Information Fusion and Resource Allocation for Accelerated Life Testing-Based System Reliability Assessment

by

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DEDICATION

I dedicate this work to my family, to my father, Mohammad Moustafa and my mother, Aliya Ibrahim, for their unconditional love, encouragement and the example of persistence. Thank you for everything. I also dedicate it to my two sisters, Malak and Rayan, for their wisdom words and for being there whenever I needed them. I further dedicate this work to all doctoral students carrying a research in Reliability Engineering.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF PROBLEM STATEMENTS

LIST OF FIGURES

NOMENCLATURE

$\mathbf{x}$	Covariate – Explanatory Variable (ALT Stress)
$\mathbf{x}_{all}$	System Stress Vector
$\mathbf{x}^{\mathbf{b}}$	Stress of Boundary Component
$\mathbf{x}^{\mathbf{b}-}$	Stress of Non-boundary Component
$\boldsymbol{\xi}$	Normalized Accelerating Factor / Stress
S_U, s_u	Upper Stress Level
S_L, s_L	Lower (nominal) Stress Level
$\mathbf{S}_i, \mathbf{s}_{\boldsymbol{i}}$	Component Stress Level
t , $\mathbf{t}$	Failure Time notation
$\mathbf{t}_i^s$	Vector of All Failure Times at All Stress Level of Component i
λ	Scale Parameter of the Weibull Distribution
ψ	Shape Parameter of the Weibull Distribution
$\boldsymbol{\alpha}(.)$	Stress Dependent Parameter
σ	Stress Independent Parameter
$\theta_0; \theta_1$	Unknown Parameters to Be Estimated
$\boldsymbol{\theta}$	Parameter Vector of $[\theta_0, \theta_1]$
$\boldsymbol{\theta}_{\mathbf{sys}}$	System Parameter Vector of $[\theta_0, \theta_1]$
$\boldsymbol{\theta}^{(i)}$	A Vector of $\theta_0; \theta_1$ of component (i)
$\sigma^{(i)}$	Stress Independent Parameter of the i^{th} Component

$\boldsymbol{\alpha}^{(i)}$	Stress Dependent Location Parameter of the i^{th} Component
N_c	Number of Components in An Engineering System
F_T	CDF of Failure Time
f_T	PDF of Failure Time
$C(:;\rho)$	Copula Function
$\boldsymbol{\rho}$	Copula Function Parameter
Pr{.}	Probability Operator
$\Phi_\rho(.)$	CDF of Standard Multivariate Normal Distribution
$\Phi_\rho^{-1}(.)$	CDF of A Standard Normal Variable
$\emptyset_\rho(.)$	PDF of Multivariate Normal Variables
x_{ij}	Normalized Stress at the j^{th} stress level
$n^{(i)}$	Number of tests at different stress levels of the i^{th} component
u_{MCS}	CDF Monte-Carlo Samples of the i^{th} Components
R_{sys}	System Reliability
R_S^{prior}	Prior Distribution of System Reliability
R_S^{post}	Posterior Distribution of System Reliability
$I\left(t_{MCS}^{(i)}\right)$	Failure Indicator Function
T_e	Failure Threshold
$X_{b^-}^i$	Normalized Stress of the Non-Boundary Component i
X_b^i	Normalized Stress of Boundary Component i
n_b	Number of Boundary Components
n_{b^-}	Number of Non-Boundary Components

$\boldsymbol{X}_{sj}$	j^{th} element of X_b-
$L_{sj}(.)$	Load Prediction Model
$\boldsymbol{\omega}^{(j)}$	Uncertainty Parameters
$\mathbf{n}_{sys}$	Number of System Level Tests at each Stress Level
$\mathbf{X}_{sys}$	Normalized System Level Stress Levels
$\mathbf{t}_{sys}$	System Failure Time
$t^{i}_{mcs,j}$	Component i Monte Carlo Simulated Failure Time at Stress Level j
$f_{time}(.)$	Component Failure Time to System Failure Time Conversion Function
$\kappa(.)$	Kernel Smoothing Function
δ	Bandwidth of The Kernel Smoothing Function
$w(k)$	Weight of Each Prior Sample
D_{kl}	Kullback-Leibler (KL) Divergence
$f_{R_0}(R)$	Prior of PDF of Rs
$f_{R_s}(.)$	Posterior PDF of Rs
$C_{ALT}\ (\mathbf{X}_{sys}, \mathbf{n}_{sys})$	Total Testing Cost for a Given Testing Plan (X_{sys}, n_{sys})
C_{total}	Total Testing Budget
e_i	Testing Cost of the i^{th} Component Per Unit Testing Time
e_{sys}	Testing Cost of The System Per Unit Testing Time
C_{sys}	Cost of a System Testing Specimen
C_i	Cost of a Testing Specimen of the i^{th} Component
$\gamma_{0,}, \gamma_1, \gamma_2$	Quadratic Baseline Hazard Function Parameters
$\boldsymbol{\gamma}$	Quadratic Baseline Hazard Function Parameters Vector of $[\gamma_{0,}, \gamma_1, \gamma_2]$

$\boldsymbol{\xi}_{sys}$	System Hazard Function Regression Coefficient
$\boldsymbol{\alpha}_{sys}$	System Regression Coefficient Vector of $[\alpha_0, \alpha_1]$
$\boldsymbol{\alpha}_i$	Component i Regression Coefficient Vector
$\boldsymbol{\beta}_{sys}$	System Regression Coefficient Vector
$\boldsymbol{\beta}_i$	Component i Regression Coefficient Vector of $[\beta_0, \beta_1]$
$\alpha_0, \alpha_1, \beta_0, \beta_1$	Regression Coefficients
υ	Variance of the Shared Frailty Factor (i.e. υ = Var(z))
z	Shared Frailty Factor
$\lambda(.)$	Hazard Function
$\Lambda(.)$	Cumulative Hazard Function
$\lambda_0(.)$	Baseline Hazard Function
U	Uniform Distribution
Gamma	Gamma Distribution
Uni	Uniform Distribution
$e^{(.)}$	Exponential Function
L_z	Laplace Transform Over z
δ_i	Censoring Indicator
L_C	Full Conditional Likelihood of parameters V
$t_{Failure}$	ALT Failure Time Data (General Notation)
m_i	Number of ALT Stress Levels of the i-th Component
VR	Variability
$Var(\cdot)$	Variance Operator

ALT	Accelerated Life Testing
EHR	Extended Hazard Regression
PH	Proportional Hazard
AFT	Accelerated Failure Time
PDF	Probability Distribution Function
CDF	Cumulative Distribution Function
SIS	Sequential Importance Sampling
PF	Particle Filter
OEM	Original Equipment Manufacturer

ABSTRACT

Accelerated life testing (ALT) has been widely used to expedite the analysis of a product's failure time when used under normal conditions and calculate its reliability. For engineering systems, ALT could be performed at two levels: component level and/or system level. Each testing level requires different resources to be performed and a specific approach to analyze the data of failure times collected in order to draw reliability conclusions. Present methodologies in ALT allow for the assessment of system reliability conclusions by testing at the system level. On the other hand, some of the available practices allows testing single components and calculate the component reliability. Both schools of thoughts do not take into account any dependence that might exist among the components of one system once put in use and the possibility of using either type of testing in order to calculate the system reliability.

In addition, each of these two levels: The component level testing and the system level testing have their own advantages and disadvantages. Systems of multiple components undergoing a system level testing could be expensive, but it takes into account the dependence of the system's components failure times. The component testing level consists of testing each component separately. Using the information collected from component level testing to assess the system reliability could be of great financial importance at the design level. being cheap and allowing testing customization. However, it does not include any of failure time correlations of components when assembled in one system.

The research aims at fusing testing information collected from component level testing and system level testing in order to draw system reliability conclusions. This research tackles the

dependence between the component failure times of a system that is caused by unobservable factors. Two novel frameworks are proposed to analyze the reliability of systems with multiple components using ALT testing. The difference between the two frameworks lies in hoe we model the dependency between the failure times of the components. We model the dependency using a Copula function in conjunction with Weibull distributions in the first framework and using shared frailty models with extended hazard model in the second. Both frameworks present a propagation of uncertainty from both testing levels: the component testing level and the system testing level to the system reliability. Firstly, the frameworks incorporate a model to calculate the system reliability using ALT component testing data, this presents a linkage method allowing uncertainty propagation by using ALT component failure time data in order to conclude the system reliability. Secondly, the latter is followed by a linkage method to show how the ALT System testing data could be used to calculate the system reliability with minimal uncertainty. Thirdly, we present the concept of information fusion a which is a method to fuse both component level testing information and system level ALT testing information (i.e. Failure Time Data) to calculate the system reliability. This research relies heavily on different statistical concepts and Bayesian inference approaches.

An optimization model that takes into consideration the cost of testing and other ALT parameters, namely stress levels and number of tests at each stress levels, is employed to find the optimal and cost-effective values of these parameters. The optimization model is applied to the framework that uses the Copula function as a way to model dependence.

A sensitivity analysis has been done to analyze the effect of the variance of the frailty factor variance on the reliability estimate. A four-arm robots and a mixed system examples are used to show the effectiveness and usefulness of the proposed Copula based ALT system reliability method and a circuit board of an autonomous vehicle is employed to demonstrate the proposed approach to estimate the system reliability using frailty model with extended hazard regression analysis method. For each example, we show results in a graphical format followed by an interpretation explaining the reduction in the uncertainty of the system reliability.

Chapter 1 Introduction

In this chapter, we provide a background about accelerated life testing (ALT), we define the term and explain its importance in reliability analysis and product design analysis. Next, we present the research objectives followed by the outline of the dissertation.

1.1 Background

Engineered products are getting more complex in terms of structure and manufacturers are competing to keep up with new inventions and technologies. Studying the life of new products that integrate new technologies is an essential component in the product development phase. The failure of a product affects the warranty terms and safety incurred costs and companies need to find ways to estimate the life of their products. Additionally, the life of a product is considered a differentiation factor and a quality indicator to many companies which drives the competition against their competitors in the market of a product. As well, the quality of a product could very well drive the pricing strategy that a manufacturer intends to follow when launching the product for sale.

Moreover, the expectation of customers placed when owning a new product is to function for as long as possible serving its purpose to the fullest. So, early failure of products not only increase the warranty cost to the manufacturer but could lead to serious loss in the customer satisfaction which in turn will affects the sales force and the company's reputation.

An old type of testing used in the industry is called the Accelerated Life Testing. The aforementioned type of testing is widely used in reliability analysis in order to study the life of products in general [1]. ALT is considered a reliability testing procedure and various methods have been developed to concentrate on the study of such type of testing with the intention of estimating the reliability via various prediction models[2].

ALT has gained enormous traction and attention to develop new observations in how to analyze the data collected from this testing and try to estimate the reliability of products and systems with minimum uncertainty and higher confidence.

In ALT, products could be individual components or systems of multi-components. ALT entails putting the product (a component or a system) at higher than use stress[2]. Use stress is defined as the stress or load that the product would experience when operating in its normal conditions. ALT testing is done in order to expedite the failure of the product reaching its maximum life.

ALT involves testing the specimen under different stress levels. At each stress level, that is often referred to as the accelerating factor, the specimen is tested multiple times. Figure 1-1 below shows the concept of the accelerated life testing used in this research where each test specimen, which could be a component or a system, is subject to different stress levels and tested n times according to the number of tests (number of test specimens) at each stress levels. So, the output of this ALT testing is experimental failure time data which is the failure times of the specimen (i) at different stress levels[3].

Another aspect of systems and products is that they impose dependence among the failure modes due to a functional and/or physical connection among the components in that system. Systems fail under different failure modes (Fracture, Corrosion, Wear etc.). That dependence, as shown in Figure 1-2, cannot be neglected or else the model could lead to a bias (i.e. overconfidence or under confidence) in the estimation of the product reliability. Dependence occurs between the different failure modes of the components of one system. This correlation among the components which produce different failure modes on the overall system adds complications to the accelerated life testing data analysis.

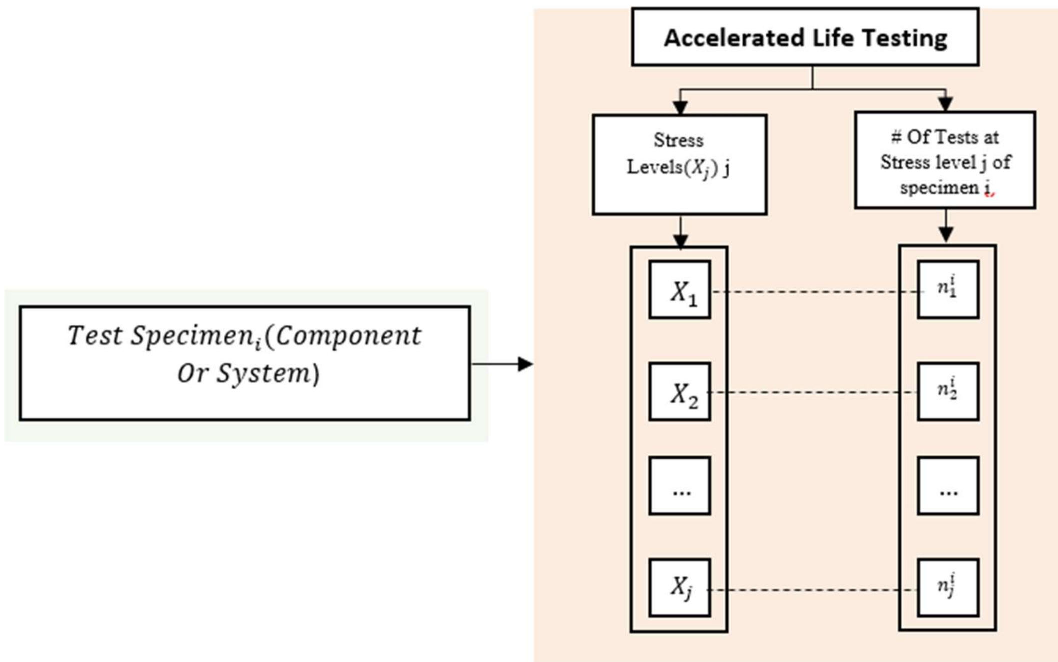


Figure 1-1 Accelerated Life Testing Concept

Systems can as well take multiple configurations; the components of a system could be mounted in different topologies making the failure of the system different for every topology. The system failure mode depends on the system topology. Components of a system could be mounted in as simple as series (Figure 1-3) and parallel (Figure 1-4) configurations or in a more complex predefined structure (Figure 1-5). Considering the structure of the system in system reliability analysis is essential as it affects its failure mode.

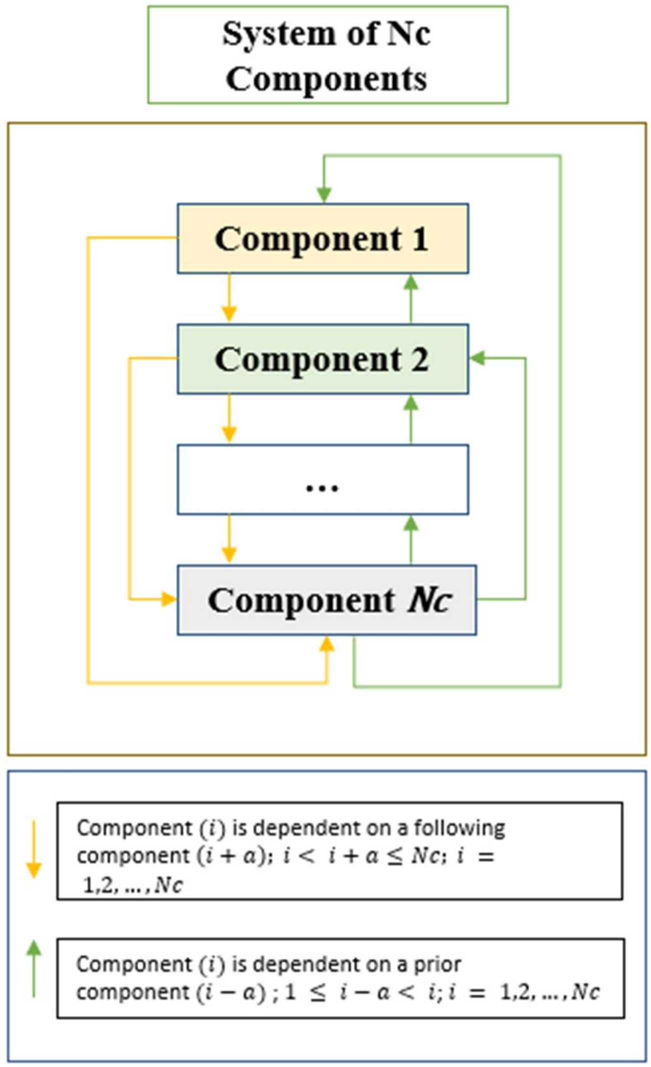


Figure 1-2 Dependence illustration among the components of a system of Nc components

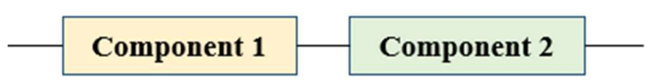


Figure 1-3 System of two components mounted in a series configuration

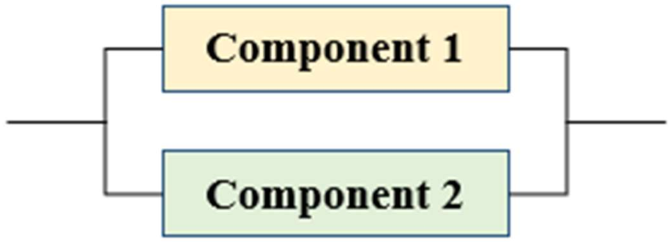


Figure 1-4 System of two components mounted in a parallel configuration

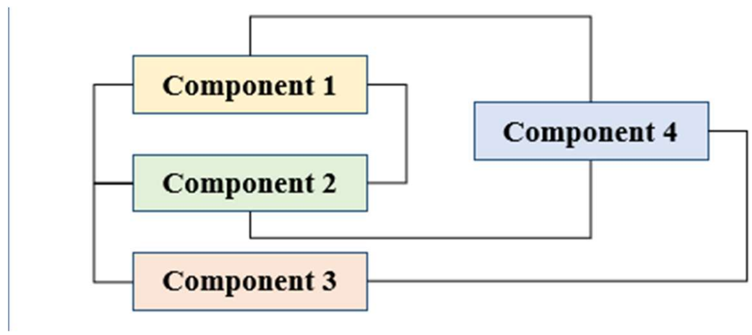


Figure 1-5 System of four components mounted in a complex configuration

A complete system reliability analysis shall consider the failure data collected from ALT. Also, it shall include the modelling of the existing dependence among the components of one system as well as it shall be applicable for the different system topographies.

ALT data could be collected at two different testing levels: component level Component Level ALT Data by testing each of the components individually and System Level ALT Data by putting the whole system at test. So, in order to use each set of testing data collected from component testing or system testing and link it to the system reliability, this research target is to pursue meeting six objectives detailed in section 1.2 below.

1.2 Research Objectives

The goal in this research is to find the reliability of systems with minimum cost and with high accuracy level. Reducing the uncertainty of a product reliability requires a robust prediction model in place. The research aims at developing a system reliability prediction model by using accelerated life testing data applicable for all system configurations in which components are configured in series, parallel, or any other specific topography decided by the product design engineers. As mentioned previously, for a system with N_C components the accelerated life testing data could be collected from two testing levels: Component -level ALT Data when testing individual components and system-level ALT Data when testing systems with multiple components. So, the goal is analyzing and modelling the collected ALT data from the aforementioned two ALT levels to minimize and estimate the system reliability with confidence.

In order to reduce the uncertainty of the system reliability six objectives are pursued. The research objectives are as follows:

The first objective is to consider the correlation between the failure time data and find a suitable method to consider the association between the failures and the components of a system. Since we are dealing with systems of multiple components connected physically or logically to serve a purpose under normal operating conditions, the dependence is an association of sharing a failure factor. Failures of components in one system can take different forms leading to multiple competing risks or failure modes, modelling the dependence of these failure modes that lead to a dependence in the failure time data is essential to remove the bias from the prediction model. This objective goes hand in hand with every objective listed below and is considered an integral part of their implementation.

The second objective is to develop a framework to estimate the system reliability by using ALT data collected from components level testing. This objective is characterized by finding a suitable linkage method to link the failure data (i.e. ALT data) of each component tested separately and to be connected in one system and integrate the dependence method in order to infer and reduce the uncertainty of the system reliability under normal operation. Mathematically, we aim at finding the likelihood and apply the Bayesian estimation method that takes prior information about the model parameters collected from experts and sample posterior distributions with reduced uncertainty then link the data of all components together to find the system reliability including the dependence among the failures of the components.

The third objective is to construct a linkage method to derive and minimize the uncertainty in the system reliability using ALT data collected from system level ALT testing. In similar fashion of objective but using a different approach in allocating the model parameter, the research aims at finding a suitable connection to fit the ALT data and reduce the uncertainty in the system reliability while modelling the dependence among the failure data.

The fourth objective of the research is referred to as information fusion which aims at combining ALT testing data from both testing levels when they are available in order to reduce the uncertainty in the system reliability. This objective is a grouping objective to combine the three prior objectives listed above.

The fifth objective is to allocate the ALT resources optimally. The research aims at finding the optimal design parameters of the mode conditioned on the testing budget and cost. This objective is pursued by developing an optimization model subject to conditions of ALT testing cost and total budget:

$$f(X,n)\ s.t.\ \ Cost\ C_{total};0\leq\xi\leq1,n\geq0 \qquad (1\text{-}1)$$

The last objective which is the sixth objective is to find a way to apply all of the above to any system of N_c components mounted in different configuration that could be components in series, parallel or other topography. This objective aims at making the methods developed versatile and not limited in application.

Pursuing the six objectives listed above allows to construct a comprehensive ALT analysis model to estimate the system reliability of product with high confidence and minimal uncertainty all while considering the dependence among the components that carries latent failure times. Finally, the model is versatile in its application to different system structures.

1.3 Dissertation Outline

The Dissertation contains eight chapters in total in order to pursue the research objectives detailed in section 1.2 above. Chapter 1 gives an overview background about the research and details the research objective.

Chapter 2 presents the literature review for the different methods and frameworks related to the accelerated life testing and are in direct relation to what is used in this research. The latter includes: 1) Statistical methods in ALT, 2) Regression survival models, 3) Dependence modelling. In each of the three sections under the State of Art chapter, we detail the different methods available. This allows identifying the gaps and helps identifying the novelty of the ALT framework enclosed in this research.

Following the literature review presented inChapter 1, the motivation of this research is given in Chapter 3 in which we talk about the advantages of this framework by presenting the intellectual merit and the broader impact of the research and its findings. The intellectual merit taps into the new observations and the novelty of the ALT model as well as the positive impact that this ALT framework could present to industries and individuals.

Chapter 4 includes the approach by which we intend to solve the problem. The chapter dissects the problem into its subcomponents and steps and describe the reason behind each of the steps we intend to follow in the following chapter. The assumptions to the model are presented, and illustrative example about autonomous vehicle leads to set the problem statement of the research. By setting the problem statement of the research which is reducing the system reliability uncertainty and the methods to solve the presented problem are given and explained:

1. Log-scale parametric distribution model
2. Copula function
3. Extended hazard model
4. Frailty model
5. Bayesian Estimation method (the concept of likelihood function)
6. Particle Filtering

Chapter 5, the previous chapters have presented and defined the terms to be used and the frameworks available to be used. This piece of the research deep dive into the mathematical formulation of propagating the uncertainty using component level testing data, then it details the steps about using the system ALT testing data and the propagation of uncertainty to the system reliability closing with the information fusion of both ALT testing data. An optimization model to find the optimal design parameters of the ALT is enclosed at the end of this chapter, this part encloses the resource allocation bounded by an ALT budget.

In Chapter 1 we identify the gaps of the method implemented in the previous chapter (Chapter 5), so the chapters suggests a novel method consisting of using distribution free regression models to explain the effect of explanatory factors on the failure time and it integrates the frailty model to explain correlation among failure times. Numerical examples are presented, and the results are interpreted to obtain the statistical inference meanings and develop investigate new observations about the model.

Chapter 1 aims to explain the effect of the frailty factor on the uncertainty of the system reliability. A sensitivity analysis is enclosed by varying the frailty factor value and the changes are presented graphically followed by an explanation of the changes. This part allows understanding the effect of using the frailty to model dependence among failure time on the system reliability confidence.

The last chapter, Chapter 7, presents the concluding remarks and proposes the future work, the major achievements achieved by conducting this research and the concluding remarks about the findings.

Chapter 2 State of Knowledge

Different models have been implemented to find the optimal design and model accelerated life testing. The advantage of accelerated life testing is to predict the life of product in an expedited format under higher than use stress that allows predicting the probability of no failure by using translation function to translate the performance of the product in an accelerated environment to the normal one. Various models have been presented in the literature and the field has been of interest for so long, some researchers presented ALT models involving parametric distribution function such that the Weibull distribution, Exponential distribution and the Lognormal distribution, along with using specific stress- failure time translation functions. Others have shown the advantage of using distribution free parameters by incorporating regression modelling in the big picture.

The different models tackle data censoring with its different types. Censoring is involved when a testing unit survives without failing at the end of the test. ALT is often timed and sometimes the testing unit survives the accelerating factor without any indication of a failure. The data for such testing units is called censored data. The most common two types of censoring are Type -I and Type-II which allows engineers to remove the non-failed units and at different times during testing.

In what follows we present some of the available work published by peers over the years. We present the models according to their overarching modelling perspective with a focus on models with statistical distributions and models with regression analysis.

2.1 Accelerated Life Testing (ALT) Statistical Models

We review in this section the different accelerated testing models available in the literature. Multiple statistical (parametric and non-parametric), regression and distribution free have been developed to estimate the system reliability using ALT data. We review some of these models and identify the gaps.

2.1.1 Accelerated Failure Time Models

An alternative method to regression models is the Accelerated Failure Time (AFT) that is classified as parametric, The method initial name is Scale-Accelerated Failure Time or SAFT [3]

The method is often characterized by linking the logarithm of the event or failure time to the stress. It assumes that the covariate factors which is the stress in the ALT case act linearly on the logarithm of the failure time or multiplicatively on the failure time. AFT is widely used in the reliability field, and the disadvantage lies in their application that necessitate finding a suitable parametric distribution [4]. Newby [5] mentions that the effect of covariates on the event time are described in the scale parameter of a parametric distribution of choice to fit the failure data.

Stute in [6] presented a methodology to estimate linear regression parameters, the method could be regarded as an accelerated failure time model as referenced in [4] and it gives the benefit of being a distribution free so it allows making inference without fitting the failure times into a probability distribution function making it a promising methodology to be used in survival analysis making it equivalent to the hazard model presented by Cox in 1972 which we will review in section 3.2.1.

AFT models have been first referenced by Pieruschka[7]. AFT models have been demonstrated robust against neglecting explanatory variables as shown by Hougaard in [8].. Xu et al [9] used the AFT model in order to investigate a general Bayesian approach for step-stress accelerated life testing is investigated for the log-location-scale distribution family and particularly the widely used parametric lifetime distributions in ALT.

Louis [10] presented a complement method to the regression method that uses hazard functions and it has the restriction of hazard rate proportionality at two different levels of the explanatory variable (covariate). The approach integrates the accelerated failure time whereas the proposed method is identified as efficient for the Weibull distribution class and does not include censored data in the formulation.

In a different approach, Kuo and Mallick [11] considered a Bayesian framework by using parametric prior information on the regression coefficients of the AFT model. They have deployed Markov chain Monte Carlo (MCMC) to sample the posterior data of the model parameters and they have concluded their work with numerical examples including censored data.

Anderson in [12] presented a non-proportional hazard Weibull accelerated failure time model where they do not use the standard Weibull AFT model with a standard linear location AFT model, instead they considered a varying location and dispersion parameters model in which the dispersion parameter is dependent on the location parameter, more information about the specifics of such model could be found in [12]. The application problem in this work is medical.

The AFT models assume that for a given covariates vector Z which is the applied stress which follows a distribution with a location parameter $\alpha(Z)$ and a constant positive scale parameter $\sigma > 0$, the logarithm of the failure time equation is linked to these parameters by the following[13]:

$$Y = \log(t) = \alpha(Z) + \sigma\varepsilon \quad (2\text{-}1)$$

ε is a random variable that has a specific distribution, and it is assumed to be independent of Z.

The AFT model has been of great use in the reliability field for estimating the life of engineered goods. However, it requires the model to be parametrized and it is often used with parametric probability distribution functions, and the linkage form is limiting as it takes the form of linear regression between the logarithm of the event time and the applied stress. Various approaches and distributions have been used to go with the accelerated failure time. We review some of them in the next section 3.1.2.

2.1.2 ALT Data Parametric Distributions

Log-location scale distributions have been extensively studied in the accelerated life testing field. Weibull, exponential and lognormal distributions have been the focus of multiple researchers. In this section we review studies available that have used the latter distributions to analyze and plan ALT.

Klein et al [14] have developed a model for a multi-component placed in series configuration in 1981. The model consists of using data collected from accelerated life testing in order to predict parameters of a function called a stress translation function. The function is then used to predict the reliability of a system when operating under normal conditions. The model fits the component failure times collected in an accelerated environment via a 2-parameters Weibull distribution. The failure times of each component are assumed to be independent from each other. Maximum likelihood has been used as the estimation method of the different parameters involved in order to conclude the reliability of the system. Van Dorp et al [15] developed a Bayes approach to model accelerated life testing with step stress and they have used the exponential distribution to fit the data collected from ALT at each stress level. They adopted a probabilistic approach that uses parametric prior distribution by assuming that information about these parameters is found by referring to expert's judgement and that the prior distributions of the model parameters preserves the ordering of the failure rates into the sampled posterior distributions.

Tang at al [16] developed a model to design accelerated life testing under k-step stress as the accelerating factor. They have presented a method to find the optimum test plans with Type I censoring, defined by removing the item within the testing stage if it fails, for two types of parametric distributions: exponential distribution and Weibull distribution. They as well use the accelerated tampered model that assumes the hazard function rate at high stress is the hazard rate function at lower stress multiplied by a modifying factor referred to as the accumulated tampered factor and determined by the each stress level (low and high) as well as it is assumed to be related to the time at which we move from a low stress level to higher level of stress. They use the concept of maximum likelihood along with the fisher matrix to estimate necessary parameters.

Wang at al [17] presented a model for a Weibull distributed failure data with a non-constant shape parameter for a constant stress ALT, it is assumed that both the shape and scale of the 2 parameters Weibull distribution are affected by the stress applied. the research uses the EM (Expectation-maximization), MLE (Maximum Likelihood Estimation) and ML (Maximum Likelihood) as estimation methods for the parameters involved. The paper as well take into consideration progressive type-II censoring defined as randomly removing some of the surviving units every time a failure is noticed during testing so if there are n testing units, one failure is noticed at time t_1, the remaining surviving units is $n-1$ at time t_1, by progressive type-II censor, their model assumes r units out of the $n-1$ units that have survived at time t_1 when the first failure has been noticed and the same concept is applied at t_2 when the second failure is noticed.

Doksum et al [18] presented the time transformed inverse Gaussian distribution model for variable stress ALT data to fit time to failure as a flexible alternative to the Weibull Distribution which is widely used to model failure times with the same shape parameters. Their model consists of a fatigue failure model in which the accumulated decay is covered by a Gaussian process considering a continuous stress increase. A failure is defined when the Gaussian process crosses certain limit. Time to failure is governed as a function of the accumulated decay where parametric functions are used to explain the effect of higher stress on the failure time and decay rate. The model presents how the decay under both normal and accelerated stresses could be found as well as the mean life under use stress.

Meeker et al [19] used log location scale distributions to model cycles to failures of components and noted that the two most used distributions are the Weibull and Lognormal distributions as special cases. The research proposed a model to predict the system reliability in the use field by using ALT data and characterization of the use field. So, they suggested a model that relate these data sets (failure time data and field data) in order to predict the life distribution of a future component operating in normal conditions. In order to estimate the model parameters, they have used the Maximum Likelihood (ML). Zhang et al [20] described the Bayesian methods Accelerated life testing and planning involving a Type II censored data from a 2 parameters Weibull distribution, where the PDF function is expressed by:

$$f(t|\eta,\beta)=\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right) \tag{2-2}$$

where η is the unknown scale parameter and β is the known shape parameter that is given.

The model allows a better planning that is based on the precision of the life distribution quantile assuming the information about the shape parameter of the Weibull distribution is known.

Miller & Nelson [21] presented a framework to obtain the optimum simple step-stress ALT plans when the failure time of the test units follows an Exponential distribution assuming they are monitored to failure without any censoring. Bai et al [22] extended the model the latter work of Miller & Nelson to include censored data.

Doksum and Hbyland [23] created models for variable-stress accelerated life testing experiments based developing a Wiener process consisting of considering fatigue failure model that includes an accumulated decay that is modelled by a continuous Gaussian process with a variable distribution that changes with the stress change point instead of using the widely used distribution which is segmented Weibull-distributions for the failure time of the ALT experimental units at increased stress at stress change points. Chaloner and Larntz [24] studied the design of accelerated life testing (ALT) assuming two distribution models for the failure times of the experimental units, namely lognormal distributions and Weibull distributions. In their paper, they assume that the increased level of the testing stress has an upper limit and they consider Type I censoring which is based on assuming that the experiment is timed over a fixed period of time and samples that do not fail upon terminating the test are referred to as censored.

Bagdonavicius et al tackled [25] special plans for the ALT design and analyzed the ALT data (i.e. failure times of experimental samples) using numerical methods and simulation using the changing shape and scale model which is a natural extension to the known Accelerated

Failure Time model (AFT), they also propose parametric and semiparametric estimation procedures for their model.

Additionally, J. Rene et al [26] developed a general Bayes Weibull inference model for ALT in which they assumed that failure times follows a Weibull distribution under a constant stress level. They have used prior information to define prior distribution for the scale parameters at different stress levels and the shape parameter. Additionally, a general Bayesian exponential inference model for accelerated life testing has been established. I-Chen Lee [27] et al has presented a method to overcome the problem of guessing values for the parameters to be estimated by introducing ta sequential Bayesian design for planning ALT. Zhang and Meeker [28] described Bayesian methods for accelerated life testing planning assuming one accelerating variable and that the acceleration model is linear in the parameter based on censored data.

Statistical methods in ALT are often combined with other methods like the Inverse Power law and the Arrhenius Relationship. In the sections below, we present available models developed that uses these methods in analyzing ALT data.

Nelson and Kielpinski [29] have presented the optimal ALT design using normal and lognormal life distributions. The model incorporates the Arrhenius relationship and they assumed that the mean of the life distribution is a function of the applied stress.

In this research, we will make use of the log-scale distribution because as it has been noted in the state of art, it is a widely used distribution and in more specific we will use the Weibull distribution with 2 parameters shape and scale and apply the AFT concept to link the accelerated factor to the failure of components and systems. More on the construction of the mathematical and statistical form could be found in Chapter 6.

2.1.3 Competing Risks in ALT Design

Nelson [30] has introduced examples of product that fails under various failure modes and discussed the ALT analysis of such products with multiple competing risks. Examples of system with multiple failure modes are the semiconductor devices and insulation systems.

Kim and Bai [31] have studied the accelerated life testing with two failure modes. They have presented a paper in which they have showed a framework to calculate the life distribution of a component at use stress when there are two failure modes: an extrinsic failure mode and an intrinsic one by using constant stress accelerated life testing data. They have used a location-scale distribution to model failure times, and they have derived the equation for a 2-parameters Weibull distribution. In their analysis they assumed that the lifetimes of the test units are independent, and that the location parameter of the life time distribution is a function of the applied stress and it is given by:

$$\mu_{jk} = \alpha_{0k} + \alpha_{1k} sj \tag{2-3}$$

where s_j designate the stress, α_{0k} and α_{1k} are coefficient parameters and k is the index that represents the failure modes $k = \{1,2\}$ because they are considering two failure modes. They also use EM methods to estimate the model's parameters by using the likelihood as a mixture pdf which is represented as the sum of two portion where each portion represents the pdf of a failure mode.

Patra et al [32] constructed a multivariate distribution of a mix of Weibull distributions and they characterize the dependence among the components by a latent random variable that is assumed to be independently distributed of the original component. They have extended the model and showed how it could be applied to model competing risks which takes place when a component potentially could fail under different failure modes and each competing risk is a mixture of either Weibull or exponential distributions. So, for r competing risks where each is a mix of k Weibull distributions, they define the following:

$$T = \{\mathbf{X}_1, \mathbf{X}_2, \dots., \mathbf{X}_r, Z\} \tag{2-4}$$

where $\mathbf{X}_i = X_{i1}, \dots, X_{ik}$, $i = 1,2, \dots, r$ is a mixture of Weibull (α, θ_{ij}) and Z is the independent latent random variable that follows a Weibull distribution $Z \sim Weibull\ (\alpha, \theta_0)$. Next, they define the survival probability as:

$$P(T > t) = P(\mathbf{X}_1 > t)P(\mathbf{X}_2 > t) \dots. P(\mathbf{X}_r > t)P(Z > t)$$

$$= \prod_{i=1}^{r} \{\sum_{j=1}^{k} a_{ij} e^{-t^{\alpha\left(\left(\frac{1}{\theta_{ij}}\right)+\left(\frac{1}{r\theta_0}\right)\right)}} = \prod_{i=1}^{r} P(T_i > t) \tag{2-5}$$

where a_{ij} is the mixing probability and $\sum_{j=1}^{k} a_{ij} = 1$ for all $i = 1,2, \dots. r.$ and by that the multivariate joint survival function is given by:

$$\mathcal{L}_T(t) = \prod_{i=1}^{r} S_{T_i}(t) \tag{2-6}$$

Ishioka et al [33] worked on constructing the maximum likelihood parameters of a Weibull distribution for two components forming a series system. They assumed the mean of the Weibull distribution of the each of the independent failure mode to be a log-linear function of the applied stress. The likelihood of sample size N for two components including censored data is given by:

$$L(\theta) = C \prod_{i=1}^{N} \{f(t_i)\}^{\delta_i} \{R(t_i)\}^{1-\delta_i} \tag{2-7}$$

where $f(.)$ Is the pdf and $R(.)$ is the survival function. C is a constant and δ_i is = 1 if the system fails at the end of the ALT test and 0 otherwise. In their framework they assume that regardless of the system configuration of the 2 components $R(t) = R_1(t)R_2(t)$ and that the total PDF is $f(t) = R_1(t)f_1(t) + R_2(t)f_2(t)$. They apply the log-likelihood and by maximizing the derivative of the$Log(L(\theta))$ they find the closed form of the distribution parameters.

Bai and Chun [34] constructed a model to find the optimum simple step-stress accelerated life test data with competing causes of failure (i.e. competing risks) where the life distribution of each failure cause is assumed to follow an exponential distribution and the failure modes are assumed to be independent of each other. Their framework assumed that each unit has p statistically independent failure modes and that the failure of the unit the smallest of the p failure times corresponding to the p potential failure modes. The log of the mean life of p failure times is a function of the applied stress. The derived likelihood for two stress levels from observations (y_{ij}, c_{ij}) representing the (failure time, cause of failure) of the test unit j at stress $x_i, i = 1,2 \; and \; j = 1,2, \dots, n_i$. The likelihood includes n_η censored data and is given by the equation below:

$$L(\alpha_1, \beta_1, \dots, \alpha_p, \beta_p) = \prod_{j=1}^{n=1} \left[\prod_{k=1}^{p} \{\lambda_{1k} \exp(-\lambda_1 . y_{1j})\}^{\delta_k(c_{1j})} \right]$$
$$\times \prod_{j=1}^{n=2} \left[\prod_{k=1}^{p} \{\lambda_{2k}\, exp(-\lambda_2 . y_{2j})\}^{\delta_k(c_{2j})} \right] \qquad (2\text{-}8)$$
$$\times \exp(-\lambda_2 . (T - \tau) n_\eta - \lambda_1 . \tau \, n_\eta)$$

where $\lambda_i = \sum_{k=1}^{p} \lambda_{ik}$ is the failure rate under the failure mode k at stress x_i and τ is the test run time and $\alpha_1, \beta_1, \dots, \alpha_p, \beta_p$ are the parameters to be inferred.

Pascual Francis [35]presents method for accelerated life testing planning under k independent failure modes. Pascual used the lognormal distribution to derive the Fisher Matrix. The method is established in conjunction with the Arrhenius relationship for temperature acceleration. On the other hand, Pascual in [36] derived a method for ALT planning for competing risks when failure times are assumed to follow a Weibull distribution. the framework assumed s −independent competing risks and the minimum latent failure time corresponding to a failure mode or competing risk is assumed to be the minimum. Klein and Basu presented worked on series of papers [37] [38] in which they have presented the analysis of ALT involving more than one failure mode. they further assumed that the competing risks or failure modes are independent for each stress level, they have used the maximum likelihood estimators when the lifetimes follow Weibull and exponential distributions and they have considered the case of having a common versus varying shape parameters, as well as 3 types of censored data being Type I, Type II and progressive censoring.

Additionally, Bunea and Mazzuchi [39] presented a Bayesian method for the analysis of ALT data with possible multiple failure modes. They have used failure rates following a Gamma distribution and the Arrhenius relationship to relate the failure rate due to a stress level to the actual failure rate under use stress.

2.1.4 Inverse Power Law in ALT

Inverse Power Law is often used in combination with the statistical models in accelerated life testing to relate the applied accelerated variable to the life of the testing unit and it has the scale-accelerated failure time form[3]. Nelson [40] developed a model using the maximum likelihood estimators (MLEs) to obtain the parameters of a Weibull distribution in combination with the inverse power law using the breakdown time data of electrical insulation.

The inverse power describes the relationship between the constant V stress and the life of the test specimen that is assumed to follow a Weibull distribution with a constant shape parameter β and the scale parameter α takes the following form [40]:

$$\alpha(V) = \left(\frac{V_0}{V}\right)^p \tag{2-9}$$

where V_0 and p are positive parameters.

The latter implies that the CDF of the failure time t of a specimen is expressed by the following equation [40]:

$$F(t; V) = 1 - \exp\left(-\left[t\left(\frac{V}{V_0}\right)^p\right]^\beta\right) \tag{2-10}$$

Allegri and Zhang [41] aimed to develop a model to provide an estimation tool of the relative accumulation of fatigue damage under random loading conditions and their work has addressed the usage of the inverse power law in accelerated fatigue testing. Escobar and Meeker [3] in their review paper explained the usefulness of the Inverse Power Law is describing the effect of some accelerating variables like voltage and pressure on the failure times of testing units. The Inverse Power Model is an empirical model and has been used because engineers emphasized its power in analyzing ALT data. Caruso et al [42] in an overview of the fundamental ALT methods lists the Inverse Power Relationship and the Arrhenius relationship, which we will review next, and describes the versatility of the different forms of these relationships in describing the relationship between the accelerating stress and the life of the testing unit.

Another representation of the inverse power relationship for the characteristic life $\eta(V)$ is given by [43]:

$$\eta(V) = 1/KV^n \tag{2-11}$$

For a general notation, V in Equation (2-11) denotes the accelerated variant or stress. K and n are referred to as characteristic parameters determined based on the material and the test procedure or method used to perform the ALT. Given the lifetime follows a Weibull distribution with shape parameter β the PDF function is given by:

$$f(t;V) = \frac{\beta}{\eta(V)}\left(\frac{t}{\eta(V)}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\eta(V)}\right)^{\beta}\right] \tag{2-12}$$

2.1.5 Arrhenius Relationship in ALT

On the same previous note of the Inverse Power Law above as a linkage method used with statistical distribution to link the failure time to the accelerating variable in ALT data analysis, the Arrhenius relationship has gained attention and has been extensively used in different researches where distribution parameters are a function of temperature as the accelerating variable. Nelson [44] present a three-part series describing statistical methods to model temperature accelerated life testing data by assuming all testing units are tested to failure. In the first part, Nelson described the Arrhenius method in combination with graphical methods to solve the problem of ALT when the accelerating factor is the temperature and highlighted that the same method could be applicable to different ALT when the accelerating variable is not necessarily temperature. His model is designed for single failure modes. The Arrhenius model has been found useful to describe the life of a component when temperature is the accelerating variable. It suggests that the mean of the lifetime distribution is a function of temperature and that the standard deviation is constant. However, the Arrhenius model does not tackle the dependence among the failure time distributions and temperature due to multiple failure modes and suggests that each failure mode could be represented by a separate Arrhenius model, but multiple failure modes is not allowed during ALT.

The general Arrhenius reaction model is given by the following equation [43] for when the accelerating variable is the temperature:

$$\upsilon = Aexp\,(-\frac{B}{T}) \tag{2-13}$$

υ is the rate to failure, the speed of a reaction, T is the absolute temperature in Kelvin units. A is a non- thermal constant factor whereas $B = \frac{E_A}{K} = \frac{Activation\ Energy}{\text{Boltzmann's constant}}$.

The use life and accelerated life relationship at nominal stress under use conditions and the accelerated conditions could be found in [43] by:

$$L_{use\ stress} = L_{ALT}\text{exp}\ [E_A/K\left(\frac{1}{T_{use}} - \frac{1}{T_{ALT}}\right) \tag{2-14}$$

in which $L_{use\ Stress}$ is the life at use temperature, L_{ALT} denoted life under ALT conditions (i.e. accelerated conditions), E_A is the activation energy, K is the Boltzmann's constant $K = 8.623\ x10^{-5}\frac{eV}{K}$. T_{use} and T_{ALT} are respectively the use and accelerated temperature.

Pascual [45] developed s-independent Weibull-Arrhenius competing risk model for accelerated life test (ALT) planning involving multiple failure modes dependent on one accelerating factor. The failure modes are assumed to have an unobservable failure times and that the minimum represents the product. The latent failure times of these failure modes are assumed to follow a Weibull distribution with a known common shape parameter. ML methods are used to obtain the planning values for the model parameters. Given $emp\ (Kelvin) = temp\ (Celsius) + 273^{o}$; the Arrhenius relationship for the location parameter when the failure time follows a Weibull distribution:

$$\mu(temp\ K) = \gamma_0 + \gamma_1 \frac{11605}{temp(K)} \tag{2-15}$$

The standardization of the experiment conditions is often used to generalize the test planning model [45], for a given accelerating factor s, with upper stress s_u and a lower stress s_L, by:

$$X = \frac{s - s_u}{s - s_L}\ ; 0 < X < 1 \tag{2-16}$$

2.1.6 Eyring Relationship in ALT

The Eyring model is used for cases using the temperature as the accelerating variable as the Arrhenius relationship. It is derived from quantum mechanics, however it is not as common as the Arrhenius relationship [43]. The relationship below represents the mean life as it related to the temperature:

$$L_{mean} = \frac{1}{T} \exp\left\{-A - \left(\frac{B}{T}\right)\right\} \tag{2-17}$$

L_{mean} is the mean life, A and B are parameters to be determined by ALT Testing and T is the temperature. For an exponential distribution, the mean life under use stress is given by the following equation:

$$L_{mean,use} = L_{mean,ALT} \left(\frac{T_{ALT}}{T_{use}}\right) \exp\left\{B\left[\left(\frac{1}{T_{use}}\right) - \frac{1}{T_{ALT}}\right]\right\} \tag{2-18}$$

2.2 Regression Survival Data Models

Another class of research analyzed survival data by applying different regression model which would explain the effect of the applied stress on the failure time or they make use of the accelerated failure method. Some models integrate log-scale statistical distributions with regression models, models like proportional hazard model, accelerated failure model and the extended hazard regression model. ALT data are considered survival data as it tests the life of products. This type of models has gained giant traction in the medical field and recently it was imported to the reliability analysis world. In this section, we review some of these researches and their findings.

2.2.1 Cox-Proportional Hazard Regression Models

Nelson [46] offered a detailed analysis of methods about regression models used in accelerated testing to analyze ALT data. Survival data could be very well fitted into statistical distribution functions as shown in section 2.1 or could be represented in terms of hazard rate functions. Cox regression model presented in 1972 [47] is one of the very used models with different variations to describe the effect of covariates (i.e. factors) on the event time that could be failure time. The hazard rate of an event time t with covariate **X** is given by:

$$\lambda(t; \mathbf{X}) = \lim_{\Delta t \to 0} P(t \leq T \leq t + \Delta t | T \geq t, \mathbf{X}) / \Delta t \tag{2-19}$$

and the Cox regression model is given by:

$$\lambda(t; \mathbf{X}) = \lambda_0(t) e^{\mathbf{X}^{\mathbf{T}}.\boldsymbol{\beta}} \tag{2-20}$$

where $\lambda_0(t)$ is defined as the baseline unspecified function which is the hazard rate when $\mathbf{X} = 0$ and $\boldsymbol{\beta}$ is a regression coefficient vector acting multiplicatively on the covariates **X**.

Ata et al studied the assumptions of that model and they showed how it can be applied to analyze lung cancer survival data [48]. Cox model is based on the main assumption of hazard rate proportionality where the hazard rate ratio at any two-event time t is constant under two covariates levels.

Cox-proportional model could take a parametric form where the baseline hazard rate function could be chosen one of a Weibull distribution as shown in [49] and [50] where θ is a positive shape parameter and σ a positive scale parameter:

$$\lambda_0(t) = \frac{\theta}{\sigma}\left(\frac{t}{\sigma}\right)^{\theta-1} \qquad (2\text{-}21)$$

Breslow in 1974 presented the application of regression models with censored data using [51] among the regression models, he made use of the non-parametric cox proportional model. Breslow is 1975 presented a method to estimate the baseline hazard function known as Breslow's estimator [52] which is regarded as a step function. Different methods have been used to estimate the parameters of the Cox regression model like Bayesian inference methods using prior information as presented in [49] or like the marginal likelihood function estimation method has been used by Kalbfleisch and Prentice [53] to obtain the cox proportional hazard model parameters. Other researchers like Anderson et al [54]used piece wise smooth estimate of the baseline hazard function where he assumes that $\lambda_0(t)$ is a quadratic spline function. Campolieti [55] proposed a Bayesian framework used to estimate and smooth the baseline hazard in a discrete time hazard model.

ElSayed et al [56]applied the proportional hazard model to find the optimal ALT design of a selection of constant stresses including Type-I censoring. They have assumed the baseline hazard to be linear with time and obtained the maximum likelihood estimates of the regression coefficient as well as the baseline hazard function parameters. The reliability function is given by:

$$R(t;\mathbf{X}) = \exp\left(-\left(\gamma_0 t + \gamma_1 \frac{t^2}{2}\right)\right) e^{\boldsymbol{\beta}^{\mathbf{T}}.\mathbf{X}} \tag{2-22}$$

ElSayed and Zhang [57]presented an approach in which they used the proportional hazard model to optimize the accelerated life testing design with multiple stress levels. The optimal stress levels are obtained based on the condition that reduces the variance in the reliability estimation over a specified period of time.

Hu et al [58] obtained the upper confidence bound of the cumulative failure probability of a unit under operation stress under use stress by using a non-parametric proportional hazard model and step stress ALT data. Furthermore, The cox proportional hazard model has been given a preference for being non-parametric in the sense that the baseline hazard function does not have to take a parametric form and hence the reliability function distribution is not necessary in order to explain the effect of covariates (i.e. explanatory variables) on the event time[4].

As well, Newby [5] they demonstrated in a comparison study between AFT models and proportional hazard model that when a Weibull distribution is picked, the distinction between the two methods is masked and cannot be distinguished due to the similarity in the model equation. Also, it has been noted in [59] that the advantage of the PH over AFT lies in being able to derive the partial likelihood to estimate the relative risk function which describes the effect of covariates on the failure time in a hazard function form without parametrizing the baseline hazard function unlike AFT models which is considered a valuable aspect of PH models if one is interested in the quantification of the effects of the covariates on the failure time.

As a conclusion of this section, one can notice that PH model has gained great attention and traction in applying the model to survival analysis and is observed as an alternative to the AFT model. However, the assumption of proportionality of the hazard rates of two event times at two levels of the covariates was limiting and has pushed the researchers to look into different models and estimation approaches to cover the possibility of having this assumption not satisfied.

2.2.2 Extended Hazard Models

A class of regression models known by the extended hazard models introduced by Amoli and Ciampi in [60]after testing the PH and AFT and their application to survival data in[61]. The model is versatile and takes the proportional hazard model and the accelerated failure time model as special cases. Authors in [60] approximated the baseline hazard function using a spline quadratic function and the maximum likelihood is used to approximate the regression parameters of the model. the importance of combining AFT and PH models in one is in the possibility of covering a large gamut of applications. the framework is useful to analyze survival data including censored data.

Other researchers have modified the EHR model like Shyur et al in [62] presented a new framework that modifies the EHR model using the partial likelihood function. The approach is developed to analyze failure data and takes into consideration time-dependent covariates. They also suggest that the proposed method is easily adopted to come up with ALT plans with varying stress loadings (Step Stress, Cyclic etc.). The model has been verified using real testing data collected from lab testing of units where the specimens are subject to a time dependent load as the accelerating stress. Then, the data collected is used in a comparative analysis between the model result and the lab data, the latter verification approach is regarded as unique to the development of reliability framework.

ElSayed et al in [63] studied the extensions of AFT and PH models in order to extended the EHR model to an Extended Linear Regression Model to provide a new framework for reliability estimation. The model is regarded as an extension to both the Proportional Linear Hazard (PHL) and EHR model. the PHL is an extension of the Cox model presented first by Hastie and Tibshirani in 1993 [64]in which the regression coefficients are allowed to vary with other variables factor (i.e. time) but the effect of the covariates is kept a linear effect. The PHL model for a single covariate where the regression coefficient $\beta = \beta_0 + \beta_1 t$ is made dependent on time and is expressed by:

$$\lambda(t;\mathbf{X}) = \lambda_0(t)\exp((\beta_0 + \beta_1 t)\mathbf{X}) \tag{2-23}$$

The ELHR model includes the reflection of three effects as follows: the proportional-hazards (PH)effect, the time-scale changing effect and last but not least the time-varying coefficients effect of the PHL. The baseline hazard function in this research is assumed a quadratic function. Researcher as well considered censored data in the model and when collecting testing data in the lab for model verification.

Neto presented an EHR model in which the spread parameters is dependent on the covariates, more details about this models could be found in [65]. The model is developed for application to the reliability analysis and survival data. Seng et al [66] presented a semiparametric form of the extended hazard model and they have obtained the estimation equation of the regression parameters using counting processes and martingale techniques. The model has been tested on medical data.

We can conclude from this review that the model presented great benefits to cover different aspects in reliability and survival analysis as it combines the assumptions of the PH and AFT models and expresses them as special cases. The model uses the hazard function form and it allows, with its extensions, studying the effect of time- dependent regression coefficients and the effect of covariates on event times. However, the application of this model in the reliability field is very limited. Conversely, some researchers applied the EHR model in the medical field to study survival data.

The attractiveness of the EHR model is being a distribution free model and using the quadratic form for the baseline hazard allows to have different distributions as special cases which will then covers a wide spectrum of data types and distribution. More on the formulation of this model and how we will apply it to ALT data will be explained in Chapter 7.

2.3 Dependence Modelling

In this section, we focus on the dependence modelling using the copula function. We review the state of art and list the work for modelling competing risks via copula function. Then, we review another dependence model which has been widely used in the medical field known as frailty models.

2.3.1 Competing Risks Via Copula Function

The interaction of failure modes between two-components system was first presented by [67] and since then multiple studies have tackled the idea of interaction between the failures, referred to as competing risks or failure modes, of a system or a component. In this section, we review some of these findings in what follows.

Nelson in his book [68] defines the Copulas as functions that takes multivariate distribution functions as inputs and joins them to their one-dimensional margins. Studying Copulas and their usefulness in statistics is fairly new and it is a growing field of study. Nelsen presented in this book the different characteristics of various Copula functions and their applicability to study dependence.

Zheng and Klein [69] talked about the difficulty of presenting the net survival function in engineering or in any other field in a competing risk framework, because if T represents the time to failure of an equipment, it is often difficult or even impossible to measure T because of the occurrence of another event at time T'. They also presented a Copula graphic estimator framework to estimate marginal distribution using Copulas to model dependence between censoring and survival times (like the failure X at time T and event Y at time T'). They used the Copula function as a nonparametric function allowing to detect dependence between two random variables. The Copula includes all information which joins the two marginal distributions of the two dependent events X and Y together to give their joint distribution.

Schweider and Skar in their book Probabilistic metric spaces [70] defined the Copula mathematically as follows:

$$C(y_1, y_2) = H\{F^{-1}(y_1), G^{-1}(y_2)\} \ ; y_i \in [0,1] \ (i = 1,2) \tag{2-24}$$

In Equation (2-24), H is the joint distribution of two events X and Y, F is the marginal distribution of X and G is the marginal distribution of Y.

Lo and Wilke [71] extended the model presented by Zhen and Klein in [69] to model the dependence between more than two competing risks using the Archimedean Copula. Rivest et al [72] assessed the proposal in [69] and constructed a martingale framework for the survival function with dependent censored times and derived a closed form expression for the copula-graphic estimator assuming the joint survival function using Archimedean Copula. Genest and Rivest [73] constructed a one dimensional empirical distribution function for two events X and Y and they have assessed the dependence between two variables X and Y for an Archimedean Copula.

Gu et al [74] constructed a reliability framework for systems by establishing a life distribution model based on a correlation analysis of the failure modes of components of a system in conjunction with Copula function. The Copula's parameters are obtained using the maximum likelihood technique. They applied the model to a crank and connecting rod mechanism of a diesel engine where the dependence among the failure modes of the same component as well as the dependence among failure modes of other components are considered to estimate the reliability of the system. The framework presents a calculation procedure of the reliability using Copula function to model dependence between failure modes.

Zu and Lu in [75] aimed at estimating the system reliability of structural systems by considering the dependence among failure modes using the Copula function. They formulated the problem based on quantitative method and assumed the system is a series components system. In order to model the dependence among failure modes, they have proposed 4 copula functions, namely: Gaussian, Clayton, Gumbel, and Frank copula. They have made use of the method of moments in order to compute the reliability of a component and estimate its parameters.

In a different scope, Peng et al [76] proposed a failure rate model that captures the dependence among the failure modes of the components. Failure rate is assumed to play a central role in systems maintainability analysis. They further analyze the influence of the maintenance on the failure rate. Limbourg et al [77] modelled spatial dependencies (i.e. physical location of a component in a system) by considering two system layouts and they have presented a framework that uses the Copula function as a mean to model dependency between failure modes of components based on their physical location in a system.

On the other hand, the power of copula modelling dependence among competing risks has been demonstrated in the work of Carrière [78]. In their work applied the concept of Copula to capture the dependence among competing risks in the medical field. The work shows how the survival probabilities could be calculated by solving a set of differential equations and that how dependence is modelled via Copula function.

2.3.2 Failure Time Dependence Via Frailty Models

In survival studies, Frailty models are widely applied in the medical field to study randomness among individuals in clusters. Different researchers have tackled the frailty and various models have introduced where some are parametric, and others are arbitrary. As well, different estimation methods have been used to obtain an estimate of the parameters of the model. The Frailty modelling has been shown powerful explaining dependence among event times and many authors hinted to its usefulness in reliability analysis, however based on our research its use and study is still very limited. Next, we review notable researches and models of frailty models.

The data in survival analysis is often a multivariate or clustered failure time data [79]. It was of great interest to develop a framework or a method to model the correlation among observations sharing certain factors appropriately. A commonly used method is the frailty that was named first after Vaupel et al [80], in general, if an individual is sought to be frailer in a population, it is more likely to die before the less frail individuals.

Clayton [81] in has studied the Frailty models for bivariate data, they applied the Cox PH model and added the modelling of association among observation via frailty.

It has been shown that frailty could be an extension of the PH hazard regression model. This modification in the model increases the precision of the study as it accounts for observable factors by the covariates and unobservable factors by assigning a frailty factor for each cluster of associated event data, such a model takes the following form provided in [82] where frailty is defined as unobserved covariates or variables affecting the event time:

$$\lambda(t; x_{ij}, z_j) = \lambda_0(t) \exp(\mathbf{x}_{ij}\boldsymbol{\beta} + z_j) \tag{2-25}$$

where z_j is defined by log of the frailty factor $(z_j = \log(w_j))$, $\boldsymbol{\beta}$ represents the relative risk factor for the variables $\mathbf{x}_{ij}$ and $\lambda_0(t)$ is the baseline hazard function when $\mathbf{x}_{ij} = 0$.

It assumed that the frailty factor follows a parametric distribution as in [83] and [84] in which the association or frailty is allowed to be negative and following a parametric distribution and the framework is assumed to work for censored data.

Multiple researchers have done extensive research to expand it to multivariate cases like in [85] for recurrent data and like Klein et al in[86] constructed a framework for multivariate data with censoring that includes correlated data using a normal distribution data.

Sidhu in [87] talks about frailty and how it is used in medical studies. They say that the individuals at test are usually clustered into groups, the cluster groups are seen to have a common factor associated with it. Survival analysis is concerned with events time. Event times in the medical field could be death or healing time. Examples of these clusters include event times (i.e. death) of individuals life suffering from a disease and exposed to the same environment or event times (i.e. healing time) of individuals receiving some sort of treatment in similar conditions. It is thought that the latter is a good reason to group these event times together.

Lambert et al [88] presented a parametric accelerated failure time model with random effects, they used the frailty to model the randomness factors of survival data. Randomness or random effects are often referred to as frailty components, to use the model, hazard functions are widely used and often fitted via parametric forms by adapting a probability distribution function, however there are situations where the PDF is limiting or not concise, so their framework allowed a mixture hazard model which permits different forms of the hazard function.

Balakrishnan1 et al [89] presented a generalized gamma distribution model. They assumed that the frailty factor follows a gamma distribution and they include other parametric distribution like Weibull and Lognormal as special cases. They suggested the maximum likelihood method to estimate the parameters of the model.

Shared Frailty Models tackles multivariate cases and includes the randomness in the reaction to the applied load is characterized by the frailty factor explaining the unobservable factors affecting the failure. The randomness is the dependence among event times and other unobservable factors [90].

Liu in [91] presented a framework applying the frailty concept to model the dependence among competing risks of a system and fitted ALT data to the model in order to find the optimal ALT plan. The frailty is assumed parametric and following a Gamma distribution. Maximum Likelihood technique has been used to estimate the model parameters.

If the frailty factor $\boldsymbol{z}$ is shared among all latent lifetime, the model is called shared Frailty model. The value of $\boldsymbol{z}$ is constant over time and is assumed common to the all components in a system. In other words, all components share the same frailty which is responsible for their dependence[92].

For all $j \in \mathbf{K}$, the cumulative hazard functions $\Lambda(t^{(j)})$ share an unobservable frailty factor z, conditional on the the frailty factor z, the latent lifetimes $[T^{(1)}, T^{(2)}, \dots, T^{(k)}]$ are assumed independent and the generalized joint survival function $S(.)$is given by[91]:

$$S\left(t^{(1)}, t^{(2)}, \dots, t^{(k)} | z\right) = e^{-z \sum_{j=1}^{k} \Lambda\left(t^{(j)}\right)} \tag{2-26}$$

The state of art showed the effectiveness of using frailty model to model dependence or association among survival data or observations belonging to one cluster. However, the frailty application in reliability analysis is very limited. The hazard function form or PH regression models and AFT models combined with frailty [93] has attracted the eyes of many scientists and researchers to use it and apply in the medical field. Compared to Copula, the number of parameters to be estimated is much less in frailty models. Copula uses correlation parameters among the all possible correlated items, while frailty assigns a single term to model the dependence or association as an unseen variable. In this research we will make use of these models, the mathematical formulations will be illustrated and explained in the context of this research in subsequent chapters.

Chapter 3 Research Motivation and Merit

The following shows the reasons this research has been carried by explaining the testing stages if products during the design and product development phase. Then we present the added value of the implemented model for statistical and engineering sciences.

3.1 Research Motivation

Often, the life of a product is a requirement established by the OEM to meet customers' expectations and to dictate warranty cost and terms requirements. ALT is a common testing method that is used in reliability prediction. ALT take place at different stages of the product development cycle. Figure 3-1 shows the different testing stages in a production environment of industrial goods.

The production of a product goes through different cycles and it is normally a joint effort between different parties called OEM (Brand manufacturer) and suppliers (supplying the OEM with components or services) and each of them have their own people responsible for the different tasks: designers, validation engineers, safety engineers, sales men, financial experts etc. Each of the parties involved are responsible for the delivery of materials, components (i.e. resistors, sensors and other), or sub-systems that are safe and that meet the quality expectations of the OEM. Part of quality and design verification and validation is testing. Given the cycles a product goes through from design to sourcing to production, a product and its original constituents are subject to testing.

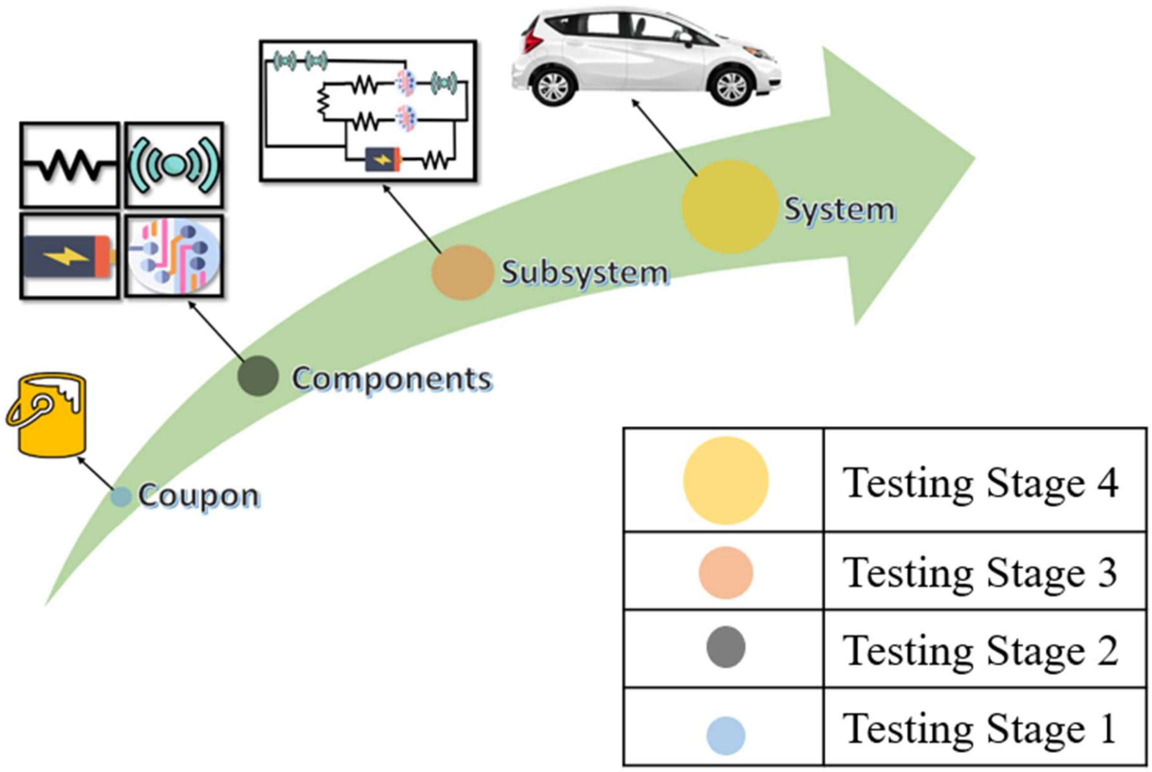


Figure 3-1 Testing Stages of a System During Production Cycle

Testing could be applied at each cycle to support the validation of the supplied components; coupons or raw materials (i.e. iron, stainless steel, zinc, aluminum and others) could be tested to derive their physical properties like the maximum tensile strength or hardness, and that is the testing stage 1. Other type of testing could be corrosion testing, fatigue testing and others. These types of testing identify the performance of the materials under certain load and their failure modes. The failure of these material is then analyzed to develop an understanding about their life and the characteristics of the environment (Temperature threshold, humidity threshold) in which they can be used and the expected life in these conditions.

Raw materials are then processed to form elements (i.e. wires). Once processed, the material properties are affected due to material mixing, welding, etc., the industry responsible of turning the coupons into components usually runs testing, testing stage 2 in Figure 3-1, to determine the physical and performance aspects of their product and in turn they would run life testing to determine the life expectancy of their elements in certain operation conditions (loads like voltage, temperature, maximum pressure and other). The testing is supported by data and results accompanies the design or element delivery to the OEM.

The cycle continues in the same fashion at each stage, the contract cost that an OEM would sign with a supplier to get components or service supply is often dictated by the different design, engineering and testing requirements to perform and execute the design. Testing occupy a major cost in the game and every component or product is expected to be tested to meet certain safety, quality and performance set of criteria. Accelerated life testing is a major component of every cycle and data can be collected at different stages from raw material to components and into subsystems, testing stage 3 in Figure 3-1, to the overall assembled system (i.e. circuit board), testing stages 3 and 4 , at each stage a new set of testing data is provided.

The research develops a new methodology to integrate these data from different testing levels in order to ensure minimum uncertainty is being propagated to estimate the system reliability. On the other hand, limited by testing cost which might limit the available ALT data, the novel framework developed in this research allows OEMs to use the different set of data collected from different ALT stages to determine the most accurate system reliability prediction.

The state of art shows the availability of different methods that could be applied directly to a system of n components assuming these components are independent, or some researchers addressed a specific topology of systems like the series systems. On the other hand, some researchers analyzed the ALT data of a single testing stage (a component or a system), so the gap is in building a bridge to navigate the ALT of one testing stage, as shown in Figure 3-1, to the next stage. Moreover, most of the time a single standalone component is manufactured to be part of a bigger system where it is put in a physical or a logical connection with other components of different or similar type to serve a sub-service within a system in order to deliver the ultimate purpose of a system (i.e. product) for which it has been designed and manufactured.

This sets the motivation of the research of building a propagation method by reducing the uncertainty between the different stages of testing during a production cycle. This motivation raises other concerns as we intend to tackle it, given the fact that at the last stage the components are brought together in a certain system structure, the configuration or the topology in which these components are placed in the system creates a dependence not only in the functionality but in the way they fail (failure modes or competing risks) due to certain conditions. So, modelling the effect of these factors along with dependence among the components of a system in order to build the bridge to propagate ALT data from once testing stage to another or navigate and infer statistical investigations and observations from any of the testing stage (go backward from one testing stage to another, go forward from one testing stage to another).

Additionally, given the availability of data at each testing stage raises the concern of fusing the ALT data collected from two different testing stages together in order to determine an estimate to the overall product. This testing effort is being paid for and available for the engineer to use, the larger the data set, the more the accuracy of the model. This is crucial at the last step when the system is too complex to be tested, imposes testing hardships like limited capabilities and limited budgets to perform the testing, limited testing results in limited available data which will then minimize the certainty of the resulting system reliability estimation. Accordingly, the latter sets the motivation of how ALT data from different stages could be combined to complement each other in reducing the uncertainty in the system reliability.

3.2 Research Advantages

Starting with the testing data from both testing level and their statistical distribution, we can tie the parameters of interest where the uncertainty resides together into one probabilistic equation for each level of testing. This will allow us to see how the uncertainties in the hyper parameters resulted from the fitting model of the ALT failure data could result in an accuracy bias if used to derive the system reliability. The goal is to minimize the uncertainty in the system reliability and allocate the testing design parameters optimally in order to minimize the testing budget while avoiding any compromise of the precision of the system reliability estimation. Fusing the information from system level testing data and component level testing data requires understanding the mathematical linkages between component level and system level. These linkages will further detail how the uncertainty is propagated and open the door for reducing it step by step then use it to optimize the design parameters. A system comprising multiple components contains dependence of the failure times among its components. Understanding the dependence is crucial as it will affect the precision of the system reliability if it is neglected. To manage the uncertainties reduction and the information fusion, the following five challenges must be resolved.

Tackling the objectives, listed above, in this research, will be of great benefit to achieve the coveted goal behind this research that has the following advantages:

1. Understand the uncertainty propagation from the component level and system level throughout the whole approach leading to the system reliability derivation.

2. Develop the linkages between a system and its components that will explain the connections in a reliability analysis context.
3. It allows versatility in the use of the available data that could be used to travel from a system level to component level or vice versa depending on the level of testing chosen and the corresponding failure time data: Component-Level versus System-Level.
4. Maximizes the return of the available data via data fusion to further tune the precision of the system reliability.
5. Adds value to the realm of Reliability Analysis by proposing a model that takes into consideration the dependence among the failure of components.
6. Achieve optimal testing design parameters that will help getting rid of irrelevant test attempts which will lead to a reduction in the testing time as well as well definition of the targeted stress levels. This will help better the Design of Experiment plan and execution.
7. Reduce the cost of the product development phase by reducing the cost of quality and reliability testing while maintaining good quality reliability assessment.

3.3 Intellectual Merit and Broader Impact

Present methodologies in the accelerated life testing allows the assessment of system reliability conclusions by testing at the system level. On the other hand, some of the available practices allows testing single components and calculate its reliability. Both schools of thoughts do not take into account any dependence that might exist among the components of one system once put in use. Using the information collected from component level testing to assess the system reliability could be of great financial importance at the design level. In addition, sometimes it is viable to use system level testing information to form reliability conclusions about the components failures. This research will address these issues by showing first, new flexible methods allowing the use of the component level testing information with the purpose of calculating the system reliability once these components are assembled into one system, second, it presents a model to link system level testing data to the system reliability by reducing the uncertainty and calculating the system reliability. Third, the method versatility allows the fusion of both information: system level and components level in order to reduce the uncertainty in the system reliability. Fourth, the proposed approach in this research takes into account the dependence that might exist between the components. Last but not least, the research at hand allows the optimization of the testing design parameters to keep the development cost at minimum.

Considering the current advancements in the technologies, assessing the failure times via testing of systems is becoming more challenging for its complexity, placing some constraint on the testing procedures, which in turn is increasing the cost of testing in the prototype phase in order to figure out the life of systems and their probability of failure. This research benefits the large OEMs, introducing new complex systems, to better assess the reliability of their products by optimizing the cost and use the testing information of components subject to an accelerated failure which is easier to achieve than testing the whole system.

Beyond OEM, the research could be of great use for Engineering Quality Consulting companies, Quality and Reliability Engineers, Data Scientists, Statistician and Probability researchers. The content of this research will be communicated at engineering and educational conference presentations as well as peer reviewed journal papers.

Chapter 4 Proposed Model Background

In Chapter 4, we enclose the background about the models used to develop the ALT models enclosed in Chapters 5 and 6. We explain the problem-solving strategy and how the models will help construct the linkages between the data collected from different testing stages to the system reliability.

4.1 Problem Solving Strategy Background

The goal of this research is to find a methodology allowing to fuse the information collected from system level testing and components level testing to improve the assessment of system reliability. As well, it aims at optimizing the accelerated testing design variables constrained by a budget. The outcome of this research is expected to be applied to any systems consisting of multiple components with testing being feasible by applying a higher than use stress to accelerate their failures.

Normally, a system consists of multiple components connected together to achieve certain targeted operation, these components often share some environmental and stress loads that are not clearly observable leading to the dependence among them in how they fail, the latter is referred to as dependence of components failure times. The dependence forces a challenge on the modelling of the information gained at each level of testing

We identify two testing levels performed at different testing stages: component-level ALT and system-level ALT, so this research is dealing with two types of information, one data set is inferred from system level testing and another is collected from component level testing and each of there before mentioned two testing levels have their own advantages and disadvantages.

1. The system-level testing defined as taking the whole system and apply an accelerating factor on the whole system. The collected data contains an implication to the dependence between the components because the system is tested as a whole and the linkage among the components functioning under this system is already established and thus considered in the results. Systems of multiple components undergoing a system level testing could be expensive, and often test customization in order to test a system with certain design in a specific operational method imposes additional costs and hardships. However, the result it takes into account the dependence of the system's components failure times which is an advantage to this level of testing.
2. The component- level testing consists of testing each component separately before it becomes a part of system or linked to any other component. This level of testing is considered inexpensive and allows testing customization because the functionality of a single component or shape is regarded simpler compared to a system of multiple components, however it does not include any of failure time correlations of components when assembled together in one system.

Using the testing design parameters, the failure time ALT data and other identified model parameters, the testing data could then be fitted to be used with the purpose of running a reliability assessment and derive the probability of no failure of the system. This analysis incorporates some uncertainty that propagates starting from the testing data and ending in the result which is the system reliability.

The research proposes a method to reduce the uncertainty while presenting a novel approach in using testing data fusion from component level and system level to reduce the uncertainties in the components parameters and the system level parameters and then derive the system reliability with high confidence.

Figure 4-1 shows a figure of a system with two components and the associated parameters to be estimated in order to reduce the uncertainty carried with each parameter using the failure times of the two components as data from accelerated life testing as well as the system level testing data. An example of such a system could be any electrical board with electronic components such as two resistors, or two sensors in a giant robotic system. Each of the components and the system, as shown in the figure, shows a set of associated statistical parameters that are used in the failure time statistical distribution to infer real (i.e. actual) time failures from accelerated life time testing.

It is shown in Figure 4-1 that component one and component two each has its observation node which refers to the failure time observations collected from putting each component under a high stress. Alternatively, the system in the middle consists of the components 1 and 2, has its own observation node which is the failure time data obtained from putting the system under higher than use stress in order to attain an early failure. As seen in the figure, the random nodes from the components is directed at the nodes of the system box meaning that the information gained from the component level could be used to reduce uncertainty in the system level parameters.

Outside the three dashed boxes in Figure 4-1 below we see three functional nodes, among which one is red, which represents the reliability of the whole system. As indicated in the figure, the input to that node are parameters from both components and the system which means that there is a fusion of information in order to derive the reliability of the system.

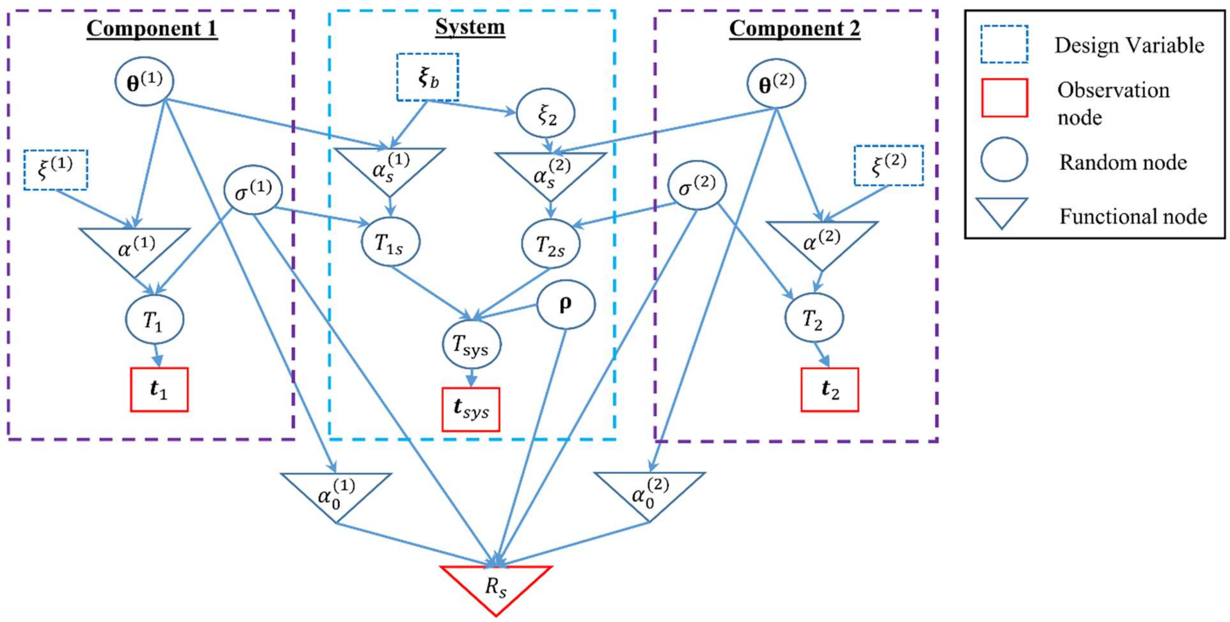


Figure 4-1 Connections between component-level ALT, system-level ALT, and system reliability (a system with two components)

Table 4-1 Nomenclature of Parameters in Figure 4.1

Symbol	Definition
$\mathbf{t_1, t_2, t_{sys}}$	Testing Data – Failure Times
$\boldsymbol{\sigma}^{\mathbf{i}}, \boldsymbol{\theta}^{\mathbf{i}}, \boldsymbol{\alpha}^{\mathbf{i}}, \boldsymbol{\alpha}_{\mathbf{s}}^{\mathbf{i}}, \boldsymbol{\rho}$	Statistically Inferred Hyper parameters
$\mathbf{X^i}$	Accelerated Life Testing Design Parameter

Referring to Figure 4-1, One node existing under the system box and does not exist under either of the boxes representing the components is a random node with the letter ρ in it indicating the dependence. The dependence is not something at the component level rather it is established at the system level. The explanation of the latter is that, the failure of an individual component by itself does not depend on anything but on its specific operation. When a component is put in linkage with another component to form a system in order to operate together on achieving a system targeted operation, the component establish a connection among each other due to a symbiotic relationship in the operation.

The components of a system share loads that could be quantified like rotational loads, electrical loads and others and some other loads that are not observable and could not be quantified due to the non-linearity of their effect on the operation. Both types of loads could create dependence in the operation and the failures of the components and hence failure of the system. In our research we focus on the dependence in the failure and not in the operation, in the meaning of one component could be affected by certain type of loads leading to an effect on the other component due to a physical connection between them in a system and in turn flagging an effect on the system. The effect could lead to a failure of either the components or the system as a whole which leads to the idea of dependence of failure time among the components.

The design variables indicated in Figure 4-1 above represent the variable that will be determined according to an optimization model and subjected to a budget in order to find the optimal values that would lead to an effective testing cost. Each level of testing has its own design parameters which are mainly two parameters in this research: The stress level and the number of tests (specimens) at each level of testing.

To target the six objectives of the research and construct the methodology, the establishment of a relationship between component level and system level failure times is inevitable. This research uses two approaches: a probabilistic approach and a regression approach to link the probability of failures of the components and the system in order to fuse the failure time data and infer the parameters needed to carry on the system survival analysis.

4.1.1 Autonomous Vehicles Example

The competition between OEMs towards automation and presenting new innovative technologies has led them to look for more ways to test the robustness of new product designs. Both OEMs and customers build certain expectations on the performance of these new products for a certain period of time which in turn has led to the birth of various prediction models and methodologies to estimate the life of these products [94]. As an example, we consider the Automated Driving has been greatly capturing the interest of the automotive industry. Countless hours of design efforts are being invested with the purpose of creating a robust vehicle in order to gain the confidence of the market. The sensing system is perhaps the most important system of all as it is responsible for watching the environment and command the vehicle accordingly. Knowing the lifetime of such system is of great importance, because the moment the sensing system in the vehicle dies, the vehicle is no longer safe to be driven. Reliability Engineers are putting enormous efforts to quantify the life of these systems especially that these systems are being newly invented, and the hardware designs imposes a level of technological complexity hindering their testing due to the increased costs and customization required to do so.

The method proposed in this research could be put in use in order to run a survival analysis of these systems using ALT. Figure 4-2 below shows an Autonomous Vehicle with its radar or sensing system on its roof. The system is equipped with multiple cameras of high resolution and advanced artificial intelligence as well as multiple sensors, this system is the brain that commands the car and its level of design safety must be highly rated and designed. For confidentially purposes we would simplify the system in Figure 4-2.

The Figure below shoes multiple components (Table 4-)three sensors, two processors, a power source and 4 resistors. The types of stress that could lead to the failure of the system are numerous and it could be electrical (excess power), informational (software), physical (crash).

Creating prototypes of the system for each type of the stresses in order to study the behavior of the system in various environments is very expensive yet complicated and requires long time of design and application as well as enormous efforts from validation engineers and quality engineers. So, the optimality of the testing design is key and important to achieve the system reliability with high confidence.

Also, not all components involved to create these systems are often testable (i.e. wires, welding). On the other hand, suppliers often test their components separately as part of the delivery of their products to OEM, so information could be collected for each component (sensor, power source, resistors etc.). The latter would create the motivation to maximize the use of the available information and mold a strategy to fuse the data from component level with the system level (if available) in order to quantify the life of the system.

Table 4-2 Components constituting the sensing system of an autonomous vehicle

Component	Description
	Resistor
	Processor
	Power Source
	Sensing Electronical Device

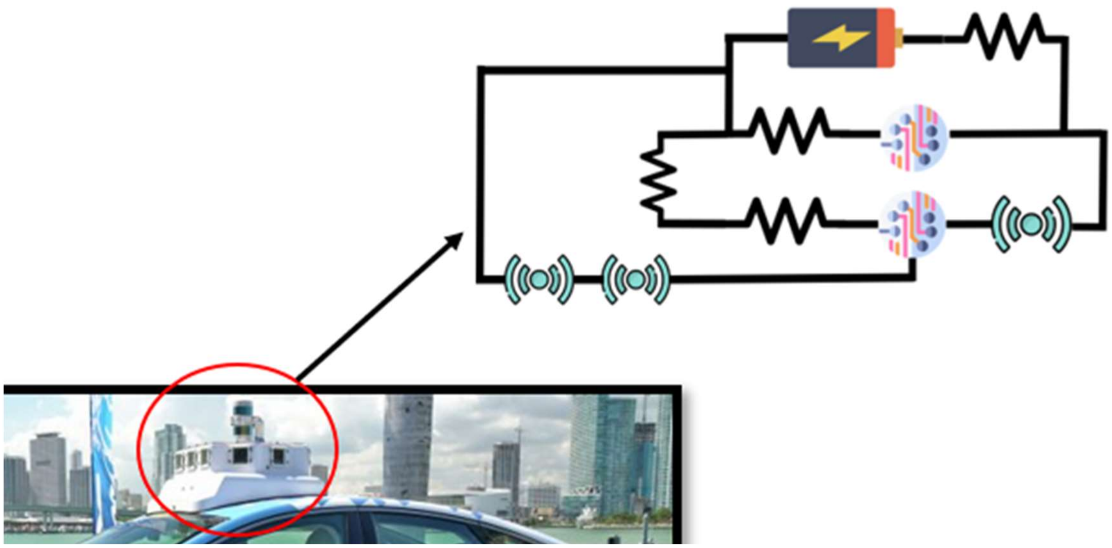


Figure 4-2 Autonomous Vehicle Sensing System

With the research goal in mind and based on Figure 4-1, the overarching research need could be generalized to be as follows:

Develop a framework to reduce the uncertainties in the ALT failure time data model parameters $(\theta, \sigma, \beta, \alpha)$ using the Accelerated Life Testing (ALT) failure time data $(t_{components}, t_{system})$ by considering dependence (ρ) and propagate the uncertainty to system reliability then integrate them into an optimization model to optimize the ALT design parameters: X $(stress\ levels)$ and n $(Number\ of\ stests\ at\ each\ stress\ level)$.

Problem Statement 4-1 Research problem statement

Accordingly, the problem-solving strategy is decomposed into two solving procedures the first one uses a statistical approach combined with a Copula function to model dependence among the failure time. The research then applies an optimization model to this approach. Another method is implemented which uses the EHR regression model in combination with frailty mode, the approach is considered a distribution free method. Both approaches could be applied to any system with any configuration. In each approach, we target five objectives out of the six. The optimization objective is applied to the statistical method only.

In the following two sections 4.1.2 and 0 we explain how each strategy is decomposed and then we detail each of the strategies in terms of mathematical formulation and numerical examples in a separate chapter (0).

4.1.2 Reliability Assessment Via Statistical Models and Copula Function

In this problem-solving strategy, we make use of the parametric distribution approach combined with the Copula function to model the dependence. We use a log-scale distribution function given by[95] :

$$Pr[T \leq t; T] = G\left[\frac{(log(t) - \alpha)}{\sigma}\right] \quad (4\text{-}1)$$

where α is the shape parameter and σ represents the scale parameter. The shape parameter is given by

$$\alpha = \theta_0 + \theta_1 X \quad (4\text{-}2)$$

in which X is the normalized accelerated stress.

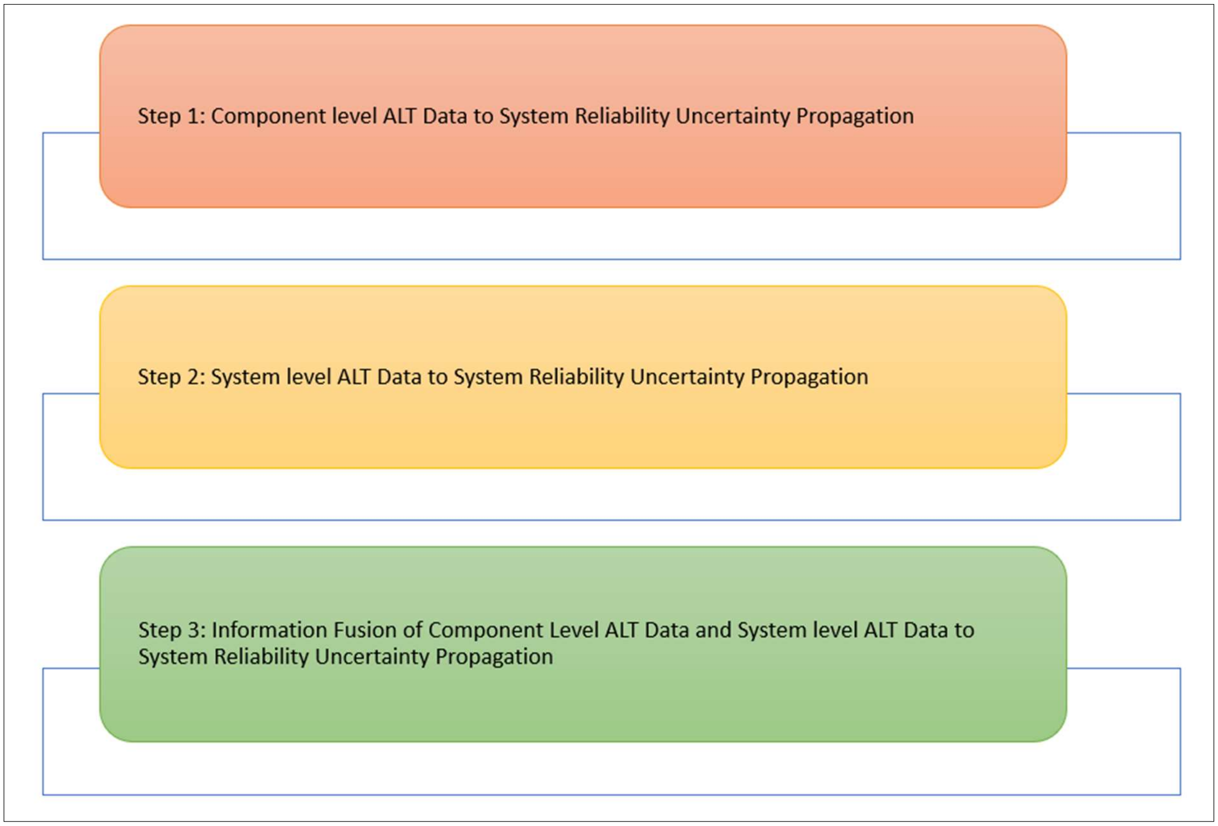


Figure 4-3 The three major steps of the problem solution strategy

The approach targets three main objectives and comprises three steps as shown in Figure 4-3. To illustrate the three steps better, we take the example of the autonomous vehicle radar system as shown in Figure 4-2. The steps as they apply to that example are as follows:

Step 1: The use of the component level information to reduce uncertainty of the system failure time hyper parameters. This step consists of using testing stage 2 according to Figure 3-1, data and migrate the uncertainties to the system reliability by estimating the reliability of the product of interest (Figure 4-4).

Step 2: The use of the system level information to reduce uncertainty of the component failure times hyper parameters. As shown in Figure 4-5 in this step, the research intends to make use of the testing stages 3 and 4 to propagate the uncertainty to the system reliability.

Step 3: The fusion of both component level and system level testing information to reduce uncertainty of the system reliability. Figure 4-6 shows the process under this step, merging the data collected from testing stage 2 and testing stages 3 and 4 is the scope in order to propagate the uncertainty to the system reliability.

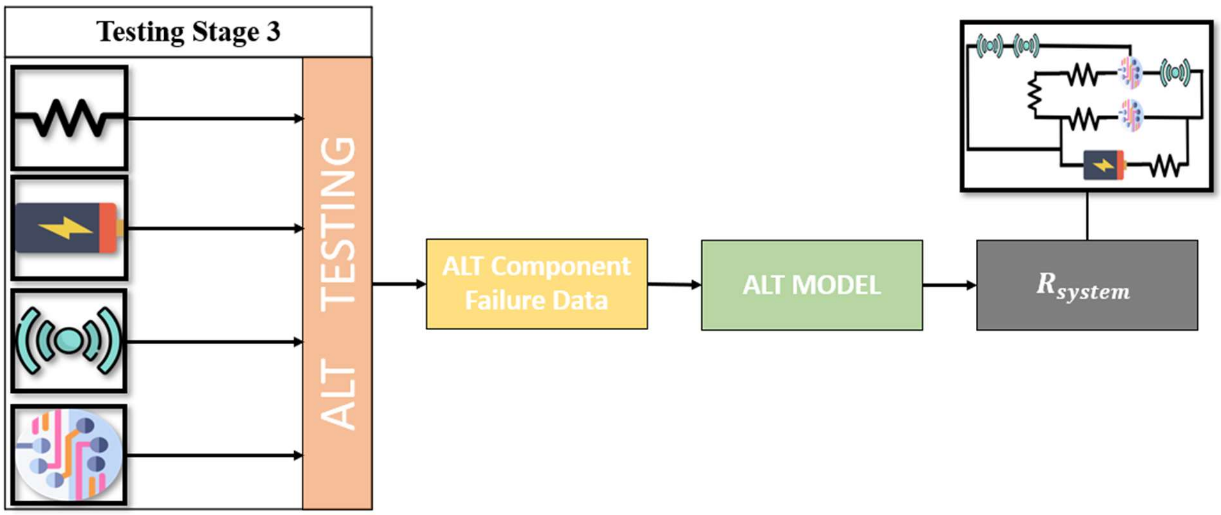


Figure 4-4 Component ALT data uncertainty propagation to system reliability

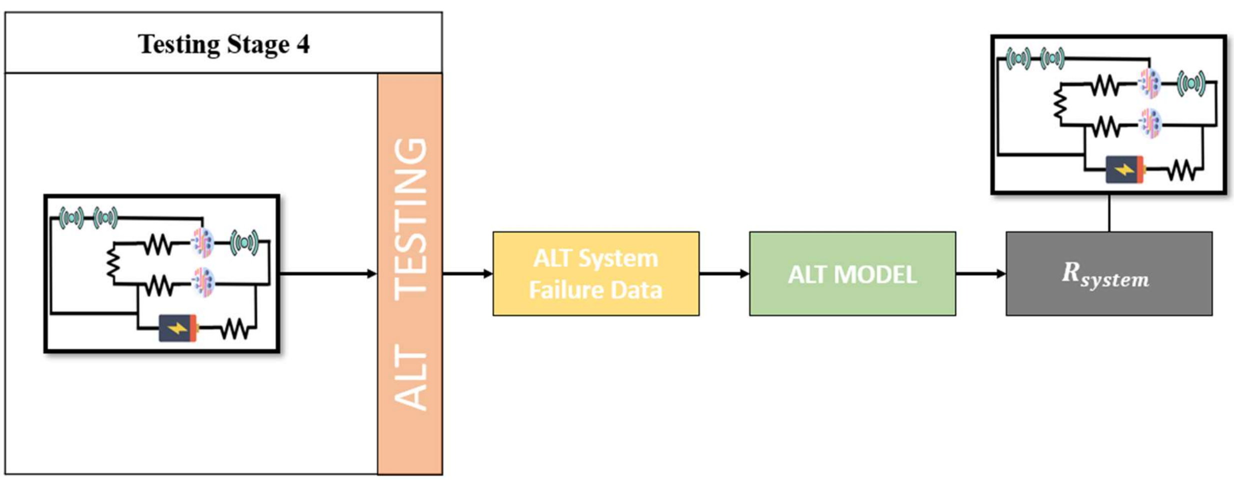


Figure 4-5 Step 2 System ALT data uncertainty propagation to system reliability

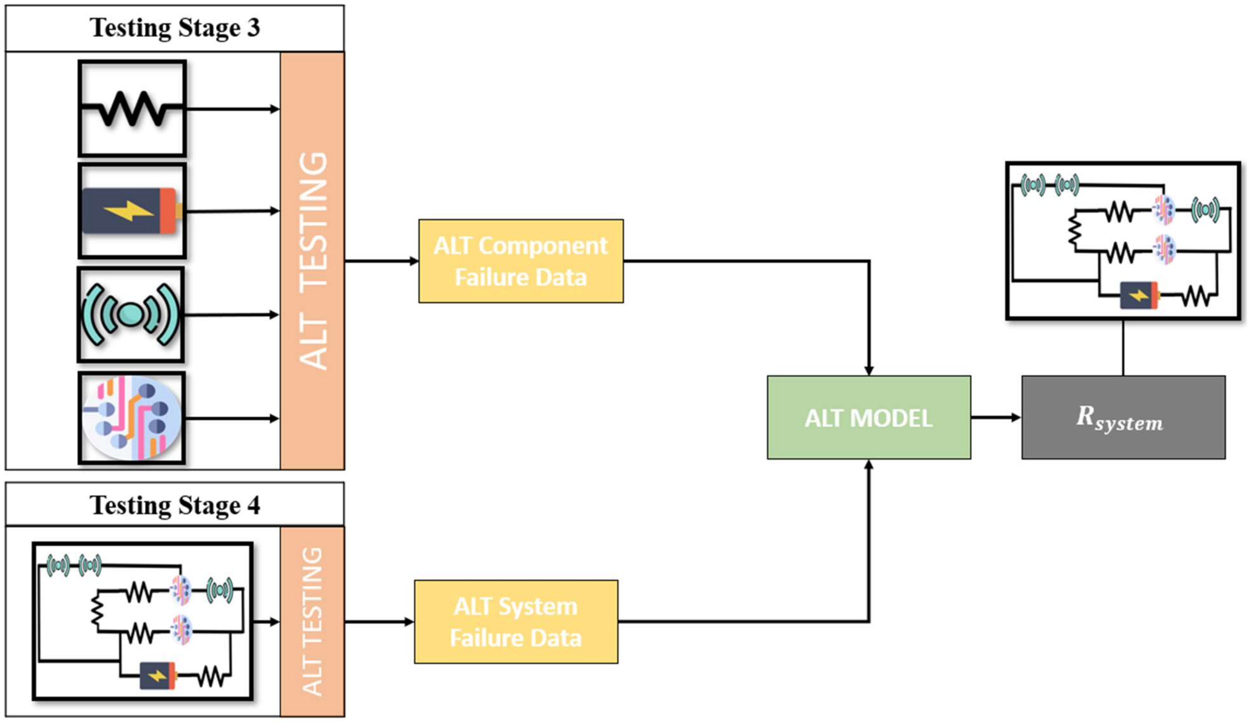


Figure 4-6 Step 3 Information Fusion of Testing Stages 2 & 3,4

The first challenge imposed by the first objective of this research and the first step above that consists of analyzing the linkage between the components and the system in order to reduce the uncertainty in the ALT data distribution parameters. The reasons behind this challenge are as follows:

1. Reducing the uncertainty in the parameters of the failure time distribution given component level testing data in order to limit the propagation of large uncertainties when reaching the system level reliability.
2. Understand the linkage between the component level failure times collected as observations, or in other words testing data, and the system level reliability, allows using the data collected from a component level in order to derive the system reliability without testing the system itself.

3. Dissecting the linkage through mathematical equations allows traceability of the dependence and understand how it might affect the system reliability. Neglecting the dependence among the failure time of the component would lead to a biased yet unrealistic system reliability evaluation.

The problem to be solved in this research challenge is as follows:

- The probabilistic distribution of component failure times
- Parameters distribution estimation
- Dependence modelling of failure time ALT data correlation
- Uncertainty propagation to system reliability

- **Given**: Failure Times $(t_{failues})$of Components at different stress levels higher than use stress level
- **Find**: The data distribution equation: $(F(t_{Failure})and\ f\left(t_{failure}\middle|v\right)\ where\ v\ is\ the\ set\ of\ parameters)$ allowing to reduce the uncertainty in the distribution parameter
- **Find**: Linkage between the component and the system reliability $R(t)$and propagate the uncertainties

Problem Statement 4-2 Problem to be solved in Step 1

Imposed by the second objective of the research and the second step that is finding an approach to link the system level testing data to the system reliability for the following reasons:

1. The linkage between system ALT data and system reliability allows the use of failure time data collected at a system level to derive the actual system reliability.
2. It helps figuring out how the uncertainty could be reduced at the component level when attempting to fuse the information later.
3. Allows reducing the uncertainty in the dependence factor given the system level failure data because the dependence is not introduced at the component level and

the reason for that is that a component tested individually instead of being an integral part of a system does not involve dependence on another component unless brought in a system through some sort of a connection.

According to the latter, the problem statement is as follows:

- **Given**: Failure Times $(t_{failues})$of System at different stress levels higher than use stress
- **Find**: Map stresses from boundary components to non-boundary components via physics-informed model
- **Find**: Linkage between the system level ALT data and the system reliability $R(t)$and propagation of the uncertainties

Problem Statement 4-3 Problem statement of Step 2

This step imposes a new challenge which is understanding the load sharing scheme of the components within a system is crucial because it will allow the modelling of the component loads cascaded from the loads applied during system level testing. Because when putting the whole system at test, the different components in the system receive different loads: some components receive the stress directly and some receive a residual stress cascaded or extrapolated from components receiving the loads, leading to different failure modes. The latter requires the use of physics methodologies combined with a statistical approach to reduce the uncertainty in the failure time's parameters which will lead to a reduction in the uncertainty of the reliability quantification. The research integrates two different approaches to model the dependence and closes the analysis with optimization methodologies so as to decrease the cost of testing and hence the product development cost by meeting a specified testing cost budget.

In the last step, we intend at fusing the information by using both data collected from the two ALT levels: component level testing data and system level data in order to reduce the uncertainty in the system reliability.

In every step above, we apply the Bayesian estimation method and we model the dependence between the failure times of a system's components failure times using the Copula function. And the last objective is to find the optimal accelerated life testing design parameters which are mainly the stress levels and the number of tests at each stress level subject to a testing budget to be spent on testing.

4.1.3 Reliability Assessment Via Distribution Free Models and Shared Frailty Models

In Section A, we explain the main concept for the distribution free models using the extended hazard regression model. In Section B, we explain the concept of the frailty models that will be used to model dependence among the ALT data.

A – Distribution Free - Extended Regression Model

While this approach uses the same decomposition of first using component level data to reduce the uncertainty in the system reliability and then use of system level ALT data to reduce the uncertainty in the system reliability and finally perform an information fusion, we do not apply an optimization model to find the optimal ALT design parameters.

This approach uses EHR model as given in [63]with baseline hazard function being a distribution free and taking the most used parametric distribution (log-scale distributions) as special cases which allows a distribution free approach to be used in the ALT field:

$$\lambda(t|\mathbf{x}) = \lambda_0\left(te^{\boldsymbol{\beta}^T\mathbf{x}}\right) e^{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}}. \tag{4-3}$$

in which $\lambda_0(t)$ is the baseline hazard function. The baseline hazard function could be parametric, non-parametric, or semi-parametric. In this research, $\lambda_0(t)$ is assumed a distribution free function. Regression methods could be used to estimate the regression coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the covariates.

The unknowns in above model are the regression coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The model suggests that both the time scale effect and the hazard multiplicative effect of the covariate $\mathbf{x}$ are contained in the model. Based on the hazard rate function in Equation (4-3), we can notice that when $\boldsymbol{\beta} = \mathbf{0}$, we get the PH model and when the $\boldsymbol{\alpha} = \boldsymbol{\beta}$, the AFT model is obtained [62].

In this research, the EHR model will be used in conjunction with frailty models to model the complicated dependence in reliability analysis. In the next section, we first introduce the concept of frailty models and then discuss how the HER model is integrated with the frailty model.

B – Frailty Models

The dependence is modelled using frailty model[87]and more specifically shared frailty model. So, for the first three steps, the approach uses a frailty factor to model the dependence among the failure time.

$$\lambda(t; X) = z.\lambda_0(t)e^{\mathbf{x}^{\mathbf{T}}.\boldsymbol{\beta}} \tag{4-4}$$

A class of frailty models is called shared frailty models. Shared frailty models have been extensively studied in various research fields as in [62] and [79]. The model is considered a shared frailty if the frailty factor z is shared among all latent lifetime. The value of z is constant over time and is assumed to be common to all components in a system. In other words, all components share the same frailty which is responsible for their dependence [92]. That is, for all components $i = 1, 2, \dots, N_C$, where N_C is the number of components in a system, the cumulative hazard functions, $\Lambda\left(t^{(i)}\right), i = 1, 2, \dots, N_C$, share an unobservable frailty factor z.

Conditioned on the frailty factor z, the latent lifetimes of the components $[T^{(1)}, T^{(2)}, \ldots, T^{(N_C)}]$ are assumed independent and the generalized joint survival function $S(\cdot)$ is given by:

$$S\left(t^{(1)}, t^{(2)}, \ldots, t^{(k)} | z\right) = e^{-z\sum_{j=1}^{k} \Lambda\left(t^{(j)}\right)}. \tag{4-5}$$

Additionally, a widely used distribution to model the frailty factor is the gamma distribution. When the gamma distribution is chosen for z, the model is referred to as gamma shared frailty model [92,96].

Alternatively, shared frailty models could be used with AFT models [63] by modifying Equation (4-3)by adding a multiplicative frailty factor as follows:

$$\lambda(t|\mathbf{x}) = z\, \lambda_0\left(te^{\boldsymbol{\beta}^{\mathbf{T}}\mathbf{x}}\right) e^{\boldsymbol{\beta}^{\mathbf{T}}\mathbf{x}}, \tag{4-6}$$

where **x** is the covariate or accelerating stress and *z* is the frailty factor.

Figure 4.7 shows how the data is used to propagate the uncertainty and reduce it and it shows the versatility of going from the data of a testing level to another testing level in order to derive the system reliability and the possibility of fusing both data from both system levels in order to assess the system reliability.

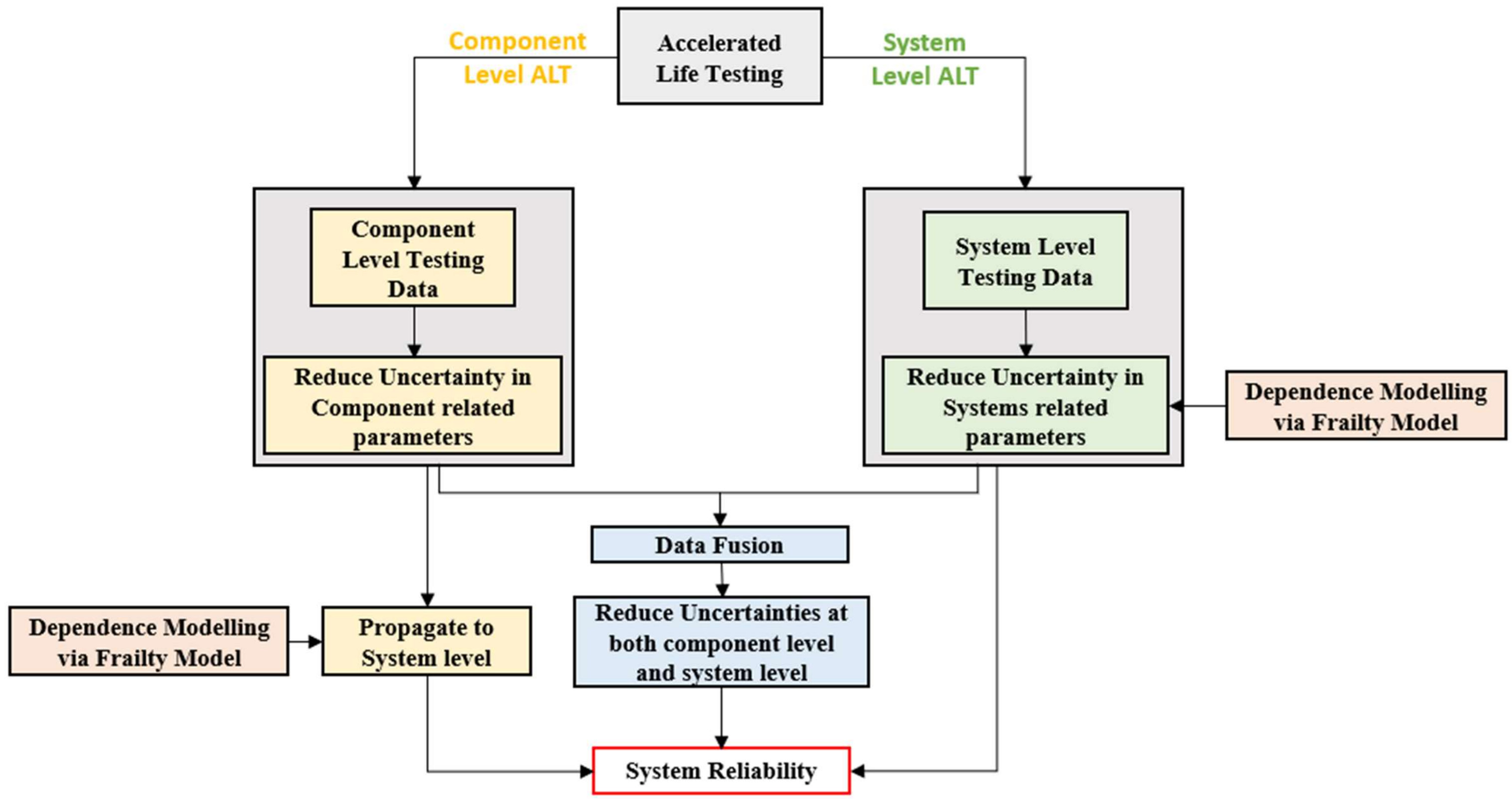


Figure 4-7 Flow of the Use of Testing Data in Reducing Uncertainties

4.1.4 Bayesian Inference and Particle Filtering

In this section we explain the Bayesian inference relationship used to sample posterior information given prior information of model parameters and the algorithm used to perform the sampling of particles which the particle filtering method used in this research.

A – Bayesian Inference

In this section we explain the Bayesian inference and its relation to the likelihood and the sampling method used in this research. Each of the main 3 steps above use the Likelihood function along with Bayesian method in order to sample posterior estimation of the model parameters. Bayesian inference and its application in the ALT field have been studied in [97] by Shaked et al and in 1988 by Blackwell et al in [98]. Bayesian inference allows the reduction of uncertainty in the parameters or variables as one gains more information through data analysis[99].

Bayesian inference is one among many statistical inference methods that is widely used in investigating data. Inference is a probabilistic explanatory state. So, it is a probabilistic approach derived from Bayes theorem. As provided with greater details in [100], the Bayes theorem suggests that if a set of n observations (i.e. data) $\mathbf{y} = (y_1, y_2, \dots, y_n)$ having a probability distribution $p(\mathbf{y}|\boldsymbol{\theta})$ depending on k parameters noted as $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_k)$ and assuming that the parameters have a distribution $p(\theta)$then:

$$p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}) = p(\mathbf{y}, \boldsymbol{\theta}) = p(\boldsymbol{\theta}|\mathbf{y})p(\mathbf{y}) \qquad (4\text{-}7\ )$$

$p(.)$ is the probability notation.

Next, given the probability distribution of the observation data $p(y)$, it is implied that the conditional probability of θ is given by Bayes' theorem:

$$p(\boldsymbol{\theta}|\mathbf{y}) = \frac{p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{y})} \tag{4-8 }$$

Also;

$$p(\mathbf{y}) \ = E[p(\mathbf{y}|\boldsymbol{\theta})] = \begin{cases} \int p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}) \ ; if \ \boldsymbol{\theta} \ is \ continuous \\ \sum p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}) \ ; if \ \boldsymbol{\theta} \ is \ discrete \end{cases} \tag{4-9}$$

where $E[p(\mathbf{y}|\boldsymbol{\theta})]$ is the mathematical expectation of $p(\mathbf{y}|\boldsymbol{\theta})$. Based on (4-9) the theorem could be written as:

$$p(\boldsymbol{\theta}|\mathbf{y}) = cp(\mathbf{y}|\boldsymbol{\theta}).p(\boldsymbol{\theta}) \tag{4-10}$$

To define the terms better, $p(\theta)$ explains what is recognized about the parameters vector θ without any given data or observations and it is referred to as the prior distribution of the parameter θ. Accordingly, $p(\theta|y)$ explains the known about the set of parameters θ given the observations seen in y and they are referred to as posterior distribution of θ. The “c” in the equation (4-10) above is a normalizing constant or vector ensuring that the posterior distribution integrates or sums up to a total of 1.

According to Fisher in 1922 [101], when $p(y|\theta)$ is regarded a function of the parameters θ rather than being a function of y, $p(y|\theta)$ is called the likelihood function and could be noted as $l(\theta|y)$ which is the likelihood of θ and the Bayes theorem becomes:

$$p(\boldsymbol{\theta}|\mathbf{y}) = l(\boldsymbol{\theta}|\mathbf{y}).p(\boldsymbol{\theta}) \tag{4-11}$$

that is the posterior distribution of θ given new knowledge is proportional to the product of the likelihood function of θ and the prior distribution of θ before gaining any observations.

$$posterior\ distribution\ \propto likelihood\ function\ x\ prior\ distribution \quad (4\text{-}12)$$

where ∝ stands for "proportional to".

The likelihood function explains the effect in the information getting gained on the parameter θ as it comes from the observations or data. It is worth noting that only the relative value of the likelihood matters so multiplying the likelihood by a constant will not change the value or its effect on the posterior distribution.

As shown in Figure 4-8, the process of the Bayesian estimation procedure is to have the prior information collected from experts as an input, the Bayesian estimation requires the use of the Bayes Law (Equation (4-11)), the theorem requires developing a likelihood function to conclude the parameters to be estimated or inferred, then apply an MCMC method like the Particle Filtering method which this research makes use of and then sample the posterior information of the model parameters upon gaining more information.

B – Particle Filtering Sampling Method

The idea of particle filtering is based on the Monte Carlo (MC) methods. The particle filtering is a sequential importance sampling method. Based on importance weights associated to particles or samples, the method aims at approximating the probability distribution function.

Candy in [102] presents a good and brief definition of the MC techniques. The original groundwork is based on the Markov chain theory which advocates that by random sampling the empirical distributions is set to converge to the target posterior distribution and that distribution is referred to as the invariant distribution of the chain. Based on a stochastic system called the state-space which is governed by a transition probability, Markov chain MC techniques are based on sampling from probability distributions based these is a stochastic (state-space) system. A crucial property of the technique is that as the number of particles or samples becomes larger, the chain is assumed to converge to the coveted posterior distribution by proper random sampling based on a set of assumptions.

Particle filtering is being integrated in estimation problems and it is an attractive method to sample the posterior samples in the Bayes theorem. Correspondingly, the method is a computational algorithm and, we make use of this algorithm in this research, every time a posterior distribution sampling is needed as part of the problem-solving strategy.

There are many types of particle filters, the Sequential Importance Sampling (SIS) is one that is considered the base of all PF MC filters constructed over the research spans[103]. The core concept of this technique lies in developing an implementation of Bayesian filter recursively through MC simulations. It is called by different names by researchers, it is referred to as Bootstrap filtering in [104].

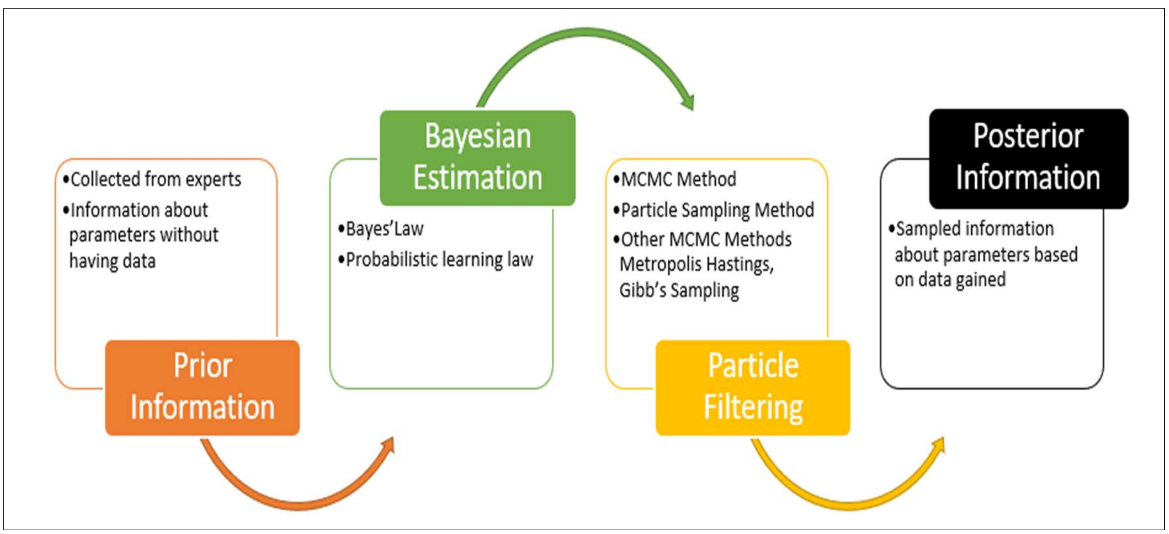


Figure 4-8 Bayesian Estimation Process

The method represents the posterior distribution of state variables in terms of samples and associated weights reflecting the importance of a samples and then estimate the posterior values using these samples and weights. The SIS algorithm approaches to the optimal

Bayesian estimate as the number of particles becomes larger and the output of the PF represents the posterior pdf of the parameters.

The PF algorithm in 3 steps is enclosed below to sample a posterior belief ($p(x)$) for an arbitrary variable x:

Step 1: Sample from the prior distribution (the prior distribution is assumed to be known parametric distributions), for illustration we use the notation $q(x)$ to represent the prior samples of a generic variable x.

Step 2: calculate the importance weight: $w = \frac{p(x)}{q(x)}$ calculated based on the importance sampling as in [105].

Step 3: Replace unlikely samples with low weights with more likely ones and that is called resampling.

Based on the tracking concept:

$$V_t = f_t(V_{t-1}, \vartheta_{t-t}) \tag{4-13}$$

where f_t is often a non-linear functionof the state variable V_{t-1}, ϑ_{t-t} is a process noise sequence or vector where $\{\vartheta_{t-t};\ t\ \in\ \mathbb{N}\}$; $\mathbb{N}$ is a set of natural numbers, and the state of variable sequence is denoted by $\{V_{t-t}\ \in t\ \in\ \mathbb{N}\}$. To recursively estimate V_t, measurements are needed and denoted by:

$$D_t = H_T(V_t, n_t) \tag{4-14}$$

H_t is as well a non-linear function and n_t ; $t\ \in\ \mathbb{N}$ is the noise terms for the measurements function.

In Bayesian inference, particle filtering is an estimation method of the belief in state V_t at time t given some data D_t up to that time. The estimates of the states are referred to as filtered estimates based on available measurements. In the ALT context, these measurements are the failure data collected from the testing. The method is composed into two major steps: 1) Prediction (prediction of priors) and 2) Update (Based on importance weights). Accordingly, V_t denotes the state variables at time t and D_t the set of measurements up to time t. The SIS algorithm estimates the posterior distribution $p(V_{0:t}|D_{1:t})$ using a set of N samples with associated weights $\{V_{0:t}^i;\ \varpi_t^i\}_t^N$ by:

$$p(V_{0:t}|D_{1:t}) \approx \sum_{i=1}^{N} \varpi_t^i\, \delta(V_{0:1} - V_{0:t}^i) \tag{4-15}$$

$\delta(.)$ is the Dirac delta measure at $V_{0:t}^i$. At time step t, the particle V_t^i of V_t is estimated based on the state at $t-1$ denoted by V_{t-1}^i by a distribution of parameters V_{t-1}^i and the measurements up to time t denoted by D_t^i, meaning that the i^{th} particle or sample V^i are generated using a proposal pdf $q(.)$:

$$V_t^i \ \approx q(V_t^i|V_{0:t}^i, D_t^i) \tag{4-16}$$

and the SIS weight is obtained by the following equation:

$$\varpi_t^i \ \propto \varpi_{t-1}^i \ \frac{p\big(D_t\big|V_t^i\big)p(V_t^i|V_{t-1}^i)}{q(V_t^i|V_{0:t-1}^i, D_{1:t})} \tag{4-17}$$

The initial state V_0^i is sampled from the prior distribution (the initial pdf of state vector V_t) ; $p(V_0|D_0)$ and D_0 is the set of no measurements available and the weight $\varpi_{t=0}^i$ is $\frac{1}{N}$. more details could be found about MCMC in [106] and [107].

Subsequently, the first step in developing the algorithm to estimating a posterior distribution is to identify the likelihood function, to do so we use the conditional PDF or probability given by the product of the hazard function and the reliability function. In general form:

$$f(y|X) = \lambda(y|X) \times R(y|X) \tag{4-18}$$

$f(y|X)$ is the conditional pdf function of y conditioned on X, $\lambda(y|X)$ is the conditional hazard function and $R(y|X)$ is the conditional reliability function of y conditional on X. Equation (4-18) will be detailed for the model in the following chapters.

Next, deciding on the appropriate prior information about the set of parameters for which one intends to sample a posterior distribution. Parameters of the model are decided based on the form of the $f(y|X)$ used.

Now that the prior belief has been established and the likelihood function is put in a closed form, the posterior distributions cannot be found in parametric form to sample from. Hence, the particle filtering method comes into play as a non-parametric representation of the posterior distribution. the way it works is by sampling from the prior distribution and adjust the belief as it gains more knowledge from the observations using a weight function.

Chapter 5 ALT Via Log-Scale Parametric Statistical Distribution and Copula Function

In Chapter 5, we develop a novel ALT model to connect component-level ALT data and system-level ALT data to the system reliability. We then work on fusing the information collected from both ALT levels together. Dependence is modelled using the copula function for this ALT model. The chapter includes an optimization model for the ALT parameters constrained by testing cost. The ALT model effectiveness is shown via numerical examples at the end.

5.1 Uncertainty Propagation of Component Level ALT Data to System Reliability

We start the uncertainty propagation by first analyzing the component to system linkage and uncertainty propagation as the first research task which is split into three main steps, the first one is modelling the distribution of the component failure time, the second step is estimation of parameters which aims at reducing the uncertainty in these parameters, and the last step uses Monte Carlo Simulation in order to propagate the information to a system level and derive the system reliability by taking into consideration the dependence between the components via Copula function.

As discussed earlier, the current research task is divided into three categories:

- The probabilistic distribution of component failure times
- Distribution /Model parameters estimation
- Uncertainty propagation to system reliability and dependence modelling

5.1.1 Framework Steps Overview

In this section of the research, the end goal is to achieve a system reliability with minimal uncertainty and an optimal Accelerated Life Testing Design by optimizing the design parameters which are the stress levels and the number of tests at each stress level for each test specimen that could be a component or a system. In order to find the optimal ALT design parameters, first we need to investigate the uncertainty propagation. We use the probability distribution of failure time data collected from the Accelerated Life Testing (ALT) in order to trace and reduce the uncertainty propagation.

The problem to be solved in this research section is as follows:

- **Given**: Failure Times $(t_{failues})$of Components at different stress levels higher than use stress level
- **Find**: The data distribution equation: $(F(t_{Failure}) and\ f\left(t_{failure} \middle| v\right)\ where\ v\ is\ the\ set\ of\ parameters)$ allowing to reduce the uncertainty in the distribution parameter
- **Find**: Linkage between the component and the system reliability $R(t)$and propagate the uncertainties

Problem Statement 5-1 Component level ALT data uncertainty propagation to system reliability

As shown in Figure 5-1 above, the experimental data follows certain distribution. The experimental data is simply failure time of specimens. Each experimental data set corresponds to a given stress level, at each stress level we have a set of data (failure time) derived from testing a specimen multiple times (the red dots) at that stress level, the data set is then corresponding to a component or a system depending on the level of testing chosen(component level versus system level).

A – 2-Parameters Statistical Distribution and Inverse Power Relationship:

For the purpose of ALT modelling, there is a need to define the parameters of interest. In order to achieve that, a statistical distribution approach has been chosen to model the failure times of the test units. In this research, the ALT model presented in [28] and [108] is employed and is summarized below. The probability distribution function for life time data collected from the ALT testing must be derived in order to capture the parameters that carries uncertainty. The ALT model thereafter takes the location parameter as a function of stress that is a stress dependent parameter and the shape parameter as stress-independent parameter.

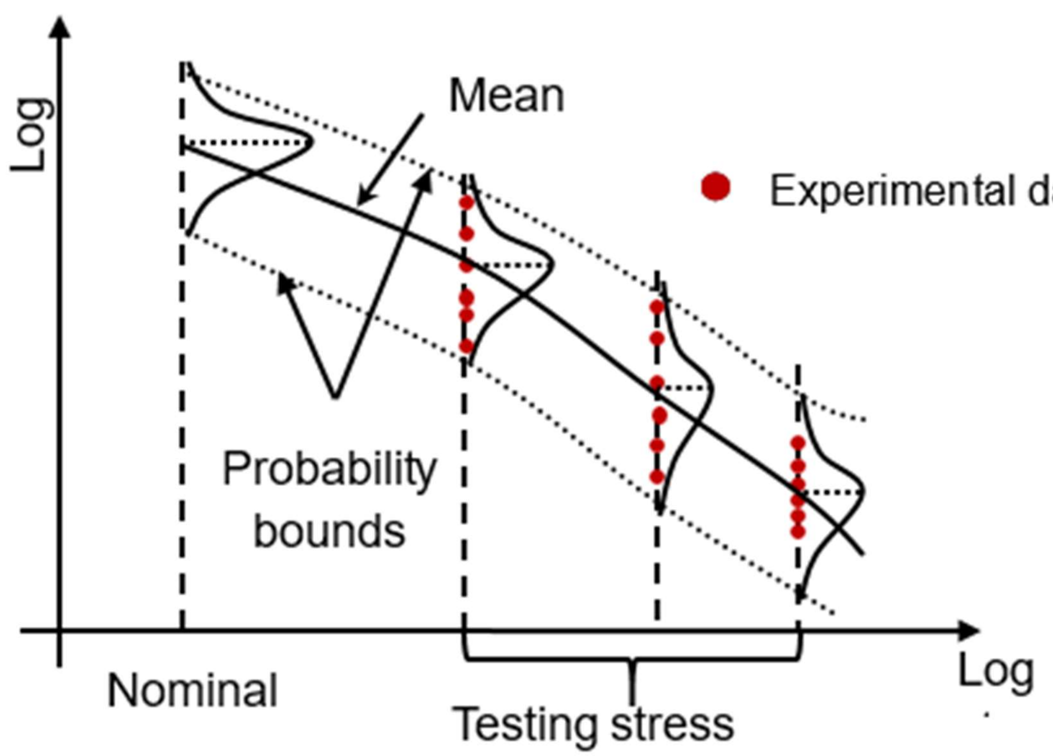


Figure 5-1 ALT experimental data distributions examples

For an engineered system with N_C components, the component-level ALT failure time t are supposed to have a log-location scale distribution and having the following CDF equation:

$$F_T(t|\alpha, \sigma) = G\left[\frac{(log(t) - \alpha)}{\sigma}\right] \tag{5-1}$$

where σ is the scale parameters and is assumed to be stress independent and α is the location parameter of the i-th component which is the stress dependent and is computed using Equation (5-2) shown below:

$$\boldsymbol{\alpha}^{(i)} = \psi_0 + \psi_1 \boldsymbol{S}^{(\boldsymbol{i})} \tag{5-2}$$

we denote the stress on a component during ALT testing by $\boldsymbol{S}^{(\boldsymbol{i})}$.

The testing stress used is bounded by an upper limit and lower limit. We use these limits in order to normalize the actual stress used S to test the specimen.

We designate by $\boldsymbol{\xi}^{(\boldsymbol{i})}$ the normalized stress level of the i-th component, S_L and S_U are respectively the lower and upper bounds of the testing stress level of the i-th component. The normalized stress is then a value between 0 and 1 $\left(\mathbf{0} \leq \boldsymbol{\xi}^{(\mathbf{i})} \leq \mathbf{1}\right)$. The ALT model proposed is valid for up to S_U, that is the validity of the model depends on the accelerated variant is falling within the range $\boldsymbol{S}^{(\boldsymbol{i})} \in [S_L, S_U]$.

The following is based on if the stress goes beyond its upper limit bound, the failure mechanism would change so this condition preserves the failure mechanism. To accommodate for that, we normalize the ALT stress by:

$$\boldsymbol{\xi}^{(\boldsymbol{i})} = \frac{\mathbf{S}^{(\boldsymbol{i})} - S_L}{S_U - S_L} \tag{5-3}$$

According to the latter, the use stress or nominal stress is the lower stress bound of the accelerated variant; $S_{nominal} = S_L$. Which transforms Equation (5-2) to:

$$\boldsymbol{\alpha}^{(i)} = \theta_0 + \theta_1 \boldsymbol{\xi}^{(i)} \tag{5-4}$$

The parameters (θ_0, θ_1) are a re-parametrization of the parameters (ψ_0, ψ_1) given by the following:

$$\begin{cases} \theta_0 = \psi_0 + \psi_1 S_U \\ \theta_1 = \psi_1 (S_U - S_L) \end{cases} \tag{5-5}$$

ALT Design approaches widely use the Weibull Distribution; more information about the distribution could be found in [109]. It is assumed that the general Weibull distribution function of lifetime data of units at test takes the following form [110]:

$$f(t|\, \beta, \lambda) = \begin{cases} \beta\lambda e^{-\lambda\beta} t^{\beta-1} & ; t > 0 \\ 0 & ; t \leq 0 \end{cases} \tag{5-6}$$

the scale parameter λ and β is the shape parameter of the Weibull distribution above that is $Weibull(\beta, \lambda)$. However, to model the ALT, the scope is to use the statistical distribution with a relationship to the accelerated variant as it acts on the failure time. With the aim of modelling the life distributions, we assume that the component level failure time follows a Weibull distribution (i.e. $G[.]$is the Type-I extreme value distribution in Equation (5-1). Based on this assumption, the generalized cumulative distribution function (CDF) is given by:

$$F_T(t|\alpha, \sigma) = 1 - \exp\left(-\left(\frac{t}{exp(\alpha)}\right)^{\frac{1}{\sigma}} \right) \tag{5-7}$$

We note the reliability function based on Equation (5-7) is given by the CDF and reliability relationship $R_T(t|\alpha,\sigma) = 1 - F_T(t|\alpha,\sigma)$, equivalently it could be expressed by the following equation:

$$R_T(t|\alpha,\sigma) = \exp\left(-\left(\frac{t}{exp(\alpha)}\right)^{\frac{1}{\sigma}}\right) \tag{5-8}$$

Accordingly, the hazard function conditioned on the scale parameter and shape parameter of the considered Weibull Distribution is:

$$\lambda_T(t|\alpha,\sigma) = \frac{1}{\sigma \exp(\alpha)}\left(\frac{t}{\exp(\alpha)}\right)^{\frac{1-\sigma}{\sigma}} \tag{5-9}$$

And hence the probability density function (PDF) of the failure time is given by:

$$f_T(t|\alpha,\sigma) = \frac{1}{\sigma \exp(\alpha)}\left(\frac{t}{\exp(\alpha)}\right)^{\frac{1-\sigma}{\sigma}} \exp\left(\left(-\left(\frac{t}{\exp(\alpha)}\right)^{\frac{1}{\sigma}}\right)\right) \tag{5-10}$$

Now that we have developed the PDF and the CDF of the component failure time, we know the parameters that requires processing for uncertainty reduction.

The statistical distribution will be used to fit the data collected from ALT component level testing. And the distribution scale parameter will be used to extrapolate the results from ALT accelerating stress to the nominal stress which the normal stress under which a product will operate in its use environment. Another aspect to be modelled is the dependence, for that the model uses a class of Copula function. More on the Copula function and how it will be used in the context of this research will be found in the following section below.

B – Copula Function

In order to make a complete and realistic reliability assessment, it is necessary to model the dependence between the failure time distributions of different components. In this section we introduce two concepts to model the dependence between the failure time distributions that we intend to use while developing the component to system linkage framework. We present the copula function as a way to model the dependence.

The definition of an n-dimensional Copula function is given by the following [111]:

1. The Copula function is a function from I^n to I : $C: I^n \rightarrow I$; I is the unit interval [0,1]
2. If at least one coordinate of u is zero, $(u = 0)$, then $C(u) = 0$
3. If all coordinates of u are one except u_k then $C(u) = u_k$
4. For every a and b such that $a < b; V_C([a,b]) \geq 0$ where $V_C([a,b] = \Delta_a^b C(t) = \Delta_{a_n}^{b_n} \Delta_{a_{n-1}}^{b_{n-1}} \dots \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} C(t)$; and

$$\Delta_{a_k}^{b_k} C(t) = C(t_1, \dots, t_{k-1}, b_k, t_{k+1}, \dots, t_n) - C(t_1, \dots, t_{k-1}, a_k, t_{k+1}, \dots, t_n).$$

In order to define the equations as used in this research, first we define Sklar's theorem and its corollary. The Sklar's theorem states the following [68]:

Let H be an n-dimensional joint distribution function with margins $F_1, F_2, \dots, F_n$ then there exists an n-copula C such that for all $x \in \boldsymbol{\mathcal{R}}^{\mathbf{n}}$**:**

$$H(x_1, x_2, \dots, x_n) = C\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big) \qquad (5\text{-}11)$$

If C is an n −Copula and $F_1, F_2, \dots, F_n$ are distribution functions, therefore H is a n −dimensional distribution function with margins $F_1, F_2, \dots, F_n$. Based on this theorem, there exists a failure distribution Copula corollary as follows:

Let $H, C, F_1, F_2, \dots, F_n$ hold the same definition as in the theorem above, and let $F_1^{-1}, F_2^{-1}, \dots, F_n^{-1}$ be the quasi-inverse functions of the $F_1, F_2, \dots, F_n$ respectively. Then for any u in I^n :

$$C(u_1, u_2, \dots, u_n) = H\left(F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_n^{-1}(u_n)\right) \tag{5-12}$$

Given the fact that H is an $n-$dimensional joint distribution function with margins $F_1, F_2, \dots, F_n$ then H is defined as:

$$H(x_1, x_2, \dots, x_n) = P[X_1 < x_1, X_2 < x_2, \dots, X_n < x_n] \tag{5-13}$$

Then according to the Sklar's theorem the failure distribution Copula is the $n-$Copula C given by:

$$H(x_1, x_2, \dots, x_n) = C\left(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\right) = P[X_1 < x_1, X_2 < x_2, \dots, X_n < x_n] \tag{5-14}$$

Let $f(x_1, x_2, \dots, x_n)$ denote the joint probability distribution function of $X_1, X_2, \dots, X_n$, the PDF is given by:

$$f(x_1, x_2, \dots, x_n) = c(x_1, x_2, \dots, x_n) \prod_{i=1}^{n} f_i(x_i) \tag{5-15}$$

where $f_i(x_i)$ PDF of x_i and $c(x_1, x_2, \dots, x_n)$ is given by:

$$c(x_1, x_2, \dots, x_n) = \frac{\partial^n C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))}{\partial F_1(x_1) \partial F_2(x_2) \dots \partial F_n(x_n)} \tag{5-16}$$

Subsequently, to put things in the research context notation and as defined previously, a copula function describes the dependence between random variables by connecting the marginal CDFs to the joint cumulative distribution function [112]. The Copula is a multivariate cumulative distribution function which is used to describe the dependence between random variables in the CDF domain. There are various parametric Copula functions with a parameter describing the strength of dependence. Using the copula function concept to the component failure time distributions, the Copula function is written as:

$$\begin{aligned} Pr\{T_1 \le t_1, T_2 \le t_2, \dots, T_{N_c} \le t_{N_c}\} &= C\left(F_{T_1}(t_1), F_{T_2}(t_2), \dots, F_{T_{N_c}}(t_{N_C}); \boldsymbol{\rho}\right) \\ &= C(u_1, u_2, \dots, u_{N_c}; \boldsymbol{\rho}) \end{aligned} \tag{5-17}$$

where Pr{.} is the probability operator, $C(.;\ \rho)$ is a Copula function that takes dependence strength parameters $\rho = \ \rho_{12}, \rho_{13}, \dots, \rho_{ij}\ , i, j = 1,2,3, \dots., N_C$ and marginal CDF values of the *i*-th component $u_i = F_{T_i}(t_i), i = 1,2,3, \dots, N_c$ as detailed in Equation (5-7). These marginal distributions are referred to sometimes as Copulae.

The corresponding joint PDF of the component failure time is given by:

$$f_T(t_1, t_2, \dots, t_{N_C}) = f_{T_1}(t_1) f_{T_2}(t_2) \dots . f_{T_{N_C}}(t_{N_c}) c(u_1, u_2, \dots, u_{N_C};\ \boldsymbol{\rho}) \tag{5-18}$$

where $f_T(.)$ is the margin PDF as given in Equation (5-10) and $c(., \rho)$ is the Copula function of the marginal CDF u_i.

Copula functions are well-studied for bivariate cases, except the Gaussian copula and student's t copula function. Even though Vine copula approach has been developed to make it possible to model the high-dimensional non-linear dependences among a large number of random variables [113], [114]the implementation procedure is complicated. In this research, for the sake of illustration, the Gaussian copula is employed to model the dependence between the failure time distributions of different components. Using Gaussian copula, the joint CDF given in Equation (5-19) is rewritten as stated by:

$$Pr\{T_1 \le t_1, T_2 \le t_2, \dots, T_{N_c} \le t_{N_c}\} = \ \Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \dots, \Phi^{-1}(u_{N_c});\ \boldsymbol{\rho}) \tag{5-19}$$

In Equation (5-19) $\Phi^{-1}(.)$ is the inverse CDF distribution of a standard normal variable and $\Phi_{\text{p}}(.)$ is the CDF of standard multivariate normal distribution.

The joint PDF $f_T\left(t_1, t_2, \dots, t_{N_C}\right)$ of the failure time of N_c components is represented using Gaussian copula as:

$$
\begin{aligned}
f_T\left(t_1, t_2, \dots, t_{N_C}\right) = {} & f_{T_1}(t_1) f_{T_2}(t_2) \dots . f_{T_{N_C}}\left(t_{N_c}\right) \\
& \times \frac{\partial \Phi^{-1}(u_1)}{\partial u_1} \frac{\partial \Phi^{-1}(u_2)}{\partial u_2} \dots \frac{\partial \Phi^{-1}(u_{N_c})}{\partial u_{N_C}} \\
& \times \phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \dots, \Phi^{-1}\left(u_{N_c}\right);\ \boldsymbol{\rho})
\end{aligned}
\tag{5-20}
$$

$\phi_{\boldsymbol{\rho}}(.)$ is the is the joint PDF of multivariate standard normal variables.

In this research and in order to use the Copula function, we assume that the parameters ρ are all stress independent as argued in reference [111]. The s −dependent factors causing the dependence among the components of the system are assumed to stay the same under normal operation stress and therefore they are assumed to hold this property under ALT stress. Hence the dependence among the components under test stays the same as stress varies because the factors causing this dependence are regarded as constant or unalterable.

5.1.2 Proposed Framework Assumptions

In this section we present the assumptions made in order to develop the ALT framework via parametric statistical distribution and the copula function. The main assumptions of the model are listed below in this separate section. Some minor assumptions are mentioned as needed while explaining the mathematical formulation. The main assumptions for this model are listed below:

A1- The Log-Location Scale Distribution Assumption: To model the life distribution which is the failure time distribution we follow a widely used assumption in the ALT design, so we assume that the component-level failure time follows a Log-Location scale distribution, this assumption is one of the most widely used assumptions in ALT design and is acceptable to both academia and industry. Correspondingly, at any stress ξ the failure time follows a Log-Location scale distribution. The distribution is then assumed to be a Weibull distribution with Stress dependent parameter α and a stress independent parameter σ. We assume that all components $i = 1,2,..,N_C$ and the system failure times follows a Weibull distribution.

A2- Stress Independent Distribution Type Assumption: The distribution family does not change when varying the stress level. That is, the cumulative distribution probability function (CDF $F(.)$) , the probability density function (PDF $f(.)$) , the reliability function ($R(T)$), and the failure rate function of any component i ; $i = 1,2 \dots, N_c$ stay the same for all $\mathbf{S}^{(i)} = \{S_1^{(i)}, S_2^{(i)}, \dots, S_j^{(i)}, .. S_{n_s}^{(i)}\}$.

A3- Shape Parameter Assumption: The shape parameter of the distribution of choice which is the Weibull distribution does not vary with the stress levels $\mathbf{S}^{(i)} = \{S_1^{(i)}, S_2^{(i)}, \dots, S_j^{(i)}, .. S_{n_s}^{(i)}\}$ of each unit. That assumption is based on that the failure mode under $S_1^{(i)}, \dots, S_{n_s}^{(i)}$ does not change and remains the same at all testing stress levels of the unit.

A4- System Topography Assumption: The first assumption is in regard to the system configuration that explains how the components are placed in a system/ The system topology (i.e. components could be assembled in series, in parallel or in any different configuration) is pre-defined and identifiable. The system topology is important to identify what contributes to the system failure (which component(s) shall fail to cause a system failure). In addition to that, in case the system topography is a custom configuration, the methodology assumes that a function representing this specific configuration could be identifiable and known to describe the system failure in terms of the components failure times.

A5- ALT Feasibility Assumption: For the accelerated life testing to be feasible, an assumption about the testing of components or systems is made. Accordingly, all components and systems are testable and ALT testing is feasible to collect failure time data. That is, the product under study for which an estimation of the reliability is to be determined, is assumed to be testable and failure data could be collected and recorded followed by a data fitting in each of the models prescribed by research. System and component are set to fail if any one of the failure modes/competing risks takes place.

A6- Load Transfer Function Assumption: When conducting a system level testing, we assume that the load distribution is identifiable and could be described mathematically via a parametric function that could take any form. For ALT system level testing, the accelerated variant (i.e. stress) that is received by components referred to as boundary components is assumed transferable via physics informed model to the non-boundary components or the components that does not receive a direct load during the ALT testing. The load transfer function to calculate the non-boundary stress is identified in the design phase or based on the component layout to form the system. Also, it could be a function describing the performance (Mechanical, chemical, electrical or other) by propagating the loads among the components.

A7- Censoring Assumption: We assume that there is no censoring and that all testing unit would fail at the end of the test. Censoring could be easily incorporated in the model and in the likelihood function. All items are removed at the end of a timed test, that is every item is assumed to fail at time $t_{end\ of\ test.}$ Equivalently, for right censoring that would be $t_i = \min(T_i, C_i)$, as shown in [87] ,where i is the testing unit and t_i denotes the survival time and C is the censoring time and T is the lifetime of a test unit:

$$\delta_i = \begin{cases} 1\ if\ T_i \leq C_i\ , That\ is\ event\ is\ not\ censored \\ 0\ if\ T_i > C_i\ that\ is\ the\ event\ is\ censored \end{cases} \tag{5-21}$$

Accordingly, the full conditional likelihood function of M testing units given some parameters denoted by a vector **V** including right censoring takes the following form:

$$L_c = \prod_{i=1}^{M} [h(t_i|\mathbf{V})]^{\delta_i} R(t_i|\mathbf{V}) \tag{5-22}$$

So, in this research we assume that the censoring indicator $\delta_i = 1$ at all time, that is the failure event of the unit is always taking place before it reaches the censoring time C. Many researchers have tackled different types of censoring, for an example of ALT with interval censoring with a statistical parametric distribution (i.e. Weibull) modelling one could refer to [115]. On the other hand, [116] presents an exponential distribution Bayesian model for step-stress ALT with progressive Type I censoring

A8- Constant ALT Accelerated Stress Assumption: Testing units (i.e. components or systems) are tested using a constant stress as the accelerating factor, the stress is not a function of time and each testing unit is tested at the same accelerating stress multiple times until it reaches failure so that the failure time is not censored.

A9- Prior Information Availability Assumption: As stated before, the Bayesian estimation requires defining prior information about the parameters that are to be inferred or estimated. Accordingly, this information is assumed accessible and available and is allowed to follow a parametric distribution. Prior distributions can be found from available resources or by referring to experts in the field.

A10- Extrapolation of ALT Failures to Normal Operation Assumption: The failure modes of the testing unit in the use field could be recreated in the ALT setting. This assumption allows us to predict the system reliability using the ALT data due to the same failure mode. It is necessary when extrapolating the information learned in ALT to make sure the failure modes are the same in both environment: the accelerated environment as well as the use environment that is the failure modes experienced by a product under normal operations could be emulated during the ALT allowing an apple to apple comparison.

A11- Copula Correlation Factor Prior Info Assumption: The model in this section uses the Copula as the dependence modelling method among the N_C components of the system. Prior information about the correlation factor $\boldsymbol{\rho}_{\boldsymbol{i},\boldsymbol{u}}; i, u = (1,2, \dots, N_c)$ between any two components or the failure modes of the same components are available. If not in a distribution form, the correlation factor is assumed to be identifiable, predefined or could be calculated.

5.1.3 Likelihood and Bayesian Inference Via Log-Scale Distribution

The symbol t denotes the failure time data collected from ALT component level testing or testing stage 2. Each component is tested at different stress levels $\mathbf{S}^{(i)}$ which will be normalized using the Equation (5-3) by setting the lower and upper limits of the testing stress, denoted respectively as S_{L_i} and S_{U_i}. With that being said, $\mathbf{S}^{(i)}$ is a vector of the stress levels at which a component of the system, which is the test unit or specimen, is tested.

Additionally, at each stress level, each of the components in a system is tested multiple times, in other words, at each designated component accelerating stress level, we use multiple test units of the same component and collect the failure times accordingly. Based on that, $\boldsymbol{S}^{(i)} = \left[S_1^{(i)}, S_2^{(i)}, \dots, S_{n_s}^{(i)}\right]$, where n_s is the total number of the stress levels for component i. Given the fact that the stresses will be normalized using the upper and lower bounds, the corresponding vector of normalized accelerated stresses for a component i is hence denoted by $\boldsymbol{\xi}^{(i)} = \left[\xi_1^{(i)}, \dots, \xi_{n_s}^{(i)}\right]$.

Each component i is then tested separately, at different stress levels j, and at each of the stress levels, the component is tested n_j^i times, where n_j^i represents the number of tests at the j^{th} stress level of component i, $i = 1,2, \dots, N_C$ and $j = 1,2, .., n_s$. Hence, for each $\xi_j^{(i)}$, component i is tested $n_j^{(i)}$ times at stress level j. So, for each component the total number of test units at each stress level is represented by the vector $\mathbf{n}^{(i)} = \left[n_1^{(i)}, \dots., n_{n_s}^{(i)}\right]$.

The data is failure times corresponding to the specimen at test and are grouped in vectors where $\mathbf{t}^{(i)}$ is the set of failure times vectors corresponding to component i, where each sub-set vector is for a stress level, that is $\mathbf{t}^{(i)} = \{\mathbf{t}_1^{(i)}, \mathbf{t}_2^{(i)}, \dots, \mathbf{t}_{n_s}^{(i)}\}$ and in turn $\mathbf{t}_j^{(i)} = \left[t_{j,1}^i, t_{j,2}^i, \dots., t_{j,n_j^{(i)}}^i\right]$.

To better illustrate the indexing of the parameters in this framework, Figure 5-2 explains the indexing and terminology of the data collected by taking a system of three components for simplicity. That is the total number of components $N_C = 3$ and $i = [1,2,3]$. Taking component 1, we see that it is tested under 3 stress levels and hence $j = 1,2,3$ where $n_s = 3$. At stress level 1, it shown that it is tested 3 times that is three test units are tested at stress level $j = 1$, so $n_1^{(1)} = 3$. Similarly, at stress level $j = 2$, the component $i = 1$ is tested three times and then $n_2^{(1)} = 3$

and last but not least the component $i = 1$ is tested two times under the accelerated stress $n_3^{(1)} = 2$. Accordingly, the set $\mathbf{n^{(1)}} = [3,3,2]$. The failure time data for component $i = 1$ are grouped in $\mathbf{t^{(1)}} = \{\mathbf{t_1^{(1)}}, \mathbf{t_2^{(1)}}, \mathbf{t_3^{(1)}}\}$ where $\mathbf{t_1^{(1)}} = \left[t_{1,1}^{(1)}, t_{1,2}^{(1)}, t_{1,3}^{(1)}\right]$ is the vector of failure times of component $i = 1$ containing the failure times of three specimen tested at stress level $j = 1.$, $\mathbf{t_2^{(1)}} = \left[t_{2,1}^{(1)}, t_{2,2}^{(1)}, t_{2,3}^{(1)}\right]$ corresponds to testing data of three specimen of component $i = 1$ tested at stress level $j = 2$ and $\boldsymbol{t_3^{(1)}} = \left[t_{3,1}^{(1)}, t_{3,2}^{(1)}\right]$ is the vector of component ALT data of two specimens of component $i = 1$ tested twice at stress level $j = 3$. For component $i = 2$ and $i = 3$, refer to Figure 5-2.

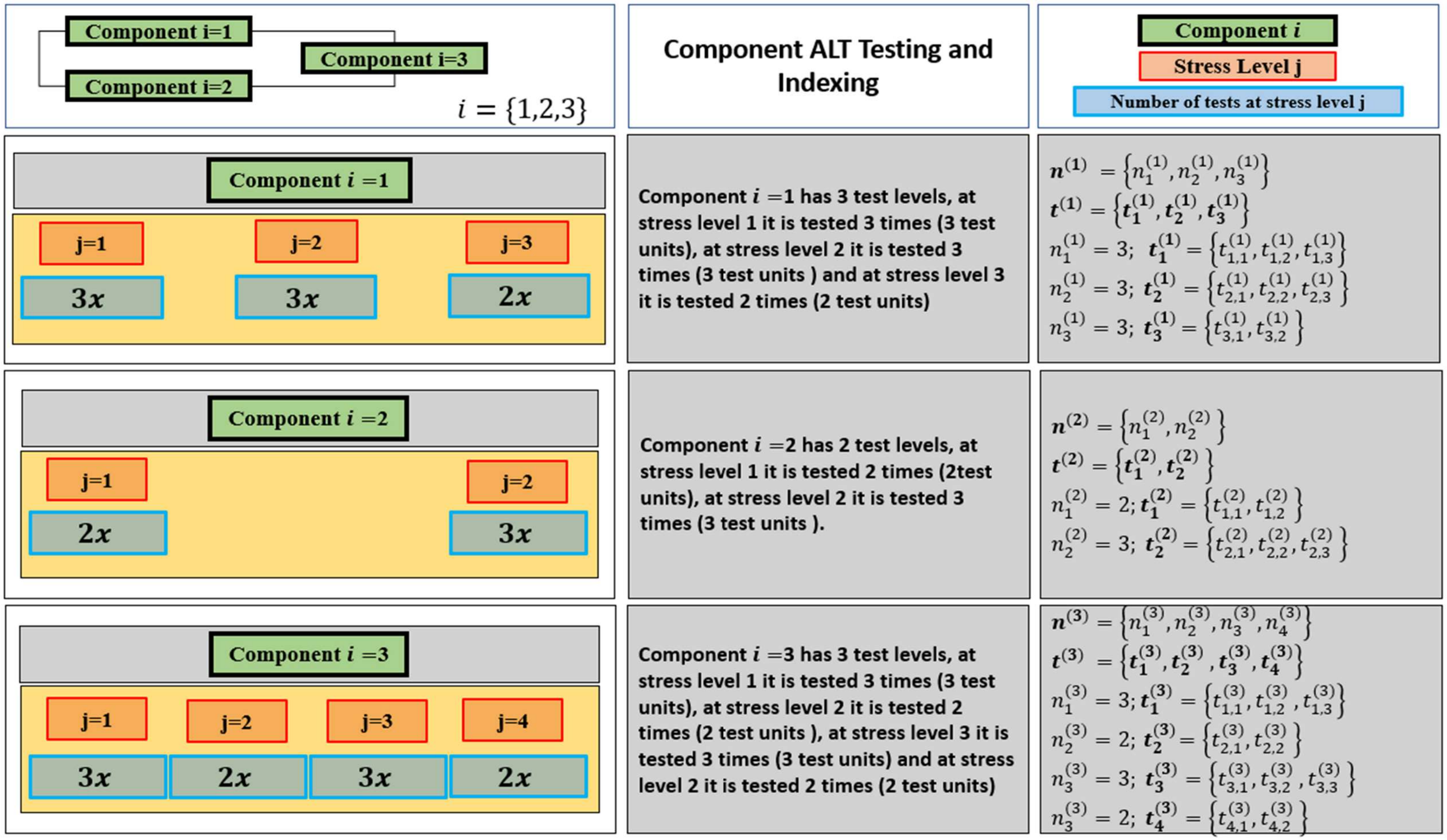


Figure 5-2 Component-Level ALT concept and indexing of a system of 3 component

A – Bayesian Estimation Formulation

As presented in Section 4.1.4A, the Bayesian estimation method will be used to reduce the uncertainty in the parameters using the prior information of the stress-dependent parameter α of the Weibull distribution as defined by Equation (5-4) and the stress independent parameter denoted by σ. In order to sample the posterior distributions of these parameters, the Bayesian method is applied based on Equation (4-11). Using the Weibull parametrized distribution this Equation becomes as:

$$f(\boldsymbol{\theta}^{(i)}, \sigma^{(i)} | \mathbf{t}^{(i)}, \boldsymbol{\xi}^{(i)}, \boldsymbol{n}^{(i)}) \\ \propto f\left(\mathbf{t}^{(i)} \middle| \boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \boldsymbol{\xi}^{(i)}, \boldsymbol{n}^{(i)}\right) f_{\theta i}\left(\boldsymbol{\theta}^{(i)}\right) f_{\sigma i}\left(\sigma^{(i)}\right) \tag{5-23}$$

In which $f\left(\mathbf{t}^{(i)} \middle| \boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \boldsymbol{\xi}^{(i)}, \boldsymbol{n}^{(i)}\right)$ represents the conditional likelihood function and $f_{\theta i}\left(\boldsymbol{\theta}^{(i)}\right)$, $f_{\sigma i}\left(\sigma^{(i)}\right)$ are the prior distributions of the $\boldsymbol{\theta}^{(i)}$ and $\sigma^{(i)}$.

Note that in Equation (5-23) the parameter $\boldsymbol{\theta}^{(i)} = [\theta_0^{(i)}, \theta_1^{(i)}]$ rather than $\boldsymbol{\alpha}^{(i)}$ because the stress dependent parameter is a function of the theta parameter and the uncertainty could be propagated from the term $\boldsymbol{\theta}^{(i)}$ to $\boldsymbol{\alpha}^{(i)}$which is the vector of parameters characterizing the stress-dependent Weibull parameters as given by :

$$\boldsymbol{\alpha}^{(i)} = \theta_0^{(i)} + \theta_1^{(i)} \boldsymbol{\xi}^{(i)} \tag{5-24}$$

So,

at each stress level $S_j^{(i)}$ for component i normalized by the upper bound and lower bound of that accelerating factor and converted to $\xi_j^{(i)}$, there exist a stress dependent parameter that could be calculated using the parameter $\boldsymbol{\theta}^{(i)}$ using $\alpha_j^{(i)} = \theta_0^{(i)} + \theta_1^{(1)} \xi_j^{(i)}$.

Accordingly, Equation (5-24) is the set of all stress dependent parameters $\boldsymbol{\alpha}^{(i)} = [\ \alpha_1^{(i)}$, $\alpha_2^{(i)}$,…, $\alpha_{n_s}^{(i)}$] (i.e. location parameter) of a component calculated using the vector of all stress levels applied to that component during ALT which is normalized by $\boldsymbol{S_{Li}}$ and $\boldsymbol{S_{Ui}}$ to give $\boldsymbol{\xi}^{(i)} = \frac{\boldsymbol{S}^{(i)} - \boldsymbol{S_{Li}}}{\boldsymbol{S_{Ui}} - \boldsymbol{S_{Li}}}$.

In the next section, the likelihood function is formulated using the parametric form a Weibull distribution probability density function.

B – Likelihood Function Formulation

In order to develop the likelihood function, the Weibull distribution is used to formulate the probability density function as given by its general form in Equation (5-6). The likelihood function is based on the conditional probability concept, where the failure time of each component is a data point conditioned on the model parameters: the stress-dependent parameter α of the Weibull distribution as defined by Equation (5-4), the stress independent parameter denoted by σ, the design parameters of the ALT: ξ and n which denotes the normalized accelerated factor or stress applied during ALT and the number of specimen needed at each stress level. The problem formulation is based on multiple components comprised under one system and as mentioned earlier, ALT design approaches widely use the Weibull Distribution to model life distributions, we assume that the component level failure time follows a Weibull distribution (i.e. $G[.]$is the Type-I extreme value distribution in Equation (5-1)). Based on this assumption, the generalized probability density function (PDF) of the failure time of any given component is given by Equation (5-17).

The term $f\left(\mathbf{t}^{(i)} \middle| \boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \boldsymbol{\xi}^{(i)}, \boldsymbol{n}^{(i)}\right)$ in Equation represents the conditional likelihood function and is given by the following product representation:

$$f\left(\mathbf{t}^{(i)} \middle| \boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \boldsymbol{\xi}^{(i)}, \mathbf{n}^{(i)}\right) = \prod_{j=1}^{n_s} \prod_{k=1}^{n_j^{(i)}} f\left(t_j^{(i)}(k) \middle| \boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \xi_j^{(i)}\right) \tag{5-25}$$

Applying the aforementioned equation to multiple components, the following equation represents the conditional probability density function of the failure time of a component i among N_C components tested $n^{(i)}$ times under a stress $\xi^{(i)}$. So the following $f\left(t_j^{(i)}(k) \middle| \boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \xi_j^{(i)}\right)$ is found by using the skeleton of Equation (5-10) and is given by:

$$f\left(t_j^{(i)}(k) \middle| \boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \xi_j^{(i)}\right) = \frac{1}{\sigma^{(i)} \exp\left(\theta_0^{(i)} + \theta_1^{(1)} \xi_j^{(i)}\right)} . M . N \tag{5-26}$$

where M and N are defined by:

$$M = \exp\left(-\left(\frac{t}{\exp\left(\theta_0^{(i)} + \theta_1^{(1)} \xi_j^{(i)}\right)}\right)^{\frac{1}{\sigma^{(i)}}}\right)$$

And,

$$N = \left(\frac{t}{\exp\left(\theta_0^{(i)} + \theta_1^{(1)} \xi_j^{(i)}\right)}\right)^{\frac{1-\sigma^{(i)}}{\sigma^{(i)}}}$$

In the above equations, following the censored data assumptions, we assume that all the components are tested to failure and hence no censored data is available. The above equations can be easily modified to include censored data following the procedure in [117].

Up until this point in the formulation of the methodology, the likelihood has been established and the Bayesian estimation relationship could be applied to sample posterior distributions for the parameters of the distribution and namely the stress dependent (location parameter) and the stress independent parameters given the failure time of the components at test. For that purpose, the particle filtering method could be applied as illustrated in Section 4.1.3A.

After estimating the parameters $\sigma^{(i)}$ and $\boldsymbol{\theta}^{(i)}$ for all components ($\forall i = 1,2, \dots, N_C$), and in turn calculate the location parameter $\boldsymbol{\alpha}^{(i)}$ using $\boldsymbol{\theta}^{(i)}$ as demonstrated previously, in the next steps, the uncertainty is propagated from the component level (testing level 4) to the system level in order to find an estimate to the system reliability R_S. In the next section, the uncertainty propagation method is illustrated.

5.1.4 Uncertainty Propagation to System Reliability Using Copula Function

After the Bayesian updating of the distribution parameters: $\boldsymbol{\theta}^{(i)}$ and $\sigma^{(i)}, \forall i = 1,2, \dots, N_c$ of the component failure time distributions parameters, the uncertainty in these parameters are propagated to the system reliability (R_s).

Now, in this section we detail the plan in how we intend to propagate the uncertainty in the updated parameters. The following steps explain the plan of linking the component-level information to the system reliability R_S for the model with copula-based dependence modelling.

Step 1 As shown earlier under the copula function dependence section B – I , we have a ρ parameters included in the Gaussian-Copula function, so for a given set of the $\boldsymbol{\rho}$ parameters, we first generate samples of component failure time CDFs and denote the generated CDF samples as $u_{MCS}^{(i)}, i = 1,2,3, \dots, N_C$ where $u_{MCS}^{(i)}$ is the CDF samples of the i^{th} component.

Step 2 In this step, we make use of the posterior samples derived for the distribution parameters updated via Bayesian Inference. We denote these posterior samples by $\boldsymbol{\theta}^{(i)}$ and $\sigma^{(i)}, \forall i = 1,2, \dots, N_c / \forall q = 1,2, \dots, n_{post}$ where n_{post} is the number of posterior samples. Using the CDF function as shown in Equation (6), we generate T_i samples at the use stress (the intended nominal stress at which the component will operate normally in its normal working conditions) using the following equation:

$$t_{MCS}^{(i)} = \exp\,(\alpha_0^{(i)}(q))(-\ln\left(1-u_{MCS}^{(i)}\right))^{\sigma^{(i)}(q)} \tag{5-27}$$

$\alpha_0^{(i)} = \theta_0^{(i)}(q)$ is the $\alpha^{(i)}$ at the nominal stress level, $u_{MCS}^{(i)} = \left[\,u_{MCS}^{(i)}(1), u_{MCS}^{(i)}, \dots, u_{MCS}^{(i)}(n_{MCS})\right]$ and n_{MCS} is the number of MCS samples.

Step 3 Now, at this step we have the samples $t_{MCS}^{(i)}$,, $\forall i = 1,2,\dots,N_c$ generated by using Equation (15) in Step 2, the system reliability $R_S$is calculated depending on the system topology. The system topology could be multiple components put together according to a standard configuration: a parallel configuration or a series configuration or defined according to a special design. The system topology defines the failure of the system according to the configuration of its components and hence it, as well, defines its probability of no failure or system reliability.

$$R_S \approx \frac{1}{n_{MCS}} \sum_{j=1}^{n_{MCS}} I\left(t_{MCS}^{(1)}(j), t_{MCS}^{(2)}(j), \dots, t_{MCS}^{(N_C)}(j)\right) \tag{5-28}$$

where $I\{.\}$ is a failure indicator function derived according to the system topology. For example, for a series system the indicator function is defined as:

$$I\left(t_{MCS}^{(1)}(j), t_{MCS}^{(2)}(j), \dots, t_{MCS}^{(N_C)}(j)\right) = \begin{cases} 1, if\ \exists i, t_{MCS}^{(i)}(j) < T_e \\ 0, Otherwise \end{cases} \tag{5-29}$$

The three steps above are repeated for all the posterior samples updated by Bayesian Inference, $\mathbf{\theta}^{(i)}$ and $\sigma^{(i)}, \forall i = 1,2, \dots, N_c / \forall q = 1,2, \dots, n_{post}$. Afterwards, we obtain Samples for R_S as $R_s(q)$, $q = 1,2, \dots, n_{post}$. The effect of uncertainty reduction could be now quantified using the R_S samples.

Figure 5-3 below is a flowchart summarizing the steps in a graphical format for the steps detailed above. Starting with the Bayesian estimation in order to update the prior data of the $\boldsymbol{\theta}^{(i)}$ and $\sigma^{(i)}$, starting with component $i = 1$ all the way to component $i = N_C$ which will allow sampling posterior data with reduced uncertainty. Once the parameters are updated, the propagation of uncertainty is performed by calculating the stress dependent parameter $\boldsymbol{\alpha}^{(i)}$ for all component, $i = 1,2, \dots, N_C$ at nominal stress which is defined as the use stress at which the component will operate under normal conditions in an environment in which it is designed to function.

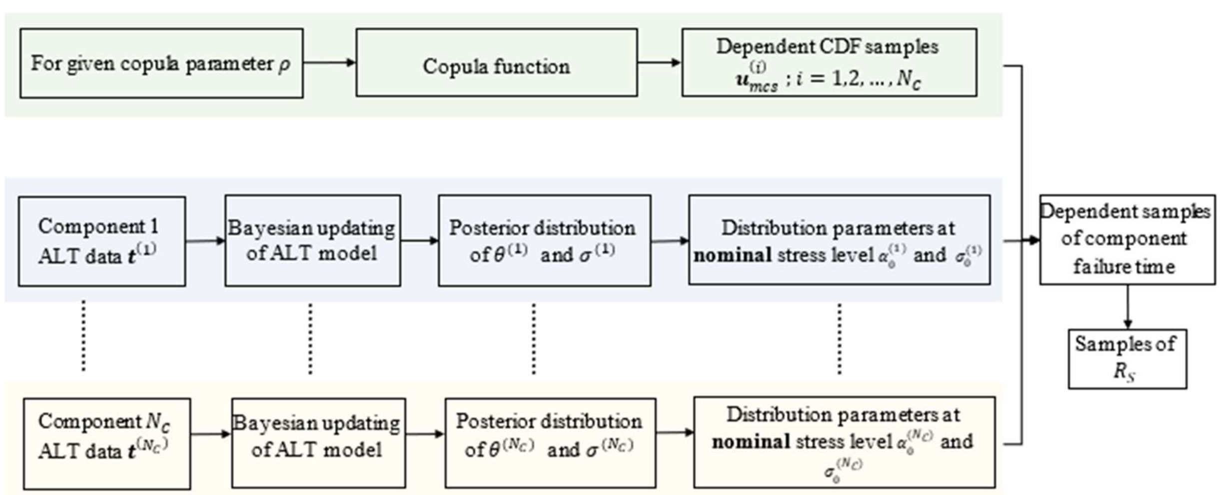


Figure 5-3 Flowchart of connecting component-level ALT data with system reliability using Copula function

Considering the concept of dependence among these components, the copula function is employed as well in order to close the loop and propagate the uncertainty to the system reliability R_S. The copula allows modelling the possible correlations among the components by generating dependent CDF samples using the posterior distribution of the distribution parameters at nominal stress.

5.2 Uncertainty Propagation of System Level ALT Data to System Reliability

In what follows, we detail the plan for connecting the testing data collected by testing the system at high stress levels to the system reliability at nominal stress levels or use stress. The uncertainty propagation in this part consists of using the system level ALT data by applying parametric Weibull distribution. The system level ALT data is the testing data collected from putting a system of multiple components under accelerated life testing which is the fourth test level as shown in Chapter 3 in Figure 3-1.

5.2.1 Framework Steps Overview

When testing a system that comprises multiple components, the analysis is subject to some complications due to the following reasons:

1- Due to the various components under one system, the failure modes of each component might be different because the load applied to the system is not evenly distributed on all components, so the stress-life relationship is not straightforward and easily derived.

2- Testing the whole system at once at higher than nominal stress to accelerate the failure, imposes stress on some component that we refer to as boundary component, which is defined as the component receiving the load directly during a system-level testing, while other components, receive a cascaded or extrapolated stress and we refer to them by non-boundary components. Having this concept of boundary versus non-boundary components requires a mapping of stresses from the boundary to non-boundary components. Mapping the stresses means finding a framework to calculate the load carried by non-boundary components by using the properties and stress load on the boundary components.

In order to resolve the above-mentioned complications, we bridge the gap to connect the ALT system level data to the actual system reliability by using physics-informed model which resolves the mapping of stresses from the boundary components to the non-boundary components.

According to the latter, the problem statement for this section is as follows:

- **Given**: Failure Times $(t_{failues})$of System at different stress levels higher than use stress
- **Find**: Map stresses from boundary components to non-boundary components via physics-informed model and develop the likelihood function to establish the Bayesian Inference
- **Find**: Linkage between the system level ALT data and the system reliability $R(t)$and propagation of the uncertainties

Problem Statement 5-2 System level ALT data uncertainty propagation to system reliability

Step 1 Map the loads from boundary components to non-boundary components using Equation (20). The mapping should be done at all stress levels to all components present in the system.

Step 2 Calculate the stress-dependent parameter $\alpha_s^{(i)}(j)$ using ξ_{bi} of stress level j if the component is identified as boundary component, and if the component is identified as non-boundary, we use the mapped stresses predicted using the physics informed model, symbolized by ξ_{bi-} of stress level j.

Step 3 Using the calculated $\alpha_s^{(i)}(j)$ and $\sigma^{(i)}, i = 1,2, \ldots, N_c$ of component failure time distributions, we then generate random MCS sample ($t_{MCS,j}^{(i)}$)of each of the component's failure time at the j-th stress level.

Step 4 At this step, we would need to generate CDF distributions to complete Step 3, because the equation for $t_{MCS,j}^{(i)}$ following the copula-based model are generated for a given ρ.

Step 5 Given the set of $t_{MCS,j}^{(i)}$ we can now convert the component failure time to system failure time using a function ($f_{time}(.)$) defined according to the system topology or configuration (i.e. series, parallel, custom configuration).

Step 6 The linkage is done by calculating the likelihood $f(t_{sys,j}|(.))$.

5.2.2 Physics-Informed Model

The Accelerated Life Testing (ALT) design has been the center of attention of a wide range of studies in the past decades for its usefulness in the reliability analysis. However, it has focused on adopting statistical strategies. On the other hand, the physics-informed prediction modelling has not been widely leveraged in the ALT design. Recently, some researchers shed light on the practicality of the physics-informed models in ALT design for the rich advantages it returns in terms of the physical information in ALT design [118][119]. Principles used in this type of modelling include, but not limited to, analytical methods [120]and data-driven approaches [121], computer simulations (i.e. Finite Element Analysis (FEA)) [122].

Taking the mechanical system given in Figure 5.4 as an example, there are six types of components in the system, namely, component 1 (i.e., crank), component 2 (i.e., rigid connecting rod 1), component 3 (i.e., connecting rod 2), component 4 (i.e., horizontal sliding bar), component 5 (i.e., slider), and component 6 (i.e., pin). In the nominal operation condition, the crank is rotating at a specific speed. The rotating crank then drives the movements of the other five components. In component-level ALT, the component reliability of these six components can be tested separately.

When the six components are assembled together and tested as a whole, the crank (i.e., component 1) is then the boundary component and the other components are non-boundary components. For this type of mechanical machine system, the "stress" or "accelerating load" is the rotating speed of the crank. Suppose that the rotating speed of the crank is 10 rad/s at the nominal condition, it could be 100 rad/s in the accelerated situation in order to induce failures. For a given accelerated rotating speed of the crank (i.e., boundary component), the rotating speed of component 2, sliding speed of component 4 and 5, and the movement of other non-boundary components can be predicted using physics-based kinetic analysis. It means that we are able to predict the testing load conditions of the non-boundary components based on the applied accelerated loads (e.g., rotating speed) of boundary components in system ALT using physics-based analysis. Note that the example given below is only used for illustration purpose. For different types of systems, different physic- based approaches For different types of systems, different physics-based approaches are needed to perform this type of load analysis in system ALT.

Defining the testing stress levels, which is the rotating speed of component 1 in Figure 5-4, of the boundary components in system-level tests denoted by $\boldsymbol{\xi}_{\boldsymbol{b}}$ the corresponding stress levels denoted by $\boldsymbol{\xi}_{\boldsymbol{b}-}$, which is the rotating speed of component 2 and sliding speed of the slider in Figure 5-4,of the non-boundary components excluding the boundary components are predicted using physics-informed load analysis.

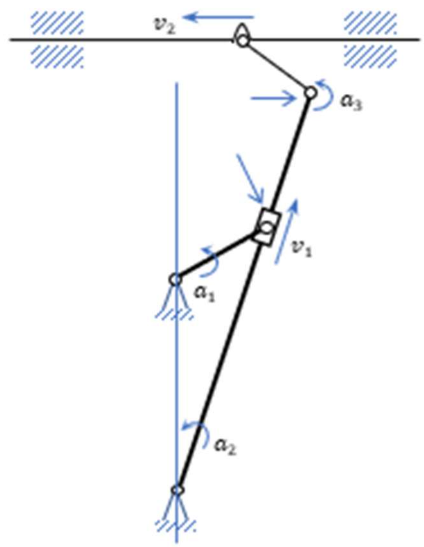


Figure 5-4 Illustration of a mechanical system

To remember, the testing stress in an ALT design are normalized and symbolized by ξ, so assuming we have n_b boundary components, we denote the stresses of the boundary components by $\boldsymbol{\xi_b} = [\xi_{b1}, \dots, \xi_{bn_b}]$ and the non-boundary components stress by $\boldsymbol{\xi_{b-}} = [\xi_{b1-}, \dots, \xi_{bn_{b-}}]$ where n_{b-} is the total number of non-boundary component. Now, the problem is to use physics-informed model to predict $\boldsymbol{\xi_{b-}} = [\xi_{b1-}, \dots, \xi_{bn_{b-}}]$ by using $\boldsymbol{\xi_b} = [\xi_{b1}, \dots, \xi_{bn_b}]$. So we can write the following equation:

$$\boldsymbol{\xi_{bj-}} = L_{bj-}(\boldsymbol{\xi_b}, \boldsymbol{\omega}^{(j)}), \forall j = 1,2, \dots, n_{b-} \tag{5-30}$$

In Equation (5-19) , $\boldsymbol{\omega}^{(i)}$ is a set of deterministic and random parameters representing uncertainty for situation in which the load prediction models cannot accurately predict the load conditions of non-boundary components [123], [124], $\boldsymbol{\xi}_{bj-}$ is the j-th element of $\boldsymbol{\xi}_{b-}$ and $n_{b-} = N_C - n_b$ represents the total of non-boundary components and $L_{bj-}(.)$is the set of load prediction models used for stress mapping from $\boldsymbol{\xi}_{\mathbf{b}}$ to $\boldsymbol{\xi}_{\mathbf{bj}-}, \forall j = 1,2, \dots, n_{b-}$. These models can be obtained via computer simulation models, analytical models or data-driven models.

At nominal stress, the physics -informed model shall satisfy the following condition:

$$0 = L_{bj-}\left(0, \boldsymbol{\omega}^{(j)}\right), \forall j = 1,2, \dots, n_{b-} \tag{5-31}$$

The latter applies for both copula-based and frailty-based dependence as part of the big model for the connection of system ALT data to actual system reliability in order to map the stresses from the boundary components in a system to the non-boundary components operating under the same system.

5.2.3 Bayesian Estimation Formulation

The fact that the whole system is put under testing, the dependence among the components is accounted for. Consequently, the system-level ALT data collected at higher than nominal stress level could be used to update the dependence factors,ρ, z, contingent to the dependence model used to quantify the correlations among failure times of the system's components. Also, since the physics informed models are used to bridge the stress from boundary component to non-boundary components, we can use the ALT system-level data to update the $\omega^{(i)}$ parameters as part of the prediction model.

To perform the parameter update, the Bayesian inference is applied to derive posterior samples to all related parameters according to the equation below:

$$f\left(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(\mathrm{Nc})}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\omega} \middle| \mathbf{t}_{\mathbf{sys}}, \boldsymbol{\xi}_{\mathbf{sys}}, \mathbf{n}_{\mathbf{sys}}\right)$$

$$\propto f\left(\mathbf{t}_{\mathbf{sys}} \middle| \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(\mathrm{Nc})}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\mathbf{sys}}, \mathbf{n}_{\mathbf{sys}}\right) f_{\rho}(\boldsymbol{\rho}) f_{\omega}(\boldsymbol{\omega}) \prod_{i=1}^{N_c} \left[f_{\theta i}\left(\boldsymbol{\theta}^{(i)} f_{\sigma i}\left(\sigma^{(i)}\right)\right]\right. \quad (5\text{-}32)$$

where $\mathbf{t}_{sys} = \left[\mathbf{t}_{sys,1}, \dots, \mathbf{t}_{sys,n_s}\right]$ where $\mathbf{t}_{sys,j} = \left[t_{sys,j}(1), \dots, t_{sys,j}\left(n_{sys}(j)\right)\right]^T$ are the observations or failure time data collected from ALT of the system at stresses higher than nominal stress and $\boldsymbol{\xi}_{sys} == \{\boldsymbol{\xi}_{\boldsymbol{b}}(1), \boldsymbol{\xi}_{\boldsymbol{b}}(2), \dots, \boldsymbol{\xi}_{\boldsymbol{b}}(n_s)\}$, and $\boldsymbol{\xi}_{\boldsymbol{b}}(i) = \left[\xi_{b1}(i), \xi_{b2}(i), \dots, \xi_{bn_b}(i)\right]$; $i = 1,2, \dots n_b$. The $\mathbf{n}_{sys} = [n_{sys}(1), n_{sys}(2), \dots, n_{sys}(n_s)]$ are the number of tests at each stress level and $f_x(.)$ denotes the prior distribution of parameter x.

Different methods could be used to run the estimation and sample the posterior data of the parameters. At this step, we have the Bayesian Inference relationship to update the parameters that could be used to connect the ALT system-level data to the system reliability as shown in the next Section. The Bayesian Estimation requires the formulation of the likelihood function in terms of the parameters of interest and the failure time data collected at the system level from the system level ALT testing which is shown in what follows.

5.2.4 Likelihood Function Formulation

The likelihood function $f\left(\mathbf{t}_{\mathbf{sys}} \middle| \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(\mathrm{Nc})}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\mathbf{sys}}, \mathbf{n}_{\mathbf{sys}}\right)$ is computed by

$$f\left(\mathbf{t}_{\mathbf{sys}} \middle| \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(\mathrm{Nc})}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\mathbf{sys}}, \mathbf{n}_{\mathbf{sys}}\right)$$
$$= \prod_{j=1}^{n_s} \prod_{k=1}^{n_{sys(j)}} f\left(t_{sys,j}(k) \middle| \boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(\mathrm{Nc})}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\boldsymbol{b}}(j)\right) \quad (5\text{-}33)$$

where $t_{sys,j}(k)$ is the k^{th} observation at the j^{th} stress level.

In order to compute $f\left(\mathbf{t_{sys}}\middle|\boldsymbol{\theta}^{(1)},\boldsymbol{\theta}^{(2)},\dots,\boldsymbol{\theta}^{(N_C)},\sigma,\boldsymbol{\rho},\boldsymbol{\omega},\boldsymbol{\xi}_{sys},\mathbf{n_{sys}}\right)$, we first map the testing load ξ_{sys} from the boundary components to the non-boundary components using the load prediction models by using Equation (5-30) as follows:

$$\xi_{bq-}(j) = L_{bj-}\left(\xi_b(j), \omega^{(q)}\right), \forall q = 1,2,\dots,n_{b-} \tag{5-34}$$

where $\boldsymbol{\xi_{bq-}(j)}$ is the load condition of the $\boldsymbol{q^{th}}$ non-boundary component at the $\boldsymbol{j^{th}}$ testing stress level in the system-level tests.

After we obtain $\xi_{bq-}(j), \forall q = 1,2,\dots,n_{b-}$ and $\boldsymbol{\xi_b}(j)$, the distribution parameters $\alpha_s^{(i)}(j)$ at the j^{th} testing stress level in the system-level tests are computed by the following equation below:

$$\alpha_s^{(i)}(j) = \begin{cases} \theta_0^{(i)} + \theta_1^{(i)}\xi_{bj}(j), \text{if component } i \text{ is a boundary component} \\ \theta_0^{(i)} + \theta_1^{(i)}\xi_{bi-}(j), \text{Otherwise} \end{cases} \tag{5-35}$$

Based on parameters, $\alpha_s^{(i)}(j)$ and $\sigma^{(i)}$ for all $i = 1,2,\dots,N_C$, of the component failure time distributions, we then generate random samples for each component-level failure time at the j^{th} testing stress level of the system-level test using Monte Carlo Simulation methods by using the Weibull CDF parametric function as follows:

$$t_{MCS,j}^{(i)} = e^{\alpha_s^{(i)}(j)}\left(-\ln\left(1-u_{MCS}^{(i)}\right)\right)^{\sigma^{(i)}} \tag{5-36}$$

where $\mathbf{t}_{\boldsymbol{MCS,j}}^{(\boldsymbol{i})} = \left[t_{MCS,j}^{(i)}(1), t_{MCS,j}^{(i)}(2), \dots, t_{MCS,j}^{(i)}(\mathrm{n_{MCS}})\right]$ are the random failure time samples of the *i*-th component at the *j*-th stress level in the system-level ALT. Whereas, the set of component random CDF distributions of the *i*-th component generated from Gaussian copula for a given copula factor vector $\boldsymbol{\rho}$ is represented by $\mathbf{u}_{\boldsymbol{MCS,j}}^{(\boldsymbol{i})} =$

$\left[u_{MCS,j}^{(i)}(1), u_{MCS,j}^{(i)}(2), \dots, u_{MCS,j}^{(i)}(\mathrm{n_{MCS}})\right]$.

Based on the component failure time samples that are generated as detailed above, the system failure time samples could be then derived using the following function:

$$t_{MCS,j}^{(i)}(h) = f_{time}\left(t_{MCS,j}^{(1)}(h), t_{MCS,j}^{(2)}(h), \dots, t_{MCS,j}^{(N_C)}(h)\right), \forall h = 1,2,..,n_{MCS} \tag{5-37}$$

where $f_{time}(.)$ is a function used to convert component failure time to system failure time and is given by:

$$f_{time}\left(t_{MCS,j}^{(1)}(h), t_{MCS,j}^{(2)}(h), \dots, t_{MCS,j}^{(N_C)}(h)\right) = \begin{cases} \min\limits_{i\in[1,N_C]} \left\{t_{MCS,j}^{(i)}(h)\right\}, \text{for series system} \\ \max\limits_{i\in[1,N_C]} \left\{t_{MCS,j}^{(i)}(h)\right\}, \text{for parallel system} \\ \text{Defined According to System} \\ \text{Topology, Oterwise} \end{cases} \tag{5-38}$$

After we obtain $t_{MCS,j}^{(i)}(h), \forall h = 1,2,..,n_{MCS}$, the function $f\left(t_{sys,j}(k)\middle|\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \dots, \boldsymbol{\theta}^{(\mathrm{N_C})}, \sigma, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\mathbf{b}}(j)\right)$ in Equation (5-34) which is a density function could be estimated by applying the concept of the kernel smoothing technique. The following section taps into kernel smoothing in order to briefly explain the technique.

A – Kernel Smoothing:

Kernel smoothing is a statistical technique used to estimate functions like regression function or probability density function [125]. It is a statistical technique that uses non-parametric estimation methods to estimate functions.

The kernel density estimator is given by the following general form [126]:

$$f_h(t) = \frac{1}{n\delta}\sum_{v=1}^{n} K\left(\frac{x_v - t}{\delta}\right) \tag{5-39}$$

K is defined as the kernel and δ is called the bandwidth. K could be any pdf function and is often chosen to be a unimodal distribution that is symmetric around zero, some of the known kernels are the Epanechnikov Kernel [126] and Biweight Kernel[127] . The bandwidth is a smoothing factor determining the smoothness of the estimated function, it is a scaling factor. The bandwidth plays a major role in the estimation as it dictates if a density function estimate is overestimated or underestimated. Detailed information about the bandwidth selection could be found in [125].

Given the form of the likelihood function developed in Section 5.2.4, and by applying the concept of kernel estimator, the following equation is developed in order to estimate the density function that is the likelihood function non-parametrically:

$$f\left(t_{sys,j}\,(k)\middle|\boldsymbol{\theta}^{(1)},\dots,\boldsymbol{\theta}^{(\mathrm{N_C})},\sigma,\boldsymbol{\rho},\boldsymbol{\omega}\right) = \frac{1}{(n_{MCS}\delta)}\sum_{i=1}^{n_{MCS}}\kappa\left(\frac{t_{sys,j}(k)-t_{MCS,j}^{sys}(i)}{\delta}\right) \tag{5-40}$$

where $\kappa(.)$is a kernel smoothing function and δ is the bandwidth or smoothing factor [128].

5.2.5 Uncertainty Propagation to System Reliability

In the previous sections, the system-level ALT testing data $\boldsymbol{t_{sys}}$ are connected to the component-level ALT models, copula function, and load prediction model, the connection is established using the Equations (5-33) through (5-40) .

Table 5-1 below, summarizes the overall procedure for the evaluation of the likelihood function which in turn established the connection between component-level models and the system level models. Based on this established connection, the system-level testing data at higher-than-nominal stress levels can be used to reduce the uncertainty in the model parameters: $\boldsymbol{\theta}^{(1)},\dots,\boldsymbol{\theta}^{(\mathrm{N_C})},\boldsymbol{\sigma},\boldsymbol{\rho}$ and $\boldsymbol{\omega}$, by using the Bayesian inference procedure in Section 4.1.4A and thus reduce the uncertainty in the system reliability estimate R_s.

Table 5-1 Summary of the evaluation procedure of Equation (5-33)

Steps	Description
1	Map the loads $\boldsymbol{\xi_b}(j)$ from boundary components to their counterparts of non-boundary components using physics-informed load prediction models (i.e. Equation (5-34)).
2	Obtain the distribution parameters $\alpha_s^{(i)}(j)$ and $\sigma^{(i)}$ for all $i = 1,2, \ldots, N_C$ at the testing stress level in the system-level ALT using Equation (5-35).
3	Generate random samples of component failure time using Equation (5-36), copula function, and the distribution parameters obtained from Step 2.
4	Convert the samples of component failure time to samples of system failure time using Equation (5-37).
5	Compute likelihood function (i.e. Equation (5-33)) using kernel smoothing function estimate based on the samples of system failure time.

5.3 Reliability Assessment via Information Fusion of Component Level ALT Data with System Level ALT Data

A major advantage of using Bayesian methods as demonstrated in Sections 5.1 and 5.2, in order to establish connections between component-level ALT data and system reliability, system-level ALT data and system reliability, and in both cases aiming at reducing the uncertainty in the system reliability estimation, is that it allows us to fuse the information from both component-level and system-level testing data collected from accelerated life testing at two different testing stages: by testing the components separately or by testing the whole system, which could turn further reduction in the uncertainty of the system reliability estimate.

The fusion of the information from both testing level: component level and system level data could be now combined to establish a connection between system reliability and both ALT component-level and system-level information data in order to reduce the uncertainty when assessing the system reliability. With the intention of merging both the component-level ALT data and the system-level ALT data, we define the testing plan as follows: the normalized stress $\boldsymbol{\xi}_{\mathbf{test}} = \{\boldsymbol{\xi}_{\mathbf{sys}}, \boldsymbol{\xi}^{(\mathbf{1})}, \boldsymbol{\xi}^{(\mathbf{2})}, \ldots, \boldsymbol{\xi}^{(\mathbf{N_C})}\}$ where $\boldsymbol{\xi}_{\mathbf{sys}} = \{\boldsymbol{\xi}_{\mathbf{b}}(\mathbf{1}), \boldsymbol{\xi}_{\mathbf{b}}(\mathbf{2}), \ldots, \boldsymbol{\xi}_{\mathbf{b}}(\mathbf{n_s})\}$, in which $\boldsymbol{\xi}_{\boldsymbol{b}}(i) = [\xi_{b1}(i), \xi_{b2}(i), \ldots, \xi_{bn_b}(i)]$; $i = 1,2, \ldots n_b$. The component normalized stress is defined by $\boldsymbol{\xi}^{(\boldsymbol{i})} = [\xi^{(i)}(1), \xi^{(i)}(2), \ldots, \xi^{(i)}(n_s)], \forall i = 1,2, \ldots, N_C$. Also, we introduce $\mathbf{n}_{\boldsymbol{test}} = \{\mathbf{n}_{\boldsymbol{sys}}, \mathbf{n}^{(\mathbf{1})}, \mathbf{n}^{(\mathbf{2})}, \ldots, \mathbf{n}^{(\boldsymbol{N_C})}\}$, where The $\mathbf{n}_{\boldsymbol{sys}} = [n_{sys}(1), n_{sys}(2), \ldots, n_{sys}(n_s)]$ and $n^{(i)} = [n^{(i)}(1), n^{(i)}(2), \ldots, n^{(i)}(n_s)]$, $\forall i = 1,2, \ldots, N_C$.

Based on the established connections, the uncertainty parameters in the ALT models and the load prediction model can be updated using the component-level and system-level testing data. At this stage, we can use the established connections in previous sections to estimate the distribution parameters, we show below in Equation (5-41)for a copula-based dependence model as follows:

$$\begin{aligned} &\boldsymbol{f}\left(\boldsymbol{\theta}^{(\mathbf{1})}, \boldsymbol{\theta}^{(\mathbf{2})}, \cdots, \boldsymbol{\theta}^{(\mathbf{N})}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\omega} \middle| \mathbf{t}^{(\mathbf{1})}, \mathbf{t}^{(\mathbf{2})}, \cdots, \mathbf{t}^{(\mathbf{N})}, \mathbf{t}_{\mathbf{sys}}, \boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}}\right) \\ &\propto \boldsymbol{f}\left(\mathbf{t}^{(\mathbf{1})}, \mathbf{t}^{(\mathbf{2})}, \cdots, \mathbf{t}^{(\mathbf{N})}, \mathbf{t}_{\mathbf{sys}} \middle| \boldsymbol{\theta}^{(\mathbf{1})}, \boldsymbol{\theta}^{(\mathbf{2})}, \cdots, \boldsymbol{\theta}^{(\mathbf{N})}, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}}\right) \\ &\qquad \times \boldsymbol{f}_{\boldsymbol{\rho}}(\boldsymbol{\rho}) \boldsymbol{f}_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \prod_{\mathbf{i}=\mathbf{1}}^{\mathbf{N_C}} \left[\boldsymbol{f}_{\boldsymbol{\theta i}}\left(\boldsymbol{\theta}^{(\mathbf{i})}\right) \boldsymbol{f}_{\boldsymbol{\sigma i}}\left(\boldsymbol{\sigma}^{(\mathbf{i})}\right)\right] \end{aligned} \tag{5-41}$$

where

$$
\begin{aligned}
\boldsymbol{f}\big(\mathbf{t}^{(1)},\ \mathbf{t}^{(2)},\ \cdots,\ \mathbf{t}^{(\mathrm{N})},\ \mathbf{t}_{\mathrm{sys}}\big|\boldsymbol{\theta}^{(1)},\ \boldsymbol{\theta}^{(2)},\ \cdots,\ \boldsymbol{\theta}^{(\mathrm{N})},\ \boldsymbol{\sigma},\ \boldsymbol{\rho},\ \boldsymbol{\omega},\ \boldsymbol{\xi}_{\mathrm{test}},\ \mathbf{n}_{\mathrm{test}}\big) \\
= \boldsymbol{f}\big(\mathbf{t}_{\mathrm{sys}}\big|\boldsymbol{\theta}^{(1)},\ \boldsymbol{\theta}^{(2)},\ \cdots,\ \boldsymbol{\theta}^{(\mathrm{N})},\ \boldsymbol{\sigma},\ \boldsymbol{\rho},\ \boldsymbol{\omega},\ \boldsymbol{\xi}_{\mathrm{sys}},\ \mathbf{n}_{\mathrm{sys}}\big) \times \\
\prod_{i=1}^{N_C} \boldsymbol{f}\big(\mathbf{t}^{(\mathrm{i})}\big|\boldsymbol{\theta}^{(\mathrm{i})},\ \sigma^{(\mathrm{i})},\ \boldsymbol{\xi}^{(\mathrm{i})},\ \mathbf{n}^{(\mathrm{i})}\big)
\end{aligned}
\tag{5-42}
$$

in which $\boldsymbol{f}\big(\mathbf{t}_{sys}\big|\boldsymbol{\theta}^{(1)},\ \boldsymbol{\theta}^{(2)},\ \cdots,\ \boldsymbol{\theta}^{(N_C)},\ \boldsymbol{\sigma},\ \boldsymbol{\rho},\ \boldsymbol{\omega},\ \boldsymbol{\xi}_{sys},\ \mathbf{n}_{sys}\big)$ is computed using Equations (5-33) through (5-40) in Section 5.1.3 and $\boldsymbol{f}\big(\mathbf{t}^{(i)}\big|\boldsymbol{\theta}^{(i)},\ \boldsymbol{\sigma}^{(i)},\ \boldsymbol{\xi}^{(i)},\ \mathbf{n}^{(i)}\big), \forall \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{N_C}$ are computed using Equations through in Section (5-25) through (5-29).

In this research, the particle filtering (PF) method is employed to perform the Bayesian inference given in Section 4.1.3A. In PF method, we first generate n_{prior} prior samples for the uncertainty parameters which are $\boldsymbol{\theta}^{(1)},\ \boldsymbol{\theta}^{(2)},\ \cdots,\ \boldsymbol{\theta}^{(N)},\ \boldsymbol{\sigma},\ \boldsymbol{\rho}$, and $\boldsymbol{\omega}$**.** After that, we compute the weights of each prior sample as:

$$
\boldsymbol{w}(k) = \frac{\boldsymbol{f}\big(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(\mathrm{N_c})}, \mathbf{t}_{\mathrm{sys}}\big|\boldsymbol{\theta}^{(1)}(\mathbf{k}), \dots, \boldsymbol{\theta}^{(\mathrm{N_C})}(\mathbf{k}), \boldsymbol{\sigma}(\mathbf{k}), \boldsymbol{\rho}(\mathbf{k}), \boldsymbol{\omega}(\mathbf{k}), \boldsymbol{\xi}_{\mathrm{test}}, \mathbf{n}_{\mathrm{test}}\big)}{\sum_{k=1}^{n_{prior}} \boldsymbol{f}\big(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(\mathrm{N_c})}, \mathbf{t}_{\mathrm{sys}}\big|\boldsymbol{\theta}^{(1)}(\mathbf{k}), \dots, \boldsymbol{\theta}^{(\mathrm{N_C})}(\mathbf{k}), \boldsymbol{\sigma}(\mathbf{k}), \boldsymbol{\rho}(\mathbf{k}), \boldsymbol{\omega}(\mathbf{k}), \boldsymbol{\xi}_{\mathrm{test}}, \mathbf{n}_{\mathrm{test}}\big)}, \quad \forall k = 1, \dots, n_{prior} \tag{5-43}
$$

Using the weights obtained from the above equation, the prior samples are then resampled to get the posterior distributions of $\boldsymbol{\theta}^{(1)},\ \boldsymbol{\theta}^{(2)},\ \cdots,\ \boldsymbol{\theta}^{(N)},\ \boldsymbol{\sigma},\ \boldsymbol{\rho}$, and $\boldsymbol{\omega}$**.** The posterior distribution of the system reliability after the uncertainty reduction using component-level and system-level testing data is then obtained by propagating the uncertainty in c to the system reliability by following the procedure discussed in Sec. 3.2.1.

The above discussions imply that the uncertainty reduction in the system reliability estimate is affected by the component-level and system-level testing data. As shown in Figure 4.1, in Chapter 4 there are several observation nodes in the network, which means that we can collect data at different locations and different levels of the system. With the limited testing resources, how to optimally allocate the resources is a challenging issue. In the next section, building upon the connections established in this section, we focus on the resource allocation for ALT-based system reliability analysis.

5.4 Accelerated Life Testing Design Optimization Model

In this section, we first formulate the objective function used for resource allocation. We then present the resource allocation optimization model based on the formulated objective function.

5.4.1 Objective Function

In order to quantify the value of information contained in the testing data to the system reliability estimate, in this paper, the Kullback–Leibler (KL) divergence [124] is employed and is given by:

$$D_{KL} = \int f_{R_S}\left(R \middle| \mathbf{t}^{(1)}, \mathbf{t}^{(2)} \ldots \mathbf{t}^{(\mathrm{N_C})}, \mathbf{t_{sys}}\right) \times \log\left(\frac{f_{R_S}\left(R \middle| \mathbf{t}^{(1)}, \mathbf{t}^{(2)} \ldots \mathbf{t}^{(\mathrm{N_C})}, \mathbf{t_{sys}}\right)}{f_{R_0}(R)}\right) dR \tag{5-44}$$

where $f_{R_S}\left(R \middle| t^{(1)}, \ldots, t_{sys}\right)$ is the posterior PDF of the system reliability R_S for given testing data $t^{(1)}, \ldots, t^{(N_C)}, t_{sys}$ and $f_{R_0}(R)$ is the prior PDF of R_S. The distributions of R_S ,required to use the Kullback-Leibler (KL) as stated by Equation (5-44) above, are obtained by propagating the uncertainty in the distribution parameters $\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(N_C)}, \boldsymbol{\rho}, \sigma, \boldsymbol{\omega}$ to the system reliability per the methodology described in Sections 5.1 through 5.3.

Defining the posterior samples of defining R_S as $R_{post}(i), i = 1,2,..,n_{post}$ using the following summation:

$$D_{KL} \approx \sum_{i=1}^{n_{post}} \log\left(\frac{f_{R_S}\left(R_{post}(i)\middle|\boldsymbol{t}^{(1)},\boldsymbol{t}^{(2)} \dots \boldsymbol{t}^{(N_C)},\boldsymbol{t_{sys}}\right)}{f_{R_0}\left(R_{post}(i)\right)}\right) \tag{5-45}$$

And the following numerator $f_{R_S}\left(R_{post}(i)\middle|\boldsymbol{t}^{(1)},\boldsymbol{t}^{(2)} \dots \boldsymbol{t}^{(N_C)},\boldsymbol{t_{sys}}\right)$ is estimated by using kernel smoothing estimate function given by:

$$f_{R_S}\left(R_{post}(i)\middle|\boldsymbol{t}^{(1)},\boldsymbol{t}^{(2)} \dots \boldsymbol{t}^{(N_C)},\boldsymbol{t_{sys}}\right) \approx \frac{1}{n_{post}\delta_{post}} \sum_{j=1}^{n_{post}} \kappa\left(\frac{R_{Post}(i) - R_{Post}(j)}{\delta_{post}}\right) \tag{5-46}$$

in which δ_{Post} is the bandwidth of the kernel density function. $f_{R_0}(R_{Post}(i))$ is computed similarly using kernel density function based on the prior samples of R_S.

In Equations (5-44) to (5-46), the testing data $\boldsymbol{t}^{(1)},\boldsymbol{t}^{(2)} \dots \boldsymbol{t}^{(N_C)},\boldsymbol{t_{sys}}$ are assumed to be known. In the ALT design stage, however, we do not have testing data. As a common practice in experimental or testing design, synthetic data are generated as the testing data using the prior distributions of $\boldsymbol{\theta}^{(1)}$, $\boldsymbol{\theta}^{(2)}$, $\cdots$, $\boldsymbol{\theta}^{(N)}$, $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$, and $\boldsymbol{\omega}$. The synthetic data are uncertain due to the uncertainty in parameters $\boldsymbol{\theta}^{(1)}$, $\boldsymbol{\theta}^{(2)}$, $\cdots$, $\boldsymbol{\theta}^{(N)}$, $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$, and $\boldsymbol{\omega}$ as well as the inherent uncertainty in the failure time distributions. As a result, the KL divergence given in Equation (5-45) is uncertain in the ALT design. So, in order to account for this uncertainty, the expected KL divergence is employed to be the objective function of the optimization model as a widely used approach in the experimental design domain and Monte Carlo Simulation (MCS) method is used in order to approximate the expected divergence $E\left(D_{KL}(\xi_{test}, n_{test})\right)$ where $\boldsymbol{\xi}_{test}, \boldsymbol{n}_{test}$ are the design parameters of the test plan (ALT Design). Accordingly, For a given testing plan characterized by the two parameters: $\boldsymbol{\xi}_{test}$ and $\boldsymbol{n}_{test}$, the expected KL divergence noted by $E\left(D_{KL}(\xi_{test}, n_{test})\right)$, corresponding to this testing plan is computed by:

$$E\left(D_{KL}(\boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{test}})\right) = \iint \dots \iint (D_{KL} | \mathbf{t}^{(1)}, \mathbf{t}^{(2)} \dots \mathbf{t}^{(\mathrm{N_C})}, \mathbf{t}_{\mathbf{sys}}) f(\mathbf{t}^{(1)} | \boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{test}}) \\ \dots f\left(\mathbf{t}^{(\mathrm{N_C})} \middle| \boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{test}}\right) f\left(\mathbf{t}_{\text{sys}} \middle| \boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{test}}\right) d\mathbf{t}^{(1)} \dots d\mathbf{t}^{(N_C)} d\mathbf{t}_{\text{sys}} \tag{5-47}$$

where $(D_{KL} | \boldsymbol{t}^{(1)}, \boldsymbol{t}^{(2)} \dots \boldsymbol{t}^{(N_C)}, \boldsymbol{t}_{\boldsymbol{sys}})$ is the KL divergence conditioned on given $\boldsymbol{t}^{(1)}, \boldsymbol{t}^{(2)} \dots \boldsymbol{t}^{(N_C)}, \boldsymbol{t}_{\boldsymbol{sys}}$ and is computed using Equation (5-44), $f\left(\boldsymbol{t}^{(i)} \middle| \boldsymbol{\xi}_{test}, \boldsymbol{n}_{test}\right), \forall i = 1,2, \dots, N_C$ is obtained by solving the following equation:

$$f\left(\mathbf{t}^{(\mathrm{i})} \middle| \boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{test}}\right) = \int \dots \int f\left(\mathbf{t}^{(\mathrm{i})} \middle| \boldsymbol{\theta}^{(\mathrm{i})}, \sigma^{(\mathrm{i})}, \boldsymbol{\xi}^{(\mathrm{i})}, \mathbf{n}^{(\mathrm{i})}\right) f_{\theta i}\left(\boldsymbol{\theta}^{(\mathrm{i})}\right) f_{\sigma i}\left(\sigma^{(i)}\right) d\boldsymbol{\theta}^{(i)} d\boldsymbol{\sigma}^{(\boldsymbol{i})} \tag{5-48}$$

in which $\boldsymbol{\xi}^{(i)} = \left[\xi^{(i)}(1), \xi^{(i)}(2), \dots, \xi^{(i)}(n_s)\right]$, $\mathbf{n}^{(i)} = \left[n^{(i)}(1), n^{(i)}(2), \dots, n^{(i)}(n_s)\right]$, $\forall i = 1,2, \dots, N_C$, and $f\left(\mathbf{t}^{(\mathrm{i})} \middle| \boldsymbol{\theta}^{(\mathrm{i})}, \sigma^{(\mathrm{i})}, \boldsymbol{\xi}^{(\mathrm{i})}, \mathbf{n}^{(\mathrm{i})}\right)$ is computed using Equation (5-25).

Additionally, $f\left(\mathbf{t}_{\text{sys}} \middle| \boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{test}}\right)$ is given by:

$$f\left(\mathbf{t}_{\text{sys}} \middle| \boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{test}}\right) \\ = \int \dots \int f\left(\mathbf{t}_{\text{sys}} \middle| \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(\mathrm{N_C})}, \sigma^{(1)} \dots, \sigma^{(N_C)}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\text{sys}}, \mathbf{n}_{\text{sys}}\right) \\ f_\rho(\boldsymbol{\rho}) f_\omega(\boldsymbol{\omega}) \prod_{i=1}^{N_C} f_{\theta i}\left(\boldsymbol{\theta}^{(i)}\right) f_{\sigma i}\left(\sigma^{(i)}\right) d\boldsymbol{\theta}^{(1)} \dots d\boldsymbol{\theta}^{(N_C)} d\sigma^{(1)} \dots d\sigma^{(N_C)} \tag{5-49}$$

where $f\left(\mathbf{t}_{sys} \middle| \boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(\mathrm{N_C})}, \sigma^{(1)} \dots, \sigma^{(N_C)}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\text{sys}}, \mathbf{n}_{\text{sys}}\right)$ is computed using Equation (5-33) , $\boldsymbol{\xi}_{\mathbf{test}} = \left\{\boldsymbol{\xi}_{\mathbf{sys}}, \boldsymbol{\xi}^{(\mathbf{1})}, \boldsymbol{\xi}^{(\mathbf{2})}, \dots, \boldsymbol{\xi}^{(\mathbf{N_C})}\right\}$, and $\mathbf{n}_{\mathbf{test}} = \{\mathbf{n}_{\mathbf{sys}}, \mathbf{n}^{(\mathbf{1})}, \mathbf{n}^{(\mathbf{2})}, \dots, \mathbf{n}^{(\mathbf{N_C})}\}$.

The above formulation, Equations (5-47) through (5-49), indicate that the evaluation of the expected KL divergence requires very complicated high-dimensional integration. In this research, the Monte Carlo simulation-based method is employed to approximate Eq. (33). In the

MCS-based method, we first generate N_{obj} groups of samples of $\mathbf{t}^{(1)}, \mathbf{t}^{(2)} \ldots \mathbf{t}^{(\mathrm{N_C})}, \mathbf{t}_{\mathbf{sys}}$ using MCS by accounting for the uncertainty in the prior distributions of $\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(\mathrm{N_C})}, \sigma^{(1)} \ldots, \sigma^{(N_C)}, \boldsymbol{\rho}$, and $\boldsymbol{\omega}$, as well as the uncertainty in the life distributions. We then compute the KL divergence using Equation (5-45) for each group of the generated MCS samples. After that, Equation (5-47) is computed as

$$E\left(D_{KL}(\boldsymbol{\xi}_{\text{test}}, \mathbf{n}_{\text{est}})\right) \approx \frac{1}{N_{obj}} \sum_{i=1}^{N_{obj}} D_{KL}(i) \tag{5-50}$$

in which $D_{KL}(i)$ is the KL divergence computed using Equation (5-45) based on the *i*-th group of the MCS samples and the number of groups of samples N_{obj} is determined such that the variance in the estimate in Equation (5-47) can satisfy our requirements.

Next, we perform resource allocation for ALT-based system reliability analysis using the above objective function.

5.4.2 Resource Allocation Optimization Model

In resource allocation, the goal is to maximize the information gain from the limited accelerated life tests. With this objective in mind, we formulate the following resource allocation optimization model:

$$\begin{aligned}
&\max_{\boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}}} E(KL(\boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}})) \\
&\textbf{Subject to}: \\
&C_{ALT}(\xi_{\text{test}}, \mathrm{n}_{\text{test}}) \leq C_{total} \\
&\mathbf{n_L} \leq \mathbf{n}_{\mathbf{test}} \leq \mathbf{n_U} \\
&\mathbf{0} \leq \boldsymbol{\xi}_{\mathbf{test}} \leq \mathbf{1}
\end{aligned} \tag{5-51}$$

where $C_{\mathbf{ALT}}(\boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}})$ is the total testing cost for a given testing plan $\boldsymbol{\xi}_{\mathbf{test}}$ and $\mathbf{n}_{\mathbf{test}}$, $C_{\boldsymbol{total}}$ is the total budget, $\mathbf{n}_{\mathbf{L}}$ and $\mathbf{n}_{\mathbf{U}}$ are respectively the lower and upper bounds of the number of tests at each stress levels, $\mathbf{n}_{\mathbf{test}} = \{\mathbf{n}_{\mathbf{sys}}, \mathbf{n}^{(\mathbf{1})}, \mathbf{n}^{(\mathbf{2})}, \ldots, \mathbf{n}^{(\mathbf{N_C})}\}$, where The $\boldsymbol{n_{sys}} = [n_{sys}(1), n_{sys}(2), \ldots, n_{sys}(n_s)]$ and $\mathbf{n}^{(\mathbf{i})} = \left[n^{(i)}(1), n^{(i)}(2), \ldots, n^{(i)}(n_s)\right]$, $\forall i = 1,2, \ldots, N_C$. And $\boldsymbol{\xi}_{\mathrm{test}}$ is the test limits $\boldsymbol{\xi}_{\mathrm{test}} = \left\{\boldsymbol{\xi}_{\mathrm{sys}}, \boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(\mathrm{N_C})}\right\}$.

The cost function could be formulated in different ways to account for various costs included in the ALT testing. Generally, the plan to formulate a cost function and link it to the parameters at hand, to do so we plan on using the cost of a testing specimen which could be cost of testing a component i , C_i ,or cost of testing a system C_{sys}. Also, we consider the cost of testing per unit time which as well is divided into two types of costs depending on the testing level, for a system level testing, we designate the cost of system per unit time by e_{sys} and the cost of component I per unit time by e_i. Having the expected testing time for each of the components and the system, the total testing cost $\boldsymbol{C_{ALT}}(\xi_{test}, n_{test})$ for given ξ_{test} and n_{test} in Equation (5-51) is computed by the following equation:

$$C_{ALT}(\boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}}) = C_{system\ testing} + C_{component\ testing} \tag{5-52}$$

where, the system testing cost is given by:

$$C_{component\ testing} = \left[C_{sys} \sum_{j=1}^{n_s} \mathbf{n}_{\,sys}(j) + e_{sys} \sum_{j=1}^{n_s} \mathbf{n}_{\,sys}(j) \left(\overline{\mathbf{T}}_{\mathbf{sys}}(\boldsymbol{\xi}_{\mathbf{b}}(j)\right) \right] \tag{5-53}$$

the component testing cost formula is:

$$C_{component\ testing} = \left[\sum_{i=1}^{N_c} C_i \left(\sum_{j=1}^{n_s} \mathbf{n}^{(i)}(j) \right) + e_i \sum_{j=1}^{n_s} \mathbf{n}^{(i)}(\boldsymbol{j}) \overline{\mathbf{T}}_{\mathbf{i}}(\boldsymbol{\xi}^{(\mathbf{i})}(j)) \right] \tag{5-54}$$

The next step includes formulation of the expected testing time per system specimen and the expected testing time per component specimen. Once we formulate the expected time ($\bar{T}_{sys}$ and $\bar{T}_i$) we then replace them back in the cost function and the optimization model is now complete.

$\bar{T}_\iota(\xi^{(i)}(j))$ is given by:

$$\bar{\boldsymbol{T}}_\iota\left(\boldsymbol{\xi}^{(i)}(j)\right) = \int_0^\infty \int_{\Omega_{\sigma^{(i)}}} \int_{\Omega_{\theta^{(i)}}} t\boldsymbol{f}\left(\mathbf{t}^{(\mathbf{i})} \middle| \boldsymbol{\theta}^{(\mathbf{i})}, \sigma^{(\mathbf{i})},\ \boldsymbol{\xi}^{(\mathbf{i})}(\mathbf{j})\right) \times \tag{5-55}$$

$$\boldsymbol{f}_{\boldsymbol{\theta i}}\left(\boldsymbol{\theta}^{(\mathbf{i})}\right)\boldsymbol{f}_{\boldsymbol{\sigma i}}\left(\sigma^{(i)}\right)\boldsymbol{d\theta}^{(i)}\boldsymbol{d\sigma}^{(i)}\boldsymbol{dt}$$

where $\Omega_{\theta^{(i)}}$ and $\Omega_{\sigma^{(i)}}$ are respectively the domain of $\boldsymbol{\theta}^{(i)}$ and $\sigma^{(i)}$, $f\left(\mathbf{t}^{(\mathrm{i})} \middle| \boldsymbol{\theta}^{(\mathrm{i})}, \sigma^{(\mathrm{i})}, \boldsymbol{\xi}^{(\mathrm{i})}(\mathbf{j})\right)$ is computed similarly to $f\left(\mathbf{t}^{(\mathrm{i})} | \boldsymbol{\theta}^{(\mathrm{i})}, \sigma^{(\mathrm{i})}, \boldsymbol{\xi}^{(\mathrm{i})}, \mathbf{n}^{(\mathrm{i})}\right)$ using Equation (5-25) and (5-26).

Referring to the analytical expression of a Weibull distribution, the Equation (5-55) is rewritten as:

$$\bar{\boldsymbol{T}}_\iota\left(\boldsymbol{\xi}^{(i)}(j)\right) = \int_{\Omega_{\sigma^{(i)}}} \int_{\Omega_{\theta^{(i)}}} \boldsymbol{\mu_T}\left(\boldsymbol{\theta}^{(\mathbf{i})}, \sigma^{(i)}, \boldsymbol{\xi}^{(i)}(\mathbf{j})\right) \\ \boldsymbol{f}_{\boldsymbol{\theta i}}\left(\boldsymbol{\theta}^{(\mathbf{i})}\right)\boldsymbol{f}_{\boldsymbol{\sigma i}}\left(\sigma^{(i)}\right) \mathbf{d}\boldsymbol{\theta}^{(\mathbf{i})}\boldsymbol{d}\sigma^{(i)}\boldsymbol{dt} \tag{5-56}$$

where $\boldsymbol{\mu_T}\left(\boldsymbol{\theta}^{(i)}, \sigma^{(i)}, \boldsymbol{\xi}^{(i)}(\boldsymbol{j})\right)$ is given by:

$$\boldsymbol{\mu_T}\left(\boldsymbol{\theta}^{(\mathbf{i})}, \boldsymbol{\sigma}^{(\mathbf{i})}, \boldsymbol{\xi}^{(\mathbf{i})}(\boldsymbol{j})\right) = \boldsymbol{e}^{\left(\boldsymbol{\theta}_0^{(i)} + \boldsymbol{\theta}_1^{(i)}(j)\right)}\boldsymbol{\Gamma}\left(\mathbf{1} + \sigma^{(i)}\right) \tag{5-57}$$

in which $\boldsymbol{\Gamma}(.)$ is a gamma distribution function generally given by:

$$\Gamma(\mathrm{z}) = \int_0^\infty e^{-t}t^{z-1}dt \tag{5-58}$$

$\bar{T}_{sys}(\xi_b(j)$ is given by:

$$\bar{\boldsymbol{T}}_{sys}(\boldsymbol{\xi}_{\boldsymbol{b}}(j)) = \int \dots \int \boldsymbol{tf}\left(\mathbf{t}_{\mathbf{sys}} \middle| \boldsymbol{\theta}^{(\mathbf{1})}, \dots, \boldsymbol{\theta}^{(\mathbf{N_C})}, \boldsymbol{\sigma}^{(\mathbf{1})} \dots, \boldsymbol{\sigma}^{(N_C)}, \boldsymbol{\rho}, \boldsymbol{\omega}, \boldsymbol{\xi}_{\mathbf{b}}(j)\right) \tag{5-59}$$

$$f_{\rho}(\boldsymbol{\rho})f_{\omega}(\boldsymbol{\omega})\prod_{i=1}^{N} f_{\theta i}\left(\boldsymbol{\theta}^{(i)}\right)f_{\sigma i}\left(\sigma^{(i)}\right)d\boldsymbol{\theta}^{(1)}\ldots d\boldsymbol{\theta}^{(Nc)}d\sigma^{(1)}\ldots d\sigma^{(Nc)}\ d\boldsymbol{\rho}\ d\boldsymbol{\omega}\ dt$$

where $f\left(\mathbf{t_{sys}}\middle|\boldsymbol{\theta}^{(\mathbf{1})},\ldots,\boldsymbol{\theta}^{(\mathrm{Nc})},\sigma^{(\mathbf{1})}\ldots,\sigma^{(Nc)},\boldsymbol{\rho},\boldsymbol{\omega},\boldsymbol{\xi}_{\mathbf{b}}(j)\right)$ is computed using the Equations discussed in Section 5.2.

Figure 5-5 summarizes the overall flowchart for the evaluation of the objective function $E(KL(\boldsymbol{\xi}_{test},\mathbf{n}_{test}))$ for a given testing plan $(\boldsymbol{\xi}_{test},\mathbf{n}_{test})$. As shown in this figure, the prior distributions of the component-level ALT model parameters and the correlation coefficients are inputs for the evaluations of the objective function. For the prior distributions of component-level ALT parameters, they can be obtained based on calibration of the component-level ALT models using historical data, previously conducted component-level ALT testing data, or expert opinions. For the correlation coefficients, their prior distributions are more difficult to get than that of component-level ALT model parameters. Their prior distributions can be obtained based on historical data, expert opinion, or physics-based failure correlation analysis. As shown in the numerical examples, relatively accurate prior distributions are assumed for component-level ALT parameters while wide and non-informative priors are assumed for the correlation coefficients. It should also be noted that the effectiveness of the resource allocation framework will be affected by the prior distributions since they are inputs for the objective function. This fact is true for all Bayesian experimental design methods.

Due to the uncertainty in the prior distributions of the ALT parameters, solving the constraint function given in Equation (5-52) to (5-54) is also challenging. Similar to the evaluation of the expected KL divergence, in this research, the MCS-based method is employed to estimate Equations (5-56) through (5-59). In the MCS-based method, we first generate prior samples of the ALT parameters. After that, random samples of the failure time are generated. Based on the generated random failure time samples, the expected component-level and system-level testing time for given testing plan ($\boldsymbol{\xi}_{test}, \mathbf{n}_{test}$) are computed. By solving the optimization model formulated above, we are able to optimize the component-level and system-level ALT plans to effectively perform system reliability analysis using ALT tests. In this research, the efficient global optimization method with constraint function [129], [130] is employed to solve the optimization given in Equation (5-51).

In the next section, two numerical examples are used to illustrate the effectiveness of the proposed resource allocation framework for ALT-based system reliability analysis using 2 parameters log-scale distribution and the copula function as a mean to model the dependence among the components of one system.

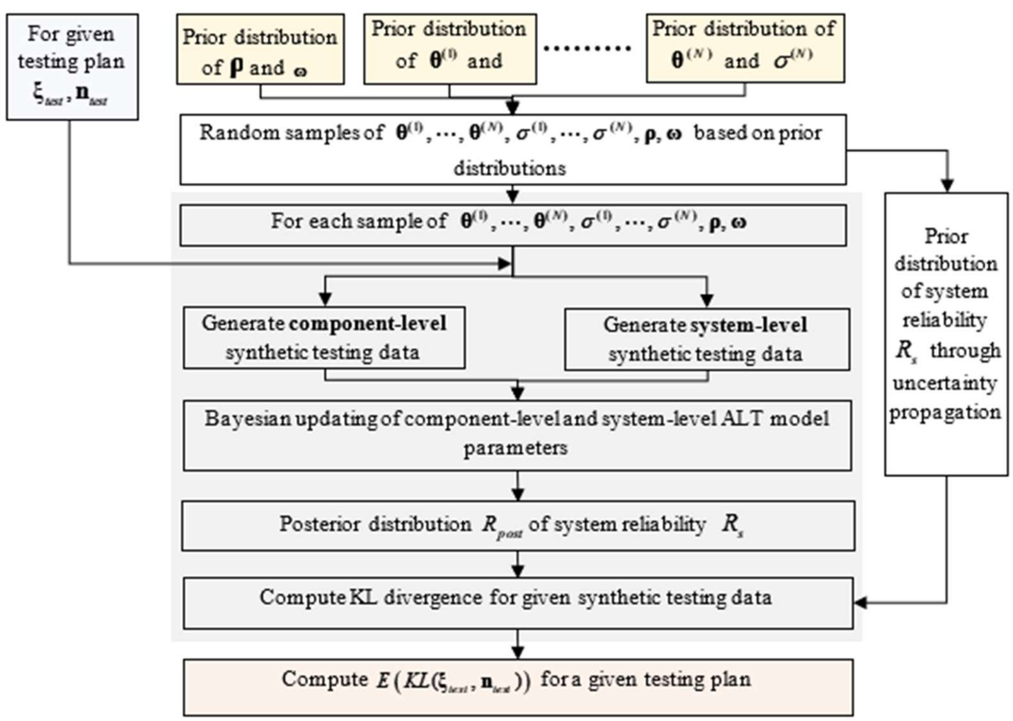


Figure 5-5 Overall flowchart for the evaluation of the objective function for a given testing plan

5.5 Numerical Examples

In this section, a mixed system and a four-joint robot system are used to illustrate the effectiveness of the proposed framework.

5.5.1 A Mixed System

A - Problem Statement

A mixed system given in Figure 5-6 is employed as the first numerical example. The system consists of three components. Amongst the three components, component 1 is the boundary component. The reliability of the system over a time period [0, 3.5×10^5] cycles at the nominal stress level needs to be estimated based on accelerated life testing.

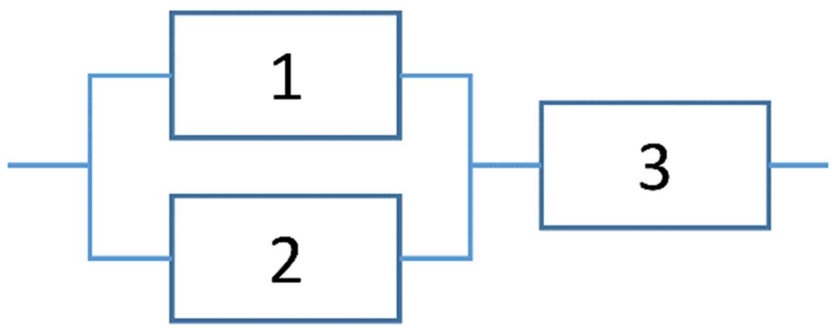


Figure 5-6 A mixed system with three components

Table 5-2 gives the true parameters ($\theta_0^{(i)}, \theta_1^{(i)}$, and $\sigma^{(i)}$, $\forall i = 1, 2, 3$) of the component-level ALT model of the three components. The true Gaussian copula parameters of the three components are given by

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix} \tag{5-60}$$

where $\rho_{12} = \rho_{21} = 0.9$, $\rho_{13} = \rho_{31} = 0.02$, and $\rho_{23} = \rho_{32} = 0.12$.

Table 5-2 True parameters of the component-level ALT models

	Component-level ALT parameters		
Component	$\theta_0^{(i)}$	$\theta_1^{(i)}$	$\sigma^{(i)}$
1	12.85	-5.3	0.06
2	13.05	-4.9	0.15
3	13.22	-4.8	0.13

Based on the above true models and parameters, the true system reliability (R_S) over the time period of interest is estimated as $R_s = 0.853$. In ALT-based system reliability assessment, we assume that the above parameters are unknown and thus the true system reliability is unknown. We need to estimate them based on accelerated life tests. Table 3 presents the required testing cost of component-level and system-level tests. The total budget of the accelerated life tests is $C_{total} = 6x10^5$. Table 4 gives the prior distributions of the ALT model parameters. For Gaussian distribution, in this Table, Parameters 1 and 2 are respectively the mean value and standard deviation. For uniform distribution, Parameters 1 and 2 respectively the lower and upper bounds of the distribution. Based on the prior information of the ALT model parameters, we then perform ALT design for system reliability assessment. In the ALT design, two-stress level tests are designed for the three components and the system. In addition, the number of the available testing chambers is 30 for each component and for the system-level tests at each testing stress level.

Table 5-3 Testing cost of component-level and system-level tests

	Component 1	**Component 2**	**Component 3**	**System**
Cost/specimen	1000	3000	2000	7000
Cost/unit time	0.02	0.01	0.01	0.03

Table 5-4 Prior distributions of the ALT model parameters

Variable	$\boldsymbol{\theta_0^{(1)}}$	$\boldsymbol{\theta_1^{(1)}}$	$\boldsymbol{\sigma^{(1)}}$	$\boldsymbol{\theta_0^{(2)}}$	$\boldsymbol{\theta_1^{(2)}}$	$\boldsymbol{\sigma^{(2)}}$	$\boldsymbol{\theta_0^{(3)}}$	$\boldsymbol{\theta_1^{(3)}}$	$\boldsymbol{\sigma^{(3)}}$	$\boldsymbol{\rho_{12}}$	$\boldsymbol{\rho_{13}}$	$\boldsymbol{\rho_{23}}$
Parameter 1	12.87	-5.35	0.055	13.03	-493	0.152	13.24	-4.83	0.132	0	0	0
Parameter 2	0.02	0.05	0.006	0.04	0.03	0.004	0.04	0.05	0.005	0.98	0.1	0.2
Distribution	**Gaussian**									**Uniform**		

With the above information, we then perform system reliability assessment based on ALTs.

B - System reliability assessment based on ALT

Assume that from the physics-informed analysis, we obtain the relationship between the load conditions of boundary component and non-boundary components as

where ξ_b is the normalized stress of the boundary element, ω_b is an uncertain parameter due to model uncertainty in the physic-informed load prediction model. The prior distribution of is

$$\begin{aligned} \boldsymbol{\xi}_{\mathbf{s2}} &= \boldsymbol{\xi}_{\mathbf{b}} \\ \boldsymbol{\xi}_{\mathbf{s3}} &= |\mathbf{sin}(\mathbf{30}\boldsymbol{\omega}_{\mathbf{b}}\boldsymbol{\xi}_{\mathbf{b}})| \end{aligned} \tag{5-61}$$

given by $\omega_b \sim Unif(0.11, 0.13)$, where $\mathrm{Unif}(\cdot)$ means uniform distribution.

In order to optimize both the component-level and system-level ALT plans, we formulate an optimization model using Equation

(5-51) as below:

$$\begin{aligned} &\max_{\boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}}} \boldsymbol{E}(\boldsymbol{KL}(\boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}})) \\ &\text{Subject to:} \\ &\boldsymbol{\xi}_{\mathbf{test}} = \{\boldsymbol{\xi}_{\mathbf{b}}(\mathbf{1}), \boldsymbol{\xi}_{\mathbf{b}}(\mathbf{2}), \boldsymbol{\xi}^{(\mathbf{i})}(\mathbf{1}), \boldsymbol{\xi}^{(\mathbf{i})}(\mathbf{2})\};\ \forall i = 1,2,3 \\ &\mathbf{n}_{\mathbf{test}} = \{\mathbf{n}_{\mathbf{sys}}(\mathbf{1}), \mathbf{n}_{\mathbf{sys}}(\mathbf{2}), \mathbf{n}^{(\mathbf{i})}(\mathbf{1}), \mathbf{n}^{(\mathbf{i})}(\mathbf{2})\};\ \forall i = 1,2,3 \\ &C_{ALT}(\boldsymbol{\xi}_{\mathbf{test}}, \mathbf{n}_{\mathbf{test}}) \le C_{total} \\ &\mathbf{1} \le n^{(i)}(j) \le 30\ , \forall i = 1,2,3\ ,;\ \forall j = 1,2 \\ &1 \le n_{sys}(j) \le 30\ , \forall i = 1,2,3\ ,;\ \forall j = 1,2 \\ &0 \le \boldsymbol{\xi}_{test} \le 1 \end{aligned} \tag{5-62}$$

We then solve the above optimization model using the approach discussed in Section 5.4.2. We first generate 300 training points for the design variables ($\boldsymbol{\xi}_{test}$ and $\boldsymbol{n}_{test}$) using Latin Hypercube sampling approach [131]. After evaluating the objective function at these 300 training points, we build a Kriging surrogate model for the objective function and refine the Kriging surrogate modeling using the efficient global optimization approach with the consideration of the budget constraint [129], [130]. In evaluating the objective function, the numbers of samples we used are $n_{prior} = 20{,}000$, $n_{mcs} = 1{,}000{,}000$ (Equation (5-28)),$N_{obj} = 500$ (Equation (5-50)). By adaptively refining the surrogate model, we obtain the optimal design corresponding to the maximum expected KL divergence.

Table 5-5 ALT Plan for Optimal Design, Design 1, Design 2 and Design 3

	Component 1				**Component 2**			
	$\xi^{(1)}(1)$	$\xi^{(1)}(2)$	$n^{(1)}(1)$	$n^{(1)}(2)$	$\xi^{(2)}(1)$	$\xi^{(2)}(2)$	$n^{(2)}(1)$	$n^{(2)}(2)$
Optimal Design	0.378	0.858	20	18	0.789	0.409	22	26
Design 1	0.443	0.465	15	4	0.973	0.595	18	14
Design 2	0.404	0.627	15	6	0.501	0.527	18	13
Design 3	0.232	0.279	3	18	0.442	0.737	6	8
	Component 3				**System**			
	$\xi^{(3)}(1)$	$\xi^{(3)}(2)$	$n^{(3)}(1)$	$n^{(3)}(2)$	$\xi_b(1)$	$\xi_b(2)$	$n_{sys}(1)$	$n_{sys}(2)$
Optimal Design	0.617	0.678	18	13	0.833	0.546	16	27
Design 1	0.546	0.398	4	26	0.929	0.380	28	26
Design 2	0.654	0.04	22	24	0.296	0.996	19	17
Design 3	0.826	0.383	25	24	0.795	0.512	25	27

Table 5-5 gives the optimal component-level and system-level ALT designs obtained from the proposed approach. In this table, for the purpose of comparison, we also provide three non-optimal designs, which have similar expected testing costs (i.e. 6×10^5) as the optimal design. Following that, Figs. 4 and 5 depict the comparisons between the posterior distributions and prior distribution of the system reliability estimate obtained using different designs. Note that, the Bayesian updating of the system reliability estimate is performed 200 times for each design (as indicated by multiple posterior distribution curves) to account for the uncertainty in the testing data. The comparison given in and Figure 5-10 illustrate that the optimal testing plan can more effectively reduce the uncertainty in the system reliability estimate than the other three non-optimal designs. This demonstrates the effectiveness of the proposed framework.

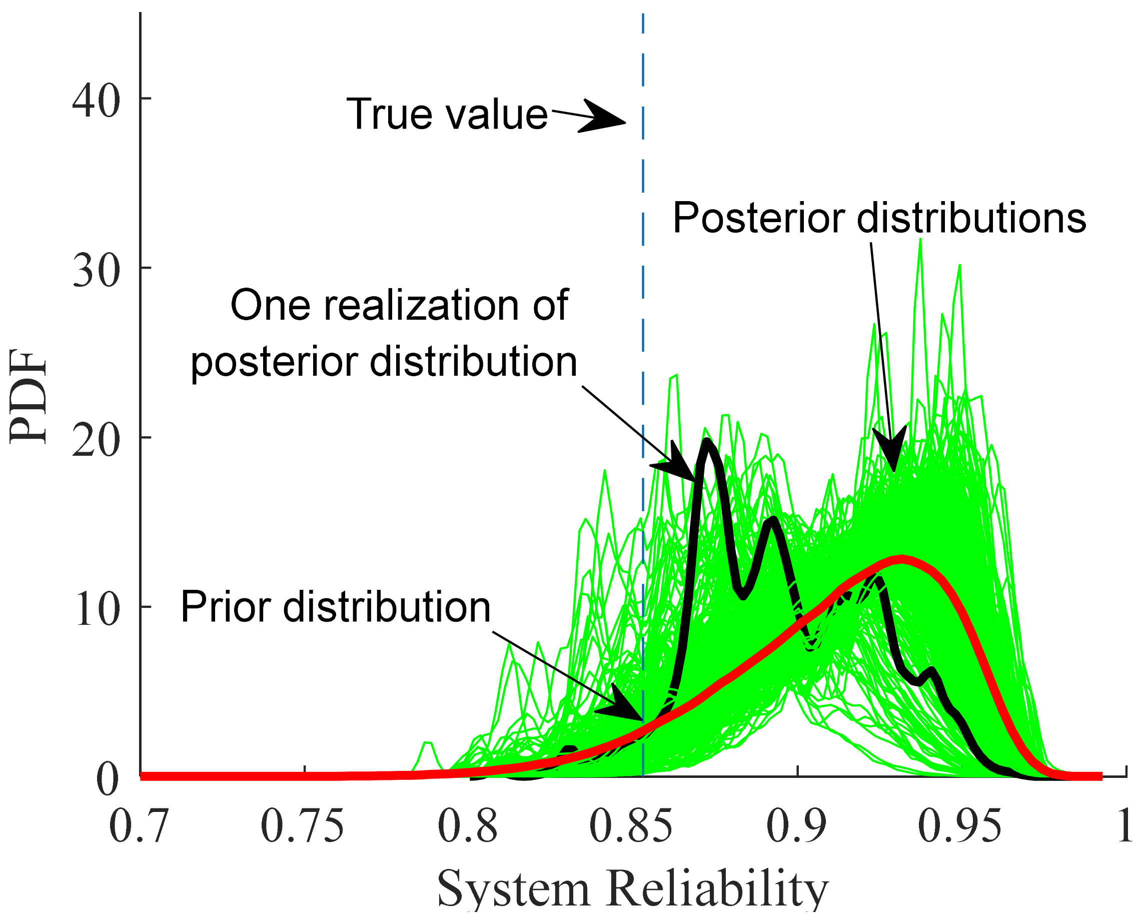


Figure 5-7 Comparison of prior and posterior distributions of the system reliability for optimal design

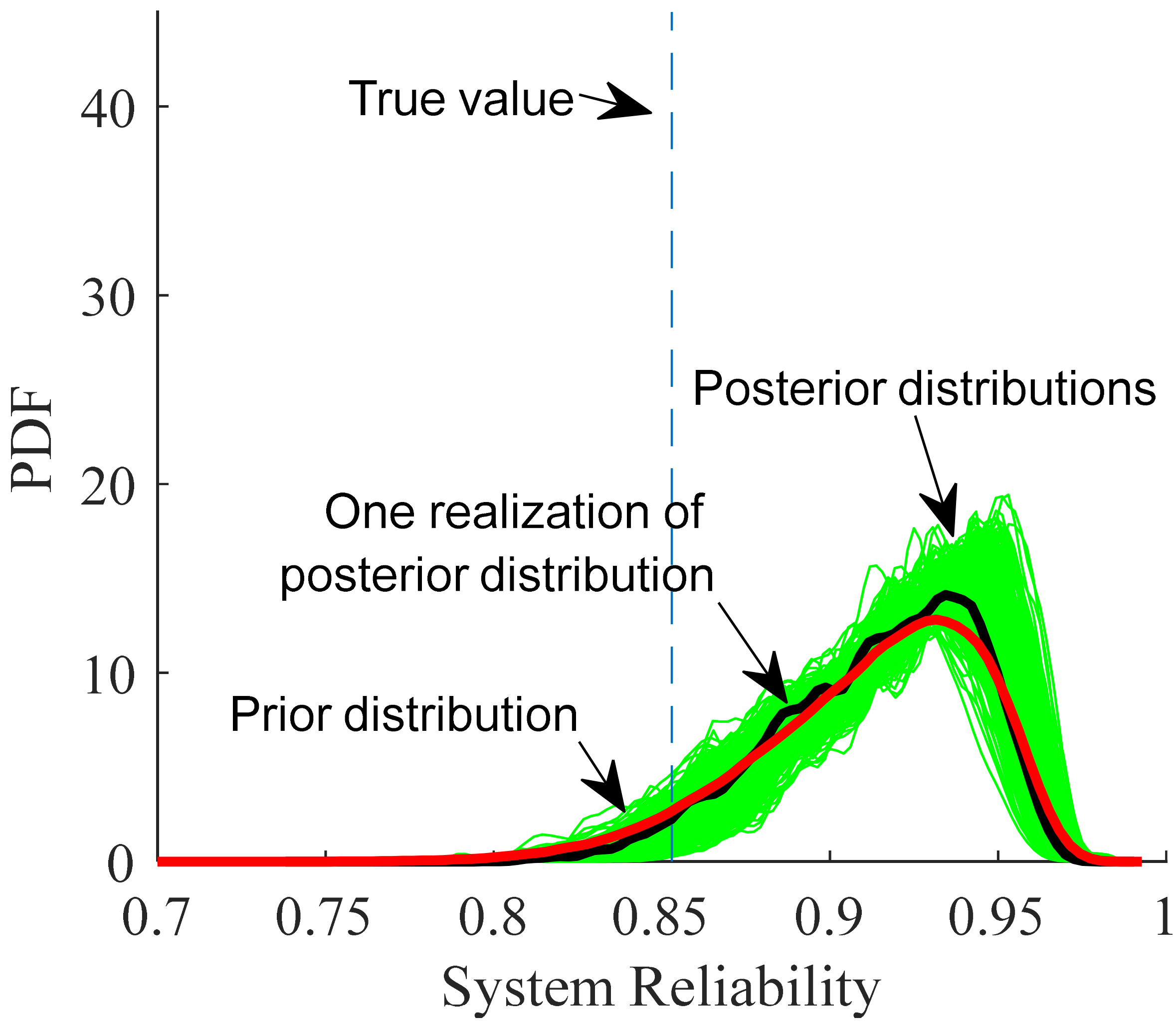


Figure 5-8 Comparison of prior and posterior distributions of the system reliability for Design 1

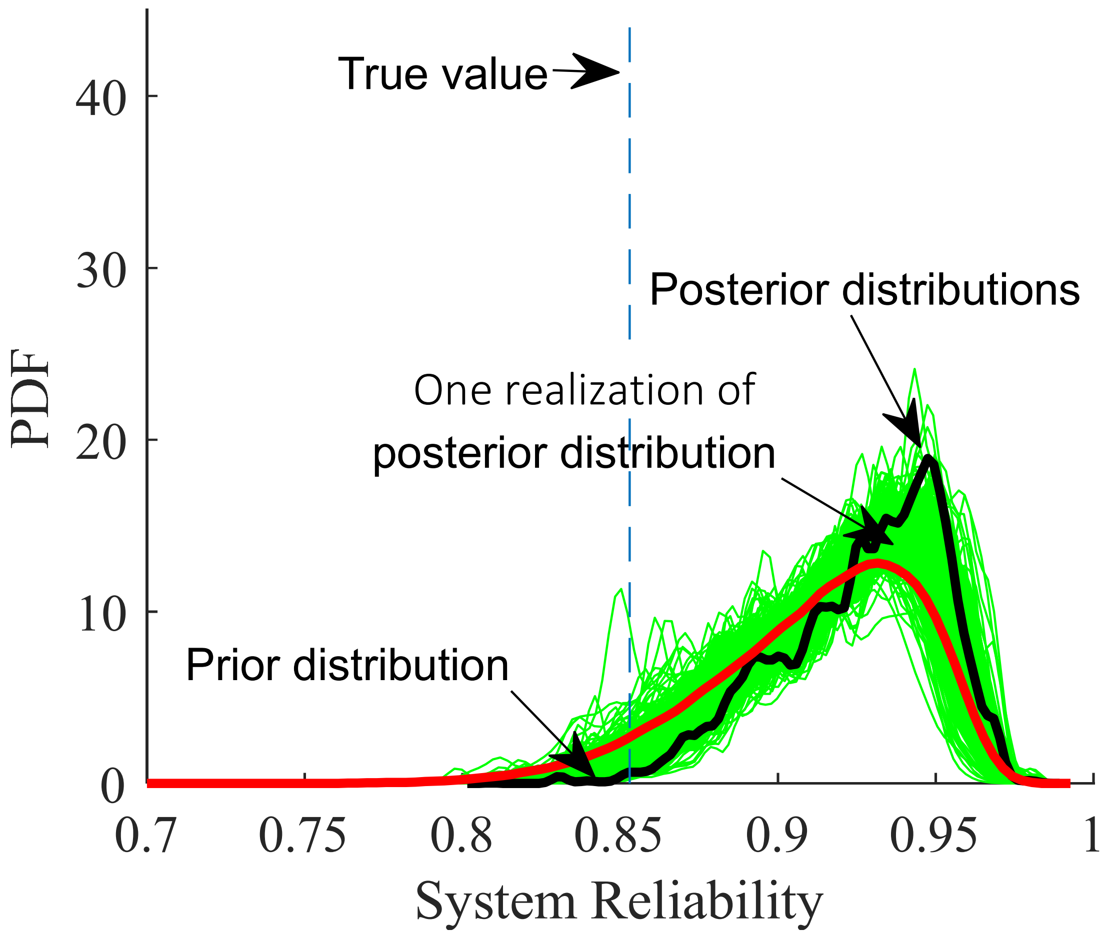


Figure 5-9 Comparison of prior and posterior distributions of the system reliability for Design 2

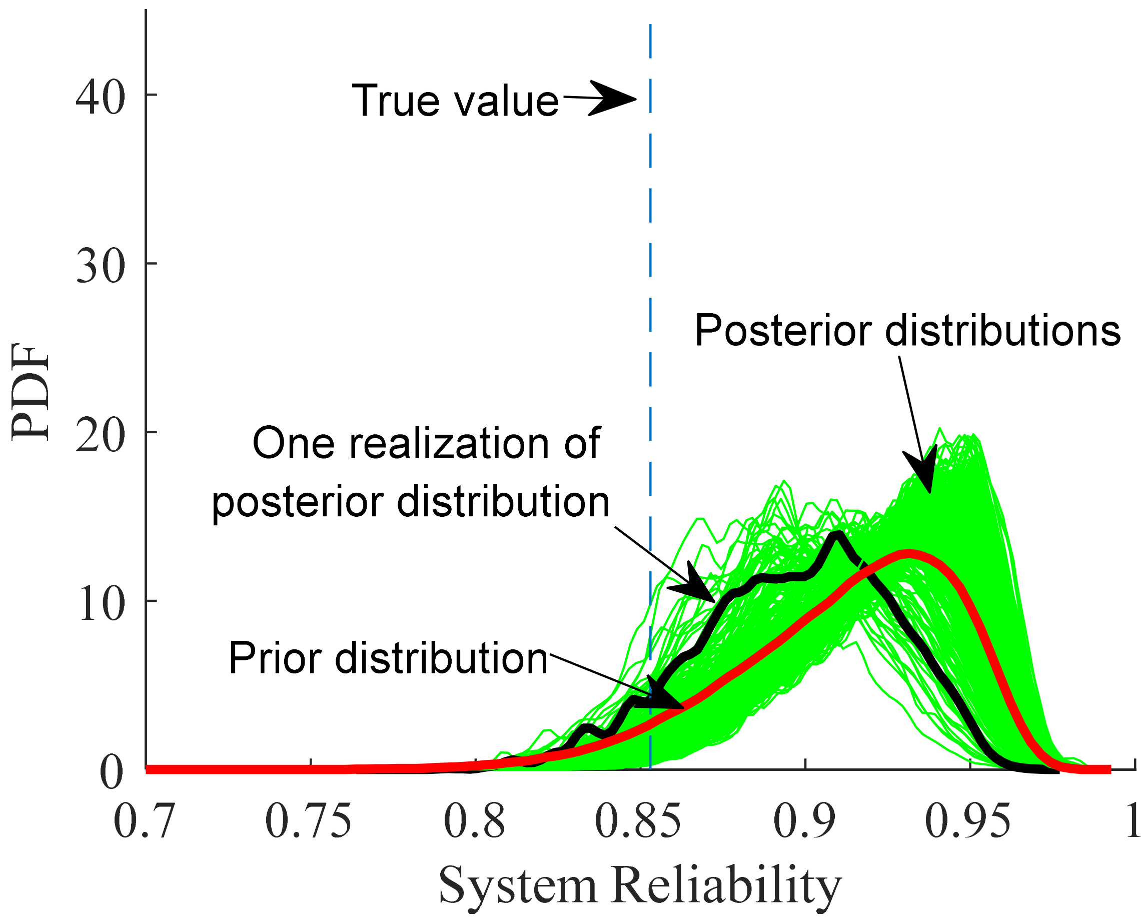


Figure 5-10 Comparison of prior and posterior distributions of the system reliability for Design 3

5.5.2 A Four-Joint Robot System

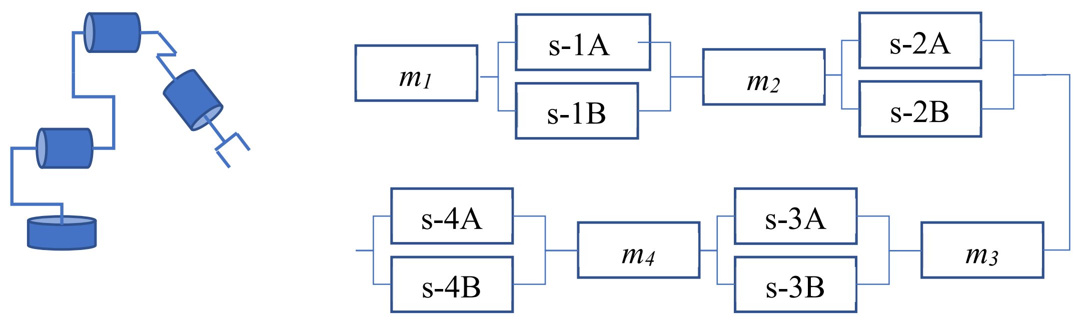
A - Problem Statement

A four-joint Unmanned Ground Vehicle (UGV) [132], [133] is employed as the second example.

Figure 5-11 Figure 5-11 gives the schematic kinematic diagram of the four-joint robot arm (Figure 5.11 (a)) and its reliability block diagram (Figure 5.11 (b)). As shown in this figure, there are four joints in total. Each joint is actuated by a motor m_i $(i = 1,\ 2,\ 3, 4)$ and the joint angle is measured by a sensor. To guarantee the reliability of the system, redundancy is added to each sensor.

The four motors ($m_1, m_2, m_3, and\ m_4$) are identical and independent. The angle sensors s-1A, s-1B, s-2A, s-2B, s-3A, s-3B, s-4A, and s-4B, are also identical components. There are mainly two types of components and twelve components in the system. Amongst the twelve components, component 10 is the boundary component. The reliability of the system over a time period $[0,\ 3 \times 10^5]$ cycles at the nominal stress level needs to be estimated based on ALT.

Table **5-6** gives the true ALT model parameters of the two types of components (motor and angle sensor).



(a) Kinematic diagram (b) Reliability block

Figure 5-11 A four-joint robot system

Table 5-6 True parameters of the component-level ALT models

	Component-level ALT parameters		
Component	$\theta_0^{(i)}$	$\theta_1^{(i)}$	$\sigma^{(i)}$
1	13.2	-4.6	0.12
2	12.8	-4.2	0.08

From experts' opinion, it is known that the failures of components 1 and 2, 1 and 3, 2 and 3, 5 and 6, 8 and 9, and 11 and 12 are dependent due to the shared load conditions. The true Gaussian copula parameters are assumed to be as shown in Table 5-7 below :

Table 5-7 Components Copula Correlation Factors

	Components Copula Correlation Factors						
Copula Factor	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$	$\rho_{5,6}$	$\rho_{8,9}$	$\rho_{11,12}$	$\rho_{i,i}$; $\forall i = 1,2,\dots,12$
Value	0.43	0.43	0.73	0.85	0.8	0.82	1

It is assumed that any other correlation parameters not listed in the table to be zeros. Based on the above assumed true models and parameters, the true system reliability (R_S) is estimated as $R_S = 0.7922$. Similar to example one, in ALT-based system reliability assessment, we assume that the above parameters are unknown and estimate them using ALT data. **Table 5-8** presents the required testing cost of component-level and system-level tests. The total budget of the accelerated life tests is $C_{total} = 2x10^5$.

Table 5-9 presents the prior distributions of the unknown parameters. In the ALT design, two-stress level tests are designed for the components and four-stress level tests are designed for the system. The number of available testing chambers is 20 for each component and for the system at each testing stress level. We then perform system reliability assessment using ALT based on the aforementioned information.

Table 5-8 Testing cost of component-level and system-level tests

	Component 1	Component 2	System
Cost/specimen	200	50	1300
Cost/unit time	0.02	0.01	0.045

Table 5-9 Prior distributions of the ALT model parameters

Variable	$\boldsymbol{\theta_0^{(1)}}$	$\boldsymbol{\theta_1^{(1)}}$	$\boldsymbol{\sigma^{(1)}}$	$\boldsymbol{\theta_0^{(2)}}$	$\boldsymbol{\theta_1^{(2)}}$	$\boldsymbol{\sigma^{(2)}}$	$\boldsymbol{\rho_{1,2}}$	$\boldsymbol{\rho_{1,3}}$	$\boldsymbol{\rho_{2,3}}$	$\boldsymbol{\rho_{5,6}}$	$\boldsymbol{\rho_{8,9}}$	$\boldsymbol{\rho_{11,12}}$
Parameter 1	13.25	-4.64	0.124	12.82	-4.22	0.083	0.1	0.1	0.1	0.1	0.1	0.1
Parameter 2	0.05	0.06	0.005	0.04	0.05	0.004	0.5	0.5	0.9	0.9	0.95	0.95
Distribution	Gaussian						Uniform					

B - System reliability assessment using ALT

Following the procedure discussed in Section 5.4.2, assume that from physics-informed load analysis, we obtain the relationships between the load conditions of boundary component and non-boundary components as follows:

Table 5-10 Physics-Informed model mapped stresses values

Mapped Stresses
$\boldsymbol{\xi_{s1} = 0.9\xi_b}$
$\boldsymbol{\xi_{s3} = \xi_{s2} = \sin^2(4\xi_b)}$
$\boldsymbol{\xi_{s7} = \xi_{s4} = \xi_b}$
$\boldsymbol{\xi_{s6} = \xi_{s5} = \left\lvert \sin(5\xi_b)\cos\left(\frac{3\xi_b}{8}\right)\right\rvert}$
$\boldsymbol{\xi_{s8} = \xi_{s9} = \lvert \sin(6\xi_b)\rvert}$
$\boldsymbol{\xi_{s12} = \xi_{s11} = \sin^2(4.5\xi_b)}$

The ALT design optimization model is formulated as:

$$
\begin{aligned}
&\max_{\boldsymbol{\xi}_{\mathbf{test}},\mathbf{n}_{\mathbf{test}}} \boldsymbol{E}(KL(\boldsymbol{\xi}_{\mathbf{test}},\mathbf{n}_{\mathbf{test}}))\\
&\textbf{Subject to}:\\
&\boldsymbol{\xi}_{\mathbf{test}} = \{\boldsymbol{\xi}_{\mathbf{b}}(\mathbf{1}),\boldsymbol{\xi}_{\mathbf{b}}(\mathbf{2}),\boldsymbol{\xi}_{\mathbf{b}}(\mathbf{3}),\boldsymbol{\xi}_{\mathbf{b}}(\mathbf{4})\ \boldsymbol{\xi}^{(\mathbf{i})}(\mathbf{1}),\boldsymbol{\xi}^{(\mathbf{i})}(\mathbf{2})\};\ \forall \boldsymbol{i} = \mathbf{1,2,3}\\
&\mathbf{n}_{\mathbf{test}} = \{\mathbf{n}_{\mathbf{sys}}(\mathbf{1}),\mathbf{n}_{\mathbf{sys}}(\mathbf{2}),\mathbf{n}_{\mathbf{sys}}(\mathbf{3}),\mathbf{n}_{\mathbf{sys}}(\mathbf{3}),\mathbf{n}^{(\mathbf{i})}(\mathbf{1}),\mathbf{n}^{(\mathbf{i})}(\mathbf{2})\};\\
&\forall \boldsymbol{i} = \mathbf{1,2,3}\\
&\boldsymbol{C}_{\boldsymbol{ALT}}(\boldsymbol{\xi}_{\mathbf{test}},\mathbf{n}_{\mathbf{test}}) \le \boldsymbol{C}_{\boldsymbol{total}}\\
&1 \le n^{(i)}(j) \le 20,\ \forall i = 1,2;\ \forall j = 1,2\\
&1 \le n_{sys}(j) \le 20,\ \forall j = 1,2,3,4\\
&0 \le \boldsymbol{\xi}_{test} \le 1
\end{aligned}
\tag{5-63}
$$

We then solve the above optimization model similarly to Example one. All results along with their interpretation are reported in what follows.

C – Results

Table 5-12 gives the optimal component-level and system-level ALT designs obtained from the proposed approach. We also compare the obtained optimal design with three other non-optimal designs given in Table 5-12. The Figures below shows the posterior distributions obtained using different designs and the prior distribution of the system reliability estimate. Table 5-11gives the expected KL divergence if ALTs are performed using these different testing plans.

Following that, Figure 5-12 through Figure 5-14 depict corresponding comparisons between posterior distributions and prior distribution of the system reliability estimate obtained using different designs. Note that Bayesian updating of the system reliability estimate is performed 200 times for each design (as indicated by multiple posterior distribution curves) to account for the uncertainty in the testing data. Figure 5-12, it also depicts the system reliability estimate using only component-level tests by ignoring the dependence.

It shows that using only the component-level ALTs for the system reliability estimate can lead to large error in the reliability estimate. The posterior distributions (i.e., Figure 5-12) obtained from the proposed method by fusing the component and system-level tests are much closer to the true value than that of using only component-level testing. This demonstrates the benefit of fusing component-level and system level testing data in system reliability analysis. Comparing the results in Figure 5-12 and Figure 5-13 through Figure 5-15, it shows that the posterior distributions obtained from the optimal design are getting closer to the true value, while the difference between the posterior distributions and the prior distribution is not as significant as the optimal design for the other non-optimal testing plans.

Table 5-11 Comparison of expected KL divergence for different testing plan designs

Design	Optimal Design	Design 1	Design 2	Design 3
$\boldsymbol{E(KL(\xi_{test}, n_{test}))}$	0.4902	0.2446	0.2570	0.3329
Expected Cost ($)	$5.9898x10^5$	$5.9890x10^5$	$5.9888x10^5$	$5.9889x10^5$

Table 5-12 ALT Plan for Optimal Design, Design 1, Design 2 and Design 3

	Component 1				**Component 2**			
	$\xi^{(1)}(1)$	$\xi^{(1)}(2)$	$n^{(1)}(1)$	$n^{(1)}(2)$	$\xi^{(2)}(1)$	$\xi^{(2)}(2)$	$n^{(2)}(1)$	$n^{(2)}(2)$
Optimal Design	0.545	0.494	16	18	0.016	0.523	18	5
Design 1	0.663	0.148	5	15	0.954	0.260	17	6
Design 2	0.267	0.094	18	9	0.820	0.733	9	19
Design 3	0.241	0.258	7	20	0.857	0.532	4	5
	System							
	$\xi_b(1)$	$\xi_b(2)$	$\xi_b(3)$	$\xi_b(4)$	$n_{sys}(1)$	$n_{sys}(2)$	$n_{sys}(3)$	$n_{sys}(4)$
Optimal Design	0.478	0.396	0.414	0.743	17	18	16	10
Design 1	0.306	0.782	0.960	0.373	18	14	17	18
Design 2	0.302	0.363	0.458	0.285	10	5	14	16
Design 3	0.847	0.380	0.919	0.700	15	20	17	12

The differences between the prior distribution and the posterior distributions for different designs are quantified quantitatively by the expected KL divergence in Table 6. The above results comparisons demonstrate the effectiveness of the proposed framework in reducing the uncertainty in the system reliability estimate.

5.6 Summary

Accelerated life testing has been widely used in product development to certify the reliability of the products in the early design stage. In order to evaluate the reliability of a complex system, components are usually tested separately and the complicated dependence between different components are ignored in practice. System-level reliability tests can effectively account for the complicated dependence between different failure modes and components. The required testing cost for system-level tests, however, are much higher than its counterpart of component-level tests. How to effectively allocate the limited testing resources to the component-level and system-level tests in ALT-based system reliability assessment is a challenge issue that need to be addressed.

Chapter 5 proposes a novel ALT design framework for system reliability assessment. In order to fuse the information from component-level and system-level tests for the purpose of system reliability estimate, connections are first established between the component-level ALT and system reliability, as well as system-level ALT and system reliability. More specifically, physics-informed load prediction models are employed to bridge the gap between the system-level tests at higher-than-nominal stress level and system reliability analysis at the nominal stress level. Building upon the established connections, an optimization model is then formulated to maximize the information gain from various tests subject to budget constraint. The results of two numerical examples including a mixed system and a four-joint robot system, demonstrate that the proposed framework can effectively reduce the uncertainty in the system reliability estimate through the information fusion of component-level and system-level tests.

Similar to many other ALT design approaches, the proposed method adopts several commonly utilized assumptions. Eliminating these assumptions is a research topic that is worth pursuing in future. The component-level ALT and system-level ALT are optimized concurrently in the proposed framework. This may result in a large number of design variables when the number of components is high. A possible way of addressing this challenge is to perform the component-level ALT design and system-level ALT design sequentially instead of concurrently. This will be investigated in our future work.

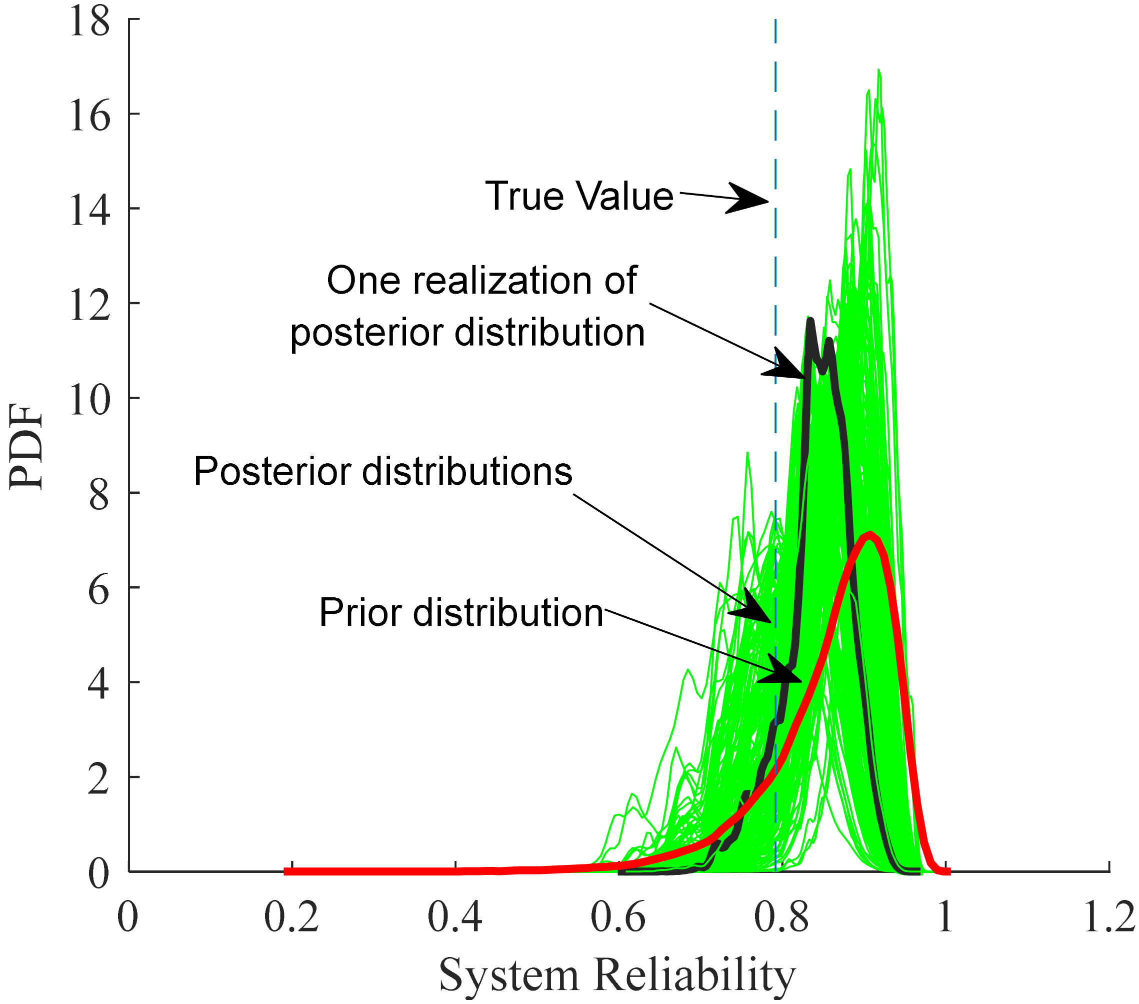


Figure 5-12 Comparison of prior and posterior distributions of the system reliability for the optimal design

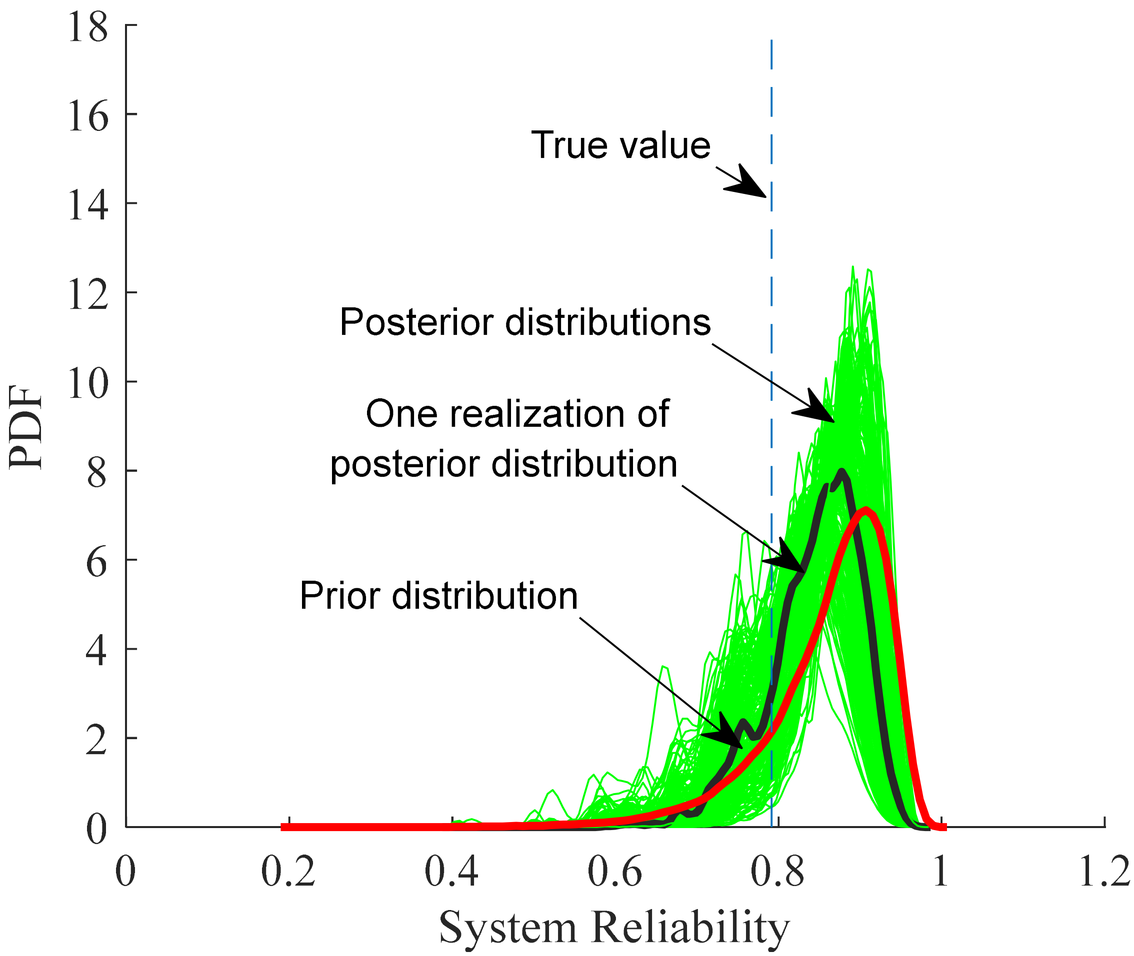


Figure 5-13 Comparison of prior and posterior distributions of the system reliability for Design 1

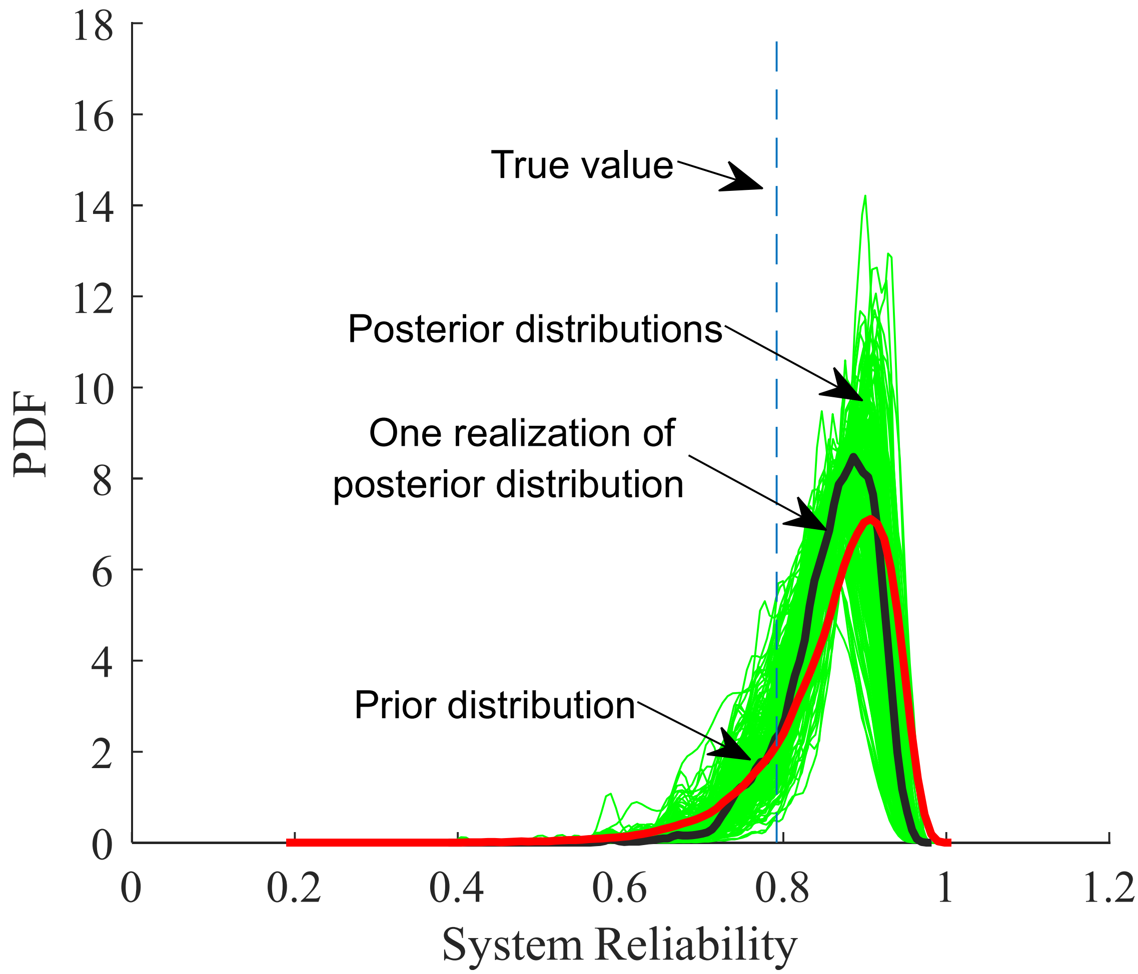


Figure 5-14 Comparison of prior and posterior distributions of the system reliability for Design 2

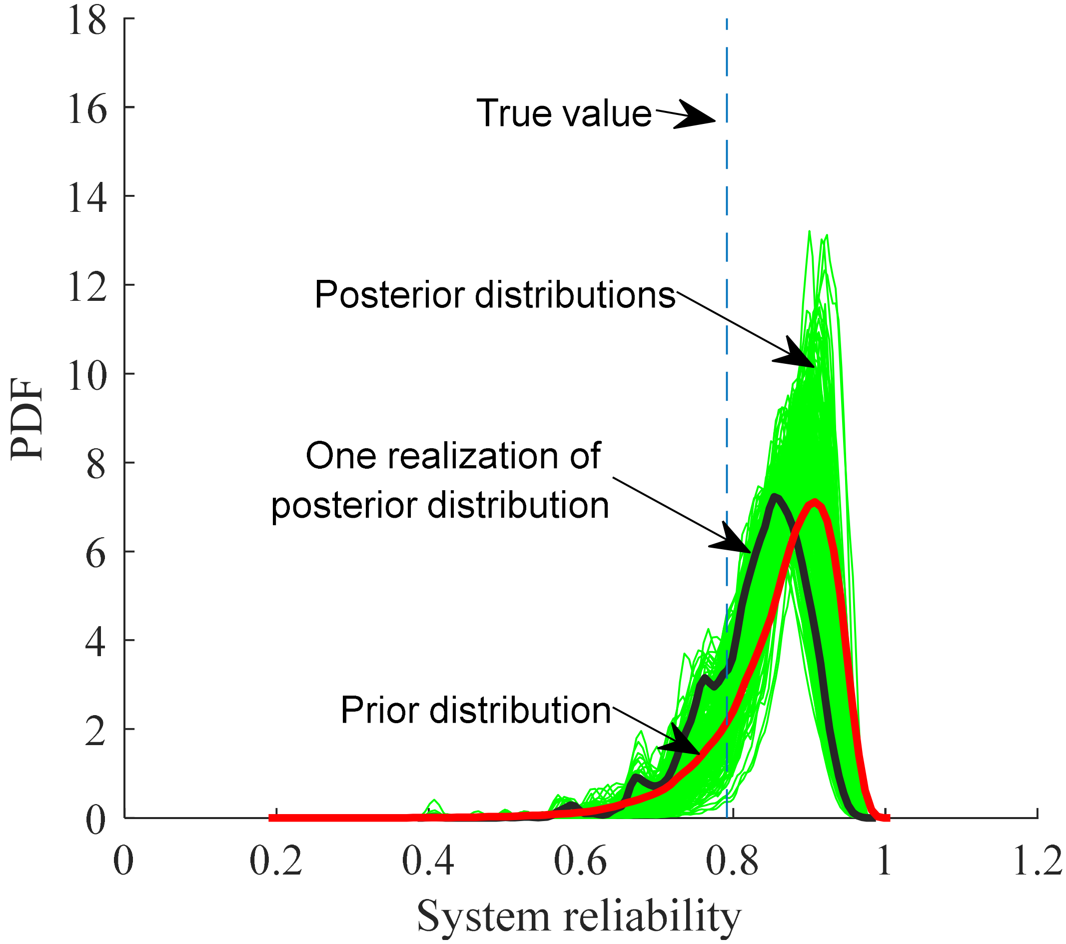


Figure 5-15 Comparison of prior and posterior distributions of the system reliability for Design 3

Chapter 6 System Reliability Assessment via Distribution Free Model and Shared Frailty Models

In this chapter, we present the proposed method to perform system reliability assessment based on ALT data collected from two different testing levels, namely component level and system level. The model is based on using the extended hazard regression (EHR) model is employed in order to model the component ALT failure time data in conjunction with Frailty models in order to model the dependence among the failure time data. The EHR model, as explained previously in Section 4.1.3, relies on using the hazard function formulation and it is a regression model that takes into account the effect of the covariate which is the applied stress during testing on the failure time of a component. Additionally, a frailty factor is incorporated in conjunction with the EHR model in order to model the dependence among the components.

It is assumed that there is no censoring and that all testing units would be tested to failure. Censoring could be easily incorporated in the model. In addition, only constant stresses are considered. Step-stress ALT design is not the focus of this research.

As shown in Figure 6-1, the model consists of three sub-models: the first is establishing a relationship between the component-level ALT data and the system reliability. Second, a model to connect system-level ALT data to the system reliability and third we aim at fusing both component-level ALT data and system-level ALT data in order establish an estimate for the system reliability.

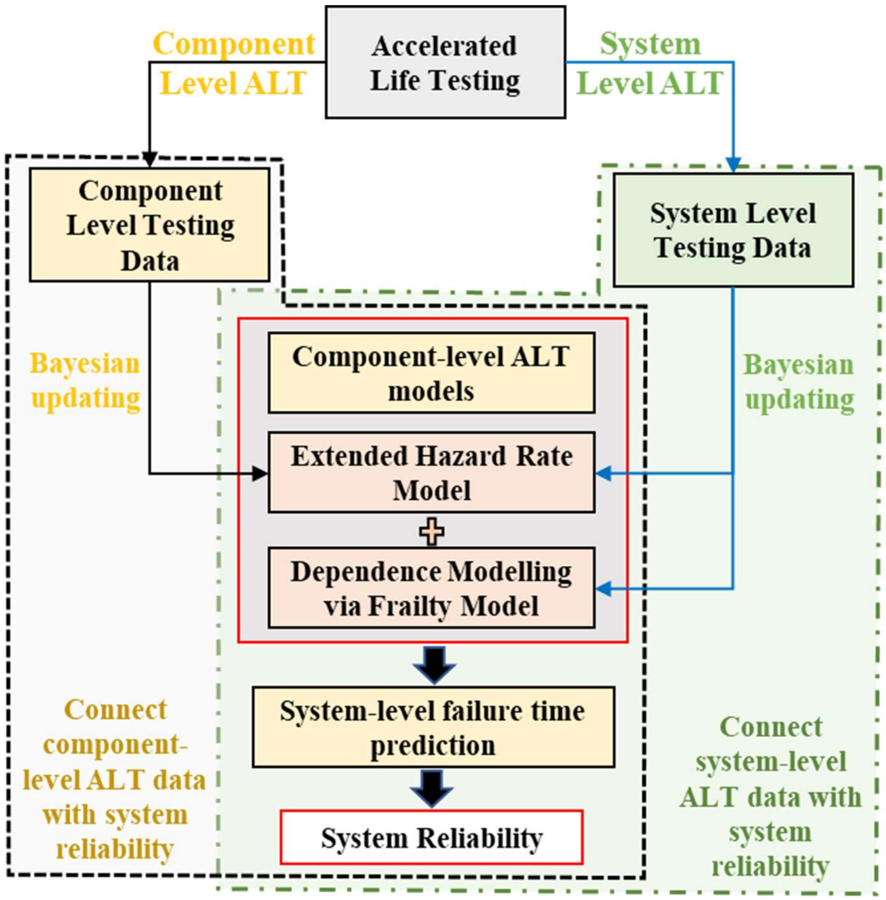


Figure 6-1 Overview of the proposed ALT framework for system reliability analysis

Next, we start by presenting the component-level ALT data framework that connect the component-level ALT data to the system reliability using EHR and Frailty models.

6.1 Uncertainty Propagation of Component Level ALT Data to System Reliability

6.1.1 Framework Steps Overview

Aiming at establishing a system reliability estimate, the goal of the component-level ALT framework is to reduce and propagate the uncertainties in the data of the component parameters using ALT data collected from component- level ALT by subjecting each of the component to a higher than nominal stress individually.

The Diagram shown in Figure 6-2 shows the steps taken in order to establish a connection between the component level ALT data and the system reliability through uncertainty propagation. First, ALT data is collected for each of the components in a system. Second, A likelihood function is formulated using EHR model. Third, the uncertainties of the EHR model parameters are reduced using Bayesian Inference for all components. Fourth, posterior distributions for all parameters for each of the components are sampled using particle filtering. Fifth, Dependence is accounted for using frailty models. Last, using the system topology (series versus mixed) the uncertainties are propagated to the system reliability.

The first section in this chapter tackles reducing the uncertainties in the prior information of the component parameters. To do so, the Bayesian relationship is developed to sample out posterior distribution with minimal uncertainty for model parameters and namely, the regression parameters of the EHR model and the parameters of the baseline quadratic hazard function. Accordingly, the Bayesian estimation relationship requires the development of the likelihood function. After formulating the likelihood function, particle filtering is applied as the sampling procedure to derive the posterior samples.

In addition, the propagation of the uncertainties in the component parameters to the system reliability imposes the correlation among the components when assembled together under one system. In order to tackle the dependence, the frailty models are integrated along with the EHR model to account for any possible dependence among the components.

Lastly, we calculate the system reliability. The model details the formulation for any pre-defined system configuration: series topology or mixed topology (i.e. series, parallel or any other custom topology).

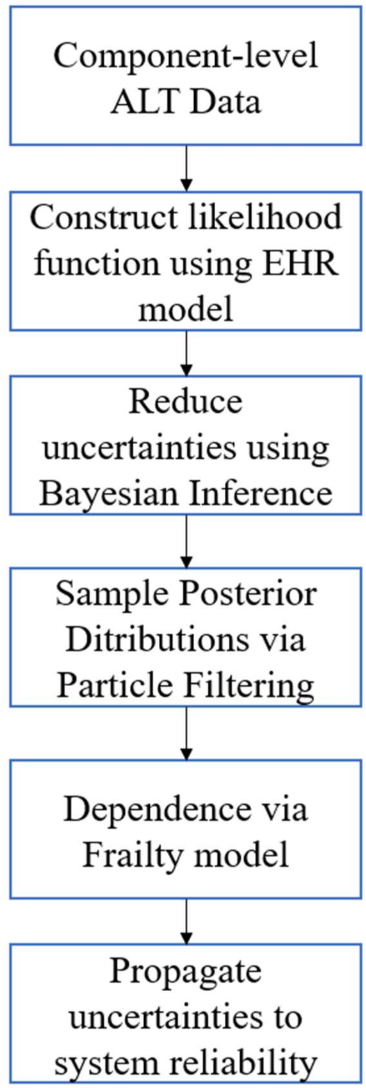


Figure 6-2 Framework overview of the connection between component-level ALT at high stress levels and system reliability model overview

6.1.2 Framework Assumptions

In this section we present the assumptions made in order to develop the ALT framework via distribution free model that includes the concept of frailty models. The following assumptions listed in section 5.1.2 in Chapter 5 applies to this model as well:

A1- System Topography Assumption

A2- ALT Feasibility Assumption

A3- Load Transfer Function Assumption

A4- Time Censoring Assumption

A5- Constant ALT Accelerated Stress Assumption

A6- Prior Information Availability Assumption

A7- Extrapolation of ALT Failures to Normal Operation Assumption

Adding to the seven assumptions that could be carried over from the ALT using Log-Scale parametric statistical distribution and copula chapter, this model requires an additional assumption and they are as follows:

A8- Frailty Factor Identifiability: The z factor denoting the frailty factor that is part of the frailty model is assumed to have an accessible and identifiable prior information. Unlike the Copula function, one factor is required for all the components of one system which is the concept of shared frailty factors. The mathematical formulation will be presented in the sections below.

As shown in Figure 6-2, ALT data is first collected for each component in the system individually. After that, the data are used to update the component-level ALT models using Bayesian inference method. The updated component-level ALT models are then connected to system reliability at the nominal stress level through frailty models. In the subsequent sections, we will discuss the major steps to establish such a connection between component-level ALT and system reliability.

6.1.3 Bayesian Updating of the Component-level ALT Models

In order to develop the Bayesian Inference relationship to sample the posterior distributions, the parameters of interests are identified for which prior information shall be available. Accordingly, as detailed in Section 4.1.3 the EHR model is employed based on Equations (4-3) through (4-4) and the Bayesian Inference Equation (4-11).

A – The EHR Model and the PDF Function Formulation

Considering a system with N_C components, we denote the number of ALT stress levels of the *i*-th component as m_i, $\forall i = 1,2,\dots,N_C$. Letting the testing stress levels of the *i*-th component be $\mathbf{s}_i = [s_{ij},\ j = 1,2,\dots,\ m_i]$, where s_{ij} is the *j*-th testing stress level of the *i*-th component, we first normalize the stress level as below:

$$\mathrm{x}_{ij} = \frac{s_{ij} - s_{L_i}}{s_{U_i} - s_{L_i}} \tag{6-1}$$

where s_{L_i} and s_{U_i} are respectively the lower and upper bounds of the testing stress level of the *i*-th component and x_{ij} is the normalized *j*-th testing stress level of the *i*-th component. The normalized stress is then a value between 0 and 1 $\left(\mathbf{0} \le \mathrm{x}_{ij} \le \mathbf{1}\right)$.

After that, we model the failure time data collected from component-level ALT using the EHR model explained in Section 2.2.3. The hazard rate function of the *i*-th component at the *j*-th testing stress level is given by:

$$\lambda\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = \lambda_{0,i}\left(te^{(\mathbf{x}_{ij}^T\boldsymbol{\beta}_i)}\right)e^{(\mathbf{x}_{ij}^T\boldsymbol{\alpha}_i)}, \forall i = 1,2,\dots,N_C; j = 1,2,\dots,m_i, \tag{6-2}$$

where $\lambda_{0,i}(\cdot)$ is the baseline hazard function of the *i*-th component and $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ are the regression parameters of the *i*-th component given by:

$$\begin{aligned} \boldsymbol{\alpha} &= [\alpha_0\ \alpha_1] \\ \boldsymbol{\beta} &= [\beta_0\ \beta_1] \end{aligned} \tag{6-3}$$

Based on the hazard function, we then have the probability density function (PDF) of failure time t at the *j*-th testing stress level as below:

$$f_{T_{ij}}\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = \lambda\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) \times R\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) \tag{6-4}$$

where $f_{T_{ij}}(\cdot)$ is the PDF of t for given $\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i$, and $\mathbf{x}_{ij}$, $\lambda\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right)$ is the hazard rate function given in Equation (6-2) and $R\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right)$ is the reliability function defined as $R\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = e^{-\Lambda\left(t; \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right)}$, in which $\Lambda\left(t; \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = \int_0^t \lambda\left(y; \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) dy$ is the cumulative hazard function.

In order to get the cumulative hazard function to compute $R\left(t \middle| \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right)$, the baseline hazard function $\lambda_{0,i}(\cdot)$ is assumed to be a quadratic function by following the method presented in [62] as follows:

$$\lambda_{0,i}(u) = \gamma_{0,i} + \gamma_{1,i} u_i + \gamma_{2,i} u_i^2 \tag{6-5}$$

where $\gamma_{0,i}$, $\gamma_{1,i}$, and $\gamma_{2,i}$ are regression coefficients and $u = te^{(\mathbf{x}_{ij}^T \boldsymbol{\beta}_i)}$.

Combining Equations (6-2) and (6-5) yields the following hazard rate function of the *i*-th component:

$$\begin{gathered}
\lambda_i\left(t \middle| \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = \gamma_{0,i} e^{\mathbf{x}_{ij}^T \boldsymbol{\alpha}_i} + \gamma_{1,i} t e^{\mathbf{x}_{ij}^T \mathbf{w}_{0,i}} + \gamma_{2,i} t^2 e^{\mathbf{X}_i^T \mathbf{w}_{1,i}}, \\
\mathbf{w}_{0,i} = \boldsymbol{\alpha}_i + \boldsymbol{\beta}_i \;\; \& \;\; \mathbf{w}_{1,i} = \boldsymbol{\alpha}_i + 2\boldsymbol{\beta}_i, \\
\boldsymbol{\gamma}_i = \{\gamma_{0,i}, \gamma_{1,i}, \gamma_{2,i}\}; \; \forall\, i = 1,2, \dots, N_c
\end{gathered} \tag{6-6}$$

We then can get the cumulative hazard function by integrating Equation (6-6) above:

$$\Lambda\left(t \middle| \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = \int_0^t \left(\gamma_{0,i} e^{\mathbf{x}_{ij}^T \boldsymbol{\alpha}_i} + \gamma_{1,i} t e^{\mathbf{x}_{ij}^T \mathbf{w}_{0,i}} + \gamma_{2,i} t^2 e^{\mathbf{X}_i^T \mathbf{w}_{1,i}}\right) dy, \tag{6-7}$$

$$\Lambda\left(t \middle| \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = \gamma_{0,i} t e^{\mathbf{x}_{ij}^T \boldsymbol{\alpha}_i} + \frac{\gamma_{1,i} t^2}{2} e^{\mathbf{x}_{ij}^T \mathbf{w}_{0,i}} + \frac{\gamma_{2,i} t^3}{3} e^{\mathbf{x}_{ij}^T \mathbf{w}_{1,i}}. \tag{6-8}$$

The reliability function $R(t|\boldsymbol{\alpha}_i, \boldsymbol{\beta}_\mathbf{i}, \mathbf{x}_{ij})$ in Equation (6-4) is then given by:

$$R(t|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_\mathbf{i}, \mathbf{x}_{ij}) = e^{-\Lambda(t|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij})},$$
$$= e^{-\left(\gamma_{0,i}te^{\mathbf{x}_{ij}^T\boldsymbol{\alpha}_i}+\frac{\gamma_{1,i}t^2}{2}e^{\mathbf{x}_{ij}^T\mathbf{w}_{0,i}}+\frac{\gamma_{2,i}t^3}{3}e^{\mathbf{x}_{ij}^T\mathbf{w}_{1,i}}\right)}. \tag{6-9}$$

Plugging Equations (6-6) and (6-9)into Equation (6-5), we have the PDF function of the failure time of the *i*-th component at the *j*-th testing stress level as:

$$f_{T_{ij}}(t|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_\mathbf{i}, \mathbf{x}_{ij}) =$$
$$\left(\gamma_{0,i}e^{\mathbf{x}_{ij}^T\boldsymbol{\alpha}_i} + \gamma_{1,i}te^{\mathbf{x}_{ij}^T\mathbf{w}_{0,i}} + \gamma_{2,i}t^2e^{\mathbf{X}_i^T\mathbf{w}_{1,i}}\right)$$
$$\times (e^{-\left(\gamma_{0,i}te^{\mathbf{x}_{ij}^T\boldsymbol{\alpha}_i}+\frac{\gamma_{1,i}t^2}{2}e^{\mathbf{x}_{ij}^T\mathbf{w}_{0,i}}+\frac{\gamma_{2,i}t^3}{3}e^{\mathbf{x}_{ij}^T\mathbf{w}_{1,i}}\right)}. \tag{6-10}$$

B – Bayesian Inference Relationship and Likelihood Function Formulation

Letting the collected failure time data of the *i*-th component at the *j*-th testing stress level be $\mathbf{t}_{ij} = \{t_{ij}(k),\ k = 1,\ 2, \ldots,\ n_{ij}\}$, where n_{ij} is the number of tests at the *j*-th testing stress level of the *i*-th component, we have the likelihood function of observing $\mathbf{t}_{ij}$ as

$$f(\mathbf{t}_{\mathrm{ij}}|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}) = \prod_{k=1}^{n_{ij}} f_{T_{ij}}(t_{ij}(k)|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}). \tag{6-11}$$

Letting the failure time data collected over all the testing stress level of the *i*-th component be $\mathbf{t}_i^s = \{t_{ij},\ j = 1,2, \ldots, m_i\}$, the likelihood function of observing $\mathbf{t}_i^s$ is then given by:

$$f(\mathbf{t}_i^s|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_i^s) = \prod_{j=1}^{m_i} f(\mathbf{t}_{ij}|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}), \tag{6-12}$$

in which $\mathbf{x}_i^s = \{\mathrm{x}_{ij},\ j = 1,2, \ldots, m_i\}$.

Plugging Equations (6-10) and (6-11)into Equation (6-12), we have the likelihood function for the i-th component as follows

$$f(\boldsymbol{t}_i^s|\boldsymbol{\gamma}_i,\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i,\mathbf{x}_i^s) = \prod_{j=1}^{m_i}\prod_{k=1}^{n_{ij}} f\left(t_{ij}(k)\middle|\boldsymbol{\gamma}_i,\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i,\mathbf{x}_{ij}\right),$$

$$= \prod_{j=1}^{m_i}\prod_{k=1}^{n_{ij}}\left(\left[[\gamma_{0,i}e^{\mathbf{x}_{ij}^{\mathrm{T}}\boldsymbol{\alpha}_i} + \gamma_{1,i}t_{ij}(k)e^{\mathbf{x}_{ij}^{\mathrm{T}}\mathbf{w}_{0,i}} + \gamma_{2,i}\left(t_j^i(k)\right)^2 e^{\mathbf{x}_{ij}^{\mathrm{T}}\mathbf{w}_{1,i}}]\right] \times \left[e^{-\left(\gamma_{0,i}(t_j^i(k))e^{X_i(j)^T\boldsymbol{\alpha_i}} + \frac{\gamma_{1,i}(t_j^i(k))^2}{2}e^{X_i(j)^T\boldsymbol{w_{0,i}}} + \frac{\gamma_{2,i}(t_j^i(k))^3}{3}e^{X_i(j)^T w_{1,i}}\right)}\right]\right) \tag{6-13}$$

where $\mathbf{w}_{0,i} = \boldsymbol{\alpha}_i + \boldsymbol{\beta}_i$ & $\mathbf{w}_{1,i} = \boldsymbol{\alpha}_i + 2\boldsymbol{\beta}_i$.

The Likelihood function given in Equation (6-13) does not include censoring and it can be simply modified to include censored data as shown in [134].

With the above likelihood function, we can then update the component-level ALT parameters, $\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i$, and $\boldsymbol{\beta}_i$ of the i-th component using Bayesian method as below

$$f(\boldsymbol{\gamma}_i,\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i|\boldsymbol{t}_i^s,\mathbf{x}_i^s) \propto f(\boldsymbol{t}_i^s|\boldsymbol{\gamma}_i,\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i,\mathbf{x}_i^s)f_{\gamma_i}(\boldsymbol{\gamma}_i)f_{\alpha_i}(\boldsymbol{\alpha}_i)f_{\beta_i}(\boldsymbol{\beta}_i), \tag{6-14}$$

where "∝" stands for "proportional to", $f_{\gamma_i}(\boldsymbol{\gamma}_i), f_{\alpha_i}(\boldsymbol{\alpha}_i)$, and $f_{\beta_i}(\boldsymbol{\beta}_i)$ are the prior distributions of $\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i$, and $\boldsymbol{\beta}_i$, respectively.

Next, we will discuss how to aggregate the component-level information to the system-level for the evaluation of system reliability.

6.1.4 Uncertainty Propagation to System Reliability

In this section, we intend to calculate the system reliability of a system in which the components are functioning together, logically or physically, causing their failures to be dependent on each other. So, in order to propagate the uncertainties from the component parameters to the system reliability, it is necessary to model the dependence between the component-level ALT failure time data of the different components. In this following section, the shared frailty model as discussed in Section 4.1.3B is employed to establish a link between the component-level ALT model parameters and system-level failure time. Next, we will discuss how to construct this kind of connections for two categories of systems, namely series system configuration and other system configurations- Dependence via Frailty Models and Gamma Distribution

A – Series System Topology

To model the dependence among components and establish the connection for a series system, we use the Gamma shared frailty model. The gamma shared frailty model assumes that the frailty factor follows a gamma distribution that is $Z \sim Gamma(v, v)$ with mean equal to 0 and variance $var(Z) = \delta^2$ where $\delta^2 = \frac{1}{v}$ and v is the shape parameter which is equal to the scale parameter of the gamma distribution of *Z*. This assumption is made to avoid the non-identifiability issue in Bayesian inference [135].

Combining the frailty model given in Section 4.1.3B with the EHR model and the quadratic baseline hazard function [63], the modified hazard function is then given by

$$\lambda\left(t \middle| z, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = z\lambda_0\left(t\ e^{\mathbf{x}_{ij}^T \boldsymbol{\beta}_i)}\right) e^{(\mathbf{x}_{ij}^T \boldsymbol{\alpha}_i)}, \tag{6-15}$$

The reliability of the *i*-th component at the *j*-th stress-level after introducing the frailty factor z is given by

$$\Pr\left(T_i > t \middle| z, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right)$$
$$= R\left(t \middle| z, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right) = e^{-z\Lambda\left(t|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{ij}\right)} \tag{6-16}$$

After integrating out the uncertain frailty factor Z for given v, we have the unconditional reliability function at nominal stress $\mathbf{x}_{i,n}, \forall i = 1,2, \dots, N_C$ as [89] :

$$\begin{aligned} R(t|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, v,\ \mathbf{x}_{i,n}) &= \int_0^\infty R(t|z, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, v,\ \mathbf{x}_{i,n}) f_{Z|v}(z|v)dz, \\ &= E\left(e^{-z\Lambda(t|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n})}\right) = L_z\left(\Lambda(t_i|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n})\right), \end{aligned} \tag{6-17}$$

where $f_{Z|v}(z|v)$ is the PDF of the random frailty factor conditioned on given distribution parameter of v, and $L_z\,(\cdot)$ represents the Laplace transform over Z.

The joint survival function of a series system (i.e. system reliability) at the nominal stress levels, $\mathbf{x}_{i,n}, \forall i = 1,2, \dots, N_C$, is given by

$$\begin{aligned} R_{sys}(t\,|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \mathbf{x}_{all}) &= \Pr(T_1 > t, \dots, T_{N_c} > t|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \mathbf{x}_{all}), \\ &= \int_0^\infty \prod_{i=1}^{Nc} \Pr(T_i > t|z, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, v,\ \mathbf{x}_{i,n})\, f_{Z|v}(z|v)dz, \end{aligned} \tag{6-18}$$

where $R_{sys}(t\,|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v,\ \mathbf{x}_{all})$ is the system reliability conditioned on $\boldsymbol{\gamma}_{sys}$, $\boldsymbol{\alpha}_{sys}$, $\boldsymbol{\beta}_{sys}$, v, and $\mathbf{x}_{all}$, $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}$, and $\mathbf{x}_{all}$ are respectively $\mathbf{x}_{all} = \{\mathbf{x}_{i,n}, \forall i = 1,2, \dots, N_C\}$, $\boldsymbol{\alpha}_{\mathbf{sys}} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_{N_C}\}$, $\boldsymbol{\beta}_{\mathbf{sys}} = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_{N_C}\}$, and $\boldsymbol{\gamma}_{\mathrm{sys}} = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \dots, \boldsymbol{\gamma}_{N_C}\}$.

The Joint Survival Function (i.e. Reliability) for given a series system is given by the following probability equation:

$$\begin{aligned} R_{sys}(t|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \mathbf{x}_{all}) &= L_z\left(\sum_{i=1}^{N_C} z\Lambda(t_i|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n})\right), \\ &= \left(1 + \frac{1}{v}\sum_{i=1}^{N_C} \Lambda(t_i|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n})\right)^{-v}. \end{aligned} \tag{6-19}$$

Combining Equation (6-8) with Equation (6-19), we have system reliability of a series system for given $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}$, and v as

$$R_{sys}(t|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \mathbf{x}_{all})$$

$$= \left[1 + \frac{1}{v}\sum_{i=1}^{N_C}\left(\gamma_{0,i}t_i e^{\mathbf{x}_{i,n}^{\mathrm{T}}\boldsymbol{\alpha}_i} + \frac{\gamma_{1,i}t_i^2}{2}e^{\mathbf{x}_{i,n}^{\mathrm{T}}\mathbf{w}_{0,i}} + \frac{\gamma_{2,i}t_i^3}{3}e^{\mathbf{x}_{i,n}^{\mathrm{T}}\mathbf{w}_{1,i}}\right)\right]^{-v}, \tag{6-20}$$

where v is the variance of Z, and $\mathbf{w}_{0,i} = \boldsymbol{\alpha}_i + \boldsymbol{\beta}_i$ & $\mathbf{w}_{1,i} = \boldsymbol{\alpha}_i + 2\boldsymbol{\beta}_i$.

B - Mixed and Non-Standard Systems Topology

Even though the above equations (6-15) through (6-20) have analytical solutions that allows linking component-level ALT models to system reliability, it is inapplicable to systems with complicated configurations, it is rather only applicable to systems with series configuration. However, some system configurations can be converted into series system expressions and then the above discussed approach can be applied. For some situation, this kind of transformation, however, is complicated. In addition, it is sometimes cumbersome to convert systems configurations (e.g. networked systems) into series ones. To overcome this challenge, this subsection will develop an approach that allows connecting component-level ALT models with system reliability for any system configuration only if the system topology can be expressed as Boolean logical functions.

Next, for given values of v, $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}$, and $\boldsymbol{\beta}_{sys}$, we sample n_Z particles for the frailty factor z using the inverse of the Gamma distribution; $Gamma^{-1}(v, v)$. We define the obtained samples of Z as z_s, $s = 1,2, \dots, N_z$. For each z_s, we then generate samples of the component failure time $t_{i,n}^s$ of the i-th component at nominal stress $\mathbf{x}_{i,n}$ using the following inverse CDF function:

$$t_{i,n}^{s} = F_{T_i}^{-1}\left(u_{i,n}|z_s, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n}\right), \tag{6-21}$$

where $u_{i,n}$ is a random CDF sample generated from MCS for the i-th component at nominal stress level, and $F_{T_i}^{-1}\left(u|z_s, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n}\right)$ is the inverse function of the following CDF function

$$\begin{aligned} F_{T_i}\left(t|z_s, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n}\right) &= 1 - R\left(t|z_s, \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,n}\right), \\ &= 1 - e^{-z_s\left([\gamma_{0,i}e^{\mathbf{x}_{i,n}^T\alpha_i} + \gamma_{1,i}te^{\mathbf{x}_{i,n}^T w_{0,i}} + \gamma_{2,i}t^2e^{\mathbf{x}_{i,n}^T w_{1,i}}]\right)}. \end{aligned} \tag{6-22}$$

For each component, we obtain the failure time samples as $\mathbf{t}_{i,n}^{s} = \{t_{i,n}^{s}(q),\ \forall q = 1,2, \dots, N_r; i = 1,2, \dots, N_C\}$, where N_r is the number of samples generated using MCS in Eq. (27).

Now, for any system topography, we define the failure function $f_{time}(\cdot)$ as a function representing the failure of a system based on its topology (i.e. configuration) as below:

$$t_{sys,n}^{s}(q) = f_{time}\left(t_{1,n}^{s}(q), t_{2,n}^{s}(q), \dots, t_{N_C,n}^{s}(q)\right), \tag{6-23}$$

$$f_{time} = \begin{cases} \min\left(t_{1,n}^{s}(q), t_{2,n}^{s}(q), \dots, t_{N_C,n}^{s}(q)\right) - \text{Series System}, \\ \max\left(t_{1,n}^{s}(q), t_{2,n}^{s}(q), \dots, t_{N_C,n}^{s}(q)\right) - \text{Parallel System}, \\ \text{Defined according to system topology} - \text{Otherwise}, \end{cases} \tag{6-24}$$

where $t_{sys,n}^{s}(q)$ is the q-th sample of system failure time at nominal stress conditioned on $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}$, and v.

Based on samples of $t_{sys,n}^{s}(q)$, $\forall q = 1,2, \dots, N_r$, obtained from the above equations, the system reliability at nominal stress level $\mathbf{x}_{all}$ conditioned on $z_s, \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}$, and v is calculated by

$$R_{sys}\left(t\,|z_s, \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{all}\right) = \frac{\sum_{q=1}^{N_r} I(t_{sys,n}^{s}(q))}{N_r}, \tag{6-25}$$

where $I\left(t_{sys,n}^{s}(q)\right) = 1$, if $t_{sys,n}^{s}(q) \geq t$; otherwise, $I\left(t_{sys,n}^{s}(q)\right) = 0$.

The system reliability conditioned on $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}$, and v is then given by

$$R_{sys}\left(t \mid \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \mathbf{x}_{all}\right) = \frac{\sum_{s}^{N_z} R_{sys}\left(t \mid z_s, \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{all}\right)}{N_z}. \tag{6-26}$$

With Equations (6-15) through (6-20), we connect component-level ALT models with system reliability with the consideration of dependence between failure time distributions of different components. By applying Equation (6-15) through (6-26) for posterior distributions of $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}$ obtained using the method presented in Section 6.1.3, the uncertainty in system reliability estimate can be reduced using component-level ALT data for any system configurations. Next, we will discuss how to connect system-level ALT data with component-level ALT models and system reliability.

Figure 6-2 below summarizes the steps of propagating the uncertainties using the component level ALT data with frailty models to model unseen factors causing dependence among the components of a system when operating under normal conditions.

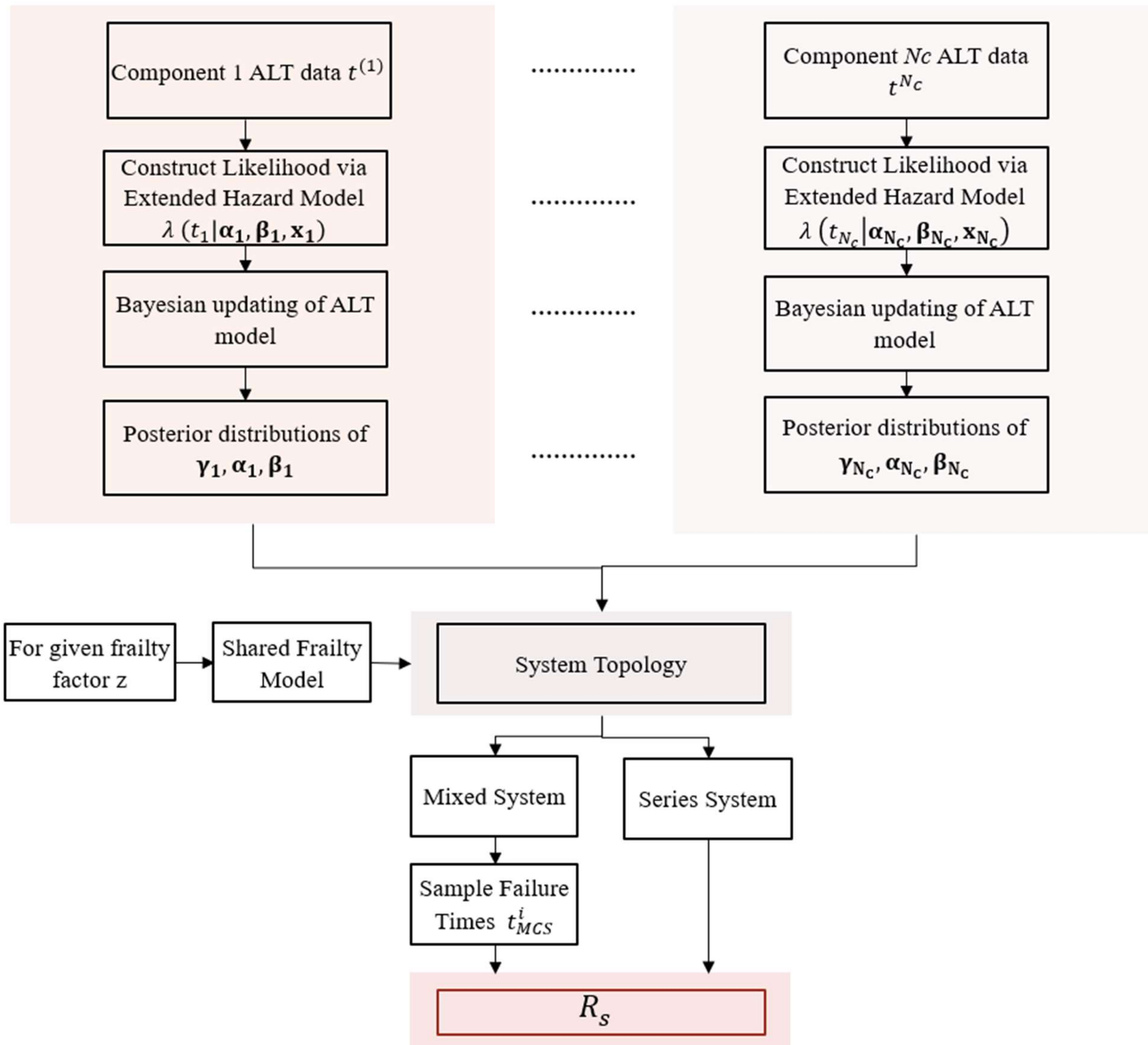


Figure 6-3 Flowchart of connecting component-level ALT data with system reliability using shared frailty models

6.2 Uncertainty Propagation of System Level ALT Data to System Reliability

In this section, we establish a connection between the system-level ALT at high stress level with system reliability estimate at nominal stress level. A direct way of achieving this goal is to test the whole system followed by applying the steps of the component-level ALT analysis method. Aiming to link system-level ALT with system reliability while preserving the connection developed in section 6.1, we leverage the use of physics-informed models, which are available in the design stage, to establish the stress relationship among components. Next, we start by introducing the physics-informed model and then discuss how it will be applied to construct the connection between system-level ALT and system reliability.

6.2.1 Framework Steps Overview

The frameworks allow propagating the uncertainties using the testing failure data collected from system level testing via accelerated life testing of a whole system. The framework consists first of mapping the stresses via the physics informed model as shown in Section 5.2.2.

Unlike the framework presented in the previous section above, Section 6.1, the model integrates the frailty factor by taking its variance into account. The shared frailty model allows modelling the unobserved factors causing dependence among the components. Including the frailty factor into the likelihood and applying Bayesian Inference to update the parameters of the model will then reduce the uncertainty included in the variance of the frailty factor.

After sampling out the posterior information of the parameter, the uncertainties are propagated using the system topology: mixed or series, depending on the system failure design. The propagation of the uncertainties will allow estimating the system reliability.

The framework is broken into three steps: first the likelihood function is derived based on the distribution free model by using the pdf based on hazard function, next the Bayesian inference is applied to sample the posterior distribution of the parameters, and last the uncertainties are propagated to estimate the system reliability based on the system topography.

Figure 6-4 below summarizes the steps of framework that will be followed in order to establish the system reliability using system-level ALT data. To construct the likelihood, first we need to map the stresses from the boundary component to the non-boundary components. Next, the dependence shall be modelled using frailty models. Adding the system-level ALT data the likelihood will be formulated. Bayesian Inference is applied to reduce uncertainties. Last, the system reliability is established based on the system topography.

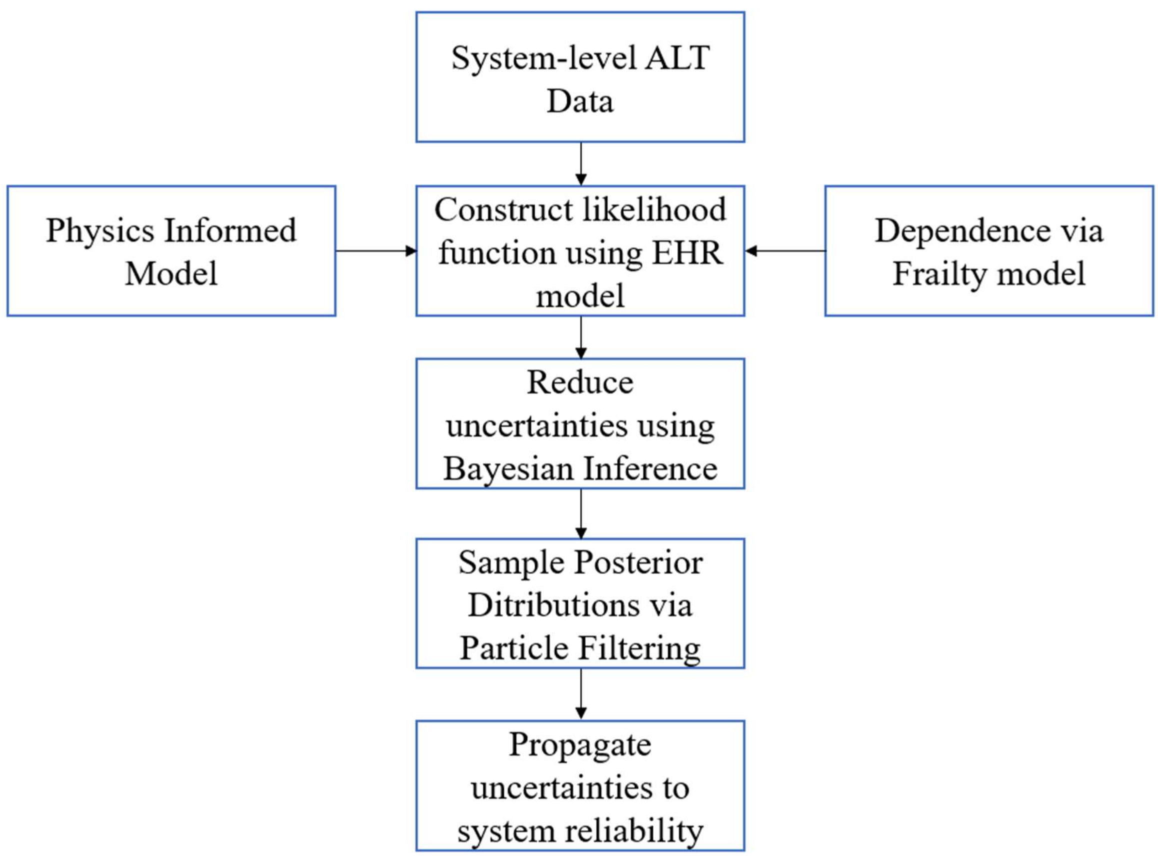


Figure 6-4 Framework Overview of the uncertainty propagation using system-level ALT data

6.2.2 Likelihood and Bayesian Inference Via EHR Model

In order to develop the likelihood function using EHR model, establishing the connection between the component stresses and the system level stress is important. First, the physics informed model is explained then the likelihood and the Bayesian inference relationship based on EHR model are presented.

A – Physics Informed Model

Let the testing stress be $\mathbf{x}$, so assuming we have n_b boundary components, we denote the stresses of boundary components as $\mathbf{x}^{\mathrm{b}} = \{\mathbf{x}_1^{\mathrm{b}}, \ldots, \mathbf{x}_{n_b}^{\mathrm{b}}\}$, and that of non-boundary components stress as $\mathbf{x}^{\mathrm{b}-} = \left\{\mathbf{x}_1^{\mathrm{b}-}, \ldots, \mathbf{x}_{n_{b-}}^{\mathrm{b}-}\right\}$where $n_{b-} = N_C - n_b$ is the total number of non-boundary component. Using the physics-informed model, the non-boundary component stress to predict $\mathbf{x}^{\mathrm{b}-}$can be predicted using $\mathbf{x}^{\mathrm{b}}$ as follows

$$\mathbf{x}_q^{\mathrm{b}-} = L_{bq-}\left(\mathbf{x}^{\mathrm{b}}, \boldsymbol{\omega}^{(q)}\right), \forall q = 1,2, \ldots, n_{b-}, \tag{6-27}$$

where , $\boldsymbol{\omega}^{(q)}$ is a set of deterministic and random parameters representing uncertainty for situation in which the load prediction models cannot accurately predict the load conditions of non-boundary components[39,40], $\mathbf{x}_q^{\mathrm{b}-}$ is the q-th non-boundary component (q-th element of $\mathbf{x}^{\mathrm{b}-}$) , and $L_{bq-}(\cdot)$; $\forall q = 1,2, \ldots, n_{b-}$ is the set of load prediction models used for stress mapping of $\mathbf{x}^{\mathrm{b}}$ to $\mathbf{x}_q^{\mathrm{b}-}$.

B – Bayesian Update and Likelihood Function Formulation

Recall that we have the parameters of the component-level ALT models as $\boldsymbol{\alpha}_{\mathrm{sys}} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{\mathrm{N_C}}\}$, $\boldsymbol{\beta}_{\mathrm{sys}} = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_{\mathrm{N_C}}\}$, and $\boldsymbol{\gamma}_{\mathrm{sys}} = \left\{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_{\mathrm{N_C}}\right\}$. We define the system-level testing stress levels as $\mathbf{x}_{\mathrm{sys}} = \left\{\mathbf{x}_1^{\mathrm{b}}, \mathbf{x}_2^{\mathrm{b}}, \dots, \mathbf{x}_{\mathrm{m_{sys}}}^{\mathrm{b}}\right\}$, where m_{sys} is the number of ALT system-level testing stress levels, $\mathbf{x}_j^b = \left[x_{1,j}^b, x_{2,j}^b, \dots, x_{n_b,j}^b\right], \forall j = 1, \dots, m_{sys}$ is the vector of normalized testing stresses applied to the boundary component at the j-th stress level. Letting the collected failure time data of the system at the j-th testing stress level be $\mathbf{t}_{\mathrm{sys}} = \{\mathbf{t}_{\mathrm{sys,1}}, \mathbf{t}_{\mathrm{sys,2}}, \dots, \mathbf{t}_{\mathrm{sys,m_{sys}}}\}$ and $\mathbf{t}_{\mathrm{sys,j}} = \{t_{sys,j}(k),\ k = 1,\ 2, \dots,\ n_{sys,j}\}$, where $n_{sys,j}$ is the number of tests at the j-th testing stress level, the likelihood function of observing $\boldsymbol{t}_{sys,j}$ is given by

$$f\left(\mathbf{t}_{sys} \middle| \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{sys}, z, \boldsymbol{\omega}\right) = \prod_{j=1}^{m_{sys}} \prod_{k=1}^{n_{sys,j}} f\left(t_{sys,j}(k) \middle| \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_j^{\mathrm{b}}, z, \boldsymbol{\omega}\right). \tag{6-28}$$

where $t_{sys,j}(k)$ is the k-th observation at the j-th stress level.

In order to evaluate $f\left(t_{sys,j}(k) \middle| \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_j^{\mathrm{b}}, z, \boldsymbol{\omega}\right)$ in above equation, we first map the accelerating factor from the boundary components to the non-boundary components using the physics-informed model as follows

$$\mathrm{x}_{q,j}^{\mathrm{b-}} = L_{bq-}\left(\mathbf{x}_j^{\mathrm{b}}, \boldsymbol{\omega}^{(q)}\right); \forall q = 1, \dots, n_{b-}, \tag{6-29}$$

in which $\mathrm{x}_{q,j}^{\mathrm{b-}}$ is the testing load of the q-th non-boundary component at the j-th stress level and $\mathbf{x}_j^{\mathrm{b}}$ is the load condition of boundary component at the j-th stress level.

We then generate random failure time samples, $t_{mcs,j}^i, \forall\, i = 1,2, \dots, N_c$, for each component at the j-th ALT stress level of the system-level ALT as below

$$t_{mcs,j}^i = F_{T_i}^{-1}\left(u_{ij} | \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,j}, z, \boldsymbol{\omega}\right), \tag{6-30}$$

where u_{ij} is a random CDF sample generated from MCS for the *i*-th component at the *j*-th stress level in the system-level ALT, and $F_{T_i}^{-1}(u|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,j}, z, \boldsymbol{\omega})$ is the inverse function of the following CDF function

$$F_{T_i}(t|\boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \mathbf{x}_{i,j}, z, \boldsymbol{\omega}) = 1 - R(t_{mcs,j}^i)$$
$$= 1 - e^{-z\left([\gamma_{0,i}e^{\mathbf{x}_{i,j}^{\mathrm{T}}\boldsymbol{\alpha}_i} + \gamma_{1,i}te^{\mathbf{x}_{i,j}^{\mathrm{T}}\mathbf{w}_{0,i}} + \gamma_{2,i}t^2e^{\mathbf{x}_{i,j}^{\mathrm{T}}\mathbf{w}_{1,i}}]\right)}, \tag{6-31}$$

where

$$\mathbf{x}_{i,j} = \begin{cases} \mathbf{x}_{r,j}^{\mathrm{b}}\ ;\ \forall r = 1,2,\ldots,n_b \text{ if } i \text{ is a boundary component} \\ \mathbf{x}_{q,j}^{\mathrm{b-}}\ ;\ \forall q = 1,2,\ldots,n_{b-} \text{ if } i \text{ is a non}-\text{boundary component} \end{cases}. \tag{6-32}$$

After we generate random failure time samples, $t_{mcs,j}^i$, $\forall\ i = 1,2,\ldots,N_c$, of the failure time (latent failure time) of each component in system-level ALT, we have the samples of the system-level failure time at the *j*-th stress level as

$$t_{mcs,j}^{sys}(h) = f_{time}\left(t_{mcs,j}^1(h), t_{mcs,j}^2(h), \ldots, t_{mcs,j}^{Nc}(h)\right); h = 1,2,\ldots,n_{mcs}, \tag{6-33}$$

in which n_{mcs} is the number of Monte Carlo samples and $f_{time}(\cdot)$ is a function defined according to the system topology as discussed in Sec. 3.2.

Based on the system-level failure time sample, $f\left(t_{sys,j}(k)\middle|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_j^{\mathrm{b}}, z, \boldsymbol{\omega}\right)$ in Equation (2-25) is then computed using the Kernel function as

$$f\left(t_{sys,j}(k)\middle|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_j^{\mathrm{b}}, z, \boldsymbol{\omega}\right) = \frac{1}{\delta_t}\sum_{h=1}^{n_{mcs}} \kappa\left(\frac{t_{sys,j}(k) - t_{mcs,j}^{sys}(h)}{\delta_t}\right), \tag{6-34}$$

where $\kappa(\cdot)$ is the kernel smoothing function and δ_t is the band width.

Using the above equation, the likelihood function $f\left(\mathbf{t}_{sys}\middle|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{sys}, z, \boldsymbol{\omega}\right)$ can be obtained. Based on that, for given parameter v of the Gamma frailty factor, we then have the unconditional likelihood function by integrating out the uncertain frailty factor *Z* as follows

$$f(\mathbf{t}_{sys}|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{sys}, v, \boldsymbol{\omega}) = \int_0^\infty f(\mathbf{t}_{sys}|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{sys}, z, \boldsymbol{\omega}) f_Z(z|v)dz, \tag{6-35}$$

where $f_Z(z|v)$ is the PDF function of the uncertain frailty factor Z for given parameter v.

Based on the likelihood formulation given in Equation (6-35), we can then reduce the uncertainty in the component-level ALT model parameters and the frailty factor distribution parameter using system-level ALT testing data $\mathbf{t}_{sys}$ using Bayesian method as below

$$f(\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \boldsymbol{\omega}|\mathbf{t}_{sys}, \mathbf{x}_{sys}) \propto f(\mathbf{t}_{sys}|\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{sys}, v, \boldsymbol{\omega}) \times f_V(v) \times f_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \times \prod_{i=1}^{N_c} f_{\gamma_i}(\boldsymbol{\gamma}_i) f_{\alpha_i}(\boldsymbol{\alpha}_i) f_{\beta_i}(\boldsymbol{\beta}_i), \tag{6-36}$$

where $f_V(v)$ and $f_{\boldsymbol{\omega}}(\boldsymbol{\omega})$ are prior distributions of v and $\boldsymbol{\omega}$, respectively.

In the next section, building upon the approaches developed in Sections 6.1 and 6.2, we discuss the fusion of the ALT data collected from component-level ALT and system-level ALT for system reliability analysis.

6.2.3 Uncertainty Propagation to System Reliability

After we have the posterior samples of the parameters $\boldsymbol{\gamma}_{sys}$, $\boldsymbol{\alpha}_{sys}$, $\boldsymbol{\beta}_{sys}$, and v, we can propagate the uncertainty to the system reliability at nominal stress level by following the same steps detailed in Section 6.1.4 depending on the system topology. If the system is in series, we would use Equation (6-20) shown in Section 6.1.4A and if it is a mixed system, we use the algorithm detailed I Section 6.1.4B to sample component failure times using Equation (6-31) the posterior distributions of the parameters and proceed as detailed in 6.1.4B.

Figure 6-5 on the next page explains in brief the steps detailed above starting with system level testing data and how this data could be used through to estimate and reduce the uncertainty in the system reliability.

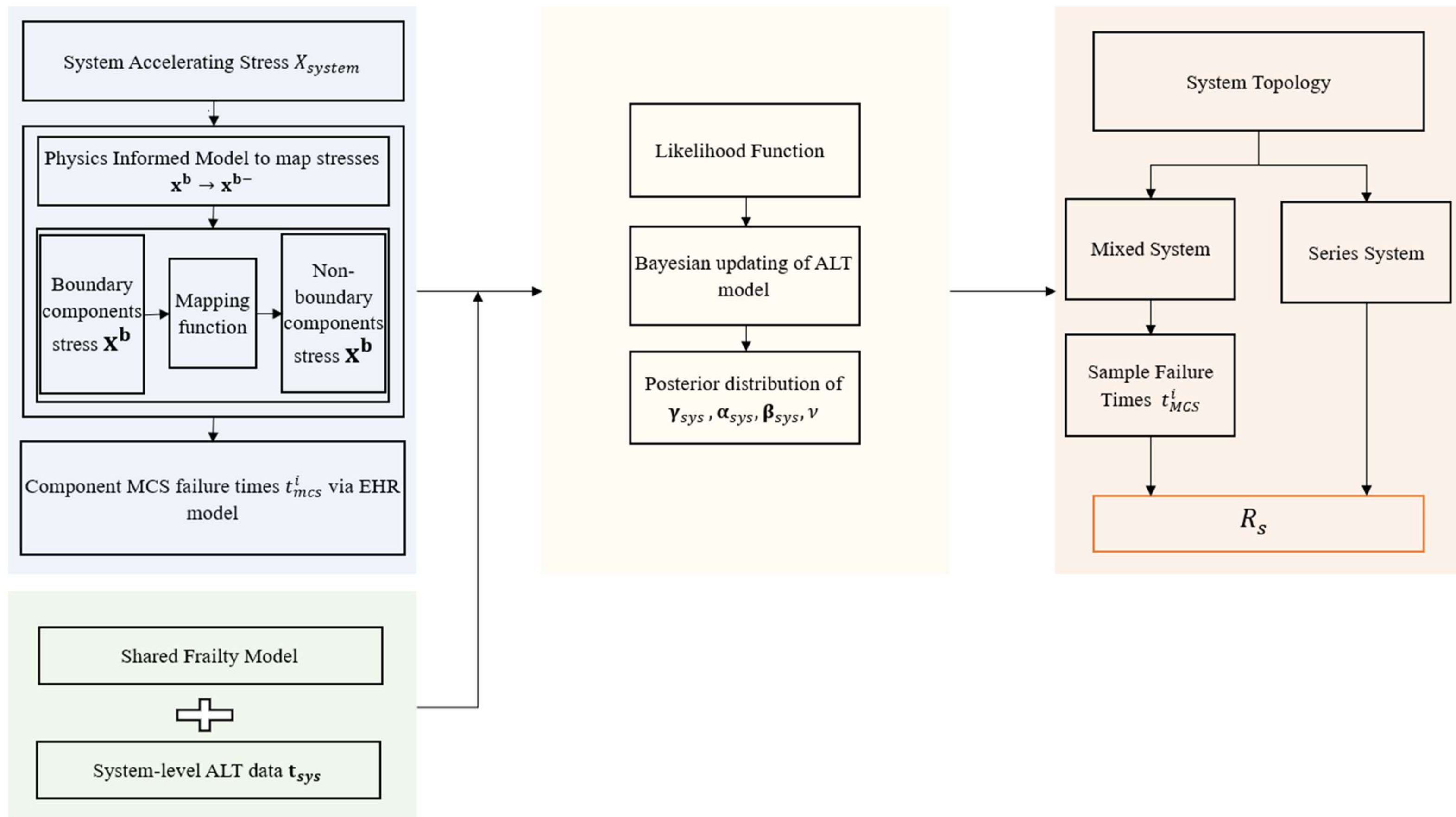


Figure 6-5 Flowchart of connecting system-level ALT data with system reliability using shared frailty models

6.3 Reliability Assessment via Information Fusion of Component Level ALT Data with System Level ALT Data

In order to fuse the information from component-level ALT and system-level ALT, we make use of Sections 6.1 and 6.2. We define the combined accelerated testing stress as $\mathbf{x}_{test} = \{\mathbf{x}_{sys}, \mathbf{x}_1^s, \mathbf{x}_2^s, \dots, \mathbf{x}_{N_C}^s\}$, in which $\mathbf{x}_{\text{sys}} = \{\mathbf{x}_1^{\text{b}}, \mathbf{x}_2^{\text{b}}, \dots, \mathbf{x}_{m_{sys}}^{\text{b}}\}$ and $\mathbf{x}_i^s = [\mathbf{x}_{i,1}, \mathbf{x}_{i,2}, \dots, \mathbf{x}_{i,m_i}], \forall i = 1,2, \dots, N_C$, where m_{sys} is the number of ALT system-level testing stress levels and m_i is the number of ALT component-level testing stress levels

Following Bayes' theorem and the Bayesian inference as the estimation method and merge the ALT testing data we get:

$$\begin{aligned} f(\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \boldsymbol{\omega} | \mathbf{t}_1^s, \mathbf{t}_2^s, \dots, \mathbf{t}_{N_C}^s, \mathbf{t}_{sys}, \mathbf{x}_{test}) \\ \propto f(\mathbf{t}_1^s, \mathbf{t}_2^s, \dots, \mathbf{t}_{N_C}^s, \mathbf{t}_{sys} | \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \boldsymbol{\omega}, \mathbf{x}_{test}) \\ \times f_V(v) \times f_\omega(\boldsymbol{\omega}) \times \prod_{i=1}^{N_C} f_{\gamma_i}(\boldsymbol{\gamma}_i) f_{\alpha_i}(\boldsymbol{\alpha}_i) f_{\beta_i}(\boldsymbol{\beta}_i), \end{aligned} \tag{6-37}$$

where $f(\mathbf{t}_1^s, \mathbf{t}_2^s, \dots, \mathbf{t}_{N_C}^s, \mathbf{t}_{sys} | \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \boldsymbol{\omega}, \mathbf{x}_{test})$ is given by

$$\begin{aligned} f(\mathbf{t}_1^s, \mathbf{t}_2^s, \dots, \mathbf{t}_{N_C}^s, \mathbf{t}_{sys} | \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \boldsymbol{\omega}, \mathbf{x}_{test}) \\ = f(\mathbf{t}_{sys} | \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{sys}, v, \boldsymbol{\omega}) \\ \times \prod_{i=1}^{N_C} f(\mathbf{t}_i^s | \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, , \mathbf{x}_i^s), \end{aligned} \tag{6-38}$$

in which $f(\mathbf{t}_{sys} | \boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, \mathbf{x}_{sys}, v, \boldsymbol{\omega})$ is computed using Equations (6-28) through (6-36) and $f(\mathbf{t}_i^s | \boldsymbol{\gamma}_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, , \mathbf{x}_i^s)$ is computed as given in Equations (6-11) through (6-14).

After the uncertainty reduction using both component-level and system-level ALT data using Equations (6-37)and (6-38), we can obtain the posterior distributions of $R_{sys}(t)$ by propagating the posterior distributions of $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}, \boldsymbol{\beta}_{sys}, v, \boldsymbol{\omega}$ to system reliability using the uncertainty propagation method presented in Section 3.2.2 for different system configurations.

In the following section, we demonstrate the effectiveness of the methodologies developed. We first take a numerical example to show the results of using two different approaches to estimate the system reliability of the same system configuration. Then, we demonstrate the effectiveness fusing the information from both component-level ALT and system-level ALT.

6.4 Numerical Examples

In this section, we use numerical examples to demonstrate the efficacy of the proposed method. It consists of two parts: (1) demonstration of the proposed method in system reliability analysis of mixed systems using ALT; (2) information fusion of component-level and system-level ALT data for system reliability analysis. Next, we will first present the example that is employed. After that, we will explain the two case studies in details.

6.4.1 Description of the Numerical Example

We take a circuit board as an example to demonstrate the developed method. The reliability of the system for time threshold of $T_e = 150$ weeks at the nominal stress is to be estimated using component-level ALT data.

The example is a circuit board of 4 electronical components and is used for illustration. We simplify the radar circuit shown above in section 4.1.1. Figure 6-6 shows the simplified circuit board of the radar system and Figure 6-7 presents the reliability block diagram of the system.

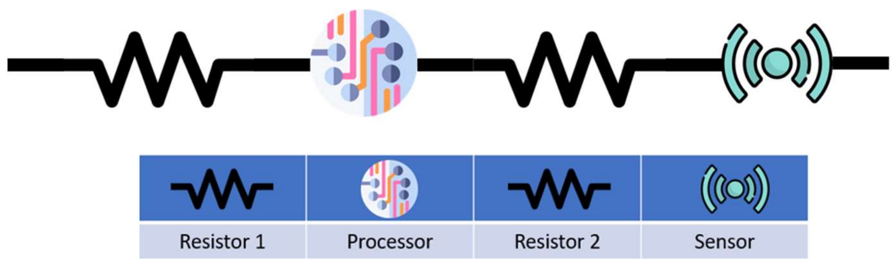


Figure 6-6 Simplified series configuration of the circuit board of a radar system

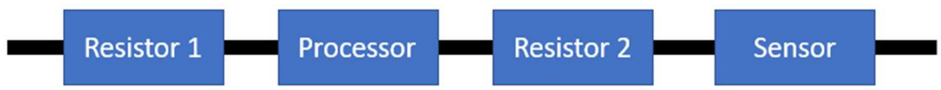


Figure 6-7 Reliability block of a radar circuit system

In this case study, Table 6-1 gives the true values of $\boldsymbol{\gamma}_{sys} = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4\}$ (i.e. the quadratic function parameters of the baseline hazard function as discussed in Section 3.2.1). Table 6-2 presents the true values of the regression parameters $\boldsymbol{\alpha}_{sys} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$, and $\boldsymbol{\beta}_{sys} = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$.

Table 6-1 True Values of baseline hazard function and the shared frailty factor variance

	Component index (i)	$\boldsymbol{\gamma}_{\mathbf{i}}$		
		$\gamma_{0,i}$ ($\times 10^{-5}$)	$\gamma_{1,i}$ ($\times 10^{-16}$)	$\gamma_{2,i}$ ($\times 10^{-08}$)
Resistor 1	1	0.97565	2.75371	5.95432
Processor	2	0.96005	2.46947	2.52201
Resistor 2	3	8.48655	3.48827	0.35832
Sensor	4	5.67227	2.96519	6.29922

The prior distribution of the parameters is assumed to be a non-informative uniform prior distribution. Uniform prior distributions (Unif$[a, b]$, where *a* and *b* are respectively the lower and upper bounds of the distribution) are used for the other ALT model parameters.

Table 6-2 True values of the regression parameters

	Component index (i)	$\boldsymbol{\alpha}_i$		$\boldsymbol{\beta}_i$	
		$\alpha_{0,i}$ ($\times 10^{-11}$)	$\alpha_{1,i}$	$\alpha_{0,i}$ ($\times 10^{-11}$)	$\alpha_{1,i}$
Resistor 1	1	1.59914	13.82604	1.31948	1.05106
Processor	2	1.54849	14.13588	2.00388	2.50621
Resistor 2	3	1.46454	14.72182	1.06629	1.15892
Sensor	4	1.24498	15.46258	0.02146	2.53921

Table 6-3 and Table 6-4 below present the prior distribution parameters of these ALT model parameters.

Table 6-3 Lower and upper bounds of the uniform prior distributions of the baseline hazard function parameters

	Componen-t index (i)	$\boldsymbol{\gamma_i \sim}$ **Unif** $[\boldsymbol{a_{\gamma_i}}, \boldsymbol{b_{\gamma_i}}]$					
		$\gamma_{0,i}(\times 10^{-04})$		$\gamma_{1,i}$ ($\times 10^{-15}$)		$\gamma_{2,i}$ ($\times 10^{-07}$)	
		$a_{\gamma_{0,i}}$	$b_{\gamma_{0,i}}$	$a_{\gamma_{1,i}}$	$b_{\gamma_{1,i}}$	$a_{\gamma_{2,i}}$	$b_{\gamma_{2,i}}$
Resistor 1	1	0.015	0.150	0.0464	0.464	0.09	0.9
Processor	2	0.019	0.19	0.0422	0.422	0.0449	0.449
Resistor 2	3	0.15	1.5	0.0465	0.465	0.0055	0.055
Sensor	4	0.105	1.05	0.0375	0.375	0.0685	0.685

Table 6-4 Lower and upper bounds of the uniform prior distributions of the regression parameters

	Comp-onent index (i)	$\boldsymbol{\alpha_i \sim}$ **Unif** $[\boldsymbol{a_{\alpha_i}}, \boldsymbol{b_{\alpha_i}}]$				$\boldsymbol{\beta_i \sim}$ **Unif** $[\boldsymbol{a_{\beta_i}}, \boldsymbol{b_{\beta_i}}]$			
		$\alpha_{0,i}$ ($\times 10^{-10}$)		$\alpha_{1,i}$ ($\times 10^{2}$)		$\beta_{0,i}(\times 10^{10})$		$\beta_{1,i}(\times 10^{2})$	
		$a_{\alpha_{0,i}}$	$b_{\alpha_{0,i}}$	$a_{\alpha_{1,i}}$	$b_{\alpha_{1,i}}$	$a_{\beta_{0,i}}$	$b_{\beta_{0,i}}$	$a_{\beta_{1,i}}$	$b_{\beta_{1,i}}$
Resistor 1	1	0.1028	0.171	0.091	0.1514	0.09	0.15	0.072	0.12
Processor	2	0.099	0.165	0.087	0.145	0.139	0.231	0.151	0.251
Resistor 2	3	0.1041	0.173	0.098	0.1635	0.069	0.115	0.075	0.125
Sensor	4	0.087	0.145	0.093	0.155	0.142	0.236	0.153	0.256

We then document the results in terms of variability percentage change. The percentage of variance reduction (VR) is employed to quantify the reduction in the uncertainty between the prior and posterior distributions of system reliability, $R_{sys}(t)$. It is given by

$$VR = \frac{Var(R_S^{prior}) - Var(R_S^{post})}{Var(R_S^{prior})} \times 100\%, \tag{6-39}$$

where R_S^{prior} and R_S^{post} are respectively the prior and posterior distributions of system reliability and $Var(\cdot)$ is a variance operator. A positive VR is an indication that the uncertainty has been effectively reduced by applying the developed algorithm.

In the following sections we present the results graphically and tabulated, we then present a comparative analysis with interpretations. We solve the problem using the three uncertainty propagation means as in Sections 6.1, 6.2 and 6.3 to check which method is more efficient in reducing the uncertainty given the variability of the frailty variance ν.

6.4.2 Case Study 1: Connecting Component-level ALT with System Reliability

This section focuses on demonstrating the approach developed in Section6.1, which connects component-level ALT with system reliability. In order to illustrate the capability of the proposed methods in handling systems with different topologies, we first obtain the system reliability of the system given in Section 4.1.1 in Figure 6-6, using the propagation method of a series system topography presented in Section 6.1.3A. After that, we will treat the same system as a mixed system configuration and analyze the system reliability using the approach proposed in Section 6.1.3B.

Using the same example presented in the previous section, Section 6.5, under sensitivity analysis we proceed to solve the example via the two different system configurations approach.

Table 6-1 gives the true values of $\boldsymbol{\gamma}_{sys} = \{\boldsymbol{\gamma}_{1,}\boldsymbol{\gamma}_{2}, \boldsymbol{\gamma}_{3,} \boldsymbol{\gamma}_{4}\}$ (i.e. the quadratic function parameters of the baseline hazard function as discussed in Section 6.1.3A). **Table 6-2** True values of the regression parameters presents the true values of the regression parameters $\boldsymbol{\alpha}_{sys}$ =\{$\boldsymbol{\alpha}_{1}$, $\boldsymbol{\alpha}_{2}, \boldsymbol{\alpha}_{3}, \boldsymbol{\alpha}_{4}\}$, and $\boldsymbol{\beta}_{ys}$ = \{$\boldsymbol{\beta}_{1}$, $\boldsymbol{\beta}_{2}, \boldsymbol{\beta}_{3}, \boldsymbol{\beta}_{4}\}$ **.** The prior information for $\boldsymbol{\gamma}_{sys}$ are as given in Table 6-3 and the prior information for the regression parameters $\boldsymbol{\alpha}_{sys}$ and $\boldsymbol{\beta}_{sys}$ are given in Table 6-4 above.

In this case study, the prior distribution of the frailty factor variance ν is assumed to be a non-informative uniform prior distribution, ν~Unif[5, 10.00] and the true value of the frailty factor variance is set as $\nu = 6$.

For a series system, we use Equations (6-15) through (6-20) to calculate the true system reliability using the true values of $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}$, and $\boldsymbol{\beta}_{sys}$. Then, we obtain the prior distribution of R_S^{prior} by propagating the uncertainties in the prior distributions of $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}$, and $\boldsymbol{\beta}_{sys}$ to system reliability using the method discussed in Section 6.2.2. Afterwards, component-level ALT data are used to reduce the uncertainty in $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}$, and $\boldsymbol{\beta}_{sys}$. For the component-level ALTs, the testing stress levels for components 1 to 4 are respectively $\mathbf{x}_1 = [0.6,0.24]$; $\mathbf{x}_2 = [0.643,0.571,0.536,]$; $\mathbf{x}_3 = [0.443,0.393,]$, and $\mathbf{x}_4 = [0.529,0.643]$. The number of tests at each stress level for each component are respectively $n_{11} = 20$; $n_{12} = 15, n_{21} = 20, n_{22} = n_{23} = 10$; $n_{31} = n_{32} = 10$; $n_{41} = 20$ and $n_{42} = 10$. The posterior distribution of the system reliability R_S^{post} is finally obtained by using the posterior distributions of $\boldsymbol{\gamma}_{sys}, \boldsymbol{\alpha}_{sys}$, and $\boldsymbol{\beta}_{sys}$. Figure 6-8 shows the prior and posterior distributions of $R_{sys}(t)$ in comparison with the true value. We note R_S^{prior} as "Prior R_S" (green line) has a greater variability compared to R_S^{post} as "Posterior R_S" (red line).

Figure 6-8 shows the posterior and prior distributions in comparison with the true value for a series system. First, the uncertainty in the priors has been propagated from the components to the system reliability by using prior distributions of the parameters as shown in Section 6.1.3 , we note this reliability "Posterior System Reliability" below plotted in dashed blue line color. We then use these sampled posterior data from Bayesian inference to propagate the uncertainties and compute the system reliability using Equation (6-20). The posterior system reliability is noted as "Posterior System Reliability" and it is represented by the orange line. The green straight vertical line is the True System Reliability referred to as "True Value" in the graph below.

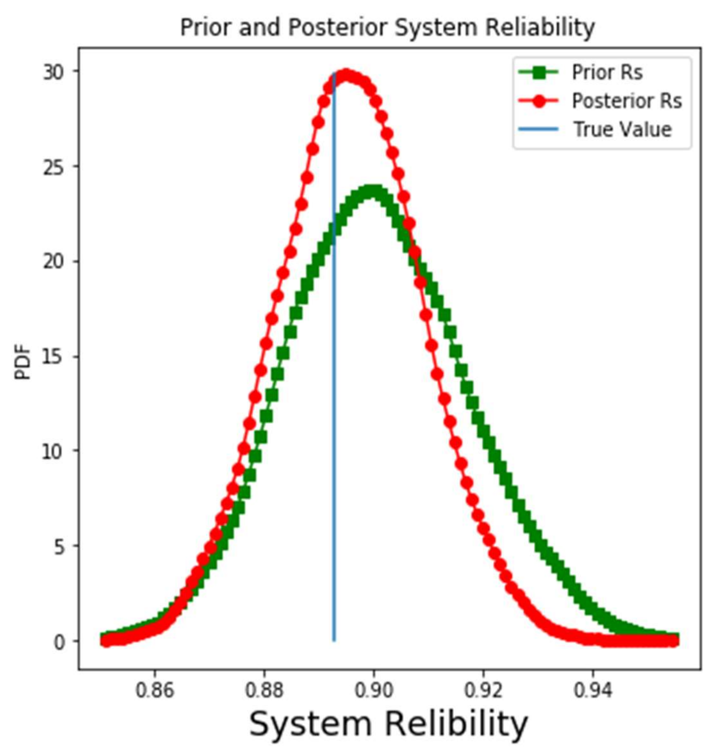


Figure 6-8 Comparison of prior and posterior distributions for the series system topology

We then treat the system shown in Figure 6-6 Simplified series configuration of the circuit board of a radar systemin Section 6.5 as a mixed system and use the approach discussed in Section 6.1.4B to evaluate the system reliability. The same number of stress levels and tests are employed as that when the system is treated as a series one. In order to apply the approach for a mixed system, we define $f_{time}(\cdot)$ in Equation (6-24) as $f_{time}(\mathbf{T}) = \min\{T_1, T_2, T_3, T_4\}$. Figure 7 below shows the comparison of the prior and posterior distributions of the system reliability using the mixed system algorithm. It shows a reduction in the uncertainty in the posterior distribution of $R_{sys}(t)$ as compared to the prior distribution. The results in Figure 6-8 and Figure 6-9 show that the proposed approach can effectively reduce the uncertainty in the system reliability using component-level ALT data with the consideration of the dependence between the failure time distributions. Note that the difference between Figure 6 and Figure 7 is caused by the uncertainty in the synthetically generated ALT data.

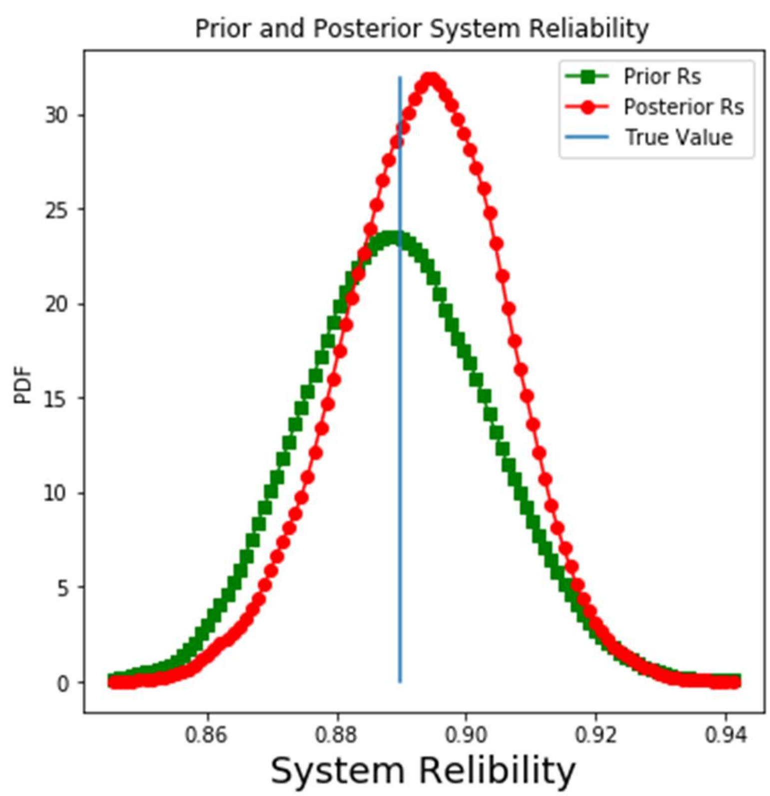


Figure 6-9 Comparison of prior and posterior distributions for the mixed system topology

We then use the metric given in Section 4.1 to quantitatively quantify the accuracy of the proposed methods. Table 6-5 shows the results of VR for a series system versus a mixed one.

Results shows that using the series system algorithm (i.e. Section 6.1.3A) leads to a reduction of 38% in the variance of system reliability estimate, and the mixed system algorithm (i.e. Section6.1.3B) results in 43% reduction in the variance. These results demonstrate the efficacy of the proposed methods in assess system reliability for different system configurations.

Table 6-5 Comparison of variability reduction for series and mixed system configurations

	Series System Topology	Mixed System Topology
VR	37.79%	43.11%

6.4.3 Case Study 2: Uncertainty Propagation to System Reliability Via Information Fusion

In this section, we continue using the same example in Section 6.5 to demonstrate the capability of the proposed method in fusing both component-level and system-level ALT data for system reliability analysis. Figure 6.10 shows the system under system ALT testing and shows the *boundary* and the *non-boundary* components model of the system which consists of 4 electronical components.

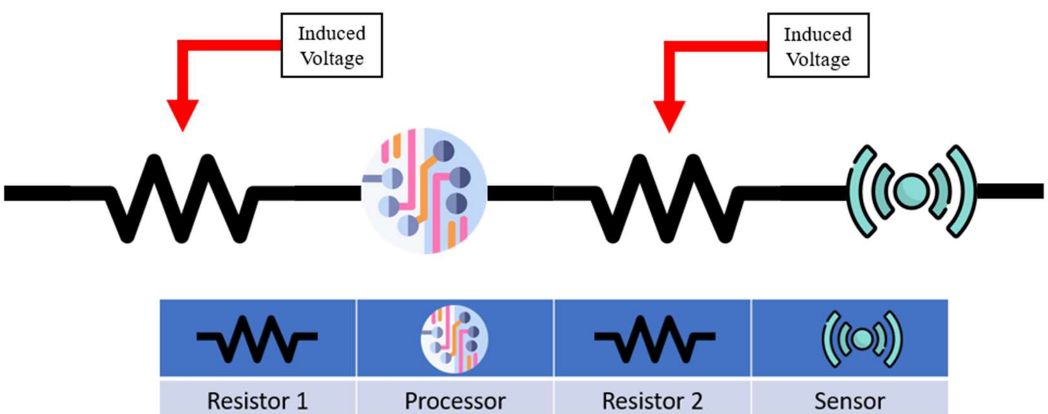


Figure 6-10 Circuit board of a radar system under system- level ALT testing

In this case study, the true values and prior distributions for $\boldsymbol{\gamma}_{sys}$, $\boldsymbol{\alpha}_{sys}$, and $\boldsymbol{\beta}_{sys}$ are the same as that of the first case study given in Table 6-1 through 6-4.

The prior distribution of the frailty factor variance ν is assumed to be a non-informative uniform prior distribution, $\nu \sim \text{Unif}[0.005, 10.00]$ and the true value of the frailty factor variance is set as $\nu = 6$.

In system-level ALT, it is assumed that resistor 1 and resistor 2 receive the induced testing voltage as shown in Figure 8. The boundary components are therefore resistor 1 and resistor 2 and $\mathbf{x}^{\mathrm{b}} = \{\mathbf{x}_1^{\mathrm{b}}, \mathbf{x}_2^{\mathrm{b}}\}$. The testing stress levels for resistor 1 and resistor 2 are respectively $\mathbf{x}_1^{\mathrm{b}} = \mathbf{x}_2^{\mathrm{b}} = [0.571, 0.315]$. The number of tests at each stress level for each component are respectively $n_{sys,1} = n_{sys,2} = 20$. The stress relationships between boundary and non-boundary components are assumed to be

$$\begin{gathered} \mathbf{x}_{Processor}^{\mathrm{b}-} = 0.98\, \mathbf{x}_1^{\mathrm{b}} + \boldsymbol{\omega}^{Processor}, \\ \mathbf{x}_{Sensor}^{\mathrm{b}-} = 0.85\, \mathbf{x}_2^{\mathrm{b}} + \boldsymbol{\omega}^{Sensor}, \end{gathered} \tag{6-40}$$

where $\mathbf{x}_1^{\mathrm{b}}$ and $\mathbf{x}_2^{\mathrm{b}}$ are the stress levels of resistor 1 and 2, respectively.

Figure 6.11 shows the compassion of the prior and posterior distributions of system reliability using only component-level ALT data. We then quantify the reduction in the variance using the metrics defined in Section 6.5. It shows that VR equals to 18.94 %. This results further demonstrate the effectiveness of the method in reducing the uncertainty of system reliability estimate using component-level ALT data. Note that the prior distribution in Figure 6.11 is different from that in the first case study in Section 6.4.1 because different prior distributions are used for the frailty factor ν.

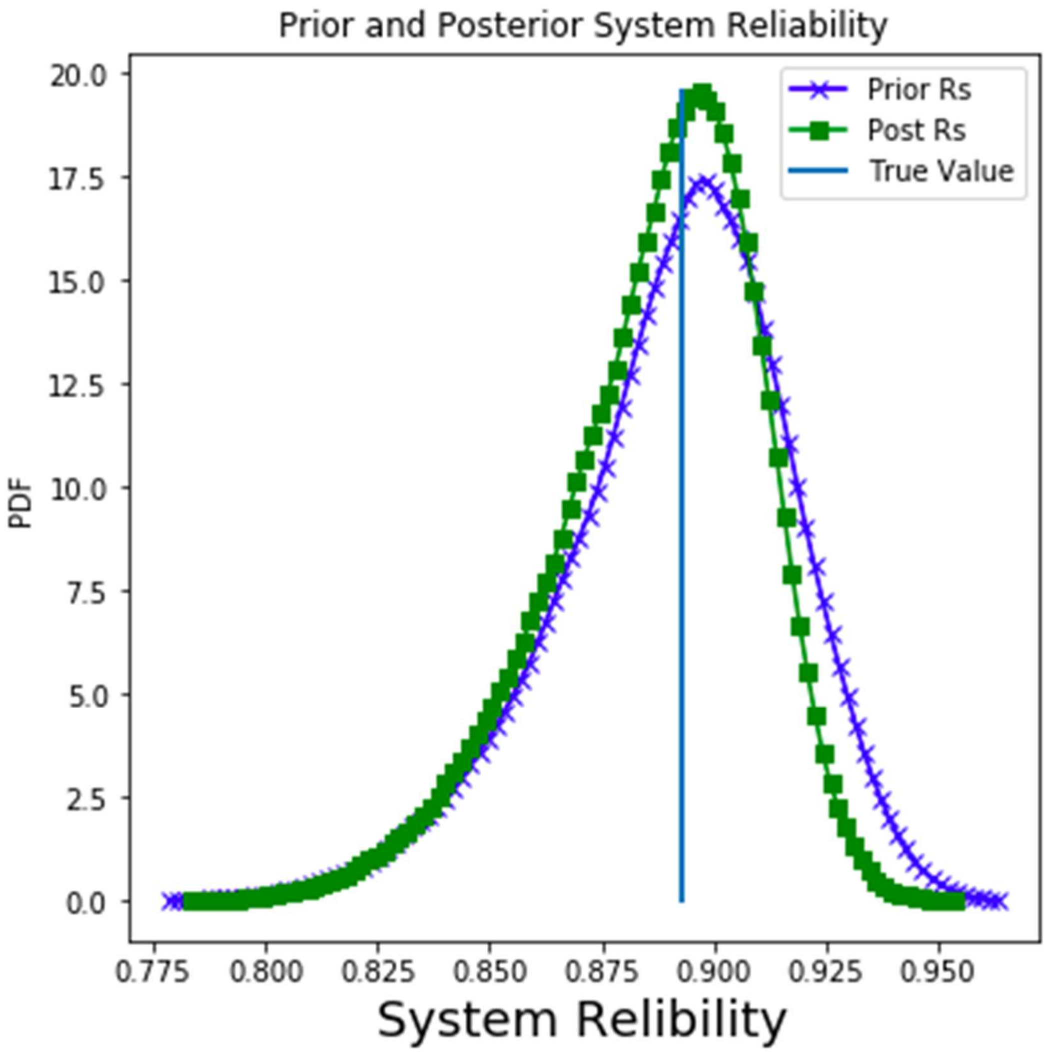


Figure 6-11 Prior and posterior distributions of system reliability using component-level ALT data

Subsequently, we fuse the component-level ALT data with the system ALT data using the method discussed in Section 6.3. Figure 6-12 shows the comparison between the posterior

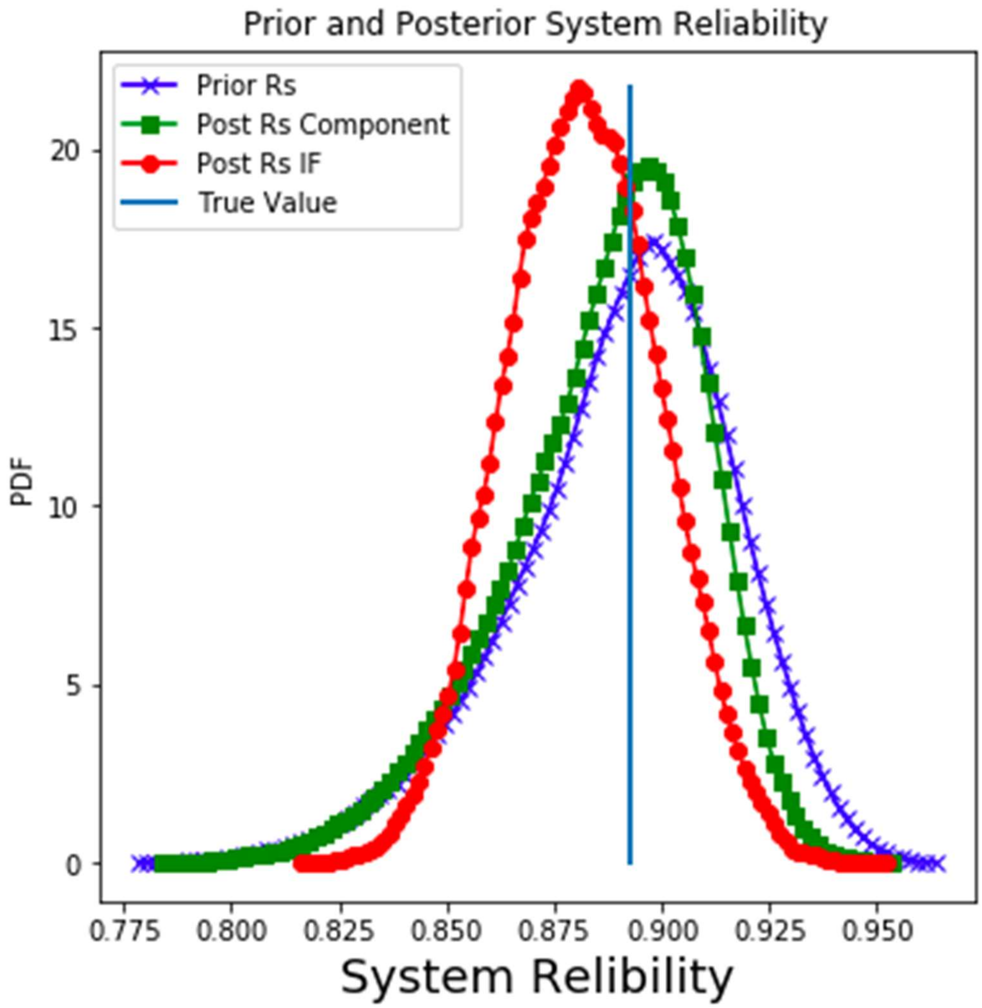


Figure 6-12 Comparison of prior and posterior distributions of the system reliability using different methods

distributions obtained by using the component-level ALT data versus the information fusion (both component-level and system-level ALT data). While both methods reduced the variability in R_s^{post} compared to the R_s^{prior}, the posterior distribution R_s^{post} of the system reliability obtained via information fusion (i.e. "Post Rs IF") shows a further reduction in the uncertainty than its counterpart obtained using the component-level ALT data (i.e. “Post Rs Component”) alone.

Table 6-6 shows a side-by-side comparison of both methods in terms of VR. The VR using information fusion (~51%) is ~2.5 times greater than that obtained from component-level data alone (~19%). It shows that combining testing data from two different testing levels (i.e. component-level and system-level) leads to a further decrease in the uncertainty of the system reliability and a higher confidence than using only component-level ALT data.

Table 6-6 Comparison of variance reduction using different methods

	Information Fusion	**Component-Level ALT data only**
$\boldsymbol{VR}$	50.84	18.94

The above results demonstrate the effectiveness of the proposed method in fusing the information from both component-level and system-level ALT data for system reliability estimate.

6.5 Sensitivity Analysis of Frailty Factor on the Uncertainty of the System Reliability

In this Section, we aim at studying the effect of the Variance of z which is the variance of the frailty factor on minimizing the uncertainty results in the system reliability between the prior and posterior distributions sampled by using a series system of 4 components as the one presented in Section 6.4.1.

As a reminder, the larger the value of ν, the stronger the dependence is among the component, so the variance of the frailty factor is the indicator of the dependence power/strength among the components of one system.

We provide a sensitivity analysis based on varying the variance of ν prior information interval. Table 6-7 shows the different prior distribution intervals (Uni $\sim[a,b]$) of the variance of ν that is assumed to be a non-informative uniform prior distribution.

We study the reliability over the time interval of [0,150] with a time unites in weeks. For component-level testing we follow the testing specifications given in Section 6.4.2 and for system-level testing we follow the given in Section 6.4.2. Table 6-1 gives the true values of $\boldsymbol{\gamma}_{sys} = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4\}$ and Table 6-2 presents the true values of the regression parameters $\boldsymbol{\alpha}_{sys}$ $=\{\boldsymbol{\alpha}_1,$ $\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$, and $\boldsymbol{\beta}_{sys} = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$.

Table **6-3** and Table **6-4** present the prior distribution parameters of these ALT model parameters: $\boldsymbol{\gamma}_{sys}$, $\boldsymbol{\alpha}_{sys}$ and $\boldsymbol{\beta}_{sys}$ respectively.

Table 6-7 True Values of Variance ν

variance of $\boldsymbol{\nu}$ Uni $\sim[\boldsymbol{a},\boldsymbol{b}]$	True Values of ν
[0.001,2.00]	1.8
[0.001,7.00]	6
[0.001,10.00]	6
[0.001,12.00]	6
[0.001,14.00]	6

6.5.1 Results and Interpretations:

The corresponding graphs are listed in Appendix A at the end. Increasing the dependence strength by increasing the variance of ν range and solving using the different methods presented in Sections 6.1,6.2 and 6.3 respectively, it is obvious that as the variance of ν is larger the methods of the uncertainty propagation using the information fusion is more effective compared to using the component- ALT data and system-ALT data propagation methods in order to estimate the system reliability.

The variability is reflected by calculating the VR according to Equation (6-39). As we increase the variability of the variance of ν, the information fusion and the system-ALT methods show an increasing VR compared to a decreasing VR for the component-ALT. Fusing the information from component-ALT data and system-ALT data allows a further reduction and becomes more important as the variability of ν increases.

In conclusion, system data becomes more important when the variance of ν increases which explains the information fusion effectiveness. Further sensitivity analysis could be applied to study the effect of the variability of the regression parameters on the result of the system reliability estimation as well as the parameters of the quadratic baseline hazard function parameters.

Table 6-8 Variability comparison in prior system reliability and posterior system reliability using the different propagation models

ν	Prior Var(ν)	True R_{sys}	VR		
			Component ALT to R_{sys}	System ALT to R_{sys}	Information Fusion
[0.001,7.00]	4.066	0.892	34.21%	24.44%	50.85%
[0.001,10.00]	8.2989		43.46%	18.93%	50.97%
[0.001,12.00]	12.08489		47.82%	15.83%	51.39%
[0.001,14.00]	16.2449		49.32%	13.21%	51.77%

6.6 Summary

This Chapter presents a novel framework to system reliability assessment using component-level and system-level ALT data. To establish connections between different testing levels and system reliability, frailty models are employed to model the dependence among components. Physics informed analysis is used to connect the system-level tests at higher-than-nominal stress to the system reliability at nominal stress.

The likelihood function and the Bayesian Inference relationship have been established using the extended hazard regression models. The model uses a baseline hazard function that is assumed to be quadratic, with three parameters of interest, and distribution free. Regression parameters to model the effect of the accelerating factor on the failure time has been considered and subject to uncertainty reduction.

The uncertainty propagation has been detailed for different ALT data collected at a system level or component level as well as the chapter shows a fusion method for both data together in order to estimate the system reliability.

The developed approach also has been investigated for different system topographies including series and mixed systems. Two case studies demonstrate the effectiveness of the proposed framework.

Numerical algorithms are developed to reduce the uncertainty in system reliability analysis using different type of testing data by integrating frailty models and Bayesian inference methods with extended hazard regression (EHR) models. Graphical and tabulated results have been enclosed showing the effectiveness in the framework in reducing the variability in the system reliability.

Studying different frailty models to model the heterogeneity among the ALT instead of shared frailty factor is an important focus to research in the future. Additionally, including shared frailty models with a different distribution rather than Gamma distribution is also worth pursuing in the future.

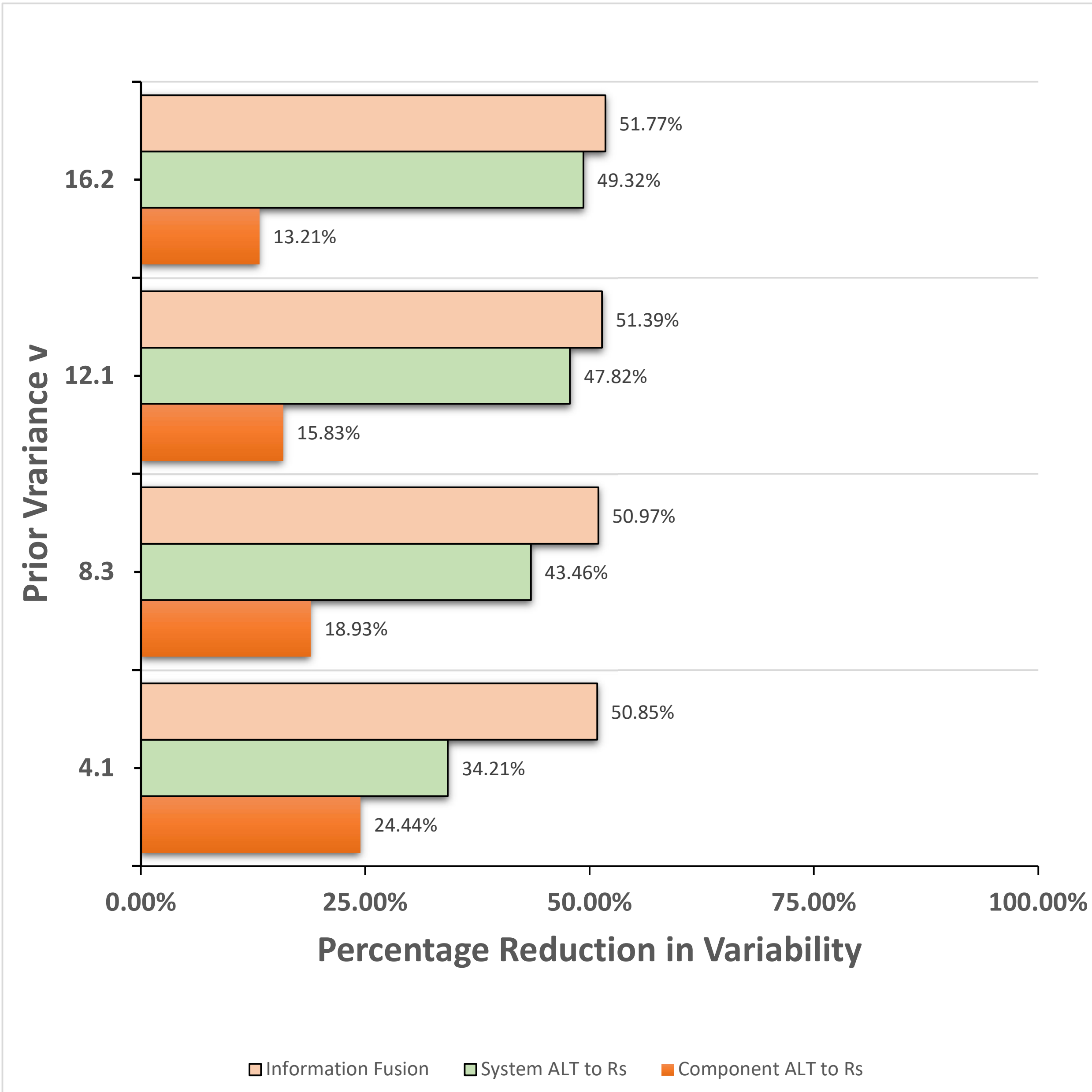


Figure 6-13 Chart showing the percentage reduction in the variability between prior Rs and posterior Rs for different Prior Variance $\boldsymbol{\nu}$ using the different propagation models

Chapter 7 Concluding Remarks and Future Work

In this Chapter, we review in summary the work done in this research and summarize the research. We then identify future work on the research topic. Finally, we list he takeaways in the concluding remarks section.

7.1 Summary of Research Achievements

The research is focusing on accelerated life testing for systems with dependent components. The research goal is to develop methods for propagating uncertainties from data collected at different testing stages: component level testing and system level testing. the research proposed a novel ALT framework combining both types of data in order to maximize the uncertainty reduction by using all available data to estimate the system reliability. The research was approached by a thorough review of the state of art. Various methods have been used in the implementation of the frameworks: Bayesian Inference, Particle Filtering, Weibull Distribution, Copula Functions and Frailty Models. The following were accomplished by carrying this research:

1- Review and Background:

Chapter 2 presented a review of the literature about the frameworks, observations and studied that tackled ALT design and modelling. Reviewing the state of art enabled the use of different techniques together in order to mold a new methodology to model the estimation of system reliability with different components. The methodology is based on propagating the uncertainties in parameters derived based on the approach chosen to fit the failure time data collected during Accelerated Life Testing which was abbreviated by ALT.

Chapter 3 showed the merit and novelty of the research by incorporating a new approach using different dependence modelling frameworks in order to model the dependence between the components of a system. Additionally, it adds to the ALT field by showing the versatility of integrating data from different ALT testing stages to derive and reduce the uncertainty in the overall system reliability.

Chapter 4 talks about the methods available in the literature that have been merged together to approach solving the ALT problem proposed in this research leading to the major two chapters; Chapter 5 and Chapter 1 where two approaches have been taken in order to estimate the system reliability.

2- Uncertainty propagation using ALT data using statistical distribution:

Chapter 5 presents a novel framework for using a 2 parameters Weibull distribution to model the ALT data collected from different testing stages: component level testing and system level testing. Three frameworks have been presented by propagating the uncertainty : one for uncertainty propagation to system reliability using component level ALT data, the second framework was intended for system level ALT data used to propagate the uncertainty to system level, and the third model combines the first two frameworks by fusing the data together collected from component testing and system testing.

3- Uncertainty propagation using ALT data using distribution free models:

Chapter 1 presents a different facet of the ALT coin by approaching the problem using a distribution free approach. The framework shows the usefulness of the gamma shared frailty models in modelling heterogeneity that is an unobserved factor that causes dependence among the components. The methodology is broken into three parts based on testing stages: component level testing and system level testing as well as the information fusion concept that is built based on fusing data from two different testing stages together in order to minimize the uncertainty in the system reliability. The method in this chapter is based on using hazard functions with a quadratic baseline hazard function that has 3 parameters and two regression parameters as the base equation, the model integrates the effect of the covariates which is the ALT accelerated factor on the failure time of the testing units through an exponential regression form.

Bayesian inference methods were applied to reduce uncertainty in the parameters based on prior information and particle filtering is used to sample the posterior information, followed by uncertainty propagation to system reliability for different system topologies. In this chapter, three frameworks are detailed similar to the one noted in point 1 above.

4- Modelling dependence among components:

Both, Chapter 5 and Chapter 1, showed how dependence models could be integrated to model any dependencies between the components of a system. In Chapter 5, the copula function is used in conjunction with the Weibull distribution in order to model the correlation among the failure times of the components, whereas Chapter 6 makes use of the frailty models that are widely used in the medical field and shows that frailty models are useful in modelling heterogeneity among components in ALT modelling.

5- ALT optimization model:

Allocating the resources optimally can greatly help in reducing the cost of the accelerated life testing. Planning ALT involves deciding on the number of specimens to be put at test, and the levels of the accelerating factor to be used. Applying the cost of testing as a constraint and using Kullback-Liebler divergence to develop the objective method, Chapter 5 presents an optimization model to estimate the optimal ALT design parameters.

7.2 Future Work

Future work includes integrating Correlated Frailty Models instead of Shared Frailty Models and make a comparison between the two models regarding the ability of reducing the uncertainties. Correlated Frailty Models allows to assume different dependencies among the components however it might make the number of the estimation parameters large similar to the Copula.

On the other hand, a different form the baseline hazard function could be a point of interest for future work. Assuming statistical forms for the baseline hazard function with shared frailty models is a room for investigation Studying the effect of the baseline hazard function form might have an impact on the end result.

A different distribution for the Frailty Factor, other than the Gamma distribution, could be studied in the future and compare the results to the Gamma shared frailty model

A different Copula Function form could be studied, and a sensitivity analysis could be done to conclude how the Copula form could affect the system reliability results

A sensitivity analysis on the parameters included in Chapter 6 is important to study the effect of the values as well as the variability in the prior data on the system reliability estimation.

7.3 Summary and Concluding Remarks

In brief, this research outcome is a new methodology for the Accelerated Life Testing solving the issue of dependence among failure time of components of one system which will increase the accuracy of the system reliability estimate. The research proposes a method showing the versatility of ALT data usage by bridging the gaps between system reliability, component-level testing data and system-level testing data. Additionally, it details an optimization model that solves for the optimal design parameters which will reduce the cost of testing during any product development phase. The latter will enable quality engineers better assess the system reliability during the design stage by optimally allocating the resources and reduce the uncertainty. Also, the research presents an approach of how frailty models could be of great use in the mechanical design environment to model and quantify dependence among failure times of components. This research will also turn benefits in the decision making and statistical studies domains.

Appendix A

In this appendix the figures showing the variability reduction between the prior information and the posterior information of the system reliability are listed as follows:

1- Figure A.1 (a) through (d) corresponds to the component-level ALT data propagation framework presented in Section 6.1, each of figures correspond to a prior υ interval.

2- Similarly, Figure A.2 (a) through (d) corresponds to the system-level ALT data propagation framework presented in Section 6.2, each of figures correspond to a prior υ interval.

3- Figure A.3 corresponds to the information fusion framework as presented in Section 6.3.

4- Last, Figure A.4 shows a comparison between the posterior distribution of the system reliability obtained via component-level ALT data propagation method versus the one obtained via information fusion. The graph shows the variability of the posterior distribution of R_{sys} against the prior information.

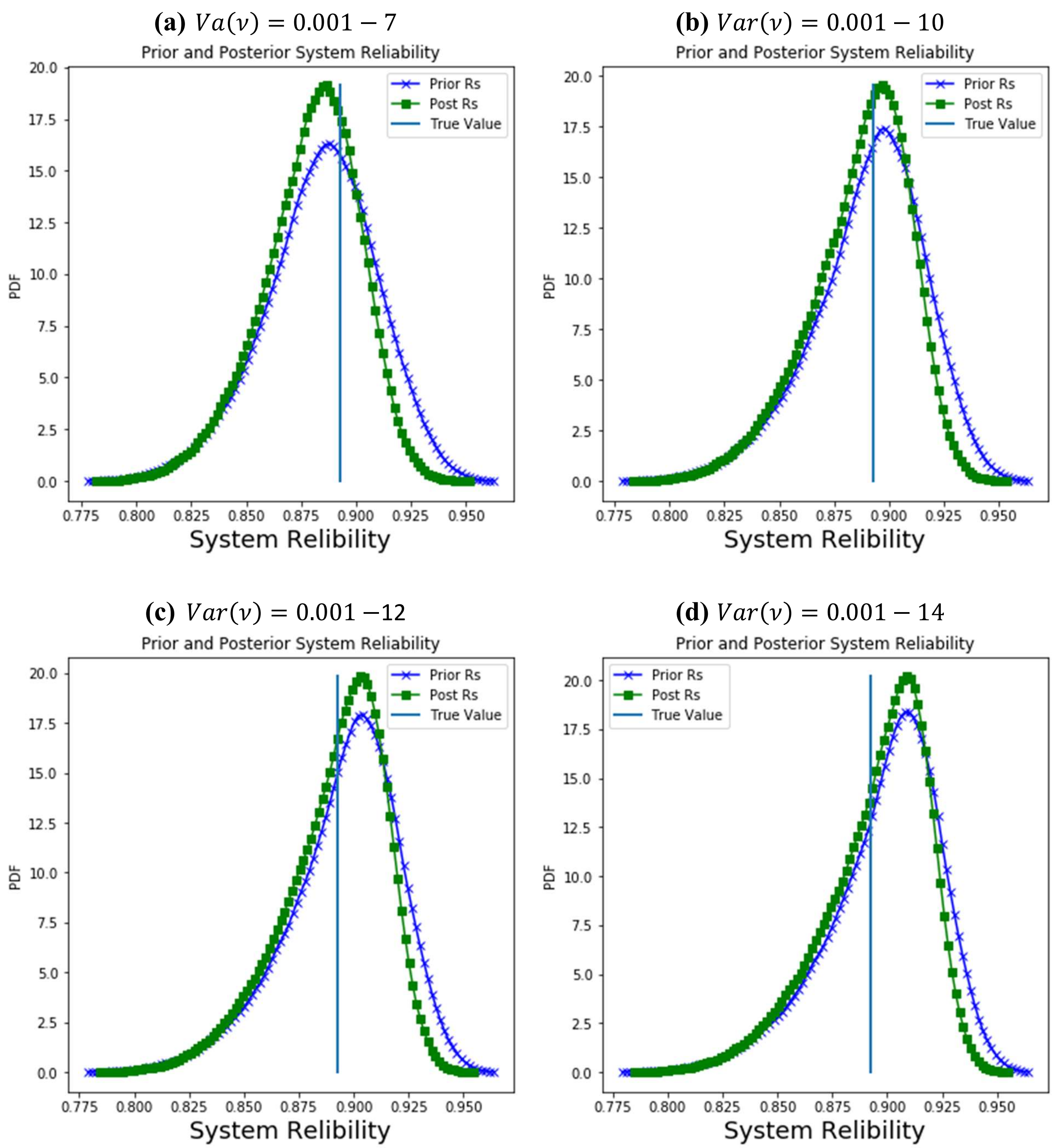
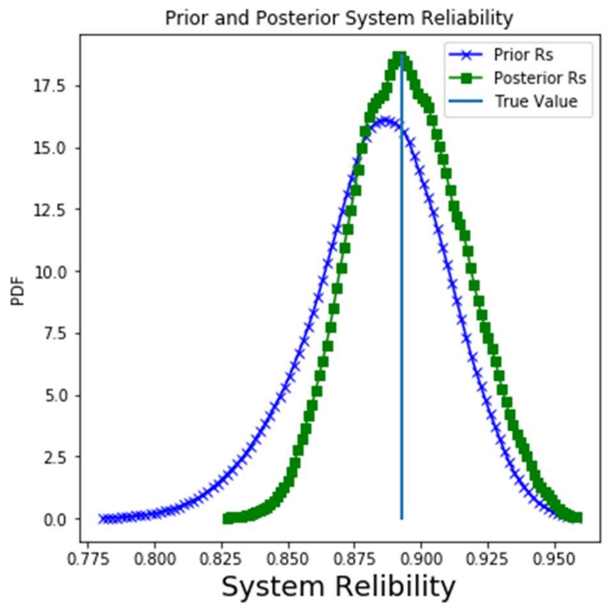
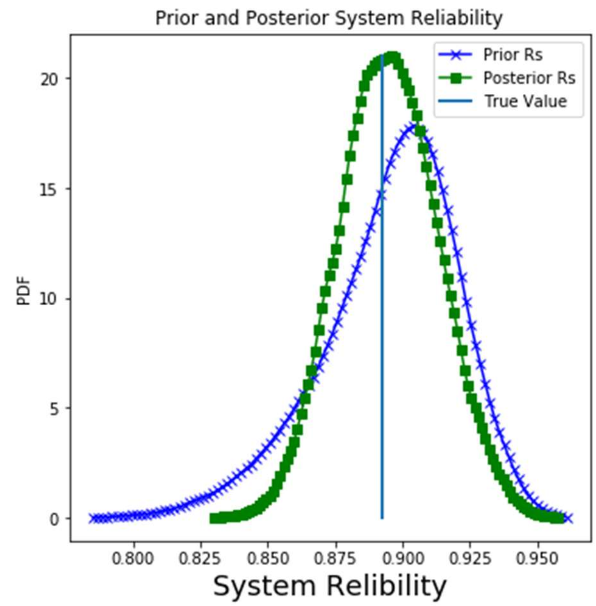


Figure A.1 Prior, Posterior and True Value of the system reliability for different variance $\boldsymbol{\nu}$ using Component Level ALT data propagation to system reliability method

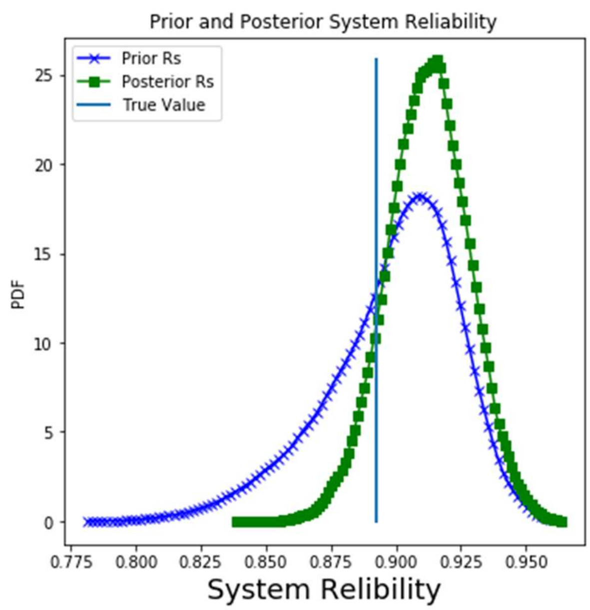
(e) $Va(v) = 0.001 - 7$



(f) $Var(v) = 0.001 - 10$



(g) $Var(v) = 0.001 -12$



(h) $Var(v) = 0.001 - 14$

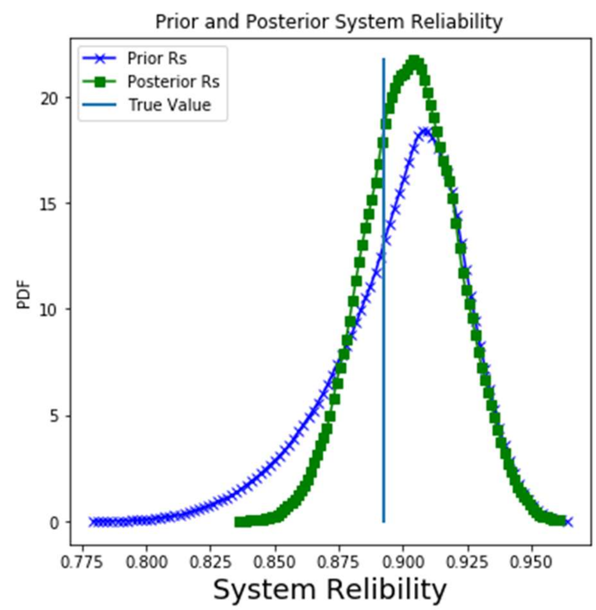


Figure A.2 Prior, Posterior and True Value of the system reliability for different variance $\boldsymbol{\nu}$ using System ALT data propagation to system reliability method

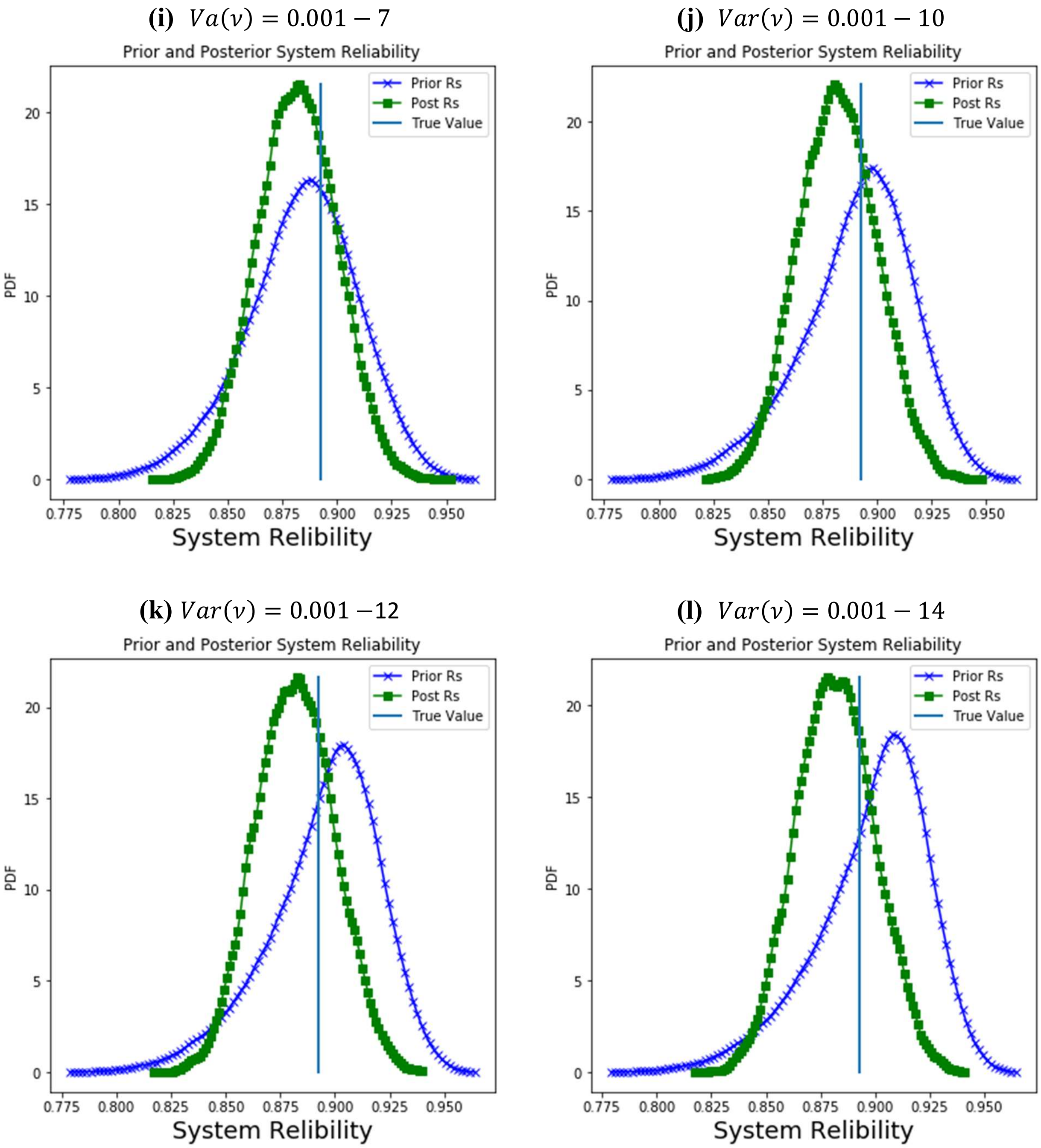


Figure A.3 Prior, Posterior and True Value of the system reliability for different variance $\boldsymbol{v}$ using Information Fusion method

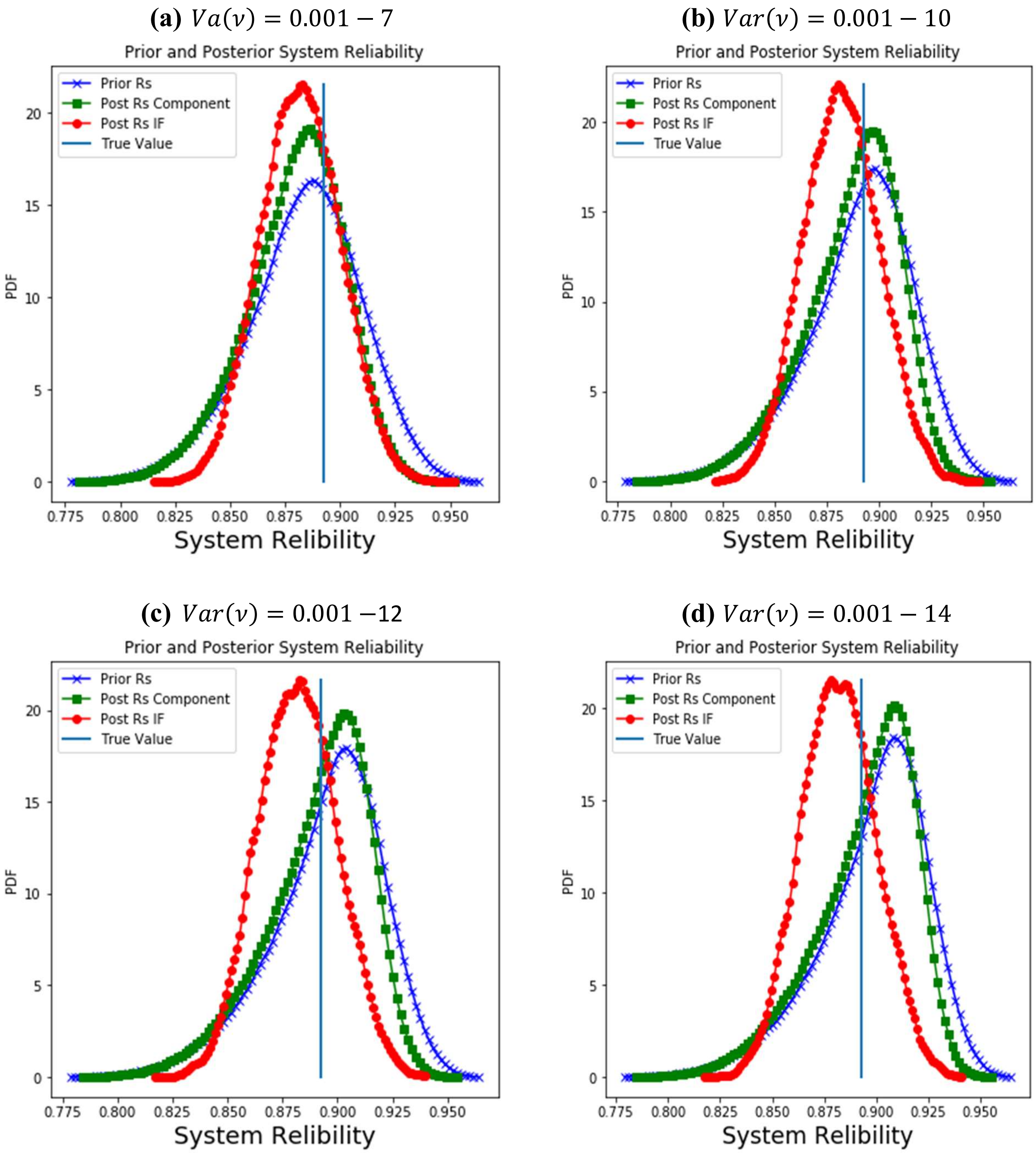


Figure A.4 Comparison of prior and posterior distributions of the system reliability using different methods

Bibliography

[1] W. H. Chen, L. Gao, J. Pan, P. Qian, and Q. C. He, "Design of Accelerated Life Test Plans—Overview and Prospect," *Chinese J. Mech. Eng. (English Ed.)*, vol. 31, no. 1, 2018.

[2] E. A. Elsayed, "Overview of reliability testing," *IEEE Trans. Reliab.*, vol. 61, no. 2, pp. 282–291, 2012.

[3] L. A. Escobar and W. Q. Meeker, "A review of accelerated test models," *Stat. Sci.*, vol. 21, no. 4, pp. 552–577, 2006.

[4] J. Orbe, E. Ferreira, and V. Núñez-Antón, "Comparing proportional hazards and accelerated failure time models for survival analysis," *Stat. Med.*, vol. 21, no. 22, pp. 3493–3510, 2002.

[5] M. Newby, "Perspective on Weibull Proportional-Hazards Models," *IEEE Trans. Reliab.*, vol. 43, no. 2, pp. 217–223, 1994.

[6] W. Stute, "Consistent estimation under random censorship when covariables are present," *Journal of Multivariate Analysis*, vol. 45, no. 1. pp. 89–103, 1993.

[7] E. Pieruschka, "Relation between lifetime distribution and the stress level causing the failures," LOCKHEED MISSILES AND SPACE CO SUNNYVALE CALIF, 1961.

[8] Hougaard P., "Fundamentals of survival data. Biometrics 1999.pdf," *Biometrics*, vol. 55, pp. 13–22, 1999.

[9] A. Xu, S. Basu, and Y. Tang, "A full bayesian approach for masked data in step-stress accelerated life testing," *IEEE Trans. Reliab.*, vol. 63, no. 3, pp. 798–806, 2014.

[10] A. T. Louis, "Nonparametric Analysis of an Accelerated Failure Time Model," *Biometrika Trust Nonparametric*, vol. 68, no. 2, pp. 381–390, 2020.

[11] L. Kuo and B. Mallick, "Bayesian semiparametric inference for the accelerated failure-time model," *Can. J. Stat.*, vol. 25, no. 4, pp. 457–472, 1997.

[12] K. M. Anderson, "A Nonproportional Hazards Weibull Accelerated Failure Time Regression Model," *Biometrics*, vol. 47, no. 1, pp. 281–288, 1991.

[13] E. Liu and K. Lim, "Using the Weibull accelerated failure time regression model to predict time to health events Keywords Weibull distribution Log-Weibull distribution,"

pp. 1–15, 2018.

[14] J. P. Klein and A. P. Basu, "Theory and Methods Weibull accelerated life tests when there are competing causes of failure," *Commun. Stat. - Theory Methods*, vol. 10, no. 20, 1981.

[15] J. Van René Dorp, T. A. Mazzuchi, G. E. Fornell, and L. R. Pollock, "A bayes approach to step-stress accelerated life testing," *IEEE Trans. Reliab.*, vol. 45, no. 3, pp. 491–498, 1996.

[16] Y. Tang, Q. Guan, P. Xu, and H. Xu, "Optimum design for type-i step-stress accelerated life tests of two-parameter weibull distributions," *Commun. Stat. - Theory Methods*, vol. 41, no. 21, pp. 3863–3877, 2012.

[17] L. Wang, "Estimation of constant-stress accelerated life test for Weibull distribution with nonconstant shape parameter," *J. Comput. Appl. Math.*, vol. 343, pp. 539–555, 2018.

[18] A. H. Kjell A. Doksum, "Models for Variable-Stress Accelerated Life Testing Experiments Based on Wiener Processes and the Inverse Gaussian Distribution," *Am. Soc. Qual.*, vol. 34, no. 1, pp. 74–82, 2020.

[19] W. Q. Meeker, L. A. Escobar, and Y. Hong, "Using accelerated life tests results to predict product field reliability," *Technometrics*, vol. 51, no. 2, pp. 146–161, 2009.

[20] Y. Zhang and W. Q. Meeker, "Bayesian life test planning for the Weibull distribution with given shape parameter," *Metrika*, vol. 61, no. 3, pp. 237–249, 2005.

[21] R. Miller and W. Nelson, "Optimum simple step-stress plans for accelerated life testing," *IEEE Trans. Reliab.*, vol. 32, no. 1, pp. 59–65, 1983.

[22] D. S. Bai, M. S. Kim, and S. H. Lee, "Optimum Simple Step-Stress Accelerated Life Tests with Censoring," *IEEE Trans. Reliab.*, vol. 38, no. 5, pp. 528–532, 1989.

[23] K. A. Doksum and A. Hbyland, "Models for variable-stress accelerated life testing experiments based on wener processes and the inverse gaussian distribution," *Technometrics*, vol. 34, no. 1, pp. 74–82, 1992.

[24] K. Chaloner and K. Larntz, "Bayesian design for accelerated life testing," *J. Stat. Plan. Inference*, vol. 33, no. 2, pp. 245–259, 1992.

[25] V. Bagdonavičius, O. Cheminade, and M. Nikulin, "Statistical planning and inference in accelerated life testing using the CHSS model," *J. Stat. Plan. Inference*, vol. 126, no. 2, pp. 535–551, 2004.

[26] J. René Van Dorp and T. A. Mazzuchi, "A general Bayes weibull inference model for accelerated life testing," *Reliab. Eng. Syst. Saf.*, vol. 90, no. 2–3, pp. 140–147, 2005.

[27] I. C. Lee, Y. Hong, S. T. Tseng, and T. Dasgupta, "Sequential Bayesian Design for Accelerated Life Tests," *Technometrics*, vol. 60, no. 4, pp. 472–483, 2018.

[28] Y. Zhang and W. Q. Meeker, "Bayesian methods for planning accelerated life tests," *Technometrics*, vol. 48, no. 1, pp. 49–60, 2006.

[29] W. Nelson and T. J. Kielpinski, "Theory for Optimum Censored Accelerated Life Tests for Normal and Lognormal Life Distributions Linked references are available on JSTOR for this article : Theory for Optimum Censored Accelerated Life Tests for Normal and Logno," *Am. Soc. Qual.*, vol. 18, no. 1, pp. 105–114, 1976.

[30] W. B. Nelson, *Accelerated Testing - Statistical models, Test Plans, and Data Analyses*, John Wiley & Sons, 1990.

[31] S. Engineering, C. M. Kim, and D. S. Bai, "Analyses of Accelerated Life Testing Data Under Two Failure Modes," *Int. J.*, vol. 9, no. 2, pp. 111–125, 2002.

[32] K. Patra and D. K. Dey, "A multivariate mixture of Weibull distributions in reliability modeling," *Stat. Probab. Lett.*, vol. 45, no. 3, pp. 225–235, 1999.

[33] T. Ishioka and Y. Nonaka, "Maximum likelihood estimation of weibull parameters for two independent competing risks," *IEEE Trans. Reliab.*, vol. 40, no. 1, pp. 71–74, 1991.

[34] Y. R. Chun and D. S. Bai, "Optimum Simple Step-Stress Accelerated Life-Tests with Competing Causes of Failure," *IEEE Trans. Reliab.*, vol. 40, no. 5, pp. 622–627, 1991.

[35] F. Pascual, "Planning and Inference Accelerated life test planning with independent lognormal competing risks," *J. Stat. Plan. Inference*, vol. 140, no. 4, pp. 1089–1100, 2010.

[36] F. Pascual, "Accelerated Life Test Planning With Independent Weibull Competing Risks," *IEEE Trans. Reliab.*, vol. 57, no. 3, pp. 435–444, 2008.

[37] Klein, J.P., Basu, A.P., "Accelerated Life Tests Under Competing Welbull Causes of Failure," *Comm. Stat. Theory Methods A10*, pp. 2073–2100, 1981.

[38] A. . Klein, J.P., Basu, "Accelerated fe Testing Under Competing Exponential Failure Distributions," *IAPQR Trans.*, vol. 7, pp. 1–20.

[39] C. Bunea and T. A. Mazzuchi, "Competing failure modes in accelerated life testing," vol. 136, pp. 1608–1620, 2006.

[40] W. Nelson, "Accelerated life testing-step-stress models and data analyses," *IEEE Trans. Reliab.*, vol. 29, no. 2, pp. 103–108, 1980.

[41] G. Allegri and X. Zhang, "On the inverse power laws for accelerated random fatigue testing," *Int. J. Fatigue*, vol. 30, no. 6, pp. 967–977, 2008.

[42] W. E. Corporation, "An Overview of Environmental Reliability Testing," pp. 102–109, 1996.

[43] D. Kececioglu and J. A. Jacks, "The Arrhenius, Eyring, inverse power law and combination models in accelerated life testing," *Reliab. Eng.*, vol. 8, no. 1, pp. 1–9, 1984.

[44] W. Nelson, "Analysis of accelerated life test data-Part I: The Arrhenius model and graphical methods," *IEEE Trans. Electr. Insul.*, vol. EI-6, no. 4, pp. 165–181, 1971.

[45] F. Pascual, "Accelerated Life Test Planning With Independent Weibull Competing Risks With Known Shape Parameter," *IEEE Trans. Reliab.*, vol. 56, no. 1, pp. 85–93, 2007.

[46] Wayne B. Nelson, *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*. John Wiley & Sons, 2009.

[47] D . R . Cox, "Regression Models and Life-Tables," *R. Stat. Soc.*, vol. 34, no. 2, pp. 187–220, 1972.

[48] N. Ata, "Cox Regression Models with Nonproportional Hazards Applied to Lung Cancer Survival Data," *Hacettepe J. Math. Stat.*, vol. 36, no. 2, pp. 157–167, 2007.

[49] J. Barrett, "Weibull-Cox proportional hazard model," vol. 0, no. July, pp. 1–7, 2014.

[50] Z. Zhang, "Parametric regression model for survival data: Weibull regression model as an example," *Ann. Transl. Med.*, vol. 4, no. 24, 2016.

[51] N. . Breslow, "Covariance Analysis of Censored Survival Data," *Biometrics*, vol. 30, no. 1, pp. 89–99, 2010.

[52] N. E. Breslow, "Analysis of Survival Data under the Proportional Hazards Model," *Int. Stat. Rev. / Rev. Int. Stat.*, vol. 43, no. 1, p. 45, 1975.

[53] J. D. Kalbfleisch and R. L. Prentice, "Marginal Likelihoods Based on Cox ' s Regression and Life Model," *Biometrika Trust Marg.*, vol. 60, no. 2, pp. 267–278, 2017.

[54] J. A. A. and A. Senthilselvan, "Smooth Estimates for the Hazard Function," *Soc. R. Stat.*, vol. 44, no. 3, pp. 287–321, 2016.

[55] M. Campolieti, "Bayesian Estimation and Smoothing of the Baseline Hazard in Discrete Time Duration Models," *Rev. Econ. Stat.*, vol. 82, no. 4, pp. 685–694, 2000.

[56] H. Zhang and E. A. Elsayed, "Optimal design of ALT plans based on proportional hazards models," *IIE Annu. Conf. Exhib. 2004*, pp. 3–4, 2004.

[57] E. A. Elsayed and H. Zhang, "Design of PH-based accelerated life testing plans under multiple-stress-type," *Reliab. Eng. Syst. Saf.*, vol. 92, no. 3, pp. 286–292, 2007.

[58] C. H. Hu, R. D. Plante, and J. Tang, "Step-stress accelerated life tests: A proportional hazards-based non-parametric model," *IIE Trans. (Institute Ind. Eng.*, vol. 44, no. 9, pp. 754–764, 2012.

[59] D. O. D.R. Cox, *Analysis of Surviva1 Data*. CRC Press, 1984.

[60] J. Etezadi-Amoli and A. Ciampi, "Extended Hazard Regression for Censored Survival Data with Covariates : A Spline Approximation for the Baseline Hazard Function," *Int. Biometric Soc.*, vol. 43, no. 1, pp. 181–192, 1987.

[61] A. Ciampi and J. Etezadi-Amoli, "A General Model for Testing The Proportional Hazards and the Accelerated Failure Time Hypotheses in the Analysis of Censored Survival Data with Covariates," *Commun. Stat. - Theory Methods*, vol. 14, no. 3, pp. 651–667, 1985.

[62] H. J. Shyur, E. A. Elsayed, and J. T. Luxhøj, "A general hazard regression model for accelerated life testing," *Ann. Oper. Res.*, vol. 91, pp. 263–280, 1999.

[63] E. A. Elsayed, H. Liao, and X. Wang, "An extended linear hazard regression model with application to time-dependent dielectric breakdown of thermal oxides," *IIE Trans. (Institute Ind. Eng.*, vol. 38, no. 4, pp. 329–340, 2006.

[64] T. Hastie and R. Tibshirani, "Varying-Coefficient Models," *Soc. R. Stat.*, vol. 55, no. 4, pp. 757–796, 2013.

[65] F. Louzada-Neto, "Extended Hazard Regression Model for Reliability and Survival Analysis," *Lifetime Data Anal.*, vol. 3, no. 4, pp. 367–381, 1997.

[66] Y.K. Tseng, K.N. Hsu, and Y.F. Yang, "A semiparametric extended hazard regression model with time-dependent covariates," *J. Nonparametr. Stat.*, vol. 26, no. 1, pp. 115–128, 2014.

[67] G. Nguyen, "Study of Two-Component System with Failure Interaction *," vol. 32, pp. 239–247, 1985.

[68] R. B. Nelsen, *An introduction to copulas*. Springer Science & Business Media, 2007.

[69] M. Zheng and J. P. . Klein, "Trust Estimates of Marginal Survival for Dependent Competing Risks Based on an Assumed," *Biometrika*, vol. 82, no. 1, pp. 127–138, 2020.

[70] B. Schweizer and A. Sklar, *Probabilistic metric spaces*. Courier Corporation, 2011.

[71] S. M. S. Lo and R. A. Wilke, "A copula model for dependent competing risks," *J. R. Stat. Soc. Ser. C Appl. Stat.*, vol. 59, pp. 359–376, 2010.

[72] L. P. Rivest and M. T. Wells, "A martingale approach to the copula-graphic estimator for the survival function under dependent censoring," *J. Multivar. Anal.*, vol. 79, no. 1, pp. 138–155, 2001.

[73] C. Genest and L. P. Rivest, "Statistical inference procedures for bivariate Archimedean copulas," *J. Am. Stat. Assoc.*, vol. 88, no. 423, pp. 1034–1043, 1993.

[74] Y. Gu, C. F. L. Liang, and J. Zhang, "Reliability calculation method based on the Copula function for mechanical systems with dependent failure," *Ann. Oper. Res.*, 2019.

[75] H. Lu and Z. Zhu, "A method for estimating the reliability of structural systems with moment-matching and copula concept," vol. 46, no. 2, pp. 196–208, 2018.

[76] W. Peng, X. Zhang, H. Huang, Y. Liu, and R. Yuan, "Notice of Retraction," pp. 14–17.

[77] P. Limbourg, H. Kochs, K. Echtle, and I. Eusgeld, "Reliability prediction in systems with correlated component failures ü an approach using copulas," in *20th International Conference on Architecture of Computing Systems 2007*, 2007, pp. 1–8.

[78] J. F. Carrière, "Removing Cancer when it is Correlated with other Causes of Death," *Biometrical Journal, 1995*, vol. 37, no. 3, pp. 339–350, 1995.

[79] A. Pickles and R. Crouchley, "A Comparison of Frailty Models For Multivariate Survival Data," *Stat. Med.*, vol. 14, no. August 1994, pp. 1447–1461, 1995.

[80] J. Vaupel, K. Manton, and E. Stallard, "The Impact of Heterogeneity In Individual Frailty On The Dynamics of Mortality," *Demography*, vol. 16, no. 3, pp. 440-454 Volume 16 Number 3, 1979.

[81] D. G. Clayton, "A Model for Association in Bivariate Life Tables and Its Application in Epidemiological Studies of Familial Tendency in Chronic Disease Incidence," *Biometrika Trust*, vol. 65, no. 1, pp. 141–151, 2020.

[82] D. J. Sargent, "Reproduced with permission of the copyright owner. Further reproduction prohibited without permission."

[83] D. G. Clayton, "A Monte Carlo Method for Bayesian Inference in Frailty Models," *Biometrics*, vol. 47, no. 2, pp. 467–485, 2020.

[84] D. Oakes and D. Oakes, "Bivariate Survival Models Induced by Frailties," *J. Am. Stat. Assoc.*, vol. 84, no. 406, pp. 487–493, 2020.

[85] M. Greenwood and U. G. Yule, "An Inquiry into the Nature of Frequency Distributions Representative of Multiple Happenings with Particular Reference to the Occurrence of Multiple Attacks of Disease," *Wiley R. Stat. Soc.*, vol. 83, no. 2, pp. 255–279, 2020.

[86] J. P. Klein, C. Pelz, and M. Zhang, "Modeling Random Effects for Censored Data by a Multivariate Normal Regression Model," no. June, pp. 497–506, 1999.

[87] S. Sidhu, K. Jain, and S. K. Sharma, "Bayesian estimation of generalized gamma shared," *Comput. Stat.*, vol. 33, no. 1, pp. 277–297, 2018.

[88] P. Lambert, D. Collett, A. Kimber, and R. Johnson, "Parametric accelerated failure time models with random effects and an application to kidney transplant survival," *Stat. Med.*, vol. 23, no. 20, pp. 3177–3192, 2004.

[89] N. Balakrishnan and Y. Peng, "Generalized gamma frailty model," no. October 2005, pp. 2797–2816, 2006.

[90] G. Yin and J. G. Ibrahim, "A class of Bayesian shared gamma frailty models with multivariate failure time data," *Biometrics*, vol. 61, no. 1, pp. 208–216, 2005.

[91] X. Liu, "Planning of accelerated life tests with dependent failure modes based on a gamma frailty model," *Technometrics*, vol. 54, no. 4, pp. 398–409, 2012.

[92] P. Hougaard, "Shared frailty models," in *Analysis of multivariate survival data*, Springer, 2000, pp. 215–262.

[93] W. Pan, "Using frailties in the accelerated failure time model," *Lifetime Data Anal.*, vol. 7, no. 1, pp. 55–64, 2001.

[94] E. A. Elsayed, "Accelerated life testing," in *Handbook of reliability engineering*, Springer, 2003, pp. 415–428.

[95] H. Ma and W. Q. Meeker, "Optimum Step-Stress Accelerated Life Test Plans for Log-Location-Scale Distributions," *Wiley Intersci.*, vol. 55, no. 6, 2008.

[96] R. G. Gutierrez, "Parametric Frailty and Shared Frailty Survival Models," *Stata J. Promot. Commun. Stat. Stata*, vol. 2, no. 1, pp. 22–44, 2002.

[97] M. Shaked and N. D. Singpurwalla, "A Bayesian approach for quantile and response probability estimation with applications to reliability," *Ann. Inst. Stat. Math.*, vol. 42, no. 1, pp. 1–19, 1990.

[98] M. . L. Blackwell and N. D. Singpurwalla, "Inference from Accelerated Life Tests Using Filtering in Coloured Noise," *J. R. Stat. Soc.*, vol. 50, no. 2, pp. 281–292, 1988.

[99] C. Li, S. MahaDeVan, Y. Ling, S. Choze, and L. Wang, "Dynamic Bayesian network for aircraft wing health monitoring digital twin," *AIAA J.*, vol. 55, no. 3, pp. 930–941, 2017.

[100] D. V. Gokhale, G. E. P. Box, and G. C. Tiao, "Bayesian Inference in Statistical Analysis," *Biometrics*, vol. 30, no. 1, p. 211, 1974.

[101] R. A. Fisher, P. Trans, and R. S. Lond, "On the Mathematical Foundations of Theoretical Statistics," *Phiolosophical Trans. R. Soc.*, no. March 2009, pp. 309–368, 1922.

[102] J. V. Candy and L. Livermore, "Bayesian approach to model-based signal processing : An overview.," *IEEE Signal Process. Mag.*, vol. 24, no. 4, pp. 73–85, 2015.

[103] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/nongaussian bayesian tracking," *Bayesian Bounds Param. Estim. Nonlinear Filtering/Tracking*, vol. 50, no. 2, pp. 723–737, 2007.

[104] N.J. Grdon, D. J. Salmond, and A. F. M. Smith, "Novel approach to nonlinear/non-Gaussian Bayesian state estimation," *IEE Proc. F - Radar signal Process.*, vol. 140, no. 2, pp. 139–176, 1993.

[105] D. Crisan, P. Del Moral, and T. Lyons, "Del Moral Nonlinear filtering Interacting particle solution.pdf," *Markov Process. Relat. Fields*, vol. 2, no. 4, pp. 555–580, 1998.

[106] A. M. Johansen, "Journal of Statistical Software SMCTC: Sequential Monte Carlo in C++," *J. Stat. Softw.*, vol. 30, no. 6, pp. 1–41, 2009.

[107] A. Doucet, F. Nando de, and N. Gordon, *An Introduction to Sequential Monte Carlo Methods*. New York, NY: Springer, 2001.

[108] Q. Guan and Y. cai Tang, "Bayesian planning of optimal step-stress accelerated life test for log-location-scale distributions," *Acta Math. Appl. Sin.*, vol. 34, no. 1, pp. 51–64, 2018.

[109] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Computational Statistics and Data Analysis*, 2nd ed., vol. 21. Elsevier B.V, 1996.

[110] D. Kundu, "Bayesian inference and life testing plan for the weibull distribution in presence of progressive censoring," *Technometrics*, vol. 50, no. 2, pp. 144–154, 2008.

[111] X. P. Zhang, J. Z. Shang, X. Chen, C. H. Zhang, and Y. S. Wang, "Statistical Inference of Accelerated Life Testing With Dependent Competing Failures Based on Copula," vol. 63, no. 3, pp. 764–780, 2014.

[112] Z. Hu and S. Mahadevan, "Probability models for data-Driven global sensitivity analysis," *Reliab. Eng. Syst. Saf.*, vol. 187, pp. 40–57, 2019.

[113] T. Bedford and R. M. Cooke, "Vines--a new graphical model for dependent random variables," *Ann. Stat.*, vol. 30, no. 4, pp. 1031–1068, 2002.

[114] K. Aas, C. Czado, A. Frigessi, and H. Bakken, "Pair-copula constructions of multiple dependence," *Insur. Math. Econ.*, vol. 44, no. 2, pp. 182–198, 2009.

[115] T. Yang and R. Pan, "A novel approach to optimal accelerated life test planning with interval censoring," *IEEE Trans. Reliab.*, vol. 62, no. 2, pp. 527–536, 2013.

[116] S. Z. Ramadan and K. Z. Ramadan, "Bayesian simple step-stress acceleration life testing plan under progressive type-I right censoring for exponential life distribution," *Mod. Appl. Sci.*, vol. 6, no. 3, pp. 91–99, 2012.

[117] D. S. Bai, S. W. Chung, and Y. R. Chun, "Optimal design of partially accelerated life tests for the lognormal distribution under type I censoring," *Reliab. Eng. Syst. Saf.*, vol. 40, no. 1, pp. 85–92, 1993.

[118] Z. Hu and S. Mahadevan, "Accelerated life testing (ALT) design based on computational reliability analysis," *Qual. Reliab. Eng. Int.*, vol. 32, no. 7, pp. 2217–2232, 2016.

[119] W. Si, Q. Yang, and X. Wu, "Material Degradation Modeling and Failure Prediction Using Microstructure Images," *Technometrics*, vol. 61, no. 2, pp. 246–258, 2019.

[120] H. N. Özgüven and D. R. Houser, “Mathematical models used in gear dynamics—a review,” *J. Sound Vib.*, vol. 121, no. 3, pp. 383–411, 1988.

[121] T. Kirchdoerfer and M. Ortiz, “Data-driven computational mechanics,” *Comput. Methods Appl. Mech. Eng.*, vol. 304, pp. 81–101, 2016.

[122] Z. Hu, H. Li, X. Du, and K. Chandrashekhara, “Simulation-based time-dependent reliability analysis for composite hydrokinetic turbine blades,” *Struct. Multidiscip. Optim.*, vol. 47, no. 5, pp. 765–781, 2013.

[123] Z. Hu and X. Du, “Saddlepoint approximation reliability method for quadratic functions in normal variables,” *Struct. Saf.*, vol. 71, pp. 24–32, 2018.

[124] Z. Hu, D. Ao, and S. Mahadevan, “Calibration experimental design considering field response and model uncertainty,” *Comput. Methods Appl. Mech. Eng.*, vol. 318, pp. 92–119, 2017.

[125] M. P. Wand and M. C. Jones, *Kernel smoothing*. Crc Press, 1994.

[126] M. Samiuddin and G. M. El-Sayyad, “On nonparametric Kernel density estimates,” *Biometrika*, vol. 77, no. 4, pp. 865–874, 1990.

[127] M. C. Jones and P. J. Foster, “A simple nonnegative boundary correction method for kernel density estimation,” *Stat. Sin.*, vol. 6, no. 4, pp. 1005–1013, 1996.

[128] M. C. Jones, “Simple boundary correction for kernel density estimation,” *Stat. Comput.*, vol. 3, no. 3, pp. 135–146, 1993.

[129] M. Schonlau, “Computer experiments and global optimization,” 1997.

[130] M. J. Sasena, P. Papalambros, and P. Goovaerts, “Exploration of metamodeling sampling criteria for constrained global optimization,” *Eng. Optim.*, vol. 34, no. 3, pp. 263–278, 2002.

[131] J. C. Helton and F. J. Davis, “Latin hypercube sampling and the propagation of uncertainty in analyses of complex systems,” *Reliab. Eng. Syst. Saf.*, vol. 81, no. 1, pp. 23–69, 2003.

[132] Z. Hu and Z. P. Mourelatos, “A Sequential Accelerated Life Testing Framework for System Reliability Assessment With Untestable Components,” *J. Mech. Des.*, vol. 140, no. 10, 2018.

[133] V. Pandey, Z. Mourelatos, E. Nikolaidis, M. Castanier, and D. Lamb, “System Failure Identification Using Linear Algebra: Application to Cost-Reliability Tradeoffs Under Uncertain Preferences,” SAE Technical Paper, 2012.

[134] R. L. Prentice, “Hazard Rate Models with Covariates,” *Int. Biometric Soc.*, vol. 35, no. 1, pp. 25–39, 1979.

[135] D. D. Hanagal and R. Sharma, "Modeling heterogeneity for bivariate survival data by shared gamma frailty regression model," *Model Assist. Stat. Appl.*, vol. 8, no. 2, pp. 85–102, 2013.