Double ramification cycles with target varieties

F. Janda, R. Pandharipande, A. Pixton, D. Zvonkine

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Abstract

Let X be a nonsingular projective algebraic variety over $\mathbb{C}$, and let $\overline{\mathcal{M}}_{g,n,\beta}(X)$ be the moduli space of stable maps

$$f : (C, x_1, \ldots, x_n) \to X$$

from genus g, n-pointed curves C to X of degree β. Let S be a line bundle on X. Let $A = (a_1, \ldots, a_n)$ be a vector of integers which satisfy

$$\sum_{i=1}^{n} a_i = \int_\beta c_1(S)\,.$$

Consider the following condition: *the line bundle f^*S has a meromorphic section with zeroes and poles exactly at the marked points x_i with orders prescribed by the integers a_i*. In other words, we require $f^*S\left(-\sum_{i=1}^n a_i x_i\right)$ to be the trivial line bundle on C.

A compactification of the space of maps based upon the above condition is given by the moduli space of stable maps to *rubber* over X and is denoted by $\overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S)$. The moduli space carries a virtual fundamental class

$$[\overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S)]^{\mathrm{vir}} \in A_*\left(\overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S)\right)$$

in Gromov-Witten theory. The main result of the paper is an explicit formula (in tautological classes) for the push-forward via the forgetful morphism of $[\overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S)]^{\mathrm{vir}}$ to $\overline{\mathcal{M}}_{g,n,\beta}(X)$. In case X is a point, the result here specializes to Pixton's formula for the double ramification cycle proven in [35]. Several applications of the new formula are given.

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Contents

0 Introduction

0.1 Double ramification cycles

Let $A = (a_1, \ldots, a_n)$ be a vector of n integers satisfying

$$\sum_{i=1}^{n} a_i = 0\,.$$

In the moduli space $\mathcal{M}_{g,n}$ of nonsingular curves of genus g with n marked points, consider the substack defined by the following classical condition:

$$\left\{ (C, x_1, \ldots, x_n) \subset \mathcal{M}_{g,n} \,\middle|\, \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i x_i\Big) \simeq \mathcal{O}_C \right\}. \tag{1}$$

From the point of view of relative Gromov-Witten theory, the most natural compactification of the substack (1) is the space $\overline{\mathcal{M}}^{\sim}_{g,A}$ of stable maps to *rubber* [27, 38]: stable maps to $\mathbb{CP}^1$ relative to 0 and ∞ modulo the $\mathbb{C}^*$ action on $\mathbb{CP}^1$.

The rubber moduli space carries a natural virtual fundamental class $\left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\text{vir}}$ of dimension $2g - 3 + n$. The push-forward via the canonical morphism

$$\epsilon : \overline{\mathcal{M}}^{\sim}_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

is the *double ramification cycle*

$$\epsilon_* \left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\mathrm{vir}} = \mathsf{DR}_{g,A} \in A_{2g-3+n}(\overline{\mathcal{M}}_{g,n})\,. \tag{2}$$

The double ramfication cycle $\mathsf{DR}_{g,A}$ can also be defined via log stable maps (and was motivated in part by Symplectic Field Theory [18]).

The classical approach to the locus (1) is via Abel-Jacobi theory for the universal curve. However, extending the Abel-Jacobi map over the boundary $\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ of the moduli space of curves is *not* straightforward. Approaches by Marcus-Wise [40] and Holmes [30] (motivated by log geometry), nevertheless, provide a partial resolution of the Abel-Jacobi which is sufficient to define a double ramification cycle. The result also agrees with definition (2).

Eliashberg posed the question of computing $\mathsf{DR}_{g,n}$ in 2001. The hope for a possible formula was strengthened in [21] where the double ramification cycle was proven to lie in the tautogical ring of $\overline{\mathcal{M}}_{g,n}$. Calculations on the open set of compact type curves were given in [28, 29]. Integrals against the double ramification cycle[1] played a fundamental role in the solution of the Gromov-Witten theory for target curves [46, 47, 48].

A complete formula for $\mathsf{DR}_{g,A}$ in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ was conjectured by Pixton in 2014 and proven in [35] via Gromov-Witten theory. Pixton's formula expresses $\mathsf{DR}_{g,A}$ directly as a sum over stable graphs Γ indexing the boundary strata of $\overline{\mathcal{M}}_{g,n}$. The contribution of each stable graph Γ is the constant term of a polynomial in r naturally associated to the combinatorics of Γ and A. The proof of [35] was obtained by studying the Gromov-Witten theory of the target $\mathbb{P}^1$ with an orbifold $B\mathbb{Z}_r$-point at $0 \in \mathbb{P}^1$ and a relative point at $\infty \in \mathbb{P}^1$ in the $r \to \infty$ limit.

Pixton's formula's opened new directions in the subject: new formulas for Hodge classes [35, Section 3], new relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ [16], new connections to the loci of meromorphic differentials [24, Appendix], and connections to new integrable hierarchies [9]. For a sampling of the subsequent study and applications, see [6, 7, 8, 10, 12, 17, 23, 31, 32, 45, 56, 57]. We refer the reader to [35, Section 0] and [50, Section 5] for more leisurely introductions to the subject.

The double ramification cycle study above concerns the Gromov-Witten

[1] Termed *rubber integrals* in the papers [46, 47, 48].

theory of a *point*.[2] Our goal here is to develop a full theory of double ramification cycles for general nonsingular projective target varieties X.

Let $\overline{\mathcal{M}}_{g,n,\beta}(X)$ be the moduli space of stable maps

$$f : (C, x_1, \ldots, x_n) \to X$$

from genus g, n-pointed curves C to X of degree β. Let S be a line bundle on X. Let $A = (a_1, \ldots, a_n)$ be a vector of integers which satisfy

$$\sum_{i=1}^{n} a_i = \int_\beta c_1(S)\,.$$

Consider the following condition analogous to (1): *the line bundle f^*S has a meromorphic section with zeroes and poles exactly at the marked points x_i with orders prescribed by the integers a_i.* In other words, we require $f^*S\,(-\sum_{i=1}^n a_i x_i)$ to be the trivial line bundle on C. Rubber maps with target X provide a natural compactification of the locus of solutions and define an X-valued double ramification cycle in $A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))$. Our main result is a complete formula for the X-valued double ramification cycle which generalizes the structure of Pixton's formula.

The formal definition of the X-valued double ramification cycle is given in Section 0.2. After a discussion of X-valued stable graphs and tautological classes in Sections 0.3 and 0.4, our formula for the X-valued double ramification cycles is presented in Section 0.6. In case X is a point, we recover our previous study [35].

The double ramification cycle construction for target varieties plays a crucial role in relative Gromov-Witten theory. Since the answer for the X-valued double ramification cycles takes such a simple form, new directions are again opened in the subject:

- The study of the tautological ring of $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is suggested by a theory of relations [5] parallel to the case of point [16].
- For a pair (X, D) where D is nonsingular divisor, the relationship between the relative Gromov-Witten theory and the orbifold Gromov-Witten theory of the root stack is beautifully settled in [58, 59]. The

[2]The moduli space of stable maps to a point is $\overline{\mathcal{M}}_{g,n}$.

study of the D-valued double ramification cycle here plays a crucial role.

- The $\mathbb{CP}^N$-valued DR-cycle in the limit $N \to \infty$, suitably interpreted, is a universal DR-cycle on the moduli space of line bundles on curves. The universal Abel-Jacobi theory on the Picard stack [6] is both motivated by and dependent upon our calculation of $\mathbb{CP}^N$-valued DR-cycles.

Of course, the X-valued formula also leads immediately to simple derivations of older results in Gromov-Witten theory. Applications are discussed at the end of the paper in Section 4.

0.2 Rubber maps with target X

Let X be a nonsingular projective variety over $\mathbb{C}$. Let $S \to X$ be a line bundle, and let

$$\mathbb{P}(\mathcal{O}_X \oplus S) \to X$$

be the canonically associated $\mathbb{CP}^1$-bundle over X. Let

$$D_0\,,\ D_\infty\ \subset\ \mathbb{P}(\mathcal{O}_X \oplus S)$$

be the divisors defined by the projectivizations of the loci $\mathcal{O}_X \oplus \{0\}$ and $\{0\} \oplus S$ respectively. We will call D_0 the 0-divisor and D_∞ the ∞-divisor.

Let C be a nonsingular curve with n marked points, and let

$$f : C \to X$$

be an algebraic map of degree $\beta \in H_2(X, \mathbb{Z})$. Furthermore, let s be a nonzero meromorphic section of f^*S over C, defined up to a multiplicative constant, with zeros and poles belonging to the set of marked points of C. We denote the orders of zeros and poles by

$$A = (a_1, \ldots, a_n)\,.$$

If the ith marking is neither a zero nor a pole, we set $a_i = 0$. We have

$$\sum_{i=1}^{n} a_i = \int_\beta c_1(S)\,.$$

The pair (f, s) defines a map to *rubber with target* X. Let

$$\overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S)$$

be the compact moduli space of stable maps to rubber with target X. A general stable map to rubber with target X is a map to a rubber chain of $\mathbb{CP}^1$-bundles $\mathbb{P}(\mathbb{C} \oplus S)$ over X attached along their 0- and ∞-divisors. The space of rubber maps with target X carries a perfect obstruction theory and a virtual fundamental class, see [38, 39, 43] for a detailed discussion. Let

$$\epsilon : \overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S) \to \overline{\mathcal{M}}_{g,n,\beta}(X)$$

be the morphism obtained by the projection of the rubber map to X (and the contraction of the resulting unstable components).

Definition 1 The X*-valued double ramification cycle* is the ϵ push-forward of the virtual fundamental class of the moduli space of stable maps to rubber over X:

$$\mathsf{DR}_{g,A,\beta}(X,S) \;=\; \epsilon_* \left[\overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S)\right]^{\mathrm{vir}} \;\in A_{\mathrm{vdim}(g,n,\beta)-g}(\overline{\mathcal{M}}_{g,n,\beta}(X))\,.$$

The virtual dimension $\mathrm{vdim}(g,n,\beta)$ of $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is determined by

$$\mathrm{vdim}(g,n,\beta) = \int_\beta c_1(X) + (g-1)(\dim_{\mathbb{C}}(X) - 3) + n\,.$$

0.3 X-valued stable graphs

We define the set $\mathsf{G}_{g,n,\beta}(X)$ of X*-valued stable graphs* as follows. A graph $\Gamma \in \mathsf{G}_{g,n,\beta}(X)$ consists of the data

$$\Gamma = (\mathrm{V}\,,\ \mathrm{H}\,,\ \mathrm{L}\,,\ \mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}\,,\ v : \mathrm{H} \to \mathrm{V}\,,\ \iota : \mathrm{H} \to \mathrm{H},\ \beta : \mathrm{V} \to H_2(X,\mathbb{Z}))$$

satisfying the properties:

(i) V is a vertex set with a genus function $\mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : \mathrm{H} \to \mathrm{V}$ and an involution ι,

(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted),

(iv) L, the set of legs, is defined by the fixed points of ι and is placed in bijective correspondence with a set of n markings,

(v) the pair (V,E) defines a *connected* graph satisfying the genus condition

$$\sum_{v\in \mathrm{V}} \mathrm{g}(v) + h^1(\Gamma) = g\,,$$

(vi) for each vertex v, the stability condition holds: if $\beta(v)=0$, then

$$2\mathrm{g}(v) - 2 + \mathrm{n}(v) > 0\,,$$

where $\mathrm{n}(v)$ is the valence of Γ at v including both edges and legs,

(vii) the degree condition holds:

$$\sum_{v\in \mathrm{V}} \beta(v) = \beta\,.$$

To emphasize Γ, the notation $\mathrm{V}(\Gamma)$, $\mathrm{H}(\Gamma)$, $\mathrm{L}(\Gamma)$, and $\mathrm{E}(\Gamma)$ will also be to used for the vertex, half-edges, legs, and edges of Γ.

An automorphism of $\Gamma \in \mathsf{G}_{g,n,\beta}(X)$ consists of automorphisms of the sets V and H which leave invariant the structures L, g, v, ι, and β. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

An X-valued stable graph Γ determines a moduli space $\overline{\mathcal{M}}_\Gamma$ of stable maps with the degenerations forced by the graph together with a canonical map,

$$j_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n,\beta}(X)\,.$$

The moduli space $\overline{\mathcal{M}}_\Gamma$ is the substack of the product

$$\overline{\mathcal{M}}_\Gamma \subset \prod_{v\in \mathrm{V}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X)$$

cut out by the inverse images of the diagonal $\Delta \subset X \times X$ under the evaluations maps associated to the edges $e=(h,h') \in \mathrm{E}$,

$$\prod_{v\in \mathrm{V}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X) \xrightarrow{\mathrm{ev}_e} X \times X\,.$$

The moduli space $\overline{\mathcal{M}}_\Gamma$ carries a natural virtual fundamental class $\left[\overline{\mathcal{M}}_\Gamma\right]^{\mathrm{vir}}$ defined by the refined intersection,

$$[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}} = \prod_{e\in \mathrm{E}} \mathrm{ev}_e^{-1}(\Delta) \ \cap \ \prod_{v\in \mathrm{V}} \left[\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X)\right]^{\mathrm{vir}} . \tag{3}$$

0.4 Tautological ψ, ξ, and η classes

The universal curve

$$\pi : \mathcal{C}_{g,n,\beta}(X) \to \overline{\mathcal{M}}_{g,n,\beta}(X)$$

carries two natural line bundles: the relative dualizing sheaf ω_π and the pull-back f^*S of the line bundle S via the universal map,

$$f : \mathcal{C}_{g,n,\beta}(X) \to X .$$

Let s_i be the ith section of the universal curve, let

$$D_i \subset \mathcal{C}_{g,n,\beta}(X)$$

be the corresponding divisor, and let

$$\omega_{\log} = \omega_\pi\Big(\sum_{i=1}^n D_i\Big)$$

be the relative logarithmic line bundle with first Chern class $c_1(\omega_{\log})$. Let

$$\xi = c_1(f^*S)$$

be the first Chern class of the pull-back of S.

Definition 2 The following classes in $\overline{\mathcal{M}}_{g,n,\beta}(X)$ are obtained from the universal curve $\mathcal{C}_{g,n,\beta}(X)$:

- $\psi_i = c_1(s_i^*\omega_\pi)$,
- $\xi_i = c_1(s_i^* f^* S)$,
- $\eta_{a,b} = \pi_*\left(c_1(\omega_{\log})^a \xi^b\right)$.

All can be viewed as either cohomology classes or as operational Chow classes in $A^*(\overline{\mathcal{M}}_{g,n,\beta}(X))$. We will use the term *Chow cohomology* for operational Chow.

Since the class $\eta_{0,2} = \pi_*(\xi^2)$ will play a prominent role in our formulas for the X-valued DR-cycle, we will use the notational convention

$$\eta = \pi_*(\xi^2)\,.$$

The standard κ classes are defined by the π push-forwards of powers of $c_1(\omega_{\log})$, so we have

$$\eta_{a,0} = \kappa_{a-1}\,.$$

Consider the moduli $\overline{\mathcal{M}}_\Gamma$ of stable maps described by a stable graph Γ. Let e be an edge of the graph composed of two half-edges h_1 and h_2. The space $\overline{\mathcal{M}}_\Gamma$ carries two natural cohomology classes ψ_{h_1} and ψ_{h_2} (associated to the two cotangent lines at the node corresponding to the edge e) and a cohomology class $\xi_e = c_1(s_e^* S)$, where s_e is the section of the universal curve determined by the same node.

Definition 3 A *decorated X-valued stable graph* $[\Gamma, \gamma]$ is an X-valued stable graph $\Gamma \in \mathsf{G}_{g,n,\beta}(X)$ together with the following decoration data γ:

- each leg $i \in \mathrm{L}$ is decorated with a monomial $\psi_i^a \xi_i^b$,
- each half-edge $h \in \mathrm{H} \setminus \mathrm{L}$ is decorated with a monomial ψ_h^a,
- each edge $e \in \mathrm{E}$ is decorated with a monomial ξ_e^a,
- each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b\geq 2}$.

Let $\mathsf{DG}_{g,n,\beta}(X)$ be the set of decorated X-valued stable graphs. To each decorated graph

$$[\Gamma, \gamma] \in \mathsf{DG}_{g,n,\beta}(X)\,,$$

we assign the cycle class $j_{\Gamma *}[\gamma]$ obtained via the push-forward via

$$j_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n,\beta}(X)$$

of the action of the product of the ψ, ξ, and η decorations on $\left[\overline{\mathcal{M}}_\Gamma\right]^{\text{vir}}$,

$$j_{\Gamma*}[\gamma] \stackrel{\text{def}}{=} j_{\Gamma*}\left(\gamma \cap \left[\overline{\mathcal{M}}_\Gamma\right]^{\text{vir}}\right) \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X)). \tag{4}$$

Our formula for the X-valued DR-cycle is a sum of cycle classes (4) assigned to decorated X-valued stable graphs.

0.5 Weightings mod r

Following the notation of Sections 0.2-0.4, let $S \to X$ be a line bundle on a nonsingular projective variety X. Fix the data

$$g \geq 0, \quad \beta \in H_2(X,\mathbb{Z}), \quad A = (a_1, \ldots, a_n)$$

subject to the condition

$$\sum_{i=1}^n a_i = \int_\beta c_1(S).$$

Let $\Gamma \in \mathsf{G}_{g,n,\beta}(X)$ be an X-valued stable graph, and let r be a positive integer.

Definition 4 A *weighting mod* r of Γ is a function on the set of half-edges,

$$w : \mathrm{H}(\Gamma) \to \{0, 1, \ldots, r-1\},$$

which satisfies the following three properties:

(i) $\forall i \in \mathrm{L}(\Gamma)$, corresponding to the marking $i \in \{1, \ldots, n\}$,

$$w(i) = a_i \mod r,$$

(ii) $\forall e \in \mathrm{E}(\Gamma)$, corresponding to two half-edges $h, h' \in \mathrm{H}(\Gamma)$,

$$w(h) + w(h') = 0 \mod r,$$

(iii) $\forall v \in \mathrm{V}(\Gamma)$,

$$\sum_{v(h)=v} w(h) = \int_{\beta(v)} c_1(S) \mod r,$$

where the sum is taken over *all* $\mathsf{n}(v)$ half-edges incident to v.

We denote by $\mathsf{W}_{\Gamma,r}$ the finite set of all possible weightings mod r of Γ. The set $\mathsf{W}_{\Gamma,r}$ has cardinality $r^{h^1(\Gamma)}$. We view r as a *regularization parameter*.

0.6 The double ramification formula

We denote by $\mathsf{P}^{d,r}_{g,A,\beta}(X,S) \in A_{\mathrm{vdim}(g,n,\beta)-d}(\overline{\mathcal{M}}_{g,n,\beta}(X))$ the degree d component of the tautological class

$$\sum_{\substack{\Gamma \in \mathsf{G}_{g,n,\beta}(X) \\ w \in \mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} j_{\Gamma*} \Bigg[\prod_{i=1}^{n} \exp\left(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\right) \prod_{v \in \mathrm{V}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v)\right)$$

$$\prod_{e=(h,h') \in \mathrm{E}(\Gamma)} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}} \Bigg].$$

Inside the push-forward in the above formula, the first product

$$\prod_{i=1}^{n} \exp\left(\frac{1}{2}a_i^2\psi_{h_i} + a_i\xi_{h_i}\right)$$

is over $h \in \mathrm{L}(\Gamma)$ via the correspondence of legs and markings. The class $\eta(v)$ is the $\eta_{0,2}$ class of Definition 2 associated to the vertex. The third product is over all $e \in \mathrm{E}(\Gamma)$. The factor

$$\frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}$$

is well-defined since

- the denominator formally divides the numerator,
- the factor is symmetric in h and h'.

No edge orientation is necessary.

The following fundamental polynomiality property of $\mathsf{P}^{d,r}_{g,A,\beta}(X,S)$ is parallel to Pixton's polynomiality in [35, Appendix] and is a consequence of [35, Proposition 3″].

Proposition 1 *For fixed g, A, β, and d, the class*

$$\mathsf{P}^{d,r}_{g,A,\beta}(X,S) \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))$$

is polynomial in r (for all sufficiently large r).

We denote by $\mathsf{P}^d_{g,A,\beta}(X,S)$ the value at $r=0$ of the polynomial associated to $\mathsf{P}^{d,r}_{g,A,\beta}(X,S)$ by Proposition 1. In other words, $\mathsf{P}^d_{g,A,\beta}(X,S)$ is the *constant* term of the associated polynomial in r.

The main result of the paper is a formula for the X-valued double ramification cycle parallel[3] to Pixton's proposal in case X is a point.

Theorem 2 *Let X be a nonsingular projective variety with line bundle*

$$S \to X\,.$$

For $g \geq 0$, double ramification data A, and $\beta \in H_2(X,\mathbb{Z})$, we have

$$\mathsf{DR}_{g,A,\beta}(X,S) = \mathsf{P}^g_{g,A,\beta}(X,S) \, \in \, A_{\mathrm{vdim}(g,n,\beta)-g}(\overline{\mathcal{M}}_{g,n,\beta}(X))\,.$$

0.7 Strategy of proof

Consider the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus S) \to X$. By applying Cadman's rth root construction [11] to the 0-divisor D_0, we obtain a bundle

$$\mathbb{P}(X,S)[r] \to X \tag{5}$$

where every fiber is a projective line with a single stacky point of stabilizer $\mathbb{Z}/r\mathbb{Z}$.

Our proof of Theorem 2 is obtained by studying the Gromov-Witten theory of the bundle (5) relative to the ∞-divisor D_∞. The virtual localization formula for the orbifold/relative geometry

$$\mathbb{P}(X,S)[r]/D_\infty$$

yields relations for every r which depend polynomially on r (for all sufficiently large r). After setting $r=0$, we obtain the equality of Theorem 2. The argument has two main parts.

(i) Let $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$ be the moduli space of stable maps

$$f : C \to X$$

[3]Our handling of the prefactor 2^{-g} in [35, Theorem 1] differs here. The factors of 2 are now placed in the definition of $\mathsf{P}^{d,r}_{g,A,\beta}$.

endowed with an rth tensor root L of the line bundle $f^*S(-\sum_{i=1}^n a_i x_i)$. Furthermore, let

$$\pi : \mathcal{C}^r_{g,A,\beta}(S) \to \overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$$

be the universal curve, and let $\mathcal{L}$ be the universal rth root over the universal curve. A crucial step is to prove that the push-forward of

$$r \cdot c_g(-R\pi_*\mathcal{L})$$

to $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is a polynomial in r (for all sufficiently large r) and that the polynomial has the *same* constant term as the polynomial $\mathsf{P}^{g,r}_{g,A,\beta}(X,S)$. Our formula for the X-valued DR-cycle therefore has a geometric interpretation in terms of the top Chern class $c_g(-R\pi_*\mathcal{L})$.

Contrary to the case of ordinary DR-cycles studied in [35], for the case of X-valued DR-cycles, we cannot use Chiodo's formulas [14] to deduce polynomiality. Instead, we adapt Chiodo's computations to our geometric setting in Sections 2.3 and 2.4.

(ii) We use the localization formula [25] for the virtual fundamental class of the moduli space of stable maps to the orbifold/relative geometry

$$\mathbb{P}(X,S)[r]/D_\infty\,.$$

The positive (respectively, negative) coefficients a_i specify the monodromy conditions over the 0-divisor (respectively, the tangency conditions along the ∞-divisor).

The moduli space $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$ appears in the localization formula. Indeed, the space of stable maps to the $\mathbb{C}^*$-invariant locus corresponding to the stacky 0-divisor D_0 is precisely $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$. The push-forward of the localization formula to $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is a Laurent series in the equivariant parameter t and in r. The coefficient of $t^{-1}r^0$ must vanish by geometric considerations. We prove that the relation obtained from the coefficient of $t^{-1}r^0$ has *only two terms*:

- The first is the constant term in r of the push-forward of $r \cdot c_g(-R\pi_*\mathcal{L})$ to $\overline{\mathcal{M}}_{g,n,\beta}(X)$.
- The second term is the X-valued double ramification cycle $\mathsf{DR}_{g,A,\beta}(X,S)$ with a minus sign.

The vanishing of the sum of the two terms yields Theorem 2.

0.8 Notation table

To help the reader, we list here the symbols used for the various spaces which arise in the paper:

- $\mathbb{P}(X, S)$ the $\mathbb{CP}^1$-bundle $\mathbb{P}(\mathcal{O}_X \oplus S)$ over X,
- $\mathbb{P}(X, S)[r]$ is the outcome of applying Cadman's rth root construction to the 0-divisor $D_0 \subset \mathbb{P}(X, S)$,
- $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is the space of stable maps $f : (C, x_1, \ldots, x_n) \to X$,
- $\overline{\mathcal{M}}^r_{g,A,\beta}(X, S)$ is the space of stable maps $f : (C, x_1, \ldots, x_n) \to X$ together with an rth root of $f^*S(-\sum_{i=1}^n a_i x_i)$,
- $\mathfrak{M}_{g,n}$ is the stack of prestable curves,
- $\mathfrak{M}^r_{g,n}$ is the stack of twisted prestable curves,
- $\mathfrak{M}^Z_{g,n}$ is the stack of prestable curves together with a degree 0 line bundle Z,
- $\mathfrak{M}^{r,Z,\mathrm{triv}}_{g,n}$ is the stack of prestable twisted curves together with a degree 0 line bundle Z where the stabilizer of every point of the twisted curve acts trivially in the fibers of the line bundle,
- $\mathfrak{M}^{r,L}_{g,n}$ is the stack of prestable twisted curves together with a degree 0 line bundle L with no conditions on the stabilizers.

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1 Curves with an rth root

1.1 Artin stacks and Chow cohomology

Let $\mathfrak{M}_{g,n}$ denote the smooth Artin stack of prestable curves. The Artin stack $\mathfrak{M}^Z_{g,n}$ of prestable curves with a line bundle Z of total degree 0 is obtained from the Picard stack of the universal curve

$$\pi : \mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$$

and is also smooth. The Artin stack $\mathfrak{M}^Z_{g,n}$ has a universal curve

$$\pi : \mathfrak{C}^Z_{g,n} \to \mathfrak{M}^Z_{g,n}$$

which carries a universal line bundle $\mathcal{Z}$ with sections $s_1, \ldots, s_n$.

Kresch [37] has developed a theory of Chow cohomology classes on Artin stacks. A basic property is that given a morphism from a scheme to the stack, Kresch's Chow cohomology class on the stack determines a Chow cohomology class on the scheme (comptatible with further pull-backs). We describe here a family of Chow cohomology classes on $\mathfrak{M}^Z_{g,n}$.

We first define the set $\mathsf{G}^Z_{g,n}$ of *prestable graphs* as follows. A prestable graph $\Gamma \in \mathsf{G}^Z_{g,n}$ consists of the data

$$\Gamma = (\mathrm{V}\,,\ \mathrm{H}\,,\ \mathrm{L}\,,\ \mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}\,,\ v : \mathrm{H} \to \mathrm{V}\,,\ \iota : \mathrm{H} \to \mathrm{H},\ \mathrm{d} : \mathrm{V} \to \mathbb{Z})$$

satisfying the properties:

(i) V is a vertex set with a genus function $\mathrm{g} : V \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : \mathrm{H} \to \mathrm{V}$ and an involution ι,

(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted),

(iv) L, the set of legs, is defined by the fixed points of ι and is placed in bijective correspondence with a set of n markings,

(v) the pair (V, E) defines a *connected* graph satisfying the genus condition

$$\sum_{v \in \mathrm{V}} \mathrm{g}(v) + h^1(\Gamma) = g\,,$$

(vi) the degree condition holds:

$$\sum_{v \in \mathrm{V}} \mathrm{d}(v) = 0\,.$$

An automorphism of $\Gamma \in \mathsf{G}^Z_{g,n}(X)$ consists of automorphisms of the sets V and H which leave invariant the structures L, g, v, ι, and d. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

Definition 5 A *decorated prestable graph* $[\Gamma, \gamma]$ is a prestable graph $\Gamma \in \mathsf{G}^Z_{g,n}$ together with the following decoration data γ:

- each leg $i \in \mathrm{L}$ is decorated with a monomial $\psi_i^a \xi_i^b$.
- each half-edge $h \in \mathrm{H} \setminus \mathrm{L}$ is decorated with a monomial ψ_h^a,
- each edge $e \in \mathrm{E}$ is decorated with a monomial ξ_e^a,
- each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b \geq 2}$.

Let $\mathsf{DG}^Z_{g,n}$ be the set of decorated prestable graphs. Every

$$[\Gamma, \gamma] \in \mathsf{DG}^Z_{g,n}$$

determines a class in the Chow cohomology of $\mathfrak{M}^Z_{g,n}$:

- Γ specifies the degeneration of the curve,
- d specifies the degree distribution of Z,
- ψ_i corresponds to the cotangent line class,
- $\xi_i = c_1(s_i^* \mathcal{Z})$,
- $\xi_e = c_1(s_e^* \mathcal{Z})$ where s_e is the node associated to e,
- $\eta_{a,b} = \pi_* \big(c_1(\omega_{\log})^a \, c_1(\mathcal{Z})^b \big)$.

We have followed here the pattern of Definition 2.

More generally, every possibly infinite linear combination of decorated prestable graphs determines a class in the Chow cohomology of $\mathfrak{M}_{g,n}^Z$. Indeed, for any morphism

$$B \to \mathfrak{M}_{g,n}^Z$$

from a scheme B of finite type, only a finite number of terms in the linear combination will contribute. We refer the reader to the Appendix of [26] for the construction of the product in the Chow cohomology algebra (see also the discussion in Section 1.7 below).

1.2 Twisted curves

Let $r \geq 1$ be an integer. The analog of $\mathfrak{M}_{g,n}$ in the context of rth roots is moduli space of $\mathfrak{M}_{g,n}^r$ of twisted prestable curves constructed in [49], see also [2, 3]. We give a short summary here.

A *twisted curve* is a prestable curve with stacky structure at the nodes[4]. Denote by $\mu_r \subset \mathbb{C}^*$ the group of rth roots of unity in the complex plane. The neighborhood of a node in a family of twisted curves is obtained from the family

$$(x, y) \mapsto z = xy$$

[4]A more complete name is *balanced twisted curve*, but we omit the word *balanced*, since these are the only twisted curves that we consider. While stacky structure can also be imposed at the markings of the curve, our twisted curves have stacky structure only at the nodes.

by taking a $\mu_r \times \mu_r$ quotient in the source and a μ_r quotient in the target.

To construct the versal deformation of the node of twisted curve, we start with the versal deformation

$$\mathbb{C}^2 \to \mathbb{C}\,, \qquad (x,y) \mapsto z = xy \tag{6}$$

of the node of a prestable curve. Let $(a,b) \in \mu_r \times \mu_r$ act on $\mathbb{C}^2$ by

$$(x,y) \mapsto (ax, by)\,,$$

and let $c \in \mu_r$ act on $\mathbb{C}$ by $z \mapsto cz$. These actions commute with (6) via the group morphism

$$\phi : \mu_r \times \mu_r \to \mu_r\,, \qquad (a,b) \mapsto c = ab\,.$$

After taking the stack quotient of both sides of (6), we obtain a family of twisted curves over $[\mathbb{C}/\mu_r]$ with one stacky μ_r-point at the origin. The fibers of the family over $t \neq 0$ are nonsingular curves isomorphic to $\mathbb{C}^*$. The fiber over the origin $t = 0$ is the union of the coordinate axes $xy = 0$ factored by the kernel of the morphism ϕ. As soon as a twisted curve acquires a node, the twisted curve simultaneously acquires an extra μ_r group of symmetries (given by the image of ϕ).

In a family of prestable curves, the neighborhood of a node is modeled by the versal deformation

$$(x,y) \mapsto z = xy\,.$$

Given two line bundles T_x and T_y over a base B, consider the tensor product

$$T_z = T_x \otimes T_y\,.$$

We construct a family of curves over the total space of T_z over B by

$$T_x \oplus_B T_y \to T_z\,, \quad (b,x,y) \mapsto (b,xy)\,.$$

Here, B is the boundary divisor and T_z is the normal line bundle to B. We can construct a family of twisted curves over the total space of T_z by applying Cadman's rth root construction to the zero section $B \subset T_z$. In particular, the normal bundle to the locus of nodal twisted curves is now $(T_x \otimes T_y)^{\otimes(1/r)}$.

A *prestable twisted curve* is a twisted curve with a prestable coarsification. Let $\mathfrak{M}^r_{g,n}$ be the moduli space of prestable twisted curves of genus g with n marked points. Since $\mathfrak{M}^r_{g,n}$ is obtained from the smooth Artin stack $\mathfrak{M}_{g,n}$ of ordinary prestable curves by applying Cadman's rth root construction to the boundary divisor, $\mathfrak{M}^r_{g,n}$ is also a smooth Artin stack. The moduli space $\mathfrak{M}^r_{g,n}$ carries three universal curves, see [14]:

(i) There is the universal twisted prestable curve

$$\mathfrak{C}^r_{g,n} \to \mathfrak{M}^r_{g,n} \, .$$

(ii) There is the fiberwise coarsification $\mathsf{C}^r_{g,n}$ of $\mathfrak{C}^r_{g,n}$. The local model of $\mathsf{C}^r_{g,n}$ is given by the quotient of the map

$$(x, y) \mapsto z = xy$$

by the kernel of the group morphism ϕ. An A_{r-1} singularity at the origin is obtained, so the universal curve $\mathsf{C}^r_{g,n}$ is singular.

(iii) There is the universal curve $\widetilde{\mathsf{C}}^r_{g,n}$ obtained by resolving the singularities of $\mathsf{C}^r_{g,n}$ by a series of blow-ups. The resolution of the A_{r-1} singularity yields a chain of $r-1$ rational exceptional curves over the origin. The rational curves correspond to the vertices of the A_{r-1} Dynkin diagram, and their intersection points correspond to the edges. We call $\widetilde{\mathsf{C}}^r_{g,n}$ the *bubbly universal curve.*

1.3 Twisted curves with a line bundle

We introduce two more Artin stacks denoted by $\mathfrak{M}^{r,Z,\mathrm{triv}}_{g,n}$ and $\mathfrak{M}^{r,L}_{g,n}$:

- The stack $\mathfrak{M}^{r,Z,\mathrm{triv}}_{g,n}$ is obtained from the stack $\mathfrak{M}^{Z}_{g,n}$ of prestable curves with a line bundle by applying Cadman's rth root construction to the boundary divisors. It is the stack of twisted prestable curves endowed with a degree 0 line bundle Z with one extra condition: *the stabilizer of every point of the twisted curve acts trivially in the fibers of the line bundle.* A Chow cohomology class on $\mathfrak{M}^{Z}_{g,n}$ determines a Chow cohomology class on $\mathfrak{M}^{r,Z,\mathrm{triv}}_{g,n}$ by pull-back.

- The stack $\mathfrak{M}_{g,n}^{r,L}$ is the stack of twisted prestable curves with a degree 0 line bundle (*with no stabilizer conditions*).

These stacks are related by three natural morphisms:

$$\mathfrak{M}_{g,n}^{r,L} \xrightarrow{p_1} \mathfrak{M}_{g,n}^{r,Z,\mathrm{triv}} \xrightarrow{p_2} \mathfrak{M}_{g,n}^{Z} \xrightarrow{p_3} \mathfrak{M}_{g,n}\,. \tag{7}$$

The morphism p_1 assigns to a pair (C, L) the pair $(C, Z = L^{\otimes r})$, where C is a twisted prestable curve and L a line bundle. The morphism p_1 is étale of degree r^{2g-1}. The morphism p_2 comes from Cadman's rth root construction. The morphism p_3 assigns to a pair (C, S) the curve C.

While we have taken both Z and L to be of degree 0 in the definitions, all of the constructions and results of Sections 1 and 2 will be valid in case

$$\deg(Z) = r\deg(L)\,.$$

1.4 Commutative diagram

Let X be a nonsingular projective variety with line bundle $S \to X$. Let $A = (a_1, \ldots, a_n)$ be a vector of integers which satisfy

$$\sum_{i=1}^{n} a_i = \int_\beta c_1(S)$$

for $\beta \in H_2(X, \mathbb{Z})$.

The moduli space $\overline{\mathcal{M}}_{g,A,\beta}^{r}(X, S)$ of stable maps

$$f : (C, x_1, \ldots, x_n) \to X$$

endowed with an rth root of the degree 0 line bundle

$$f^*S\Big(-\sum_{i=1}^{n} a_i x_i\Big)$$

plays a central role in the proof of Theorem 2. The moduli space $\overline{\mathcal{M}}_{g,A,\beta}^{r}(X, S)$ is defined as the fiber product of the following two maps:

(i) $\pi_Z : \overline{\mathcal{M}}_{g,n,\beta}(X) \to \mathfrak{M}^Z_{g,n}$ assigns to a stable map $f : C \to X$ the pair

$$\Big(C, f^*S\Big(-\sum_{i=1}^{n} a_i x_i\Big)\Big)\,.$$

(ii) $\epsilon : \mathfrak{M}^{r,L}_{g,n} \to \mathfrak{M}^Z_{g,n}$ is the composition $\epsilon = p_2 \circ p_1$.

The moduli space $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$ is the fiber product of π_Z and ϵ:

$$\begin{array}{ccc} \overline{\mathcal{M}}^r_{g,A,\beta}(X,S) & \xrightarrow{\epsilon} & \overline{\mathcal{M}}_{g,n,\beta}(X) \\ \Big\downarrow{\scriptstyle \pi_L} & & \Big\downarrow{\scriptstyle \pi_Z} \\ \mathfrak{M}^{r,L}_{g,n} & \xrightarrow{\epsilon} & \mathfrak{M}^Z_{g,n} \end{array}\,, \tag{8}$$

where we denote top arrow also by ϵ.

1.5 The pull-back map π_Z^*

We describe here the pull-back map π_Z^* for Chow cohomology classes defined by decorated graphs. Let

$$[\Gamma, \gamma] \in \mathsf{DG}^Z_{g,n}$$

be a decorated prestable graph representing a Chow cohomology class in $\mathfrak{M}^Z_{g,n}$ following the conventions of Section 1.1.

Lemma 3 *The pull-back $\pi_Z^*[\Gamma, \gamma]$ is obtained in terms of Chow cohomology classes of decorated X-valued stable graphs by applying the following procedure:*

- *Replace the degree $\mathrm{d}(v) \in \mathbb{Z}$ of each vertex with effective classes*

$$\beta(v) \in H_2(X, \mathbb{Z})$$

satisfying

$$\int_{\beta(v)} c_1(S) - \sum_{i \vdash v} a_i = \mathrm{d}(v)\,,$$

where the sum is over the legs i incident to v. Sum over all choices of $\beta(v)$.

- *Replace each ξ_i with $\xi_i + a_i\psi_i$.*
- *Replace each class $\eta_{0,b}$ at each vertex v with*

$$\eta_{0,b} - \sum_{i \vdash v} \sum_{k=1}^{b} \binom{b}{k} a_i^k \psi_i^{k-1} \xi_i^{b-k},$$

where the first sum is again over the legs i incident to v.

All other decorations are kept the same.

Proof. Given a stable map $f : C \to X$, the degree of $f^*S(-\sum_{i=1}^n a_i x_i)$ on the component of the curve C corresponding a vertex v of the dual graph equals

$$\int_{\beta(v)} c_1(S) - \sum_{i \vdash v} a_i \,,$$

which justifies the first operation.

Recall the divisor D_i corresponds to the ith section,

$$D_i \subset \mathfrak{C}^Z_{g,n} \xrightarrow{\pi} \mathfrak{M}^Z_{g,n} \,,$$

and $\omega_{\log} = \omega_\pi(\sum_{i=1}^n D_i)$. The first Chern class of $f^*S(-\sum_{i=1}^n a_i x_i)$ on the universal curve equals

$$c_1(S) - \sum_{i=1}^n a_i D_i \,.$$

The pull-back $\pi_Z^*(\eta_{a,b})$ is the push-forward from the universal curve of the product

$$K^a \Big(c_1(S) - \sum_{i=1}^n a_i D_i \Big)^b \,, \tag{9}$$

where $K = c_1(\omega_{\log})$. Since $KD_i = 0$, the product (9) is equal to $K^a \xi^b$ if $a > 0$. Hence, for $a > 0$,

$$\pi_Z^*(\eta_{a,b}) = \eta_{a,b} \,,$$

In case $a = 0$, we expand $(c_1(S) - \sum_{i=1}^n a_i D_i)^b$ and take the push-forward from the universal curve to the moduli space. We find

$$\eta_{0,b} - \sum_{i \vdash v} \sum_{k=1}^{b} \binom{b}{k} a_i^k \psi_i^{k-1} \xi_i^{b-k}$$

in the notation of decorated X-valued stable graphs. $\Diamond$

1.6 Chow cohomology classes on $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$ and $\mathfrak{M}^{r,L}_{g,n}$

1.6.1 Overview

Recall the commutative diagram (8):

$$\begin{array}{ccc} \overline{\mathcal{M}}^r_{g,A,\beta}(X,S) & \xrightarrow{\epsilon} & \overline{\mathcal{M}}_{g,n,\beta}(X) \\ \Big\downarrow \pi_L & & \Big\downarrow \pi_Z \\ \mathfrak{M}^{r,L}_{g,n} & \xrightarrow{\epsilon} & \mathfrak{M}^{Z}_{g,n} \end{array}$$

We now define Chow cohomology classes via decorated graphs on

$$\overline{\mathcal{M}}^r_{g,A,\beta}(X,S) \quad \text{and} \quad \mathfrak{M}^{r,L}_{g,n}$$

and describe the pull-back map π_L^*. Except for the additional data recording the twisted structure, the discussion is almost identical to our treatment of

$$\pi_Z : \overline{\mathcal{M}}_{g,n,\beta}(X,S) \to \mathfrak{M}^{Z}_{g,n}\,.$$

1.6.2 The moduli space $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$

We define the set $\mathsf{G}^r_{g,A,\beta}(X,S)$ of *X-valued r-twisted stable graphs* as follows. A graph $\Gamma \in \mathsf{G}^r_{g,A,\beta}(X,S)$ consists of the data

$$\Gamma = (\mathrm{V}\,,\ \mathrm{H}\,,\ \mathrm{L}\,,\ \mathrm{g},\ v\,,\ \iota\,,\ \beta : \mathrm{V} \to H_2(X,\mathbb{Z}), \mathrm{tw} : \mathrm{H} \to \{0,\ldots,r-1\})$$

satisfying the properties:

(i-vii) exactly as for X-valued stable graphs in Section 0.3,

(viii) the twist conditions hold:

$$\begin{aligned}
&\text{(L)}\ \forall i \in \mathrm{L} \implies \mathrm{tw}(i) = 0\,,\\
&\text{(E)}\ \forall e = (h', h'') \in \mathrm{E} \implies \mathrm{tw}(h') + \mathrm{tw}(h'') = 0 \bmod r\,,\\
&\text{(V)}\ \forall v \in \mathrm{V} \implies \sum_{\nu(h)=v} \mathrm{tw}(h) = \int_{\beta(v)} c_1(S) - \sum_{i \vdash v} a_i \mod r\,.
\end{aligned}$$

The line bundle $S \to X$ and vector A only appear in property (viii).

The universal curve

$$\pi : \mathcal{C}^r_{g,A,\beta} \to \overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$$

carries the log relative dualizing sheaf $\omega_{\log}$ and the pull-back f^*S of the line bundle S via the universal map,

$$f : \mathcal{C}^r_{g,A,\beta} \to X\,.$$

Following Section 0.4, we define Chow cohomology classes on $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$:

- $\psi_i = c_1(s_i^* \omega_\pi)\,,$
- $\xi_i = c_1(s_i^* f^* S)\,,$
- $\eta_{a,b} = \pi_* \left(c_1(\omega_{\log})^a \, c_1(f^*S)^b \right)\,.$

Definition 6 A *decorated X-valued r-twisted stable graph* $[\Gamma, \gamma]$ is an X-valued stable graph $\Gamma \in \mathsf{G}^r_{g,A,\beta}(X)$ together with the following decoration data γ:

- each leg $i \in \mathrm{L}$ is decorated with a monomial $\psi_i^a \xi_i^b$,
- each half-edge $h \in \mathrm{H} \setminus \mathrm{L}$ is decorated with a monomial ψ_h^a,
- each edge $e \in \mathrm{E}$ is decorated with a monomial ξ_e^a,
- each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b \geq 2}$.

Let $\mathsf{DG}^r_{g,A,\beta}(X,S)$ be the set of decorated X-valued r-twisted stable graphs. Every

$$[\Gamma,\gamma] \in \mathsf{DG}^r_{g,A,\beta}(X,S)$$

determines a class in the Chow cohomology of $\overline{\mathcal{M}}^r_{g,A,\beta}(X,S)$:

- Γ specifies the degeneration of the curve with twisted structure w at the nodes on the rth root of

$$f^*S\Big(-\sum_{i=1}^{n} a_i x_i\Big)\,,$$

- β specifies the curve class distribution of the map f,
- the decorations ψ_i, ξ_i, ξ_e, and $\eta_{a,b}$ specify Chow cohomology classes,

$$\xi_e = c_1(s_e^* f^* S)\,.$$

1.6.3 The Artin stack $\mathfrak{M}^{r,L}_{g,n}$

Let $\mathsf{G}^{r,L}_{g,n}$ be the set of *r-twisted prestable graphs* defined as follows. A graph

$$\Gamma \in \mathsf{G}^{r,L}_{g,n}$$

consists of the data

$$\Gamma = (\mathrm{V}\,,\ \mathrm{H}\,,\ \mathrm{L}\,,\ \mathrm{g}\,,\ v\,,\ \iota\,,\ \mathrm{d} : \mathrm{V} \to \mathbb{Z}\,, \mathrm{tw} : \mathrm{H} \to \{0,\ldots,r-1\})$$

satisfying the properties:

(i-vi) exactly as for prestable graphs in Section 1.1,

(vii) the twist conditions hold:

(L) $\forall i \in \mathrm{L} \implies \mathrm{tw}(i) = 0\,,$
(E) $\forall e = (h', h'') \in \mathrm{E} \implies \mathrm{tw}(h') + \mathrm{tw}(h'') = 0 \bmod r\,,$
(V) $\forall v \in \mathrm{V} \implies \sum_{\nu(h)=v} \mathrm{tw}(h) = \mathrm{d}(v) \bmod r\,.$

Definition 7 A *decorated r-twisted prestable graph* $[\Gamma, \gamma]$ is a graph

$$\Gamma \in \mathsf{G}_{g,n}^{r,L}$$

together with the following decoration data γ:

- each leg $i \in \mathrm{L}$ is decorated with a monomial $\psi_i^a \xi_i^b$,
- each half-edge $h \in \mathrm{H} \setminus \mathrm{L}$ is decorated with a monomial ψ_h^a,
- each edge $e \in \mathrm{E}$ is decorated with a monomial ξ_e^a,
- each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b\geq 2}$.

Let $\mathsf{DG}_{g,n}^{r,L}$ be the set of decorated r-weighted prestable graphs. Every

$$[\Gamma, \gamma] \in \mathsf{DG}_{g,n}^{r,L}$$

determines a class in the Chow cohomology of $\mathfrak{M}_{g,n}^{r,L}$:

- Γ specifies the degeneration of the curve with twisted structure w at the nodes on L,
- d specifies the degree distribution of $L^{\otimes r}$,
- ψ_i corresponds to the cotangent line class,
- $\xi_i = c_1(s_i^* L^{\otimes r})$,
- $\xi_e = c_1(s_e^* L^{\otimes r})$ where s_e is the node associated to e,
- $\eta_{a,b} = \pi_*\big(c_1(\omega_{\log})^a \, c_1(L^{\otimes r})^b\big)$.

1.6.4 The pull-back map π_L^*

Let $[\Gamma, \gamma] \in \mathsf{DG}_{g,n}^{r,L}$ be a decorated prestable graph representing a Chow cohomology class in $\mathfrak{M}_{g,n}^{r,L}$. The pull-back π_L^* along

$$\pi_L : \overline{\mathcal{M}}^r_{g,A,\beta}(X, S) \to \mathfrak{M}_{g,n}^{r,L}$$

is computed by exactly the same rules governing π_Z^*. The proof is identical.

Lemma 4 *The pull-back $\pi_L^*[\Gamma, \gamma]$ is obtained in terms of Chow cohomology classes of decorated X-valued r-twisted stable graphs by applying the following procedure:*

- *Replace the degree $\mathrm{d}(v) \in \mathbb{Z}$ of each vertex with all effective classes*
$$\beta(v) \in H_2(X, \mathbb{Z})$$
satisfying
$$\int_{\beta(v)} c_1(S) - \sum_{i \vdash v} a_i = \mathrm{d}(v)\,,$$
where the sum is over the legs i incident to v. Sum over all choices of $\beta(v)$.

- *Replace each ξ_i with $\xi_i + a_i\psi_i$.*

- *Replace each class $\eta_{0,b}$ at each vertex v with*
$$\eta_{0,b} - \sum_{i \vdash v} \sum_{k=1}^{b} \binom{b}{k} a_i^k \psi_i^{k-1} \xi_i^{b-k},$$
where the sum is again over the legs i incident to v.

All other decorations are kept the same.

1.7 Multiplication in the Chow cohomology of $\mathfrak{M}_{g,n}^{r,L}$

The product in Chow cohomology of the classes of two decorated r-twisted prestable graphs in $\mathsf{DG}_{g,n}^{r,L}$ is defined in a very similar way to the product of stable graphs carefully described in the appendix of [26]. We briefly sketch the construction and highlight the differences.

An *edge contraction* in an r-twisted prestable graph is defined in the natural way:

- if the edge contraction merges two vertices, the corresponding genera and degrees are summed,

- if the contracted edge is a loop, the genus of the base vertex increases by 1 and the degree remains unchanged.

The total degree and twisting conditions are still satisfied after edge contraction in an r-twisted prestable graph.

We define the product of two decorated r-twisted prestable graphs

$$[\Gamma_A, \gamma_A]\,,\ [\Gamma_B, \gamma_B]\ \in \mathsf{DG}^{r,L}_{g,n}$$

as follows. The product is a (possibly infinite) linear combination of decorated r-twisted prestable graphs.

We first consider prestable graphs

$$\Gamma \in \mathsf{G}^{r,L}_{g,n}$$

with edges colored by A, B, or both A and B, satisfying the conditions:

(i) after contracting the edges not colored A, we obtain Γ_A,

(ii) after contracting the edges not colored B, we obtain Γ_B.

For each such Γ, we add decorations by the following rules.

- The monomials $\psi^a\xi^b$ on the legs of the graph Γ are obtained by multiplying the corresponding leg monomials on the graphs Γ_A and Γ_B.
- The monomial ψ^a on a half-edge colored A only or colored B only is inherited from the graph Γ_A or Γ_B respectively. On an edge $e = (h', h'')$ colored both A and B, we take the product of the monomials on the corresponding edge in the graphs Γ_A and Γ_B and include an extra factor

  $$-\frac{1}{r}(\psi_{h'} + \psi_{h''})$$

  corresponding to the excess intersection.
- The monomial ξ^a on an edge colored A only or colored B only is inherited from the graph Γ_A or Γ_B respectively. On an edge colored both A and B, we take the product of the monomials on the corresponding edge in the graphs Γ_A and Γ_B.

- The factors $\eta_{a,b}$ of the monomials assigned to each vertex v of Γ_A (and Γ_B) are distributed in all possible ways among the vertices which collapse to v as Γ is contracted to Γ_A (and Γ_B). In other words, each factor is assigned to a unique vertex, and we sum over all such assignments. Each vertex of Γ is then marked with two monomials in the variables $\eta_{a,b}$, which we multiply together.

The product $[\Gamma_A, \gamma_A] \cdot [\Gamma_B, \gamma_B]$ is then the sum over all decorated r-twisted prestable graphs $[\Gamma, \gamma]$ produced by the above construction.

It is important to note that a product of two decorated r-twisted prestable graphs can be an *infinite* linear combination of decorated r-twisted prestable graphs. For instance, if we take the square of the graph with a single vertex of degree 0 and a single loop (and no decorations), we obtain, among other terms, a sum over all graphs with two vertices of degrees d and $-d$ connected by two edges. Since the integer d can be chosen arbitrarily, the result is an infinite linear combination. However, the product of two decorated r-twisted prestable graphs or even of infinite linear combinations of such graphs, is always well-defined because the coefficient of each graph in the product only involves a finite number of graphs in the factors. Therefore, the product rule transforms the vector space of possibly infinite linear combinations of decorated r-twisted prestable graphs into an algebra which agrees with the product in the Chow cohomology of $\mathfrak{M}^{r,L}_{g,n}$.

2 GRR for the universal line bundle

2.1 Universal bundles on universal curves over $\mathfrak{M}^{r,L}_{g,n}$

The universal twisted curve

$$\pi : \mathfrak{C}^{r,L}_{g,n} \to \mathfrak{M}^{r,L}_{g,n}$$

carries a universal line bundle $\mathcal{L}$. Consider a node in a singular fiber of the universal curve. The kernel of the group morphism ϕ of Section 1.2 acts on the fiber of $\mathcal{L}$ over the node. If the generator

$$(1, -1) \in \mu_r \times \mu_r$$

of the kernel acts on the fiber of L at the node by $e^{2\pi a/r}$, we assign the remainders a and $-a$ mod r to the branches of the curve at the node. Every node in the universal curve therefore acquires a *type*: a pair of remainders mod r assigned to the branches meeting at the node (with vanishing sum mod r). For the line bundle

$$\mathcal{Z} = \mathcal{L}^{\otimes r}\,,$$

the action of the kernel of ϕ is always trivial.

The push-forward L of the sheaf of invariant sections of $\mathcal{L}$ to the coarse universal curve $\mathsf{C}^{r,L}_{g,n}$, is a rank 1 torsion free sheaf which is described in detail in [15]. The push-forward of the sheaf of sections of $\mathcal{L}^{\otimes r}$ to $\mathsf{C}^{r,L}_{g,n}$ is just a line bundle $\widetilde{\mathcal{Z}}$.

On the bubbly universal curve $\widetilde{\mathsf{C}}^{r,L}_{g,n}$, we may pull-back the line bundle $\widetilde{\mathcal{Z}}$ from the coarse curve $\mathsf{C}^{r,L}_{g,n}$. The situation is more interesting for $\mathcal{L}$. Chiodo [14] has proven that there exists a line bundle

$$\widetilde{\mathcal{L}} \to \widetilde{\mathsf{C}}^{r,L}_{g,n}$$

for which the push-forward of the associated sheaf of sections to the coarse curve $\mathsf{C}^{r,L}_{g,n}$ is L. However, instead of the simple isomorphism

$$\mathcal{Z} = \mathcal{L}^{\otimes r} \quad \text{on} \quad \mathfrak{C}^{r,L}_{g,n}\,,$$

we have the more complicated relation

$$\widetilde{\mathcal{Z}} = \widetilde{\mathcal{L}}^{\otimes r}(D) \quad \text{on} \quad \widetilde{\mathsf{C}}^{r,L}_{g,n}\,,$$

where D is a linear combination of the exceptional divisors of the desingularization

$$\widetilde{\mathsf{C}}^{r,L}_{g,n} \to \mathsf{C}^{r,L}_{g,n}\,.$$

More precisely, a node of type

$$(a,b) \quad \text{with} \quad a+b=0 \pmod r\,,$$

gives rise to a chain of $r-1$ rational curves in the fiber of the bubbly universal curve. These rational curves appear in D with coefficients

$$a,\ 2a,\ \ldots,\ (b-1)a,\ ab,\ (a-1)b,\ \ldots,\ 2b,\ b,$$

see [14].

2.2 Polynomial classes

Ehrhart's theory states that if we take an n-dimensional polytope Δ with integer vertices and a polynomial $P(x_1, \ldots, x_n)$ in n variables, then the sum of values of P over the integer points inside $r\Delta$ is a polynomial in r. Actually, the same claim holds for a polynomial $P(x_1, \ldots, x_n; r)$ in $n+1$ variables depending explicitly on r. Similarly to the discussion in the Appendix of [35], we will use Ehrhart's theory to prove that a family of cohomology classes on $\mathfrak{M}^{r,L}_{g,n}$ projected to $\mathfrak{M}^{Z}_{g,n}$ forms a cohomology-valued Laurent polynomial in r.

Consider a family of polynomials

$$\Big\{ P_\Gamma \in \mathbb{C}[\mathrm{tw}_h, r, r^{-1}] \Big\}_{\Gamma \in \mathsf{DG}^{Z}_{g,n}}.$$

For each graph $\Gamma \in \mathsf{DG}^{Z}_{g,n}$, P_Γ is a polynomial in the variables tw_h, where h runs over the half-edges of Γ which *are not legs* (and P_Γ is a Laurent polynomial in r). The formal variables tw_h play the role of the variables x_i in the previous paragraph.

The family P_Γ determines a family of Chow cohomology classes on $\mathfrak{M}^{r,L}_{g,n}$ for all r:

$$\alpha = \sum_{\Gamma} \sum_{\mathrm{tw}} P_\Gamma\left(\mathrm{tw}(h), r, r^{-1}\right) \cdot [\Gamma, \mathrm{tw}]\,,$$

where the summation is over all r-twistings tw of all decorated prestable graphs $\Gamma \in \mathsf{DG}^{Z}_{g,n}$ which, equivalently, is the set $\mathsf{DG}^{r,L}_{g,n}$. Note that we have substituted the value of the twist $\mathrm{tw}(h) \in \{0, \ldots, r-1\}$ in place of the formal variable tw_h.

We call such families of classes *polynomial*. We let

$$\mathrm{val}(\alpha) = \inf_{\Gamma \in \mathsf{DG}^{Z}_{g,n}} \Big[\mathrm{val}_r(P_\Gamma) - |\mathrm{E}(\Gamma)|\Big]\,.$$

Proposition 5 *If α and β are two polynomial families of classes of valuations* $\mathrm{val}(\alpha)$ *and* $\mathrm{val}(\beta)$*, then the product*[5] *$\alpha\beta$ is a polynomial family of classes satisfying*

$$\mathrm{val}(\alpha\beta) \geq \mathrm{val}(\alpha) + \mathrm{val}(\beta)\,.$$

[5]The product of two families of Chow cohomology classes is defined by taking the products of the corresponding Chow cohomology classes on $\mathfrak{M}^{r,L}_{g,n}$ for all r.

Proof. Let Γ be a decorated prestable graph and tw an r-twisting. The coefficient of Γ in the product $\alpha\beta$ is a finite linear combination of coefficients of contractions of Γ in α and β. More precisely, the coefficient is a sum over all ways to label the edges of Γ with A, B, or AB and to split every monomial on a leg, edge, or vertex of Γ into a product of two factors labeled A and B while keeping a factor

$$-\frac{1}{r}(\psi' + \psi'')$$

aside for every edge labeled AB. For every such labelling, the contribution to the coefficient of Γ is the product of the two corresponding polynomials in $\mathrm{tw}(h)$, r, and r^{-1} times a factor of r^{-1} for each AB-edge. Therefore the coefficient of Γ is also a polynomial in $\mathrm{tw}(h)$, r, and r^{-1}.

The lowest possible order in r is given by summing the following three degrees:

- $\mathrm{val}(\alpha) +$ (number of A-edges and AB-edges) from the coefficient of the A-contraction of Γ in α,
- $\mathrm{val}(\beta) +$ (number of B-edges and AB-edges) from the coefficient of the B-contraction of Γ in β,
- $-$(number of AB-edges) from the excess intersection factor

$$-\frac{1}{r}(\psi' + \psi'')\,.$$

The sum of these three degrees is, indeed, $\mathrm{val}(\alpha) + \mathrm{val}(\beta) + |E(\Gamma)|$, so the valuation of $\alpha\beta$ is the sum of valuations of α and β (or larger if the lowest degree terms cancel out). $\Diamond$

Proposition 6 *Let α be a polynomial family of classes in the strata algebra of $\mathfrak{M}^{r,L}_{g,n}$. Let β be the push-forward of α to the strata algebra of $\mathfrak{M}^{Z}_{g,n}$. Then, for any decorated prestable graph $\Gamma \in \mathsf{DG}^Z_{g,n}$, the coefficient of Γ in β is a Laurent polynomial in r of valuation at least* $\mathrm{val}(\alpha)+2g-1$*, for all sufficiently large r.*

Proof. The projection from $\mathfrak{M}^{r,L}_{g,n}$ to $\mathfrak{M}^{Z}_{g,n}$ sums over all possible twists, which is the analog of summing the values of a polynomial over integer points in a polytope in Ehrhart's theory.

For every r-twisting tw of Γ, the push-forward from the stratum $[\Gamma, \mathrm{tw}]$ of $\mathfrak{M}^{r,L}_{g,n}$ to the stratum Γ of $\mathfrak{M}^{Z}_{g,n}$ has degree

$$r^{\sum_{v\in V(\Gamma)}(2\mathrm{g}(v)-1)} = r^{2g-2h^1(\Gamma)-|V(\Gamma)|}$$

see the proof of [35, Corollary 4].

The coefficient of Γ in β is obtained by summing the coefficients of $[\Gamma, \mathrm{tw}]$ in α over all r-twistings tw of Γ. According to Proposition $3''$ of the Appendix of [35], the sum is a Laurent polynomial in r, for r sufficiently large, with valuation at least

$$\mathrm{val}(\alpha) + |E(\Gamma)| + h^1(\Gamma)\,.$$

Multiplying the Laurent polynomial by the degree of the push-forward map, we obtain, again, a Laurent polynomial in r with valuation at least

$$\mathrm{val}(\alpha) + |E(\Gamma)| + h^1(\Gamma) + 2g - 2h^1(\Gamma) - V(\Gamma) = \mathrm{val}(\alpha) + 2g - 1$$

as claimed. ◊

2.3 Applying the GRR formula to $\widetilde{\mathsf{C}}^{r,L}_{g,n} \to \mathfrak{M}^{r,L}_{g,n}$

The bubbly universal curve

$$\pi : \widetilde{\mathsf{C}}^{r,L}_{g,n} \to \mathfrak{M}^{r,L}_{g,n}$$

is representable and proper over $\mathfrak{M}^{r,L}_{g,n}$. We may apply the Grothendieck-Riemann-Roch (GRR) formula to compute $R\pi_*\widetilde{\mathcal{L}}$,

$$\mathrm{ch}\left(R\pi_*\widetilde{\mathcal{L}}\right) = \pi_*\left(\mathrm{ch}(\widetilde{\mathcal{L}})\cdot \mathrm{td}(\pi)\right).$$

As before, let D_i be the class of the divisor of $\widetilde{\mathsf{C}}^{r,L}_{g,n}$ corresponding to the ith section. Let

$$K = c_1(\omega_{\log}) = c_1\Big(\omega\Big(\sum_{i=1}^{n} D_i\Big)\Big)\,,\quad \xi = c_1\big(\widetilde{\mathcal{Z}}\big)\,.$$

Following the notation of Section 2.1, let D be defined by

$$\widetilde{\mathcal{Z}} = \widetilde{\mathcal{L}}^{\otimes r}(D)\,.$$

Let j be the double covering of the locus of nodes in the singular fibers,

$$j : \Delta \to \widetilde{\mathsf{C}}^{r,L}_{g,n}\,.$$

The sheets of the covering correspond to the two ways of numbering the branches of the curve at the node. The domain Δ carries two cotangent line bundles corresponding to the two branches of the node. We denote by ν_1 and ν_2 first Chern classes of the two cotangent lines.

Using the above notation, we have:

$$\mathrm{ch}(\widetilde{\mathcal{L}}) = e^{\xi/r}e^{-D/r},$$

$$\mathrm{td}(\pi) = \frac{K}{e^K-1}\prod_{i=1}^{n}\frac{D_i}{1-e^{-D_i}}\left(1 + j_*\frac{1}{2}\sum_{m\geq 1}\frac{B_{2m}}{(2m)!}\frac{\nu_1^{2m-1}+\nu_2^{2m-1}}{\nu_1+\nu_2}\right),$$

see [14, 20].

Lemma 7 *We have*

$$\pi_*\Big(\mathrm{ch}(\widetilde{\mathcal{L}})\mathrm{td}(\pi)\Big) =$$

$$\pi_*\left[e^{\xi/r}\frac{K}{e^K-1}\prod_{i=1}^{n}\frac{D_i}{1-e^{-D_i}} \;+\; e^{\xi/r-D/r}\left(1 + j_*\frac{1}{2}\sum_{m\geq 1}\frac{B_{2m}}{(2m)!}\frac{\nu_1^{2m-1}+\nu_2^{2m-1}}{\nu_1+\nu_2}\right)\right].$$

Proof. The following intersections vanish in $\widetilde{\mathsf{C}}^{r,L}_{g,n}$ by the definition of $\omega_{\log}$ and the fact that the markings are disjoint from the nodes (before the bubbly resolution and, therefore, also after the bubbly resolution):

$$K\cdot\Delta = D_i\cdot\Delta = D_i\cdot D = 0\,.$$

Moreover, since the the bubble curve is crepant (see [14]), we have

$$K\cdot D = 0\,.$$

The push-forward of the constant term vanishes

$$\pi_*(1) = 0\,.$$

The formula is then easily obtained from the above vanishing from GRR. ◊

2.4 GRR and polynomiality in r

Proposition 8 *The GRR formula for the Chern character* $\mathrm{ch}_k(R\pi_*\widetilde{\mathcal{L}})$ *is a polynomial family with valuation* $-(k+1)$*. Its coefficients of lowest degree in* r *equal*

$$r^{-(k+1)}\,\eta_{0,k+1} \;+\; r^{-k} \sum_{\Gamma \text{ with one edge } (h,h')} \;\sum_{i+j+m=k-1} \frac{\xi_e^m}{m!}\frac{\mathrm{tw}(h)^{k+1-m}}{(k+1-m)!}(-1)^i\psi_h^i\psi_{h'}^j\,.$$

Proof. The first term in Lemma 7 only involves decorated graphs with a single vertex and with a single (trivial) weighting. The dependence on r in the first term is only through $e^{\xi/r}$. The degree k part of the class, obtained by π push-forward of the degree $k+1$ part, is a Laurent polynomial in r with lowest degree term

$$r^{-(k+1)}\cdot\pi_*(\xi^{k+1}) = r^{-(k+1)}\eta_{0,k+1}\,.$$

Since the second term in Lemma 7 is supported on Δ before π push-forward, we can write the result after π push-forward as a sum of contributions of graphs Γ with a single edge $e=(h,h')$. Since the factor $e^{\xi/r}$ is then the π pull-back of ξ_e, the crucial calculation is

$$\pi_*\left[e^{-D/r}\left(1+j_*\frac{1}{2}\sum_{m\geq 1}\frac{B_{2m}}{(2m)!}\frac{\nu_1^{2m-1}+\nu_2^{2m-1}}{\nu_1+\nu_2}\right)\right]. \tag{10}$$

Fortunately, the π push-forward (10) was computed by Chiodo in Step 3 of [14, Section 3]. The codimension k part of (10) is equal to a sum over r-weighted prestable decorated graphs Γ with a single edge $e=(h,h')$ with coefficient

$$r\,\frac{B_{k+1}(\mathrm{tw}(h)/r)}{(k+1)!}\sum_{i+j=k-1}(-1)^i\psi_h^i\psi_{h'}^j\,. \tag{11}$$

Here, $B_k(x)$ is the Bernoulli polynomial defined by

$$\frac{t\,e^{xt}}{e^t-1} = \sum_{k\geq 0} B_k(x)\frac{t^k}{k!}\,.$$

We see (11) is a a polynomial in $\mathrm{tw}(h)$, r and r^{-1} with lowest degree term in r given by

$$r^{-k}\,\frac{\mathrm{tw}(h)^{k+1}}{(k+1)!}\sum_{i+j=k-1}(-1)^i\psi_h^i\psi_{h'}^j\,. \tag{12}$$

The formula for the second term in Lemma 7 is then obtained by multiplying by $e^{\xi_e/r}$. ◊

Proposition 9 *The GRR formula for the Chern class $c_k(-R\pi_*\widetilde{\mathcal{L}})$ is a polynomial family with valuation $-2k$. The coefficient of lowest degree in r equals the degree k part of the mixed degree class*

$$\sum_{\Gamma\in\mathsf{G}_{g,n}^{r,L}}\frac{r^{-2k+|E(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|}\,j_{\Gamma*}\Bigg[\prod_{v\in\mathrm{V}(\Gamma)}\exp\left(-\frac{1}{2}\eta(v)\right)$$
$$\prod_{e=(h,h')\in\mathrm{E}(\Gamma)}\frac{1-\exp\left(-\frac{\mathrm{tw}(h)\mathrm{tw}(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\Bigg],$$

where $\eta=\eta_{0,2}$.

Proof. The Chern class c_k can be expressed as a quasi-homogeneous polynomial in the Chern characters,

$$c_k=\frac{1}{k!}\mathrm{ch}_1^k+\dots\,. \tag{13}$$

By Proposition 5, each monomial M on the right side of (13) is a polynomial family of classes with valuation at least $-(k+\deg(M))$. Since the monomial of largest degree is

$$M=\frac{1}{k!}\mathrm{ch}_1^k\,,$$

the Chern class c_k is a polynomial family with valuation at least $-2k$. Since only the lowest degree coefficient in r of the polynomial class ch_1 contributes to the lowest degree in r of the polynomial family c_k, the valuation of the latter is exactly $-2k$.

More precisely, the lowest degree coefficient in

$$1 + c_1(-R\pi_*\widetilde{\mathcal{L}}) + c_2(-R\pi_*\widetilde{\mathcal{L}}) + \dots$$

is obtained by exponentiating the formula for $\mathrm{ch}_1(-R\pi_*\widetilde{\mathcal{L}})$ given by Proposition 8 after changing the sign. A parallel exponentiation is taken in [35, Section 1]. ◊

We will now push-forward the GRR formula of Proposition 9 along the morphism

$$\epsilon : \mathfrak{M}^{r,L}_{g,n} \to \mathfrak{M}^{Z}_{g,n}$$

from the commutative diagram (8).

Corollary 10 *The push-forward $\epsilon_* c_k(-R\pi_*\widetilde{\mathcal{L}})$ is a Laurent polynomial in r with valuation $2g-2k-1$ for r sufficiently large. The coefficient of $r^{2g-2k-1}$ is obtained by substituting $r = 0$ into the polynomial*

$$\sum_{\Gamma \in \mathsf{G}^Z_{g,n}} \sum_{r\text{-twist tw}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \, j_{\Gamma *}\Bigg[\prod_{v \in \mathrm{V}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v)\right) \prod_{e=(h,h') \in \mathrm{E}(\Gamma)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \Bigg]$$

and extracting the part of degree k.

Proof. The claim follows directly from Proposition 6, Proposition 9, and the calculation:

$$-2k + |\mathrm{E}(\Gamma)| + 2g - 2h^1(\Gamma) - |\mathrm{V}(\Gamma)| - (2g - 2k - 1) = -h^1(\Gamma) \, .$$

While graphs $\Gamma \in \mathsf{G}^{r,L}_{g,n}$ in Proposition 9 correspond to classes in the Chow cohomology of $\mathfrak{M}^{r,L}_{g,n}$, graphs

$$\Gamma \in \mathsf{G}^Z_{g,n}$$

here correspond to classes in the Chow cohomology of $\mathfrak{M}^Z_{g,n}$. ◊

Finally, as proven by Chiodo [13, Proposition 4.3.3] and [14, Lemma 2.2.5], all of the following three push-forwards yield the same complex on $\mathfrak{M}^{r,L}_{g,n}$:

- $R\pi_*\mathcal{L}$ via $\pi : \mathfrak{C}^{r,L}_{g,n} \to \mathfrak{M}^{r,L}_{g,n}$,
- $R\pi_*\mathsf{L}$ via $\pi : \mathsf{C}^{r,L}_{g,n} \to \mathfrak{M}^{r,L}_{g,n}$,
- $R\pi_*\widetilde{\mathcal{L}}$ via $\pi : \widetilde{\mathsf{C}}^{r,L}_{g,n} \to \mathfrak{M}^{r,L}_{g,n}$.

Hence, Corollary 10 also holds for $R\pi_*\mathcal{L}$.

Corollary 11 *The push-forward $\epsilon_* c_k(-R\pi_*\mathcal{L})$ is a Laurent polynomial in r with valuation $2g-2k-1$ for r sufficiently large. The coefficient of $r^{2g-2k-1}$ is obtained by substituting $r=0$ into the polynomial*

$$\sum_{\Gamma\in\mathsf{G}^Z_{g,n}} \sum_{r\text{-twist tw}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} j_{\Gamma*}\Bigg[\prod_{v\in\mathrm{V}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v)\right) \prod_{e=(h,h')\in\mathrm{E}(\Gamma)} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\Bigg]$$

and extracting the part of degree k.

3 Localization analysis

3.1 Overview

Let X be a nonsingular projective variety over $\mathbb{C}$. Let $S \to X$ be a line bundle. Let

$$A = (a_1, \ldots, a_n)$$

be a vector of double ramification data as defined in Section 0.2,

$$\sum_{i=1}^n a_i = \int_\beta c_1(S)\,.$$

The double ramification cycle

$$\mathsf{DR}_{g,A,\beta}(X) \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))$$

is defined via the moduli space of stable maps to rubber $\overline{\mathcal{M}}^{\sim}_{g,A,\beta}(X,S)$. We prove here the claim of Theorem 2,

$$\mathsf{DR}_{g,A,\beta}(X) = \mathsf{P}^g_{g,A,\beta} \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))\,.$$

Our path follows [35, Section 2].

Theorem 2 for an arbitrary vector A can be deduced from the case where every a_i is nonzero by forgetting the markings with $a_i = 0$. Our proof of Theorem 2 has no mathematical difficulties when A has zeros, but the discussion then requires separating markings into three types instead of just positive and negative. For simplicity, we assume every a_i is nonzero.

3.2 Target geometry

Denote by $\mathbb{P}(X,S)$ the $\mathbb{CP}^1$-bundle $\mathbb{P}(\mathcal{O}_X \oplus S) \to X$. Following the notation of Section 0.2, let

$$D_0\,,\; D_\infty \subset \mathbb{P}(X,S)$$

be the divisors defined by the projectivizations of the loci

$$\mathcal{O}_X \oplus \{0\} \quad \text{and} \quad \{0\} \oplus S$$

respectively. After applying Cadman's rth root construction [11] to D_0, we obtain a bundle

$$\mathbb{P}(X,S)[r] \to X$$

with fiber given by the orbifold projective line $\mathbb{CP}^1[r]$ with single orbifold point with stabilizer $\mathbb{Z}/r\mathbb{Z}$.

Denote by $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$ the moduli space of stable maps to the orbifold $\mathbb{P}(X,S)[r]$ relative to D_∞. The moduli space parametrizes connected, nodal, twisted curves $(C, x_1, \ldots, x_n)$ of genus g with n markings[6] together with a map

$$f : C \to P(X,S)[r]\,,$$

where $P(X,S)[r]$ is an expansion[7] of $\mathbb{P}(X,S)[r]$ along D_∞. The following conditions are required to hold over

$$D_0, D_\infty \subset P(X,S)[r]\ :$$

[6] As always, the markings are distinct and away from the nodes.

[7] The expansion has a canonical $D_0 \subset P(X,S)[r]$ from the bulk and a canonical $D_\infty \subset P(X,S)[r]$ from the last component of the expansion.

(i) The stack structure of the domain curve C occurs only at the nodes over D_0 and at the markings corresponding to positive elements of A (which must be mapped to D_0). The monodromies associated to the latter markings are specified by the parts a_i of A at these markings. More precisely, the monodromies are a_i mod r.

(ii) The only points mapped by f to D_∞ are the markings x_i with negative a_i. The multiplicity of D_∞ at x_i is $-a_i$. The map f satisfies the ramification matching condition over the internal nodes of the expansion P.

(iii) Stability requires the full data $(C, x_1, \ldots, x_n, f, P(X,S)[r])$ to have only finitely many automorphisms.

The moduli space $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$ has a perfect obstruction theory and a virtual class of dimension

$$\dim_{\mathbb{C}} [\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)]^{\mathrm{vir}} = \dim_{\mathbb{C}} [\overline{\mathcal{M}}_{g,n,\beta}(X,S)]^{\mathrm{vir}} - (g-1) + \sum_{i | a_i > 0} \lfloor \frac{a_i}{r} \rfloor \,, \quad (14)$$

see [36, Section 1.1].

We will be most interested in the case where $r > \sum_{i=1}^n |a_i|$. The last term in (14) then vanishes. We refer the reader to [1, 36, 43] for a more detailed definition of the moduli space of stable relative maps.

3.3 The $\mathbb{C}^*$-fixed loci

3.3.1 The $\mathbb{C}^*$-action

The standard $\mathbb{C}^*$-action over X on the projective bundle $\mathbb{P}(\mathcal{O} \oplus S) \to X$, defined by

$$\xi \cdot [x, z, s] = [x, z, \xi s] \,,$$

lifts canonically to $\mathbb{C}^*$-actions on $\mathbb{P}(X,S)[r]$ and $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$.

3.3.2 Graphs

The $\mathbb{C}^*$-fixed loci of $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$ are in bijective correspondence with decorated graphs

$$\Phi = (\mathrm{V}\,,\ \mathrm{E}\,,\ \mathrm{L}\,,\ \mathrm{g}:\mathrm{V}\to\mathbb{Z}_{\geq 0}\,,\ \beta:\mathrm{V}\to H_2(X,\mathbb{Z})\,,\ \ell:\mathrm{V}\to\{0,\infty\}\,,\ d:\mathrm{E}\to\mathbb{Z}_{>0})$$

which satisfy the following six properties:

(i) V is a vertex set with a genus function g, a degree function β, and a label ℓ. For $v\in\mathrm{V}$, the degree $\beta(v)$ must be an effective[8] curve class. We also require the genus and degree conditions to hold:

$$g = \sum_{v\in\mathrm{V}} \mathrm{g}(v) + h^1(\Phi) \quad \text{and} \quad \beta = \sum_{v\in\mathrm{V}} \beta(v)\,.$$

(ii) L, the set of legs, is placed in bijective correspondence with the n markings:

- legs marked i with $a_i > 0$ are incident to vertices labeled 0,
- legs marked i with $a_i < 0$ are incident to vertices labeled ∞.

(iii) E is the edge set. For $e\in\mathrm{E}$, the edge degree d_e corresponds to the d_e-th power map

$$\mathbb{CP}^1[r]\to\mathbb{CP}^1[r]\,.$$

(iv) Φ is a connected graph, and Φ is bipartite with respect to labeling ℓ: every edge is incident to a 0-labeled vertex and an ∞-labeled vertex.

(v) If $\ell(v)=0$, denote by $A(v)$ the list of integers formed by the values a_i for the legs i incident to v *and* by the values $-d_e$ for the edges e incident to v. For every such vertex v, we impose the condition

$$|A(v)| = \int_{\beta(v)} c_1(S) \mod r\,,$$

where $|A(v)|$ is the sum of the elements of $A(v)$.

[8]Effective here includes the class $0\in H^2(X,\mathbb{Z})$.

(vi) If $\ell(v) = \infty$, denote by $A(v)$ the list of integers formed by the values a_i for the legs i incident to v *and* by the values d_e for the edges e incident to v. For every such vertex, we impose the condition

$$|A(v)| = \int_{\beta(v)} c_1(S) \ .$$

To every 0-labeled vertex v of Φ, we assign the space $\overline{\mathcal{M}}^r_{\mathrm{g}(v),A(v),\beta(v)}(X,S)$. We will use the notation

$$\overline{\mathcal{M}}^r_v \ = \overline{\mathcal{M}}^r_{\mathrm{g}(v),A(v),\beta(v)}(X,S)\,.$$

As explained in Section 1.3, the forgetful map

$$\overline{\mathcal{M}}^r_v \to \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X)$$

is a finite map of degree r^{2g-1}. The virtual fundamental class $\left[\overline{\mathcal{M}}^r_v\right]^{\mathrm{vir}}$ of $\overline{\mathcal{M}}^r_v$ is of dimension

$$\begin{aligned} \dim_{\mathbb{C}} \left[\overline{\mathcal{M}}^r_v\right]^{\mathrm{vir}} &= \dim_{\mathbb{C}} \left[\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X)\right]^{\mathrm{vir}} \\ &= (3-\dim(X))(\mathrm{g}(v)-1) + \mathrm{n}(v) + \int_{\beta(v)} c_1(X)\,. \end{aligned}$$

A $\mathbb{C}^*$-fixed relative stable map with vector $A = (a_1, \ldots, a_n)$,

$$f : (C, x_1, \ldots, x_n) \to \mathbb{P}(X,S)[r]/D_\infty$$

takes two basic forms:

- If the target does not expand, then the stable map has a finite number of preimages of D_∞ which correspond precisely to the markings i with $a_i < 0$. Each such preimage is then described by an unstable vertex of Φ decorated by ∞.

- If the target expands, then C contains a possibly disconnected subcurve mapping to the rubber $\mathbb{P}(X,S)$ – with no orbifold structure at D_0 in the rubber. The ramification in the rubber over D_0 is specified by the degrees d_e of the edges of Φ, and the ramification over D_∞ is specified by the negative elements of A. The ramification in the rubber occurs over D_0 and is specified by the negative elements of A over D_∞ Every ∞-labeled vertex v describes a connected component of the rubber map. We will denote the moduli space of stable maps to rubber by $\overline{\mathcal{M}}^{\sim}_{\infty}$.

In the second case above, let $\mathrm{g}(\infty)$ be the genus of the possibly disconnected domain of the rubber map, $\mathrm{n}(\infty)$ the total number of legs and edges adjacent to ∞-labeled vertices, and $\beta(\infty)$ their total degree. The virtual fundamental class $\left[\overline{\mathcal{M}}^{\sim}_{\infty}\right]^{\mathrm{vir}}$ has dimension

$$\begin{aligned}\dim_{\mathbb{C}} \left[\overline{\mathcal{M}}^{\sim}_{\infty}\right]^{\mathrm{vir}} &= \dim_{\mathbb{C}} \left[\overline{\mathcal{M}}_{\mathrm{g}(\infty),\mathrm{n}(\infty),\beta(\infty)}(X)\right]^{\mathrm{vir}} - \mathrm{g}(\infty) \\ &= (3-\dim(X))(\mathrm{g}(\infty)-1) + \mathrm{n}(\infty) + \int_{\beta(\infty)} c_1(X) - g(\infty)\ .\end{aligned}$$

The image of the virtual fundamental class $\left[\overline{\mathcal{M}}^{\sim}_{\infty}\right]^{\mathrm{vir}}$ in the moduli space of (not necessarily connected) stable maps to X is denoted by DR_∞.

3.3.3 Unstable vertices

A vertex $v \in \mathsf{V}(\Phi)$ is *unstable* if $\beta(v) = 0$ and $2\mathrm{g}(v) - 2 + \mathrm{n}(v) \leq 0$. There are four types of unstable vertices:

(i) $\ell(v) = 0$, $\mathrm{g}(v) = 0$, v carries no markings and one incident edge,

(ii) $\ell(v) = 0$, $\mathrm{g}(v) = 0$, v carries no markings and two incident edges,

(iii) $\ell(v) = 0$, $\mathrm{g}(v) = 0$, v carries one marking and one incident edge,

(iv) $\ell(v) = \infty$, $\mathrm{g}(v) = 0$, v carries one marking and one incident edge.

The target of the stable map expands if and only if there is at least one ∞-labeled stable vertex.

A stable map in the $\mathbb{C}^*$-fixed locus corresponding to Φ is obtained by gluing together maps associated to the vertices $v \in \mathsf{V}(\Phi)$ with Galois covers associated to the edges. Denote by $\mathsf{V}^0_{\rm st}(\Phi)$ the set of 0-labeled stable vertices of Φ. Then the $\mathbb{C}^*$-fixed locus corresponding to Φ is isomorphic to the product

$$\overline{\mathcal{M}}_\Phi = \begin{cases} \prod_{v\in \mathsf{V}^0_{\rm st}(\Phi)} \overline{\mathcal{M}}^r_v \;\times\; \overline{\mathcal{M}}^\sim_\infty\,, & \text{if the target expands,}\\ \prod_{v\in \mathsf{V}^0_{\rm st}(\Phi)} \overline{\mathcal{M}}^r_v\,, & \text{if the target does not expand,} \end{cases}$$

quotiented by the automorphism group of Φ *and* the product of cyclic groups $\mathbb{Z}_{d_e}$ associated to the Galois covers of the edges.

The natural morphism corresponding to Φ,

$$\iota : \overline{\mathcal{M}}_\Phi \to \overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)\,,$$

is of degree

$$|\mathrm{Aut}(\Phi)| \cdot \prod_{e\in \mathsf{E}(\Phi)} d_e$$

onto the image $\iota(\overline{\mathcal{M}}_\Phi)$.

Lemma 12 *For r sufficiently large, the unstable vertices of type (i) and (ii) can* not *occur.*

Proof. We define $\beta' \in H_2(X,\mathbb{Z})$ to be an *effective summand* of β if both β' and $\beta - \beta'$ are effective cycle classes (including 0). Let b be the maximum of $\left| \int_{\beta'} c_1(S) \right|$ over all effective summands of β. Further, let

$$a_+ = \sum_{i | a_i > 0} a_i$$

be the sum of the positive elements of the vector A. Assume

$$r > 2(a_+ + b)\,.$$

Let β_0 (respectively, β_∞) be the sum of degrees of all vertices of Φ with label 0 (respectively, ∞), so

$$\beta = \beta_0 + \beta_\infty\,.$$

We then have

$$a_+ - \int_{\beta_0} c_1(S) = \sum_{e \in \mathrm{E}(\Phi)} d_e \,.$$

By our choice of r, we have $\sum_{e \in \mathrm{E}(\Phi)} d_e < r/2$.

At each 0-labeled stable vertex $v \in \mathsf{V}(\Phi)$, the condition

$$\sum_{i \vdash v} a_i - \int_{\beta(v)} c_1(S) = \sum_{e \vdash v} d_e \mod r$$

holds by the conditions on the graph Φ. By our choice of r, both the absolute value of $\sum_{i \vdash v} a_i - \int_{\beta(v)} c_1(S)$ and $\sum_{e \vdash v} d_e$ are less than $r/2$. Therefore, the equality mod r is actually an exact equality:

$$\sum_{i \vdash v} a_i - \int_{\beta(v)} c_1(S) = \sum_{e \vdash v} d_e \,. \tag{15}$$

For an unstable vertex of type (i) or (ii), we have both $\sum_{i \vdash v} a_i = 0$ and $\beta(v) = 0$. The sum of the degrees of edges adjacent to such a vertex then vanishes by (15). However, the degree of every edge is a positive integer and the graph Φ is connected. The resulting contradiction implies that there are no unstable vertices of types (i) and (ii). $\diamondsuit$

3.4 Localization formula

We write the $\mathbb{C}^*$-equivariant Chow ring of a point as

$$A^*_{\mathbb{C}^*}(\bullet) = \mathbb{Q}[t] \,,$$

where t is the first Chern class of the standard representation.

For the localization formula, we will require the inverse of the $\mathbb{C}^*$-equivariant Euler class of the virtual normal bundle in $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$ to the $\mathbb{C}^*$-fixed locus corresponding to Φ. Let

$$f : (C, x_1, \ldots, x_n) \to P(X,S)[r] \,, \qquad [f] \in \overline{\mathcal{M}}_\Phi \,,$$

where $P(X,S)[r]$ is a possible expansion of $\mathbb{P}(X,S)[r]$ along D_∞. Denote by

$$T \to \mathbb{P}(X,S)[r]$$

the tangent line bundle to the fiber of $\mathbb{P}(X,S)[r] \to X$. For simplicity, we will also denote by T the pull-back of T from $\mathbb{P}(X,S)[r]$ to the expansion $P(X,S)[r]$. The formula for the inverse Euler class can then be written as:

$$\frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})} = \frac{e(H^1(C, f^*T(-D_\infty)))}{e(H^0(C, f^*T(-D_\infty)))} \frac{1}{\prod_i e(N_i)} \frac{1}{e(N_\infty)} . \tag{16}$$

Several aspects of Formula (16) require explanation. To start, we assume that r is sufficiently large (using Lemma 12) to exclude the presence of unstable vertices of types (i) and (ii) in Φ. To compute the leading factor of (16),

$$\frac{e(H^1(C, f^*T(-D_\infty)))}{e(H^0(C, f^*T(-D_\infty)))} , \tag{17}$$

we use the normalization exact sequence for the domain C tensored with the line bundle $f^*T(-D_\infty)$. The associated long exact sequence in cohomology decomposes the leading factor into a product of vertex, edge, and node contributions:

- Let $v \in \mathsf{V}(\Phi)$ be a stable vertex over $D_0 \subset \mathbb{P}(X,S)[r]$ corresponding to a moduli space
$$\overline{\mathcal{M}}^r_v = \overline{\mathcal{M}}^r_{\mathrm{g}(v),A(v),\beta(v)}(X,S) .$$
The orbifold universal curve[9]
$$\pi : \mathcal{C}^{r,\mathrm{orb}}_{\mathrm{g}(v),A(v),\beta(v)} \to \overline{\mathcal{M}}^r_{\mathrm{g}(v),A(v),\beta(v)}$$
carries an orbifold line bundle $\mathcal{L}^{\mathrm{orb}}$ (the rth root of the pull-back of S) which is the pull-back of T to the universal curve. Therefore, the contribution
$$\frac{e(H^1(C_v, f^*T(-D_\infty))}{e(H^0(C_v, f^*T(-D_\infty))}$$
yields the class
$$e\left((-R\pi_*\mathcal{L}^{\mathrm{orb}}) \otimes \mathcal{O}^{(1/r)}\right) = c_{\mathrm{rk}}\left((-R\pi_*\mathcal{L}^{\mathrm{orb}}) \otimes \mathcal{O}^{(1/r)}\right)$$

[9]The moduli space $\overline{\mathcal{M}}^r_{g,A,\beta}(X)$ may be considered with the universal curve $\mathcal{C}^r_{g,A,\beta}$ of Section 1.6.2 or with the orbifold universal curve $\mathcal{C}^{r,\mathrm{orb}}_{g,A,\beta}$. While $\mathcal{C}^r_{g,A,\beta}$ has orbifold structure only at the nodes of the fibers, $\mathcal{C}^{r,\mathrm{orb}}_{g,A,\beta}$ has orbifold structure both at the markings x_i *and* at the nodes. A full discussion of the differences will be given in Section 3.5.4.

in $A^*(\overline{\mathcal{M}}^r_v) \otimes \mathbb{Q}\left[t, \frac{1}{t}\right]$, where $\mathcal{O}^{(1/r)}$ is a trivial line bundle with a $\mathbb{C}^*$-action of weight $\frac{1}{r}$ and

$$\mathrm{rk} = \mathrm{g}(v) - 1 + |\mathsf{E}(v)|$$

is the virtual rank of $-R\pi_*\mathcal{L}^{\mathrm{orb}}$.

Unstable 0-labeled vertices of type (iii) contribute factors of 1. Since the restriction of $T(-D_\infty)$ to D_∞ is trivial, the ∞-labeled vertices also contribute factors of 1.

- The edge contribution is trivial since the degree $\frac{d_i}{r}$ of $f^*T(-D_\infty)$ is less than 1, see [36, Section 2.2].

- The contribution of a node N over D_0 is trivial. Indeed, the space of sections $H^0(N, f^*T(-D_\infty))$ vanishes because N must be stacky, and $H^1(N, f^*T(-D_\infty))$ is trivial for dimension reasons. Nodes over D_∞ contribute 1.

Consider next the last two factors of (16),

$$\frac{1}{\prod_i e(N_i)} \frac{1}{e(N_\infty)} \,.$$

- The product $\prod_i e(N_i)^{-1}$ is over the nodes that correspond to half-edges of the graph Φ adjacent to a 0-labeled vertex. If N is a node corresponding to an edge $e \in \mathsf{E}(\Phi)$ and the associated vertex v is stable, then

$$e(N) = \frac{t + \mathrm{ev}_e^*(c_1(S))}{r\, d_e} - \frac{\psi_e}{r} \,. \tag{18}$$

  The factor corresponds to the smoothing of the node N of the domain curve: $e(N)$ is the first Chern class of the normal line bundle of the divisor of nodal domain curves. The first Chern classes of the tangent lines to the branches at the node are divided by r because of the orbifold twist, see Section 1.2.

  In the case of an unstable vertex of type (iii), the associated edge does *not* produce a node of the domain. The type (iii) edge incidences do *not* appear in $\prod_i e(N_i)^{-1}$.

- N_∞ corresponds to the expansion of the target $\mathbb{P}(X,S)[r]$ over D_∞. The factor $e(N_\infty)$ is 1 if the target $(\mathbb{P}(X,S)[r]/D_\infty)$ does not expand and
$$e(N_\infty) = -\frac{t+\Psi_\infty}{\prod_{e\in \mathsf{E}(\Phi)} d_e}$$
if the target expands.

  Here, Ψ_∞ is the first Chern class of a line bundle defined as follows. Consider a point of the moduli space $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$ where the target expands. For the target over the point, the divisor along which the target expands carries tangent line bundles to the two components of the target. The tensor product of these two line bundles is a trivial line bundle. Thus the tensor product is the pull-back of a line bundle N over the divisor of $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$ where the target expands, and Ψ_∞ is the first Chern class of N. See [43] for more details.

The virtual class of $\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)$ can be written in terms of the $\mathbb{C}^*$-fixed point loci by the virtual localization formula [25]:

$$\left[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)\right]^{\mathrm{vir}} = \sum_{\Phi} \frac{1}{|\mathrm{Aut}(\Phi)|}\frac{1}{\prod_{e\in\mathsf{E}(\Phi)} d_e}\cdot \iota_*\left(\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right) \tag{19}$$

in $A^*\left(\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)\right)\otimes\mathbb{Q}[t,\frac{1}{t}]$. Our analysis of the inverse Euler class of the virtual normal bundle yields the following contributions to $\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}$ associated to the graph Φ :

- a factor
$$\prod_{e\in\mathsf{E}(v)} \frac{r}{\frac{t+\mathrm{ev}_e^*(c_1(S))}{d_e}-\psi_e}\cdot\sum_{d\geq 0} c_d(-R\pi_*\mathcal{L}^{\mathrm{orb}})\left(\frac{t}{r}\right)^{\mathrm{g}(v)-1+|\mathrm{E}(v)|-d}$$
for each stable vertex $v\in\mathsf{V}(\Phi)$ over 0, where $\mathrm{E}(v)$ is the set of edges incident to v,

- a factor
$$-\frac{\prod_{e\in\mathsf{E}(\Phi)} d_e}{t+\psi_\infty}\cdot\mathsf{DR}_\infty$$
if the target expands, where DR_∞ is the virtual class of the moduli space of map to the rubber over D_∞.

3.5 The formula for the DR-cycle

3.5.1 Three operations

We will now perform three operations on the localization formula (19) for the virtual class $[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)]^{\text{vir}}$:

(i) the $\mathbb{C}^*$-equivariant push-forward via

$$\epsilon : \overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty) \to \overline{\mathcal{M}}_{g,n,\beta}(X) \tag{20}$$

to the moduli space $\overline{\mathcal{M}}_{g,n,\beta}(X)$ of stable maps to X with trivial $\mathbb{C}^*$-action,

(ii) extraction of the coefficient of t^{-1} after push-forward by ϵ_*,

(iii) extraction of the coefficient of r^0.

After push-forward by ϵ_*, the coefficient of t^{-1} is equal to 0 because

$$\epsilon_*[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r]/D_\infty)]^{\text{vir}} \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X)) \otimes \mathbb{Q}[t]\,.$$

Using Proposition 9, all terms of the t^{-1} coefficient will be seen to be polynomials in r, so operation (iii) will be well-defined. After operations (i-iii), only two nonzero terms will remain. The cancellation of the two remaining terms will prove Theorem 2.

To perform (i-iii), we multiply the ϵ-push-forward of the localization formula (19) by t and extract the coefficient of t^0r^0. To simplify the computations, we introduce the new variable

$$s = tr\,.$$

Then, instead extracting the coefficient of t^0r^0, we extract the coefficient of s^0r^0.

3.5.2 Push-forward to $\overline{\mathcal{M}}_{g,n,\beta}(X)$

For each vertex $v \in \mathsf{V}(\Phi)$, following the notation of Section 3.3.2, we have

$$\overline{\mathcal{M}}^r_v = \overline{\mathcal{M}}^r_{\mathrm{g}(v),A(v),\beta(v)}\,.$$

As in diagram (8), we denote by

$$\epsilon : \overline{\mathcal{M}}_v^r \to \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X) \tag{21}$$

the morphism obtained by forgetting the rth root line bundle. The maps ϵ in (20) and (21) are compatible. Denote by

$$\widehat{c}_d \;=\; r^{2d-2\mathrm{g}(v)+1}\epsilon_* c_d(-R\pi_*\mathcal{L}^{\mathrm{orb}}) \;\in\; A^d(\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X))\,. \tag{22}$$

By Corollary 10, $\widehat{c}_d$ is a polynomial in r for r sufficiently large.

We now write the inverse Euler class of the virtual normal bundle for the $\mathbb{C}^*$-fixed point locus associated to the graph Φ *after* push-forward along

$$\epsilon : \overline{\mathcal{M}}_{g,n,\beta}(\mathbb{P}(X,S)[r]/D_\infty) \to \overline{\mathcal{M}}_{g,n,\beta}(X)$$

in terms of $s = rt$. The analysis of Section 3.4 yields the following contributions to $\epsilon_*\iota_*\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}$:

- a factor

  $$\frac{r}{s}\cdot\prod_{e\in\mathsf{E}(v)}\frac{d_e}{1+\frac{r}{s}\mathrm{ev}_e^*(c_1(S))-\frac{rd_e}{s}\psi_e}\cdot\sum_{d\geq 0}\widehat{c}_d\, s^{\mathrm{g}(v)-d}\cdot\left[\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X)\right]^{\mathrm{vir}}$$

  $$\in A_*(\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X))\otimes\mathbb{Q}\left[s,\frac{1}{s}\right]$$

  for each stable vertex $v\in\mathsf{V}(\Phi)$ over 0,

- a factor

  $$-\frac{r}{s}\cdot\frac{\prod_{e\in\mathsf{E}(\Phi)}d_e}{1+\frac{r}{s}\Psi_\infty}\cdot\mathsf{DR}_\infty$$

  if the target degenerates.

For the first factor, we have used the compatibility of the virtual classes

$$\epsilon_*\left[\overline{\mathcal{M}}_v^r\right]^{\mathrm{vir}} = r^{2g-1}\left[\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)}(X)\right]^{\mathrm{vir}}$$

proven in [4, Theorem 6.8].

3.5.3 Extracting coefficients

From (19) and the contribution calculus for Φ presented in Section 3.5.2, we have a complete formula for the $\mathbb{C}^*$-equivariant push-forward of t times the virtual class:

$$\epsilon_*\Big(t[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r],D_\infty)]^{\mathrm{vir}}\Big) = \frac{s}{r}\cdot\sum_{\Phi}\frac{1}{|\mathrm{Aut}(\Phi)|}\frac{1}{\prod_{e\in\mathsf{E}(\Phi)}d_e}\cdot\epsilon_*\left(\frac{[\overline{\mathcal{M}}_\Phi]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right). \quad (23)$$

Extracting the coefficient of r^0. By Corollary 10, the classes $\widehat{c}_d$ are polynomial in r for r sufficiently large. We have an r in the denominator in the prefactor on the right side of (23) which comes from the multiplication by t on the left side. However, in all other factors, we only have positive powers of r, with at least one r per 0-labeled vertex of the graph and one more r if the target degenerates. The *only* graphs Φ which contribute to the coefficient of r^0 are those with *exactly one r in the numerator*. There are only two graphs which have exactly one r factor in the numerator:

- the graph Φ' with a 0-labeled stable vertex of full genus g and an ∞-labeled unstable vertex of type (iv) for each negative element of A,
- the graph Φ'' with a stable ∞-labeled vertex of full genus g and a 0-labeled unstable type (iii) vertex for each positive element of A.

No terms involving $\mathrm{ev}^*(c_1(S))$, ψ or Ψ_∞ classes contribute to the r^0 coefficient of either Φ' or Φ'' since every ψ class in the localization formula comes with an extra factor of r. The homology class associated with Φ'' is, by definition, the double ramification cycle $\mathsf{DR}_{g,A,\beta}(X)$. We can now write the r^0 coefficient of the right side of (23) as

$$|\mathrm{Aut}|\cdot\mathrm{Coeff}_{r^0}\left[\epsilon_*\Big(t[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r],D_\infty)]^{\mathrm{vir}}\Big)\right] = \quad (24)$$

$$\mathrm{Coeff}_{r^0}\left\{\sum_{d\geq 0}\widehat{c}_d\, s^{g-d}\cdot\big[\overline{\mathcal{M}}_{g,n,\beta}(X)\big]^{\mathrm{vir}}\right\} - \mathsf{DR}_{g,A,\beta}(X)$$

in $A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))\otimes\mathbb{Q}[s,\frac{1}{s}]$. Here, $|\mathrm{Aut}| = |\mathrm{Aut}(\Phi')| = |\mathrm{Aut}(\Phi'')| = 1$.

Extracting the coefficient of s^0. The remaining powers of s in (24) appear only in the classes $\widehat{c}_d$ (in the contribution of the graph Φ'). In order to obtain s^0, we must take $d = g$,

$$|\mathrm{Aut}| \cdot \mathrm{Coeff}_{s^0 r^0}\left[\epsilon_*\Big(t[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r], D_\infty)]^{\mathrm{vir}}\Big)\right] = \tag{25}$$

$$\mathrm{Coeff}_{r^0}\left\{\widehat{c}_g \cdot \left[\overline{\mathcal{M}}_{g,n,\beta}(X)\right]^{\mathrm{vir}}\right\} - \mathsf{DR}_{g,A,\beta}(X)$$

in $A_*(\overline{\mathcal{M}}_{g,n,\beta}(X)$.

3.5.4 Proof of Theorem 2

Since $\mathrm{Coeff}_{s^0 r^0}\left[\epsilon_*\Big(t[\overline{\mathcal{M}}_{g,A,\beta}(\mathbb{P}(X,S)[r], D_\infty)]^{\mathrm{vir}}\Big)\right]$ vanishes, we can rewrite equality (25) as

$$\mathsf{DR}_{g,A,\beta}(X) = \\ \mathrm{Coeff}_{r^0}\left[r\epsilon_* c_g(-R\pi_*\mathcal{L}^{\mathrm{orb}}) \cdot \left[\overline{\mathcal{M}}_{g,n,\beta}(X)\right]^{\mathrm{vir}}\right] \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X))\,. \tag{26}$$

Here, we have used the definition (22) of $\widehat{c}_g$,

$$\widehat{c}_g = r\epsilon_* c_g(-R\pi_*\mathcal{L}^{\mathrm{orb}})\,.$$

What is the relationship between the line bundle

$$\mathcal{L} \;\text{ on }\; \pi : \mathcal{C}^r_{g,A,\beta} \to \overline{\mathcal{M}}^r_{g,A,\beta}(X)$$

considered in Section 1.6.2 and the line bundle

$$\mathcal{L}^{\mathrm{orb}} \;\text{ on }\; \pi : \mathcal{C}^{r,\mathrm{orb}}_{g,A,\beta} \to \overline{\mathcal{M}}^r_{g,A,\beta}(X)$$

which appears in (26) here? The definitions are slightly different:

- $\mathcal{C}^r_{g,A,\beta}$ has orbifold structure only at the nodes of the fibers,
- $\mathcal{L}^{\otimes r} = f^*S(-\sum_{i=1}^n a_i x_i)$,
- $\mathcal{C}^{r,\mathrm{orb}}_{g,A,\beta}$ has orbifold structure both at the markings x_i *and* at the nodes,

- $\mathcal{L}^{\text{orb}} = f^*T$.

The universal curve $\mathcal{C}^r_{g,A,\beta}$ is the coarsification along the markings x_i of $\mathcal{C}^{r,\text{orb}}_{g,A,\beta}$. By considering the sheaf of invariant sections of $\mathcal{L}^{\text{orb}}$ on $\mathcal{C}^r_{g,A,\beta}$, we obtain an rth root of

$$f^*S\Big(-\sum_{i|a_i>0} a_i x_i - \sum_{e\in\mathrm{E}}(r-d_e)x_e\Big) = f^*S\Big(-\sum_{i=1}^n a_i x_i\Big)\otimes\mathcal{O}_C\Big(-\sum_{i|a_i<0} x_i\Big)^{\otimes r}.$$

So the rth roots corresponding to $\mathcal{L}$ and the coarsification of $\mathcal{L}^{\text{orb}}$ are related simply by the factor $\mathcal{O}_C\Big(-\sum_{i|a_i<0} x_i\Big)$ which yields a shift

$$\xi \;\mapsto\; \xi - r\sum_{i|a_i<0} D_i$$

in the study of $\mathcal{L}$ in Section 2. The lowest r terms of

$$c_g(-R\pi_*\mathcal{L}) \;\text{ and }\; c_g(-R\pi_*\mathcal{L}^{\text{orb}})$$

are therefore equal, and we can apply Corollary 11 to calculate

$$\mathrm{Coeff}_{r^0}\left[r\epsilon_* c_g(-R\pi_*\mathcal{L}^{\text{orb}})\cdot\left[\overline{\mathcal{M}}_{g,n,\beta}(X)\right]^{\text{vir}}\right] \in A_*(\overline{\mathcal{M}}_{g,n,\beta}(X)).$$

Corollary 11 gives the coefficient of r^{-1} of $\epsilon_* c_g(-R\pi_*\mathcal{L})$ in the Artin stack $\mathfrak{M}^Z_{g,n}$. The answer is obtained by the $r=0$ restriction of the degree g part of

$$\sum_{\Gamma\in\mathsf{G}^Z_{g,n}}\sum_{r\text{-twist tw}}\frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\, j_{\Gamma*}\Bigg[\prod_{v\in\mathrm{V}(\Gamma)}\exp\left(-\frac{1}{2}\eta(v)\right) \prod_{e=(h,h')\in\mathrm{E}(\Gamma)}\frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\Bigg].$$

By applying Lemma 3, we can calculate the pull-back of the class in $\overline{\mathcal{M}}_{g,n,\beta}(X)$. After interpreting the twists as weights, we obtain the $r=0$ restriction of the degree g part of

$$\sum_{\substack{\Gamma \in \mathsf{G}_{g,n,\beta}(X) \\ w \in \mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} j_{\Gamma *} \Bigg[\prod_{i=1}^{n} \exp\left(\frac{1}{2} a_i^2 \psi_i + a_i \xi_i\right) \prod_{v \in \mathrm{V}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v)\right)$$

$$\prod_{e=(h,h') \in \mathrm{E}(\Gamma)} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \Bigg] .$$

The result is exactly the formula for the X-valued DR-cycle claimed in Theorem 2. $\Diamond$

4 Applications

4.1 A topological view

Let X be a nonsingular projective variety with a line bundle $S \to X$, and let

$$\mathbb{P}(\mathcal{O}_X \oplus S) \to X$$

be the canonically associated $\mathbb{CP}^1$-bundle over X with 0-divisor D_0 and ∞-divisor D_∞.

Localization with respect to the fiberwise $\mathbb{C}^*$-action immediately leads to a calculation of the Gromov-Witten theory of $\mathbb{P}(\mathcal{O}_X \oplus S)$ in terms of the Gromov-Witten theory of X and the class

$$c_1(S) \in H_2(X, \mathbb{Z}) .$$

In [43], an effective procedure was given to compute the Gromov-Witten invariants of the associated rubber geometry and the three relative geometries

$$\mathbb{P}(\mathcal{O}_X \oplus S)/D_0 \ , \quad \mathbb{P}(\mathcal{O}_X \oplus S)/D_\infty \ , \quad \mathbb{P}(\mathcal{O}_X \oplus S)/D_0 \cup D_\infty$$

in terms of the Gromov-Witten theory of X and the class $c_1(S)$. The results may be viewed as analogues in Gromov-Witten theory of the Leray-Hirsch Theorem.

A basic consequence of the X-valued DR-cycle formula of Theorem 2 is a much stronger result on the level of Gromov-Witten classes (not invariants).

Proposition 13 *The X-valued* DR*-cycle formula calculates the push-forward to the moduli space of maps to X of the virtual fundamental classes of the moduli spaces of stable maps to*

$$\mathbb{P}(\mathcal{O}_X \oplus S)/D_0 \ , \quad \mathbb{P}(\mathcal{O}_X \oplus S)/D_\infty \ , \quad \mathbb{P}(\mathcal{O}_X \oplus S)/D_0 \cup D_\infty \tag{27}$$

in terms of tautological classes and $c_1(S) \in A^1(X)$.

Proof. Theorem 2 provides a formula for the push-forward to the moduli space of maps to X of the virtual fundamental classes of moduli space of maps to rubber in terms of tautological classes and

$$c_1(S) \in A^1(X)\,.$$

To apply Theorem 2, we localize the equivariant virtual fundamental classes of the moduli spaces of stable maps to the three relative geometries (27) with respect to the fiberwise $\mathbb{C}^*$-action. The $\mathbb{C}^*$-fixed contributions are either absolute or relative. The virtual localization formula [25] on the absolute side is already of the desired form. On the relative side, after removing the cotangent line via the rubber calculus [43, Section 1.5], the desired form is provided by Theorem 2. $\Diamond$

While [43] provides an algorithm for calculating the Gromov-Witten invariants of the relative geometries (27), the complexity of the method is not practical for calculations[10]. On the other hand, Theorem 2 may be used effectively for calculation.

In case X is a point, exact calculations using Pixton's formula were presented in [35, Section 3] for Hodge classes and Hodge integrals. In Section 4.2 below, an application of Theorem 2 is presented where X is the resolution of the surface A_ℓ-singularity.

4.2 Resolution of surface singularities

4.2.1 Gromov-Witten invariants of X_ℓ

Maulik [41] computed the Gromov-Witten invariants of the toric surface X_ℓ obtained by resolving the surface A_ℓ-singularity. We briefly review the geom-

[10]The results, however, have been used for theoretical purposes, see [51, 53].

etry of the problem and refer the reader to [41] for a more detailed treatment.

The resolution of the A_ℓ-singularity is a nonsingular quasi-projective surface X_ℓ with ℓ exceptional divisors. The intersection pairing of the divisors is given by the Cartan matrix C of the Lie algebra A_ℓ. For a simply laced Lie algebra, the Cartan matrix is given by

$$C_{ii} = -2\,, \quad C_{ij} = 1$$

if vertices i and j in the Dynkin diagram are connected by an edge.

There is a $(\mathbb{C}^*)^2$-action on the resolution X_ℓ of the A_ℓ-singularity which leaves every exceptional divisor invariant. We denote by t_1 and t_2 the corresponding equivariant weights. Because X_ℓ is a holomorphic symplectic variety, the ordinary Gromov-Witten invariants of X_ℓ vanish except in degree 0. Maulik computed the *reduced* Gromov-Witten invariants which correspond to $(t_1 + t_2)$-coefficient of the $(\mathbb{C}^*)^2$-equivariant Gromov-Witten invariants of X_ℓ. The $(t_1 + t_2)$-coefficient is the lowest nonvanishing coefficient.

Theorem 14 (Maulik [41]) *Let α be a root, let $\beta = d\alpha \in H_2(X_\ell, \mathbb{Z})$ be a nonzero curve class, and let*

$$\omega_1, \ldots, \omega_p \in H^2(X_\ell, \mathbb{Z})$$

be divisor classes. Let $b_1, \ldots, b_p \geq 0$ and $c_1, \ldots, c_q > 0$ be integers subject to the dimensional constraint

$$\sum_{i=1}^{p} b_i + \sum_{j=1}^{q} c_j = g + q\,.$$

Then, we have

$$\left\langle \prod_{i=1}^{p} \tau_{b_i}(\omega_i) \, \prod_{j=1}^{q} \tau_{c_j}(1) \right\rangle_{g,p+q,\beta}^{X_\ell,\,\mathrm{red}} = \frac{(2g+p+q-3)!}{(2g+p-3)!}\, d^{2g+p-3} \cdot \prod_{i=1}^{p} \frac{b_i!}{(2b_i+1)!} \left(-\frac{1}{2}\right)^{b_i} (\alpha, \omega_i) \cdot \prod_{j=1}^{q} \frac{(c_j-1)!}{(2c_j-1)!} \left(-\frac{1}{2}\right)^{c_j-1}.$$

If β is not a multiple of a root or if the dimensional constraint is not satisfied, then the invariant vanishes.

4.2.2 The DR-cycle

The rank of the Cartan matrix for A_ℓ is ℓ. The dimension of the equivariant cohomology of X_ℓ is $\ell+1$, and the Poincaré intersection form is given by an extension of the Cartan matrix C :

$$\eta = \left(\begin{array}{c|c} \frac{1}{(p+1)t_1t_2} & 0 \\ \hline 0 & C \end{array} \right). \tag{28}$$

Let $A = (a_1, \ldots, a_n)$ be a vector of integers satisfying $\sum_{i=1}^{n} a_i = 0$. Let $\beta \neq 0$. We can compute the reduced rubber Gromov-Witten invariant

$$\left\langle \prod_{i=1}^{n} \tau_0(\omega_i) \cdot \mathsf{DR}_{g,A,\beta}(X_\ell, \mathcal{O}_{X_\ell}) \right\rangle^{\text{red}} \tag{29}$$

using the formula of Theorem 2 for the X_ℓ-valued DR-cycle. The intersection number (29) is, by definition, the coefficient of $t_1 + t_2$ in the corresponding equivariant intersection (the lowest nonvanishing coefficient).

Lemma 15 *The only X_ℓ-valued stable graphs which contribute to the coefficient of $t_1 + t_2$ are graphs with one vertex.*

Proof. The $(\mathbb{C}^*)^2$-equivariant Gromov-Witten invariant associated to a vertex v of the graph Γ with $\beta(v) \neq 0$ is a polynomial divisible by t_1+t_2 since X_ℓ is holomorphic symplectic. By formula (28) for η, an edge of Γ contributes either a constant, if the markings at the half-edges are divisors, or a factor of

$$(p+1)t_1t_2 \,,$$

if the markings at the half-edges are equal to 1.

For a vertex v of Γ with $\beta(v) = 0$, a more careful study using the $(\mathbb{C}^*)^2$-localization formula for the degree 0 Gromov-Witten invariants of X_ℓ is required. The $(\mathbb{C}^*)^2$-invariant locus in X_ℓ consists of $\ell+1$ points. The tangent weights at the kth point are

$$\alpha_k(t_1,t_2) = (p+2-k)t_1 - (k-1)t_2 \,, \quad -\alpha_{k+1}(t_1,t_2) = (k-p-1)t_i + kt_2 \tag{30}$$

and satisfy the equation

$$\alpha_k - \alpha_{k+1} = t_1 + t_2 \,.$$

For a divisor class ω of X_ℓ, we denote by $\omega^{(k)}(t_1, t_2)$ the restriction of ω to the kth invariant point in $(\mathbb{C}^*)^2$-equivariant cohomology. The restriction $\omega^{(k)}(t_1, t_2)$ is a linear combination of α_k and α_{k+1}.

The $(\mathbb{C}^*)^2$-fixed locus of $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v),\beta(v)=0}(X_\ell)$ is the union of $\ell+1$ copies of $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$ corresponding to constant maps to the $\ell+1$ invariant points. The contribution of the kth copy to the $(\mathbb{C}^*)^2$-equivariant integral

$$\left\langle \prod_{i=1}^{p} \tau_{b_i}(\omega_i) \prod_{j=1}^{q} \tau_{c_j}(1) \right\rangle^{X_\ell}_{\mathrm{g}(v),\, \mathrm{n}(v)=p+q,\, \beta(v)=0}$$

equals

$$-\frac{\prod_{i=1}^{p} \omega_i^{(k)}}{\alpha_k\, \alpha_{k+1}} \int_{\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)=p+q}} \prod_{i=1}^{p} \psi_i^{b_i} \prod_{j=1}^{q} \psi_{p+j}^{c_j} \cdot \Lambda^\vee(\alpha_k)\Lambda^\vee(-\alpha_{k+1}) \,. \qquad (31)$$

Here, we use the notation

$$\Lambda^\vee(\alpha) = \alpha^{\mathrm{g}(v)} - \alpha^{\mathrm{g}(v)-1}\lambda_1 + \cdots + (-1)^{\mathrm{g}(v)}\lambda_{\mathrm{g}(v)} \,.$$

The weights α_k, α_{k+1}, and $\omega_i^{(k)}$ are linear forms in t_1 and t_2.

By formula (30), the weights α_k and α_{k+1} are never proportional to t_1+t_2. Thus, the (t_1+t_2)-valuation of the rational function (31) is nonnegative. In other words, every vertex v of Γ of nonzero degree has (t_1+t_2)-valuation at least one, while every edge and degree zero vertex has (t_1+t_2)-valuation at least zero. Since we are interested in the coefficient of t_1+t_2 of the result, there can be only one vertex of nonzero degree, and we can restrict ourselves to the (t_1+t_2)-valuation zero part of every degree zero vertex contribution.

We can extract the (t_1+t_2)-valuation zero part of (31) by substituting

$$t_1 = t\,, \quad t_2 = -t\,.$$

In particular, then $\alpha_k = -\alpha_{k+1}$ and hence, by Mumford's identity [44] for Hodge classes,

$$\Lambda^\vee(\alpha_k)\Lambda^\vee(-\alpha_{k+1}) = (-1)^{\mathrm{g}(v)}\alpha_k^{2\mathrm{g}(v)} \,.$$

Thus, the contribution (31) simplifies to

$$(-1)^{\mathrm{g}(v)-1}\alpha_k^{2\mathrm{g}(v)-2}\prod_{i=1}^{p}\omega_i^{(k)}\int_{\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)=p+q}}\prod_{i=1}^{p}\psi_i^{b_i}\prod_{j=1}^{q}\psi_{p+j}^{c_j}\,, \qquad (32)$$

where α_k and $\omega_i^{(k)}$ are now linear forms in t.

The contribution (32) is a Laurent monomial in t of degree $2g-2+p$. In the total contribution of the graph Γ, we will have a product of these monomials over the genus 0 vertices and also a product of monomials

$$-(p+1)t^2$$

over the edges carrying the class 1 on both half-edges. We distribute the edge factor t^2 to the two adjacent vertices, one factor of t for each vertex. For the invariant (29), all legs of Γ carry divisors. Hence, every half-edge carrying the class 1 contributes a factor of t to the monomial. Every degree 0 vertex therefore contributes a factor of

$$t^{2\mathrm{g}(v)-2+p+q}=t^{2\mathrm{g}(v)-2+\mathrm{n}(v)}\,.$$

By the stability condition, for a degree 0 vertex, the integer $2\mathrm{g}(v)-2+\mathrm{n}(v)$ is positive. Every degree 0 vertex thus contributes a monomial factor of positive degree. A product of such factors can never have a constant term. We conclude that there are no degree 0 vertices and, as claimed, only one vertex of nonzero degree. ◊

We are ready now to compute the reduced rubber Gromov-Witten invariant

$$\left\langle \prod_{i=1}^{n}\tau_0(\omega_i)\cdot \mathsf{DR}_{g,A,\beta}(X_\ell,\mathcal{O}_{X_\ell})\right\rangle^{\mathrm{red}}.$$

For the computation, we will need two identities.

Lemma 16 *We have*

$$\sum_{i+j=n}\frac{1}{(2i+1)!(2j+1)!}=\frac{2^{2n+1}}{(2n+2)!}\,.$$

Proof. After multiplying the left side by $(2n+2)!$, we obtain the well-known sum of odd binomial coefficients. ◊

The Bernoulli number are defined by the following generating series:

$$\sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \frac{t}{e^t - 1}\,.$$

We define the functions $\mathcal{S}(t)$ and $\mathcal{G}(t)$ by

$$\begin{aligned} \mathcal{S}(t) &= \frac{\sin(t/2)}{t/2} = \sum_{b\geq 0} \frac{(-1)^b}{(2b+1)!\,4^b}\, t^{2b}\,, \\ \mathcal{G}(t) &= -\frac{1}{2}\sum_{c\geq 1} \frac{B_{2c}}{2c} \sum_{c'+c''=c-1} \frac{(-1)^{c'}}{(2c'+1)!\,4^{c'}} \frac{(-1)^{c''}}{(2c''+1)!\,4^{c''}}\, t^{2c} \\ &= \sum_{c\geq 1} (-1)^c \frac{B_{2c}}{2c} \frac{t^{2c}}{(2c)!}\,. \end{aligned}$$

The last equality follows from Lemma 16.

Lemma 17 *We have* $\mathcal{S}(t) = e^{\mathcal{G}(t)}$.

Proof. The verification is straightforward:

$$\begin{aligned} t\left(\log(\mathcal{S}(t))\right)' &= t\left(\log \frac{\sin(t/2)}{t/2}\right)' \\ &= t \cdot \frac{t/2}{\sin(t/2)} \cdot \left(\frac{\frac{1}{2}\cos(t/2)}{t/2} - \frac{\frac{1}{2}\sin(t/2)}{(t/2)^2} \right) \\ &= \frac{t}{2}\cot(t/2) - 1 \\ &= \frac{it}{e^{it}-1} + \frac{it}{2} - 1 \\ &= \sum_{b\geq 1} (-1)^b B_{2b} \frac{t^{2b}}{(2b)!} \\ &= t\mathcal{G}'(t)\,. \end{aligned}$$

◊

We now substitute the formula of Theorem 2 for the DR-cycle in (29). Since only graphs with a single vertex contribute, we obtain the following sum over the number k of loops:

$$\sum_{k\geq 0}\frac{1}{2^k\,k!}\sum_{\substack{b_1,\dots,b_n\\ c_1,\dots,c_k}}\sum_{\substack{c_1'+c_1''=c_1-1\\ \cdots\\ c_k'+c_k''=c_k-1}}\sum_{\substack{\mu_1,\nu_1\\ \cdots\\ \mu_k,\nu_k}}\prod_{j=1}^{k}\left(-\eta^{\mu_j\nu_j}\frac{B_{2c_j}}{2c_j}\right)$$
$$\times\prod_{i=1}^{n}\frac{(a_i^2/2)^{b_i}}{b_i!}\prod_{j=1}^{k}\frac{(1/2)^{c_j'}}{c_j'!}\frac{(1/2)^{c_j''}}{c_j''!}\left\langle\prod_{i=1}^{n}\tau_{b_i}(\omega_i)\prod_{j=1}^{k}\tau_{c_j'}(\omega_{\mu_j})\tau_{c_j''}(\omega_{\nu_j})\right\rangle_{g-k,\beta}^{\mathrm{red}}.$$

In the above sum, c_j' and c_j'' are the powers of the ψ-classes at the branches of the jth node. The indices μ and ν run over a basis of $(\mathbb{C}^*)^2$-equivariant divisor classes of X_ℓ and $\eta^{\mu\nu}$ is the inverse of the Poincaré intersection form $\eta_{\mu\nu}$.

In fact, we can restrict the range of μ_j and ν_j in the diagonal splitting by one. Indeed,

$$\eta^{1,1}=(p+1)t_1t_2\,,$$

while we are interested in the coefficient of t_1+t_2. We can therefore replace the inverse of η by the inverse of the Cartan matrix C and write $(C^{-1})^{\mu\nu}$ instead of $\eta^{\mu\nu}$.

Next, we apply Maulik's formula of Theorem 14 to the reduced Gromov-Witten invariants which appear and use our definitions of the generating series $\mathcal{S}$ and $\mathcal{G}$, We conclude that (29) equals

$$d^{2g+n-3}\prod_{i=1}^{n}(\alpha,\omega_i)$$
$$\times\sum_{k\geq 0}\frac{1}{2^k\,k!}\,[t^{2g}]\left(\prod_{i=1}^{n}\mathcal{S}(a_it)\right)\left(2\mathcal{G}(t)\sum_{\mu,\nu}(\alpha,\omega_\mu)(C^{-1})^{\mu\nu}(\alpha,\omega_\nu)\right)^k.$$

Since $\sum_{\mu,\nu}(\alpha,\omega_\mu)(C^{-1})^{\mu\nu}(\alpha,\omega_\nu)=(\alpha,\alpha)=-2$, the formula simplifies to

$$d^{2g+n-3}\prod_{i=1}^{n}(\alpha,\omega_i)\cdot\sum_{k\geq 0}\frac{1}{2^k\,k!}\,[t^{2g}]\left(\prod_{i=1}^{n}\mathcal{S}(a_it)\right)(-4\mathcal{G}(t))^k$$

$$= d^{2g+n-3} \prod_{i=1}^{n} (\alpha, \omega_i) \cdot [t^{2g}] \left(\prod_{i=1}^{n} \mathcal{S}(a_i t) \right) \exp(-2\mathcal{G}(t))$$

$$= d^{2g+n-3} \prod_{i=1}^{n} (\alpha, \omega_i) \cdot [t^{2g}] \frac{\prod_{i=1}^{n} \mathcal{S}(a_i t)}{\mathcal{S}(t)^2} .$$

The final equality coincides[11] with [41, Proposition 3.6] except for the automorphism factors of the partitions (omitted here since we have numbered our marked points).

The same calculation as above using the formula of Theorem 2 for the DR-cycle yields the following more general evaluation for the rubber theory over X_ℓ.

Theorem 18 *Let $S \to X_\ell$ be a $(\mathbb{C}^*)^2$-equivariant line bundle on X_ℓ. Let α be a root, let $\beta = d\alpha \in H_2(X_\ell, \mathbb{Z})$ be a nonzero curve class, and let*

$$A = (a_1, \ldots, a_n)$$

be a vector of integers satisfying

$$\sum_{i=1}^{n} a_i = \int_\beta c_1(S) .$$

Then, we have the evaluation

$$\left\langle \prod_{i=1}^{n} \tau_0(\omega_i) \cdot \mathsf{DR}_{g,A,\beta}(X_\ell, S) \right\rangle^{\mathrm{red}} = d^{2g+n-3} \prod_{i=1}^{n} (\alpha, \omega_i) \cdot [t^{2g}] \frac{\prod_{i=1}^{n} \mathcal{S}(a_i t)}{\mathcal{S}(t)^2} .$$

Proof. Since $S \to X$ is now not necessarily trivial, Theorem 2 has additional ξ_i terms at the markings and $\pi_*(\xi^2)$ terms at the vertices. Lemma 15 still holds since the changes in the DR-cycle formula due to the line bundle S play no role in the argument. The k-loop summation for

$$\left\langle \prod_{i=1}^{n} \tau_0(\omega_i) \cdot \mathsf{DR}_{g,A,\beta}(X_\ell, S) \right\rangle^{\mathrm{red}}$$

[11] The result is also stated in [41, Proposition 3.1] where the factor $\prod_{i=1}^{n} (\alpha, \omega_i)$ is forgotten.

is again

$$\sum_{k\geq 0}\frac{1}{2^k\,k!}\sum_{\substack{b_1,\dots,b_n\\ c_1,\dots,c_k}}\sum_{\substack{c'_1+c''_1=c_1-1\\ \cdots\\ c'_k+c''_k=c_k-1}}\sum_{\substack{\mu_1,\nu_1\\ \cdots\\ \mu_k,\nu_k}}\prod_{j=1}^{k}\left(-\eta^{\mu_j\nu_j}\frac{B_{2c_j}}{2c_j}\right)$$
$$\times\prod_{i=1}^{n}\frac{(a_i^2/2)^{b_i}}{b_i!}\prod_{j=1}^{k}\frac{(1/2)^{c'_j}}{c'_j!}\frac{(1/2)^{c''_j}}{c''_j!}\left\langle\prod_{i=1}^{n}\tau_{b_i}(\omega_i)\prod_{j=1}^{k}\tau_{c'_j}(\omega_{\mu_j})\tau_{c''_j}(\omega_{\nu_j})\right\rangle_{g-k,\beta}^{\mathrm{red}}.$$

The extra ξ_i and $\pi_*(\xi^2)$ terms produce additional factors of the equivariant parameters (and hence do not affect the reduced invariants). The evaluation of the k-loop formula is then just as before. $\Diamond$

4.2.3 Remarks

Maulik's evaluation of the reduced rubber invariants (29) played a crucial role in establishing the GW/DT/PT correspondences for toric 3-folds, see [42, 52]. His calculation of (29) in the case of A_1 relied upon the evaluation of the stationary theory of $\mathbb{CP}^1$ in [46, 47]. Using Maulik's A_1 argument in the reverse direction, Theorem 2 via Theorem 18 provides a completely new DR derivation of the stationary Gromov-Witten theory of $\mathbb{CP}^1$.

Theorem 18 is also new. The DR-cycle for the rubber

$$\mathbb{P}(\mathcal{O}_{X_\ell}\oplus S)\to X_\ell$$

constructed from the line bundle $S\to X_\ell$ had not been considered before. The DR perspective puts all the rubber theories over X_ℓ on the same footing.

4.3 The tautological ring of the moduli of stable maps

Pixton's formula for the standard DR-cycle leads to relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ first conjectured by Pixton [55] and later proven by Clader and Janda [16]. After defining an appropriate strata algebra and tautological ring for the moduli space of stable maps $\overline{\mathcal{M}}_{g,n,\beta}(X)$, Bae [5] uses the formula of Theorem 2 for the X-valued DR-cycle to construct tautological relations in the Chow theory of $\overline{\mathcal{M}}_{g,n,\beta}(X)$, a rich new direction of study.

4.4 Universal Abel-Jacobi theory on the Picard stack

The classifying space of the group $\mathbb{C}^*$ is $\mathbb{CP}^\infty$. If we take the $\mathbb{CP}^N$-valued DR-cycle in the limit $N \to \infty$, we may hope that the result, suitably interpreted, is a universal DR-cycle on the moduli space of line bundles on curves. The required universal Abel-Jacobi theory on the Picard stack is developed in [6]. The calculation of the universal DR-cycle there is both motivated by and dependent upon our calculation of $\mathbb{CP}^N$-valued DR-cycles. The circle of ideas, also using [33], leads to the proof of the formulas for the loci of holomorphic and meromorphic differentials in $\overline{\mathcal{M}}_{g,n}$ conjectured in the Appendix of [24].

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Department of Mathematics, Univ. of Michigan
janda@umich.edu

Departement Mathematik, ETH Zürich
rahul@math.ethz.ch

Department of Mathematics, MIT
apixton@mit.edu

CNRS, Versailles Mathematics Laboratory
dimitri.zvonkine@uvsq.fr