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Finite-time distributed control with time transformation

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Summary

In this article, we propose distributed control algorithms for first- and second-order multiagent systems for addressing finite-time control problem with a priori given, user-defined finite-time convergence guarantees. The proposed control frameworks are predicated on a recently developed time transformation approach. Specifically, our contribution is twofold: First, a generalized time transformation function is proposed that converts the user-defined finite-time interval to a stretched infinite-time interval, where one can design a distributed control algorithm on this stretched interval and then transform it back to the original finite-time interval for achieving a given multiagent system objective. Second, for a specific time transformation function, we analytically establish the robustness properties of the resulting finite-time distributed control algorithms against vanishing and nonvanishing system uncertainties. By contrast to existing finite-time approaches, it is shown that the proposed algorithms can preserve a priori given, user-defined finite-time convergence regardless of the initial conditions of the multiagent system, the graph topology, and without requiring a knowledge of the upper bounds of the considered class of system uncertainties. Illustrative numerical examples are included to further demonstrate the efficacy of the presented results.

KEYWORDS

finite-time distributed control, multiagent systems, robustness, system stability, time transformation

1 | INTRODUCTION

Networked multiagent systems consist of interacting agents that locally exchange information, energy, or matter.[1,2] In order to achieve high system performance, reliability, and operation in the presence of system uncertainties, it is required to design resilient distributed control architectures for these systems.[3,4] In the most of existing distributed control system design architectures, asymptotic convergence of the agents is guaranteed. In other words, the general focus is on the synthesis and analysis of distributed controllers such that the agent states asymptotically converge over an infinite-time interval to static equilibrium points of interest.[1,5,6] However, this asymptotic convergence guarantees can degrade to uniform ultimate boundedness in presence of vanishing (ie, state-dependent) and nonvanishing (ie, time-dependent) system uncertainties (see, eg, References 3,7)

By contrast to the aforementioned studies and from a practical point of view in many time-critical multiagent system applications, it is essential for agents to complete a given task in finite time. To this end, several attempts have been carried out on achieving finite-time convergence guarantees (see, eg, References 8-11). In Reference 12, finite-time convergence

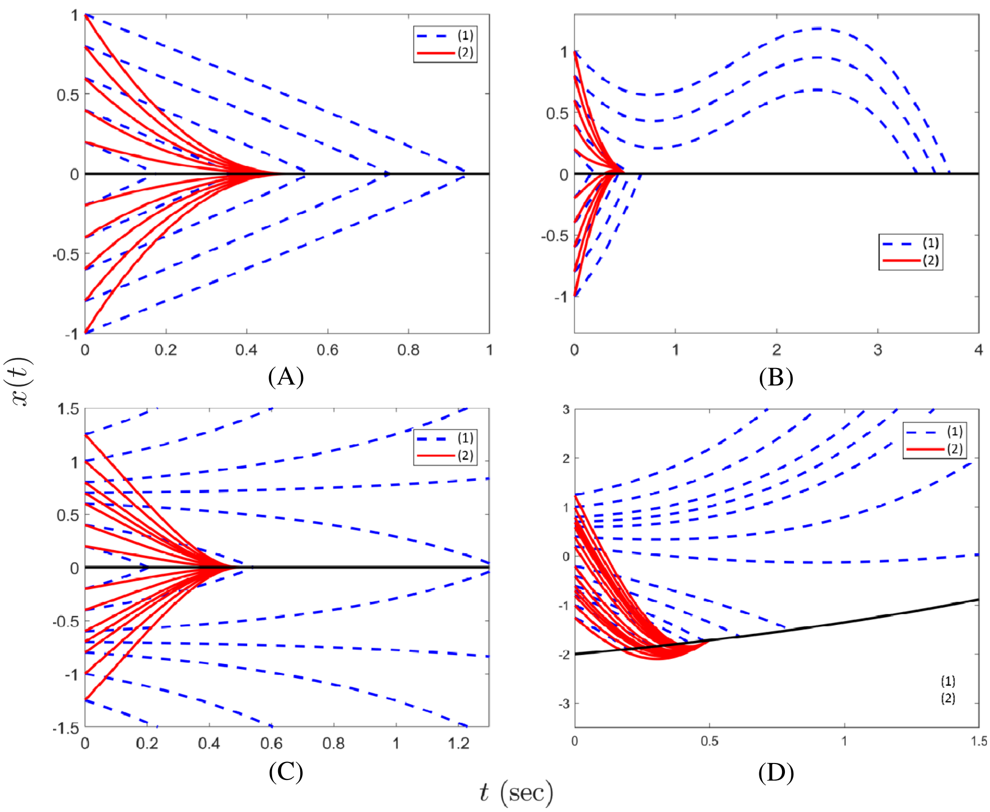

FIGURE 1 The comparison of the controller in (1) with the controller predicated on time transformation method in (2) for A, stabilization problem; B, stabilization problem with nonvanishing uncertainty as $\dot{x}(t) = u(t) + 1.5\sin(t)$; C, stabilization problem with vanishing uncertainty as $\dot{x}(t) = u(t) + 1.5x(t)$; and D, command following problem, that is, $x(t)$ in (1) and (2) is replaced with $x(t) - c(t)$, where $c(t) = -3 + e^{0.5t}$ is the command, and the system dynamics is subject to vanishing and nonvanishing uncertainties as $\dot{x}(t) = u(t) + 1.5x(t) + 1.5\sin(t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

control barrier functions are introduced for finite-time convergence to the required communication graph structure. These approaches guarantee that the system trajectories converge in less than some calculated finite-time value. Yet, a limitation of these approaches is that their finite-time convergence depends on the initial conditions of agents and the graph topology[13] (see Figure 1 and the corresponding discussions in Section 1.2 for details). This limitation is partially resolved in References 13-16 in which the proposed approach, referred to as *fixed-time* control architecture, results in an upper bound on the convergence time that is independent of the initial conditions. However, the calculated bounds do not hold globally for all initial conditions and/or they can be still conservative.[13] Thus, it may not be possible to assign a priori, user-defined finite-time (referred to as *prescribed* finite-time) necessary for many practical networked multiagent systems applications while using these algorithms. Furthermore, in the *fixed-time* control architectures, the finite-time convergence value can depend on the magnitude of the system uncertainties.

Notable contributions for achieving *prescribed* finite-time convergence guarantees include.[17-24] Specifically, in References 17,18 the convergence time of the agents is upper bounded by a prescribed time. Yet, the actual convergence can be reached prior to this upper bound, and therefore, it cannot be assigned by the user. In References 19,20 sliding mode control designs are utilized to achieve a prescribed finite-time convergence; however, the proposed control architecture applies only to single systems, and more importantly, their proposed control design require the knowledge of the upper bound on the system uncertainties. The proposed method in Reference 21 does not account for any system uncertainties and is also applied to single systems. The control architectures in References 22-24 utilize an optimal control platform. However, the control architecture in Reference 22 is not in the context of multiagent systems and applies only to single systems. The authors of References 23,24 utilize a sampling time sequence technique, where the results in Reference 23 do not include the robustness to system uncertainties or convergence to a time-varying leader trajectories, and the control algorithm in Reference 24 only tolerates impulsive system disturbances.

A novel idea, namely, the time transformation approach, is utilized in the recent results documented in References 25,26 for achieving a priori, user-defined smooth finite-time convergence, where these results consider distributed control algorithms for multiagent systems. The key feature of this method is to perform analysis of a given finite-time distributed control algorithm in a stretched infinite-time interval, where one can readily utilize tools from, for example, standard Lyapunov stability theory, to conclude stability and convergence guarantees on the original user-defined finite-time interval. While the authors of References 27-33 use similar distributed control algorithms to the ones presented in References 25,26 (note that the studies in References 27-29 can be considered as a multiagent systems generalization of the idea documented in Reference 19), the results documented in Reference 25 are not limited to time-invariant graph topologies and the results documented in Reference 26 are not limited to multiagent systems with static equilibrium points[1]. More importantly, due to a lack of complete systematic design and analysis framework, the results in References 27-31 may not readily be extendable and system-theoretic robustness tradeoffs of these algorithms are unknown. For instance, the authors of Reference 31 do not consider system uncertainties for the dynamics of the agents, and by using a time-varying scaling function that switches to a constant value, the continuity of the control signal is ensured at the prescribed time. However, for uncertain system dynamics, the control signal at the end of the prescribed time interval may be nonzero and unknown due to the effects of the uncertainties. Hence, a switching remedy alone may not help to preserve the continuity of the control signal. The authors of Reference 32 study single systems in normal form with nonvanishing system uncertainty. Furthermore, the results in Reference 33 are limited to single systems and boundedness analysis for the control signal are not performed. The notion of time-base generator is introduced in Reference 35 and is used to achieve predefined finite-time convergence using time-varying control gains, for example, in References 36-39. Specifically, the results in Reference 36 only applies to single systems with nonvanishing system uncertainty in normal form, similar to Reference 33. Although the presented results in Reference 37 is promising for consensus of multiagent systems, the robustness of the presented control design to system uncertainties is not studied. The authors of Reference 38 consider agents with nonvanishing system uncertainties; however, the robustness of the presented result in presence of vanishing system uncertainties is not established. Finally, the authors of Reference 39 recently establish conditions on the Lyapunov function for achieving fixed and prescribed finite-time stability for single systems, where the proposed structure can be viewed as a general form of, for example, the control structure used in Reference 33. Yet, the robustness analysis presented in Reference 39 require the knowledge of the upper bound on the system uncertainties and, more importantly, the boundedness of the system state derivative (ie, the control signal) is not addressed.

1.1 | Contribution

In this article, we propose distributed control algorithms for first- and second-order multiagent systems, where they are predicated on a time transformation approach based on the results in References 25,26 for achieving the a priori given, user-defined finite-time convergence guarantees. Specifically, our contribution is twofold: First, a generalized time transformation function is proposed that converts the user-defined finite-time interval to a stretched infinite-time interval, where one can design a distributed control algorithm on this stretched interval and then transform it back to the original finite-time interval for achieving a given multiagent system objective. Second, for a specific time transformation function, we analytically establish the robustness properties of the resulting finite-time distributed control algorithms against vanishing and nonvanishing system uncertainties in our systematic time transformation framework. By contrast to existing finite-time approaches, it is shown that the proposed algorithms can preserve a priori given, user-defined finite-time convergence regardless of the initial conditions of the multiagent system, the graph topology, and without requiring a knowledge of the upper bounds of the considered class of system uncertainties. Illustrative numerical examples are further included to demonstrate the presented results. Note that preliminary conference versions of this article are appeared in References 40,41, where the present article significantly goes beyond these conference versions not only by providing detailed proofs of all the key theoretical results, but also by focusing on all the theoretical developments necessary for the generalization of the results with detailed examples and motivations.

1.2 | A motivational example

To elucidate the fundamental advantages of the time transformation-based technique proposed in References 25,26 over well-studied finite-time control architectures, consider the first-order dynamical system given by $\dot{x}(t) = u(t),\ x(0) = x_0$, where $x(t) \in \mathbb{R}$ is the system state and $u(t) \in \mathbb{R}$ is the control signal. First, we use the well-studied finite-time control design that utilizes non-Lipschitz control tools and methods generally with signum functions, for example, by extending the seminal results in References 10,11, and consider the representative control signal given by

$$u(t) = -\text{sgn}(x(t))|x(t)|^{\alpha_1-1}, \tag{1}$$

[1]The authors of Reference 34 provide extensions of References 27-29; however, the boundedness of their proposed control signals is not addressed.

where $\alpha_1 \in (0,1)$ and $\mathrm{sgn}(\cdot)$ is the signum function. Note that (1) serves as the building brick for many existing distributed controllers with finite-time convergence property (see, eg, References 10,11). Next, based on the time transformation-based finite-time distributed controller proposed in References 25,26, we consider the finite-time interval $[0,T)$, where $T \in \mathbb{R}_+$ is a priori given, user-defined finite time, and utilize a strictly increasing and smooth time transformation function given by $t \triangleq \theta(s)$ with $s \in [0,\infty)$ being the infinite (ie, stretched) time interval. In this case, the corresponding control signal can be given by

$$u(t) = -\alpha_2(\mathrm{d}s/\mathrm{d}t)x(t), \tag{2}$$

where α_2 is chosen such that $\alpha_2 > (\mathrm{d}s/\mathrm{d}t)^{-2}(\mathrm{d}(\mathrm{d}s/\mathrm{d}t)/\mathrm{d}t)$ holds. Figure 1 compares the performance of the controller in (1) using $\alpha_1 = 0.95$ with the controller in (2) using the time transformation function $t = T(1-e^{-s})$ and $\alpha_2 = 2$, where the a priori given, user-defined finite time is $T = 0.5$ sec. In particular, Figure 1A shows that the stabilization time with the controller in (1) varies depending on the initial conditions. Figure 1B (respectively, Figure 1C) shows that the finite-time convergence value with the controller in (1) changes with respect to the nonvanishing (respectively, vanishing) system uncertainties. Finally, Figure 1D represents a tracking problem in presence of nonvanishing system uncertainty, in which, once again, the tracking performance of the controller in (1) depends on the initial conditions as well as the system uncertainty. Thus, for all of the stabilization and tracking scenarios given in Figure 1, one can see that, in contrast to the controller in (1), the finite-time control algorithm with time transformation in (2) is able to reach the equilibrium at the a priori given, user-defined finite time T in all cases and independent of the initial conditions and the presence of different types of system uncertainties[2].

2 | NOTATION AND MATHEMATICAL PRELIMINARIES

The notation used in this article is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{R}_+^{n\times n}$ (resp., $\overline{\mathbb{R}}_+^{n\times n}$) denotes the set of $n \times n$ positive-definite (resp., nonnegative-definite) real matrices, $\mathbb{Z}_+$ (resp., $\overline{\mathbb{Z}}_+$) denotes the set of positive (resp., nonnegative) integers, $\mathbf{0}_n$ denotes the $n \times 1$ zero vector, $\mathbf{1}_n$ denotes the $n \times 1$ ones vector, $\mathbf{0}_{n\times m}$ denotes the $n \times m$ zero matrix, and "$\triangleq$" denotes equality by definition. In addition, we write $(\cdot)^{\mathrm{T}}$ for the transpose, $(\cdot)^{-1}$ for the inverse, $\det(\cdot)$ for the determinant, and $\|\cdot\|_2$ for the Euclidean norm. We also write $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) for the minimum (resp., maximum) eigenvalue of the square matrix A, $\lambda_i(A)$ for the ith eigenvalue of the square matrix A (with eigenvalues ordered from minimum to maximum value), $[A]_{ij}$ for the (i,j)th entry of the matrix A, and $\underline{x}$ (resp., $\overline{x}$) for the lower bound (resp., upper bound) of a bounded signal $x(t) \in \mathbb{R}^n$, that is, $\underline{x} \le \|x(t)\|_2$ (resp., $\|x(t)\|_2 \le \overline{x}$).

Next, we recall some basic notions from graph theory, where we refer the reader to References 2,42 for further details. Specifically, an undirected graph $\mathfrak{G}$ is defined by a set $\mathcal{V}_{\mathfrak{G}} = \{1, \dots, N\}$ of *nodes* and a set $\mathcal{E}_{\mathfrak{G}} \subset \mathcal{V}_{\mathfrak{G}} \times \mathcal{V}_{\mathfrak{G}}$ of *edges*. If $(i,j) \in \mathcal{E}_{\mathfrak{G}}$, then nodes i and j are neighbors and the neighboring relation is indicated by $i \sim j$. The *degree* of a node is given by the number of its neighbors. Letting d_i denote the degree of node i, then the *degree matrix* of a graph $\mathfrak{G}$, denoted by $\mathcal{D}(\mathfrak{G}) \in \mathbb{R}^{N\times N}$, is given by $\mathcal{D}(\mathfrak{G}) \triangleq \mathrm{diag}[d]$, where $d = [d_1, \dots, d_N]^{\mathrm{T}}$. A *path* $i_0 i_1 \dots i_L$ of a graph $\mathfrak{G}$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \dots, L$, and if every pair of distinct nodes has a path, then the graph $\mathfrak{G}$ is connected. We write $\mathcal{A}(\mathfrak{G}) \in \mathbb{R}^{N\times N}$ for the *adjacency matrix* of a graph $\mathfrak{G}$ defined by $[\mathcal{A}(\mathfrak{G})]_{ij} \triangleq 1$ if $(i,j) \in \mathcal{E}_{\mathfrak{G}}$, and $[\mathcal{A}(\mathfrak{G})]_{ij} \triangleq 0$ otherwise, and $\mathcal{B}(\mathfrak{G}) \in \mathbb{R}^{N\times M}$ for the (node-edge) *incidence matrix* of a graph $\mathfrak{G}$ defined by $[\mathcal{B}(\mathfrak{G})]_{ij} \triangleq 1$ if node i is the head of edge j, $[\mathcal{B}(\mathfrak{G})]_{ij} \triangleq -1$ if node i is the tail of edge j, and $[\mathcal{B}(\mathfrak{G})]_{ij} \triangleq 0$ otherwise, where M is the number of edges, i is an index for the node set, and j is an index for the edge set.

The *graph Laplacian matrix*, $\mathcal{L}(\mathfrak{G}) \in \overline{\mathbb{R}}_+^{N\times N}$, is defined by $\mathcal{L}(\mathfrak{G}) \triangleq \mathcal{D}(\mathfrak{G}) - \mathcal{A}(\mathfrak{G})$ or equivalently $\mathcal{L}(\mathfrak{G}) = \mathcal{B}(\mathfrak{G})\mathcal{B}(\mathfrak{G})^{\mathrm{T}}$ and the spectrum of the graph Laplacian of a connected, undirected graph $\mathfrak{G}$ can be ordered as $0 = \lambda_1(\mathcal{L}(\mathfrak{G})) < \lambda_2(\mathcal{L}(\mathfrak{G})) \le \cdots \le \lambda_N(\mathcal{L}(\mathfrak{G}))$ with $\mathbf{1}_N$ being the eigenvector corresponding to the zero eigenvalue $\lambda_1(\mathcal{L}(\mathfrak{G}))$, and $\mathcal{L}(\mathfrak{G})\mathbf{1}_N = \mathbf{0}_N$ and $e^{\mathcal{L}(\mathfrak{G})}\mathbf{1}_N = \mathbf{1}_N$. In this article, we model a given multiagent system as a connected, undirected graph $\mathfrak{G}$, where nodes and edges, respectively, represent agents and interagent communication links.

[2]While the analysis performed in Reference 26 can also be utilized for nonvanishing system uncertainties, this prior work does not make any attempts in showing robustness against state-dependent vanishing system uncertainties. Note that such vanishing system uncertainties can destabilize dynamical systems unlike the nonvanishing ones; hence, they are more critical to the overall system stability and convergence.

The following remarks and lemmas are needed for the results of this article.

Lemma 1 ([43, lemma 3.3]). *Let* $K = \mathrm{diag}(k)$, $k = [k_1, k_2, \dots, k_N]^{\mathrm{T}}$, $k_i \in \overline{\mathbb{Z}}_+$, $i = 1, \dots, N$, *and assume that at least one element of k is nonzero. Then,* $\mathcal{F}(\mathfrak{G}) \triangleq \mathcal{L}(\mathfrak{G}) + K \in \mathbb{R}_+^{N\times N}$ *and* $\det(\mathcal{F}(\mathfrak{G})) \neq 0$ *for the Laplacian of a connected, undirected graph.*

Remark 1. From Lemma 1, $-\mathcal{F}(\mathfrak{G})$ is clearly a symmetric and Hurwitz matrix. As a consequence, $-\mathcal{F}(\mathfrak{G})$ satisfies the Lyapunov equation $R = \mathcal{F}(\mathfrak{G})P + P\mathcal{F}(\mathfrak{G})$ for a given $R \in \mathbb{R}_+^{N\times N}$.

Lemma 2. *Consider an energy function* $\mathcal{V} \in \mathbb{R}_+$ *satisfying the inequality given by*

$$\mathcal{V}'(s) + (a_0 - b_0 e^{-s})\mathcal{V}(s) \le c_0 e^{-s}, \tag{3}$$

where $a_0 \in \mathbb{R}_+$, $b_0, \in \overline{\mathbb{R}}_+$, *and* $c_0 \in \overline{\mathbb{R}}_+$ *are constants, and* $\mathcal{V}'(s) = \mathrm{d}\mathcal{V}(s)/\mathrm{d}s$ *with* $s \in [0, \infty)$. *Then, this energy function is bounded for all* $s \in [0, \infty)$ *and* $\lim_{s\to\infty} \mathcal{V}(s) = 0$.

Proof. Defining the integrating factor $\mu(s) \triangleq \exp(a_0 s + b_0 e^{-s}) \in \mathbb{R}_+$ and multiplying both sides of (3) with this factor, one obtains $\mu(s)\mathcal{V}'(s) + \mu(s)(a_0 - b_0 e^{-s})\mathcal{V}(s) \le \mu(s)e^{-s}c_0$ or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mu(s)\mathcal{V}(s)) \le \mu(s)e^{-s}c_0. \tag{4}$$

Next, we integrate both sides of (4) over the s domain that yields $\mu(s)\mathcal{V}(s) \le c_0 \int_0^s e^{-\tau} \exp(a_0\tau + b_0 e^{-\tau})\mathrm{d}\tau + \mathcal{V}_0$, where $\mathcal{V}_0 \triangleq \mu(0)\mathcal{V}(0)$. We now set $\nu(\tau) \triangleq e^{-\tau} \in \mathbb{R}_+$ with the derivative $\mathrm{d}\nu(\tau) = -e^{-\tau}\mathrm{d}\tau$ to get

$$\begin{aligned}
\mu(s)\mathcal{V}(s) &\le -c_0 \int_1^{e^{-s}} \exp(-a_0 \ln(\nu(\tau)) + b_0\nu(\tau))\mathrm{d}\nu(\tau) + \mathcal{V}_0,\\
&\le c_0 \int_{e^{-s}}^1 (\nu(\tau))^{-a_0} e^{b_0\nu(\tau)}\mathrm{d}\nu(\tau) + \mathcal{V}_0,\\
&\le d_0 \int_{e^{-s}}^1 (\nu(\tau))^{-a_0}\mathrm{d}\nu(\tau) + \mathcal{V}_0,
\end{aligned} \tag{5}$$

where $d_0 \triangleq c_0 e^{b_0} \in \mathbb{R}_+$ is the upper bound on $c_0 e^{b_0\nu(\tau)}$. Note that since $\tau \in [0, s)$ with $s \in [0, \infty)$, we have $\nu(\tau) \in (0, 1]$ showing the existence of d_0. We can now solve the integral in (5) for two possible a_0 cases. In particular, if $a_0 = 1$, we have

$$\begin{aligned}
\mu(s)\mathcal{V}(s) &\le d_0 \int_{e^{-s}}^1 (\nu(\tau))^{-1}\mathrm{d}\nu(\tau) + \mathcal{V}_0 = d_0 \ln \nu(\tau)\Big|_{e^{-s}}^1 + \mathcal{V}_0,\\
&\le d_0 s + \mathcal{V}_0.
\end{aligned} \tag{6}$$

Introducing the integral factor $\mu(s) = \exp(s + b_0 e^{-s})$ in (6), the bound on the energy function is now given by

$$\mathcal{V}(s) \le \frac{d_0 s}{e^{s + b_0 e^{-s}}} + \mathcal{V}_0 e^{-s - b_0 e^{-s}}. \tag{7}$$

We are interested in the limit of this bound when $s \to \infty$ given by

$$\begin{aligned}
\lim_{s\to\infty} \mathcal{V}(s) &\le \lim_{s\to\infty} \frac{d_0 s}{e^{s + b_0 e^{-s}}} + \lim_{s\to\infty} \mathcal{V}_0 e^{-s - b_0 e^{-s}},\\
&= \lim_{s\to\infty} \frac{d_0}{(1 - b_0 e^{-s})e^{s + b_0 e^{-s}}} = 0,
\end{aligned} \tag{8}$$

where the L'Hospital's Rule is applied to the first limit term. Next, since $\mathcal{V}(s)$ is a positive definite function we can conclude $\lim_{s\to\infty} \mathcal{V}(s) = 0$. For the case when $a_0 \neq 1$, one can write $\mu(s)\mathcal{V}(s) \le d_0 \int_{e^{-s}}^1 (\nu(\tau))^{-a_0}\mathrm{d}\nu(\tau) + \mathcal{V}_0 = d_0(\nu(\tau))^{(1-a_0)}/(1 -$

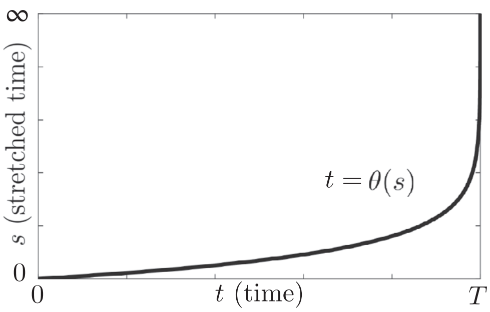

FIGURE 2 The time transformation function in (10)

$a_0)\Big|_{e^{-s}}^{1} + \mathcal{V}_0 = \frac{d_0}{1-a_0}(1 - e^{(a_0-1)s}) + \mathcal{V}_0$. Introducing the integral factor $\mu(s)$, the bound on the energy function is now given by

$$\mathcal{V}(s) \leq \frac{d_0}{1-a_0}(e^{-a_0 s - b_0 e^{-s}} - e^{-s - b_0 e^{-s}}) + \mathcal{V}_0 e^{-a_0 s - b_0 e^{-s}}. \tag{9}$$

Once again, we take the limit on the bound on the energy function in (9) when $s \to \infty$ as $\lim_{s\to\infty} \mathcal{V}(s) \leq \lim_{s\to\infty} \frac{d_0}{1-a_0}(e^{-a_0 s - b_0 e^{-s}} - e^{-s - b_0 e^{-s}}) + \lim_{s\to\infty} \mathcal{V}_0 e^{-a_0 s - b_0 e^{-s}} = 0$. Therefore, in all cases we have $\lim_{s\to\infty} \mathcal{V}(s) = 0$. ∎

Remark 2. We now use a notion from section 1.1.1.4 of Reference 44. Specifically, let $\xi(t)$ denote a solution to the dynamical system $\dot{x}(t) = f(t, x(t)),\ x(0) = x_0$. In addition, let $t = \theta(s)$ denote a time transformation, where $\theta(s)$ is a strictly increasing and continuously differentiable function, and define $\chi(s) = \xi(t)$. Then, $\chi'(s) = \theta'(s) f(\theta(s), \chi(s)),\ \chi(\theta^{-1}(0)) = x_0$, where $\chi'(s) \triangleq \mathrm{d}\chi(s)/\mathrm{d}s$, and $\theta'(s) \triangleq \mathrm{d}\theta(s)/\mathrm{d}s$.

Remark 3. For the sake of simplicity, considering the time transformation $t = \theta(s)$ and any signal $\eta(t)$, we write $\eta_s(s)$ to denote the transformed signal in the infinite interval s; that is, $\eta_s(s) \triangleq \eta(\theta(s))$.

Remark 4. Consider the time transformation function $t = \theta(s)$ that is strictly increasing and continuously differentiable with respect to s. A candidate time transformation function is given by

$$t = \theta(s) \triangleq T(1 - e^{-s}), \tag{10}$$

where $T \in \mathbb{R}_+$ is the a priori given, user-defined finite time.[25,26] Note that this time transformation function converts the infinite-time interval $s \in [0, \infty)$ to the finite-time interval $t \in [0, T)$ with $T \in \mathbb{R}_+$ being the user-defined finite time (see Figure 2).

3 | FINITE-TIME CONTROL OF FIRST-ORDER MULTIAGENT SYSTEMS

In this section, we introduce the leader-follower problem for multiagent systems with agents having first-order dynamics. While we consider this benchmark problem, this is without loss of generality in the sense that the presented distributed control architecture can be readily extended to other multiagent control problems. Specifically, consider a multiagent system that consists of N agents exchanging information based on a connected and undirected graph $\mathfrak{G}$. We consider that the dynamics of the agents are in the form given by

$$\dot{x}_i(t) = u_i(t) + \omega_i x_i(t) + \rho_i(t), \quad x_i(0) = x_{i0}, \quad i \in \{1, 2, \dots, N\}, \tag{11}$$

where $x_i(t) \in \mathbb{R}, i \in \{1, 2, \dots, N\}$ and $u_i(t) \in \mathbb{R}, i \in \{1, 2, \dots, N\}$ are the position and the control signal of each agent, respectively. In (11), $\omega_i \in \mathbb{R}, i \in \{1, 2, \dots, N\}$, and $\rho_i(t) \in \mathbb{R}, i \in \{1, 2, \dots, N\}$, represent vanishing and nonvanishing uncertainties in each agent's dynamics. Here, while we assume that the unknown term ω_i is bounded and the unknown term $\rho_i(t)$ is bounded and piecewise continuous for the well-posedness of the considered problem, we do not require the knowledge of their upper bounds. Furthermore, consider that a subset of the agents have access to the position of a time-varying leader given by

$$\dot{x}_0(t) = v_0(t), \quad x_0(0) = x_{00}, \tag{12}$$

where $v_0(t) \in \mathbb{R}$, denotes the bounded and piecewise continuous velocity (with unknown bound) of the leader. Our objective is to design a distributed control algorithm for achieving finite-time convergence with a priori given, user-defined finite time $T \in \mathbb{R}_+$; that is, $\lim_{t\to T}(x_i(t) - x_0(t)) = 0,\ i \in \{1, 2, \ldots, N\}$.

Remark 5. The considered leader-following consensus problem in this article is motivated by, for example, cooperative engagement applications. In particular, the control objective is defined for the agents to reach to a time-varying target in a given time T. Therefore, all of the analyses are focused on the time interval $[0, T)$. For other applications, where it is required for the agents to continue following the leader after the time T, one may resort to switching-type control designs.

3.1 | Finite-time distributed control algorithm with time transformation

We propose a generalized time transformation function that converts the user-defined finite-time interval to a stretched infinite-time interval, where one can design a distributed control algorithm on this stretched interval. Specifically, let $t = \theta(s)$ denote this generalized time transformation function. Here, $\theta(s)$ is strictly increasing and continuously differentiable with respect to s, which converts the infinite-time interval $s \in [0, \infty)$ to the finite-time interval $t \in [0, T)$ with $T \in \mathbb{R}_+$ being the user-defined finite time[3]. To this end, we propose the distributed control algorithm

$$u_i(t) = -\alpha \frac{\mathrm{d}s}{\mathrm{d}t}\left(\sum_{i\sim j}(x_i(t) - x_j(t)) + k_i(x_i(t) - x_0(t))\right), \quad i \in \{1, 2, \ldots, N\}, \tag{13}$$

defined on $t \in [0, T)$ with $\alpha \in \mathbb{R}_+$ and $\mathrm{d}s/\mathrm{d}t = (\theta'(\theta^{-1}(t)))^{-1}$. In (13), $k_i = 1$ for the subset of the agents having access to the position of a time-varying leader in (12) and $k_i = 0$ for other agents.

Let $\tilde{x}_i(t) \triangleq x_i(t) - x_0(t), i \in \{1, 2, \ldots, N\}$, be the error between the position of each agent and that of the leader. Based on the proposed distributed control signal given by (13), the resulting error dynamics becomes

$$\dot{\tilde{x}}_i(t) = -\alpha \frac{\mathrm{d}s}{\mathrm{d}t}\left(\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_j(t)) + k_i\tilde{x}_i(t)\right) + \omega_i(\tilde{x}_i(t) + x_0(t)) + \rho_i(t) - v_0(t), \quad \tilde{x}_i(0) = \tilde{x}_{i0},$$
$$i \in \{1, 2, \ldots, N\}. \tag{14}$$

Defining now the augmented error state as $\tilde{x}(t) \triangleq [\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$, the error dynamics in (14) can be written in the compact form given by

$$\begin{aligned}\dot{\tilde{x}}(t) &= -\alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G})\tilde{x}(t) + \Omega(\tilde{x}(t) + \mathbf{1}_N x_0(t)) + \rho(t) - \mathbf{1}_N v_0(t),\\ &= \left(-\alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G}) + \Omega\right)\tilde{x}(t) + h(t), \quad \tilde{x}(0) = \tilde{x}_0,\end{aligned} \tag{15}$$

where $\Omega \triangleq \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N) \in \mathbb{R}^{N\times N}$, $\rho(t) \triangleq [\rho_1(t), \rho_2(t), \ldots, \rho_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$, and $h(t) \triangleq \Omega\mathbf{1}_N x_0(t) + \rho(t) - \mathbf{1}_N v_0(t)$.

Considering the time transformation function $t = \theta(s)$, let $\xi(t) \in \mathbb{R}^N, t \in [0, T)$, be a solution to the dynamical system given by (15) such that $\tilde{x}_s(s) = \xi(t), s \in [0, \infty)$. It now follows from Remark 2 that $\tilde{x}_s'(s) = (-\alpha\mathcal{F}(\mathfrak{G}) + \Omega\theta'(s))\tilde{x}_s(s) + \theta'(s)h_s(s),\ \tilde{x}_s(\theta^{-1}(0)) = \tilde{x}_0$, where $h_s(s) = h(\theta(s))$ based on Remark 3. Note that $h_s(s)$ consists of bounded terms; hence, it is a bounded function. Using $\theta'(s) = \mathrm{d}\theta(s)/\mathrm{d}s = \mathrm{d}t/\mathrm{d}s$ now yields

$$\tilde{x}_s'(s) = \left(-\alpha\mathcal{F}(\mathfrak{G}) + \Omega\frac{\mathrm{d}t}{\mathrm{d}s}\right)\tilde{x}_s(s) + \frac{\mathrm{d}t}{\mathrm{d}s}h_s(s), \quad \tilde{x}_s(0) = \tilde{x}_0. \tag{16}$$

[3]It is considered that $s = \theta^{-1}(t)$ exists for the results of this article, where this is clearly possible by properly selecting the generalized time transformation function in the control design process.

As noted in Remark 2, the solution of (15) and (16) are equivalent with different argument domains; that is, $\tilde{x}_s(s) = \tilde{x}(t)$. We now write the introduced control signal (13) in the compact form given by

$$u(t) = -\alpha \frac{\mathrm{d}s}{\mathrm{d}t} \mathcal{F}(\mathfrak{G})\tilde{x}(t), \tag{17}$$

which has the following time derivative with respect to t

$$\begin{aligned} \dot{u}(t) &= -\alpha \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \mathcal{F}(\mathfrak{G})\tilde{x}(t) - \alpha \frac{\mathrm{d}s}{\mathrm{d}t} \mathcal{F}(\mathfrak{G})\dot{\tilde{x}}(t), \\ &= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \frac{\mathrm{d}t}{\mathrm{d}s} u(t) - \alpha \frac{\mathrm{d}s}{\mathrm{d}t} \mathcal{F}(\mathfrak{G})(u(t) + \Omega(\tilde{x}(t) + \mathbf{1}_N x_0(t)) + \rho(t) - \mathbf{1}_N v_0(t)), \\ &= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \frac{\mathrm{d}t}{\mathrm{d}s} u(t) - \alpha \frac{\mathrm{d}s}{\mathrm{d}t} \mathcal{F}(\mathfrak{G})(u(t) + \Omega\tilde{x}(t) + h(t)), \quad u(0) = u_0. \end{aligned} \tag{18}$$

Similar to how we obtain (16) from (15), one can rewrite (18) in the infinite-time interval $s \in [0, \infty)$ as

$$u_s'(s) = -\alpha \mathcal{F}(\mathfrak{G})\Omega\tilde{x}_s(s) - \left(\alpha \mathcal{F}(\mathfrak{G}) - \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-2} I_N\right) u_s(s) - \alpha \mathcal{F}(\mathfrak{G}) h_s(s), \quad u_s(0) = u_0. \tag{19}$$

One can now augment the state error dynamics in (16) and control signal dynamics in (19) as

$$\begin{bmatrix} \tilde{x}_s'(s) \\ u_s'(s) \end{bmatrix} = \begin{bmatrix} -\alpha \mathcal{F}(\mathfrak{G}) + \Omega \frac{\mathrm{d}t}{\mathrm{d}s} & 0 \\ -\alpha \mathcal{F}(\mathfrak{G})\Omega & -\alpha \mathcal{F}(\mathfrak{G}) + \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-2} I_N \end{bmatrix} \begin{bmatrix} \tilde{x}_s(s) \\ u_s(s) \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{d}t}{\mathrm{d}s} I_N \\ -\alpha \mathcal{F}(\mathfrak{G}) \end{bmatrix} h_s(s),$$

$$\begin{bmatrix} \tilde{x}_s(0) \\ u_s(0) \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ u_0 \end{bmatrix}. \tag{20}$$

Remark 6. The time transformation function $t = \theta(s)$ should be chosen by the control user such that (i) the state error dynamics and control signal dynamics given by (20) are stable, which results in bounded error state $\tilde{x}_s(s)$ and control signal $u_s(s)$, and (ii) the asymptotic stability for the error state $\tilde{x}_s(s)$ is achieved (ie, $\lim_{s\to\infty} \tilde{x}_s(s) = 0$). By Remark 2, the above discussion implies that the error state $\tilde{x}(t)$ and control signal $u(t)$ are bounded in the original time interval $t \in [0, T)$ and $\lim_{t\to T}(x_i(t) - x_0(t)) = 0$. The latter result implies that the agents converge to the position of the time-varying leader at the user-defined finite time T.

As discussed in Remark 6, the selection of the time transformation function $t = \theta(s)$ plays an important role on the stability of the closed-loop system dynamics given in (20). Adopted from the previous work of the authors in References 25,26, a candidate time transformation function satisfying the conditions in Remark 6, along with rigorous and detailed analysis on stability of the closed-loop system dynamics, is presented in the following section.

3.2 | Robustness to vanishing and nonvanishing system uncertainties

In this section, we use a time transformation function candidate to analytically evaluate the robustness properties of the overall multiagent system with the time transformation-based finite-time distributed control algorithm in (13) in presence of vanishing and nonvanishing system uncertainties (ie, $\omega_i \in \mathbb{R}, i \in \{1, 2, \dots, N\}$, and $\rho_i(t) \in \mathbb{R}, i \in \{1, 2, \dots, N\}$, in (11)). To this end, consider the time transformation candidate function given by (10) that has the derivative with respect to $s \in [0, \infty)$ given by

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \theta'(s) = Te^{-s} = T - t. \tag{21}$$

Introducing (21) in (13) results in the control signal

$$u_i(t) = -\alpha\lambda(t)\left(\sum_{i\sim j}(x_i(t) - x_j(t)) + k_i(x_i(t) - x_0(t))\right), \quad i \in \{1, 2, \dots, N\}, \tag{22}$$

where $\lambda(t) \triangleq 1/(T-t)$, or in the compact form $u(t) = -\alpha\lambda(t)\mathcal{F}(\mathfrak{G})\tilde{x}(t)$. Furthermore, using (21) in the error dynamics given by (14) yields

$$\dot{\tilde{x}}(t) = (-\alpha\lambda(t)\mathcal{F}(\mathfrak{G}) + \Omega)\tilde{x}(t) + h(t), \quad \tilde{x}(0) = \tilde{x}_0. \tag{23}$$

Similar to the steps shown in the previous section, one can now write (23) in the infinite-time interval $s \in [0, \infty)$ as

$$\tilde{x}_s'(s) = (-\alpha\mathcal{F}(\mathfrak{G}) + \Omega Te^{-s})\tilde{x}_s(s) + Te^{-s}h_s(s), \quad \tilde{x}_s(0) = \tilde{x}_0. \tag{24}$$

The augmented form of the state error and the control signal dynamics can also be obtained similar to (20) as

$$\begin{bmatrix} \tilde{x}_s'(s) \\ u_s'(s) \end{bmatrix} = \begin{bmatrix} -\alpha\mathcal{F}(\mathfrak{G}) + \Omega Te^{-s} & 0 \\ -\alpha\mathcal{F}(\mathfrak{G})\Omega & -\mathcal{S} \end{bmatrix} \begin{bmatrix} \tilde{x}_s(s) \\ u_s(s) \end{bmatrix} + \begin{bmatrix} Te^{-s}I_N \\ -\alpha\mathcal{F}(\mathfrak{G}) \end{bmatrix} h_s(s), \quad \begin{bmatrix} \tilde{x}_s(0) \\ u_s(0) \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ u_0 \end{bmatrix}, \tag{25}$$

where

$$\mathcal{S} \triangleq \alpha\mathcal{F}(\mathfrak{G}) - I_N \in \mathbb{R}^{N\times N}. \tag{26}$$

Remark 7. Letting $\mathcal{M}(t) \triangleq -\alpha\lambda(t)\mathcal{F}(\mathfrak{G}) + \Omega$, the error dynamics given in (23) can be rewritten in the form $\dot{\tilde{x}}(t) = \mathcal{M}(t)\tilde{x}(t) + h(t)$, $\tilde{x}(0) = \tilde{x}_0$. Now, since $\mathcal{M}(t)$ and $h(t)$ are integrable functions of t over the finite-time interval $t \in [0, T-\delta]$ for every small positive constant δ, it follows from [45, p. 97] that the error dynamics given in (23) has a unique solution on the finite-time interval $[0, T)$. Alternatively, by analyzing the transferred error dynamics in the infinite-time interval given in (24), one can conclude the existence and uniqueness of the solution $\tilde{x}_s(s)$ over the infinite-time interval $s \in [0, \infty)$. Hence, there exist a unique solution for the error dynamics given in (23) over the finite-time interval $[0, T)$.

Theorem 1. *Consider the multiagent system that consists of N agents on a connected, undirected graph $\mathfrak{G}$, where the uncertain dynamics of agent $i \in \{1, \ldots, N\}$ is given by (11). In addition, assume that there exists at least one agent sensing the position of the time-varying leader given by (12), which has bounded but unknown velocity. Considering the local control algorithm $u_i(t)$, $i = 1, \ldots, N$, for each agent given by (22), let the design parameter α be chosen to make $\mathcal{S} = \alpha\mathcal{F}(\mathfrak{G}) - I_N$ positive definite. Then, the closed-loop system signals including the control signals remain bounded and all agents converge to the position of the leader in the a priori given, user-defined finite time T (ie, $\lim_{t\to T}\tilde{x}(t) = 0$) for all initial conditions of agents and for all finite pairs $(\omega_i, \rho_i(t))$, $i \in \{1, \ldots, N\}$.*

Proof. To show boundedness of the closed-loop system signals and convergence of all agents to the position of the leader at the user-defined finite time T, we consider the closed-loop system dynamics after applying the time transformation function as given in (25). We first consider the system error dynamics in the infinite-time interval $s \in [0, \infty)$ and let $\mathcal{V}_1(\tilde{x}_s(s)) \in \overline{\mathbb{R}}_+$ be an energy function given by

$$\mathcal{V}_1(\tilde{x}_s(s)) = \tilde{x}_s^{\mathrm{T}}(s)\tilde{x}_s(s) = \|\tilde{x}_s(s)\|_2^2. \tag{27}$$

The derivative of (27) with respect to the stretched time $s \in [0, \infty)$ along the trajectories of (24) is given by

$$\begin{aligned} \mathcal{V}_1'(\tilde{x}_s(s)) &= 2\tilde{x}_s^{\mathrm{T}}(s)\tilde{x}_s'(s), \\ &= -2\alpha\tilde{x}_s^{\mathrm{T}}(s)\mathcal{F}(\mathfrak{G})\tilde{x}_s(s) + 2Te^{-s}\tilde{x}_s^{\mathrm{T}}(s)\Omega\tilde{x}_s(s) + 2Te^{-s}\tilde{x}_s^{\mathrm{T}}(s)h_s(s), \\ &\leq -2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G}))\|\tilde{x}_s(s)\|_2^2 + 2Te^{-s}\omega_{\max}\|\tilde{x}_s(s)\|_2^2 + 2Te^{-s}\|\tilde{x}_s(s)\|_2\|h_s(s)\|_2, \end{aligned} \tag{28}$$

where $\omega_{\max} \triangleq \max\{\omega_1, \ldots, \omega_N\}$. Using $2\|\tilde{x}_s(s)\|_2\|h_s(s)\|_2 \leq \|\tilde{x}_s(s)\|_2^2 + \|h_s(s)\|_2^2$ on the last term and replacing $\|\tilde{x}_s(s)\|_2^2$ with $\mathcal{V}_1(\tilde{x}_s(s))$ in (28) results in

$$\begin{aligned} \mathcal{V}_1'(\tilde{x}_s(s)) &\leq \mathcal{V}_1(\tilde{x}_s(s))(-2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G})) + 2Te^{-s}\omega_{\max}) + Te^{-s}(\mathcal{V}_1(\tilde{x}_s(s)) + \|h_s(s)\|_2^2), \\ &\leq \mathcal{V}_1(\tilde{x}_s(s))(-2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G})) + Te^{-s}(2\omega_{\max} + 1)) + Te^{-s}\|h_s(s)\|_2^2. \end{aligned} \tag{29}$$

For the sake of simplicity of the rest of the analysis, we define $a_0 \triangleq 2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G})) \in \mathbb{R}_+$ by Lemma 1, $b_0 \triangleq \max\{0, T(2\omega_{\max} + 1)\} \in \mathbb{R}_+$, and $c_0 \in \mathbb{R}_+$ to be the upper bound on $T\|h_s(s)\|_2^2$, that is, $T\|h_s(s)\|_2^2 \leq c_0$. This simplifies

(29) to $\mathcal{V}_1'(\tilde{x}_s(s)) + (a_0 - b_0 e^{-s})\mathcal{V}_1(\tilde{x}_s(s)) \le e^{-s}c_0$. It now follows from Lemma 2 that $\mathcal{V}_1(\tilde{x}_s(s))$ is bounded for $s \in [0, \infty)$ and $\lim_{s\to\infty} \mathcal{V}_1(\tilde{x}_s(s)) = 0$ resulting in $\lim_{s\to\infty} \tilde{x}_s(s) = 0$. Since $\tilde{x}_s(s) = \tilde{x}(t)$ by Remark 2 and $t \to T$ as $s \to \infty$, one can obtain $\lim_{t\to T} \tilde{x}(t) = 0$. Furthermore, it follows from the boundedness of the energy function $\mathcal{V}_1(\tilde{x}_s(s))$ that the state error vector $\tilde{x}_s(s)$ also remains bounded for all $s \in [0, \infty)$. Equivalently, the state error vector $\tilde{x}(t)$, remains bounded in the finite-time interval for all $t \in [0, T)$.

Finally, to show the boundedness of the control signal $u_s(s)$, consider the second equation in (25) given by

$$u_s'(s) = -\mathcal{S}u_s(s) - \alpha\mathcal{F}(\mathfrak{G})\Omega\tilde{x}_s(s) - \alpha\mathcal{F}(\mathfrak{G})h_s(s), \quad u(0) = u_0. \tag{30}$$

As noted earlier, from the boundedness of the position and velocity of the leader as well as the boundedness of the system uncertainties, it follows that $h_s(s)$ is a bounded function. Now, since the last two terms in (30) are bounded, and $-\mathcal{S}$ is Hurwitz by the assumption of the theorem, then $u_s(s)$ is bounded. Equivalently, $u(t)$ remains bounded in the finite interval $t \in [0, T)$ which concludes the proof. ∎

Remark 8. The notable feature of the proposed distributed control algorithm in this section for handling system uncertainties while achieving an a priori given, user-defined finite-time multiagent system performance arises mainly from the utilization of the time transformation function in (10). Specifically, the aforementioned time transformation method converts the problem under study from its finite interval $t \in [0, T)$ to the infinite-time interval $s \in [0, \infty)$. This then enables a control user to exploit any standard (ie, over infinite horizon) system-theoretic tools for synthesis and analysis purposes. The finite-time stability and convergence guarantees are then immediate by transforming the time to the original interval.

4 | FINITE-TIME CONTROL OF SECOND-ORDER MULTIAGENT SYSTEMS

We next consider the leader-follower problem for multiagent systems with agents having second-order dynamics. Specifically, we focus on the leader-follower problem in a multiagent system with N agents exchanging information based on a connected, undirected graph $\mathfrak{G}$. Mathematically speaking, we consider the agents having second-order dynamics

$$\dot{x}_i(t) = v_i(t), \quad x_i(0) = x_{i0}, \tag{31}$$

$$\dot{v}_i(t) = u_i(t) + \omega_i x_i(t) + \rho_i(t), \quad v_i(0) = v_{i0}, \tag{32}$$

where $\omega_i \in \mathbb{R}$ and $\rho_i(t) \in \mathbb{R}$, respectively, represent vanishing and nonvanishing uncertainties in each agent's dynamics. For well-posedness of the considered problem, we here consider that ω_i, $\rho_i(t)$, and $\dot{\rho}_i(t)$ are bounded but we do not require a knowledge of their upper bounds.

Next, we consider a leader with second-order dynamics

$$\dot{x}_0(t) = v_0(t), \quad x_0(0) = x_{00}, \tag{33}$$

$$\dot{v}_0(t) = a_0(t), \quad v_0(0) = v_{00}, \tag{34}$$

where $x_0(t) \in \mathbb{R}$ and $v_0(t) \in \mathbb{R}$, respectively, denote the position and velocity of the leader. In addition, $a_0(t) \in \mathbb{R}$ in (34) stands for a time-varying, bounded, and piecewise continuous acceleration signal of the leader (with unknown bound) under the consideration that this signal results in bounded position and velocity of the leader. Our objective here is to design a distributed control algorithm for achieving finite-time convergence with a priori given, user-defined finite time $T \in \mathbb{R}_+$; that is, $\lim_{t\to T}(x_i(t) - x_0(t)) = 0, \quad i \in \{1, 2, \ldots, N\}$.

4.1 | Finite-time distributed control algorithm with time transformation

We propose a distributed finite-time control algorithm using a generalized time transformation for addressing the leader-follower problem. We note that finite-time control of second-order multiagent systems using the time-transformation approach has not been addressed in Reference 26 (even without any system uncertainties). Hence, in this section we start with a simplified version of the agent's dynamics in (31) and (32) as

$$\dot{x}_i(t) = v_i(t), \quad x_i(0) = x_{i0}, \tag{35}$$

$$\dot{v}_i(t) = u_i(t), \quad v_i(0) = v_{i0}. \tag{36}$$

The robustness properties of the proposed algorithm against vanishing and nonvanishing system uncertainties are then proposed in Section 4.2.

To this end, let $t = \theta(s)$ denote a generalized time transformation function. Here, $\theta(s)$ is strictly increasing and continuously differentiable with respect to s, which converts the infinite-time interval $s \in [0, \infty)$ to the finite-time interval $t \in [0, T)$ with $T \in \mathbb{R}_+$ being the user-defined finite time. To this end, we propose the distributed control algorithm given by

$$\begin{aligned} u_i(t) = &-\alpha\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + k_\epsilon\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right)\left(\sum_{i\sim j}(x_i(t) - x_j(t)) + k_i(x_i(t) - x_0(t))\right) - \alpha\frac{\mathrm{d}s}{\mathrm{d}t}\left(\sum_{i\sim j}(v_i(t) - v_j(t))\right. \\ &\left.+k_i(v_i(t) - v_0(t))\right) - k_\epsilon\frac{\mathrm{d}s}{\mathrm{d}t}v_i(t), \quad i \in \{1, 2, \ldots, N\}, \end{aligned} \tag{37}$$

where $k_\epsilon \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+$ are design parameters. Note that in (37), we use $\mathrm{d}s/\mathrm{d}t = (\theta'(\theta^{-1}(t)))^{-1}$ and $\mathrm{d}^2s/\mathrm{d}t^2 = \mathrm{d}(\theta'(\theta^{-1}(t)))^{-1}/\mathrm{d}t$ for simplicity. Note also that we use $k_i = 1$ for the subset of the agents having access to the states of the time-varying leader in (33) and (34), and $k_i = 0$ for other agents. Now, let $\tilde{x}_i(t) \triangleq x_i(t) - x_0(t)$, $i \in \{1, 2, \ldots, N\}$, and $\tilde{v}_i(t) \triangleq v_i(t) - v_0(t)$, $i \in \{1, 2, \ldots, N\}$, be the position and the velocity error states, respectively. Furthermore, define an auxiliary state of the form

$$\begin{aligned} \epsilon_i(t) &\triangleq v_i(t) + \alpha\frac{\mathrm{d}s}{\mathrm{d}t}\left(\sum_{i\sim j}(x_i(t) - x_j(t)) + k_i(x_i(t) - x_0(t))\right) \\ &= v_i(t) + \alpha\frac{\mathrm{d}s}{\mathrm{d}t}\left(\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_j(t)) + k_i\tilde{x}_i(t)\right). \end{aligned} \tag{38}$$

One can now use (38) to write the position error dynamics as

$$\dot{\tilde{x}}_i(t) = -\alpha\frac{\mathrm{d}s}{\mathrm{d}t}\left(\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_j(t)) + k_i\tilde{x}_i(t)\right) + \epsilon_i(t) - v_0(t), \quad \tilde{x}_i(0) = \tilde{x}_{i0}. \tag{39}$$

The derivative of the auxiliary state in (38) satisfies

$$\dot{\epsilon}_i(t) = u_i(t) + \alpha\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\left(\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_j(t)) + k_i\tilde{x}_i(t)\right) + \alpha\frac{\mathrm{d}s}{\mathrm{d}t}\left(\sum_{i\sim j}(\tilde{v}_i(t) - \tilde{v}_j(t)) + k_i\tilde{v}_i(t)\right), \quad \epsilon_i(0) = \epsilon_{i0}. \tag{40}$$

Using the proposed control signal in (37), (40) can be equivalently written as

$$\begin{aligned} \dot{\epsilon}_i(t) &= -\alpha k_\epsilon\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\left(\sum_{i\sim j}(x_i(t) - x_j(t)) + k_i(x_i(t) - x_0(t))\right) - k_\epsilon\frac{\mathrm{d}s}{\mathrm{d}t}v_i(t), \\ &= -k_\epsilon\frac{\mathrm{d}s}{\mathrm{d}t}\epsilon_i(t), \quad \epsilon_i(0) = \epsilon_{i0}. \end{aligned} \tag{41}$$

Next, we define the augmented position error state, velocity error state, and the auxiliary state, respectively, as $\tilde{x}(t) \triangleq [\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$, $\tilde{v}(t) \triangleq [\tilde{v}_1(t), \tilde{v}_2(t), \ldots, \tilde{v}_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$, and $\epsilon(t) \triangleq [\epsilon_1(t), \epsilon_2(t), \ldots, \epsilon_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$. One can then write the system dynamics in (39) and (41) in the compact form

$$\dot{\tilde{x}}(t) = -\alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G})\tilde{x}(t) + \epsilon(t) - \mathbf{1}_N v_0(t), \quad \tilde{x}(0) = \tilde{x}_0, \tag{42}$$

$$\dot{\epsilon}(t) = -k_\epsilon \frac{\mathrm{d}s}{\mathrm{d}t}\epsilon(t), \quad \epsilon(0) = \epsilon_0. \tag{43}$$

In addition, the distributed control algorithm can be rewritten as

$$\begin{aligned} u(t) &= -\alpha \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathcal{F}(\mathfrak{G})\tilde{x}(t) - \alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G})\tilde{v}(t) - k_\epsilon \frac{\mathrm{d}s}{\mathrm{d}t}\epsilon(t), \\ &= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}(v(t) - \epsilon(t)) - \alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G})\tilde{v}(t) - k_\epsilon \frac{\mathrm{d}s}{\mathrm{d}t}\epsilon(t). \end{aligned} \tag{44}$$

The derivative of (44) is now given by

$$\begin{aligned} \dot{u}(t) = {} & \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}\left(u(t)k_\epsilon \frac{\mathrm{d}s}{\mathrm{d}t}\epsilon(t)\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}\right)(v(t) - \epsilon(t)) - \alpha \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathcal{F}(\mathfrak{G})\tilde{v}(t) - \alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G})\dot{\tilde{v}}(t) \\ & - k_\epsilon \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\epsilon(t) - k_\epsilon \frac{\mathrm{d}s}{\mathrm{d}t}\left(-k_\epsilon \frac{\mathrm{d}s}{\mathrm{d}t}\epsilon(t)\right), \quad u(0) = u_0. \end{aligned} \tag{45}$$

Using $\tilde{v}(t) = v(t) - \mathbf{1}_N v_0(t)$ and $\dot{\tilde{v}}(t) = u(t) - \mathbf{1}_N a_0(t)$ in (45) yields

$$\begin{aligned} \dot{u}(t) = {} & \left[\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s} - \alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G})\right]u(t) + \left[k_\epsilon^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}\right)\right]\epsilon(t) + \alpha \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathcal{F}(\mathfrak{G})\mathbf{1}_N v_0(t) \\ & - \left[\alpha \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathcal{F}(\mathfrak{G}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}\right)\right]v(t) + \alpha \frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_0(t), \quad u(0) = u_0. \end{aligned} \tag{46}$$

Similar to how we obtain (16) from (15) in Section 3.1 and based on the time transformation function $t = \theta(s)$, one can now rewrite (42), (43), and (46) as

$$\tilde{x}_s'(s) = -\alpha\mathcal{F}(\mathfrak{G})\tilde{x}_s(s) + \frac{\mathrm{d}t}{\mathrm{d}s}\epsilon_s(s) - \frac{\mathrm{d}t}{\mathrm{d}s}\mathbf{1}_N v_{0_s}(s), \; \tilde{x}_s(0) = \tilde{x}_0, \tag{47}$$

$$\epsilon_s'(s) = -k_\epsilon \epsilon_s(s), \quad \epsilon_s(0) = \epsilon_0, \tag{48}$$

$$\begin{aligned} u_s'(s) = {} & \left[\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-2} - \alpha\mathcal{F}(\mathfrak{G})\right]u_s(s) + \left[k_\epsilon^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) - \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}\right)\right]\epsilon_s(s) + \alpha \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathcal{F}(\mathfrak{G})\mathbf{1}_N v_{0_s}(s) \\ & - \left[\alpha \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\mathcal{F}(\mathfrak{G}) - \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}s}\right)\right]v_s(s) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_{0_s}(s), \quad u_s(0) = u_0. \end{aligned} \tag{49}$$

where the subscript s is used; see Remark 3.

Remark 9. Since $k_\epsilon \in \mathbb{R}_+$, the error dynamics in (48) is stable and $\lim_{s\to\infty}\epsilon_s(s) = 0$. Now, similar to Remark 9, the time transformation function $t = \theta(s)$ should be chosen by the control user such that (i) the state error dynamics and control signal dynamics given by (47) and (49) are stable, which results in bounded error states $\tilde{x}_s(s)$, $\tilde{v}_s(s)$, and the control signal $u_s(s)$, and (ii) the asymptotic stability for the error state $\tilde{x}_s(s)$ is achieved (ie, $\lim_{s\to\infty}\tilde{x}_s(s) = 0$). By Remark 2, the above discussion implies that the error states $\tilde{x}(t)$, $\tilde{v}(t)$, and the control signal $u(t)$ are bounded in the original time interval $t \in [0, T)$ and $\lim_{t\to T}(x_i(t) - x_0(t)) = 0$. The latter result implies that the agents converge to the position of the time-varying leader at the user-defined finite time T.

Once again, the selection of the time transformation function $t = \theta(s)$ plays a crucial role on the stability of the closed-loop system dynamics given in (47), (48), and (49) as discussed in Remark 9. In what follows, a candidate time transformation function satisfying the conditions in Remark 9, along with rigorous and detailed analysis on stability of the closed-loop system dynamics, is presented.

To this end, consider the time transformation function given by (10) that converts the infinite-time interval $s \in [0, \infty)$ to the finite-time interval of interest $t \in [0, T)$. Note that the derivative of this time transformation function with respect to $s \in [0, \infty)$ is given by $\mathrm{d}t/\mathrm{d}s = \theta'(s) = Te^{-s} = T - t$ that is $\frac{\mathrm{d}s}{\mathrm{d}t} = \lambda(t)$. In addition, one can write $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \lambda^2(t)$. Hence, the proposed distributed control signal given by (37) now can be written as

$$\begin{aligned} u_i(t) = &-\alpha\lambda^2(t)(k_\epsilon + 1)\left(\sum_{i\sim j}(x_i(t) - x_j(t)) + k_i(x_i(t) - x_0(t))\right) \\ &- \alpha\lambda(t)\left(\sum_{i\sim j}(v_i(t) - v_j(t)) + k_i(v_i(t) - v_0(t))\right) - k_\epsilon\lambda(t)v_i(t), \quad i \in \{1, 2, \dots, N\}, \end{aligned} \tag{50}$$

where $k_\epsilon \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+$ are design parameters. Furthermore, the auxiliary state can be written in the form

$$\epsilon_i(t) = v_i(t) + \alpha\lambda(t)\left(\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_j(t)) + k_i\tilde{x}_i(t)\right). \tag{51}$$

Similar to writing (43) and (42), one can then write the position error and auxiliary state error dynamics as

$$\dot{\tilde{x}}(t) = -\alpha\lambda(t)\mathcal{F}(\mathfrak{G})\tilde{x}(t) + \epsilon(t) - \mathbf{1}_N v_0(t), \quad \tilde{x}(0) = \tilde{x}_0, \tag{52}$$

$$\dot{\epsilon}(t) = -k_\epsilon\lambda(t)\epsilon(t), \quad \epsilon(0) = \epsilon_0. \tag{53}$$

It now follows from Remark 2 that

$$\tilde{x}_s'(s) = -\alpha\mathcal{F}(\mathfrak{G})\tilde{x}_s(s) + Te^{-s}(\epsilon_s(s) - \mathbf{1}_N v_{0_s}(s)),\ \tilde{x}_s(0) = \tilde{x}_0, \tag{54}$$

$$\epsilon_s'(s) = -k_\epsilon\epsilon_s(s), \quad \epsilon_s(0) = \epsilon_0. \tag{55}$$

As a consequence, the solution of the auxiliary state given by (55) in the infinite-time horizon can be written as $\epsilon_s(s) = \epsilon_0 e^{-k_\epsilon s}$. Using this in (54) yields

$$\tilde{x}_s'(s) = -\alpha\mathcal{F}(\mathfrak{G})\tilde{x}_s(s) + e^{-s}\beta(s), \quad \tilde{x}_s(0) = \tilde{x}_0, \tag{56}$$

where $\beta(s) \triangleq T(\epsilon_0 e^{-k_\epsilon s} - \mathbf{1}_N v_{0_s}(s))$ is a bounded function (ie, $\|\beta(s)\|_2 \le \overline{\beta}$). For the following result, we also define

$$\mathcal{R} \triangleq \alpha\mathcal{F}(\mathfrak{G}) - 2I_N \in \mathbb{R}^{N\times N}. \tag{57}$$

Theorem 2. *Consider the multiagent system that consists of N agents on a connected, undirected graph $\mathfrak{G}$, where the agents have the second-order dynamics given by (35) and (36). Furthermore, assume that there exists at least one agent exchanging information with the leader with bounded position and velocity given by (33) and (34), where this leader is subject to a bounded but otherwise unknown acceleration signal. Based on the distributed control algorithm $u_i(t)$, $i = 1, \dots, N$, for each agent given by (50), let the design parameter α be chosen to make $\mathcal{R}$ in (57) positive definite and $k_\epsilon > 1$. Then, the closed-loop system signals including the control and internal signals remain bounded and all agents' positions converge to the position of the leader in the a priori given, user-defined finite time T (ie, $\lim_{t\to T}\tilde{x}(t) = 0$) for all initial conditions of agents.*

Proof. We first show that all agents' positions converge to the position of the leader in the user-defined finite time T. Then, to demonstrate the feasibility of the proposed algorithm, we respectively, show that the velocity of the agents and the control signals remain bounded.

Step 1. Consider the system error dynamics in the infinite-time interval $s \in [0, \infty)$ given by (56). Let $\mathcal{V}_1(s) \in \mathbb{R}_+$ be an energy function of the form

$$\mathcal{V}_1(s) = \tilde{x}_s^{\mathrm{T}}(s)\tilde{x}_s(s) = \|\tilde{x}_s(s)\|_2^2. \tag{58}$$

The derivative of (58) with respect to the stretched time $s \in [0, \infty)$ along the trajectories of (56) is given by

$$\mathcal{V}_1'(s) = -2\alpha\tilde{x}_s^{\mathrm{T}}(s)\mathcal{F}(\mathfrak{G})\tilde{x}_s(s) + 2\tilde{x}_s^{\mathrm{T}}(s)\beta(s)e^{-s}, \\ \leq -2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G}))\|\tilde{x}_s(s)\|_2^2 + 2\|\tilde{x}_s(s)\|_2\|\beta(s)\|_2 e^{-s}. \tag{59}$$

Using $2\|\tilde{x}_s(s)\|_2\|\beta(s)\|_2 \leq \|\tilde{x}_s(s)\|_2^2 + \|\beta(s)\|_2^2$ yields

$$\mathcal{V}_1'(s) \leq -2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G}))\mathcal{V}_1(s) + (\mathcal{V}_1(s) + \|\beta(s)\|_2^2)e^{-s}, \\ \leq (-2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G})) + e^{-s})\mathcal{V}_1(s) + \overline{\beta}^2 e^{-s}. \tag{60}$$

Moreover, one can rearrange (60) to obtain $\mathcal{V}_1'(s) + (a_1 - b_1 e^{-s})\mathcal{V}_1(s) \leq c_1 e^{-s}$, where $a_1 \triangleq 2\alpha \ \lambda_{\min}(\mathcal{F}(\mathfrak{G})) \in \mathbb{R}_+$ by Remark 1, $b_1 = 1$, and $c_1 \triangleq \overline{\beta}^2 \in \overline{\mathbb{R}}_+$. It now follows from Lemma 2 that $\mathcal{V}_1(s)$ is bounded for $s \in [0, \infty)$ and $\lim_{s\to\infty}\mathcal{V}_1(s) = 0$ resulting in $\lim_{s\to\infty}\tilde{x}_s(s) = 0$. Since $\tilde{x}_s(s) = \tilde{x}(t)$ by definition (see Remark 2) and $t \to T$ as $s \to \infty$, one obtains $\lim_{t\to T}\tilde{x}(t) = 0$. From the boundedness of $\mathcal{V}_1(s)$, it also follows that the state error vector $\tilde{x}_s(s)$ is bounded for all $s \in [0, \infty)$. By Remark 2, in addition, the state error vector $\tilde{x}(t)$ remains bounded, or equivalently, the position states of the agents $x(t)$ are bounded in the finite-time interval $t \in [0, T)$.

Step 2. Next, to show that the velocity of agents remain bounded, we take the derivative of (52) with respect to t given by

$$\ddot{\tilde{x}}(t) = -k_\epsilon\lambda(t)\epsilon(t) - \alpha\lambda^2(t)\mathcal{F}(\mathfrak{G})\tilde{x}(t) - \alpha\lambda(t)\mathcal{F}(\mathfrak{G})\tilde{v}(t) - \mathbf{1}_N a_0(t), \\ = -k_\epsilon\lambda(t)\epsilon(t) + \lambda(t)(\dot{\tilde{x}}(t) - \epsilon(t) + \mathbf{1}_N v_0(t)) - \alpha\lambda(t)\mathcal{F}(\mathfrak{G})\dot{\tilde{x}}(t) - \mathbf{1}_N a_0(t), \\ = -(k_\epsilon + 1)\lambda(t)\epsilon(t) - \mathcal{S}\lambda(t)\dot{\tilde{x}}(t) + \mathbf{1}_N v_0(t)\lambda(t) - \mathbf{1}_N a_0(t), \quad \dot{\tilde{x}}(0) = \dot{\tilde{x}}_0, \tag{61}$$

where $\mathcal{S} \triangleq \alpha\mathcal{F}(\mathfrak{G}) - I_N \in \mathbb{R}^{N\times N}$. Similar to writing (16) from (15), one can replace $\dot{\tilde{x}}(t)$ by $\tilde{v}(t)$ and use the time transformation function in (10) to obtain

$$\tilde{v}_s'(s) = -\mathcal{S}\tilde{v}_s(s) - (k_\epsilon + 1)\epsilon_s(s) + h_1(s), \quad \tilde{v}_s(0) = \tilde{v}_0, \tag{62}$$

where $h_1(s) \triangleq \mathbf{1}_N v_{0_s}(s) - Te^{-s}\mathbf{1}_N a_{0_s}(s)$ consists of bounded terms; hence, it is bounded. We now write (55) and (62) in the augmented form given by

$$\begin{bmatrix} \tilde{v}_s'(s) \\ \epsilon_s'(s) \end{bmatrix} = \begin{bmatrix} -\mathcal{S} & -(k_\epsilon + 1)I_N \\ \mathbf{0}_{N\times N} & -k_\epsilon I_N \end{bmatrix} \begin{bmatrix} \tilde{v}_s(s) \\ \epsilon_s(s) \end{bmatrix} + \begin{bmatrix} h_1(s) \\ \mathbf{0}_N \end{bmatrix}, \quad \begin{bmatrix} \tilde{v}_s(0) \\ \epsilon_s(0) \end{bmatrix} = \begin{bmatrix} \tilde{v}_0 \\ \epsilon_0 \end{bmatrix}. \tag{63}$$

Based on the upper triangular structure of the system matrix in (63) and since $-\mathcal{S} = -\mathcal{R} - I_N$ is Hurwitz by the assumption of the theorem, it then follows that $\tilde{v}_s(s)$ is bounded. Since $\tilde{v}_s(s) = \tilde{v}(t)$ by definition, one can conclude that $\tilde{v}(t)$ (and the velocity state $v(t)$) is bounded in the finite-time interval $t \in [0, T)$.

Step 3. Finally, we show that the control algorithm given by (50) also remains bounded over $t \in [0, T)$. To this end, we write the derivative of (53) with respect to t given by

$$\ddot{\epsilon}(t) = -k_\epsilon\lambda^2(t)\epsilon(t) - k_\epsilon\lambda(t)\dot{\epsilon}(t) = \dot{\epsilon}(t)\lambda(t) - k_\epsilon\lambda(t)\dot{\epsilon}(t), \\ = -(k_\epsilon - 1)\lambda(t)\dot{\epsilon}(t), \quad \dot{\epsilon}(0) = \dot{\epsilon}_0. \tag{64}$$

We now let $\psi(t) \triangleq \dot{\epsilon}(t)$ and use the time transformation function in (10) to obtain

$$\psi_s'(s) = -(k_\epsilon - 1)\psi_s(s), \quad \psi_s(0) = \psi_{s0}. \tag{65}$$

Since $k_\epsilon > 1$ based on the assumption of the theorem, it is clear from (65) that $\psi_s(s)$ (and $\psi(t) = \dot{\epsilon}(t)$) is bounded (and converges to zero). Furthermore, we respectively, write the control signal in (50) and the auxiliary state in (51) in the compact form given by

$$u(t) = -\alpha\lambda^2(t)\mathcal{F}(\mathfrak{G})\tilde{x}(t) - \alpha\lambda(t)\mathcal{F}(\mathfrak{G})\tilde{v}(t) - k_\epsilon\lambda(t)\epsilon(t), \tag{66}$$

$$\epsilon(t) = \nu(t) + \alpha\lambda(t)\mathcal{F}(\mathfrak{G})\tilde{x}(t). \tag{67}$$

Using (67) in (66) yields

$$\begin{aligned} u(t) &= \lambda(t)(\nu(t) - \epsilon(t) - \alpha\mathcal{F}(\mathfrak{G})\tilde{\nu}(t) - k_\epsilon\epsilon(t)), \\ &= \lambda(t)(\nu(t) - \epsilon(t) - \alpha\mathcal{F}(\mathfrak{G})\nu(t) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N\nu_0(t) - k_\epsilon\epsilon(t)), \\ &= z(t)\lambda(t), \end{aligned} \tag{68}$$

where $z(t) \triangleq -\mathcal{S}\nu(t) - (k_\epsilon + 1)\epsilon(t) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N\nu_0(t)$ and has the derivative with respect to t given by

$$\dot{z}(t) = -\mathcal{S}u(t) - (k_\epsilon + 1)\psi(t) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_0(t), \; z(0) = z_0. \tag{69}$$

One can also obtain the derivative of (68) with respect to t as

$$\begin{aligned} \dot{u}(t) &= \lambda(t)\dot{z}(t) + \lambda^2(t)z(t) = \lambda(t)\dot{z}(t) + \lambda(t)u(t), \\ &= \lambda(t)(-\mathcal{S}u(t) - (k_\epsilon + 1)\psi(t) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_0(t)) + \lambda(t)u(t), \\ &= -\mathcal{R}\lambda(t)u(t) + \lambda(t)(\alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_0(t) - (k_\epsilon + 1)\psi(t)), \quad u(0) = u_0. \end{aligned} \tag{70}$$

Finally, using the time transformation function in (10) one can rewrite (70) as $u_s'(s) = -\mathcal{R}u_s(s) + n_1(s), \; u_s(0) = u_0$, where $n_1(s) \triangleq \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_{0_s}(s) - (k_\epsilon + 1)\psi_s(s)$. Now, since $n_1(s)$ is a bounded function and $-\mathcal{R}$ is Hurwitz by the assumption of the theorem, then $u_s(s)$ is bounded. Consequently, $u(t)$ remains bounded in the finite interval $t \in [0, T)$ and this concludes the proof of this theorem. ■

4.2 | Robustness to vanishing and nonvanishing system uncertainties

In this section, we analyze the proposed distributed control algorithm proposed in the previous section in the presence of vanishing and nonvanishing system uncertainties and analytically show that it can still preserve the user-defined finite-time convergence at $t = T$ seconds. To this end, consider the second-order agent dynamics given by (31) and (32). One can now write the derivative of the auxiliary state in (51) as

$$\begin{aligned} \dot{\epsilon}_i(t) = u_i(t) &+ \alpha\lambda^2(t)\left(\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_j(t)) + k_i\tilde{x}_i(t)\right) + \alpha\lambda(t)\left(\sum_{i\sim j}(\tilde{\nu}_i(t) - \tilde{\nu}_j(t)) + k_i\tilde{\nu}_i(t)\right) \\ &+ \omega_i x_i(t) + \rho_i(t), \quad \epsilon_i(0) = \epsilon_{i0}. \end{aligned} \tag{71}$$

Utilizing the proposed control algorithm (50) in (71) yields $\dot{\epsilon}_i(t) = -k_\epsilon\lambda(t)\epsilon_i(t) + \omega_i x_i(t) + \rho_i(t)$, $\epsilon_i(0) = \epsilon_{i0}$ that can also be compactly written as

$$\dot{\epsilon}(t) = -k_\epsilon\lambda(t)\epsilon(t) + \Omega\tilde{x}(t) + \Omega\mathbf{1}_N x_0(t) + \rho(t), \quad \epsilon(0) = \epsilon_0, \tag{72}$$

where $\Omega \triangleq \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N) \in \mathbb{R}^{N\times N}$ and $\rho(t) \triangleq [\rho_1(t), \rho_2(t), \ldots, \rho_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$. Using the time transformation function (10), one can convert (72) to the infinite-time interval as

$$\epsilon_s'(s) = -k_\epsilon\epsilon_s(s) + Te^{-s}\Omega\tilde{x}_s(s) + Te^{-s}(\Omega\mathbf{1}_N x_{0_s}(s) + \rho_s(s)), \quad \epsilon_s(0) = \epsilon_0. \tag{73}$$

We can now augment the system dynamics in (54) and (73) as

$$\begin{bmatrix} \tilde{x}_s'(s) \\ \epsilon_s'(s) \end{bmatrix} = \begin{bmatrix} -\alpha\mathcal{F} & Te^{-s}I_N \\ Te^{-s}\Omega & -k_\epsilon I_N \end{bmatrix} \begin{bmatrix} \tilde{x}_s(s) \\ \epsilon_s(s) \end{bmatrix} + Te^{-s}\begin{bmatrix} -\mathbf{1}_N\nu_{0_s}(s) \\ \Omega\mathbf{1}_N x_{0_s}(s) + \rho_s(s) \end{bmatrix}, \quad \begin{bmatrix} \tilde{x}_s(0) \\ \epsilon_s(0) \end{bmatrix} = \begin{bmatrix} \tilde{x}_0 \\ \epsilon_0 \end{bmatrix}. \tag{74}$$

Theorem 3. *Consider the multiagent system that consists of N agents on a connected, undirected graph $\mathfrak{G}$, where the agents have the second-order dynamics as given by (31) and (32) having vanishing and nonvanishing system uncertainties. Furthermore, assume that there exists at least one agent exchanging information with the leader with bounded position and velocity given by (33) and (34), where this leader is subject to a bounded but otherwise unknown acceleration signal. Based on the distributed control algorithm $u_i(t)$, $i=1, \ldots, N$, for each agent given by (50), let the design parameter α be chosen to make $\mathcal{R}$ in (57) positive definite and $k_\epsilon > 1$. Then, the closed-loop system signals including the control signals remain bounded and all agents' positions converge to the position of the leader in the a priori given, user-defined finite time T (ie, $\lim_{t\to T}\tilde{x}(t) = 0$) for all initial conditions of agents and for all finite $\rho_i(t)$, $i \in \{1, \ldots, N\}$.*

Proof. We follow similar steps in the proof of Theorem 2.

Step 1. Consider the system error dynamics in the infinite-time interval $s\in[0,\infty)$ given by (74) and let $\mathcal{V}_2(s) \in \mathbb{R}_+$ be an energy function of the form

$$\mathcal{V}_2(s) = \mathcal{V}_1(s) + \epsilon_s^{\mathrm{T}}(s)\epsilon_s(s) = \|\tilde{x}_s(s)\|_2^2 + \|\epsilon_s(s)\|_2^2. \tag{75}$$

The derivative of (75) with respect to the stretched time $s\in[0,\infty)$ along the trajectories of (74) is now given by

$$\begin{aligned}\mathcal{V}_2'(s) &= 2\tilde{x}_s^{\mathrm{T}}(s)\tilde{x}_s'(s) + 2\epsilon_s^{\mathrm{T}}(s)\epsilon_s'(s),\\ &= -2\alpha\tilde{x}_s^{\mathrm{T}}(s)\mathcal{F}(\mathfrak{G})\tilde{x}_s(s) + 2Te^{-s}\tilde{x}_s^{\mathrm{T}}(s)\epsilon_s(s) - 2Te^{-s}\tilde{x}_s^{\mathrm{T}}(s)\mathbf{1}_N v_{0_s}(s) - 2k_\epsilon\epsilon_s^{\mathrm{T}}(s)\epsilon_s(s)\\ &\quad + 2Te^{-s}\epsilon_s^{\mathrm{T}}(s)\Omega\tilde{x}_s(s) + 2Te^{-s}\epsilon_s^{\mathrm{T}}(s)(\Omega\mathbf{1}_N x_{0_s}(s) + \rho_s(s)),\\ &\leq -2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G}))\|\tilde{x}_s(s)\|_2^2 - 2k_\epsilon\|\epsilon_s(s)\|_2^2 + 2Te^{-s}\|\tilde{x}_s(s)\|_2\|\epsilon_s(s)\|_2(1+\omega_{\max})\\ &\quad + 2Te^{-s}\|\mathbf{1}_N v_{0_s}(s)\|_2\|\tilde{x}_s(s)\|_2 + 2Te^{-s}\|\Omega\mathbf{1}_N x_{0_s}(s) + \rho_s(s)\|_2\|\epsilon_s(s)\|_2,\end{aligned} \tag{76}$$

where $\omega_{\max} \triangleq \max\{1, \ldots, N\}$. Now, using $2\|x(s)\|_2\|y(s)\|_2 \leq \|x(s)\|_2^2 + \|y(s)\|_2^2$ for any vectors, one can write

$$\begin{aligned}\mathcal{V}_2'(s) &\leq -a_2(\|\tilde{x}_s(s)\|_2^2 + \|\epsilon_s(s)\|_2^2) + Te^{-s}(1+\omega_{\max})(\|\tilde{x}_s(s)\|_2^2 + \|\epsilon_s(s)\|_2^2)\\ &\quad + Te^{-s}(\|\mathbf{1}_N v_{0_s}(s)\|_2^2 + \|\tilde{x}_s(s)\|_2^2) + Te^{-s}(\|\Omega\mathbf{1}_N x_{0_s}(s) + \rho_s(s)\|_2^2 + \|\epsilon_s(s)\|_2^2),\end{aligned} \tag{77}$$

where $a_2 \triangleq \min\{2\alpha\lambda_{\min}(\mathcal{F}(\mathfrak{G})), 2k_\epsilon\} \in \mathbb{R}_+$ by Remark 1. Substituting the energy function $\mathcal{V}_2(s)$ given in (75) yields $\mathcal{V}_2'(s) + (a_2 - b_2e^{-s})\mathcal{V}_2(s) \leq c_2e^{-s}$, where $b_2 \triangleq \max\{0, T(2+\omega_{\max})\} \in \overline{\mathbb{R}}_+$ and $c_2 \in \mathbb{R}_+$ is the upper bound on $T(\|\Omega\mathbf{1}_N x_{0_s}(s) + \rho_s(s)\|_2^2 + \|\mathbf{1}_N v_{0_s}(s)\|_2^2)$. Similar to the step 1 given in the proof of Theorem 2, it follows that the pair $(\tilde{x}(t), \epsilon(t))$ remains bounded in the finite-time interval for all $t\in[0,T)$ and $\lim_{t\to T}(\tilde{x}(t), \epsilon(t)) = (0,0)$.

Step 2. For showing the boundedness of the velocity of the agents, we take the derivative of (52) with respect to t and use (72) to obtain

$$\begin{aligned}\ddot{\tilde{x}}(t) &= -k_\epsilon\lambda(t)\epsilon(t) + \Omega\tilde{x}(t) + \Omega\mathbf{1}_N x_0(t) + \rho(t) - \alpha\lambda^2(t)\mathcal{F}(\mathfrak{G})\tilde{x}(t) - \alpha\lambda(t)\mathcal{F}(\mathfrak{G})\tilde{v}(t) - \mathbf{1}_N a_0(t),\\ &= -k_\epsilon\lambda(t)\epsilon(t) + \Omega\tilde{x}(t) + \Omega\mathbf{1}_N x_0(t) + \rho(t) + \lambda(t)(\dot{\tilde{x}}(t) - \epsilon(t) + \mathbf{1}_N v_0(t)) - \alpha\lambda(t)\mathcal{F}(\mathfrak{G})\dot{\tilde{x}}(t)\\ &\quad - \mathbf{1}_N a_0(t),\\ &= -(k_\epsilon+1)\lambda(t)\epsilon(t) - \mathcal{S}\lambda(t)\dot{\tilde{x}}(t) + \mathbf{1}_N v_0(t)\lambda(t) + \Omega\tilde{x}(t) + \Omega\mathbf{1}_N x_0(t) + \rho(t) - \mathbf{1}_N a_0(t),\ \dot{\tilde{x}}(0) = \dot{\tilde{x}}_0.\end{aligned} \tag{78}$$

Replacing $\dot{\tilde{x}}(t)$ by $\tilde{v}(t)$ and using the time transformation function in (10) yields $\tilde{v}_s'(s) = -\mathcal{S}\tilde{v}_s(s) + h_2(s)$, $\tilde{v}_s(0) = \tilde{v}_0$, where $h_2(s) \triangleq -(k_\epsilon+1)\epsilon_s(s) + \mathbf{1}_N v_{0_s}(s) + Te^{-s}(\Omega\tilde{x}_s(s) + \Omega\mathbf{1}_N x_{0s}(s) + \rho_s(s) - \mathbf{1}_N a_{0_s}(s))$ consists of bounded terms; hence, it is a bounded function. Since $-\mathcal{S} = -\mathcal{R} - I_N$ is Hurwitz by the assumption of the theorem, it then follows that $\tilde{v}_s(s)$ is bounded. Since $\tilde{v}_s(s) = \tilde{v}(t)$ by definition, one can conclude that $\tilde{v}(t)$ (and the velocity state $v(t)$) is bounded in the finite-time interval $t\in[0,T)$.

Step 3. We finally show that the control algorithm given in (50) also remains bounded in presence of considered vanishing and nonvanishing system uncertainties over the finite-time interval. For this purpose, we write the derivative of (72) with respect to t given by

$$\ddot{\epsilon}(t) = -k_\epsilon \lambda^2(t)\epsilon(t) - k_\epsilon \lambda(t)\dot{\epsilon}(t) + \Omega\dot{\tilde{x}}(t) + \Omega \mathbf{1}_N v_0(t) + \dot{\rho}(t),$$
$$= \lambda(t)(\dot{\epsilon}(t) - \Omega\tilde{x}(t) - \Omega \mathbf{1}_N x_0(t) - \rho(t)) - k_\epsilon \lambda(t)\dot{\epsilon}(t) + \Omega\dot{\tilde{x}}(t) + \Omega \mathbf{1}_N v_0(t) + \dot{\rho}(t), \ \dot{\epsilon}(0) = \dot{\epsilon}_0. \tag{79}$$

Letting $\psi(t) = \dot{\epsilon}(t)$, $\tau(t) \triangleq \dot{\rho}(t)$, and using the time transformation function in (10) yields

$$\psi_s'(s) = -(k_\epsilon - 1)\psi_s(s) + h_\epsilon(s), \quad \psi_s(0) = \psi_{s0}, \tag{80}$$

where $h_\epsilon(s) \triangleq -(\Omega\tilde{x}_s(s) + \Omega \mathbf{1}_N x_{0_s}(s) + \rho_s(s)) + Te^{-s}(\Omega\dot{\tilde{x}}_s(s) + \Omega \mathbf{1}_N v_{0_s}(s) + \tau_s(s))$ contains all the bounded terms in (79). Now, since $k_\epsilon > 1$ based on the assumption of the theorem, it follows from (80) that $\psi_s(s)$ (and $\psi(t) = \dot{\epsilon}(t)$) is bounded. Next, similar to Step 3 in proof of Theorem 2, by writing the control signal (50) in the compact form, one can obtain

$$u(t) = \lambda(t)z(t), \tag{81}$$

where $z(t) \triangleq -\mathcal{S}v(t) - (k_\epsilon + 1)\epsilon(t) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N v_0(t)$ with the derivative with respect to t given by

$$\dot{z}(t) = -\mathcal{S}(u(t) + \Omega x(t) + \rho(t)) - (k_\epsilon + 1)\psi(t) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_0(t), \quad z(0) = z_0. \tag{82}$$

One can obtain the derivative of (81) with respect to t as

$$\dot{u}(t) = \lambda(t)\dot{z}(t) + \lambda^2(t)z(t) = \lambda(t)\dot{z}(t) + \lambda(t)u(t),$$
$$= \lambda(t)(-\mathcal{S}(u(t) + \Omega x(t) + \rho(t)) - (k_\epsilon + 1)\psi(t) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_0(t)) + \lambda(t)u(t),$$
$$= -\mathcal{R}\lambda(t)u(t) + \lambda(t)(-\mathcal{S}(\Omega x(t) + \rho(t)) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_0(t) - (k_\epsilon + 1)\psi(t)), \quad u(0) = u_0. \tag{83}$$

Finally, using the time transformation function in (10) one can rewrite (83) as $u_s'(s) = -\mathcal{R}u_s(s) + h_u(s)$, $u_s(0) = u_0$, where $h_u(s) \triangleq -\mathcal{S}(\Omega\tilde{x}_s(s) + \Omega\mathbf{1}_N x_{0_s}(s) + \rho_s(s)) + \alpha\mathcal{F}(\mathfrak{G})\mathbf{1}_N a_{0_s}(s) - (k_\epsilon + 1)\psi_s(s)$. Now, since $h_u(s)$ is a bounded function and $-\mathcal{R}$ is Hurwitz by the assumption of the theorem, then $u_s(s)$ is bounded. Equivalently, $u(t)$ remains bounded in the finite interval $t \in [0, T)$ and this concludes the proof of this theorem. ■

Remark 10. The proposed distributed control algorithms in this article are not limited to the considered leader-following setting and the time transformation method can be also used for achieving finite-time convergence in other multiagent control problems. In particular, the proposed distributed control algorithms in (13) and (37) can serve as building bricks for control design of various multiagent system problems such as consensus, formation control, and containment control. As an example, consider the multiagent system considered in Section 3 consisting of N agents with first-order dynamics given by (11). To address formation control, one can use a change of variable $z_i(t) = x_i(t) - \zeta_i$, $i \in \{1, 2, \dots, N\}$, where $\zeta_i \in \mathbb{R}$, $i \in \{1, 2, \dots, N\}$ denotes the desired distance from the leader in the formation of the multiagent system [2, section 6.3]. Similar to (22), the control signal in this case can be designed as

$$u_i(t) = -\alpha\lambda(t)\left(\sum_{i\sim j}(z_i(t) - z_j(t)) + k_i(z_i(t) - x_0(t))\right), \quad i \in \{1, 2, \dots, N\}. \tag{84}$$

Letting $\tilde{z}(t) \triangleq [\tilde{z}_1(t), \tilde{z}_2(t), \dots, \tilde{z}_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$, where $\tilde{z}_i(t) \triangleq z_i(t) - x_0(t), i \in \{1, 2, \dots, N\}$, denotes the new error signal, one can write the error dynamics in the compact form as

$$\dot{\tilde{z}}(t) = \left(-\alpha\frac{\mathrm{d}s}{\mathrm{d}t}\mathcal{F}(\mathfrak{G}) + \Omega\right)\tilde{z}(t) + h_\zeta(t), \quad \tilde{z}(0) = \tilde{z}_0, \tag{85}$$

where $h_\zeta(t) \triangleq \Omega(\zeta + \mathbf{1}_N x_0(t)) + \rho(t) - \mathbf{1}_N v_0(t)$ with $\zeta \triangleq [\zeta_1, \zeta_2, \dots, \zeta_N]^{\mathrm{T}} \in \mathbb{R}^N$. Note that (85) has the same form as (15). It now follows from the presented analyses in Section 3 (particularly, Theorem 1) that $\lim_{t\to T}(z_i(t) - x_0(t)) = 0$, that is, $\lim_{t\to T}(x_i(t) - \zeta_i - x_0(t)) = 0$. This equivalently means that the required formation is achieved and the position of the

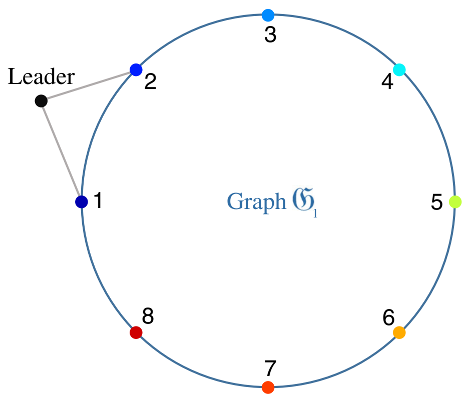

FIGURE 3 An example multiagent system on an undirected, connected circle graph $\mathfrak{G}_1$ considered in Example 1 [Colour figure can be viewed at wileyonlinelibrary.com]

agent i reaches to a distance from the position of the leader within the user-defined finite time T, and this distance can be characterized by ζ_i.

5 | ILLUSTRATIVE NUMERICAL EXAMPLES

We now present illustrative numerical examples to demonstrate the efficacy of the presented distributed control algorithms in Sections 3 and 4 for addressing finite-time control of multiagent systems with first- and second-order agent dynamics.

5.1 | Example 1: First-order multiagent system

Consider a multiagent systems that consists of $N = 8$ agents having the dynamics given by (11) and exchanging information based on an undirected, connected circle graph $\mathfrak{G}_1$ as shown in Figure 3, where the first two agents have access to the position of the time-varying leader given by $x_0(t) = 2.5 + 5\sin(0.5t) + 0.5\sin(5t)$ (ie, $k_i = 1$ for $i \in \{1, 2\}$, and $k_i = 0$ for $i \in \{3, 4, \dots, 8\}$ in (22)). Furthermore, the vanishing and nonvanishing system uncertainties are selected for this numerical example, respectively, as $w_i = -0.3, i \in \{1, \dots, 8\}$, and $\rho_i(t) = \sin(2t), i \in \{1, \dots, 8\}$. Finally, we note that the initial positions of the agents are selected randomly using the `randn` function in MATLAB.

For the proposed distributed control algorithm in Section 3, we use the time transformation function given in (10) with $T = 4$ in order to enforce the finite-time convergence value equal to 4 seconds and we set $\alpha = 10$ that results in a positive definite matrix $\mathcal{S}$ in (26). Figure 4 shows the performance of the proposed distributed control algorithm in presence of vanishing and nonvanishing system uncertainties. As expected from Theorem 1, the position of each agent converges to that of the leader at the chosen user-defined finite time with bounded agent states and control signals. Moreover, for further illustrating the robustness of the proposed distributed control algorithm in (22) to different vanishing and nonvanishing system uncertainties, consider four additional uncertainty scenarios as $A : w_i = 0,\ \rho_i(t) = 0$, $B : w_i = -0.1,\ \rho_i(t) = 2\sin(4t)$, $C : w_i = -0.5,\ \rho_i(t) = -4\sin(0.5t)$, $D : w_i = 0.5,\ \rho_i(t) = -6\sin(2t)$. The initial condition of the agents are selected randomly in the interval $[14, 16]$ for scenario A, $[8, 10]$ for scenario B, $[-1, 1]$ for scenario C, and $[-6, -4]$ for scenario D. Figure 5 shows that in all cases the proposed algorithm in (22) can preserve the user-defined finite-time convergence, regardless of the initial conditions of the agents and without requiring the knowledge of the upper bounds of the system uncertainties.

The proposed algorithm can also be utilized for a priori given, user-defined finite-time convergence in higher dimensions. For demonstrating this fact in a two-dimensional case, we next consider the same multiagent system having the same vanishing and nonvanishing system uncertainties for both dimensions, where the first two agents have access to the position of the time-varying leader given by $x_0(t) = [4\sin(1.5t),\ \ 4\cos(1.5t)]^{\mathrm{T}}$. Figure 6 shows the performance of the proposed distributed control algorithm applied to both dimensions simultaneously. Once again, the position of each agent converges to that of the leader at the user-defined finite time.

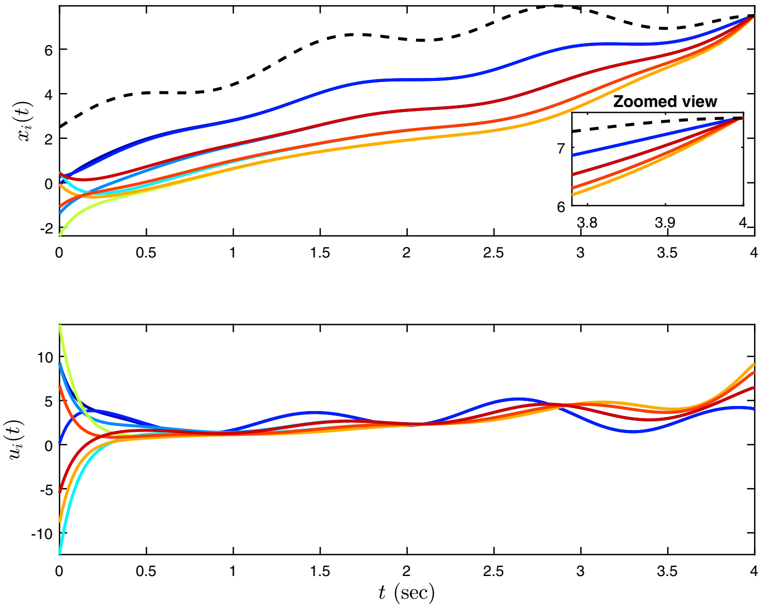

FIGURE 4 Leader-follower performance with the proposed finite-time control algorithm in (22) ($T = 4$ and $\alpha = 10$) in the presence of vanishing and nonvanishing uncertainties (dashed line shows the position of the leader and solid lines show the position of agents) [Colour figure can be viewed at wileyonlinelibrary.com]

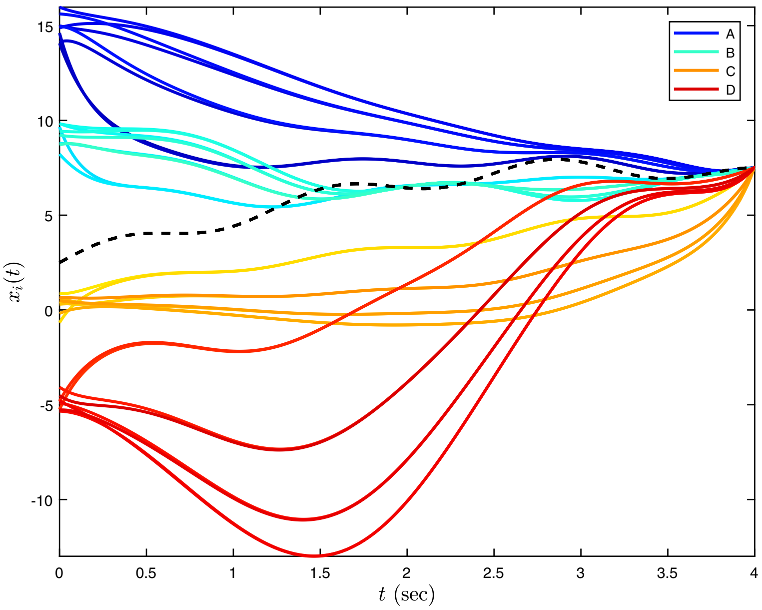

FIGURE 5 Leader-follower performance with the proposed finite-time control algorithm in (22) ($T = 4$ and $\alpha = 10$) in the presence of different system uncertainty scenarios (dashed line shows the position of the leader and solid lines show the position of agents) [Colour figure can be viewed at wileyonlinelibrary.com]

5.2 | Example 2: Second-order multiagent system

Consider a multiagent systems that consists of $N = 8$ agents having the dynamics given by (31) and (32), and exchanging information based on an undirected, connected line graph $\mathfrak{G}_2$ as shown in Figure 7, where agents 1, 5, and 6 have access to the position and velocity of the leader given by $\dot{x}_0(t) = v_0(t), \dot{v}_0(t) = -2x_0(t)$ (ie, $k_i = 1$ for $i \in \{1, 5, 6\}$, and $k_i = 0$ for the rest of the agents in (50)). We note that the initial positions and velocities of the agents are respectively, selected randomly in the intervals [0,2] and [−2,0].

For the proposed distributed control algorithm in Section 4, we use the time transformation function given in (10) with $T = 5$ in order to enforce the finite-time convergence value equal to 5 seconds and we set $k_c = 10$ and $\alpha = 15$ that results in a positive definite matrix $\mathcal{R}$ in (57). Figure 8 show the performance of the proposed distributed control algorithm in the absence of system uncertainties. As expected from Theorem 2, the position of each agent converges to that of the leader at the chosen user-defined finite time with bounded agent states and control signals.

In order to demonstrate the robustness of the proposed algorithm in (50) to system uncertainties, we now consider that the vanishing and nonvanishing system uncertainties, respectively, satisfy $\omega_i = 4$ and $\rho_i(t) = 0.5\sin(2t), i \in \{1, \ldots, 8\}$.

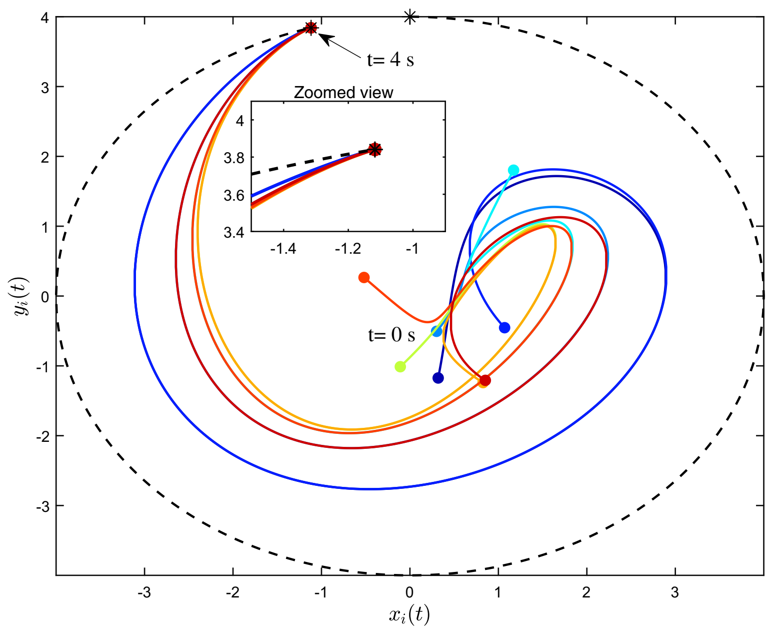

FIGURE 6 Two-dimensional leader-follower performance with the proposed finite-time control algorithm in (22) ($T = 4$ and $\alpha = 10$) in the presence of vanishing and nonvanishing uncertainties (dashed line shows the position of the leader and solid lines show the position of agents) [Colour figure can be viewed at wileyonlinelibrary.com]

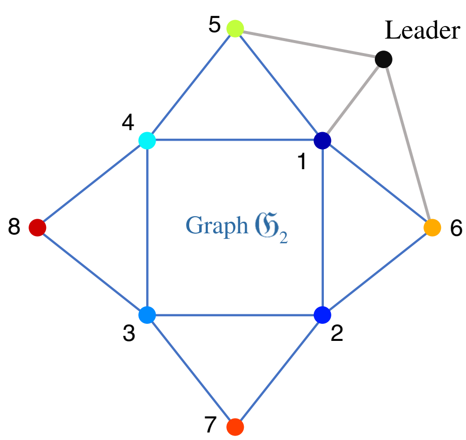

FIGURE 7 An example multiagent system on an undirected, connected line graph $\mathfrak{G}_2$ considered in Example 2 [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 9 show the performance of the proposed distributed control algorithm in the presence of system uncertainties. As expected from Theorem 3, the position of each agent once again converges to that of the leader at the chosen user-defined finite time with bounded agent states and control signals.

Finally, for illustrating the robustness of the proposed distributed control algorithm to different initial conditions and different vanishing and nonvanishing system uncertainties, consider four additional uncertainty scenarios as $A : w_i = 0,\ \rho_i(t) = 0$, $B : w_i = -0.5,\ \rho_i(t) = 2\sin(4t)$, $C : w_i = -3,\ \rho_i(t) = 3\sin(0.5t)$, $D : w_i = 1,\ \rho_i(t) = -5\sin(2t)$. Furthermore, the initial positions and velocity of the agents are selected randomly, respectively in the intervals $[6, 8]$ and $[-8, -6]$ for scenario A, $[2, 4]$ and $[-4, -2]$ for scenario B, $[-4, -2]$ and $[2, 4]$ for scenario C, and $[-8, -6]$ and $[6, 8]$ for scenario D. Figure 10 show that in all of these scenarios the proposed algorithm in (50) can preserve the user-defined finite-time convergence, regardless of the initial conditions of the agents and without requiring the knowledge of the upper bounds of the system uncertainties.

6 | CONCLUSION

This article presented a new distributed control algorithm for addressing user-defined finite-time convergence guarantees in first- and second-order multiagent systems. Specifically, the proposed generalized time transformation function converts the user-defined finite-time interval of interest to a stretched infinite-time interval, in which a distributed control

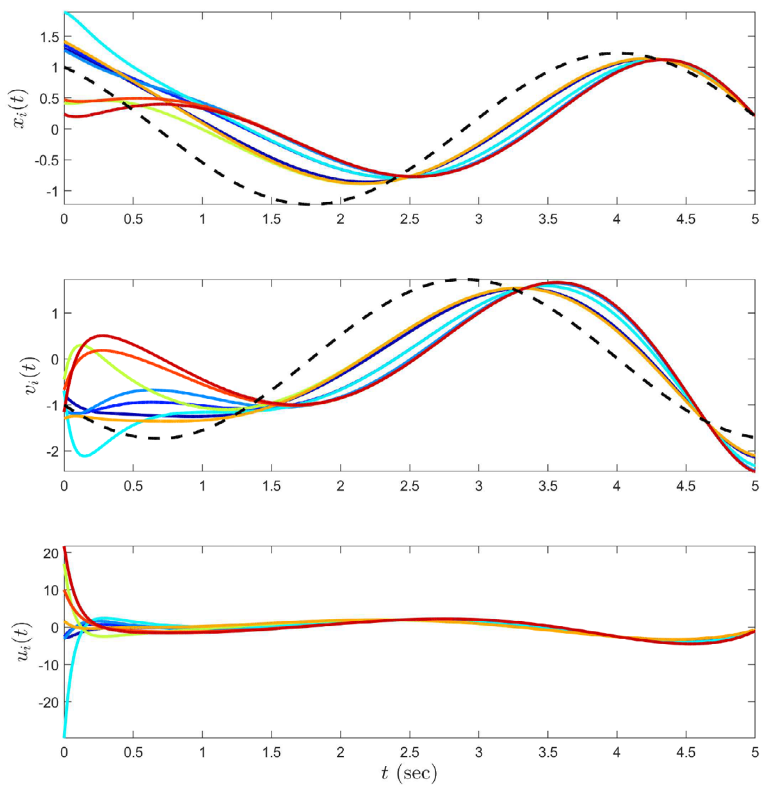

FIGURE 8 Leader-follower performance with the proposed finite-time control algorithm in (50) ($T = 5$, $k_\epsilon = 10$, and $\alpha = 15$) in the absence of system uncertainties (dashed line shows the position and velocity of the leader and solid lines show those of the agents) [Colour figure can be viewed at wileyonlinelibrary.com]

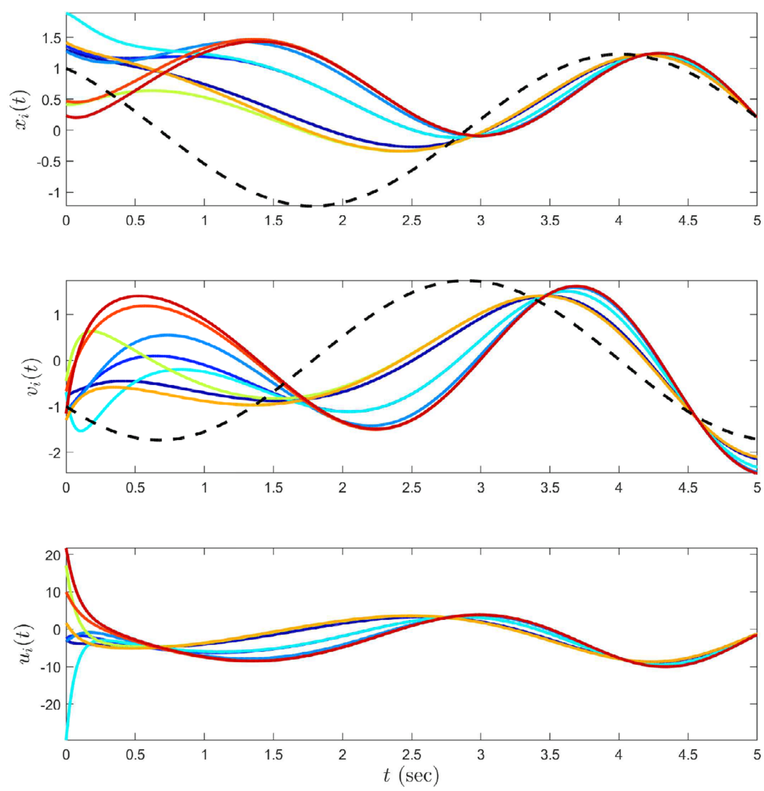

FIGURE 9 Leader-follower performance with the proposed finite-time control algorithm in (50) ($T = 5$, $k_\epsilon = 10$, and $\alpha = 15$) in the presence of vanishing and nonvanishing uncertainties (dashed line shows the position and velocity of the leader and solid lines show those of the agents) [Colour figure can be viewed at wileyonlinelibrary.com]

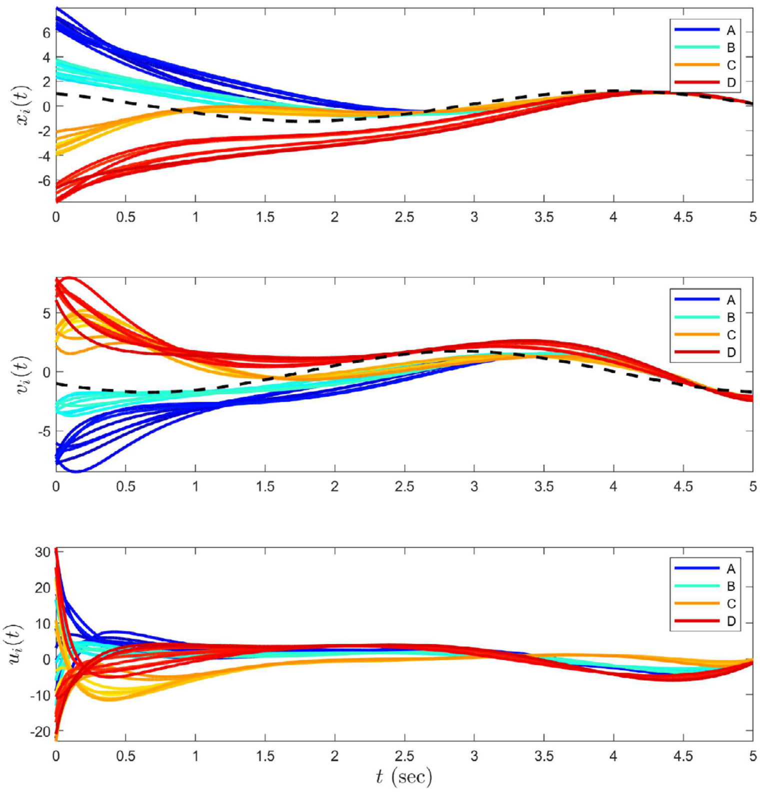

FIGURE 10 Leader-follower performance with the proposed finite-time control algorithm in (50) ($T = 5$, $k_\epsilon = 10$, and $\alpha = 15$) in the presence of different system uncertainty scenarios (dashed line shows the position and velocity of the leader and solid lines show those of the agents) [Colour figure can be viewed at wileyonlinelibrary.com]

algorithm can be designed on this stretched interval. The results were then transformed back to the original finite-time interval for achieving a given multiagent system objective. Second, robustness properties of the resulting finite-time distributed control algorithms were established analytically using a candidate time transformation function. Unlike the existing distributed controllers with finite-time convergence property, the key feature of the proposed algorithms can preserve a priori given, user-defined finite-time convergence regardless of the initial conditions of the multiagent system, the graph topology, and without requiring a knowledge of the upper bounds of the considered class of system uncertainties.

As discussed before, this article focused on control over the time interval $[0, T)$. For applications, where it is desired for the agents to continue following the leader after the time T, a potential future research direction may be to resort to switching-type control designs. To address the adverse effects of vanishing and nonvanishing uncertainties after the time T, adaptive control algorithms can be used as an example. To this end, a candidate algorithm is the set-theoretic model reference adaptive control approach in References 46-48, where it has the capability to enforce a user-defined performance guarantee without requiring the strict knowledge of the bounds on the system uncertainties.

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