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CALCULUS 2 WORKSHOPS

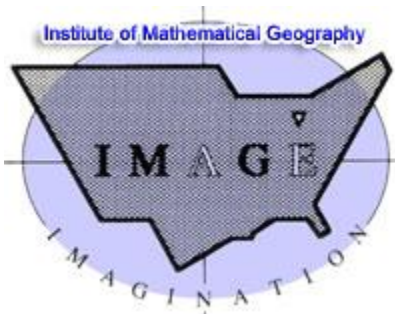
Six Semester-long Series

Derived from Experience in the

Department of Mathematics and Computer Science

Lawrence Technological University

Southfield, Michigan



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with input from

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Professor James Nanny

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Table of Contents

Acknowledgments.....	iii
Introduction	iv
Series 1.....	6
Series 2.....	35
Series 3.....	63
Series 4.....	92
Series 5.....	120
Series 6.....	148

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Most of all, however, I thank the hundreds of students at Lawrence Technological University who participated in these workshops and offered constructive feedback.

Finally, I thank Sandra L. Arlinghaus, Ph.D., of the Institute of Mathematical Geography for her major role in bringing this book to light.

The collaborative experience developed in the production of this book harks back to the collaborative experience we know that many of our students enjoyed!

William C. Arlinghaus
Ann Arbor, MI

INTRODUCTION

In 1997, a group of three of us worked to develop workshops in support of Calculus 2 lectures. My colleagues, Mike Merscher and Jim Nanny, in the Department of Mathematics and Computer Science, assisted me in creating problem sets that would both foster critical thinking skills about calculus and develop an appreciation for the benefits of interactive and collaborative work.

My original suggestions were for the following topics:

Week 1 Inverse Functions

Week 2 Exponential and Logarithmic Functions

Week 3 Inverse Trigonometric Functions

Week 4 l'Hospital's Rule

Week 5 Integration by Parts

Week 6 Trigonometric Integral

Week 7 Partial Fractions

Week 8 Approximate Integration

Week 9 Improper Integrals

Week 10 Arc Length

Week 11 Sequences or Sequences and Series

Week 12 Series or Series Tests

Week 13 Test Comparisons or Power Series

Week 14 Power Series or Taylor Series

Over time, we developed variations on these topics, in six different series, from 1997 through 2002. Variations on whether logarithms or exponentials came first, dictated differing ordering of the workshops, as did other concerns that we noted in the process of using the documents. Some decisions were made based on the current choice of textbook. In this document, comments that in the past made reference to a specific place in a specific textbook have been replaced with general phrases such as 'It was established elsewhere...' or 'Earlier...' Flexible insertion of other references might suit institutional variation.

In addition, so that students (from semester to semester) would have a consistent experience from one offering of Calculus 2 to another, but probably not an identical experience to what

their roommates might earlier have had, we deliberately altered material from one series to the next. There are obvious merits to such jumbling of materials that a prudent instructor follows in order to optimize materials actually learned.

These series of workshops were classroom/laboratory tested over a period of six years with feedback from hundreds of students. Those who participated fully in the workshop experience felt that they learned more and did better in the course than if they had not gone to the extra workshops. We all hope that your students might have a similar experience and that you, as an instructor, might derive some benefit today from our success in the past.

William C. Arlinghaus
Ann Arbor, Michigan
December, 2015.

Calculus 2 Workshop for a Single Semester

SERIES 1

Classroom tested at Lawrence Technological University
Department of Mathematics and Computer Science

Prof. William C. Arlinghaus, Ph.D.
with input from
Prof. Michael Mersher
Prof. James Nanny

Note: This particular series has no workshop for Week 11. It serves as a model for how to adjust the set when holidays, or other events, remove a workshop from the sequence.

The goal of the calculus workshop is to provide each student with (1) a deeper understanding of the important mathematical concepts encountered in calculus and (2) an opportunity to develop beneficial problem-solving

skills within a collaborative group setting. While the workshop program is designed to be essentially independent of a traditional lecture portion of a course, the work done in the workshop will generally be related to many materials covered in lectures. Instructors may make a judicious selection from the various series that appear below, depending on how an individual course might be structured.

The calculus workshop is designed to be completed in 1 hour and 15 minutes, once a week throughout a semester. Students should work in small groups on a special set of problems, distributed at the beginning of the workshop session. Workshop facilitators should be available to answer questions and offer limited suggestions during each workshop session. Work groups might be reassigned periodically during the semester.

Members of each group are encouraged to work together, share ideas, and divide tasks when appropriate in order to complete as many of the assigned exercises as possible during the scheduled workshop session. Groups are also encouraged to voluntarily exchange e-mail usernames and to set up additional meeting times as needed during the week in order to complete unfinished solutions and to share all final results.

Each student is required to turn in an individual written report following each weekly workshop. Each graded workshop report should be assigned a score ranging from 0 to 10 points (or other scheme at instructor discretion).

The overall semester grade for the workshop will equal the sum of the highest ten report scores (maximum 100 points, using the ten-point scale suggested above) and will be considered equivalent to one in-class exam.

One final note: Students are also encouraged to consider forming independent study groups to work on regularly assigned homework and to prepare for exams throughout the semester. (Keep in mind that the workshop session is not to be used for this purpose.)

Calculus Workshop Report Guidelines

Weekly workshop reports should be submitted to a fixed regular location by a fixed time. Late reports should not be considered. Graded reports should be returned the following week.

Each student must submit a complete workshop report for each workshop attended. Every member of a group is expected to record the solutions collaboratively obtained by that group either during the workshop or, if necessary, at a subsequent group meeting. Once the group work is completed, each student is expected to independently prepare a final workshop report recounting the computational details of the solutions obtained and providing all final answers and required graphs. In some cases it may also be appropriate to include a brief summary of the methods used and any general conclusions that can be drawn from the results.

Mathematical solutions should be complete, precise, well organized, and notationally correct. In all written explanations, proper attention should be given to legibility, grammar, spelling, and punctuation. Each solution should be clearly labeled with the appropriate problem number.

Each week students should receive a Workshop Report cover page which must be included as the first page of the individual report. On it students must provide

- Name and student ID number
- Date
- Lecture instructor
- Names of all other group members

Additional sheets should be attached in order with a staple in the upper left corner. Use only one side of each sheet. Partial sheets and pages torn from spiral notebooks or ring binders are not acceptable. The use of graph paper is recommended for hand-drawn graphs.

Excessive or habitual tardiness by any member of a group is unfair to the other members and puts the whole group at a disadvantage. For this reason, groups with fewer than four members will generally be combined with other groups. Any student arriving late will be reassigned to a different group as space is available. Anyone arriving over 20 minutes late will be excluded from participating in that session and will be marked absent. There will be no opportunity to “makeup” a missed workshop session.

Calculus Workshop Series 1
Calculus 2
Week 1

1. Let

$$f(x) = \begin{cases} 4x + 4, & x \leq -2 \\ \frac{5x-14}{6}, & -2 \leq x \leq 4 \\ \frac{3x-7}{5}, & x \geq 4 \end{cases}$$

- a) Plot the 4 points corresponding to $x = -3$, $x = -2$, $x = 4$, and $x = 9$.
- b) Draw the graph of $f(x)$. (Don't plot any more points.)
- c) As you can see, the graph consists of portions of 3 lines. What are the slopes of these three portions of lines?
- d) Plot the mirror images (across $y=x$) of the 4 points you found in part a).
- e) Draw the graph of $f^{-1}(x)$.
- f) What are the slopes of the three portions of lines that comprise the graph of $f^{-1}(x)$?
- g) Give an explicit formula for $f^{-1}(x)$.

2. Let $f(x) = 7 - (x-5)^2$ for $x \leq 5$, the left half of a parabola with vertex $(5, 7)$.

- a) Show f is 1-1 by showing that f is increasing throughout its domain.
- b) Find the point(s) where $y=x$ and $y=f(x)$ intersect.
- c) Find the intercepts of the graph of $f(x)$.
- d) Draw the graph of $f(x)$.
- e) What are the domain and range of $f^{-1}(x)$?

- f) Draw the graph of $f^{-1}(x)$.
- g) Find the value of the derivative of $f^{-1}(x)$ at the points where $x = 0$ and $x = -18$. Use your knowledge of the derivative of $f(x)$ to perform this calculation.
- h) Find an explicit formula for $f^{-1}(x)$.
- i) Calculate the derivative of $f^{-1}(x)$. Evaluate this derivative at $x = 0$ and $x = 18$ to verify that your answers in part g) are correct.

Calculus Workshop Series 1**Calculus 2****Week 2**

1. Let R be the region bounded by $y = \ln x$ and by a line L which passes through the points $(1,0)$ and $(e,1)$.

- Draw a graph of $y = \ln x$ and L , and shade the region R .
- Set up the definite integral which to calculate the area of R by integration with respect to x .
- Why can't you evaluate this integral?
- Draw the mirror image of R across the line $y = x$.
- What is the equation of the mirror image of L ?
- Set up the integral to evaluate the area of the mirror image of R by integration with respect to x .
- Evaluate this integral.
- What is the area of R ?

2. Now consider the integral $\int_1^{e^2} \ln x \, dx$.

- Graph the region whose area is represented by this integral.
- Besides $y = \ln x$, what are the other two boundaries of this region?
- Draw the mirror image of this region across the line $y = x$.
- What are the equations of the two straight line boundaries of this region?
- Set up the integral to evaluate the area of this region by integration with respect to x .
- Evaluate this integral.

- g) What is the value of the original integral $\int_1^{e^2} \ln x \, dx$?
- h) Now look a table of integrals and find an antiderivative of $\ln x$. Use this to verify that the answer you obtained in g) is correct.

Calculus Workshop Series 1
Calculus 2
Week 3

1. Let R be the region in the first quadrant bounded by $y = \sin x$, $y = \sin^{-1} x$, and by the straight lines $x = \pi/2$ and $y = \pi/2$.
 - a) Draw R (recall that the graph of $y = \sin x$ lies below the graph of $y = x$ in the first quadrant, and use reflection to draw the graph of $y = \sin^{-1} x$).
 - b) Set up, but do not evaluate, the integrals (with respect to x) which represent the area of R .
 - c) Now draw the line $y = x$ through this region R , dividing the region into two regions. Call the upper region S and the lower region T .
 - d) Why do you know that the area of $S =$ the area of T ?
 - e) Set up and evaluate the integral (with respect to x) to calculate the area of T .
 - f) What are the areas of R , S , and T respectively? Give both exact answers and numerical approximations.

NOTE: The following antiderivative formulas will be useful to you in calculating the integrals in problem 2.

$$\int \sin^2 x dx = x/2 - \sin x \cos x/2 + C$$

$$\int x \sin x dx = \sin x - x \cos x + C$$

2. Now rotate the region R about the y -axis, obtaining a solid of volume V .
 - a) Set up, but do not evaluate, the integrals to calculate this volume V by using cylindrical shells.

b) Set up, but do not evaluate, the integrals to calculate this volume V by using washers.

NOTICE that neither of these integrals can be calculated by methods we know so far.

c) Suppose that S and T were rotated about the y -axis, giving rise to solids of volume V_S and V_T , respectively. Does $V = V_S + V_T$? Does $V_S = V_T$? Why or why not?

In parts d, e, f give both exact answers and numerical approximations.

d) Calculate V_S by using washers.

e) Calculate V_T by using cylindrical shells.

f) Calculate V .

g) Verify, by differentiating the antiderivatives of the note, that the antiderivative formulas given in the note are correct.

Calculus Workshop Series 1

Calculus 2

Week 4

1. a) Let $F(t) = \int_1^2 x^t dx$

i) Compute $F(2), F(1), F(0), F(-1/2)$.

ii) Compute $F(t)$ for $t \neq -1$.

iii) Use l'Hospital's Rule to compute $L = \lim_{t \rightarrow -1} F(t)$.

iv) Does $L = F(-1)$? What does this say about F ?

b) Let $F_r(t) = \int_1^r x^t dx$

i) Compute $F_r(t)$ for $t \neq -1$.

ii) Compute $L_r = \lim_{t \rightarrow -1} F_r(t)$.

iii) Does $L_r = F_r(-1)$?

c) What does this tell you about $\int_1^r \frac{1}{x} dx$ in relation to the power rule?

2. Consider the graph of $f(x) = \frac{x^2 - 3}{e^x(2x - 3)}$.

a) Find all intercepts.

b) Find all vertical asymptotes.

c) Use l'Hospital's Rule to compute $\lim_{x \rightarrow -\infty} e^x(2x - 3)$.

d) Find all horizontal asymptotes.

e) Calculate $f'(x)$.

f) What are the critical points?

g) Verify that $f''(x) = \frac{4x^4 - 20x^3 + 33x^2 - 24x + 3}{e^x(2x-3)^3}$.

h) What are the local maxima and minima of f ?

i) Calculate $f''(.15)$, $f''(.2)$, $f''(2.75)$, $f''(2.8)$.

j) What are the inflection points of f (approximately)?

k) Graph $f(x)$, plotting both x - and y -coordinates of all relevant points and identifying all of the above features carefully.

Calculus Workshop Series 1
Calculus 2
Week 5

1. Recall that not all quadratic polynomials have real roots. Indeed, when in the quadratic formula $b^2 - 4ac < 0$, the roots are complex numbers, the polynomial is said to be non-factorable over the real numbers, and its graph lies either entirely above or entirely below the x -axis.

a) Find the solutions of $13x^2 - 4x + 1 = 0$.

b) If a complex number is of the form $z = a + bi$, a is called the real part of z , and b is called the imaginary part of z . We sometimes write this $\operatorname{Re}(z) = a$, $\operatorname{Im}(z) = b$. Two complex numbers are called equal if their real parts are equal, and their imaginary parts are equal.

i) What is $\operatorname{Re}(7 + 3i)$? What is $\operatorname{Im}(8 - 11i)$?

ii) What are s and t if $(3s + 1) + (8 - 3t)i = 7 - 4i$?

c) Since $i^2 = -1$, we can use the "FOIL" law to conclude that $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.

i) Compute $(2 + 3i)(2 + 3i)$.

ii) Compute $(2 + 3i)(2 - 3i)$.

iii) Complex numbers of the form $a + bi$ and $a - bi$ are called complex conjugates of each other. Noticing that the product of complex conjugates is always real, we can compute $\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(a + bi)(c - di)}{c^2 + d^2}$, turning division into multiplication by using the complex conjugate of the denominator.

Compute $\frac{1}{2 + 3i}$ and $\frac{1}{2 - 3i}$.

d) Later in the term we will derive Euler's famous formula relating complex numbers and trigonometry, $e^{i\theta} = \cos \theta + i \sin \theta$. Use this formula to compute $e^{i\pi}$.

2. a) Use integration by parts to verify that $\int e^{2x} \cos 3x \, dx = e^{2x} \left(\frac{2}{13} \cos 3x + \frac{3}{13} \sin 3x \right)$

b) Now compute $I = \int e^{2x} e^{3ix} \, dx$ by using elementary integration.

c) Verify that $\operatorname{Re}(I) = \int e^{2x} \cos 3x \, dx$.

d) Use the result of part b) to compute $\int e^{2x} \sin 3x \, dx$.

e) (For your final report) use integration by parts to verify the result in part d).

3. a) Now use integration by parts to compute $\int x e^{2x} e^{3ix} \, dx = \int x e^{(2+3ix)} \, dx$.

b) What are $\int x e^{2x} \cos 3x \, dx$ and $\int x e^{2x} \sin 3x \, dx$

Note that these would be extremely difficult to do using integration by parts.

Calculus Workshop Series 1

Calculus 2

Week 6

Recall that if a function has two antiderivatives, then those two antiderivatives must differ by a constant.

This workshop is designed to reinforce that fact.

1. a) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \sin x$.
 - b) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \cos x$.
 - c) Calculate $\int_0^{\pi/2} \sin x \cos x \, dx$ twice, once using each of the above antiderivatives.
 - d) Now use a trigonometric identity to notice that the antiderivative you found in part a) differs from the one you found in part b) by a constant. What is the constant?
2. a) Calculate $\int \sin^3 x \cos^3 x \, dx$ using the substitution $u = \sin x$.
 - b) Calculate $\int \sin^3 x \cos^3 x \, dx$ using the substitution $u = \cos x$.
 - c) Take the antiderivative you found in part a), use a trigonometric identity to change all the occurrences of $\sin x$ to occurrences of $\cos x$, and show that the antiderivative of part a) differs from that of part b) by a constant. What is the constant?
3. a) Calculate $\int \sin^4 x \, dx$ with the help of the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$.
 - b) Calculate $\int \sin^4 x \, dx$ with the help of the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$
 - c) Use trigonometric identities to show that the antiderivatives you obtained in parts a) and b) differ from each other by a constant. What is the constant?

Hint: it may be useful to change the antiderivative of part a) using the identities

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1 \text{ and}$$

$$\sin 2\theta = 2\sin \theta \cos \theta$$

Calculus Workshop Series 1

Calculus 2

Week 7

In mathematics, it is common to find new methods to extend knowledge from what is known to what is unknown, but similar to what is known. Both of this week's problems illustrate that; in fact, the new methods will even provide another way to do the problem whose solution was already known.

1. a) Use the substitution $u = 4 - x^2$ to help you calculate $\int x \sqrt{4 - x^2} dx$.

Notice that the integral obtained is a simple power rule integral, which was possible to do in Calculus 1.

- b) Now use the trigonometric substitution $x = 2 \sin \theta$ to calculate the same integral. This time notice that a so-called "easy" trigonometric integral is obtained; that is, one involving powers of the sine and cosine with at least one of the exponents an odd integer.

- c) Use this same trigonometric substitution $x = 2 \sin \theta$ to calculate $\int x^3 \sqrt{4 - x^2} dx$.

Note that another "easy" trigonometric integral is obtained.

- d) Evaluate the integral of part c) using the substitution $u = 4 - x^2$. Notice that this time it is more difficult, and that you need to write x^3 as $x^2(x dx)$ and then set $x^2 = 4 - u$.

- e) Now consider the integral $\int x^2 \sqrt{4 - x^2} dx$. This time the substitution $u = 4 - x^2$ doesn't help at all, and the trigonometric substitution $x = 2 \sin \theta$ yields a more difficult trigonometric integral than the ones in parts b) and c). Evaluate the integral using this trigonometric substitution; be sure your final answer is in terms of x .

One of the key things we see from this example is that trigonometric substitutions handle all of these integrals, and that Calculus 1 methods only work in special cases, in this case when the power of x is odd, and even then they require a lot of manipulation.

2. a) Earlier we learned the formula $\int 1/(u^2 + a^2) du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$. Use this formula to help you calculate $\int \frac{1}{x^2+8x+25} dx$.

b) Calculate this same integral using the trigonometric substitution $x + 4 = 3 \tan \theta$.

c) Use this same trigonometric substitution $x + 4 = 3 \tan \theta$ to help you calculate $\int \frac{1}{(x^2+8x+25)^2} dx$.

d) Finally use this same trigonometric substitution $x + 4 = 3 \tan \theta$ to help you calculate $\int \frac{1}{(x^2+8x+25)^3} dx$.

Remember to undo all substitutions, so that all your answers are in terms of x .

Calculus Workshop Series 1**Calculus 2****Week 8**

1. Consider $\int \frac{49x^2 + 9}{x^5 - 3x^4 + x^3 - 3x^2} dx$.

- Factor the denominator of the integrand.
- Expand the integrand as a sum of partial fractions.
- Evaluate the constants in the expansion.
- Evaluate the integral.

2. Now consider the integral $\int \frac{8x^3 + 35x^2 + 119x + 34}{(x^2 + 4x + 13)^2} dx$.

- Expand the integrand as a sum of partial fractions.
- Write the system of 4 equations in 4 unknowns whose solution will give you the constants.
- Evaluate the constants.
- Use a trigonometric substitution to convert the integral to one we can do.
- Evaluate the integral. Be sure to undo the substitution so that the final answer is in terms of x .

Calculus Workshop Series 1

Calculus 2

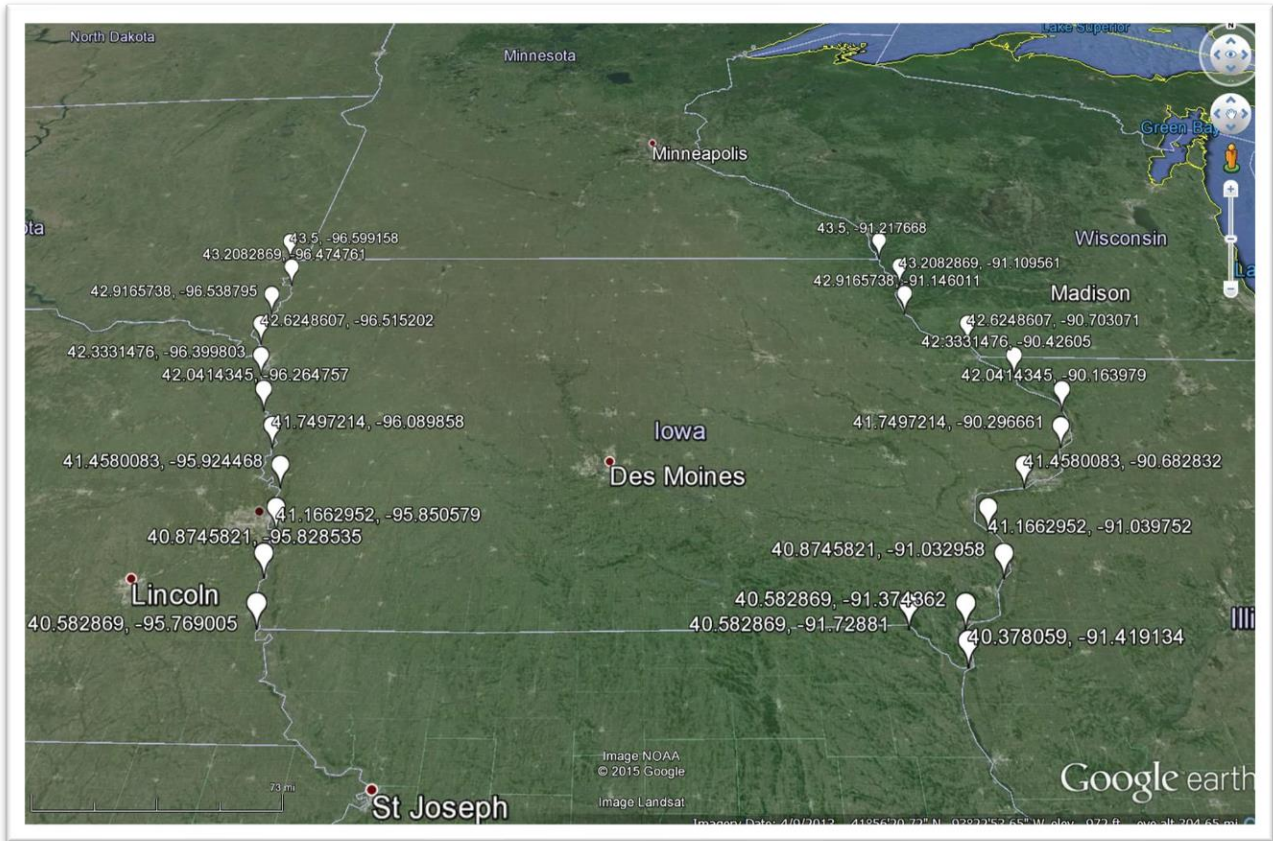
Week 9

Simpson’s Rule allows us to approximate areas under curves even if a formula for the curve is not known, but the values of the function at equally spaced points along the axis are known, since only those values and the length Δx of the spacing are needed. We would like to use this technique to help us measure the area of Iowa. A map of Iowa is provided on your table, along with the following table of values of latitude and longitude for selected points.

latitude		west boundary longitude			east boundary longitude
43.5		-96.5992			-91.2177
43.20829		-96.4748			-91.1096
42.91657		-96.5388			-91.146
42.62486		-96.5152			-90.7031
42.33315		-96.3998			-90.4261
42.04143		-96.2648			-90.164
41.74972		-96.0899			-90.2967
41.45801		-95.9245			-90.6828
41.1663		-95.8506	extra point		-91.0398
40.87458		-95.8285	on s bdry		-91.033
40.58287		-95.769	-91.7288		-91.7288
40.37806		point at south tip		-91.4191	
latitude				longitude	

The following facts are needed to help compute the area:

- a) The circumference of the earth is approximately 24,800 miles.
 - b) The circumference of the circle at latitude α is $24800 \cos \alpha$. This circle is divided into 360 degrees of longitude. For example, the length of the line at the northern boundary between the west boundary and the east boundary is $(-91.2177 + 96.5992)24800 \cos 43.5^\circ / 360$.
 - c) The distance between points of equal longitude is measured along great circles, which have the same circumference as the equator. So in our case Δx is $(43.5 - 43.20829)24800 / 360$.
1. Use Simpson's rule with $n=10$ to calculate that portion of the area of Iowa which does not include the small triangular piece at the southeast edge.
 2. Approximate the base and height of the small triangular piece, and then approximate its area.
 3. Calculate the area of Iowa. How close did you come to the published area of 56,363.3 sq. mi.? How large a percentage error is this?



<p>Calculus Workshop Series 1</p> <p>Calculus 2</p> <p>Week 10</p>

1. Consider $\int_2^{\infty} \frac{22x+6}{2x^3-x^2+8x-4} dx$.

- a) As the first step in evaluating this integral, use the method of partial fractions to write the integrand as a sum of two fractions.
- b) Find the antiderivative of the integrand.
- c) Combine the logarithmic terms of the antiderivative.
- d) Find the limit of the antiderivative as $x \rightarrow \infty$. Note that one term of the limit can be calculated with the help of l'Hospital's rule, and the other term can be calculated directly.
- e) Evaluate the integral.

2. Now consider $\int_0^{\infty} e^{-x^2} dx$. Since this is an integrand for which we are unable to find an antiderivative, other methods must be used.

- a) First note that $\int_0^1 e^{-x^2} dx$ is known to exist, since it represents an area under a curve which in fact lies above the x -axis in the region of integration. If we wanted to know its value, we could approximate it using some method such as Simpson's Rule.
- b) Note that if $x \geq 1$, then $x^2 \geq x$. What does this say about the relationship between e^{-x^2} and e^{-x} ?
- c) Show that $\int_1^{\infty} e^{-x} dx$ converges.
- d) Conclude that $\int_1^{\infty} e^{-x^2} dx$ converges. This method is called comparison. By comparing one integral (known to be positive) to another positive integral which is larger and yet still convergent, we are able to conclude that the smaller integral also converges.

e) Why do the results of parts a) and d) tell us that $\int_0^{\infty} e^{-x^2} dx$ converges?

f) Do you know whether or not $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges? Why?

Calculus Workshop Series 1
Calculus 2
Week 12

1. Recall that a geometric series with first term a , ratio r has sum $S = \frac{a}{1-r}$ and n th partial sum

$$s_n = \frac{a(1-r^n)}{1-r}.$$

- a) Find the sum of the geometric series with first term 3, ratio $4/5$.
- b) Find the first term of the geometric series with sum 14, ratio $3/7$.
- c) Find the ratio for the geometric series with first term 2, sum 18.
- d) Suppose a geometric series has first term 5, ratio $7/8$, and $s_n = 35.86765$. Find n .
2. When one pays back a loan, the loan payments are made after the loan is made, so they only serve to pay back what is called the PRESENT VALUE of the payment amounts. For example, if money is lent at 12% yearly interest compounded quarterly, or 3% interest per quarter, a payment of \$1000 3 months from now has PRESENT VALUE $V = 1000(1.03)^{-1}$, since an amount V invested for 3 months would at the end of that 3 months be worth $V(1.03) = 1000(1.03)^{-1}(1.03) = 1000$.

Thus payments of \$1000 at the end of 3 months, 6 months, 9 months, and 12 months would have a PRESENT VALUE of

$$1000(1.03)^{-1} + 1000(1.03)^{-2} + 1000(1.03)^{-3} + 1000(1.03)^{-4} = 3717.10$$

Note that this is the sum of the first 4 terms of a geometric series with $a = 1000(1.03)^{-1}$ and $r = (1.03)^{-1}$.

- a) Show that in a geometric series with $r = (1+i)^{-1}$ and first term $a(1+i)^{-1}$,

$$s_n = \frac{a}{i}(1-r^n) = \frac{a}{i}(1-(1+i)^{-n}).$$

Notice that s_n calculates the size of loan that n payments of size a will pay off if the interest rate per period is i .

- b) Verify that if $a = 1000, i = .03$, then $s_4 = 3717.10$. Calculate s_8 .
- c) Suppose that, after making 4 payments on the loan of size s_8 of part b), you decided to pay off the whole loan. How much would you have to pay at that time.

Hint: You have paid off 3717.10, so what you owe is the value of $(s_8 - 3717.10) = L$, 4 periods from the beginning of the loan. That is, you owe $L(1.03)^4$.

3. Now consider the geometric series with $r = (1.005)^{-1}$, first term $700(1.005)^{-1}$.
- a) What is the sum of this series?
- b) If you took out a 15-year loan at 6% interest compounded monthly, how big a loan could you pay off with payments of \$700 per month (6% per year is .005 per month)?
- c) How big a loan could you pay off if you made payments for 30 years?
- d) If you took out the 30-year loan of part c), how much would you owe after 15 years?
- e) Recently, banks in Japan have started giving 100-year loans. At the interest rates of this problem, how big a loan could you pay off in 100 years?
- f) What percentage of the sum of the infinite series is s_{1200} ?
- g) After 30 years of payments on the 100-year loan, how much would you still owe?

Calculus Workshop Series 1
Calculus 2
Week 13

1. Consider the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Call its sum L .

a) In the proof of the integral test, we established the inequality

$$\int_1^{\infty} \frac{1}{x^4} dx \leq L \leq a_1 + \int_1^{\infty} \frac{1}{x^4} dx.$$

Use this inequality to establish limits for L .

b) Since this is a series with all positive terms, we also know $a_1 \leq L$.

Use this fact to improve your limits for L .

c) It was established earlier that if the n th partial sum s_n is used as an estimate for L , the error

$$R_n \text{ satisfies } \int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx.$$

i) Use s_3 as an estimate for L . How big could the error be?

ii) What is the smallest n you could use to be sure the error $< .001$? Find this estimate for L .

d) Now consider $\sum_{n=1}^{\infty} \frac{1}{n^4 + 2n + 3}$, a series known to converge by comparison to the series

above. Earlier it was established that the error in using s_n as an estimate for the sum has error no more than the error for the corresponding partial sum in part c. Find an estimate for the sum of this series which has error $< .001$.

e) Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$.

i) Use Leibniz' Test to show this series converges.

- ii) Recall that the error in using s_n for an estimate of the sum is less than the absolute value of the $(n+1)$ st term. What is the smallest n which will make the error $< .001$? Find the estimated sum using this value of n .

2. The following are related alternating series, whose relationship we shall investigate later.

- a) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms of the series did you have to add?
- b) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(.08)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
- c) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2.5)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
- d) Calculate $\sin 1$, $\sin .08$, $\sin 2.5$. Are the answers related to what you found above?

Calculus Workshop Series 1

Calculus 2

Week 14

Notation: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with sum L .

Then $s_n = a_0 + a_1 x + \dots + a_n x^n$ and $R_n = L - s_n = a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$

1. Consider the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$, convergent for $-1 < x < 1$.

a) Use s_2 for this series to estimate $1/.98$

b) Show this estimate is accurate to 4 decimal places by showing that R_2 is itself a geometric series with sum $< .00001$

c) Find a series $\sum a_n x^n$ for $\frac{1}{1+x} = \frac{1}{1-(-x)}$.

d) Find a series for $\ln(1+x) = \int \frac{1}{1+x} dx = C + \int \sum a_n x^n dx$.

i) Set $x=0$ to calculate C .

ii) Does this series converge for $x=1$? Why?

iii) How many terms would you have to add to estimate $\ln 2$ with error $< .001$?

e) Calculate $\int \frac{1}{1-x^2} dx$ using partial fractions.

f) Set $x=1/3$ in this series. To what value does the series converge?

g) Use $2s_4$ to estimate $\ln 2$.

h) Show this is a good estimate by showing that

$$R_4 < \frac{(1/3)^9}{9} + \frac{(1/3)^{11}}{9} + \dots = \frac{(1/3)^9}{7(1-1/9)}$$

2. a) Find a series $\sum b_n x^n$ for $\frac{1}{1+x^2}$.
- b) Find a series for $\tan^{-1}x = \int \frac{1}{1+x^2} dx = C + \int \sum b_n x^n$.
- c) Does this series converge for $x=1$? Why?
- d) How many terms would be needed to estimate $\pi = 4(\pi/4) = 4 \tan^{-1} 1$ with error $< .001$?
- e) Use the addition formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ to show that
$$\tan(\tan^{-1} 1/2 + \tan^{-1} 1/3) = 1 = \tan(\pi/4)$$
 and thus that $\pi/4 = \tan^{-1} 1/2 + \tan^{-1} 1/3$.
- f) Estimate π using the formula of e) and the series for $\tan^{-1} x$. Show the error $< .001$

Calculus 2 Workshop for a Single Semester

SERIES 2

Classroom tested at Lawrence Technological University
Department of Mathematics and Computer Science

Prof. William C. Arlinghaus, Ph.D.
with input from
Prof. Michael Mersher
Prof. James Nanny

The goal of the calculus workshop is to provide each student with (1) a deeper understanding of the important mathematical concepts encountered in calculus and (2) an opportunity to develop beneficial problem-solving skills within a collaborative group setting. While the workshop program is designed to be essentially independent of a traditional lecture portion of a course, the work done in the workshop will generally be related to many materials covered in lectures. Instructors may make a judicious selection from the various series that appear below, depending on how an individual course might be structured.

The calculus workshop is designed to be completed in 1 hour and 15 minutes, once a week throughout a semester. Students should work in small groups on a special set of problems, distributed at the beginning of the workshop session. Workshop facilitators should be available to answer questions and offer limited suggestions during each workshop session. Work groups might be reassigned periodically during the semester.

Members of each group are encouraged to work together, share ideas, and divide tasks when appropriate in order to complete as many of the assigned exercises as possible during the scheduled workshop session. Groups are also encouraged to voluntarily exchange e-mail usernames and to set up additional meeting times as needed during the week in order to complete unfinished solutions and to share all final results.

Each student is required to turn in an individual written report following each weekly workshop. Each graded workshop report should be assigned a score ranging from 0 to 10 points (or other scheme at instructor discretion).

The overall semester grade for the workshop will equal the sum of the highest ten report scores (maximum 100 points, using the ten-point scale suggested above) and will be considered equivalent to one in-class exam.

One final note: Students are also encouraged to consider forming independent study groups to work on regularly assigned homework and to prepare for exams throughout the semester. (Keep in mind that the workshop session is not to be used for this purpose.)

Calculus Workshop Report Guidelines

Weekly workshop reports should be submitted to a fixed regular location by a fixed time. Late reports should not be considered. Graded reports should be returned the following week.

Each student must submit a complete workshop report for each workshop attended. Every member of a group is expected to record the solutions collaboratively obtained by that group either during the workshop or, if necessary, at a subsequent group meeting. Once the group work is completed, each student is expected to independently prepare a final workshop report recounting the computational details of the solutions obtained and providing all final answers and required graphs. In some cases it may also be appropriate to include a brief summary of the methods used and any general conclusions that can be drawn from the results.

Mathematical solutions should be complete, precise, well organized, and notationally correct. In all written explanations, proper attention should be given to legibility, grammar, spelling, and punctuation. Each solution should be clearly labeled with the appropriate problem number.

Each week students should receive a Workshop Report cover page which must be included as the first page of the individual report. On it students must provide

- Name and student ID number
- Date
- Lecture instructor
- Names of all other group members

Additional sheets should be attached in order with a staple in the upper left corner. Use only one side of each sheet. Partial sheets and pages torn from spiral notebooks or ring binders are not acceptable. The use of graph paper is recommended for hand-drawn graphs.

Excessive or habitual tardiness by any member of a group is unfair to the other members and puts the whole group at a disadvantage. For this reason, groups with fewer than four members will generally be combined with other groups. Any student arriving late will be reassigned to a different group as space is available. Anyone arriving over 20 minutes late will be excluded from participating in that session and will be marked absent. There will be no opportunity to “makeup” a missed workshop session.

Calculus Workshop Series 2**Calculus 2****Week 1**

1. a) Use the substitution $u = 4 - x^2$ to help you calculate $\int x\sqrt{4 - x^2} dx$. Notice that the integral obtained is a simple power rule integral.
b) Use the same substitution to calculate $\int_0^1 x\sqrt{4 - x^2} dx$. Undo the substitution before calculating the integral
c) Use this same substitution to calculate $\int_0^1 x\sqrt{4 - x^2} dx$ again, only this time change the limits of integration when you make the substitution.

2. a) Use the same substitution to calculate $\int x^3\sqrt{4 - x^2} dx$. Notice that you will have to think of x^3 as $x^2 \cdot x$, expressing the first factor in terms of u and using the second as part of du .
b) Calculate $\int_0^{\sqrt{3}} x^3\sqrt{4 - x^2} dx$.

3. Evaluate $\int_0^4 \frac{x}{\sqrt{1 + 2x}} dx$. Notice that this time using the substitution $u = 1 + 2x$ and changing the limits of integration makes this problem a power rule problem with easy calculations.

Calculus Workshop Series 2

Calculus 2

Week 2

$$1. \text{ Let } f(x) = \begin{cases} \frac{2x-4}{3} & x \leq -2 \\ \frac{4x-17}{5} & -2 \leq x \leq 8 \\ \frac{2x+5}{7} & x \geq 8 \end{cases}$$

- a) Plot the 4 points corresponding to $x = -4$, $x = -2$, $x = 8$, and $x = 15$.
- b) Draw the graph of $f(x)$. (Don't plot any more points.)
- c) As you can see, the graph consists of portions of 3 lines. What are the slopes of these three portions of lines?
- d) Plot the mirror images (across $y=x$) of the 4 points you found in part a).
- e) Draw the graph of $f^{-1}(x)$.
- f) What are the slopes of the three portions of lines that comprise the graph of $f^{-1}(x)$?
- g) Give an explicit formula for $f^{-1}(x)$.

$$2. \text{ Let } f(x) = 5 - (x-4)^2 \text{ for } x \leq 4, \text{ the left half of a parabola with vertex } (4,5).$$

- a) Show f is 1-1 by showing that f is increasing throughout its domain.
- b) Find the point(s) where $y=x$ and $y=f(x)$ intersect.
- c) Find the intercepts of the graph of $f(x)$.
- d) Draw the graph of $f(x)$.
- e) What are the domain and range of $f^{-1}(x)$?
- f) Draw the graph of $f^{-1}(x)$.

- g) Find the value of the derivative of $f^{-1}(x)$ at the points where $x = 0$ and $x = -11$. Use your knowledge of the derivative of $f(x)$ to perform this calculation.
- h) Find an explicit formula for $f^{-1}(x)$.
- i) Calculate the derivative of $f^{-1}(x)$. Evaluate this derivative at $x = 0$ and $x = -11$ to verify that your answers in part g) are correct.

Calculus Workshop Series 2

Calculus 2

Week 3

1. Let R be the region bounded by $y = \ln x$ and by a line L which passes through the points $(1,0)$ and $(e^2,2)$.

- a) Draw a graph of $y = \ln x$ and L , and shade the region R .
- b) Set up the definite integral to calculate the area of R by integration with respect to x .
- c) Why can't you evaluate this integral?
- d) Draw the mirror image of R across the line $y = x$.
- e) What is the equation of the mirror image of L ?
- f) Set up the integral to evaluate the area of the mirror image of R by integration with respect to x .
- g) Evaluate this integral.
- h) What is the area of R ?

2. Now consider the integral $\int_1^{\sqrt{e}} \ln x \, dx$.

- a) Graph the region whose area is represented by this integral.
- b) Besides $y = \ln x$, what are the other two boundaries of this region?
- c) Draw the mirror image of this region across the line $y = x$.
- d) What are the equations of the two straight line boundaries of this region?
- e) Set up the integral to evaluate the area of this region by integration with respect to x .
- f) Evaluate this integral.

g) What is the value of the original integral $\int_1^{\sqrt{e}} \ln x \, dx$?

- h) Now look in a table of integrals, and find an antiderivative of $\ln x$. Use this to verify that the answer you obtained in g) is correct.

Calculus Workshop Series 2

Calculus 2

Week 4

1. Let R be the region in the first quadrant bounded by $y = \sin x$, $y = \sin^{-1} x$, and by the straight lines $x = \pi/3$ and $y = \pi/3$.
 - a) Draw R (recall that the graph of $y = \sin x$ lies below the graph of $y = x$ in the first quadrant, and use reflection to draw the graph of $y = \sin^{-1} x$).
 - b) Set up, but do not evaluate, the integrals (with respect to x) which represent the area of R .
 - c) Now draw the line $y = x$ through this region R , dividing the region into two regions. Call the upper region S and the lower region T .
 - d) Why do you know that the area of $S =$ the area of T ?
 - e) Set up and evaluate the integral (with respect to x) to calculate the area of T .
 - f) What are the areas of R , S , and T respectively? Give both exact answers and numerical approximations.

NOTE: The following antiderivative formulas will be useful to you in calculating the integrals in problem 2.

$$\int \sin^2 x dx = x/2 - \sin x \cos x / 2 + C$$

$$\int x \sin x dx = \sin x - x \cos x + C$$

2. Now rotate the region R about the y -axis, obtaining a solid of volume V .
 - a) Set up, but do not evaluate, the integrals to calculate this volume V by using cylindrical shells.
 - b) Set up, but do not evaluate, the integrals to calculate this volume V by using washers.

NOTICE that neither of these integrals can be calculated by methods we know so far.

- c) Suppose that S and T were rotated about the y -axis, giving rise to solids of volumes V_S and V_T , respectively. Does $V = V_S + V_T$? Does $V_S = V_T$? Why or why not?

In parts d,e,f give both exact answers and numerical approximations.

- d) Calculate V_S by using washers.
- e) Calculate V_T by using cylindrical shells.
- f) Calculate V .
- g) Verify, by differentiating the antiderivatives of the note, that the antiderivative formulas given in the note are correct.

Calculus Workshop Series 2

Calculus 2

Week 5

1. a) Let $F(t) = \int_1^2 x^t dx$.

i) Compute $F(2), F(1), F(0), F(-1/2)$.

ii) Compute $F(t)$ for $t \neq -1$.

iii) Use l'Hospital's Rule to compute $L = \lim_{t \rightarrow -1} F(t)$.

iv) Does $L = F(-1)$? What does this say about F ?

b) Let $F_r(t) = \int_1^r x^t dx$

i) Compute $F_r(t)$ for $t \neq -1$.

ii) Compute $L_r = \lim_{t \rightarrow -1} F_r(t)$.

iii) Does $L_r = F_r(-1)$?

c) What does this tell you about $\int_1^r \frac{1}{x} dx$ in relation to the power rule?

2. Consider the graph of $f(x) = \frac{x^2 - 5}{e^x(3x - 5)}$.

a) Find all intercepts.

b) Find all vertical asymptotes.

c) Use l'Hospital's Rule to compute $\lim_{x \rightarrow -\infty} e^x(3x - 5)$.

d) Find all horizontal asymptotes.

e) Calculate $f'(x)$.

f) What are the critical points?

g) Verify that $f''(x) = \frac{19x^4 - 48x^3 + 70x^2 - 40x - 15}{e^x (3x - 5)^3}$

h) What are the local maxima and minima of f ?

i) Calculate $f''(-.3)$, $f''(-.2)$, $f''(3.5)$, $f''(3.6)$.

j) What are the inflection points of f (approximately)?

k) Graph $f(x)$, plotting both x - and y -coordinates of all relevant points and identifying all of the above features carefully.

Calculus Workshop Series 2

Calculus 2

Week 6

Notice that, even in integration by parts, the obvious substitution may not be correct.

1. a) Use integration by parts to calculate $\int x^6 \tan^{-1} x \, dx$. Note that because only the derivative, not the integral of $\tan^{-1} x$ is known, we must let $u = \tan^{-1} x$. Only one application of integration by parts is needed, but long division is needed to calculate the resulting integral.

b) Use integration by parts to calculate $\int x^6 (\ln x)^2 \, dx$. Again, we must use the non-obvious substitution. This time we must apply integration by parts twice.

2. Let $I = \int e^{6x} \cos 3x \, dx$ and $J = \int e^{6x} \sin 3x \, dx$.

a) Use integration by parts with $u = \cos 3x$ to show that $I = \frac{1}{6} e^{6x} \cos 3x + \frac{1}{2} J$.

b) Use integration by parts with $u = \sin 3x$ to show that $J = \frac{1}{6} e^{6x} \sin 3x - \frac{1}{2} I$.

c) Substitute the value of J from b) into the equation for I from a) to calculate I .

d) Substitute the value of I from a) into the equation for J from b) to calculate J .

Notice that this time we applied integration by parts twice but were only able to calculate I, J indirectly.

3. Consider the integral $\int x^6 \cos x \, dx$. This time the obvious substitution works, but we have to apply integration by parts six times. Notice, however, that the functions that were du and v in the first application become u and dv in the second, and that this continues in all the applications. So make a table as follows:

u / du	dv / v
x^6	$\cos x$
$6x^5$	$\sin x$
$30x^4$	$-\cos x$

0

Note that the first two values of uv appear as products along the diagonal lines in the table, given that the signs (+ and -) are included.

- Complete the table.
- Write the integral, using this table. This method is called tabular integration and can be used any time that the original u is a polynomial and the original dv can be integrated as many times as may be required.

Calculus Workshop Series 2

Calculus 2

Week 7

Recall that if a function has two antiderivatives, then those two antiderivatives must differ by a constant.

This workshop is designed to reinforce that fact.

1. a) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \sin x$.
- b) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \cos x$.
- c) Calculate $\int_0^{\pi/2} \sin x \cos x \, dx$ twice, once using each of the above antiderivatives.
- d) Now use a trigonometric identity to notice that the antiderivative you found in part a) differs from the one you found in part b) by a constant. What is the constant?

2. a) Calculate $\int \sin^5 x \cos^3 x \, dx$ using the substitution $u = \sin x$.
- b) Calculate $\int \sin^5 x \cos^3 x \, dx$ using the substitution $u = \cos x$.
- c) Take the antiderivative you found in part a), use a trigonometric identity to change all the occurrences of $\sin x$ to occurrences of $\cos x$, and show that the antiderivative of part a) differs from that of part b) by a constant. What is the constant?

3. a) Calculate $\int \cos^4 x \, dx$ with the help of the identity $\cos^2 x = \frac{1 + \cos 2x}{2}$.
- b) Calculate $\int \cos^4 x \, dx$ with the help of the reduction formula

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

- c) Use trigonometric identities to show that the antiderivatives you obtained in parts a) and b) differ from each other by a constant. What is the constant?

Hint: it may be useful to change the antiderivative of part a) using the identities

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1 \quad \text{and}$$

$$\sin 2\theta = 2\sin \theta \cos \theta$$

Calculus Workshop Series 2

Calculus 2

Week 8

In mathematics, it is common to find new methods to extend knowledge from what is known to what is unknown, but similar to what is known. Both of this week's problems illustrate that; in fact, the new methods will even provide another way to do the problem whose solution was already known.

1. a) Use the substitution $u = 9 - x^2$ to help you calculate $\int x\sqrt{9 - x^2} dx$. Notice that the integral obtained is a simple power rule integral, which was possible to do in Calculus 1.
- b) Now use the trigonometric substitution $x = 3\sin \theta$ to calculate the same integral. This time notice that a so-called "easy" trigonometric integral is obtained; that is, one involving powers of the sine and cosine with at least one of the exponents an odd integer.
- c) Use this same trigonometric substitution $x = 3\sin \theta$ to calculate $\int x^3\sqrt{9 - x^2} dx$. Note that another "easy" trigonometric integral is obtained.
- d) Evaluate the integral of part c) using the substitution $u = 9 - x^2$. Notice that this time it is more difficult, and that you need to write x^3 as $x^2(x dx)$ and then set $x^2 = 9 - u$.
- e) Now consider the integral $\int x^2\sqrt{9 - x^2} dx$. This time the substitution $u = 9 - x^2$ doesn't help at all, and the trigonometric substitution $x = 3\sin \theta$ yields a more difficult trigonometric integral than the ones in parts b) and c). Evaluate the integral using this trigonometric substitution; be sure your final answer is in terms of x .

One of the key things we see from this example is that trigonometric substitutions handle all of these integrals, and that Calculus 1 methods only work in special cases, in this case when the power of x is odd, and even then they require a lot of manipulation.

2. a) Earlier we learned the formula $\int \frac{1}{u^2 + a^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$. Use this formula to help you calculate $\int \frac{1}{x^2 + 6x + 25} dx$.

- b) Calculate this same integral using the trigonometric substitution $x + 3 = 4 \tan \theta$.

c) Use this same trigonometric substitution $x + 3 = 4 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2 + 6x + 25)^2} dx.$$

d) Finally use this same trigonometric substitution $x + 3 = 4 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2 + 6x + 25)^3} dx.$$

Remember to undo all substitutions, so that all your answers are in terms of x .

Calculus Workshop Series 2**Calculus 2****Week 9**

1. Consider $\int \frac{17x^2-20}{x^6-x^4-4x^3+4x^2} dx$

- Factor the denominator of the integrand.
- Expand the integrand as a sum of partial fractions.
- Evaluate the constants in the expansion.
- Evaluate the integral.

2. Now consider the integral $\int \frac{(3x^3+23x^2+71x+72)}{(x^2+6x+13)^2} dx$

- Expand the integrand as a sum of partial fractions.
- Write the system of 4 equations in 4 unknowns whose solution will give you the constants.
- Evaluate the constants.
- Use a trigonometric substitution to convert the integral to one we can do.
- Evaluate the integral. Be sure to undo the substitution so that the final answer is in terms of x .

Calculus Workshop Series 2**Calculus 2****Week 10**

1. Consider $\int_3^{\infty} \frac{30x}{2x^3 - 3x^2 + 18x - 27} dx$.

- As the first step in evaluating this integral, use the method of partial fractions to write the integrand as a sum of two fractions.
- Find the antiderivative of the integrand.
- Combine the logarithmic terms of the antiderivative.
- Find the limit of the antiderivative as $x \rightarrow \infty$. Note that one term of the limit can be calculated with the help of l'Hospital's rule, and the other term can be calculated directly.
- Evaluate the integral.

2. Now consider $\int_0^{\infty} e^{-x^2} dx$. Since this is an integrand for which we are unable to find an antiderivative, other methods must be used.

- First note that $\int_0^1 e^{-x^2} dx$ is known to exist, since it represents an area under a curve which in fact lies above the x -axis in the region of integration. If we wanted to know its value, we could approximate it using some method such as Simpson's Rule.
- Note that if $x \geq 1$, then $x^2 \geq x$. What does this say about the relationship between e^{-x^2} and e^{-x} ?
- Show that $\int_1^{\infty} e^{-x} dx$ converges.
- Conclude that $\int_1^{\infty} e^{-x^2} dx$ converges. This method is called comparison. By comparing one integral (known to be positive) to another positive integral which is larger and yet still convergent, we are able to conclude that the smaller integral also converges.

- e) Why do the results of parts a) and d) tell us that $\int_0^{\infty} e^{-x^2} dx$ converges?
- f) Do you know whether or not $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges? Why?

Calculus Workshop Series 2

Calculus 2

Week 11

1. Calculate the limits of the following sequences.

a) $1, \frac{28}{29}, \dots, \frac{7n^2 - 3n + 6}{5n^2 + 4n + 1}, \dots$

b) $1, \frac{56}{29}, \dots, \frac{7n^3 - 3n + 6}{5n^2 + 4n + 1}, \dots$

c) $\frac{\pi}{4}, \tan^{-1} \frac{56}{29}, \dots, \tan^{-1} \frac{7n^3 - 3n + 6}{5n^2 + 4n + 1}, \dots$

2. Recall that a geometric series with first term a , ratio r has sum $S = \frac{a}{1-r}$ and n th partial

$$\text{sum } s_n = \frac{a(1-r^n)}{1-r}.$$

a) Find the sum of the geometric series with first term 5, ratio $4/9$.

b) Find the first term of the geometric series with sum 11, ratio $3/5$.

c) Find the ratio for the geometric series with first term 6, sum 18.

d) Suppose a geometric series has first term 5, ratio $7/8$, and $s_n = 35.86765$. Find n .

3. When one pays back a loan, the loan payments are made after the loan is made, so they only serve to pay back what is called the PRESENT VALUE of the payment amounts. For example, if money is lent at 6% yearly interest compounded monthly, or 0.5% interest per month, a payment of \$1000 one month from now has PRESENT VALUE

$V = 1000(1.005)^{-1}$, since an amount V invested for 1 month would at the end of that month be worth $V(1.005) = 1000(1.005)^{-1}(1.005) = 1000$.

Thus payments of \$1000 at the end of each month for a year, starting in one month, would have a PRESENT VALUE of

$1000(1.005)^{-1} + 1000(1.005)^{-2} + 1000(1.005)^{-3} + \dots + 1000(1.005)^{-12}$ Note that this is the sum of the first 12 terms of a geometric series with $a = 1000(1.005)^{-1}$ and $r = (1.005)^{-1}$.

a) Show that in a geometric series with $r = (1+i)^{-1}$ and first term $a(1+i)^{-1}$,

$s_n = \frac{a}{i}(1-r^n) = \frac{a}{i}(1-(1+i)^{-n})$. Notice that s_n calculates the size of loan that n payments of size a will pay off if the interest rate per period is i .

b) Find the sum of the series if $a = 1000, i = .005$.

c) Verify that if $a = 1000, i = .005$, then $s_{12} = 11618.93$.

d) How large a mortgage could you pay off in 15 years with payments of \$1000 per month, with interest rate 6% compounded monthly? What partial sum is this?

e) How large a mortgage could you pay off in 30 years with payments of \$1000 per month, with interest rate 6% compounded monthly? What partial sum is this?

Calculus Workshop Series 2

Calculus 2

Week 12

1. Consider the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Call its sum L .

a) In the proof of the integral test, we established the inequality

$$\int_1^{\infty} \frac{1}{x^4} dx \leq L \leq a_1 + \int_1^{\infty} \frac{1}{x^4} dx.$$

Use this inequality to establish limits for L .

b) Since this is a series with all positive terms, we also know $a_1 \leq L$. Use this fact to improve your limits for L .

c) We have established elsewhere that if the n th partial sum s_n is used as an estimate for L ,

$$\text{the error } R_n \text{ satisfies } \int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx.$$

i) Use s_3 as an estimate for L . How big could the error be?

ii) What is the smallest n you could use to be sure the error $< .001$? Find this estimate for L .

d) It was also established elsewhere that by adding s_n to the inequality of part c), one obtains $s_n + \int_{n+1}^{\infty} f(x) dx \leq L \leq s_n + \int_n^{\infty} f(x) dx$. Taking the average of the left-hand and right-hand sides of this inequality gives an estimate of L which is in error by at most half the difference between them.

i) Use the s_3 of part c)i) and the above inequality to get a new estimate for L . How large could the error be?

ii) Now use s_n for the n of part c)ii) and the above inequality to get a new estimate for L . How large could the error be?

2. Now consider $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2 + 3}$, a series known to converge by comparison to the series above. This time it was established previously that the error in using s_n as an estimate for the sum has error no more than the error for the corresponding partial sum in part c. Find an estimate for the sum of this series which has error $< .001$.

Calculus Workshop Series 2

Calculus 2

Week 13

1. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$.
 - a) Use the Alternating Series Test to show this series converges.
 - b) Recall that the error in using s_n for an estimate of the sum is less than the absolute value of the $(n+1)$ st term. What is the smallest n which will make the error $< .001$? Find the estimated sum using this value of n .

2. The following are related alternating series, whose relationship we shall investigate later.
 - a) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms of the series did you have to add?
 - b) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(.13)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
 - c) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2.3)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
 - d) Calculate $\sin 1$, $\sin .13$, $\sin 2.3$. Are the answers related to what you found above?

Notation: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with sum L .

Then $s_n = a_0 + a_1 x + \dots + a_n x^n$ and $R_n = L - s_n = a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$

3. Consider the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$, convergent for $-1 < x < 1$.
- Use s_2 for this series to estimate $1/.98$
 - Show this estimate is accurate to 4 decimal places by showing that R_2 is itself a geometric series with sum $< .00001$
 - Find a series $\sum a_n x^n$ for $\frac{1}{1+x} = \frac{1}{1-(-x)}$. If a series has radius of convergence R , so does the integral or derivative of the series. So we consider the integral of the series in part c.
 - Find a series for $\ln(1+x) = \int \frac{1}{1+x} dx = \int \sum a_n x^n dx$.
 - Set $x=0$ to calculate C .
 - Does this series converge for $x=1$? Why?
 - How many terms would you have to add to estimate $\ln 2$ with error $< .001$?
 - Calculate $\int \frac{1}{1-x^2} dx$ both using partial fractions and using power series.
 - Set $x=1/3$ in both calculations. To what value does the series converge?
 - Use $2s_7$ to estimate $\ln 2$. Notice that this requires adding only 4 non-zero terms.
 - Show this is a good estimate by showing that

$$R_7 < \frac{(1/3)^9}{9} + \frac{(1/3)^{11}}{9} + \dots = \frac{(1/3)^9}{9(1-1/9)}.$$

Calculus Workshop Series 2

Calculus 2

Week 14

1. a) Recall the series representation $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
- b) How many terms would be needed to estimate $\pi = 4(\pi/4) = 4 \tan^{-1} 1$ with error $< .001$?
- c) Use the addition formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ to show that
- $$\tan(\tan^{-1} 1/2 + \tan^{-1} 1/3) = 1 = \tan(\pi/4) \text{ and thus that } \pi/4 = \tan^{-1} 1/2 + \tan^{-1} 1/3.$$
- d) Estimate π using the formula of c) and the series for $\tan^{-1} x$. Show the error $< .001$
2. Find a Taylor series for $\sqrt[3]{x-3}$ near $x=11$.
3. a) Find a Maclaurin series for $\cosh x$, using the definition of Maclaurin series.
- b) Find this same series by using the definition of $\cosh x$ and the Maclaurin series for e^x .

Calculus 2 Workshop for a Single Semester

SERIES 3

Classroom tested at Lawrence Technological University
Department of Mathematics and Computer Science

Prof. William C. Arlinghaus, Ph.D.
with input from
Prof. Michael Mersher
Prof. James Nanny

The goal of the calculus workshop is to provide each student with (1) a deeper understanding of the important mathematical concepts encountered in calculus and (2) an opportunity to develop beneficial problem-solving skills within a collaborative group setting. While the workshop program is designed to be essentially independent of a traditional lecture portion of a course, the work done in the workshop will generally be related to many materials covered in lectures. Instructors may make a judicious selection from the various series that appear below, depending on how an individual course might be structured.

The calculus workshop is designed to be completed in 1 hour and 15 minutes, once a week throughout a semester. Students should work in small groups on a special set of problems, distributed at the beginning of the workshop session. Workshop facilitators should be available to answer questions and offer limited suggestions during each workshop session. Work groups might be reassigned periodically during the semester.

Members of each group are encouraged to work together, share ideas, and divide tasks when appropriate in order to complete as many of the assigned exercises as possible during the scheduled workshop session. Groups are also encouraged to voluntarily exchange e-mail usernames and to set up additional meeting times as needed during the week in order to complete unfinished solutions and to share all final results.

Each student is required to turn in an individual written report following each weekly workshop. Each graded workshop report should be assigned a score ranging from 0 to 10 points (or other scheme at instructor discretion).

The overall semester grade for the workshop will equal the sum of the highest ten report scores (maximum 100 points, using the ten-point scale suggested above) and will be considered equivalent to one in-class exam.

One final note: Students are also encouraged to consider forming independent study groups to work on regularly assigned homework and to prepare for exams throughout the semester. (Keep in mind that the workshop session is not to be used for this purpose.)

Calculus Workshop Report Guidelines

Weekly workshop reports should be submitted to a fixed regular location by a fixed time. Late reports should not be considered. Graded reports should be returned the following week.

Each student must submit a complete workshop report for each workshop attended. Every member of a group is expected to record the solutions collaboratively obtained by that group either during the workshop or, if necessary, at a subsequent group meeting. Once the group work is completed, each student is expected to independently prepare a final workshop report recounting the computational details of the solutions obtained and providing all final answers and required graphs. In some cases it may also be appropriate to include a brief summary of the methods used and any general conclusions that can be drawn from the results.

Mathematical solutions should be complete, precise, well organized, and notationally correct. In all written explanations, proper attention should be given to legibility, grammar, spelling, and punctuation. Each solution should be clearly labeled with the appropriate problem number.

Each week students should receive a Workshop Report cover page which must be included as the first page of the individual report. On it students must provide

- Name and student ID number
- Date
- Lecture instructor
- Names of all other group members

Additional sheets should be attached in order with a staple in the upper left corner. Use only one side of each sheet. Partial sheets and pages torn from spiral notebooks or ring binders are not acceptable. The use of graph paper is recommended for hand-drawn graphs.

Excessive or habitual tardiness by any member of a group is unfair to the other members and puts the whole group at a disadvantage. For this reason, groups with fewer than four members will generally be combined with other groups. Any student arriving late will be reassigned to a different group as space is available. Anyone arriving over 20 minutes late will be excluded from participating in that session and will be marked absent. There will be no opportunity to “makeup” a missed workshop session.

<p>Calculus Workshop Series 3</p> <p>Calculus 2</p> <p>Week 1</p>
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1. Let $f(x) = \begin{cases} \frac{2x-8}{3} & x \leq -2 \\ \frac{5x-14}{5} & -2 \leq x \leq 4 \\ \frac{3x-7}{5} & x \geq 4 \end{cases}$

- a) Plot the 4 points corresponding to $x = -3$, $x = -2$, $x = 4$, and $x = 9$.
- b) Draw the graph of $f(x)$. (Don't plot any more points.)
- c) As you can see, the graph consists of portions of 3 lines. What are the slopes of these three portions of lines?
- d) Plot the mirror images (across $y=x$) of the 4 points you found in part a).
- e) Draw the graph of $f^{-1}(x)$.
- f) What are the slopes of the three portions of lines that comprise the graph of $f^{-1}(x)$?
- g) Give an explicit formula for $f^{-1}(x)$.

2. Let $f(x) = 7 - (x-5)^2$ for $x \leq 5$, the left half of a parabola with vertex $(5,7)$.

- a) Show f is 1-1 by showing that f is increasing throughout its domain.
- b) Find the point(s) where $y=x$ and $y=f(x)$ intersect.
- c) Find the intercepts of the graph of $f(x)$.
- d) Draw the graph of $f(x)$.
- e) What are the domain and range of $f^{-1}(x)$?
- f) Draw the graph of $f^{-1}(x)$.

- g) Find the value of the derivative of $f^{-1}(x)$ at the points where $x = 0$ and $x = -18$. Use your knowledge of the derivative of $f(x)$ to perform this calculation.
- h) Find an explicit formula for $f^{-1}(x)$.
- i) Calculate the derivative of $f^{-1}(x)$. Evaluate this derivative at $x = 0$ and $x = -18$ to verify that your answers in part g) are correct.

Calculus Workshop Series 3**Calculus 2****Week 2**

1. Let R be the region bounded by $y = \ln x$ and by a line L which passes through the points $(1,0)$ and $(e^{1.5}, 1.5)$
 - a) Draw a graph of $y = \ln x$ and L , and shade the region R .
 - b) Set up the definite integral to calculate the area of R by integration with respect to x .
 - c) Why can't you evaluate this integral?
 - d) Draw the mirror image of R across the line $y = x$.
 - e) What is the equation of the mirror image of L ?
 - f) Set up the integral to evaluate the area of the mirror image of R by integration with respect to x .
 - g) Evaluate this integral.
 - h) What is the area of R ?

2. Now consider the integral $\int_1^{e^{\sqrt{2}}} \ln x \, dx$.
 - a) Graph the region whose area is represented by this integral.
 - b) Besides $y = \ln x$, what are the other two boundaries of this region?
 - c) Draw the mirror image of this region across the line $y = x$.
 - d) What are the equations of the two straight line boundaries of this region?
 - e) Set up the integral to evaluate the area of this region by integration with respect to x .
 - f) Evaluate this integral.

g) What is the value of the original integral $\int_1^{e^{\sqrt{2}}} \ln x \, dx$?

h) Now look a table of integrals and find an antiderivative of $\ln x$. Use this to verify that the answer you obtained in g) is correct.

Calculus Workshop Series 3

Calculus 2

Week 3

1. Let R be the region in the first quadrant bounded by $y = \sin x/2$, $y = 2\sin^{-1} x$, and by the straight lines $x = \pi$ and $y = \pi$.
 - a) Draw R (note that the graph of $y = \sin x/2$ lies below the graph of $y = x$ in the first quadrant; then use properties of reflection and inverse functions to draw the graph of $y = 2\sin^{-1} x$).
 - b) Set up, but do not evaluate, the integrals (with respect to x) which represent the area of R .
 - c) Now draw the line $y = x$ through this region R , dividing the region into two regions. Call the upper region S and the lower region T .
 - d) Why do you know that the area of $S =$ the area of T ?
 - e) Set up and evaluate the integral (with respect to x) to calculate the area of T .
 - f) What are the areas of R , S , and T respectively? Give both exact answers and numerical approximations.

NOTE: The following antiderivative formulas will be useful to you in calculating the integrals in problem 2.

$$\int \sin^2 u \, du = \frac{u}{2} - \sin u \cos \frac{u}{2} + C$$

$$\int u \sin u \, du = \sin u - u \cos u + C$$

2. Now rotate the region R about the y -axis, obtaining a solid of volume V .
 - a) Set up, but do not evaluate, the integrals to calculate this volume V by using cylindrical shells.
 - b) Set up, but do not evaluate, the integrals to calculate this volume V by using washers.

NOTICE that neither of these integrals can be calculated by methods we know so far.

- c) Suppose that S and T were rotated about the y -axis, giving rise to solids of volumes V_S and V_T , respectively. Does $V = V_S + V_T$? Does $V_S = V_T$? Why or why not?

In parts d,e,f give both exact answers and numerical approximations.

- d) Calculate V_S by using washers.
- e) Calculate V_T by using cylindrical shells.
- f) Calculate V .
- g) Verify, by differentiating the antiderivatives of the note, that the antiderivative formulas given in the note are correct.

Calculus Workshop Series 3

Calculus 2

Week 4

1. a) Let $F(t) = \int_1^2 x^t dx$.

i) Compute $F(2), F(1), F(0), F(-1/2)$.

ii) Compute $F(t)$ for $t \neq -1$.

iii) Use l'Hospital's Rule to compute $L = \lim_{t \rightarrow -1} F(t)$.

iv) Does $L = F(-1)$? What does this say about F ?

b) Let $F_r(t) = \int_1^r x^t dx$

i) Compute $F_r(t)$ for $t \neq -1$.

ii) Compute $L_r = \lim_{t \rightarrow -1} F_r(t)$.

iii) Does $L_r = F_r(-1)$?

c) What does this tell you about $\int_1^r \frac{1}{x} dx$ in relation to the power rule?

2. Consider the graph of $f(x) = \frac{x^2 - 7}{e^x(4x - 7)}$.

a) Find all intercepts of f .

b) Find all vertical asymptotes of f .

c) Use l'Hospital's Rule to compute $\lim_{x \rightarrow -\infty} e^x(4x - 7)$.

d) Find all horizontal asymptotes of f .

e) Calculate $f'(x)$.

f) What are the critical points for f ?

g) Verify that $f''(x) = \frac{16x^4 - 88x^3 + 105x^2 - 28x - 77}{e^x(4x-7)^3}$.

h) What are the local maxima and minima of f ?

i) Calculate $f''(-0.7)$, $f''(-0.6)$, $f''(4)$, $f''(4.1)$.

j) What are the inflection points of f (approximately)?

k) Graph $f(x)$, plotting both x - and y -coordinates of all relevant points and identifying all of the above features carefully.

Calculus Workshop Series 3
Calculus 2
Week 5

Notice that, even in integration by parts, the obvious substitution may not be correct.

1. a) Use integration by parts to calculate $\int x^3 \tan^{-1} 2x \, dx$. Note that because only the derivative, not the integral of $\tan^{-1} 2x$ is known, we must let $u = \tan^{-1} 2x$. Only one application of integration by parts is needed, but long division is needed to calculate the resulting integral.

b) Use integration by parts to calculate $\int x^3 (\ln x)^2 \, dx$. Again, we must use the non-obvious substitution. This time we must apply integration by parts twice.

2. Let $I = \int e^{4x} \cos 3x \, dx$ and $J = \int e^{4x} \sin 3x \, dx$.

a) Use integration by parts with $u = \cos 3x$ to show that $I = \frac{1}{4} e^{4x} \cos 3x + \frac{3}{4} J$.

b) Use integration by parts with $u = \sin 3x$ to show that $J = \frac{1}{4} e^{4x} \sin 3x - \frac{3}{4} I$.

c) Substitute the value of J from b) into the equation for I from a) to calculate I .

d) Substitute the value of I from a) into the equation for J from b) to calculate J .

Notice that this time we applied integration by parts twice but were only able to calculate I, J indirectly.

3. Consider the integral $\int x^6 \sin x \, dx$. This time the obvious substitution works, but we have to apply integration by parts six times. Notice, however, that the functions that were du and v in the first application become u and dv in the second, and that this continues in all the applications. So make a table as follows:

u / du	dv / v
x^6	$\sin x$
$6x^5$	$-\cos x$
$30x^4$	$-\sin x$

0

Note that the first two values of uv appear as products along the diagonal lines in the table, given that the signs (+ and -) are included.

- Complete the table.
- Write the integral, using this table. This method is called tabular integration and can be used any time that the original u is a polynomial and the original dv can be integrated as many times as may be required.

Calculus Workshop Series 3
Calculus 2
Week 6

Recall that if a function has two antiderivatives, then those two antiderivatives must differ by a constant.

This workshop is designed to reinforce that fact.

1. a) Calculate $\int \sin 3x \cos 3x \, dx$ using the substitution $u = \sin 3x$.
- b) Calculate $\int \sin 3x \cos 3x \, dx$ using the substitution $u = \cos 3x$.
- c) Calculate $\int_0^{\pi/6} \sin 3x \cos 3x \, dx$ twice, once using each of the above antiderivatives.
- d) Now use a trigonometric identity to notice that the antiderivative you found in part a) differs from the one you found in part b) by a constant. What is the constant?

2. a) Calculate $\int \sin^3 x \cos^5 x \, dx$ using the substitution $u = \sin x$.
- b) Calculate $\int \sin^3 x \cos^5 x \, dx$ using the substitution $u = \cos x$.
- c) Take the antiderivative you found in part a), use a trigonometric identity to change all the occurrences of $\sin x$ to occurrences of $\cos x$, and show that the antiderivative of part a) differs from that of part b) by a constant. What is the constant?

3. a) Calculate $\int \sin^4 x \, dx$ with the help of the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$.
- b) Calculate $\int \sin^4 x \, dx$ with the help of the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

c) Use trigonometric identities to show that the antiderivatives you obtained in parts a) and b) differ from each other by a constant. What is the constant?

Hint: it may be useful to change the antiderivative of part a) using the identities

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1 \quad \text{and}$$

$$\sin 2\theta = 2\sin \theta \cos \theta$$

Calculus Workshop Series 3

Calculus 2

Week 7

In mathematics, it is common to find new methods to extend knowledge from what is known to what is unknown, but similar to what is known. Both of this week's problems illustrate that; in fact, the new methods will even provide another way to do the problem whose solution was already known.

1. a) Use the substitution $u = 1 - 4x^2$ to help you calculate $\int x\sqrt{1 - 4x^2} dx$. Notice that the integral obtained is a simple power rule integral, which was possible to do in Calculus 1.
- b) Now use the trigonometric substitution $2x = \sin \theta$ to calculate the same integral. This time notice that a so-called "easy" trigonometric integral is obtained; that is, one involving powers of the sine and cosine with at least one of the exponents an odd integer.
- c) Use this same trigonometric substitution $2x = \sin \theta$ to calculate $\int x^3\sqrt{1 - 4x^2} dx$. Note that another "easy" trigonometric integral is obtained.
- d) Evaluate the integral of part c) using the substitution $u = 1 - 4x^2$. Notice that this time it is more difficult, and that you need to write x^3 as $x^2(x dx)$ and then set $x^2 = (1 - u) / 4$.
- e) Now consider the integral $\int x^2\sqrt{1 - 4x^2} dx$. This time the substitution $u = 1 - 4x^2$ doesn't help at all, and the trigonometric substitution $2x = \sin \theta$ yields a more difficult trigonometric integral than the ones in parts b) and c). Evaluate the integral using this trigonometric substitution; be sure your final answer is in terms of x .

One of the key things we see from this example is that trigonometric substitutions handle all of these integrals, and that Calculus 1 methods only work in special cases, in this case when the power of x is odd, and even then they require a lot of manipulation.

2. a) Earlier we learned the formula $\int 1/(u^2 + a^2) du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$. Use this formula to help you calculate $\int \frac{1}{x^2 + 6x + 18} dx$.
- b) Calculate this same integral using the trigonometric substitution $x + 3 = 3 \tan \theta$.
- c) Use this same trigonometric substitution $x + 3 = 3 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2+6x+18)^2} dx.$$

d) Finally use this same trigonometric substitution $x + 3 = 3 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2+6x+18)^3} dx.$$

Remember to undo all substitutions, so that all your answers are in terms of x .

Calculus Workshop Series 3**Calculus 2****Week 8**

1. Consider $\int \frac{-46x^2+16}{x^5-4x^4-x^3+4x^2} dx$.

- Factor the denominator of the integrand.
- Expand the integrand as a sum of partial fractions.
- Evaluate the constants in the expansion.
- Evaluate the integral.

2. Now consider the integral $\int \frac{2x^3+15x^2+71x+139}{(x^2+4x+20)^2} dx$.

- Expand the integrand as a sum of partial fractions.
- Write the system of 4 equations in 4 unknowns whose solution will give you the constants.
- Evaluate the constants.
- Use a trigonometric substitution to convert the integral to one we can do.
- Evaluate the integral. Be sure to undo the substitution so that the final answer is in terms of x .

Calculus Workshop Series 3**Calculus 2****Week 9**

1. Consider $\int_3^{\infty} \frac{30x}{2x^3 - 3x^2 + 18x - 27} dx$.
 - a) As the first step in evaluating this integral, use the method of partial fractions to write the integrand as a sum of two fractions.
 - b) Find the antiderivative of the integrand.
 - c) Combine the logarithmic terms of the antiderivative.
 - d) Find the limit of the antiderivative as $x \rightarrow \infty$. Note that one term of the limit can be calculated with the help of l'Hospital's rule, and the other term can be calculated directly.
 - e) Evaluate the integral.

2. Now consider $\int_0^{\infty} e^{-x^2} dx$. Since this is an integrand for which we are unable to find an antiderivative, other methods must be used.
 - a) First note that $\int_0^1 e^{-x^2} dx$ is known to exist, since it represents an area under a curve which in fact lies above the x -axis in the region of integration. If we wanted to know its value, we could approximate it using some method such as Simpson's Rule.
 - b) Note that if $x \geq 1$, then $x^2 \geq x$. What does this say about the relationship between e^{-x^2} and e^{-x} ?
 - c) Show that $\int_1^{\infty} e^{-x} dx$ converges.
 - d) Conclude that $\int_1^{\infty} e^{-x^2} dx$ converges. This method is called comparison. By comparing one integral (known to be positive) to another positive integral which is larger and yet still convergent, we are able to conclude that the smaller integral also converges.

- e) Why do the results of parts a) and d) tell us that $\int_0^{\infty} e^{-x^2} dx$ converges?
- f) Do you know whether or not $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges? Why?

Calculus Workshop Series 3

MCS1424 Calculus 2

Week 10

1. Calculate the limits of the following sequences.

a) $1, \frac{28}{29}, \dots, \frac{7n^2 - 3n + 6}{5n^2 + 4n + 1}, \dots$

b) $1, \frac{56}{29}, \dots, \frac{7n^3 - 3n + 6}{5n^2 + 4n + 1}, \dots$

c) $\frac{\pi}{4}, \tan^{-1} \frac{56}{29}, \dots, \tan^{-1} \frac{7n^3 - 3n + 6}{5n^2 + 4n + 1}, \dots$

2. For each of the following sequences, i) list the values of the first 5 terms ii) find the limit of the sequence

a) $a_n = \int_1^n \frac{1}{x} dx$

b) $a_n = \int_1^n \frac{1}{x^2} dx$

c) $a_n = \int_1^{n+5} \frac{1}{x} dx$

d) $a_n = \int_1^\infty \frac{1}{x^{n+1}} dx$

e) $a_n = \int_n^\infty \frac{1}{x^2} dx$

3. For each of the following series $\sum a_n$, i) compute the n th partial sum s_n for $n=1,2,3,4,5$. ii)

find a general expression for s_n . iii) compute $\lim_{n \rightarrow \infty} s_n$.

a) $a_n = \frac{1}{n} - \frac{1}{n+1}$

b) $a_n = \frac{6}{4n^2 - 1}$ (hint: use partial fractions)

Calculus Workshop Series 3

Calculus 2

Week 11

1. Recall that a geometric series with first term a , ratio r has sum $S = \frac{a}{1-r}$ and n th partial

$$\text{sum } s_n = \frac{a(1-r^n)}{1-r}.$$

- a) Find the sum of the geometric series with first term 3, ratio $4/5$.
 - b) Find the first term of the geometric series with sum 14, ratio $3/7$.
 - c) Find the ratio for the geometric series with first term 2, sum 18.
 - d) Suppose a geometric series has first term 5, ratio $7/8$, and $s_n = 35.86765$. Find n .
2. When one pays back a loan, the loan payments are made after the loan is made, so they only serve to pay back what is called the PRESENT VALUE of the payment amounts. For example, if money is lent at 12% yearly interest compounded quarterly, or 3% interest per quarter, a payment of \$1000 3 months from now has PRESENT VALUE $V = 1000(1.03)^{-1}$, since an amount V invested for 3 months would at the end of that 3 months be worth $V(1.03) = 1000(1.03)^{-1}(1.03) = 1000$.

Thus payments of \$1000 at the end of 3 months, 6 months, 9 months, and 12 months would have a PRESENT VALUE of

$$1000(1.03)^{-1} + 1000(1.03)^{-2} + 1000(1.03)^{-3} + 1000(1.03)^{-4} = 3717.10$$

Note that this is the sum of the first 4 terms of a geometric series with $a = 1000(1.03)^{-1}$ and $r = (1.03)^{-1}$

- a) Show that in a geometric series with $r = (1+i)^{-1}$ and first term $a(1+i)^{-1}$,

$$s_n = \frac{a}{i}(1-r^n) = \frac{a}{i}(1-(1+i)^{-n}).$$

Notice that s_n calculates the size of loan that n payments of size a will pay off if the interest rate per period is i .

- b) Verify that if $a = 1000$, $i = .03$, then $s_4 = 3717.10$. Calculate s_8 .

- c) Suppose that, after making 4 payments on the loan of size s_8 of part b), you decided to pay off the whole loan. How much would you have to pay at that time. Hint: You have paid off 3717.10, so what you owe is the value of $(s_8 - 3717.10) = L$ 4 periods from the beginning of the loan. That is, you owe $L(1.03)^4$.
3. Now consider the geometric series with $r = (1.005)^{-1}$, first term $700(1.005)^{-1}$.
- a) What is the sum of this series?
- b) If you took out a 15-year loan at 6% interest compounded monthly, how big a loan could you pay off with payments of \$700 per month (6% per year is .005 per month)?
- c) How big a loan could you pay off if you made payments for 30 years?
- d) If you took out the 30-year loan of part c), how much would you owe after 15 years?
- e) Recently, banks in Japan have started giving 100-year loans. At the interest rates of this problem, how big a loan could you pay off in 100 years?
- f) What percentage of the sum of the infinite series is s_{1200} ?
- g) After 30 years of payments on the 100-year loan, how much would you still owe?

Calculus Workshop Series 3

Calculus 2

Week 12

1. Consider the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Call its sum L .

a) In the proof of the integral test, we established the inequality

$$\int_1^{\infty} \frac{1}{x^4} dx \leq L \leq a_1 + \int_1^{\infty} \frac{1}{x^4} dx.$$

Use this inequality to establish limits for L .

b) Since this is a series with all positive terms, we also know $a_1 \leq L$. Use this fact to improve your limits for L .

c) It can also be established that if the n th partial sum s_n is used as an estimate for L , the error

$$R_n \text{ satisfies } \int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx.$$

i) Use s_3 as an estimate for L . How big could the error be?

ii) What is the smallest n you could use to be sure the error $< .001$? Find this estimate for L .

d) It can also be established that by adding s_n to the inequality of part c), one obtains $s_n + \int_{n+1}^{\infty} f(x) dx \leq L \leq s_n + \int_n^{\infty} f(x) dx$. Taking the average of the left-hand and right-hand sides of this inequality gives an estimate of L which is in error by at most half the difference between them.

i) Use the s_3 of part c)i) and the above inequality to get a new estimate for L . How large could the error be?

ii) Now use s_n for the n of part c)ii) and the above inequality to get a new estimate for L . How large could the error be?

2. Now consider $\sum_{n=1}^{\infty} \frac{1}{n^4 + 2n^2 + 2}$, a series known to converge by comparison to the series above. It is established elsewhere that the error in using s_n as an estimate for the sum

has error no more than the error for the corresponding partial sum in part c. Find an estimate for the sum of this series which has error $< .001$.

Calculus Workshop Series 3

Calculus 2

Week 13

1. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$. i) Use the Alternating Series Test to show this series converges. ii) Recall that the error in using s_n for an estimate of the sum is less than the absolute value of the $(n+1)$ st term. What is the smallest n which will make the error $< .001$? Find the estimated sum using this value of n .
2. The following are related alternating series, whose relationship we shall investigate later.
 - a) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms of the series did you have to add?
 - b) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(.11)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
 - c) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2.4)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
 - d) Calculate $\sin 1$, $\sin .11$, $\sin 2.4$. Are the answers related to what you found above?

Notation: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with sum L .

Then $s_n = a_0 + a_1 x + \dots + a_n x^n$ and $R_n = L - s_n = a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$

3. Consider the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$, convergent for $-1 < x < 1$.

- a) Use s_2 for this series to estimate $1/.985$

b) Show this estimate is accurate to 4 decimal places by showing that R_2 is itself a geometric series with sum $< .00001$

c) Find a series $\sum a_n x^n$ for $\frac{1}{1+x} = \frac{1}{1-(-x)}$.

If a series has radius of convergence R , so does the integral or derivative of the series. So we consider the integral of the series in part c.

d) Find a series for $\ln(1+x) = \int \frac{1}{1+x} dx = C + \int \sum a_n x^n dx$.

i) Set $x=0$ to calculate C .

ii) Does this series converge for $x=1$? Why?

iii) How many terms would you have to add to estimate $\ln 2$ with error $< .001$?

e) Calculate $\int \frac{1}{1-x^2} dx$ both using partial fractions and using power series.

f) Set $x=1/3$ in both calculations. To what value does the series converge?

g) Use $2s_7$ to estimate $\ln 2$. Notice that this requires adding only 4 non-zero terms.

h) Show this is a good estimate by showing that

$$R_7 < \frac{(1/3)^9}{9} + \frac{(1/3)^{11}}{9} + \dots = \frac{(1/3)^9}{9(1-1/9)} = \frac{(1/3)^9}{8}.$$

Calculus Workshop Series 3

Calculus 2

Week 14

1. a) Recall the series representation $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
- b) How many terms would be needed to estimate $\pi = 4(\pi/4) = 4 \tan^{-1} 1$ with error $< .001$?
- c) Use the addition formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ to show that
- $$\tan(\tan^{-1} 1/2 + \tan^{-1} 1/3) = 1 = \tan(\pi/4) \text{ and thus that } \pi/4 = \tan^{-1} 1/2 + \tan^{-1} 1/3.$$
- d) Estimate π using the formula of c) and the series for $\tan^{-1} x$. Show the error $< .001$
2. Find a Taylor series for $\sqrt[3]{x+8}$ near $x=19$.
3. a) Define $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$.
- b) Show that $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$.
- c) Find a Maclaurin series for $\cosh x$, using the definition of Maclaurin series.
- d) Find this same series by using the definition of $\cosh x$ and the Maclaurin series for e^x .
- e) Find a Maclaurin series for $\sinh x$ by differentiating the series of parts c) and d).
- f) Integrate the series of parts c) and d) instead. See that if you evaluate the constant correctly, you get the same series as in part e).

Calculus 2 Workshop for a Single Semester

SERIES 4

Classroom tested at Lawrence Technological University
Department of Mathematics and Computer Science

Prof. William C. Arlinghaus, Ph.D.
with input from
Prof. Michael Mersher
Prof. James Nanny

Note: This particular series has no workshop for Week 2. It serves as a model for how to adjust the set when holidays, or other events, remove a workshop from the sequence.

The goal of the calculus workshop is to provide each student with (1) a deeper understanding of the important mathematical concepts encountered in calculus and (2) an opportunity to develop beneficial problem-solving skills within a collaborative group setting. While the workshop program is designed to be essentially independent of a traditional lecture portion of a course, the work done in the workshop will generally be related to many materials covered in lectures. Instructors may make a judicious selection from the various series that appear below, depending on how an individual course might be structured.

The calculus workshop is designed to be completed in 1 hour and 15 minutes, once a week throughout a semester. Students should work in small groups on a special set of problems, distributed at the beginning of the workshop session. Workshop facilitators should be available to answer questions and offer limited suggestions during each workshop session. Work groups might be reassigned periodically during the semester.

Members of each group are encouraged to work together, share ideas, and divide tasks when appropriate in order to complete as many of the assigned exercises as possible during the scheduled workshop session. Groups are also encouraged to voluntarily exchange e-mail usernames and to set up additional meeting times as needed during the week in order to complete unfinished solutions and to share all final results.

Each student is required to turn in an individual written report following each weekly workshop. Each graded workshop report should be assigned a score ranging from 0 to 10 points (or other scheme at instructor discretion).

The overall semester grade for the workshop will equal the sum of the highest ten report scores (maximum 100 points, using the ten-point scale suggested above) and will be considered equivalent to one in-class exam.

One final note: Students are also encouraged to consider forming independent study groups to work on regularly assigned homework and to prepare for exams throughout the semester. (Keep in mind that the workshop session is not to be used for this purpose.)

Calculus Workshop Report Guidelines

Weekly workshop reports should be submitted to a fixed regular location by a fixed time. Late reports should not be considered. Graded reports should be returned the following week.

Each student must submit a complete workshop report for each workshop attended. Every member of a group is expected to record the solutions collaboratively obtained by that group either during the workshop or, if necessary, at a subsequent group meeting. Once the group work is completed, each student is expected to independently prepare a final workshop report recounting the computational details of the solutions obtained and providing all final answers and required graphs. In some cases it may also be appropriate to include a brief summary of the methods used and any general conclusions that can be drawn from the results.

Mathematical solutions should be complete, precise, well organized, and notationally correct. In all written explanations, proper attention should be given to legibility, grammar, spelling, and punctuation. Each solution should be clearly labeled with the appropriate problem number.

Each week students should receive a Workshop Report cover page which must be included as the first page of the individual report. On it students must provide

- Name and student ID number
- Date
- Lecture instructor
- Names of all other group members

Additional sheets should be attached in order with a staple in the upper left corner. Use only one side of each sheet. Partial sheets and pages torn from spiral notebooks or ring binders are not acceptable. The use of graph paper is recommended for hand-drawn graphs.

Excessive or habitual tardiness by any member of a group is unfair to the other members and puts the whole group at a disadvantage. For this reason, groups with fewer than four members will generally be combined with other groups. Any student arriving late will be reassigned to a different group as space is available. Anyone arriving over 20 minutes late will be excluded from participating in that session and will be marked absent. There will be no opportunity to “makeup” a missed workshop session.

Calculus Workshop Series 4**Calculus 2****Week 1**

1. a) Use the substitution $u = 4 - x^2$ to help you calculate $\int x\sqrt{4 - x^2} dx$. Notice that the integral obtained is a simple power rule integral.
b) Use the same substitution to calculate $\int_0^1 x\sqrt{4 - x^2} dx$. Undo the substitution before calculating the integral.
c) Use this same substitution to calculate $\int_0^1 x\sqrt{4 - x^2} dx$ again, only this time change the limits of integration when you make the substitution.

2. a) Use the same substitution to calculate $\int x^3\sqrt{4 - x^2} dx$. Notice that you will have to think of x^3 as $x^2 \cdot x$, expressing the first factor in terms of u and using the second as part of du .
b) Calculate $\int_0^{\sqrt{3}} x^3\sqrt{4 - x^2} dx$.

3. Evaluate $\int_0^4 \frac{x}{\sqrt{1 + 2x}} dx$. Notice that this time using the substitution $u=1+2x$ and changing the limits of integration makes this problem a power rule problem with easy calculations.

Calculus Workshop Series 4**Calculus 2****Week 3**

1. Let R be the region bounded by $y = \ln x$ and by a line L which passes through the points $(1,0)$ and $(e,1)$.
 - a) Draw a graph of $y = \ln x$ and L , and shade the region R .
 - b) Set up the definite integral to calculate the area of R by integration with respect to x .
 - c) Why can't you evaluate this integral?
 - d) Draw the mirror image of R across the line $y = x$.
 - e) What is the equation of the mirror image of L ?
 - f) Set up the integral to evaluate the area of the mirror image of R by integration with respect to x .
 - g) Evaluate this integral.
 - h) What is the area of R ?

2. Now consider the integral $\int_1^{e^2} \ln x \, dx$.
 - a) Graph the region whose area is represented by this integral.
 - b) Besides $y = \ln x$, what are the other two boundaries of this region?
 - c) Draw the mirror image of this region across the line $y = x$.
 - d) What are the equations of the two straight line boundaries of this region?
 - e) Set up the integral to evaluate the area of this region by integration with respect to x .
 - f) Evaluate this integral.

- g) What is the value of the original integral $\int_1^{e^2} \ln x \, dx$?
- h) Now look in a table of integrals and find an antiderivative of $\ln x$. Use this to verify that the answer you obtained in g) is correct.

Calculus Workshop Series 4
Calculus 2
Week 4

1. Let R be the region in the first quadrant bounded by $y = \sin x$, $y = \sin^{-1} x$, and by the straight lines $x = \pi/2$ and $y = \pi/2$.
 - a) Draw R (recall that the graph of $y = \sin x$ lies below the graph of $y = x$ in the first quadrant, and use reflection to draw the graph of $y = \sin^{-1} x$).
 - b) Set up, but do not evaluate, the integrals (with respect to x) which represent the area of R .
 - c) Now draw the line $y = x$ through this region R , dividing the region into two regions. Call the upper region S and the lower region T .
 - d) Why do you know that the area of $S =$ the area of T ?
 - e) Set up and evaluate the integral (with respect to x) to calculate the area of T .
 - f) What are the areas of R , S , and T respectively? Give both exact answers and numerical approximations.

NOTE: The following antiderivative formulas will be useful to you in calculating the integrals in problem 2.

$$\int \sin^2 x dx = x/2 - \sin x \cos x/2 + C$$

$$\int x \sin x dx = \sin x - x \cos x + C$$

2. Now rotate the region R about the y -axis, obtaining a solid of volume V .
 - a) Set up, but do not evaluate, the integrals to calculate this volume V by using cylindrical shells.
 - b) Set up, but do not evaluate, the integrals to calculate this volume V by using washers.

NOTICE that neither of these integrals can be calculated by methods we know so far.

- c) Suppose that S and T were rotated about the y -axis, giving rise to solids of volumes V_S and V_T , respectively. Does $V = V_S + V_T$? Does $V_S = V_T$? Why or why not?

In parts d, e, f give both exact answers and numerical approximations.

- d) Calculate V_S by using washers.
- e) Calculate V_T by using cylindrical shells.
- f) Calculate V .
- g) Verify, by differentiating the antiderivatives of the note, that the antiderivative formulas given in the note are correct.

Calculus Workshop Series 4

Calculus 2

Week 5

1. a) Let $F(t) = \int_1^2 x^t dx$.

i) Compute $F(2), F(1), F(0), F(-1/2)$.

ii) Compute $F(t)$ for $t \neq -1$.

iii) Use l'Hospital's Rule to compute $L = \lim_{t \rightarrow -1} F(t)$.

iv) Does $L = F(-1)$? What does this say about F ?

b) Let $F_r(t) = \int_1^r x^t dx$

i) Compute $F_r(t)$ for $t \neq -1$.

ii) Compute $L_r = \lim_{t \rightarrow -1} F_r(t)$.

iii) Does $L_r = F_r(-1)$?

c) What does this tell you about $\int_1^r \frac{1}{x} dx$ in relation to the power rule?

2. Consider the graph of $f(x) = \frac{x^2 - 3}{e^x(2x - 3)}$.

a) Find all intercepts of f .

b) Find all vertical asymptotes of f .

c) Use l'Hospital's Rule to compute $\lim_{x \rightarrow -\infty} e^x(2x - 3)$.

d) Find all horizontal asymptotes.

e) Calculate $f'(x)$.

f) What are the critical points for f ?

g) Verify that $f''(x) = \frac{4x^4 - 20x^3 + 33x^2 - 24x + 3}{e^x(2x-3)^3}$.

h) What are the local maxima and minima of f ?

i) Calculate $f''(.15)$, $f''(.2)$, $f''(2.75)$, $f''(2.8)$.

j) What are the inflection points of f (approximately)?

k) Graph $f(x)$, plotting both x - and y -coordinates of all relevant points and identifying all of the above features carefully.

Calculus Workshop Series 4

Calculus 2

Week 6

Notice that, even in integration by parts, the obvious substitution may not be correct.

1. a) Use integration by parts to calculate $\int x^3 \tan^{-1} 2x \, dx$. Note that because only the derivative, not the integral of $\tan^{-1} 2x$ is known, we must let $u = \tan^{-1} 2x$. Only one application of integration by parts is needed, but long division is needed to calculate the resulting integral.

b) Use integration by parts to calculate $\int x^3 (\ln x)^2 \, dx$. Again, we must use the non-obvious substitution. This time we must apply integration by parts twice.

2. Let $I = \int e^{4x} \cos 3x \, dx$ and $J = \int e^{4x} \sin 3x \, dx$.

a) Use integration by parts with $u = \cos 3x$ to show that $I = \frac{1}{4} e^{4x} \cos 3x + \frac{3}{4} J$.

b) Use integration by parts with $u = \sin 3x$ to show that $J = \frac{1}{4} e^{4x} \sin 3x - \frac{3}{4} I$.

c) Substitute the value of J from b) into the equation for I from a) to calculate I .

d) Substitute the value of I from a) into the equation for J from b) to calculate J .

Notice that this time we applied integration by parts twice but were only able to calculate I, J indirectly.

3. Consider the integral $\int x^6 \sin x \, dx$. This time the obvious substitution works, but we have to apply integration by parts six times. Notice, however, that the functions that were du and v in the first application become u and dv in the second, and that this continues in all the applications. So make a table as follows:

u / du	dv / v
x^6	$\sin x$
$6x^5$	$-\cos x$
$30x^4$	$-\sin x$

0

Note that the first two values of uv appear as products along the diagonal lines in the table, given that the signs (+ and -) are included.

- Complete the table.
- Write the integral, using this table. This method is called tabular integration and can be used any time that the original u is a polynomial and the original dv can be integrated as many times as may be required.

Calculus Workshop Series 4
Calculus 2
Week 7

Recall that if a function has two antiderivatives, then those two antiderivatives must differ by a constant.

This workshop is designed to reinforce that fact.

1. a) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \sin x$.
- b) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \cos x$.
- c) Calculate $\int_0^{\pi/2} \sin x \cos x \, dx$ twice, once using each of the above antiderivatives.
- d) Now use a trigonometric identity to notice that the antiderivative you found in part a) differs from the one you found in part b) by a constant. What is the constant?

2. a) Calculate $\int \sin^3 x \cos^3 x \, dx$ using the substitution $u = \sin x$.
- b) Calculate $\int \sin^3 x \cos^3 x \, dx$ using the substitution $u = \cos x$.
- c) Take the antiderivative you found in part a), use a trigonometric identity to change all the occurrences of $\sin x$ to occurrences of $\cos x$, and show that the antiderivative of part a) differs from that of part b) by a constant. What is the constant?

3. a) Calculate $\int \sin^4 x \, dx$ with the help of the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$.
- b) Calculate $\int \sin^4 x \, dx$ with the help of the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

c) Use trigonometric identities to show that the antiderivatives you obtained in parts a) and b) differ from each other by a constant. What is the constant?

Hint: it may be useful to change the antiderivative of part a) using the identities

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1 \quad \text{and}$$

$$\sin 2\theta = 2\sin \theta \cos \theta$$

Calculus Workshop Series 4

Calculus 2

Week 8

In mathematics, it is common to find new methods to extend knowledge from what is known to what is unknown, but similar to what is known. Both of this week's problems illustrate that; in fact, the new methods will even provide another way to do the problem whose solution was already known.

1. a) Use the substitution $u = 4 - x^2$ to help you calculate $\int x \sqrt{4 - x^2} dx$. Notice that the integral obtained is a simple power rule integral, which was possible to do in Calculus 1.
- b) Now use the trigonometric substitution $x = 2 \sin \theta$ to calculate the same integral. This time notice that a so-called "easy" trigonometric integral is obtained; that is, one involving powers of the sine and cosine with at least one of the exponents an odd integer.
- c) Use this same trigonometric substitution $x = 2 \sin \theta$ to calculate $\int x^3 \sqrt{4 - x^2} dx$. Note that another "easy" trigonometric integral is obtained.
- d) Evaluate the integral of part c) using the substitution $u = 4 - x^2$. Notice that this time it is more difficult, and that you need to write x^3 as $x^2(x dx)$ and then set $x^2 = 4 - u$.
- e) Now consider the integral $\int x^2 \sqrt{4 - x^2} dx$. This time the substitution $u = 4 - x^2$ doesn't help at all, and the trigonometric substitution $x = 2 \sin \theta$ yields a more difficult trigonometric integral than the ones in parts b) and c). Evaluate the integral using this trigonometric substitution; be sure your final answer is in terms of x .

One of the key things we see from this example is that trigonometric substitutions handle all of these integrals, and that Calculus 1 methods only work in special cases, in this case when the power of x is odd, and even then they require a lot of manipulation.

2. a) Earlier we learned the formula $\int 1/(u^2 + a^2) du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$. Use this formula to help you calculate $\int \frac{1}{x^2 + 8x + 25} dx$.
- b) Calculate this same integral using the trigonometric substitution $x + 4 = 3 \tan \theta$.
- c) Use this same trigonometric substitution $x + 4 = 3 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2+8x+25)^2} dx.$$

d) Finally use this same trigonometric substitution $x + 4 = 3 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2+8x+25)^3} dx.$$

Remember to undo all substitutions, so that all your answers are in terms of x .

Calculus Workshop Series 4**Calculus 2****Week 9**

1. Consider $\int \frac{49x^2 + 9}{x^5 - 3x^4 + x^3 - 3x^2} dx$.

- Factor the denominator of the integrand.
- Expand the integrand as a sum of partial fractions.
- Evaluate the constants in the expansion.
- Evaluate the integral.

2. Now consider the integral $\int \frac{8x^3 + 35x^2 + 119x + 34}{(x^2 + 4x + 13)^2} dx$.

- Expand the integrand as a sum of partial fractions.
- Write the system of 4 equations in 4 unknowns whose solution will give you the constants.
- Evaluate the constants.
- Use a trigonometric substitution to convert the integral to one we can do.
- Evaluate the integral. Be sure to undo the substitution so that the final answer is in terms of x .

Calculus Workshop Series 4

Calculus 2

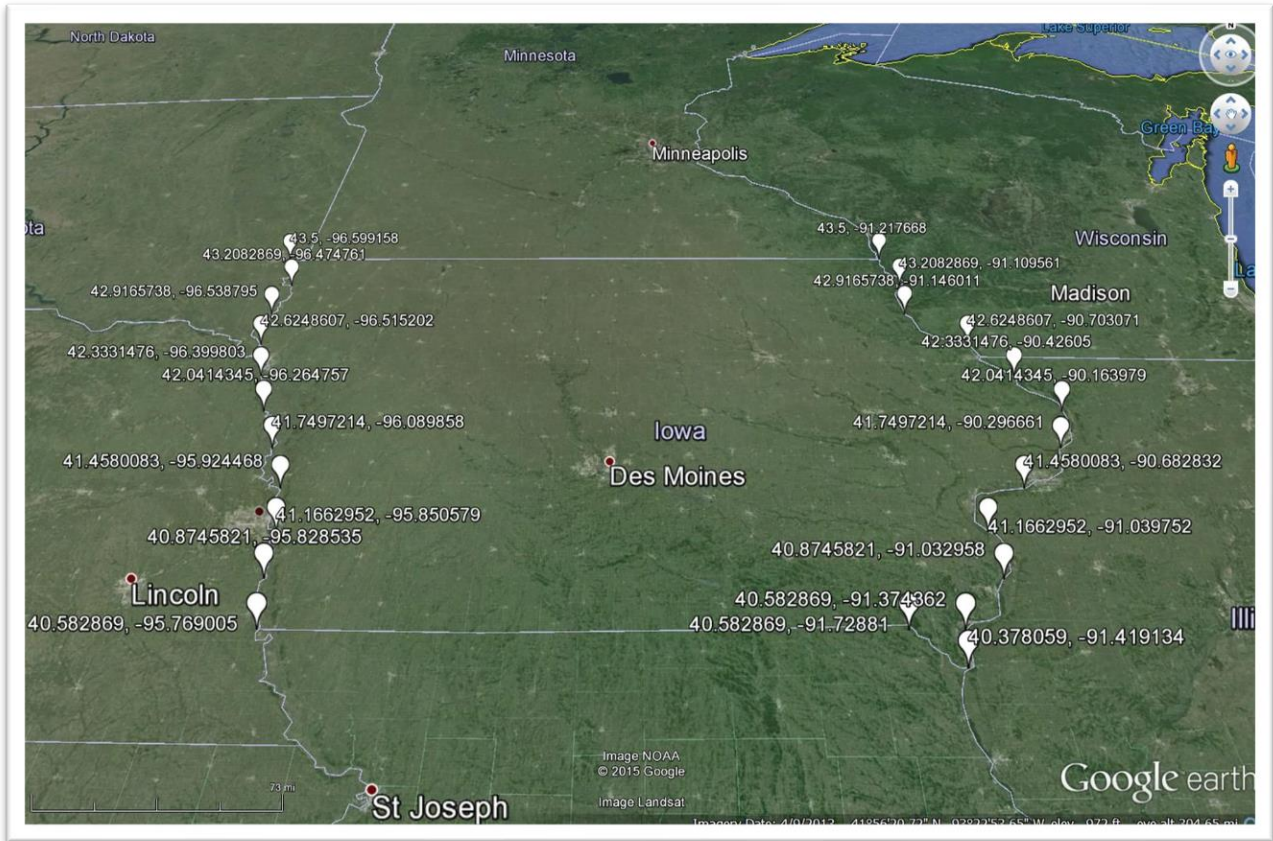
Week 10

Simpson’s Rule allows us to approximate areas under curves even if a formula for the curve is not known, but the values of the function at equally spaced points along the axis are known, since only those values and the length Δx of the spacing are needed. We would like to use this technique to help us measure the area of Iowa. A map of Iowa is provided on your table, along with the following table of values of latitude and longitude for selected points.

latitude		west boundary longitude				east boundary longitude	
43.5		-96.5992				-91.2177	
43.20829		-96.4748				-91.1096	
42.91657		-96.5388				-91.146	
42.62486		-96.5152				-90.7031	
42.33315		-96.3998				-90.4261	
42.04143		-96.2648				-90.164	
41.74972		-96.0899				-90.2967	
41.45801		-95.9245				-90.6828	
41.1663		-95.8506		extra point		-91.0398	
40.87458		-95.8285		on s bdry		-91.033	
40.58287		-95.769		-91.7288		-91.7288	
40.37806		point at south tip				-91.4191	
latitude					longitude		

The following facts are needed to help compute the area:

- a) The circumference of the earth is approximately 24,800 miles.
 - b) The circumference of the circle at latitude α is $24800 \cos \alpha$. This circle is divided into 360 degrees of longitude. For example, the length of the line at the northern boundary between the west boundary and the east boundary is $(-91.2177 + 96.5992)24800 \cos 43.5^\circ / 360$.
 - c) The distance between points of equal longitude is measured along great circles, which have the same circumference as the equator. So in our case Δx is $(43.5 - 43.20829)24800 / 360$.
1. Use Simpson's rule with $n=10$ to calculate that portion of the area of Iowa which does not include the small triangular piece at the southeast edge.
 2. Approximate the base and height of the small triangular piece, and then approximate its area.
 3. Calculate the area of Iowa. How close did you come to the published area of 56,363.3 sq. mi.? How large a percentage error is this?



Calculus Workshop Series 4**Calculus 2****Week 11**

1. Consider $\int_2^{\infty} \frac{22x+6}{2x^3-x^2+8x-4} dx$.

- As the first step in evaluating this integral, use the method of partial fractions to write the integrand as a sum of two fractions.
- Find the antiderivative of the integrand.
- Combine the logarithmic terms of the antiderivative.
- Find the limit of the antiderivative as $x \rightarrow \infty$. Note that one term of the limit can be calculated with the help of l'Hospital's rule, and the other term can be calculated directly.
- Evaluate the integral.

2. Now consider $\int_0^{\infty} e^{-x^2} dx$. Since this is an integrand for which we are unable to find an antiderivative, other methods must be used.

- First note that $\int_0^1 e^{-x^2} dx$ is known to exist, since it represents an area under a curve which in fact lies above the x-axis in the region of integration. If we wanted to know its value, we could approximate it using some method such as Simpson's Rule.
- Note that if $x \geq 1$, then $x^2 \geq x$. What does this say about the relationship between e^{-x^2} and e^{-x} ?
- Show that $\int_1^{\infty} e^{-x} dx$ converges.

d) Conclude that $\int_1^{\infty} e^{-x^2} dx$ converges. This method is called comparison. By comparing one integral (known to be positive) to another positive integral which is larger and yet still convergent, we are able to conclude that the smaller integral also converges.

e) Why do the results of parts a) and d) tell us that $\int_0^{\infty} e^{-x^2} dx$ converges?

f) Do you know whether or not $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges? Why?

Calculus Workshop Series 4

Calculus 2

Week 12

1. For each of the following sequences, i) list the values of the first 5 terms ii) find the limit of the sequence

a) $a_n = \int_1^n \frac{1}{x} dx$

b) $a_n = \int_1^n \frac{1}{x^2} dx$

c) $a_n = \int_n^{n+5} \frac{1}{x} dx$

d) $a_n = \int_1^\infty \frac{1}{x^{n+1}} dx$

e) $a_n = \int_n^\infty \frac{1}{x^2} dx$

2. For each of the following series $\sum a_n$, i) compute the n th partial sum s_n for $n=1,2,3,4,5$. ii)

find a general expression for s_n . iii) compute $\lim_{n \rightarrow \infty} s_n$.

a) $a_n = \frac{1}{n} - \frac{1}{n+1}$

b) $a_n = \frac{6}{4n^2 - 1}$ (hint: use partial fractions)

3. Recall that a geometric series with first term a , ratio r has sum $S = \frac{a}{1-r}$ and n th partial

$$\text{sum } s_n = \frac{a(1-r^n)}{1-r}.$$

- a) Find the sum of the geometric series with first term 3, ratio $4/5$.
- b) Find the first term of the geometric series with sum 14, ratio $3/7$.
- c) Find the ratio for the geometric series with first term 2, sum 18.
- d) Suppose a geometric series has first term 5, ratio $7/8$, and $s_n = 35.86765$. Find n .

Calculus Workshop Series 4

Calculus 2

Week 13

1. Consider the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Call its sum L .

a) In the proof of the integral test, we established the inequality

$$\int_1^{\infty} \frac{1}{x^4} dx \leq L \leq a_1 + \int_1^{\infty} \frac{1}{x^4} dx.$$

Use this inequality to establish limits for L .

b) Since this is a series with all positive terms, we also know $a_1 \leq L$. Use this fact to improve your limits for L .

c) It is established elsewhere that if the n th partial sum s_n is used as an estimate for L , the error R_n satisfies

$$\int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx.$$

i) Use s_3 as an estimate for L . How big could the error be?

ii) What is the smallest n you could use to be sure the error $< .001$? Find this estimate for L .

d) It is also established elsewhere that by adding s_n to the inequality of part c), one obtains

$s_n + \int_{n+1}^{\infty} f(x) dx \leq L \leq s_n + \int_n^{\infty} f(x) dx$. Taking the average of the left-hand and right-hand sides of this inequality gives an estimate of L which is in error by at most half the difference between them.

i) Use the s_3 of part c)i) and the above inequality to get a new estimate for L . How large could the error be?

ii) Now use s_n for the n of part c)ii) and the above inequality to get a new estimate for L . How large could the error be?

2. Now consider $\sum_{n=1}^{\infty} \frac{1}{n^4 + 2n^2 + 2}$, a series known to converge by comparison to the series above. It is established elsewhere that the error in using s_n as an estimate for the sum

has error no more than the error for the corresponding partial sum in part c. Find an estimate for the sum of this series which has error $< .001$.

Calculus Workshop Series 4

Calculus 2

Week 14

1. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$. i) Use the Alternating Series Test to show this series converges. ii) Recall that the error in using s_n for an estimate of the sum is less than the absolute value of the $(n+1)$ st term. What is the smallest n that will make the error $< .001$? Estimate the sum using this value of n .
2. The following are related alternating series, whose relationship we shall investigate later.
- a) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms of the series did you have to add?
- b) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(.13)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
- c) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2.3)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
- d) Calculate $\sin 1$, $\sin .13$, $\sin 2.3$. Are the answers related to what you found above?

Notation: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with sum L .

$$\text{Then } s_n = a_0 + a_1 x + \dots + a_n x^n \quad \text{and} \quad R_n = L - s_n = a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$$

3. Consider the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$, convergent for $-1 < x < 1$.

- a) Use s_2 for this series to estimate $1/.987$

b) Show this estimate is accurate to 4 decimal places by showing that R_2 is itself a geometric series with sum $< .00001$

c) Find a series $\sum a_n x^n$ for $\frac{1}{1+x} = \frac{1}{1-(-x)}$.

If a series has radius of convergence R , so does the integral or derivative of the series. So we consider the integral of the series in part c.

d) Find a series for $\ln(1+x) = \int \frac{1}{1+x} dx = C + \int \sum a_n x^n dx$.

i) Set $x=0$ to calculate C .

ii) Does this series converge for $x=1$? Why?

iii) How many terms would you have to add to estimate $\ln 2$ with error $< .001$?

e) Calculate $\int \frac{1}{1-x^2} dx$ both using partial fractions and using power series.

f) Set $x=1/3$ in both calculations. To what value does the series converge?

g) Use $2s_7$ to estimate $\ln 2$. Notice that this requires adding only 4 non-zero terms.

h) Show this is a good estimate by showing that

$$R_7 < \frac{(1/3)^9}{9} + \frac{(1/3)^{11}}{9} + \dots = \frac{(1/3)^9}{9(1-1/9)} = \frac{(1/3)^9}{8}.$$

Calculus 2 Workshop for a Single Semester

SERIES 5

Classroom tested at Lawrence Technological University
Department of Mathematics and Computer Science

Prof. William C. Arlinghaus, Ph.D.
with input from
Prof. Michael Mersher
Prof. James Nanny

The goal of the calculus workshop is to provide each student with (1) a deeper understanding of the important mathematical concepts encountered in calculus and (2) an opportunity to develop beneficial problem-solving skills within a collaborative group setting. While the workshop program is designed to be essentially independent of a traditional lecture portion of a course, the work done in the workshop will generally be related to many materials covered in lectures. Instructors may make a judicious selection from the various series that appear below, depending on how an individual course might be structured.

The calculus workshop is designed to be completed in 1 hour and 15 minutes, once a week throughout a semester. Students should work in small groups on a special set of problems, distributed at the beginning of the workshop session. Workshop facilitators should be available to answer questions and offer limited suggestions during each workshop session. Work groups might be reassigned periodically during the semester.

Members of each group are encouraged to work together, share ideas, and divide tasks when appropriate in order to complete as many of the assigned exercises as possible during the scheduled workshop session. Groups are also encouraged to voluntarily exchange e-mail usernames and to set up additional meeting times as needed during the week in order to complete unfinished solutions and to share all final results.

Each student is required to turn in an individual written report following each weekly workshop. Each graded workshop report should be assigned a score ranging from 0 to 10 points (or other scheme at instructor discretion).

The overall semester grade for the workshop will equal the sum of the highest ten report scores (maximum 100 points, using the ten-point scale suggested above) and will be considered equivalent to one in-class exam.

One final note: Students are also encouraged to consider forming independent study groups to work on regularly assigned homework and to prepare for exams throughout the semester. (Keep in mind that the workshop session is not to be used for this purpose.)

Calculus Workshop Report Guidelines

Weekly workshop reports should be submitted to a fixed regular location by a fixed time. Late reports should not be considered. Graded reports should be returned the following week.

Each student must submit a complete workshop report for each workshop attended. Every member of a group is expected to record the solutions collaboratively obtained by that group either during the workshop or, if necessary, at a subsequent group meeting. Once the group work is completed, each student is expected to independently prepare a final workshop report recounting the computational details of the solutions obtained and providing all final answers and required graphs. In some cases it may also be appropriate to include a brief summary of the methods used and any general conclusions that can be drawn from the results.

Mathematical solutions should be complete, precise, well organized, and notationally correct. In all written explanations, proper attention should be given to legibility, grammar, spelling, and punctuation. Each solution should be clearly labeled with the appropriate problem number.

Each week students should receive a Workshop Report cover page which must be included as the first page of the individual report. On it students must provide

- Name and student ID number
- Date
- Lecture instructor
- Names of all other group members

Additional sheets should be attached in order with a staple in the upper left corner. Use only one side of each sheet. Partial sheets and pages torn from spiral notebooks or ring binders are not acceptable. The use of graph paper is recommended for hand-drawn graphs.

Excessive or habitual tardiness by any member of a group is unfair to the other members and puts the whole group at a disadvantage. For this reason, groups with fewer than four members will generally be combined with other groups. Any student arriving late will be reassigned to a different group as space is available. Anyone arriving over 20 minutes late will be excluded from participating in that session and will be marked absent. There will be no opportunity to “makeup” a missed workshop session.

Calculus Workshop Series 5**Calculus 2****Week 1**

1. a) Use the substitution $u = 4 - x^2$ to help you calculate $\int x\sqrt{4-x^2} dx$. Notice that the integral obtained is a simple power rule integral.
- b) Use the same substitution to calculate $\int_0^1 x\sqrt{4-x^2} dx$. Undo the substitution before calculating the integral
- c) Use this same substitution to calculate $\int_0^1 x\sqrt{4-x^2} dx$ again, only this time change the limits of integration when you make the substitution.
2. a) Use the same substitution to calculate $\int x^3\sqrt{4-x^2} dx$. Notice that you will have to think of x^3 as $x^2 \cdot x$, expressing the first factor in terms of u and using the second as part of du .
- b) Calculate $\int_0^{\sqrt{3}} x^3\sqrt{4-x^2} dx$.
3. Evaluate $\int_0^4 \frac{x}{\sqrt{1+2x}} dx$. Notice that this time using the substitution $u=1+2x$ and changing the limits of integration makes this problem a power rule problem with easy calculations.

Calculus Workshop Series 5

Calculus 2

Week 2

1. Let $f(x) = \begin{cases} \frac{2x-8}{3} & x \leq -2 \\ 5x-14 & -2 \leq x \leq 4 \\ \frac{6}{3x-7} & x \geq 4 \end{cases}$

- Plot the 4 points corresponding to $x = -3$, $x = -2$, $x = 4$, and $x = 9$.
- Draw the graph of $f(x)$. (Don't plot any more points.)
- As you can see, the graph consists of portions of 3 lines. What are the slopes of these three portions of lines?
- Plot the mirror images (across $y=x$) of the 4 points you found in part a).
- Draw the graph of $f^{-1}(x)$.
- What are the slopes of the three portions of lines that comprise the graph of $f^{-1}(x)$?
- Give an explicit formula for $f^{-1}(x)$.

2. Let $f(x) = 7 - (x-5)^2$ for $x \leq 5$, the left half of a parabola with vertex $(5,7)$.

- Show f is 1-1 by showing that f is increasing throughout its domain.
- Find the point(s) where $y=x$ and $y=f(x)$ intersect.
- Find the intercepts of the graph of $f(x)$.
- Draw the graph of $f(x)$.
- What are the domain and range of $f^{-1}(x)$?

- f) Draw the graph of $f^{-1}(x)$.
- g) Find the value of the derivative of $f^{-1}(x)$ at the points where $x = 0$ and $x = -18$. Use your knowledge of the derivative of $f(x)$ to perform this calculation.
- h) Find an explicit formula for $f^{-1}(x)$.
- i) Calculate the derivative of $f^{-1}(x)$. Evaluate this derivative at $x = 0$ and $x = -18$ to verify that your answers in part g) are correct.

Calculus Workshop Series 5**Calculus 2****Week 3**

1. Let R be the region bounded by $y = \ln x$ and by a line L which passes through the points $(1,0)$ and $(e,1)$.
 - a) Draw a graph of $y = \ln x$ and L , and shade the region R .
 - b) Set up the definite integral to calculate the area of R by integration with respect to x .
 - c) Why can't you evaluate this integral?
 - d) Draw the mirror image of R across the line $y = x$.
 - e) What is the equation of the mirror image of L ?
 - f) Set up the integral to evaluate the area of the mirror image of R by integration with respect to x .
 - g) Evaluate this integral.
 - h) What is the area of R ?

2. Now consider the integral $\int_1^{e^2} \ln x dx$.
 - a) Graph the region whose area is represented by this integral.
 - b) Besides $y = \ln x$, what are the other two boundaries of this region?
 - c) Draw the mirror image of this region across the line $y = x$.
 - d) What are the equations of the two straight line boundaries of this region?
 - e) Set up the integral to evaluate the area of this region by integration with respect to x .
 - f) Evaluate this integral.

g) What is the value of the original integral $\int_1^{e^2} \ln x dx$?

h) Now look in a table of integrals and find an antiderivative of $\ln x$. Use this to verify that the answer you obtained in g) is correct.

Calculus Workshop Series 5

Calculus 2

Week 4

1. Let R be the region in the first quadrant bounded by $y = \sin x$, $y = \sin^{-1} x$, and by the straight lines $x = \pi/2$ and $y = \pi/2$.
 - a) Draw R (recall that the graph of $y = \sin x$ lies below the graph of $y = x$ in the first quadrant, and use reflection to draw the graph of $y = \sin^{-1} x$).
 - b) Set up, but do not evaluate, the integrals (with respect to x) which represent the area of R .
 - c) Now draw the line $y = x$ through this region R , dividing the region into two regions. Call the upper region S and the lower region T .
 - d) Why do you know that the area of $S =$ the area of T ?
 - e) Set up and evaluate the integral (with respect to x) to calculate the area of T .
 - f) What are the areas of R , S , and T respectively? Give both exact answers and numerical approximations.

NOTE: The following antiderivative formulas will be useful to you in calculating the integrals in problem 2.

$$\int \sin^2 x dx = x/2 - \sin x \cos x/2 + C$$

$$\int x \sin x dx = \sin x - x \cos x + C$$

2. Now rotate the region R about the y -axis, obtaining a solid of volume V .
 - a) Set up, but do not evaluate, the integrals to calculate this volume V by using cylindrical shells.
 - b) Set up, but do not evaluate, the integrals to calculate this volume V by using washers.

NOTICE that neither of these integrals can be calculated by methods we know so far.

- c) Suppose that S and T were rotated about the y -axis, giving rise to solids of volumes V_S and V_T , respectively. Does $V = V_S + V_T$? Does $V_S = V_T$? Why or why not?

In parts d, e, f give both exact answers and numerical approximations.

- d) Calculate V_S by using washers.
- e) Calculate V_T by using cylindrical shells.
- f) Calculate V .
- g) Verify, by differentiating the antiderivatives of the note, that the antiderivative formulas given in the note are correct.

Calculus Workshop Series 5

Calculus 2

Week 5

1. a) Let $F(t) = \int_1^2 x^t dx$.

i) Compute $F(2), F(1), F(0), F(-1/2)$.

ii) Compute $F(t)$ for $t \neq -1$.

iii) Use l'Hospital's Rule to compute $L = \lim_{t \rightarrow -1} F(t)$.

iv) Does $L = F(-1)$? What does this say about F ?

b) Let $F_r(t) = \int_1^r x^t dx$

i) Compute $F_r(t)$ for $t \neq -1$.

ii) Compute $L_r = \lim_{t \rightarrow -1} F_r(t)$.

iii) Does $L_r = F_r(-1)$?

c) What does this tell you about $\int_1^r \frac{1}{x} dx$ in relation to the power rule?

2. Consider the graph of $f(x) = \frac{x^2 - 3}{e^x(2x - 3)}$.

a) Find all intercepts of f .

b) Find all vertical asymptotes of f .

c) Use l'Hospital's Rule to compute $\lim_{x \rightarrow -\infty} e^x(2x - 3)$.

d) Find all horizontal asymptotes.

e) Calculate $f'(x)$.

f) What are the critical points for f ?

g) Verify that $f''(x) = \frac{4x^4 - 20x^3 + 33x^2 - 24x + 3}{e^x(2x-3)^3}$.

h) What are the local maxima and minima of f ?

i) Calculate $f''(.15)$, $f''(.2)$, $f''(2.75)$, $f''(2.8)$.

j) What are the inflection points of f (approximately)?

k) Graph $f(x)$, plotting both x - and y -coordinates of all relevant points and identifying all of the above features carefully.

Calculus Workshop Series 5

Calculus 2

Week 6

Notice that, even in integration by parts, the obvious substitution may not be correct.

1. a) Use integration by parts to calculate $\int x^3 \tan^{-1} 2x dx$. Note that because only the derivative, not the integral of $\tan^{-1} 2x$ is known, we must let $u = \tan^{-1} 2x$. Only one application of integration by parts is needed, but long division is needed to calculate the resulting integral.

b) Use integration by parts to calculate $\int x^3 (\ln x)^2 dx$. Again, we must use the non-obvious substitution. This time we must apply integration by parts twice.

2. Let $I = \int e^{4x} \cos 3x dx$ and $J = \int e^{4x} \sin 3x dx$.

a) Use integration by parts with $u = \cos 3x$ to show that $I = \frac{1}{4} e^{4x} \cos 3x + \frac{3}{4} J$.

b) Use integration by parts with $u = \sin 3x$ to show that $J = \frac{1}{4} e^{4x} \sin 3x - \frac{3}{4} I$.

c) Substitute the value of J from b) into the equation for I from a) to calculate I .

d) Substitute the value of I from a) into the equation for J from b) to calculate J .

Notice that this time we applied integration by parts twice but were only able to calculate I, J indirectly.

3. Consider the integral $\int x^6 \sin x dx$. This time the obvious substitution works, but we have to apply integration by parts six times. Notice, however, that the functions that were du and v in the first application become u and dv in the second, and that this continues in all the applications. So make a table as follows:

u / du	dv / v
x^6	$\sin x$
$6x^5$	$-\cos x$
$30x^4$	$-\sin x$
0	

Note that the first two values of uv appear as products along the diagonal lines in the table, given that the signs (+ and -) are included.

- a) Complete the table.
- b) Write the integral, using this table. This method is called tabular integration and can be used any time that the original u is a polynomial and the original dv can be integrated as many times as may be required.

Calculus Workshop Series 5**Calculus 2****Week 7**

Recall that if a function has two antiderivatives, then those two antiderivatives must differ by a constant.

This workshop is designed to reinforce that fact.

1. a) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \sin x$.
 - b) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \cos x$.
 - c) Calculate $\int_0^{\pi/2} \sin x \cos x \, dx$ twice, once using each of the above antiderivatives.
 - d) Now use a trigonometric identity to notice that the antiderivative you found in part a) differs from the one you found in part b) by a constant. What is the constant?
-
2. a) Calculate $\int \sin^3 x \cos^3 x \, dx$ using the substitution $u = \sin x$.
 - b) Calculate $\int \sin^3 x \cos^3 x \, dx$ using the substitution $u = \cos x$.
 - c) Take the antiderivative you found in part a), use a trigonometric identity to change all the occurrences of $\sin x$ to occurrences of $\cos x$, and show that the antiderivative of part a) differs from that of part b) by a constant. What is the constant?
-
3. a) Calculate $\int \sin^4 x \, dx$ with the help of the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$.
 - b) Calculate $\int \sin^4 x \, dx$ with the help of the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

- c) Use trigonometric identities to show that the antiderivatives you obtained in parts a) and b) differ from each other by a constant. What is the constant?

Hint: it may be useful to change the antiderivative of part a) using the identities

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1 \quad \text{and}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

Calculus Workshop Series 5**Calculus 2****Week 8**

In mathematics, it is common to find new methods to extend knowledge from what is known to what is unknown, but similar to what is known. Both of this week's problems illustrate that; in fact, the new methods will even provide another way to do the problem whose solution was already known.

1. a) Use the substitution $u = 4 - x^2$ to help you calculate $\int x\sqrt{4 - x^2} dx$. Notice that the integral obtained is a simple power rule integral, which was possible to do in Calculus 1.
- b) Now use the trigonometric substitution $x = 2\sin \theta$ to calculate the same integral. This time notice that a so-called "easy" trigonometric integral is obtained; that is, one involving powers of the sine and cosine with at least one of the exponents an odd integer.
- c) Use this same trigonometric substitution $x = 2\sin \theta$ to calculate $\int x^3\sqrt{4 - x^2} dx$. Note that another "easy" trigonometric integral is obtained.
- d) Evaluate the integral of part c) using the substitution $u = 4 - x^2$. Notice that this time it is more difficult, and that you need to write x^3 as $x^2(x dx)$ and then set $x^2 = 4 - u$.
- e) Now consider the integral $\int x^2\sqrt{4 - x^2} dx$. This time the substitution $u = 4 - x^2$ doesn't help at all, and the trigonometric substitution $x = 2\sin \theta$ yields a more difficult trigonometric integral than the ones in parts b) and c). Evaluate the integral using this trigonometric substitution; be sure your final answer is in terms of x .

One of the key things we see from this example is that trigonometric substitutions handle all of these integrals, and that Calculus 1 methods only work in special cases, in this case when the power of x is odd, and even then they require a lot of manipulation.

2. a) Earlier we learned the formula $\int \frac{1}{u^2 + a^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$. Use this formula to help you

calculate $\int \frac{1}{x^2 + 8x + 25} dx$.

b) Calculate this same integral using the trigonometric substitution $x + 4 = 3 \tan \theta$.

c) Use this same trigonometric substitution $x + 4 = 3 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2 + 8x + 25)^2} dx.$$

d) Finally use this same trigonometric substitution $x + 4 = 3 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2 + 8x + 25)^3} dx.$$

Remember to undo all substitutions, so that all your answers are in terms of x .

Calculus Workshop Series 5**Calculus 2****Week 9**

1. Consider $\int \frac{49x^2 + 9}{x^5 - 3x^4 + x^3 - 3x^2} dx$.

- Factor the denominator of the integrand.
- Expand the integrand as a sum of partial fractions.
- Evaluate the constants in the expansion.
- Evaluate the integral.

2. Now consider the integral $\int \frac{8x^3 + 35x^2 + 119x + 34}{(x^2 + 4x + 13)^2} dx$.

- Expand the integrand as a sum of partial fractions.
- Write the system of 4 equations in 4 unknowns whose solution will give you the constants.
- Evaluate the constants.
- Use a trigonometric substitution to convert the integral to one we can do.
- Evaluate the integral. Be sure to undo the substitution so that the final answer is in terms of x .

Calculus Workshop Series 5**Calculus 2****Week 10**

1. Consider $\int_2^{\infty} \frac{22x+6}{2x^3-x^2+8x-4} dx$.

- As the first step in evaluating this integral, use the method of partial fractions to write the integrand as a sum of two fractions.
- Find the antiderivative of the integrand.
- Combine the logarithmic terms of the antiderivative.
- Find the limit of the antiderivative as $x \rightarrow \infty$. Note that one term of the limit can be calculated with the help of l'Hospital's rule, and the other term can be calculated directly.
- Evaluate the integral.

2. Now consider $\int_0^{\infty} e^{-x^2} dx$. Since this is an integrand for which we are unable to find an antiderivative, other methods must be used.

- First note that $\int_0^1 e^{-x^2} dx$ is known to exist, since it represents an area under a curve which in fact lies above the x -axis in the region of integration. Use Simpson's Rule with $n=6$ to approximate it.
- Note that if $x \geq 1$, then $x^2 \geq x$. What does this say about the relationship between e^{-x^2} and e^{-x} when $x \geq 1$?
- Show that $\int_1^{\infty} e^{-x} dx$ converges. What is its value? What does this tell you about the value of the integral $\int_1^{\infty} e^{-x^2} dx$.

Note: the method used in part c) is called comparison. By comparing one integral (known to be positive) to another positive integral which is larger and yet still convergent, we are able to conclude that the smaller integral also converges.

- d) Why do the results of parts a) and c) tell us that $\int_0^{\infty} e^{-x^2} dx$ converges? Give an approximate value of this integral, based on your previous results.
- e) Use Simpson's Rule, with $n=6$, to estimate $\int_1^4 e^{-x^2} dx$. Use this estimate, along with the value of $\int_4^{\infty} e^{-x} dx$, to improve your approximation of $\int_0^{\infty} e^{-x^2} dx$.
- f) Do you know whether or not $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges? Why? Estimate its value, if it converges.

Calculus Workshop Series 5**Calculus 2****Week 11**

1. For each of the following sequences, i) list the values of the first 5 terms, ii) find the limit of the sequence

a) $a_n = \int_1^n \frac{1}{x} dx$

b) $a_n = \int_1^n \frac{1}{x^2} dx$

c) $a_n = \int_n^{n+5} \frac{1}{x} dx$

d) $a_n = \int_1^\infty \frac{1}{x^{n+1}} dx$

e) $a_n = \int_n^\infty \frac{1}{x^2} dx$

2. For each of the following series $\sum a_n$, i) compute the n th partial sum s_n for $n=1,2,3,4,5$. ii) find a general expression for s_n . iii) compute $\lim_{n \rightarrow \infty} s_n$.

a) $a_n = \frac{1}{n} - \frac{1}{n+1}$

b) $a_n = \frac{6}{4n^2 - 1}$ (hint: use partial fractions)

3. Recall that a geometric series with first term a , ratio r has sum $S = \frac{a}{1-r}$ and n th partial

$$\text{sum } s_n = \frac{a(1-r^n)}{1-r}.$$

- a) Find the sum of the geometric series with first term 3, ratio $4/5$.
- b) Find the first term of the geometric series with sum 14, ratio $3/7$.
- c) Find the ratio for the geometric series with first term 2, sum 18.
- d) Suppose a geometric series has first term 5, ratio $7/8$, and $s_n = 35.86765$. Find n .

Calculus Workshop Series 5

Calculus 2

Week 12

1. Consider the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Call its sum L .

a) In the proof of the integral test, we established the inequality

$$\int_1^{\infty} \frac{1}{x^4} dx \leq L \leq a_1 + \int_1^{\infty} \frac{1}{x^4} dx.$$

Use this inequality to establish limits for L .

b) Since this is a series with all positive terms, we also know $a_1 \leq L$. Use this fact to improve your limits for L .

c) Earlier it was established that if the n th partial sum s_n is used as an estimate for L , the error

$$R_n \text{ satisfies } \int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx.$$

i) Use s_3 as an estimate for L . How big could the error be?

ii) What is the smallest n you could use to be sure the error $< .001$? Find this estimate for L .

d) It is established elsewhere that by adding s_n to the inequality of part c), one obtains

$s_n + \int_{n+1}^{\infty} f(x) dx \leq L \leq s_n + \int_n^{\infty} f(x) dx$. Taking the average of the left-hand and right-hand sides of this inequality gives an estimate of L which is in error by at most half the difference between them.

i) Use the s_3 of part c) i) and the above inequality to get a new estimate for L . How large could the error be?

ii) Now use s_n for the n of part c)ii) and the above inequality to get a new estimate for L . How large could the error be?

2. Now consider $\sum_{n=1}^{\infty} \frac{1}{n^4 + 2n^2 + 2}$, a series known to converge by comparison to the above series. It is established elsewhere that the error in using s_n as an estimate for the sum has error no more than the error for the corresponding partial sum in part c. Find an estimate for the sum of this series which has error $< .001$.
3. Show that $\sum_{n=1}^{\infty} \frac{8}{n^2 + 14n + 33} = \frac{32891}{27720}$ as follows.
- Use partial fractions to write the fraction as a difference of two fractions.
 - Write s_8 as $A - R_8$, where A is the sum of the positive terms in the summation.
 - Note that $R_8 < 8/12$.
 - Show that $s_n = A - R_n$, where $R_n < 8/(n+4)$
 - Show that $A = 32891/27720$

Calculus Workshop Series 5

Calculus 2

Week 13

1. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$. i) Use the Alternating Series Test to show this series converges. ii) Recall that the error in using s_n for an estimate of the sum is less than the absolute value of the $(n+1)$ st term. What is the smallest n that will make the error $< .001$? Estimate the sum using this value of n .
2. The following are related alternating series, whose relationship we shall investigate later.
- a) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms of the series did you have to add?
- b) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(.13)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
- c) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2.3)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?
- d) Calculate $\sin 1$, $\sin .13$, $\sin 2.3$. Are the answers related to what you found above?

Notation: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with sum L .

$$\text{Then } s_n = a_0 + a_1 x + \dots + a_n x^n \quad \text{and} \quad R_n = L - s_n = a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$$

3. Consider the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$, convergent for $-1 < x < 1$.

- a) Use s_2 for this series to estimate $1/.987$

b) Show this estimate is accurate to 4 decimal places by showing that R_2 is itself a geometric series with sum $< .00001$

c) Find a series $\sum a_n x^n$ for $\frac{1}{1+x} = \frac{1}{1-(-x)}$.

If a series has radius of convergence R , so does the integral or derivative of the series. So we consider the integral of the series in part c.

d) Find a series for $\ln(1+x) = \int \frac{1}{1+x} dx = C + \int \sum a_n x^n dx$.

i) Set $x=0$ to calculate C .

ii) Does this series converge for $x=1$? Why?

iii) How many terms would you have to add to estimate $\ln 2$ with error $< .001$?

e) Calculate $\int \frac{1}{1-x^2} dx$ both using partial fractions and using power series.

f) Set $x=1/3$ in both calculations. To what value does the series converge?

g) Use $2s_7$ to estimate $\ln 2$. Notice that this requires adding only 4 non-zero terms.

h) Show this is a good estimate by showing that

$$R_7 < \frac{(1/3)^9}{9} + \frac{(1/3)^{11}}{9} + \dots = \frac{(1/3)^9}{9(1-1/9)} = \frac{(1/3)^9}{8}.$$

Calculus Workshop Series 5

Calculus 2

Week 14

1. a) Recall the series representation $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
- b) How many terms would be needed to estimate $\pi = 4(\pi/4) = 4 \tan^{-1} 1$ with error $< .001$?
- c) Use the addition formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ to show that

$$\tan(\tan^{-1} 1/2 + \tan^{-1} 1/3) = 1 = \tan(\pi/4)$$
 and thus that $\pi/4 = \tan^{-1} 1/2 + \tan^{-1} 1/3$.
- d) Estimate π using the formula of c) and the series for $\tan^{-1} x$. Show the error $< .001$
2. a) Find a Taylor series for $\sqrt[3]{x+8}$ near $x=19$.
- b) Use the second Taylor polynomial to estimate $\sqrt[3]{27.03}$. How accurate is your estimate?
- c) How many terms of the Taylor series are necessary to estimate $\sqrt[3]{27.48}$ correct to four decimal places? Find this estimate.
3. a) Define $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$.
- b) Show that $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$.
- c) Find a Maclaurin series for $\cosh x$, using the definition of Maclaurin series.
- d) Find this same series by using the definition of $\cosh x$ and the Maclaurin series for e^x .
- e) Find a Maclaurin series for $\sinh x$ by differentiating the series of parts c) and d).
- f) Integrate the series of parts c) and d) instead. See that if you evaluate the constant correctly, you get the same series as in part e).
- g) Calculate $\sinh(.12)$ using this series, accurate to three decimal places.

Calculus 2 Workshop for a Single Semester

SERIES 6

Classroom tested at Lawrence Technological University
Department of Mathematics and Computer Science

Prof. William C. Arlinghaus, Ph.D.
with input from
Prof. Michael Mersher
Prof. James Nanny

The goal of the calculus workshop is to provide each student with (1) a deeper understanding of the important mathematical concepts encountered in calculus and (2) an opportunity to develop beneficial problem-solving skills within a collaborative group setting. While the workshop program is designed to be essentially independent of a traditional lecture portion of a course, the work done in the workshop will generally be related to many materials covered in lectures. Instructors may make a judicious selection from the various series that appear below, depending on how an individual course might be structured.

The calculus workshop is designed to be completed in 1 hour and 15 minutes, once a week throughout a semester. Students should work in small groups on a special set of problems, distributed at the beginning of the workshop session. Workshop facilitators should be available to answer questions and offer limited suggestions during each workshop session. Work groups might be reassigned periodically during the semester.

Members of each group are encouraged to work together, share ideas, and divide tasks when appropriate in order to complete as many of the assigned exercises as possible during the scheduled workshop session. Groups are also encouraged to voluntarily exchange e-mail usernames and to set up additional meeting times as needed during the week in order to complete unfinished solutions and to share all final results.

Each student is required to turn in an individual written report following each weekly workshop. Each graded workshop report should be assigned a score ranging from 0 to 10 points (or other scheme at instructor discretion).

The overall semester grade for the workshop will equal the sum of the highest ten report scores (maximum 100 points, using the ten-point scale suggested above) and will be considered equivalent to one in-class exam.

One final note: Students are also encouraged to consider forming independent study groups to work on regularly assigned homework and to prepare for exams throughout the semester. (Keep in mind that the workshop session is not to be used for this purpose.)

Calculus Workshop Report Guidelines

Weekly workshop reports should be submitted to a fixed regular location by a fixed time. Late reports should not be considered. Graded reports should be returned the following week.

Each student must submit a complete workshop report for each workshop attended. Every member of a group is expected to record the solutions collaboratively obtained by that group either during the workshop or, if necessary, at a subsequent group meeting. Once the group work is completed, each student is expected to independently prepare a final workshop report recounting the computational details of the solutions obtained and providing all final answers and required graphs. In some cases it may also be appropriate to include a brief summary of the methods used and any general conclusions that can be drawn from the results.

Mathematical solutions should be complete, precise, well organized, and notationally correct. In all written explanations, proper attention should be given to legibility, grammar, spelling, and punctuation. Each solution should be clearly labeled with the appropriate problem number.

Each week students should receive a Workshop Report cover page which must be included as the first page of the individual report. On it students must provide

- Name and student ID number
- Date
- Lecture instructor
- Names of all other group members

Additional sheets should be attached in order with a staple in the upper left corner. Use only one side of each sheet. Partial sheets and pages torn from spiral notebooks or ring binders are not acceptable. The use of graph paper is recommended for hand-drawn graphs.

Excessive or habitual tardiness by any member of a group is unfair to the other members and puts the whole group at a disadvantage. For this reason, groups with fewer than four members will generally be combined with other groups. Any student arriving late will be reassigned to a different group as space is available. Anyone arriving over 20 minutes late will be excluded from participating in that session and will be marked absent. There will be no opportunity to “makeup” a missed workshop session.

Calculus Workshop Series 6

Calculus 2

Week 1

1. Let $f(x) = \begin{cases} \frac{2x-5}{3} & x \leq -2 \\ \frac{5x-8}{6} & -2 \leq x \leq 4 \\ \frac{3x-2}{5} & x \geq 4 \end{cases}$

- a) Plot the 4 points corresponding to $x = -3$, $x = -2$, $x = 4$, and $x = 9$.
- b) Draw the graph of $f(x)$. (Don't plot any more points.)
- c) As you can see, the graph consists of portions of 3 lines. What are the slopes of these three portions of lines?
- d) Plot the mirror images (across $y=x$) of the 4 points you found in part a).
- e) Draw the graph of $f^{-1}(x)$.
- f) What are the slopes of the three portions of lines that comprise the graph of $f^{-1}(x)$?
- g) Give an explicit formula for $f^{-1}(x)$.

2. Let $f(x) = 5 - (x-4)^2$ for $x \leq 4$, the left half of a parabola with vertex $(4,5)$.

- a) Show f is 1-1 by showing that f is increasing throughout its domain.
- b) Find the point(s) where $y=x$ and $y=f(x)$ intersect.
- c) Find the intercepts of the graph of $f(x)$.
- d) Draw the graph of $f(x)$.
- e) What are the domain and range of $f^{-1}(x)$?

- f) Draw the graph of $f^{-1}(x)$.
- g) Find the value of the derivative of $f^{-1}(x)$ at the points where $x = 0$ and $x = -11$. Use your knowledge of the derivative of $f(x)$ to perform this calculation.
- h) Find an explicit formula for $f^{-1}(x)$.
- i) Calculate the derivative of $f^{-1}(x)$. Evaluate this derivative at $x = 0$ and $x = -11$ to verify that your answers in part g) are correct.

Calculus Workshop Series 6**Calculus 2****Week 2**

1. Let R be the region bounded by $y = \ln x$ and by a line L which passes through the points $(1,0)$ and $(e^2,2)$.
 - a) Draw a graph of $y = \ln x$ and L , and shade the region R .
 - b) Set up the definite integral to calculate the area of R by integration with respect to x .
 - c) Why can't you evaluate this integral?
 - d) Draw the mirror image of R across the line $y = x$.
 - e) What is the equation of the mirror image of L ?
 - f) Set up the integral to evaluate the area of the mirror image of R by integration with respect to x .
 - g) Evaluate this integral.
 - h) What is the area of R ?

2. Now consider the integral $\int_1^{\sqrt{e}} \ln x \, dx$.

- a) Graph the region whose area is represented by this integral.
- b) Besides $y = \ln x$, what are the other two boundaries of this region?
- c) Draw the mirror image of this region across the line $y = x$.
- d) What are the equations of the two straight line boundaries of this region?
- e) Set up the integral to evaluate the area of this region by integration with respect to x .

f) Evaluate this integral.

g) What is the value of the original integral $\int_1^{\sqrt{e}} \ln x \, dx$?

h) Now look in a table of integrals and find an antiderivative of $\ln x$. Use this to verify that the answer you obtained in g) is correct.

Calculus Workshop Series 6

Calculus 2

Week 3

1. Let R be the region in the first quadrant bounded by $y = \sin(x/2)$, $y = 2\sin^{-1}x$, and by the straight lines $x = \pi$ and $y = \pi$.
 - a) Draw R (note that the graph of $y = \sin(x/2)$ lies below the graph of $y = x$ in the first quadrant; then use properties of reflection and inverse functions to draw the graph of $y = 2\sin^{-1}x$).
 - b) Set up, but do not evaluate, the integrals (with respect to x) which represent the area of R .
 - c) Now draw the line $y = x$ through this region R , dividing the region into two regions. Call the upper region S and the lower region T .
 - d) Why do you know that the area of $S =$ the area of T ?
 - e) Set up and evaluate the integral (with respect to x) to calculate the area of T .
 - f) What are the areas of R , S , and T respectively? Give both exact answers and numerical approximations.

NOTE: The following antiderivative formulas will be useful to you in calculating the integrals in problem 2.

$$\int \sin^2 u \, du = u/2 - \sin u \cos u / 2 + C$$

$$\int u \sin u \, du = \sin u - u \cos u + C$$

2. Now rotate the region R about the y -axis, obtaining a solid of volume V .
 - a) Set up, but do not evaluate, the integrals to calculate this volume V by using cylindrical shells.
 - b) Set up, but do not evaluate, the integrals to calculate this volume V by using washers.

NOTICE that neither of these integrals can be calculated by methods we know so far.

- c) Suppose that S and T were rotated about the y -axis, giving rise to solids of volumes V_S and V_T , respectively. Does $V = V_S + V_T$? Does $V_S = V_T$? Why or why not?

In parts d,e,f give both exact answers and numerical approximations.

- d) Calculate V_S by using washers.
- e) Calculate V_T by using cylindrical shells.
- f) Calculate V .
- g) Verify, by differentiating the antiderivatives of the note, that the antiderivative formulas given in the note are correct.

Calculus Workshop Series 6

Calculus 2

Week 4

1. a) Let $F(t) = \int_1^2 x^t dx$.

i) Compute $F(2), F(1), F(0), F(-1/2)$.

ii) Compute $F(t)$ for $t \neq -1$.

iii) Use l'Hospital's Rule to compute $L = \lim_{t \rightarrow -1} F(t)$.

iv) Does $L = F(-1)$? What does this say about F ?

b) Let $F_r(t) = \int_1^r x^t dx$

i) Compute $F_r(t)$ for $t \neq -1$.

ii) Compute $L_r = \lim_{t \rightarrow -1} F_r(t)$.

iii) Does $L_r = F_r(-1)$?

c) What does this tell you about $\int_1^r \frac{1}{x} dx$ in relation to the power rule?

2. Use the methods of Calculus to help you graph $f(x) = \frac{x^2 - 7}{e^x(4x - 7)}$.

a) Find all intercepts of f .

b) Find all vertical asymptotes of f .

c) Use l'Hospital's Rule to compute $\lim_{x \rightarrow -\infty} e^x(4x - 7)$.

d) Now compute $\lim_{x \rightarrow -\infty} \frac{x^2 - 7}{e^x(4x - 7)}$.

e) What does this tell you about whether there is a horizontal asymptote on the left?

f) Now find all horizontal asymptotes of f .

g) Calculate $f'(x)$.

h) What are the critical points for f ?

i) Verify that $f''(x) = \frac{16x^4 - 88x^3 + 105x^2 - 28x - 77}{e^x(4x - 7)^3}$.

j) Use the formula of i) to help determine which critical points are local maxima and minima.

k) Use i) to help determine inflection points for f (if you have trouble finding the zeroes of the second derivative, try calculating $f''(-0.7)$, $f''(-0.6)$, $f''(4)$, $f''(4.1)$).

l) Graph $f(x)$, plotting both x - and y -coordinates of all relevant points and identifying all of the above features carefully.

Calculus Workshop Series 6

Calculus 2

Week 5

Notice that, even in integration by parts, the obvious substitution may not be correct.

1. a) Use integration by parts to calculate $\int x^6 \tan^{-1} x dx$. Note that because only the derivative, not the integral of $\tan^{-1} x$ is known, we must let $u = \tan^{-1} x$. Only one application of integration by parts is needed, but long division is needed to calculate the resulting integral.
- b) Use integration by parts to calculate $\int x^6 (\ln x)^2 dx$. Again, we must use the non-obvious substitution. This time we must apply integration by parts twice.

2. Let $I = \int e^{6x} \cos 3x dx$ and $J = \int e^{6x} \sin 3x dx$.

- a) Use integration by parts with $u = \cos 3x$ to show that $I = \frac{1}{6} e^{6x} \cos 3x + \frac{1}{2} J$.
- b) Use integration by parts with $u = \sin 3x$ to show that $J = \frac{1}{6} e^{6x} \sin 3x - \frac{1}{2} I$.
- c) Substitute the value of J from b) into the equation for I from a) to calculate I .
- d) Substitute the value of I from a) into the equation for J from b) to calculate J .

Notice that this time we applied integration by parts twice but were only able to calculate I, J indirectly.

3. Consider the integral $\int x^7 \sin x dx$. This time the obvious substitution works, but we have to apply integration by parts six times. Notice, however, that the functions that were du and v in the first application become u and dv in the second, and that this continues in all the applications. So make a table as follows:

u/du	dv / v
x^7	$\sin x$
$7x^6$	$-\cos x$
$42x^5$	$-\sin x$

0

Note that the first two values of uv appear as products along the diagonal lines in the table, given that the signs (+ and -) are included.

- a) Complete the table.
- b) Write the integral, using this table. This method is called tabular integration and can be used any time that the original u is a polynomial and the original dv can be integrated as many times as may be required.

Calculus Workshop Series 6
Calculus 2
Week 6

Recall that if a function has two antiderivatives, then those two antiderivatives must differ by a constant.

This workshop is designed to reinforce that fact.

1. a) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \sin x$.
 - b) Calculate $\int \sin x \cos x \, dx$ using the substitution $u = \cos x$.
 - c) Calculate $\int_0^{\pi/2} \sin x \cos x \, dx$ twice, once using each of the above antiderivatives.
 - d) Now use a trigonometric identity to notice that the antiderivative you found in part a) differs from the one you found in part b) by a constant. What is the constant?
2. a) Calculate $\int \sin^5 x \cos^3 x \, dx$ using the substitution $u = \sin x$.
 - b) Calculate $\int \sin^5 x \cos^3 x \, dx$ using the substitution $u = \cos x$.
 - c) Take the antiderivative you found in part a), use a trigonometric identity to change all the occurrences of $\sin x$ to occurrences of $\cos x$, and show that the antiderivative of part a) differs from that of part b) by a constant. What is the constant?
3. a) Calculate $\int \cos^4 x \, dx$ with the help of the identity $\cos^2 x = \frac{1 + \cos 2x}{2}$.
 - b) Calculate $\int \cos^4 x \, dx$ with the help of the reduction formula

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

c) Use trigonometric identities to show that the antiderivatives you obtained in parts a) and b) differ from each other by a constant. What is the constant?

Hint: it may be useful to change the antiderivative of part a) using the identities

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1 \quad \text{and}$$

$$\sin 2\theta = 2\sin \theta \cos \theta$$

Calculus Workshop Series 6

Calculus 2

Week 7

In mathematics, it is common to find new methods to extend knowledge from what is known to what is unknown, but similar to what is known. Both of this week's problems illustrate that; in fact, the new methods will even provide another way to do the problem whose solution was already known.

1. a) Use the substitution $u = 9 - x^2$ to help you calculate $\int x\sqrt{9 - x^2} dx$. Notice that the integral obtained is a simple power rule integral, which was possible to do in Calculus 1.
- b) Now use the trigonometric substitution $x = 3\sin \theta$ to calculate the same integral. This time notice that a so-called "easy" trigonometric integral is obtained; that is, one involving powers of the sine and cosine with at least one of the exponents an odd integer.
- c) Use this same trigonometric substitution $x = 3\sin \theta$ to calculate $\int x^3\sqrt{9 - x^2} dx$. Note that another "easy" trigonometric integral is obtained.
- d) Evaluate the integral of part c) using the substitution $u = 9 - x^2$. Notice that this time it is more difficult, and that you need to write x^3 as $x^2(x dx)$ and then set $x^2 = 9 - u$.
- e) Now consider the integral $\int x^2\sqrt{9 - x^2} dx$. This time the substitution $u = 9 - x^2$ doesn't help at all, and the trigonometric substitution $x = 3\sin \theta$ yields a more difficult trigonometric integral than the ones in parts b) and c). Evaluate the integral using this trigonometric substitution; be sure your final answer is in terms of x .

One of the key things we see from this example is that trigonometric substitutions handle all of these integrals, and that Calculus 1 methods only work in special cases, in this case when the power of x is odd, and even then they require a lot of manipulation.

2. a) Earlier we learned the formula $\int \frac{1}{u^2 + a^2} du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$. Use this formula to help you calculate $\int \frac{1}{x^2 + 6x + 25} dx$.

b) Calculate this same integral using the trigonometric substitution $x + 3 = 4 \tan \theta$.

c) Use this same trigonometric substitution $x + 3 = 4 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2 + 6x + 25)^2} dx.$$

d) Finally use this same trigonometric substitution $x + 3 = 4 \tan \theta$ to help you calculate

$$\int \frac{1}{(x^2 + 6x + 25)^3} dx.$$

Remember to undo all substitutions, so that all your answers are in terms of x .

Calculus Workshop Series 6**Calculus 2****Week 8**

1. Consider $\int \frac{-46x^2 + 16}{x^5 - 4x^4 - x^3 + 4x^2} dx$.

- Factor the denominator of the integrand.
- Expand the integrand as a sum of partial fractions.
- Evaluate the constants in the expansion.
- Evaluate the integral.

2. Now consider the integral $\int \frac{2x^3 + 15x^2 + 71x + 139}{(x^2 + 4x + 20)^2} dx$.

- Expand the integrand as a sum of partial fractions.
- Write the system of 4 equations in 4 unknowns whose solution will give you the constants.
- Evaluate the constants.
- Use a trigonometric substitution to convert the integral to one we can do.
- Evaluate the integral. Be sure to undo the substitution so that the final answer is in terms of x .

Calculus Workshop Series 6**Calculus 2****Week 9**

1. Consider $\int_4^{\infty} \frac{22x+6}{2x^3-x^2+8x-4} dx$.

- As the first step in evaluating this integral, use the method of partial fractions to write the integrand as a sum of two fractions.
- Find the antiderivative of the integrand.
- Combine the logarithmic terms of the antiderivative.
- Find the limit of the antiderivative as $x \rightarrow \infty$. Note that one term of the limit can be calculated with the help of l'Hospital's rule, and the other term can be calculated directly.
- Evaluate the integral.

2. Now consider $\int_0^{\infty} e^{-x^2} dx$. Since this is an integrand for which we are unable to find an antiderivative, other methods must be used.

- First note that $\int_0^1 e^{-x^2} dx$ is known to exist, since it represents an area under a curve which in fact lies above the x-axis in the region of integration. Use Simpson's Rule with $n=6$ to approximate it.
- Note that if $x \geq 1$, then $x^2 \geq x$. What does this say about the relationship between e^{-x^2} and e^{-x} when $x \geq 1$?
- Show that $\int_1^{\infty} e^{-x} dx$ converges. What is its value? What does this tell you about the value of the integral $\int_1^{\infty} e^{-x^2} dx$.

Note: the method used in part c) is called comparison. By comparing one integral (known to be positive) to another positive integral which is larger and yet still convergent, we are able to conclude that the smaller integral also converges.

- d) Why do the results of parts a) and c) tell us that $\int_0^{\infty} e^{-x^2} dx$ converges? Give an approximate value of this integral, based on your previous results.
- e) Use Simpson's Rule, with $n=6$, to estimate $\int_1^3 e^{-x^2} dx$. Use this estimate, along with the value of $\int_3^{\infty} e^{-x} dx$, to improve your approximation of $\int_0^{\infty} e^{-x^2} dx$.
- f) Do you know whether or not $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges? Why? Estimate its value, if it converges.

Calculus Workshop Series 6**Calculus 2****Week 10**

1. For each of the following sequences, i) list the values of the first 5 terms, ii) find the limit of the sequence

a) $a_n = \int_1^n \frac{1}{x} dx$

b) $a_n = \int_1^n \frac{1}{x^2} dx$

c) $a_n = \int_n^{n+5} \frac{1}{x} dx$

d) $a_n = \int_1^\infty \frac{1}{x^{n+1}} dx$

e) $a_n = \int_n^\infty \frac{1}{x^2} dx$

2. For each of the following series $\sum a_n$, i) compute the 5th partial sum s_5 . ii) find a general expression for s_n . iii) compute $\lim_{n \rightarrow \infty} s_n$.

a) $a_n = \frac{1}{n} - \frac{1}{n+1}$

b) $a_n = \frac{10}{4n^2 - 16n + 15}$ (hint: use partial fractions)

3. Recall that a geometric series with first term a , ratio r has sum $S = \frac{a}{1-r}$ and n th partial

$$\text{sum } s_n = \frac{a(1-r^n)}{1-r}.$$

- a) Find the sum of the geometric series with first term 4, ratio $3/5$.
- b) Find the first term of the geometric series with sum 21, ratio $4/7$.
- c) Find the ratio for the geometric series with first term 5, sum 22.
- d) Suppose a geometric series has first term 5, ratio $7/8$, and $s_n = 35.86765$. Find n .

Calculus Workshop Series 6

Calculus 2

Week 11

1. Consider the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Call its sum L .

a) In the proof of the integral test, we established the inequality

$$\int_1^{\infty} \frac{1}{x^4} dx \leq L \leq a_1 + \int_1^{\infty} \frac{1}{x^4} dx.$$

Use this inequality to establish limits for L .

b) Since this is a series with all positive terms, we also know $a_1 \leq L$. Use this fact to improve your limits for L .

c) It is established elsewhere that if the n th partial sum s_n is used as an estimate for L , the error R_n satisfies $\int_{n+1}^{\infty} \frac{1}{x^4} dx \leq R_n \leq \int_n^{\infty} \frac{1}{x^4} dx$.

i) Use s_3 as an estimate for L . How big could the error be?

ii) What is the smallest n you could use to be sure the error $< .001$? Find this estimate for L .

d) It is also established elsewhere that by adding s_n to the inequality of part c), one obtains

$s_n + \int_{n+1}^{\infty} f(x)dx \leq L \leq s_n + \int_n^{\infty} f(x)dx$. Taking the average of the left-hand and right-hand sides of this inequality gives an estimate of L which is in error by at most half the difference between them.

i) Use the s_3 of part c) i) and the above inequality to get a new estimate for L . How large could the error be?

ii) Now use s_n for the n of part c)ii) and the above inequality to get a new estimate for L . How large could the error be?

4. Compute $\sum_{n=1}^{\infty} \frac{4}{n^2 + 10n + 21}$ as follows.
- Use partial fractions to write the fraction as a difference of two fractions.
 - Write s_4 as $A - B_4$, where A is the sum of the positive terms in the summation.
 - Write s_8 as $A - B_8$. How many fractions are added together in B_8 ?
 - Show $B_8 < 4/12$.
 - If $s_n = A - B_n$, what is B_n ?
 - Show $B_n < 4/(n+4)$.
 - Show $\sum_{n=1}^{\infty} \frac{4}{n^2 + 10n + 21} = A$.
 - Write A as a simple fraction.

Calculus Workshop Series 6

Calculus 2

Week 12

1. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$.

a) Use the Alternating Series Test to show this series converges.

b) Recall that the error in using s_n for an estimate of the sum is less than the absolute value of the $(n+1)$ st term. What is the smallest n that will make the error $< .001$? Estimate the sum using this value of n .

2. The following are related alternating series, whose relationship we shall investigate later.

a) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms of the series did you have to add?

b) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(.11)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?

c) Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2.5)^{2n-1}}{(2n-1)!}$. Show this series converges, and find an estimate of the sum accurate to 3 decimal places. How many terms were required this time?

d) Calculate $\sin 1$, $\sin .11$, $\sin 2.5$. Are the answers related to what you found above?

Notation: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with sum L .

Then $s_n = a_0 + a_1 x + \dots + a_n x^n$ and $R_n = L - s_n = a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$

3. Consider the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$, convergent for $-1 < x < 1$.

a) Use s_2 for this series to estimate $1/.983$

b) Show this estimate is accurate to 4 decimal places by showing that R_2 is itself a geometric series with sum $< .00001$

c) Find a series $\sum a_n x^n$ for $\frac{1}{1+x} = \frac{1}{1-(-x)}$.

If a series has radius of convergence R , so does the integral or derivative of the series. So we consider the integral of the series in part c.

d) Find a series for $\ln(1+x) = \int \frac{1}{1+x} dx = C + \int \sum a_n x^n dx$.

i) Set $x=0$ to calculate C .

ii) Does this series converge for $x=1$? Why?

iii) How many terms would you have to add to estimate $\ln 2$ with error $< .001$?

e) Calculate $\int \frac{1}{1-x^2} dx$ both using and using power series.

f) Set $x=1/3$ in both calculations. To what value does the series converge?

g) Use $2s_7$ to estimate $\ln 2$. Notice that this requires adding only 4 non-zero terms.

h) Show this is a good estimate by showing that

$$R_7 < \frac{(1/3)^9}{9} + \frac{(1/3)^{11}}{9} + \dots = \frac{(1/3)^9}{9(1-1/9)} = \frac{(1/3)^9}{8}.$$

Calculus Workshop Series 6

Calculus 2

Week 13

1. a) Recall the series representation $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
- b) How many terms would be needed to estimate $\pi = 4(\pi/4) = 4 \tan^{-1} 1$ with error $< .001$?
- c) Use the addition formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ to show that $\tan(\tan^{-1} 1/2 + \tan^{-1} 1/3) = 1 = \tan(\pi/4)$ and thus that $\pi/4 = \tan^{-1} 1/2 + \tan^{-1} 1/3$.
- d) Estimate π using the formula of c) and the series for $\tan^{-1} x$. Show the error $< .001$
2. a) Find a Taylor series for $\sqrt[3]{x+9}$ near $x=18$.
- b) Use the second Taylor polynomial to estimate $\sqrt[3]{27.025}$. How accurate is your estimate?
- c) How many terms of the Taylor series are necessary to estimate $\sqrt[3]{27.47}$ correct to four decimal places? Find this estimate.
3. a) Define $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$.
- b) Show that $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$.
- c) Find a Maclaurin series for $\cosh x$, using the definition of Maclaurin series.
- d) Find this same series by using the definition of $\cosh x$ and the Maclaurin series for e^x .
- e) Find a Maclaurin series for $\sinh x$ by differentiating the series of parts c) and d).
- f) Integrate the series of parts c) and d) instead. See that if you evaluate the constant correctly, you get the same series as in part e).

g) Calculate $\sinh(.11)$ using this series, accurate to three decimal places.

Calculus Workshop Series 6

Calculus 2

Week 13, variant

1. a) Recall the series representation $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
- b) How many terms would be needed to estimate $\pi = 4(\pi/4) = 4 \tan^{-1} 1$ with error $< .001$?
- c) Use the addition formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ to show that
 $\tan(\tan^{-1} 1/2 + \tan^{-1} 1/3) = 1 = \tan(\pi/4)$ and thus that $\pi/4 = \tan^{-1} 1/2 + \tan^{-1} 1/3$.
- d) Estimate π using the formula of c) and the series for $\tan^{-1} x$. Show the error $< .001$
2. a) Find a Taylor series for $\sqrt[3]{x+9}$ near $x=18$.
- b) Use the second Taylor polynomial to estimate $\sqrt[3]{27.025}$. How accurate is your estimate?
- c) How many terms of the Taylor series are necessary to estimate $\sqrt[3]{27.47}$ correct to four decimal places? Find this estimate.
3. a) Given the Maclaurin series for $\cos x$, find a power series representing $f(x) = x^2 \cos x$.
- b) Given the Maclaurin series for $\sin x$, find a power series representing $g(x) = \frac{\sin x}{x}$.
- c) Find a power series representing $h(x) = f(x) + g(x)$. Be sure to find a formula for the coefficient of x^n in this series.
- d) Use this series to estimate $h(0.4)$ correct to six decimal places. What enables you to estimate the error easily for this series?