Hilbert Domains, Conics, and Rigidity

by

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ABSTRACT

A class of compact projective manifolds can be viewed as a convex set in projective space modulo a discrete group of isometries. This thesis explores the circumstances under which this convex set is a symmetric convex cone. The irreducible symmetric convex cones are analogous to symmetric spaces in Riemannian geometry and consist of hyperbolic space and positive definite Hermitian matrices. Having a properly embedded conic in the boundary of the convex set is equivalent to the existence of a subspace isometric to the hyperbolic plane. When enough of these conics exist, I will show that the convex set is a symmetric convex cone. This demonstrates how the shape of the boundary of the convex set determines its isometry class. Further, if enough twice differentiable curves are found in the boundary of the convex set, I will show that it must be hyperbolic space. This result also has applications to affine spheres.

CHAPTER 1

Introduction

1.1 Motivation

Can two dimensional affine slices of the universal cover $\widetilde{M}$ of a compact manifold M determine when M has a lot of symmetries? This question will be explored for when $\widetilde{M}$ can be identified with a particular type of convex subset Ω of a projective space. The space Ω, and hence also M, is given the Hilbert metric which is determined entirely by the shape of the boundary of Ω. For instance when the boundary of Ω has some regularity, e.g. C^2 or $C^{1+\alpha}$ for some $\alpha \in (0,1)$ (Benoist [8]), the manifold M has hyperbolic properties. When Ω lacks these regularity properties, M will have properties similar to a Euclidean space with the hexagonal norm. This is similar to the potential presense of flats in Riemannian geometry. We will primarily look for slices of Ω that are isometric to the hyperbolic plane $\mathbb{H}^2$ and use their existence to determine when Ω is a *symmetric convex cone*; definitions and details about these cones will follow a discussion of motivation from Riemannian geometry.

A manifold M with Riemannian metric d is a symmetric space if for every point

$x \in M$, there is an isometry g_x that is an involution. This means that $g_x(x) = x$ and the derivative of g_x, $dg_x : T_xM \to T_xM$, is minus the identity. See [30] for more information on symmetric spaces.

In Riemannian geometry one can use the sectional curvature of M to determine if M is locally a symmetric space, or if $\widetilde{M}$ is a symmetric space. When M has constant sectional curvature κ, the situation is ideal; by the Killing-Hopf theorem, $\widetilde{M}$ is isometric to

- a sphere if $\kappa > 0$,
- a Euclidean space if $\kappa = 0$ or
- a hyperbolic space if $\kappa < 0$.

The third case where $\kappa < 0$ has the most connections to Hilbert geometry due to the hyperbolic behavior of Ω when its boundary is C^2 or $C^{1+\alpha}$. If the curvature κ is not constant, the Killing-Hopf theorem is not applicable; for negative sectional curvature κ that is bounded away from zero, some conclusions can be made about $\widetilde{M}$. These claims require some additional assumptions on M which will be discussed now.

In particular, one looks for conditions for when a Riemannian manifold M is locally a *rank one* symmetric space, since in that case M has no subspaces that are flat planes. Roughly, this means that M has totally geodesic strips of sectional curvature -1 which is analogous to the property in Hilbert geometry of Ω having a lot of subspaces isometric to $\mathbb{H}^2$. This is a natural choice when classifying spaces of negative curvature since symmetric spaces with negative curvature are rank one.

The manifold M is said to have *higher hyperbolic rank* if every geodesic lies in a strip of sectional curvature -1. A full definition for higher hyperbolic rank can be found in [15]. Intuitively, this means that every geodesic locally lies in a subspace isometric to $\mathbb{H}^2$.

Here we recall some history of results involving higher hyperbolic rank. The first is Hamenstädt's hyperbolic rank rigidity theorem:

Theorem 1.1 (Hamenstädt, [28])**.** *Let M be a closed manifold with higher hyperbolic rank and sectional curvature $\kappa \leq -1$. Then M is a locally rank one symmetric space.*

Another from Connell-Nguyen-Spatzier looks at κ when it is $\frac{1}{4}$-pinched:

Theorem 1.2 (Connell, Nguyen, Spatzier [15])**.** *Let M be a closed Riemannian manifold with higher hyperbolic rank and sectional curvature κ $\frac{1}{4}$-pinched: $-1 \leq \kappa \leq -\frac{1}{4}$. Then M is a rank one locally symmetric space.*

This theorem pairs well with Hamenstädt's due to its constrasting bound on κ.

Further, one can analogously define spherical rank and Euclidean rank when $\kappa = 1$ and $\kappa = 0$ respectively. In the next section, we will explore how Hilbert geometry is connected to hyperbolic geometry through a review of each of their histories.

1.2 Ancient History

The Greek mathematician Euclid created an axiomic system for geometry in his textbook *The Elements*. These axioms are members of a list of five statements, or postulates, that are assumed to be true in Euclidean geometry. The first four postulates are intuitive. The fifth statement is the following:

5. Given a line and a point not on that line, there is exactly one line (in the same plane) that passes through the point and is parallel to the line.

and is called the *parallel* axiom. The parallel axion is necessary for Euclidean geometry; disregarding it allows one to study spherical or hyperbolic geometry. There are two ways of breaking the fifth postulate: either parallel lines never exist, or parallel lines are not unique.

On a sphere, parallel lines do not exist. The lines of shortest path, or geodesics, are the great circles. These are the curves that split the sphere into two hemispheres. Any two great circles will intersect.

In hyperbolic geometry, parallel lines are not unique. This is demonstrated in Figure 1.1 for $\mathbb{H}^2$ with the Klein model of hyperbolic space which is characterized by its straight line geodesics. A geodesic and a point not on that geodesic are shown in green. Many geodesics pass through the point but do not intersect the line.

1.3 Klein's Erlangen Program

By 1872, mathematicians' view of geometry had evolved which was reflected in Klein's work 'A comparative review of recent researches in geometry' (see [34] for more details). This work was the introduction to *Klein's Erlangen program* and was more of a manifesto than a research paper. Klein asserted that for a manifold under some geometry and a group of transformations of the manifold, the goal is to research the invariants of the manifold under the group. Further, he considered projective

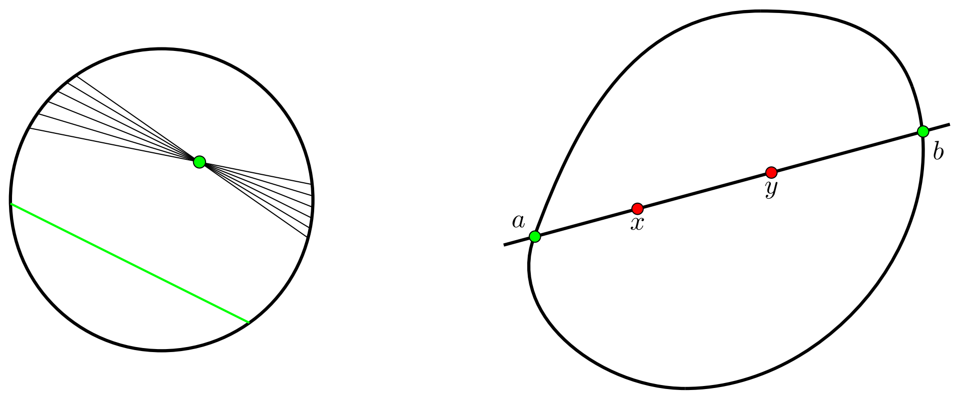

Figure 1.1: The Klein model of $\mathbb{H}^2$ and the Hilbert metric on Ω

geometry to be the most general form of geometry as it contains all others. Klein's original description of the program was not precise; he did not define a manifold and instead discussed *manifoldness* and inverses of transformations of spaces were not mentioned.

Additionally, the Erlangen program contains a description of isomorphic group actions. Klein gives an example with the projective line and conics: the projective line has a one to one correspondence with any conic, which means that the projective line and a conic have equivalent group actions which are the Möbius transformations. Hence geometric ideas about the projective line can be translated to geometric ideas about conics. This theme will be used in further sections.

1.4 Hyperbolic Space

Here, we will further discuss the hyperbolic plane. The Klein model of hyperbolic space consists of the points $\{(x_1, x_2, \ldots, x_n, 1) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots x_n^2 < 1\}$ with

Riemannian metric for this model given by

$$ds = \sqrt{\frac{dx_1^2 + \cdots + dx_n^2}{1 - x_1^2 - \cdots - x_n^2} + \frac{(x_1 dx_1 + \cdots + x_n dx_n)^2}{(1 - x_1^2 - \cdots - x_n^2)^2}}.$$

Since geodesics are straight lines, calculating the distance between two points can be reduced to a one dimensional problem. In this case, the metric is

$$ds = \sqrt{\frac{dx^2}{1 - x^2} + \frac{x^2 dx^2}{(1 - x^2)^2}} = \frac{dx}{(1 - x^2)}$$

so that the distance between the point $x = p_1$ and $x = p_2$ where $-1 < p_1 < p_2 < 1$ is

$$\int_{x=p_1}^{x=p_2} \frac{1}{(1 - x^2)} dx = \frac{1}{2} \log \left(\frac{(1 - p_1)(1 + p_2)}{(1 + p_1)(1 - p_2)} \right).$$

This distance can be applied not only to $\mathbb{H}^n$, but to any convex set Ω (see Figure 1.1); extend a line segment between the points $x, y \in \Omega$ to find the intersection points $a, b \in \partial\Omega$ of the line with the boundary of Ω, as shown in Figure 2.3.

Hence we define the Hilbert metric on a convex set Ω as

$$d_\Omega(x, y) = \frac{1}{2} \log \left(\frac{|ay||xb|}{|ax||yb|} \right)$$

where $|\cdot|$ is the Euclidean distance between the two points. Note that $\frac{|ax||yb|}{|ay||xb|} = [x; y; a; b]$ is the projective cross ratio of four points. The value of $[x; y; a; b]$ is invariant under projective transformations.

1.5 Hilbert Geometry and Convex Cones

In this thesis, Ω will be a type of convex set arising from convex cones. Let $\mathcal{C}$ be a convex cone in $\mathbb{R}^{m+1}$ with the property that if v is a ray in $\mathcal{C}$, then $-v \notin \mathcal{C}$. This

$\mathcal{C}$ is called *properly convex*. Let Ω be the projection p of $\mathcal{C}$ into $\mathbb{S}^m = \mathbb{R}^{m+1}/\mathbb{R}_{>0}$, i.e. the set of half lines in $\mathbb{R}^{m+1}$. To apply the Hilbert metric, we view Ω in any affine chart where Ω is convex. The Hilbert metric is well defined since Ω is convex and d_Ω is projectively invariant. The set Ω is called *properly convex* if $\Omega = p(\mathcal{C})$ and $\mathcal{C}$ is a properly convex cone. If there is an affine chart of Ω such that the the boundary $\partial\Omega$ does not contain any line segments, then Ω is called *strictly convex*.

If $\mathcal{C} = \mathrm{Con}(\mathcal{C}_1 \oplus \mathcal{C}_2)$ where $\mathrm{Con}(\mathcal{C}')$ is the convex hull of $\mathcal{C}'$ in $\mathbb{R}^{m+1}$, then $\mathcal{C}$ is called *reducible*. Then Ω is reducible if its associated cone $\mathcal{C}$ is reducible. For example, if Ω is a line segment with cone $\mathcal{C} = \{x, y \in \mathbb{R}^2 | x, y > 0\}$, then Ω is reducible since $\mathcal{C} = \mathrm{Con}(\{\mathbb{R}^+ \cdot e_1\} \oplus \{\mathbb{R}^+ \cdot e_2\})$. If $\mathcal{C}$ is not the convex hull of the sum of convex cones, then $\mathcal{C}$ and $\Omega = p(\mathcal{C})$ are *irreducible*.

Additionally, any $g \in \mathrm{SL}(m+1, \mathbb{R})$ with $g \cdot \Omega = \Omega$ is an isometry of Ω. For example, when $\Omega \cong \mathbb{H}^2$, the isometry group of Ω is $\mathrm{SO}(2,1) \subset \mathrm{SL}(m+1, \mathbb{R})$. If there is a discrete subgroup Γ of the isometry group of Ω and Ω/Γ is compact, then we say that Ω is *divisible*. This will allow us to work in the compact setting of the manifold $M \cong \Omega/\Gamma$ when convenient.

Let $g \in \Gamma \subset \mathrm{SL}(m+1, \mathbb{R})$ have the property that its eigenvalues λ_i ordered such that $|\lambda_1| \geq \cdots \geq |\lambda_{m+1}|$ have the property that $|\lambda_1| > |\lambda_2|$ and $|\lambda_m| > |\lambda_{m+1}|$. Then g is called *biproximal*. The eigenline associated with λ_1 is called x_g^+ and the eigenline associated with λ_{m+1} is x_g^-. The (projective) line segment between the points a and b is denoted (a, b). The line $\gamma_g = (x_g^+, x_g^-)$ is an *axis* of g. If the eigenvalues λ_i of g

are such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_m| > |\lambda_{m+1}|$ then g is called *loxodromic*.

Put Ω into an affine chart so that $\Omega \subset \mathbb{R}^m$. Let H be a two dimensional plane and take $R = H \cap \Omega$. If $\partial R \subset \partial\Omega$ and the convex hull $\mathrm{Con}(R)$ of R is such that $\mathrm{Con}(R) \subset \Omega$, then R is called a *properly embedded* region. Typically we will discuss *properly embedded* conics which are subspaces isometric to $\mathbb{H}^2$. Properly embedded triangles are isometric to $\mathbb{R}^2$ with the hexagonal norm (see [21]).

Two points $a, b \in \partial\Omega$ are said to be part of a *half triangle* if there exists a point $c \in \partial\Omega$ such that $(a, c) \subset \partial\Omega$ and $(b, c) \subset \partial\Omega$. If $(a, b) \subset \Omega$, then a and b are extreme points of a properly embedded triangle. An element $g \in \Gamma$ is called a *rank one isometry* if g is biproximal and x_g^+, x_g^- are not contained contained in any half triangle

Naturally, properties of Ω hugely influence what we can know about M. In the Riemannian case, we asked if $\widetilde{M}$ is a symmetric space. In Hilbert geometry, we will ask if Ω is a symmetric convex cone. Some examples, including symmetric convex cones, are discussed in the next section.

1.6 Examples

The first example is $\mathcal{C} = \{x \in \mathbb{R}^3 | x_1, x_2, x_3 > 0\}$, i.e an octant in $\mathbb{R}^3$. Here $\Omega = p(\mathcal{C}) \cong T$ where T is a triangle. The automorphism group of T is $\mathrm{diag}(3, \mathbb{R}) \rtimes S_3$ where $\mathrm{diag}(3, \mathbb{R})$ is the group of 3×3 diagonal matrices over $\mathbb{R}$ and S_3 permutes the vertices e_1, e_2, and e_3.

We will choose $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$ so that Ω/Γ is a torus;

$$\Gamma = \left\langle g_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, g_2 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\rangle.$$

The orbits of $g_1 \cdot T$ and $g_2 \cdot T$ are shown in Figure 1.2. The blue and green lines are representing dotted lines since Γ acts discretely on T. One of the fundamental domains of the $\Gamma \cdot T$ action is traced out in red; in the quotient T/Γ, opposite edges are identified. Hence T/Γ is the two dimensional torus T^2.

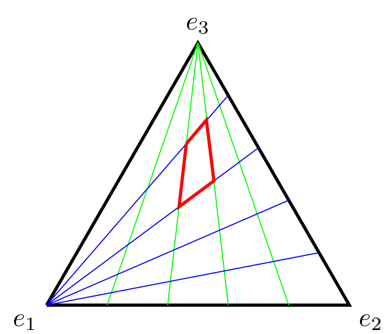

Figure 1.2: The fundamental domain of the action of Γ on the triangle T. The orbits of $g_1 \cdot T$ are represented in green and the orbits of $g_2 \cdot T$ are represented in blue. Each orbit is discrete, so the solid lined orbits are representing dotted lines.

We will primarily be discussing irreducible sets Ω; the triangle T is not irreducible since $\mathcal{C} = \mathrm{Con}(\mathbb{R}_{>0}e_1 \oplus (\mathbb{R}_{>0}e_2 \oplus \mathbb{R}_{>0}e_3))$; in fact any simplex is reducible. However, the triangle is very important in characterizing different Ω. Since $T/\Gamma \cong T^2$, the presense of a properly embedded triangle in any Ω indicates a flat. The geometry on a flat is not well behaved. In Hilbert geometry, geodesics are not unique on the triangle- not even locally. An example of this is shown in Figure 1.6.

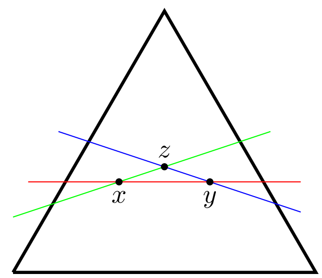

Figure 1.3: A cross ratio argument shows that since the red, green, and blue lines intersect exactly one boundary edge of T on each side, $d_\Omega(x, y) = d_\Omega(x, z) + d_\Omega(z, y)$. Small variations of z show that geodesics are not locally unique on T.

The triangle T is an example of an n-simplex with $n = 2$. The cone associated to the n-simplex is

$$\{(x_1, \ldots, x_{n+1}) \subset \mathbb{R}^{n+1} | x_1, \ldots, x_{n+1} > 0\}$$

which project to convex Ω and are hence (reducible) Hilbert geometries.

Since simplices in Hilbert geometry play an analogous role to flat subspaces in Riemannian geometry, they provide a method to define a rank. A properly convex set $\Omega \subset \mathbb{S}^m$ has *higher rank* if for every $p, q \in \Omega$, there exists a properly embedded simplex where (p, q), the line between p and q, is contained in S. This definition came from Zimmer whose results in [44] will be discussed later.

A convex divisible Ω for which the automorphism group acts transitively is called *homogeneous*; its cone $\mathcal{C}$ is also homogeneous. If for each $x \in \Omega$ there exists an automorphism of Ω of order 2 such that x is the only fixed point in Ω, then Ω and its cone $\mathcal{C}$ are called *symmetric* (these definitions are from [8]).

The irreducible symmetric convex cones were classified by Koecher in 1965 (see [8], [43] for Vinberg on homogeneous convex cones, [35]). Any irreducible symmetric convex cone is one of the following:

- the half line consisting of all $x \in \mathbb{R}_{>0}$,
- hyperbolic space $\mathbb{H}^n = \{x_1^2 - x_2^2 - \cdots x_n^2 > 0 \text{ with } x \in \mathbb{R}^n \text{ and } x_1 > 0\}$,
- positive definite $n \times n$ Hermitian (or symmetric for $k = \mathbb{R}$) matrices over $k = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$
- positive definite 3×3 Hermitian matrices over the octonions $\mathbb{O}$.

Any simplex can be decomposed into the convex hull of the sum of half lines. Properly embedded simplices and conics occur in every positive definite $n \times n$ Hermitian matrix; an illustrative example for simplices will be discussed now and the case for conics is the focus of Chapter 3. The image of the cone of positive definite $n \times n$ Hermitian matrices in $\mathbb{S}^N$ for some N under π is called $\mathrm{POS}(n, k)$. The isometry group for each $\mathrm{POS}(n, k)$ is $\mathrm{GL}(n, k)$ where $g \in \mathrm{GL}(n, k)$ acts on $X \in \mathrm{POS}(n, k)$ by

$$gXg^* \text{ where } g^* \text{ is the conjugate transpose of } g.$$

Consider $X \in \mathrm{POS}(3, \mathbb{R})$ where X is diagonal and is in the affine chart $\mathrm{Trace}(X) = 1$. Then

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-\mu \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1-\sigma \end{pmatrix}$$

define the three edges of a properly embedded triangle for $\lambda, \mu, \sigma \in (0, 1)$ and the three vertices when $\lambda = \mu = \sigma = 0$. By the spectral theorem, any X in the cone associated to $\mathrm{POS}(3,\mathbb{R})$ is diagonalizable, so every point in $\partial\mathrm{POS}(3,\mathbb{R})$ is either a vertex or along an edge of a triangle. For $n > 3$, the boundary of POS contains simplices. Further information can be found in Section 3 when we discuss properly embedded conics in $\mathrm{POS}(n,k)$ for $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.

1.7 Previous Results in Hilbert Geometry

Convex divisible Hilbert geometries Ω fall in two categories:

- either Ω is strictly convex with C^1 boundary or
- Ω is not strictly convex, i.e. there exists a line segment in $\partial\Omega$.

In the first case, Ω is has properties associated with hyperbolic manifolds. For example,

Theorem 1.3 (Benoist [5]). *Let Γ be a discrete group which divides some properly convex open set $\Omega \subset \mathbb{S}^m$. Then Ω is strictly convex if and only if the group Γ is Gromov hyperbolic.*

This means that the Cayley graph of Γ has negative curvature. Additionally, Benoist proves:

Theorem 1.4 (Benoist [5]). *Let Γ be a torsion free discrete group which divides some strictly convex open set $\Omega \subset \mathbb{S}^m$. Then the geodesic flow φ_t of the Hilbert metric on the quotient manifold $M = \Omega/\Gamma$ is Anosov.*

However, the second case when line segments lie in the boundary of Ω is more complicated and less studied. It follows from Benzecri (see [11], Section 5) that if there exists a line segment in $\partial\Omega$, then Ω contains a properly embedded triangle. Hence Ω contains at least one flat.

For any divisible properly convex set Ω, one can ask: when is Ω an irreducible symmetric convex cone? For example, a corollary of a theorem of Benzecri determines

Corollary 1.1 (Benzecri [11] or [8]). *The only divisible properly convex open set in $\mathbb{S}^m$ whose boundary is of class C^2 is the hyperbolic space $\mathbb{H}^m$.*

Such Ω with C^2 boundary must be strictly convex.

Now we will return to the notion of rank. The *real rank* of $\mathrm{SL}(m+1, \mathbb{R})$ is m. Recall that Ω has higher rank if every line $(x, y) \subset \Omega$ is contained in a properly embedded triangle. In the case where Ω is irreducible and not strictly convex, Zimmer has a result about different types of rank [44] which is more extensive than given here:

Theorem 1.5 (Zimmer [44] Theorem 1.4). *Suppose that $\Omega \subset \mathbb{S}^m$ is an irreducible properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a discrete group which divides Ω. Then the following are equivalent:*

- *Ω is symmetric with real rank at least two,*
- *Ω has higher rank*
- *$(x_g^+, x_g^-) \subset \partial\Omega$ for every biproximal element $g \in \Lambda$.*

If Ω is symmetric with real rank at least two, then Ω is $\mathrm{POS}(n, \mathbb{K})$ for $n > 2$ and $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathrm{POS}(3, \mathbb{O})$ where $\mathbb{O}$ is the octonions.

Recall that $g \in \Gamma$ is a rank one isometry if g is biproximal and x_g^+, x_g^- are not part of any half triangle. A corollary of Theorem 1.5 is

Corollary 1.2 (Zimmer [44]). *With Ω defined as in Theorem 1.5, the following are equivalent:*

- *Ω does not have higher rank*
- *Γ contains a rank one isometry.*

This implies that if Ω is $\mathrm{POS}(n, \mathbb{K})$ for $n > 2$ or $\mathrm{POS}(3, \mathbb{O})$, then any Γ dividing Ω cannot have a rank one isometry.

1.8 Results

Let $\Lambda_\Gamma^\Omega = \overline{\{x_g^+, g \in \Gamma\}}$. We will explore the classification of Ω when Ω has embedded hyperbolic planes. The first result is about the hyperbolic planes embedded in symmetric convex cones of higher rank.

Theorem 1. *The symmetric convex cones* $\mathrm{POS}(n, \mathbb{K})$ *with* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *($n \geq 2$) and* $\mathrm{POS}(3, \mathbb{O})$ *have a properly embedded conic through every boundary point.*

The following is the main result and uses Theorem 1.5 and Theorem 1.

Theorem 2. *Let Γ be a discrete subgroup of* $\mathrm{SL}(m+1, \mathbb{R})$ *which divides a properly convex set Ω that is irreducible. Assume that for all $x \in \Lambda_\Gamma^\Omega$ there is a properly embedded conic $C_x \subset \partial\Omega$. If Ω has rank one, then Ω is projectively equivalent to $\mathbb{H}^n$. If Ω has higher rank, then Ω is projectively equivalent to* $\mathrm{POS}(n, \mathbb{K})$ *with $n > 2$ and* $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ *or* $\mathrm{POS}(3, \mathbb{O})$ *where $\mathbb{O}$ is the octonions.*

An alternate to Theorem 2 is the following:

Theorem 3. *Let* Γ *be a discrete subgroup of of* $\mathrm{SL}(m+1,\mathbb{R})$ *which divides a strictly convex irreducible set* Ω*. If for all* $x \in \Lambda^{\Omega}_{\Gamma}$*, there exists a* C^2 *curve through* x *with positive second derivative, then* Ω *is projectively equivalent to* $\mathbb{H}^n$*.*

Notice that Ω is required to be strictly convex in Theorem 3 which is necessary as some of the tools used in the proof require strict convexity. We need that the C^2 curve has positive second derivative at $x \in \partial\Omega$ because this ensures that Ω is strictly convex in any neighborhood around $x \in \partial\Omega$.

In the last section, Theorem 2 is applied to classical objects called *affine spheres.* For a hypersurface H with a normal vector field ξ pointing towards the convex side of H, H is an affine sphere if the lines through ξ meet at a point. If this point is on the concave side of H, then H is a *hyperbolic affine sphere.* An example would be one component of a hyperboloid of two sheets. The Cheng-Yau correspondence gives the connection between hyperbolic affine spheres and convex sets Ω:

Theorem 1.6 (Cheng-Yau [14])**.** *For any properly convex domain* Ω *in* $\mathbb{S}^m$*, there is a unique hyperbolic affine sphere* H *asymptotic to* Ω*.*

For example, a component of a hyperboloid of two sheets is asymptotic to hyperbolic space. The classification of symmetric convex cones was applied to the problem of classifying hyperbolic affine spheres as a result of this correspondence. To apply Theorem 2 to affine spheres, we will translate some of the hypotheses.

1.8.1 Outline of Proof

Here we will briefly outline the proof of Theorem 2. An important step is the application of the following theorem of Benoist.

Theorem 1.7 (Benoist [5])**.** *Let Ω be a properly convex irreducible set in $\mathbb{S}^m$ which is divided by the discrete group Γ. If Ω is not symmetric, then Γ is Zariski dense in* $\mathrm{SL}(m+1,\mathbb{R})$.

The final step will be to show that for Ω that satisfy the hypotheses of Theorem 2, a discrete $\Gamma \subset \mathrm{SL}(m+1,\mathbb{R})$ cannot be Zariski dense in $\mathrm{SL}(m+1,\mathbb{R})$. This creates a contradiction.

When $g \in \Gamma$ is the axis of an embedded $\mathbb{H}^2$, the eigenvalues of g must satisfy

$$1 = 2\frac{\log|\lambda_1| - \log|\lambda_k|}{\log|\lambda_1| - \log|\lambda_{m+1}|}. \tag{1.1}$$

for some $1 < k < m+1$. We will show that the set of $g \in \Gamma$, under the hypotheses of Theorem 2, that have axis γ_g in an embedded $\mathbb{H}^2$ must Zariski dense in Γ. We will show that Equation 1.1 is an algebraic equation on a subset of Γ which is Zariski dense in $\mathrm{SL}(m+1,\mathbb{R})$. This creates a contradiction, so Ω must be a symmetric convex cone.

Now, we will outline the proof of Theorem 3 which has some overlapping ideas with Theorem 2. For a point $x \in \partial\Omega$ where Ω is strictly convex, the quantity

$$\alpha = \frac{\log|\lambda_1| - \log|\lambda_{m+1}|}{\log|\lambda_1| - \log|\lambda_k|}$$

will measure the shape of the boundary of Ω around x for $1 < k < m+1$. When there is a C^2 curve with positive second derivative passing through x, there exists a k where $\alpha = 2$. Since we have a dense set of $g \in \Gamma$ whose x_g^+ are part of C^2 curves in the boundary, we can follow the argument from Theorem 2 to finish the proof.

CHAPTER 2

Background

Let $\mathcal{C}$ be an open convex cone in $\mathbb{R}^{m+1}$. Let $\mathbb{S}^m$ be the space of half lines through the origin in $\mathbb{R}^{m+1}$ with $p : \mathbb{R}^{m+1} \to \mathbb{S}^m$. Then $\Omega = p(\mathcal{C})$. We will assume that Ω is properly convex, i.e. $\mathcal{C}$ does not contain an affine line. In either case, the boundaries are defined as $\partial\mathcal{C} = \overline{C} \setminus C$ and $\partial\Omega = \overline{\Omega} \setminus \Omega$.

The set Ω is called *divisible* if there exists a discrete $\Gamma \subset \mathrm{SL}(m+1, \mathbb{R})$ such that Ω/Γ is comapct.

2.1 Zariski Density

The goal of these first few sections is to define Zariski dense subsemigroups of reductive groups. These subsemigroups have special properties when projected into a linear space (see [3] and [10]). We will start with the Zariski topology and Zariski density of algebraic sets and we will work our way up to groups and finally, to results of Benoist.

The first step is to define the Zariski topology on k^n, where k is a field. Any subset $X \subset k^n$ will be given the subspace topology. Intuitively, the closed sets of k^n are the algebraic sets; we define the *closed sets* in the Zariski topology on k^n by

$$V(P) = \{x \in k^n | f(x) = 0 \text{ for all } f \in P\}$$

where P is a finite set of polynomials of n variables over k.

Intersections and unions of algebraic sets are defined as follows. Let I and J be two sets of polynomials of n variables over k. Then $IJ = \{fg | f \in I, g \in J\}$ and $I + J = \{f + g | f \in I, g \in J\}$. Using these, it follows that

$$V(I) \cap V(J) = V(I + J) \qquad \text{and} \qquad V(I) \cup V(J) = V(IJ)$$

and these statements imply that finite unions $\bigcup_{i=1}^{k} V_i$ and infinite intersections $\bigcap_{i=1}^{\infty} V_i$ are closed algebraic subsets.

Some families of polynomials and their associated set are given below.

Family of Polynomials	Associated Subvariety
$\{0\}$	V
$\{1\}$	$\emptyset$
$\{x = 0\}$ when $V = \mathbb{R}^2$	the vertical axis of $\mathbb{R}^2$
$\det X = 1$ when $X \in \mathrm{Mat}_{n \times n}(k)$	$\mathrm{SL}(n, k)$

Intuitively, closed sets in the Zariski topology are intersections of hyperplanes and hypersurfaces which have zero Lebesgue measure in $V = k^n$. Hence the open sets of the Zariski topology are very large; if $U \subset \mathbb{C}^n$ is open in the Zariski topology on $\mathbb{C}^n$ then U is dense in the standard topology on $\mathbb{C}^n$. For $X \subset V$, let $I(X)$ be the set of functions on V that vanish on X. This is called the *ideal* associated to X. The ideal is used to define Zariski closure.

Let X be a subset of V. The *Zariski closure* of X is the set

$$\overline{X} = \{v \in V | \text{for all f} \in I(X),\ f(v) = 0\} \subset V.$$

For example, if $V = \mathbb{R}$ and X is the set of integers, then $\overline{X} = \mathbb{R}$.

Finally, if $X \subset k^n$ and $Y \subset k^m$ are two algebraic sets, then a map $f : X \to Y$ is continuous in the Zariski topology, or *regular*, if f can be written as a restriction of a polynomial map $k^n \to k^m$.

2.2 Algebraic Groups

An algebraic group is a group that is also an algebraic set such that multiplication and inversion are regular maps. Our focus is on affine algebraic groups, i.e. $\mathrm{GL}(n, k)$.

A reductive group G is an algebraic group whose representations are all semi-simple. The groups $\mathrm{GL}(n, k)$, $\mathrm{SL}(n, k)$ and $\mathrm{SO}(n)$ are reductive groups while any unipotent group is not.

Each reductive group has an associated root system, which is a special set of vectors in a vector space V. Abstract root systems will be introduced in the next subsection and their association with reductive groups, along with examples, will appear following. It is a theorem of Chevalley ([20]) that reductive groups are completely classified by their root systems. Here, our motivation for introducing root systems will be to project G into a subset of V called the Weyl chamber $\mathfrak{a}^+$. This

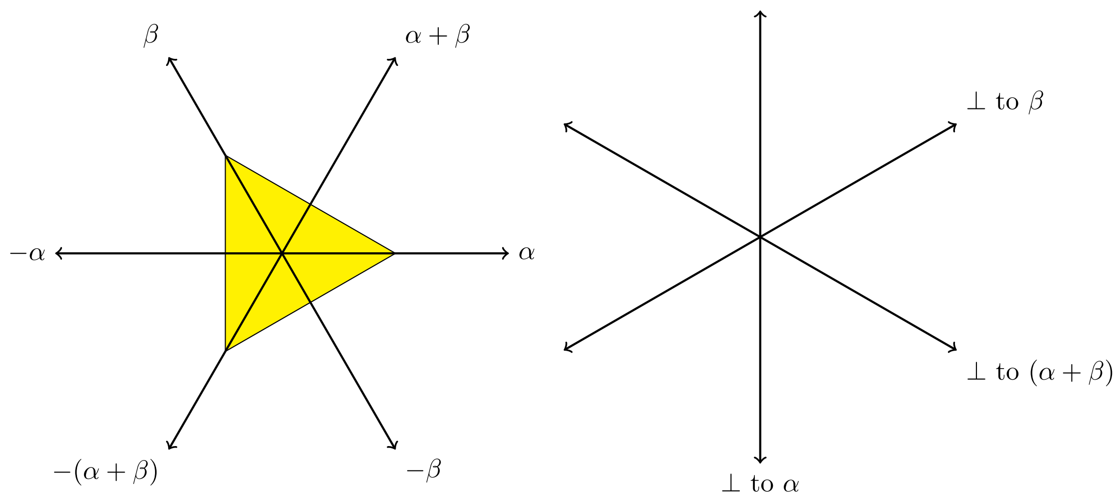

Figure 2.1: The root system A_2

will allow us to further exploit the algebraic properties of G.

2.2.1 Root Systems

Several definitions of root systems exist and the one presented here is most similar to the one given in [13].

Let V be a finite dimensional vector space with a Euclidean inner product $(\cdot,\cdot)$. The map

$$v \mapsto v - 2\left(\frac{v \cdot n}{n \cdot n}\right) n$$

defines a reflection of the vector v across a hyperplane passing through the origin with normal vector n. If $v \in V$ and $f^* \in V^*$, define $f^*(v) = \langle f^*, v\rangle$. A generalization of reflection where a Euclidean inner product is not required is

$$v \mapsto v - \langle f^*, v\rangle f$$

where $v, f \in V$ and $f^* \in V^*$ with $\langle f^*, f\rangle = 2$.

Reductive groups are classified by their root systems, which consist of the following data:

- the pairings (V, Σ) and (V^*, Σ^*) where V is a finite dimensional vector space over $\mathbb{R}$, V^* is the dual space of linear functions on V, Σ is a finite subset of $V \setminus \{0\}$ and Σ^* is a finite subset of $V^* \setminus \{0\}$,
- a bijection $\alpha \mapsto \alpha^*$ between Σ and Σ^*

with the following conditions on this data:

- for each $\alpha \in \Sigma$, $\langle \alpha^*, \alpha \rangle = 2$,
- for each $\alpha, \beta \in \Sigma, \langle \alpha^*, \beta \rangle \in \mathbb{Z}$
- for each $\alpha \in \Sigma$, the reflection

$$\sigma_\alpha : v \mapsto v - \langle \alpha^*, v \rangle \, \alpha$$

  in V preserves Σ and the reflection

$$\sigma_{\alpha^*} : v \mapsto v - \lambda^*(v)\lambda^*$$

  in V^* preserves Σ^*.

With these conditions, one can classify root systems for a vector space V. The most basic root system exists in $V = \mathbb{R}$ and consists of the pair $\{v, -v\}$ with $\langle v^*, v \rangle = 2$.

The roots for another system denoted A_2 are shown in Figure 2.1. Each vector in the A_2 system has coordinates $\left(\sqrt{2}\cos\left(\frac{n\pi}{3}\right), \sqrt{2}\sin\left(\frac{n\pi}{3}\right)\right)$ for some integer n, and

$$\begin{aligned}\left(\sqrt{2}\cos\left(\frac{n\pi}{3}\right), \sqrt{2}\sin\left(\frac{n\pi}{3}\right)\right) \cdot \left(\sqrt{2}\cos\left(\frac{m\pi}{3}\right), \sqrt{2}\sin\left(\frac{m\pi}{3}\right)\right) \\ = 2\cos\left(\frac{(n-m)\pi}{2}\right) \in \mathbb{Z} \quad \text{and} \\ = 2 \text{ when } n = m\end{aligned}$$

as required in the definition of root systems.

For the A_2 example, the reflections across hyperplanes perpendicular to each root are shown in Figure 2.1; these correspond to the reflections of a triangle and generate the dihedral group of order six. For any root system $(V, \Sigma, V^*, \Sigma^*)$, the group generated by σ_α for $\alpha \in \Sigma$ is called the Weyl group; we have just shown that the Weyl group for the root system A_2 is the dihedral group of order six.

The hyperplanes of reflection of A_2, shown in Figure 2.1, subdivide $\mathbb{R}^2$ into six disconnected regions called *Weyl chambers*. In any root system a Weyl chamber is a fundamental domain of the action of the Weyl group on V. We can choose a particular Weyl chamber, called the *principal Weyl chamber* $\mathfrak{a}^+$ by

$$\mathfrak{a}^+ = \{v \in V \,|\, \langle v, \alpha \rangle > 0 \text{ for all } \alpha \in \Sigma\}.$$

2.2.2 Root Systems and Algebraic Groups

In future sections we will restrict the group G to the case where $G = \mathrm{SL}(n,k)$ where k is a field. This group is algebraic because it is a closed subset under the Zariski topology of all $n \times n$ matrices since $\mathrm{SL}(n,k) = \{X \in \mathrm{Mat}(n) | \det(X) = 1\}$.

Let $A = (a_i)$ be the subgroup of $\mathrm{SL}(n,k)$ of diagonal matrices; this group is a subgroup of the group $(k^\times)^n$ under multiplication. Let $X(A)$ be the group of algebraic homomorphisms from A to $k^\times$. We can map A to $X(A)$ via $\epsilon_i : a \mapsto a_i$. In fact, the ϵ_i generate $X(A)$.

The group A acts on the Lie algebra $\mathfrak{sl}(n,k)$ of $\mathrm{SL}(n,k)$ by the adjoint action; here this is just conjugation i.e. $\mathrm{Ad}(a)(X) = aXa^{-1}$. Explicitly, when $X = (x_{ij})$,

$$aXa^{-1} = \begin{pmatrix} x_{11} & \frac{a_1}{a_2}x_{12} & \cdots & \frac{a_1}{a_n}x_{1n} \\ \frac{a_2}{a_1}x_{21} & x_{22} & \cdots & \frac{a_2}{a_n}x_{2n} \\ \vdots & & \ddots & \vdots \\ \frac{a_n}{a_1}x_{n1} & \cdots & \cdots & x_{nn} \end{pmatrix}$$

From this you can see that the adjoint action of A on $\mathfrak{sl}(n,k)$ splits $\mathfrak{sl}(n,k)$ into a sum of eigenspaces E_{ij} where each E_{ij} has a 1 in the (i,j) position and zero elsewhere. Each E_{ij} has associated eigenvalue λ_{ij} given by $\lambda_{ij}(a) = \frac{a_i}{a_j}$.

2.3 The Benoist Cone for Zariski Dense Subsemigroups

Suppose that $\Gamma \subset G$ is a lattice in G. This means that

- G is a linear, semisimple Lie group (with finitely many connected components,
- Γ is a discrete subgroup of G, and
- G/Γ has finite volume.

When Γ is a lattice, the space $M = G/\Gamma$ has a Riemannian metric that comes from the Haar measure. The Borel density theorem says that in many cases, the lattice Γ

is also Zariski dense in G.

Theorem 2.1 (Borel Density Theorem)**.** *A lattice* Γ *in a linear semisimple Lie group* G *is Zariski dense as long as* G *has no compact factors.*

Rather than working with lattice subgroups, we will work in the similar situation where $G = \mathrm{SL}(m+1, \mathbb{R})$ acts on a convex set Ω and has a discrete subgroup Γ such that Ω/Γ is compact. In this case, we say that Ω is *divided* by Γ. The subgroup Γ may or may not be Zariski dense in $\mathrm{SL}(m+1, \mathbb{R})$; this will affect the geometric properties of Ω/Γ.

2.3.1 Zariski Dense Semigroups

In the work of Benoist and Quint in [10], results are given for the more general semigroups rather that groups. The set G is a semigroup if there is a multiplication $*$ on G such that

- for all $g, h \in G$, $g * h \in G$ and
- there exists an element e such that for all $g \in G$, $e * g = g * e = g$.

Essentially, the difference bewteen how groups and semigroups are defined is that inverses are not required in semigroups. Benoist and Quint work in this generality as it is necessary in proofs related to the Law of Large Numbers for random walks in reductive groups. However, in all of our applications, G is a subgroup of $\mathrm{SL}(m, \mathbb{R})$ for some integer $m > 0$ and the symbol $*$ for multiplication will be omitted.

From this point, our discussion is concerned with Zariski dense subsemigroups of

$\mathrm{SL}(n, \mathbb{R})$. We do not need to worry about the group structure of the Zariski closure of subgroups due to the following lemma:

Lemma 2.2 (See Lemma 6.15 in [10])**.** *Let* Γ *be a Zariski dense subsemigroup of* $\mathrm{GL}(n, \mathbb{R})$*. Then the Zariski closure of* Γ *in* $\mathrm{GL}(n, \mathbb{R})$ *is a group.*

2.3.2 The Jordan Projection

When G is a reductive group, every element $g \in G$ can be decomposed uniquely into its *Jordan decomposition*. This is a set of commuting elements g_e, g_h, and g_u in G where $g = g_e g_h g_u$ and g_e is semisimple with eigenvalues of modulus one, g_h is semisimple with positive eigenvalues, and g_u is unipotent. When $G = \mathrm{SL}(n, \mathbb{R})$, g_h is a diagonal matrix with positive eigenvalues. This decomposition ensures the existence of a map called the *Jordan projection* μ for any $g \in \mathrm{SL}(n, \mathbb{R})$ as

$$\mu : G \to \mathfrak{a}^+ \qquad \text{where} \qquad \mu(g) = (\log|\lambda_1|, \ldots, \log|\lambda_n|)$$

and where $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ are the moduli of the eigenvalues of g. The Jordan projection is uniquely determined due to the Jordan decomposition. The element $\mu(g)$ lies on the wall of a Weyl chamber if and only if there is some i where $|\lambda_i| = |\lambda_{i+1}|$.

2.3.3 Loxodromic Elements

Let $g \in \mathrm{SL}(n, k)$ and $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ be the modulus of the eigenvalues of g in decreasing order. The element g is said to be *proximal* if $|\lambda_1| > |\lambda_2|$ and *biproximal* if both g and g^{-1} are proximal. An element g is called *loxodromic* if

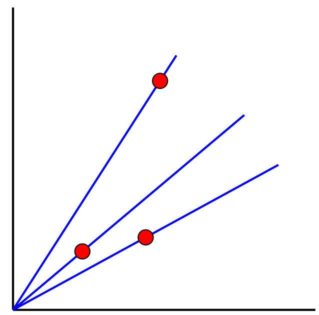

Figure 2.2: The limit cone of $\lambda(g_1)$, $\lambda(g_2)$, and $\lambda(g_3)$ for some $g_1, g_2, g_3 \in \Gamma$.

$|\lambda_i| > |\lambda_{i+1}|$ for all $0 < i < n$. Loxodromic elements g have the property that the Jordan projection $\mu(g)$ belong to the interior of $\mathfrak{a}^+$. In most cases we can assume any $g \in G$ is loxodromic due to the following result:

Theorem 2.3 (Benoist [10])**.** *Let G be a connected algebraic semisimple real Lie group and let Γ be a Zariski dense subsemigroup of G. Then the set Γ_{lox} of loxodromic elements of Γ is still Zariski dense.*

2.3.4 Benoist results

In this section we discuss the Jordan projection $\mu(\Gamma)$ of a Zariski dense subgroup Γ of G. For each $g \in \Gamma_{\text{lox}}$, $\mu(g)$ maps into what we will be the *limit cone* of Γ. The limit cone consists of $t \cdot \chi(g)$ for all $g \in \Gamma_{\text{lox}}$ with $t > 0$; this is shown in Figure 2.2 Then the *limit cone* L_Γ of Γ is defined as the smallest closed cone in the Weyl chamber containing $\mu(\Gamma_{\text{lox}})$.

When G is a connected algebraic semisimple Lie group and Γ is a Zariski dense subsemigroup of G, one can apply the following very important theorem of Benoist:

Theorem 2.4 (Benoist [2])**.** *The limit cone L_Γ is convex with nonempty interior.*

We will return to a discussion of the limit cone when $G = \mathrm{SL}(3, \mathbb{R})$ in Chapter 5.

2.4 The Hilbert Metric

The Klein model of hyperbolic space consists of the points $\{(x_1, x_2, \ldots, x_n, 1) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots x_n^2 < 1\}$ where geodesics are straight lines. This can be checked using the Riemannian metric for this model which is given by

$$ds = \sqrt{\frac{dx_1^2 + \cdots + dx_n^2}{1 - x_1^2 - \cdots - x_n^2} + \frac{(x_1 dx_1 + \cdots + x_n dx_n)^2}{(1 - x_1^2 - \cdots - x_n^2)^2}}$$

Since geodesics are straight lines, calculating the distance between two points reduces to a one dimensional picture. In this case, the metric is

$$ds = \sqrt{\frac{dx^2}{1 - x^2} + \frac{x^2 dx^2}{(1 - x^2)^2}} = \frac{dx}{(1 - x^2)}$$

so that the distance between the point $x = p_1$ and $x = p_2$ where $-1 < p_1, p_2 < 1$ is

$$\int_{x=p_1}^{x=p_2} \frac{1}{(1 - x^2)} dx = \frac{1}{2} \log\left(\frac{1 - p_1}{1 + p_1}\right) - \frac{1}{2} \log\left(\frac{1 - p_2}{1 + p_2}\right) = \frac{1}{2} \log\left(\frac{(1 - p_1)(1 + p_2)}{(1 + p_1)(1 - p_2)}\right).$$

This metric can be rewritten in terms of Euclidean distances. Therefore this distance can be applied not only to $\mathbb{H}^n$, but to any convex set Ω; extend a line segment between the points $x, y \in \Omega$ to find the intersection points $a, b \in \partial\Omega$ of the line with the boundary of Ω, as shown in Figure 2.3.

Hence we define the Hilbert metric on a convex set Ω as

$$d_\Omega(x, y) = \frac{1}{2} \log\left(\frac{|ay||xb|}{|ax||yb|}\right)$$

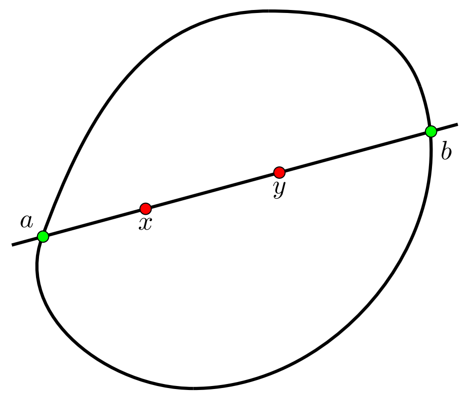

Figure 2.3: The Hilbert Metric

where $|\cdot|$ is the Euclidean distance. Note that $\frac{|ay||xb|}{|ax||yb|} = [x; y; a; b]$ is the projective cross ratio of four points. The value of $[x; y; a; b]$ is invariant under projective transformations.

Let $x \in \Omega$, let $\xi \in T_x\Omega$ and consider $t \in \mathbb{R}$ such that $x + t\xi \in \Omega$. The distance d_Ω can be seen as arising from a Finsler norm F in the following sense:

$$F(x, \xi) = \frac{d}{dt}(d_\Omega(x, x + t\xi))_{t=0} = \frac{|\xi|}{2}\left(\frac{1}{|xx^+|} + \frac{1}{|xx^-|}\right)$$

where x^+ is the intersection of the line through $(x, x + t\xi)$ and $\partial\Omega$ when $t > 0$ and x^- is the intersection of the line through $(x, +t\xi)$ and $\partial\Omega$ when $t < 0$. Note that this norm may be asymmetrical due to the shape of the boundary of Ω.

2.5 Lyapunov Exponents

Let N be a Riemannian manifold and let $\phi_t : N \to N$ be a C^1 flow on N. The *Lyapunov exponents* of the flow ϕ_t measure the spread of infinitesimally close geodesics on N under the action of ϕ_t. For example, a vector $v \in TN$ approximates the initial displacement between two close geodesics. As t increases under the flow ϕ_t, the distance between each orbit changes by approximately $||d\phi_t(v)||$. The exponential growth rate, averaged for all t, of $||d\phi_t(v)||$ is

$$\lim_{t\to+\infty} \frac{1}{t} \log ||d\phi_t(v)|| = \chi(v)$$

whenever $\chi(v)$ exists. The existence of $\chi(v)$ is guaranteed for a subset of full measure of TN by the multiplicative ergodic theorem (MET) when N has a ϕ_t invariant probability measure μ. In particular, the MET guarantees more; there is a set of full measure with respect to μ of points $v \in TN$ and a decomposition

$$TN = \mathbb{R} \cdot X \oplus E_1 \oplus \cdots \oplus E_p$$

where X is a vector field that generates the flow ϕ_t. Further, there are real numbers $\chi_1(v) < \cdots < \chi_p(v)$ where for any vector $v_i \in E_i \setminus \{0\}$, the Lyapunov exponents are

$$\chi_i(v) = \lim_{t\to+\infty} \frac{1}{t} \log ||d\phi_t(v_i)||. \tag{2.1}$$

Finally, the MET ensures that $\lim_{t\to\pm\infty} \frac{1}{t} \log |\det d\phi_t| = \sum_{i=1}^{p} \chi_i(v) \dim E_i$.

In the following sections, we will examine the case where ϕ_t is the geodesic flow φ_t on SM, the unit tangent space of M where M is either $\Gamma\backslash\mathbb{H}^2$ or Ω/Γ under the Hilbert metric. When $\gamma(t) \in M$ is a geodesic with $\gamma(0) = x \in M$ and $\gamma'(0) = v \in S_xM$, then $\varphi_t((x, v)) = (\gamma(t), \gamma'(t)) \in SM$.

Note that $d\varphi_t$ is a map $d\varphi_t : TSM \to TSM$ and the Lyapunov decomposition given by the MET is a decomposition of TSM. In the next section, we will study the geodesic flow on $\mathbb{H}^2$ using algebraic techniques (see [23]) and afterwards follow Crampon in [18], [17], [19], and [16] to study the Lyapunov exponents in Hilbert geometry from multiple points of view.

2.5.1 Lyapunov Exponents for the Geodesic Flow on $\Gamma\backslash\mathbb{H}^2$

The Lyapunov exponents for the geodesic flow on compact quotients of $\mathbb{H}^2$ will be used in Chapter 4 so we will give an outline of how they are calculated. Much of the setup can be found in further detail in [23]. To simplify calculations, we will use the upper half space model of $\mathbb{H}^2$;

$$\mathbb{H}^2 = \{x + iy = z \in \mathbb{C} | \mathrm{Im}(z) = y > 0\}$$

which has the metric $ds = \sqrt{\frac{dx^2+dy^2}{y}}$. Under this metric, the positive imaginary axis, i.e. $\{x + iy = z \in \mathbb{H}^2 | x = 0\}$ is a geodesic. Since the isometry group of $\mathbb{H}^2$ is transitive on $S\mathbb{H}^2$, the full description of all geodesics can be obtained by considering the orbit of the positive imaginary axis under the isometry group.

The isometry group of $\mathbb{H}^2$ is $\mathrm{PSL}(2,\mathbb{R})$ which acts on the upper half space model by Mobius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Lete Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ such that $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ is compact. The Mobius transformations are generated by the maps $z \mapsto az$, $z \mapsto z + b$ and $z \mapsto \frac{1}{z}$. The orbit of the positive imaginary contains lines parallel to the imaginary axis and half circles which meet the real axis $\{z = x + iy | y = 0\}$ orthogonally. Identifying z with $\begin{pmatrix} z \\ 1 \end{pmatrix} \in \mathbb{C}P^1$, one can formulate the action as matrix multipliation;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} \in \mathbb{C}P^1.$$

The stabilizer of the point $z = i \in \mathbb{H}^2$ is $PSO(2) \subset \mathrm{PSL}(2, \mathbb{R})$. Thus $\mathbb{H}^2$ can be identified with $\mathrm{PSL}(2, \mathbb{R})/\mathrm{PSO}(2)$ by sending $z \in \mathbb{H}^2$ to $g\mathrm{PSO}(2)$ where $g \cdot i = z$. In addition to acting on $\mathbb{H}^2$, the element $g \in \mathrm{PSL}(2, \mathbb{R})$ acts on $x = (z, v) \in TM$ by $g \cdot x = (g \cdot z, g'(z)v)$. In coordinates, this map is

$$(dg)(z, v) = \left(\frac{az+b}{cz+d}, \frac{1}{(cz+d)^2} v \right).$$

The map dg preserves the length of the vector v under the norm $ds^2 = \frac{dx^2+dy^2}{y}$. Hence we get a restricted map $dg : S\mathbb{H}^2 \to S\mathbb{H}^2$ which defines an action of $\mathrm{PSL}(2, \mathbb{R})$ on $S\mathbb{H}^2$. This action is simply transitive, so we conclude that

$$\mathrm{PSL}(2, \mathbb{R}) \cong S\mathbb{H}^2.$$

To determine a map $\mathrm{PSL}(2, \mathbb{R}) \to \mathbb{H}^2$, we simply pick a basepoint $(z, v) \in \mathbb{H}^2$ and let $g \mapsto dg \cdot (z, v) = (g(z), g'(z)v)$. For all further calculations, this basepoint is $(z, v) = (i, i)$ where $v = i$ indicates a vector of unit length in the positive vertical direction.

The motivation for this process is that under the identification $S\mathbb{H}^2 \cong \mathrm{PSL}(2, \mathbb{R})$, the geodesic flow $\varphi_t : S\mathbb{H}^2 \to S\mathbb{H}^2$ can be defined entirely in terms of actions on

PSL$(2,\mathbb{R})$. When working with these group actions, the Lyapunov exponents for a quotient of hyperbolic space is a more attainable calculation. The first step will be to define the geodesic flow on $S\mathbb{H}^2$ and then interpret the map for $\varphi_t : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ using the identification of PSL$(2,\mathbb{R})$ with $S\mathbb{H}^2$.

To define the map $\varphi_t : S\mathbb{H}^2 \to S\mathbb{H}^2$ it is only necessary to determine $\varphi_t(i,i)$. Note that the action of PSL$(2,\mathbb{R})$ preserves geodesics. Then, if $x = (z,v) = dg \cdot (i,i)$, it follows that $\varphi_t(x) = dg \cdot \varphi_t(i,i)$. It can be shown that $\varphi_t(i,i) = (e^t i, e^t i)$.

In order to compose the action of φ_t on (i,i) with the action of dg, we will define φ_t so that $\varphi_t(z,v) = dh_t \cdot (z,v) = (h_t(z), h_t'(z)v)$ for some Mobius transformation h. We only need that $dh_t \cdot (i,i) = (e^t i, e^t i)$ since dh_t will only be directly acting on the basepoint (i,i). More simply, we need h where

$$dh_t \cdot (i,i) = (h_t(i), h_t'(i)i) = (e^t i, e^t i).$$

A choice of $h_t(z)$ is $h_t(z) = e^t z$. A representative of the Mobius transformation $h_t(z)$ in PSL$(2,\mathbb{R})$ is a_t^{-1} where

$$a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}. \tag{2.2}$$

The inverse of a_t is used because the geodesic flow will be a right action on PSL$(2,\mathbb{R})$. Thus

$$\varphi_t(z,v) = dg \cdot d(a_t^{-1})(i,i) = d(ga_t^{-1})(i,i)$$

where $dg \cdot (i,i) = (z,v)$. The geodesic flow on PSL$(2,\mathbb{R})$ can now be defined;

$$\varphi_t : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R}) \quad \text{where} \quad g \mapsto R_{a_t}(g) \tag{2.3}$$

where R_{a_t} is right multiplication by a_t^{-1}.

Recall from Equation 2.1 that Lyapunov exponents of the geodesic flow φ_t on $\Gamma\backslash\mathbb{H}^2$ measure the exponential growth rate of $d\varphi_t$. Although the multiplicative ergodic theorem only guarantees existence of Lyapunov exponents for spaces with a probability measure, in this example we can compute this growth rate on $\mathbb{H}^2$ rather than $\Gamma\backslash\mathbb{H}^2$. On $\mathbb{H}^2$, $d\varphi_t$ maps $TS\mathbb{H}^2$ to itself. Since we have defined φ_t as a right translation by a_t^{-1} on $\mathrm{PSL}(2,\mathbb{R})$, this growth rate can be explored by treating $\mathrm{PSL}(2,\mathbb{R})$ as a Lie group.

Let G be a closed linear group with tangent bundle $TG \cong G \times \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G. If $\alpha(t) : [0,1] \to G$ is a differentiable curve, then define its derivative to be

$$d\alpha(t_0) = (\alpha(t_0), \alpha(t_0)^{-1}\alpha'(t_0)) \in G \times \mathfrak{g} \tag{2.4}$$

with $t_0 \in [0,1]$. Defining a new curve $\alpha(t)h^{-1}$ allows us to see the effect of right translation on tangent spaces. Calculating the derivative,

$$d(\alpha(t)h^{-1})(t_0) = (\alpha(t_0)h^{-1}, (\alpha(t)h^{-1})^{-1}\alpha'(t_0)h^{-1}) = ((\alpha(t_0)h^{-1}, h\alpha(t_0)^{-1}\alpha'(t_0)h^{-1})$$

and comparing with Equation 2.4 we conclude that the derivative of R_h is

$$dR_h : T_gG \to T_{gh^{-1}}G \qquad dR_h(g,v) = (gh^{-1}, hvh^{-1}). \tag{2.5}$$

Now we can return to the geodesic flow $\varphi_t : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ which was defined by right multiplication by a_t^{-1} in Equation 2.3. From Equation 2.5, we have that $d\varphi_t(g,v) = dR_{a_t}(g,v) = (ga_t^{-1}, a_t v a_t^{-1})$; to calculate Lyapunov exponents for φ_t we will need $||d\varphi_t(v)||$. The value of $||d\varphi_t(v)||$ will depend on v. Note that $v \in \mathfrak{g}$, so we will choose $v \in \mathfrak{sl}(2,\mathbb{R})$. The space $\mathfrak{sl}(2,\mathbb{R})$ is a three dimensional vector space with basis

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Next,

$$a_t v_1 a_t^{-1} = v_1 \qquad a_t v_2 a_t^{-1} = e^{-t} v_2 \qquad a_t v_3 a_t^{-1} = e^t v_3.$$

Finally, it follows that for any norm $||\cdot||$ on $\mathfrak{g}$, the Lyapunov exponents for φ_t are

1. $\chi(v_1) = 0$,
2. $\chi(v_2) = -1$, and
3. $\chi(v_3) = 1$.

Now that we have calculated the Lyapunov exponents of the geodesic flow, we can discuss what they mean. First, the case where $\chi = 0$ corresponds to the flow direction. Notice that $\exp\left(\frac{t}{2}v_1\right) = a_t^{-1}$ and that the geodesic flow is given by R_{a_t}.

The case where $\chi = -1$ describes a direction where vectors in $S\mathbb{H}^2$ move closer together under the geodesic flow in positive time.

Next, let $s \in \mathbb{R}$ and consider the points (i,i) and $(s+i,i)$ in $S\mathbb{H}^2$. Then

$$\varphi_t(i,i) = (e^t i, e^t i) \text{ and } \varphi_t(s+i,i) = (s + e^t i, e^t i)$$

and note that $d(e^t i, s+e^t i) \leq \frac{|s|}{e^t}$ since $\frac{|s|}{e^t}$ is the length of the horizontal line between $z = e^t i$ and $z = s+e^t i$. Thus the distance between $\varphi_t(i,i)$ and $\varphi_t(s+i,i)$ goes to zero as t goes to infinity. With some work, one can show that these points $\{(s+i,i)|s \in \mathbb{R}\}$ form the *stable manifold* for the geodesic flow through the point (i,i).

Since $(z,v) = (i,i)$ is the basepoint for the identification of $\mathbb{SH}^2$ with $\mathrm{PSL}(2,\mathbb{R})$, we can define the stable manifold for the geodesic flow through any point (z,v) which will be the orbit of the *stable horocycle flow*. Let

$$u^-(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

and define the stable horocycle flow H_s^- on $\mathrm{PSL}(2,\mathbb{R}) \cong S\mathbb{H}^2$ by

$$H_s^- : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R}) \quad \text{with} \quad h \mapsto R_{u^-(s)} = hu^-(-s).$$

This is analogous to the definition of the geodesic flow on $\mathrm{PSL}(2,\mathbb{R})$.

We can now interpret the Lyapunov exponent $\chi(v_2) = -1$. This suggests that along the direction of $v_2 \in \mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, the geodesic flow compresses geodesics and that this exponential decay is -1. Note that $\exp(-sv_2) = u^-(-s)$ and that $u^-(s)$ generates stable manifolds, which confirms this interpretation.

Now, we can also define the *unstable manifold* for the basepoint (i,i). These are the points (z,v) on $S\mathbb{H}^2$ such that the distance between $\varphi_t(i,i)$ and $\varphi_t(z,v)$ goes to zero as t goes to *negative* infinity. These points are given by the orbit of (i,i) under the action of

$$u^+(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

More generally, the *unstable horocycle flow* $H_s^+ : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is defined by $h \mapsto hu^+(-s)$. Finally, $\chi(v_3) = 1$ and $\exp(-2v_3) = u^+(-s)$ which generates the unstable manifolds for the geodesic flow.

2.6 Lyapunov Exponents in Hilbert Geometry

The background in this section is primarily work from Crampon, who applied some of Foulon's constructions in [25] to Hilbert geometry. A summary of this work is in his survey [18] with most of the details found in [16] and [19]. The relationship between Lyapunov exponents and the shape of the boundary of Ω is covered in [17]. The notation used in this section is inspired by these papers.

The concepts that Crampon introduces require Ω to be strictly convex with C^1 boundary (see discussion in Section 3.6 in [18]). These concepts include geodesic flow, parallel transport, and stable/unstable sets. Note that parallel transport will be defined as an operator on the *double* tangent space and will only be defined along geodesics, unlike in Riemannian geometry (see Section 3.3 in [18] for motivation). Crampon also introduces a notion of curvature, which will not be discussed here.

A summary of this background is given in Subsection 2.6.1. In Subsection 2.6.2, we describe the relationship between geodesic flow and parallel transport (see [16] or [19]) on $M = \Omega/\Gamma$. Finally, the different notions of Lyapunov exponents on Ω/Γ are

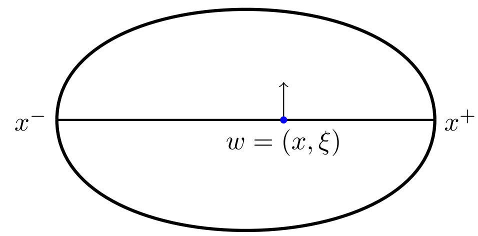

Figure 2.4: $w \in H\Omega$ and $x^+, x^- \in \partial\Omega$

described in Subsection 2.6.3.

2.6.1 Parallel Transport and Geodesic Flow

Let $HM = (TM \setminus \{0\})/\mathbb{R}_{>0}$ be the space of pairs $w = (x, [\xi])$ where $x \in M$ and $[\xi]$ is a direction in TM. The geodesic flow φ_t is a map $\varphi_t : HM \to HM$ that projects to geodesics under $\pi : HM \to M$ and is defined as follows. If $l_w(t)$ is a speed one (with respect to d_Ω) geodesic line leaving $x \in M$ at $t = 0$ in the direction of $[\xi] \in H_xM$, then $\varphi_t(w) = (l_w(t), [l'_w(t)])$.

In addition to geodesic flow, a well defined parallel tranport exists on M. M. Crampon constructs these maps in [16] using the dynamical formalism introduced by Foulon in [25]. Let $X : HM \to THM$ be the vector field that generates the geodesic flow φ_t. The method starts by finding a splitting of THM which will depend on the vector field X. This relies on some regularity of X and HM. The next step is to use an analogue of covariant differentiation, D_X, which is defined so that a vector field $Z \in THM$ is parallel when $D_X(Z) = 0$ and Z projects to the geodesic flow on HM. This allows a link between geodesic flow and parallel transport that will be essential in the calculation of Lyapunov exponents.

In Foulon's dynamical formalism, a vector field X^0 and a function m must be chosen so that X^0 is a smooth second order differential equation on M and

$$m \in C^1_{X^0}(HM) = \{f : HM \to HM | \mathcal{L}_{X^0} f \text{ exists.}\}$$

where $\mathcal{L}$ is the Lie derivative. We will take X^0 to be the generator of the Euclidean geodesic flow on HM, where straight lines are also geodesics. Hence X^0 is well defined and smooth. We will take m to be the function that satisfies $X = mX_0$ where X is the generator of the Hilbert geodesic flow. To see what m should be, recall from Section 2.4 that the Finsler norm on TM is

$$F(x, \xi) = \frac{|\xi|}{2}\left(\frac{1}{|xx^+|} + \frac{1}{|xx^-|}\right) \tag{2.6}$$

where $|\cdots|$ is a Euclidean norm in an affine chart on Ω and x^+, x^- can be seen in Figure 2.6.1 on the boundary of Ω. Therefore to ensure that each vecotr $\pi(X(w))$ has unit Finsler norm while each vector $\pi(X^0(w))$ has unit Euclidean norm, we take $m = 2\left(\frac{1}{|xx^+|} + \frac{1}{|xx^-|}\right)^{-1} = 2\left(\frac{|xx^+||xx^-|}{|xx^+|+|xx^-|}\right)$. In some affine chart on Ω, $\mathcal{L}_{X^0}$ is just differentiation along the direction of (x^+x^-), so that

$$\mathcal{L}_{X^0} = 2\left(\frac{|xx^+| - |xx^-|}{|x^+x^-|}\right), \ \mathcal{L}^2_{X^0} = -\frac{4}{|x^+x^-|} \text{ and } \mathcal{L}^n_{X^0} = 0$$

for $n > 2$. Therefore the conditions of Foulon's dynamical formalism are satifised and we will apply their results.

First, recall that π is the projection $\pi : HM \to M$ and that $d\pi$ is a map $d\pi : THM \to TM$. The *vertical distribution* $VHM \subset THM$ is the kernel of $d\pi$ and this is the standard vertical bundle of THM. If M has dimension m, then T_wHM has dimension $2m-1$ and V_wHM has dimension $m-1$. There is a decomposition

$$T_wHM = \mathbb{R} \cdot X \oplus V_wHM \oplus h^XHM;$$

it follows that h^XHM has dimension $m-1$. The distribution h^XHM is called the *horizontal distribution*. The horizontal distribution depends on X and has basis $[X, Y_i]$ where Y_i is one of the basis elements of V_wHM.

Remarkably, this construction allows us to define a pseudo-complex structure J^X on $VHM \oplus h^XHM$:

$$J^X : VHM \oplus h^XHM \to VHM \oplus h^XHM$$

where $J^X : VHM \to h^XHM$, $J^X : h^XM \to VHM$, and $J^X \circ J^X = -\mathrm{Id}$. Later, we will see that the relationship between the Lyapunov exponents of φ_t and the Lyapunov exponents of the parallel transport T^t relies on the existence of J^X.

Finally, M. Crampon defines an analogue of covariant differentiation, $D^X : THM \to THM$, which is used to define parallel transport. We can just state how it acts on each distribution:

$$D^X(X) = 0, D^X(Y) = -\frac{1}{2}v_X([X, [X, Y]]), [D^X, H_X] = 0$$

where $v_X : THM \to VHM$ is the vertical operator defined by

$$v_X(X) = v_X(Y) = 0, v_X([X, Y]) = -Y$$

and $H_X : VHM \to THM$ is the horizontal operator defined by

$$H_X(Y) = -[X, Y] - \frac{1}{2} v_X([X, [X, Y]]).$$

The properties of V_X and H_X can be found in Crampon's [16].

A vector field $Z \in THM$ is called *parallel* along the generator of the flow X if $D^X(Z) = 0$ and Z projects to the geodesics flow on HM. Then the *parallel transport* of the vector $Z(w) \in T_w HM$ at time t is defined as $T^t(Z(w)) = Z(\varphi_t(w))$ where $Z(\varphi_t(w))$ is the parallel vector at $\varphi_t(w)$. Note that without a Riemannian connection, the parallel transport T^t is not necessarily an isometry.

2.6.2 Relationship Between Geodesic Flow and Parallel Transport

Let X be the vector field that generates the Hilbert geodesic flow. In [16], Crampon shows that $TH\Omega$ has a splitting that depends on X where

$$THM = \mathbb{R} \cdot X \oplus E^s \oplus E^u$$

and

$$E^s = \left\{ Z \in TH\Omega, \lim_{t \to \infty} ||d\varphi_t(Z)|| = 0 \right\} \text{ and } E^u = \left\{ Z \in TH\Omega, \lim_{t \to -\infty} ||d\varphi_t(Z)|| = 0 \right\}$$

and $|| \cdot ||$ is a Finsler norm. Now, in terms of dynamical formalism,

$$E^u = \{Y + J^X(Y), Y \in VHM\} \quad \text{and} \quad E^s = \{Y - J^X(Y), Y \in VHM\} = J^X(E^u)$$

which have the property that for $Z^u \in E^u$ and $Z^s \in E^s$,

$$d\varphi_t(Z^u) = e^t T^t(Z^u) \quad \text{and} \quad d\varphi_t(Z^s) = e^{-t} T^t(Z^s). \tag{2.7}$$

which demonstrates a relationship between the lengths of vectors under orbits of $d\varphi_t$ and under orbits of T^t.

A horosphere $\mathcal{H}_{x^+}(x)$ based at $x^+ \in \partial\Omega$ through $x \in \Omega$ is the subset

$$\mathcal{H}_{x^+}(x) = \{y \in \Omega, \lim_{p \to x^+} d_\Omega(x,p) - d_\Omega(y,p) = 0\}.$$

The horospheres based at x^+ and x^- provide the basepoints of the stable and unstable sets of the geodesic flow φ^t on $H\Omega$ as Crampon proves in [19]:

Theorem 2.5 (Crampon [19]). *Let $w = (x, [\xi]) \in H\Omega, x^\pm = \varphi^{\pm\infty}(w) \in \partial\Omega$. The stable an unstable sets of w are the C^1 submanifolds*

$$W^s(w) = \{v \in H\Omega, \varphi^{+\infty}(v) = x^+, \pi(v) \in \mathcal{H}_{x^+}(x)\}$$

and

$$W^u(w) = \{v \in H\Omega, \varphi^{-\infty}(v) = x^-, \pi(v) \in \mathcal{H}_{x^-}(x)\}$$

where $\pi : H\Omega \to \Omega$ is the projection.

Later, in Section 2.7, we will need $T\mathcal{H}_{x^+}$ which is defined as

$$T\mathcal{H}_{x^+} = \{T_x H_w | w = (x, [\xi]) \in W^s(x^+)\} \tag{2.8}$$

where H_w is the horosphere at w.

2.6.3 Lyapunov Exponents of Geodesic Flow and Parallel Transport

Recall that parallel transport T^t is a map on THM. We will use T^t to define a map $T^t_{x^+}$ on HM which can be thought of as parallel transport in the direction of x^+. Recall the projection $\pi : HM \to M$ so that $d\pi : THM \to HM$. Define $d\pi_{x^+}^{-1}(x) = (x, [xx^+]) \in THM$. Then $T_{x^+}^t = d\pi \circ T^t \circ d\pi_{x^+}^{-1}$.

We call a point $w \in HM$ weakly regular if for all $Z \in T_w HM$, the limit

$$\lim_{t \to \pm\infty} \frac{1}{t} \log ||d\varphi_t(Z)||$$

exists. Whenever w is weakly regular, these numbers $\chi(Z) = \lim_{t\to\pm\infty} \frac{1}{t} \log ||d\varphi_t(Z)||$ for $Z \in T_w HM$ can only take a finite number of values, called the Lyapunov exponents χ_i of the geodesic flow φ_t.

When w is weakly regular for the geodesic flow, w is also weakly regular for the parallel transport. A Lyapunov exponent for $T_{x^+}^t$ is defined to be

$$\eta(v) = \lim_{t \to \infty} \frac{1}{t} \log F(T_{x^+}^t v)$$

where F is the Finsler norm from Equation 2.6. These $\eta(Z)$ for $Z \in T_w HM$ also take only a finite number of values for weakly regular w and are called the parallel Lyapunov exponents.

For $Z_i \in THM$, let

$$\chi_i^+ = \chi(Z_i) \text{ for } Z_i \in Z^u,\ \chi_i^- = \chi(Z_i) \text{ for } Z_i \in Z^s, \text{ and } \eta_i = \eta(Z_i).$$

Working from Equation 2.7, Crampon shows that

1. $\chi_i^+ = 1 + \eta_i$
2. $\chi_i^- = -1 + \eta_i$
3. $\chi_i^+ = \chi_i^- + 2$, and
4. $-2 \leq \chi_i^+ \leq 0 \leq \chi_i^- \leq 2$.

2.6.4 Lyapunov Exponents for Periodic Orbits

Let γ be a periodic orbit of the geodesic flow on M. Lifted to Ω, the geodesic $\tilde{\gamma}$ is associated to an element $g \in \Gamma$ for which $\tilde{\gamma}$ is its axis. Since γ is loxodromic we can identify a highest modulus eigenvalue $|\lambda_1|$ and a lowest modulus eigenvalue $|\lambda_{m+1}|$. Then

Lemma 2.6. *The length of the periodic orbit of γ is*

$$l_g = \frac{1}{2}(\log|\lambda_1| - \log|\lambda_{m+1}|)$$

Proof. Choose an affine chart for Ω where the eigenvector associated to λ_1 is $\begin{bmatrix} 0 & \dots & 1 \end{bmatrix}^T$ and the eigenvector associated to λ_{m+1} is $\begin{bmatrix} 1 & \dots & 0 \end{bmatrix}^T$. Let

$$p = \begin{bmatrix} \mu & 0 & \dots & 0 & 1-\mu \end{bmatrix}^T \in \tilde{\gamma}.$$

Then

$$g \cdot p = \frac{1}{\lambda_1\mu + \lambda_{m+1}(1-\mu)} \begin{bmatrix} \lambda_1\mu \\ \vdots \\ \lambda_{m+1}(1-\mu) \end{bmatrix} = \begin{bmatrix} gp_1 \\ \vdots \\ gp_{m+1} \end{bmatrix}.$$

Then from a calculation it follows that

$$\begin{aligned} d_\Omega(p, g\cdot p) &= \frac{1}{2}\log\left(\frac{d((0,1),(gp_1, gp_{m+1}))d((1,0),(\mu,1-\mu))}{d((0,1),(\mu,1-\mu))d((1,0),(gp_1,gp_{m+1}))}\right) \\ &= \frac{1}{2}\log\sqrt{\frac{\lambda_1^2}{\lambda_{m+1}^2}} \end{aligned}$$

where d is the Euclidean distance in the affine chart. ∎

Let $Z \in T_w H\Omega$ be such that $d\pi(Z) = v_i$, an eigenvector with eigenvalue λ_i. Then in [16], Crampon shows that

$$||T^{nl_g}|| \approx \left|\frac{\lambda_1}{\lambda_2}\right|^n e^{-nl_\gamma}.$$

To calculate the parallel lyapunov exponent η_i, we can take

$$\eta_i = \lim_{n\to\infty} \frac{1}{nl_\gamma}||T^{nl_\gamma}(Z)|| = -1 + 2\frac{\log|\lambda_1| - \log|\lambda_i|}{\log|\lambda_1| - \log|\lambda_{m+1}|}$$

Finally, using the equations from 2.6.3, we also get that

$$\chi_i^+ = 2\frac{\log|\lambda_1| - \log|\lambda_i|}{\log|\lambda_1| - \log|\lambda_{m+1}|} \quad \text{and } \chi_i^- = -2 + 2\frac{\log|\lambda_1| - \log|\lambda_i|}{\log|\lambda_1| - \log|\lambda_{m+1}|}. \tag{2.9}$$

These formulas for the Lyapunov exponents of periodic orbits will become very important in the proof of Theorem 2.

2.7 Approximate Regularity

In this subsection we will define an alternative definition of Lyapunov exponents for points $p \in \partial\Omega$ where the boundary $\partial\Omega$ around p has a Hölder-like regularity. This property is called *approximate regularity* and was introduced by M. Crampon in [17] and [19]; we will first define approximate regularity for functions on $\mathbb{R}^n$ and then extend the definition to $\partial\Omega$.

A convex C^1 function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$ is called *approximately α-regular* (Crampon, [19]) if there exists an $\alpha \in [1, \infty]$ such that

$$\lim_{t\to 0} \frac{\log\left(\frac{f(t)+f(-t)}{2}\right)}{\log|t|} = \alpha.$$

For example, the function $t \mapsto |t|^{\alpha'}$ is approximately regular with $\alpha = \alpha'$ (note that $\alpha' > 1$ since $t \mapsto |t|^{\alpha'}$ must have continuous derivative). In particular, if f is approximately α-regular, then for any ϵ and small enough t, (Lemma 4.3 in [19])

$$|t|^{\alpha+\epsilon} \leq \frac{f(t)+f(-t)}{2} \leq |t|^{\alpha-\epsilon}.$$

Hence whenever f is approximately α-regular, the function $\frac{f(t)+f(-t)}{2}$ has a shape similar to $|t|^{\alpha}$ around $t = 0$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex C^1 function with $f(0) = 0$ and $\frac{\partial f}{\partial x_i}(0) = 0$ for $1 \leq i \leq n$. Then f is called *approximately regular* (Crampon, [19]) if for any vector $v \in \mathbb{R}^n \setminus \{0\}$, there exists $\alpha(v) \in [1, \infty]$ such that

$$\lim_{t \to 0} \frac{\log\left(\frac{f(tv)+f(-tv)}{2}\right)}{\log|t|} = \alpha(v).$$

For example, the function $(x, y) \mapsto |x|^{\alpha_1} + |y|^{\alpha_2}$ is approximately regular with $\alpha(e_1) = \alpha_1$ and $\alpha(e_2) = \alpha_2$.

The next step is to extend the notion of approximate regularity the boundary of a convex set Ω of dimension m. Pick a point $p \in \partial\Omega$. We will model the shape of $\partial\Omega$ around p with a function $f : T_x\partial\Omega \cong \mathbb{R}^{m-1} \to \mathbb{R}$. In particular, for a small subset $U \subset T_p\partial\Omega$ containing p, define

$$f : U \to \mathbb{R} \quad x \mapsto \mu$$

where μ is the scalar such that $(x + \mu n_p)$ intersects $\partial\Omega$ and n_p is the normal unit vector to $\partial\Omega$ at p that points inside of Ω. In this construction, $f(0) = 0$ and all of the first partial derivatives of f vanish at 0.

As an example we will explore the approximate regularity of the boundary of $\mathbb{H}^n$. Take a ball model for $\mathbb{H}^n$ where the center of the ball is at $(0, \cdots, 1)$:

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_{n-1}^2 + (x_n - 1)^2 < 1\}.$$

The boundary of $\mathbb{H}^n$ around the origin is modeled by the function

$$(x_1, \ldots, x_{n-1}) \mapsto 1 - \sqrt{1 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}$$

and a calculation reveals that $\alpha(e_i) = 2$ for $1 \leq i \leq n-1$.

The next goal is to relate approximate regularity α to the parallel Lyapunov exponent η. However, since $\alpha(v)$ is defined for a direction $v \in T_p\partial\Omega$ and $\eta(Z)$ is defined for a vector $Z \in T_wHM$, we will need to construct a relationship between vectors in T_wHM and vectors in $T_p\partial\Omega$.

In Theorem 4.11 in [19], Crampon shows that there is a correspondence between weakly forward regular points $w = (x, [\xi])$ and approximately regular points $\varphi^\infty(w) = x^+ \in \partial\Omega$ which we will discuss now. Define

$$p_{x^+} : T_x\mathcal{H}_w(x) \to T_{x^+}\partial\Omega$$

be the projection in the direction $[xx^+]$ from the tangent space to the horosphere at $w = (x, [\xi])$ to the tangent space of $\partial\Omega$ at the point x^+ as shown in Figure 2.7.

Finally, we can restate Crampon's Theorem 4.11 in [19]:

Theorem 2.7 (Crampon [19]). *A point in $w = (x, [\xi]) \in H\Omega$ is weakly forward regular if and only if $\partial\Omega$ is approximately regular at $x^+ = \varphi^\infty(w)$. The Lyapunov*

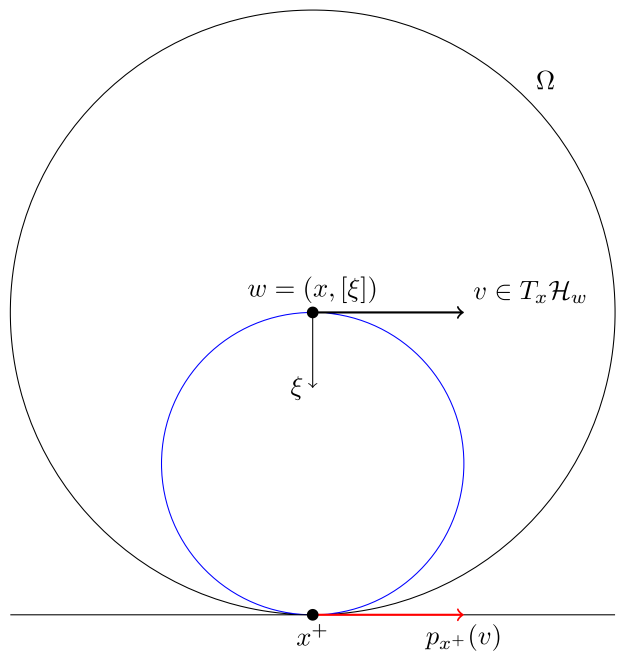

Figure 2.5: Relationship between weakly regular $w = (x, [\xi])$ and approximately regular $x^+ = \varphi^\infty(w)$

exponents are related by

$$\eta(v) = \frac{2}{\alpha(p_{x^+}(v))} - 1, \quad v \in T_x\mathcal{H}_w(x).$$

CHAPTER 3

Conics in the Boundary of $\mathrm{POS}(n)$

In this section, we explore $\mathrm{POS}(n)$, one type of symmetric cone in Hilbert geometry. The goal is to prove Theorem 1:

Theorem 1. *The symmetric convex cones* $\mathrm{POS}(n, \mathbb{K})$ *with* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *with* $n \geq 2$ *and* $\mathrm{POS}(3, \mathbb{O})$ *have a properly embedded conic through every boundary point.*

The method is that we will prove Theorem 1 by considering each case for $\mathbb{K}$ individually in the following sections.

3.1 Introduction

Let M be an $m \times m$ matrix with entries in $\mathbb{K}$ where $\mathbb{K}$ will be $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$, or the octonions $\mathbb{O}$. If $\mathbb{K} = \mathbb{R}$, then M is a *symmetric* matrix. Assume $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. For $q \in \mathbb{K}$, let $\overline{q}$ be the conjugate of q. Let M be an $m \times m$ matrix with entries in $\mathbb{K}$ and let M^* be its conjugate transpose. M is Hermitian if $M^* = M$. For $x \in \mathbb{K}^n$

and a Hermitian matrix $M = (a_{ik})$,

$$x^* M x = \sum_{i,k=1}^{n} \overline{x_i} a_{ik} x_k \tag{3.1}$$

which is in $\mathbb{R}$ since

$$\overline{\sum_{i,k=1}^{n} \overline{x_i} a_{ik} x_k} = \sum_{i,k=1}^{n} x_k \overline{a_{ik} x_i}$$

and $\overline{a_{ik}} = a_{ki}$.

Let $\mathbb{K} = \mathbb{R}$. The matrix M is called *positive definite* if for all $x \in \mathbb{K}^m \setminus \{0\}$, $x^T M x > 0$ and *positive semi-definite* if for all $x \in \mathbb{K}^m$, $x^T M x \geq 0$. Further, any positive semi-definite symmetric $m \times m$ matrix can be diagonalized into the form $\text{diag}(1, \dots, 1, 0, \dots, 0)$ by the spectral theorem which will be discussed in the next section. The number of 1's is called the *rank* and the number of zeros, $m - (\text{rank})$, is called the *co-rank*. Rank is preserved when projecting M to $\mathbb{S}^N$ by equivalently defining rank as the number of nonzero eigenvalues.

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. We can say that a Hermitian matrix M is positive definite if $x^* M x > 0$ for all $x \in \mathbb{K}^n$ and positive semi-definite if $x^* M x \geq 0$ for all $x \in \mathbb{K}^n$.

Recall that $\mathcal{C}$ is a convex cone in $\mathbb{R}^{N+1}$ and $\Omega = p(\mathcal{C})$ is the projection of $\mathcal{C}$ into $\mathbb{S}^N$ which is the set of half lines in $\mathbb{R}^{N+1}$. Throughout this section, $\mathcal{C} = \{\text{positive definite Hermitian } m \times m \text{ matrices over } \mathbb{K}\}$ and $\text{POS}(m, \mathbb{K}) = p(\mathcal{C})$. Note that if $M \in \mathcal{C}$, the set $p^{-1}(p(M) \setminus \{0\})$ is in $\mathcal{C}$. Hence techniques from linear algebra can be applied to $\mathcal{C}$ and then projected to $\text{POS}(m, \mathbb{K})$.

3.2 The Real Case

Let $\mathbb{K} = \mathbb{R}$ and let M be symmetric.

Another way to determine if M is positive definite is by the following remark which follows from Sylvester's Law.

Remark 3.1. A symmetric matrix M is positive definite if and only if the determinants of its leading minors are positive.

In order to move from the projective $\mathrm{POS}(m, \mathbb{K})$ to $\mathcal{C}$, we will sometimes take an affine chart of $\mathrm{POS}(m, \mathbb{K})$ by setting $\mathrm{Tr}(M) = 1$. The boundary of $\mathrm{POS}(m, \mathbb{K})$ is

$$\partial\mathrm{POS}(m, \mathbb{K}) = p(\{\text{positive } \textit{semi-definite} \text{ matrices over } \mathbb{K}\}) \setminus (\mathrm{POS}(m, \mathbb{K})).$$

Let $X \in \mathrm{POS}(m, \mathbb{K})$ and $g \in \mathrm{GL}(m, \mathbb{K})$. Then g acts on X by gXg^T. This action is transitive on all of $\mathrm{POS}(m, \mathbb{K})$ and is transitive on the sets $\{M \in \partial\mathrm{POS}(m, \mathbb{K}) : M \text{ has rank } r\}$ for $0 \leq r \leq m$.

The positive definite symmetric matrices have properly embedded conics through every point on the boundary:

Proposition 3.2. *Let $x \in \partial\mathrm{POS}(m, \mathbb{R})$. Then there is a conic $C_x \subset \partial\mathrm{POS}(m, \mathbb{R})$ such that $\mathrm{Con}(C_x) \setminus C_x$ is contained in the interior of $\mathrm{POS}(m, \mathbb{R})$.*

We will need some facts about diagonalizing symmetric matrices to prove this proposition.

Theorem 3.1 (Spectral Theorem, Theorem 4.1.5 in [32])**.** *Let A be an $n \times n$ matrix* *over $\mathbb{R}$. Then A is symmetric if and only if there is an orthogonal $n \times n$ matrix P and a real diagonal matrix D such that $A = PDP^T$.*

Further, suppose the matrix D from Theorem 3.1 is of the form

$$D = \operatorname{diag}(a_1, \dots, a_k, 0, \dots, 0).$$

Let $g = \operatorname{diag}(\frac{1}{\sqrt{|a_1|}}, \dots, \frac{1}{\sqrt{|a_k|}}, 0, \dots, 0)$. Then

$$gDg^T = \operatorname{diag}(\pm 1, \dots, \pm 1, 0 \dots, 0)$$

where the ith coordinate is $+1$ if $a_i > 0$, and -1 if $a_i < 0$.

Proof. Assume without loss of generality (see Theorem 3.1 and the discussion that follows) that $x = \operatorname{diag}(\overbrace{1, \dots, 1}^{p \text{ times}}, \overbrace{0, \dots, 0}^{q \text{ times}})$. Let $x_d = \operatorname{diag}(1, \overbrace{0, \dots, 0}^{q \text{ times}})$. Any properly embedded conic C_{x_d} passing through $x_d \in \partial\mathrm{POS}(q+1, \mathbb{R})$ can be mapped to a properly embedded conic passing through x with the following block diagonals:

$$\left(\begin{array}{c|c} I_{p-1} & 0 \\ \hline 0 & C_{x_d} \end{array}\right)$$

Let $y_d = (0, \overbrace{1, \dots, 1}^{q \text{ times}})$. If $q = 1$, then the orbit

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} b^2 & bd \\ bd & d^2 \end{pmatrix} \tag{3.2}$$

is parametrized by $(b, d) \in \mathbb{R}^2 \setminus \{0\}$ with $b^2 + d^2 = 1$ in the affine chart Trace $= 1$. Setting $b = 1, d = 0$, this conic passes through x_d. Note that $x_d \in \partial\mathrm{POS}(2, \mathbb{R})$

and has dimension 1. Since $\mathrm{POS}(2,\mathbb{R})$ has dimension 2, the orbit in Equation 3.2 is exactly the boundary of $\mathrm{POS}(2,\mathbb{R})$. Hence C_{x_d} is properly embedded.

Now let $q > 1$. The following orbit will parametrize a conic:

(3.3)
$$\begin{pmatrix} a & c & \cdots & c \\ & d & & \\ & & \ddots & \\ & & & d \end{pmatrix} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a & & & \\ c & d & & \\ \vdots & & \ddots & \\ c & & & d \end{pmatrix} = \begin{pmatrix} (n-1)c^2 & cd & \cdots & cd \\ cd & d^2 & & \\ \vdots & & \ddots & \\ cd & & & d^2 \end{pmatrix}$$

which we will call $\mathrm{Orb}(y_d)$.

Taking the affine chart, Trace $= 1$, we get the equation $c^2 + d^2 = \frac{1}{n-1}$, $d \neq 0$. This is a conic with the point x_d missing. Letting $d \to 0$, the orbit converges to the point x_d. Hence $\overline{\mathrm{Orb}(y_d)}$ is a conic passing through x_d on the boundary of $\mathrm{POS}(q+1,\mathbb{R})$.

It remains to show that $\overline{\mathrm{Orb}(y_d)}$ is properly embedded. Let

$$Y = \begin{pmatrix} \alpha & \beta & \cdots & \beta \\ \beta & \gamma & & \\ \vdots & & \ddots & \\ \beta & & & \gamma \end{pmatrix} \cap \mathrm{POS}(m,\mathbb{R});$$

this is nonempty by choosing α, β, γ so that the determinants of the leading minors of Y are positive. The set Y has dimension 2 and is preserved by the action in Equation 3.3. The set $\overline{\mathrm{Orb}(y_d)}$ is in the boundary of Y and has dimension 1. Hence $\partial Y = \overline{\mathrm{Orb}(y_d)}$.

$\square$

3.3 The Complex Case

Recall that for $\mathbb{K} = \mathbb{C}$, the action of $g \in G = \mathrm{GL}(m, \mathbb{K})$ on $M \in \mathrm{POS}(m, \mathbb{C})$ is gMg^*.

We will need the spectral theorem for $n \times n$ complex Hermitian matrices:

Theorem 3.2 (Spectral Theorem, Theorem 4.1.5 in [32])**.** *Let A be an $n \times n$ matrix over $\mathbb{C}$. Then A is Hermitian if and only if there is a unitary $n \times n$ matrix U and a real diagonal matrix D such that $A = UDU^*$.*

This will allow us to prove the following proposition directly.

Proposition 3.3. *Let $x \in \partial\mathrm{POS}(m, \mathbb{C})$. Then there is a properly embedded conic $C_x \subset \partial\mathrm{POS}(m, \mathbb{C})$.*

Proof. Given some point on $x \in \partial\mathrm{POS}(m, \mathbb{C})$, you can diagonalise x it with some unitary matrix U to get a diagonal matrix D and the rest follows from Proposition 3.2 □

3.4 The Quaternion Case

Let $\mathbb{K} = \mathbb{H}$ and $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k = q \in \mathbb{H}$. The conjugate of q is $\overline{q} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$. M is called normal if $M^*M = MM^*$. Note that Hermitian $\Rightarrow$ normal.

Let $\mathrm{POS}(m, \mathbb{H})$ be the set of all Hermitian $m \times m$ matrices M over $\mathbb{H}$ such that $x^*Mx > 0$ for all $x \in \mathbb{H}^n$. We will apply a quaternionic version of the spectral theorem to show that there is a properly embedded conic through every point on the boundary of $\mathrm{POS}(m, \mathbb{H})$.

Theorem 3.3 ([24] (Theorem 3.3 and Proposition 3.8))**.** *If M is an $m \times m$ normal matirx with entries in $\mathbb{H}$, then there exists matrices D and U with U unitary, D diagonal, and $U^*MU = D$. Further if M is Hermitian, all of its right eigenvalues are real and hence D has real entries.*

Proposition 3.4. *Let $x \in \partial\mathrm{POS}(m, \mathbb{H})$. Then there is a properly embedded conic $C_x \subset \partial\mathrm{POS}(m, \mathbb{H})$.*

Proof. Apply Theorem 3.3 to any point in the boundary of $\mathrm{POS}(m, \mathbb{H})$ and then follow the real case. □

3.5 The Octonionic Case

In Vinberg's classification of homogoneous convex cones (see Section 1.6), the only example of $\mathrm{POS}(m, \mathbb{K})$ with $\mathbb{K} = \mathbb{O}$ is $\mathrm{POS}(3, \mathbb{O})$. Our exploration of this space, or rather its associated convex cone in $\mathbb{R}^{27}$, is covered in more detail by Baez in [1] and our notation is primarily from this source. A 3×3 octonionic Hermitian matrix is of the form

$$\begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$ and $x, y, x \in \mathbb{O}$. Its automorphism group is $e_{6(-26)}$ with maximal compact subgroup F_4 which can be used to diagonalize any element.

Let V be a vector space over $\mathbb{R}$ of dimension n with inner product $\langle \cdot, \cdot \rangle$ and associated norm $|\cdot|$ such that the orthogonal group $\mathrm{O}(p,q)$ acts on V. The Clifford algebra $C(p,q)$, or $C(V)$, is the tensor algebra $\otimes V$ quotiented out by elements of the form $x \otimes x + \langle x, x \rangle$ with $x \in V$. This is equivalent to quotienting out by

$$x \otimes y + y \otimes x + 2 \langle x, y \rangle .$$

The Clifford algebras contain the groups Spin and Pin which we will construct now. Let $C^*(V)$ be the group of invertible elements in $C(V)$. The group $\mathrm{Pin}(V)$ is generated by the unit vectors in V, i.e.

$\mathrm{Pin}(V) = \{a \in C^*(V) : a = u_1 \cdots u_r$ with nonnegative integer r, $u_i \in V$ and $|u_i| = 1.\}$.

The group $\mathrm{Spin}(V)$ is generated by the product of pairs of unit vectors in V, i.e.

$\mathrm{Spin}(V) = \{a \in C^*(V) : a = u_1 \cdots u_{2r}$ with nonnegative integer r, $u_i \in V$ and $|u_i| = 1\}$.

Next, return to the 3×3 octonionic Hermitian matrices, which we will call $\mathfrak{h}_3(\mathbb{O})$; they are of the form

$$\begin{pmatrix} \alpha & z^* & y^* \\ z & \beta & x \\ y & x^* & \gamma \end{pmatrix}. \tag{3.4}$$

We will use the decomposition

$$\mathfrak{h}_3(\mathbb{O}) \ni X = \begin{pmatrix} \alpha & \varphi_1 & \varphi_2 \\ \varphi_1^* & a+\beta & \gamma^* \\ \varphi_2^* & \gamma & -a+\beta \end{pmatrix}$$

so that $\mathfrak{h}_3(\mathbb{O}) \cong \mathbb{R}^2 \oplus V_9 \oplus (S_8^+ \oplus S_8^-)$ where $(\alpha, \beta) \in \mathbb{R}^2$, $h = \begin{pmatrix} a & \gamma^* \\ \gamma & -a \end{pmatrix} \in V_9 \cong \mathbb{R}^9$ (traceless elements of $\mathfrak{h}_2(\mathbb{O})$) and $(\varphi_1, \varphi_2) = \varphi \in \mathbb{O}^2 \cong S_8^+ \oplus S_8^- \cong \mathbb{R}^{16}$. The goal is to find an element of the orbit of X under F_4 with all real entries so that we can apply the spectral theorem from the $\mathbb{R}$ case.

First, $\mathrm{Spin}(9,0)$ is a double cover of $\mathrm{SO}(9)$, so $\mathrm{Spin}(9,0)$ acts transitively on the unit sphere in $\mathbb{R}^9 \cong V_9$. Hence you can map Equation 3.4 to an element where the entry x is real. Call this new matrix Y. It remains to make z and y real while fixing x. The group that fixes a vector in V_9 is $\mathrm{Spin}(8,0)$. We will use the following result from Theorem 14.69 in [29]:

Theorem 3.4 (Theorem 14.69 in [29])**.** *Consider the spin representation $\rho_+ \oplus \rho_-$ of $\mathrm{Spin}(7,0)$ on $S_8^+ \oplus S_8^- \cong \mathbb{R}^8 \oplus \mathbb{R}^8$. The orbit through the point $(a,b) \in S_+ \oplus S_-$ is*

$$\{(x,y) \in S_+ \oplus S_- | |x| = a \quad \textit{and} \quad |y| = b\}.$$

Let $(a,b) = (\varphi_1, \varphi_2)$. Since $\mathrm{Spin}(7,0) \subset \mathrm{Spin}(8,0)$, we let $\mathrm{Spin}(7,0)$ act on (φ_1, φ_2). By the preceding theorem, the orbit of (φ_1, φ_2) contains

$$(\underbrace{(|\varphi_1|, 0, \ldots, 0)}_{x}, \underbrace{(|\varphi_2|, 0, \ldots, 0)}_{y}),$$

so that when we identify $S_8^+ \oplus S_8^-$ with $\mathbb{O} \oplus \mathbb{O}$, each coordinate is real. Thus Y (and hence X) has been conjugated to a matrix in $\mathrm{POS}(3, \mathbb{R})$ which we can diagonalize with the spectral theorem over $\mathbb{R}$.

The above discussion provides a proof for the following.

Proposition 3.5. *Let* $x \in \partial\mathrm{POS}(3, \mathbb{O})$*. Then there is a properly embedded conic* $C_x \subset \partial\mathrm{POS}(3, \mathbb{O})$*.*

CHAPTER 4

Conics in the Boundary of Rank One Hilbert Geometries

Let Ω be a convex divisible set with dividing group $\Gamma \subset \mathrm{SL}(m+1,\mathbb{R})$. The set Ω has *higher rank* if for every $a, b \in \Omega$ the line (a,b) is contained in a properly embedded simplex. If Ω does not have higher rank, then Ω has *rank one*. Let $\Lambda_\Gamma^\Omega = \overline{\{x_g^+ : g \in \Gamma\}}$. The goal of this section is to prove the following theorem.

Theorem 2. *Let Γ be a discrete subgroup of $\mathrm{SL}(m+1,\mathbb{R})$ which divides a properly convex set Ω that is irreducible. Assume that for all $x \in \Lambda_\Gamma^\Omega$ there is a properly embedded conic $C_x \subset \partial\Omega$. If Ω has rank one, then Ω is projectively equivalent to $\mathbb{H}^n$. If Ω has higher rank, then Ω is projectively equivalent to $\mathrm{POS}(n,\mathbb{K})$ with $n > 2$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathrm{POS}(3,\mathbb{O})$ where $\mathbb{O}$ is the octonions.*

We will first prove the following theorem:

Theorem 4.1. *Let Γ be a discrete subgroup of $\mathrm{SL}(m+1,\mathbb{R})$ which divides a properly convex irreducible set Ω. Assume that for all $x \in \Lambda_\Gamma^\Omega$ there is a properly embedded conic $C_x \subset \partial\Omega$. If there exists a $g \in \Gamma$ whose axis $\gamma_g \subset \Omega$, then Ω is projectively* *equivalent to* $\mathbb{H}^n$.

In order to prove Theorem 4.1, we need to collect some basic facts in 4.1 about Ω and Γ. This will allow us to show that for each $g \in \Gamma_{\text{lox}} = \{g \in \Gamma | g \text{ is loxodromic}\}$, Ω contains an embedded $\mathbb{H}^2$ which is fixed by g. Then, in Section 4.2, we introduce flag manifolds to show that if we have one geodesic in Ω with endpoints in Λ_Γ^Ω, we get a Zariski dense set of $g \in \Gamma$, such that $(x_g^-, x_g^+) = \gamma_g \subset \Omega$. In particular, each γ_g is *not* contained in $\partial\Omega$.

An important step of the proof is an application of the following theorem of Benoist:

Theorem 4.2 (Benoist, [5])**.** *Let Ω be a properly convex irreducible set divisible by the discrete group Γ. If Ω is not symmetric, then Γ is Zariski dense in* $\mathrm{SL}(m+1, \mathbb{R})$.

Our goal will be to show that if $\Omega \subset \mathbb{S}^m$ and Γ have all of the properties listed in Theorem 2, then Γ cannot be Zariski dense in $\mathrm{SL}(m+1, \mathbb{R})$ and hence Ω must be symmetric.

4.1 Embedded $\mathbb{H}^2$

Lemma 4.3 (See discussions in [6])**.** *Let Ω be a properly convex set divided by a discrete group Γ. Then for all $g \in \Gamma - \{1\}$, there exist $x_g^+, x_g^- \in \partial\Omega$ which are fixed by the action of g. In particular, g fixes the geodesic between x_g^+ and x_g^-.*

The idea behind this lemma is that all $g \in \Gamma - \{1\}$ are *biproximal*. This means that for the eigenvalues $\{\lambda_i\}_{i=1}^{m+1}$ (with corresponding eigenvectors $\{p_i\}_{i=1}^{m+1}$) of g with

$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{m+1}|$, we have that $|\lambda_1| > |\lambda_2|$ and $|\lambda_{m+1}| < |\lambda_m|$. The eigenvectors p_i are points in $\mathbb{S}^m$ and from Lemma 4.3, $p_1, p_{m+1} \in \partial\Omega$. The action of g on Ω moves points in Ω towards $p_1 = x_g^+$ and the action of g^{-1} on Ω moves points towards $p_{m+1} = x_g^-$.

Now, fix $g \in \Gamma$ and assume that $\gamma_g = (x_g^-, x_g^+) \subset \Omega$.

Lemma 4.4. *Let $g \in \Gamma - \{1\}$ with $x_g^+, x_g^- \in \partial\Omega$ as in Lemma 4.3 and assume that $\gamma_g = (x_g^+, x_g^-) \subset \Omega$. Assume that there exists a properly embedded conic $C_{x_g^-} \subset \partial\Omega$ passing through x_g^-. Then there exists a properly embedded conic $C_{(x_g^+, x_g^-)} \subset \partial\Omega$ that passes through the points x_g^+ and x_g^-. Further, this conic must be the boundary of an embedded $\mathbb{H}^2$ and contain the line γ_g.*

Proof. Observe that g fixes γ_g, x_g^+ and x_g^- and moves all other elements of $\overline{\Omega}$ towards x_g^+. Hence a candidate for $C_{(x_g^+, x_g^-)}$ is $\lim_{k\to\infty} g^k C_{x_g^-} = C$. We must have that $x_g^+ \in C$ because x_g^+ is the sink of g. It remains to verify that C does not collapse into a segment in the boundary $\partial\Omega$. The convex hull of C, $\mathrm{Con}(C)$ contains $\gamma_g \setminus \{x_g^+, x_g^-\}$ which is contained in the interior of Ω. So $\mathrm{Con}(C)$ is not in $\partial\Omega$. $\square$

4.2 Axes in Ω

Theorem 4.5 (Theorem 6.3 in [10])**.** *Let G be a connected algebraic semisimple real Lie group and let Γ be a Zariski dense subsemigroup of G. Then the set Γ_{lox} of loxodromic elements of Γ is also Zariski dense in G.*

From the hypotheses in Theorem 4.1, we will assume that there is an isometry $g \in \Gamma$ with axis $(x^+, x^-) \subset \Omega$. By definition, $x^+, x^- \in \Lambda_\Gamma^\Omega$. Using this line (x^+, x^-), we will find a Zariski dense subset $\Gamma' \subset \Gamma_{\text{lox}} \subset \Gamma$ where each $g \in \Gamma'$ has line (x_g^-, x_g^+) inside of Ω. Note that Lemma 4.3 and convexity of Ω already guarantee that $(x_g^-, x_g^+) \subset \overline{\Omega}$.

Let z be a *full flag* of $\mathbb{R}^{m+1}$, which is a nested collected of subspaces $\{0\} \subset V_1 \subset \cdots \subset V_m \subset \mathbb{R}^{m+1}$ where $\dim(V_i) = i$. The *full flag manifold* X is the set of all full flags z. Following Guivarc'h in [27], the eigenvectors $\{p_i\}_{i=1}^{m+1}$ of g map to X by

$$\{p_i\}_{i=1}^{m+1} \mapsto \left(\{0\} \subset \mathbb{R} \cdot p_1 \subset \cdots \subset \sum_{i=1}^{m} \mathbb{R} \cdot p_i \subset \mathbb{R}^{m+1}\right) = z_g^+.$$

The set Λ_Γ^X is a subset of X defined by $\Lambda_\Gamma^X = \overline{\{z_g^+ | g \in \Gamma_{\text{lox}}\}}$. Note that $z_g^- = z_{g^{-1}}^+$. Distances in X are measured as the supremum of distances between the subspaces of corresponding dimension (see [2] for more details). The following lemma will allow us to start to restrict Γ_{lox} to a Zariski dense subset, which will be the goal of Lemma 4.8.

Lemma 4.6 (Lemma 2.6 in [3])**.** *Let $G = \mathrm{SL}(m+1, \mathbb{R})$, X the flag manifold of $\mathbb{R}^{m+1}$ and let Γ be a Zariski dense subsemigroup in G. Then for all $\epsilon > 0$, $z^+, z^- \in \Lambda_\Gamma^X$, the set*

$$\{g \in \Gamma_{\text{lox}}, d(z_g^+, z^+) \leq \epsilon, d(z_g^-, z^-) \leq \epsilon\}$$

is Zariski dense in Γ.

Next, we apply this result to the boundary of Ω by projection.

Lemma 4.7. *Let $G = \mathrm{SL}(m+1, \mathbb{R})$, Ω a convex set and Γ a Zariski dense subgroup of G. Let $\Lambda_\Gamma^\Omega = \overline{\{x_g^+, g \in \Gamma\}}$ and let $x^-, x^+ \in \Lambda_\Gamma^\Omega$. Let Γ_{lox} be the set of loxodromic elements in Γ. Then the set*

$$\{g \in \Gamma_{\mathrm{lox}}, d(x_g^+, x^+) \leq \epsilon, d(x_g^-, x^-) \leq \epsilon\}$$

is Zariski dense in Γ.

Proof. Let z_g^+ be the flag corresponding to g. Then z_g^+ is the flag $\{0\} \subset V_1 \subset \cdots \subset V_{m+1} = \mathbb{R}^{m+1}$ invariant under g defined as follows. The subspace V_1 is identified with the projective point x_g^+ and the other V_i are invariant under g. Similarly, z_g^- is the flag invariant under g^{-1}. Then we can apply Lemma 4.6.

Now, let q be the projection from the full flag to V_1 which is a one dimensional subspace. Then $q(z_g^+) = x_g^+$ and $q(z_g^-) = x_g^-$. Note also that $d_X \geq d_{\mathbb{S}^m}$ since d_X is the supremum of distances in each subspace in the flag. Then by choosing some $z^+ = q^{-1}(x^+)$ and $z^- = q^{-1}(x^-)$, we get Lemma 4.7.

$\square$

Finally, we almost have the conclusion to this section:

Lemma 4.8. *With the assumptions from Lemma 4.7, there exists an $\epsilon > 0$ so that the set*

$$\Gamma' = \{g \in \Gamma_{\mathrm{lox}}, d(x_g^+, x^+) \leq \epsilon, d(x_g^-, x^-) \leq \epsilon, (x_g^-, x_g^+) \subset \Omega\}$$

is Zariski dense in Γ.

Proof. Since $(x^+, x^-) \subset \Omega$, we can choose an ϵ so that all points (x_g^+, x_g^-) in both Λ_Γ^Ω and such that $d(x_g^+, x^+) \leq \epsilon$ and $d(x_g^-, x^-) \leq \epsilon$ have the property that $(x_g^+, x_g^-) \subset \Omega$ i.e. (x_g^+, x_g^-) is not contained in the boundary of Ω. □

4.3 Technical Lemma

The following lemma is used in the proof of Theorem 4.1.

Lemma 4.9. *The set of all loxodromic $(m+1) \times (m+1)$ matrices that have eigenvalues $\lambda_i, \lambda_j, \lambda_k$ with $i \neq j$ that satisfy the equation*

$$\prod_k \prod_{i \neq j} \left(\lambda_k^4 - \lambda_i^2 \lambda_j^2\right) = 0 \tag{4.1}$$

is an algebraic set.

In order to prove this lemma, we will introduce the *tensor product* and the *exterior product* of finite dimensional vector spaces V and W over $\mathbb{R}$. Let $v_i \in V$, $w_i \in W$ and $\mu_i \in \mathbb{R}$ and let H be the subspace of $V \times W$ generated by elements of the form

$$(v_i+v_j, w_k)-(v_i, w_k)-(v_j, w_k), (v_i, w_j+w_k)-(v_i, w_j)-(v_i, w_k), \text{ and } (\mu_i v_j, w_k)-(v_j, \mu_i w_k).$$

Suppose V has basis $\{u_i\}$ for $1 \leq i \leq n$ and W has basis $\{d_i\}$ for $1 \leq i \leq m$. The *tensor product* $V \otimes W$ is defined to be $(V \times W)/H$ and has basis $\{u_i \otimes d_j\}$. If g and h are linear maps on V and W respectively, then we get a map $g \otimes h$ on $V \otimes W$ defined by

$$(g \otimes h)(v_i \otimes w_j) = (g \cdot v_i) \otimes (h \cdot w_j).$$

The map $g \otimes h$ can be written as an $nm \times nm$ matrix called the *Kronecker product* using the entries of the matrices associated to g and h. We can gain some understanding of the behavior of $g \otimes h$ with the following theorem.

Theorem 4.10 (Theorem 4.2.12 in [31])**.** *Let $A \in \mathrm{Mat}(n)$ and $B \in \mathrm{Mat}(m)$. Let λ be an eigenvalue of A with corresponding eigenvector x and let μ be an eigenvalue of B with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$ and any eigenvalue of $A \otimes B$ arises as such a product of eigenvalues of A and B.*

In the proof of Lemma 4.9 we will be using Theorem 4.10 to construct a matrix whose eigenvalues are products of the eigenvalues of g. We can also construct a matrix whose eigenvalues are pairwise sums of the eigenvalues of g;

Theorem 4.11 (Theorem 4.4.5 in [31])**.** *Let $A \in \mathrm{Mat}(n)$ and $B \in \mathrm{Mat}(m)$. Let λ be an eigenvalue of A with corresponding eigenvector x and let μ be an eigenvalue of B with corresponding eigenvector y. Then $\lambda + \mu$ is an eigenvalue of the sum $(I_n \otimes B) + (A \otimes I_m)$ and $x \otimes y$ is a corresponding eigenvector. Every eigenvector of the sum $(I_n \otimes B) + (A \otimes I_m)$ arises as a sum of the eigenvectors of A and B.*

Consider $V \otimes V$, the tensor product of V with itself. Let J be the subset of $V \otimes V$ generated by elements of the form $v \otimes v$. Define the *exterior product* of V, denoted $\wedge^2 V$, to be $(V \otimes V)/J$. Note that in the exterior product, $v_i \wedge v_j = -v_j \wedge v_i$. Since $\wedge^2(V)$ is a vector space, we can use it as part of a tensor product. In the proof of Lemma 4.9, we will construct the space $V \otimes \wedge^2(V)$.

Proof of Lemma 4.9. Let $V = \mathbb{R}^{m+1}$. The map $g \otimes g$ is a map on $W = V \otimes V$. Let W have the basis B where

$$B = \{v_i \otimes v_j \text{ such that } v_i,\, v_j \text{ are eigenvectors of } g.\}$$

Note that W has dimension $(m+1)^2$. From Theorem 4.10, the linear map $g \otimes g$ has eigenvalues $\lambda_i \lambda_j$ with associated eigenvectors $v_i \otimes v_j$. Let U be the subspace of W with basis $C = \{m \otimes m, m \in \mathbb{R}^{m+1}\}$. Then $g \otimes g$ induces a linear map on W/U which we will call $g \wedge g$.

Note that for $i \neq j$,

$$(g \wedge g)(v_i \otimes v_j + U) = \lambda_i \lambda_j (v_i \otimes v_j) + U,$$

so $\lambda_i \lambda_j$ with $i \neq j$ are eigenvalues of $g \wedge g$. Since W/U has dimension $\binom{m+1}{2}$, these are the only eigenvalues of $g \wedge g$.

Note that if A is an $n \times n$ matrix with eigenvalues μ_i, then the matrix $(-A)$ has eigenvalues $-\mu_i$. Let $g \wedge g \in \text{Mat}\left(\binom{m+1}{2}\right)$. Then by Theorem 4.11 the map

$$(I_{m+1} \otimes -(g \wedge g)) + (g^2 \otimes (I_{\binom{m+1}{2}})$$

has eigenvalues $\lambda_k^2 - \lambda_i \lambda_j$ for $i \neq j$. Replacing g with g^2, we get that the following algebraic condition

$$\det((I_{m+1} \otimes -(g^2 \wedge g^2)) + (g^4 \otimes (I_{\binom{m+1}{2}})) = 0.$$

is equivalent to Equation 4.1.

□

4.4 Proof of Theorem 4.1 and 2

Here we combine the previous sections with background from Section 2.5 and Section 2.3 to finish proving Theorem 4.1.

Proof of Theorem 4.1. Assume for contradiction that Γ is Zariski dense in $\mathrm{SL}(m+1,\mathbb{R})$. Once a contradition is obtained, we will apply Theorem 4.2; this will prove that Ω is a symmetric convex cone.

Since Γ is Zariski dense in $\mathrm{SL}(m+1,\mathbb{R})$, the set $\Gamma' \subset \Gamma$ (see Lemma 4.8 for information about Γ') must also be Zariski dense in $\mathrm{SL}(m+1,\mathbb{R})$ (Lemma 4.8). Choose $g \in \Gamma'$. By Lemma 4.4, there exists a conic C in $\partial\Omega$ whose interior contains the line (x_g^-, x_g^+). This conic C is the boundary of an embedded $\mathbb{H}^2 \subset \Omega$. Thus, for some k, we must have that $\mu_k(g) = 1$, i.e.

$$1 = 2\frac{\log|\lambda_1| - \log|\lambda_k|}{\log|\lambda_1| - \log|\lambda_{m+1}|}. \tag{4.2}$$

Repeating this process for all of $g \in \Gamma'$, we see that all $g \in \Gamma'$ must satisfy Equation 4.2.

Equation 4.2 can be rearranged to $|\lambda_k|^2 - |\lambda_{m+1}||\lambda_1| = 0$. Since g is loxodromic, each $\lambda_i \in \mathbb{R}$ so any g satisfying Equation 4.2 must also satisfy

$$\lambda_k^4 - \lambda_1^2\lambda_{m+1}^2 = 0 \tag{4.3}$$

Let

$$c(g) = \prod_k \prod_{i \neq j} (\lambda_k^4 - \lambda_i^2 \lambda_j^2) \tag{4.4}$$

for $1 \leq i, j, k \leq m+1$ and $i \neq j$. The set of $g \in \mathrm{SL}(m+1, \mathbb{R})$ satisfying Equation 4.4 cannot be all of $\mathrm{SL}(m+1, \mathbb{R})$; let p_i be the ith prime starting with $p_1 = 2$ and construct

$$g = \left(\frac{1}{p_1 \cdots p_{m+1}}\right) \mathrm{diag}(p_1, p_2, \ldots, p_{m+1}).$$

Then for g to satisfy Equation 4.4, there must be primes p_i, p_j, p_k with $i \neq j$ and $p_k^2 = p_i p_j$, which is not possible. By Lemma 4.9, Equation 4.4 is algebraic. Since all of $g \in \Gamma'$ must satisfy Equation 4.4, Γ' cannot be Zariski dense in $\mathrm{SL}(m+1, \mathbb{R})$ and hence by Theorem 4.2, Ω must be a symmetric convex cone.

□

Two points $c, d \in \partial\Omega$ are said to be part of a *half triangle* if there exists a point $f \in \partial\Omega$ such that $(c, f) \subset \partial\Omega$ and $(d, f) \subset \partial\Omega$. An element $g \in \Gamma$ is called a *rank one isometry* if g is biproximal and x_g^+, x_g^- are not contained contained in any half triangle.

The proof of Theorem 2 will require the following result by Zimmer ([44]):

Theorem 4.12 ([44] Theorem 1.4)**.** *Suppose that $\Omega \subset \mathbb{S}^m$ is an irreducible properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a discrete group which divides Ω. Then the following are equivalent:*

- *Ω is symmetric with real rank at least two,*

- Ω *has higher rank*
- $(x_g^+, x_g^-) \subset \partial\Omega$ *for every biproximal element* $g \in \Gamma$.

And a corollary of Theorem 4.12:

Corollary 4.13. *With* Ω *defined as in Theorem 4.12, the following are equivalent:*

- Ω *does not have higher rank*
- Γ *contains a rank one isometry.*

Proof of Theorem 2. We know from Corollary 4.13 that if Ω does not have higher rank, Γ contains a rank one isometry. Then by Theorem 4.12, there exists an element $g \in \Gamma$ such that $(x_g^+, x_g^-) \subset \Omega$. Then apply Theorem 4.1. If Ω has higher rank, Theorem 4.12 implies that Ω is symmetric with real rank at least 2, which makes Ω one of $\mathrm{POS}(n, \mathbb{K})$ for $n > 2$ and $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathrm{POS}(3, \mathbb{O})$. Note that by Theorem 1, such spaces have properly embedded conics through every boundary point. $\square$

CHAPTER 5

Strictly Convex Hilbert Geometries

In this section, the main focus will be on strictly convex Ω and C^2 curves in the boundary of Ω and a major tool here will be α:

$$\alpha(v) = \lim_{t \to 0} \frac{\log\left(\frac{f(tv)+f(-tv)}{2}\right)}{\log|t|}$$

where$f$ is a function whose graph models the shape of the boundary of Ω at $x \in \partial\Omega$ (see Section 2.5). Recall that α indicates how much the graph of f at x looks like the graph of $|t|^\alpha$ at $t = 0$. Define $g(t) = f(tv)$. We will see that when $g(t)$ is C^2 and $g''(t) > 0$, then $\alpha = 2$.

For loxodromic elements g, the corresponding α at $x_g^+ \in \partial\Omega$ are given by

$$\alpha(v_i) = \frac{\log|\lambda_1| - \log|\lambda_i|}{\log|\lambda_1| - \log|\lambda_{m+1}|} \quad i = 2, \dots, m$$

so the possible values α can take depend on the projection of Γ into the cone L_Γ (See Section 2.3.4). In this first section, we describe this relationship for when $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$.

In the second section, we prove the theorem:

Theorem 3. *Let Γ be a discrete subgroup of of* $\mathrm{SL}(m+1,\mathbb{R})$ *which divides a strictly convex irreducible set Ω. Assume that Γ contains a rank one isometry. If for all $x \in \Lambda_\Gamma^\Omega$, there exists a C^2 curve through x with positive second derivative, then Ω is projectively equivalent to $\mathbb{H}^n$.*

5.1 Bounds on α

For a loxodromic $g \in \mathrm{SL}(3,\mathbb{R})$, its Jordan projection is given by

$$g \mapsto (\log|\lambda_1|, \log|\lambda_2|, \log|\lambda_3|)$$

where the λ_i are the eigenvalues of g with $|\lambda_1| > |\lambda_2| > |\lambda_3|$. We can view these coordinates as $(\log|\lambda_1|, \log|\lambda_2|, \log|\lambda_3|) = (x, y, z)$ with $x > y > z$ and $x+y+z = 0$. The quantity α is given by

$$\alpha = \frac{\log|\lambda_1| - \log|\lambda_3|}{\log|\lambda_1| - \log|\lambda_2|} = \frac{x-z}{x-y}.$$

The goal is to find bounds for α for $g \in \Gamma \subset \mathrm{SL}(3,\mathbb{R})$ in terms of the limit cone L_Γ.

Since (x, y, z) lies in the plane $x+y+z = 0$, it is more convenient to use two coordinates instead of three coordinates. Additionally, the points (x, y, z) and $(\mu x, \mu y, \mu z)$ with $\mu \in \mathbb{R}$ produce the same α, so a further dimension can be removed.

Note that for fixed x and z, $y \to x$ means $\alpha \to \infty$ and $y \to z$ means $\alpha \to 1$. The boundary of the Weyl chamber contained in the plane $x + y + z = 0$ is two lines:

- one where $y = z$ which has boundary vector $\frac{1}{\sqrt{6}}\langle 2, -1, -1\rangle$ and
- another where $x = y$ which has boundary vector $\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle$.

Making these two vectors part of a circle of radius 1, the circle is parametrized by

$$r(\theta) = \begin{pmatrix} \frac{2}{\sqrt{6}}\cos\theta \\ -\frac{1}{\sqrt{6}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta \\ -\frac{1}{\sqrt{6}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta \end{pmatrix}$$

which you can substitute into the equation for α to get

$$\alpha(\theta) = \frac{\sqrt{3}\cos\theta + \sin\theta}{\sqrt{3}\cos\theta - \sin\theta}.$$

The function $\alpha(\theta)$ has the property that $\alpha(0) = 1$, $\alpha(\theta) \to \infty$ as $\theta \to \frac{\pi}{3}$ and α is always increasing on $0 < \theta < \frac{\pi}{3}$. Hence if you know the values of θ that are possible for the Jordan projection of $g \in \Gamma$, you can get bounds for α.

Let the limit cone L_Γ be such that

$$\theta_l = \inf_{r(\theta)\in L_\Gamma} \theta \qquad \text{and} \qquad \theta_u = \sup_{r(\theta)\in L_\Gamma} \theta.$$

Then $\alpha_{\min} = \alpha(\theta_1)$ and $\alpha_{\max} = \alpha(\theta_2)$.

5.2 Smooth Curves in the Boundary

The purpose of this section is to prove the following theorem:

Theorem 3. *Let Γ be a discrete subgroup of of $\mathrm{SL}(m+1,\mathbb{R})$ which divides a strictly convex irreducible set Ω. If for all $x \in \Lambda_\Gamma^\Omega = \overline{\{x_g^+ : g \in \Gamma\}}$, there exists a C^2 curve through x with positive second derivative, then Ω is projectively equivalent to $\mathbb{H}^n$.*

We will first prove the following lemma:

Lemma 5.1. *Let* Γ *be a discrete subgroup of* $\mathrm{SL}(m+1,\mathbb{R})$ *which divides a strictly convex set* Ω. *Assume that there exists an element* $g \in \Gamma$ *such that the axis of* g, *denoted* γ_g, *is not contained in the boundary of* Γ. *If for all* $x \in \Lambda_\Gamma^\Omega$ *there exists a direction* $v \in T_x\Omega$ *for which* $\alpha(v) = 2$, *then* Ω *is projectively equivalent to the symmetric convex cone* $\mathbb{H}^n$.

Proof. From Equation 2.9 in Section 2.5, we have that

$$\alpha(v_i) = \frac{\log|\lambda_1| - \log|\lambda_{m+1}|}{\log|\lambda_1| - \log|\lambda_i|}$$

for some $v_i \in T_x\Omega$. Then the argument proceeds in the same way as in the proof of Theorem 2.

□

To extend the results of the lemma to C^2 curves, we need Taylor's theorem:

Theorem 5.2 (Taylor's theorem, see [36])**.** *Suppose* f *is a* k *times differentiable function in an open interval* $(-a, a)$. *Then in this interval,*

$$f(t) = \sum_{j=0}^{k} \frac{f^j(0)}{j!} t^j + r_k(t)$$

where $r_k(t)$ *is the remainder function and has the property that*

$$\lim_{t\to 0} \frac{r_k(t)}{t^k} = 0.$$

Proof of Theorem 3. The goal is to show that a C^2 curve $c(t)$, with $c(0) = x$ and positive second derivative, has the property that, for $c'(0) = v_i$, $\alpha(v_i) = 2$. As in Section 2.5, we can model the boundary of Ω at x as the graph of a function $f : \mathbb{R}^N \to \mathbb{R}$ where $f(0) = 0$ and $Df \cdot v = 0$ for all $v \in \mathbb{R}^N$. By assumption, $f''(0) > 0$. Recall the definition of α:

$$\alpha(v_i) = \lim_{t \to 0} \frac{\log\left(\frac{f(tv_i)+f(-tv_i)}{2}\right)}{\log|t|}$$

and define

$$g(t) = \frac{f(tv_i) + f(-tv_i)}{2}$$

Then $g(0) = 0$ and $g'(0) = 0$ and $g''(0) > 0$. By Taylor's theorem, g can be written as

$$g(t) = g''(0)t^2 + r_2(t) \text{ with } \lim_{t \to 0} \frac{r_2(t)}{t^2} = 0 \ .$$

Then,

$$\begin{aligned}
\alpha &= \lim_{t \to 0} \frac{\log g(t)}{\log|t|} \\
&= \lim_{t \to 0} \frac{\log\left(\frac{g(t)}{g''(0)t^2}\right) + \log(g''(0)t^2)}{\log|t|} \\
&= \lim_{t \to 0} \frac{\log\left(1 + \frac{r_2(t)}{g''(0)t^2}\right) + 2\log|t| + \log(g''(0))}{\log|t|} \\
&= 2
\end{aligned}$$

The result follows from Lemma 5.1 $\square$

CHAPTER 6

Application to Affine Spheres

6.1 Introduction

Let $\mathcal{C}$ be the convex cone associated to hyperpolic space $\mathbb{H}^2$, i.e. $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 < z^2\}$ with isometry group $\mathrm{PSO}(2, 1)$. The orbit of the point $(0, 0, 1)$ is the component of the hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$ with $z > 0$ and shares an isometry group with the cone $\mathcal{C}$. The hyperboloid sits inside the cone $\mathcal{C}$ (see Figure 6.1) and is *asymptotic* to the boundary of $\mathcal{C}$. It is an example of a collection of classical convex objects called *affine spheres* which will be defined in the next section. In particular, it is a *hyperbolic* affine sphere which means that lines transverse to the surface meet on the concave side of the sphere at a point called the *center*. There is a direct correspondence between convex cones and a subset of affine spheres, which was conjectured by Calabi and proved by Cheng-Yau:

Theorem 6.1 (Cheng-Yau [14])**.** *For any properly convex domain Ω in $\mathbb{S}^m$, there is a unique hyperbolic affine sphere H asymptotic to Ω.*

Further, any hyperbolic affine sphere H whose center is the origin is asymptotic to a properly convex cone by projecting H to $\mathbb{S}^m$.

One motivation for working with affine spheres instead of convex sets is that affine spheres have a Riemannian metric arising from the second fundamental form of H called the *Blaschke metric*. In [9], Benoist and Hulin prove that the Hilbert metric d_Ω and the Blaschke metric d_B are uniformly comparable with

$$\frac{1}{c(n)} d_\Omega \leq d_B \leq c(n) d_\Omega \tag{6.1}$$

where $c(n)$ depends on the dimension n of the space. Tholozan extends this result in [42] to

$$d_B(x,y) < d_\Omega(x,y) + 1 \tag{6.2}$$

for $x, y \in \Omega$. The usefulness of Equation 6.2 is limited to when d_Ω and d_B are large, e.g. Tholozan uses it to compare volume entropy.

However, the goal of this section is to translate results about convex sets into results about affine spheres. The first step is to provide the necessary definitions; a more comprehensive description can be found in Loftin's survey on affine spheres ([39]).

6.2 Preliminaries

6.2.1 Definitions

A smooth hypersurface H in $\mathbb{R}^{n+1}$ has a transverse normal vector field called an affine normal. When this vector field ξ is invariant under affine transformations, then the hypersurface H can be studied with tools from affine geometry. The set of these transformations is the special affine group,

$$\mathrm{SA}(n+1,\mathbb{R}) = \{\phi : x \mapsto Ax + b | A \in \mathrm{SL}(n+1,\mathbb{R}), b \in \mathbb{R}^{n+1}\}$$

and the vector field ξ is affine invariant if

$$\phi_* \xi_H(x) = \xi_{\phi(H)}(\phi(x))$$

for $\phi \in \mathrm{SA}(n+1,\mathbb{R})$ and $x \in H$. This means that while ϕ might not fix H, ϕ must map the affine normal vector field ξ_H of H to the affine normal vector field $\xi_{\phi(H)}$ of $\phi(H)$. The described vector field ξ is not unique (e.g. $-\xi$) but we will later add conditions on ξ that will make it unique.

For a hypersurface H and its affine normal vector field ξ, let ξ_x be the vector in ξ at the point x. Extend all ξ_x to lines $\mathcal{L} = \{x + \mu\xi_x | \mu \in \mathbb{R}\}$. If every line in $\mathcal{L}$ meets at a point, then H is called a *proper affine sphere* and the meeting point is called the *center* of the affine sphere H. If the lines in $\mathcal{L}$ are parallel, i.e. meet at infinity, then H is called an *improper affine sphere* or a *parabolic affine sphere*. By symmetry, n-spheres in $\mathbb{R}^{n+1}$ are affine spheres and further, ellipsoids must be affine spheres by affine invariance.

Choose ξ so that the transverse normal vectors point to the convex side of a proper affine sphere H. If the ξ_x point towards the center, then H is an *elliptic affine sphere*. If the ξ_x point away from the center, then H is a *hyperbolic affine sphere*. An ellipsoid is an elliptic affine sphere and one component of a hyperboloid of two sheets is a hyperbolic affine sphere. The typical examples of affine spheres are quadric hypersurfaces; when H is a smooth quadric hypersurface, it is either an elliptic, a hyperbolic, or a parabolic affine sphere.

Any elliptic affine sphere must be an ellipsoid, which was shown by Blaschke in $\mathbb{R}^3$ in [12] and for any dimension by Deicke in [22]. Further, any parabolic affine sphere must be an elliptic paraboloid which is a corollary of Cheng-Yau's work in [14] (see Section 6 in [39] for more details.) However, the classification of hyperbolic affine spheres is more complicated. These are the affine spheres that are asymptotic to a convex cone, as stated in Theorem 6.1. In addition to the hyperbolic affine sphere described in the introduction, there is a family of hyperbolic affine spheres asymptotic to the simplex whose cone is $\mathcal{C} = \{(x, y, z)|x, y, z > 0\}$ given by $xyz = K$ for $K > 0$. For a general simplex with cone $\mathcal{C} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n|x_1, x_2, \ldots, x_n > 0\}$, the associated affine sphere is

$$x_1 x_2 \cdots x_n = K$$

for $K > 0$. In the next section, we will study some of the affine geometry associated to affine spheres.

Affine Structure Equations and the Blaschke metric

Let H be a strictly convex hypersurface and let ξ be a transverse vector field on H that is not necessarily an affine transverse vector field. There is a splitting of $T\mathbb{R}^{n+1}$ at each $x \in H$ given by

$$T_x\mathbb{R}^{n+1} = T_xH \oplus \mathbb{R} \cdot \xi$$

If X and Y are two vector fields tangent to H, the standard connection on $\mathbb{R}^{m+1}$ $\nabla_X Y$ is not necessarily tangent to H. The relationship is described via the Gauss-Weingarten structure equations

$$\nabla_X Y = D_X Y + h(X,Y)\xi$$

$$\nabla_X \eta = -S(X) + \tau(X)\xi$$

where D is a torsion free connection on TH, h is a symmetric tensor, S is an endomorphism on TM and τ is a one form on TM. Note that these equations are not specific to affine spheres, but are defined for hypersurfaces H and a transverse vector field to H. The transverse vector field ξ is considered an *affine transverse vector field* when each ξ_x points to the convex side of H, h is positive definite, $\tau = 0$, and $|\det(X_1, \ldots, X_n, \xi)| = 1$ for any h-orthonormal frame $(X_1, \ldots, X_n)$ of TH (see [39] for more details).

Any smooth, strictly convex hypersurface has a well defined affine normal (see Theorem 1 [39]). Since h is positive definite, it is a Riemannian metric which is called the *Blaschke metric*. The *affine mean curvature* is $K = \frac{1}{n}\mathrm{tr}S$. Affine spheres can be classified by their mean curvatures; if $K > 0$, the affine sphere is elliptic, if $K = 0$, the affine sphere is an elliptic paraboloid and if $K < 0$ the affine sphere is hyperbolic. The metric of an elliptic or a hyperbolic affine sphere can be scaled so that the mean curvature is $K = \pm 1$.

6.2.2 The Cheng-Yau Correspondence

The existence of a hyperbolic affine sphere asymptotic to the boundary of the cone over Ω comes from solving the Monge-Ampère equation in the form

$$\det(u_{ij}) = \left(\frac{K}{u}\right)^{n+2} \text{ in } \Omega \quad \text{and} \quad (u_{ij}) > 0 \quad \text{and} \quad u|_{\partial\Omega} = 0 \tag{6.3}$$

where u_{ij} is the (ij)th entry in the Hessian of the function u and K is a chosen affine mean curvature. This method is described by Gigena in [26]. We will describe this procedure for the example where $\mathcal{C}$ is the cone $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq z^2\}$ and $K = -1$. In this case, we will take an affine chart for $\Omega = \pi(\mathcal{C})$ to be parametrized by $X = \{(x, y, 1) : x^2 + y^2 \leq 1\}$.

The first step is to solve the differential equation

- $u_{xx}u_{yy} - (u_{xy})^2 = \left(\frac{1}{u}\right)^4$ and $u > 0$ for $x^2 + y^2 \leq 1$
- $u = 0$ on $x^2 + y^2 = 1$.

One can check that $u(x, y) = \sqrt{1 - x^2 - y^2}$ satisfies this differential equation and Cheng-Yau guarantees its uniqueness. Then the affine sphere is $Y = \frac{1}{u}X$ which is given by

$$\begin{aligned} Y &= \left\{ \frac{1}{\sqrt{1 - x^2 - y^2}}(x, y, 1) | x^2 + y^2 \leq 1 \right\} \\ &= \{(x, y, z) | x^2 + y^2 - z^2 \leq -1\} \end{aligned}$$

where the second equality comes from relabeling.

In the 1970s, Cheng-Yau solved these equations and gave us the following theorem:

Theorem 6.2 (Theorem 3 in [39]). *For any proper, open, convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$, there is a unique convex properly embedded hyperbolic affine sphere $H \subset \mathbb{R}^{n+1}$ which has affine mean curvature -1, has the vertex of $\mathcal{C}$ as its center, and is asymptotic to the boundary of $\mathcal{C}$.*

The reverse is much easier. If you start with the affine sphere H, then the convex cone $\mathcal{C}$ is just the convex hull of H and its center. This correspondence allows us to apply Theorem 2 to affine spheres.

6.2.3 Homogeneous Affine Spheres

An affine sphere H is called *homogeneous* if its automorphism group $G \subset \mathrm{SA}(n+1, \mathbb{R})$ acts transitively on H. Due to the Cheng-Yau correspondence, the classification of homogeneous convex cones given by Vinberg (see Section 1.6) carries over to a classification of homogeneous hyperbolic affine spheres which we will discuss further now.

Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a convex cone and let dx' be a measure on $\mathbb{R}^{n+1}$ that is invariant under the special affine group. Define the *characteristic function* $\phi_{\mathcal{C}}$ on $\mathcal{C}$ as

$$\phi_{\mathcal{C}}(x) = \int_{\mathcal{C}'} e^{-\langle x, x' \rangle} dx'$$

for every $x \in \mathcal{C}$. It is called the characteristic function because it characeterizes $\mathcal{C}$; if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two convex cones such that $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ and $\phi_{\mathcal{C}_1} = \phi_{\mathcal{C}_2}$ on $\mathcal{C}_1 \cap \mathcal{C}_2$, then $\mathcal{C}_2 = \mathcal{C}_2$. This can be proved by considering that when y approaches the boundary of $\mathcal{C}$, the quantity $\phi_{\mathcal{C}}(y)$ approaches infinity. Further facts about the characteristic function are given by Vinberg in [43].

In [41], Sasaki defines the *characteristic surface* S_d of $\phi_{\mathcal{C}}$ as

$$S_d = \{x \in \mathcal{C} : \phi_{\mathcal{C}}(x) = d\}$$

which is a noncompact submanifold of $\mathcal{C}$. This surface is an affine sphere:

Theorem 6.3 ([41] Theorem 4a)**.** *Let $\mathcal{C}$ be a homogeneous convex cone with characteristic function $\phi_{\mathcal{C}}$. Every characteristic surface S_d is a complete hyperbolic affine sphere with mean curvature $a \cdot d^{\frac{2}{n+2}}$ where a is a negative constant.*

Additionally, this affine sphere is asymptotic to the boundary of $\mathcal{C}$ which follows from the following theorem:

Theorem 6.4 ([41] Theorem 4b)**.** *Let H be a complete homogeneous hyperbolic affine sphere whose center is at the origin. Then there exists a homogeneous convex cone $\mathcal{C}$ such that H is asyptoptic to the boundary of $\mathcal{C}$. Further, $\mathcal{C} = \bigcup_d S_d$ where S_d are characteristic surfaces of Ω and $H = S_d$ for some $d > 0$.*

Hence the homogeneous hyperbolic affine spheres are in correspondence with the homogeneous convex cones.

6.3 Results

In this section we will extend Theorem 4.1 to apply to hyperbolic affine spheres. We will say that an affine sphere H is *divisible* if there exists a discrete subgroup $\Gamma \subset \mathrm{SA}(n+1, \mathbb{R})$ such that H/Γ is compact.

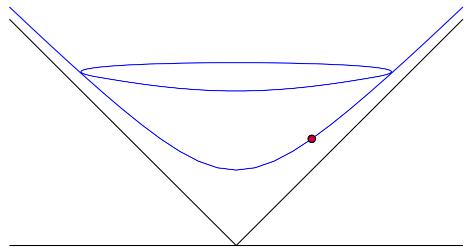

Figure 6.1: The affine sphere asymptotic to the cone over $\mathbb{H}^2$ is one sheet of a hyperboloid of two sheets.

Corollary 6.1. *Let* Γ *be a discrete subgroup of* $\mathrm{SA}(m+1, \mathbb{R})$ *which divides a hyperbolic affine sphere* $H \subset \mathbb{R}^{n+1}$ *with center at the origin. If for every* $g \in \Gamma_{\text{lox}}$*, the axis* l_g *lies in an embedded hyperboloid, then* H *is a hyperbolic space.*

Proof. From Theorem 6.2, the affine sphere H has a corresponding convex set $\Omega \subset \mathbb{S}^m$. The first obstacle is that we need Ω (and its convex cone $\mathcal{C}$) to be divisible, i.e. there is a discrete group Γ of projective transformations that acts cocompactly on Ω. A transformation $g \in \mathrm{SA}(n+1, \mathbb{R})$ that preserves an affine sphere with center at the origin must also lie in $\mathrm{SL}(n+1, \mathbb{R})$. Since Γ, which divides H, also preserves the line passing through each point in H and the origin, Γ preserves the convex cone $\mathcal{C}$. Since $\Omega = \pi(\mathcal{C})$, Γ fixes Ω and it only remains to show that Ω/Γ is compact.

Let F be a compact fundamental domain for the action of Γ on H and construct F_Ω by taking the convex hull of F. Clearly F_Ω is compact. Since $\{\gamma(F)|\gamma \in \Gamma\}$ covers H, $\{\gamma(F_\Omega)|\gamma \in \Gamma\}$ must cover Ω (take the convex hull of each $\gamma(F)$). Hence H divisible implies that Ω is divisible.

Take an element $g \in \Gamma_{\text{lox}}$, its corresponding axis l_g, and the embedded hyperboloid

S. This corresponds to a conic $C_{x_g^+}$ in the boundary of Ω with axis $(x_g^+, x_g^-) \subset \mathrm{Con}(C_{x_g^+})$. Then follow the proof of Theorem 4.1 starting at Lemma 4.8 where we construct a Zariski dense subset Γ' of Γ_{lox} where each $g \in \Gamma'$ has axis in the interior of Ω.

$\square$

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