Some New Directions in Teichmüller Theory

by

Mark Greenfield

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2021

Doctoral Committee:

Professor Lizhen Ji, Chair
Professor Christine Aidala
Professor Richard Canary
Associate Professor Sarah Koch
Professor Ralf Spatzier

Mark Greenfield

markjg@umich.edu

ORCID iD: 0000-0001-7676-0557

ACKNOWLEDGEMENTS

First and foremost I am grateful to my advisor, Lizhen Ji, for supporting me throughout my graduate career. He has introduced me to many different areas of mathematics, and always had confidence in my ability to make meaningful contributions. I would like to thank my doctoral committee for working with me through this process during a pandemic. I am indebted to many professors at the University of Michigan, especially Sarah Koch and Richard Canary, for numerous helpful conversations. Dmitri Gekhtman deserves special mention, as Chapter V would not exist without our fruitful collaboration and friendship. I would like to acknowledge Bradley Zykoski for many fascinating mathematical and non-mathematical discussions over the years, and for providing helpful comments on early drafts of this thesis.

There are numerous others who deserve to be listed here, but I hope a general expression of gratitude to the wonderful people in the department and beyond, including but not limited to my fellow graduate students, postdoctoral scholars, administrative staff, lecturers, friends from my undergraduate years, as well as my students, will suffice. Most of this work was completed while I was funded by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE#1256260. I am also grateful for the generous support of the mathematics department and the Rackham Graduate School for funding several conference trips.

Finally, none of this work would have been possible without the constant love and support from my family, especially Meg, for which I am forever grateful.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS . . . . **ii**

ABSTRACT . . . . **iv**

CHAPTER

I. Introduction 1

II. Teichmüller Theory 14

2.1 Riemann Surfaces and Tools to Study Them 14
2.2 Teichmüller Spaces 19

III. Symmetric Spaces 29

3.1 Lie Theory and Symmetric Spaces 30
3.2 Compactifications 34

IV. Teichmüller Spaces of Flat n-Tori 38

4.1 The Teichmüller Spaces of Flat n-Tori 38
4.2 Extremal Lipschitz Maps Between Tori 41
4.3 Thurston's Metric for n-Dimensional Flat Tori 45
4.4 Teichmüller Metric for Higher-Dimensional Tori 51
4.5 The Hilbert Metric on $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ 55
4.6 The Weil-Petersson Metric 56
4.7 Horofunction and Satake Compactifications 58
4.8 The Thurston Compactification of $\mathcal{T}(n)$ 61

V. Holomorphic Isometric Submersions Between Teichmüller Spaces 68

5.1 Embeddings of Spaces of Quadratic Differentials 69
5.2 Infinitesimal Geometry 72
5.3 Using Theorem V.1 to Prove Theorem I.7 78
5.4 Infinitely Punctured Surfaces 85

BIBLIOGRAPHY **93**

ABSTRACT

In this thesis we will extend the study of Teichmüller spaces in two relatively unexplored new directions. First, beginning with the Teichmüller space of the flat 2-torus, rather than increasing the genus, we will explore higher dimensional tori. This yields Riemannian symmetric spaces with very different, yet analogous, behavior to classically studied Teichmüller spaces of hyperbolic surfaces. Second, in the setting of hyperbolic surfaces, we study a certain kind of rigidity for maps between different Teichmüller spaces. We will classify most of the possible cases of holomorphic isometric submersions between Teichmüller spaces of finite-type hyperbolic surfaces and begin exploration in the case of infinite-type.

CHAPTER I

Introduction

Since the 19th century, and in many ways much earlier, classifying all possible surfaces and exploring the resulting collections has been a central theme across mathematics. The most well-known approach to this problem is the study of the moduli spaces of Riemann surfaces, initiated by Riemann in the mid-19th century.

Building up the foundations of complex analysis, Riemann surfaces were first defined in order to give domains on which certain complex functions could be univalent, such as $f(z) = \sqrt{z}$. Riemann built his moduli space to better understand analytic functions, but it turned out to be a fundamental object across mathematics [37]. Fixing a topological type of the underlying surface (numbers of handles, punctures, and boundary components), Riemann first gave a count of the number of parameters (deemed "moduli") needed to specify a surface, which we now see as the dimension of the moduli space as a complex manifold.

It turned out that the moduli spaces contain singularities which prevent it from being realized as a smooth manifold. Indeed, even in the case of the moduli space of the flat torus, there are cone points. In the 1930s, Teichmüller was motivated to precisely understand Riemann's moduli space as a smooth manifold, and recognized that the presence of singularities prevented this. He realized that the singularities arose due to nontrivial automorphisms of Riemann surfaces (e.g. if a Riemann surface is equivalent to itself flipped over), and that a way around these singularities was to "unfold" the moduli space by tracking both the structure of the Riemann surface and a privileged homotopy class. This is most clearly defined by using equivalence classes of marked Riemann surfaces. The resulting collections are known as Teichmüller spaces. Studying the (complex, Riemannian, etc.) geometry of Teichmüller space has occupied generations of mathematicians, and this thesis continues that story.

In a thesis focused on Teichmüller theory, it seems appropriate to mention that the personal views and political activities of Oswald Teichmüller himself were unacceptable (see [49], pages 442 – 451 for a brief but illuminating biography by Sanford Segal). Segal claims that "Teichmüller's dedication to the Nazi cause and ideology seems complete. . . ." He further rejects the view that Teichmüller's dedication to Nazism was due to naïveté. Teichmüller's activities include leading the November 2nd, 1933 boycott of Edmund Landau's calculus class, which led to Landau's early retirement. The associated letter he wrote to Landau is rife with anti-Semitism and xenophobia. Beyond being complicit, he was an active proponent of the Nazi ideals which continue to have damaging impacts on the world. Perhaps the most poignant perspective on Teichmüller's role in mathematics and modern history comes from Lipman Bers' 1960 article [7], quoting Plutarch (Life of Pericles, 2.2): "It does not of necessity follow that, if the work delights you with its grace, the one who wrought it is worthy of your esteem."

The Teichmüller space of a closed oriented surface S_g of genus g, denoted $\mathcal{T}(S_g)$ (or $\mathcal{T}(S_{g,n})$ for a surface of genus g with n punctures), is the moduli space of marked complex structures on the surface. By the uniformization theorem, each such marked complex structure possesses a canonical Riemannian metric of constant curvature.

An important consequence is that one can equivalently view the Teichmüller spaces as classifying marked surfaces with a (constant-curvature) Riemannian metric. The structures on the underlying surfaces enable one to define different kinds of structures on the Teichmüller space itself.

Several different metrics have been defined for $\mathcal{T}(S_g)$, some of which are built directly from structures on the underlying surfaces. The earliest such construction is the classical Teichmüller metric d_{Teich}, defined in terms of extremal quasiconformal distortion between two marked complex structures. Another well-known metric on $\mathcal{T}(S_g)$ is the Weil-Petersson metric, introduced by Weil [55], which is an incomplete Riemannian metric. In [53], Thurston defined an asymmetric metric on $\mathcal{T}(S_g)$, $g \geq 2$, using the extremal Lipschitz constant for marking-preserving maps between hyperbolic surfaces. This metric is natural for Teichmüller spaces of hyperbolic surfaces as it uses only the canonical Riemannian metric associated to each complex structure. In Chapter II we will review the necessary background on Teichmüller theory.

Symmetric spaces are another class of spaces that are very well-studied. Symmetric spaces are (Riemannian) manifolds which admit an inversion symmetry at every point, and further all such spaces admit the isometric action of a Lie group. We will review some basics of symmetric spaces in Chapter III. The action of the mapping class group (which changes markings on marked Riemann surfaces) on Teichmüller space has been compared to the action of a Lie group on an associated symmetric space. This has helped motivate a great deal of work studying analogies between Teichmüller spaces and symmetric spaces throughout their long histories. Usually, questions, results, and properties about the latter motivate those about the former; for example, one might ask if Teichmüller spaces admit inversion symmetries (famously, they do not: see [46]). Our first new direction in Teichmüller theory will reverse this pattern and study certain symmetric spaces through the lens of Teichmüller theory.

The moduli space and the Teichmüller space of the flat 2-dimensional torus are well-understood as the locally symmetric space $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, respectively. The Teichmüller spaces of higher-genus 2-dimensional surfaces have been studied extensively (including in the final chapter of this thesis). In Chapter IV, we will instead focus on higher-dimensional flat tori, where we will leverage the modular interpretation of the symmetric spaces $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ to define and interpret new and old metrics and compactifications on them.

While there are similarities between the action of mapping class groups on Teichmüller spaces and the action of arithmetic subgroups of Lie groups on associated symmetric spaces, Teichmüller spaces are very different from symmetric spaces. For example, a corollary of Royden's theorem [46] shows that there are no symmetric points of Teichmüller spaces of hyperbolic surfaces, and Royden's theorem itself shows that the automorphism groups of finite-dimensional Teichmüller spaces are discrete. Despite important departures from symmetric space behavior for the case of hyperbolic surfaces, in the case of flat n-tori of unit volume, the Teichmüller spaces are precisely symmetric spaces. After defining the Teichmüller spaces of unit volume flat n-tori, denoted by $\mathcal{T}(n)$, we will define analogs of the three metrics for $\mathcal{T}(n)$ described earlier. The natural bijection $\mathcal{T}(n) \leftrightarrow \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ (reviewed in Section 4.1) is utilized throughout. The first of our main results is the following characterization of these metrics:

Theorem I.1. *For $\mathcal{T}(2)$, the Thurston metric, Teichmüller metric, Weil-Petersson metric, and hyperbolic metric all coincide. For $\mathcal{T}(n)$ with $n \geq 3$, we have:*

1. *The Thurston metric is an asymmetric polyhedral Finsler metric which can be computed explicitly (Theorem IV.14, Proposition IV.37).*

2. *The Teichmüller metric is the symmetrization of the Thurston metric by maximum (Theorem IV.29).*

3. *The Weil-Petersson metric is equal to the natural Riemannian metric on the symmetric space* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ *(Proposition IV.35).*

In addition, the Teichmüller metric on $\mathcal{T}(n)$ has been studied in a very different context before: in [40], the same metric on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ was found to be a generalization of the Hilbert projective metric. The Teichmüller metric on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ has also been studied in the context of conformal structures on vector spaces (see [42] Appendix A.1).

Our main tool for understanding the Thurston metric is Proposition IV.6, where we show that the minimal Lipschitz constant is realized by the unique affine map between two marked tori. Recall that the extremal quasiconformal map realizing the Teichmüller distance is unique (see Theorem 11.9 of [20], originally in [51]). Interestingly, this is not the case for extremal Lipschitz maps. We give a construction for an infinite family of extremal Lipschitz maps in Proposition IV.8.

Beyond metric structures, compactifications of symmetric spaces and Teichmüller spaces have been studied from many perspectives. A compactification of a topological space is in some sense a way to affix a boundary in order to, among other things, understand the ways in which sequences in the space can diverge. We make this more precise in Chapter III. For example, to compactify the real number line $\mathbb{R}$, one could add two "endpoints" and obtain the space $[-\infty,\infty]$ and study the real numbers using tools intended for closed intervals. Another compactification of $\mathbb{R}$ is the one-point compactification, where we "glue" both infinite ends together "at infinity" and obtain the circle S^1. Understanding different compactifications and their relationships to other structures can give new insights into various spaces. In the latter part of Chapter IV, we continue our study of Teichmüller spaces of flat tori by introducing and studying several compactifications.

One of the most important compactifications for symmetric spaces is the Satake compactification associated to a representation of the isometry group, first studied in [48]. For the broader class of Finsler manifolds, one has the horofunction compactification with respect to the Finsler metric, first defined by Gromov in [28]. For Teichmüller spaces, Thurston's compactification and its geometric interpretation using projective measured foliations (see [22]) is the most well-known. We briefly review this idea in Chapter IV. In [54], Walsh showed that the horofunction compactification with respect to the Thurston metric is equivalent to Thurston's compactification.

Haettel in [30] defined and studied a Thurston-type compactification of the space of marked lattices in $\mathbb{R}^n$ via an embedding in the projective space $\mathbb{P}(\mathbb{R}_+^{\mathbb{Z}^n})$. This mimics the original construction of Thurston. Theorem 3.1 in [30] shows that this compactification is $\mathrm{SL}(n,\mathbb{R})$-equivariantly isomorphic to the minimal Satake compactification induced by the standard representation of $\mathrm{SL}(n,\mathbb{R})$.

In Section 4.8, we introduce a related compactification of $\mathcal{T}(n)$, analogous to the geometric description of Thurston's compactification. In particular, we define an analog of projective measured foliations on n-tori to construct a Thurston boundary of $\mathcal{T}(n)$.

Theorem I.2. *For the Teichmüller space $\mathcal{T}(n) = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ of unit volume flat n-tori, the following compactifications are $\mathrm{SL}(n,\mathbb{R})$-equivariantly isomorphic:*

1. *Thurston compactification via measured foliations on n-tori*

2. *Horofunction compactification with respect to the Thurston metric*

3. Minimal Satake compactification associated to the standard representation of $\mathrm{SL}(n,\mathbb{R})$

Corollary I.3. *The Thurston boundary for* $\mathcal{T}(n)$ *is a topological sphere.*

The equivalence (1)$\leftrightarrow$(2) is analogous to the case of hyperbolic surfaces, while (1)$\leftrightarrow$(3) is related to Theorem 3.1 in [30], and gives a geometric interpretation of the boundary points of the compactification in [30]. Theorem I.2 is the combination of Proposition IV.37 and Theorem IV.49. Corollary I.3 again mimics the case of Teichmüller spaces of hyperbolic surfaces, and follows immediately from Theorem I.2 in light of some past work on Satake compactifications. We also show the following for the Teichmüller metric:

Theorem I.4. *The horofunction compactification of* $\mathcal{T}(n)$ *with the Teichmüller metric is* $\mathrm{SL}(n,\mathbb{R})$*-equivariantly isomorphic to the generalized Satake compactification associated to the sum of the standard and dual representations of* $\mathrm{SL}(n,\mathbb{R})$.

Finally, as an immediate corollary to Theorem I.1(3) and well-known facts about compactifications of nonpositively-curved Riemannian symmetric spaces, we have:

Corollary I.5. *The horofunction compactification of* $\mathcal{T}(n)$ *with respect to the Weil-Petersson metric is the visual compactification of* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

A further avenue of study would be to explore the Teichmüller theory of the Siegel upper-half space consisting of symmetric complex matrices whose imaginary part is positive definite. This is the moduli space of marked abelian varieties. The Siegel upper-half space is an alternative way to generalize the hyperbolic upper-half plane, which is the 1×1 matrix case. This direction may allow for an analog of the complex structure of Teichmüller spaces, which is lacking in the case of real tori.

Another interesting result comparing Teichmüller spaces and symmetric spaces is contained in the thesis of S. Antonakoudis [4]. He proved that there is no holomorphic and Kobayshi-isometric submersion between a finite-dimensional Teichmüller space and a bounded symmetric domain, provided each is of complex dimension at least two. It is also an especially interesting connection to the last part of this thesis: in Chapter V we study holomorphic isometric submersions between two Teichmüller spaces, instead of between a Teichmüller space and a bounded symmetric domain.

The work of Chapter IV began by considering the Thurston metric on Teichmüller spaces of 2-tori, following [6]. By defining a new analog of Thurston's metric and extending to higher dimensions, this work (especially Theorem IV.14) gives an answer to Problem 5.3 in W. Su's list of problems on the Thurston metric [50] from the AIM workshop "Lipschitz metric on Teichmüller space" in 2012.

Comparisons to symmetric spaces provide an interesting perspective from which one can discover structural properties of Teichmüller space. Another direction in Teichmüller theory is to see when various properties of Teichmüller spaces reflect the surfaces they classify. A central theme here is the interplay between the analytic structure of $\mathcal{T}(S_{g,n})$ and the topology and geometry of the underlying finite-type surface $S_{g,n}$.

This theme is exemplified by the result of Royden [46] asserting that every biholomorphism of $\mathcal{T}(S_g)$ with $g \geq 2$ arises from the action of a mapping class of S_g. The fascinating idea is that the intrinsic structure of $\mathcal{T}(S_g)$ as a complex manifold reflects the underlying topology of the surfaces it classifies. To prove this, Royden first established that the Teichmüller metric is an invariant of the complex structure on $\mathcal{T}(S_g)$ – it coincides with the intrinsically defined Kobayashi metric. Thus, any biholomorphism of $\mathcal{T}(S_g)$ is an isometry for the Teichmüller metric. Then, by analyzing the infinitesimal properties of the Teichmüller norm, Royden showed that any holomorphic isometry is induced by a mapping class. Earle and Kra [16] later extended Royden's result to the finite-dimensional Teichmüller spaces $\mathcal{T}(S_{g,n})$. Finally, Markovic [41] generalized to the infinite-dimensional case, proving for any Teichmüller space of complex dimension ≥ 2, that the biholomorphisms are induced by quasiconformal self-maps of the underlying Riemann surface.

Royden, Earle-Kra, and Markovic characterized holomorphic isometries between Teichmüller spaces - except in a few low-complexity cases, these are induced by identifications of the underlying surfaces. While these results do not require a priori the Teichmüller spaces to be classifying the same surfaces, they conclude that way. Maps between distinct Teichmüller spaces are not very well-studied. Our second new direction in Teichmüller theory is to study rigidity properties for maps which may be between distinct Teichmüller spaces. Weakening the assumption of maps being biholomorphic, we will generalize the celebrated result of Royden.

In Chapter V, we first detail joint work of the author with Dmitri Gekhtman on this topic [25]. In particular, we characterize a broader class of maps between finite-type Teichmüller spaces - the holomorphic and isometric submersions. Recall that a C^1 map between Finsler manifolds is an *isometric submersion* if the derivative maps the unit ball of each tangent space onto the unit ball of the target tangent space.

Consider the *forgetful maps*. These are maps $\mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{g,m})$ with $m < n$ which simply "forget" the punctures. That is, given an inclusion map $S_{g,n} \hookrightarrow S_{g,m}$ between the underlying smooth surfaces, one can define a map between Teichmüller spaces by inducing a map between marked surfaces. It turns out that these are a motivating example of a class of holomorphic isometric submersions between Teichmüller spaces; we will quickly verify this in Chapter V:

Proposition I.6. *Forgetful maps between Teichmüller spaces are holomorphic isometric submersions.*

Our main result in Chapter V is that the holomorphic isometric submersions between Teichmüller spaces are all of geometric origin - with some low genus exceptions, these submersions are precisely the forgetful maps $\mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{g,m})$.

Theorem I.7 (Theorem 1.1 from [25])**.** *Let $F : \mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{k,m})$ be a holomorphic map which is an isometric submersion with respect to the Teichmüller metrics on the domain and range. Assume (k, m) satisfies the following conditions:*

(1.1) *The type (k, m) is* non-exceptional*: $2k + m \geq 5$.*

(1.2) *The genus k is positive: $k \geq 1$.*

Then $g = k$, $n \geq m$, and up to pre-composition by a mapping class, $F : \mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{g,m})$ is the forgetful map induced by filling in the last $n - m$ punctures of $S_{g,n}$.

Remark I.8. Recall that we have isomorphisms $\mathcal{T}(S_{2,0}) \cong \mathcal{T}(S_{0,6})$ and $\mathcal{T}(S_{1,2}) \cong \mathcal{T}(S_{0,5})$ induced by hyperelliptic quotients. Thus, our hypothesis on the type (k, m) can be rephrased as follows: $\mathcal{T}(S_{k,m})$ is of complex dimension at least 2 and is not biholomorphic to a genus zero Teichmüller space $\mathcal{T}(S_{0,m})$. We expect that it is possible to remove the genus condition:

Conjecture I.9. *Any holomorphic and isometric submersion between finite-dimensional Teichmüller spaces of complex dimension at least 2 is a composition of*

1. *Forgetful maps $\mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{g,m})$ with $m < n$.*
2. *Mapping classes $\mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{g,n})$.*
3. *The isomorphisms $\mathcal{T}(S_{2,0}) \cong \mathcal{T}(S_{0,6})$ and $\mathcal{T}(S_{1,2}) \cong \mathcal{T}(S_{0,5})$.*

Remark I.10. The complex dimension 1 Teichmüller spaces $\mathcal{T}(S_{0,4})$, $\mathcal{T}(S_{1,0})$, and $\mathcal{T}(S_{1,1})$ are all biholomorphic to the unit disk $\mathbb{D}$. There are many isometric submersions $\mathcal{T}(S_g) \to \mathbb{D}$; the diagonal entries of the canonical period matrix are examples. Theorem 5.2 and its corollaries in [42] show that for each diagonal element of the period matrix, one can define a $\mathrm{GL}_2^+(\mathbb{R})$-invariant foliation of $\mathcal{T}(S_g)$, the leaves of which are determined by the value of that diagonal element of the period matrix at that point in $\mathcal{T}(S_g)$. One can then define the map $\mathcal{T}(S_g) \to \mathbb{D}$ sending each point to the value associated to that leaf, and the $\mathrm{GL}_2^+(\mathbb{R})$-invariance implies that these maps are holomorphic isometric submersions.

Theorem I.7 generalizes Royden's theorem on isometries by studying isometric submersions between Teichmüller spaces. Dually, one can attempt to generalize Royden's theorem by classifying the holomorphic and isometric *embeddings* between Teichmüller spaces. A claimed result of S. Antonakoudis states that the isometric embeddings all arise from covering constructions. This is another example of studying maps between distinct Teichmüller spaces.

Our result on holomorphic isometric submersions in the finite-type setting complements a classic theorem of Hubbard [34] asserting that there are no holomorphic sections of the forgetful map $\mathcal{T}(S_{g,1}) \to \mathcal{T}(S_g)$, except for the six sections in genus 2 obtained by marking fixed points of the hyperelliptic involution. Earle and Kra [16] later extended the result to the setting of forgetful maps between finite-type Teichmüller spaces $\mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{g,m})$. Combined, Theorem I.7 and the theorem of Hubbard-Earle-Kra have the following interpretation:

1. Holomorphic and isometric submersions between finite-dimensional Teichmüller spaces are of geometric origin. (They are forgetful maps.)

2. These submersions do not admit holomorphic sections, unless there is a geo-

metric reason (fixed points of elliptic involutions in genus 1 and hyperelliptic involutions in genus 2).

We mention also a result of Antonakoudis-Aramayona-Souto [5] stating that any holomorphic map $\mathcal{M}_{g,n} \to \mathcal{M}_{k,m}$ between moduli spaces is forgetful, as long as $g \geq 6$ and $k \leq 2g-2$. One can see this is as a parallel of our result, with our metric constraint (isometric submersion) replaced by an equivariance condition (preservation by the mapping class group action).

Markovic resolved a longstanding conjecture in [41] by generalizing Royden's theorem to all infinite-type surfaces, and the tools developed therein form the foundation of our approach to Theorem I.7. This motivates the extension of Theorem I.7 to infinite-type surfaces. In Section 5.4, we will consider the special case of infinite punctures but finite genus, and we will give a few steps towards generalizing Theorem I.7. In particular, we will show the following partial results:

Theorem I.11. *Let X and Y be Riemann surfaces of non-exceptional type with positive (finite) genus, possibly with (infinitely many) punctures. Let $F : \mathcal{T}(X) \to \mathcal{T}(Y)$ be a holomorphic isometric submersion with respect to the Teichmüller metric, and assume that the derivative maps dF_τ for $\tau \in \mathcal{T}(X)$ are weak*-sequentially continuous. Then:*

1. *X and Y have the same genus*
2. *If Y has finitely many punctures, then at each point $\tau \in \mathcal{T}(Y)$, there exists a holomorphic inclusion map h on the underlying surfaces which induces dF_τ.*

Remark I.12. In the case of finite punctures for both domain and range spaces, Theorem I.7 is a stronger version of the above (fully concluding that the maps are forgetful) without the assumption of weak*-sequential continuity.

This is a new result beyond those contained in [25]. We utilize tools of Earle-Gardiner in [15] which allow us to obtain a map between spaces of quadratic differentials even in this infinite-dimensional case where simply taking the dual of the derivative is not sufficient. The added assumption of weak*-sequential continuity is used in order to generalize the Earle-Gardiner Adjointness Theorem for isometric submersions. It may be the case that this assumption is not necessary. Further, proving Theorem I.11 does not require all of the machinery in [41], which enabled Markovic to generalize Royden's theorem to surfaces even of infinite genus. Perhaps by utilizing the full thrust of Markovic's methods, further results about surfaces of infinite type will be achievable in future work.

Remark I.13. While many of the tools in [41] are likely to generalize to the case of isometric submersions, the main tools used there to reduce the problem of isometries of Teichmüller spaces to a problem about maps between spaces of quadratic differentials are less likely to work without substantial modifications. One of the main tools used is the Uniqueness Theorem from Earle-Gardiner [15], which essentially states that a holomorphic automorphism of Teichmüller space is determined by the image and derivative at a single point. The proof involves inverse maps and the Cartan Uniqueness Theorem, neither of which immediately work for isometric submersions.

CHAPTER II

Teichmüller Theory

In this chapter, we will review some background on Teichmüller spaces and the surfaces they classify, as well as higher-dimensional tori which will be the objects of study in Chapter IV. Some additional references for this material include [13] for Riemann surfaces; for Teichmüller theory, [34], [20], and [35]; and for flat tori, see e.g. [11].

2.1 Riemann Surfaces and Tools to Study Them

2.1.1 Riemann Surfaces

A *Riemann surface* is a 2-dimensional topological surface with an atlas of charts mapping to the complex plane whose transition maps are biholomorphisms. In other words, Riemann surfaces are 1-dimensional complex manifolds. The atlas of complex-valued charts is called the surface's *complex structure*. There are many ways to specify or construct a Riemann surface. One particularly enlightening viewpoint is seen via the holomorphic universal cover.

The universal cover of a Riemann surface S is a simply-connected Riemann surface $\tilde{S}$ of which S is a quotient by the action of a discrete subgroup G of the automorphism group $\mathrm{Aut}(\tilde{S})$ of $\tilde{S}$. The celebrated Uniformization Theorem has the following immediate corollary (statement from [13]):

Theorem II.1 (Uniformization)**.** *Any connected Riemann surface is biholomorphic to one of the following:*

- *the Riemann sphere* $\hat{\mathbb{C}}$
- $\mathbb{C}$*,* $\mathbb{C}/\mathbb{Z}$*, or* $\mathbb{C}/\Lambda$*, where* Λ *is a lattice in* $\mathbb{C}$
- *a quotient* $\mathbb{H}/\Gamma$ *where* $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ *is a discrete subgroup acting freely on* $\mathbb{H}$

The idea is that the groups Λ or Γ in the above statement are representations of the fundamental group $\pi_1(S)$ of the Riemann surface in the automorphism group of the holomorphic universal cover (respectively, $\mathbb{C}$ for tori and $\mathbb{H}$ for higher-genus surfaces). That is to say, the gluing data for a fundamental domain in $\mathbb{C}$ or $\mathbb{H}$, determined in particular by a generating set for the (properly discontinuous and free) action of $\pi_1(S)$ on $\mathbb{H}$ or $\mathbb{C}$ uniquely determines a Riemann surface, and any Riemann surface can be described in this manner.

Recall that the Euler characteristic of a surface S with genus g and n punctures is given by $\chi(S) = 2 - 2g - n$. By the Gauss-Bonnet theorem (see e.g. [12] §4-5), a torus with no punctures may be endowed with a flat metric, and a torus with one or more punctures or any surface of genus at least two may be endowed with a hyperbolic metric. A Riemann surface with negative Euler characteristic will be called *hyperbolic*. See [35] §1 for a discussion on the beautiful relationship between conformal, complex, and metric structures.

Particularly in Chapter V, we will be interested in punctured surfaces. We occasionally refer to punctures as marked points. If X is a Riemann surface with punctures, we write $\hat{X}$ for the Riemann surface obtained from X by filling in the punctures. In particular, any local coordinate at the puncture of X can be given by a map to $\mathbb{D} - \{0\}$; in defining the complex structure of $\hat{X}$ we just extend this map by defining a point of $\hat{X}$ at the corresponding puncture and sending it to 0 in each such coordinate chart.

2.1.2 Quasiconformal Maps

Let ϕ be an orientation-preserving almost-everywhere real differentiable map $\phi : D_1 \to D_2$ between domains in $\mathbb{C}$. We say the *quasiconformal dilatation* of ϕ is given by:

$$K_\phi = \sup \frac{|\phi_z| + |\phi_{\bar{z}}|}{|\phi_z| - |\phi_{\bar{z}}|}, \tag{2.1}$$

where the supremum is over all points where ϕ is real-differentiable. If $K_\phi \leq K < \infty$, we say ϕ is a *K-quasiconformal map.* This relaxes the condition of holomorphicity by allowing $\phi_{\bar{z}}$ to be nonzero, provided it remains smaller than ϕ_z.

Several basic facts about quasiconformal maps will be important for our discussion. The following is a summary of several statements from [20] §11.1.

Lemma II.2 (Basic properties of quasiconformal maps)**.** *Let $\phi : D_1 \to D_2$ and $\psi : D_2 \to D_3$ be quasiconformal maps between domains in $\mathbb{C}$.*

- *The map ϕ is 1-quasiconformal if and only if it is holomorphic.*
- $K_{\psi\circ\phi} \leq K_\psi K_\phi$
- *If ϕ has an inverse, then $K_{\phi^{-1}} = K_\phi$.*
- *If ψ is conformal, then $K_{\psi\circ\phi} = K_\phi$; if ϕ is conformal, then $K_{\psi\circ\phi} = K_\psi$.*

Because the definition is local and conformal maps do not alter the dilatation, one can see that the definition of quasiconformal maps extends to maps between Riemann surfaces and the dilatation is independent of the choice of coordinates. Quasiconformal homeomorphisms between Riemann surfaces are central in Teichmüller theory because they enable one to directly compare non-equivalent complex structures on the same underlying topological surface.

2.1.3 Quadratic Differentials

Spaces of quadratic differentials on Riemann surfaces encode a great deal of information about the underlying surfaces, and as it turns out, are central to understanding the infinitesimal structure of Teichmüller space. Let X be a hyperbolic Riemann surface. Good references for this part are [20] §11.3 and [34] §5.3.

Formally, a *holomorphic quadratic differential* q on X is a holomorphic section of the symmetric square of the cotangent bundle of X. We occasionally leave off the word holomorphic, and we occasionally allow for meromorphic quadratic differentials, generalizing the holomorphic case. We provide an alternative description, perhaps more intuitively palatable, as follows.

If q is a holomorphic quadratic differential on X and z is a local coordinate defined on a neighborhood $U \subseteq X$, then where z is defined we may write q as $q_U(z)dz^2$ where $q_U : U \to \mathbb{C}$ is holomorphic in the coordinate z. If w is another local coordinate defined in a neighborhood V, then on $U \cap V$:

$$q_U(z(w))\left(\frac{dz}{dw}\right)^2 = q_V(w)$$

where $z = z(w)$ defines the holomorphic change of coordinates. The fact that we change coordinate systems by multiplying by a conformal map gives quadratic differentials several interesting properties.

Notice that the location and order of zeros (or poles in the meromorphic case) is independent of the coordinate system, so we may speak of the zeros or poles of a quadratic differential without ambiguity. Given quadratic differentials q_1 and q_2 on X, one can define the ratio q_1/q_2. This object is then simply a meromorphic function $X \to \hat{\mathbb{C}}$, with poles at the zeros of q_2 or poles of q_1 (depending on where the zeros and poles match up, in the obvious way). To see why, first notice that if we pick a coordinate system in which to express q_1 and q_2, locally the ratio is a ratio of holomorphic (meromorphic) functions. To change coordinates, we must apply the change-of-coordinate transformation $\left(\frac{dz}{dw}\right)^2$ to both the numerator and denominator, so the meromorphic function is independent of the choice of coordinates.

Another property of quadratic differentials is that one can integrate them across the surface. If q is a quadratic differential on X, then we define a norm as follows:

$$||q|| := \int_X |q|.$$

This is known as the 1-norm on the collection of quadratic differentials. If $||q|| < \infty$, we say q is an *integrable* quadratic differential. We have the following characterization:

Lemma II.3. *Let q be a holomorphic quadratic differential on a punctured surface. Then q is integrable if and only if all all poles at the punctures are simple.*

Because the integrals are defined locally, Lemma II.3 follows from the basic theory of integration in $\mathbb{C}$, since closed surfaces (of finite genus) are compact, and meromorphic functions are locally integrable in norm only when the poles are simple.

We will primarily be interested in the structure of the set of all integrable holomorphic quadratic differentials on a surface X, denoted by $Q(X)$. Quadratic differentials can be added together and multiplied by scalars in $\mathbb{C}$ in an obvious way, thus endowing $Q(X)$ with a $\mathbb{C}$-vector space structure.

If X is a surface with punctures, the space of integrable quadratic differentials $Q(\hat{X})$ on $\hat{X}$, the filled-in surface, is related to $Q(X)$ in a straightforward way. We have:

$$Q(X) = Q(\hat{X}) \cup \{\text{quadratic differentials on } X \text{ with simple poles at some of the punctures}\}.$$

Notice also that if $\rho : X \to Y$ is a holomorphic covering map of Riemann surfaces and $q \in Q(Y)$, then we can define the pullback differential $\rho^* q \in Q(X)$. Given a neighborhood $U \subseteq X$ of $x \in X$, we may (by restricting) assume ρ is univalent on U, so the restriction of ρ to U is biholomorphic. Thus $\rho^* q$ on U may be defined by simply looking at q on $\rho(U)$. This also provides a way to define an embedding $Q(Y) \hookrightarrow Q(X)$.

We will also need the dimension of $Q(X)$. Let $X \cong S_{g,k}$ be finite type. It is a consequence of the Riemann-Roch theorem that the complex dimension of $Q(X)$ is given by $\dim_{\mathbb{C}} Q(X) = 3g - 3 + k$ (see e.g. Proposition III.5.2 of [21] for a proof).

If X is of infinite type, then the dimension of $Q(X)$ is infinite. To see why, if X has infinite genus, then it can be written as a cover of arbitrarily high-genus finite surfaces, and so $Q(X)$ contains arbitrarily high-dimensional subspaces. If X has infinite punctures, for each puncture p there is some $q_p \in Q(X)$ with a simple pole at p and no poles at any other puncture. This gives infinitely many linearly independent quadratic differentials.

2.2 Teichmüller Spaces

2.2.1 Defining the Space

Fixing the underlying topological type of a surface (that is, the genus, number of punctures, and boundary components), one can consider the collection of all possible complex structures on the surface. This is called the *moduli space* and has been well-studied since the 19th century.

Denote by $S_{g,k}$ a surface of genus g with k punctures. We also write S_g for a closed surface of genus g, and we will sometimes suppress the subscripts to mean any surface (possibly infinite-type).

Definition II.4. The Teichmüller space $\mathcal{T}(S_{g,k})$ of $S_{g,k}$ is defined as the set of equivalence classes of Riemann surfaces of genus g with k punctures, marked via orientation-preserving homeomorphisms:

$$\mathcal{T}(S_{g,k}) = \{[X, f] : X \text{ a Riemann surface}, f : S_{g,k} \to X \text{ o.p. homeomorphism}\}/\sim$$

where $[X, f] \sim [X', f']$ if and only if there exists a biholomorphism h such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} & X \arrow[d, "h"] \\ S_{g,k} \arrow[ur, "f"] \arrow[r, "f'"'] & X' \end{tikzcd}$$

The equivalence relation is sometimes called *Teichmüller equivalence* of marked Riemann surfaces.

Remark II.5. By forgetting the maps f and f', we forget the markings and the condition reduces to conformal equivalence. The resulting collection defines the moduli space $\mathcal{M}_{g,k}$ of complex structures on $S_{g,k}$. More formally, the moduli space is realized as the quotient $\mathcal{M}(S_{g,k}) = \mathrm{Mod}_{g,k} \backslash \mathcal{T}(S_{g,k})$, where $\mathrm{Mod}_{g,k} = \mathrm{Diff}^+(S_{g,k})/\mathrm{Diff}_0(S_{g,k})$ is the mapping class group of $S_{g,k}$, and the action is given by

$$\varphi \cdot [S, f] = [S, f \circ \varphi^{-1}].$$

One can think of $\mathrm{Mod}_{g,k}$ as a "change-of-marking" group acting on $\mathcal{T}(S_{g,k})$.

Remark II.6. Occasionally we will reverse the direction of the arrows in the definition, namely we will consider points of Teichmüller space as equivalence classes of maps $f : S \to S_{g,k}$. The mapping class group action then becomes $\varphi \cdot [S, f] = [S, \varphi \circ f]$. This is less common in the literature, but especially for the purposes of Chapter IV it results in more transparent notation. Fortunately, either definition yields the same space.

Remark II.7. Another way to specify a Teichmüller space is by using a Riemann surface X as the "reference surface" rather than a topological surface $S_{g,k}$. In this case we simply write $\mathcal{T}(X)$, and the rest of the definition is the same. For surfaces of finite topological type, the definition only depends on the topological type (i.e. g and k). For surfaces of infinite type, Definition II.4 will not yield a manifold. To remedy this, we must specify both the topological type and a quasiconformal class (i.e. a collection of marked surfaces related by quasiconformal maps of finite dilatation). If X is infinite type, this is done by stipulating that representatives $f : X \to S$ of the elements $[S, f]/\sim$ must not only be homeomorphisms, but must also be quasiconformal. See [34] §6.4 for a more complete treatment of the definition in the case of infinite-type surfaces.

Recall next the correspondence between complex structures and constant-curvature metrics via the uniformization theorem, mentioned in Section 2.1.

Proposition II.8. *For each $g \geq 2$, there is a canonical bijection*

$$\mathcal{T}(S_g) \cong Met_g^{-1}/Diff_0(S_g)$$

where Met_g^{-1} is the collection of hyperbolic metrics on S_g, and $Diff_0(S_g)$ is the collection of diffeomorphisms of S_g isotopic to the identity.

This is a special case of Theorem 1.8 in [35]. We can thus also view elements of Teichmüller space as equivalence classes of marked hyperbolic surfaces.

We will next define the Teichmüller metric. Let $[S, f], [S', f'] \in \mathcal{T}(S_g)$. Then the map $f' \circ f^{-1}$ is an orientation-preserving homeomorphism from S to S'. Recall the quasiconformal dilatation K_ϕ of ϕ. The *Teichmüller metric* on $\mathcal{T}(S_g)$ is defined as:

$$d_{Teich}([S, f], [S', f']) = \frac{1}{2} \log \inf_{\phi \in [f' \circ f^{-1}]} (K_\phi) \tag{2.2}$$

where the infimum is taken over all homeomorphisms ϕ in the homotopy class $[f' \circ f^{-1}]$ which are almost-everywhere real-differentiable. As a consequence of Lemma II.2, this defines a metric on $\mathcal{T}(S_g)$ (see also §5.1 of [35]).

For $g = 1$, the Teichmüller metric was determined by Teichmüller in [51] (see also the translation and commentary in [2]):

Proposition II.9. *Under the identification* $\mathbb{H}^2 \xrightarrow{\sim} \mathcal{T}(S_1)$ *defined by* $\tau \mapsto \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, *the Teichmüller metric is equal to the hyperbolic metric.*

Thurston's (asymmetric) metric [53] utilizes the hyperbolic structure on surfaces. If $[S, f], [S', f'] \in \mathcal{T}(S_g)$, then the Thurston distance between them is defined:

$$d_{Th}([S, f], [S', f']) = \frac{1}{2} \inf_{\phi \in [f' \circ f^{-1}]} \log(\mathcal{L}(\phi))$$

where the infimum is over all Lipschitz maps $\phi : S \to S'$ in the homotopy class $[f' \circ f^{-1}]$, and

$$\mathcal{L}(\phi) = \sup_{x \neq y} \frac{d_{S'}(\phi(x), \phi(y))}{d_S(x, y)}$$

is the Lipschitz constant for ϕ, and $d_{S'}$, d_S are the induced hyperbolic metrics.

2.2.2 Teichmüller Space as a Complex Manifold

One of the central topics in Teichmüller theory is the study of $\mathcal{T}(S_{g,n})$ as a complex manifold. Many have contributed to this study, especially Ahlfors and Bers starting in the 1960s. There are several different ways to construct a complex manifold structure, and the details get quite involved. Here, we will briefly state some of the main ideas in the approach outlined in [34] §6.5 (as well as tools from §4.8 and §6.4). Another good source is [35], §6.1.

First, we will need the language of Beltrami forms. A *Beltrami form* on a Riemann surface X is a measurable $\mathbb{C}$-antilinear bundle map $\mu : TX \to TX$ with

$$||\mu||_\infty := \operatorname*{ess\,sup}_{x \in X} ||\mu(x)|| < 1$$

where $\mu(x)$ denotes the $\mathbb{C}$-antilinear map $\mu(x) : T_x X \to T_x X$ for $x \in X$, and the norm on $\mu(x)$ is the usual operator norm for (anti)linear maps.

Following [34] Definition 4.8.11, the space of Beltrami forms on X is denoted $\mathrm{Bel}(X)$. This is the unit ball in the infinite-dimensional Banach space given by the collection of such forms without the norm restriction, and so $\mathrm{Bel}(X)$ inherits the structure of an analytic Banach manifold. In other words, it is an infinite-dimensional complex manifold.

One can intuitively think of each $\mu \in \mathrm{Bel}(X)$ as a field of infinitesimal ellipses on X, the idea being that each ellipse represents a local (quasiconformal) deformation of X, and the essential boundedness means that, apart from a measure zero subset, the ellipses have bounded eccentricity. In local coordinates, Beltrami forms may be written as $\mu(z)\frac{d\bar{z}}{dz}$, with a transformation rule similar to that for quadratic differentials. Let $\mu_1(z_1)\frac{d\bar{z}_1}{dz_1}$ and $\mu(z)\frac{d\bar{z}}{dz}$ be two (overlapping) local realizations of $\mu \in \mathrm{Bel}(X)$. If $z_1 = g(z)$ is a change-of-coordinates function, then:

$$\mu_1(z_1) = \mu(z)\frac{g'(z)}{\overline{g'(z)}}.$$

Remark II.10. Given $\mu \in \mathrm{Bel}(X)$ and $q \in Q(X)$, it is possible to integrate the pair against each other over X, that is, one can compute $\int_X \mu q$. The notation already suggests as much: in local coordinates, we can think of μq as

$$\mu(z)\frac{d\bar{z}}{dz} \cdot q(z)dz^2 = \mu(z)q(z)|dz|^2.$$

This defines a bilinear pairing between $\mathrm{Bel}(X)$ and $Q(X)$ which we will return to later.

Now, we will explain the statement of Proposition and Definition 4.8.12 in [34], in which a Beltrami form μ on X is used to build a new Riemann surface X_μ. Intuitively, this is the surface obtained by performing the deformations specified by the field of ellipses.

Let $(\varphi_i : U_i \to V_i)_{i\in I}$ be an atlas of charts for X, with the U_i's an open cover of X, φ_i's biholomorphic maps, and V_i's domains in $\mathbb{C}$. Then there are functions $\mu_i : V_i \to \mathbb{C}$ such that

$$\mu|_{U_i} = \varphi_i^*(\mu_i \frac{d\bar{z}}{dz})$$

found by essentially looking at what μ does on X and transporting that to $\mathbb{C}$. By the Measurable Riemann Mapping Theorem (see e.g. [20] Theorem 11.16), there exist mappings $\psi_i : V_i \to \mathbb{C}$ which are solutions of the Beltrami equation

$$\frac{\partial \psi_i}{\partial \bar{z}} = \mu_i \frac{\partial \psi_i}{\partial z},$$

which are homeomorphisms onto their images. Finally, we claim that the composite maps $(\psi_i \circ \varphi_i : U_i \to \mathbb{C})_{i\in I}$ form an atlas of charts defining a complex structure on the topological surface underlying X, which defines a new Riemann surface X_μ.

This construction allows us to consider families of Riemann surfaces built from a basepoint surface X, by considering the surfaces X_μ as we let μ vary across the space of Beltrami forms.

Given a Riemann surface X of topological type $S_{g,n}$, fix a point in $\mathcal{T}(S_{g,n})$ represented by $\varphi : S_{g,n} \to X$. We can define a map $\Phi : \mathrm{Bel}(X) \to \mathcal{T}(S_{g,n})$:

$$\mu \mapsto [\mathrm{Id}_\mu \circ \varphi : S_{g,n} \to X_\mu].$$

where Id_μ is the canonical (quasiconformal) map $X \to X_\mu$ given by the identity on the underlying set of points.

Combining Propositions 6.1.4 and 6.4.11 in [34], the map Φ is surjective (and in fact is a *split submersion*), and further we can characterize the lack of injectivity. In particular, $\Phi(\mu_1) = \Phi(\mu_2)$ if and only if there exists a homeomorphism $f : S_{g,n} \to S_{g,n}$ isotopic to the identity map such that

$$\mathrm{Id}_{\mu_1} \circ \varphi \circ f = \mathrm{Id}_{\mu_2} \circ \varphi.$$

We observe that this is Teichmüller equivalence.

It follows that Teichmüller space may be viewed as a quotient of $\mathrm{Bel}(X)$. By Theorem 6.5.1 in [34], there is a unique complex manifold structure on $\mathcal{T}(S_{g,n})$ such that the map Φ is analytic, and this structure is independent of the choices of φ and X. This completes the search for a complex structure on Teichmüller space. While several methods for defining a complex structure are known, they lead to the same structure. The method described above also generalizes to infinite-dimensional Teichmüller spaces.

Remark II.11. The *Kobayashi metric*, first studied in 1967 in [38], is an intrinsic metric one can define on complex manifolds. In the case of the upper half-plane, it coincides with the usual hyperbolic metric. One way to define it for a complex manifold M is given as follows. Denote by $d_{\mathbb{H}}$ the hyperbolic metric on $\mathbb{D}$. Then the Kobayashi pseudometric d is defined to be the maximal pseudometric on M with $d(f(x), f(y)) \leq d_{\mathbb{H}}(x, y)$ for every holomorphic map $f : \mathbb{H} \to M$. This depends only on the complex structure of M and in this sense is intrinsic. Royden [46] showed that the Kobayashi metric is equal to the Teichmüller metric. This is a remarkable because the Kobayashi metric depends only on the complex structure of $\mathcal{T}(S_g)$, while the Teichmüller metric is defined explicitly in terms of the marked surfaces parametrized by $\mathcal{T}(S_g)$.

2.2.3 Infinitesimal Geometry of Teichmüller Spaces

In Chapter V, we will study the local structure of Teichmüller space to prove Theorem I.7. We briefly review the main elements at play and state the main results. See §6.5–6.6 of [34] for the full details and proofs, especially Proposition 6.6.2.

Beltrami forms play a very central role in the geometry of Teichmüller spaces. A detailed understanding of the map $\Phi : \mathrm{Bel}(X) \to \mathcal{T}(S)$ defined in Section 2.2.2, and in particular the derivative and its kernel, enables one to define a pairing between the tangent spaces to Teichmüller space and the spaces of quadratic differentials on the surfaces represented. Recall that if $\tau \in \mathcal{T}(S)$ is represented by $\varphi : S \to X$, then there is a pairing $\mathrm{Bel}(X) \times Q(X) \to \mathbb{C}$ defined by $(\mu, q) \mapsto \int_X \mu q$.

In order for this pairing to descend from $\mathrm{Bel}(X)$ to $\mathcal{T}(S)$ via Φ, we must have that $(\mu, q) = 0$ for all q whenever $\mu \in \ker(D\Phi)$; that is, deformations of the surface X which yield Teichmüller-equivalent surfaces should yield zero upon pairing with any quadratic differential. This is indeed the case, and there is an isomorphism $T_\tau \mathcal{T}(S) \to (Q(X))^\perp$.

It follows that the cotangent space $T^*_\tau \mathcal{T}(S)$ may be identified with $Q(X)$ (notably, by considering the *pre-dual* in order to work in the infinite-dimensional case). This is a central feature in the infinitesimal geometry of Teichmüller spaces. Because $\dim_\mathbb{C} Q(X) = 3g - 3 + n$ for $X \cong S_{g,n}$ a finite-type hyperbolic surface, it follows that $\dim_\mathbb{C} \mathcal{T}(S_{g,n}) = 3g - 3 + n$ as well. The cotangent spaces of $\mathcal{T}(S)$ also inherit the L^1-norm from $Q(X)$ as well; by duality a Finsler norm is induced on the tangent spaces $T_\tau \mathcal{T}(S)$, and it is a remarkable result (Theorem 6.6.5 of [34]) that this Finsler metric induces the Teichmüller metric (up to a choice of scaling).

With the geometry of the cotangent space in hand, we are in a position to recall next the Weil-Petersson metric [55]. See also [35] Chapter 7 or [34] §7.7. Let $[S, f] \in$

$\mathcal{T}_g$, and let $Q(S)$ be the vector space of holomorphic quadratic differentials on S, identified with the cotangent space of $\mathcal{T}(S_g)$. For $q_1, q_2 \in Q(S)$ define a Hermitian metric on $Q(S)$ by

$$\langle q_1, q_2 \rangle_{WP} = \int_S \bar{q}_1 q_2 (ds^2)^{-1},$$

where ds^2 is the hyperbolic metric on the Riemann surface. This induces an inner product on the tangent space $T_{[S,f]}\mathcal{T}(S_g)$ by taking the real part, known as the Weil-Petersson metric.

2.2.4 Royden's Theorem and Generalizations

Two of the most important results in Teichmüller theory are an understanding of the isometry group of Teichmüller space and the equivalence of the Kobayashi metric and the Teichmüller metric. In Royden's celebrated paper [46], both of these are established, thereby solidifying the connections between the complex geometry of Teichmüller space, the Teichmüller metric, and as we will see, the mapping class group. Proofs of both results in more recent language for the case of finite-type surfaces can be found in [34] §7.4. We will focus on the former result, but the latter is of independent interest. By the latter result, the isometry group of Teichmüller space with the Teichmüller metric is the same as the automorphism group of Teichmüller space as a complex manifold.

Royden's theorem states that the isometry group of $\mathcal{T}(S_g)$ is exactly the mapping class group Mod_g (with the action described in Section 2.2) for $g > 2$, and quotient $\mathrm{Mod}_g/(\mathbb{Z}/2\mathbb{Z})$ for $g = 2$ (where the quotient is generated by the hyperelliptic involution, or informally, "turning the surface over"). This is a type of rigidity result, where we consider the space of maps $\mathcal{T}(S_g) \to \mathcal{T}(S_g)$ with some condition (isometric, or equivalently, biholomorphic) and see what must result.

Royden's theorem has been generalized several times. One interesting generalization in Earle-Kra [19], shows that if there exists a biholomorphic map between arbitrary Teichmüller spaces, they must be the same Teichmüller space, and that the map must be induced by the action of an element of the mapping class group. Most importantly for us is the generalization by Markovic in [41] to all surfaces of *non-exceptional type*, that is, with $2g + n \geq 5$. Markovic developed new methods to handle the case of infinite-type surfaces. It turned out that a simplified version of these methods provided a new proof in the finite-type case, which is explained in [17].

One of our main results, Theorem I.7, is a generalization of Royden's theorem in a new direction: rigidity of non-biholomorphic maps between different Teichmüller spaces. We will study *holomorphic isometric submersions* and prove Theorem I.7 in Chapter V.

The proofs of Royden's theorem and its generalizations (including ours) hinge on the analysis of the infinitesimal geometry of the Teichmüller norm.

Let $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ be a holomorphic isometry. Then by taking the coderivative, F induces for each $X \in \mathcal{T}_{g,n}$ a bijective, $\mathbb{C}$-linear isometry of quadratic differential spaces $Q\,(F(X)) \to Q(X)$. The core step in the proof of Royden's theorem is showing that, up to scale by a constant $e^{i\theta}$, any such isometry is pullback by a biholomorphism $X \to F(X)$. We will continue this theme in Chapter V, where the proof of Theorem I.7 similarly begins by taking coderivatives and analyzing the rigidity of maps between spaces of quadratic differentials.

CHAPTER III

Symmetric Spaces

We briefly review some relevant classical ideas about symmetric spaces and compactifications. The main references are [32], [10], [9], [29], and [31]. Helgason's text [33] covers much more material on the relationship between symmetric spaces and Lie groups in much more depth.

For brevity, we do not go into many details on the motivations and proofs of what follows, which can be found in the references. Fortunately, the theory is very well-developed and the cases we need are very well-behaved, so we can quickly hone in on the tools we need. We will utilize them in Chapter IV to more deeply understand the symmetric spaces $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. In the following, fix $n \geq 2$ and let $G = \mathrm{SL}(n,\mathbb{R})$, $K = \mathrm{SO}(n)$, and $X = G/K$.

Proposition III.1. *There is a natural bijective correspondence between the quotient* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ *and the space* $\mathcal{P}_n$ *consisting of* $n \times n$ *real symmetric positive-definite matrices of determinant 1.*

Proof. Let $X \in \mathcal{P}_n$. $\mathrm{SL}(n,\mathbb{R})$ acts on $\mathcal{P}_n$ by $g \cdot X = gXg^T$, where g^T is the transpose. This is transitive with the stabilizer of the identity matrix precisely $\mathrm{SO}(n)$. Hence $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is identified with $\mathcal{P}_n$ as homogeneous spaces of $\mathrm{SL}(n,\mathbb{R})$ by the map $gK \mapsto gg^T$. □

A quick dimension count gives that $\dim \mathcal{P}_n = (n^2+n)/2 - 1$.

3.1 Lie Theory and Symmetric Spaces

Recall that the Lie algebra of a Lie group may be viewed as the tangent space to the identity. The Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ consisting of traceless matrices, which decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k}$ is the Lie algebra of K, consisting of traceless anti-symmetric matrices, and $\mathfrak{p}$ consists of traceless symmetric matrices. This is the *Cartan decomposition* of $\mathfrak{g}$, with respect to the involution of taking the negative of the matrix transpose. The subspace $\mathfrak{p}$ has the property that the Killing form $B(X,Y) = 2n\mathrm{Tr}(XY)$ is positive-definite on $\mathfrak{p}$.

Fix a *Cartan subalgebra* $\mathfrak{a} \subseteq \mathfrak{p}$ consisting of traceless diagonal matrices; this is a maximal abelian subalgebra. The dimension of $\mathfrak{a}$ is the *rank* of G and of X. Here, the rank is $r = n-1$. Utilizing the exponential map from the tangent space at the identity to the Lie group itself, which in the case of G and $\mathfrak{g}$ is simply the matrix exponential, denote $A = \exp(\mathfrak{a})$, the subgroup of G corresponding to the subalgebra $\mathfrak{a}$. This defines a totally geodesic submanifold which turns out to be isometric to $\mathbb{R}^r$. A totally geodesic copy of $\mathbb{R}^r$ embedded in the symmetric space X is called a *maximal flat* when r is the rank of the Lie group.

We next recall a few important examples of representations of G and $\mathfrak{g}$.

Example III.2. The *standard representation* of G is the inclusion

$$\Pi : \mathrm{SL}(n, \mathbb{R}) \hookrightarrow \mathrm{GL}(n, \mathbb{C}).$$

This is a faithful representation. The standard representation of $\mathfrak{g}$ is the inclusion

$$\pi : \mathfrak{sl}(n, \mathbb{R}) \hookrightarrow \mathrm{M}_n(\mathbb{C}).$$

Composing Π with the quotient map $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{PGL}(n, \mathbb{C})$ defines a *projective faithful* representation

$$\Pi_P : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{PGL}(n, \mathbb{C}).$$

Example III.3. The *adjoint representation* of the Lie algebra $\mathfrak{g}$ is defined by

$$\mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),\ \ A \mapsto [A, \cdot] \text{ for } A \in \mathfrak{g}$$

The adjoint representation expresses the action of the Lie algebra's bracket operation as a linear operator on $\mathfrak{g}$.

The *dual* of a representation Π of G is the representation Π^* defined by

$$\Pi^*(g) = \Pi(g^{-1})^T$$

where A^T is the transpose of A. The dual of a representation π of a Lie algebra is defined by

$$\pi^*(A) = -\pi(A)^T.$$

The direct sum of two representations $\tau_1 : G \to \mathrm{GL}(n, \mathbb{C})$ and $\tau_2 : G \to \mathrm{GL}(m, \mathbb{C})$ is the representation $\tau_1 \oplus \tau_2 : G \to \mathrm{GL}(n + m, \mathbb{C})$ with the diagonal action. A representation is said to be *irreducible* if there are no nontrivial invariant subspaces of the space on which the Lie algebra acts.

We next recall weights and roots associated to $\mathfrak{a}$. A natural inner product on $\mathfrak{a}$ is given by

$$\langle A, B \rangle = \mathrm{tr}(\overline{A}^T B)$$

where $\overline{A}$ is the complex conjugate matrix. This inner product identifies $\mathfrak{a}$ with the dual space $\mathfrak{a}^*$. Let π be a nonzero representation of $\mathfrak{g}$ acting on $\mathbb{R}^m$. We say $\mu \in \mathfrak{a}$ is a *weight* for π if there exists a nonzero $v \in \mathbb{R}^m$ such that

$$\pi(H) \cdot v = \langle \mu, H \rangle v \tag{3.1}$$

for all $H \in \mathfrak{a}$. In particular, a weight allows us to use elements of $\mathfrak{a}$ itself to express the behavior of the representation (restricted to $\mathfrak{a}$) as scalar multiplication. The weight space of μ, denoted V_μ, is the subspace of all $v \in \mathbb{R}^m$ for which Equation 3.1 holds. Each representation of a Lie group has an associated representation of its Lie algebra. The weights of a Lie group representation are defined to be the weights of the associated Lie algebra representation.

Example III.4. Let π be the standard representation for $\mathfrak{sl}(n, \mathbb{R})$. Then the weights are given by the standard basis e_i, so $\langle e_i, \cdot \rangle$ returns the ith diagonal element of a matrix, and the weight space for e_i is the line $\{\lambda e_i \ : \ \lambda \in \mathbb{R}\}$.

Let π^* be the dual of the standard representation. Then the weights are $-e_i$ with corresponding weight spaces generated by e_i after identifying $\mathbb{R}^n$ with its dual.

Let $\Pi_1 \oplus \Pi_2$ be a direct sum of two representations acting on $V \oplus W$, and let

$$\mathcal{W}_1 = \{\mu_i \ : \ i = 1, \ldots, n\} \text{ and } \mathcal{W}_2 = \{\nu_j \ : \ j = 1, \ldots, m\}$$

be the weights of Π_1 and Π_2 respectively, with corresponding weight spaces $V_i \subseteq V$ and $W_j \subseteq W$. Then the weights of $\Pi_1 \oplus \Pi_2$ are $\mathcal{W}_1 \cup \mathcal{W}_2$ with weight spaces $V_i \oplus \{0\}$ and $\{0\} \oplus W_j$ when $\mu_i \notin \mathcal{W}_2$ and $\nu_j \notin \mathcal{W}_1$. If some $\mu_i = \nu_j$, then its (common) weight space is $V_i \oplus W_j$.

The set of *roots* of $\mathfrak{g}$ relative to $\mathfrak{a}$, denoted Σ, are the weights of the adjoint representation. A set Δ of *simple roots* is a basis of $\mathfrak{a}$ made up of roots such that any root for $\mathfrak{a}$ can be expressed as an integer linear combination of elements of Δ where all coefficients are non-positive or non-negative.

Example III.5. A set of simple roots for $\mathfrak{sl}(n, \mathbb{R})$ with the Cartan subalgebra $\mathfrak{a}$ defined above is given by

$$\alpha_1 = (1, -1, 0, \ldots, 0),\ \alpha_2 = (0, 1, -1, 0, \ldots, 0), \ldots,\ \alpha_{n-1} = (0, \ldots, 0, 1, -1).$$

The root space for α_j is spanned by the matrix $E^{j,j+1}$ which has a 1 in the $(j, j+1)$ spot and 0 elsewhere.

Given a representation of $\mathfrak{g}$, a choice of simple roots endows the set of weights with a partial ordering (§8.8 in [32]). If $\{\alpha_1, \ldots, \alpha_n\}$ is the set of simple roots of $\mathfrak{g}$ and λ_1, λ_2 are weights of a representation, we say $\lambda_2 \succeq \lambda_1$ if there exist non-negative real numbers $c_1, \ldots, c_n$ such that

$$\lambda_2 - \lambda_1 = c_1\alpha_1 + \cdots + c_n\alpha_n.$$

It is a fundamental result (Theorems 9.4 and 9.5 in [32]) that irreducible, finite-dimensional representations of semisimple Lie algebras (including $\mathfrak{sl}(n, \mathbb{R})$) are classified by their highest weights (which always exist).

To each root α of $\mathfrak{g}$ is associated a hyperplane $P_\alpha = \ker(\langle \alpha, \cdot \rangle)$. The complement of these hyperplanes, $\mathfrak{a} \setminus \cup_{\alpha \in \Sigma} P_\alpha$, is a set of open polytopes, each connected component of which is called a *Weyl chamber*. A choice of a set of simple roots corresponds to distinguishing a *positive* Weyl chamber. The *Weyl group* W is the group of reflections across the hyperplanes P_α, and acts simply transitively on the set of Weyl chambers. In the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the Weyl group is the permutation group on n elements.

Now, we define a special type of Finsler metric built from Minkowski norms which plays a major role in the theory of compactifications of symmetric spaces.

Definition III.6. A *polyhedral Finsler metric* on a symmetric space is a Finsler metric such that for each tangent space, the induced unit ball is a polytope.

The following theorem of Planche [45] shows how polyhedral Finsler metrics relate to several fundamental structures in symmetric spaces. This result applies more broadly to real semisimple Lie groups with finite center, but we will only need it in the special case of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

Theorem III.7 ([45], Theorem 6.2.1)**.** *The following are in natural bijection:*

1. *the W-invariant convex closed balls in $\mathfrak{a}$*
2. *the $Ad(K)$-invariant convex closed balls of $\mathfrak{p}$*
3. *the G-invariant Finsler metrics on $X = G/K$*

The idea of this theorem is that, given a Finsler metric on a maximal flat F of G/K, if it is invariant under the Weyl group action, it can be extended to all of G/K by enforcing G-invariance. This defines a G-invariant Finsler metric.

3.2 Compactifications

Let X be a locally compact space. A *compactification* of X is a pair $(\overline{X}, i)$ where $\overline{X}$ is a compact space and $i : X \to \overline{X}$ is a dense topological embedding. If $(\overline{X}_1, i_1)$ and $(\overline{X}_2, i_2)$ are compactifications of X, we say they are *isomorphic* if there exists a homeomorphism $\phi : \overline{X}_1 \to \overline{X}_2$ such that $\phi \circ i_1 = i_2$. If ϕ is only continuous, then it is necessarily surjective, and $(\overline{X}_1, i_1)$ is said to *dominate* $(\overline{X}_2, i_2)$. Domination puts a partial order on the set of compactifications of a space.

In the case of symmetric spaces $X = G/K$, we are also interested in compactifications that admit a continuous G-action. The relations of G-*isomorphism* and G-*compactification* are extensions of the above definitions with the added condition of equivariance under the G action.

Horofunction compactifications are a special type of compactification defined for certain metric spaces. They were introduced by Gromov in the setting of geometric group theory [28] and has seen applications in various areas of mathematics. Walsh [54] has studied horofunction compactifications of Teichmüller spaces, to which we will return later.

Let (X, d) be a (possibly asymmetric) proper metric space with $C(X)$ the set of continuous real-valued functions on X endowed with the compact-open topology. Denote by $\tilde{C}(X)$ the quotient of $C(X)$ by constant functions (additively). We embed X into $\tilde{C}(X)$ as follows:

$$\psi : X \to \tilde{C}(X),\ z \mapsto [\psi_z] \text{ where } \psi_z(x) = d(x, z).$$

Definition III.8. The *horofunction compactification* $X \cup \partial_{hor} X$ of X is the topological closure of the image of ψ:

$$\overline{X}^{hor} := \mathrm{cl}\{[\psi_z] | z \in X\} \subseteq \tilde{C}(X)$$

Another type of compactification for nonpositively-curved Riemannian manifolds is known as the *visual compactification*. Briefly, this is obtained by affixing a visual boundary consisting of equivalence classes of geodesic rays, where two rays are equivalent if for some parametrization they remain within a bounded distance of each other. This definition can also be generalized to CAT(0) spaces. See §II.8 in [10] for details.

It is known that the horofunction compactification of a non-positively curved, complete, simply-connected Riemannian symmetric space G/K with its G-invariant metric is naturally isomorphic to its visual compactification. This holds more generally for CAT(0) spaces (Theorem 8.13, §II.8 in [10]).

Next, we briefly review Satake compactifications of symmetric spaces, first defined in [48]. See also Chapter IV of [29] and Chapter I.4 of [9], and [31] §5.1 for generalized Satake compactifications.

Let $X = G/K$ be a symmetric space associated to a semisimple Lie group G with maximal compact subgroup K. Let $\tau : G \to \mathrm{PSL}(m, \mathbb{C})$ be an irreducible projective faithful representation such that $\tau(K) \subseteq \mathrm{PSU}(m)$. This induces a map

$$\tau_X : X \to \mathbb{P}(\mathcal{H}_n)$$

where $\mathbb{P}(\mathcal{H}_n)$ is the projective space of Hermitian matrices, defined by

$$\tau_X(gK) = \tau(g)\overline{\tau(g)}^T.$$

This is a topological embedding (Lemma 4.36 in [29]).

Definition III.9. The *Satake compactification of* X *associated to* τ is the closure of $\tau_X(X)$ in $\mathbb{P}(\mathcal{H}_n)$ and is denoted by $\overline{X}^S_\tau$.

Two Satake compactifications are G-isomorphic if and only if the highest weights of their representations lie in the same Weyl chamber face, so there are only finitely many different G-isomorphism types (Chapter IV, [29]).

Definition III.10. The *maximal Satake compactification* of a symmetric space is a Satake compactification whose highest weight lies in the interior of the positive Weyl chamber. A *minimal Satake compactification* of a symmetric space is a Satake compactification whose highest weight lies in an edge of the Weyl chamber.

It is known that there is a unique (up to G-isomorphism) maximal Satake compactification which dominates all other Satake compactifications, and many minimal Satake compactifications. For $\mathrm{SL}(n, \mathbb{R})$, it is known that the standard representation induces a minimal Satake compactification [9, Proposition I.4.35].

We will also need *generalized Satake compactifications*, the definition of which differs only in that the assumption that τ is irreducible is dropped.

In [31], Haettel, Schilling, Walsh, and Wienhard related generalized Satake compactifications of a symmetric space to horofunction compactifications of polyhedral Finsler metrics.

Theorem III.11 ([31] Theorem 5.5)**.** *Let $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$ be a projective faithful representation, and $X = G/K$ be the associated symmetric space, where X is of non-compact type. Let $\mu_1, \ldots, \mu_k$ be the weights of τ. Let d be the polyhedral Finsler metric whose unit ball in a Cartan subalgebra is*

$$B = -D^\circ = -conv(\mu_1, \ldots, \mu_k)$$

where conv is the convex hull. Then $\overline{X}^S_\tau$ is G-isomorphic to $\overline{X}^{hor}_d$.

Example III.12. The horofunction compactification of $X = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ with respect to the standard $\mathrm{SL}(n, \mathbb{R})$-invariant Riemannian metric is not isomorphic to a generalized Satake compactification because the unit ball in a flat is a Euclidean ball, which is not the convex hull of finitely many points.

Finally, we recall the following very special case of a result of L. Ji [36, Theorem 2.4]. This topological result will allow us to compare the topology of compactified Teichmüller spaces of flat n-tori with that of compatified Teichmüller spaces of hyperbolic surfaces.

Proposition III.13. *Every Satake compactification $\overline{X}^S_\tau$ of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is homeomorphic to a closed topological ball.*

CHAPTER IV

Teichmüller Spaces of Flat n-Tori

4.1 The Teichmüller Spaces of Flat n-Tori

In this chapter, we describe and prove the results of the author and Lizhen Ji [27]. We will start by introducing the Teichmüller spaces of unit volume flat n-tori, denoted $\mathcal{T}(n)$, where $n \geq 2$. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the square torus of dimension n.

Definition IV.1. The Teichmüller space $\mathcal{T}(n)$ is defined as the set of equivalence classes of marked flat tori of dimension n and unit volume:

$$\mathcal{T}(n) = \{[S, f] : S \text{ a flat } n\text{-torus of volume } 1,\ f : S \to \mathbb{T}^n \text{ orientation-preserving homeo}\}/\sim$$

where $[S, f] \sim [S', f']$ if and only if there exists an isometry $h : S \to S'$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} S \arrow[d, "h"] \arrow[dr, "f"] & \\ S' \arrow[r, "f'"'] & \mathbb{T}^n \end{tikzcd}$$

As mentioned in Chapter II, this convention where the arrows in the marking are reversed defines the same Teichmüller space, but for our present purposes several aspects of the notation simplify considerably. We now recall a few classical facts.

Proposition IV.2. *There is a natural bijective correspondence:* $\mathcal{T}(n) \leftrightarrow \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

Proof. We use methods similar to §10.2 of [20]. Given a marked unit volume torus $f : S \to \mathbb{T}^n$, write $S = \mathbb{R}^n/\Lambda$ for a lattice Λ of unit covolume. Lift the map f to $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$, and let $\zeta_i = \tilde{f}^{-1}(e_i)$ for $i = 1, \ldots, n$, where the e_i are the standard basis vectors of $\mathbb{R}^n$. These form an ordered generating set (i.e. a marking) for the lattice Λ, the coordinates of which form the columns of a matrix in $\mathrm{SL}(n, \mathbb{R})$. The original choice of Λ was unique up to the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$, and so this specifies an element of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. Homotopic markings give the same lattice by Lemma IV.5 below.

Conversely, given a matrix in $\mathrm{SL}(n, \mathbb{R})$, the columns form an ordered generating set for a unit covolume lattice Λ. Now, there exists a linear map $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ which sends the ordered generating set for Λ to the standard basis of $\mathbb{R}^n$. This map descends to a map $\phi : \mathbb{R}^n/\Lambda \to \mathbb{T}^n$ which defines a marked flat torus. Two matrices will give the same marked flat torus if and only if they represent the same coset in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. ∎

Remark IV.3. The symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is a complete, simply-connected manifold of non-positive curvature, and hence is diffeomorphic to the Euclidean space of the same dimension $\mathbb{R}^{(n^2+n)/2-1}$ by the Cartan-Hadamard theorem. The Teichmüller spaces of hyperbolic surfaces are also diffeomorphic to Euclidean spaces (see e.g. [1] §3.2).

Corollary IV.4. *There is a natural bijective correspondence*

$$\mathcal{T}(2) \leftrightarrow \mathbb{H}^2.$$

Proof. We need only the identification $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) \leftrightarrow \mathbb{H}^2$, which follows from the fact that $\mathrm{SL}(2, \mathbb{R})$ acts transitively on $\mathbb{H}^2$ by fractional linear transformations with point stabilizers isomorphic to $\mathrm{SO}(2)$. ∎

The following properties will enable us to more easily translate between elements of $\mathcal{P}_n$ and marked flat n-tori. See [39, Lemma V.6.2, Theorem IV.3.5] for the dimension 2 case of Lemma IV.5, whose proofs generalize immediately.

Lemma IV.5. *1. The group of isometries of a flat n-torus acts transitively.*

2. If two homeomorphisms $\varphi_i : S \to S'$, $i = 0, 1$, between flat n-tori are homotopic, then they induce the same isomorphism of deck transformation groups acting on $\mathbb{R}^n$.

Henceforth we will interchangeably refer to points of $\mathcal{T}(n)$ as either marked flat n-tori, coset (representatives) $gK \in \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, or as elements of $\mathcal{P}_n$. Next, we consider the metric perspective on $\mathcal{T}(n)$.

While the columns of a matrix representative of a point gK determine a marked lattice Λ which descends to a marked flat torus $\mathbb{R}^n/\Lambda$, the corresponding point $gg^T \in \mathcal{P}_n$ also has a concrete interpretation in the language of flat tori. The matrix gg^T is an explicit realization of the metric tensor for $\mathbb{R}^n/\Lambda$. To see this, use Euclidean coordinates on the standard torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The inner product between two vectors $v_1, v_2 \in \mathbb{R}^n \cong T_pX$ for any $p \in \mathbb{R}^n/\Lambda$ is given by:

$$\langle v_1, v_2 \rangle_p = \langle v_1 g, v_2 g \rangle_{\mathbb{R}^n} = \langle v_1 gg^T, v_2 \rangle_{\mathbb{R}^n}.$$

This defines a Riemannian metric on the standard torus $\mathbb{R}^n/\mathbb{Z}^n$ which is isometric to $\mathbb{R}^n/\Lambda$. If $\gamma : [0, 1] \to \mathbb{R}^n/\mathbb{Z}^n$ is a smooth closed curve and $X \in \mathcal{P}_n$ is the metric tensor, then the length $\ell_X(\gamma)$ is computed as follows:

$$\ell_X(\gamma) = \int_0^1 \sqrt{\langle \gamma'(t)X, \gamma'(t) \rangle} dt$$

This formula behaves nicely with the action $g \cdot \gamma = \gamma g$ for $g \in \mathrm{SL}(n, \mathbb{R})$:

$$\ell_X(g \cdot \gamma) = \int_0^1 \sqrt{\langle (\gamma'(t)g)X, \gamma'(t)g \rangle} dt = \int_0^1 \sqrt{\langle \gamma'(t)(gXg^T), \gamma'(t) \rangle} dt = \ell_{g \cdot X}(\gamma).$$

4.2 Extremal Lipschitz Maps Between Tori

Let $[S, f], [S', f'] \in \mathcal{T}(n)$, with $S = \mathbb{R}^n/\Lambda$ and $S' = \mathbb{R}^n/\Lambda'$. Our main result in this section is the following:

Proposition IV.6. *The map* $\psi : S \to S'$ *which lifts to the unique affine map* $\tilde{\psi} : \mathbb{R}^n \to \mathbb{R}^n$ *realizes the minimal Lipschitz constant in* $[f'^{-1} \circ f]$.

Proof. Let $S = \mathbb{R}^n/\Lambda$ and $S' = \mathbb{R}^n/\Lambda'$ be tori of volume 1 with markings f and f'. Because affine self-maps on flat tori are isometric and transitive we may assume lifts of maps $\varphi : S \to S'$ to $\mathbb{R}^n$ have the property that $\tilde{\varphi}(0) = 0$. Let $\mathcal{F}$ denote the class of all such lifts whose quotients are homotopic to $f'^{-1} \circ f$. For $g \in \mathcal{F}$, let $\bar{g}$ denote the induced map $S \to S'$.

Let q and q' be the quotient maps for S and S', respectively. Then for all $g \in \mathcal{F}$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{g} & \mathbb{R}^n \\ \downarrow q & & \downarrow q' \\ S & \xrightarrow{\bar{g}} & S' \end{array}$$

Let $\{\omega_1, \dots, \omega_n\}$ be a basis of Λ. For any $g_1, g_2 \in \mathcal{F}$, it follows that $g_1(\omega_i) = g_2(\omega_i) + \lambda_i$ for some $\lambda_i \in \Lambda$ for each of $i = 1, \dots, n$. By Lemma IV.5, it follows that $\lambda_i = 0$ for $i = 1, \dots, n$ since g_1 and g_2 are homotopic. One then obtains a basis $\{\zeta_1, \dots, \zeta_n\}$ of Λ' such that $\mathcal{F}$ is the class of homeomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$ with

$$g(0) = 0, \; g(x + \sum_i^n m_i\omega_i) = g(x) + \sum_i^n m_i\zeta_i \tag{4.1}$$

for all $x \in \mathbb{R}^n$. Notice that any homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ satisfying Equation 4.1 descends to a map $S \to S'$ homotopic to $f'^{-1} \circ f$. The condition of being affine uniquely determines such a map inside a fundamental domain of Λ, and hence on all of $\mathbb{R}^n$. This proves uniqueness of the affine map; let $w \in \mathcal{F}$ be the affine map.

Now we show w has the least Lipschitz constant. Let $g \in \mathcal{F}$ be a K-Lipschitz map, i.e.

$$K \geq \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|}. \tag{4.2}$$

Define $g_k(x) = g(kx)/k$ for $k = 1, 2, \ldots$. These maps are all K-Lipschitz and satisfy Equation 4.1, so $g_k \in \mathcal{F}$ for all k. By Lemma IV.7 below, $g_k \xrightarrow{k \to \infty} w$ uniformly on $\mathbb{R}^n$. It is a standard fact from real analysis that the pointwise limit of a sequence of K-Lipschitz functions is also K-Lipschitz. Hence w is K-Lipschitz. In other words, $K \geq \mathcal{L}(w)$. Because this holds for any Lipschitz map $g \in \mathcal{F}$, it follows that w has minimal Lipschitz constant. $\square$

Lemma IV.7. *In the proof of Proposition IV.6, the sequence $g_k \to w$ uniformly.*

Proof. Pick $\epsilon > 0$ and let $x_0 \in \mathbb{R}^n$. Since $\omega_1, \ldots, \omega_n$ are linearly independent, x_0 may be written as

$$x_0 = \sum_{i=1}^{n} r_i \omega_i$$

for some $r_i \in \mathbb{R}$, $i = 1, \ldots, n$. Let

$$M = \sup_{(a_1, \ldots, a_n) \in [0,1]^n} \left| g(\sum_{i=1}^{n} a_i \omega_i) \right| + \sum_{i}^{n} |\zeta_i|.$$

This is finite since g is continuous and this domain is compact. Then for any integer $k > M/\epsilon$, we have:

$$\big| g_k(x_0) - w(x_0) \big| = \frac{1}{k} \big| g(k \sum_{i=1}^{n} r_i \omega_i) - (k \sum_{i=1}^{n} r_i \zeta_i) \big| \tag{4.3}$$

since w is affine. Write $kr_i = m_i + t_i$, where $t_i \in [0, 1)$ and $m_i \in \mathbb{Z}$, for $i = 1, \ldots, n$. In Equation 4.3, the integer part m_i of each term kr_i factors through g. We then compute:

$$\big| g_k(x_0) - w(x_0) \big| = \frac{1}{k} | g(\sum_{i=1}^{n} t_i \omega_i) - \sum_{i=1}^{n} t_i \zeta_i | \leq \frac{1}{k} M < \epsilon.$$

$\square$

It is also known that the extremal quasiconformal map for the Teichmüller distance is unique (see [39], Theorem 6.3). Interestingly, in the case of $\mathcal{T}(n)$, there are many extremal Lipschitz maps, at least in some cases.

Proposition IV.8. *There exists a pair of marked flat 2-tori with an infinite family of distinct homeomorphisms respecting the markings, all of which realize the extremal Lipschitz constant.*

Proof. Let S be the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ and T be the rectangle $[0,r] \times [0,1/r]$. These regions S and T represent fundamental domains for two flat tori. An extremal Lipschitz map is given by $(x,y) \mapsto (rx, y/r)$ with Lipschitz constant r. Fix $r > 1$. Choose $\epsilon \in (-1/2, 1/2)$ and δ such that

$$\max\{0, \frac{1}{r} - \frac{r}{2} + \epsilon r\} < \delta < \min\{\frac{1}{r}, \frac{r}{2} + \epsilon r\}.$$

Define the map $F : S \to T$ by:

$$F(x,y) = \begin{cases} \left(rx, \frac{1/r-\delta}{1/2-\epsilon} y\right) & y \leq 1/2 - \epsilon \\ \left(rx, \left(\frac{1}{r} - \delta\right) + \frac{y-(1/2-\epsilon)}{1/2+\epsilon}\delta\right. & y \geq 1/2 - \epsilon \end{cases}$$

See the figure for an explanation of these values.

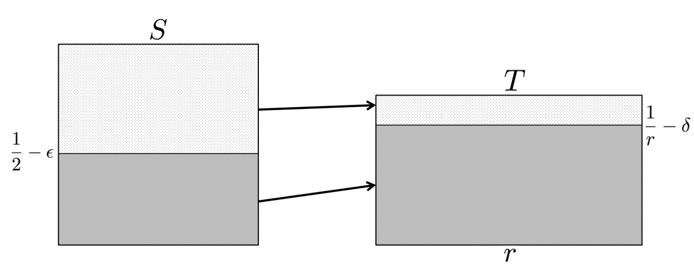

Figure 4.1: The map F sends the two portions of the square linearly to the two similarly-shaded portions of the rectangle.

This map is linear in the x-direction (the direction of maximum stretch), but only piecewise linear in the y-direction. The affine map occurs at $\epsilon = 0$ and $\delta =$

$(2r)^{-1}$. This map projects onto a homeomorphism of the corresponding tori since it respects the boundaries. The map F is differentiable almost everywhere, and the total derivatives on the top and bottom halves of the domain are respectively given by:

$$D_{\text{bottom}} = \begin{pmatrix} r & 0 \\ 0 & \frac{1/r-\delta}{1/2-\epsilon} \end{pmatrix},\ D_{\text{top}} = \begin{pmatrix} r & 0 \\ 0 & \frac{\delta}{1/2+\epsilon} \end{pmatrix}$$

With the above constraints on ϵ and δ, one can see from D_{top} and D_{bottom} that the Lipschitz constant for F is r, as desired. $\square$

In contrast to the case of the affine map, the inverses of the maps constructed in Proposition IV.8 are not Lipschitz-extremal. The above construction generalizes easily to the case of higher dimensions.

Corollary IV.9. *There exists a pair of flat tori in any dimension $n \geq 2$ with infinitely many homotopic homeomorphisms respecting the markings which all realize the extremal Lipschitz constant.*

Proof. Let S and T be the two marked flat 2-tori from Proposition IV.8 and let $S' = S \times (S^1)^{n-2}$ and $T' = T \times (S^1)^{n-2}$ with the product metrics, where each new copy of S^1 is isometric to a unit circle. An infinite extremal family is given by using the family from Proposition IV.8 on the S and T components, and the identity on the remaining components. $\square$

Remark IV.10. It is straightforward to generalize the above construction for any two rectangular tori, but it is unclear whether all pairs of tori admit many distinct Lipschitz-extremal maps, and if not, under what conditions they are unique.

4.3 Thurston's Metric for n-Dimensional Flat Tori

Definition IV.11. *Thurston's metric* d_{Th} on $\mathcal{T}(n)$ is defined as follows:

$$d_{Th}([S,f],[S',f']) = \frac{1}{2}\log \inf_{\phi\in[f'^{-1}\circ f]} \mathcal{L}(\phi)$$

$$\mathcal{L}(\phi) = \sup_{x,y\in S,\ x\neq y} \frac{d_{S'}(\phi(x),\phi(y))}{d_S(x,y)}$$

where the infimum is over all Lipschitz homeomorphisms homotopic to $f'^{-1}\circ f$.

This is identical to the definition for hyperbolic surfaces. Proposition 2.1 in [53] gives a geometric proof that the Thurston metric is positive-definite for $\mathcal{T}(S_g)$, which works similarly for our case.

Proposition IV.12. *For all points $[S,f],[S'f']\in\mathcal{T}(n)$, we have*

$$d_{Th}([S,f],[S',f'])\geq 0,$$

with equality only if $[S,f]=[S',f']$.

Proof. Suppose we have $[S,f],[S',f']$ such that $d_{Th}([S,f],[S',f'])\leq 0$. Then by compactness there exists a homeomorphism $\phi: S\to S'$ in the appropriate homotopy class with realizing the extremal Lipschitz constant $L\leq 1$.

Under ϕ every sufficiently small ball of radius r in the domain space is mapped to a subset of a ball of radius $\leq r$ in the target. However, both tori have unit volume. If we cover the domain space by a disjoint union of balls of full measure, one sees that each ball must map surjectively onto a ball of the same size. This procedure works for arbitrarily small balls, and so ϕ is an isometry. $\square$

Because composing Lipschitz maps with constants L_1 and L_2 gives a Lipschitz map with constant at most L_1L_2, the triangle inequality for d_{Th} follows. Together with Proposition IV.12, we have that d_{Th} is a (possibly asymmetric) metric. We will need a quick classical fact before we can state a formula for d_{Th}.

Lemma IV.13. *The Lipschitz constant of a linear map $M : \mathbb{R}^n \to \mathbb{R}^n$ is given by*

$$\max\{\sqrt{|\lambda|} \ : \ \lambda \textit{ is an eigenvalue of } M^T M\}.$$

Proof. First, recall $\mathcal{L}(M) = ||M||_{op}$, the operator norm of M:

$$\mathcal{L}(M) = \sup_{x \neq y} \frac{||Mx - My||}{||x - y||} = \sup_{x \neq 0} \frac{||Mx||}{||x||} = ||M||_{op}.$$

Since the operator norm of a diagonalizable matrix is the absolute value of the largest eigenvalue, using $||M^T M||_{op} = ||M||_{op}^2$ the result follows. □

Next, we will derive a formula for easy computation using the structure of the symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

Theorem IV.14. *Let Y, X be positive-definite symmetric matrices corresponding to points of $\mathcal{T}(n)$. Thurston's metric d_{Th} on $\mathcal{T}(n) = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is given by the following formula:*

$$d_{Th}(Y, X) = \frac{1}{2} \max\{\log|\lambda| \ : \ \lambda \textit{ is an eigenvalue of } XY^{-1}\} \tag{4.4}$$

Proof. Let $h\,\mathrm{SO}(n)$ and $g\,\mathrm{SO}(n)$ be points in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ corresponding to Y and X. The linear map between them is given by gh^{-1}, which by Proposition IV.6 is an extremal Lipschitz map. By Lemma IV.13, the Lipschitz constant is given by

$$\lambda_0 := \max\{|\lambda| \ : \ \lambda \text{ is an eigenvalue of } (h^{-1})^T g^T g h^{-1}\}$$

Because

$$XY^{-1} = g^T g h^{-1} (h^T)^{-1} \sim (h^{-1})^T g^T g h^{-1}$$

are similar matrices, they have the same eigenvalues, and the result follows. □

Note that if $Y = I$, the absolute values in Equation 4.4 are redundant since X is positive-definite.

Corollary IV.15. *The Thurston metric is* $\mathrm{SL}(n,\mathbb{R})$ *invariant for the action on* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n) \cong \mathcal{T}(n)$.

Proof. This is immediate from the formula and the definition of the action $g \cdot X = gXg^T$. □

Corollary IV.16. *The Thurston metric on* $\mathcal{T}(2)$ *is equal to the Riemannian symmetric metric on* $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$*, and hence matches the Teichmüller metric and hyperbolic metric up to scaling.*

Proof. The distance formula for the Riemannian symmetric metric on

$$\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n) \cong \mathcal{P}_n$$

is given by (see e.g. [52], Theorem 1.1.1):

$$d(Y,X) = \sqrt{\sum_i (\log \lambda_i)^2}$$

where the sum is over the eigenvalues of YX^{-1}. In the case of 2×2 positive-definite matrices of determinant one, there are precisely two eigenvalues whose product is 1. Write the eigenvalue with absolute value at least 1 as λ. Then the formula becomes:

$$d(Y,X) = \sqrt{(\log \lambda)^2 + (\log 1/\lambda)^2} = \sqrt{2\log(\lambda)^2} = \sqrt{2}|\log \lambda|.$$

But λ is also the maximum eigenvalue of YX^{-1}, and XY^{-1} has the same eigenvalues, so up to a choice of scaling, these are the same metrics. □

Remark IV.17. A proof of Corollary IV.16 is obtained in the unpublished work [26] by the author and L. Ji using an explicit computation of the Lipschitz distortion in a realization of the fundamental domains as parallelograms in $\mathbb{C}$.

Remark IV.18. Another proof of Corollary IV.16 is possible using work of Belkhirat-Papadopoulos-Troyanov [6], where the Thurston metric is defined on $\mathcal{T}(2)$, but $\mathcal{T}(2)$ is defined using a different normalization. A fixed curve is set to length 1 via the marking, as opposed to here, where we choose volume 1. Using the usual identification of $\mathcal{T}(2) \leftrightarrow \mathbb{H}^2$, it is shown that the resulting Thurston metric, denoted here by $\hat{\kappa}$, can be computed by the following formula ([6], Theorem 3):

$$\hat{\kappa}(\zeta, \zeta') = \log \sup_{\alpha \in \mathcal{S}} \left(\frac{\ell_{\zeta'}(\alpha)/\ell_{\zeta'}(\epsilon)}{\ell_{\zeta}(\alpha)/\ell_{\zeta}(\epsilon)} \right) = \log \left(\frac{|\zeta' - \overline{\zeta}| + |\zeta' - \zeta|}{|\zeta - \overline{\zeta}|} \right)$$

where the supremum is over homotopy classes of closed curves, $\ell_\zeta(\alpha)$ is the length of α in the metric associated to $\zeta \in \mathbb{H}^2$, and ϵ is the normalizing curve. In order to recover our d_{Th}, we normalize using $\sqrt{\mathrm{Im}\zeta}$, the volume. Using the identification $\mathcal{T}(2) \leftrightarrow \mathbb{H}^2$ for d_{Th}, we obtain:

$$d_{Th}(\zeta, \zeta') = \hat{\kappa}(\zeta, \zeta') + \log \left(\frac{\sqrt{\mathrm{Im}\zeta}}{\sqrt{\mathrm{Im}\zeta'}} \right) = \log \left(\frac{|\zeta' - \overline{\zeta}| + |\zeta' - \zeta|}{\sqrt{|\zeta - \overline{\zeta}||\zeta' - \overline{\zeta'}|}} \right)$$

$$= \frac{1}{2} \log \left(\frac{|\zeta' - \overline{\zeta}| + |\zeta' - \zeta|}{|\zeta' - \overline{\zeta}| - |\zeta' - \zeta|} \right)$$

where the last equality follows from Lemma 2 (an identity for complex numbers) from [6]. This is exactly the Poincaré metric.

Next, as in [53], we define another asymmetric metric, κ, on $\mathcal{T}(n)$. Let $\mathcal{S}(\mathbb{T}^n)$ denote the set of homotopy classes of essential closed curves on the n-torus. For $\alpha \in \mathcal{S}(\mathbb{T}^n)$ and h a metric on $\mathbb{T}^n$, denote by $\ell_h(\alpha)$ the shortest length of any curve in the homotopy class α. For the flat torus, while the curve realizing this length is not unique, the shortest length is well-defined and positive. As above, let $[S, f], [S', f'] \in \mathcal{T}(n)$ with h and h' the corresponding unit-volume flat metrics on $\mathbb{T}^n$. Now, κ is defined as:

$$\kappa([S, f], [S', f']) = \log \sup_{\alpha \in \mathcal{S}(T^n)} \left(\frac{l_{h'}(\alpha)}{l_h(\alpha)} \right) \tag{4.5}$$

That is, κ is a measure of the maximum stretch along a geodesic. As in [53], we show:

Proposition IV.19. *The two metrics κ and d_{Th} are equal on $\mathcal{T}(n)$.*

Proof. It is immediate that

$$\kappa([S,f],[S',f']) \leq d_{Th}([S,f],[S',f'])$$

for all $[S,f],[S',f'] \in \mathcal{T}(n)$, since the latter involves a supremum over all geodesic segments rather than only closed geodesics. For the opposite inequality, we will utilize a geometric argument. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be the (lift of the) affine marking-preserving map between S and S'.

There exists a line L containing the origin along which the maximal stretch of φ is realized. If there are two lattice points on L, then the segment connecting them descends to a geodesic whose length is stretched by the Lipschitz constant, yielding $\kappa \geq d_{Th}$, and we are done.

Suppose now 0 is the only lattice point on L. One can find a sequence of lattice points $p_n \in \Lambda$, $n = 1, 2, \ldots$ which approach L. By continuity, under φ the corresponding sequence of closed geodesics will have stretch factors approaching the Lipschitz constant of the map φ. After taking the supremum of the stretches, we conclude $\kappa \geq d_{Th}$, as required. $\square$

4.3.1 The Finsler Structure of the Thurston Metric

Finsler metrics are important in classical Teichmüller theory since both the Teichmüller metric and Thurston metric are Finsler but not Riemannian. Here, we will give a formula for the Finsler metric on $\mathcal{T}(n)$ associated to the Thurston metric d_{Th}.

Definition IV.20. A *Finsler metric* on a manifold M is a continuous function

$$F : TM \to [0, \infty)$$

on the tangent bundle such that for each $p \in M$, the restriction $F|_{T_pM} : T_pM \to [0, \infty)$ is a norm (i.e. positive-definite, subadditive, linear under scaling by *positive* scalars).

Our formula for the Finsler metric for d_{Th} is very similar to the Finsler metric discussed in [40] Theorem 3 (see also Section 4.5 of this paper). Recall first that the tangent space of $\mathcal{T}(n) = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ at the identity is identified with the space of traceless symmetric matrices. One obtains any other tangent space by left translation via elements of $\mathrm{SL}(n, \mathbb{R})$.

Proposition IV.21. *The Finsler structure on the tangent space at $Z \in \mathcal{T}(n)$ for the Thurston metric d_{Th} is given by*

$$|X|_{Th(Z)} = \frac{1}{2} \max\{\lambda : \lambda \text{ is an eigenvalue of } XZ^{-1}\}$$

where $X \in T_Z\mathcal{T}(n) \cong \mathfrak{sl}(n, \mathbb{R})$.

Proof. It suffices to show the case of $Z = I$. First, note that this is always non-negative since the trace of X is zero. Let $\gamma : [0, 1] \to \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ be a smooth path from I to A. Since A is symmetric, its operator norm coincides with the maximum eigenvalue, and so

$$d_{Th}(I, A) = \frac{1}{2} \sup_{0 \neq v \in \mathbb{R}^n} \log \frac{\langle \gamma(1)v, v\rangle}{\langle v, v\rangle}$$

comes from the maximum eigenvalue. We then compute:

$$\begin{aligned} d_{Th}(I, A) &= \frac{1}{2} \sup_{0 \neq v \in \mathbb{R}^n} \int_0^1 \frac{d}{dt} \log\langle \gamma(t)v, v\rangle dt = \frac{1}{2} \sup_{0 \neq v \in \mathbb{R}^n} \int_0^1 \frac{\langle \gamma'(t)v, v\rangle}{\langle \gamma(t)v, v\rangle} dt \\ &\leq \frac{1}{2} \int_0^1 \sup_{0 \neq v \in \mathbb{R}^n} \frac{\langle \gamma'(t)v, v\rangle}{\langle \gamma(t)v, v\rangle} dt = \frac{1}{2} \int_0^1 |\gamma'(t)|_{Th(\gamma(t))} dt \end{aligned}$$

where the final equality follows because the supremum on the left-hand side yields the operator norm, which matches the Finsler norm inside the integral on the right-hand side. This is the Finsler length of γ. Thus d_{Th} is bounded above by the Finsler distance of any path.

Next, choose X such that $e^X = A$, which exists because $A \in \mathcal{P}_n$. The Finsler length of the path $\gamma(t) = e^{tX}$ for $t \in [0,1]$ is computed as follows:

$$\ell(\gamma) = \frac{1}{2}\int_0^1 \sup_{0\neq v\in\mathbb{R}^n} \frac{\langle Xe^{tX}v, v\rangle}{\langle e^{tX}v, v\rangle} dt = \frac{1}{2}\int_0^1 \sup_{0\neq v\in\mathbb{R}^n} \frac{d}{dt}\log\langle e^{tX}v, v\rangle dt$$
$$= \frac{1}{2}\sup_{0\neq v\in\mathbb{R}^n} \frac{\langle Av, v\rangle}{\langle v, v\rangle} = d_{Th}(I, A)$$

Thus the Thurston distance is realized by the Finsler length of a path, as desired. □

Corollary IV.22. *For $U, V \in \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, if $e^X = UV^{-1}$, the path given by $t \mapsto e^{tX}V$ for $t \in [0,1]$ is a geodesic path from V to U with respect to d_{Th}.*

4.4 Teichmüller Metric for Higher-Dimensional Tori

Here, we utilize the definition of quasiconformal maps in higher dimensions from [24] to define the Teichmüller metric on $\mathcal{T}(n)$ for $n \geq 2$ and explore its properties. The Teichmüller metric on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ has been studied for $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ as a Finsler metric on the space of conformal structures on vector spaces; see [42] Appendix A.1. Here, we review this metric in the context of quasiconformal maps between n-tori and compare it to our other metrics on $\mathcal{T}(n)$.

4.4.1 Definitions and Useful Facts on Quasiconformal Maps

We will first state as concisely as possible the definition of K-quasiconformal maps between domains D and D' in $\mathbb{R}^n$ under the assumption that they are also diffeomorphisms, from Chapter 4 of [24].

For a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, define the following:

$$L(T) = \max_{|x|=1} |T(x)|, \quad \ell(T) = \min_{|x|=1} |T(x)|.$$

These are the maximal and minimal stretching of T, respectively.

Definition IV.23. Let $f : D \to D'$ be a diffeomorphism of domains in $\mathbb{R}^n$. Define the *inner*, *outer*, and *maximal dilatations* respectively as follows:

$$K_I(f) = \sup_{x \in D} \frac{|J_f(x)|}{\ell(f'(x))^n}$$

$$K_O(f) = \sup_{x \in D} \frac{L(f'(x))^n}{|J_f(x)|}$$

$$K(f) = \max(K_I(f), K_O(f))$$

where $f'(x)$ is the total derivative of f at $x \in D$ and J_f is the Jacobian. The map f is said to be K-quasiconformal if $K(f) \leq K < \infty$.

The above definition is local, so it applies immediately to flat tori by lifting any map to its universal cover.

Next, we list a few basic properties of quasiconformal maps which will be essential to the definition of the Teichmüller metric. They are direct analogs of the 2-dimensional case (compare to Lemma II.2). These come from Lemma 6.1.1 and Theorem 6.8.4 of [24]:

Proposition IV.24. *Let $f : D \to D'$ and $g : D' \to D''$ be quasiconformal homeomorphisms of domains in $\mathbb{R}^n$. Then the following hold:*

1. $K(g \circ f) \leq K(g)K(f)$
2. $K(f) \geq 1$ *with equality if and only if f is a Möbius transformation*
3. $K(f^{-1}) = K(f)$

We will need one more property of quasiconformal maps in order to prove that the extremal quasiconformal constant is realized by the affine map. This is a very special case of Theorem 6.6.18 in [24].

Proposition IV.25. *Let $(f_k)_{k\in\mathbb{N}} : \mathbb{R}^n \to \mathbb{R}^n$ be a sequence of K-quasiconformal homeomorphisms. Suppose $f_k \to f$ locally uniformly. Then $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal homeomorphism as well.*

We now prove the quasiconformal analog of Proposition IV.6.

Proposition IV.26. *The extremal quasiconformal constant for a homeomorphism between two flat n-tori in a specified homotopy class is given by the unique affine map.*

Proof. Recall the proof of Proposition IV.6; in particular, recall the collection $\mathcal{F}$ of homeomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(0) = 0,\ g(x + \sum_i^n m_i\omega_i) = g(x) + \sum_i^n m_i\zeta_i$$

for all $x \in \mathbb{R}^n$. This is precisely the collection of lifts of marking-preserving homeomorphisms. Let $g \in \mathcal{F}$ be K-quasiconformal, and define $g_k(x) = g(kx)/k$ for $k = 1, 2, \ldots$. The maps g_k are also K-quasiconformal since they are built from g by pre- and post-composition with dilations. Further $g_k \in \mathcal{F}$, and the sequence of maps uniformly converges to the affine map (by Lemma IV.7). By Proposition IV.25, the affine map has dilatation at most K. This holds for all $g \in \mathcal{F}$, so the result follows. □

We are now ready to define the Teichmüller metric. The scaling factor of $1/2n$ in the definition is a choice similar to a factor of $1/2$ which appears in some definitions of the Teichmüller metric for hyperbolic surfaces, and enables several analogous properties to work out more nicely.

Definition IV.27. Let $[S, f], [S', f'] \in \mathcal{T}(n)$. The *Teichmüller metric* on $\mathcal{T}(n)$ is defined as:

$$d_{Teich}([S,f],[S',f']) = \frac{1}{2n} \log \inf_{g \in [f'^{-1} \circ f]} K(g)$$

where the infimum is taken over quasiconformal maps homotopic to $f'^{-1} \circ f$.

Proposition IV.28. *The function d_{Teich} above is a metric.*

Proof. Proposition IV.24 (1) and (3) give symmetry and the triangle inequality, and (2) shows $d_{Teich} \geq 0$. Now suppose $d_{Teich}([S,f],[S',f']) = 0$. Then there exists a 1-quasiconformal map $g : S \to S'$ preserving the marking. By Proposition IV.24 (2), it must be a Möbius transformation. Since it preserves the marking, it must be orientation-preserving and not include inversions in spheres. Thus it is generated by an even number of reflections over hyperplanes, so it is (the quotient of) an orientation-preserving isometry of $\mathbb{R}^n$. We conclude $[S,f] = [S',f']$. $\square$

Next, we exhibit a significant departure from Teichmüller spaces of hyperbolic surfaces.

Theorem IV.29. *For all $[S,f],[S',f'] \in \mathcal{T}(n)$, we have:*

$$d_{Teich}([S,f],[S',f']) = \max(d_{Th}([S,f],[S',f']), d_{Th}([S',f'],[S,f]))$$

Proof. Recall from Corollary IV.26 that the extremal quasiconformal constant between two marked flat n-tori is realized by the unique affine map. The Jacobian of an affine map is equal to its determinant, which must be 1, since it is volume-preserving. Definition IV.23 then gives

$$K(g) = \max\left(\sup_{x \in \mathbb{R}^n} L(g'(x))^n, \sup_{x \in \mathbb{R}^n} \frac{1}{\ell(g'(x))^n}\right),$$

but g is affine, so $L(g'(x))$ is the Lipschitz constant of g, and $\ell(g'(x))^{-1}$ is the Lipschitz constant of the inverse map. $\square$

Corollary IV.30. *The Teichmüller metric on $\mathcal{T}(n)$ is given by:*

$$d_{Teich}(X,Y) = \frac{1}{2}\max\{\big|\log|\lambda|\big| \ : \ \lambda \text{ is an eigenvalue of } XY^{-1}\}$$

Proof. This is precisely the symmetrization of the formula from Theorem IV.14 by maximum, since the eigenvalues of YX^{-1} are the reciprocals of the eigenvalues of XY^{-1}. $\square$

4.5 The Hilbert Metric on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$

Liverani and Wojtkowski [40] defined a generalization of Hilbert's projective metric for the symmetric space $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Their metric s arises naturally during the study of the symplectic geometry of $\mathbb{R}^n \times \mathbb{R}^n$, and measures the distance between pairs of Lagrangian subspaces. An explicit formula for the Finsler metric on the tangent space $T_Z X$ at a point $Z \in X$ associated to their Hilbert metric is also computed, along with examples of geodesics.

Consider the standard symplectic vector space $\mathbb{R}^n \times \mathbb{R}^n$, where the symplectic form is given by:

$$\omega((x,y),(w,z)) = \langle x,z\rangle_{\mathbb{R}^n} - \langle w,y\rangle_{\mathbb{R}^n}$$

A subpsace V of $(\mathbb{R}^n \times \mathbb{R}^n, \omega)$ is called *Lagrangian* if it is a maximal subspace such that $\omega|_V \equiv 0$. These subspaces must be n-dimensional. A Lagrangian subspace is *positive* if it is the graph of a positive-definite symmetric linear map $U : \mathbb{R}^n \to \mathbb{R}^n$. The collection of positive Lagrangian subspaces is parametrized by the space $\mathcal{P}_n$.

The metric s is defined as the supremum of the symplectic angle between vectors in two positive Lagrangian subspaces. A useful result is the following formula.

Proposition IV.31 (Proposition 5, Theorem 3, [40])**.** *For two positive Lagrangian* *subspaces defined by* $U, W : \mathbb{R}^n \to \mathbb{R}^n$, s *is given by*

$$s(U,W) = \max\left\{\frac{\big|\log|\lambda|\big|}{2} \;:\; \lambda \textit{ is an eigenvalue of } UW^{-1}\right\}. \tag{4.6}$$

The Finsler norm $|A|_Z$ *for* $A \in T_Z X$ *is given by*

$$|A|_Z = \frac{1}{2}\max\{|\lambda| \;:\; \lambda \textit{ is an eigenvalue of } AZ^{-1}\}. \tag{4.7}$$

and the paths $t \mapsto e^{tX}$ *for* $t \in [0,1]$ *and* X *of trace zero are geodesic paths.*

Notice that Equation 4.6 matches the formula in Corollary IV.30, so we conclude:

Proposition IV.32. *By the identification* $\mathcal{T}(n) \leftrightarrow \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, d_{Teich} *is equal to the Hilbert projective metric, and* d_{Teich} *is a Finsler metric with norm defined by Equation 4.7. The paths* $t \mapsto e^{tX}$ *for* $t \in [0,1]$ *and* X *of trace zero are geodesics.*

The significance of Proposition IV.32 is that the same metric d_{Teich} on $\mathcal{T}(n)$ arises in a natural way in a very different context. This provides further evidence of the usefulness and richness of the study of this Finsler metric on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

Remark IV.33. The Hilbert metric, defined on open convex subsets $C \subseteq \mathbb{R}^n$ not containing a line, is based on the cross-ratio of two points a, b and the points where the line $\overline{ab}$ meets the boundary ∂C. When C is the positive orthant of $\mathbb{R}^n$, one obtains a Finsler metric with many properties similar to the metric s.

4.6 The Weil-Petersson Metric

In this section, we will define the Weil-Petersson metric on $\mathcal{T}(n)$. Fischer-Tromba [23] show the classical Weil-Petersson metric is recovered using a L^2-pairing between metrics on hyperbolic surfaces. In [56], Yamada gives an exposition of this approach, including a definition of the Weil-Petersson metric for the Teichmüller space of the flat 2-torus. We will follow Yamada's presentation and explain how this quickly generalizes to the case of flat tori in all dimensions.

Write $K = \mathrm{SO}(n)$, $G = \mathrm{SL}(n, \mathbb{R})$. Recall first the tangent space of G/K at the basepoint eK is the vector space of $n \times n$ symmetric matrices of trace 0. The $\mathrm{SL}(n, \mathbb{R})$-invariant metric at this point is defined by:

$$\langle X, Y \rangle_{eK} = \mathrm{tr}(XY).$$

By translation, at other points $gK \in G/K$ for $g \in \mathrm{SL}(n, \mathbb{R})$ the metric is given by

$$\langle X, Y \rangle_{gK} = \mathrm{tr}(g^{-1} X g^{-1} Y). \tag{4.8}$$

Now, recall that $\mathcal{T}(n) \cong \mathcal{P}_n$ is also the space of unit-volume flat metrics on $\mathbb{T}^n$.

The tangent space to the set of Riemannian metrics on a manifold is naturally the space of smooth symmetric $(0, 2)$-tensors ([56], §3). There is a natural L^2 pairing $\langle\langle , \rangle\rangle_{L^2(g)}$ at a metric g defined by:

$$\langle\langle h_1, h_2 \rangle\rangle_{L^2(g)} = \int_M \langle h_1(x), h_2(x) \rangle_{g(x)} d\mu_g(x) \tag{4.9}$$

using the volume form $d\mu_g$ of g. Using local coordinates g^{ij} for g and $(h_k)_{lm}$ for h_k, $k = 1, 2$, we can rewrite the integrand as:

$$\langle h_1(x), h_2(x) \rangle_{g(x)} = \sum_{1 \le i,j,k,l \le 2} g^{ij} g^{kl} (h_1)_{ik} (h_2)_{jl} = \mathrm{Tr}(g^{-1} h_1 g^{-1} h_2).$$

In §3.2 of [56], two conditions are imposed on the deformations of a metric in order to ensure that each tensor h is tangent to the Teichmüller space and not merely the space of all possible metrics: (1) the deformations must be L^2-perpendicular to the action of the identity component of the diffeomorphism group $\mathrm{Diff}_0(M)$, and (2) the deformations must preserve curvature. It is shown there that these conditions are equivalent to being divergence-free and trace-free.

Finally, we arrive at the definition of the Weil-Petersson metric on Teichmüller space with the viewpoint of deformations of Riemannian metrics.

Definition IV.34 ([23], Theorem 0.8)**.** The L^2-pairing in Equation 4.9 restricted to the trace-free, divergence-free tensors is called the *Weil-Petersson metric.*

We apply the above definitions to $\mathcal{T}(n)$. Deformations of flat metrics which remain in the Teichmüller space define a subspace of all $(0, 2)$-tensors. Maintaining unit volume restricts to traceless tensors, while the restriction to flat metrics implies the tensors have constant $\mathbb{R}^n$-coordinates. These are trace-free and divergence-free tensors. Thus the integrand in Equation 4.9 is constant and given globally by the local coordinates. The volume of each metric is 1, so the L^2-pairing simplifies to:

$$\langle\langle h_1, h_2 \rangle\rangle_{L^2(g)} = \operatorname{Tr}(g^{-1}h_1 g^{-1}h_2).$$

This matches precisely the usual symmetric metric for $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ given in Equation 4.8. We now have for all $n \geq 2$:

Proposition IV.35. *The Teichmüller space* $\mathcal{T}(n)$ *with the Weil-Petersson metric is isometric to* $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ *with the* $\mathrm{SL}(n, \mathbb{R})$*-invariant Riemannian metric.*

Remark IV.36. The Weil-Petersson metric for Teichmüller spaces of hyperbolic surfaces is also a Riemannian metric, but it is not complete. This leads to an interesting theory of bordifications and nodal surfaces. Here, we see another interesting departure from the hyperbolic surface setting in that the Weil-Petersson metric on $\mathcal{T}(n)$ is complete.

4.7 Horofunction and Satake Compactifications

In this section, we will describe horofunction compactifications of $\mathcal{T}(n)$ with the Thurston and Teichmüller metrics defined in Sections 4.3 and 4.4.

4.7.1 The Thurston Metric

Recall that the standard representation of $\mathrm{SL}(n,\mathbb{R})$ induces a minimal Satake compactification of $\mathcal{T}(n)$. It has the following metric realization.

Proposition IV.37. *The following compactifications are G-isomorphic:*

$$\overline{\mathcal{T}(n)}^{hor}_{d_{Th}} \cong_G \overline{\mathcal{T}(n)}^{S}_{\Pi}$$

where Π *is the standard representation of* $G = \mathrm{SL}(n,\mathbb{R})$.

Proof. The weights of the standard representation are simply the standard basis e_i, $i = 1, \dots, n$, for $\mathbb{R}^n$. Projecting them onto the hyperplane in $\mathbb{R}^n$ corresponding to $\mathfrak{a}$, the set of weights is given by:

$$\mu_i := e_i - \sum_{j=1}^{n} \frac{1}{n} e_j,\ \ i = 1, \dots, n.$$

Following [31], consider the convex hull $D := \mathrm{conv}(\mu_1, \dots, \mu_n)$. This lies within the codimension 1 hyperplane $\sum_i x_i = 0$ in $\mathbb{R}^n$. In order to utilize Theorem III.11, we now compute the negative of the dual polytope of D. If $\{a_1, \dots, a_k\} \subseteq \mathbb{R}^n$ are the vertices of a convex polytope, then the dual polytope is given by:

$$\{y \in \mathbb{R}^n : \langle a_i, y\rangle \geq -1\ \forall i\}.$$

The extremal points are those where equality holds. By symmetry, the μ_i's are extremal points for the convex hull D, and the dual must live in the same hyperplane, so this becomes:

$$\begin{aligned} B_0 := -D^\circ &= -\{(y_1, \dots, y_n) \in \mathbb{R}^n : y_1 + \cdots + y_n = 0,\ y_i - \frac{1}{n}\sum_j y_j \geq -1\ \forall i\} \\ &= \{(y_1, \dots, y_n) \in \mathbb{R}^n : y_1 + \cdots + y_n = 0,\ y_i \leq 1\ \forall i\} \end{aligned}$$

By Theorem III.11, this is a unit ball for a polyhedral Finsler metric whose horofunction compactification is the Satake compactification of the standard representation.

To complete the proof, we compute the unit ball of the Finsler metric d_{Th} in the Cartan subalgebra. Using the formula in Proposition IV.21, this is relatively straightforward:

$$B = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_1 + \cdots + y_n = 0,\ y_i \leq 2\ \forall i\}$$

Because $B_0 = B$ up to scaling, we are done. □

By Proposition III.13, it follows that the boundary of the compactification $\overline{\mathcal{T}(n)}_{\pi}^{S}$ is homeomorphic to the sphere $S^{(n^2+n)/2-2}$.

4.7.2 The Teichmüller Metric

We have a similar result for d_{Teich}.

Proposition IV.38. *Let* Π *be the standard representation of* $G = \mathrm{SL}(n, \mathbb{R})$*. Then the following compactifications are* G*-isomorphic:*

$$\overline{\mathcal{T}(n)}_{d_{Teich}}^{hor} \cong_G \overline{\mathcal{T}(n)}_{\Pi \oplus \Pi^*}^{S}$$

Proof. Consider the faithful representation

$$\Pi \oplus \Pi^* : \mathrm{SL}(n, \mathbb{R}) \hookrightarrow \mathrm{SL}(2n, \mathbb{C}),$$

using the standard and dual representations as a block diagonal acting on the direct sum of the vector spaces.

The collection of weights, viewed as elements of $\mathbb{R}^n$, is the union of the weights for the standard and dual representations. We project them onto the hyperplane $P \subseteq \mathbb{R}^n$ defined by $\sum_i y_i = 0$ to obtain the weights in $\mathfrak{a}$. After projection, two of the weights are given by

$$a_1 := \left(\frac{n-1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}\right),\ b_1 = \left(-\frac{n-1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right),$$

and the others are similar, with $\pm(1-1/n)$ in the ith component and $\mp 1/n$ in the remaining components. We consider the convex hull D of these points. This defines a polyhedron in $\mathbb{R}^n$, of which we compute the negative of the dual.

Lemma IV.39. *The negative of the dual to the polyhedron $D = conv(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is given by:*

$$-D^\circ = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \ : \ \sum_i y_i = 0, \ |y_i| \leq 1 \ \forall i = 1, \ldots, n\}$$

We prove this lemma below. Now, using Equation 4.7 for the Finsler metric associated of d_{Teich}, we see that the ball $-D^\circ$ is, up to a choice of scaling, the same as the unit ball for the Teichmüller metric. Theorem III.11 completes the proof. $\square$

Proof of Lemma IV.39. Since all points $a_1, \ldots, b_n$ lie in the hyperplane $\sum_i y_i = 0$, the dual polyhedron must as well. Now, choose some $i \in \{1, \ldots, n\}$ and consider the condition $\langle (y_1, \ldots, y_n) | a_i \rangle \geq -1$. Expanding, this becomes:

$$-\frac{1}{n}(y_1 + \cdots + y_n) + y_i \geq -1$$

But since $\sum_i y_i = 0$, this simplifies to $y_i \geq -1$. For b_i, we obtain $1 \geq y_i$. $\square$

4.8 The Thurston Compactification of $\mathcal{T}(n)$

Inspired by Thurston's compactification for Teichmüller spaces of hyperbolic surfaces using projective measured laminations on the underlying surfaces, we define a natural Thurston-type compactification of $\mathcal{T}(n)$. It is closely related to T. Haettel's compactification of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ built from the closure of a projective embedding into $\mathbb{P}(\mathbb{R}_+^{\mathbb{Z}^n})$ in [30], but we provide a new construction utilizing a geometric interpretation of quadratic forms.

Recall the Satake compactification of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ with respect to the standard representation of $\mathrm{SL}(n, \mathbb{R})$, whose boundary points correspond to projective classes of positive-semidefinite matrices. After relating this compactification to the Thurston compactification, we have a geometric interpretation of the Satake compactification $\overline{\mathcal{T}(n)}^S_\pi$.

Recall (see [22]) that a measured foliation on a surface is a (singular) foliation with an arc measure in the transverse direction that is invariant under holonomy (translations along leaves). Our goal is to develop an analogous Thurston-type boundary for $\mathcal{T}(n)$. With that in mind, we start with the following definition.

Definition IV.40. A *measured flat foliation* on $\mathbb{R}^n/\mathbb{Z}^n$ is a non-singular measured foliation (F, μ) with the following requirements:

- The leaves of F are given by parallel hyperplanes.
- The measure μ is invariant under isometries of the torus.
- In the lift to $\mathbb{R}^n$, if V_0 is the leaf containing the origin, then there exists an orthogonal decomposition

$$V_0^\perp = V_1 \oplus \cdots \oplus V_k$$

  and positive constants λ_i, $i = 1, \ldots, k$, such that the lift of an arc γ contained in subspace V_i has measure $\mu(\gamma) = \lambda_i \ell_I(\gamma)$, where ℓ_I is the Euclidean length.

This is a simple higher-dimensional analog of measured foliations for surfaces where the leaves are totally geodesic submanifolds. Invariance under isometries implies that we may assume any arc to be measured has a lift that begins at the origin in $\mathbb{R}^n$. There is an obvious action by $\mathbb{R}^+$ on the set of measured flat foliations by scaling the measure. Denote the set of projective classes of measured flat foliations by $\mathcal{PMFF}$. In addition to building a Thurston boundary, we will relate it to the compactifications studied in Section 4.7.

Lemma IV.41. *The collection $\mathcal{PMFF}$ is in a natural one-to-one correspondence with the boundary of the minimal Satake compactification associated to the standard representation.*

Proof. Let Q be a matrix representative of the class $[Q] \in \partial\overline{\mathcal{T}(n)}^S_\pi$. Define the leaves of a foliation of $\mathbb{R}^n$ by all parallel translations of $\ker(Q)$. This descends to the quotient $\mathbb{R}^n/\mathbb{Z}^n$. Arc length with respect to Q defines a transverse measure, which for an arc $\gamma : [0,1] \to \mathbb{R}^n/\mathbb{Z}^n$ is given by

$$\ell_Q(\gamma) = \int_0^1 \sqrt{\langle \gamma'(t)Q, \gamma'(t)\rangle}dt.$$

Because the quadratic form Q is constant across $\mathbb{R}^n/\mathbb{Z}^n$ and diagonalizable, the measure satisfies the conditions in Definition IV.40.

In this way, Q endows $\mathbb{R}^n/\mathbb{Z}^n$ with a measured foliation. Taking the projective class gives us the projective measured flat foliation associated to $[Q]$.

Conversely, given $(F, [\mu]) \in \mathcal{PMFF}$, we can obtain the associated $[Q] \in \partial\overline{\mathcal{T}(n)}^S_\pi$ as follows. Take any representative (F, μ) of the projective class. Then:

1. Lift the measured foliation to $\mathbb{R}^n$
2. Let $v_1, \ldots, v_m$ be an orthonormal basis of the subspace V_0 spanned by the leaf through the origin
3. For each subspace V_j in the direct sum $V_0^\perp = V_1 \oplus \cdots \oplus V_k$ from Definition IV.40, choose an orthonormal basis. Label these vectors $v_{m+1}, \ldots, v_n$
4. Let λ_i be the measure of a straight line segment of Euclidean length 1 extending from the origin in the direction of v_i for $i = 1, \ldots, n$
5. Let P be the matrix whose columns are v_i for $i = 1, \ldots, n$ and let D be the diagonal matrix whose diagonal entries are λ_i for $i = 1, \ldots, n$.

6. Let $Q = P^{-1}DP$. This is a positive-semidefinite symmetric matrix which induces the same measured foliation we began with.

Taking the projective class of the matrix gives us the associated element of the Satake compactification. This establishes maps in both directions which are inverses, as required. $\square$

The viewpoint of Lemma IV.41 gives a geometric way to interpret quadratic forms as measured foliations. Next, we will give the collection $\mathcal{T}(n) \cup \mathcal{PMFF}$ a topology. We do so by defining a notion of convergence to points of $\mathcal{PMFF}$ by sequences of points in $\mathcal{T}(n) = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$. Let $(F, [\mu]) \in \mathcal{PMFF}$, where F is the foliation of $\mathbb{T}^n$ and $[\mu]$ is the projective class of the transverse measure. Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of elements of $\mathcal{T}(n)$.

Definition IV.42. We say the sequence $(X_i)_{i \in \mathbb{N}}$ *converges* to (F, μ) if for

$$r_i = 1 / \max\{\lambda \ : \ \lambda \text{ is an eigenvalue of } X_i\} \tag{4.10}$$

the following holds: there exists a representative $\mu_0 \in [\mu]$ such that for all simple closed curves $\gamma \subseteq \mathcal{T}^n$, we have

$$\ell_{r_i X_i}(\gamma) \xrightarrow{i \to \infty} \mu_0(\gamma)$$

where $\ell_Q(\gamma)$ denotes the length of the curve γ with the metric Q.

Remark IV.43. Convergence to points of $\mathcal{PMFF}$ may also be viewed geometrically: we can also define convergence to $\mathcal{PMFF}$ by requiring that the Hausdorff distance between unit balls goes to 0. This is essentially convergence of metrics while allowing some directions to degenerate.

Lemma IV.44. *The collection $\mathcal{T}(n) \cup \mathcal{PMFF}$ is compact.*

Proof. We show that every sequence has a convergent subsequence. First, suppose $(X_i)_{i\in\mathbb{N}}$ consists only of elements of $\mathcal{T}(n)$, but no subsequence converges to a point of $\mathcal{T}(n)$. Consider then the sequence of matrices r_iX_i, where r_i is defined in Equation 4.10. Now, the set of positive-definite symmetric matrices with eigenvalues bounded above by 1 is compact, so we may assume r_iX_i converges to a positive-semidefinite matrix M. By Lemma IV.41 and by construction, M corresponds to an element of $\mathcal{PMFF}$ which satisfies the conditions of Definition IV.42.

Now suppose that some $X_k \in \mathcal{PMFF}$ for some (perhaps infinitely many) $k \in \mathbb{N}$. Pick a sequence $(Y_j^k)_{j\in\mathbb{N}} \in \mathcal{T}(n)$ which converges to X_k. Then replace X_k with Y_k^k in the sequence $(X_i)_{i\in\mathbb{N}}$, and use the first case to find a limit for the new sequence. The original sequence also must converge to this same limit. $\square$

We are now prepared to make the following definition.

Definition IV.45. The *Thurston compactification* of $\mathcal{T}(n)$ is

$$\overline{\mathcal{T}(n)}^{Th} := \mathcal{T}(n) \cup \mathcal{PMFF}.$$

By Lemmas IV.41 and IV.44, we see that the Thurston compactification $\overline{\mathcal{T}(n)}^{Th}$ is a compactification of $\mathcal{T}(n)$ built from measured foliations on the underlying structures, as in Thurston's compactification for Teichmüller spaces of hyperbolic surfaces.

Lemma IV.46. *Let $(F,[\mu]) \in \mathcal{PMFF}$, and let $[Q] \in \partial\overline{\mathcal{T}(n)}^{Th}$ be the quadratic form associated to $(F,[\mu])$. For a sequence $(X_i)_{i\in\mathbb{N}} \in \mathcal{T}(n)$, we have*

$$(X_i)_{i\in\mathbb{N}} \xrightarrow{i\to\infty} (F,[\mu]) \textit{ if and only if } (X_i)_{i\in\mathbb{N}} \xrightarrow{i\to\infty} [Q]$$

where on the right-hand side the convergence is with respect to the topology on the Satake compactification.

Proof. Notice that convergence on the right-hand side is equivalent to the following: if $1/r_i$ is the maximal eigenvalue of X_i for each i, then

$$r_i X_i \xrightarrow{i\to\infty} Q$$

for some representative $Q \in [Q]$ as matrices. Let μ_0 be the representative of $[\mu]$ associated to the semidefinite form Q. Then $\ell_Q(\gamma) = \mu_0(\gamma)$ for all simple closed curves γ, and so from Lemma IV.41 we have

$$X_i \xrightarrow{i\to\infty} (F, [\mu]).$$

The reverse implication is nearly identical. $\square$

Immediately following from Lemmas IV.41 and IV.46 is the following:

Corollary IV.47. *The identity map on $\mathcal{T}(n)$ extends to a homeomorphism*

$$\overline{\mathcal{T}(n)}^{Th} \cong \overline{\mathcal{T}(n)}^{S}_{\pi}.$$

Proof. Lemma IV.46 shows that the bijection from Lemma IV.41 preserves convergence in both directions. $\square$

Next, we endow $\mathcal{PMFF}$ with a $\mathrm{SL}(n, \mathbb{R})$-action. For $g \in \mathrm{SL}(n, \mathbb{R})$, define:

$$g \cdot (F, [\mu]) = (Fg, [g^{-1} * \mu]).$$

One can verify that this defines a $\mathrm{SL}(n, \mathbb{R})$-action on $\mathcal{PMFF}$.

Lemma IV.48. *This* $\mathrm{SL}(n, \mathbb{R})$*-action is equivariant with respect to the bijection of Lemma IV.41.*

Proof. Recall from Lemma IV.41 that for any smooth arc γ, if (F, μ_0) is a representative of the projective class of $(F, [\mu])$ associated to $Q \in [Q]$, then

$$\ell_Q(\gamma) = \mu_0(\gamma).$$

Now, for $g \in \mathrm{SL}(n, \mathbb{R})$ we have

$$\ell_{g \cdot Q}(\gamma) = \ell_Q(g \cdot \gamma) = \mu_0(g \cdot \gamma) = g^{-1} * \mu_0(\gamma).$$

Finally, if γ is a curve contained in a single leaf, then $g \cdot \gamma = \gamma g$ is then contained in a leaf of $g \cdot F = Fg$. $\square$

Combining Lemma IV.48 and Corollary IV.47, we arrive at:

Theorem IV.49. *The Thurston compactification* $\overline{\mathcal{T}(n)}^{Th}$ *is* $\mathrm{SL}(n, \mathbb{R})$*-isomorphic to the Satake compactification with respect to the standard representation* $\overline{\mathcal{T}(n)}^{S}_{\pi}$.

Theorem I.2 is then the combined results of Theorem IV.49 and Proposition IV.37, and Corollary I.3 is immediate.

CHAPTER V

Holomorphic Isometric Submersions Between Teichmüller Spaces

This chapter is devoted first to the proofs of Theorems I.7 and V.1, which originally occurred in joint work of the author with Dmitri Gekhtman [25], and second to an introduction to the case of infinite punctures, which is new. Theorem I.11 is a summary of the progress on the infinite-type surfaces. Broadly speaking, this work generalizes Royden's theorem, which states that automorphisms of Teichmüller spaces must be induced by the mapping class group, to the case of holomorphic isometric submersions between Teichmüller spaces, which in the finite-dimensional case we will show must be forgetful maps (possibly excepting a small number of cases conjectured to have the same property).

Our study of isometric submersions between Teichmüller spaces follows a similar theme to many classical results about maps between Teichmüller spaces. In particular, we find the restriction to forgetful maps by considering the induced map between cotangent spaces. The key observation is that an isometric submersion induces isometric embeddings of cotangent spaces (see Section 5.2.1). While we will follow this approach in both the finite-type and infinite-type surface cases, we will need to bring in some new tools for the case of infinite punctures, and several of the methods break down.

We will begin with the case of finite-type surfaces. Sections 5.1 through 5.3 contain the results from [25], and Section 5.4 contains the initial exploration of the case of infinite punctures.

5.1 Embeddings of Spaces of Quadratic Differentials

Beginning with a holomorphic isometric submersion $F : \mathcal{T}(Y) \to \mathcal{T}(X)$, the coderivative at the basepoint is a map $T : Q(X) \to Q(Y)$. The majority of the proof of Theorem I.7 comes from analysis of the resulting map T. We prove the following classification result, which is of independent interest.

Theorem V.1. *Let X and Y be finite-type Riemann surfaces. Let $\widehat{X}$ and $\widehat{Y}$ be the compact surfaces obtained by filling the punctures of X and Y. Assume the type (k, m) of X is non-exceptional: $2k + m \geq 5$. Let $T : Q(X) \hookrightarrow Q(Y)$ be a $\mathbb{C}$-linear isometric embedding. Then there is a holomorphic map $h : \widehat{Y} \to \widehat{X}$ and a constant $c \in \mathbb{C}$ of magnitude deg$(h)^{-1}$ so that $T = c \cdot h^*$.*

Remark V.2. Suppose X is of exceptional type (k, m), so $2k + m \leq 4$. Then one of the following holds:

1. $\dim_{\mathbb{C}} Q(X) \leq 1$
2. (k, m) is $(2, 0)$ or $(1, 2)$, in which case $Q(X)$ identifies naturally with the quadratic differential space of a surface of non-exceptional type $(0, 6)$ or $(0, 5)$, respectively.

Thus, Theorem V.1 amounts to a complete classification of $\mathbb{C}$-linear isometric embeddings $Q(X) \to Q(Y)$ for X and Y of finite type.

To prove Theorem V.1, we use methods developed by V. Markovic [41] in his proof of the infinite-dimensional generalization of Royden's theorem. (See also the paper of Earle-Markovic [17] and the thesis of S. Antonakoudis [4].) Recall the *bi-canonical map* $\widehat{X} \to \mathbb{P}Q(X)^*$ sending each $x \in \widehat{X}$ to the hyperplane in $Q(X)$ of quadratic differentials vanishing at x. The idea is to relate the bi-canonical images of X and Y using a result of Rudin [47] on isometries of L^p spaces. The fact that $T : Q(X) \to Q(Y)$ is an isometric embedding implies via Rudin's theorem that $T^* : \mathbb{P}Q(Y)^* \to \mathbb{P}Q(X)^*$ carries the bi-canonical image of $\widehat{Y}$ onto the bi-canonical image of $\widehat{X}$. So, there is a unique $h : \widehat{Y} \to \widehat{X}$ making the following diagram commute:

$$\begin{tikzcd} \mathbb{P}Q(Y)^* \arrow[r, "T^*"] & \mathbb{P}Q(X)^* \\ \widehat{Y} \arrow[u] \arrow[r, "h"] & \widehat{X} \arrow[u] \end{tikzcd}$$

In fact, Rudin's result gives us more: for any $\phi \in Q(X)$, the map h pushes the $|T\phi|$-measure on $\widehat{Y}$ to the $|\phi|$-measure on $\widehat{X}$. Thus, we obtain the following intermediate result:

Proposition V.3. *Let X and Y be finite-type Riemann surfaces, with X of non-exceptional type. Suppose $T : Q(X) \hookrightarrow Q(Y)$ is a $\mathbb{C}$-linear isometric embedding. Then there is a holomorphic map $h : \widehat{Y} \to \widehat{X}$ with the following property: For any $\phi \in Q(X)$ and any measurable $K \subset \widehat{X}$,*

$$\int_K |\phi| = \int_{h^{-1}(K)} |T\phi| \, .$$

We then use Proposition V.3 to derive the classification result Theorem V.1.

5.1.1 Infinitesimal to Global

The last step is to obtain the global result, Theorem I.7, from the infinitesimal Theorem V.1. We are given a holomorphic and isometric submersion $F : \mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{k,m})$, with (k, m) satisfying hypotheses (1.1) and (1.2). Since (k, m) is assumed non-exceptional, Theorem V.1 gives for each $Y \in \mathcal{T}(S_{g,n})$ a holomorphic branched cover $h_Y : \widehat{Y} \to \widehat{F(Y)}$. By a dimension count, it is not the case that every Riemann surface of genus g is a branched cover of a surface of genus k with $1 \le k < g$. We then obtain that $g = k$. Finally, an argument involving the universal families over $\mathcal{T}(S_{g,n})$ and $\mathcal{T}(S_{g,m})$ shows that the map $h_Y : Y \to F(Y)$ varies continuously in $Y \in \mathcal{T}(S_{g,n})$. Thus, the topological type of h_Y is constant in Y. We conclude that the map F is induced by a (fixed) mapping class composed with the inclusion map on the underlying surfaces, filling in punctures.

Section 5.2 focuses on the infinitesimal geometry of isometric submersions between Teichmüller spaces. In 5.2.1, we recall basic facts on isometric submersions between Finsler manifolds. We first establish that forgetful maps between Teichmüller spaces are holomorphic and isometric submersions. Next, we review a theorem of Rudin concerning isometries between L^p spaces and discuss the bi-canonical embedding $X \hookrightarrow \mathbb{P}Q(X)^*$ of a Riemann surface. Then, we follow the argument of [41] to obtain Proposition V.3. Finally, we obtain the classification Theorem V.1 of isometric embeddings between quadratic differential spaces.

Section 5.3 focuses on the global geometry of isometric submersions $F : \mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{k,m})$ and the proof of the main result on holomorphic isometric submersions between finite-type Teichmüller spaces, Theorem I.7. To complete the proof, we first use Theorem V.1 to obtain for each $Y \in \mathcal{T}(S_{g,n})$ a non-constant holomorphic map $h_Y : \widehat{Y} \to \widehat{F(Y)}$. Then we use a dimension-counting argument to show that $g = k$. We next use properties of the universal family to show that the collection of maps $h_Y : Y \to X$ varies continuously in the parameter $Y \in \mathcal{T}(S_{g,n})$, after which we finish the proof.

To conclude, in Section 5.4 we introduce the problem of classifying holomorphic isometric submersions between Teichmüller spaces of finite-genus surfaces with infinite punctures. While we do not achieve a proof of the full generalization, we have several partial results, including that with an additional technical assumption, such a map can only exist when the Teichmüller spaces classify surfaces of the same genus.

5.2 Infinitesimal Geometry

5.2.1 Isometric Submersions of Finsler Manifolds

We review basic properties of isometric submersions, following [3]. First, we recall the relevant notion from linear algebra. An *isometric submersion* between normed vector spaces V and W is a linear map $V \to W$ so that the image of the closed unit ball in V is the closed unit ball in W. Isometric submersions and isometric embeddings of normed vector spaces are dual in the following sense.

Lemma V.4. *Let $T : V \to W$ be a linear map between normed vector spaces.*

1. *If T is an isometric submersion, then the dual map $T^* : W^* \to V^*$ is an isometric embedding.*
2. *If T is an isometric embedding, then $T^* : W^* \to V^*$ is an isometric submersion.*

The proof of the first assertion of the Lemma is elementary. The second assertion is a restatement of the Hahn-Banach theorem.

An *isometric submersion between Finsler manifolds M, N* is a C^1 submersion $F : M \to N$ such that the derivative $dF_m : T_m M \to T_{F(m)} N$ is an isometric submersion between tangent spaces with respect to the Finsler norms, for each $m \in M$. We will use the characterization of isometric submersions in terms of isometric embeddings of cotangent spaces.

Corollary V.5. *Let $F : M \to N$ be a C^1 map of Finsler manifolds. Then F is an isometric submersion if and only if for each $m \in M$, the coderivative*

$$dF_m^* : T^*_{F(m)} N \to T^*_m M$$

is an isometric embedding of cotangent spaces with respect to the dual Finsler norms.

5.2.2 Forgetful Maps Between Teichmüller Spaces

We recall basic properties of forgetful maps between Teichmüller spaces, and in particular observe that these maps are holomorphic and isometric submersions. Let $F : \mathcal{T}_{g,1} \to \mathcal{T}(S_g)$ be the forgetful map; for each $X \in \mathcal{T}(S_{g,1})$, $F(X)$ is the marked Riemann surface obtained by filling in the puncture of X. The cotangent space $T^*_X\mathcal{T}(S_{g,1}) = Q(X)$ consists of holomorphic quadratic differentials on X with at worst a simple pole at the puncture, while $T^*_{F(X)}\mathcal{T}(S_g) = Q(F(X)) = Q(\hat{X})$ consists of those quadratic differentials on X which extend holomorphically over the puncture. The co-derivative dF^*_X is the inclusion $Q(F(X)) \hookrightarrow Q(X)$, which is clearly isometric and complex-differentiable. Thus, F is a holomorphic and isometric submersion. The same reasoning shows that any forgetful map $\mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{g,m})$ is an isometric submersion. We have just shown:

Lemma V.6. *Forgetful maps between finite-dimensional Teichmüller spaces are holomorphic isometric submersions.*

5.2.3 Rudin's Equimeasurability Theorem

We will need a general result of Rudin concerning isometries between subspaces of L^p spaces. Markovic [41] used this result in the $p = 1$ case to extend Royden's theorem to Teichmüller spaces of infinite dimension, and Earle-Markovic [17] used the result to give a new and illuminating proof of Royden's theorem in the finite-dimensional case.

Proposition V.7 (Rudin [47], Theorem 1)**.** *Let p be a positive real number which is not an even integer. Let X and Y be sets with finite positive measures μ and ν respectively. Let l be a positive integer. Suppose $f_1, \ldots, f_l$ in $L^p(\mu, \mathbb{C})$, and $g_1, \ldots, g_l$*

in $L^p(\nu, \mathbb{C})$ satisfy the following condition:

$$\int_X \left|1 + \sum_{j=1}^{l} \lambda_j f_j\right|^p d\mu = \int_Y \left|1 + \sum_{j=1}^{l} \lambda_j g_j\right|^p d\nu, \quad \textit{for all } (\lambda_1, \ldots, \lambda_l) \in \mathbb{C}^l. \tag{5.1}$$

If $F = (f_1, \ldots, f_l)$ and $G = (g_1, \ldots, g_l)$, then the maps $F : X \to \mathbb{C}^l$ and $G : Y \to \mathbb{C}^l$ satisfy the following equimeasurability condition:

$$\mu(F^{-1}(E)) = \nu(G^{-1}(E)) \quad \textit{for each Borel set } E \subseteq \mathbb{C}^l. \tag{5.2}$$

Equation (5.1) is an assumption on the moments of the $\mathbb{C}^l$-valued random variables F and G. The conclusion (5.2) is that F and G have the same distribution. In other words, the pushforward measures $F_*(\mu)$ and $G_*(\nu)$ on $\mathbb{C}^l$ are equal.

5.2.4 Projective Embeddings of Riemann Surfaces

In this section, we establish the setting for our application of Rudin's theorem. Let L be a holomorphic line bundle over a compact Riemann surface $\widehat{X}$, and let $\mathcal{O}(L)$ denote the space of holomorphic sections of L. There is a holomorphic map $\widehat{X} \to \mathbb{P}\mathcal{O}(L)^*$ sending $x \in \widehat{X}$ to the hyperplane in $\mathcal{O}(L)$ consisting of sections which vanish at x. An argument using the Riemman-Roch theorem (see [44] p. 55) shows that if the degree of L is at least $2g+1$, then the map $\widehat{X} \to \mathbb{P}\mathcal{O}(L)^*$ is an embedding.

Now, let X be a Riemann surface of type (g, n). Denote by $\widehat{X}$ the compact, genus g Riemann surface obtained by filling in the punctures of X. The space $Q(X)$ consists of quadratic differentials which are holomorphic on X and have at most simple poles at the punctures $\widehat{X} \setminus X$. Thus, elements of $Q(X)$ correspond to sections of a line bundle on $\widehat{X}$ of degree $4g-4+n$. By the preceding discussion, the associated *bi-canonical map* $\widehat{X} \to \mathbb{P}Q(X)^*$ is an embedding provided $4g - 4 + n \geq 2g + 1$, or $2g + n \geq 5$. Thus, the surfaces X of non-exceptional type are precisely those for which $\widehat{X} \to \mathbb{P}Q(X)^*$ is an embedding.

5.2.5 Applying the Equimeasurability Theorem

In this section, we apply the methods of [41] to prove Proposition V.3. We acknowledge some overlap with [4] Section 5, particularly in the proof of the fact that the surface $\widehat{Y}$ covers the surface $\widehat{X}$ if there is a $\mathbb{C}$-linear isometric embedding $Q(X) \hookrightarrow Q(Y)$.

Proof of Proposition V.3. Let X and Y be Riemann surfaces of finite type. Assume X is of non-exceptional type, and denote by $\Phi : \widehat{X} \hookrightarrow \mathbb{P}Q(X)^*$ the bi-canonical embedding associated to X. Let $T : Q(X) \to Q(Y)$ be a $\mathbb{C}$-linear isometric embedding. Denote by Ψ the composition $\widehat{Y} \to \mathbb{P}Q(Y)^* \to \mathbb{P}Q(X)^*$ of the bi-canonical map of Y with the dual of T. To describe the maps Φ and Ψ more concretely, fix a basis $\phi_0, \ldots, \phi_k$ for $Q(X)$ and let $\psi_i = T\phi_i$ denote the images in $Q(Y)$. In terms of local coordinates z, w for $\widehat{X}$ and $\widehat{Y}$, respectively, the maps $\Phi : \widehat{X} \to \mathbb{P}^l$ and $\Psi : \widehat{Y} \to \mathbb{P}^l$ are given by

$$\Phi(z) = [\phi_0(z) : \ldots : \phi_l(z)], \;\; \Psi(w) = [\psi_0(w) : \ldots : \psi_l(w)].$$

Now, consider the rational functions $f_i = \frac{\phi_i}{\phi_0}$ on $\widehat{X}$ and $g_i = \frac{\psi_i}{\psi_0}$ on $\widehat{Y}$, with $i = 1, \ldots, l$. Form the $\mathbb{C}^l$-valued maps $F = (f_1, \ldots, f_l)$ and $G = (g_1, \ldots, g_l)$. The maps F and G are just Φ and Ψ viewed as rational maps to $\mathbb{C}^l$.

Let μ denote the $|\phi_0|$-measure on $\widehat{X}$; that is,

$$\mu(K) = \int_K |\phi_0|$$

for any measurable $K \subset \widehat{X}$. Similarly, let ν denote the $|\psi_0|$-measure on $\widehat{Y}$. Then f_i and g_i are L^1 functions with with respect to the measures μ and ν. The assumption that T is isometric and $\mathbb{C}$-linear translates precisely to the hypothesis (5.2) of Rudin's theorem:

$$\int_{\widehat{X}} \left| 1 + \sum_{i=1}^{l} \lambda_i f_i \right| d\mu = \int_{\widehat{X}} \left| \phi_0 + \sum_{i=1}^{l} \lambda_i \phi_i \right|$$
$$= \int_{\widehat{Y}} \left| \psi_0 + \sum_{i=1}^{l} \lambda_i \psi_i \right| = \int_{\widehat{Y}} \left| 1 + \sum_{i=1}^{l} \lambda_i g_i \right| d\nu.$$

Note that we used $\mathbb{C}$-linearity of T in the second equality. We conclude that the measures $F_*(\mu)$ and $G_*(\nu)$ on $\mathbb{C}^l$ are equal. What amounts to the same thing, the measures $\Phi_*(\mu)$ and $\Psi_*(\nu)$ on $\mathbb{P}^k$ are equal.

We now show that Φ and Ψ have the same image. To this end, note that the measure $\Psi_*(\nu) = \Phi_*(\mu)$ has as its support the compact set $\Phi(\widehat{X})$. Since Ψ is continuous and since ν assigns nonzero measure to each open set of $\widehat{Y}$, we conclude $\Psi(\widehat{Y}) \subset \Phi(\widehat{X})$. Thus, there is a unique holomorphic map $h : \widehat{Y} \to \widehat{X}$ so that $\Psi = \Phi \circ h$. Obviously, Ψ is not constant and so neither is h. In particular, h is a branched cover and $\Psi(\widehat{Y}) = \Phi(\widehat{X})$.

In terms of the map h, the equimeasurability condition $\Psi_*(\nu) = \Phi_*(\mu)$ becomes simply $h_*(\nu) = \mu$. Thus, for any measurable $K \subset \widehat{X}$ we have

$$\int_K |\phi_0| = \mu(K) = \nu\left(h^{-1}(K)\right) = \int_{h^{-1}(K)} |T\phi_0|\,.$$

Since ϕ_0 was chosen arbitrarily, we have the desired equality

$$\int_K |\phi| = \int_{h^{-1}(K)} |T\phi|$$

for any $\phi \in Q(X)$ and any measurable $K \subset \widehat{X}$. This completes the proof of Proposition V.3. $\square$

5.2.6 Completing the Classification of Isometric Embeddings

Let $\phi \in Q(X)$ and write $\psi = T\phi$. Proposition V.3 says

$$\int_{h^{-1}(K)} |\psi| = \int_K |\phi| \tag{5.3}$$

for any measurable $K \subset \widehat{X}$. To complete the proof of Theorem V.1, we must show that ψ is a scalar multiple of the pullback $h^*\phi$. By working over an appropriate coordinate chart in X, we will reduce the proof to the following elementary lemma.

Lemma V.8. *Let g be a real-valued function defined on a domain in $\mathbb{C}$. If both g and e^g are harmonic, then g is constant.*

Proof. Compute

$$0 = (e^g)_{z\bar{z}} = e^g\left(g_z g_{\bar{z}} + g_{z\bar{z}}\right) = e^g g_z g_{\bar{z}}.$$

Thus, g is either holomorphic or anti-holomorphic. Since g is real-valued, it follows that it is constant. □

Returning to the proof of Theorem V.1, fix a coordinate chart (U, z) in X on which $\phi = (dz)^2$. (Recall that one achieves this by integrating a local square root of ϕ.) Shrinking U if necessary, assume U is evenly covered by h and that ψ has no zeros or poles in $h^{-1}(U)$. Write $h^{-1}(U)$ as a disjoint union of coordinate charts (U_i, z_i), with coordinate functions chosen so that $h : (U_i, z_i) \to (U, z)$ is the identity function:

$$z(h(y)) = z_i(y)$$

Let $\psi_i(z_i)(dz_i)^2$ denote the local expression for ψ in U_i. Let $K \subset U$ be measurable. Then equation (5.3) yields

$$\int_K \left(\sum_{i=1}^{\deg(h)} |\psi_i(z)|\right) |dz| = \int_K |dz|\,.$$

Since K was arbitrary, we have

$$\sum_{i=1}^{\deg(h)} |\psi_i(z)| = 1,$$

identically on U. Recall that the absolute value of a holomorphic function of one variable is subharmonic. So the function

$$|\psi_1(z)| = 1 - \sum_{i=2}^{\deg(h)} |\psi_i(z)|$$

is both subharmonic and superharmonic. That is, $|\psi_1(z)|$ is harmonic. But, since $\psi_1(z)$ is holomorphic and non-vanishing, $\log|\psi_1(z)|$ is also harmonic. By Lemma V.8, $\psi_1(z)$ is identically equal to some constant c. In other words,

$$\psi = c \cdot h^*\phi$$

on the open set U_1 and thus on all of X. Since $\phi \in Q(X)$ was arbitrary and $T : Q(X) \to Q(Y)$ is linear, we have

$$T\phi = c \cdot h^*\phi$$

for all $\phi \in Q(X)$, with c independent of ϕ. Since T is an isometric embedding, we have

$$|c| = \frac{\|\phi\|}{\|h^*\phi\|} = \deg(h)^{-1}.$$

This completes the proof of Theorem V.1.

5.3 Using Theorem V.1 to Prove Theorem I.7

5.3.1 The Setup

With the tools and results established in the previous section, we begin the proof of Theorem I.7. Let $F : \mathcal{T}(S_{g,n}) \to \mathcal{T}(S_{k,m})$ be a holomorphic and isometric submersion of Teichmüller spaces. Assume $2k+m \geq 5$ and $k \geq 1$. By Corollary V.5, we have for each $Y \in \mathcal{T}(S_{g,n})$ that the induced map of cotangent spaces $Q(F(Y)) \to Q(Y)$ is an isometric embedding. Since $2k+m \geq 5$, Theorem V.1 tell us that the embedding is, up to scale, pull-back by a holomorphic branched cover of compact surfaces

$$h_Y : \widehat{Y} \to \widehat{F(Y)}.$$

We conclude in particular that every Riemann surface of genus g admits a holomorphic branched cover of a surface of genus h. We now use our assumption that $k \geq 1$. The following elementary lemma implies that $g = k$.

Lemma V.9. *Suppose $g \geq 2$. It is not the case that every $X \in \mathcal{T}(S_g)$ admits a holomorphic cover of a surface of genus k with $1 \leq k < g$.*

Proof. The proof is by a dimension comparison. Suppose $1 \leq k < g$ and let $f : S_g \to S_k$ be a degree d branched cover. Recall the Riemann-Hurwitz formula:

$$2 - 2g = d \cdot (2 - 2k) - b,$$

where b is the total branch order of the cover.

We distinguish the cases $k = 1$ and $k \geq 2$. If $k \geq 2$, we have $\dim \mathcal{T}_g = 3g - 3$ and $\dim \mathcal{T}_k = 3k - 3$, so we get

$$\dim \mathcal{T}_g = d \cdot \dim \mathcal{T}_k + \frac{3}{2} b.$$

On the other hand, for a fixed topological type of branched cover, the space of surfaces in $Y \in \mathcal{T}_g$ which admit a holomorphic cover $Y \to X$ of that type has dimension at most

$$\dim \mathcal{T}_k + b,$$

which is less than $\dim \mathcal{T}(S_g)$ since $g > k$ and thus $d > 1$.

If $k = 1$, then $\dim \mathcal{T}(S_g) = \frac{3}{2}b$ and the dimension of the locus of $X \in \mathcal{T}_g$ which admit a holomorphic cover of the given type is at most b. Since $g > k = 1$, the cover must have $b > 0$ and so $b < \frac{3}{2}b = \dim \mathcal{T}(S_g)$.

Thus, the locus of $X \in \mathcal{T}_g$ covering a surface of genus less than g and greater than 0 is a countable union of lower-dimensional subvarieties. The lemma follows. $\square$

Remark V.10. The locus of $X \in \mathcal{T}(S_g)$ which cover the square torus (i.e. the collection of square-tiled surfaces) is dense. This follows from the fact that the locus of abelian differentials with rational period coordinates is dense in the Hodge bundle over $\mathcal{T}(S_g)$ [57].

We conclude that $g = k$, so our submersion F maps from $\mathcal{T}(S_{g,n})$ to $\mathcal{T}(S_{g,m})$ with $m \leq n$. We are almost done: If $g \geq 2$, the covering maps $h_Y : \widehat{Y} \to \widehat{F(Y)}$ must be biholomorphisms. If $g = 1$, we know a priori only that h_Y are (unbranched) holomorphic covers. Since the pullback h_Y^* sends $Q\left(F(Y)\right)$ into $Q(Y)$, each preimage of a puncture p in $F(Y)$ must be a puncture of Y. (Otherwise, h_Y pulls a differential with a pole at p back to a differential which is not in $Q(Y)$.) Thus, h_Y restricts to a map between the (potentially punctured) surfaces Y and X. The map $h_Y : Y \to X$ and the markings $S_{g,n} \to Y$, $S_{g,m} \to X$ fit into a diagram

$$\begin{array}{ccc} S_{g,n} & \longrightarrow & S_{g,m} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{h_Y} & X \end{array} .$$

It remains to establish two facts.

1. The maps h_Y are biholomorphisms in the $g = 1$ case.
2. The isotopy class of $S_{g,n} \to S_{g,m}$, is independent of $Y \in \mathcal{T}(S_{g,n})$.

The key to establishing both is showing that the family $h_Y : \widehat{Y} \to \widehat{F(Y)}$ varies continuously in the variable Y. To make this precise, we observe that the maps

$h_Y : \widehat{Y} \to \widehat{F(Y)}$ fit together into a map of universal curves $H : \mathcal{C}_{g,n} \to \mathcal{C}_{g,m}$ covering the map $F : \mathcal{T}(S_{g,m}) \to \mathcal{T}(S_{g,m})$ of Teichmüller spaces:

$$\begin{array}{ccc} \mathcal{C}_{g,n} & \xrightarrow{H} & \mathcal{C}_{g,m} \\ \downarrow & & \downarrow \\ \mathcal{T}(S_{g,n}) & \xrightarrow{F} & \mathcal{T}(S_{g,m}) \end{array}$$

We will show in the next section that H is continuous. Recall h_Y was constructed using the maps $X \to \mathbb{P}Q(X)^*$ and $Y \to \mathbb{P}Q(Y)^*$. We will leverage properties of the bundle of quadratic differentials over Teichmüller space to prove that H is in fact holomorphic.

5.3.2 The Universal Curve and the Cotangent Bundle

We start by recalling the properties of the universal curve $\pi : \mathcal{C}_{g,n} \to \mathcal{T}(S_{g,n})$. A good reference for this material is [43].

The map $\pi : \mathcal{C}_{g,n} \to \mathcal{T}(S_{g,n})$ is a holomorphic submersion whose fiber over $X \in \mathcal{T}(S_{g,n})$ is exactly the compact Riemann surface $\widehat{X}$. The locations of the punctures are encoded by canonical holomorphic sections

$$s_i : \mathcal{T}(S_{g,n}) \to \mathcal{C}_{g,n} \; i = 1, \ldots, n.$$

The point $s_i(X) \in \widehat{X}$ is the ith puncture of X. Moreover, there is a canonical topological trivialization

$$\mathcal{F}_{g,n} : \mathcal{T}(S_{g,n}) \times S_{g,n} \to \mathcal{C}_{g,n} \setminus \bigcup_{i=1}^{n} s_i(\mathcal{T}(S_{g,n})),$$

unique up to fiberwise isotopy, so that the induced marking of each fiber

$$S_{g,n} \to \{X\} \times S_{g,n} \stackrel{\mathcal{F}_{g,n}}{\to} X$$

agrees with the marking defining X as a point of $\mathcal{T}(S_{g,n})$. The family $(\pi, \{s_i\}_{i=1}^{n}, \mathcal{F}_{g,n})$ is universal among n-pointed marked holomorphic families of genus g Riemann surfaces (see [43]).

Now, let $\mathcal{Q}_{g,n} \to \mathcal{T}(S_{g,n})$ denote the bundle of integrable holomorphic quadratic differentials over Teichmüller space. Let $\mathbb{P}\mathcal{Q}^*_{g,n} \to \mathcal{T}(S_{g,n})$ denote the associated holomorphic bundle of projectivized dual spaces. The bi-canonical maps $\widehat{X} \to \mathbb{P}Q(X)^*$ fit into a map

$$\Psi : \mathcal{C}_{g,n} \to \mathbb{P}\mathcal{Q}^*_{g,n}$$

covering the projections to Teichmüller space. We need to show that this map of bundles is holomorphic.

Proposition V.11. *The fiberwise bi-canonical map $\Psi : \mathcal{C}_{g,n} \to \mathbb{P}\mathcal{Q}^*_{g,n}$ is holomorphic. If the type (g, n) is non-exceptional, then the map is a biholomorphism onto its image.*

Proof. Since π is a holomorphic submersion, $\mathcal{C}_{g,n}$ is covered by product neighborhoods $U \times V$, with U open in $\mathcal{T}(S_{g,n})$ and V open in $\mathbb{C}$. Each $U \times V$ maps biholomorphically to an open neighborhood of $\mathcal{C}_{g,n}$ by a map commuting with the projections:

$$\begin{array}{ccc} U \times V & \longrightarrow & \mathcal{C}_{g,n} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{T}(S_{g,n}) \end{array}$$

Given $X \in U$, the slice $\{X\} \times V$ is a holomorphic coordinate chart for the Riemann surface $\widehat{X}$. For this reason, the product neighborhoods $U \times V$ are called *relative coordinate charts* for the family $\mathcal{C}_{g,n}$.

Recall $\mathcal{Q}_{g,n} \to \mathcal{T}(S_{g,n})$, the bundle of integrable holomorphic quadratic differentials over Teichmüller space. A section $q : \mathcal{T}(S_{g,n}) \to \mathcal{Q}_{g,n}$ can be thought of as a fiberwise quadratic differential on $\mathcal{C}_{g,n}$. In a relative coordinate chart $U \times V$, the differential q takes the form $q(X, z)(dz)^2$. It follows by a result of Bers [8] that a section $q : \mathcal{T}(S_{g,n}) \to \mathcal{Q}_{g,n}$ is holomorphic if and only if $(X, z) \mapsto q(X, z)$ is meromorphic in each relative chart $U \times V$.

Now, let $U \times V$ be a relative coordinate chart for $\mathcal{C}_{g,n}$ and let $q_0, \ldots, q_k$ be a holomorphic frame for $\mathcal{Q}_{g,n} \to \mathcal{T}(S_{g,n})$ over U. With respect to the choice of coordinates and frame, the fiberwise bi-canonical map $\mathcal{C}_{g,n} \to \mathbb{P}Q^*_{g,n}$ is expressed as the map $U \times V \to \mathbb{P}^k$ given by

$$(X, z) \mapsto [q_0(X, z) : q_1(X, z) : \cdots : q_k(X, z)], \tag{5.4}$$

which is holomorphic since the $q_i(X, z)$ are meromorphic.

We conclude that $\Psi : \mathcal{C}_{g,n} \to \mathbb{P}Q^*_{g,n}$ is holomorphic, as claimed. If (g, n) is non-exceptional, then Ψ restricts to an embedding on the fibers of $\mathcal{C}_{g,n} \to \mathcal{T}(S_{g,n})$. Since the fibers are compact, Ψ is a biholomorphism onto its image. $\square$

We now prove the main result of this subsection.

Proposition V.12. *The map $H : \mathcal{C}_{g,n} \to \mathcal{C}_{g,m}$ defined in the last section is holomorphic.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
\mathcal{C}_{g,n} \arrow[r, "\Psi"] \arrow[dr] & \mathbb{P}Q^*_{g,n} \arrow[r, "F_*"] \arrow[d] & \mathbb{P}Q^*_{g,m} \arrow[d] & \mathcal{C}_{g,m} \arrow[l, "\Phi"'] \arrow[dl] \\
& \mathcal{T}(S_{g,n}) \arrow[r, "F"] & \mathcal{T}(S_{g,m}) &
\end{tikzcd}$$

Here, Ψ and Φ denote the fiberwise bi-canonical maps, which are holomorphic by Proposition V.11. The map F_* can be viewed in two ways.

1. F_* is the projectivization of the derivative of the holomorphic map F.

2. On the fiber over $Y \in \mathcal{T}(S_{g,n})$, F_* is the dual of the isometric embedding $dF^*_Y : Q\left(F(Y)\right) \hookrightarrow Q(Y)$.

The first interpretation shows that F_* is holomorphic. The second interpretation, combined with the results of Section 5.2.5, shows that $F_* \circ \Psi$ has the same image as

Φ. Moreover, $H : \mathcal{C}_{g,n} \to \mathcal{C}_{g,m}$ is the unique map so that

$$F_* \circ \Psi = \Phi \circ H.$$

But since (g,m) is non-exceptional, Φ is a biholomorphism onto its image. Thus, H can be expressed as the composition of holomorphic maps

$$\mathcal{C}_{g,n} \overset{F_*\circ\Psi}{\to} \Phi\left(\mathcal{C}_{g,m}\right) \overset{\Phi^{-1}}{\to} \mathcal{C}_{g,m}.$$

$\square$

5.3.3 Completing the Proof of Theorem I.7

As discussed at the end of Section 5.3.1, each map $h_Y : \widehat{Y} \to \widehat{X}$ sends Y to X. Thus, there is a unique map $G : \mathcal{T}(S_{g,n}) \times S_{g,n} \to \mathcal{T}(S_{g,m}) \times S_{g,m}$ fitting into the diagram

$$\begin{tikzcd} \mathcal{T}(S_{g,n}) \times S_{g,n} \arrow[r, "G"] \arrow[d, "\mathcal{F}_{g,n}"] & \mathcal{T}(S_{g,m}) \times S_{g,m} \arrow[d, "\mathcal{F}_{g,m}"] \\ \mathcal{C}_{g,n} \arrow[r, "H"] & \mathcal{C}_{g,m}, \end{tikzcd}$$

where the vertical maps are the canonical trivializations discussed in the last section. Since H is continuous, the maps $S_{g,n} \to S_{g,m}$ obtained by restricting G to fibers are all isotopic. Restricting the above commutative square to fibers, we conclude that there is a fixed $f : S_{g,n} \to S_{g,m}$ so that

$$\begin{tikzcd} S_{g,n} \arrow[r, "f"] \arrow[d] & S_{g,m} \arrow[d] \\ Y \arrow[r, "h_Y"] & F(Y) \end{tikzcd}$$

commutes up to isotopy for all $Y \in \mathcal{T}(S_{g,n})$. By construction, the vertical arrows are the markings defining Y and $F(Y)$ as points of Teichmüller space. If $g \geq 2$, we already know that $f : S_{g,n} \to S_{g,m}$ is one-to-one. Thus, up to pre-composition by a mapping class, $Y \mapsto F(Y)$ is the forgetful map filling in the last $n - m$ punctures. This completes the proof when $g \geq 2$.

To finish the proof in the case $g = 1$, it suffices to establish that $f : S_{1,n} \to S_{1,m}$ is one-to-one. We prove this by another dimension argument. The point is that, if the degree of f is greater than 1, then not every $X \in \mathcal{T}(S_{1,n})$ admits a non-constant holomorphic map to a $Y \in \mathcal{T}(S_{1,m})$.

In more detail: Let d denote the degree of the cover $S_1 \to S_1$ obtained by extending f over the punctures. Then f factors through a degree d (unbranched) cover $S_{1,dm} \to S_{1,m}$.

$$\begin{tikzcd} S_{1,n} \arrow[rr, "f"] \arrow[dr] & & S_{1,m} \\ & S_{1,dm} \arrow[ur] & \end{tikzcd}$$

The covering $S_{1,dm} \to S_{1,m}$ induces an isometric embedding of Teichmüller spaces $\mathcal{T}(S_{1,m}) \hookrightarrow \mathcal{T}(S_{1,dm})$, while the injective map $S_{1,n} \to S_{1,dm}$ induces a forgetful map $\mathcal{T}(S_{1,n}) \twoheadrightarrow \mathcal{T}(S_{1,dm})$. These fit into the diagram

$$\begin{tikzcd} \mathcal{T}(S_{1,n}) \arrow[rr, "F"] \arrow[dr, two heads] & & \mathcal{T}(S_{1,m}) \arrow[dl, hook'] \\ & \mathcal{T}(S_{1,dm}) & \end{tikzcd}$$

Thus, $\mathcal{T}(S_{1,m}) \hookrightarrow \mathcal{T}(S_{1,dm})$ is surjective, which implies $d = 1$. $\square$

5.4 Infinitely Punctured Surfaces

In this section, we take the first few steps towards generalizing Theorem I.7 to infinitely-punctured surfaces of finite genus. We will start by introducing a few tools which allow us to recycle methods from the proof of Theorem I.7 in the new case of infinite-dimensional Teichmüller spaces. Then, we will prove our main results of this section, stated as Theorem I.11.

5.4.1 Tools for Infinite-Dimensional Teichmüller Spaces

Let $F : \mathcal{T}(X) \to \mathcal{T}(Y)$ be a Teichmüller-metric holomorphic isometric submersion whose derivative maps are weak*-sequentially continuous. Then $dF_\tau : T_\tau\mathcal{T}(X) \to T_{F(\tau)}\mathcal{T}(Y)$ is a linear isometric submersion between the tangent spaces for $\tau \in \mathcal{T}(X)$. In this section, we will need the following technical assumption: all derivative maps dF_τ for $\tau \in \mathcal{T}(X)$ are weak*-sequentially continuous.

In the finite-type case, the first step is to take the coderivative and observe that by Lemma V.4 it must be an isometric embedding of spaces of quadratic differentials. For infinite-type surfaces, we cannot simply take a coderivative of an isometric submersion to obtain an embedding of quadratic differential spaces, since the dual of the tangent space to infinite-dimensional Teichmüller spaces is not $Q(X)$ (in fact, $Q(X)$ is the pre-dual). However, tools of Earle-Gardiner [15] generalize for our purposes.

Recall that convergence in the weak* topology on $Q^*(X)$ is equivalent to pointwise convergence of functionals in $Q^*(X)$ viewed as functions $Q(X) \to \mathbb{C}$. Recall also that the norm $||\cdot||$ on $Q^*(X)$ is given by

$$||v|| = \sup\{|v(\varphi)| : \varphi \in Q(X) \text{ and } ||\varphi|| = 1\}$$

with the 1-norm on $Q(X)$. First, we will need the following fact from [14] and [18]. The statement is based on the presentation in §6 of [15].

Lemma V.13. *Let X be any Riemann surface. There exists a subspace $Q^*(X)_0 \subseteq Q^*(X)$ such that $(Q^*(X)_0)^*$ is isometrically isomorphic to $Q(X)$. Further, $Q^*(X)_0$ is weak*-dense in $Q^*(X)$; that is, for all $v \in Q^*(X)$, there exists a sequence $(v_n) \in Q^*(X)_0$ such that $v_n \to v$ in the weak* sense.*

We adopt the notation and description of the result as described in [15] which includes a characterization of the subspace $Q^*(X)_0$. We will utilize our assumption of weak* sequential continuity in the following.

Proposition V.14 (Earle-Gardiner Adjointness Theorem generalized to isometric submersions)**.** *If $S : Q^*(X) \to Q^*(Y)$ is a weak*-sequentially continuous $\mathbb{C}$-linear isometric submersion, then there is a $\mathbb{C}$-linear isometric embedding $T : Q(Y) \to Q(X)$ such that $T^* = S$.*

Proof. Consider the map $(S|_{Q^*(X)_0})^*$ defined as the restriction of the adjoint of S to $Q^*(X)_0$. By Lemma V.13, this is a map $Q(Y)^{**} \to Q(X)$. Define $T = (S|_{Q^*(X)_0})^*|_{Q(Y)}$. Being (a restriction of) the adjoint of an isometric submersion, this is a $\mathbb{C}$-linear isometric embedding $T : Q(Y) \to Q(X)$. We have that

$$v(T\varphi) = (Sv)(\varphi) \tag{5.5}$$

for all $\varphi \in Q(Y)$ and $v \in Q^*(X)_0$. What remains is to show that $S = T^*$. To show this, we must show that Equation 5.5 holds not just for all $v \in Q^*(X)_0$ but for all of $Q^*(X)$.

Choose any $\varphi \in Q(Y)$ and $v \in Q^*(X)$. Let $(v_n) \in Q^*(X)_0$ be a sequence with $v_n \to v$ (weak*). By assumption, $Sv_n \to Sv$ (weak*). And so

$$v(T\varphi) = \lim_{n\to\infty} v_n(T\varphi) = \lim_{n\to\infty} (Sv_n)(\varphi) = (Sv)(\varphi),$$

with the middle equality coming from Equation 5.5 and the outer ones by weak*-sequential continuity. □

Remark V.15*.* Proposition V.14 allows us to recover an associated embedding of spaces of quadratic differentials even in the infinite-dimensional case. For the finite-dimensional case, we simply take the dual of the derivative since pre- and post-duals are equivalent in that case. In a sense, this is allowing us to find what amounts to a canonical pre-dual of the derivative map.

Next, we verify that forgetful maps are indeed isometric submersions with the required sequential continuity, strengthening Lemma V.6.

Lemma V.16. *Let X be a Riemann surface (possibly with punctures) and let $S \subset T \subset X$ be countable sets of points. Let $\mathcal{F} : \mathcal{T}(X - T) \to \mathcal{T}(X - S)$ be the forgetful map filling in the punctures in $T - S$. Then $\mathcal{F}$ is an isometric submersion whose derivative maps are weak* sequentially continuous.*

Proof. The forgetful map $\mathcal{F}$ is induced by the inclusion map $X - T \hookrightarrow X - S$. On the level of quadratic differentials, this induces the inclusion $i : Q(X - S) \hookrightarrow Q(X - T)$, which is an isometric embedding. The dual of this map $i^* : Q(X - T)^* \to Q(X - S)^*$ is thus an isometric submersion with respect to the dual norm, which is known to coincide with the infinitesimal Teichmüller metric.

Now, we show i^* is weak*-sequentially continuous. Let $(v_n) \in Q^*(X - T)$ be a sequence which weak*-converges to $v \in Q^*(X - T)$. We show that $i^* v_n \to i^* v$ (weak*) in $Q^*(X - S)$. Let $\varphi \in Q(X - S)$. Then

$$i^* v_n(\varphi) = v_n(i\varphi) \to v(i\varphi) = i^* v(\varphi) \tag{5.6}$$

by weak* convergence of (v_n). The map i^* is a (linear) projection map. Consider the derivative of $\mathcal{F}$ at the basepoint:

$$d\mathcal{F}_{X-T} : T_{X-T}\mathcal{T}(X - T) \to T_{X-S}\mathcal{T}(X - S).$$

Because $\mathcal{F}$ is forgetful, the map $d\mathcal{F}_{X-T}$ must be dual to the embedding $Q(X - S) \hookrightarrow Q(X - T)$. Thus $d\mathcal{F}_{X-T} = i^*$, and so $\mathcal{F}$ satisfies the conclusion at the basepoint, and similarly it will be the case at all other points. ∎

Remark V.17. Here, showing weak*-sequential continuity of i^* is straightforward since we already have a pre-dual, namely the map i itself, allowing us in Equation

5.6 to simply look at the quadratic differential $i(\varphi)$ and immediately use weak*-convergence of the sequence (v_n). Our technical assumption is useful in the proof of Lemma V.14 since we do not a priori know if the map $S : Q^*(X) \to Q^*(Y)$ comes from the dual of a map $Q(Y) \to Q(X)$ on the underlying space of quadratic differentials.

Now, Proposition I.6 is simply part of Lemma V.16. Looking ahead, we will only need the case of Lemma V.16 where the surfaces are finite genus and the set S is empty (i.e. the forgetful map is filling in all the punctures). Proving Theorem I.11 will require further study of isometric embeddings of spaces of quadratic differentials – in particular, we start by generalizing Theorem V.1.

5.4.2 Finding Maps Between Underlying Surfaces

We first prove part 2 of Theorem I.11. To begin, we will study embeddings of spaces of quadratic differentials.

Lemma V.18. *Let X be a finite genus Riemann surface of non-exceptional type, possibly with infinitely many punctures, and Y be a non-exceptional-type Riemann surface of finite type. Let $T : Q(Y) \to Q(X)$ be a $\mathbb{C}$-linear isometric embedding. Then there exists a holomorphic map $h : X \to Y$ and some $c \in \mathbb{C}$ with $|c| = \deg(h)^{-1}$ so that $T = c \cdot h^*$.*

This generalizes Theorem V.1, which was our main intermediate step along the way to Theorem I.7. Recall that $\hat{X}$ is the filled-in surface obtained by forgetting the punctures of X.

Proof. We will give an outline of the main steps, since the proof is nearly identical to that of Theorem V.1. Let $q_0, q_1, \ldots, q_l$ be a basis of $Q(Y)$, which is assumed to be finite. Write $r_i = T(q_i)$ for each i. While the dimension of $Q(X)$ is infinite, as in the finite case, we need only consider the subspace consisting of the image of T.

By the machinery of Section 5.2.4, the map $\Psi := (r_1/r_0, \ldots, r_l/r_0) : X \to \hat{\mathbb{C}}^l$ is a holomorphic map, while $\Phi := (q_1/q_0, \ldots, q_l/q_0) : Y \to \hat{\mathbb{C}}^l$ is a holomorphic embedding. Let $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$. The following integrals are equal by linearity of T and the same change-of-variables approach as in the proof of Proposition V.3:

$$\int_{\hat{X}} \left| r_0 + \sum_{i=1}^{l} \lambda_i r_i \right| = \int_{\hat{Y}} \left| q_0 + \sum_{i=1}^{l} \lambda_i q_i \right|$$

This is the key step in reaching the hypotheses of the Rudin result about the measures μ and ν on Y and X coming from the integrals of q_0 and r_0 respectively.

By Proposition V.7, the induced measures $\Phi_*(\mu)$ and $\Psi_*(\nu)$ are equal, and the support is the compact set $\Phi(\hat{Y})$, since $\hat{Y}$ is finite-type and Φ is continuous. By the same argument as in Section 5.2.5, we find that $\Psi(\hat{X}) \subseteq \Phi(\hat{Y})$. Thus we obtain the desired map

$$\hat{X} \to \Psi(\hat{X}) \to \Phi(\hat{Y}) \to \hat{Y},$$

which we will denote by $h : \hat{X} \to \hat{Y}$. It is not (necessarily) bijective but it is holomorphic.

Because we assumed $\hat{X}$ is finite-type (i.e. the filled-in surface has finite genus), the rest of the proof that this map h is an inclusion map which induces T (and that X and Y have the same genus) is identical to the case of X having at most finitely many punctures proven in Theorem V.1. It is also identical to before to show that h restricts to a map on the punctured surfaces, $h : X \to Y$. $\square$

Now to prove the second assertion of Theorem I.11, let $F : \mathcal{T}(X) \to \mathcal{T}(Y)$ be a holomorphic isometric submersion whose derivatives are weak*-sequentially continuous. Let $\tau \in \mathcal{T}(X)$ be represented by $\varphi : X \to X'$, and let $F(\tau)$ be represented by $\psi : Y \to Y'$. By Proposition V.14, there exists an isometric embedding of the cotangent spaces $T : Q(Y') \to Q(X')$ which satisfies the conditions of Lemma V.18. Thus there exists a map $h_\tau : X \to Y$ between the underlying surfaces which induces T, and by construction the dual map matches the derivative: $T^* = dF_\tau$ as desired.

5.4.3 Infinite Punctures in the Codomain

Now, we complete the proof of Theorem I.11 by studying the case of Y having infinitely many punctures. Let $T \subseteq X$ and $S \subseteq Y$ be countable subsets of closed, hyperbolic surfaces X and Y of finite genus, with $X - T$ and $Y - S$ of non-exceptional type. Let $F : \mathcal{T}(X - T) \to \mathcal{T}(Y - S)$ be an isometric submersion whose derivative maps are weak*-sequentially continuous.

We may post-compose with a forgetful map $G : \mathcal{T}(Y - S) \to \mathcal{T}(Y)$ to obtain $G \circ F : \mathcal{T}(X - T) \to \mathcal{T}(Y)$, which by the work of Section 5.4.2 and Lemma V.16 must satisfy the conclusion of Theorem I.11(2). This means that at each point, $G \circ F$ is induced by a holomorphic inclusion map on underlying Riemann surfaces. It follows that for each representative $\varphi : X - T \to Z$ of a point $\tau \in \mathcal{T}(X - T)$, we must have that the marked surface $G \circ F(\tau)$ is biholomorphic to $\hat{Z}$. In particular, $G \circ F$ cannot change the conformal type of the underlying filled-in surface. Because G is a forgetful map, it must be the case that F also cannot change the conformal type of the underlying filled-in surface. We conclude that X and Y have the same genus. □

We end with the following lemma, which shows that generalizing Theorem I.7 to the case of infinite punctures in both the domain and codomain hinges on only showing it for the case of infinite punctures in the domain space. For clarity, below we use the notation (X, S) where $S \subseteq X$ for a surface X with S the set of marked points.

Lemma V.19. *Let $F : \mathcal{T}(X,T) \to \mathcal{T}(X,S)$ be a map. For each finite subset $A \subseteq T$, denote by G_A the forgetful map $G_A : \mathcal{T}(X,S) \to \mathcal{T}(X,A)$. Suppose we have that for all such A, $G_A \circ F : \mathcal{T}(X,T) \to \mathcal{T}(X,A)$ is forgetful. Then F is also a forgetful map.*

Proof. Let $f : (X,S) \to (X', S_{X'})$ represent a point in $\mathcal{T}(X,S)$, and consider the fiber

$$F^{-1}(f) \subseteq \mathcal{T}(X,T).$$

It must be contained in the fiber $(G_A \circ F)^{-1} G_A(f)$ obtained by sending f to $\mathcal{T}(X,A)$ and then pulling back to $\mathcal{T}(X,T)$. We have

$$F^{-1}(f) \subseteq (G_A \circ F)^{-1}(G_A(f)).$$

In particular, since $G_A \circ F$ is forgetful, the fiber of F over f only contains marked surfaces of the same conformal type (for the filled-in surface) and the punctures of A are all in the same place. Next, if $A' \subseteq A$, then

$$(G_A \circ F)^{-1}(G_A(f)) \subseteq (G_{A'} \circ F)^{-1}(G_{A'}(f))$$

because the left-hand side corresponds to a fiber where all points in A are specified. We further have that

$$F^{-1}(f) \subseteq \bigcap_{A \subseteq S} (G_A \circ F)^{-1}(G_A(f))$$

with the intersection over all finite subsets of S. It follows all $g \in F^{-1}(f)$ correspond to surfaces with the same conformal type and all punctures of S are in the same place, which means the fibers over f in $\mathcal{T}(X,T)$ are equal to the fibers of a forgetful map. This holds for all $f \in \mathcal{T}(Y,S)$, and so we conclude that F is itself a forgetful map.

∎

BIBLIOGRAPHY

[1] William Abikoff. *The real analytic theory of Teichmüller space*, volume 820. Springer-Verlag, 1980.

[2] Vincent Alberge, Athanase Papadopoulos, and Weixu Su. A commentary on Teichmüller's paper "Extremale quasikonforme abbildungen und quadratische differentiale". In Athanase Papadopoulos, editor, *Handbook of Teichmüller Theory, Volume V*, pages 485–531. European Mathematical Society Publishing House, 2015.

[3] Juan-Carlos Álvarez Paiva and Carlos Durán. Isometric submersions of finsler manifolds. *Proceedings of the American Mathematical Society*, 129(8):2409–2417, 2001.

[4] Stergios Antonakoudis. *The complex geometry of Teichmüller space*. PhD thesis, 2014.

[5] Stergios Antonakoudis, Javier Aramayona, and Juan Souto. Holomorphic maps between moduli spaces. *Annales de l'Institut Fourier*, 68(1):217–228, 2018.

[6] Abdelhadi Belkhirat, Athanase Papadopoulos, and Marc Troyanov. Thurston's weak metric on the Teichmüller space of the torus. *Transactions of the American Mathematical Society*, 357(8):3311–3324, 2005.

[7] Lipman Bers. Quasiconformal mappings and Teichmüller's theorem. *Princeton Math. Ser.*, 24:89–119, 1960.

[8] Lipman Bers. Holomorphic differentials as functions of moduli. *Bulletin of the American Mathematical Society*, 67(2):206–210, 1961.

[9] Armand Borel and Lizhen Ji. *Compactifications of symmetric and locally symmetric spaces.* Mathematics: Theory & Applications, Birkäuser Boston Inc., 2006.

[10] Martin R Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[11] Manfredo Perdigao Do Carmo. *Riemannian geometry.* Birkhäuser, 1992.

[12] Manfredo Perdigao Do Carmo. *Differential geometry of curves and surfaces: revised and updated second edition.* Courier Dover Publications, 2016.

[13] Simon Donaldson. *Riemann surfaces.* Oxford University Press, 2011.

[14] Clifford Earle. The integrable automorphic forms as a dual space. In Thomas A Metzger, editor, *Modular Functions in Analysis and Number Theory*, pages 30–40. Department of Mathematics and Statistics, University of Pittsburgh, 1983.

[15] Clifford Earle and Frederick Gardiner. Geometric isomorphisms between infinite dimensional Teichmüller spaces. *Transactions of the American Mathematical Society*, 348(3):1163–1190, 1996.

[16] Clifford Earle and Irwin Kra. On sections of some holomorphic families of closed Riemann surfaces. *Acta Mathematica*, 137(1):49–79, 1976.

[17] Clifford Earle and Vladimir Markovic. Isometries between the spaces of L^1 holomorphic quadratic differentials on Riemann surfaces of finite type. *Duke Mathematical Journal*, 120(2):433–440, 2003.

[18] Clifford J Earle. The integrable automorphic forms as a dual space, II. *Complex Variables and Elliptic Equations*, 12(1-4):153–158, 1989.

[19] Clifford J Earle and Irwin Kra. On holomorphic mappings between Teichmüller spaces. In *Contributions to analysis*, pages 107–124. Elsevier, 1974.

[20] Benson Farb and Dan Margalit. *A primer on mapping class groups (pms-49)*. Princeton University Press, 2011.

[21] Hershel M Farkas and Irwin Kra. Riemann surfaces. In *Riemann surfaces*, pages 9–31. Springer, 1992.

[22] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Thurston's Work on Surfaces*, volume 48. Princeton University Press, 2012.

[23] Arthur Fischer and Anthony Tromba. On the Weil-Petersson metric on Teichmüller space. *Transactions of the American Mathematical Society*, 284(1):319–335, 1984.

[24] Frederick W Gehring, Gaven J Martin, and Bruce P Palka. *An introduction to the theory of higher-dimensional quasiconformal mappings*, volume 216. American Mathematical Soc., 2017.

[25] Dmitri Gekhtman and Mark Greenfield. Isometric submersions of Teichmüller spaces are forgetful. *Preprint, arXiv:1901.02586*, 2019. To appear.

[26] Mark Greenfield and Lizhen Ji. A new modular characterization of the hyperbolic plane. *Preprint arXiv:1707.00818*, 2017.

[27] Mark Greenfield and Lizhen Ji. Metrics and compactifications of Teichmüller spaces of flat tori. *Preprint, arXiv:1903.10655*, 2020. To appear.

[28] Misha Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, NY, 1978)*, volume 97, pages 183–213, 1981.

[29] Yves Guivarc'h, Lizhen Ji, and John C Taylor. *Compactifications of symmetric spaces*, volume 156. Springer, 2012.

[30] Thomas Haettel. Compactification de Thurston despaces de réseaux marqués et de lespace de Torelli. *Groups Geom. Dyn.*, 9(2):331–368, 2015.

[31] Thomas Haettel, Anna-Sofie Schilling, Cormac Walsh, and Anna Wienhard. Horofunction compactifications of symmetric spaces. *Preprint arXiv:1705.05026*, 2018.

[32] Brian Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, 2015.

[33] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*. American Mathematical Society, 2001.

[34] John H Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics, Volume I: Teichmüller theory*. 2006.

[35] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer, 2012.

[36] Lizhen Ji. Satake and Martin compactifications of symmetric spaces are topological balls. *Mathematical Research Letters*, 4(1):79–89, 1997.

[37] Lizhen Ji and Athanase Papadopoulos. Historical development of Teichmüller theory. *Archive for History of exact Sciences*, 67(2):119–147, 2013.

[38] Shoshichi Kobayashi. Intrinsic metrics on complex manifolds. *Bulletin of the American Mathematical Society*, 73(3):347–349, 1967.

[39] Olli Lehto. *Univalent functions and Teichmüller spaces*, volume 109. Springer, 2012.

[40] Carlangelo Liverani and Maciej Wojtkowski. Generalization of the Hilbert metric to the space of positive definite matrices. *Pacific Journal of Mathematics*, 166(2):339–355, 1994.

[41] Vladmir Markovic. Biholomorphic maps between Teichmüller spaces. *Duke Mathematical Journal*, 120(2):405–431, 2003.

[42] Curtis McMullen. Foliations of Hilbert modular surfaces. *American Journal of Mathematics*, 129(1):182–215, 2007.

[43] Subhashis Nag. *The complex analytic theory of Teichmüller spaces*. Wiley-Interscience, 1988.

[44] Raghavan Narasimhan. *Compact Riemann Surfaces*. Birkhäuser, 1992.

[45] Pierre Planche. *Géométrie de Finsler sur les espaces symétriques*. PhD thesis, Atelier de reproduction de la Section de physique, 1995.

[46] Halsey Royden. Automorphisms and isometries of Teichmüller space. In *Advances in the Theory of Riemann Surfaces*, volume 66 of *Annals of Mathematics Studies*, pages 369–383. Princeton University Press, 1971.

[47] Walter Rudin. L^p-isometries and equimeasurability. *Indiana University Mathematics Journal*, 25(3):215–228, 1976.

[48] Ichiro Satake. On representations and compactifications of symmetric Riemannian spaces. *Annals of Mathematics*, pages 77–110, 1960.

[49] Sanford L Segal. *Mathematicians under the Nazis*. Princeton University Press, 2014.

[50] Weixu Su. Problems on Thurston metric. *Handbook of Teichmüller Theory, Volume V*, pages 55–72, 2016.

[51] Oswald Teichmuller. Extremale quasikonforme abbildungen und quadratische differentialen. *Abh. Preuss. Akad. Wiss.*, 22:3–197, 1939.

[52] Audrey Terras. *Harmonic analysis on symmetric spaces: higher rank spaces, positive definite matrix space and generalizations*. Springer, 2016.

[53] William P Thurston. Minimal stretch maps between hyperbolic surfaces. *Preprint arxiv:9801039*, 1998.

[54] Cormac Walsh. The horoboundary and isometry group of Thurston's Lipschitz metric. In Athanase Papadopoulos, editor, *Handbook of Teichmüller Theory, Volume IV*, IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2014.

[55] André Weil. Modules des surfaces de Riemann. *Seminare N. Bourbaki*, 168:413–419, 1958.

[56] Sumio Yamada. Local and global aspects of Weil-Petersson geometry. In *Handbook of Teichmüller theory. Vol. IV, 43–111, IRMA Letc. Math.* 2014.

[57] Anton Zorich. Flat surfaces. In *Frontiers in Number Theory, Physics and Geometry. Volume 1: On random matrices, zeta functions and dynamical systems*, pages 439–586. Spring-Verlag, Berlin, 2006.