS_2 because the $\hat{\mu}=\alpha$ index incorporates spin-$\frac{3}{2}$ components on S^2 while the $\hat{\mu}=\mu$ index only includes spin-$\frac{1}{2}$ components. To present the equations we define

$$\hat{\psi}^{(L/R)}_{A\mu} \equiv (\psi_{A\mu+} \pm \gamma_A \psi_{A\mu-}) + i\,(\psi_{A\mu-} \pm \gamma_A \psi_{A\mu+}) \ , \tag{3.103}$$
$$\widetilde{\psi}^{(L/R)}_{A\mu} \equiv (\psi_{A\mu+} \pm \gamma_A \psi_{A\mu-}) - i\,(\psi_{A\mu-} \pm \gamma_A \psi_{A\mu+}) \ , \tag{3.104}$$

in analogy with the variables $\hat{\lambda}^{(L/R)}_A$ and $\widetilde{\lambda}^{(L/R)}_A$ introduced for the gaugino block in (3.90). This gives the three coupled equations

$$\gamma^{\mu\nu} D_\mu \widetilde{\psi}^{(L)}_{A\nu} - \frac{1}{2}(l+1)\gamma^\mu \widetilde{\psi}^{(L)}_{A\mu} + \frac{\sqrt{3}}{2} e^{-i\frac{\pi}{4}} \hat{\lambda}^{(L)}_A = 0 \ , \tag{3.105}$$
$$-\gamma^\mu \widetilde{\psi}^{(L)}_{A\mu} + \gamma^\mu D_\mu \widetilde{\psi}^{(L)}_A - \frac{1}{2} i e^{i\frac{\pi}{4}} \epsilon_{AB} \widetilde{\psi}^{(R)}_B = 0 \ , \tag{3.106}$$
$$(l+1)\widetilde{\psi}^{(L)}_{A\rho} - \frac{1}{2}((l+1)^2-1)\gamma_\rho \widetilde{\psi}^{(L)}_A + \frac{\sqrt{3}}{2} e^{-i\frac{\pi}{4}} \hat{\lambda}^{(L)}_A - \frac{1}{2} e^{-i\frac{\pi}{4}} \epsilon_{AB} \widetilde{\psi}^{(R)}_{B\rho} = 0 \ , \tag{3.107}$$

as well as the three coupled equations

$$\gamma^{\mu\nu} D_\mu \hat{\psi}^{(L)}_{A\nu} + \frac{1}{2}(l+1)\gamma^\mu \hat{\psi}^{(L)}_{A\mu} - \frac{\sqrt{3}}{2} e^{-i\frac{\pi}{4}} \widetilde{\lambda}^{(L)}_A = 0 \ , \tag{3.108}$$
$$-\gamma^\mu \hat{\psi}^{(L)}_{A\mu} + \gamma^\mu D_\mu \hat{\psi}^{(L)}_A - \frac{1}{2} i e^{i\frac{\pi}{4}} \epsilon_{AB} \hat{\psi}^{(R)}_B = 0 \ , \tag{3.109}$$
$$-(l+1)\hat{\psi}^{(L)}_{A\rho} - \frac{1}{2}((l+1)^2-1)\gamma_\rho \hat{\psi}^{(L)}_A - \frac{\sqrt{3}}{2} e^{-i\frac{\pi}{4}} \widetilde{\lambda}^{(L)}_A - \frac{1}{2} e^{-i\frac{\pi}{4}} \epsilon_{AB} \hat{\psi}^{(R)}_{B\rho} = 0 \ . \tag{3.110}$$

Similarly, starting out by acting with the left projection operator $\frac{1}{2}(1+\Gamma_5)$ on the 4D equations of motion we find the complex conjugate of the preceding six equations.

Gravitino block: gaugino equation

Acting with the right projection operator $\frac{1}{2}(1-\Gamma_5)$ on the 4D equations of motion for gaugini (3.100) and expanding it in partial waves, we get

$$\begin{aligned}
&\Big[\gamma^\mu D_\mu (\lambda_{A-} + \gamma_A \lambda_{A+}) + i(l+1)(\lambda_{A+} + \gamma_A \lambda_{A-}) + i e^{-i\frac{\pi}{4}} \epsilon_{AB} (\lambda_{B-} - \gamma_A \lambda_{B+}) \\
&- \frac{\sqrt{3}}{4} i e^{i\frac{\pi}{4}} \gamma_\mu \left(\psi^\mu_{A+} + \gamma_A \psi^\mu_{A-}\right)\Big] \otimes \eta \, + \\
&\Big[\gamma^\mu D_\mu (\lambda_{A+} + \gamma_A \lambda_{A-}) - i(l+1)(\lambda_{A-} + \gamma_A \lambda_{A+}) + i e^{-i\frac{\pi}{4}} \epsilon_{AB} (\lambda_{B+} - \gamma_A \lambda_{B-}) \\
&- \frac{\sqrt{3}}{4} i e^{i\frac{\pi}{4}} \gamma_\mu \left(\psi^\mu_{A-} + \gamma_A \psi^\mu_{A+}\right)\Big] \otimes \gamma_S \eta = 0 \ .
\end{aligned} \tag{3.111}$$

Again, orthogonality implies that each square bracket vanishes by itself. After introduction of the variables (3.90) the 2D equation of motion become

$$\gamma^\mu D_\mu \hat{\lambda}_A^{(L)} - (l+1)\hat{\lambda}_A^{(L)} + ie^{-i\frac{\pi}{4}}\epsilon_{AB}\hat{\lambda}_B^{(R)} + \frac{\sqrt{3}}{4}e^{i\frac{\pi}{4}}\gamma^\mu \widetilde{\psi}_{A\mu}^{(L)} = 0 \ , \tag{3.112}$$

$$\gamma^\mu D_\mu \widetilde{\lambda}_A^{(L)} + (l+1)\widetilde{\lambda}_A^{(L)} + ie^{-i\frac{\pi}{4}}\epsilon_{AB}\widetilde{\lambda}_B^{(R)} - \frac{\sqrt{3}}{4}e^{i\frac{\pi}{4}}\gamma^\mu \hat{\psi}_{A\mu}^{(L)} = 0 \ . \tag{3.113}$$

Similarly, starting out by acting with the left projection operator $\frac{1}{2}(1+\Gamma_5)$ on the 4D equations of motion we find the complex conjugate of these two equations.

We next proceed to solve the 2D equations of motion and compute the mass spectrum of gravitino block. We first discuss modes with $l \geq 1$ modes and then deal with the special case $l = 0$.

Gravitino block: $l \geq 1$ ***modes***

We begin by considering (3.105, 3.106, 3.107, 3.112) and the equations conjugate to (3.108, 3.109, 3.110,3.113) since only $\widetilde{\psi}_{A\mu}^{(L/R)}$, $\widetilde{\psi}_A^{(L/R)}$ and $\hat{\lambda}_A^{(L/R)}$ are involved.in this system.

Inspection of (3.107) with $A = 1$ and the conjugate of (3.110) with $A = 2$ shows that the 2D gravitino $\widetilde{\psi}_{A\mu}^{(L/R)}$ is not an independent field. It can be expressed by $\widetilde{\psi}_A^{(L/R)}$ and $\hat{\lambda}_A^{(L/R)}$ as

$$\begin{pmatrix} \widetilde{\psi}_{1\mu}^{(L)} \\ \widetilde{\psi}_{2\mu}^{(R)} \end{pmatrix} = \begin{pmatrix} \frac{4(l+1)}{4(l+1)^2-1} & \frac{2e^{-i\frac{\pi}{4}}}{4(l+1)^2-1} \\ \frac{2e^{i\frac{\pi}{4}}}{4(l+1)^2-1} & \frac{4(l+1)}{4(l+1)^2-1} \end{pmatrix} \gamma_\mu \left[\frac{1}{2}((l+1)^2-1) \begin{pmatrix} \widetilde{\psi}_1^{(L)} \\ \widetilde{\psi}_2^{(R)} \end{pmatrix} - \frac{\sqrt{3}}{2} \begin{pmatrix} e^{-i\frac{\pi}{4}}\hat{\lambda}_1^{(L)} \\ e^{i\frac{\pi}{4}}\hat{\lambda}_2^{(R)} \end{pmatrix} \right] . \tag{3.114}$$

Inserting this into (3.106, 3.112) with $A = 1$ and the conjugates of (3.109, 3.113) with $A = 2$ we find

$$\begin{aligned} \gamma^\mu D_\mu \begin{pmatrix} \hat{\lambda}_1^{(L)} \\ \hat{\lambda}_2^{(R)} \\ \widetilde{\psi}_1^{(L)} \\ \widetilde{\psi}_2^{(R)} \end{pmatrix} &= \begin{pmatrix} l+1 & -e^{i\frac{\pi}{4}} & 0 & 0 \\ -e^{-i\frac{\pi}{4}} & l+1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}e^{-i\frac{\pi}{4}} \\ 0 & 0 & -\frac{1}{2}e^{i\frac{\pi}{4}} & 0 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_1^{(L)} \\ \hat{\lambda}_2^{(R)} \\ \widetilde{\psi}_1^{(L)} \\ \widetilde{\psi}_2^{(R)} \end{pmatrix} + \gamma^\mu \begin{pmatrix} -\frac{\sqrt{3}}{4}e^{i\frac{\pi}{4}}\widetilde{\psi}_{1\mu}^{(L)} \\ -\frac{\sqrt{3}}{4}e^{-i\frac{\pi}{4}}\widetilde{\psi}_{2\mu}^{(R)} \\ \widetilde{\psi}_{1\mu}^{(L)} \\ \widetilde{\psi}_{2\mu}^{(R)} \end{pmatrix} \\ &= \begin{pmatrix} (l+1) + \frac{3(l+1)}{4(l+1)^2-1} & \left(\frac{3(l+1)}{8(l+1)^2-2} - 1\right)e^{i\frac{\pi}{4}} & -\frac{\sqrt{3}(l+1)((l+1)^2-1)}{4(l+1)^2-1}e^{i\frac{\pi}{4}} & -\frac{\sqrt{3}((l+1)^2-1)}{8(l+1)^2-2} \\ \left(\frac{3(l+1)}{8(l+1)^2-2} - 1\right)e^{-i\frac{\pi}{4}} & (l+1) + \frac{3(l+1)}{4(l+1)^2-1} & -\frac{\sqrt{3}((l+1)^2-1)}{8(l+1)^2-2} & -\frac{\sqrt{3}(l+1)((l+1)^2-1)}{4(l+1)^2-1}e^{-i\frac{\pi}{4}} \\ \frac{4\sqrt{3}(l+1)}{4(l+1)^2-1}e^{-i\frac{\pi}{4}} & -\frac{2\sqrt{3}}{4(l+1)^2-1} & (l+1) - \frac{3(l+1)}{4(l+1)^2-1} & -\frac{3}{8(l+1)^2-2}e^{-i\frac{\pi}{4}} \\ -\frac{2\sqrt{3}}{4(l+1)^2-1} & \frac{4\sqrt{3}(l+1)}{4(l+1)^2-1}e^{i\frac{\pi}{4}} & -\frac{3}{8(l+1)^2-2}e^{i\frac{\pi}{4}} & (l+1) - \frac{3(l+1)}{4(l+1)^2-1} \end{pmatrix} \begin{pmatrix} \hat{\lambda}_1^{(L)} \\ \hat{\lambda}_2^{(R)} \\ \widetilde{\psi}_1^{(L)} \\ \widetilde{\psi}_2^{(R)} \end{pmatrix} . \end{aligned} \tag{3.115}$$

The matrix on the right hand side appears very complicated but, remarkably, it has simple eigenvalues: $l - \frac{1}{2}$, $l + \frac{1}{2}$, $l + \frac{3}{2}$, and $l + \frac{5}{2}$.

Similarly, taking equations (3.108, 3.109, 3.110, 3.113) with $A = 1$ and the conjugates of (3.105, 3.106, 3.107, 3.112) with $A = 2$ we find a matrix equation for $\widetilde{\lambda}_1^{(L)}, \widetilde{\lambda}_2^{(R)}, \hat{\psi}_1^{(L)}, \hat{\psi}_2^{(R)}$. The matrix again has simple eigenvalues: $-\left(l - \frac{1}{2}\right)$, $-\left(l + \frac{1}{2}\right)$, $-\left(l + \frac{3}{2}\right)$, and $-\left(l + \frac{5}{2}\right)$. Thus the complete result for the eigenvalues of $\gamma^\mu D_\mu$ can be expressed as eight AdS_2 spinor masses

$$m = \pm(l - \frac{1}{2}) \ , \ \pm(l + \frac{1}{2}) \ , \ \pm(l + \frac{3}{2}) \ , \ \pm(l + \frac{5}{2}) \ . \tag{3.116}$$

For spinors we use the relations $j_{\text{spinor}} = l + \frac{1}{2}$ for the $SU(2)$ quantum number and $h_{\text{spinor}} = |m| + \frac{1}{2}$ for the conformal weight and so our result for the spectrum of the gravitino block becomes

$$h = 2 \times (j - \frac{1}{2}) \ , \ 2 \times (j + \frac{1}{2}) \ , \ 2 \times (j + \frac{3}{2}) \ , \ 2 \times (j + \frac{5}{2}) \ , \tag{3.117}$$

where "2×" denotes multiplicity 2, not the normal multiplication. This result is valid for $j \geq \frac{3}{2}$.

Gravitino block: $l = 0$ ***modes***

The $l = 0$ mode is special for gravitini because the helicity $\pm\frac{3}{2}$ components in the partial wave expansion (3.19) vanish identically. Therefore, the fields $\psi_{A\pm}$ are not defined for $l = 0$.

The manipulations giving (3.114) for the AdS$_2$ gravitini remain valid for $l = 0$ and we see that, in this special case, the term involving the nonexistent $\psi_{A\pm}$ has vanishing coefficient. Therefore, all components of the 2D gravitini $\psi_{A\mu}$ are determined by the gaugini λ_A. Accordingly, the first equation in (3.115) depends only on gaugini for $l = 0$. The resulting equation of motion can be read off from the upper left 2×2 submatrix of the second equation in (3.115) by taking $l = 0$:

$$\gamma^\mu D_\mu \begin{pmatrix} \hat{\lambda}_1^{(L)} \\ \hat{\lambda}_2^{(R)} \end{pmatrix} = \begin{pmatrix} 2 & -\frac{1}{2}e^{i\frac{\pi}{4}} \\ -\frac{1}{2}e^{-i\frac{\pi}{4}} & 2 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_1^{(L)} \\ \hat{\lambda}_2^{(R)} \end{pmatrix} \ . \tag{3.118}$$

The matrix on the right hand side has eigenvalues $\frac{3}{2}$ and $\frac{5}{2}$ and the analogous equations for $\widetilde{\lambda}_A^{(L/R)}$ similarly give $-\frac{3}{2}$ and $-\frac{5}{2}$. This corresponds to two modes with conformal weight $h = 2$ and another two with $h = 3$. These modes extend the towers $h = 2\times(j+\frac{3}{2}),\ 2\times(j+\frac{5}{2})$ in (3.117) so they apply for all $j \geq \frac{1}{2}$.

3.3.7 Boundary Modes

Boundary modes are harmonic modes on AdS$_2$ which are formally pure gauge but in fact physical because the gauge functions that generate them are non-normalizable. There are no boundary modes for the scalar block or gaugino block because they involve no gauge symmetries. Thus all boundary modes come from vector blocks, the KK block, and gravitino blocks. This subsection determines the boundary modes of these three types of blocks in turns.

Since boundary modes are somewhat subtle we proceed with special care. In each case we add gauge fixing terms and compute the full off-shell spectrum, along with the appropriate ghosts. This requires some additional effort. On the other hand, since the gauge functions underlying boundary modes are harmonic, they generally do not couple to bulk modes so the relevant field content remains manageable.

Boundary modes in vector blocks

For boundary modes in vector blocks we add the gauge fixing term $\left(\nabla_{\hat{\mu}} a^{\hat{\mu}}\right)^2$ to the Lagrangian (3.27), with hatted variables denoting 4D indices as in previous sections. We can consistently ignore the scalar field in the vector block because it couples to $\nabla_\mu b^\mu$ which vanishes in the boundary sector due to the harmonic condition. The effective Lagrangian of the boundary modes in the vector block becomes

$$\mathcal{L}^{\text{bndy}}_{\text{vector}} = b^\mu \left(\nabla_A^2 + 1 - l(l+1)\right) b_\mu \ . \tag{3.119}$$

Harmonic vector modes satisfy $\nabla_A^2 + 1$ so this is equivalent to a tower of nondynamical fields with $m^2 = l(l+1)$, $l \geq 0$ with degeneracy $2l+1$. This result is unsurprising because the residual gauge transformations underlying these modes satisfy the massless Klein-Gordon equation in 4D $\nabla_4^2 \Lambda = 0$.

Boundary modes in the KK block

In this sector we must consider gravity as well as the KK vector field. We add the gauge fixing term

$$\mathcal{L}^{\text{g.f.}}_{\text{KK}} = -\left(\nabla^{\hat{\mu}} h_{\hat{\mu}\hat{\rho}} - \frac{1}{2}\nabla_{\hat{\rho}} h\right)\left(\nabla^{\hat{\nu}} h_{\hat{\nu}}{}^{\hat{\rho}} - \frac{1}{2}\nabla^{\hat{\rho}} h\right) - 4\left(\nabla_{\hat{\mu}} a^{\hat{\mu}}\right)^2 \ , \tag{3.120}$$

to the 4D Lagrangian of the KK block. We then compute the corresponding 2D Lagrangian

$$\begin{aligned}
\mathcal{L}^{\text{bndy}}_{\text{KK}} &= H'^{\{\mu\nu\}} \left(\nabla_x^2 + 2 - l(l+1)\right) H'_{\{\mu\nu\}} \qquad (3.121)\\
&+ \begin{pmatrix} b'_\mu & B'_{2\mu} & B'_{1\mu} \end{pmatrix} \begin{pmatrix} \left(\nabla_A^2 + 1 - l(l+1)\right)\delta^\mu_\nu & -\sqrt{2l(l+1)}\delta^\mu_\nu & -\sqrt{2l(l+1)}\epsilon^\mu{}_\nu \\ -\sqrt{2l(l+1)}\delta^\mu_\nu & \left(\nabla_A^2 + 1 - l(l+1)\right)\delta^\mu_\nu & -2\epsilon^\mu{}_\nu \\ \sqrt{2l(l+1)}\epsilon^\mu{}_\nu & 2\epsilon^\mu{}_\nu & \left(\nabla_A^2 + 1 - l(l+1)\right)\delta^\mu_\nu \end{pmatrix} \begin{pmatrix} b'^\nu \\ B'^\nu_2 \\ B'^\nu_1 \end{pmatrix} ,
\end{aligned}$$

where we introduced conveniently normalized fields $H'_{\{\mu\nu\}} = \frac{1}{\sqrt{2}} H_{\{\mu\nu\}}$, $b'_\mu = 2b_\mu$ and $B'^\mu_{1/2} = \sqrt{l(l+1)} B^\mu_{1/2}$. We consistently ignored scalar fields because their couplings to AdS_2 vectors all contain $\nabla_\mu b^\mu$ or $\nabla_\mu B^\mu_{1/2}$ which vanish for boundary modes.

We can diagonalize the mass matrix in $\mathcal{L}^{\text{bndy}}_{\text{KK}}$ above and so determine the eigenvalues of ∇_A^2. The scalars that are equivalent to AdS_2 tensors and vectors have masses given by the eigenvalues of $\nabla_A^2 + 2$ and $\nabla_A^2 + 1$, respectively. The eigenvectors and the corresponding scalar masses become:

$$m^2 = l(l+1) \quad : \quad H'_{\{\mu\nu\}} \ , \ l \geq 0 \ , \tag{3.122}$$

$$m^2 = l(l-1) \quad : \quad \sqrt{\frac{1+l}{1+2l}} b'_\mu - \sqrt{\frac{l}{1+2l}} \frac{1}{\sqrt{2}} \left(B'_{2\mu} + \epsilon_{\mu\nu} B'^{\nu}_1 \right) \ , \ l \geq 0 \ , \tag{3.123}$$

$$m^2 = (l+1)(l+2) \quad : \quad \sqrt{\frac{l}{1+2l}} b'_\mu + \sqrt{\frac{1+l}{1+2l}} \frac{1}{\sqrt{2}} \left(B'_{2\mu} + \epsilon_{\mu\nu} B'^{\nu}_1 \right) \ , \ l \geq 1 \ , \tag{3.124}$$

$$m^2 = l(l+1) + 2 \quad : \quad \frac{1}{\sqrt{2}} \left(B'_{1\mu} + \epsilon_{\mu\nu} B'^{\nu}_2 \right) \ , \ l \geq 1 \ . \tag{3.125}$$

The last three entries deserve some comments: the matrix in the second line of (3.121) acts on AdS_2 vectors so it is 6×6 and it allows 6 eigenvectors. However, harmonic modes satisfy a duality condition so, in order to avoid overcounting we should take either the eigenvector or the dual eigenvector, not both. Since the formalism is off-shell there is some scheme-dependence to this choice. The analogous treatment of boundary modes for BPS black holes by Sen [8] includes all contributions and divide by two in the end. We elect instead to pick an orthogonal set that diagonalizes the off-diagonal terms in (3.121) and cancels the ghost tower determined below. This choice seems more physical to us but the final on-shell results are at any rate independent of scheme.

Tensors $H_{\{\mu\nu\}}$ have degeneracy three [8, 75], therefore the boundary modes in the KK block are 3 towers of $m^2 = l(l+1)$ with $l \geq 0$, 1 tower of $m^2 = l(l-1)$ with $l \geq 0$, 1 tower of $m^2 = (l+1)(l+2)$ with $l \geq 1$, and 1 tower of $m^2 = l(l+1) + 2$ with $l \geq 1$.

In the off-shell formalism that we apply for boundary modes we must consider also the contribution from the ghost that generates the diffeomorphism $\delta H_{\{\mu\nu\}} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - g_{\mu\nu} \nabla^\rho \xi_\rho$. The ghost equation of motion follows by variation of the gauge condition under a diffeomorphism:

$$\delta \left(\nabla^{\hat{\mu}} h_{\hat{\mu}\hat{\rho}} - \frac{1}{2} \nabla_{\hat{\rho}} h \right) = 0 \quad \Rightarrow \quad \left(\nabla_A^2 - 1 - l(l+1) \right) \xi_\rho = 0 \ . \tag{3.126}$$

Since eigenvalues of $\nabla_A^2 + 1$ acting on a vector can be identified with the mass of the dual scalar we find that the boundary ghosts have $m^2 = l(l+1) + 2, l \geq 0$ with degeneracy $2l + 1$. These contributions effectively cancel one of the 6 towers of KK boundary modes.

Boundary modes in gravitino blocks

In the off-shell formalism that we apply to boundary modes we add the gauge fixing term $\frac{1}{2} \left(\bar{\Psi}_{A\hat{\mu}} \Gamma^{\hat{\mu}} \right) \Gamma^{\hat{\nu}} D_{\hat{\nu}} \left(\Gamma^{\hat{\rho}} \Psi_{A\hat{\rho}} \right)$ to the gravitino Lagrangian (3.98) and redefine the field as

$\Phi_{A\hat{\mu}} = \Psi_{A\hat{\mu}} - \frac{1}{2}\Gamma_{\hat{\mu}}\Gamma^{\hat{\rho}}\Psi_{A\hat{\rho}}$. The Lagrangian for the gravitini then becomes

$$e^{-1}\delta^2\mathcal{L}^{\text{bndy}}_{\text{gravitino}} = -\bar{\Phi}_{A\hat{\mu}} H^{\hat{\mu}\hat{\nu}}_{AB}\Phi_{B\hat{\nu}} = -\bar{\Phi}_{A\hat{\mu}}\left[\Gamma^{\hat{\rho}}D_{\hat{\rho}}g^{\hat{\mu}\hat{\nu}} - \frac{1}{2}\epsilon_{AB}\left(G^{\hat{\mu}\hat{\nu}} + \Gamma_5\widetilde{G}^{\hat{\mu}\hat{\nu}}\right)\right]\Phi_{B\hat{\nu}} \ ,$$

after consistently ignoring the gaugini which do not couple to harmonic modes.

The square of the quadratic fluctuation operator $H^{\hat{\mu}\hat{\nu}}_{AB}$ is

$$\Lambda^{\hat{\mu}\hat{\nu}}_{AB} = H^{\hat{\mu}\hat{\rho}\dagger}_{AC} H_{\hat{\rho}\ CB}^{\ \hat{\nu}} \ = \ -\left(\Gamma^{\hat{\rho}}D_{\hat{\rho}}\Gamma^{\hat{\sigma}}D_{\hat{\sigma}} + \frac{1}{4}\right) g^{\hat{\mu}\hat{\nu}}\delta_{AB} \ . \tag{3.127}$$

Only the components of $\Phi^{\hat{\mu}}_A$ with $\hat{\mu} = \mu$ have boundary modes. Using the partial wave expansion $\Phi^\mu_A = \phi^\mu_{A+}\otimes\eta + \phi^\mu_{A-}\otimes\gamma_S\eta$, the equation of motion $\Lambda^{\mu\hat{\nu}}_{AB}\Phi_{B\hat{\nu}} = 0$ can be expanded as

$$-\left(\gamma^\rho D_\rho\gamma^\sigma D_\sigma - (l+1)^2 + \frac{1}{4}\right)\phi^\mu_{A+}\otimes\eta - \left(\gamma^\rho D_\rho\gamma^\sigma D_\sigma - (l+1)^2 + \frac{1}{4}\right)\phi^\mu_{A-}\otimes\gamma_S\eta = 0 \ .$$

Thus we find that there are 4 towers of gravitino boundary modes with identical mass squared $m^2 = (l+1)^2 - \frac{1}{4}$, each with the degeneracy $2l+2$.

3.3.8 Summary of the Mass Spectrum

As conclusion to this long section we summarize our results.

Blocks	Bulk Modes Spectrum (h,j)
Scalar	$(k+1,k)$
Gaugino	$2(k+2,k+\frac{1}{2})$ $\quad$ $2(k+1,k+\frac{1}{2})$
Vector	$(k+2,k)$ $\quad$ $(k+2,k+1)$ $\quad$ $(k+1,k+1)$
Gravitino	$2(k+3,k+\frac{1}{2})$ $\quad$ $2(k+2,k+\frac{1}{2})$ $\quad$ $2(k+2,k+\frac{3}{2})$ $\quad$ $2(k+1,k+\frac{3}{2})$
KK	$(k+3,k)$ $\quad$ $(k+3,k+1)$ $\quad$ $(k+2,k+1)$ $\quad$ $(k+2,k+2)$ $\quad$ $(k+1,k+2)$

Table 3.2: Mass spectrum of bulk modes in nonBPS blocks. The label $k = 0, 1, \ldots$

The mass spectrum (h,j) for the bulk modes is given in table 3.2. In all cases the conformal weight h is related to an effective scalar mass as $m^2 = h(h-1)$ for bosons or $m^2 = (h-\frac{1}{2})^2$ for fermions. For scalar fields m^2 is the on-shell eigenvalue of ∇^2_A. However, for vectors and tensors we identify m^2 as the eigenvalue of $\nabla^2_A + 1$ and $\nabla^2_A + 2$, respectively. This is justified by the action of the operators $\nabla^2_A + 1$, $\nabla^2_A + 2$ on vectors and tensors being equivalent to the action of ∇_A^2 on the corresponding scalar field obtained by the appropriate dualization in AdS_2. Thus we can use the formula for the conformal weight (3.1) for all bosons.

A notable feature of the results recorded in table 3.2 is that the conformal weight h is integral in all cases. Since h is determined for each entry by solving the quadratic $m^2 = h(h-1)$ or $m^2 = (h-\frac{1}{2})^2$ this is a rather nontrivial result. It requires that all effective masses are such that the discriminant of the quadratic is a perfect square. This property would be expected if the spectrum was organized in supermultiplets but, in the present context, supersymmetry is entirely broken by the background. We will develop this point further in sections 3.5 and 3.6.

The angular momentum quantum number j labels the irreducible representation of $SU(2)$. There are $(2j+1)$ states for each value of j. The values of j are integral (half-integral) for bosons (fermions), as expected. It is interesting that, in contrast, the conformal weight h is integral for both bosons and fermions. In our context states are not organized in supermultiplets so there is no general expectation that h must be half-integral for fermions but the result seems surprising nonetheless. We will also develop this point further in section 3.5.

The scalars in the vector block generally mix with the vector field. However, the vector field does not include a spherically symmetric mode so the $j = 0$ sector has just one mode, an effective 2D scalar with $h = 2$. A minimally coupled scalar would have $h = 1$, as for the scalar block, so these scalars are non-minimally coupled even in the spherically symmetric sector. This is an aspect of the attractor mechanism which determines the horizon value of the scalars in the vector block as a function of the charges and therefore inhibits their fluctuations around the preferred attractor value. This is a nonBPS version of the mechanism familiar from BPS black holes where these fields are known as fixed scalars [91].

The analogous scalar mode in the $j = 0$ sector of the KK block is also interesting. It has conformal weight $h = 3$. Thus the coupling between the KK scalar and gravity is stronger than the analogous coupling between gauge fields and their scalar partners. This effect has no analogue on the BPS branch but the $h = 3$ mode was previously identified for rotating black holes [70].

Boundary modes are more subtle since they are based on harmonic modes which have no bulk kinetic term. For these modes we worked out the full off-shell spectrum, to circumvent any ambiguity. The result is present in terms of "masses" in table 3.3. The mass indicates the departure from a true zero-mode so m^2 is the eigenvalue appropriate for computing functional determinants.

Blocks	Masses of Boundary Modes	Degeneracy	Multiplicity	Range
KK	$m^2 = l(l+1)$	$2l+1$	3	$l \geq 0$
KK	$m^2 = (l-1)l$	$2l+1$	1	$l \geq 0$
KK	$m^2 = (l+1)(l+2)$	$2l+1$	1	$l \geq 1$
KK	$m^2 = l(l+1)+2$	$2l+1$	1	$l \geq 1$
KK	$m^2 = l(l+1)+2$	$2l+1$	-1	$l \geq 0$
Vector	$m^2 = l(l+1)$	$2l+1$	1	$l \geq 0$
Gravitino	$m^2 = (l+1)^2 - \frac{1}{4}$	$2l+2$	4	$l \geq 0$

Table 3.3: Mass spectrum of boundary modes in nonBPS blocks. (Multiplicity -1 denotes contribution from ghosts.)

3.4 Heat Kernels

In this section we use the mass spectrum determined in the previous section to compute the 4D heat kernel and the associated logarithmic corrections to black hole entropy. There are three distinct contributions:

- *Bulk modes*: The propagating degrees of freedom summarized in table 3.2.
- *Boundary modes*: The global degrees of freedom due to the harmonic modes of AdS_2 vectors, gravitini, and tensors. They are summarized in table 3.3.
- *Zero mode corrections*: On-shell boundary modes that were already counted as boundary modes need corrections to their counting weights.

3.4.1 Heat Kernel Preliminaries

The action for quadratic fluctuations around a background has the generic form

$$\mathcal{S} = -\int d^4x\sqrt{-g}\ \phi_n \Lambda^n_m \phi^m\ , \tag{3.128}$$

where $\{\phi_n\}$ is a complete set of fields and Λ^n_m is a matrix that encodes the action of quadratic fluctuations around the background. The heat kernel of the operator Λ is then defined by

$$K_4(s) = \text{Tr}\ e^{-s\Lambda} = \sum_i e^{-s\lambda_i}\ , \tag{3.129}$$

where $\{\lambda_i\}$ is the set of eigenvalues of Λ.

We denote the heat kernel of a massless field on AdS_2 by $K_A(s)$. The result for a mode with effective 2D mass m^2 is suppressed by an additional factor $e^{-m^2 s}$ so, upon summing over a complete tower of states with masses m_j^2 and $SU(2)$ quantum number j, we find

$$K_4(s) = K_A(s)K_S(s) = K_A(s)\,\frac{1}{4\pi}\sum_j (2j+1)e^{-m_j^2 s}\;, \tag{3.130}$$

for the 4D heat kernel on $\mathrm{AdS}_2 \times S^2$. The sum over the tower can be interpreted as a field on S^2 and so we divide it by the area 4π of the unit S^2 and denote it by $K_S(s)$. The masses are related to conformal weights as $m^2 = h(h-1)$ for bosons and $m^2 = (h-\frac{1}{2})^2 = h(h-1)+\frac{1}{4}$ for fermions.

The Laurent expansion of the heat kernel $K_4(s)$ around $s=0$ generally has poles of order s^{-2} and s^{-1}, followed by a constant that we denote K_4^{const}. It is related to the central charge a of the 4D conformal anomaly by

$$a = 2\pi^2 K_4^{\mathrm{const}}\;. \tag{3.131}$$

The other central charge c is immaterial here because the $\mathrm{AdS}_2 \times S^2$ background is conformally flat.

3.4.2 Bulk Modes

All bulk bosons in 2D are represented as scalars. A massless scalar on Euclidean AdS_2 has continuous eigenvalues $\lambda_A = p^2 + \frac{1}{4},\ p \in \mathbb{R}$ weighted by the Plancherel measure $\mu(p) = p\tanh(\pi p)$. It has heat kernel [74]:

$$\begin{aligned} K_A^b(s) &= \frac{1}{2\pi}\int_0^\infty dp\; p\tanh(\pi p)\exp\left[-s\left(p^2+\frac{1}{4}\right)\right] \\ &= \frac{1}{4\pi s}\left(1-\frac{1}{3}s+\frac{1}{15}s^2+\mathcal{O}(s^3)\right)\;. \end{aligned} \tag{3.132}$$

For sums over towers of modes an essential benchmark is the heat kernel of a minimally coupled scalar on S^2. The standard result from introductory quantum mechanics is that the eigenvalues of $-\nabla_S^2$ are $l(l+1)$ with degeneracy $2l+1$ and range $l = 0, 1. \ldots$. This gives the heat kernel:

$$K_S^b(s) = \frac{1}{4\pi}\sum_{k=0}^\infty (2k+1)e^{-k(k+1)s} = \frac{1}{4\pi s}\left(1+\frac{1}{3}s+\frac{1}{15}s^2+\mathcal{O}(s^3)\right)\;. \tag{3.133}$$

With these results the spectrum for bulk bosons given in table 3.2 yields the following heat kernels:

Scalar block

The scalar block is just a minimal scalar with spectrum $(h, j) = (k+1, k)$ for $k \geq 0$. For bosons, we have $m^2 = h(h-1)$ with degeneracy $2j+1$, therefore

$$\begin{aligned} K_4^{\text{scalar}} &= K_A^b K_S^{\text{scalar}} = K_A^b \frac{1}{4\pi} \sum_{k=0} (2k+1) e^{-k(k+1)s} \\ &= \frac{1}{16\pi^2 s^2} \left(1 + \frac{1}{45} s^2 + \mathcal{O}(s^3) \right) . \end{aligned} \tag{3.134}$$

where we used (3.133) for the sum over the tower. The constant term $K_4^{\text{const}} = \frac{1}{720\pi^2}$ corresponds to the conformal anomaly $a^{\text{scalar.bulk}} = \frac{1}{360}$ according to (3.131). This is the standard answer for a minimally coupled scalar (1 d.o.f.) [92, 93].

Vector block

The spectrum of the vector block has 3 towers: $(h, j) = (k+2, k)$, $(k+2, k+1)$, $(k+1, k+1)$ for $k \geq 0$. Therefore we have

$$\begin{aligned} K_4^{\text{vector}} &= K_A^b K_S^{\text{vector}} \\ &= \frac{K_A^b}{4\pi} \left(\sum_{k=0} (2k+1) e^{-(k+1)(k+2)s} + \sum_{k=1} (2k+1) \left(e^{-k(k+1)s} + e^{-(k-1)ks} \right) \right) \\ &= K_A^b \frac{1}{4\pi} \left(3 \sum_{k=0} (2k+1) e^{-k(k+1)s} \right) = 3 K_4^{\text{scalar}} . \end{aligned} \tag{3.135}$$

Thus the vector block (3 d.o.f.) has the same heat kernel as 3 minimally coupled scalars: $a^{\text{vector.bulk}} = \frac{1}{120}$.

KK block

The spectrum of the KK block has 5 towers: $(h, j) = (k+3, k)$, $(k+3, k+1)$, $(k+2, k+1)$,

$(k+2, k+2)$, $(k+1, k+2)$ for $k \geq 0$. Therefore we have

$$
\begin{aligned}
K_4^{\text{gravity}} &= K_A^b K_S^{\text{gravity}} \\
&= \frac{K_A^b}{4\pi}\left(\sum_{k=0}(2k+1)e^{-(k+2)(k+3)s} + \sum_{k=1}(2k+1)\left(e^{-(k+1)(k+2)s} + e^{-k(k+1)s}\right)\right. \\
&\qquad \left. + \sum_{k=2}(2k+1)\left(e^{-(k-1)ks} + e^{-(k-2)(k-1)s}\right)\right) \\
&= K_A^b \, \frac{1}{4\pi}\left(5\sum_{k=0}(2k+1)e^{-k(k+1)s}\right) = 5K_4^{\text{scalar}} \ . \qquad (3.136)
\end{aligned}
$$

Thus the KK block (5 d.o.f.) has the same heat kernel as 5 minimally coupled scalars: $a^{\text{KK.bulk}} = \frac{1}{72}$.

The heat kernel of a massless minimally coupled spinor (1 d.o.f.) on AdS_2 is given by [94]

$$
\begin{aligned}
K_A^f(s) &= -\frac{1}{2\pi}\int_0^\infty dp \; p \coth(\pi p) \exp\left(-sp^2\right) \\
&= -\frac{1}{4\pi s}\left(1 + \frac{1}{6}s - \frac{1}{60}s^2 + \mathcal{O}(s^3)\right) \ , \qquad (3.137)
\end{aligned}
$$

where the overall sign incorporates fermionic statistics. With this result as starting point, the spectrum for bulk fermions given in table 3.2 yields:

Gaugino block

The spectrum of the gaugino block has 4 towers: two copies of $(h, j) = (k+2, k+\frac{1}{2})$, $(k+1, k+\frac{1}{2})$ for $k \geq 0$. For fermions we have $m^2 = (h-\frac{1}{2})^2 = h(h-1) + \frac{1}{4}$ with degeneracy $2j+1$, therefore

$$
\begin{aligned}
K_4^{\text{gaugino}} &= K_A^f K_S^{\text{gaugino}} \\
&= K_A^f \, \frac{2}{4\pi}e^{-\frac{1}{4}s}\left(\sum_{k=0}(2k+2)e^{-k(k+1)s} + \sum_{k=1}2ke^{-k(k+1)s}\right) \\
&= K_A^f \, \frac{4}{4\pi}e^{-\frac{1}{4}s}\sum_{k=0}(2k+1)e^{-k(k+1)s} \\
&= -\frac{1}{4\pi^2 s^2}\left(1 + \frac{1}{4}s + \frac{17}{1440}s^2 + \mathcal{O}(s^3)\right) \ . \qquad (3.138)
\end{aligned}
$$

We used (3.133) for the sum over the tower, as for bosons. The constant term $K_4^{\text{const}} =$

$-\frac{17}{5760\pi^2}$ and (3.131) give the conformal anomaly $a^{\text{gaugino.bulk}} = -\frac{17}{2880}$ for the gaugino block (4 d.o.f.).

Gravitino block

The spectrum of the gravitino block has 8 towers: two copies of $(h,j) = (k+3, k+\frac{1}{2})$, $(k+2, k+\frac{1}{2})$, $(k+2, k+\frac{3}{2})$, $(k+1, k+\frac{3}{2})$ for $k \geq 0$. It gives the heat kernel

$$\begin{aligned}
K_4^{\text{gravitino}} &= K_A^f K_S^{\text{gravitino}} \\
&= K_A^f \, \frac{2}{4\pi} e^{-\frac{1}{4}s} \left(\sum_{k=0} (2k+4) e^{-k(k+1)s} + \sum_{k=1} (2k+2) e^{-k(k+1)s} \right. \\
&\qquad \left. + \sum_{k=1} 2k e^{-k(k+1)s} + \sum_{k=2} (2k-2) e^{-k(k+1)s} \right) \\
&= K_A^f \, \frac{8}{4\pi} e^{-\frac{1}{4}s} \sum_{k=0} (2k+1) e^{-k(k+1)s} = 2K_4^{\text{gaugino}} \; .
\end{aligned} \tag{3.139}$$

Thus the gravitino block (8 d.o.f.) has the same heat kernel as 2 gaugino blocks: $a^{\text{gravitino.bulk}} = -\frac{17}{1440}$.

It is interesting that in all cases the results are equivalent to free massless bosons or fermions with the appropriate number of degrees of freedom. This amounts to a delicate conspiracy between non-minimal couplings and ranges of partial wave towers. The origin of these simplifications is not clear to us.

For $\mathcal{N}=8$ SUGRA, there are 1 KK block, 27 vector blocks, 42 minimally coupled scalars, 4 gravitino blocks, and 24 gaugino blocks. In this case the total contribution from the bulk modes becomes:

$$a^{\text{bulk}}_{\mathcal{N}=8} = (5 + 27\times 3 + 42) \times \frac{1}{360} - (4\times 2 + 24) \times \frac{17}{2880} = \frac{1}{6} \; . \tag{3.140}$$

For $\mathcal{N}=4$ SUGRA with n_V matter multiplets, there are 1 KK block, (n_V+5) vector blocks, $(5n_V-4)$ minimally coupled scalars, 2 gravitino blocks, and $2n_V$ gaugino blocks, which give the bulk contribution $a^{\text{bulk}}_{\mathcal{N}=4} = \frac{n+2}{96}$.

3.4.3 Boundary Modes

As discussed in subsection 3.3.7, boundary modes are due to the harmonic modes on AdS_2 of vectors, gravitini, and tensors. The scalar and gaugino blocks do not have boundary modes. These modes are constant on the AdS_2 space with (renormalized) volume 2π.

Therefore, the heat kernel for a single boundary mode is given by

$$K_A^{\mathrm{zero}}(s) = \pm\frac{1}{2\pi} \ , \tag{3.141}$$

where $\pm$ is for bosons/fermions. The contributions to the heat kernel from the entire towers of boundary modes are then computed as follows.

Vector block

The spectrum of boundary modes for the vector block given in table 3.3 is $m_l^2 = l(l+1)$ with integral $l \geq 0$. This is equivalent to a single scalar field on the S^2. Their contribution to the heat kernel become

$$\begin{aligned} K_4^{\mathrm{vector.bndy}} &= K_A^{\mathrm{zero}} K_S \\ &= \frac{1}{2\pi}\frac{1}{4\pi}\sum_{k=0}(2k+1)e^{-k(k+1)s} \\ &= \frac{1}{8\pi^2 s}\left(1+\frac{1}{3}s+\mathcal{O}(s^2)\right) \ , \end{aligned} \tag{3.142}$$

where we used the sum (3.133).

According to (3.131) the constant term in this expression gives conformal anomaly $a^{\mathrm{vector.bndy}} = \frac{1}{12}$, so, adding the bulk contribution of a single vector block $a^{\mathrm{vector.bulk}} = \frac{1}{120}$ from table 3.2, our explicit sum over modes gives $a^{\mathrm{vector.bulk+bndy}} = \frac{11}{120}$. This agrees with the result found in [27] using a very different method.

KK block

The boundary modes listed for the KK block in table 3.3 comprise 6 towers as well as a single ghost tower. Their heat kernel becomes

$$\begin{aligned} K_4^{\mathrm{KK.bndy}} &= \frac{1}{2\pi}\frac{1}{4\pi}\left(3\sum_{k=0}^{\infty}(2k+1)e^{-sk(k+1)} + \sum_{k=0}^{\infty}(2k+1)e^{-s(k-1)k} \right. \\ &\left. + \sum_{k=1}^{\infty}(2k+1)e^{-s(k+1)(k+2)} + e^{-2s}\sum_{k=1}^{\infty}(2k+1)e^{-sk(k+1)} - e^{-2s}\sum_{k=0}^{\infty}(2k+1)e^{-sk(k+1)}\right) \\ &= \frac{1}{8\pi^2}\left(5\sum_{k=0}^{\infty}(2k+1)e^{-sk(k+1)} + 2 - 2e^{-2s}\right) \\ &= \frac{1}{8\pi^2}\left(\frac{5}{s}+\frac{5}{3}+\frac{13}{3}s+\mathcal{O}(s^2)\right) \ . \end{aligned} \tag{3.143}$$

Reading off the constant term $K_4^{\mathrm{KK.bndy}}$ we find $a^{\mathrm{KK.bndy}} = \frac{5}{12}$ from (3.131). Adding the bulk contribution $a^{\mathrm{KK.bulk}} = \frac{1}{72}$, we get $a^{\mathrm{KK.bulk+bndy}} = \frac{31}{72}$, which also agrees with the result in

[27].

Gravitino block

According to table 3.3, the gravitino block comprises 4 towers of boundary modes with $m_l^2 = (l+1)^2 - \frac{1}{4}$, $l \geq 0$, each with degenracy $2l+2$. This spectrum gives the heat kernel

$$\begin{aligned} K_4^{\text{gravitino.bndy}} &= K_A^{\text{zero}} K_S \\ &= -\frac{1}{2\pi}\,\frac{4}{4\pi}\sum_{k=0}(2k+2)e^{-\left((k+1)^2-\frac{1}{4}\right)s} \\ &= -\frac{4}{8\pi^2 s}\left(1-\frac{1}{6}s+\mathcal{O}(s^2)\right)e^{\frac{1}{4}s} \\ &= -\frac{4}{8\pi^2 s}\left(1+\frac{1}{12}s+\mathcal{O}(s^2)\right)\,, \end{aligned} \tag{3.144}$$

corresponding to $a^{\text{gravitino.bndy}} = -\frac{1}{12}$. With the bulk contribution $a^{\text{gravitino.bulk}} = -\frac{17}{1440}$. Again, the sum $a^{\text{gravitino.bulk+bndy}} = -\frac{137}{1440}$ agrees with that of [27].

For $\mathcal{N}=8$ SUGRA, there are 1 KK block, 27 vector blocks, and 4 gravitino blocks. In this case the total contribution from the boundary modes becomes:

$$a_{\mathcal{N}=8}^{\text{boundary}} = 27\times\frac{1}{12}+\frac{5}{12}-4\times\frac{1}{12}=\frac{7}{3}\,. \tag{3.145}$$

For $\mathcal{N}=4$ SUGRA with n_V matter multiplets, there are 1 KK block, (n_V+5) vector blocks and 2 gravitino blocks, which give the boundary modes contribution $a_{\mathcal{N}=4}^{\text{boundary}} = \frac{n_V+8}{12}$.

3.4.4 Zero Mode Corrections

Almost all of the modes we encounter are suppressed in the heat kernel (3.129): their eigenvalue is strictly positive. The zero modes are the exceptions: they are constant on the AdS_2 like all boundary modes but they are also constant on the S^2; so they are zero-modes on the full spacetime $\text{AdS}_2 \times S^2$. The canonical relation between the heat kernel and the effective action which is implicitly presumed in the formula (3.131) for the anomaly coefficient a requires damping for large s of an integral over the Feynman parameter s and this assumption fails in the case of zero-modes.

The correct treatment of zero-modes takes advantage of their relation to symmetries which means their contributions to the path integral are given by integrals over the volume of the appropriate symmetry group, rather than Gaussian integrals over damped modes [8]. Therefore, the correct contribution to the conformal anomaly a depends on the dimension of the symmetry parameter which is $\Delta = 1, \frac{3}{2}, 2$ for vectors, gravitini, tensors. The heat kernel

(3.129) includes all modes with weight 1 but the correct scaling dimension is Δ for bosons and 2Δ for fermions. The zero mode *correction* takes this effect into account.

Gauge symmetry generators have $\Delta = 1$ so their zero-modes are, by chance, already accounted for correctly in the naïve heat kernel, in the sense that the formula for a (3.131) can be trusted. Moreover, on the nonBPS branch the gravitino has no zero-modes, because supersymmetry is entirely broken. Therefore, the KK-block is the only one affected by zero mode corrections. For diffeomorphisms $\Delta = 2$ so, since they were already counted with weight one, the contributions of these zero modes should be doubled.

In the KK block, there are in total 6 zero modes from non-normalizable diffeomorphisms that need zero mode corrections: 3 zero modes from the AdS_2 tensor $H_{\{\mu\nu\}}$ and 3 more from the mixed vector modes (3.123) with $l = 1$. This gives the zero mode correction

$$a^{\text{KK.zero}} = 2\pi^2 \times 6 \times \frac{1}{8\pi^2} = \frac{3}{2} \tag{3.146}$$

This is the same as the contribution from $m^2 = 0$ modes to the sum (3.143) over KK boundary modes. Thus, by adding this zero mode *correction* their contribution is doubled, as it should be.

3.4.5 Summary of Anomaly Coefficients

Blocks	d.o.f.	a^{bulk}	a^{bndy}	a^{zero}	$a^{\text{bulk}}_{+\text{bndy}}$	a^{total}
Scalar	1	$\frac{1}{360}$	0	0	$\frac{1}{360}$	$\frac{1}{360}$
Gaugino	4	$-\frac{17}{2880}$	0	0	$-\frac{17}{2880}$	$-\frac{17}{2880}$
Vector	3	$\frac{1}{120}$	$\frac{1}{12}$	0	$\frac{11}{120}$	$\frac{11}{120}$
Gravitino	8	$-\frac{17}{1440}$	$-\frac{1}{12}$	0	$-\frac{137}{1440}$	$-\frac{137}{1440}$
KK	5	$\frac{1}{72}$	$\frac{5}{12}$	$\frac{3}{2}$	$\frac{31}{72}$	$\frac{139}{72}$
$\mathcal{N} = 4$	$32 + 16n_V$	$\frac{n_V+2}{96}$	$\frac{n_V+8}{12}$	$\frac{3}{2}$	$\frac{3n_V+22}{32}$	$\frac{3n_V+70}{32}$
$\mathcal{N} = 8$	256	$\frac{1}{6}$	$\frac{7}{3}$	$\frac{3}{2}$	$\frac{5}{2}$	4

Table 3.4: Anomaly Coefficients of the NonBPS Blocks.

As summary of this section we give our results for the anomaly coefficients a in table 3.4. The entry for boundary modes a^{bndy} includes naïve zero modes and a^{zero} denotes the corrections determined by the more careful treatment. The sum $a^{\text{bulk}}_{+\text{bndy}}$ is of interest since it can be compared with results from the local method [27]. We find agreement for each of the

5 type of blocks. This gives great confidence in all our computations.

3.5 Compactifications with an AdS$_3$ Factor

In this section we consider the special case where the AdS$_2 \times S^2$ geometry arises from AdS$_3 \times S^2$ with $(0,4)$ supersymmetry through a reduction along a direction that is nearly null. We recover the black hole spectrum on the BPS (or nonBPS) branch depending on whether the reduction is along the "0" (or the "4") direction.

3.5.1 String Theory on AdS$_3 \times S^2 \times \mathcal{M}$

We consider M-theory compactified to 5D on a Calabi-Yau manifold $\mathcal{M}$ in the supergravity limit. The 5D $\mathcal{N}=2$ content of this theory was worked out in [95]. We include cases with enhanced holonomy $\mathcal{M}=K3\times T^2$ and $\mathcal{M}=T^6$ so, in the long distance approximation, we effectively study 5D SUGRA with $\mathcal{N}\geq 2$ supersymmetry. It is useful to describe this theory as $\mathcal{N}=2$ SUGRA coupled to $n_S=\mathcal{N}-2$ gravitino multiplets (corresponding to supersymmetry extended beyond $\mathcal{N}=2$) and also to $\mathcal{N}=2$ matter in n_V vector multiplets and n_H hypermultiplets.

In the setting of these 5D theories we consider field configurations with magnetic fluxes through an S^2. They correspond to black string solutions in 5D that are interesting for our purposes because, after further compactification of the string on a circle, they correspond to black holes in 4D [96]. We focus on fluxes such that the world-volume of the 5D black strings preserve $(0,4)$ supersymmetry while their gravitational description features an AdS$_3 \times S^2$ near horizon geometry. Supergravity fluctuations in this background can be classified by the quantum numbers of primary fields $(h_L, h_R; j_R)$, where h_L is the scaling dimension with respect to an $SL(2)_L$ isometry of AdS$_3$ and h_R, j_R are the quantum number under $SL(2)_R$ and $SU(2)_R$ isometries of AdS$_3$ and S^2, respectively.

Because the 5D black string solution preserves $(0,4)$ supersymmetry we can organize its spectrum into supermultiplets. The supergravity fluctuations are all in short multiplets characterized by chiral primaries (states with $h_R=j_R$ but any h_L) and their descendants under the preserved $\mathcal{N}=2$ supersymmetry are

$$(h_L, h_R; j_R)\ , \quad 2(h_L, h_R+\frac{1}{2}; j_R-\frac{1}{2})\ , \quad (h_L, h_R+1; j_R-1)\ , \tag{3.147}$$

with appropriate truncations of the multiplet for small values of j_R. The short multiplet numerically has $h_R=j_R$ but we retain both notations to emphasize that these are quantum numbers of two distinct operators. The short multiplet structure applies to all fluctuations in the supergravity approximation so it is common to present the black hole spectrum in terms of the chiral primaries, with descendants under supersymmmetry (3.147) implied. A standard computation (see e.g. [97]) yields the spectrum of chiral primaries for the $AdS_3 \times S^2$ compactification of 5D supergravity given in table 3.5. We want to deduce the implications of this spectrum on $\mathrm{AdS}_3 \times S^2$ for theories on $\mathrm{AdS}_2 \times S^2$.

5D multiplets	Spectrum (h_L, h_R, j_R) of chiral primaries ($h_R = j_R$)
Hyper	$2(k+1, k+\frac{1}{2}; k+\frac{1}{2})$
Vector	$(k+2, k+1; k+1)$ $(k+1, k+1; k+1)$
Gravitino	$(k+2, k+\frac{1}{2}; k+\frac{1}{2})$ $(k+2, k+\frac{3}{2}; k+\frac{3}{2})$ $(k+1, k+\frac{3}{2}; k+\frac{3}{2})$
Gravity	$(k+3, k+1; k+1)$ $(k+2, k+1; k+1)$ $(k+2, k+2; k+2)$ $(k+1, k+2; k+2)$

Table 3.5: The spectrum of chiral primaries on $AdS_3 \times S^2 \times \mathcal{M}$. The label $k = 0, 1, \ldots$

3.5.2 nNull Reduction: Thermodynamics

Many versions of the reduction from $\mathrm{AdS}_3/\mathrm{CFT}_2$ to $\mathrm{AdS}_2/\mathrm{CFT}_1$ have appeared in the literature over the years, including [85, 98–101]. However, the recent advent of $\mathrm{nAdS}_2/\mathrm{nCFT}_1$ correspondence [76, 102] justifies renewed scrutiny of this point.

We first describe the dimensional reduction from a thermodynamic point of view, that is more familiar. Because of the chiral nature of CFT_2's it is useful to introduce two independent "temperatures" $T_{L,R}$ that incorporate both "the" temperature T (the thermodynamic potential for energy $E = (h_L + h_R)/\ell_3$)

$$\frac{1}{T} = \frac{1}{2}\left(\frac{1}{T_L} + \frac{1}{T_R}\right) , \tag{3.148}$$

and an independent chemical potential (the difference of "temperatures") for the spin $s = h_L - h_R$.

Implementing the low temperature limit $T \to 0$ by taking $T_R \to 0$ with T_L fixed, the semiclassical entropy of the theory takes the form

$$S = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right)\ell_3 = S_0 + \frac{1}{2}\pi T \mathcal{L} + \mathcal{O}(T^2) , \tag{3.149}$$

where the extremal entropy $S_0 = \frac{\pi^2}{3} c_L T_L \ell_3$ is independent of the temperature and the length scale $\mathcal{L} = \frac{2\pi}{3} c_R \ell_3$ that characterizes the linear term in the temperature is proportional to the inverse mass gap of the theory [76, 103, 104]. Our normalization for the length scale $\mathcal{L}$

follows [87] and ensures that it agrees with the "long string scale" that is characteristic of the $(0, 4)$ models underlying microscopics of 4D black holes.

The *strict* extremal limit $T \to 0$ clearly retains states of the form $|\text{anything}, \text{gs}\rangle$ where the R-sector is in its ground state (except perhaps for a finite ground state multiplicity) and "anything" is the origin of the extremal entropy S_0. In the standard BPS limit "anything" are the states counted by the elliptic genus.

The *near* extremal limit is qualitatively different: it is the theory of *excitations above the strict extremal limit* $T \to 0$. If focusses on states that take the schematic form $|\text{anything}, \delta\text{gs}\rangle$. The right-moving excitations $|\delta\text{gs}\rangle$ are responsible for the term in the entropy (3.149) that is linear in T. It is the spectrum of these excitations that we study.

The upshot of our discussion of near-extreme thermodynamics is that reduction from $\text{AdS}_3 \times S^2$ to $\text{AdS}_2 \times S^2$ amounts to a basic prescription: simply *disregard* the left moving weight h_L corresponding to the "anything" that specifies the extremal state and retain the right moving weight h_R that characterizes the excitation. Simple as this algorithm may be, it is quite unusual. The canonical set-up for Kaluza-Klein compactification considers a small Kaluza-Klein circle S^1 and finds that the low energy approximation retains only modes that are *constant* on the compactification circle because higher Fourier modes on the S^1 are "heavy". In contrast, our prescription keeps *all* modes on the Kaluza-Klein circle, we *omit* a "momentum" quantum number rather than insisting that it vanishes.

The nNull reduction is chiral in that it (nearly) projects to either the L(eft) or the R(ight) moving sector, depending on whether we study $T_L \to 0$ or $T_R \to 0$. Its two versions are equivalent a priori but, when we apply the construction to the $(0, 4)$ CFT_2's that we have in mind, there is an asymmetry between the two chiralities. In this subsection, we elected to focus on the nNull reduction $T_R \to 0$ that (nearly) projects on the *BPS branch*, since that facilitates comparison with the literature. However, our interest in this chapter will ultimtaely is primarily in the analogous discussion for the nonBPS branch. It follows by interchanging L and R labels.

3.5.3 nNull Reduction: Kinematics

The thermodynamic reasoning above establishes features that reduction from $\text{AdS}_3/\text{CFT}_2$ to $\text{AdS}_2/\text{CFT}_1$ must exhibit in order to describe the facts we have established by explicit computations in $\text{AdS}_2 \times S^2$. They are not consistent with standard Kalaza-Klein reduction on a spatial circle so their geometrical implementation must be nonstandard. In the folliowing we show that they can be recovered from *null reduction*, *i.e.* "compactification" on a null circle. The details will not only prove illuminating conceptually but also yield precise consequences that we can test.

A Lorentzian CFT_2 on a spatial circle with radius R is obviously invariant under simultaneous shift of the two null coordinates $x_{R,L} = t \pm x$ by $\pm 2\pi R$. However, due to invariance under a boost (with rapidity η) it is also invariant under shifts of these null coordinates by unequal amounts $\pm 2\pi R e^{\pm\eta}$. Therefore, as in the DLCQ description of M(atrix)-theory [85, 105], there is a family of equivalent theories that all have the same fixed periodicity of the coordinate x_L but variable periodicity of x_R. As this periodicity get smaller, states with large "momentum" h_R become heavy, as the intuition from standard Kaluza-Klein compactification suggests. However, in contrast to the standard construction, the value of the "momentum" h_L is inconsequential in this limit.

In the language of effective quantum field theory, the nNull-reduction presents operators in the theory as

$$\left(\frac{p_R}{\Lambda}\right)^{h_R-1} \mathcal{O}^{(h_L,h_R;j_R)}(x_L, x_R) \ , \tag{3.150}$$

where p_R is the typical frequency corresponding to the x_R dependence and Λ is the "R" cutoff. The dependence on x_L is inconsequential. The strict IR limit takes the cut-off $\Lambda \to \infty$ with the physical momenta $p_{L,R}$ fixed so only operators with $h_R = 1$ remain. These ground states of the R sector are the BPS states in the case of a CFT_2 with $(0,4)$ supersymmetry. These important operators form the chiral ring of the CFT_2 and they are counted by the the elliptic genus. However, the *near* IR limit describes the *approach to* the IR limit by operators (3.150) with $h_R > 1$. Geometrically, this corresponds to compactification along a direction that is *nearly* null. We refer to this construction as a nNull reduction.

In the nNull reduction procedure, the wave functions on AdS_3 generally depend on the x_L coordinate but we are instructed to ignore this dependence and instead focus exclusively on the R direction. Therefore, the effective 2D wave functions that follow from nNull reduction depend on the position in AdS_2. We interpret our computations directly in 2D as the identification of this dependence.

The nNull reduction thus ignores the L sector and describes the dynamics of the R sector as a self-contained theory. It is a consistency condition on this procedure that operators with identical x_R dependence but distinct x_L dependence realize physics that is largely independent of the latter. This is indeed the expectation: the L sector is in a thermal state characterized by temperature T_L and, according to standard arguments in statistical mechanics, the precise state of this thermal background is inconsequential.

The situation is similar to the well-known description of quasiparticles in the effective field theory of Fermi liquids. In that context the vast majority of the electrons reside deep under the Fermi surface but these "typical" electrons are not the interesting ones: the nontrivial dynamics is captured by the quasiparticles corresponding to low energy excitations on top of the Fermi surface. It is consistent that the Fermi liquid theory ignores the vast number of states under the Fermi surface as long as the quasiparticles are long lived, a condition that is satisfied at low temperature. Similarly, in our black hole context, the coupling between left- and right-moving sectors will also be suppressed thermally. We can interpret the small residual interaction as the origin of Hawking radiation from the black hole [106].

3.5.4 Explicit Comparison Between $\mathrm{AdS}_3 \times S^2$ and $\mathrm{AdS}_2 \times S^2$

We can use the prescription from the preceding subsection to compare results from explicit computations in 4D with dimensional reductions from 5D. It is important to distinguish two cases from the 4D point of view: the BPS branch that was already discussed in the literature [107, 108] and the nonBPS branch that this chapter analyzes in detail. They correspond to two distinct dimensional reductions of the spectrum on $\mathrm{AdS}_3 \times S^2$. In terms of the labels $(h_L, h_R; j_R)$ employed in table 3.5 for bulk 5D representations they are:

- The *BPS branch*: the dimensional reduction removes the h_L quantum number. It is manifest that the spectrum is organized into short multiplets of the form (3.147) also after reduction. Starting from the 5D spectrum in table 3.5 we recover the bulk BPS spectrum on $\mathrm{AdS}_2 \times S^2$ presented in table 3.6 for reference and comparison.

- The *nonBPS branch*: the reduction removes the h_R quantum number from the labels $(h_L, h_R; j_R)$. Thus, to find the spectrum on the nonBPS branch of $\mathrm{AdS}_2 \times S^2$ we first augment the chiral primaries in table 3.5 with the structure of short multiplets (3.147) and only then omit the index h_R. The spectrum of primaries that follows from this procedure retains no simplifications that can be obviously traced to supersymmetry. Nonetheless, the result for primaries identified this way agree with our explicit computations on $\mathrm{AdS}_2 \times S^2$ presented in table 3.2.

In the discussion of CFT_2's in this chapter we have assigned the theory $(0, 4)$ supersymmetry. This convention implies no loss of generality by itself but, once we have it, it is consequential that in subsections 3.5.2 and 3.5.3 we discussed reduction along the null-direction with label L, corresponding to the thermodynamic limit $T_R \to 0$. This choice preserves supersymmetry so it amounts to focus the BPS branch of $\mathrm{AdS}_2 \times S^2$. The discussion of the nonBPS branch is entirely analogous but, as noted in the end of subsection 3.5.2, the labels L and R must be interchanged throughout. In the introduction we similarly opted to assign labels L, R such that they are appropriate for the more familiar BPS branch.

With these potential confusions in mind, we spell out the details for each 5D $\mathcal{N} = 2$ multiplet at a time:

4D supermultiplet	Spectrum (h,j) of BPS solutions			$SU(6)$
Hypermultiplet	$2(k+\frac{1}{2},k+\frac{1}{2})$	$4(k+1,k)$	$2(k+\frac{5}{2},k+\frac{1}{2})$	**20**
Vector multiplet	$2(k+1,k+1)$	$4(k+\frac{3}{2},k+\frac{1}{2})$	$2(k+2,k)$	**15**
Gravitino multiplet	$2(k+\frac{3}{2},k+\frac{3}{2})$	$4(k+2,k+1)$	$2(k+\frac{5}{2},k+\frac{1}{2})$	**6**
Gravity multiplet	$2(k+2,k+2)$	$4(k+\frac{5}{2},k+\frac{3}{2})$	$2(k+3,k+1)$	**1**

Table 3.6: Bulk spectrum of BPS solutions. The integral label $k \geq 0$. In each line the first entry is the chiral primary and the remaining entries reflect the structure (3.147) of a short multiplet.

- *Hypermultiplet*

  The on-shell field content of a 5D hypermultiplet in $\mathcal{N}=2$ supergravity is two gaugini (2×2 d.o.f), and four scalars (4×1 d.o.f.). On the BPS branch this amounts precisely to a 4D hypermultiplet but on the nonBPS branch the fields split so fermions are in one gaugino block (with two gaugini) and the bosons are in four scalar blocks (each with one real scalar).

  Table 3.5 indicates that on $\mathrm{AdS}_3\times S^2$ an $\mathcal{N}=2$ hypermultiplet is organized in two towers of chiral primaries that both have $(h_L,h_R;j_R)=(k+1,k+\frac{1}{2};k+\frac{1}{2})$ where $k=0,1,\ldots$. The structure of short multiplets given in (3.147) then yields 8 towers of primary fields with $(h_L,h_R;j_R)=2(k+1,k+\frac{1}{2};k+\frac{1}{2}),4(k+1,k+1;k),2(k+1,k+\frac{3}{2};k-\frac{1}{2})$. In the last towers the entry with $k=0$ is empty so we may replace these tower with $2(k+2,k+\frac{5}{2};k+\frac{1}{2})$ with $k=0,1,\ldots$.

  Dimensional reduction to the BPS branch of $\mathrm{AdS}_2\times S^2$ simply omits h_L. The resulting 8 towers indeed reproduce the BPS spectrum found directly in 4D that is summarized in table 3.6 [97, 109].

  On the nonBPS branch we must instead remove the quantum number h_R. This results in 4 bosonic towers with the quantum numbers given in table 3.2 for a scalar block, i.e. a minimally coupled scalar field. Importantly, it also gives 4 fermion towers with the assignments previously found for a gaugino block on the nonBPS branch.

- *Vector multiplet*

  The on-shell field content of a 5D vector multiplet in $\mathcal{N}=2$ supergravity is one 5D vector field (3 d.o.f.), two gaugini (2×2 d.o.f), and one scalar (1 d.o.f.). Dimensional reduction of a 5D vector field gives a 4D vector field and a *real* scalar so an $\mathcal{N}=2$

vector multiplet in 5D corresponds to an $\mathcal{N}=2$ vector multiplet in 4D on the BPS branch, comprising one 4D vector, two gaugini and a *complex* scalar. On the nonBPS branch these 8 degrees of freedom are organized into one vector block (a 4D vector plus one real scalar), one gaugino block (two gaugini), and one scalar block (one real scalar).

On $\mathrm{AdS}_3 \times S^2$ an $\mathcal{N}=2$ vector multiplet gives chiral primaries that, according to table 3.5, are organized in two towers with $(h_L, h_R; j_R) = (k+2, k+1; k+1)$ and $(k+1, k+1; k+1)$ where $k = 0, 1, \ldots$. The structure of short multiplets given in (3.147) then yields 8 towers of primary fields.

On the BPS branch our algorithm instructs us to omit the h_L index so it is immediately clear that the reduction of the 5D spectrum to $\mathrm{AdS}_2 \times S^2$ yields two copies of $(h_R; j_R) = (k+1; k+1)$, each with the descendants prescribed by (3.147). This agrees with the BPS result exhibited in table 3.6.

The nonBPS branch is less familiar, but equally simple. Upon omission of the quantum number h_R, the 8 aforementioned towers of primary fields each give unambiguous values for the pair (h_L, j_R). The quantum numbers found by this procedure can be organized into the sum of the spectra presented in table 3.2 for a vector block, a gaugino block, and a scalar block.

- *Gravitino multiplet*

  The 5D gravitino multiplet consists of one 5D gravitino (4 d.o.f.), two 5D vectors (2×3 d.o.f.) and a gaugino (2 d.o.f.). Dimensional reduction of a 5D gravitino gives a gravitino and a gaugino in 4D. An $\mathcal{N}=2$ gravitino multiplet in 5D therefore corresponds to one 4D gravitino, two 4D vectors, two gaugini, and two scalars. On the BPS branch these fields amount to the sum of an $\mathcal{N}=2$ gravitino multiplet and an $\mathcal{N}=2$ $\frac{1}{2}$-hypermultiplet in 4D. However, on the nonBPS branch, they decompose as the sum of half a gravitino block (one gravitino plus one gaugino in 4D), two vector blocks (two vectors plus two scalars in 4D) and half a gaugino block (one gaugino).

  The 5D quantum numbers on $\mathrm{AdS}_3 \times S^2$ given in table 3.5 indeed reduce to the sum of a gravitino multiplet and half a hypermultiplet entries given for the 4D BPS branch in table 3.6, upon omission of the h_L index. After omission of the h_R index they similarly agree with the sum of half a gravitino block, two vector blocks, and half a gaugino block given for the 4D nonBPS branch in table 3.2 .

- *Gravity multiplet*

  The gravity multiplet in 5D $\mathcal{N}=2$ SUGRA consists of the 5D graviton (5 d.o.f.), two

5D gravitini (2×4 d.o.f), and the 5D graviphoton (3 d.o.f). On the BPS branch these fields are represented in 4D as the sum of an $\mathcal{N} = 2$ gravity multiplet ($4+4$ d.o.f.) and an $\mathcal{N} = 2$ vector multiplet ($4+4$ d.o.f.). On the nonBPS branch, they are represented instead as the sum of a KK-block (5 d.o.f.), one gravitino block (2×4 d.o.f), and a 4D vector block (3 d.o.f).

The 5D quantum numbers on $\mathrm{AdS}_3 \times S^2$ given in table 3.5 for the gravity multiplet indeed reduce to the sum of the gravity and hypermultiplet entries given for the 4D BPS branch in table 3.6, upon omission of the h_L index. After omitting the h_R index they similarly agree with the sum of a KK block, two gravitino blocks, and a vector block given for the 4D nonBPS branch in table 3.2.

It is interesting that the decomposition into decoupled blocks on the nonBPS branch faithfully reflect their 5D origin: the 5D graviton reduces to the KK block, the two 5D gravitini reduce to a gravitino block, and the 5D vector field reduces to the vector block.

The dimensional reduction from 5D to 4D illuminates the unsettling feature that fermions on the nonBPS branch all have integral conformal weight in AdS_2. A 5D spinor on $\mathrm{AdS}_3 \times S^2$ has half-integral spin on AdS_3 and S^2 independently. Projection of the half-integral spin vector in AdS_3 on to the periodic spatial coordinate give a half-integral value of $s = h_L - h_R$. Since h_R is tied by supersymmetry to the half-integral spin j_R on S^2 it must be that h_L is integral. Since the reduction from AdS_3 to AdS_2 on the nonBPS branch omits h_R we see that "the" conformal weight on AdS_2 is the integral h_L. The integral weights in 2D are therefore perfectly consistent with the spin-statistics theorem. Indeed, on the nonBPS branch they are required by its 5D version.

Theories on $\mathrm{AdS}_2 \times S^2$ that arise through dimensional reduction from $\mathrm{AdS}_3 \times S^2$ are not the most general ones, specific assumptions on the moduli of the 5D theory must be imposed. However, for the purpose of computing primary fields in supergravity, this situation does not imply any limitations. This is obvious from a practical point of view: there is a canonical equivalence between the allowed supermultiplets of $\mathcal{N} = 2$ supergravity in 4D and in 5D to the extent that, allowing ourselves some abuse of terminology, we apply identical names to analogous representations in 4D and in 5D: supergravity, gravitino, vector, hyper. Therefore, since consistency requires that the black hole spectrum agrees for the $\mathrm{AdS}_2 \times S^2$ theories that descend from $\mathrm{AdS}_3 \times S^2$, it must in fact agree for all black holes. A more abstract approach reaches the same conclusion: since chiral primaries are robust under motions in moduli space it is sufficient to establish the correspondence when $\mathrm{AdS}_2 \times S^2$ descends from $\mathrm{AdS}_3 \times S^2$ and then we can conclude that the chiral primaries determined these two ways must agree. From either point of view our explicit computation of the black hole spectrum on the nonBPS branch at some level amounts to a consistency check, albeit a rather nontrivial one.

3.6 Global Supersymmetry

Although our focus is on black holes that do not preserve any supersymmetry it is significant that they are solutions to supergravity. One aspect of this setting is that a remnant of the symmetry persists in the spectrum where it acts as a global supersymmetry.

3.6.1 Global Supercharges: the BPS Branch of N=8 Theory

Recall that on the BPS branch there are two spinors $\epsilon_{1,2}$ such that the supersymmetry transformation (3.8) vanishes. This indicates preserved *local* supersymmetry and forces the black hole spectrum into short multiplets with the structure (3.147). The nonBPS branch has no analogous symmetries and so its spectrum is not organized into short multiplets. However, on both branches we can exploit the *global* part of supersymmetry, i.e. the actions of the transformations (3.8) (and analogous actions on the bosons) that do not depend on spacetime position.

On the BPS branch of $\mathcal{N}=8$ SUGRA the R-symmetry is partially broken as $SU(8)_R \to SU(2)_R \times SU(6)$. The 2 preserved and the 6 broken supersymmetries transform as $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{6})$ under the unbroken $SU(2)_R \times SU(6)$. In this section we write the generators of the broken supersymmetry as $Q_A^{(\frac{1}{2},\frac{1}{2})}$ where superscripts refer to (h_R, j_R) and A is an $SU(6)$ index. These global supersymmetries (anti)commute with the preserved ones so they leave the structure (3.147) of short multiplets intact.

The chiral primaries are the first entries in each line of table 3.6. Their multiplicities **20**, **15**, **6**, **1** can be identified with dimensions of $SU(6)$ representations. For example, the towers of hypermultiplets are in the antisymmetric 3-tensor of $SU(6)$ and their chiral primaries are gaugini with quantum numbers $(h_R, j_R) = (k+\frac{1}{2}, k+\frac{1}{2})$ that we can write as $\Lambda^{ABC}_{(k+\frac{1}{2},k+\frac{1}{2})}$. With this notation the obvious contractions

$$V^{AB}_{(k+1,k+1)} = Q_C^{(\frac{1}{2},\frac{1}{2})}\Lambda^{ABC}_{(k+\frac{1}{2},k+\frac{1}{2})} , \tag{3.151}$$

$$S^{A}_{(k+\frac{3}{2},k+\frac{3}{2})} = \frac{1}{2}Q_B^{(\frac{1}{2},\frac{1}{2})}Q_C^{(\frac{1}{2},\frac{1}{2})}\Lambda^{ABC}_{(k+\frac{1}{2},k+\frac{1}{2})} , \tag{3.152}$$

$$G_{(k+2,k+2)} = \frac{1}{6}Q_A^{(\frac{1}{2},\frac{1}{2})}Q_B^{(\frac{1}{2},\frac{1}{2})}Q_C^{(\frac{1}{2},\frac{1}{2})}\Lambda^{ABC}_{(k+\frac{1}{2},k+\frac{1}{2})} , \tag{3.153}$$

reproduce the remaining chiral primaries in table 3.6. In each case indices indicate (h_R, j_R) so note that, while generally an $SU(2)$ quantum number j can combine with the $j_R = \frac{1}{2}$ of the supercharge and give $j \pm \frac{1}{2}$, for the broken supersymmetry we select just the upper sign. This defines global supersymmetry as an operator in the ring of chiral primary fields.

3.6.2 Global Supercharges: the nonBPS Branch of N=8 Theory

We now apply the analogous considerations to the nonBPS branch of $\mathcal{N} = 8$ SUGRA. In this case the local supersymmetry is entirely broken but we can exploit the global supersymmetry that remains. Its manifestation is a set of global charges $Q_A^{(0,\frac{1}{2})}$ where the index A denotes the fundamental representation of the preserved global $USp(8)$ symmetry and, as usual, $(h_L, j_R) = (0, \frac{1}{2})$ denote the $SL(2) \times SU(2)$ quantum numbers of the $\mathrm{AdS}_2 \times S^2$ isometries.

We start with the 42 moduli, the minimally coupled real scalar fields assembled in a **42** of the global $USp(8)$. We denote this antisymmetric four-tensor of $USp(8)$ as $W^{ABCD}_{(k+1,k)}$. Upon action with the global supercharges we find

$$Q_A^{(0,\frac{1}{2})} W^{ABCD}_{(k+1,k)} = \Lambda^{BCD}_{(k+1,k+\frac{1}{2})} \oplus \Lambda^{BCD}_{(k+2,k+\frac{1}{2})} \ . \tag{3.154}$$

In this formula, and generally on the nonBPS branch, we refer by definition to an entire tower with indices $k = 0, 1, \ldots$. In other words, for a given value of k the product of the $SU(2)$ representations $j_R = k$ and $j_R = \frac{1}{2}$ generally allows the values $j_R = k \pm \frac{1}{2}$. However, in the special case of $k = 0$ the option of "-" is absent so, for the second tower in (3.154), we must shift the indices $k \to k+1$. We stress that, on the nonBPS branch, we take towers for both the "+" and "−" of $j_R = k \pm \frac{1}{2}$. This is in contrast with the BPS branch where multiplets are shortened so that only the "−" applies for preserved supersymmetries and only the "+" is active for broken supersymmetries. In the context of the global symmetry group $USp(8)$, the contraction of the antisymmetric four-tensor **42** with the supercharge yields an antisymmetric three-tensor **48**. Thus the gaugino spectrum (3.154) agrees with the one we find by explicit computation in section 3.3 and summarized in table 3.2.

Action with two global supercharges on the minimal scalar fields similarly gives

$$Q_A^{(0,\frac{1}{2})} Q_B^{(0,\frac{1}{2})} W^{ABCD}_{(k+1,k)} = V^{CD}_{(k+2,k)} \oplus V^{CD}_{(k+2,k+1)} \oplus V^{CD}_{(k+1,k+1)} \ . \tag{3.155}$$

Since supercharges anticommute and the fields are antisymmetric in the indices $A, B, \ldots$, the product of the global supersymmetries is effectively symmetric and so corresponds to spin 1. Generically the product of spin 1 and spin k gives three towers with spin $k+1$, k, and $k-1$. However, for $k = 0$ there is obviously just one tower in this product so, according to our convention that the index k has range $k = 0, 1, \ldots$, we redefined the label $k \to k+1$ in the first two towers of (3.155). Since the two $USp(8)$ indices of the fields V^{CD} place the fields in the **27** of $USp(8)$ we recover the spectrum of a vector block reported in table 3.2, as claimed.

For three global supercharges we similarly reason that, when acting on an antisymmetric representation, we effectively multiply spin k of the scalar field with spin $\frac{3}{2}$ of the generators. This gives the decomposition

$$Q_A^{(0,\frac{1}{2})} Q_B^{(0,\frac{1}{2})} Q_C^{(0,\frac{1}{2})} W_{(k+1,k)}^{ABCD} = S^D_{(k+3,k+\frac{1}{2})} \oplus S^D_{(k+2,k+\frac{1}{2})} \oplus S^D_{(k+2,k+\frac{3}{2})} \oplus S^D_{(k+1,k+\frac{3}{2})} . \quad (3.156)$$

The smallest values are easily checked by hand: the $h_L = 1$ state in $W_{(k+1,k)}^{ABCD}$ has $j_R = 0$ so, after taking the product with spin $\frac{3}{2}$ of the generators, we find that the $h_L = 1$ level has just one state and that state has $j_R = \frac{3}{2}$. The only $h_L = 1$ on the right hand side is the fourth term for $k = 0$ and this term indeed has $j_R = \frac{3}{2}$. Similarly, the $h_L = 2$ states on the left hand side arise from the spin composition $\frac{3}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2} \oplus \frac{5}{2}$, in agreement with the j_R values of the $k = 0$ states in the 2nd and 3rd tower and the $k = 1$ state in the 4th tower. The result for the spectrum (3.156) generated by global supersymmetry agrees with that given in table 3.2 for half a gravitino block.

Finally, we act with four global supercharges and get

$$Q_A^{(0,\frac{1}{2})} Q_B^{(0,\frac{1}{2})} Q_C^{(0,\frac{1}{2})} Q_D^{(0,\frac{1}{2})} W_{(k+1,k)}^{ABCD} = G_{(k+3,k)} \oplus G_{(k+3,k+1)} \oplus G_{(k+2,k+1)} \oplus G_{(k+2,k+2)} \oplus G_{(k+1,k+2)} .$$

We find the structure of the right hand side by multiplication of spin 2 and spin k, and then adjust the indices on states with $h_L = 1$ and $h_L = 2$ following the model from the preceding paragraph. Our result matches the spectrum of the KK block given in table 3.2, as expected.

3.6.3 Global Supercharges in AdS$_3$

We have shown that the black hole spectrum on the BPS branch is generated by global supercharges $Q_A^{(\frac{1}{2},\frac{1}{2})}$ while on the nonBPS branch it is organized by $Q_A^{(0,\frac{1}{2})}$. It is interesting to inquire whether these charges acting on the AdS$_2$ spectra can descend from AdS$_3$.

The AdS$_3 \times S^2$ near horizon geometry of triply self-intersecting strings in 5D $\mathcal{N} = 8$ theory [56, 110] features a supercharge of the form $Q_A^{(0,\frac{1}{2};\frac{1}{2})}$ where $(h_L, h_R; j_R) = (0, \frac{1}{2}; \frac{1}{2})$. According to the rules for dimensional reduction introduced in section 3.5.2 omission of h_L yields the BPS branch while omission of h_R gives the nonBPS branch. Therefore, a single AdS$_3$ supercharge gives appropriate supercharges on both branches of the AdS$_2$ theory. This construction explains the unusual feature that the supercharge on the nonBPS branch has $h = 0$. This is possible because the energy h_R is unimportant after the reduction to the nonBPS branch and is closely related to the reason that fermions have integral conformal weights.

However, the global symmetry encoded in the index A is not entirely clear. The moduli space of AdS$_3 \times S^2$ vacua in 5D $\mathcal{N} = 8$ SUGRA is $F_{4(4)}/USp(2) \times USp(6)$ [110] and from this perspective the index A transforms according to the $USp(2) \times USp(6)$ group in the denominator. Upon dimensional reduction to $\mathrm{AdS}_2 \times S^2$ this global symmetry must be enhanced to $SU(2) \times SU(6)$ (on the BPS branch) or $USp(8)$ (on the nonBPS branch). It is unsurprising that the global symmetry is enhanced upon restriction to one sector or the other but the details have confusing aspects (see [56, 57] for discussion).

3.6.4 Global Supersymmetry in the $\mathcal{N} = 4$ Theory: the nonBPS Branch

It is also interesting to determine the global supersymmetry realized by the spectrum of nonBPS black holes in $\mathcal{N} = 4$ SUGRA with n_V matter multiplets. The situation is similar to $\mathcal{N} = 8$ SUGRA but for $\mathcal{N} = 4$ SUGRA the entire spectrum is not unified into a single representation so we encounter several distinct multiplets.

The structure of global symmetries for the nonBPS branch of $\mathcal{N} = 4$ SUGRA with n_V matter multiplets was summarized in table 3.1. The black hole breaks the global symmetry group of the theory $SU(4)_R \times SO(n_V)_{\mathrm{matter}}$ to $USp(4) \times SO(n_V - 1)_{\mathrm{matter}}$ so the global supercharges $Q_A^{(0,\frac{1}{2})}$ have $USp(4)$ index A.

- $\mathcal{N} = 4$ *superKK Vector Blocks*

  There are $n_V - 1$ decoupled blocks in the fundamental of the $SO(n_V - 1)$ global symmetry. Each superKK vector block has field content of **5** scalar blocks, **4** $\frac{1}{2}$ gaugino blocks, and **1** vector block. Table 3.2 gives their spectrum as

$$
\begin{gathered}
5(k+1, k) \\
4(k+2, k+\tfrac{1}{2})\ ,\ 4(k+1, k+\tfrac{1}{2}) \\
(k+2, k)\ ,\ (k+2, k+1)\ ,\ (k+1, k+1)\ .
\end{gathered}
\tag{3.157}
$$

  We can fit this spectrum into a supermultiplet generated by global supercharges $Q_A^{(0,\frac{1}{2})}$ acting once or twice on a scalar block W^{AB} in the **5** of $USp(4)$. The spin-1 $USp(4)$ singlet $\Omega^{AB} Q_A^{(0,\frac{1}{2})} Q_B^{(0,\frac{1}{2})}$ acts trivially in this representation.

- *The* $\mathcal{N} = 4$ *SuperKK Gravity Block.*

  This is the minimal theory with a KK solution: $\mathcal{N} = 4$ SUGRA with $n_V = 1$ vector multiplets. Our discussion in section 3.2 decomposes the $\mathcal{N} = 4$ matter content into fields that decouple in the KK background: **1** KK block, **4** $\frac{1}{2}$ gravitino blocks, **6** vector blocks, **4** $\frac{1}{2}$ gaugino blocks, and **1** scalar block. Boldfaced letters refers not only to the multiplicity but also to the $USp(4)$ representation. These fields are all singlets of $SO(n_V - 1)$ so there is just one $\mathcal{N} = 4$ superKK-block, as expected because gravity is unique. Table 3.2 gives their spectrum as

$$
\begin{gathered}
(k+1,k) \\
4(k+2,k+\tfrac{1}{2})\ ,\ 4(k+1,k+\tfrac{1}{2}) \\
6(k+2,k)\ ,\ 6(k+2,k+1)\ ,\ 6(k+1,k+1) \\
4(k+3,k+\tfrac{1}{2})\ ,\ 4(k+2,k+\tfrac{1}{2})\ ,\ 4(k+2,k+\tfrac{3}{2})\ ,\ 4(k+1,k+\tfrac{3}{2}) \\
(k+3,k)\ ,\ (k+3,k+1)\ ,\ (k+2,k+1)\ ,\ (k+2,k+2)\ ,\ (k+1,k+2)\ .
\end{gathered}
$$

We can fit all these fields into a tower of supermultiplets generated by supercharges $Q_A^{(0,\frac{1}{2})}$. Antisymmetric representations formed by tensoring $0,1,2,3,4$ vectors under the global $USp(4)$ (labelled by $0,1,2,3,4$ indices $A,B,\ldots$) account for the degeneracies $\mathbf{1},\mathbf{4},\mathbf{6},\mathbf{4},\mathbf{1}$. The middle entry is reducible as an $USp(4)$ representation $\mathbf{6}=\mathbf{5}\oplus\mathbf{1}$. However, both components are kept when the singlet $\Omega^{AB}Q_A^{(0,\frac{1}{2})}Q_B^{(0,\frac{1}{2})}$ is represented nontrivially. Moreover, symmetric combinations of $0,1,2,3,4$ supercharges of this form transform as spin $0,\frac{1}{2},1,\frac{3}{2},2$. These spins act on the first line of the equation using the standard product rule of angular momenta and, after compensating for missing entries with small spin by adjusting the index k so $k=0,1,\ldots$ in all cases, the remaining lines follow precisely.

CHAPTER IV

AdS$_5$ Black Hole Entropy near the BPS Limit

4.1 Introduction and Summary

The microscopic understanding of black hole entropy is the linchpin for progress in quantum gravity. However, studies of black hole entropy have been quantitatively successful only in a few systems with a high degree of supersymmetry, e.g. [5, 111]. Such settings offer great control but they appear far removed from the studies of black hole dynamics that is focus of much current research, such as the study of the SYK model, e.g. [76, 112, 113] or other fashionable spects of holography e.g. [114–116]. The main motivation for this chapter is to develop an arena that has the potential to bridge this gap. Our strategy is to leverage results on BPS ground states established in 4d $\mathcal{N} = 4$ SYM to study the nearBPS properties of these systems as well.

Supersymmetric black holes in AdS$_5$ offer a particularly important setting for the study of holography. The relevant classical geometries have been known for quite some time [20–23]. It is an unfortunate technical complication that these black hole solutions are necessarily somewhat complicated. For example, all supersymmetric AdS$_5$ black holes with regular event horizon must rotate. On the other hand, the entropy of these black holes is relatively simple [117]:

$$S = 2\pi\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{1}{2}N^2(J_1 + J_2)}\,, \tag{4.1}$$

where Q_I (with $I = 1, 2, 3$) denote the R-charges and J_i (with $i = 1, 2$) the angular momenta within AdS$_5$. The conserved charges of supersymmetric AdS black holes must satisfy not only a conventional BPS mass condition

$$M = \sum_{I=1}^{3} Q_I + \sum_{i=1}^{2} J_i\,, \tag{4.2}$$

but also a certain nonlinear constraint.

$$Q_1Q_2Q_3+\frac{1}{2}N^2J_1J_2=\left(\frac{1}{2}N^2+Q_1+Q_2+Q_3\right)\left(Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{1}{2}N^2(J_1+J_2)\right). \tag{4.3}$$

The full significance of the constraint is somewhat mysterious. On the gravity side, some researchers impose it as the condition that closed timelike curves are absent. In subsection 4.2.2 we show that it follows from the BPS condition (4.2) without any additional assumptions.

It was long thought that supersymmetric indices fail at counting the BPS states underlying the entropy of supersymmetric AdS_5 black holes. For example, all possible indices were constructed [118] and their growth was estimated as $\mathcal{O}(1)$. This result was interpreted as due to large cancellations in the partition function that preclude the $\mathcal{O}(N^2)$ growth that is needed to account for the black hole entropy. Further work included [119–121].

There has been progress on this research direction over the last few years. A central insight was the recasting of the entropy (4.1) and the constraint (4.3) in terms of the free energy [122]:

$$\ln Z=-\frac{1}{2}N^2\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2}\,, \tag{4.4}$$

with the potentials satisfying a complex constraint. A straightforward Legendre transform of this expression from the potentials for R-charge Δ_I and the rotational velocities ω_i gives the entropy (4.1). Moreover, the constraint (4.3) on charges follows in the process, from a reality condition on the entropy. Subsequently, the free energy (4.4) and its accompanying constraint were derived from the on-shell action of supergravity [24]. These results (and their generalizations to other dimensions) appear deeply interrelated with the supersymmetric Casimir energy [123, 124] and its relation to anomalies [125, 126].

Over the last few months several microscopic derivations of the BPS entropy of AdS_5 black holes have been presented:

- *Supersymmetric Localization* [24]. This ab initio computation is principled: the central point is the deformation of the path integral away from the physical surface while preserving fermion boundary conditions consistent with supersymmetry. However, the argument relies on a somewhat mysterious "generalized supersymmetric Casimir energy" that relates the supersymmetric partition function (which is of order $\mathcal{O}(N^2)$) and the supersymmetric index (which is of order $\mathcal{O}(1)$).

- *Free Field Construction* [25]. This is the simplest derivation by far, involving nothing but the free field representation of $\mathcal{N}=4$ SYM at vanishing coupling. However, it is not clear to us that the computation is justified in the regime where it is applied.

- *Superconformal Index* [26]. This computation relies on the earlier rewriting of the $\mathcal{N} = 4$ SYM localization on $S^1 \times S^3$ in terms of a complex integral over a circle, with poles inside the disc determined by certain Bethe Ansatz equations [127]. Manipulating the contour and taking advantage of known poles of the integrand [128, 129], the free energy (4.4) is extracted from the supersymmetric index. This derivation is the most rigorous but it is technically more involved and was so far completed only for simplified values of the angular momenta.

A key technical feature of all these derivations is that they invoke complexified potentials Δ_I, ω_i in an essential manner. However, it is not obvious to us that the details of these papers are consistent with one another [1]. Further research prompted by [24–26] has generalized details of the results in various directions [131–138] but no consensus has emerged yet.

Summary of this Chapter

In this chapter we take the substantiation of the BPS free energy (4.4) for granted. Given that basis, we make a significant leap and study the entropy of AdS_5 black holes *away from the supersymmetric limit*. We develop both gravitational and microscopic considerations, and we find several new agreements between these holographically dual descriptions. Our success in this direction develops the emerging microscopics of AdS_5 black holes directly in the regime that is physically relevant.

It is important to recognize that AdS_5 black holes allow *two distinct deformations* away from the BPS limit. Conceptually, the extremal limit $M = M_{\rm ext}$ indicates the lowest possible mass for given conserved charges. The low lying excitations with energy M in the range $0 < M - M_{\rm ext} \ll M_{\rm ext}$ are characterized quantitatively by the low temperature behavior

$$M - M_{\rm ext} = \frac{1}{2}\left(\frac{C_T}{T}\right) T^2\,, \tag{4.5}$$

where C_T is the specific heat.[2] The regime where this mass formula applies is also studied in research inspired by the SYK model [112, 113], such as [76, 87, 102, 104].

However, in the context of AdS_5 black holes, it is equally possible to consider excitations away from the BPS limit that remain on the extremal surface $M = M_{\rm ext}$ but with the

[1] For some discussions see [24, 130].

[2] The specific heat C_T at low temperature is proportional to the temperature so the combination $\frac{C_T}{T}$ is a constant in the regime we study. In our notation $\frac{C_T}{T}$ is the coefficient in front of $\frac{1}{2}T^2$, evaluated with all charges Q_I, J_i kept fixed. Some might refer to this variable as $C_{Q,J}$ but that is not what we do.

conserved charges Q_I, J_i taking values that violate the constraint (4.3). We find that such excitations have mass

$$M - M_{\rm BPS} = \frac{1}{2}\left(\frac{C_\varphi}{T}\right)\left(\frac{\varphi}{2\pi\ell_5}\right)^2, \tag{4.6}$$

where φ is a potential parametrizing departures from the BPS surface that preserve extremality and so have $T = 0$. The coefficient C_φ is the *capacitance* of the black hole. [3] We give the precise relation between the deformations φ within the extremal surface and the generic potentials Δ_I, ω_i in (4.137) with $T = 0$. To complete the set of parameters describing linear response we also introduce a thermoelectric coefficient C_E that quantifies the interplay between the temperature T and the potential φ.

The computation of the response parameters C_T, C_φ, and C_E is conceptually straightforward on the supergravity side, albeit not trivial from a technical point of view. Surprisingly, we find that the heat capacity and the capacitance are identical

$$C_T = C_\varphi. \tag{4.7}$$

Each of these physical quantities are quite nontrivial functions of black hole parameters so this agreement can hardly be an accident. They may be related by the broken supersymmetry but the realization of this mechanism in supergravity is not entirely straightforward and we did not pursue it here. Whatever its origin, this type of relation is novel and interesting in $\mathcal{N} = 4$ SYM.

On the microscopic side any progress may seem implausible because advances in the BPS limit rely heavily on supersymmetry, not to mention that they face some unresolved questions. However, the nonrenormalization due to supersymmetry may well generalize to the linear order that we study. Certainly the low temperature heat capacity is subject to a nAttractor mechanism [87] so $\frac{C_T}{T}$ can be interpreted as a symmetry breaking parameter of scale invariance and $\frac{C_\varphi}{T}$ as its partner under broken extended supersymmetry. It is therefore reasonable to expect that the parameters we compute are protected. Although we have not worked out a detailed argument in the present context we note that the mass terms depending quadratically on potentials were previously derived from BPS considerations in the case of asymptotically flat black holes [56, 139].

In our microscopic computations we proceed pragmatically. The BPS free energy (4.4) is justified only in the strict supersymmetric limit but smoothness is sufficient to find the linear dependence on temperature even away from the BPS limit. Moreover, our gravitational

[3] It is admittedly $\frac{C_\varphi}{T}$ that is the capacitance according to the definition (4.5) and standard terminology in electrodynamics. We find this abuse of language an acceptable price for making the symmetry between C_T and C_φ manifest.

study motivates relaxing the constraint imposed by supersymmetry as well. It is satisfying that the two independent deformations combine nicely, to a complex parameter $\varphi + 2\pi i T$. With these minimal and conservative ingredients we recover gravitational results through a Legendre transform.

This chapter is organized as follows. In section 4.2 we develop the gravitational thermodynamics of nearBPS AdS_5 black holes. We carefully distinguish between the near-extremal limit and the extremal nearBPS limit, and we study their interplay. In section 4.3 we review the partition function of $\mathcal{N} = 4$ SYM in the free field limit, which can be written as a matrix model. We derive the free energy (4.4) from its leading contribution in the large-N and low-temperature limit. In section 4.4 we study the resulting AdS_5 black hole entropy function. We derive some of its implications beyond the regime where it was originally derived, by relaxing constraints on potentials and exploiting general principles such as the first law of thermodynamics. This leads to statistical physics of nearBPS AdS_5 black holes that agrees with results from the gravity side. Some open questions and future directions are discussed in section 4.5.

4.2 Black Hole Thermodynamics

In this section we develop the thermodynamics of AdS_5 nearBPS black holes. As benchmarks, we first review the general AdS_5 thermodynamics and its BPS limit. We then study small but finite deviations from the strict BPS limit by allowing for temperature and for physical charges that violate the constraint. Specifically, in subsection 4.2.3 we consider the near-extremal regime (the mass exceeding the BPS value but the constraint between charges retained), in subsection 4.2.4 we study the extremal nearBPS regime (keep vanishing temperature but allow for small violations of the charge constraint), and then in subsection 4.2.5 we take on the general two-parameter deviations from the BPS surface.

The general AdS_5 black hole solution (for any M, Q_I, J_i) is known [23] but has not been analyzed in much detail [140]. In this section we follow most of the literature and focus on diagonal R-charges $Q_1 = Q_2 = Q_3$.

4.2.1 General AdS_5 Black Holes

We consider the most general charged rotating black holes in five-dimensional minimal gauged supergravity [22]. The solutions are identified by the mass M, the charge Q, and two independent angular momenta J_i (with $i = 1, 2$). These physical quantum numbers are parametrized by four variables (m, q, a, b) as

$$M = \frac{\pi}{4G_5}\frac{m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2qabg^2(\Xi_a + \Xi_b)}{\Xi_a^2\Xi_b^2} \ , \tag{4.8}$$

$$Q = \frac{\pi}{4G_5}\frac{q}{\Xi_a\Xi_b} \ , \tag{4.9}$$

$$J_1 = \frac{\pi}{4G_5}\frac{2ma + qb(1 + a^2g^2)}{\Xi_a^2\Xi_b} \ , \tag{4.10}$$

$$J_2 = \frac{\pi}{4G_5}\frac{2mb + qa(1 + b^2g^2)}{\Xi_a\Xi_b^2} \ . \tag{4.11}$$

Here G_5 is Newton's gravitational constant in five dimensions, the coupling of gauged supergravity $g = \ell_5^{-1}$ with ℓ_5 the AdS_5 radius, and

$$\Xi_a \equiv 1 - a^2g^2 \, , \quad \Xi_b \equiv 1 - b^2g^2 \, . \tag{4.12}$$

The coordinate r_+ that locates the event horizon is the largest real root of $\Delta_r = 0$, where Δ_r is given by

$$\Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 + g^2r^2) + q^2 + 2abq}{r^2} - 2m \ . \tag{4.13}$$

We can expresses m in terms of r_+^2 and other variables using this equation. Thus the temperature and entropy are presented conveniently in terms of (r_+, q, a, b). They are

$$T = \frac{r_+^4[1 + g^2(2r_+^2 + a^2 + b^2)] - (ab + q)^2}{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]} \ , \tag{4.14}$$

$$S = 2\pi\frac{\pi}{4G_5}\frac{(r_+^2 + a^2)(r_+^2 + b^2) + abq}{(1 - a^2g^2)(1 - b^2g^2)r_+} \ . \tag{4.15}$$

The electric potential and angular velocities on the horizon are given in the same notation by

$$\begin{aligned}
\Phi &= \frac{3qr_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \ , \\
\Omega_1 &= \frac{a(r_+^2 + b^2)(1 + g^2r_+^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \ , \\
\Omega_2 &= \frac{b(r_+^2 + a^2)(1 + g^2r_+^2) + aq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \ .
\end{aligned} \tag{4.16}$$

4.2.2 BPS AdS$_5$ Black Holes

The general black holes introduced in the previous subsection depend on independent physical variables (M, Q, J_1, J_2). Supersymmetry of the theory guarantees that their mass satisfies

$$M - (3Q + gJ_1 + gJ_2) \geq 0 \ .$$

The BPS black holes saturate this inequality so their mass M is given by the BPS condition

$$M^* = 3Q^* + g(J_1^* + J_2^*) \ . \tag{4.17}$$

We use the starred symbols (M^*, Q^*, J_1^*, J_2^*) instead of (M, Q, J_1, J_2) when we stress that the variables refer to the BPS case. [4]

The general formula for the mass can be written as

$$M - (3Q + gJ_1 + gJ_2) = \frac{\pi}{4G_5} \frac{3 + (a+b)g - abg^2}{(1-ag)(1+ag)^2(1-bg)(1+bg)^2} \left[m - q(1 + g(a+b)\right] \ , \tag{4.18}$$

in terms of the parameters (m, q, a, b). The coefficient in front of the square bracket is always positive, so the physical BPS condition (4.17) is equivalent to the relation

$$q^* = \frac{m^*}{1 + ag + bg} \ . \tag{4.19}$$

It turns out that once we saturate the BPS bound we must further impose a constraint between the black hole charges. This constraint takes the form

$$q^* = \frac{1}{g}(a+b)(1+ag)(1+bg) \ , \tag{4.20}$$

when expressed in terms of the parameters (q, a, b). When the BPS conditions (4.19-4.20) are imposed, the largest root r_+ of the horizon equation $\Delta_r(r_+) = 0$ where Δ_r is given in (4.13) is a double root that has the simple value:

$$r^* \equiv r_+ = \sqrt{\frac{1}{g}(a + b + abg)} \ . \tag{4.21}$$

It is important to stress that the constraint (4.20) is a *consequence* of the BPS mass formula (4.17) rather than an additional requirement due to an independent physical principle.

[4] The star must not be confused with complex conjugation.

To see this, we rewrite the horizon equation $\Delta_r(r_+) = 0$ as

$$m - q[1 + g(a+b)] = \frac{1}{2r_+^2}(r_+^2 - r_*^2)^2 + \frac{1}{2}\left[(1 + g(a+b))^2(r_+^2 - r_*^2) - (q - q_*)\right]^2 \ , \quad (4.22)$$

without any restrictions on black hole parameters (m, q, a, b). It follows from (4.18) that the left hand side of this equation vanishes when the physical BPS bound is saturated. Since the right hand side is the sum of two complete squares we then establish the constraint (4.20) without further assumptions, along with finding the coordinate position of the horizon (4.21). Thus saturation of the BPS bound implies both the BPS mass formula (4.17) and the constraint. [5]

The two conditions (4.19-4.20) that must be satisfied by BPS black holes together yield starred variables (m^*, q^*) that are specified functions of the "rotation" parameters (a, b). It is therefore convenient to pick (a, b) as the two independent coordinates on the BPS surface. This choice gives the physical variables:

$$M^* = \frac{\pi}{4G_5}\frac{(3(a+b) - (a^3+b^3)g^2 - ab(a+b)^2g^3)}{g(1-ag)^2(1-bg)^2} \ , \quad (4.23)$$

$$Q^* = \frac{\pi}{4G_5}\frac{a+b}{g(1-ag)(1-bg)} \ , \quad (4.24)$$

$$J_1^* = \frac{\pi}{4G_5}\frac{(a+b)(2a+b+abg)}{g(1-ag)^2(1-bg)} \ , \quad (4.25)$$

$$J_2^* = \frac{\pi}{4G_5}\frac{(a+b)(a+2b+abg)}{g(1-ag)(1-bg)^2} \ . \quad (4.26)$$

These expressions satisfy the BPS condition (4.17) for any (a, b), as they must.

For BPS black holes the mass M^* is never an independent parameter and in our setting it is the linear function (4.17) of the other charges. It is a less familiar feature that the condition (4.20) imposes an additional relation. This is the reason that the three conserved charges Q^*, $J_{1,2}^*$ (4.24 - 4.26) are expressed in terms of just two coordinates (a, b) on the BPS surface. It means these charges are not independent in the BPS limit, they satisfy the *constraint*:

$$Q^{*3} + \frac{\pi}{4G_5}J_1^*J_2^* = \left(\frac{\pi}{4g^2G_5} + 3Q^*\right)\left(3Q^{*2} - \frac{\pi}{4gG_5}(J_1^* + J_2^*)\right) \ . \quad (4.27)$$

[5] Specifically, we do not appeal to the absence of closed time-like curves (CTCs). It was shown in [22] that black holes satisfying both the BPS condition (4.19) and the constraint (4.20) do not have CTCs outside the event horizon. On the other hand, supersymmetric "solutions" satisfying $M = \pm 3Q \pm gJ_1 \pm gJ_2$ with positive charges $Q, J_{1,2}$ do have CTCs unless all signs are "+" [141]. Regular black holes correspond to "all +" so CTCs do not play any role and the supersymmetry algebra is sufficient to impose the constraint.

This is the special case of the constraint (4.3) with diagonal R-charges $Q_1 = Q_2 = Q_3$.

The general formulae (4.16) for the electric potential and the angular velocities are nearly trivial in the BPS limit: they give $\Phi^* = 3$ and $\Omega_1^* = \Omega_2^* = g$ for all BPS black holes, independent of the values of (a, b). These values are guaranteed by the first law of thermodynamics

$$T^* dS^* = dM^* - \Phi^* dQ^* - \Omega_1^* dJ_1^* - \Omega_2^* dJ_2^* \ , \tag{4.28}$$

because in the BPS case $T^* = 0$ and the mass $M = M^*$ is given by (4.17).

As we have noted already, when the BPS conditions (4.19-4.20) are imposed, the largest root r_+ of the horizon equation $\Delta_r(r) = 0$ is a double root. This situation corresponds to temperature $T = 0$ and is expected for any extremal black hole. Moreover, with the value for r_+ given in (4.21) and the BPS value for q^* given in (4.19), the general formula (4.15) gives the black hole entropy entirely in terms of the rotation parameters (a, b):

$$S^* = 2\pi \cdot \frac{\pi}{4G_5} \frac{a+b}{g(1-ag)(1-bg)} \sqrt{\frac{1}{g}(a+b+abg)} \, . \tag{4.29}$$

In our manipulations we will often need the inverse of (4.24-4.26) and so express (a, b) in terms of the physical variables Q^* and $J_{1,2}^*$ of the BPS black hole. The resulting formulae are not unique, because the BPS charges are subject to the constraint (4.27). A simple version is

$$a = \frac{2gQ^{*2} - \frac{\pi}{4G_5} J_2^*}{Q^*(2g^2 Q^* + \frac{\pi}{4G_5})} \ , \tag{4.30}$$

$$b = \frac{2gQ^{*2} - \frac{\pi}{4G_5} J_1^*}{Q^*(2g^2 Q^* + \frac{\pi}{4G_5})} \ . \tag{4.31}$$

As an example, we can use these equations and the constraint (4.27) to recast the BPS entropy (4.29) as [117]

$$S^* = 2\pi \sqrt{3(\ell_5 Q^*)^2 - \frac{\pi}{4G_5} \ell_5^3 \left(J_1^* + J_2^* \right)} \ . \tag{4.32}$$

In this section we have so far made the effort to retain the dimensionful scales G_5 and $g = \ell_5^{-1}$ in all our equations. This is common in supergravity equations that are written in terms of parametric variables (m, q, a, b), and not terribly inconvenient. However, the practice becomes cumbersome when rewriting formulae in terms of conserved charges M^*,

Q^*, and $J^*_{1,2}$. It is better to employ the dimensionless quantity

$$\frac{1}{2}N^2 = \frac{\pi}{4G_5}\,\ell_5^3\,, \tag{4.33}$$

where N is the rank of the dual $SU(N)$ gauge group. For example, it is superior to express the BPS entropy (4.32) as

$$S^* = 2\pi\sqrt{3(\ell_5 Q^*)^2 - \frac{1}{2}N^2\left(J_1^* + J_2^*\right)}\,. \tag{4.34}$$

To compare with microscopic results we additionally record that the supergravity charge Q (with dimension length^{-1}) and the (dimensionless) angular momenta $J_{1,2}$ are normalized such that $Q\ell_5$ and $J_{1,2}$ can be identified with the quantized charges in the microscopic theory. For clarity, we will retain the scale $g = \ell_5^{-1}$ explicitly in the remainder of this section, but these units will be dropped later in the chapter.

4.2.3 Near-Extremal Limit

Conceptually, the simplest way to deform away from the BPS surface is by adding energy to the BPS black hole, while *keeping electric charge and angular momenta fixed*. Such deviations take the black hole away from extremality and lead to nonzero temperature. Since all charges are kept fixed the constraint is satisfied in this region of parameter space. This is the situation we consider in this subsection.

Since we consider charges that are fixed at their BPS values Q^*, $J^*_{1,2}$ we can write the first law of thermodynamics in the near-extremal limit as

$$TdS = dM - \Phi dQ^* - \Omega_1 dJ_1^* - \Omega_2 dJ_2^*\,. \tag{4.35}$$

Subtracting the corresponding BPS expression (4.28) we have

$$TdS = d(M - M^*) - (\Phi - \Phi^*)dQ^* - (\Omega_1 - \Omega_1^*)dJ_1^* - (\Omega_2 - \Omega_2^*)dJ_2^*\,. \tag{4.36}$$

Variations *along* the BPS surface have $M = M^*$ identically and correspond to the limit $T \to 0$ so they are described by [142]

$$dS = -(\partial_T\Phi)dQ^* - (\partial_T\Omega_1)dJ_1^* - (\partial_T\Omega_2)dJ_2^*\,. \tag{4.37}$$

This formula describes the dependence on charges of the BPS entropy (4.34). That is interesting but not our focus in this subsection.

Instead, we keep the charges strictly fixed and consider the heat added to the black hole as we raise the temperature from $T = 0$ [103]. After dividing by the small temperature T, we have

$$dS = \left(\frac{\partial S}{\partial T}\right)_{Q,J_{1,2}} dT = \left.\frac{C_T}{T}\right|_{\text{nearExt}} dT\,, \tag{4.38}$$

where C_T is the heat capacity that was introduced already in (4.5), along with comments on our notation. The ratio $\frac{C_T}{T}$ is constant in the near-extremal regime so:

$$dM = TdS = \frac{1}{2}\left.\frac{C_T}{T}\right|_{\text{nearExt}} d\left(T^2\right)\ . \tag{4.39}$$

After integration we have the leading behavior at small temperature

$$M - M^* = \frac{1}{2}\left.\frac{C_T}{T}\right|_{\text{nearExt}} T^2\ , \tag{4.40}$$

where $M^* = 3Q^* + g(J_1^* + J_2^*)$ is our reference point on the BPS surface.

We can compute C_T explicitly from the definition (4.38) by brute force. We first use the general entropy formula (4.15) to evaluate the dependence of the entropy on black hole parameters through $\frac{\partial S}{\partial(r_+,q,a,b)}$. We similarly compute the entries of the matrix $\frac{\partial(T,Q,J_1,J_2)}{\partial(r_+,q,a,b)}$, from the general formulae for temperature (4.14) and charges (4.9-4.11), with the parameter m eliminated in terms of (r_+, q, a, b) using the horizon equation (4.13). Inversion of this matrix (using Mathematica) gives the Jacobian $\frac{\partial(r_+,q,a,b)}{\partial(T,Q,J_1,J_2)}$ and then we can form

$$\begin{aligned}\left.\frac{C_T}{T}\right|_{\text{nearExt}} &= \left(\frac{\partial S}{\partial r_+}\right)_{q,a,b}\left(\frac{\partial r_+}{\partial T}\right)_{Q,J_{1,2}} + \left(\frac{\partial S}{\partial q}\right)_{r_+,a,b}\left(\frac{\partial q}{\partial T}\right)_{Q,J_{1,2}} \\ &\quad + \left(\frac{\partial S}{\partial a}\right)_{r_+,q,b}\left(\frac{\partial a}{\partial T}\right)_{Q,J_{1,2}} + \left(\frac{\partial S}{\partial b}\right)_{r_+,q,a}\left(\frac{\partial b}{\partial T}\right)_{Q,J_{1,2}} \\ &= \frac{\pi}{4G_5}\frac{\pi^2(a+b)^2(3+(a+b)g-abg^2)}{g^2(1-ag)(1-bg)\left(1+3(a+b)g+(a^2+3ab+b^2)g^2\right)}\ .\end{aligned} \tag{4.41}$$

In the final step we exploited (4.20) and (4.21) to eliminate q and r_+ .

The result for the heat capacity (4.41) can be expressed as a function of BPS physical charges Q^* and $J_{1,2}^*$. We first rewrite (a, b) using (4.30-4.31) and then simplify using the constraint (4.27) between charges. The result is not unique, because of the constraint, but we find the manageable expression

$$\left.\frac{C_T}{T\ell_5}\right|_{\text{nearExt}} = \pi^2\frac{8(Q^*\ell_5)^3 + \frac{1}{4}N^4(J_1^* + J_2^*)}{3(Q^*\ell_5)^2 - \frac{1}{2}N^2\left(J_1^* + J_2^*\right) + \left(3Q^*\ell_5 + \frac{1}{2}N^2\right)^2}\ . \tag{4.42}$$

It is this form of the heat capacity that we can compare with microscopic considerations.

There is an alternative computation that leads to the heat capacity (4.41) with less effort and more insight. It is known as the nAttractor mechanism [87]. The key observation is that for fixed conserved charge the derivative with respect to temperature T and the horizon coordinate r_+ are equivalent. Therefore, it is sufficient to consider the effect caused by the change of r_+ for computation of the heat capacity. Moreover, the departure from the BPS mass $M - M^*$ given in (4.40) is quadratic in the temperature while the entropy is only linear. Therefore, at the linear order, it is sufficient to consider the BPS geometry, there is no need for the general black hole solution. This leads to the economical computation

$$\left.\frac{C_T}{T}\right|_{\text{nearExt}} = \left(\frac{\partial S}{\partial T}\right)_{Q,J_{1,2}} = \left(\frac{\partial S}{\partial r_+}\right)_{q,a,b}\left(\frac{\partial T}{\partial r_+}\right)^{-1}_{q,a,b} . \tag{4.43}$$

This expression can be evaluated by hand in a few lines and gives the same result as (4.41).

It is similarly useful to think of the electric potentials Φ and rotational velocities Ω_i as radially dependent "attractor flows" that take their fixed values $\Phi^* = 3$, $\Omega_1^* = \Omega_2^* = g$ at the horizon. The final approach to the horizon is determined in each case by a radial derivative along the flow. For the electric potential we have

$$\begin{aligned}
\left(\frac{\partial \Phi}{\partial T}\right)_{Q,J_{1,2}} &= \left(\frac{\partial \Phi}{\partial r_+}\right)_{q,a,b}\left(\frac{\partial T}{\partial r_+}\right)^{-1}_{q,a,b} \\
&= -\frac{3\pi(a+b)\left(1-abg^2\right)}{g\sqrt{\frac{abg+a+b}{g}}\left(1+3(a+b)g+\left(a^2+3ab+b^2\right)g^2\right)} \\
&= -\frac{\pi^2 N^2 \ell_5}{S^*}\;\frac{(J_1+J_2)\left(3Q\ell_5+\frac{1}{2}N^2\right)-2\left(\frac{S^*}{2\pi}\right)^2}{\left(\frac{S^*}{2\pi}\right)^2+\left(3Q\ell_5+\frac{1}{2}N^2\right)^2} .
\end{aligned} \tag{4.44}$$

In the final formula we used the BPS entropy S^* given in (4.34) as a preferred combination of charges, in order to avoid an expression that is overly unwieldy.

For the temperature dependence of the rotational velocity we similarly find

$$\begin{aligned}
\left(\frac{\partial \Omega_1}{\partial T}\right)_{Q,J_{1,2}} &= \left(\frac{\partial \Omega_1}{\partial r_+}\right)_{q,a,b}\left(\frac{\partial T}{\partial r_+}\right)^{-1}_{q,a,b} \\
&= -\frac{\pi(1-ag)(a+2b+(2a+b)bg)}{\sqrt{\frac{abg+a+b}{g}}\left(1+3(a+b)g+\left(a^2+3ab+b^2\right)g^2\right)} \\
&= -\frac{\pi^2 N^2}{S^*}\;\frac{J_2\left(3Q\ell_5+\frac{1}{2}N^2\right)-\left(\frac{S^*}{2\pi}\right)^2}{\left(\frac{S^*}{2\pi}\right)^2+\left(3Q\ell_5+\frac{1}{2}N^2\right)^2} .
\end{aligned} \tag{4.45}$$

There is an analogous formula for $\partial_T \Omega_2$ given by exchanging $a \leftrightarrow b$ (or $J_1 \leftrightarrow J_2$).

The temperature dependence of the potentials given in (4.44-4.45) is such that the BPS limit of the first law (4.37) is satisfied. It is also interesting that

$$g \left(\frac{\partial \Phi}{\partial T} \right)_{Q,J_{1,2}} = \left(\frac{\partial \Omega_1}{\partial T} \right)_{Q,J_{1,2}} + \left(\frac{\partial \Omega_2}{\partial T} \right)_{Q,J_{1,2}} . \tag{4.46}$$

We will see shortly that this is a consequence of temperature *respecting* the constraint on charges (4.27).

4.2.4 Extremal NearBPS Limit

In this subsection we consider departures from the BPS surface that preserve extremality $T = 0$. This situation is somewhat unusual. The nearBPS black holes we study remain extremal in the conventional sense of the black hole attaining its minimal possible mass for given charges. However, charges are modified such that the constraint (4.27) is violated. We recall that preserved supersymmetry requires BPS saturation which in turn implies the constraint on charges. We therefore conclude that these black holes break supersymmetry and their the mass must exceed the BPS bound (4.17).

The most important challenge will be to understand the extremal surface in more detail. According to (4.18), the mass M generally exceeds the BPS mass M^* by an amount that is proportional to (4.22) which we now write as

$$\begin{aligned} & m - (1 + ag + bg)q \\ &= \frac{g^2 r_+^2 (q - q^*)^2 + \Big(\left[(1 + ag + bg)^2 + g^2 r_+^2 \right] (r_+^2 - r^{*2}) - (1 + ag + bg)(q - q^*) \Big)^2}{2\, r_+^2} . \end{aligned} \tag{4.47}$$

We previously applied this expression to show that the BPS mass formula implies the constraint on charges. Presently we consider instead the extremal surface which is characterized by vanishing temperature, not by the BPS condition. Near the BPS surface we can take

$$r_+^2 - r^{*2} \sim q - q^* \sim \epsilon \,, \tag{4.48}$$

small and approximate the temperature T (4.14) as

$$T = \frac{\Big[1 + 3(a + b)g + (a^2 + b^2 + 3ab)g^2 \Big] (r_+^2 - r^{*2}) - (1 + (a + b)g)\, (q - q^*)}{\pi r^* q^*} . \tag{4.49}$$

Thus extremality corresponds to a correlation between the magnitudes of $r_+^2 - r^{*2}$ and $q - q^*$ such that the two terms in the temperature (4.49) cancel at linear order. In this regime the second term in the numerator of (4.47) vanishes, since it is proportional to the temperature, but the first term does not. Thus the BPS condition is preserved at linear order in ϵ, but it is broken at quadratic order. This structure is reminiscent of the nAttractor arguments reviewed in the preceding subsection, but with departures from the BPS surface now due to charges that violate the constraint, rather than nonzero temperature. [6]

We previously determined that the potentials are constants $\Phi = \Phi^* = 3$ and $\Omega_{1,2} = \Omega_{1,2}^* = g$ on the BPS surface. The combination of potentials,

$$\varphi = \Phi - \frac{\Omega_1 + \Omega_2}{g} - 1 \ , \tag{4.50}$$

therefore vanishes there and otherwise gives a physical measure of the "distance" away from the BPS configurations. Departure of the potentials from their BPS values due to a small *temperature* cancels from the expression due to (4.46) so the variable φ measures distance *along* the extremal surface, maintaining vanishing temperature. The general expressions for Φ, $\Omega_{1,2}$ (4.16) give:

$$\varphi = \frac{(3r_+^2 - r^{*2})(q - q^*) - (1 + ag + bg)(r_+^2 - r^{*2})^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \ , \tag{4.51}$$

after some rewriting. The second term in the numerator is negligible near the BPS surface where (4.48) instructs us to take $q - q^*$ and $r_+^2 - r^{*2}$ small and of the same order. We find

$$\varphi = \frac{2(q - q^*)}{q^*} \ , \tag{4.52}$$

at leading order. Therefore the difference $q - q^*$ is a good measure of departures from the BPS surface that preserve extremality.

To summarize so far, within the extremal surface we can take

$$m = (1 + ag + bg)q \ , \tag{4.53}$$

at the linear order and we can relate the horizon coordinate $r_+^2 - r_*^2$ to $q - q^*$ through the condition that the temperature (4.49) vanishes. This leaves $q - q^*$ as the only variable that is sensitive to the deviation from the BPS surface. It is equivalent to φ through (4.52). The

[6] The formula (4.49) facilitates explicit comparison with Silva's early study of thermodynamics above the BPS limit [142]. It corresponds to $\mu \equiv m - (1 + ag + bg)$ and $q = q^*$. It thus involved temperature alone.

remaining variables (a,b) parametrize the base point on the BPS surface, equivalent to Q, $J_{1,2}$ subject to the constraint (4.27).

After this lengthy discussion of principles, we can return to the general formulae (4.16) for the potentials Φ, $\Omega_{1,2}$. In each expression we take fixed (a,b) and expand around $r^2 = r_*^2$, $q = q^*$. We then eliminate $r_+^2 - r_*^2$ in favor of $q - q^*$ by imposing vanishing temperature and introduce φ through (4.52). The resulting deviations away from the BPS values $\Phi^* = 3$ and $\Omega^*_{1,2} = g$ can be presented at linear order in φ as derivatives of the potentials with respect to φ:

$$\frac{\partial \Phi}{\partial \varphi} = \frac{3(a+b)(2+(a+b)g)}{2[1+3(a+b)g+(a^2+3ab+b^2)g^2]} \tag{4.54}$$

$$= 1 - \frac{1}{4}N^2 \frac{J_1 + J_2 + 2(3Q\ell_5 + \frac{1}{2}N^2)}{\left(\frac{S^*}{2\pi}\right)^2 + (3Q\ell_5 + \frac{1}{2}N^2)^2}\,, \tag{4.55}$$

and

$$\frac{\partial \Omega_1 \ell_5}{\partial \varphi} = -\frac{(1-a^2g^2)g}{2[1+3(a+b)g+(a^2+3ab+b^2)g^2]} \tag{4.56}$$

$$= -\frac{1}{4}N^2 \frac{J_2 + 3Q\ell_5 + \frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2 + (3Q\ell_5 + \frac{1}{2}N^2)^2}\,. \tag{4.57}$$

The analogous expression for $\partial_\varphi \Omega_2$ is obtained by the substitutions $a \leftrightarrow b$ and $J_1 \leftrightarrow J_2$. As a consistency check, we have

$$\partial_\varphi \Phi - \partial_\varphi(\Omega_1 + \Omega_2)\ell_5 = 1 \tag{4.58}$$

as expected from the definition (4.50).

Having computed the potentials $\Phi - \Phi^*$ and $\Omega_{1,2} - \Omega^*_{1,2}$ at linear order in φ, we turn to the first law of thermodynamics

$$0 = TdS = d(M - M^*) - (\Phi - \Phi^*)dQ - (\Omega_1 - \Omega_1^*)dJ_1 - (\Omega_2 - \Omega_2^*)dJ_2\,, \tag{4.59}$$

where $T = 0$ because we consider extremal black holes. The charges are given by the general expressions (4.9-4.11) with m eliminated in favor of q through (4.53). The differentials dQ, $dJ_{1,2}$ therefore become linear combinations of dq and da, db. We are only interested in the first of these, because the others correspond to motion within the BPS surface. Introducing

φ though (4.52) we then find:

$$-(\Phi-\Phi^*)dQ-(\Omega_1-\Omega_1^*)dJ_1-(\Omega_2-\Omega_2^*)dJ_2=-\frac{C_\varphi}{T}\frac{\varphi\, d\varphi}{(2\pi\ell_5)^2}\ , \tag{4.60}$$

with the temperature independent combination $\frac{C_\varphi}{T}$ given by

$$\frac{C_\varphi}{T}=\frac{\pi}{4G_5}\frac{\pi^2(a+b)^2}{g^2(1-ag)(1-bg)}\frac{3+(a+b)g-abg^2}{1+3(a+b)g+(a^2+b^2+3ab)g^2}\ . \tag{4.61}$$

After integration of the first law (4.59) we have

$$M-M^*=\frac{1}{2}\frac{C_\varphi}{T}\left(\frac{\varphi}{2\pi\ell_5}\right)^2\ , \tag{4.62}$$

to the leading order. The identical result can be derived directly from the general mass formula (4.18). Indeed, the central manipulation is given already in (4.22), with the 2nd term in the numerator absent when the temperature vanishes $T=0$.

We refer to C_φ as the *capacitance*. This terminology is motivated by standard electromagnetism insofar as φ can be identified with the electric potential. The capacitance quantifies the energy (4.62) required to violate the constraint by an amount measured by the potential φ. Physically, this is quite distinct from the heat capacity C_T, a measure of the energy needed to increase the temperature. It is therefore surprising that

$$C_\varphi=C_T\ ,$$

for the black holes we consider.

We have repeatedly invoked the intuition that the potential φ introduced in (4.52) measures the violation of the constraint (4.27) that must be satisfied by all BPS configurations. To make this precise we define the height function:

$$h\equiv\left(3Q\ell_5+\frac{1}{2}N^2\right)\left(3(Q\ell_5)^2-\frac{1}{2}N^2(J_1+J_2)\right)-(Q\ell_5)^3-\frac{1}{2}N^2J_1J_2\,, \tag{4.63}$$

that quantifies the distance from the constraint surface $h=0$ explicitly. The differential form

$$dh=3\left(8(Q\ell_5)^2+N^2Q\ell_5-\frac{1}{2}N^2(J_1+J_2)\right)dQ\ell_5-\frac{1}{2}N^2\left(J_2+3Q\ell_5+\frac{1}{2}N^2\right)dJ_1+J_1\leftrightarrow J_2\,, \tag{4.64}$$

realizes the surfaces $h=$ const as 2D planes that are generated by the three one-forms

dQ, dJ_1, dJ_2 subject to the constraint $dh = 0$. In this construction h literally measures the distance along the normal to the 2D constraint surface $h = 0$. The BPS surface can be viewed as the intersection of the constraint surface $h = 0$ with the extremal surface $T = 0$.

Near the BPS surface we can eliminate the parameter m through (4.53) and then (4.9-4.11) express the three charges Q, $J_{1,2}$ as functions of the parameters (q, a, b). On general grounds, the differential form dh above becomes a linear combination of dq, da, db after inserting these formulae for the charges. Explicit computation gives

$$dh = \left[8(Q\ell_5)^3 + \frac{1}{4}N^4(J_1 + J_2)\right]\frac{dq}{q^*} \ .$$

The absence of da, db in this formula shows that the curve dh has no components along the BPS surface, as expected. The nontrivial component along dq relates the normalization of h and φ through (4.52). We find

$$h = \frac{1}{2}\left[8(Q\ell_5)^3 + \frac{1}{4}N^4(J_1 + J_2)\right]\varphi \,. \tag{4.65}$$

In the next subsection we will uncover a term in the entropy that is proportional to φ with a positive coefficient. Stability therefore motivates us to focus on the halfline where both height functions are nonnegative

$$\varphi \geq 0 \ . \tag{4.66}$$

4.2.5 General NearBPS Limit

In this subsection we consider the general nearBPS regime where deviations from the BPS surface may have neither fixed charge nor vanishing temperature. Some aspects of this amount to reconsidering the effects discussed in the previous two subsections at the same time. However, their interplay gives important new insights.

The energy of excitations is the thermodynamic quantity that is conceptually most straightforward in the nearBPS limit. A good starting point is the general mass formula (4.18). It depends on the combination of parameters $m - (1 + (a + b)g)q$ that was rewritten in (4.22) without invoking any assumptions black hole variables. Inserting the first order expressions (4.49) for the temperature T and (4.52) for the potential φ we immediately find the excitation energy above the BPS bound:

$$M - M^* = \frac{1}{2}\frac{C_T}{T}\bigg|_{\text{nearExt}}\left[T^2 + \left(\frac{\varphi}{2\pi\ell_5}\right)^2\right] \ , \tag{4.67}$$

at quadratic order. We have highlighted the identity of the heat capacity and the electric capacitance by avoiding reference to the latter altogether. The new fact featured in the general nearBPS limit is the *absence* of any cross-terms $T\varphi$ in the mass formula (4.67). This demonstrates some sort of rotational symmetry that must be present in the regime we explore. The existence of a continuous structure is much stronger than the equality of C_T and C_φ that we stressed in the preceding subsection.

We next consider the additional entropy due to simultaneously allowing small temperature and violation of the constraint. Starting from the general expression (4.15) for the entropy, we apply the procedure explained around (4.53). Thus we first eliminate the parameter m using (4.53) and expand to linear order in $q-q^*$ and $r_+^2 - r_*^2$. We then take appropriate linear combinations so those two variables are eliminated in favor of the temperature T (4.49) and the potential φ (4.52). These steps give an entropy of the excitations taking the form

$$S - S^* = \frac{C_T}{T}T + \frac{C_E}{T}\frac{\varphi}{2\pi\ell_5}\,, \tag{4.68}$$

where the heat capacity C_T agrees with the expression (4.41) found previously by considering temperature on its own and

$$\begin{aligned}\frac{C_E}{T} &= \frac{2\pi(Q\ell_5)^2}{S^*}\frac{\pi^2}{g}\frac{(1+2(a+b)g+abg^2)(3+(a+b)g-abg^2)}{1+3(a+b)g+g^2(a^2+b^2+3ab)} \\ &= \frac{2\pi}{S^*}\left(\frac{C_T}{T}\right)(3Q+\frac{1}{2}N^2)\ .\end{aligned} \tag{4.69}$$

The value of C_E is subject to a subtle ambiguity. The expression (4.68) gives the entropy $S - S^*$ that is *in excess* of the BPS entropy S^*. However, the BPS entropy is not a proper function of charges, it is only defined modulo the constraint (4.27). This caveat is inconsequential on the BPS surface where the constraint is satisfied identically. In contrast, the constraint is proportional to φ so, for the additional entropy $S - S^*$ the ambiguity can shift the coefficient of C_E arbitrarily, potentially rendering this quantity unphysical.

This issue must be addressed consistently in computations. For example, we variously express the BPS entropy S^* as a function of parameters (a, b) (4.29) or as a function of the charges Q, $J_{1,2}$ (4.34). The differential dS^* computed from the former only gives terms proportional to da and db but when it is evaluated from the latter we get terms of the form dQ, $dJ_{1,2}$ that, because charges depend on all of the parameters (q, a, b), also yield the differential dq. The two forms of the BPS entropy therefore give different coefficients in front of the term $d(q-q^*) = 2q^*d\varphi$ even though they agree if we impose the BPS relation $q = q^*$ before computing the differentials.

We "gauge fix" the ambiguity by insisting that the BPS entropy takes the canonical form

(4.34) in terms of Q, $J_{1,2}$, rather than something that is equivalent to this formula upon imposing the constraint. Our result for C_E (4.69) is predicated on this convention.

The final aspects of the general nearBPS limit that we consider are the potential terms in the first law of thermodynamics:

$$TdS = d(M - M^*) - (\Phi - \Phi^*)dQ - (\Omega_1 - \Omega_1^*)dJ_1 - (\Omega_2 - \Omega_2^*)dJ_2 \,. \qquad (4.70)$$

We can compute the potentials using the procedure described around (4.53), as in the previous examples. However, for the potentials the results for the general nearBPS limit can equally be inferred from the near extremal ($T \neq 0$ and $\varphi = 0$) and nearBPS extremal ($T = 0$ and $\varphi \neq 0$) special cases that we studied in the last two subsections. For example, combining (4.45) and (4.57) we find the correct result

$$(\Omega_1 - \Omega_1^*)\ell_5 \;=\; \frac{1}{4}N^2 \, \frac{\left[J_2 + (3Q\ell_5 + \frac{1}{2}N^2)\right]\varphi - \frac{(2\pi)^2}{S^*}\left[J_2\left(3Q\ell_5 + \frac{1}{2}N^2\right) - \left(\frac{S^*}{2\pi}\right)^2\right]T}{\left(\frac{S^*}{2\pi}\right)^2 + \left(3Q\ell_5 + \frac{1}{2}N^2\right)^2} \qquad (4.71)$$

for the angular velocity in the general nearBPS limit. The analogous equation for $\Omega_2 - \Omega_2^*$ follows by taking $J_1 \leftrightarrow J_2$. The one for $\Phi - \Phi^*$ can be computed similarly from (4.44) and (4.55) or by invoking the sum rule

$$\left(T\partial_T + \varphi\partial_\varphi\right)\left[\Phi - (\Omega_1 + \Omega_2)\ell_5\right] = \varphi \;, \qquad (4.72)$$

that consolidates (4.46) and (4.58) for derivatives with respect to T and φ, respectively.

The explicit formula (4.71) for $\Omega_1 - \Omega_1^*$ and its analogues for other nearBPS potentials are somewhat lengthy and not very illuminating independently. However, they become more instructive when considered together, as a vector in the space of charges. The first law of thermodynamics (4.70) motivates construction of the differential form

$$\begin{aligned} & TdS^* + (\Phi - \Phi^*)dQ + (\Omega_1 - \Omega_1^*)dJ_1 + (\Omega_2 - \Omega_2^*)dJ_2 \\ = \; & \left(\frac{2\pi}{S^*}(3Q\ell_5 + \frac{1}{2}N^2)T + \frac{\varphi}{2\pi\ell_5}\right)\frac{\pi dh}{\left(\frac{S^*}{2\pi}\right)^2 + (3Q\ell_5 + \frac{1}{2}N^2)^2} \;, \end{aligned} \qquad (4.73)$$

where dh is the one-form (4.64) generated by the height function. Thus the relative strength of the three potentials is precisely the same as the one appearing in the height function h, except for the components taken into account by dS^* that are within the BPS surface and only relevant for BPS physics.

Introducing φ using (4.65) the potential terms become

$$TdS^* + (\Phi - \Phi^*)dQ + (\Omega_1 - \Omega_1^*)dJ_1 + (\Omega_2 - \Omega_2^*)dJ_2 = \left[\left(\frac{C_E}{T}\right)T + \left(\frac{C_T}{T}\right)\frac{\varphi}{2\pi\ell_5}\right] d\left(\frac{\varphi}{2\pi\ell_5}\right) , \tag{4.74}$$

where C_T and C_E agree with the functions of charges defined in (4.61) and (4.69). This is precisely what is needed to satisfy the first law of thermodynamics. Importantly, the values for C_E match only when using the canonical form (4.34) of the BPS entropy S^* when evaluating TdS^* in (4.74). This illustrates the necessity of treating the ambiguity discussed below (4.69) consistently.

4.3 The BPS Partition Function

In this section, we review the recent progress on the microscopic origin of the AdS_5 black hole entropy. We focus on the free field approach applied to $\mathcal{N} = 4$ SYM and closely follow [25, 118].

In this and subsequent sections we adopt microscopic units with $\ell_5 = g^{-1} = 1$ and eliminate all references to Newton's constant G_5 in favor of the rank of the gauge group N through (4.33).

4.3.1 The Partition Function

Following recent work, we seek to compute the physical partition function

$$Z(\beta,\, \Delta_I,\, \omega_i) = \mathrm{Tr}\left[e^{-\beta E} e^{\Delta_I Q_I + \omega_i J_i}\right] . \tag{4.75}$$

The electric charges and the angular momenta are denoted by Q_I and J_i and the corresponding chemical potentials are Δ_I and ω_i. Sums over repeated indices I and i are implied. We stress that we do not consider the supersymmetric index since no grading $(-1)^F$ has been inserted.

The fermionic symmetries $\mathcal{Q}$ and $\mathcal{S}$ transform as spinors under both the $SU(4)_R$ symmetry and the $SO(4)$ little group. We denote the quantum numbers of the supercharge $\mathcal{Q} \equiv \mathcal{Q}^{s_1,s_2,s_3}_{-t_1,-t_2}$ with respect to these groups $\frac{1}{2}s_I$ and $-\frac{1}{2}t_i$ where $s_I = \pm 1$ and $t_i = \pm 1$. The signs of the spinorial indices are explicitly flipped for the conformal supercharges $\mathcal{S} \equiv \mathcal{S}^{-s_1,-s_2,-s_3}_{t_1,t_2}$ which have opposite chirality so we can take the products $\prod s_I \prod_i t_i = +1$ in both cases.

We write the superalgebra in component form as

$$\{\mathcal{Q},\,\mathcal{S}\} = E - \sum_{I=1}^{3} s_I Q_I - \sum_{i=1}^{2} t_i J_i\,. \tag{4.76}$$

The anticommutator $\{\mathcal{Q},\,\mathcal{S}\} = 0$ when acting on $\frac{1}{16}$-BPS states so we have the BPS condition

$$E = \sum_I Q_I + \sum_i J_i\ , \tag{4.77}$$

when we restrict to the sector where all $s_I, t_i = +1$ without loss of generality. We can adapt the physical partition function (4.75) to this sector as

$$Z(\beta,\,\Delta_I,\,\omega_i) = \mathrm{Tr}\left[e^{-\beta(E-\sum_I Q_I - \sum_i J_i)} e^{\widetilde{\Delta}_I Q_I + \widetilde{\omega}_i J_i}\right]\ , \tag{4.78}$$

where we introduced

$$\widetilde{\Delta}_I \equiv \Delta_I - \beta\,,\quad \widetilde{\omega}_i \equiv \omega_i - \beta\ . \tag{4.79}$$

Then only the states satisfying the BPS condition (4.77) contribute to the partition function in the limit $\beta \to \infty$ with fixed $\widetilde{\Delta}_I$ and $\widetilde{\omega}_i$.

The quantum numbers Q_I, J_i are half-integer valued for all states in the theory so the corresponding chemical potentials satisfy the periodicity conditions

$$\Delta_I \equiv \Delta_I + 4\pi i\,,\quad \omega_i \equiv \omega_i + 4\pi i\,. \tag{4.80}$$

However, the operator $\mathcal{O} = e^{\tilde{\Delta}\cdot Q + \tilde{\omega}\cdot J}$ that is inserted in the partition function (4.78) generally does not anticommute with the supercharge

$$\mathcal{Q}\mathcal{O} = e^{-\frac{1}{2}s\cdot\tilde{\Delta} + \frac{1}{2}t\cdot\tilde{\omega}}\,\mathcal{O}\mathcal{Q}\,. \tag{4.81}$$

Anticommutation imposes the additional condition:

$$s\cdot\tilde{\Delta} - t\cdot\tilde{\omega} = 2\pi i \pmod{4\pi i}\,. \tag{4.82}$$

Therefore supersymmetry demands that the potentials satisfy

$$\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 - \tilde{\omega}_1 - \tilde{\omega}_2 = 2\pi i \pmod{4\pi i}\,, \tag{4.83}$$

for projection to the BPS sector we focus on.

The complex condition (4.83) on the potentials is essential. It is closely related to the supersymmetric index, since insertion of $(-)^F$ in the partition function is equivalent to the shift $\omega_i \to \omega_i + 2\pi i$ for either $i = 1$ or $i = 2$. However, following recent literature we will maintain reference to the partition function rather than the supersymmetric index and we will consider complex potentials. In this terminology the partition function counts protected states (and so is independent of β) on the surface defined by the complex supersymmetry condition (4.83).

4.3.2 Single Particle Enumeration

We now turn to the central problem of computing the partition function (4.75) of $\mathcal{N} = 4$ SYM on $S^1 \times S^3$. We impose anti-periodic boundary conditions for fermions along S^1, as usual for any partition function but in contradistinction to the supersymmetric index. We work perturbatively at weak coupling and express the result as a matrix model following [118, 143, 144].

The first step is to enumerate "single-particle states", in the terminology of the bulk AdS_5 theory. In the quantum field theory description these are the individual operators that generate the operator algebra. They can be realized as elementary fields, possibly with derivatives since those do not change the particle number. It is useful to decompose the field content of $\mathcal{N} = 4$ SYM under an $\mathcal{N} = 1$ subalgebra and represent is matter as an $\mathcal{N} = 1$ vector multiplet, three $\mathcal{N} = 1$ chiral multiplets, and three $\mathcal{N} = 1$ anti-chiral multiplets. In components, a $\mathcal{N} = 1$ vector multiplet contains a gauge boson and a real spinor, while a $\mathcal{N} = 1$ chiral multiplet contains a complex scalar and a real chiral spinor.

We first consider a chiral multiplet with the spectrum:

Fields	$(E,\ J_1,\ J_2;\ Q_1,\ Q_2,\ Q_3)$
$X = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)$	$(1,\ 0,\ 0;\ 1,\ 0,\ 0)$
$Y = \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4)$	$(1,\ 0,\ 0;\ 0,\ 1,\ 0)$
$Z = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)$	$(1,\ 0,\ 0;\ 0,\ 0,\ 1)$
$\bar{\psi}_{\dot{\alpha},\, a}$	$(\frac{3}{2},\ 0,\ \pm\frac{1}{2};\ -\frac{1}{2},\ +\frac{1}{2},\ +\frac{1}{2})$
	$(\frac{3}{2},\ 0,\ \pm\frac{1}{2};\ +\frac{1}{2},\ -\frac{1}{2},\ +\frac{1}{2})$
	$(\frac{3}{2},\ 0,\ \pm\frac{1}{2};\ +\frac{1}{2},\ +\frac{1}{2},\ -\frac{1}{2})$

The corresponding partition functions become

$$f^c_B(\beta,\,\Delta_I,\,\omega_i) = \sum_{I=1}^{3} e^{\Delta_I} \frac{e^{-\beta}(1-e^{-2\beta})}{(1-e^{-\beta+\omega_1})(1-e^{-\beta+\omega_2})(1-e^{-\beta-\omega_1})(1-e^{-\beta-\omega_2})}\,, \tag{4.84}$$

$$f^c_F(\beta,\,\Delta_I,\,\omega_i) = \sum_{I=1}^{3} e^{\Delta_I} \frac{e^{-\frac{3}{2}\beta-\Delta}\left((e^{\omega_+}+e^{-\omega_+})-e^{-\beta}(e^{\omega_-}+e^{-\omega_-})\right)}{(1-e^{-\beta+\omega_1})(1-e^{-\beta+\omega_2})(1-e^{-\beta-\omega_1})(1-e^{-\beta-\omega_2})}\,, \tag{4.85}$$

where

$$\Delta \equiv \frac{\Delta_1+\Delta_2+\Delta_3}{2}\,, \quad \omega_\pm \equiv \frac{\omega_1 \pm \omega_2}{2}\,. \tag{4.86}$$

The common denominators of (4.84-4.85) incorporate the quantum numbers of the four components in the gradient ∇_μ, each resummed as a geometric series to take into account any number of derivatives. The positive terms in the numerators encode the data from the table, while the negative ones remove operators that satisfy their equations of motion.

The anti-chiral multiplet can be obtained from the chiral multiplet by

$$J_1 \leftrightarrow J_2\,, \quad Q_I \to -Q_I\,. \tag{4.87}$$

For bosons this just changes the overall factor giving the R-charge, but for fermions the Lorentz indices of the fields and the equation of motion exchange their chirality $\omega_+ \leftrightarrow \omega_-$, in addition to the R-charges changing sign:

$$f^a_B(\beta,\,\Delta_I,\,\omega_i) = \sum_{I=1}^{3} e^{-\Delta_I} \frac{e^{-\beta}(1-e^{-2\beta})}{(1-e^{-\beta+\omega_1})(1-e^{-\beta+\omega_2})(1-e^{-\beta-\omega_1})(1-e^{-\beta-\omega_2})}\,, \tag{4.88}$$

$$f^a_F(\beta,\,\Delta_I,\,\omega_i) = \sum_{I=1}^{3} e^{-\Delta_I} \frac{e^{-\frac{3}{2}\beta+\Delta}\left((e^{\omega_-}+e^{-\omega_-})-e^{-\beta}(e^{\omega_+}+e^{-\omega_+})\right)}{(1-e^{-\beta+\omega_1})(1-e^{-\beta+\omega_2})(1-e^{-\beta-\omega_1})(1-e^{-\beta-\omega_2})}\,. \tag{4.89}$$

The vector multiplet has the spectrum:

Fields	$(E,\,J_1,\,J_2;\,Q_1,\,Q_2,\,Q_3)$
$A_\mu\,(\mu=1,\,\cdots,\,4)$	$(1,\,\pm1,\,\pm1;\,0,\,0,\,0)$
ψ^a_α	$(\frac{3}{2},\,\pm\frac{1}{2},\,0;\,+\frac{1}{2},\,+\frac{1}{2},\,+\frac{1}{2})$
$\bar\psi_{\dot\alpha,\,a}$	$(\frac{3}{2},\,0,\,\pm\frac{1}{2};\,-\frac{1}{2},\,-\frac{1}{2},\,-\frac{1}{2})$

with the single particle partition functions:

$$f_B^v(\beta,\,\Delta_I,\,\omega_i) = \frac{e^{-\beta}(e^{\omega_1}+e^{\omega_2}+e^{-\omega_1}+e^{-\omega_2})-1-e^{-2\beta}}{(1-e^{-\beta+\omega_1})(1-e^{-\beta+\omega_2})(1-e^{-\beta-\omega_1})(1-e^{-\beta-\omega_2})}(1-e^{-2\beta})+1\,, \quad (4.90)$$

$$f_F^v(\beta,\,\Delta_I,\,\omega_i) = \frac{e^{-\frac{3}{2}\beta}(e^{\Delta}-e^{-\Delta}e^{-\beta})(e^{\omega_+}+e^{-\omega_+})+e^{-\frac{3}{2}\beta}(e^{-\Delta}-e^{\Delta}e^{-\beta})(e^{\omega_-}+e^{-\omega_-})}{(1-e^{-\beta+\omega_1})(1-e^{-\beta+\omega_2})(1-e^{-\beta-\omega_1})(1-e^{-\beta-\omega_2})}\,. \quad (4.91)$$

The terms in the boson numerator indicate the vector field A_μ (that has the same quantum numbers as ∇_μ in the denominator), with subtractions for the neutral scalar gauge function Λ and Lorentz condition $\nabla_\mu A^\mu = 0$. The overall factor $(1-e^{-2\beta})$ imposes the Klein-Gordon equation on all operators and the additional $+1$ corrects for the fact we erroneously counted Λ with no derivative acting on it as a pure gauge degree of freedom. The gaugini follow from the fermions in chiral and anti-chiral multiplets, upon omission of the overall $\sum_{I=1}^3 e^{-\Delta_I}$ and subsequent reversal of the correlation between chirality and R-charge.

In order to focus on states that satisfy the BPS condition (4.77) we rewrite the physical partition function as (4.78) by introducing the shifted "tilde" potentials (4.79). The low temperature limit $\beta\to\infty$ taken with these variables fixed is exceptionally simple, because only a few single particle operators contribute. Considering bosons and fermions separately, we have

$$\begin{aligned} f_B &= f_B^v+f_B^c+f_B^a = \frac{\sum_I e^{-\tilde\Delta_I}+e^{-\tilde\omega_1-\tilde\omega_2}}{(1-e^{-\tilde\omega_1})(1-e^{-\tilde\omega_2})}\,, \\ f_F &= f_F^v+f_F^c+f_F^a = \frac{\sum_I e^{\tilde\Delta_I-\tilde\Delta}e^{-\tilde\omega_+}+e^{-\tilde\Delta+\tilde\omega_+}}{(1-e^{-\tilde\omega_1})(1-e^{-\tilde\omega_2})} - e^{-\tilde\Delta+\tilde\omega_+}\,. \end{aligned} \quad (4.92)$$

The analogous expressions for general β are much more complicated. However, if we impose the constraint (4.83) by taking $e^{-\tilde\Delta+\tilde\omega_+}=-1$, the dependence on β simplifies and in fact the total single particle partition function

$$f_B+f_F = 1-\frac{\prod_I(1-e^{-\tilde\Delta_I})}{(1-e^{-\tilde\omega_1})(1-e^{-\tilde\omega_2})}\,, \quad (4.93)$$

becomes *independent* of β. Equivalently, the supersymmetric index f_B-f_F is temperature independent on the constraint surface $e^{-\tilde\Delta+\tilde\omega_+}=+1$. In fact, it is identical to (4.93). These results verify by explicit computations the expectations from the general arguments familiar from the Witten index. The demonstate that the partition function computed at vanishing temperature $\beta=\infty$ applies at any temperature when the constraint is satisfied.

4.3.3 Multiparticle Enumeration

All of the "letters" realized by single particle operators can be multiplied together to form "words". Starting from any of the operators enumerated by the single particle partition function $f(\beta,\, \Delta_I,\, \omega_i)$ we can form n-particle products counted by $f(n\beta,\, n\Delta_I,\, n\omega_i)$. An overall trace must be imposed to ensure gauge invariance but then n-fold cyclicity of the trace must be taken account so, after summing over any number of operators, the single trace partition function becomes

$$Z_{\rm ST} = \sum_{k=1}^{\infty} \frac{f(n\beta,\, n\Delta_I,\, n\omega_i)}{n}\,. \tag{4.94}$$

Furthermore, having determined all single trace operators, taking multi-trace operators into account exponentiates the counting. The full partition function becomes

$$Z_{\rm MP}(\beta,\, \Delta_I,\, \omega_i) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n}\left[f_B(n\beta,\, n\Delta_I,\, n\omega_i) + (-1)^{n+1}\, f_F(n\beta,\, n\Delta_I,\, n\omega_i)\right]\right]\,. \tag{4.95}$$

This result is purely combinatorial so the statistics of fermions only enter through the exclusion principle. This accounts for the prefactor $(-1)^{n+1}$ in front of the fermionic terms.

The brief justification of the multiparticle partition function (4.95) presented in this subsection has been cavalier about the combinatorics. The relative ordering of operators within traces is important for the detailed enumeration and somewhat elaborate combinatorics (Polya theory) must be invoked. However, for large N the correct result is in fact (4.95) and the heuristic arguments given in this subsection serve to motivate the key features of the formula.

4.3.4 The $\mathcal{N} = 4$ SYM Perturbative Matrix Model

One additional feature must taken into account. All of the quantum fields transform in the adjoint of $U(N)$ and we must incorporate their gauge indices. We (somewhat prematurely) imposed a singlet condition on the operators already in the preceding subsection. Incorporation of the full gauge structure gives the unitary matrix model [118, 143, 144]

$$Z(\beta,\, \Delta_I,\, \omega_i) = \int [dU] \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} f_n(n\beta,\, n\Delta_I,\, n\omega_i)\, \chi_{\rm Adj}(U^n)\right\}\,, \tag{4.96}$$

where U denotes a $U(N)$ matrix, $\chi_{\rm Adj}$ the character in the adjoint representation, and

$$f_n(n\beta,\, n\Delta_I,\, n\omega_i) = f_B(n\beta,\, n\Delta_I,\, n\omega_i) + (-1)^{n+1}\, f_F(n\beta,\, n\Delta_I,\, n\omega_i) \ . \tag{4.97}$$

For completeness, we recall that in the case of weakly coupled $\mathcal{N}=4$ SYM the single particle partition functions are

$$\begin{aligned} f_B &= f_B^v + f_B^c + f_B^a \ , \\ f_F &= f_F^v + f_F^c + f_F^a \ , \end{aligned} \tag{4.98}$$

where the constituent functions were given in (4.84, 4.88, 4.90) and (4.85, 4.89, 4.91), respectively. Their explicit expressions are not illuminating in general but for $\beta=\infty$ they greatly simplify and reduce to (4.92).

The standard strategy for integrating over all unitary matrices is to represent them in terms of their eigenvalues $e^{i\alpha_a}$ (so those of the adjoint representation become $e^{i\alpha_{ab}}$ where $\alpha_{ab}\equiv\alpha_a-\alpha_b$) and change variables to an integral over eigenvalues

$$Z(\beta,\, \Delta_I,\, \omega_i) = \frac{1}{N!}\oint \prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\prod_{a<b}\Big(2\sin\frac{\alpha_{ab}}{2}\Big)^2 \exp\Big[\sum_{a,b=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}f_n(n\beta,\, n\Delta_I,\, n\omega_i)\, e^{in\alpha_{ab}}\Big] . \tag{4.99}$$

The factor involving $\sin\frac{\alpha_{ab}}{2}$ is the van der Monde determinant that arises as a Jacobian due to the change of variables.

We stress that the BPS constraint (4.83) has not yet been imposed on the matrix model so the chemical potentials Δ_I and ω_i are still independent parameters and the partition function (4.99) includes contributions from nonBPS states.

4.3.5 The BPS Limit

The analysis of matrix models is a highly developed science [145, 146]. The established intuition is that the eigenvalues of the matrices experience a universal repulsion because of the van der Monde determinant that may be balanced by attraction due to a model-dependent potential, which in the current context is closely related to the single particle distribution f. The repulsion favors a uniform eigenvalue distribution that corresponds to a confined phase with free energy of order $\mathcal{O}(1)$ while attraction can prompt localization that gives rise to a deconfined phase where free energy increases to order $\mathcal{O}(N^2)$ [147].

Early studies of the matrix model for $\mathcal{N}=4$ SYM failed to identify a deconfined phase appropriate for the description of macroscopic black holes [118], but recent research makes claims to the contrary [131]. We will refrain from a nuanced discussion of the evidence one way or another and simply assume a deconfined phase where the phases $e^{in\alpha_{ab}}$ do not give cancellations at the leading order. Then the matrix model (4.99) yields the partition function

$$\ln Z(\beta,\, \Delta_I,\, \omega_i) = N^2 \sum_{n=1}^{\infty} \frac{1}{n} f_n(n\beta,\, n\Delta_I,\, n\omega_i) \ .$$

The BPS limit requires vanishing temperature and we must also impose the BPS constraint (4.83). We already computed the single particle partition function f in this limit with the result (4.93). The fermion phase factor $(-)^{n+1}$ in (4.97) conspires with signs from the constraint $e^{-\tilde{\Delta}+\tilde{\omega}_+} = -1$ such that the multiparticle generalization becomes

$$f_n(n\beta,\, n\Delta_I,\, n\omega_i) \;\; = \;\; 1 - \frac{\prod_I (1 - e^{-n\tilde{\Delta}_I})}{(1 - e^{-n\tilde{\omega}_1})(1 - e^{-n\tilde{\omega}_2})} \ . \tag{4.100}$$

This is exactly the result reported in [25]. There the analysis focussed on the high temperature limit $\beta \to 0$ while we have discussed the low temperature regime $\beta \to \infty$. The agreement of the results is due to the temperature independence along the BPS surface.

We further restrict the discussion to the Cardy limit $|\widetilde{\omega}_i| \ll 1$. In this situation it has been argued that all significant contributions to the sum over n are from sufficiently small n that $n|\widetilde{\omega}_i| \ll 1$. Therefore

$$\begin{aligned} \ln Z(\beta,\, \Delta_I,\, \omega_i) \;\; &= \;\; \frac{N^2}{\widetilde{\omega}_1 \widetilde{\omega}_2} \sum_{n=1}^{\infty} \frac{1}{n^3} \prod_I (1 - e^{-n\tilde{\Delta}_I}) \\ &= \;\; \frac{N^2}{\widetilde{\omega}_1 \widetilde{\omega}_2} \sum_{s_1 s_2 s_3 = +1} \left[\mathrm{Li}_3 \left(-e^{\frac{s_I \tilde{\Delta}_I}{2}} \right) - \mathrm{Li}_3 \left(-e^{-\frac{s_I \tilde{\Delta}_I}{2}} \right) \right] \ . \end{aligned} \tag{4.101}$$

The identity

$$\mathrm{Li}_3(-e^x) - \mathrm{Li}_3(-e^{-x}) = -\frac{x^3}{6} - \frac{\pi^2 x}{6} \quad \text{for } \; -\pi < \mathrm{Im}(x) < \pi \,, \tag{4.102}$$

finally gives the free energy

$$\begin{aligned} \ln Z(\beta,\, \Delta_I,\, \omega_i) \;\; &= \;\; -\frac{N^2}{6\, \widetilde{\omega}_1 \widetilde{\omega}_2} \sum_{s_1 s_2 s_3 = +1} \left[\frac{1}{8} \left(s_I \widetilde{\Delta}_I \right)^3 + \frac{1}{2} \pi^2 \left(s_I \widetilde{\Delta}_I \right) \right] \\ &= \;\; -\frac{1}{2} N^2 \frac{\widetilde{\Delta}_1 \widetilde{\Delta}_2 \widetilde{\Delta}_3}{\widetilde{\omega}_1 \widetilde{\omega}_2} \,. \end{aligned} \tag{4.103}$$

4.3.6 Discussion

The partition function (4.103) with the complex constraint on potentials (4.83) undoubtedly describes $\mathcal{N} = 4$ SYM in the regime relevant for comparison with AdS_5 BPS black holes. It was first inferred from black hole thermodynamics [122] and recently derived from the Euclidean path integral and supersymmetric localization [24], from free field analysis [25], and from computation of the superconformal index using a sum over Bethe vacua [26]. All these works pursue similar ideas but some aspects of the computations and their interrelation remains unclear, at least to the author.

The simple computations we presented in this section following [25] give an interesting free field representation that appears to capture some aspects of physics in the strongly coupled region. The situation is akin to a model that represents a CFT_2 with central charge c as a free model such as c free bosons. Such a toy model of CFT_2 generally misses many detailed features of the theory, but it captures some aspects of the CFT_2 robustly (such as the Casimir energy and the entropy) and so it serves as a useful benchmark.

In other words, we are not committed to the free field derivation, but the upshot one way or another is that the partition function for the BPS limit is

$$\ln Z = -\frac{1}{2} N^2 \frac{\widetilde{\Delta}_1 \widetilde{\Delta}_2 \widetilde{\Delta}_3}{\widetilde{\omega}_1 \widetilde{\omega}_2} \, . \tag{4.104}$$

Our goal in the subsequent section is to leverage this result, whichever way it came about, to account also for nonBPS physics.

4.4 Black Hole Statistical Physics

In this section we discuss thermodynamics of black holes starting from the BPS result for the microscopic free energy (4.104). Our emphasis is on thermodynamic variables for nearBPS black holes.

4.4.1 Studying nearBPS using BPS data: Introduction

The microscopic considerations in section 4.3 studied the partition function (4.78) which we reproduce here for convenience: [7]

$$Z(\beta,\, \Delta_I,\, \omega_i) = \mathrm{Tr}\left[e^{-\beta(E-E^*)} e^{\widetilde{\Delta}_I Q_I + \widetilde{\omega}_i J_i} \right] \, . \tag{4.105}$$

[7]The potentials (Δ_I, ω_i) in the microscopic theory can be identified with (Φ_I, Ω_i) in supergravity. Using different symbols let us stress the distinct provenance of various results and allows for easier comparison with some key references.

The notation strongly suggests that this partition function depends on all the variables $\beta, \Delta_I, \omega_i$ but in fact the computations in section 4.3 were restricted to the BPS limit, a surface of real codimension two. The restriction on the domain of Z can be specified by the constraints $\beta = \infty$ and $\sum_I \widetilde{\Delta}_I - \sum_i \widetilde{\omega}_i = 2\pi i$.

The goal of this section is to generalize the microscopic description beyond BPS and so allow deformations of both these constraints by small amounts. Our strategy is to make mild smoothness assumptions on the partition function, which are then validated by comparison with our results from gravity. To clarify how it is possible to learn anything interesting from such minimal assumptions, it is instructive to consider a simple example.

As a start, we must address an issue of conventions. In our gravitational computations, the thermodynamic variables satisfy the first law (4.70). Comparison with the microscopic partition function (4.78) indicates a relative factor of β in the potentials, leading to the identifications $\beta(\Phi_I - \Phi_I^*) \sim \widetilde{\Delta}_I$ and $\beta(\Omega_i - \Omega_i^*) \sim \widetilde{\omega}_i$.[8] The shift of the gravitational potentials by their BPS values Φ_I^*, Ω_i^* corresponds precisely to the definition (4.79) of microscopic potentials with a tilde from those without tilde.

The meticulous tracking of conventions gives an immediate payoff in the extremal limit $T \to 0$. The limit is taken such that variables with tilde are kept fixed. Therefore, the potentials that appear in the microscopic partition function are identified with the *thermal derivative* of the potentials employed in the gravity description [117, 142]:

$$\begin{aligned} \mathrm{Re}\ \tilde{\Delta}_I &= \partial_T \Phi_I \ , \\ \mathrm{Re}\ \tilde{\omega}_i &= \partial_T \Omega_i \ . \end{aligned} \tag{4.106}$$

The microscopic partition function (4.105) gives values for the left hand side that should coincide with the gravitational results on the right hand side. We will verify this expectation in subsection 4.4.3.

The identifications (4.106) also illustrate the strategy for going beyond BPS. These equations were found by taking the extremal limit $T \to 0$ but supersymmetry was not invoked. This provenance suggests their validity also when the constraint is violated. We will confirm this expectation below. This successful comparison is a simple example of leveraging BPS results to study the nearBPS regime.

Recall that the supersymmetry condition (4.83) is complex and by continuity potentials remain complex in the entire nearBPS region. In our identifications (4.106) we identified the

[8] The gravitational computations in section 4.2 were restricted to the diagonal case where the three potentials are equal but this limitation does not apply to formulae in this section. When we "compare" with section 4.2 we only literally compare for diagonal charges. The generic formula we derive in this section constitute microscopic *predictions* for the gravitational side.

real part of the potentials with the corresponding physical field in spacetime. This is justified because the physical charges are real and so their conjugate potentials are real, according to the partition function (4.105). We will later find more data about the nearBPS by exploiting both the real and imaginary parts of the potential, as well as their interplay.

BPS configurations have energy $E = E^*$ and so the saddlepoint approximation to the partition function (4.105) gives the entropy

$$S = \ln Z - \widetilde{\Delta}_I Q_I - \widetilde{\omega}_i J_i - \Lambda\Big(\sum_I \widetilde{\Delta}_I - \sum_i \widetilde{\omega}_i - 2\pi i\Big) \,. \tag{4.107}$$

The constraint on potentials (4.83) was imposed by introducing a Lagrange multiplier Λ. The expression (4.107) is referred to as the *entropy function*. Unlike the usual entropy it is a function of potentials but, after they are extremized over, it gives the black hole entropy as function of charges. In the present context there is little more to the terminology than a basic change of thermodynamic ensemble through Legendre transform (but there are impressive generalizations [148]). In the next subsection we will extremize the entropy function (4.107) for BPS black holes explicitly. Later, in subsection 4.4.4, we will generalize the entire entropy function to the nearBPS regime using minimal assumptions.

4.4.2 Entropy Extremization for BPS Black Holes

In this subsection we review the computation of BPS black hole entropy. We start from the entropy function S (4.107) derived from the BPS partition function (4.104). Our analysis mainly follow [24].

Extremization of the entropy function (4.107) with $\ln Z$ given by (4.104) gives

$$\frac{\partial S}{\partial \widetilde{\Delta}_I} = -\frac{1}{2}N^2 \frac{1}{\widetilde{\Delta}_I} \frac{\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3}{\widetilde{\omega}_1\widetilde{\omega}_2} - (Q_I + \Lambda) = 0\,, \tag{4.108}$$

$$\frac{\partial S}{\partial \widetilde{\omega}_i} = \frac{1}{2}N^2 \frac{1}{\widetilde{\omega}_i} \frac{\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3}{\widetilde{\omega}_1\widetilde{\omega}_2} - (J_i - \Lambda) = 0\,, \tag{4.109}$$

$$\frac{\partial S}{\partial \Lambda} = \sum_I \widetilde{\Delta}_I - \sum_i \widetilde{\omega}_i - 2\pi i = 0\,. \tag{4.110}$$

The last equation imposes the constraint (4.83) on $\widetilde{\Delta}_I$ and $\widetilde{\omega}_i$.

We obtain the BPS entropy by simplifying the entropy function (4.107) using the extremization conditions (4.108, 4.109):

$$S^* = 2\pi i \Lambda\,. \tag{4.111}$$

It is therefore essential to find the Lagrange multiplier Λ. To do so, we combine (4.108) and (4.109) to find

$$\prod_I (Q_I + \Lambda) + \frac{1}{2} N^2 \prod_i (J_i - \Lambda) = 0 \,. \tag{4.112}$$

This is a cubic equation that yields the Lagrange multiplier Λ, and so the BPS entropy S^*, as a function of the black hole charges Q_I and J_i. The explicit form of the cubic equation is

$$\Lambda^3 + A\Lambda^2 + B\Lambda + C = 0 \ , \tag{4.113}$$

where

$$\begin{aligned} A &= Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2 \,, \\ B &= Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2} N^2 (J_1 + J_2) \,, \\ C &= Q_1 Q_2 Q_3 + \frac{1}{2} N^2 J_1 J_2 \,. \end{aligned} \tag{4.114}$$

Imposing reality of the physical entropy (4.111) demands a purely imaginary Λ. Since the charges Q_I and J_i are all real, the purely imaginary roots of (4.113) appear in pairs and the cubic equation must factorize as

$$(\Lambda^2 + B)(\Lambda + A) = 0 \ . \tag{4.115}$$

Thus coefficients in the cubic satisfy $C - AB = 0$ or

$$\left(Q_1 Q_2 Q_3 + \frac{1}{2} N^2 J_1 J_2 \right) - \left(Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2 \right) \left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2} N^2 (J_1 + J_2) \right) = 0 \,. \tag{4.116}$$

This is the constraint on black hole charges (4.27) that must be satisfied on the BPS surface, as a consequence of the BPS formula for the mass. It generalizes the constraint (4.27) previously found from gravity, with perfect agreement when the three charges are identical. When the constraint is satisfied, the root for Λ with negative imaginary part gives the BPS entropy

$$S^* = 2\pi i \Lambda = 2\pi \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2} N^2 (J_1 + J_2)} \ . \tag{4.117}$$

This formula similarly generalizes the result from gravity (4.34) with agreement when the three charges are identical.

We also need the potentials $\widetilde{\Delta}_I$ and $\widetilde{\omega}_i$ at the extremum of the entropy function. We consider $\Lambda = \frac{S^*}{2\pi i}$ the known function of the charges given through (4.117). The extremization conditions (4.108, 4.109) give the ratios

$$\frac{\widetilde{\Delta}_I}{\widetilde{\omega}_i} = -\frac{J_i-\Lambda}{Q_I+\Lambda} , \tag{4.118}$$

for any $I = 1, 2, 3$ and $i = 1, 2$. Comparing (4.109) with $i = 1$ and $i = 2$ we also find the ratio

$$\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2} = \frac{J_2-\Lambda}{J_1-\Lambda} . \tag{4.119}$$

The constraint (4.83), i.e. the equation of motion for Λ (4.110), now gives

$$\frac{\widetilde{\omega}_1+\widetilde{\omega}_2+2\pi i}{\widetilde{\omega}_i} = \frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2+\widetilde{\Delta}_3}{\widetilde{\omega}_i} = -(J_i-\Lambda)\left(\frac{1}{Q_1+\Lambda}+\frac{1}{Q_2+\Lambda}+\frac{1}{Q_3+\Lambda}\right) , \tag{4.120}$$

where we have used the ratios $\tilde{\Delta}_I/\tilde{\omega}_i$ given in (4.118). The ratio between the $\tilde{\omega}_i$'s (4.119) let us reorganize as

$$\frac{2\pi i}{\widetilde{\omega}_i} = -(J_i-\Lambda)\left(\frac{1}{J_1-\Lambda}+\frac{1}{J_2-\Lambda}+\frac{1}{Q_1+\Lambda}+\frac{1}{Q_2+\Lambda}+\frac{1}{Q_3+\Lambda}\right) , \tag{4.121}$$

$$\frac{2\pi i}{\widetilde{\Delta}_I} = (Q_I+\Lambda)\left(\frac{1}{J_1-\Lambda}+\frac{1}{J_2-\Lambda}+\frac{1}{Q_1+\Lambda}+\frac{1}{Q_2+\Lambda}+\frac{1}{Q_3+\Lambda}\right) . \tag{4.122}$$

The second line was found by invoking the ratio (4.118). The inverses of these equations give

$$\frac{\widetilde{\omega}_i}{2\pi i} = \frac{1}{2}N^2\frac{\prod_k(J_k-\Lambda)}{J_i-\Lambda}\frac{1}{2\Lambda(\Lambda+Q_1+Q_2+Q_3+\frac{1}{2}N^2)} , \tag{4.123}$$

$$\frac{\widetilde{\Delta}_I}{2\pi i} = \frac{\prod_K(Q_K+\Lambda)}{Q_I+\Lambda}\frac{1}{2\Lambda(\Lambda+Q_1+Q_2+Q_3+\frac{1}{2}N^2)} . \tag{4.124}$$

We used (4.112) to simplify the algebra. These are the explicit results for the potentials on the BPS surface expressed in terms of charges. We recall that $\Lambda = \frac{S^*}{2\pi i}$ is the known function of the charges given through (4.117).

4.4.3 NearBPS Microscopics

We now want to leverage the microscopic results derived in the BPS limit to study nearBPS black holes as well.

The real part of the potentials (4.123) are

$$\text{Re }\widetilde{\omega}_1 = -\frac{\pi^2 N^2}{S^*}\frac{J_2(Q_1+Q_2+Q_3+\frac{1}{2}N^2)-\left(\frac{S^*}{2\pi}\right)^2}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2}\,, \tag{4.125}$$

with an analogous expression for $\text{Re }\widetilde{\omega}_2$, and the real part of the potentials (4.124) similarly are

$$\text{Re }\widetilde{\Delta}_1 = -\frac{2\pi^2}{S^*}\frac{(Q_2Q_3-\left(\frac{S^*}{2\pi}\right)^2)(Q_1+Q_2+Q_3+\frac{1}{2}N^2)+\left(\frac{S^*}{2\pi}\right)^2(Q_2+Q_3)}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2}\,, \tag{4.126}$$

with analogous equations for $\text{Re}\,\widetilde{\Delta}_2$ and $\text{Re}\,\widetilde{\Delta}_3$. These formulae for the potentials $\text{Re }\widetilde{\omega}_i$, $\text{Re }\widetilde{\Delta}_I$ in the microscopic theory all agree precisely with the corresponding gravitational formulae for $\partial_T\Phi_I$ and $\partial_T\Omega_i$ given in (4.44-4.45). This confirms the identifications (4.106).

It is significant that these comparisons *all* agree. The differentials ($d\text{Re }\widetilde{\omega}_i, d\text{Re }\widetilde{\Delta}_I$) form a vector in the five dimensional space that is generated (locally) by the direct sum of the tangent space to the BPS surface and its normal, and the latter violates the constraint. It is also noteworthy that the comparisons agree *precisely*. Formulae in the BPS limit are only defined modulo the constraint on charges but the agreements found here apply with no need for the constraint. These facts suggest that the microscopic description goes beyond BPS.

We can make these observations quantitative by forming the differential

$$dS^* + \text{Re}\,\widetilde{\Delta}_I\,dQ_I + \text{Re}\,\widetilde{\omega}_i\,dJ_i \;=\frac{2\pi^2}{S^*}\frac{(Q_1+Q_2+Q_3+\frac{1}{2}N^2)\,dh}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)}\,, \tag{4.127}$$

where the height function h defined through

$$h=\left(Q_1{+}Q_2{+}Q_3{+}\frac{1}{2}N^2\right)\left(Q_1Q_2{+}Q_2Q_3{+}Q_3Q_1{-}\frac{1}{2}N^2(J_1{+}J_2)\right)-\left(Q_1Q_2Q_3{+}\frac{1}{2}N^2J_1J_2\right), \tag{4.128}$$

is a measure of the violation of the constraint (4.116) on black hole charges. The addition of the differential dS^* on the left hand side of (4.127) removes the terms that are attributable to BPS physics. We can interpret the remainder as a formula for the entropy in excess of the BPS entropy

$$S-S^* = \frac{2\pi^2 h}{S^*}\frac{Q_1+Q_2+Q_3+\frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)}\,, \tag{4.129}$$

due to the violation of the constraint $h=0$. This formula is consistent with the gravitational result (4.73). Our definition (4.128) of the height function is such that $h\geq 0$ corresponds to positive entropy.

It is also well worth the effort to extract the imaginary parts of the complex potentials (4.123-4.124) found in the microscopic computation. Representative components are

$$\frac{\operatorname{Im}\widetilde{\omega}_1}{2\pi} = -\frac{1}{4}N^2\frac{J_2+Q_1+Q_2+Q_3+\frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2}\,, \tag{4.130}$$

$$\frac{\operatorname{Im}\widetilde{\Delta}_1}{2\pi} = \frac{(Q_2+Q_3)(2Q_1+Q_2+Q_3+\frac{1}{2}N^2)-\frac{1}{2}N^2(J_1+J_2)}{2\Big[\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2\Big]}\,, \tag{4.131}$$

and the complete set of potentials follow by appropriate permutation of indices. We have mentioned that the real parts of these potentials correspond to thermal derivatives of physical potentials in the near-extremal limit, now we point out that their imaginary parts correspond to φ derivatives of physical potentials in the extremal nearBPS limit, that is,

$$\frac{\operatorname{Im}\widetilde{\omega}_1}{2\pi} = \frac{\partial\Omega_1}{\partial\varphi}\,, \qquad \frac{\operatorname{Im}\widetilde{\Delta}_1}{2\pi} = \frac{\partial\Phi_1}{\partial\varphi}\,. \tag{4.132}$$

It is instructive to collect the entire vector of imaginary potentials as a one-form

$$\frac{\operatorname{Im}\widetilde{\Delta}_I dQ_I + \operatorname{Im}\widetilde{\omega}_i dJ_i}{2\pi} = -\frac{dh}{2[\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2]}\,. \tag{4.133}$$

The entropy function formalism for the BPS black holes in AdS_5 [122] employs complex potentials and identify their real part as the physical potential, but their imaginary part is enigmatic. It is therefore satisfying that the imaginary potentials computed in the microscopic description of BPS black holes vanish on the constraint surface $h=0$ but increase proportionally to violations of the constraint.

Combining the real and imaginary parts of the potentials (4.127, 4.133), we can write the first law of thermodynamics as a complex-valued expression:

$$\begin{aligned}&TdS^* + (T\operatorname{Re}\widetilde{\Delta}_I + \frac{\varphi}{2\pi i}\operatorname{Im}\widetilde{\Delta}_I)dQ_I + (T\operatorname{Re}\widetilde{\omega}_i + \frac{\varphi}{2\pi i}\operatorname{Im}\widetilde{\omega}_i)dJ_i\\ &= \left(\frac{2\pi}{S^*}(Q_1+Q_2+Q_3+\frac{1}{2}N^2)T - \frac{\varphi}{2\pi i}\right)\frac{\pi\, dh}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2}\,.\end{aligned} \tag{4.134}$$

It is not a coincidence that this formula is a close analogue of (4.73), the consolidated formula for all the potentials we computed from gravity in the nearBPS region. In the following subsection we relate them quantitatively.

4.4.4 Extremizing Free Energy near BPS

In this subsection, we generalize the BPS entropy extremization reviewed in subsection 4.4.2 to an extremization principle that describes the nearBPS region.

4.4.4.1 The Free Energy Function

As we stressed repeatedly in section 4.3, all microscopic computations of the BPS arrive at the partition function (4.104) subject to the subsidiary condition (4.83) and the black hole entropy is subsequently extracted therefrom through the extremization procedure reviewed in subsection 4.4.2. The variables of the BPS partition function are potentials that must be interpreted as thermal derivatives of physical potentials. Our discussion in subsection 4.4.1 arrived at this identification in the course of comparing microscopic conventions with those of gravity, but it applies equally as a general relation between the extremal and near-extremal partition functions. Thus the adaptation of the BPS partition function (4.104) to a notation appropriate for the nearBPS regime is

$$\ln Z(\beta, \Delta_I, \omega_i) = -\frac{N^2}{2T}\frac{(\Delta_1 - \Delta_1^*)(\Delta_2 - \Delta_2^*)(\Delta_3 - \Delta_3^*)}{(\omega_1 - \omega_1^*)(\omega_2 - \omega_2^*)} \ . \tag{4.135}$$

It is significant to note that there are *no* tilde's on the variables in this formula: Δ_I and ω_i refer to the full potentials rather than their thermal derivatives. The BPS reference values Δ_I^* and ω_i^* all equal 1 numerically, because the BPS mass is the sum of the conjugate conserved charges with coefficient 1 (in units where $\ell_5 = 1$). We maintain the more elaborate notation for conceptual clarity. The original BPS partition function (4.104) is recovered in the extremal limit $T \to 0$, as it should be.

The microscopic partition function (4.105) was introduced generally, without restricting to the BPS (or even the nearBPS) regime. In the saddle point approximation it gives

$$\ln Z = S - \beta(M - M^*) + \beta(\Delta_I - \Delta_I^*)Q_I + \beta(\omega_i - \omega_i^*)J_i \ . \tag{4.136}$$

It is tempting to identify $\ln Z$ in this formula with the BPS partition function (4.135) but that is incorrect even in the strict BPS limit where it is crucial that we require the potentials to satisfy the complex constraint (4.83). Moreover, in the BPS limit the real part of the potentials satisfy $\sum_I(\Delta_I - \Delta_I^*) - \sum_i(\omega_i - \omega_i^*) = 0$ by definition but this equation must be relaxed in the nearBPS region. We impose the constraint

$$\sum_I(\Delta_I - \Delta_I^*) - \sum_i(\omega_i - \omega_i^*) = \varphi + 2\pi i T \ . \tag{4.137}$$

The imaginary part of this equation reformulates the BPS condition (4.83) in a manner that is meaningful also for small but nonvanishing temperature. The real part allows potentials to depart from their BPS values and parametrize this violation by φ, in conformity with the convention in gravity (4.50).

We now combine the general saddle point approximation (4.136) with the microscopic computation of the free energy (4.135) and subject the result to the complex constraint (4.137). This gives the free energy in the nearBPS regime

$$\begin{aligned}\mathcal{F} &\equiv (M-M^*) - TS \\ &= \frac{1}{2}N^2\frac{(\Delta_1-\Delta_1^*)(\Delta_2-\Delta_2^*)(\Delta_3-\Delta_3^*)}{(\omega_1-\omega_1^*)(\omega_2-\omega_2^*)} + (\Delta_I-\Delta_I^*)Q_I + (\omega_i-\omega_i^*)J_i \\ &\quad + \Lambda\Big(\sum_I(\Delta_I-\Delta_I^*) - \sum_i(\omega_i-\omega_i^*) - \varphi - 2\pi i T\Big)\,. \end{aligned} \tag{4.138}$$

We have not attempted to argue that there can be no additional contributions to the free energy in the nearBPS regime. On the contrary, we conservatively claim that it *at least* includes the ingredients incorporated in (4.138).

An *effective* theory of nearBPS black holes can be found be extremizing the free energy (4.138) over the potentials Δ_I, ω_i, and the Lagrange multiplier Λ. It will depend on the remaining potentials φ, T, the dynamical fields in the effective description. As usual, the effective theory will also feature dimensionful parameters that are unspecified *a priori* but computable from the UV completion in principle. In the present context a complete microscopic theory relates the effective parameters (C_T, C_φ, C_E) to conserved charges.

4.4.4.2 Extremization of Free Energy

The free energy (4.138) differs from the entropy function (4.107) only by an overall factor $-T$ and the addition of a simple term $-\Lambda\varphi$. Therefore the extremization is nearly unchanged from the BPS computation in subsection 4.4.2. The equations of motion (4.108-4.109) are entirely unchanged so the ratios (4.118-4.119) remain, and so the steps needed for finding the potentials explicitly are exactly the same. The key modification is the equation of motion for Λ, ie. the constraint (4.137). For the potentials the constraint enters in (4.120) where its role is to provide an overall normalization for the potentials that are otherwise determined by relations between their ratios. The new (complex) normalization modifies the potentials from (4.123 - 4.124) to

$$\frac{\omega_i-\omega_i^*}{\varphi+2\pi i T}=\frac{1}{2}N^2\frac{\prod_k(J_k-\Lambda)}{J_i-\Lambda}\frac{1}{2\Lambda(\Lambda+Q_1+Q_2+Q_3+\frac{1}{2}N^2)}\,, \tag{4.139}$$

$$\frac{\Delta_I-\Delta_I^*}{\varphi+2\pi i T}=\frac{\prod_K(Q_K+\Lambda)}{Q_I+\Lambda}\frac{1}{2\Lambda(\Lambda+Q_1+Q_2+Q_3+\frac{1}{2}N^2)}\,. \tag{4.140}$$

Consider ω_1 for definiteness. For small φ the constraint on the charges is only violated mildly so we assume that Λ is unchanged to leading order, and then the right hand of the first equation is $\frac{1}{2\pi i}(\mathrm{Re}\,\widetilde{\omega}_1+i\mathrm{Im}\,\widetilde{\omega}_1)$ where $\mathrm{Re}\,\widetilde{\omega}_1$ and $\mathrm{Im}\,\widetilde{\omega}_1$ are the BPS values for these variables given in (4.125, 4.130). The physical potential is the real part so the leading dependence on the constraint violation φ is due to the *imaginary* part of $\widetilde{\omega}_1$:

$$\mathrm{Re}(\omega_1-\omega_1^*)|_{\varphi\text{ dependence}}=\varphi\,\frac{\mathrm{Im}\,\widetilde{\omega}_1}{2\pi}=-\frac{1}{4}N^2\frac{J_2+Q_1+Q_2+Q_3+\frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2}\varphi\,. \tag{4.141}$$

This agrees exactly with the gravitational computation of the change of the rotational velocity Ω_1 due to small violations of the constraint (4.57). The dependences of all other potentials on φ similarly agree with gravitational results.

At this point we have showed that the extremization principle based on the free energy (4.138) reproduces the dependence of all physical potentials on temperature and on violations of the constraint parametrized by the field deformation φ. We now turn to to the mass and entropy in the nearBPS regime.

As we mentioned earlier in this subsection, the equations of motion are entirely unchanged from the BPS case (4.108-4.109), except for the equation of motion for the Lagrange multiplier Λ, ie. the constraint (4.137). Therefore the cubic equation (4.112) for Λ still holds. However, since the potentials Δ_I,ω_i do not satisfy the BPS constraint (4.83) in the nearBPS theory, the charges Q_I,J_i may also violate their constraint (4.116).

We already anticipated this situation in the preceding subsection when introducing a height function h (4.128) parametrizing the violation of the constraint (4.116). In the schematic notation introduced in (4.113-4.114) the height function $h\equiv C-AB$ deforms the cubic equation to

$$\Lambda^3+A\Lambda^2+B\Lambda+C=(\Lambda^2+B)(\Lambda+A)+h=0\,. \tag{4.142}$$

Modifying charges and Λ away from $h=0$ at first order in perturbation theory

$$(2\Lambda\delta\Lambda+\delta B)(\Lambda+A)+h=0\ ,$$

and recalling that Λ is purely imaginary at leading order we find

$$\text{Im }\delta\Lambda = \frac{Ah}{2\text{Im}\Lambda(B+A^2)} + \frac{\delta B}{2\text{Im}\Lambda} \ . \tag{4.143}$$

The second term

$$\frac{\delta B}{2\text{Im}\Lambda} = -\frac{\delta S_*}{2\pi} \ , \tag{4.144}$$

takes into account the change of the BPS entropy due to changes of the conserved charges. Therefore the entropy in excess of the BPS entropy due to the violation of the constraint becomes

$$\delta(S-S^*) = -2\pi\text{Im }\delta\Lambda = \frac{2\pi^2 h}{S^*}\frac{Q_1+Q_2+Q_3+\frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2+(Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2} \ . \tag{4.145}$$

This formula agrees with the expression (4.129) inferred from applying the thermodynamic potentials derived in the BPS case also off the BPS surface. Moreover, the same expression was found also from gravity considerations (4.73). The considerations here are based on a minimal generalization of the *microscopic* theory.

4.4.4.3 Parameters in Effective Field Theory

We have expressed possible violations of the constraint in two distinct ways. We defined the height function h (4.128) that measures departures from the constraint $h=0$ on charges, and the effective potential φ introduced through the deformed constraint (4.137) that is required to take the value $\varphi=0$ by the conditions for supersymmetry (4.83). The "charge" h and the "potential" φ are proportional near the BPS surface we computed their constant of proportionality (4.65) on the gravity side. It is interesting to understand their relation more generally.

The height function h (4.128) is fairly elaborate so the expression for small departures $\delta h = \partial_{Q_I} h \delta Q_I + \partial_{J_i} h \delta J_i$ from the constraint $h=0$ due to general variations δQ_I, $\delta\omega_i$ of the charges is lengthy and not illuminating. However, variations that are *proportional* to the charges themselves $\delta Q_I = \lambda Q_I$, $\delta J_i = \lambda J_i$ yield a manageable formula

$$\delta h = \Big[\big(Q_1^2(Q_2+Q_3)+\text{cyclic}\big)+2Q_1Q_2Q_3+\frac{1}{4}N^4(J_1+J_2)\Big]\lambda \ , \tag{4.146}$$

after simplifications using the constraint $h=0$. However, the transformation of the effective potential is canonical (with dimension 2 relative to the charge):

$$\delta\varphi = 2\lambda\varphi \ . \tag{4.147}$$

Comparison of the two preceding equations gives

$$h = \frac{1}{2}\Big[\big(Q_1^2(Q_2+Q_3) + \text{cyclic}\big) + 2Q_1Q_2Q_3 + \frac{1}{4}N^4(J_1+J_2)\Big]\varphi\ . \tag{4.148}$$

This generalizes the constant of proportionality (4.65) computed in the gravity to the case of three distinct charges.

We can now consolidate our results by presenting the first law of thermodynamics in a coherent manner. We collected most of them already in the complex form of the first law (4.134). In the course of this subsection we have rederived each of the terms in this equation from free energy extremization. Additionally, the imaginary part of the potentials acquired a more satisfying interpretation through its relation with the deformation parameter φ given in (4.141) and its analogues for other potentials. The conversion (4.148) between h with φ finally let us rewrite (4.134) as

$$TdS^* + \mathrm{Re}(\Delta_I - \Delta_I^*)dQ_I + \mathrm{Re}(\omega_i - \omega_i^*)dJ_i = \frac{C_T}{T}\left[\frac{2\pi}{S^*}(Q_1+Q_2+Q_3+\frac{1}{2}N^2)T + \frac{\varphi}{2\pi}\right]\frac{d\varphi}{2\pi}\ , \tag{4.149}$$

where C_T is the heat capacity that is linear in temperature with proportionality constant

$$\left.\frac{C_T}{T}\right|_{\text{nearExt}} = \pi^2 \frac{Q_1^2(Q_2+Q_3) + Q_2^2(Q_3+Q_1) + Q_3^2(Q_1+Q_2) + 2Q_1Q_2Q_3 + \frac{1}{4}N^4(J_1+J_2)}{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{1}{2}N^2\,(J_1+J_2) + (Q_1+Q_2+Q_3+\frac{1}{2}N^2)^2}\ .$$

We previously computed this expression in gravity (4.42). In some parts of this chapter we introduced the effective field theory parameters C_φ and C_E in addition to C_T. They all have different physical significance but numerically $C_\varphi = C_T$ and

$$\left(\frac{C_E}{T}\right) = \frac{2\pi}{S^*}\left(\frac{C_T}{T}\right)(Q_1+Q_2+Q_3+\frac{1}{2}N^2)\ , \tag{4.150}$$

so in the presentation of these final results we opt for writing the formulae more explicitly.

The first law of thermodynamics

$$TdS = d(M - M^*) - (\Delta_I - \Delta_I^*)dQ_I - (\omega_i - \omega_i^*)dJ_i\ , \tag{4.151}$$

now gives the energy and the entropy of the excitations above the BPS ground state as

$$M - M^* = \frac{1}{2}\frac{C_T}{T}\left(\frac{\varphi}{2\pi}\right)^2\ , \tag{4.152}$$

$$S - S^* = \frac{2\pi}{S^*}\frac{C_T}{T}(Q_1+Q_2+Q_3+\frac{1}{2}N^2)\frac{\varphi}{2\pi}\ . \tag{4.153}$$

These expressions agree with results earlier in the chapter, including computations in gravity (4.62, 4.68).

Our microscopic discussion in this section has not at all touched on the conventional heat capacity, ie. the term in the entropy that is linear in temperature (and correlated with a mass term that is quadratic in the temperature). What is needed to get this term is an equation of motion where the height parameter h in the cubic equation (4.142) is traded for φ through (4.148) and subsequently complexified to $\varphi + 2\pi i T$. The last step is very natural in that much of the theory apparently depends holomorphically on a complex symmetry breaking parameter. However, we are not yet able to present a principled argument based on microscopic theory.

4.5 Summary and Outlook

In this chapter we have discussed the thermodynamics and the statistical physics of AdS_5 black holes, addressing both the BPS and the nearBPS configurations. In the nearBPS region, we made an important distinction between the near-extremal ($T \neq 0$, $\varphi = 0$), the extremal nearBPS ($T = 0$, $\varphi \neq 0$), and the general nearBPS ($T \neq 0$, $\varphi \neq 0$) black holes. The unfamiliar potential φ parametrizes the possible violation of a constraint on charges that must be imposed in the strict BPS limit.

In the gravitational theory we studied all thermodynamic potentials in great detail, especially their interrelation through the first law of thermodynamics, in the entire BPS and nearBPS domain. In the holographically dual theory, we reviewed an elementary version of the microscopic description, based on the free field representation of $\mathcal{N} = 4$ SYM. It yields the semiclassical partition function found in [122] that is common to all the recent proposals for a microscopic theory of BPS black holes in AdS_5 [24–26].

We found that, with minor additional assumptions, the same semiclassical partition function also describes aspects of nearBPS black holes. This approach to the theory without supersymmetry is particularly successful for extremal nonBPS black holes. It is the main basis for our claim of a microscopic description of the nearBPS black holes. Our generalization of the entropy function to a free energy function captures the effective field theory of the nearBPS region succinctly and in a manner that relates directly to the microscopic theory.

The recent progress towards a statistical description of BPS black holes [24–26] is not yet fully satisfactory. It is not even clear that the reports are consistent with one another. Our study gives general support for these advances. For example, we find that the BPS limit is robust in that it can be approached from any direction.

However, there are still many open questions, particularly on the microscopic side. The free energy function could surely be improved. The more important open problem is why the microscopic description of nearBPS black holes is even possible. It appears that some non-renormalization due to supersymmetry persists also away from the BPS limit. We suspect this follows from softly broken superconformal symmetry but we have not developed a principled argument.

Furthermore, we expect that the additional microscopic degrees of freedom in the nearBPS region can equally be modeled by a simple gas of free particles, much like the free model of the BPS limit that we review. However, it might well be necessary to invoke other sectors of $\mathcal{N} = 4$ SYM that are BPS, but preserving different supersymmetries than the ground state. This is the structure of successful microscopic models for near-extremal black holes with flat asymptotic space, such as the D1-D5 system [12].

There are also problems in gravitational physics that we leave for the future. We found that the potential for constraint violations φ exhibits entropic preference for $\varphi \geq 0$, reminiscent of the behavior of conventional temperature T. It would be interesting to develop the geometric underpinnings of φ. Additionally, the near horizon AdS_2 expected for any near-extremal black hole must have an analogue for extremal nearBPS configurations and the full nearBPS region of parameter space promises an interesting interplay between the low temperature limit and a mildly violated constraint.

We look forward to pursue these and related directions in future research.

APPENDICES

APPENDIX A

Computations of Seeley-DeWitt Coefficients

In this appendix, we give the details on the computation of the Seeley-DeWitt coefficients for Kaluza Klein black holes and their embeddings in $\mathcal{N} \geq 2$ supergravity. Most of the computations were done using the Mathematica package `xAct`. We present our results according to the organization of quadratic fluctuations into blocks that was introduced in section 2.4.

The basic steps of our implementation are:

1. We expand the Lagrangian to second order.[1] This was done in sections 2.4 and 2.6 for the supergravity theories of interest. The bosonic Lagrangian can also be expanded using `xPert`.

2. We gauge-fix and add the corresponding ghosts. The gauge-fixing and the ghosts were detailed for each block in sections 2.4 and 2.6. In this appendix, we highlight and record their contributions to the heat kernel.

3. We rearrange the fluctuation operator Λ^n_m so that it takes the canonical form (2.120). We then read off the operators ω_μ and P and compute the operators E and $\Omega_{\mu\nu}$. These are the most cumbersome steps so they are executed primarily using Mathematica. Since some expressions are rather lengthy for the matrix operators due to the non-minimal couplings, we mostly present the traces of these operators.

4. We compute the Seeley-DeWitt coefficient $a_4(x)$ using formula (2.121). This also includes the ghosts from the second step.

[1] For fermions we always write the quadratic fluctuations with Majorana spinors, following the conventions of [17].

5. We simplify $a_4(x)$ using the equations of motion, tensor and gamma matrix identities. This brings $a_4(x)$ to its minimal form (2.125), where we can read off the coefficients c and a.

A.1 Preliminaries

We use the following formula to compute the Seeley-DeWitt coefficient

$$(4\pi)^2 a_4(x) = \mathrm{Tr}\left[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{360}(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu})\right] . \tag{A.1}$$

This object further simplifies due to the equations of motion, Bianchi, and Schouten identities. These simplifications imply that we can cast (A.1) in the form

$$a_4(x) = \frac{c}{16\pi^2}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2}E_4 , \tag{A.2}$$

where the square of the Weyl tensor is

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 , \tag{A.3}$$

and the Euler density is

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 . \tag{A.4}$$

For each block, as summarized in Table 2.3, we will report both (A.1) and (A.2). The identities used to simplify (A.1) to its minimal form (A.2) are listed below. For fermionic fluctuations, we also use many gamma matrix identities which are well known and not repeated here.

On-shell conditions: The equations of motion background with constant dilaton are

$$\begin{aligned} F_{\mu\alpha}F_\nu{}^\alpha = 2R_{\mu\nu} , \qquad R = 0 , \\ F_{\mu\nu}F^{\mu\nu} = 0 , \qquad D_\mu F^{\mu\nu} = 0 . \end{aligned} \tag{A.5}$$

Bianchi identities: Starting from

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0 , \qquad R_{\mu[\nu\alpha\beta]} = 0 , \tag{A.6}$$

where $\tilde{F}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ we find

$$\begin{aligned}
R_{\mu\nu\alpha\beta}R^{\mu\alpha\nu\beta} &= \frac{1}{2}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \ , \\
(D_\alpha F_{\mu\nu})(D^\nu F^{\mu\alpha}) &= \frac{1}{2}(D_\alpha F_{\mu\nu})(D^\alpha F^{\mu\nu}) \ , \\
F^{\alpha\nu}(D_\alpha F_{\mu\nu}) &= \frac{1}{2}F^{\nu\alpha}(D_\mu F_{\nu\alpha}) \ , \\
R_{\mu\alpha\nu\beta}F^{\mu\nu}F^{\alpha\beta} &= \frac{1}{2}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \ , \\
\epsilon_{\mu\nu\alpha\beta}D^\alpha F^{\rho\beta} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}D^\rho F^{\alpha\beta} \ .
\end{aligned} \tag{A.7}$$

Schouten identities: The Schouten identity is $g^{\mu[\nu}\epsilon^{\rho\sigma\tau\lambda]} = 0$. From this, we can derive

$$\tilde{F}_{\mu\alpha}F_\nu{}^\alpha = \frac{1}{4}g_{\mu\nu}\tilde{F}_{\alpha\beta}F^{\alpha\beta} \tag{A.8}$$

Derivative relations: The following identity is also useful

$$(D_\alpha F_{\mu\nu})(D^\alpha F^{\mu\nu}) = -2R_{\mu\nu}F^{\mu\alpha}F^\nu{}_\alpha + R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \tag{A.9}$$

and holds up to a total derivative.

A.2 KK Block

The quadratic Lagrangian is given in (2.53). To evaluate the Seeley-DeWitt coefficient, the kinetic term of $h_{\mu\nu}$ is analytically continued to

$$h_{\mu\nu}^{\text{new}} = -\frac{i}{2}h_{\mu\nu} \ , \tag{A.10}$$

for the kinetic term to have the right sign. In addition, in order to project onto the traceless part of a symmetric tensor, we define

$$G^{\mu\nu}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu{}_\rho\delta^\nu{}_\sigma + \delta^\mu{}_\sigma\delta^\nu{}_\rho - \frac{1}{2}g^{\mu\nu}g_{\rho\sigma}\right) \ . \tag{A.11}$$

Traces of operators must be taken after contraction with this tensor. For example, for a four index operator O we use

$$\mathrm{Tr}\, O = G^{\mu\nu}_{\rho\sigma}O^{\rho\sigma}_{\mu\nu} \ . \tag{A.12}$$

The relevant traces that appear in (A.1) for the KK block are

$$
\begin{aligned}
\operatorname{Tr} E &= 3F_{\mu\nu}F^{\mu\nu} - 7R \ , \qquad \text{(A.13)} \\
\operatorname{Tr} E^2 &= \frac{33}{16}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \frac{21}{16}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - 5R_{\mu\nu}R^{\mu\nu} \\
&\quad -\frac{5}{2}R_{\mu\nu}F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{2}RF_{\mu\nu}F^{\mu\nu} + 5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
&\quad +2R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma} - 2F^{\mu\nu}{}_{;\mu}F_{\nu}{}^{\rho}{}_{;\rho} + \frac{1}{2}F_{\mu\rho;\nu}F^{\mu\nu;\rho} + \frac{1}{2}F_{\mu\nu;\rho}F^{\mu\nu;\rho} \ , \\
\operatorname{Tr} \Omega_{\mu\nu}\Omega^{\mu\nu} &= -\frac{7}{8}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{23}{8}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 2R_{\mu\nu}F^{\mu}{}_{\rho}F^{\nu\rho} \\
&\quad +RF_{\mu\nu}F^{\mu\nu} + 3R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
&\quad -F^{\mu\nu}{}_{;\mu}F_{\nu}{}^{\rho}{}_{;\rho} + 4F_{\mu\rho;\nu}F^{\mu\nu;\rho} - 8F_{\mu\nu;\rho}F^{\mu\nu;\rho} \ .
\end{aligned}
$$

The gauge-fixing also introduces ghosts with the Lagrangian

$$
e^{-1}\mathcal{L}_{\text{ghosts}} = 2b_{\mu}\left(\Box g^{\mu\nu} + R^{\mu\nu}\right)c_{\nu} + 2b\Box c - 4bF^{\mu\nu}D_{\mu}c_{\nu} \ , \qquad \text{(A.14)}
$$

where b_{μ}, c_{μ} are vector ghosts associated to the graviton and b, c are scalar ghosts associated to the graviphoton. The contribution of the ghosts are

$$
\begin{aligned}
\operatorname{Tr} E &= 2R \ , \qquad \text{(A.15)} \\
\operatorname{Tr} E^2 &= 2R_{\mu\nu}R^{\mu\nu} \ , \\
\operatorname{Tr} \Omega_{\mu\nu}\Omega^{\mu\nu} &= -2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ .
\end{aligned}
$$

The total ghost contribution is

$$
(4\pi)^2 a_4^{\text{ghost}}(x) = \frac{1}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{17}{18}R_{\mu\nu}R^{\mu\nu} - \frac{17}{36}R^2 \ . \qquad \text{(A.16)}
$$

Combining the contributions (A.13) and (A.16) gives

$$
\begin{aligned}
(4\pi)^2 a_4(x) &= \frac{23}{24}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \frac{5}{12}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{127}{36}R_{\mu\nu}R^{\mu\nu} - \frac{13}{12}R_{\mu\nu}F^{\mu}{}_{\rho}F^{\nu\rho} \\
&\quad +\frac{1}{3}RF_{\mu\nu}F^{\mu\nu} + \frac{77}{72}R^2 + \frac{1}{4}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{11}{18}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R_{\mu\rho\nu\sigma}R^{\mu\nu\rho\sigma} \\
&\quad -\frac{13}{12}F^{\mu\nu}{}_{;\mu}F_{\nu}{}^{\rho}{}_{;\rho} + \frac{7}{12}F_{\mu\rho;\nu}F^{\mu\nu;\rho} - \frac{5}{12}F_{\mu\nu;\rho}F^{\mu\nu;\rho} \ . \qquad \text{(A.17)}
\end{aligned}
$$

We use the identities listed in (A.5-A.8) to obtain

$$
(4\pi)^2 a_4(x) = \frac{10}{9}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{49}{36}R_{\mu\nu}R^{\mu\nu} \ , \qquad \text{(A.18)}
$$

and from here we find

$$a_{\rm KK} = \frac{31}{72} \ , \qquad c_{\rm KK} = \frac{37}{24} \ . \tag{A.19}$$

A.3 Vector Block

The vector block in its minimal form is described by the quadratic Lagrangian (2.103) and for the matter content of $\mathcal{N} = 8$ by (2.55). The matrices that appear in the quadratic fluctuation operator are

$$\begin{aligned}
E &= \begin{pmatrix} \frac{1}{4}F_\mu{}^\rho F_{\nu\rho} - R_{\mu\nu} & \frac{1}{2}F_\nu{}^\rho{}_{;\rho} \\ \frac{1}{2}F_\mu{}^\rho{}_{;\rho} & -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} \end{pmatrix} , \\
\Omega_{\rho\sigma} &= \begin{pmatrix} R_{\mu\nu\rho\sigma} + \frac{1}{4}F_{\mu\sigma}F_{\nu\rho} - \frac{1}{4}F_{\mu\rho}F_{\nu\sigma} & \frac{1}{2}F_{\mu\sigma;\rho} - \frac{1}{2}F_{\mu\rho;\sigma} \\ -\frac{1}{2}F_{\nu\sigma;\rho} + \frac{1}{2}F_{\nu\rho;\sigma} & 0 \end{pmatrix} ,
\end{aligned}$$

where the first row/column corresponds to the vector field and the second row/column to the scalar field. The relevant traces are

$$\begin{aligned}
\mathrm{Tr}\, E &= -R \ , & \text{(A.20)} \\
\mathrm{Tr}\, E^2 &= \frac{1}{16}F^\mu{}_\rho F^{\nu\rho}F_{\mu\sigma}F_\nu{}^\sigma + \frac{1}{16}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu}R^{\mu\nu} & \text{(A.21)} \\
&\quad -\frac{1}{2}R_{\mu\nu}F^\mu{}_\rho F^{\nu\rho} - \frac{1}{2}F^{\mu\nu}{}_{;\mu}F_\nu{}^\rho{}_{;\rho} \ , \\
\mathrm{Tr}\, \Omega_{\mu\nu}\Omega^{\mu\nu} &= \frac{1}{8}F^\mu{}_\rho F^{\nu\rho}F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} & \text{(A.22)} \\
&\quad -R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + F_{\mu\rho;\nu}F^{\mu\nu;\rho} - F_{\mu\nu;\rho}F^{\mu\nu;\rho} \ .
\end{aligned}$$

The ghosts for the vector block are two minimally coupled scalars with fermionic statistics. Their contribution to the Seeley-DeWitt coefficient is

$$(4\pi)^2 a_4^{\rm ghost}(x) = -\frac{1}{180}(2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + 5R^2) \ . \tag{A.23}$$

We combine the contributions of the vector block and its associated ghosts and get

$$
\begin{aligned}
(4\pi)^2 a_4(x) \;=\;& \frac{1}{24} F^{\mu}{}_{\rho} F^{\nu\rho} F_{\mu\sigma} F_{\nu}{}^{\sigma} + \frac{1}{48} F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + \frac{29}{60} R_{\mu\nu} R^{\mu\nu} \\
& - \frac{1}{4} R_{\mu\nu} F^{\mu}{}_{\rho} F^{\nu\rho} - \frac{1}{8} R^2 + \frac{1}{12} R_{\mu\rho\nu\sigma} F^{\mu\nu} F^{\rho\sigma} - \frac{1}{15} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
& - \frac{1}{4} F^{\mu\nu}{}_{;\mu} F_{\nu}{}^{\rho}{}_{;\rho} + \frac{1}{12} F_{\mu\rho;\nu} F^{\mu\nu;\rho} - \frac{1}{12} F_{\mu\nu;\rho} F^{\mu\nu;\rho}
\end{aligned} \tag{A.24}
$$

After using the identities (A.5-A.8), we obtain

$$
(4\pi)^2 a_4(x) = -\frac{1}{15} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{19}{60} R_{\mu\nu} R^{\mu\nu} \; . \tag{A.25}
$$

This leads to

$$
a_{\text{vector}} = \frac{11}{120} \; , \qquad c_{\text{vector}} = \frac{1}{40} \; . \tag{A.26}
$$

When the vector block contains a pseudoscalar instead of a scalar, such as in (2.56), the result remains the same because of simplifications due to our background.

A.4 Gravitino Block

The gravitino block is characterized by the quadratic Lagrangian (2.61). After using gamma matrix identities, the relevant traces are

$$
\operatorname{Tr} E \;=\; \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - 10 R \; , \tag{A.27}
$$

$$
\begin{aligned}
\operatorname{Tr} E^2 \;=\;& -\frac{105}{128} F^{\mu}{}_{\rho} F^{\nu\rho} F_{\mu\sigma} F_{\nu}{}^{\sigma} + \frac{81}{128} F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + \frac{43}{64} F^{\mu\nu} F^{\rho\sigma} \tilde{F}_{\mu\rho} \tilde{F}_{\nu\sigma} \\
& - \frac{13}{32} F^{\mu}{}_{\rho} F^{\nu\rho} \tilde{F}_{\mu}{}^{\sigma} \tilde{F}_{\nu\sigma} + \frac{7}{128} \tilde{F}^{\mu}{}_{\rho} \tilde{F}^{\nu\rho} \tilde{F}_{\mu}{}^{\sigma} \tilde{F}_{\nu\sigma} - \frac{21}{64} F_{\mu\nu} F^{\mu\nu} \tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma} \\
& + \frac{9}{128} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \tilde{F}_{\rho\sigma} \tilde{F}^{\rho\sigma} - \frac{1}{4} R F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} R \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{5}{2} R^2 \\
& - \frac{3}{2} R_{\mu\rho\nu\sigma} F^{\mu\nu} F^{\rho\sigma} + \frac{3}{2} R_{\mu\rho\nu\sigma} \tilde{F}^{\mu\nu} \tilde{F}^{\rho\sigma} + 4 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
& - \frac{7}{2} F_{\mu\rho;\nu} F^{\mu\nu;\rho} + 3 F_{\mu\nu;\rho} F^{\mu\nu;\rho} + \frac{3}{2} \tilde{F}_{\mu\rho;\nu} \tilde{F}^{\mu\nu;\rho} - 2 \tilde{F}_{\mu\nu;\rho} \tilde{F}^{\mu\nu;\rho} \; ,
\end{aligned} \tag{A.28}
$$

$$
\begin{aligned}
\operatorname{Tr}\Omega_{\mu\nu}\Omega^{\mu\nu} \;=\;& \frac{185}{64}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{185}{64}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{27}{32}F^{\mu\nu}F^{\rho\sigma}\tilde{F}_{\mu\rho}\tilde{F}_{\nu\sigma} \\
& -\frac{3}{16}F^{\mu}{}_{\rho}F^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} + \frac{9}{64}\tilde{F}^{\mu}{}_{\rho}\tilde{F}^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} + \frac{33}{32}F_{\mu\nu}F^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} \\
& -\frac{9}{64}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} + 7R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - 3R_{\mu\rho\nu\sigma}\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma} - 13R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
& +7F_{\mu\rho;\nu}F^{\mu\nu;\rho} - 7F_{\mu\nu;\rho}F^{\mu\nu;\rho} - 3\tilde{F}_{\mu\rho;\nu}\tilde{F}^{\mu\nu;\rho} + 3\tilde{F}_{\mu\nu;\rho}\tilde{F}^{\mu\nu;\rho} \;.
\end{aligned} \tag{A.29}
$$

The gauge-fixing produces fermionic ghosts b_A, c_A, e_A with Lagrangian

$$
e^{-1}\mathcal{L}_{\text{ghost}} = \bar{b}_A\gamma^{\mu}D_{\mu}c_A + \bar{e}_A\gamma^{\mu}D_{\mu}e_A \;, \tag{A.30}
$$

where $A = 1, 2$ is the flavor index. This simply corresponds to six minimally coupled Majorana fermions which contribute with an opposite sign. Their Seeley-DeWitt contribution is

$$
(4\pi)^2 a_4^{\text{ghost}}(x) = -\frac{1}{120}\left(7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 8R_{\mu\nu}R^{\mu\nu} - 5R^2\right) \;. \tag{A.31}
$$

Combining (A.27) and (A.31) gives

$$
\begin{aligned}
(4\pi)^2 a_4(x) \;=\;& \frac{65}{768}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{29}{768}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{17}{128}F^{\mu\nu}F^{\rho\sigma}\tilde{F}_{\mu\rho}\tilde{F}_{\nu\sigma} \\
& +\frac{7}{64}F^{\mu}{}_{\rho}F^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} - \frac{5}{256}\tilde{F}^{\mu}{}_{\rho}\tilde{F}^{\nu\rho}\tilde{F}_{\mu}{}^{\sigma}\tilde{F}_{\nu\sigma} + \frac{5}{128}F_{\mu\nu}F^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} \\
& -\frac{3}{256}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma} + \frac{2}{45}R_{\mu\nu}R^{\mu\nu} + \frac{1}{48}RF_{\mu\nu}F^{\mu\nu} + \frac{1}{48}R\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \\
& -\frac{1}{36}R^2 + \frac{1}{12}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - \frac{1}{4}R_{\mu\rho\nu\sigma}\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma} - \frac{113}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
& +\frac{7}{12}F_{\mu\rho;\nu}F^{\mu\nu;\rho} - \frac{11}{24}F_{\mu\nu;\rho}F^{\mu\nu;\rho} - \frac{1}{4}\tilde{F}_{\mu\rho;\nu}\tilde{F}^{\mu\nu;\rho} + \frac{3}{8}\tilde{F}_{\mu\nu;\rho}\tilde{F}^{\mu\nu;\rho} \;.
\end{aligned} \tag{A.32}
$$

Using the identities (A.5-A.8) gives

$$
(4\pi)^2 a_4(x) = -\frac{113}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{767}{720}R_{\mu\nu}R^{\mu\nu} \;, \tag{A.33}
$$

and this leads to

$$
a_{\text{gravitino}} = -\frac{137}{1440}, \qquad c_{\text{gravitino}} = -\frac{347}{480} \;. \tag{A.34}
$$

A.5 Gaugino Block

The gaugino block is given by the Lagrangian (2.62). In this case, the relevant traces are

$$
\begin{aligned}
\mathrm{Tr}\, E &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2R \,, && \text{(A.35)}\\
\mathrm{Tr}\, E^2 &= -\frac{1}{32}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} + \frac{3}{128}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{8}RF^{\mu\nu}F_{\mu\nu} + \frac{1}{2}R^2 && \text{(A.36)}\\
&\quad -\frac{1}{2}F_{\mu\rho;\nu}F^{\mu\nu;\rho} + \frac{1}{4}F_{\mu\nu;\rho}F^{\mu\nu;\rho} \,, \\
\mathrm{Tr}\, \Omega_{\mu\nu}\Omega^{\mu\nu} &= \frac{1}{8}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{8}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} - R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} && \text{(A.37)}\\
&\quad +F_{\mu\rho;\nu}F^{\mu\nu;\rho} - F_{\mu\nu;\rho}F^{\mu\nu;\rho} \,.
\end{aligned}
$$

The Seeley-DeWitt coefficient is

$$
\begin{aligned}
(4\pi)^2 a_4(x) &= \frac{1}{384}F^{\mu}{}_{\rho}F^{\nu\rho}F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{1536}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \frac{1}{45}R_{\mu\nu}R^{\mu\nu} + \frac{1}{96}RF^{\mu\nu}F_{\mu\nu} && \text{(A.38)}\\
&\quad -\frac{1}{72}R^2 - \frac{1}{24}R_{\mu\rho\nu\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{7}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{1}{12}F_{\mu\rho;\nu}F^{\mu\nu;\rho} - \frac{1}{48}F_{\mu\nu;\rho}F^{\mu\nu;\rho} \,.
\end{aligned}
$$

and gives after simplification

$$
(4\pi)^2 a_4(x) = \frac{7}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{73}{1440}R_{\mu\nu}R^{\mu\nu} \,, \tag{A.39}
$$

which leads to

$$
a_{\text{gaugino}} = -\frac{17}{2880} \,, \qquad c_{\text{gaugino}} = \frac{13}{960} \,. \tag{A.40}
$$

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