Bernstein-Sato Theory in Positive Characteristic

by

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To my parents.

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ABSTRACT

Given a holomorphic function f, its Bernstein-Sato polynomial is a classical invariant that detects the singularities of the zero locus of f in very subtle ways; for example, its roots recover the log-canonical threshold of f and the eigenvalues of the monodromy action on the cohomology of the Milnor fibre. In this thesis we continue the work of Bitoun and Mustaţă to develop an analogue of this invariant in positive characteristic. More concretely, we develop a notion of Bernstein-Sato polynomial for arbitrary ideals (which, over the complex numbers, was done by Budur, Mustaţă and Saito), we show that its roots are always rational and negative and that they encode some information about the F-jumping numbers. We also prove that for monomial ideals we can recover the roots of the classical Bernstein-Sato polynomial from this characteristic-p version.

CHAPTER I
Introduction

Let k be an algebraically closed field (for example, the complex numbers or $\overline{\mathbb{F}_p}$). An algebraic variety in k^n is the set of points defined by the vanishing of a given collection of polynomials in n variables. These varieties can have "non-smooth" points, called singularities, whose study is crucial across mathematics. This gives rise to the following fundamental problem: given polynomials $f_1, \dots, f_r$ with coefficients in k that define an algebraic variety X, find ways of quantifying how singular X is.

Therefore we want to attach to the ideal $\mathfrak{a} = (f_1, \dots, f_r)$ a positive number that measures the singularities of X. When our base field is $\mathbb{C}$, the log-canonical threshold of $\mathfrak{a}$, which is the supremum over all s for which the function $1/(|f_1|^2 + \cdots + |f_r|^2)^s$ is locally integrable, provides such an invariant. Indeed, when X is smooth of codimension r we have $\operatorname{lct}(\mathfrak{a}) = r$ and smaller values of $\operatorname{lct}(\mathfrak{a})$ correspond to worse singularities. Since its inception, the log-canonical threshold has become ubiquitous across mathematics; it appears, for example, in the minimal model program [Bir07] and the study of local zeta functions [Igu00].

Example I.1. Let $\mathfrak{a} = (f)$ where $f = x^2 + y^3$. Then $\operatorname{lct}(f) = 5/6$.

A differential operator P is an operator on the space of complex polynomials that can be written as $P = \sum_\alpha g_\alpha(x)\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where ∂_i is partial differentiation with respect to x_i and the $g_\alpha(x) = g_\alpha(x_1, \dots, x_n)$ are complex polynomials. The collection of all differential operators forms a ring D, and the theory of modules over D (so-called D-module theory) has provided powerful tools for the study of singularities. In particular, it allows us to approach the log-canonical threshold from a different point of view, on which we now elaborate.

Suppose that X is defined as the vanishing locus of a single polynomial f. A deep fact, proved by Bernstein and Sato independently and in different contexts [Ber72] [Sat90], tells us that there is a nonzero polynomial $b_f(s) \in \mathbb{C}[s]$ that satisfies the functional equation

$$b_f(s) f^s = P(s) \cdot f^{s+1}$$

for some operator $P(s) \in D[s]$; when we take $b_f(s)$ to be monic of least degree for which such an operator $P(s)$ exists, we say that $b_f(s)$ is the Bernstein-Sato polynomial of f.

Example I.2. Take $f = x^2 + y^3$ as before. Then we have

$$\left(\frac{1}{12}y\partial_x^2\partial_y + \frac{1}{27}\partial_y^3 + \frac{1}{4}\partial_x s + \frac{3}{8}\partial_x^2\right) \cdot f^{s+1} = (s+\frac{5}{6})(s+1)(s+\frac{7}{6})f^s$$

and, in fact, $b_f(s) = (s+5/6)(s+1)(s+7/6)$.

A theorem of Kashiwara tells us that the roots of $b_f(s)$ are rational and negative [Kas77], and another theorem of Kollár states that the largest amongst these roots is $-\operatorname{lct}(f)$ [Kol96]; observe the agreement with Example I.2. Since we can recover the log-canonical threshold from the Bernstein-Sato polynomial, we can say that the Bernstein-Sato polynomial provides a refinement of the log-canonical threshold.

When X is defined by a collection of polynomials $f_1, \dots, f_r$, Budur, Mustaţă and Saito defined a notion of the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$ of the ideal $\mathfrak{a} = (f_1, \dots, f_r)$ and they showed that the theorems of Kashiwara and Kollár remain true in this more general setting [BMS06a].

Let us return to the problem of quantifying singularities. Recall that the log-canonical threshold was defined in terms of an integrability condition, and hence we made crucial use of the fact that our base field was $\mathbb{C}$. This lack of "analysis" in a general base field k can be overcome when k has characteristic zero by using resolutions of singularities, but when k has characteristic $p > 0$ the problem is much more severe.

In positive characteristic we can use the Frobenius endomorphism to define another invariant called F-pure threshold of $\mathfrak{a}$, denoted by $\operatorname{fpt}(\mathfrak{a})$. For example, when $\mathfrak{a} = (f)$ and for some polynomial $f \in k[x_1, \dots, x_n]$ with an isolated singularity at the origin, the F-pure threshold of $\mathfrak{a}$ is the infimum over all m/p^e for which f^m is in the ideal $(x_1^{p^e}, \dots, x_n^{p^e})$.

The F-pure threshold of $\mathfrak{a}$ provides a characteristic p analogue of the log-canonical threshold. This is far from obvious, and was established by deep work of Hara, Takagi, Watanabe and Yoshida two decades ago [HY03] [TW04]. To get a feeling for this analogy, let us go back to our running example

Example I.3. Suppose $p > 3$, and let $\mathfrak{a} = (f)$ where $f = x^2 + y^3 \in \mathbb{F}_p[x, y]$. Then

$$\operatorname{fpt}(\mathfrak{a}) = \begin{cases} \frac{5}{6} \text{ if } p \equiv 1 \mod 3 \\ \frac{5}{6} - \frac{1}{6p} \text{ if } p \equiv 2 \mod 3. \end{cases}$$

Comparing Example I.1 and Example I.3, we note that the F-pure threshold agrees with the log-canonical threshold whenever $p \equiv 1 \mod 3$. In general, the following long-standing conjecture predicts a precise relationship between these two invariants.

Conjecture I.4. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal defined over $\mathbb{Z}$. Then the log-canonical threshold of $\mathfrak{a}$ equals the F-pure threshold of $\mathfrak{a}_p$ for infinitely many primes p, where $\mathfrak{a}_p$ is the mod-p reduction of $\mathfrak{a}$.*

Conjecture I.4 is known to hold only in a few cases, which include monomial ideals [HY03], very general complete interchapters [Tak13] (see [Her16] for the hypersurface case) and cones over elliptic curves [BS15] [Pag18].

Recall that, over $\mathbb{C}$, we can use differential operators to define the Bernstein-Sato polynomial, which provides a refinement of the log-canonical threshold. Since the F-pure threshold provides a characteristic p analogue of the log-canonical threshold, we can ask whether there is an analogue of the Bernstein-Sato polynomial in characteristic p.

This line of research was started by Mustaţă [Mus09] and then continued by Bitoun [Bit18][1]. Their approaches deal only with the case of principal ideals (i.e. ideals of the form $\mathfrak{a} = (f)$), and the main goal of this thesis is to contribute to the development of this invariant by considering the case of arbitrary ideals $\mathfrak{a} = (f_1, \dots, f_r)$. Along the way we also obtain methods and proofs that are new even in the case of principal ideals.

Let us give a brief description of how this characteristic p analogue of the Bernstein-Sato polynomial is defined and explain some of its properties.

We begin by trying to mimic the theory over $\mathbb{C}$. To do so, the best point of departure is not the description of the Bernstein-Sato polynomial given above, but rather an alternative description due to Malgrange and then generalized by Budur, Mustaţă and Saito. In this description, the Bernstein-Sato polynomial arises as the minimal polynomial of an operator s acting on a certain D-module $N_\mathfrak{a}$ (the description of $N_\mathfrak{a}$ is given in Chapter 4, but we omit it for now as it is tedious and not enlightening). The fact that the action of s on $N_\mathfrak{a}$ admits a minimal polynomial entails that the module $N_\mathfrak{a}$ splits as a direct sum of generalized eigenspaces

$$N_\mathfrak{a} = \bigoplus_{\lambda \in \mathbb{C}} (N_\mathfrak{a})_\lambda,$$

and the roots of the minimal polynomial $b_\mathfrak{a}(s)$ can be extracted from this decomposition as follows

$$\{ \text{ Roots of } b_\mathfrak{a}(s) \ \} = \{\lambda \in \mathbb{C} : (N_\mathfrak{a})_\lambda \neq 0\}.$$

In particular, the above direct sum is actually finite.

[1] We also note that Mustaţă's construction was generalized to F-regular Cartier modules by Blickle and Stäbler [BS16].

Suppose now that R is a polynomial ring over a perfect field of characteristic $p > 0$, and let $\mathfrak{a} \subseteq R$ be a nonzero ideal. The module $N_\mathfrak{a}$ can still be constructed in this setting, but the action of the operator s is naturally replaced by an action of the algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ of continuous functions from $\mathbb{Z}_p$ (the p-adics) to $\mathbb{F}_p$. We think of this algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ as playing the role of $\mathbb{C}[s]$.

The algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ and its modules behave in interesting ways and, in fact, we dedicate a whole chapter to the study of this algebra. For now, let us note that it behaves like $\mathbb{C}[s]$ in the following way: just like the maximal ideals of $\mathbb{C}[s]$ correspond to the elements of $\mathbb{C}$ via the correspondence $\lambda \leftrightarrow (s - \lambda)$, the maximal ideals of $C(\mathbb{Z}_p, \mathbb{F}_p)$ correspond to the elements of $\mathbb{Z}_p$ via the correspondence $\alpha \leftrightarrow \mathfrak{m}_\alpha = (\varphi | \varphi(\alpha) = 0)$.

We have the following result, which should be interpreted as an analogue to the existence of a minimal polynomial on $N_\mathfrak{a}$.

Theorem I.5 (Thm. IV.4)**.** *The module $N_\mathfrak{a}$ splits as a direct sum*

$$N_\mathfrak{a} = \bigoplus_{\alpha \in \mathbb{Z}_p} (N_\mathfrak{a})_\alpha$$

where $(N_\mathfrak{a})_\alpha = \operatorname{Ann}_{N_\mathfrak{a}}(\mathfrak{m}_\alpha)$ and, moreover, we have $(N_\mathfrak{a})_\alpha \neq 0$ for only a finite number of α.

Over $\mathbb{C}$ it is not true that $(N_\mathfrak{a})_\lambda = \operatorname{Ann}_{N_\mathfrak{a}}(s - \lambda)$ in general; this reflects the fact that the roots of $b_\mathfrak{a}(s)$ could have high multiplicity. In characteristic p, however, we have $\mathfrak{m}_\alpha = \mathfrak{m}_\alpha^2$ for all $\alpha \in \mathbb{Z}_p$ and the information of multiplicity is therefore lost. We do not know how to recover the information of multiplicity in positive characteristic, and therefore there is no notion of "Bernstein-Sato polynomial" in characteristic p. Instead, we define a characteristic p analogue of the *roots* of the Bernstein-Sato polynomial as follows.

Definition I.6. A p-adic integer $\alpha \in \mathbb{Z}_p$ is a *Bernstein-Sato root* of $\mathfrak{a}$ if $(N_\mathfrak{a})_\alpha$ is nonzero. The set of Bernstein-Sato roots of $\mathfrak{a}$ is denoted by $\mathrm{BSR}(\mathfrak{a})$.

As a consequence of Theorem I.5, an ideal $\mathfrak{a} \subseteq R$ has finitely many Bernstein-Sato roots. If we wanted to compute these roots at this point, we would have to understand the module $N_\mathfrak{a}$ and study its decomposition. This study is not very technical, but it is a bit tedious. Luckily, we are able to provide another characterization of the Bernstein-Sato roots of $\mathfrak{a}$ in terms of the ν-invariants of Mustaţă, Takagi and Watanabe.

Theorem I.7 (Thm. IV.17)**.** *For every $e \geq 0$ let $\nu_\mathfrak{a}^\bullet(p^e)$ denote the set of ν-invariants of level e for $\mathfrak{a}$. A p-adic integer α is a Bernstein-Sato root of $\mathfrak{a}$ if and only if there is a sequence $(\nu_e) \subseteq \mathbb{Z}_{\geq 0}$ where $\nu_e \in \nu_\mathfrak{a}^\bullet(p^e)$ whose p-adic limit is α.*

We refer the reader to Chapter IV.3 for a description of the sets $\nu_{\mathfrak{a}}^{\bullet}(p^e)$. Here we only remark that they are easily computable in simple examples (much more than the decomposition of $N_{\mathfrak{a}}$). Let us illustrate their behavior by going back to our example.

Example I.8. Suppose $p > 3$ and let $\mathfrak{a} = (f)$ where $f = x^2 + y^3 \in \mathbb{F}_p[x, y]$.

(i) If $p \equiv 1 \mod 3$ then for all integers $e \geq 0$ we have

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) = \{\frac{5}{6}p^e - \frac{5}{6}, p^e - 1\} + p^e\mathbb{Z}_{\geq 0}.$$

Taking the sequence $\nu_e = (5/6)p^e - (5/6)$ we obtain $-5/6 \in \mathrm{BSR}(\mathfrak{a})$, taking $\nu_e = p^e - 1$ we obtain $-1 \in \mathrm{BSR}(\mathfrak{a})$ and, in fact, there are no more Bernstein-Sato roots: $\mathrm{BSR}(\mathfrak{a}) = \{-5/6, -1\}$.

(ii) If $p \equiv 2 \mod 3$ then for all integers $e \geq 2$ we have

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) = \{\frac{5}{6}p^e - \frac{1}{6}p^{e-1} - 1, p^e - 1\} + p^e\mathbb{Z}_{\geq 0},$$

and thus $\mathrm{BSR}(\mathfrak{a}) = \{-1\}$.

The importance of Theorem I.7 cannot be emphasized enough: as illustrated in the above example, it gives us an easier way of computing Bernstein-Sato roots, but it also gives another powerful point of view that is useful for proving things. Indeed, we are able to combine Theorem I.7 together with a careful study of the behavior of ν-invariants to give the following characteristic p analogue of Kashiwara's theorem. Recall that a p-adic number is rational if it lies in the subring $\mathbb{Z}_{(p)}$ of $\mathbb{Z}_p$.

Theorem I.9 (Thm. V.10, Prop. V.14)**.** *The Bernstein-Sato roots of $\mathfrak{a}$ are rational and lie in the interval $[-r, 0)$, where r is the number of generators of $\mathfrak{a}$*[2].

The ν-invariants of $\mathfrak{a}$ were introduced by Mustaţă, Takagi and Watanabe in order to study of the F-jumping numbers of $\mathfrak{a}$, which we denote by $\mathrm{FJN}(\mathfrak{a})$ [MTW05]. These F-jumping numbers are another collection of rational numbers that we can attach to the ideal $\mathfrak{a}$ in order to study its singularities (for example, the smallest F-jumping number of $\mathfrak{a}$ is the F-pure threshold). The fact that the ν-invariants are also intimately linked to the Bernstein-Sato roots of $\mathfrak{a}$ suggests that there could be some relationship between the Bernstein-Sato roots and the F-jumping numbers of $\mathfrak{a}$.

This relationship is most apparent in the case of principal ideals, as shown by Bitoun in the following theorem.

[2]We get a stronger lower bound, but we use the number of generators here for simplicity.

Theorem I.10 ([Bit18]). *Suppose $\mathfrak{a} = (f)$ is principal. Then*

$$\mathrm{BSR}(\mathfrak{a}) = -\big(\,\mathrm{FJN}(f) \cap \mathbb{Z}_{(p)} \cap (0,1]\big).$$

We give another proof of Theorem I.10 by using the alternative characterization of Bernstein-Sato roots given in Theorem I.7 (see Theorem V.17). We also generalize Bitoun's theorem to the case of arbitrary ideals as follows.

Theorem I.11 (Thm. V.20). *Suppose that $\mathfrak{a}$ is generated by r elements.*

(i) If α is a Bernstein-Sato root of $\mathfrak{a}$ then there is some $m \in \{\lfloor \alpha \rfloor + 1, \lfloor \alpha \rfloor + 2, \dots, \lfloor \alpha \rfloor + r\}$ such that $m - \alpha$ is an F-jumping number of $\mathfrak{a}$.

(ii) If $\lambda \in \mathbb{Z}_{(p)}$ is an F-jumping number of $\mathfrak{a}$ then there is some $m \in \{\lceil \lambda \rceil - r, \lceil \lambda \rceil - r + 1, \dots, \lceil \lambda \rceil - 1\}$ such that $m - \lambda$ is a Bernstein-Sato root of $\mathfrak{a}$.

Corollary I.12 (Cor. V.21). *We have an equality $\mathrm{BSR}(\mathfrak{a}) + \mathbb{Z} = -\big(\,\mathrm{FJN}(\mathfrak{a}) \cap \mathbb{Z}_{(p)}\big) + \mathbb{Z}$ of subsets of $\mathbb{Z}_{(p)}$.*

Finally, this thesis studies the behavior of Bernstein-Sato polynomials under reduction mod-p. More precisely, if $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal defined over $\mathbb{Z}$ and for every prime p we let $\mathfrak{a}_p$ denotes its mod-p reduction, our goal is to understand how the roots of $b_{\mathfrak{a}}(s)$ relate to the Bernstein-Sato roots of $\mathfrak{a}_p$ for various p. We prove the following general result.

Theorem I.13 (Thm. VI.3). *Suppose that $\alpha \in \mathbb{Q}$ is such that $\alpha \in \mathrm{BSR}(\mathfrak{a}_p)$ for infinitely many p. Then α is a root of $b_{\mathfrak{a}}(s)$.*

In the case where $\mathfrak{a}$ is a monomial ideal (that is, an ideal generated by monomials), we can say more. Using results of Budur, Mustaţă and Saito [BMS06b], we are able to prove the following.

Theorem I.14 (Thm. VI.6). *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a nonzero monomial ideal. Then the Bernstein-Sato roots of $\mathfrak{a}_p$ agree with the roots of $b_{\mathfrak{a}}(s)$ for all primes p large enough.*

Combining this with Theorem V.10 we are able to provide the following purely characteristic zero result.

Corollary I.15 (Cor. VI.12). *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a nonzero monomial ideal generated by r elements. Then all roots of $b_{\mathfrak{a}}(s)$ lie in the interval $[-r, 0)$*[1].

Contributions. The goal of this thesis is to explain the results of my two papers [QG21b] and [QG21a]. Since the publication of these papers I have been working on two closely related projects, one with Thomas Bitoun and one with Jack Jeffries and Luis Núñez-Betancourt, and these projects have yielded improvements of many of the results in [QG21b] and [QG21a]. These improvements are included in this thesis, and a quick summary of the contributions of each project is as follows:

- Chapter II is background; none of it is original except possibly Section II.5.
- Chapter III is joint work with Thomas Bitoun.
- Chapter IV is in [QG21b], although I include some simplifications in the exposition that are joint work with Jack Jeffries and Luis Núñez-Betancourt.
- Any unattributed result from Chapter V that does not appear in [QG21b] is joint work with Jack Jeffries and Luis Núñez-Betancourt. All proofs given in this chapter are also joint work with them.
- From Chapter VI all unattributed results are either original or appear in [QG21a], with the exception of Corollary VI.12, which is joint work with Jack Jeffries and Luis Núñez-Betancourt.

Notation. Given a prime p, let $\mathbb{Z}_p$ denote the set of p-adic integers. We assume all rings have 1 and are usually commutative; the only noncommutative rings we will encounter are endomorphism rings (usually denoted as $\operatorname{End}_B(M)$ or similarly) and rings of differential operators (denoted with the letter D, and some decorations). In particular, all rings denoted A, B, R, S are assumed to be commutative. Given an ideal $\mathfrak{a} \subseteq R$, we will take $\mathfrak{a}^0 = R$ by convention (even if $\mathfrak{a} = (0)$).

CHAPTER II
Background

II.1: Basics of commutative algebra in prime characteristic

Let R be a commutative ring and p be a prime number. We say that R has characteristic p if R is nonzero and $p = 0$ in R or, equivalently, if R is an $\mathbb{F}_p$-algebra. If R has characteristic p then the function $F : R \to R$ that sends $F(x) = x^p$ is additive, and therefore a ring homomorphism; we call F the Frobenius endomorphism of R.

This natural endomorphism makes for an extremely powerful tool to study the ring R, since its behavior is closely related to the singularities of R. The most obvious example of this relationship is the following theorem of Kunz.

Theorem II.1 (Kunz)**.** *Let R be a noetherian ring of characteristic p. Then R is regular if and only if $F : R \to R$ is flat.*

Given an integer $e \geq 0$ we will denote by $F^e : R \to R$ the e-th iteration of F; more concretely, we have $F^e(x) = x^{p^e}$ for all $x \in R$. Since F^e is a ring homomorphism, we may equip the ring R with an exotic R-module structure that comes from restriction of scalars via F^e, and the resulting R-algebra is denoted by $F^e_* R$. Given an element $x \in R$, we will sometimes denote it by $F^e_* x$ whenever we want to think of it as an element of $F^e_* R$ and, similarly, given an ideal $I \subseteq R$ we will sometimes write it as $F^e_* I$ whenever we want to think of it as an ideal of $F^e_* R$. With this notation, the R-module structure on $F^e_* R$ is given by $y \cdot F^e_* x = F^e_*(y^{p^e} x)$ for all $x, y \in R$, and the map F^e is an R-module map when viewed as a morphism $F^e : R \to F^e_* R$.

If $W \subseteq R$ is a multiplicative subset, there is a natural isomorphism $W^{-1}R \otimes_R F^e_* R \cong F^e_*(W^{-1}R)$. Indeed, there is a natural map $W^{-1}R \otimes_R F^e_* R \to F^e_*(W^{-1}R)$ whose inverse is given $F^e_*(g/w) = (1/w) \otimes F^e_*(g w^{p^e - 1})$.

Definition II.2. A commutative ring R is said to be *F-finite* if R is noetherian, has prime characteristic p, and $F_* R$ is a finitely generated R-module.

If R is F-finite then $F^e_* R$ is a finitely generated R-module for every $e \geq 0$. One can easily construct many examples of F-finite rings by iterating the following operations:

Example II.3.

(i) Every perfect field of positive characteristic is F-finite.

(ii) If R is F-finite then so is the polynomial ring $R[x]$.

(iii) If R is F-finite then so is the power series ring $R[[x]]$.

(iv) If R is F-finite then so is every quotient of R.

(v) If R is F-finite then so is every localization of R.

One may think of Example II.3 as saying that, when we do algebraic geometry over a perfect field, most rings that we encounter are F-finite.

Definition II.4. Let R be a commutative ring of prime characteristic p, $\mathfrak{b} \subseteq R$ be an ideal and $e \geq 0$ be an integer. The *e-th Frobenius power of* $\mathfrak{b}$ is the ideal

$$\mathfrak{b}^{[p^e]} = F^e(\mathfrak{b})R = (f^{p^e} : f \in \mathfrak{b}).$$

Lemma II.5. *Let R be a ring of prime characteristic p, $\mathfrak{b} = (f_1, \ldots, f_r)$ be a finitely generated ideal and $e \geq 0$ be an integer. Then:*

1. *The ideal $\mathfrak{b}^{[p^e]}$ is generated by the elements $f_1^{p^e}, \ldots, f_r^{p^e}$.*

2. *For all integers $n \geq 0$ we have $(\mathfrak{b}^n)^{[p^e]} = (\mathfrak{b}^{[p^e]})^n$.*

3. *For all integers $n, m \geq 0$ with $n \geq mp^e + (r-1)(p^e - 1)$ we have $\mathfrak{b}^n = \mathfrak{b}^{n-mp^e}\mathfrak{b}^{m[p^e]}$.*

4. *We have $\mathfrak{b}^{[p^e]} \subseteq \mathfrak{b}^{p^e}$ and $\mathfrak{b}^{r(p^e-1)+1} \subseteq \mathfrak{b}^{[p^e]}$; in particular, the families of ideals $\{\mathfrak{b}^n\}_{n=0}^{\infty}$ and $\{\mathfrak{b}^{[p^e]}\}_{e=0}^{\infty}$ are cofinal.*

Proof. Recall that if $G : R \to S$ is a morphism of rings then the extension $\mathfrak{b}S$ is generated by $G(f_1), \ldots, G(f_r)$, and that $(\mathfrak{b}S)^n = \mathfrak{b}^n S$. One gets statements (i) and (ii) by specializing to the case $G = F^e$. Part (iii) reduces to the case $m = 1$ by using part (ii) and an easy induction, and in this case the statement follows from the observation that any monomial $f_1^{a_1} \cdots f_r^{a_r}$ with $a_1 + \cdots + a_r > r(p^e - 1)$ must have some $a_i \geq p^e$. In part (iv), the inclusion $\mathfrak{b}^{[p^e]} \subseteq \mathfrak{b}^{p^e}$ is clear and the other inclusion follows from part (iii). $\square$

Proposition II.6. *Let R be a regular F-finite ring. For every integer $e \geq 0$ there is an R-module homomorphism $\sigma : F^e_* R \to R$ that splits the e-th iterated Frobenius $F^e : R \to F^e_* R$.*

Proof. The existence of a splitting is equivalent to the surjectivity of the natural map $\mathrm{Hom}_R(F^e_* R, R) \to \mathrm{Hom}_R(R, R)$, which can be checked locally. If $\mathfrak{m} \subseteq R$ is a maximal ideal then we have natural homomorphisms

$$R_\mathfrak{m} \otimes_R \mathrm{Hom}_R(R, R) \cong \mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m}, R_\mathfrak{m})$$

and

[Sta21, 087R]
$$\begin{aligned} R_\mathfrak{m} \otimes_R \mathrm{Hom}_R(F^e_* R, R) &\cong \mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m} \otimes_R F^e_* R, R_\mathfrak{m}) \\ &\cong \mathrm{Hom}_{R_\mathfrak{m}}(F^e_* R_\mathfrak{m}, R_\mathfrak{m}). \end{aligned}$$

We thus restrict to the case were R is local. In this setting, Kunz's theorem (Theorem II.1) entails that $F^e_* R$ is free over R and, in particular, it contains at least one copy of R as a direct summand. We conclude that there is an element $a \in R$ and an R-linear map $\pi : F^e_* R \to R$ such that $\pi(F^e_* a) = 1$. Let σ be the composition

$$F^e_* R \xrightarrow{F^e_* x \mapsto F^e_*(xa)} F^e_* R \xrightarrow{\quad \pi \quad} R.$$

We then have that $\sigma(F^e_* 1) = 1$, and therefore σ gives a splitting. $\square$

Remark II.7. Rings satisfying the conclusion of Proposition II.6 are called F-split; see [SZ15] for more on this class of rings.

II.2: Rings of differential operators

Let B be a commutative ring. We describe a construction of Grothendieck that attaches to every B-algebra R a noncommutative ring $D_{R|B}$, called the ring of B-linear differential operators on R [Gro65], and we prove some of its basic properties.

We begin by noting that the endomorphism ring $\mathrm{End}_B(R)$ has a natural $R \otimes_B R$-module structure given by $((r \otimes s) \cdot \varphi)(x) = r\varphi(sx)$. In other words, the left copy of R acts by postmultiplication and the right copy of R acts by premultiplication.

There is a natural "multiplication map" $\mu : R \otimes_B R \to R$ given by $\mu(r \otimes s) = rs$ whose kernel we denote by $J_{R|B}$, or simply by J_R or J whenever R and B are understood. Given some $r \in R$, we denote by $dr \in R \otimes_B R$ the element $dr = 1 \otimes r - r \otimes 1$. It is easy to observe that the elements dr lie in $J_{R|B}$, and indeed $J_{R|B}$ is generated by these elements; that is, $J_{R|B} = (dr : r \in R)$.

Definition II.8. Let B be a commutative ring, R be a commutative B-algebra and $\varphi \in \mathrm{End}_B(R)$ be a B-linear endomorphism of R. Given an integer $n \geq 0$, we say that φ is a *B-linear differential operator of order $\leq n$* if $J^{n+1} \cdot \varphi = 0$. We say that φ is a *B-linear differential operator* if it is a B-linear differential operator of order $\leq n$ for n large enough. We denote by $D^n_{R|B}$ the set of all B-linear differential operators of order $\leq n$, and by $D_{R|B}$ the set of all B-linear differential operators on R. If the ring B and the B-algebra structure of R are understood from the context, we will write these simply as D^n_R and D_R.

In other words, $\varphi \in \mathrm{End}_B(R)$ is a differential operator if and only if there is some large enough n so that $J^n \cdot \varphi = 0$. The reader who is familiar with local cohomology may like to think of this observation in the following way:

$$D_{R|B} = H^0_J \ \mathrm{End}_B(R).$$

Example II.9.

(i) Let $f \in R$ be an element and let $\widetilde{f} : R \to R$ be the map that multiplies by f; that is, $\widetilde{f}(g) = fg$. Then $\widetilde{f}$ is a B-linear (and even R-linear) endomorphism of R, and we claim that $J\widetilde{f} = 0$. Indeed, note that $\big(dr\widetilde{f}\big)(g) = frg - rfg = 0$. We conclude that $\widetilde{f}$ is a differential operator of order ≤ 0.

(ii) Suppose that $R = \mathbb{C}[x]$, and consider the $\mathbb{C}$-linear operator $\partial : R \to R$ which sends $\partial(f) = \partial f / \partial x$. Given some $s \in R$, observe that

$$\begin{aligned}\big(ds\partial\big)(f) &= \partial(sf) - s\partial(f)\\ &= s\partial(f) + f\partial(s) - s\partial(f)\\ &= f\partial(s).\end{aligned}$$

In other words, we have shown that $ds\partial = \widetilde{\partial(s)}$, using the notation from Example (i). From Example (i) we conclude that $J^2 \cdot \partial = 0$; i.e. that ∂ is a differential operator of order ≤ 1.

(iii) The key property that made Example (ii) work is the fact that ∂ satisfies the product rule: $\partial(fg) = f\partial(g) + g\partial(f)$. More generally, if R is a B-algebra we say that $\theta : R \to R$ is a B-linear derivation whenever θ is B-linear and $\theta(fg) = f\theta(g) + g\theta(f)$ for all $f, g \in R$. The same computation as in Example (ii) then shows that every derivation is a differential operator of order ≤ 1.

(iv) In Proposition II.15 we show that the composition of two differential operators is a differential operator, and therefore many more examples can be built from the above ones.

Remark II.10. By Example II.9 (i) there is a natural algebra homomorphism $R \to D_{R|B}$ that sends f to $\widetilde{f}$, which is injective because $f = \widetilde{f}(1)$. By a standard abuse of notation we will identify R with its homomorphic image in $D_{R|B}$, and therefore from this point onwards we identify every $f \in R$ with the operator $\widetilde{f}$.

Definition II.11. Let B be a commutative ring and R be a B-algebra. Given an integer $n \geq 0$, the *n-th module $P^n_{R|B}$ of principal parts of R* is given by $P^n_{R|B} = R \otimes_B R/J^{n+1}$, which is always thought of as an R-module via the action on the left unless otherwise specified. Whenever the ring B and the B-algebra structure of R are understood we will write these as P^n_R.

Example II.12. Suppose $R = B[t_1, \ldots, t_m]$ is a polynomial ring over B. Then $R \otimes_B R$ is a polynomial ring in the variables $t_1, \ldots, t_m, dt_1, \ldots, dt_m$, and the ideal J_R is given by $J_R = (dt_1, \ldots, dt_m)$. We conclude that

$$P^n_R = \frac{B[t_1, \ldots, t_m, dt_1, \ldots, dt_m]}{(dt_1, \ldots, dt_m)^{n+1}} = \bigoplus_{\underline{a}} R\ dt_1^{a_1} \cdots dt_m^{a_m}$$

where the sum ranges over all tuples $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r_{\geq 0}$ such that $a_1 + \cdots + a_r \leq n$. In particular, P^n_R is finitely generated and free over R.

Given some $\varphi \in \operatorname{End}_B(R)$ there is a unique homomorphism $\varphi' : R \otimes_B R \to R$ that sends $\varphi'(r \otimes s) = r\varphi(s)$, which is now R-linear. One can recover φ from φ' because $\varphi(r) = \varphi'(1 \otimes r)$, and the assignment $\varphi \mapsto \varphi'$ gives a $R \otimes_B R$-module isomorphism

$$(-)' : \operatorname{End}_B(R) \xrightarrow{\sim} \operatorname{Hom}_B(R \otimes_B R, R)$$

which is $R \otimes_B R$-linear. In particular, $J^{n+1}\varphi = 0$ if and only if $\varphi'(J^{n+1}) = 0$, which shows that $(-)'$ restricts to a canonical isomorphism

$$(-)' : D^n_{R|B} \xrightarrow{\sim} \operatorname{Hom}_R(P^n_{R|B}, R).$$

Recall that $\operatorname{End}_B(R)$ has a ring structure given by composition, which we can carry over via $(-)'$ to the module $\operatorname{Hom}_B(R \otimes_B R, R)$. It turns out that this ring structure on $\operatorname{Hom}_B(R \otimes_B R, R)$ comes from a *comultiplication* $\Delta : R \otimes_B R \to (R \otimes_B R) \otimes_R (R \otimes_B R)$ on

$R \otimes_B R$. If we identify $(R \otimes_B R) \otimes_R (R \otimes_B R) = R \otimes_B R \otimes_B R$, the comultiplication is given by $\Delta(r \otimes s) = r \otimes 1 \otimes s$, and the ring structure on $\mathrm{Hom}_B(R \otimes_B R, R)$ is given as follows:

Lemma II.13. *For all $\varphi', \psi' \in \mathrm{Hom}_B(R \otimes_B R, R)$, the product $\varphi'\psi'$ is given by the composition*

$$R \otimes_B R \xrightarrow{\Delta} R \otimes_B R \otimes_B R \xrightarrow{R \otimes_B \psi'} R \otimes_B R \xrightarrow{\varphi'} R.$$

Proof. We let $\varphi \in \mathrm{End}_B(R)$ and $\psi \in \mathrm{End}_B(R)$ be the elements corresponding to φ' and ψ' under $(-)'$ respectively. Our goal is to show that $(\varphi\psi)' = \sigma$, where σ is the composition given above. Since both of these morphisms are R-linear, it suffices to check that the outputs agree on elements of the form $1 \otimes r$. On the one hand we have $(\varphi\psi)'(1 \otimes r) = (\varphi\psi)(r) = \varphi(\psi(r))$, and on the other hand we compute

$$\begin{aligned} \sigma(1 \otimes r) &= \varphi' \circ (R \otimes \psi')(1 \otimes 1 \otimes r) \\ &= \varphi'\big(1 \otimes \psi'(1 \otimes r)\big) \\ &= \varphi'(1 \otimes \psi(r)) \\ &= \varphi(\psi(r)). \end{aligned}$$

□

Note that given an ideal $K \subseteq R \otimes_B R$ there are natural maps $R \otimes_B K \to R \otimes_B R \otimes_B R$ and $K \otimes_B R \to R \otimes_B R \otimes_B R$. If R is not flat over B these need not be injective, but nonetheless let us abuse notation and denote by $R \otimes_B K$ and $K \otimes_B R$ the images of these maps in $R \otimes_B R \otimes_B R$.

Lemma II.14. *For any integers $n, m \geq 0$ we have*

$$\Delta(J^{n+m+1}) \subseteq R \otimes_B J^{m+1} + J^{n+1} \otimes_B R.$$

Proof. Since $J = (dr : r \in R)$, it suffices to show that $\Delta(dr_1 \cdots dr_{n+m+1}) \in R \otimes_B J^{m+1} + J^{n+1} \otimes_B R$ for any choice of elements $r_1, \dots r_{n+m+1} \in R$. We begin by noting that $\Delta(dr) = 1 \otimes 1 \otimes r - r \otimes 1 \otimes 1 = 1 \otimes dr + dr \otimes 1$. Therefore,

$$\Delta(dr_1 \cdots dr_{n+m+1}) = (1 \otimes dr_1 + dr_1 \otimes 1) \cdots (1 \otimes dr_{n+m+1} - dr_{n+m+1} \otimes 1),$$

and the statement follows by expanding the product: every resulting summand will be an element of the form $dr_{i_1} \cdots dr_{i_k} \otimes dr_{j_1} \cdots dr_{j_l}$ with $k + l = n + m + 1$, and thus $k \geq n + 1$ or $l \geq m + 1$. □

Proposition II.15. *Let B be a commutative ring and R be a commutative B-algebra. For all integers $n, m \geq 0$ we have $D^n_{R|B} \circ D^m_{R|B} \subseteq D^{n+m}_{R|B}$. In particular, $D_{R|B}$ is a subring of* $\mathrm{End}_B(R)$.

Proof. Let $\xi \in D^n_{R|B}$ and $\eta \in D^m_{R|B}$. By Lemma II.14 the map $(\xi\eta)'$ is given by the composition

$$R \otimes_B R \xrightarrow{\Delta} R \otimes_B R \otimes_B R \xrightarrow{R\otimes_B \eta'} R \otimes_B R \xrightarrow{\xi'} R,$$

and our goal is to show that $(\xi\eta)'(J^{n+m+1}) = 0$. By Lemma II.14, it is enough to show that the composition

$$R \otimes_B R \otimes_B R \xrightarrow{R\otimes_B \eta'} R \otimes_B R \xrightarrow{\xi'} R$$

kills the ideal $R \otimes_B J^{m+1} + J^{n+1} \otimes_B R$. Since $\eta \in D^m_{R|B}$, $\eta'(J^{m+1}) = 0$ and therefore $(R \otimes \eta')(R \otimes_B J^{m+1}) = 0$. Moreover, if we give $R \otimes_B R \otimes_B R$ the $R \otimes_B R$-module structure that comes from the morphism $R \otimes_B R \to R \otimes_B R \otimes_B R$ $[r \otimes s \mapsto r \otimes s \otimes 1]$ then $R \otimes \eta'$ is $R \otimes_B R$-linear and therefore $(R \otimes \eta')(J^{n+1} \otimes_B R) \subseteq J^{n+1}$. Since $\xi \in D^n_{R|B}$, $\xi'(J^{n+1}) = 0$, which finishes the proof. $\square$

The ring R has a natural left $\mathrm{End}_B(R)$-module structure, which induces a left $D_{R|B}$-module structure. We denote this action by "$\cdot$"; this means that if $\xi \in D_{R|B}$ is a differential operator and $f \in R$ is an element then $\xi \cdot f$ will denote $\xi \cdot f = \xi(f)$. Our next goal is to show that $D_{R|B}$ also acts on the local cohomology modules of R.

II.2.1: Behavior under localization

Let B be a commutative ring, R be a B-algebra and $W \subseteq R$ be a multiplicative subset. Note that we have a natural map $R \otimes_B R \to W^{-1}R \otimes_B W^{-1}R$ along which the ideal J_R expands to the ideal $J_{W^{-1}R}$. This induces an R-module homomorphism $P^n_R \to P^n_{W^{-1}R}$, which in turn gives a natural $W^{-1}R$-module homomorphism $W^{-1}R \otimes_R P^n_R \to P^n_{W^{-1}R}$; these homomorphisms are compatible as n varies.

Proposition II.16. *The maps*

$$W^{-1}R \otimes_R P^n_R \to P^n_{W^{-1}R}$$

described above are $W^{-1}R$-module isomorphisms.

Proof. Note that the map in question is a morphism of R-modules when we equip the P^n with their respective actions in the right, and that the induced morphism $W^{-1}R \otimes_R P^n_R \otimes_R W^{-1}R \to P^n_{W^{-1}R}$ is an isomorphism.

It thus suffices to show that the natural map $W^{-1}R \otimes_R P^n_R \to W^{-1}R \otimes_R P^n_R \otimes_R W^{-1}R$ is an isomorphism or, equivalently, that every element of the form $1 \otimes 1 \otimes w \in W^{-1}R \otimes_R P^n_R$ for $w \in W$ is invertible. To see this, observe that in P^n_R we have $(w \otimes 1 - 1 \otimes w)^{n+1} = 0$ and therefore

$$\begin{aligned} w^{n+1} \otimes 1 &= \sum_{i=1}^{n+1} (-1)^{i+1} \binom{n+1}{i} w^{n+1-i} \otimes w^i \\ &= \Big(\sum_{i=1}^{n+1} (-1)^{i+1} \binom{n+1}{i} w^{n+1-i} \otimes w^{i-1} \Big)(1 \otimes w). \end{aligned} \qquad \square$$

Corollary II.17. *Every differential operator $\xi \in D_R$ admits a unique extension $\widetilde{\xi} \in D_{W^{-1}R}$, and if ξ has order $\leq n$ then so does $\widetilde{\xi}$. The assignment $\xi \mapsto \widetilde{\xi}$ gives an algebra homomorphism $D_R \to D_{W^{-1}R}$.*

Proof. The existence and uniqueness of extensions follows from the fact that every R-linear homomorphism $P_R^n \to R$ admits a unique $W^{-1}R$-linear extension $W^{-1}R \otimes_R P_R^n \to W^{-1}R$. In order to check that the induced map $D_R \to D_{W^{-1}R}$ is an algebra homomorphism it suffices to check that it respects the multiplication, which in turn follows from the uniqueness of the extensions. $\square$

Corollary II.18. *The module $W^{-1}R$ has a natural D_R-module structure such that the localization map $R \to W^{-1}R$ is D_R-linear.*

Proof. We know that $W^{-1}R$ admits a left $D_{W^{-1}R}$-module structure, which induces a D_R-module structure by restriction of scalars along the algebra homomorphism $D_R \to D_{W^{-1}R}$. Since the image $\widetilde{\xi} \in D_{W^{-1}R}$ of an operator $\xi \in D_R$ under the map $D_R \to D_{W^{-1}R}$ is an extension of ξ, the localization map is D_R-linear. $\square$

The algebra homomorphism $D_R \to D_{W^{-1}R}$ induces a left $W^{-1}R$-module homomorphism $W^{-1}R \otimes_R D_R \to D_{W^{-1}R}$. Our next goal is to show that, under suitable hypotheses, this map is an isomorphism.

Lemma II.19. *Suppose that B is noetherian and that R is essentially of finite type over B. Then P_R^n is a finitely generated R-module.*

Proof. By Proposition II.16 we reduce to the case where R is of finite type. Write $R = Q/I$ where $Q = B[x_1, \ldots, x_n]$ is a polynomial ring over B and $I \subseteq Q$ is an ideal. Then P_R^n is a quotient of P_Q^n, and the result follows because P_Q^n is finitely generated over Q (see Example II.12). $\square$

Proposition II.20. *Let B be a noetherian ring, R be a B-algebra that is essentially of finite type and $W \subseteq R$ be a multiplicative subset. Then there are compatible isomorphisms*

$$W^{-1}R \otimes_R D_R^n \cong D_{W^{-1}R}^n$$

of left $W^{-1}R$-modules, which give rise to a $(W^{-1}R, D_R)$-bimodule isomorphism

$$W^{-1}R \otimes_R D_R \cong D_{W^{-1}R}.$$

Proof. Using Proposition II.16 and Lemma II.19 we get the following chain of isomorphisms of $W^{-1}R$-modules.

$$\begin{aligned} W^{-1}R \otimes_R D^n_R &\cong W^{-1}R \otimes_R \mathrm{Hom}_R(P^n_R, R) \\ \text{[Sta21, 087R]} \qquad &\cong \mathrm{Hom}_{W^{-1}R}(W^{-1}R \otimes_R P^n_R, W^{-1}R) \\ &\cong \mathrm{Hom}_{W^{-1}R}(P^n_{W^{-1}R}, W^{-1}R) \\ &\cong D^n_{W^{-1}R}. \end{aligned}$$

The compatibility of the resulting isomorphisms comes from the compatibility of the isomorphisms from Proposition II.16. It is clear that the resulting isomorphism $W^{-1}R \otimes_R D_R \cong D_{W^{-1}R}$ is $W^{-1}R$-linear, and the D_R-linearity follows because the D_R-module structure of $D_{W^{-1}R}$ comes from the algebra map $D_R \to D_{W^{-1}R}$ given in Corollary II.17, which agrees with the composition $D_R \to W^{-1}R \otimes_R D_R \to D_{W^{-1}R}$. □

Corollary II.21. *Let B be a noetherian ring, R be a B-algebra that is essentially of finite type and $W \subseteq R$ be a multiplicative subset. Suppose M is a left D_R-module. Then $W^{-1}M$ has a natural left $D_{W^{-1}R}$-module structure, and thus a natural left D_R-module structure. The localization map $M \to W^{-1}M$ is D_R-linear.*

Proof. By Proposition II.20 we have an isomorphism $W^{-1}M \cong D_{W^{-1}R} \otimes_{D_R} M$, across which we may transfer the $D_{W^{-1}R}$-module structure of $D_{W^{-1}R} \otimes_{D_R} M$. The composition $M \to W^{-1}M \xrightarrow{\sim} D_{W^{-1}R} \otimes_{D_R} M$ sends $u \mapsto 1 \otimes u$, and thus the localization map is D_R-linear. □

Remark II.22. Proposition II.20 can be generalized. We can remove the noetherian assumption on B if we assume that R is finitely presented over B [Gro65], and we can also replace $W^{-1}R$ with any essentially étale R-algebra S [Más91].

II.2.2: Behavior under base change

Let R and B' be B-algebras, and let $R' = B' \otimes_B R$. We view R' as a B'-algebra, and our goal is to compare the rings $D_{R|B}$ and $D_{R'|B'}$.

We begin by observing that there is an algebra isomorphism

$$\Phi : R' \otimes_R (R \otimes_B R) \xrightarrow{\sim} R' \otimes_{B'} R'$$

given by $\Phi\big((a\otimes s)\otimes(r_1\otimes r_2)\big) = (a\otimes sr_1)\otimes(1\otimes r_2)$. We let $\mu_R : R\otimes_B R\to R$ and $\mu_{R'} : R'\otimes_{B'} R'\to R'$ denote the multiplication maps.

Proposition II.23. *Let R and B' be B-algebras, and let $R' = B'\otimes_B R$. For every $n\geq 0$, the map Φ induces an algebra isomorphism*

$$R'\otimes_R P^n_{R|B}\xrightarrow{\sim} P^n_{R'|B'}$$

Proof. It suffices to show that the ideal $J_{R'|B'}$ is given by the expansion of $J_{R|B}$ along the composition

$$R\otimes_B R\longrightarrow R'\otimes_B(R\otimes_B R)\xrightarrow{\Phi} R'\otimes_{B'} R'.$$

To prove this, observe that we have a commutative diagram

$$\begin{tikzcd} R'\otimes_R J_{R|B} \arrow[r] & R'\otimes_R(R\otimes_B R) \arrow[r, "R'\otimes\mu_R"] \arrow[d, "\Phi", "\wr"'] & R'\otimes_R R \arrow[r] \arrow[d, "\wr"'] & 0 \\ & R'\otimes_B R' \arrow[r, "\mu_{R'}"] & R' & \end{tikzcd}$$

where the top row is exact, and therefore $J_{R'|B'}$ is the image of $R'\otimes_R J_{R|B}$ under Φ. $\square$

Corollary II.24. *Every $\xi\in D_{R|B}$ admits a unique extension $\bar{\xi}\in D_{R'|B'}$, and if ξ has order $\leq n$ then so does $\bar{\xi}$. The assignment $\xi\mapsto\bar{\xi}$ gives an algebra homomorphism $D_{R|B}\to D_{R'|B'}$.*

Proof. Analogous to Corollary II.17. $\square$

Proposition II.25. *Let B be a noetherian ring, R be a B-algebra that is essentially of finite type, B' be an arbitrary B-algebra, $n\geq 0$ be an integer, and let $R' = B'\otimes_B R$. Assume that one of the following holds:*

1. *The module $P^n_{R|B}$ is projective over R.*
2. *The algebra R' is flat over R.*

Then the map $R'\otimes_R D^n_{R|B}\to D^n_{R'|B'}$ is an isomorphism.

Proof. Recall that, since R is essentially of finite type, $P^n_{R|B}$ is finitely generated over R (Lemma II.19). We conclude that, under any of the conditions stated, we have isomorphisms

$$\begin{aligned} &\text{[Sta21, 087R]} && R'\otimes_R \operatorname{Hom}_R(P^n_R, R)\cong \operatorname{Hom}_{R'}(R'\otimes_R P^n_{R|B}, R')\\ &\text{(Prop. II.23)} && \cong \operatorname{Hom}_{R'}(P^n_{R'|B'}, R'). \end{aligned}$$

and the result follows. $\square$

Note that condition (ii) is satisfied whenever B' is flat over B; indeed, the functors $R' \otimes_R (-) : R\text{-}\mathrm{Mod} \to \mathrm{Ab}$ and $B' \otimes_B (-) : R\text{-}\mathrm{Mod} \to \mathrm{Ab}$ are naturally isomorphic, where Ab denotes the category of abelian groups.

II.2.3: Differential operators in characteristic p

Throughout this section, we let R be an F-finite ring of characteristic p and $k \subseteq R$ be a perfect field (note that we can always take $k = \mathbb{F}_p$). In this situation the ring $D_{R|k}$ of k-linear differential operators on R admits another nice description due to Yekutieli [Yek92], which we now explain.

Lemma II.26. *Let $J = J_{R|k}$ for an F-finite ring R and a perfect field $k \subseteq R$. The families of ideals $\{J^n\}_{n=0}^{\infty}$ and $\{J^{[p^e]}\}_{e=0}^{\infty}$ are cofinal.*

Before we give the proof of Lemma II.26, we note that the statement is clear whenever the ideal J is finitely generated. This happens, for example, when R is essentially of finite type over k, since in this case the algebra $R \otimes_k R$ is also essentially of finite type over k, and therefore noetherian. However, we observe that it is not true in general that $R \otimes_k R$ is noetherian whenever R is noetherian: borrowing the example of Smith and Van den Bergh [SVdB97], if $R = k(x_1, x_2, \dots)$ is a rational function field in infinitely many variables then $R \otimes_k R$ is not noetherian.

Proof. The inclusion $J^{[p^e]} \subseteq J^{p^e}$ is clear. Since R is F-finite, R is finitely generated as a R^p-module, and we fix R^p-module generators $x_1, \dots, x_s \in R$. We note that the elements $x_1, \dots, x_s$ also generate R as an algebra (although no longer as a module) over R^{p^e} for every $e \geq 0$.

We claim that, given e, we have an inclusion $J^{[p^e]} \supseteq J^{s(p^e-1)+1}$. For this consider the natural algebra homomorphism

$$R \otimes_k R \to (R \otimes_k R)/J^{[p^e]} = R \otimes_{R^{p^e}} R.$$

This map sends the ideal J into the ideal $\widetilde{J} = J_{R|R^{p^e}}$, and $\widetilde{J}$ is generated by the elements $dx_1, \dots, dx_s$. In particular, $\widetilde{J}^{s(p^e-1)+1} \subseteq \widetilde{J}^{[p^e]} = 0$. We conclude that the ideal $J^{s(p^e-1)}$ maps to zero and therefore $J^{s(p^e-1)} \subseteq J^{[p^e]}$ as claimed. $\square$

Proposition II.27. *Let R be an F-finite ring of characteristic p and let $k \subseteq R$ be a perfect field. Then the ring $D_{R|k}$ of k-linear differential operators on R is given by*

$$D_{R|k} = \bigcup_{e=0}^{\infty} \mathrm{End}_{R^{p^e}}(R).$$

Proof. Recall that a k-linear endomorphism $\varphi \in \mathrm{End}_k(R)$ is a differential operator if and only if it is killed by some power of $J = J_{R|k}$. By Lemma II.26, this is equivalent to being killed by some Frobenius power $J^{[p^e]}$. Note that $J^{[p^e]} = (1 \otimes r^{p^e} - r^{p^e} \otimes 1 : r \in R)$, and that $(1 \otimes r^{p^e} - r^{p^e} \otimes 1)\varphi = 0$ if and only if φ commutes with multiplication by r^{p^e}. We conclude that $J^{[p^e]}\varphi = 0$ if and only if φ is R^{p^e}-linear, which gives the result. $\square$

Corollary II.28. *The ring $D_{R|k}$ of k-linear differential operators on R is independent of the choice of perfect ground field k and, in particular, $D_{R|k} = D_{R|\mathbb{F}_p}$.*

From this point onwards, given an F-finite ring R of characteristic p we will denote by $D_{R|\mathbb{F}_p}$ simply by D_R.

Definition II.29. Let R be an F-finite ring and $e \geq 0$ be an integer. The ring $\mathrm{End}_{R^{p^e}}(R)$ is called the *ring of differential operators of level e on R*, and it is denoted by $D_R^{(e)}$. Observe that we can also view $D_R^{(e)}$ as $D_R^{(e)} = \mathrm{End}_R(F_*^e R)$.

This description also allows us to extend Proposition II.20 to F-finite rings.

Proposition II.30. *Let R be an F-finite ring, $W \subseteq R$ be a multiplicative subset and $e \geq 0$ be an integer. Then there are compatible isomorphisms*

$$W^{-1}R \otimes_R D_R^{(e)} \cong D_{W^{-1}R}^{(e)}$$

of left $W^{-1}R$-modules, which give rise to a $(W^{-1}R, D_R)$-bimodule isomorphism.

$$W^{-1}R \otimes_R D_R \cong D_{W^{-1}R}.$$

Proof. By assumption the module $F_*^e R$ is finitely generated. We thus have isomorphisms

$$\begin{aligned} W^{-1}R \otimes_R D_R^{(e)} &\cong W^{-1}R \otimes_R \mathrm{Hom}_R(F_*^e R, F_*^e R) \\ &\cong \mathrm{Hom}_{W^{-1}R}(W^{-1}R \otimes_R F_*^e R, W^{-1}R \otimes_R F_*^e R) \\ &\cong \mathrm{Hom}_{W^{-1}R}(F_*^e W^{-1}R, F_*^e W^{-1}R) \\ &\cong D_{W^{-1}R}^{(e)}, \end{aligned}$$

which gives the result. $\square$

Corollary II.31. *Let R be an F-finite ring and $W \subseteq R$ be a multiplicative subset. Suppose M is a left D_R-module. Then $W^{-1}M$ has a natural left $D_{W^{-1}R}$-module structure, and thus a natural left D_R-module structure. The localization map $M \to W^{-1}M$ is D_R-linear.*

Proof. Analogous to Corollary II.21. $\square$

We note that Proposition II.27 also gives a different proof of the fact that D_R is a ring (Proposition II.15) in the case where R is F-finite.

Definition II.32. Given an ideal $\mathfrak{b} \subseteq R$ and an integer $e \geq 0$, we let $D_R^{(e)} \cdot \mathfrak{b}$ be the $D_R^{(e)}$-submodule of R generated by $\mathfrak{b}$; in other words, $D_R^{(e)} \cdot \mathfrak{b}$ is generated by elements of the form $\xi(g)$ where $\xi \in D_R^{(e)}$ and $g \in \mathfrak{b}$. An ideal $I \subseteq R$ which is closed under the action of $D_R^{(e)}$ is called a *$D_R^{(e)}$-ideal.* With this terminology, $D_R^{(e)} \cdot \mathfrak{b}$ is the smallest $D_R^{(e)}$-ideal that contains $\mathfrak{b}$.

It will be useful to have the following precise comparison between the order and the level filtrations on D_R.

Proposition II.33. *Let k be a perfect field and R be a k-algebra generated by elements $x_1, \dots, x_n \in R$. Then for all integers $e \geq 0$ we have*

$$D_R^{p^e-1} \subseteq D_R^{(e)} \subseteq D_R^{n(p^e-1)}$$

Proof. The elements $dx_1, \dots, dx_n$ generate the ideal J_R. By Lemma II.5 we have inclusions

$$J_R^{n(p^e-1)+1} \subseteq J_R^{[p^e]} \subseteq J_R^{p^e},$$

which yield the result. $\square$

II.3: Local cohomology modules and their D-module structures

A crucial step in the construction of Bernstein-Sato polynomials involves considering a local cohomology module and its natural D-module structure. Here we briefly review the definition of local cohomology modules, and explain why they carry such D-module structures. We refer the reader to [ILL+07] for details.

Let X be a topological space. We denote by Ab the category of abelian groups and by Ab(X) the category of sheaves of abelian groups on X. Given such a sheaf F and a point $x \in X$ we denote by F_x the stalk of F at x, and given a global chapter $s \in F(X)$ we denote by s_x its image in F_x.

Given a closed subset $Z \subseteq X$, we denote by $\Gamma_Z(X, F)$ the global chapters of F that are supported in Z; that is,

$$\Gamma_Z(X, F) = \{s \in F(X) \mid s_x = 0 \text{ for all } x \notin Z\}.$$

This construction defines a functor $\Gamma_Z(X, -) : \mathrm{Ab}(X) \to \mathrm{Ab}$, which is left exact, and given an integer $i \geq 0$ we denote by $H^i_Z(X, -)$ its i-th derived functor.

Suppose that $X = \operatorname{Spec} S$ for some ring S and that $Z = V(I)$ for some ideal $I \subseteq S$. Recall that an S-module M induces a quasicoherent sheaf $\widetilde{M}$ on X. In this situation, we denote $H^i_I(M) := H^i_Z(X, \widetilde{M})$, and call it the i-th local cohomology module of M with support in I. Given an element $g \in S$, multiplication by g induces an endomorphism of sheaves $\widetilde{M} \to \widetilde{M}$ and therefore an endomorphism $H^i_I(M) \to H^i_I(M)$ of abelian groups; this gives $H^i_I(M)$ a natural S-module structure. Whenever I is finitely generated, the functor $\Gamma_I = H^0_I$ is given by

$$\Gamma_I(M) = \{u \in M \mid I^k\, u = 0 \text{ for some } k \gg 0\}.$$

If we further assume that S is noetherian then $\widetilde{E}$ is a flasque sheaf (and thus acyclic for $\Gamma_Z(X, -)$) whenever E is an injective S-module [Har77, Prop. 3.4], which shows that $H^i_I(M)$ can be computed by using an injective resolution of M. More precisely, if $M \to E^\bullet$ is an injective resoluton of M then $H^i_I(M)$ is the i-th cohomology group of $\Gamma_I(E^\bullet)$.

Working with injective modules is certainly easier than working with injective sheaves; however, injective modules are still too large to make any computations beyond the easiest examples. Luckily, the Čech complex provides a tool that greatly simplifies this task.

Fix generators $I = (f_1, \dots, f_r)$ for the ideal I, and let M be an S-module. The Čech complex for M with respect to $\underline{f} = (f_1, \dots, f_r)$ is the complex $\check{C}^\bullet(\underline{f}; M)$ given by

$$\check{C}^n(\underline{f}; M) = \bigoplus_{1 \le i_1 < \cdots < i_n \le r} M_{f_{i_1} \cdots f_{i_n}}$$

and whose differential $d : \check{C}^n(\underline{f}; M) \to \check{C}^{n+1}(\underline{f}; M)$ is given by the alternating sum of maps

$$M_{f_{i_1} \cdots f_{i_n}} \to M_{f_{j_1} \cdots f_{j_{n+1}}}$$

which are localizations whenever $\{i_1, \dots, i_n\} \subseteq \{j_1, \dots, j_{n+1}\}$ and zero otherwise.

Theorem II.34 ([ILL+07, Thm. 7.13])**.** *Let S be a noetherian ring, $f_1, \dots, f_r$ be elements of S that generate an ideal I. Let M be an S-module. Then there are S-module isomorphisms*

$$H^i_I(M) \cong H^i \check{C}^\bullet(\underline{f}; M),$$

which are functorial in M.

This allows us to compute many examples of local cohomology modules; the following ones will make an appearance in the definition of Bernstein-Sato roots (Chapter IV).

Example II.35. Let R be a noetherian ring and consider the ideal (u) in the polynomial ring $R[u]$. We have the Čech complex

$$\check{C}^\bullet(u; R[u]) = [R[u] \to R[u]_u]$$

with $R[u]$ placed in degree 0 and $R[u]_u$ placed in degree 1. We conclude that

$$H^1_{(u)} R \cong R[u]_u / R[u];$$

in particular, we have an R-module decomposition

$$H^1_{(u)} R \cong \bigoplus_{i=1}^{\infty} R\ u^{-i}.$$

Example II.36. More generally, suppose $\underline{u} = (u_1, \ldots, u_r)$ is a set of variables and that $R[\underline{u}] = R[u_1, \ldots, u_r]$ is a polynomial ring over them. The tail of the Čech complex takes the form

$$\check{C}^\bullet(\underline{u}; R[\underline{u}]) = [\cdots \to \bigoplus_{i=1}^{r} R[\underline{u}]_{u_1 \cdots \widehat{u_i} \cdots u_r} \to R[\underline{u}]_{u_1 \cdots u_r} \to 0]$$

and therefore we have

$$H^r_{(\underline{u})} R[\underline{u}] = R[\underline{u}]_{u_1 \cdots u_r} / \sum_{i=1}^{r} R[\underline{u}]_{u_1 \cdots \widehat{u_i} \cdots \widehat{u}_r}.$$

We thus have an R-module decomposition

$$H^r_{(\underline{u})} R[\underline{u}] \cong \bigoplus_{\underline{i} \in (\mathbb{Z}_{>0})^r} R\ \underline{u}^{-\underline{i}}.$$

Setup II.37. We fix a noetherian ring B and a B-algebra S such that one of the following holds:

1. The algebra S is essentially of finite type over B.

2. We have $B = \mathbb{F}_p$ and S is F-finite.

Let $f_1, \ldots, f_r \in S$ be elements that generate an ideal $I \subseteq S$, and let M be a left D_S-module. We can then view M as an S-module and consider the Čech complex $\check{C}^\bullet(\underline{f}; M)$. All the terms $\check{C}^n(\underline{f}; M)$ in the Čech complex are direct sums of localizations of M. By Corollaries II.21 and II.31, the terms $\check{C}^n(\underline{f}; M)$ have natural left D_S-module structures, and these D_S-module structures extend the already-existing S-module structures. Finally, observe that the differentials of the Čech complex are given by localizations, which are D_S-linear. We conclude the following.

Proposition II.38. *Let S be a B-algebra as in Setup II.37, M be a left D_S-module and $f_1, \ldots, f_r \in S$ be elements that generate an ideal $I \subseteq S$. Then the Čech complex $\check{C}^\bullet(\underline{f}; M)$ is a complex of left D_S-modules, and therefore the local cohomology modules $H^i_I(M)$ admit left D_S-module structures.*

This implies, for example, that $H^i_I(S)$ admits a left D_S-module structure for any $i \geq 0$ and any ideal $I \subseteq S$. The existence of such a structure is a powerful tool in the study of local cohomology modules; for example, when S is a regular local ring containing a field of characteristic zero, Lyubeznik used it to prove that $H^i_I(S)$ has finitely many associated primes [Lyu93].

For many applications the description of the D_S-module structure on $H^i_I(M)$ via the Čech complex given above is sufficient, but it is somewhat unsatisfactory: it is not clear that the resulting D_S-module structure is independent of the choice of generators $f_1, \ldots, f_r$ of the ideal I. To remedy this, let us give an alternative construction of the D_S-module structure on local cohomology that does not rely on the Čech complex. We also learned about this description from Lyubeznik's work [Lyu93, Examples 2.1].

This additional description relies on the fact that the functors $H^i_I(-)$ admit extensions to the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on $\operatorname{Spec} S$ and, in fact, works in the following great generality.

Proposition II.39. *Let S be a B-algebra as in Setup II.37, $X = \operatorname{Spec} S$, and $G : \mathrm{Ab}(X) \to \mathrm{Ab}$ be a covariant functor. If M is a left D_S-module then $G(\widetilde{M})$ acquires a natural structure of left D_S-module. Moreover, if $G' : \mathrm{Ab}(X) \to \mathrm{Ab}$ is another covariant functor and $G \implies G'$ is a natural transformation then the morphism $G(\widetilde{M}) \to G'(\widetilde{M})$ is D_S-linear.*

Proof. By Corollaries II.21 and II.31, D_S acts on every localization of M, and therefore D_S acts on $\widetilde{M}$. This means that there is a ring homomorphism $D_S \to \operatorname{End}_{\mathrm{Ab}(X)}(\widetilde{M})$ which, after composition with the natural map $\operatorname{End}_{\mathrm{Ab}(X)}(\widetilde{M}) \to \operatorname{End}_{\mathrm{Ab}}(G(\widetilde{M}))$, yields a ring homomorphism $D_S \to \operatorname{End}_{\mathrm{Ab}}(G(\widetilde{M}))$.

Given a natural transformation $G \implies G'$, we get a commutative diagram of ring homomorphisms

$$\begin{tikzcd} & & \operatorname{End}_{\mathrm{Ab}}(G(\widetilde{M})) \arrow[d] \\ D_S \arrow[r] & \operatorname{End}_{\mathrm{Ab}(X)}(\widetilde{M}) \arrow[ur] \arrow[dr] & \\ & & \operatorname{End}_{\mathrm{Ab}}(G'(\widetilde{M})), \end{tikzcd}$$

which shows that $G(\widetilde{M}) \to G(\widetilde{M'})$ is D_S-linear. $\square$

The above proposition, applied to the case $G = H^i_Z(X, -)$, yields a natural D_S-module structure on $H^i_I(M)$.

II.4: Frobenius descent and test ideals

We let R be an F-finite ring of characteristic p, and we will now work under the additional assumption that R is regular. Kunz's theorem (Theorem II.1) tells us that $F^e_* R$ is flat over R, and since it is finitely generated as an R-module by assumption, it is locally free. Morita theory then gives as equivalence between the category R- Mod of R-modules and the category $D^{(e)}_R$- Mod of left $D^{(e)}_R$-modules — recall that $D^{(e)}_R = \operatorname{End}_R(F^e_* R)$. This equivalence is loosely referred to as Frobenius descent (see [Bli01], [ÀMBL05] and the references therein). We now spell out this equivalence in more detail.

First observe that $\operatorname{Hom}_R(R, F^e_* R)$ is a $(D^{(e)}_R, R)$-bimodule, where $D^{(e)}_R$ acts by postcomposition and R acts by precomposition. An element $\varphi \in \operatorname{Hom}_R(R, F^e_* R)$ is uniquely determined by $\varphi(1)$, which gives an isomorphism $\operatorname{Hom}_R(R, F^e_* R) \cong F^e_* R$. Similarly, we define the following:

Definition II.40. [[Bli13]] Let R be an F-finite ring and $e \geq 0$ be an integer. The collection of *p^{-e}-linear operators on R* is given by $\mathcal{C}^{(e)}_R = \operatorname{Hom}_R(F^e_* R, R)$; in other words, $\mathcal{C}^{(e)}_R$ is the collection of additive operators $\varphi : R \to R$ such that $\varphi(f^{p^e} g) = f\varphi(g)$ for all $f, g \in R$. Given an ideal $J \subseteq R$ we will call $\mathcal{C}^{(e)}_R \cdot J$ the ideal generated by elements of the form $\varphi(g)$ where $\varphi \in \mathcal{C}^{(e)}_R$ and $g \in J$.

Note that $\mathcal{C}^{(e)}_R$ has a natural $(R, D^{(e)}_R)$-bimodule structure.

Lemma II.41. *Let S be a noetherian commutative ring, let F_i ($i = 1, 2, 3$) be nonzero finitely generated locally free S-modules and, for $i = 1, 2, 3$, let $E_i = \operatorname{End}_S(F_i)$. Then the natural map*

$$\operatorname{Hom}_R(F_2, F_3) \otimes_{E_2} \operatorname{Hom}_R(F_1, F_2) \longrightarrow \operatorname{Hom}_R(F_1, F_3)$$

that sends $[\varphi \otimes \psi \mapsto \varphi \circ \psi]$ is an isomorphism of (E_3, E_1)-bimodules.

Proof. One readily checks that the morphism given is (E_3, E_1)-linear. By working locally it suffices to prove the statement when all the F_i are free. We will use the notation $(-)^* = \operatorname{Hom}_S(-, S)$, and we will identify $\operatorname{Hom}_S(S, F_i) = F_i$. Before tackling the statement, we focus on two special cases.

Observe that the case where $F_2 = S$ (that is, the fact that $F_3 \otimes_S F_1^* \to \operatorname{Hom}_S(F_1, F_3)$ is an isomorphism) follows from the fact that Hom commutes with finite direct sums.

We next consider the case when $F_3 = F_1 = S$. Now we want to show that the map $F_2^* \otimes_{E_2} F_2 \to S$ given by $[\varphi \otimes v \mapsto \varphi(v)]$ is an isomorphism. Since F_2 is nonzero it contains a copy of S as a direct summand; that is, we may find homomorphisms $i : S \to F_2$ and $\pi : F_2 \to S$ such that $\pi \circ i = \mathrm{id}_S$. We claim that the map $S \to F_2^* \otimes_{E_2} F_2$ given by $[g \mapsto \pi \otimes i(g)]$ is provides an inverse. One direction is easy: $\pi(i(g)) = \mathrm{id}_S(g) = g$, and for the other direction we observe that given $\varphi \in F_2^*$ we have $i \circ \varphi \in E_2$ and therefore

$$\begin{aligned} \pi \otimes i\big(\varphi(v)\big) &= \pi \otimes (i \circ \varphi)(v) \\ &= \pi(i \circ \varphi) \otimes v \\ &= \varphi \otimes v. \end{aligned}$$

For the general case we combine the two previous cases and compute:

$$\begin{aligned} \mathrm{Hom}_S(F_2, F_3) \otimes_{E_2} \mathrm{Hom}_S(F_1, F_2) &\cong F_3 \otimes_S F_2^* \otimes_{E_2} F_2 \otimes F_1^* \\ &\cong F_3 \otimes_S F_1^* \\ &\cong \mathrm{Hom}_S(F_1, F_3). \end{aligned}$$

Since all the isomorphisms are given by composing homomorphisms, the above composition agrees with the one in the statement. □

Proposition II.42 (Frobenius descent). *Let R be a regular F-finite ring of characteristic p and $e \geq 0$ be an integer. The functors*

$$R\text{-}\mathrm{Mod} \underset{\kappa_e}{\overset{\varphi_e}{\rightleftarrows}} D_R^{(e)}\text{-}\mathrm{Mod}$$

given by $\varphi_e = F_*^e R \otimes_R (-)$ and $\kappa_e = \mathcal{C}_R^{(e)} \otimes_{D_R^{(e)}} (-)$ give an equivalence of categories $R\text{-}\mathrm{Mod} \simeq D_R^{(e)}\text{-}\mathrm{Mod}$.

Proof. The composition $\kappa_e \circ \varphi_e$ is given by tensoring with the bimodule $\mathrm{Hom}_R(F_*^e R, R) \otimes_R \mathrm{Hom}_R(R, F_*^e R)$ which, by Lemma II.41, is naturally isomorphic to the bimodule $\mathrm{Hom}_R(R, R) = R$. We conclude that $\kappa_e \circ \varphi_e \simeq R \otimes_R (-) \simeq \mathrm{id}_{R\text{-}\mathrm{Mod}}$.

Similarly, $\varphi_e \circ \kappa_e$ is given by tensoring with the bimodule

$$\mathrm{Hom}_R(R, F_*^e R) \otimes_R \mathrm{Hom}_R(F_*^e R, R) \cong \mathrm{Hom}_R(F_*^e R, F_*^e R) = D_R^{(e)}$$

and thus $\varphi_e \circ \kappa_e \simeq D_R^{(e)} \otimes_{D_R^{(e)}} (-) \simeq \mathrm{id}_{D_R^{(e)}\text{-}\mathrm{Mod}}$. □

Notice that, under the above equivalence of categories, the module $R \in R$-Mod corresponds to $F^e_* R \in D^{(e)}_R$-Mod. In particular, we get a one-to-one correspondence between the ideals of R and the $D^{(e)}_R$-submodules of $F^e_* R$. In order to unpack this correspondence more explicitly, we prove the following.

Lemma II.43. *Let $\mathfrak{b} \subseteq R$ be an ideal and $J \subseteq F^e_* R$ be a $D^{(e)}_R$-submodule of $F^e_* R$.*

1. *Under the identification $\varphi_e(R) \cong F^e_* R$, $\varphi_e(\mathfrak{b})$ gets identified with $F^e_* \mathfrak{b}^{[p^e]}$.*

2. *Under the identification $\kappa_e(F^e_* R) \cong R$, $\kappa_e(J)$ gets identified with $\mathcal{C}^{(e)}_R \cdot J$.*

Proof. Note that the identification $\varphi_e(R) = F^e_* R \otimes_R R \xrightarrow{\sim} F^e_* R$ is given by $[F^e_* r \otimes s \mapsto F^e_*(rs^{p^e})]$. The ideal $\varphi_e(\mathfrak{b})$ corresponds to the image of $F^e_* R \otimes_R \mathfrak{b}$ under this map, which is $F^e_* \mathfrak{b}^{[p^e]}$, thus proving (i). For (ii), note that the identification $\kappa_e(F^e_* R) = \mathcal{C}^{(e)}_R \otimes_R F^e_* R \xrightarrow{\sim} R$ comes from $\varphi \otimes F^e_* r = \varphi(F^e_* R)$. The ideal $\kappa_e(J)$ corresponds to the image of $\mathcal{C}^{(e)}_R \otimes J$, which is $\mathcal{C}^{(e)}_R \cdot J$. ∎

Proposition II.44. *Let R be a regular F-finite ring and $e \geq 0$ be an integer. Then there is an inclusion preserving one-to-one correspondence*

$$\left\{ \begin{matrix} \textit{Ideals} \\ \mathfrak{b} \subseteq R \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} D^{(e)}_R\textit{-ideals} \\ J \subseteq R \end{matrix} \right\}$$

which identifies an ideal $\mathfrak{b} \subseteq R$ with the $D^{(e)}_R$-ideal $\mathfrak{b}^{[p^e]}$, and a $D^{(e)}_R$-ideal $J \subseteq R$ with the ideal $\mathcal{C}^{(e)}_R \cdot J \subseteq R$.

Proof. Recall that, as a ring, $F^e_* R$ is just R. Consequently, a $D^{(e)}_R$-submodule of $F^e_* R$ is just a $D^{(e)}_R$-ideal of R. The result then follows from Proposition II.42 and Lemma II.43. ∎

Corollary II.45. *Every $D^{(e)}_R$-ideal $J \subseteq R$ has the form $J = \mathfrak{b}^{[p^e]}$ for some ideal $\mathfrak{b} \subseteq R$.*

Corollary II.46. *Let $\mathfrak{b}, \mathfrak{b}' \subseteq R$ be ideals. Then:*

1. *We have $D^{(e)}_R \cdot \mathfrak{b} = (\mathcal{C}^{(e)}_R \cdot \mathfrak{b})^{[p^e]}$.*

2. *We have $\mathcal{C}^{(e)}_R \cdot \mathfrak{b} \subseteq \mathfrak{b}'$ if and only if $\mathfrak{b} \subseteq \mathfrak{b}'^{[p^e]}$.*

3. *We have $D^{(e)}_R \cdot \mathfrak{b} = D^{(e)}_R \cdot \mathfrak{b}'$ if and only if $\mathcal{C}^{(e)}_R \cdot \mathfrak{b} = \mathcal{C}^{(e)}_R \cdot \mathfrak{b}'$.*

In the case when $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a perfect field k the ideals $\mathcal{C}^{(e)}_R \cdot \mathfrak{b}$ can be described very explicitly. First note that, in this situation, the module $F^e_* R$ is a free R-module with basis $F^e_*(x_1^{a_1} \cdots x_n^{a_n})$, as $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ ranges through all tuples with $0 \leq a_i < p^e$. It follows that, given some $g \in R$ there are some $\widetilde{g}_{\underline{a}} \in R$ such that $g = \sum_{0 \leq a_i < p^e} \widetilde{g}_{\underline{a}}^{p^e} x_1^{a_1} \cdots x_n^{a_n}$, and that these $\widetilde{g}_{\underline{a}}$ are completely determined by g.

Proposition II.47. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field k, $\mathfrak{b} = (g_1, \ldots, g_r) \subseteq R$ be an ideal and $e \geq 0$ be an integer. For each integer $1 \leq j \leq r$ and each tuple $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ with $0 \leq a_i < p^e$ let $\widetilde{g}_{j\underline{a}} \in R$ be the elements of R given by $g_j = \sum_{0 \leq a_i < p^e} \widetilde{g}_{j\underline{a}}^{p^e} x_1^{a_1} \cdots x_n^{a_n}$. Then*

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{b} = \left(\widetilde{g}_{j,\underline{a}} \mid 0 \leq j \leq r, 0 \leq a_i < p^e\right).$$

Proof. For the inclusion ($\subseteq$), notice that an arbitrary element of $\mathfrak{b}$ is written in the form $h = \sum_j h_j g_j = \sum_{j,\underline{a}} \widetilde{g}_{j,\underline{a}}^{p^e} h_j x_1^{a_1} \cdots x_n^{a_n}$ for some $h_j \in R$. Given some $\psi \in \mathcal{C}_R^{(e)}$ we have $\psi(h) = \sum_{j,\underline{a}} \widetilde{g}_{j,\underline{a}} \psi(h_j x_1^{a_1} \cdots x_n^{a_n})$, which is in the right-hand side.

For the inclusion ($\supseteq$), fix some $0 \leq k \leq r$ and some $\underline{b}$ with $0 \leq b_i < p^e$. Let $\sigma_{\underline{b}}^{(e)} \in \mathcal{C}_R^{(e)}$ be the unique operator with the property that, for all $\underline{a}$ with $0 \leq a_i < p^e$, we have

$$\sigma_{\underline{b}}^{(e)}(x_1^{a_1} \cdots x_n^{a_n}) = \begin{cases} 1 \text{ if } \underline{a} = \underline{b} \\ 0 \text{ otherwise.} \end{cases}$$

We then have $\widetilde{g}_{k,\underline{b}} = \sigma_{\underline{b}}^{(e)}(g_k) \in \mathcal{C}_R^{(e)} \cdot \mathfrak{b}$, as required. $\square$

Finally, we discuss a few more properties of the ideals $\mathcal{C}_R^{(e)} \cdot \mathfrak{b}$.

Proposition II.48. *Let R be a regular F-finite ring and $\mathfrak{b} \subseteq R$ be an ideal. For all integers $e, d \geq 0$:*

1. *We have $\mathcal{C}_R^{(e)} \cdot \mathcal{C}_R^{(d)} \cdot \mathfrak{b} = \mathcal{C}_R^{(e+d)} \cdot \mathfrak{b}$.*
2. *We have $\mathcal{C}_R^{(e)} \cdot \mathfrak{b} = \mathcal{C}_R^{(e+d)} \cdot \mathfrak{b}^{[p^d]}$.*

Proof. For part (i), notice that the left hand side is generated by elements of the form $\varphi(\psi(g)) = (\varphi \circ \psi)(g)$ where $\varphi \in \mathcal{C}_R^{(e)}$, $\psi \in \mathcal{C}_R^{(d)}$ and $g \in \mathfrak{b}$. One easily checks that $\varphi \circ \psi \in \mathcal{C}_R^{(e+d)}$, and therefore all these elements are in the right hand side.

On the other hand, the right hand side is generated by elements of the form $\eta(g)$ where $\eta \in \mathcal{C}_R^{(e+d)}$ and $g \in R$. Fix an R-module splitting $\sigma : F_*^d R \to R$ of the d-th iterated Frobenius $F^d : R \to F_*^d R$ (which exists by Proposition II.6). We then have $\eta(g) = (\sigma \circ F^d \circ \eta)(g) = \sigma((F^d \circ \eta)(g))$. We have $\sigma \in \mathcal{C}_R^{(d)}$, and we can easily check that $F^d \circ \eta \in \mathcal{C}_R^{(e)}$; we conclude that $\eta(g)$ is in the left hand side.

For part (ii), we first claim that $\mathcal{C}_R^{(d)} \cdot \mathfrak{b}^{[p^d]} = \mathfrak{b}$; the inclusion ($\subseteq$) is clear, and the inclusion ($\supseteq$) is proved by observing that $g = \sigma(g^{p^d})$ where $\sigma : F_*^d R \to R$ is a splitting of F^d. Using this claim and part (i) we conclude that $\mathcal{C}_R^{(e+d)} \cdot \mathfrak{b}^{[p^d]} = \mathcal{C}_R^{(e)} \cdot \mathcal{C}_R^{(d)} \cdot \mathfrak{b}^{[p^d]} = \mathcal{C}_R^{(e)} \cdot \mathfrak{b}$. $\square$

Our next goal will be to define, for a given ideal $\mathfrak{a} \subseteq R$ and a real number $\lambda \geq 0$, the test ideal $\tau(\mathfrak{a}^\lambda)$. Although this notion has a rich history which starts in the tight closure theory of Hochster and Huneke, we will try to motivate this notion by using Frobenius descent with the following example.

Example II.49. Let $p = 2$, $R = \mathbb{F}_2[x]$ and pick an element $f \in R$. We consider the correspondence of Proposition II.44 in the case $e = 1$. Under this correspondence, the ideal $\mathfrak{b}_1 = (f)$ corresponds to the $D_R^{(1)}$-ideal $J_1 = (f^2)$ and the ideal $\mathfrak{b}_2 = (f^2)$ corresponds to the $D_R^{(1)}$-ideal $J_2 = (f^4)$. The ideal (f^3) is not a $D_R^{(1)}$-ideal in general, but notice that the $D_R^{(1)}$-ideal $D_R^{(1)} \cdot f^3$ sits naturally in between J_1 and J_2; let us call it $J_{3/2}$ for this reason.

Under our correspondence, the chain of $D_R^{(1)}$-ideals $J_1 \supseteq J_{3/2} \supseteq J_2$ corresponds to a chain of ideals $(f) \supseteq \mathfrak{b}_{3/2} \supseteq (f^2)$, where $\mathfrak{b}_{3/2} = \mathcal{C}_R^{(1)} \cdot f^3$. The ideal $\mathfrak{b}_{3/2}$ is the so-called test ideal $\tau(f^{3/2})$ of f with exponent $3/2$ (see Definition II.50, Proposition II.51). One can compute it easily for some simple examples by using Proposition II.47; for example, for $f = x$ we get $\tau(f^{3/2}) = (x)$ and for $f = x^2$ we get $\tau(f^{3/2}) = (x^3)$.

For an arbitrary prime p and integer $e \geq 0$, the test ideal $\tau(f^{n/p^e})$ is the ideal that corresponds to the $D_R^{(e)}$-ideal $D_R^{(e)} \cdot f^n$ under the correspondence of Proposition II.44, and is therefore given by $\tau(f^{n/p^e}) = \mathcal{C}_R^{(e)} \cdot f^n$. To get a notion of $\tau(f^\lambda)$ for an arbitrary $\lambda \in \mathbb{R}_{\geq 0}$, a reasonable guess is to approximate λ with some rational number of the form n/p^e; for example, $\lceil \lambda p^e \rceil$. As proved by Blickle, Mustaţă and Smith [BMS08], this approach indeed recovers the test ideal as defined by Hara and Yoshida [BMS08] and, in fact, works for arbitrary ideals.

Definition II.50 ([BMS08])**.** Let R be a regular F-finite ring, $\mathfrak{a} \subseteq R$ be an ideal and $\lambda \geq 0$ be a real number. The *test ideal* $\tau(\mathfrak{a}^\lambda)$ *for* $\mathfrak{a}$ *with exponent* λ is given by

$$\tau(\mathfrak{a}^\lambda) = \bigcup_{e=0}^{\infty} \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}.$$

We observe that the right-hand side is an increasing union of ideals and, since R is noetherian, we have $\tau(\mathfrak{a}^\lambda) = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$ for some e large enough.

When $\mathfrak{a}$ is principal, say $\mathfrak{a} = (f)$, we denote $\tau(\mathfrak{a}^\lambda)$ by $\tau(f^\lambda)$. We show that for principal ideals we recover the notion discussed above.

Proposition II.51. *Let R be a regular F-finite ring and $f \in R$ be an element. Then for all integers $e, n \geq 0$ we have $\mathcal{C}_R^{(e)} \cdot f^n = \tau(f^{n/p^e})$.*

Proof. Using Proposition II.48, we observe that for all $d \geq 0$ we have

$$\mathcal{C}_R^{(e)} \cdot f^n = \mathcal{C}_R^{(e)} \cdot \mathcal{C}_R^{(d)} \cdot f^{np^d} = \mathcal{C}_R^{(e+d)} \cdot f^{(n/p^e)p^{e+d}}$$ ∎

If $\lambda \leq \mu$ then $\tau(\mathfrak{a}^\lambda) \supseteq \tau(\mathfrak{a}^\mu)$ and, given some $\lambda \in \mathbb{R}_{\geq 0}$, there exists some $\epsilon > 0$ such that $\tau(\mathfrak{a}^\lambda) = \tau(\mathfrak{a}^\mu)$ for all $\mu \in [\lambda, \lambda + \epsilon)$ [BMS08, Cor. 2.16].

Definition II.52. We say λ is an *F-jumping number of* $\mathfrak{a}$ if for all $\epsilon > 0$ we have $\tau(\mathfrak{a}^{\lambda-\epsilon}) \neq \tau(\mathfrak{a}^\lambda)$. We denote by $\mathrm{FJN}(\mathfrak{a})$ the set of F-jumping numbers of $\mathfrak{a}$.

The set of F-jumping numbers forms a discrete subset of $\mathbb{R}_{\geq 0}$ and all F-jumping numbers are rational; see [BMS08, Thm 3. 1] for the case where R is essentially of finite type over an F-finite field, [BMS09, Thm 1.1] for the case of a principal ideal in an arbitrary regular F-finite ring, and [ST14, Thm. B] for the general case.

Lemma II.53. *Suppose that $\mathfrak{a} \subseteq R$ is generated by r elements, and fix an integer $e \geq 0$. Let $n, m \geq 0$ be integers with $n \geq mp^e + (r-1)(p^e - 1)$. Then we have $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n = \mathfrak{a}^m\, \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n-mp^e}$.*

Proof. Observe that if $\mathfrak{b}, \mathfrak{c} \subseteq R$ are ideals then $\mathcal{C}_R^{(e)} \cdot \mathfrak{b}\mathfrak{c}^{[p^e]} = \mathfrak{c}\, \mathcal{C}_R^{(e)} \cdot \mathfrak{b}$, and by using Lemma II.5 we get

$$\begin{aligned} \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n &= \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n-mp^e} \mathfrak{a}^{m[p^e]} \\ &= \mathfrak{a}^m \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n-mp^e}. \end{aligned}$$

□

Proposition II.54 (Skoda-type theorem). *Let R be a regular F-finite ring, $\mathfrak{a} \subseteq R$ be an ideal generated by r elements and λ be a real number. Then:*

(i) If $\lambda \geq r$ then $\tau(\mathfrak{a}^\lambda) = \mathfrak{a}\ \tau(\mathfrak{a}^{\lambda-1})$.

(ii) If $\lambda > r$ is an F-jumping number for $\mathfrak{a}$ then so is $\lambda - 1$.

Proof. Pick e large enough so that $\tau(\mathfrak{a}^\lambda) = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$ and $\tau(\mathfrak{a}^{\lambda-1}) = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil (\lambda-1)p^e \rceil}$. Part (i) then follows by applying Lemma II.53. For part (ii) note that, whenever $\epsilon > 0$ is small enough so that $\lambda - \epsilon > r$, we get

$$\mathfrak{a}\ \tau(\mathfrak{a}^{\lambda-1-\epsilon}) = \tau(\mathfrak{a}^{\lambda-\epsilon}) \neq \tau(\mathfrak{a}^\lambda) = \mathfrak{a}\ \tau(\mathfrak{a}^{\lambda-1})$$

and thus $\tau(\mathfrak{a}^{\lambda-1-\epsilon}) \neq \tau(\mathfrak{a}^\lambda)$. □

II.5: The V-filtration on $D_{R[t]}$

Let R be a B-algebra, $\underline{t} = (t_1, \ldots, t_r)$ be a set of variables, and let $R[\underline{t}] := R[t_1, \ldots, t_r]$ be a polynomial ring over R. In this section we collect a few facts about the relationship between $D_{R|B}$ and $D_{R[\underline{t}]|B}$ that will be needed later. Let us now point out that we will always work relative to B, and hence we omit the subscript "$|B$" from all notations.

We have an algebra isomorphism $R \otimes_B B[\underline{t}] \cong R[\underline{t}]$, and therefore also an algebra isomorphism

$$\Phi : (R \otimes_B R) \otimes_B (B[\underline{t}] \otimes_B B[\underline{t}]) \xrightarrow{\sim} R[\underline{t}] \otimes_B R[\underline{t}].$$

which sends $\Phi\big((r \otimes s) \otimes (f \otimes g)\big) = rf \otimes sg$.

We let $\widetilde{J}_R$ be the expansion of J_R to $R[\underline{t}] \otimes_B R[\underline{t}]$ via the composition

$$R \otimes_B R \to (R \otimes_B R) \otimes_B (B[\underline{t}] \otimes_B B[\underline{t}]) \xrightarrow{\Phi} R[\underline{t}] \otimes_B R[\underline{t}],$$

and similarly we let $\widetilde{J}_{B[\underline{t}]}$ be the expansion of $J_{B[\underline{t}]}$.

Lemma II.55. *We have $\widetilde{J}_R + \widetilde{J}_{B[\underline{t}]} = J_{R[\underline{t}]}$.*

Proof. The inclusion ($\subseteq$) is clear. For ($\supseteq$) we note that $J_{R[\underline{t}]}$ is generated by elements of the form $d(rf)$ for $r \in R$ and $f \in B[\underline{t}]$, and that such an element $d(rf)$ can be written as

$$\begin{aligned} d(rf) &= 1 \otimes rf - rf \otimes 1 \\ &= (1 \otimes r)(1 \otimes f) - (r \otimes 1)(f \otimes 1). \end{aligned}$$

We have $1 \otimes r \equiv r \otimes 1 \mod \widetilde{J}_R$ and $1 \otimes f \equiv f \otimes 1 \mod \widetilde{J}_{B[\underline{t}]}$, and thus $d(rf) =\equiv 0 \mod \widetilde{J}_R + \widetilde{J}_{B[\underline{t}]}$. □

For all integers $n \geq 0$ we let $J^{\{n\}}_{R[\underline{t}]} \subseteq R[\underline{t}] \otimes_B R[\underline{t}]$ denote the ideal

$$J^{\{n\}}_{R[\underline{t}]} = \widetilde{J}^n_R + \widetilde{J}^n_{B[\underline{t}]}.$$

By Lemma II.55 we conclude that the families of ideal $\{J^n_{R[\underline{t}]}\}_{n=0}^\infty$ and $\{J^{\{n\}}_{R[\underline{t}]}\}_{n=0}^\infty$ are cofinal, and therefore we have

$$D_{R[\underline{t}]} = \bigcup_{n=0}^{\infty} D^{\{n\}}_{R[\underline{t}]}$$

where

$$D^{\{n\}}_{R[\underline{t}]} = \{\varphi \in \operatorname{End}_B(R[\underline{t}]) \mid J^{\{n+1\}}_{R[\underline{t}]} \cdot \varphi = 0\}.$$

Arguing as in Section II.2, for every $n \geq 0$ we have an $R[\underline{t}]$-module isomorphism

$$D^{\{n\}}_{R[\underline{t}]} \cong \operatorname{Hom}_{R[\underline{t}]}\left(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}]\right)$$

where

$$P^{\{n\}}_{R[\underline{t}]} = \frac{R[\underline{t}] \otimes_B R[\underline{t}]}{J^{\{n+1\}}_{R[\underline{t}]}}.$$

Note that Φ induces an algebra isomorphism

$$P_R^n \otimes_B P_{B[\underline{t}]}^n \cong P_{R[\underline{t}]}^{\{n\}}.$$

Lemma II.56. *Let R and S be B-algebras, W be an R-module, and U and V be S-modules. Suppose one of the following holds:*

(1) The module U is finitely generated and free over S, or

(2) The module W is flat over B.

Then there is a natural $R \otimes_B S$-module isomorphism

$$W \otimes_B \operatorname{Hom}_S(U, V) \cong \operatorname{Hom}_{R\otimes_B S}(R \otimes_B U, W \otimes_B V),$$

which sends $w \otimes \varphi$ to the map $[r \otimes u \mapsto rw \otimes \varphi(u)]$.

Proof. Under any of the conditions given we have $W \otimes_B \operatorname{Hom}_S(U, V) \cong \operatorname{Hom}_S(U, W \otimes_B V)$ and, by the adjunction of extension and restriction of scalars, we have $\operatorname{Hom}_S(U, W \otimes_B V) \cong \operatorname{Hom}_{R\otimes_B S}(R \otimes_B U, W \otimes_B V)$. The resulting composition is readily checked to agree with the morphism in the statement. $\square$

Lemma II.57. *Let R and S be B-algebras, M and N be R-modules, and F and G be S-modules. Assume that F is finitely generated and free over S, and that G is flat over B. Then there is an $R \otimes_B S$-module isomorphism*

$$\operatorname{Hom}_R(M, N) \otimes_B \operatorname{Hom}_S(F, G) \cong \operatorname{Hom}_{R\otimes_B S}(M \otimes_B F, N \otimes_B G),$$

which sends $\varphi \otimes \psi$ to the map $[u \otimes a \mapsto \varphi(u) \otimes \psi(a)]$.

Proof. To declutter notation, let us write "$\otimes_B$" simply as "$\otimes$", and let $T = R \otimes S$. By using Lemma II.56 twice, followed by the tensor-Hom adjunction, we get

$$\begin{aligned}
\operatorname{Hom}_R(M, N) \otimes \operatorname{Hom}_S(F, G) &\cong \operatorname{Hom}_T(R \otimes F, \operatorname{Hom}_R(M, N) \otimes G) \\
&\cong \operatorname{Hom}_T(R \otimes F, \operatorname{Hom}_T(M \otimes S, N \otimes G)) \\
&\cong \operatorname{Hom}_T\big((R \otimes F) \otimes_T (M \otimes S), N \otimes G\big).
\end{aligned}$$

Finally, we have a map $(R \otimes F) \otimes_T (M \otimes S) \to M \otimes F$ given by $[r \otimes u \otimes v \otimes s \mapsto rv \otimes su]$ and a map $M \otimes F \to (R \otimes F) \otimes_T (M \otimes S)$ given by $[v \otimes u \mapsto 1 \otimes u \otimes v \otimes 1]$. These are mutual inverses, and thus $(R \otimes F) \otimes_T (M \otimes S) \cong M \otimes F$.

The resulting composition is readily checked to agree with the one given in the statement. $\square$

Suppose that $\xi \in D_R$ is a B-linear differential operator on R. Then ξ induces an operator $\widetilde{\xi}$ on $R[\underline{t}]$ given by $\widetilde{\xi}(s\ t_1^{a_1}\cdots t_r^{a_r}) = \xi(s)\ \underline{t}_1^{a_1}\cdots t_r^{a_r}$ for all $s \in R$ and $(a_1,\dots,a_r) \in \mathbb{Z}^r_{\geq 0}$. We note that $\widetilde{\xi}$ commutes with multiplication by all the variables t_i, and we therefore have $\widetilde{J}_{B[\underline{t}]} \cdot \widetilde{\xi} = 0$. Moreover, if ξ has order $\leq n$ then $J_R^{n+1}\cdot \xi = 0$ and therefore $\widetilde{J}_R^{n+1}\cdot\widetilde{\xi} = 0$. We conclude that $J_{R[\underline{t}]}^{\{n+1\}}\cdot\widetilde{\xi} = 0$ and, in particular, $\widetilde{\xi} \in D_{R[\underline{t}]}^{\{n\}}$ is a B-linear differential operator on $R[\underline{t}]$.

Similarly, given a differential operator $\eta \in D_{B[\underline{t}]}$ on $B[\underline{t}]$, we get an operator $\widetilde{\eta}$ on $R[\underline{t}]$ given by $\widetilde{\eta}(s\ t_1^{a_1}\cdots t_r^{a_r}) = s\ \eta(t_1^{a_1}\cdots t_r^{a_r})$ for all $s \in R$ and $(a_1,\dots,a_r)\in\mathbb{Z}^r_{\geq 0}$. If η has order $\leq n$ then $\widetilde{\eta}\in D_{R[\underline{t}]}^{\{n\}}$.

Proposition II.58. *Let R be a B-algebra and $R[\underline{t}] = R[t_1,\dots,t_r]$ be a polynomial ring over R. The assignment $\xi\otimes\eta \mapsto \widetilde{\xi}\ \widetilde{\eta}$ gives an algebra isomorphism*

$$D_R \otimes_B D_{B[\underline{t}]} \xrightarrow{\sim} D_{R[\underline{t}]}.$$

When B is a perfect field of characteristic $p > 0$, this restricts to isomorphisms

$$D_R^{(e)} \otimes_B D_{B[\underline{t}]}^{(e)} \xrightarrow{\sim} D_{R[\underline{t}]}^{(e)}$$

for all $e \geq 0$.

Proof. The assignments $\xi\mapsto\widetilde{\xi}$ and $\eta\mapsto\widetilde{\eta}$ respect the multiplication and, given $\xi\in D_R$ and $\eta\in D_{B[\underline{t}]}$, the operators $\widetilde{\xi}$ and $\widetilde{\eta}$ commute. We conclude that the assignment given in the statement is indeed an algebra homomorphism, and it remains to check that it is bijective.

Recall that the module $P^n_{B[\underline{t}]}$ is finitely generated and free over B (see Example II.12). For every integer $n\geq 0$ we have the following chain of isomorphisms, where "$\otimes$" denotes "$\otimes_B$".

$$\begin{aligned}
D_R^n \otimes D_{B[\underline{t}]}^n &\cong \operatorname{Hom}_R(P_R^n, R)\otimes \operatorname{Hom}_{B[\underline{t}]}(P^n_{B[\underline{t}]}, B[\underline{t}]) \\
\text{(Lemma II.57)} \qquad &\cong \operatorname{Hom}_{R[\underline{t}]}(P_R^n \otimes P^n_{B[\underline{t}]}, R[\underline{t}]) \\
&\cong \operatorname{Hom}_{R[\underline{t}]}(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}]) \\
&\cong D^{\{n\}}_{R[\underline{t}]}.
\end{aligned}$$

This composition is checked to agree with the one given in the statement and, since $D_{R[\underline{t}]} = \varinjlim_n D^{\{n\}}_{R[\underline{t}]}$ and $D_R\otimes D_{B[\underline{t}]} = \varinjlim_n D_R^n\otimes D^n_{B[\underline{t}]}$, the first result follows.

Suppose now that B is a perfect field of characteristic $p > 0$, and let $e \geq 0$ be an integer. Using Lemma II.57 once again, we observe that

$$\begin{aligned}\mathrm{Hom}_R(F^e_* R, F^e_* R) \otimes_B \mathrm{Hom}_{B[\underline{t}]}(F^e_* B[\underline{t}], F^e_* B[\underline{t}]) &\cong \mathrm{Hom}_{R[\underline{t}]}(F^e_* R \otimes_B F^e_* B[\underline{t}], F^e_* R \otimes_B F^e_* B[\underline{t}]) \\ &\cong \mathrm{Hom}_{R[\underline{t}]}(F^e_* R[\underline{t}], F^e_* R[\underline{t}]),\end{aligned}$$

which yields an isomorphism $D^{(e)}_R \otimes_B D^{(e)}_{B[\underline{t}]} \xrightarrow{\sim} D^{(e)}_{R[\underline{t}]}$. We then check that this isomorphism agrees with the one in the statement. □

Definition II.59. Let I denote the ideal $I = (t_1, \ldots, t_r) \subseteq R[\underline{t}]$. For every $i \in \mathbb{Z}$ we denote

$$V^i D_{R[\underline{t}]} = \{\xi \in D_{R[\underline{t}]} \mid \xi \cdot I^j \subseteq I^{j+i} \text{ for all } j \in \mathbb{Z}\},$$

where we adopt the convention that $I^j = R[\underline{t}]$ for all $j \leq 0$. The filtration $\{V^i D_{R[\underline{t}]}\}_{i \in \mathbb{Z}}$ on $D_{R[\underline{t}]}$ is called the *V-filtration on $D_{R[\underline{t}]}$ with respect to I.*

The goal for the remainder of this section is to prove that the V-filtration admits another description in terms of degrees.

We give $R[\underline{t}]$ the grading that places R in degree zero and gives each variable t_i degree 1. Given some integer d we let $(R[\underline{t}])_d$ denote the set of homogeneous elements of degree d, and we let $R[\underline{t}]_{\geq d} = \bigoplus_{i=d}^{\infty} R[\underline{t}]_i$; in particular, we have $I^j = R[\underline{t}]_{\geq j}$ for all integers j.

This grading on $R[\underline{t}]$ induces a grading on $R[\underline{t}] \otimes_B R[\underline{t}]$, in which $\deg(1 \otimes t_i) = \deg(t_i \otimes 1) = 1$. For all integers $n \geq 0$, the ideal $J^{\{n+1\}}_{R[\underline{t}]}$ is homogeneous with respect to this grading, and thus $P^{\{n\}}_{R[\underline{t}]}$ acquires an induced grading, which makes it a graded $R[\underline{t}]$-module.

Given an integer d we let $\mathrm{Hom}_{R[\underline{t}]}(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}])_d$ denote the $R[\underline{t}]$-module homomorphisms $\varphi : P^{\{n\}}_{R[\underline{t}]} \to R[\underline{t}]$ which are homogeneous of degree d; that is, those for which $\varphi\big((P^{\{n\}}_{R[\underline{t}]})_i\big) \subseteq R[\underline{t}]_{i+d}$.

Lemma II.60. *We have*

$$\mathrm{Hom}_{R[\underline{t}]}\left(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}]\right) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{R[\underline{t}]}\left(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}]\right)_d,$$

and thus $\mathrm{Hom}_{R[\underline{t}]}(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}])$ *is a graded* $R[\underline{t}]$*-module.*

Proof. When $R = B$ the statement follows because $P^{\{n\}}_{B[\underline{t}]} = P^n_{B[\underline{t}]}$ is finitely generated and free over B (Example II.12). In general, we have a graded $R[\underline{t}]$-module isomorphism

$$\mathrm{Hom}_{R[\underline{t}]}\left(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}]\right) \cong \mathrm{Hom}_R(P^n_R, R) \otimes_B \mathrm{Hom}_{B[\underline{t}]}(P^n_{B[\underline{t}]}, B[\underline{t}])$$

(see proof of Proposition II.58), and the statement follows from the $R = B$ case. □

Given integers d and n with $n \geq 0$, we let $(D^{\{n\}}_{R[\underline{t}]})_d$ denote the image of $\operatorname{Hom}_{R[\underline{t}]}\left(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}]\right)_d$ along the isomorphism $\operatorname{Hom}_{R[\underline{t}]}\left(P^{\{n\}}_{R[\underline{t}]}, R[\underline{t}]\right) \cong D^{\{n\}}_{R[\underline{t}]}$, and we let $(D_{R[\underline{t}]})_d = \bigcup_{d=0}^{\infty}(D^{\{n\}}_{R[\underline{t}]})_d$. Given an integer i, we write $(D_{R[\underline{t}]})_{\geq i}$ for $(D_{R[\underline{t}]})_{\geq i} = \bigoplus_{d=i}^{\infty}(D_{R[\underline{t}]})_d$.

Lemma II.61. *For all integers d:*

1. *We have*
$$(D_{R[\underline{t}]})_d = \{\xi \in D_{R[\underline{t}]} \mid \xi \cdot R[\underline{t}]_j \subseteq R[\underline{t}]_{j+d} \text{ for all } j \in \mathbb{Z}\}.$$

2. *The decomposition $D_{R[\underline{t}]} = \bigoplus_{d\in\mathbb{Z}}(D_{R[\underline{t}]})_d$ makes $D_{R[\underline{t}]}$ into a $\mathbb{Z}$-graded noncommutative ring.*

3. *The isomorphism $D_R \otimes_B D_{B[\underline{t}]} \xrightarrow{\sim} D_{R[\underline{t}]}$ of Proposition II.58 induces an isomorphism $D_R \otimes_B (D_{B[\underline{t}]})_d \xrightarrow{\sim} (D_{R[\underline{t}]})_d$.*

Proof. For part (i), suppose that $\xi \in D_{R[\underline{t}]}$ is a differential operator. Pick n large enough so that $\xi \in D^{\{n\}}_{R[\underline{t}]}$ and let us denote by $\xi' : P^{\{n\}}_{R[\underline{t}]} \to R[\underline{t}]$ be the corresponding $R[\underline{t}]$-linear map (we recall that the correspondence is given by $\xi(r) = \xi'(1 \otimes r)$). Since we view $P^{\{n\}}_{R[\underline{t}]}$ as an $R[\underline{t}]$-module via the left structure, ξ' is homogeneous of degree d if and only if $\xi'(1 \otimes R[\underline{t}]_j) \subseteq R[\underline{t}]_{j+d}$ for all $j \in \mathbb{Z}$, and the result follows.

For part (ii), we only need to check that for all integers d_1, d_2 we have $(D_{R[\underline{t}]})_{d_1}(D_{R[\underline{t}]})_{d_2} \subseteq (D_{R[\underline{t}]})_{d_1+d_2}$, which follows from the description given in part (i).

For part (iii), we note that the image of D_R in $D_{R[\underline{t}]}$ consists of operators of degree zero, and that the image of $(D_{B[\underline{t}]})_d$ consists of operators of degree d. We conclude that the isomorphism $D_R \otimes_B D_{B[\underline{t}]} \xrightarrow{\sim} (D_{R[\underline{t}]})$ decomposes as a direct sum of maps $D_R \otimes_B (D_{B[\underline{t}]})_d \to (D_{R[\underline{t}]})_d$, each of which must be an isomorphism. $\square$

Proposition II.62. *Let R be a B-algebra and $R[\underline{t}] = R[t_1, \dots, t_r]$ be a polynomial ring over R. Then:*

1. *For all integers i we have $V^i D_{R[\underline{t}]} = (D_{R[\underline{t}]})_{\geq i}$.*

2. *For all integers $i \geq 0$ we have $V^i D_{R[\underline{t}]} = (D_{R[\underline{t}]})_0\ I^i$.*

Proof. We start with part (i). Suppose $\xi \in D_{R[\underline{t}]}$ is a differential operator. By definition, ξ is in $V^i D_{R[\underline{t}]}$ if and only if $\xi \cdot R[\underline{t}]_{\geq j} \subseteq R[\underline{t}]_{\geq i+j}$ for all integers j. We conclude that if ξ if homogeneous of degree $\geq i$, then $\xi \in V^i D_{R[\underline{t}]}$. Conversely, suppose that $\xi \in V^i D_{R[\underline{t}]}$, and let $\xi = \sum_d \xi_d$ where ξ_d is homogeneous of degree d. Observe that if $g \in R[\underline{t}]$ is homogeneous of degree k then $\xi_d \cdot g$ is the degree $d+k$ homogeneous component of $\xi \cdot g$. Since $\xi \cdot g \in R[\underline{t}]_{\geq i+k}$, we conclude that $\xi_d = 0$ whenever $d < i$.

For part (ii), it suffices to prove that $(D_{R[\underline{t}]})_i = (D_{R[\underline{t}]})_0 \cdot R[\underline{t}]_i$, where the inclusion $(\supseteq)$ is clear. By using Lemma II.61 we reduce to the case where $R = B$, and we then prove the claim by induction on the order, with the case of order zero being clear.

Pick an integer $n > 0$. Recall that the module $P^n_{B[\underline{t}]}$ is free over B in the basis $dt_1^{a_1} \cdots dt_r^{a_r})$ where $\underline{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ ranges along all tuples with $0 \leq a_i \leq n$ (Example II.12). For all such tuples $\underline{a}$ we denote by $\underline{\partial}^{[\underline{a}]} \in \operatorname{Hom}_{B[\underline{t}]}(P^n_{B[\underline{t}]}, B[\underline{t}])$ be the dual of $dt_1^{a_1} \cdots dt_r^{a_r}$; the module $\operatorname{Hom}_{B[\underline{t}]}(P^n_{B[\underline{t}]}, B[\underline{t}])$ is then generated by the $\underline{\partial}^{[\underline{a}]}$ and, in particular, it is generated by homogeneous elements of nonpositive degree. We thus have $(D^n_{R[\underline{t}]})_i = R[\underline{t}]_i (D^n_{R[\underline{t}]})_0$, and thus we need to show that $R[\underline{t}]_i (D^n_{R[\underline{t}]})_0 \subseteq (D_{R[\underline{t}]})_0 R[\underline{t}]_i$.

Hence suppose that $g \in R[\underline{t}]_i$ and that $\xi \in (D^n_{R[\underline{t}]})_0$, and observe that $g\xi = \xi g - [\xi, g]$. On the one hand, we have $\xi g \in (D_{R[\underline{t}]})_0 R[\underline{t}]_i$ and, on the other hand, we have that $[\xi, g] \in (D^{n-1}_{R[\underline{t}]})_i$. By the induction hypothesis, we have $[\xi, g] \in (D_{R[\underline{t}]})_0 R[\underline{t}]_i$ and hence we are done. $\square$

CHAPTER III

The Algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ and its Modules

Over $\mathbb{C}$, the Bernstein-Sato polynomial is the minimal polynomial of an operator s on a certain module $N_{\mathfrak{a}}$; in other words, we see $N_{\mathfrak{a}}$ as a $\mathbb{C}[s]$-module where s acts by the operator s, and the Bernstein-Sato polynomial is the monic generator of $\mathrm{Ann}_{\mathbb{C}[s]}(N_{\mathfrak{a}})$ (see Section IV.1). In characteristic $p > 0$ the role of $\mathbb{C}[s]$ is not played by $\mathbb{F}_p[s]$, but rather by the algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ of continuous functions from $\mathbb{Z}_p$ to $\mathbb{F}_p$. In this chapter we collect a few general facts about this algebra, some of which already appears in Bitoun's previous work [Bit18]. We begin with the definition.

Definition III.1. Let $\varphi : \mathbb{Z}_p \to \mathbb{F}_p$ be a function[1]. Given an integer $e \geq 0$, we say that φ is *continuous of level* e if $\varphi(\alpha) = \varphi(\beta)$ whenever $\alpha \equiv \beta \mod p^e\mathbb{Z}_p$. We say that φ is *continuous* if it is continuous of level e for some $e \geq 0$. We denote by $C^e(\mathbb{Z}_p, \mathbb{F}_p)$ the algebra of continuous functions of level e, and by $C(\mathbb{Z}_p, \mathbb{F}_p) = \bigcup_{e=0}^{\infty} C^e(\mathbb{Z}_p, \mathbb{F}_p)$ the algebra of continuous functions.

Note that $\varphi : \mathbb{Z}_p \to \mathbb{F}_p$ is continuous in the above sense if and only if it is continuous as a map of topological spaces, where $\mathbb{Z}_p$ has the p-adic topology and $\mathbb{F}_p$ has the discrete topology.

The algebra structure of $C(\mathbb{Z}_p, \mathbb{F}_p)$ (resp. $C^e(\mathbb{Z}_p, \mathbb{F}_p)$) is given by pointwise addition and multiplication; that is, given $\varphi, \psi \in C(\mathbb{Z}_p, \mathbb{F}_p)$ (resp. $C^e(\mathbb{Z}_p, \mathbb{F}_p)$), we have $(\varphi + \psi)(\alpha) = \varphi(\alpha) + \psi(\alpha)$ and $(\varphi\psi)(\alpha) = \varphi(\alpha)\psi(\alpha)$ for all $\alpha \in \mathbb{Z}_p$.

Remark III.2. Bhatt has given us an observation which proves and generalizes many of the results in this chapter; we paraphrase this observation as follows. Given a profinite set X (e.g. $X = \mathbb{Z}_p$) we let $C(X, \mathbb{F}_p)$ denote the algebra of continuous functions from X to $\mathbb{F}_p$, where $\mathbb{F}_p$ has the discrete topology. When X is finite, X also has the discrete topology, and therefore $C(X, \mathbb{F}_p)$ consists of all functions from X to $\mathbb{F}_p$ and, in particular, there is a natural homeomorphism $X \cong \mathrm{Spec}(C(X, \mathbb{F}_p))$. More generally, if (X_i) is a diagram of finite discrete spaces with inverse limit X, we get a homeomorphism $X \cong \varprojlim \mathrm{Spec}(C(X_i, \mathbb{F}_p))$ (where

[1] We do not assume that φ is additive, or that it respects any algebraic structure.

the inverse limit is taken in the category of topological spaces) and, since the natural map $\operatorname{Spec}(C(X,\mathbb{F}_p)) \to \varprojlim \operatorname{Spec}(C(X_i,\mathbb{F}_p))$ is also a homeomorphism, we get a homeomorphism $X \cong \operatorname{Spec}(C(X,\mathbb{F}_p))$ for every profinite set X. If we equip X with the constant sheaf $\underline{\mathbb{F}_p}$ associated to $\mathbb{F}_p$ we see, by comparing stalks, that the homeomorphism $X \cong \operatorname{Spec}(C(X,\mathbb{F}_p))$ becomes an isomorphism of locally ringed spaces.

Let us explain how the algebra $C(\mathbb{Z}_p,\mathbb{F}_p)$ arises in rings of differential operators. Let $\underline{t} = (t_1,\dots,t_r)$ be a set of variables and consider the polynomial ring $\mathbb{F}_p[\underline{t}] = \mathbb{F}_p[t_1,\dots,t_r]$. Given a function $\varphi : \mathbb{Z}_p \to \mathbb{F}_p$ we let $\widetilde{\varphi} : \mathbb{F}_p[\underline{t}] \to \mathbb{F}_p[\underline{t}]$ be the $\mathbb{F}_p$-linear operator given by $\widetilde{\varphi} \cdot t_1^{k_1}\cdots t_n^{k_n} = \varphi(-|\underline{k}|-r)t_1^{k_1}\cdots t_r^{k_r}$, where $|\underline{k}| = k_1 + \cdots + k_r$.

Lemma III.3. *Suppose $\varphi : \mathbb{Z}_p \to \mathbb{F}_p$ is continuous of level e. Then $\widetilde{\varphi}$ is a differential operator of level e and degree zero.*

Proof. We need to show that $\widetilde{\varphi}$ commutes with multiplication by elements of $\mathbb{F}_p[t_1^{p^e},\dots,t_r^{p^e}]$ and, using symmetry, it is enough to show that it commutes with multiplication by $t_1^{p^e}$. Let $\underline{k} \in \mathbb{Z}^r_{\geq 0}$ be a tuple and note that $|\underline{k} + (p^e,0,\dots,0)| = |\underline{k}| + p^e$. We conclude that

$$\begin{aligned}\widetilde{\varphi}\cdot t_1^{p^e}t_1^{k_1}\cdots t_r^{k_r} &= \varphi(-|\underline{k}|-p^e-r)t_1^{p^e}t_1^{k_1}\cdots t_r^{k_r}\\ &= \varphi(-|\underline{k}|-r)t_1^{p^e}t_1^{k_1}\cdots t_r^{k_r}\\ &= t_1^{p^e}\widetilde{\varphi}\cdot t_1^{k_1}\cdots t_r^{k_r}.\end{aligned}$$

The fact that $\widetilde{\varphi}$ has degree zero is clear from the definition. ☐

We conclude that for every integer $e \geq 0$ there is a map $\Delta^e : C^e(\mathbb{Z}_p,\mathbb{F}_p) \to (D^{(e)}_{\mathbb{F}_p[\underline{t}]})_0$ which sends $\Delta^e(\varphi) = \widetilde{\varphi}$, and we readily check that it is an algebra homomorphism. The maps Δ^e are compatible as e varies, in the sense that they induce an algebra homomorphism $\Delta : C(\mathbb{Z}_p,\mathbb{F}_p) \to (D_{\mathbb{F}_p[\underline{t}]})_0$.

Remark III.4. Over $\mathbb{C}$, when we consider the V-filtration of left D-modules along a smooth divisor given locally by $t_1 = \cdots = t_r = 0$ we work with the operator $s = -\sum_{i=1}^r \partial_{t_i} t_i$ by convention [Bud05]. Observe that there is a natural quotient map $\pi : \mathbb{Z}_p \to \mathbb{F}_p$ and that, in our setting, we have $\Delta(\pi) = -\sum_{i=1}^r \partial_{t_i} t_i$. This is the reason why we take $\varphi(-|\underline{k}|-r)$ as opposed to the more natural $\varphi(|\underline{k}|)$ when defining the map Δ: it is forced on us by the characteristic zero convention.

Proposition III.5. *For a fixed integer $r \geq 1$ let $\mathbb{F}_p[\underline{t}]$ be the polynomial ring $\mathbb{F}_p[\underline{t}] = \mathbb{F}_p[t_1,\dots,t_r]$. The maps $\Delta^e : C^e(\mathbb{Z}_p,\mathbb{F}_p) \to (D_{\mathbb{F}_p[\underline{t}]})_0$ and $\Delta : C(\mathbb{Z}_p,\mathbb{F}_p) \to (D_{\mathbb{F}_p[\underline{t}]})_0$ defined above are injective, and isomorphisms in the case $r = 1$.*

Proof. It suffices to prove the statements about Δ^e for a fixed e. Note that a function $\varphi \in C^e(\mathbb{Z}_p, \mathbb{F}_p)$ is uniquely determined by its values on p^e consecutive integers. By definition, $\Delta^e(\varphi) \cdot t_1^k = \varphi(-1-k)t_1^k$ for all integers $k \geq 0$, and therefore we can recover the values $\varphi(a)$ for every integer $a \leq -1$ from the operator $\Delta^e(\varphi)$. The injectivity follows.

We prove surjectivity when $r = 1$. Let $\xi \in (D^{(e)}_{\mathbb{F}_p[t]})_0$ and observe that, since ξ has degree zero, there is a function $\psi : \mathbb{Z}_{\geq 0} \to \mathbb{F}_p$ such that $\xi \cdot t^k = \psi(k)t^k$ for every integer $k \geq 0$. Moreover, the fact that ξ commutes with multiplication by t^{p^e} entails that $\psi(k) = \psi(k + p^e)$ for every integer $k \geq 0$. We conclude that ψ extends to a function $\mathbb{Z}_p \to \mathbb{F}_p$ which is continuous of level e; we abuse notation and call this extension by ψ as well. If we now consider the function $\varphi \in C^e(\mathbb{Z}_p, \mathbb{F}_p)$ given by $\varphi(\alpha) = \psi(-1-\alpha)$ then $\Delta^e(\varphi) = \xi$. $\square$

Our next goal is to describe the ideals of $C^e(\mathbb{Z}_p, \mathbb{F}_p)$ and of $C(\mathbb{Z}_p, \mathbb{F}_p)$.

Let $e \geq 0$ be an integer. Given a p-adic integer $\alpha \in \mathbb{Z}_p$ we denote by $\chi^{(e)}_\alpha \in C^e(\mathbb{Z}_p, \mathbb{F}_p)$ the function given by

$$\chi^{(e)}_\alpha(\beta) = \begin{cases} 1 \text{ if } \beta \equiv \alpha \mod p^e\mathbb{Z}_p \\ 0 \text{ otherwise.} \end{cases}$$

Observe that $\chi^{(e)}_\alpha = \chi^{(e)}_\beta$ whenever $\alpha \equiv \beta \mod p^e\mathbb{Z}_p$.

A function $\varphi \in C^e(\mathbb{Z}_p, \mathbb{F}_p)$ is uniquely determined by its values on the set $\{0, 1, \dots, p^e - 1\}$, and therefore we have algebra isomorphisms

$$C^e(\mathbb{Z}_p, \mathbb{F}_p) \cong \operatorname{Fun}(\{0, \dots, p^e - 1\}, \mathbb{F}_p) \cong \mathbb{F}_p\chi^{(e)}_0 \times \cdots \times \mathbb{F}_p\chi^{(e)}_{p^e-1}$$

and the $\chi^{(e)}_i$ are pairwise orthogonal idempotents; in particular, $\operatorname{Spec}\big(C^e(\mathbb{Z}_p, \mathbb{F}_p)\big)$ is a disjoint union of p^e points.

Given a p-adic integer $\alpha \in \mathbb{Z}_p$ we let $\mathfrak{m}^{(e)}_\alpha \subseteq C^e(\mathbb{Z}_p, \mathbb{F}_p)$ denote the ideal

$$\mathfrak{m}^{(e)}_\alpha = \{\varphi \in C^e(\mathbb{Z}_p, \mathbb{F}_p) \mid \varphi(\alpha) = 0\}.$$

Note that whenever $\alpha \equiv \beta \mod p^e\mathbb{Z}_p$ we have $\mathfrak{m}^{(e)}_\alpha = \mathfrak{m}^{(e)}_\beta$, and that $\mathfrak{m}^{(e)}_0, \mathfrak{m}^{(e)}_1, \dots, \mathfrak{m}^{(e)}_{p^e-1}$ are all the maximal ideals of $C^e(\mathbb{Z}_p, \mathbb{F}_p)$, each of them defining one of the points of $\operatorname{Spec}\big(C^e(\mathbb{Z}_p, \mathbb{F}_p)\big)$. Every ideal of $C^e(\mathbb{Z}_p, \mathbb{F}_p)$ is a product of these maximal ideals, and if M is a $C^e(\mathbb{Z}_p, \mathbb{F}_p)$-module then M splits as a direct sum $M = \bigoplus_{a=0}^{p^e-1} M_a$ where $M_a = \operatorname{Ann}_M(\mathfrak{m}^{(e)}_a)$.

Remark III.6. Note that the submodule M_a can also be identified with the quotient $M/\mathfrak{m}^{(e)}_a$. If $N \subseteq M$ is a submodule then there is a natural inclusion $N_a \subseteq M_a$, and therefore the natural map $N/\mathfrak{m}^{(e)}_a \to M/\mathfrak{m}^{(e)}_a$ is injective. We conclude that the modules $C^e(\mathbb{Z}_p, \mathbb{F}_p)/\mathfrak{m}^{(e)}_a$ are flat over $C^e(\mathbb{Z}_p, \mathbb{F}_p)$.

Let us now turn our attention to the algebra $C(\mathbb{Z}_p, \mathbb{F}_p) = \bigcup_{e=0}^{\infty} C^e(\mathbb{Z}_p, \mathbb{F}_p)$ and its ideal structure. If $I \subseteq C(\mathbb{Z}_p, \mathbb{F}_p)$ is an ideal we let $Z_I \subseteq \mathbb{Z}_p$ be the subset $Z_I = \{\alpha \in \mathbb{Z}_p \mid \varphi(\alpha) = 0 \text{ for all } \varphi \in I\}$ and, conversely, if $Z \subseteq \mathbb{Z}_p$ is a closed subset we let $I_Z \subseteq C(\mathbb{Z}_p, \mathbb{F}_p)$ be the ideal $I_Z = (\varphi \in C(\mathbb{Z}_p, \mathbb{F}_p) : \varphi(Z) = 0)$.

Lemma III.7. *If $I \subseteq C(\mathbb{Z}_p, \mathbb{F}_p)$ is an ideal then Z_I is a closed subset of $\mathbb{Z}_p$ in the p-adic topology.*

Proof. We show that $\mathbb{Z}_p \setminus Z_I$ is open. If $\beta \notin Z_I$ there is some $\varphi \in I$ such that $\varphi(\beta) \neq 0$. Pick $e \geq 0$ such that $\varphi \in C^{(e)}(\mathbb{Z}_p, \mathbb{F}_p)$, and note that for all $\gamma \in \mathbb{Z}_p$ with $\gamma \equiv \beta \mod p^e\mathbb{Z}_p$ we get $\varphi(\gamma) = \varphi(\beta) \neq 0$, and thus $\gamma \notin Z_I$. □

Proposition III.8. *The assignments $[I \mapsto Z_I]$ and $[Z \mapsto I_Z]$ give a one-to-one inclusion reversing correspondence*

$$\left\{ \textit{Ideals of } C(\mathbb{Z}_p, \mathbb{F}_p) \right\} \longleftrightarrow \left\{ \textit{Closed subsets of } \mathbb{Z}_p \right\}$$

Proof. We begin by letting $I \subseteq C(\mathbb{Z}_p \mathbb{F}_p)$ be an ideal, and $Z := \{\alpha \in \mathbb{Z}_p \mid \varphi(\alpha) = 0 \text{ for all } \varphi \in I\}$. Our goal is to show that $I = (\varphi \in C(\mathbb{Z}_p, \mathbb{F}_p) \mid \varphi(Z) = 0)$. The inclusion ($\subseteq$) is clear, and to prove ($\supseteq$) we let φ belong to the right hand side. Pick $e \geq 0$ large enough so that $\varphi \in C^e(\mathbb{Z}_p, \mathbb{F}_p)$.

We claim that for all $\alpha \in \mathbb{Z}_p$ there is some $\psi_\alpha \in I$ such that $\psi_\alpha(\alpha) = \varphi(\alpha)$. This is clear whenever $\varphi(\alpha) = 0$, and if $\varphi(\alpha) \neq 0$ then $\alpha \notin Z$, so there is some $\psi'_\alpha \in I$ such that $\psi'_\alpha(\alpha) \neq 0$; we then obtain ψ_α by multiplying ψ'_α with the appropriate unit of $\mathbb{F}_p$.

For all $\alpha \in \mathbb{Z}_p$ choose $e_\alpha \geq e$ such that $\psi_\alpha \in C^{(e_\alpha)}(\mathbb{Z}_p, \mathbb{F}_p)$. The union $\bigcup_{\alpha \in \mathbb{Z}_p}(\alpha + p^{e_\alpha}\mathbb{Z}_p)$ gives an open cover of $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is compact, this cover admits a finite subcover $\mathbb{Z}_p = \bigcup_{i=1}^{n}(\alpha_i + p^{e_{\alpha_i}}\mathbb{Z}_p)$, which we may assume to be disjoint. We claim that $\varphi = \sum_{i=1}^{n} \psi_{\alpha_i}\chi_{\alpha_i}^{(e_{\alpha_i})}$, which will give that $\varphi \in I$. Indeed, given some $\beta \in \mathbb{Z}_p$ there is a unique j such that $\beta \equiv \alpha_j \mod p^{e_{\alpha_j}}\mathbb{Z}_p$ and therefore $\sum_{i=1}^{n} \psi_{\alpha_i}\chi_{\alpha_i}^{(e_{\alpha_i})}(\beta) = \psi_{\alpha_j}(\beta) = \psi_{\alpha_j}(\alpha_j) = \varphi(\alpha_j) = \varphi(\beta)$.

To finish, let $Z \subseteq \mathbb{Z}_p$ be closed and let $I = (\varphi \in C(\mathbb{Z}_p, \mathbb{F}_p) \mid \varphi(Z) = 0)$; we want to show that $Z = \{\alpha \in \mathbb{Z}_p \mid \varphi(\alpha) = 0 \text{ for all } \varphi \in I\}$, with the inclusion ($\subseteq$) being clear. For ($\supseteq$), suppose that $\alpha \notin Z$. Since Z is closed, there is some $e \geq 0$ large enough so that $\alpha + p^e\mathbb{Z}_p \subseteq \mathbb{Z}_p \setminus Z$. It follows that $\chi_\alpha^{(e)} \in I$, while $\chi_\alpha^{(e)}(\alpha) = 1 \neq 0$, and therefore α is not on the right-hand side. □

Proposition III.9 ([Bit18, Thm. 1.1.8])**.** *Given an ideal $I \subseteq C(\mathbb{Z}_p, \mathbb{F}_p)$, the following are equivalent:*

(a) The ideal I is prime.

(b) The subspace Z_I is a one-element set.

(c) The ideal I is maximal.

Proof. We show that (a) implies (b) by contraposition. Suppose Z_I is not a point. If Z_I is empty then $I = (1)$, so I is not prime. If Z_I is nonempty then pick distinct points $\alpha, \beta \in Z_I$ and, by picking disjoint open neighbourhoods of these points and considering their complements, write $Z_I = V \cup W$ where V, W are proper closed subsets of Z_I (implicitly, we are proving that Z_I cannot be irreducible). By Proposition III.8, we have $I \subsetneq I_V$ and $I \subsetneq I_W$, so we may pick $\varphi \in I_V \setminus I$ and $\psi \in I_W \setminus I$. The product $\varphi\psi$ vanishes on Z_I which, by Proposition III.8, entails that $\varphi\psi \in I$. We conclude that I is not prime.

The fact that (b) implies (c) follows from Proposition III.8, together with the observation that the points are the minimal nonempty closed subsets of $\mathbb{Z}_p$.

For (c) implies (a), recall the standard fact that maximal ideals in commutative rings are prime. $\square$

Given a p-adic integer α we denote by $\mathfrak{m}_\alpha \subseteq C(\mathbb{Z}_p, \mathbb{F}_p)$ the ideal given by

$$\mathfrak{m}_\alpha = (\varphi \in C(\mathbb{Z}_p, \mathbb{F}_p) \mid \varphi(\alpha) = 0).$$

Note that $\mathfrak{m}_\alpha$ is a maximal ideal.

Proposition III.10 ([Bit18, Prop. 1.1.10])**.** *Every proper ideal I of $C(\mathbb{Z}_p, \mathbb{F}_p)$ is an interchapter of (possibly infinitely many) maximal ideals.*

Proof. By Proposition III.8, we have $I = I_{Z_I}$, and from the description of I_{Z_I} we get that $I_{Z_I} = \bigcap_{\alpha \in Z_I} \mathfrak{m}_\alpha$. Alternatively, from the fact that $I = I_{Z_I}$ we get that I must be radical, and a standard fact then tells us that I is the interchapter of all the prime ideals that contain it, and the result follows from Proposition III.9. $\square$

We now describe a few properties of modules over the algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$. Given a $C(\mathbb{Z}_p, \mathbb{F}_p)$-module M and a p-adic integer $\alpha \in \mathbb{Z}_p$, we denote by M_α the quotient

$$M_\alpha := M/\mathfrak{m}_\alpha M.$$

If $N \subseteq M$ is a submodule, N_α is naturally a submodule of M_α by the following result.

Lemma III.11. *The module $C(\mathbb{Z}_p, \mathbb{F}_p)/\mathfrak{m}_\alpha C(\mathbb{Z}_p, \mathbb{F}_p)$ is flat over $C(\mathbb{Z}_p, \mathbb{F}_p)$.*

Proof. For simplicity of notation, let us denote the algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ (resp. $C^e(\mathbb{Z}_p, \mathbb{F}_p)$) by C (resp. C^e). Note that $C/\mathfrak{m}_\alpha = \varinjlim_e C^e/\mathfrak{m}_\alpha^{(e)}$, and that if N is a C-module then there is a natural map

$$\varinjlim_e (C^e/\mathfrak{m}_\alpha^{(e)} \otimes_{C^e} N) \longrightarrow (C/\mathfrak{m}_\alpha) \otimes_C N,$$

which we claim is an isomorphism. Indeed, giving an C-multilinear map $C/\mathfrak{m}_\alpha \times N \to W$ is equivalent to giving a compatible collection of C^e-multilinear maps $C^e/\mathfrak{m}_\alpha^{(e)} \times N \to W$, which shows that both objects have the same universal property.

We know that $C^e/\mathfrak{m}_\alpha^{(e)}$ is flat over C^e (see Remark III.6); since taking limits is an exact operation the result follows. $\square$

Remark III.12. As pointed out to us by Bhatt, all the local rings of $C(\mathbb{Z}_p, \mathbb{F}_p)$ are fields (see Remark III.2), and thus every $C(\mathbb{Z}_p, \mathbb{F}_p)$-module is flat. An algebra for which every module is flat is called *absolutely flat* or *Von Neumann regular.*

Lemma III.13. *Let M be a $C(\mathbb{Z}_p, \mathbb{F}_p)$-module. If $M_\alpha = 0$ for all p-adic integers $\alpha \in \mathbb{Z}_p$ then $M = 0$.*

Proof. First observe that a function $\varphi \in C(\mathbb{Z}_p, \mathbb{F}_p)$ belongs to $\mathfrak{m}_\alpha$ if and only if $\chi_\alpha^{(e)}\varphi = 0$ for a sufficiently large e; indeed, it suffices to take e large enough so that $\varphi \in C^e(\mathbb{Z}_p, \mathbb{F}_p)$. We conclude that, given an element $u \in M$ and a p-adic integer $\alpha \in \mathbb{Z}_p$ there exists some large e_α such that $\chi_\alpha^{(e_\alpha)} u = 0$ or, equivalently, $(1 - \chi_\alpha^{(e_\alpha)})u = u$.

The union $\bigcup_{\alpha \in \mathbb{Z}_p} (\alpha + p^{e_\alpha}\mathbb{Z}_p)$ forms an open cover of $\mathbb{Z}_p$ which, by the compactness of $\mathbb{Z}_p$, admits a finite subcover $\mathbb{Z}_p = \bigcup_{i=1}^n (\alpha_i + p^{e_{\alpha_i}}\mathbb{Z}_p)$. We conclude that

$$u = (1 - \chi^{e_{\alpha_1}}) \cdots (1 - \chi^{e_{\alpha_n}})u = 0. \qquad \square$$

Proposition III.14. *Let M be a $C(\mathbb{Z}_p, \mathbb{F}_p)$-module. Suppose that there are only finitely many $\alpha \in \mathbb{Z}_p$ such that $M_\alpha \neq 0$, say $\alpha_1, \ldots, \alpha_n$. Then the natural map $M \xrightarrow{\sim} \bigoplus_{i=1}^n M_{\alpha_i}$ is an isomorphism which identifies M_{α_i} with $\mathrm{Ann}_M(\mathfrak{m}_\alpha)$.*

Proof. Let K (resp. Q) be the kernel (resp. cokernel) of the map $M \to \bigoplus_{i=1}^n M_{\alpha_i}$. We thus have an exact sequence

$$0 \to K \to M \to \bigoplus_{i=1}^n M_{\alpha_i} \to Q \to 0.$$

We claim that for all $\beta \in \mathbb{Z}_p$ we have $K_\beta = Q_\beta = 0$. Indeed, if $\beta \neq \alpha_i$ for any i then applying the functor $(-)_\beta$ (which, by Lemma III.11, is exact) to the above exact sequence yields

$$0 \to K_\beta \to 0 \to 0 \to Q_\beta \to 0,$$

and if $\beta = \alpha_i$ then we get

$$0 \to K_\beta \to M_\beta \xrightarrow{\mathrm{id}} M_\beta \to Q_\beta \to 0.$$

From Lemma III.13, we conclude that $K = Q = 0$. $\square$

CHAPTER IV
The Splitting of $N_{\mathfrak{a}}$ in Positive Characteristic

IV.1: Motivation from characteristic zero

We briefly review the notion of the Bernstein-Sato polynomial of an ideal, as developed by Budur, Mustaţă and Saito [BMS06a]. Let R be a regular and essentially of finite type $\mathbb{C}$-algebra, set $X = \operatorname{Spec} R$ and let $\mathfrak{a} \subseteq R$ is an ideal. We fix generators $\mathfrak{a} = (f_1, \ldots, f_r)$ for $\mathfrak{a}$, and we consider the associated graph $\gamma : X \to X \times \mathbb{C}^r$ given by $\gamma(x) = (x, f_1(x), \ldots, f_r(x))$. At the level of algebras, γ is given by the R-algebra morphism $R[\underline{t}] = R[t_1, \ldots, t_r] \to R$ that sends $t_i \mapsto f_i$. One then considers the module $H = \gamma_+ R$, the D-module pushforward of R under γ, which admits an easy description in terms of local cohomology [HTT08, Prop. 1.7.1]:

$$H = H^r_{(f_1 - t_1, \ldots, f_r - t_r)} R[\underline{t}].$$

Note that, after the change of coordinates $u_i = f_i - t_i$, this is the module described in Example II.36. Recall that, since H is a local cohomology module of the $D_{R[\underline{t}]}$-module $R[\underline{t}]$, it acquires a $D_{R[\underline{t}]}$-module structure, and that this $D_{R[\underline{t}]}$-module structure can be realized by using the Čech complex on the given generators (see Section II.3). We let δ_1 be the class of $(f_1 - t_1)^{-1} \cdots (f_r - t_r)^{-1}$. We consider the module

$$N_{\mathfrak{a}} = \frac{V^0 D_{R[\underline{t}]} \cdot \delta_1}{V^1 D_{R[\underline{t}]} \cdot \delta_1},$$

where the V-filtrations are defined as in Section II.5. The module $N_{\mathfrak{a}}$ carries an action of the operator $s = -\sum_{i=0}^r \partial_{t_i} t_i$, and the minimal polynomial $b_{\mathfrak{a}}(s)$ for this action is called the Bernstein-Sato polynomial of the ideal $\mathfrak{a}$.

We remark $N_{\mathfrak{a}}$ is not finite dimensional over $\mathbb{C}$ in general, and that the existence of such a minimal polynomial relies on the existence of a so-called V-filtration on H [BMS06a, §2.1]. Note that the construction of $N_{\mathfrak{a}}$ depends on the chosen generators $f_1, \ldots, f_r$ for $\mathfrak{a}$, but it turns out that the resulting invariant $b_{\mathfrak{a}}(s)$ does not [BMS06a, Thm. 2.5]. The roots of

$b_{\mathfrak{a}}(s)$ are rational and negative, and the largest of them is the negative of the log-canonical threshold of $\mathfrak{a}$ [BMS06a, Thm. 2]. Let us make a simple observation that will be important later.

Remark IV.1. The fact that the action of s on $N_{\mathfrak{a}}$ admits a minimal polynomial entails that $N_{\mathfrak{a}}$ splits as a direct sum $N_{\mathfrak{a}} = \bigoplus_{\lambda \in \mathbb{C}} (N_{\mathfrak{a}})_\lambda$, where $(N_{\mathfrak{a}})_\lambda$ is the generalized eigenspace for λ. Moreover, we can recover the roots of $b_{\mathfrak{a}}(s)$ from this decomposition as follows:

$$\Big\{ \text{ Roots of } b_{\mathfrak{a}}(s) \Big\} = \Big\{ \lambda \in \mathbb{C} \mid (N_{\mathfrak{a}})_\lambda \neq 0 \Big\}.$$

IV.2: The module $N_{\mathfrak{a}}$ in positive characteristic

Notation Given a tuple of integers $\underline{a} \in \mathbb{Z}^r$ and an integer $i \in \{0, 1, \ldots, r\}$ we denote by a_i the i-th entry of $\underline{a}$. With this notation, we have $\underline{a} = (a_1, a_2, \ldots, a_r)$. The symbol $|\underline{a}|$ will denote the sum of the entries of $\underline{a}$, so that $|\underline{a}| = a_1 + a_2 + \cdots + a_r$. Given integers $a \leq b$ we write $\{a, \ldots, b\}$ for the set $\{a, \ldots, b\} = [a, b] \cap \mathbb{Z}$.

Our goal is to develop the construction outlined in Section IV.1 in a positive characteristic setting. Let R be a regular and F-finite ring of characteristic p, $\mathfrak{a} \subseteq R$ be an ideal and fix generators $\mathfrak{a} = (f_1, \ldots, f_r)$ for $\mathfrak{a}$. We give a different description of the local cohomology module H that will allow us to exploit the existence of Frobenius.

For every integer $e \geq 0$ let $H^e_{\mathfrak{a}}$ denote the module

$$H^e_{\mathfrak{a}} = \delta_{p^e} = \frac{R[\underline{t}]}{(f_1 - t_1)^{p^e} \cdots (f_r - t_r)^{p^e}} \; \delta_{p^e}$$

where δ_{p^e} is just a formal symbol to denote the generator. In particular, $H^e_{\mathfrak{a}}$ is the quotient of $R[\underline{t}]$ by a $D^{(e)}_{R[\underline{t}]}$-ideal, and therefore it is a $D^{(e)}_{R[\underline{t}]}$-module itself. Note that, as an R-module, we have a decomposition

$$H^e_{\mathfrak{a}} = \bigoplus_{\underline{a} \in \{0, \ldots, p^e - 1\}^r} R \; (f_1 - t_1)^{a_1} \cdots (f_r - t_r)^{a_r} \; \delta_{p^e},$$

where a_i denotes the i-th component of $\underline{a}$.

We let $\varphi^e : H^e_{\mathfrak{a}} \to H^{e+1}_{\mathfrak{a}}$ be the map induced by multiplication by $(f_1 - t_1)^{p^e(p-1)} \cdots (f_r - t_r)^{p^e(p-1)}$, which is $D^{(e)}_{R[\underline{t}]}$-linear. We let $H_{\mathfrak{a}}$ be the direct limit

$$H_{\mathfrak{a}} = \lim_{\rightarrow} (H^0_{\mathfrak{a}} \xrightarrow{\varphi^0} H^1_{\mathfrak{a}} \xrightarrow{\varphi^1} H^2_{\mathfrak{a}} \to \cdots),$$

which acquires the structure of a $D_{R[\underline{t}]}$-module. We note that the maps φ^e are injective, and therefore each $H^e_{\mathfrak{a}}$ is isomorphic to its image in $H_{\mathfrak{a}}$. From this point onwards, we identify each module $H^e_{\mathfrak{a}}$ with its image in $H_{\mathfrak{a}}$, and we think of every element of $H^e_{\mathfrak{a}}$ as an element of $H_{\mathfrak{a}}$. For example, for all $e \geq 0$ we have

$$\delta_1 = (f_1 - t_1)^{p^e-1} \cdots (f_r - t_r)^{p^e-1}\ \delta_{p^e}.$$

Lemma IV.2. *There is an isomorphism*

$$H_{\mathfrak{a}} \cong H^r_{(f_1-t_1,\dots,f_r-t_r)} R[\underline{t}]$$

of $D_{R[\underline{t}]}$-modules.

Proof. We let L denote the local cohomology module on the right hand side, which we realize via the Čech complex on the given generators (see Section II.3). We have an R-module decomposition

$$L = \bigoplus_{\underline{a}\in(\mathbb{Z}_{>0})^r} R\delta'_{\underline{a}}$$

where $\delta'_{\underline{a}}$ denotes the class of $(f_1 - t_1)^{-a_1} \cdots (f_r - t_r)^{-a_r}$ (see Example II.36).

We let $\psi^e : H^e_{\mathfrak{a}} \to L$ be the unique R-linear map with $\psi^e\big((f_1 - t_1)^{a_1} \cdots (f_r - t_r)^{a_r}\ \delta_{p^e}\big) = \delta'_{(p^e-a_1,\dots,p^e-a_r)}$, which gives an isomorphism of $H^e_{\mathfrak{a}}$ onto the submodule $\bigoplus_{\underline{a}\in\{1,\dots,p^e\}^r} R\ \delta'_{\underline{a}}$ of L. For every $e \geq 0$, the diagram

$$\begin{tikzcd}
H^e_{\mathfrak{a}} \arrow[r, "\sim", "\psi^e"'] \arrow[d, hook] & \bigoplus_{\underline{a}\in\{1,\dots,p^e\}^r} R\ \delta'_{\underline{a}} \arrow[d, hook] \\
H^{e+1}_{\mathfrak{a}} \arrow[r, "\sim", "\psi^{e+1}"'] & \bigoplus_{\underline{a}\in\{1,\dots,p^{e+1}\}^r} R\ \delta'_{\underline{a}}
\end{tikzcd}$$

commutes, and therefore we get an R-module isomorphism $\psi : H_{\mathfrak{a}} \xrightarrow{\sim} L$.

It remains to check that this isomorphism is $D_{R[\underline{t}]}$-linear; i.e. that it is $D^{(e)}_{R[\underline{t}]}$-linear for every e. Since $H^i_{\mathfrak{a}}$ is a $D^{(e)}_{R[\underline{t}]}$-submodule of $H_{\mathfrak{a}}$ for every $i \geq e$, the $D^{(e)}_{R[\underline{t}]}$-module structure on $H_{\mathfrak{a}}$ is uniquely determined by the fact that $\xi \cdot (g\delta_{p^i}) = (\xi \cdot g)\delta_{p^i}$ for all $\xi \in D^{(e)}_{R[\underline{t}]}$, all $g \in R[\underline{t}]$ and all $i \geq e$. Now recall that δ'_{p^i} is the class of $(f_1 - t_1)^{-p^i} \cdots (f_r - t_r)^{-p^i}$, which is a p^i-th power. It follows that for every $\xi \in D^{(e)}_R$ and every $i \geq e$, ξ commutes with multiplication by $(f_1 - t_1)^{-p^i} \cdots (f_r - t_r)^{-p^i}$ in the localization $R[\underline{t}]_{(f_1-t_1)\cdots(f_r-t_r)}$. Therefore, in L we have $\xi \cdot (g\delta'_{p^i}) = (\xi \cdot g)\delta'_{p^i}$ for all $\xi \in D^{(e)}_{R[\underline{t}]}$, all $g \in R[\underline{t}]$ and all $i \geq e$. □

It follows from the proof that the isomorphism we have constructed identifies $\delta_{p^0} = \delta_1$ with the class of $(f_1 - t_1)^{-1} \cdots (f_r - t_r)^{-1}$, when the local cohomology module is viewed via the Čech complex.

The characteristic zero theory leads us to consider the module

$$N_{\mathfrak{a}} = \frac{V^0 D_{R[\underline{t}]} \cdot \delta_1}{V^1 D_{R[\underline{t}]} \cdot \delta_1}.$$

In the following lemma, we give a description that is more useful for our purposes. Recall that we denote by $(D_{R[\underline{t}]})_0$ the differential operators of degree zero, with the grading induced by $\deg t_i = 1$.

Lemma IV.3. *We have*

$$N_{\mathfrak{a}} = \frac{(D_{R[\underline{t}]})_0 \cdot \delta_1}{(D_{R[\underline{t}]})_0 \cdot \mathfrak{a}\delta_1}.$$

Proof. Let I denote the ideal $I = (t_1, \dots, t_r)$. Since $(f_i - t_i)\ \delta_1 = 0$, we have that $I\delta_1 = \mathfrak{a}\delta_1$. By using Proposition II.62 we get

$$V^0 D_{R[\underline{t}]} \cdot \delta_1 = (D_{R[\underline{t}]})_0\ R[\underline{t}] \cdot \delta_1 = (D_{R[\underline{t}]})_0 \cdot \delta_1,$$

and similarly

$$V^1 D_{R[\underline{t}]} \cdot \delta_1 = (D_{R[\underline{t}]})_0\ I \cdot \delta_1 = (D_{R[\underline{t}]})_0 \cdot \mathfrak{a}\delta_1. \qquad \square$$

In particular, we conclude that $N_{\mathfrak{a}}$ is a $(D_{R[\underline{t}]})_0$-module. Recall that, in characteristic zero, we obtain the Bernstein-Sato polynomial of $\mathfrak{a}$ by considering the action of the operator $s = -\sum_{i=1}^n \partial_{t_i} t_i$ on $N_{\mathfrak{a}}$ or, in other words, by considering the $\mathbb{C}[s]$-module structure of $N_{\mathfrak{a}}$. In characteristic p, the subalgebra $\mathbb{C}[s]$ of $(D_{R[\underline{t}]})_0$ is replaced by the algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ of continuous functions from $\mathbb{Z}_p$ to $\mathbb{F}_p$ (which we view as a subalgebra of $(D_{R[\underline{t}]})_0$ as described in Proposition III.5). Our goal is then to study the $C(\mathbb{Z}_p, \mathbb{F}_p)$-module structure of $N_{\mathfrak{a}}$.

Recall that, given a p-adic integer $\alpha \in \mathbb{Z}_p$, $(N_{\mathfrak{a}})_\alpha$ denotes the quotient

$$(N_{\mathfrak{a}})_\alpha = N_{\mathfrak{a}} / \mathfrak{m}_\alpha N_{\mathfrak{a}}.$$

The following theorem, whose proof will take up this chapter, is one of the main results of this thesis.

Theorem IV.4. *Let R be a regular F-finite ring of characteristic p, $\mathfrak{a} \subseteq R$ be an ideal and $N_{\mathfrak{a}}$ be the module defined above using a choice of generators $\mathfrak{a} = (f_1, \dots, f_r)$ for the ideal $\mathfrak{a}$. Then there are only finitely many p-adic integers $\alpha \in \mathbb{Z}_p$ such that $(N_{\mathfrak{a}})_\alpha \neq 0$. Furthermore, the natural map $N_{\mathfrak{a}} \to \bigoplus_{\alpha \in \mathbb{Z}_p} (N_{\mathfrak{a}})_\alpha$ is an isomorphism which identifies $(N_{\mathfrak{a}})_\alpha$ with the submodule $\mathrm{Ann}_{N_{\mathfrak{a}}}(\mathfrak{m}_\alpha)$.*

After giving some preliminary results and setting up some notation, we give the proof of Theorem IV.4 in Section IV.6.

Recall that over $\mathbb{C}$ we can recover the roots of the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$ from the generalized eigenspace decomposition of $N_{\mathfrak{a}}$ (see Remark IV.1). This motivates the following definition.

Definition IV.5. Let R be a regular, F-finite ring of characteristic p, $\mathfrak{a} \subseteq R$ be an ideal. We say that a p-adic integer $\alpha \in \mathbb{Z}_p$ is a *Bernstein-Sato root of* $\mathfrak{a}$ if $(N_{\mathfrak{a}})_\alpha$ is nonzero. We denote the set of Bernstein-Sato roots of $\mathfrak{a}$ by $\mathrm{BSR}(\mathfrak{a})$.

Remark IV.6.

(i) The Bernstein-Sato roots of $\mathfrak{a}$ are characteristic p analogues of the roots of the Bernstein-Sato polynomial. As a consequence of Theorem IV.4, an ideal has finitely many Bernstein-Sato roots.

(ii) A priori, the Bernstein-Sato roots of $\mathfrak{a}$ depend on the generators chosen to construct the module $N_{\mathfrak{a}}$. We will see that this is not the case by giving an alternative characterization of the Bernstein-Sato roots of $\mathfrak{a}$, which will also turn out to be much more useful than the one given above (see Theorem IV.17).

(iii) In characteristic zero, we can recover the multiplicity of a root λ of $b_{\mathfrak{a}}(s)$ from the generalized eigenspace decomposition of $N_{\mathfrak{a}}$: it is the smallest k for which $(s-\lambda)^k (N_{\mathfrak{a}})_\lambda = 0$. Carrying over this notion to characteristic p is not possible because $\mathfrak{m}_\alpha = \mathfrak{m}_\alpha^2$. As a consequence, there is no notion of Bernstein-Sato polynomial in characteristic p.

We now begin working towards the proof of Theorem IV.4. Let us note that it is enough to prove the statement that $\mathfrak{a}$ has finitely many Bernstein-Sato roots; the rest follows formally from Proposition III.14.

Given a positive integer $e \geq 0$ we define

$$N_{\mathfrak{a}}^e = \frac{(D^{(e)}_{R[\underline{t}]})_0 \cdot \delta_1}{(D^{(e)}_{R[\underline{t}]})_0 \cdot \mathfrak{a}\delta_1},$$

where $(D^{(e)}_{R[\underline{t}]})_0 = (D_{R[\underline{t}]})_0 \cap D^{(e)}_{R[\underline{t}]}$.

Note that $N_{\mathfrak{a}}^e$ is a $(D^{(e)}_{R[\underline{t}]})_0$-module, and therefore has a natural $C^e(\mathbb{Z}_p, \mathbb{F}_p)$-module structure (see Chapter III). Given a p-adic integer $\alpha \in \mathbb{Z}_p$ we denote by $\mathfrak{m}_\alpha^{(e)} := \mathfrak{m}_\alpha \cap C^e(\mathbb{Z}_p, \mathbb{F}_p)$ and $(N_{\mathfrak{a}}^e)_\alpha = N_{\mathfrak{a}}^e / \mathfrak{m}_\alpha^{(e)} N_{\mathfrak{a}}^e$. We have $N_{\mathfrak{a}} = \varinjlim_e N_{\mathfrak{a}}^e$ and $(N_{\mathfrak{a}})_\alpha = \varinjlim_e (N_{\mathfrak{a}}^e)_\alpha$. The strategy will be to understand the nonvanishing of the modules $(N_{\mathfrak{a}}^e)_\alpha$, and to ultimately understand how taking the limit affects the nonvanishing.

Lemma IV.7. *Let $\underline{x} = (x_1, \dots, x_r)$ and $\underline{y} = (y_1, \dots, y_r)$ be two sets of variables and $e \geq 0$ be an integer. In the ring $\mathbb{F}_p[\underline{x}, \underline{y}]$ we have*

$$(x_1 - y_1)^{p^e-1} \cdots (x_r - y_r)^{p^e-1} = \sum_{\underline{b} \in \{0,1,\dots,p^e-1\}^r} x_1^{b_1} \cdots x_r^{b_r} y_1^{p^e-1-b_1} \cdots y_r^{p^e-1-b_r}.$$

Proof. Observe that

$$(x_1 - y_1) \left(\sum_{b_1=0}^{p^e-1} x_1^{b_1} y_1^{p^e-1-b_1} \right) = x_1^{p^e} - y_1^{p^e} = (x_1 - y_1)^{p^e}$$

and, since $\mathbb{F}_p[x_1, y_1]$ is a domain, we get

$$\sum_{b_1=0}^{p^e-1} x_1^{b_1} y_1^{p^e-1-b_1} = (x_1 - y_1)^{p^e-1},$$

which proves the case $r = 1$. For the general case, we now observe

$$\begin{aligned}(x_1 - y_1)^{p^e-1} \cdots (x_r - y_r)^{p^e-1} &= \left(\sum_{b_1=0}^{p^e-1} x_1^{b_1} y_1^{p^e-1-b_1} \right) \cdots \left(\sum_{b_r=0}^{p^e-1} x_1^{b_r} y_1^{p^e-1-b_r} \right) \\ &= \sum_{\underline{b} \in \{0,1,\dots,p^e-1\}^r} x_1^{b_1} \cdots x_r^{b_r} y_1^{p^e-1-b_1} \cdots y_r^{p^e-1-b_r}. \qquad \square\end{aligned}$$

Given an integer $e \geq 0$ and a tuple $\underline{a} \in \{0, \dots, p^e - 1\}^r$ we define $Q^e_{\underline{a}}$ to be the following element of $H^e_{\mathfrak{a}}$:

$$Q^e_{\underline{a}} = t_1^{p^e-1-a_1} \cdots t_r^{p^e-1-a_r} \; \delta_{p^e}.$$

Note we have an R-module decomposition

$$H^e_{\mathfrak{a}} = \bigoplus_{\underline{a} \in \{0,\dots,p^e-1\}^r} R \; Q^e_{\underline{a}}.$$

Using the isomorphism $D^{(e)}_R \otimes D^{(e)}_{\mathbb{F}_p[\underline{t}]} \xrightarrow{\sim} D^{(e)}_{R[\underline{t}]}$ from Proposition II.58, we identify $D^{(e)}_R$ and $D^{(e)}_{\mathbb{F}_p[\underline{t}]}$ with subrings of $D^{(e)}_{R[\underline{t}]}$. With this identification we have $(D^{(e)}_{R[\underline{t}]})_0 = D^{(e)}_R \; (D^{(e)}_{\mathbb{F}_p[\underline{t}]})_0$ (see Lemma II.61).

Lemma IV.8. *For ever integer $e \geq 0$ we have the following equality of submodules of $H^e_{\mathfrak{a}}$:*

$$(D^{(e)}_{\mathbb{F}_p[\underline{t}]})_0 \cdot \delta_1 = \bigoplus_{\underline{a} \in \{0,\dots,p^e-1\}^r} \mathfrak{a}^{|\underline{a}|} \; Q^e_{\underline{a}}.$$

Proof. We begin by noting that, by Lemma IV.7, we have

$$\delta_1 = (f_1 - t_1)^{p^e-1} \cdots (f_r - t_r)^{p^e-1}\ \delta_{p^e} = \sum_{\underline{a}\in\{0,\ldots,p^e-1\}^r} f_1^{a_1}\cdots f_r^{a_r}\ Q^e_{\underline{a}}.$$

Given tuples $\underline{b}, \underline{c} \in \mathbb{Z}^r$ with $0 \leq b_i < p^e$ and $c_i < p^e$ we denote by $\sigma^{(e)}_{\underline{b}\to\underline{c}}$ the unique element of $D^{(e)}_{\mathbb{F}_p[\underline{t}]}$ such that for all $\underline{k} \in \{0, 1, \ldots, p^e - 1\}^r$ we have

$$\sigma^{(e)}_{\underline{b}\to\underline{c}} \cdot t_1^{k_1}\cdots t_r^{k_r} = \begin{cases} t_1^{p^e-1-c_1}\cdots t_r^{p^e-1-c_r} \text{ if } \underline{k} = (p^e-1-b_1, \ldots, p^e-1-b_r) \\ 0 \text{ otherwise.}\end{cases}$$

These operators form an $\mathbb{F}_p$-basis for $D^{(e)}_{\mathbb{F}_p[\underline{t}]}$, and since $\sigma^{(e)}_{\underline{b}\to\underline{c}}$ is homogeneous of degree $|\underline{b}| - |\underline{c}|$, the subcollection for which $|\underline{b}| = |\underline{c}|$ is an $\mathbb{F}_p$-basis for $(D^{(e)}_{\mathbb{F}_p[\underline{t}]})_0$.

Let $\underline{b}, \underline{c} \in \mathbb{Z}^r$ be tuples with $0 \leq b_i < p^e$ and $c_i < p^e$ such that $|\underline{b}| = |\underline{c}|$. Let $\underline{c}', \underline{c}'' \in \mathbb{Z}^r$ be the unique tuples with $0 \leq c'_i < p^e$, $0 \leq c''_i$ and $c_i = c'_i - p^e c''_i$ for all i. We then have $\sigma^{(e)}_{\underline{b}\to\underline{c}} \cdot Q^e_{\underline{a}} = 0$ for $\underline{a} \neq \underline{b}$ and

$$\begin{aligned}\sigma^{(e)}_{\underline{b}\to\underline{c}} \cdot Q^e_{\underline{b}} &= t_1^{p^e-1-c'_1+p^ec''_1}\cdots t_r^{p^e-1-c'_r+p^ec''_r}\ \delta_{p^e} \\ &= f_1^{p^ec''_1}\cdots f_r^{p^ec''_r}\ Q^e_{\underline{c}'},\end{aligned}$$

where in the last equality we use the fact that $(f_i^{p^e} - t_i^{p^e})\ \delta_{p^e} = (f_i - t_i)^{p^e}\ \delta_{p^e} = 0$. Therefore,

$$\sigma^{(e)}_{\underline{b}\to\underline{c}} \cdot \delta_1 = f_1^{p^ec''_1+b_1}\cdots f_r^{p^ec''_r+b_r}\ Q^e_{\underline{c}'} \in \bigoplus_{\underline{a}\in\{0,\ldots p^e-1\}^r} \mathfrak{a}^{|\underline{a}|} Q^e_{\underline{a}},$$

where the containment follows because $p^e|\underline{c}''| + |\underline{b}| = |\underline{c}'|$.

For the other inclusion, let $\underline{b}, \underline{a} \in \mathbb{Z}^r$ be tuples such that $0 \leq b_i$, $0 \leq a_i < p^e$ and $|\underline{b}| = |\underline{a}|$, and we show that $f_1^{b_1}\cdots f_r^{b_r}\ Q^e_{\underline{a}} \in (D^{(e)}_{\mathbb{F}_p[\underline{t}]})_0$. Let $\underline{b}', \underline{b}'' \in \mathbb{Z}^r$ be the unique tuples such that $0 \leq b'_i < p^e$, $0 \leq b''_i$ and $b_i = b'_i + p^e b''_i$ for all i. We then have

$$\begin{aligned}\sigma^{(e)}_{\underline{b}'\to\underline{a}-p^e\underline{b}''} \cdot \delta_1 &= \sigma^{(e)}_{\underline{b}'\to\underline{a}-p^e\underline{b}''} \cdot f_1^{b'_1}\cdots f_r^{b'_r}\ Q^e_{\underline{b}'} \\ &= f_1^{b_1}\cdots f_r^{b_r}\ Q^e_{\underline{a}},\end{aligned}$$

and since $|\underline{b}'| = |\underline{a}| - p^e|\underline{b}''|$, $\sigma^{(e)}_{\underline{b}'\to\underline{a}-p^e\underline{b}''} \in (D^{(e)}_{\mathbb{F}_p[\underline{t}]})_0$, which proves the claim. $\square$

Proposition IV.9. *Let R be a regular F-finite ring, $\mathfrak{a} \subseteq R$ be an ideal and $e \geq 0$ be an integer. Let $N^e_{\mathfrak{a}}$ be the module defined above using a choice of generators $\mathfrak{a} = (f_1, \ldots, f_r)$.*

(i) As a $D_R^{(e)}$-module, $N_{\mathfrak{a}}^e$ decomposes as a direct sum

$$N_{\mathfrak{a}}^e = \bigoplus_{\underline{a}\in\{0,\dots,p^e-1\}^r} \frac{D_R^{(e)}\cdot \mathfrak{a}^{|\underline{a}|}}{D_R^{(e)}\cdot \mathfrak{a}^{|\underline{a}|+1}}$$

(ii) If $\alpha\in\mathbb{Z}_p$ is a p-adic integer then $(N_{\mathfrak{a}}^e)_\alpha$ consists of the summands indexed by those $\underline{a}$ for which $|\underline{a}|\equiv\alpha \mod p^e\mathbb{Z}_p$.

Proof. Lemma IV.8 together with the fact that $(D_{R[\underline{t}]}^{(e)})_0 = D_R^{(e)} \ (D_{\mathbb{F}_p[\underline{t}]}^{(e)})_0$ imply that the submodule $(D_{R[t]}^{(e)})_0\cdot\delta_1$ of $H_{\mathfrak{a}}^e$ is given by

$$(D_{R[t]}^{(e)})_0\cdot\delta_1 = \bigoplus_{\underline{a}\in\{0,\dots,p^e-1\}^r} (D_R^{(e)}\cdot \mathfrak{a}^{|\underline{a}|})\ Q_{\underline{a}}^e.$$

Similarly, using Lemma IV.8 we observe that $(D_{\mathbb{F}_p[\underline{t}]}^{(e)})_0\cdot\mathfrak{a}\delta_1 = \mathfrak{a}(D_{\mathbb{F}_p[\underline{t}]}^{(e)})_0\cdot\delta_1 = \bigoplus_{\underline{a}\in\{0,\dots,p^e-1\}^r} \mathfrak{a}^{|\underline{a}|+1}Q_{\underline{a}}^e$, and by once again applying the fact that $(D_{R[\underline{t}]}^{(e)})_0 = D_R^{(e)}\otimes_{\mathbb{F}_p}(D_{\mathbb{F}_p[\underline{t}]}^{(e)})_0$ we conclude that

$$(D_{R[t]}^{(e)})_0\cdot\mathfrak{a}\delta_1 = \bigoplus_{\underline{a}\in\{0,\dots,p^e-1\}^r} (D_R^{(e)}\cdot \mathfrak{a}^{|\underline{a}|+1})\ Q_{\underline{a}}^e,$$

and part (i) follows.

For part (ii), note that $\varphi\in C^e(\mathbb{Z}_p,\mathbb{F}_p)$ acts on $Q_{\underline{a}}^e$ by the scalar $\varphi(-(r(p^e-1)-|\underline{a}|)-r) = \varphi(|\underline{a}|-rp^e) = \varphi(|\underline{a}|)$. We conclude that

$$\mathfrak{m}_\alpha^{(e)}Q_{\underline{a}}^e = \begin{cases} 0 \text{ if } |\underline{a}|\equiv\alpha \mod p^e\mathbb{Z}_p \\ \mathbb{F}_p\ Q_{\underline{a}}^e \text{ otherwise.}\end{cases}$$ ☐

Corollary IV.10. *The module $(N_{\mathfrak{a}}^e)_\alpha$ is a direct sum of the modules from the list*

$$\left\{\frac{D_R^{(e)}\cdot\mathfrak{a}^n}{D_R^{(e)}\cdot\mathfrak{a}^{n+1}}\ \middle|\ 0\le n\le r(p^e-1) \text{ and } n\equiv\alpha \mod p^e\mathbb{Z}_p\right\},$$

and every module from the list appears in the decomposition.

Proof. The result follows from part (ii), together with the observation that

$$\{|\underline{a}| \mid 0\le a_i<p^e \text{ and } |\underline{a}|\equiv\alpha \mod p^e\mathbb{Z}_p\} = \{0\le n\le r(p^e-1)\mid n\equiv\alpha \mod p^e\mathbb{Z}_p\}.$$ ☐

IV.3: The ν-invariants of Mustaţă, Takagi and Watanabe

Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be an ideal and, for a p-adic integer $\alpha \in \mathbb{Z}_p$ and an integer $e \geq 0$ let $(N_{\mathfrak{a}}^e)_\alpha$ be the module defined above using a choice of generators $\mathfrak{a} = (f_1, \ldots, f_r)$. Recall that we want to understand the nonvanishing of $(N_{\mathfrak{a}}^e)_\alpha$. As we will see later, this nonvanishing can be phrased in terms of some invariants introduced by Mustaţă, Takagi and Watanabe in their study of F-jumping numbers [MTW05] (Proposition IV.15). We call these ν-invariants, and in this section we discuss some of the basic properties that will be used later on.

Definition IV.11. Given a proper ideal $J \subseteq R$ containing $\mathfrak{a}$ in its radical and an integer $e \geq 0$ we define $\nu_{\mathfrak{a}}^J(p^e) = \max\{n \geq 0 \mid \mathfrak{a}^n \not\subseteq J^{[p^e]}\}$. The set of all such $\nu_{\mathfrak{a}}^J(p^e)$, as we range over all possible J, will be denoted by $\nu_{\mathfrak{a}}^\bullet(p^e)$; that is,

$$\nu_{\mathfrak{a}}^\bullet(p^e) = \{\nu_{\mathfrak{a}}^J(p^e) \mid J \text{ proper with } \mathfrak{a} \subseteq \sqrt{J}\}.$$

We call $\nu_{\mathfrak{a}}^\bullet(p^e)$ the set of *ν-invariants of level e*.

Recall that, by convention, we have $\mathfrak{a}^0 = R$ (even if $\mathfrak{a} = (0)$), and therefore the set $\{n \geq 0 \mid \mathfrak{a}^n \not\subseteq J^{[p^e]}\}$ is never empty.

In [MTW05] it is shown that if we fix J as above then the sequence $(\nu_{\mathfrak{a}}^J(p^e)/p^e)_{e=0}^\infty$ is increasing and bounded. The limit

$$c^J(\mathfrak{a}) = \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

is called the F-threshold of $\mathfrak{a}$ with respect to J. The set of F-thresholds coincides with the set of F-jumping numbers [MTW05, Prop. 2.7] [BMS08, Cor. 2.30].

We give alternative descriptions of the set of ν-invariants of level e in terms of the ideals $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n$ and $D_R^{(e)} \cdot \mathfrak{a}^n$ (see Definitions II.40 and II.32); these descriptions are well known to experts.

Proposition IV.12. *Let R be a regular F-finite ring, $\mathfrak{a} \subseteq R$ be an ideal and $e \geq 0$ be an integer. Then the set of ν-invariants of level e for $\mathfrak{a}$ is given by*

$$\begin{aligned}\nu_{\mathfrak{a}}^\bullet(p^e) &= \left\{n \geq 0 \;\middle|\; \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n \neq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1}\right\} \\ &= \left\{n \geq 0 \;\middle|\; D_R^{(e)} \cdot \mathfrak{a}^n \neq D_R^{(e)} \cdot \mathfrak{a}^{n+1}\right\}.\end{aligned}$$

Proof. The second equality follows from Corollary II.46, so it suffices to prove the first equality. Throughout the rest of the proof we will repeatedly use Corollary II.46 without further mention.

To prove the inclusion ($\subseteq$), suppose that $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1}$ and let J be a proper ideal containing $\mathfrak{a}$ in its radical so that $\mathfrak{a}^n \not\subseteq J^{[p^e]}$. We conclude that $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n \not\subseteq J$, and therefore $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1} \not\subseteq J$. This implies $\mathfrak{a}^{n+1} \not\subseteq J^{[p^e]}$, and therefore $n \neq \nu_{\mathfrak{a}}^J(p^e)$.

For ($\supseteq$), suppose that $n \geq 0$ is such that $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n \neq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1}$. Let $J = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1}$. Observe that $J \neq (1)$; otherwise $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1}$. Given $f \in \mathfrak{a}$ we note that $(f^m) = \mathcal{C}_R^{(e)} \cdot f^{mp^e}$ and thus $f^{mp^e} \in \mathfrak{a}^{n+1}$ for m large enough, thus $f \in \sqrt{J}$.

We claim that $n = \nu_{\mathfrak{a}}^J(p^e)$ and this will complete the proof. First, we have $\mathfrak{a}^{n+1} \subseteq J^{[p^e]}$ and therefore it suffices to show that $\mathfrak{a}^n \not\subseteq J^{[p^e]}$. To see this, note that if $\mathfrak{a}^n \subseteq J^{[p^e]}$ then $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n \subseteq J$, and thus $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1}$, giving a contradiction. □

Corollary IV.13. *Given integers $m \geq n \geq 0$ and $e \geq 0$, the following are equivalent:*

(a) We have $\nu_{\mathfrak{a}}^{\bullet}(p^e) \cap [n, m) = \emptyset$.

(b) We have $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^m$.

(c) We have $D_R^{(e)} \cdot \mathfrak{a}^n = D_R^{(e)} \cdot \mathfrak{a}^m$.

Proof. The equivalence of (b) and (c) follows from Corollary II.46, so it suffices to show the equivalence between (a) and (b).

We have $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^m$ if and only if all the inclusions in the chain

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1} \supseteq \cdots \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{m-1} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^m$$

are equalities which, by Proposition IV.12, happens precisely when none of the integers in the set $\{n, \ldots, m-1\}$ are ν-invariants of level e. □

Corollary IV.14. *Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be an ideal generated by r elements. The ν-invariants of $\mathfrak{a}$ satisfy the following properties:*

(i) They form a descending chain

$$\nu_{\mathfrak{a}}^{\bullet}(p^0) \supseteq \nu_{\mathfrak{a}}^{\bullet}(p^1) \supseteq \nu_{\mathfrak{a}}^{\bullet}(p^2) \supseteq \cdots$$

(ii) If $rp^e \leq n \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ then $n - p^e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$.

Proof. By using the alternative characterization of the ν-invariants given in Proposition IV.12, part (i) follows from the fact that $D_R^{(e)} \subseteq D_R^{(e+1)}$, and part (ii) follows from Lemma II.53. □

IV.4: Bernstein-Sato roots and the ν-invariants

As promised, we can now phrase the nonvanishing of the modules $(N^e_{\mathfrak{a}})_\alpha$ in terms of the ν-invariants of $\mathfrak{a}$.

Proposition IV.15. *Let R be a regular F-finite ring, $\mathfrak{a} \subseteq R$ be an ideal, $e \geq 0$ be an integer and $\alpha \in \mathbb{Z}_p$ be a p-adic integer. Let $(N^e_{\mathfrak{a}})_\alpha$ be the module defined above using a choice of generators $\mathfrak{a} = (f_1, \dots, f_r)$. The following are equivalent:*

(a) The module $(N^e_{\mathfrak{a}})_\alpha$ is nonzero.

(b) The image of δ_1 in $(N^e_{\mathfrak{a}})_\alpha$ is nonzero.

(c) There is a ν-invariant $n \in \nu^\bullet_{\mathfrak{a}}(p^e)$ with $n \equiv \alpha \mod p^e\mathbb{Z}_p$.

Proof. It is clear that (b) implies (a). To observe that (a) implies (b), note that the subalgebra $C^e(\mathbb{Z}_p, \mathbb{F}_p)$ of $(D^{(e)}_{R[\underline{t}]})_0$ is central and therefore $(N^e_{\mathfrak{a}})_\alpha$ is a cyclic left $(D^{(e)}_{R[\underline{t}]})_0$-module generated by the image of δ_1.

We show that (a) implies (c). By Corollary IV.10, if $(N^e_{\mathfrak{a}})_\alpha$ is nonzero we have $D^{(e)}_R \cdot \mathfrak{a}^n \neq D^{(e)}_R \cdot \mathfrak{a}^{n+1}$ for some n with $0 \leq n \leq r(p^e - 1)$ and $n \equiv \alpha \mod p^e\mathbb{Z}_p$. By Corollary II.46, $\mathcal{C}^{(e)}_R \cdot \mathfrak{a}^n \neq \mathcal{C}^{(e)}_R \cdot \mathfrak{a}^{n+1}$ and therefore $n \in \nu^\bullet_{\mathfrak{a}}(p^e)$.

To show that (c) implies (a), suppose that we are given a ν-invariant n as in part (c). By Corollary IV.14 we can subtract p^e enough times to assume that $0 \leq n \leq r(p^e - 1)$. By Corollary II.46, we have that $D^{(e)}_R \cdot \mathfrak{a}^n \neq D^{(e)}_R \cdot \mathfrak{a}^{n+1}$ and the result follows once again by applying Corollary IV.10 once again. □

Corollary IV.16. *Suppose that $(N^e_{\mathfrak{a}})_\alpha = 0$ for some $e \geq 0$. Then $(N^i_{\mathfrak{a}})_\alpha = 0$ for all $i \geq e$.*

Proof. If the image of δ_1 in $(N^e_{\mathfrak{a}})_\alpha$ is zero, then it remains zero in $(N^i_{\mathfrak{a}})_\alpha$ for all $i \geq e$. □

We now give an alternative characterization of the Bernstein-Sato roots of $\mathfrak{a}$. This provides a simpler point of view to work with, which will also allow us to compute the Bernstein-Sato roots of an ideal more easily.

Theorem IV.17. *Let R be a regular and F-finite ring of characteristic p and let $\mathfrak{a} \subseteq R$ be an ideal. A p-adic integer $\alpha \in \mathbb{Z}_p$ is a Bernstein-Sato root of $\mathfrak{a}$ if and only if α is the p-adic limit of a sequence $(\nu_e)_{e=0}^\infty \subseteq \mathbb{Z}_{\geq 0}$ with $\nu_e \in \nu^\bullet_{\mathfrak{a}}(p^e)$.*

Proof. Suppose that $\alpha \in \mathbb{Z}_p$ is the p-adic limit of such a sequence (ν_e). If $(\mu_j) = (\nu_{e_j})$ is a subsequence of (ν_e) then, by Corollary IV.14, we have $\mu_j \in \nu^\bullet_{\mathfrak{a}}(p^j)$. We may therefore pass to a subsequence to assume that $\nu_e \equiv \alpha \mod p^e\mathbb{Z}_p$. By Proposition IV.15, the image of δ_1

in $(N_{\mathfrak{a}}^{e})_{\alpha}$ is nonzero for every integer $e \geq 0$. We conclude that the image of δ_1 in $(N_{\mathfrak{a}})_{\alpha}$ is nonzero, and thus $(N_{\mathfrak{a}})_{\alpha}$ is nonzero.

For the other direction, suppose that $(N_{\mathfrak{a}})_{\alpha}$ is nonzero. By Corollary IV.16, we must have that $(N_{\mathfrak{a}}^{e})_{\alpha}$ is nonzero for every $e \geq 0$. By Proposition IV.15 we conclude that for every $e \geq 0$ there is a ν-invariant $\nu_e \in \mathcal{B}_{\mathfrak{a}}^{\bullet}(p^e)$ with $\nu_e \equiv \alpha \mod p^e\mathbb{Z}_p$, and therefore α is the p-adic limit of (ν_e). □

Remark IV.18. As observed in the proof of Theorem IV.17, the condition $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ passes to subsequences, and whenever $\alpha \in \mathbb{Z}_p$ is a Bernstein-Sato root of $\mathfrak{a}$ we can choose the sequence (ν_e) such that $\nu_e \equiv \alpha \mod p^e\mathbb{Z}_p$ and $0 \leq \nu_e \leq r(p^e - 1)$ for every $e \geq 0$.

This alternative characterization allows us to settle a concern we had with us since we defined Bernstein-Sato roots.

Corollary IV.19. *The Bernstein-Sato roots of the ideal $\mathfrak{a}$ are independent of the choice of generators used in the construction of $N_{\mathfrak{a}}$.*

Theorem IV.17 also has the following surprising consequence.

Corollary IV.20. *The ideal $\mathfrak{a}$ and the ideal $\mathfrak{a}^{[p]}$ have the same Bernstein-Sato roots.*

Proof. For all integers $e, n \geq 0$ we have $\mathcal{C}_R^{(e+1)} \cdot (\mathfrak{a}^{[p]})^n = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n$ (see Lemma II.5, Proposition II.48), and therefore $\nu_{\mathfrak{a}^{[p]}}^{\bullet}(p^{e+1}) = \nu_{\mathfrak{a}}^{\bullet}(p^e)$. □

IV.5: Bounding the number of ν-invariants

Theorem IV.17 will be one of the two main results that go into the proof of Theorem IV.4. The other one is the following.

Proposition IV.21. *Let R be a regular F-finite ring of characteristic p and let $\mathfrak{a} \subseteq R$ be an ideal generated by r elements. There is a constant $K > 0$ such that for all integers $e \geq 0$ we have*

$$\#\Big(\nu_{\mathfrak{a}}^{\bullet}(p^e) \cap [0, rp^e)\Big) \leq K.$$

We first give two proofs that work in restricted settings.

Proof of Proposition IV.21 when $r = 1$. Suppose that $\mathfrak{a} = (f)$. Recall that for all integers $e, n \geq 0$ we have $\mathcal{C}_R^{(e)} \cdot f^n = \tau(f^{n/p^e})$ (see Proposition II.51). We conclude that n is a ν-invariant of level e for $\mathfrak{a}$ if and only if there is an F-jumping number of $\mathfrak{a}$ in the interval $(n/p^e, (n+1)/p^e]$. The F-jumping numbers of $\mathfrak{a}$ form a discrete set (see [BMS09]), and if we let K be the number of F-jumping numbers of $\mathfrak{a}$ contained in $(0, 1]$ then it follows that $\#(\nu_{\mathfrak{a}}^{\bullet}(p^e) \cap [0, 1)) \leq K$. □

Our next proof will be in the case when the ring R is a polynomial ring, where we can take the advantage of the grading (even if we don't assume that the ideal $\mathfrak{a} \subseteq R$ is homogeneous). This technique was already exploited by Blickle, Mustaţă and Smith to prove the discreteness and rationality of F-jumping numbers on polynomials rings (which they then prove in greater generality) [BMS08].

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field k, which we equip with the standard grading. Given an integer i we let R_i denote the space of homogeneous polynomials of degree i, and we let $R_{\leq D} = \bigoplus_{i=0}^{D} R_i$. We say that an ideal $\mathfrak{b} \subseteq R$ is generated in degrees $\leq D$ if there is a generating set $\mathfrak{b} = (g_1, \ldots, g_r)$ for $\mathfrak{b}$ with $g_i \in R_{\leq D}$ (note: we do not assume that the g_i are homogeneous). An ideal $\mathfrak{b} \subseteq R$ is generated in degrees $\leq D$ if and only if $\mathfrak{b} = (\mathfrak{b} \cap R_{\leq D})R$; consequently, if $\mathfrak{b}' \subseteq R$ is another ideal generated in degrees $\leq D$ then $\mathfrak{b} = \mathfrak{b}'$ if and only if $\mathfrak{b} \cap R_{\leq D} = \mathfrak{b}' \cap R_{\leq D}$.

Lemma IV.22. *Suppose that $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a perfect field k. If $\mathfrak{b}$ is generated in degrees $\leq D$ then $\mathcal{C}_R^{(e)} \cdot \mathfrak{b}$ is generated in degrees $\leq \lfloor D/p^e \rfloor$.*

Proof. This follows from the description of $\mathcal{C}_R^{(e)} \cdot \mathfrak{b}$ given in Proposition II.47. □

Proof of Proposition IV.21 when R is a polynomial ring. Note that our goal is to bound the number of jumps in the chain

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^0 \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^1 \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^2 \supseteq \cdots \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{rp^e}$$

uniformly in e.

We pick an integer D such that the ideal $\mathfrak{a}$ is generated in degrees $\leq D$. For every integer $n \geq 0$ the ideal $\mathfrak{a}^n$ is generated in degrees $\leq Dn$ and, if $n \leq rp^e$, Lemma IV.22 tells us that $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n$ is generated in degrees $\leq \lfloor Dn/p^e \rfloor \leq Dr$. Therefore, it suffices to bound the number of jumps in the chain

$$\left(\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^0\right) \cap R_{\leq rD} \supseteq \left(\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^1\right) \cap R_{\leq rD} \supseteq \left(\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^2\right) \cap R_{\leq rD} \supseteq \cdots \supseteq \left(\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{rp^e}\right) \cap R_{\leq rD}.$$

The number of jumps in this chain is bounded by the k-dimension of $R_{\leq rD}$, which is finite. □

To prove the general case we will use the discreteness of F-jumping numbers for arbitrary ideals on arbitrary F-finite regular rings (see [ST14]). The main difficulty that arises in this case is that, for an arbitrary ideal $\mathfrak{a}$, we do not have $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^n = \tau(\mathfrak{a}^{n/p^e})$ in general (if this were the case, the proof for principal ideals given above would work verbatim). We get around this issue by considering the following notion, inspired by Sato's stabilization exponent [Sat19].

Definition IV.23. Given $\mathfrak{a} \subseteq R$ we call an integer $s \geq 0$ a *stable exponent* for $\mathfrak{a}$ if for all $n \in \mathbb{Z}_{\geq 0}$ we have

$$\tau(\mathfrak{a}^n) = \mathcal{C}_R^{(s)} \cdot \mathfrak{a}^{np^s}.$$

From the definition it follows that s is stable if and only if for all $e \geq 0$ we have $\mathcal{C}_R^{(s)} \cdot \mathfrak{a}^{np^s} = \mathcal{C}_R^{(s+e)} \cdot \mathfrak{a}^{np^{s+e}}$. Note that if s is stable and $e > s$ then e is also stable. By Lemma II.53 and Proposition II.54, for s to be stable it suffices to have the equality $\tau(\mathfrak{a}^n) = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{np^s}$ for all integers n with $0 \leq n < r$. In particular, stable exponents exist.

Lemma IV.24. *Suppose s is a stable exponent. Then for all $e \geq 0$ we have*

$$\tau(\mathfrak{a}^{n/p^e}) = \mathcal{C}_R^{(e+s)} \cdot \mathfrak{a}^{np^s}.$$

Proof. For all $e \geq 0$ we have, using Proposition II.48,

$$\begin{aligned}
\mathcal{C}_R^{(e+s)} \cdot \mathfrak{a}^{np^s} &= \mathcal{C}_R^{(e)} \cdot \mathcal{C}_R^{(s)} \cdot \mathfrak{a}^{np^s} \\
&= \mathcal{C}_R^{(e)} \cdot \mathcal{C}_R^{(s+d)} \cdot \mathfrak{a}^{np^{s+d}} \\
&= \mathcal{C}_R^{(e+s+d)} \cdot \mathfrak{a}^{np^s+d}.
\end{aligned}$$ □

Recall that $\mathrm{FJN}(\mathfrak{a})$ denotes the set of F-jumping numbers of $\mathfrak{a}$ (Definition II.52).

Lemma IV.25. *Let $\mathfrak{a} \subseteq R$ be an ideal with stable exponent s and let $e \geq 0$ be an integer. For all integers $n \geq 0$, we have $\nu_{\mathfrak{a}}^{\bullet}(p^{e+s}) \cap [np^s, (n+1)p^s) \neq \emptyset$ if and only if $\mathrm{FJN}(\mathfrak{a}) \cap (n/p^e, (n+1)/p^e] \neq \emptyset$.*

Proof. The statement $\nu_{\mathfrak{a}}^{\bullet}(p^{e+s}) \cap [np^s, (n+1)p^s) \neq \emptyset$ says precisely that there is a jump in the chain

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{np^s} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{np^s+1} \supseteq \cdots \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{(n+1)p^s}$$

or, equivalently, that the first and last ideals in the chain differ. By Lemma IV.24, the first ideal is $\tau(\mathfrak{a}^{n/p^e})$ and the last ideal is $\tau(\mathfrak{a}^{(n+1)/p^e})$, so the result follows. □

Proof of Proposition IV.21 in the general case. The F-jumping numbers of $\mathfrak{a}$ are known to form a discrete set [ST14]. If we let M denote the number of F-jumping numbers of $\mathfrak{a}$ in the interval $(0, r]$, then there are at most M subintervals of $(0, r]$ of the form $(n/p^e, (n+1)/p^e]$ that contain an F-jumping number.

Let s be a stable exponent for $\mathfrak{a}$. By Lemma IV.25, we conclude that there are at most M subintervals of $[0, rp^{e+s})$ of the form $[np^s, (n+1)p^s)$ which contain a ν-invariant of level

$e+s$. Since each of these subintervals contains p^s elements, we conclude that

$$\#\left(\nu_{\mathfrak{a}}^{\bullet}(p^{e+s}) \cap [0, rp^{e+s})\right) \leq Mp^s,$$

which concludes the proof. $\square$

IV.6: Proof of Theorem IV.4

We recall the statement.

Theorem. *Let R be a regular F-finite ring of characteristic p, $\mathfrak{a} \subseteq R$ be an ideal and $N_{\mathfrak{a}}$ be the module defined above using a choice of generators $\mathfrak{a} = (f_1, \dots, f_r)$ for the ideal $\mathfrak{a}$. Then there are only finitely many p-adic integers $\alpha \in \mathbb{Z}_p$ such that $(N_{\mathfrak{a}})_\alpha \neq 0$. Furthermore, the natural map $N_{\mathfrak{a}} \to \bigoplus_{\alpha \in \mathbb{Z}_p} (N_{\mathfrak{a}})_\alpha$ is an isomorphism which identifies $(N_{\mathfrak{a}})_\alpha$ with the submodule $\operatorname{Ann}_{N_{\mathfrak{a}}}(\mathfrak{m}_\alpha)$.*

Recall also that a p-adic integer $\alpha \in \mathbb{Z}_p$ for which $(N_{\mathfrak{a}})_\alpha \neq 0$ is called a Bernstein-Sato root of $\mathfrak{a}$.

Proof. It is enough to show that there is a finite number of Bernstein-Sato roots; the rest of the statement follows formally from Proposition III.14. Suppose that $\mathfrak{a}$ is generated by r elements.

Given a Bernstein-Sato root α, we claim that for every integer $e \geq 0$ there is some ν-invariant $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e) \cap [0, rp^e)$ such that $\alpha \equiv n \mod p^e\mathbb{Z}_p$. Indeed, by Theorem IV.17 we know that α is the p-adic limit of a sequence (ν_e) with $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$. By passing to a subsequence we may assume that $\nu_e \equiv \alpha \mod p^e\mathbb{Z}_p$ and that $0 \leq \nu_e < rp^e$ (see Remark IV.18).

By Proposition IV.21 there is an integer K that bounds $\nu_{\mathfrak{a}}^{\bullet}(p^e) \cap [0, rp^e)$ uniformly in e, and we claim that there can be at most K Bernstein-Sato roots. We prove this by contradiction. Suppose there were $K+1$ distinct Bernstein-Sato roots $\alpha_1, \alpha_2, \dots, \alpha_{K+1}$. Let e be large enough so that $\alpha_i \not\equiv \alpha_j \mod p^e\mathbb{Z}_p$ for $i \neq j$. By using the previous claim we can find ν-invariants $\nu^{(1)}, \dots, \nu^{(K+1)}$ with $\nu^{(i)} \equiv \alpha_i \mod p^e\mathbb{Z}_p$ and $0 \leq \nu^{(i)} < rp^e$. It follows that $\nu^{(1)}, \dots, \nu^{(K+1)}$ must be distinct, contradicting the bound K. $\square$

CHAPTER V
Properties of Bernstein-Sato Roots

V.1: Local behavior

In this section we show that Bernstein-Sato roots are local invariants (see Proposition V.2 for a precise statement). We do this by showing that the construction of the modules $N_{\mathfrak{a}}$ and $(N_{\mathfrak{a}})_\alpha$ is compatible with localization in a suitable sense. As usual, we work with a regular F-finite ring R and an ideal $\mathfrak{a} \subseteq R$.

Lemma V.1. *Let $\alpha \in \mathbb{Z}_p$ be a p-adic integer. Choose generators $\mathfrak{a} = (f_1, \dots, f_r)$ and let $(N_{\mathfrak{a}})$ and $(N_{\mathfrak{a}})_\alpha$ be the modules defined in Chapter IV.2 using this choice. Let $W \subseteq R$ be a multiplicative subset, let $R' = W^{-1}R$, let $\mathfrak{a}' = \mathfrak{a}R'$ and let $(N_{\mathfrak{a}'})$ and $(N_{\mathfrak{a}'})_\alpha$ be the modules defined in Chapter IV.2 using the corresponding generators in R'. Then there are canonical isomorphisms $R' \otimes_R N_{\mathfrak{a}} \cong N_{\mathfrak{a}'}$ and $R' \otimes_R (N_{\mathfrak{a}})_\alpha \cong (N_{\mathfrak{a}'})_\alpha$.*

Proof. We can identify $R' \otimes_R (\mathfrak{m}_\alpha N_{\mathfrak{a}})$ with its image in $R' \otimes_R N_{\mathfrak{a}}$, which is $\mathfrak{m}_\alpha(R' \otimes_R N_{\mathfrak{a}})$. We conclude that there are canonical isomorphisms

$$R' \otimes_R \frac{N_{\mathfrak{a}}}{\mathfrak{m}_\alpha N_{\mathfrak{a}}} \cong \frac{R' \otimes_R N_{\mathfrak{a}}}{R' \otimes_R (\mathfrak{m}_\alpha N_{\mathfrak{a}})} \cong \frac{R' \otimes_R N_{\mathfrak{a}}}{\mathfrak{m}_\alpha (R' \otimes_R N_{\mathfrak{a}})},$$

so it suffices to prove the statement about $N_{\mathfrak{a}}$. To do this, we will use the description of $N_{\mathfrak{a}}$ given in Lemma IV.3.

There is a canonical isomorphism $R' \otimes_R H_{\mathfrak{a}} \cong H_{\mathfrak{a}'}$ which identifies $1 \otimes \delta_1$ with δ_1'. Note that $(D_{R[\underline{t}]})_0 \cdot \delta_1$ is the image of the $(D_{R[\underline{t}]})_0$-linear map $(D_{R[\underline{t}]})_0 \to H_{\mathfrak{a}}$ that sends $1 \mapsto \delta_1$, and thus $R' \otimes_R ((D_{R[\underline{t}]})_0 \cdot \delta_1)$ can be identified with the image of the composition $(D_{R'[\underline{t}]})_0 \cong R' \otimes_R (D_{R[\underline{t}]})_0 \to R' \otimes_R H_{\mathfrak{a}} \xrightarrow{\sim} H_{\mathfrak{a}'}$ (the first isomorphism coming from Proposition II.30 and Lemma II.61). This composition maps $1 \mapsto \delta_1'$ and is $(D_{R'[\underline{t}]})_0$-linear, and therefore its image is $(D_{R'[\underline{t}]})_0 \cdot \delta_1'$.

Similarly, $(D_{R[\underline{t}]})_0 \cdot \mathfrak{a}\delta_1$ is the image of the $(D_{R[\underline{t}]})_0$-linear map $\bigoplus_{i=i}^r (D_{R[\underline{t}]})_0 \ e_i \to H_{\mathfrak{a}}$ which sends $e_i \mapsto f_i\delta_1$, and therefore $R' \otimes_R ((D_{R[\underline{t}]})_0 \cdot \mathfrak{a}\delta_1$ is identified with the image of the composition $\bigoplus_{i=1}^{r}(D_{R'[\underline{t}]})_0\ e_i \cong R' \otimes_R \bigoplus_{i=1}^{r}(D_{R[\underline{t}]})_0\ e_i \to R' \otimes_R H_{\mathfrak{a}} \cong H_{\mathfrak{a}'}$, which is $(D_{R'[\underline{t}]})_0 \cdot \mathfrak{a}'\delta_1'$. We conclude that

$$R' \otimes_R N_{\mathfrak{a}} \cong \frac{R' \otimes_R ((D_{R[\underline{t}]})_0 \cdot \delta_1)}{R' \otimes_R ((D_{R[\underline{t}]})_0 \cdot \mathfrak{a}\delta_1)} \cong \frac{(D_{R'[\underline{t}]})_0 \cdot \delta_1'}{(D_{R[\underline{t}]})_0 \cdot \mathfrak{a}'\delta_1'} = N_{\mathfrak{a}'}. \qquad \square$$

Proposition V.2. *Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be an ideal. Let $g_1, \dots, g_k \in R$ be such that $(g_1, \dots, g_k) = R$. Then we have*

$$\mathrm{BSR}(\mathfrak{a}) = \bigcup_{i=1}^{k} \mathrm{BSR}(\mathfrak{a}R_{g_i}) = \bigcup_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{BSR}(\mathfrak{a}R_{\mathfrak{p}}) = \bigcup_{\mathfrak{m}\in\mathrm{Max}(R)} \mathrm{BSR}(\mathfrak{a}R_{\mathfrak{m}}).$$

Proof. Whether a p-adic integer $\alpha \in \mathbb{Z}_p$ is a Bernstein-Sato root is given by the nonvanishing of the module $(N_{\mathfrak{a}})_\alpha$. The construction of this module is compatible with localization, as explained in Lemma V.1, and the result follows from the fact that the nonvanishing of a module is a local condition. $\square$

V.2: Homogeneous ideals

Let $R = \bigoplus_{n=0}^{\infty} R_n$ is a graded, regular, F-finite algebra over a perfect field $k = R_0$, and let $\mathfrak{m}$ denote its homogeneous maximal ideal. The properties of a homogeneous ideal $\mathfrak{a} \subseteq R$ can often be related to the properties of the ideal $\mathfrak{a}R_{\mathfrak{m}}$ in the local ring $R_{\mathfrak{m}}$. In this section we show that this general philosophy also applies to Bernstein-Sato roots.

Lemma V.3. *Let $\mathfrak{b} \subseteq R$ be a homogeneous ideal and $e \geq 0$ be an integer. Then the ideal $D_R^{(e)} \cdot \mathfrak{b}$ is also homogeneous.*

Proof. The grading of R induces a grading on its subring R^{p^e} of p^e-th powers, which makes R into a graded R^{p^e}-module. Since R is F-finite, the module $D_R^{(e)} = \mathrm{Hom}_{R^{p^e}}(R, R)$ acquires a natural grading, and hence so does the module $D_R^{(e)} \otimes_{R^{p^e}} R$.

With respect to these gradings the natural map $\varphi : D_R^{(e)} \otimes_{R^{p^e}} R \to R$ given by $\varphi(\xi \otimes r) = \xi(r)$ is homogeneous of degree zero. To conclude the proof, note that $D_R^{(e)} \cdot \mathfrak{b}$ is the image of the graded submodule $D_R^{(e)} \otimes_R \mathfrak{b}$ under φ. $\square$

Proposition V.4. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded, regular, F-finite algebra over a perfect field $k = R_0$, let $\mathfrak{m}$ denote its homogeneous maximal ideal and let $\mathfrak{a} \subseteq R$ be a homogeneous ideal. Then* $\mathrm{BSR}(\mathfrak{a}) = \mathrm{BSR}(\mathfrak{a}R_{\mathfrak{m}})$.

Proof. Fix some choice of generators for $\mathfrak{a}$ and, in what follows, the modules $N_{\mathfrak{a}}^e$ and $N_{\mathfrak{a}R_{\mathfrak{m}}}^e$ are defined using this choice.

Recall that $\alpha \in \mathbb{Z}_p$ is a Bernstein-Sato root of $\mathfrak{a}$ if and only if $(N^e_{\mathfrak{a}})_\alpha \neq 0$ for all $e \geq 0$ (see Proposition IV.15, and proof of Theorem IV.17). We claim that, for all integers $e \geq 0$ and p-adic integers $\alpha \in \mathbb{Z}_p$, we have $(N^e_{\mathfrak{a}})_\alpha = 0$ if and only if $(N^e_{\mathfrak{a}R_{\mathfrak{m}}})_\alpha = 0$, which will prove the result.

Lemma V.3 tells us that the R-module $D^{(e)}_R \cdot \mathfrak{a}^k / D^{(e)}_R \cdot \mathfrak{a}^{k+1}$ admits a grading for all integers $k \geq 0$. By using the R-module decomposition given in Proposition IV.9, we conclude that $(N^e_{\mathfrak{a}})_\alpha$ admits a graded R-module structure. By the graded version of Nakayama's Lemma, we have $(N^e_{\mathfrak{a}})_\alpha = 0$ if and only if $(N^e_{\mathfrak{a}})_\alpha \otimes_R R/\mathfrak{m} = 0$ which, by the regular version of Nakayama's Lemma, is equivalent to $(N^e_{\mathfrak{a}})_\alpha \otimes_R R_{\mathfrak{m}} = 0$. Since $(N^e_{\mathfrak{a}})_\alpha \otimes_R R_{\mathfrak{m}} \cong (N^e_{\mathfrak{a}R_{\mathfrak{m}}})_\alpha$ by Lemma V.1, the claim is proven. $\square$

V.3: Minimal reductions

We review the basics of minimal reductions, and refer the reader to Huneke and Swanson's book for details [HS06, Ch. 8]. Let $(R, \mathfrak{m})$ be a noetherian local ring and $\mathfrak{a} \subseteq R$ be an ideal. The *Rees algebra of* $\mathfrak{a}$ is given by

$$R[\mathfrak{a}t] = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n t^n = R \oplus \mathfrak{a}t \oplus \mathfrak{a}^2 t^2 \oplus \cdots,$$

which we view as a subring of $R[t]$. The *fiber cone of* $\mathfrak{a}$ is then given by

$$\mathcal{F}_{\mathfrak{a}}(R) = \frac{R[\mathfrak{a}t]}{\mathfrak{m}R[\mathfrak{a}t]} = \frac{R}{\mathfrak{m}} \oplus \frac{\mathfrak{a}}{\mathfrak{m}\mathfrak{a}} \oplus \frac{\mathfrak{a}^2}{\mathfrak{m}\mathfrak{a}^2} \oplus \cdots$$

The Krull dimension of $\mathcal{F}_{\mathfrak{a}}(R)$ is called the *analytic spread of* $\mathfrak{a}$, and is denoted by $\ell(\mathfrak{a})$. We have $\ell(\mathfrak{a}) \leq \dim R$ and, if $\mathfrak{a}$ can be generated by r elements, then $\ell(\mathfrak{a}) \leq r$.

A *reduction* of the ideal $\mathfrak{a} \subseteq R$ is another ideal $\mathfrak{b} \subseteq R$ with $\mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{b}\mathfrak{a}^k = \mathfrak{a}^{k+1}$ for k large enough; the smallest such k is then called the *reduction number of* $\mathfrak{a}$ *with respect to* $\mathfrak{b}$. An ideal $\mathfrak{b} \subseteq \mathfrak{a}$ is a reduction of $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \bar{\mathfrak{b}}$. If $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ then $\mathfrak{a}$ is integral over $\mathfrak{b}$, and $\mathfrak{b}$ requires at least $\ell(\mathfrak{a})$ generators. If the residue field of R is infinite then there is a reduction of $\mathfrak{a}$ that is generated by $\ell(\mathfrak{a})$ elements.

Lemma V.5. *Let R be an F-finite local ring, $\mathfrak{a} \subseteq R$ be an ideal and $\mathfrak{b} \subseteq \mathfrak{a}$ be a reduction of $\mathfrak{a}$ with reduction number k. Then:*

(i) For all integers $e \geq 0$ we have $\nu^\bullet_{\mathfrak{a}}(p^e) \subseteq \bigcup_{i=0}^k \nu^\bullet_{\mathfrak{b}}(p^e) + i$ and $\nu^\bullet_{\mathfrak{b}}(p^e) \subseteq \bigcup_{i=0}^k \nu^\bullet_{\mathfrak{a}}(p^e) - i$.

(ii) We have $\mathrm{BSR}(\mathfrak{a}) \subseteq \bigcup_{i=0}^k \mathrm{BSR}(\mathfrak{b}) + i$ and $\mathrm{BSR}(\mathfrak{b}) \subseteq \bigcup_{i=0}^k \mathrm{BSR}(\mathfrak{a}) - i$.

Proof. Part (i) follows by considering the following chains of ideals:

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{b}^{n-k} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{b}^{n+1},$$

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{b}^{n} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{b}^{n+1} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+k+1}.$$

For part (ii), we use the alternative characterization of Bernstein-Sato roots given in Theorem IV.17. Suppose that $\alpha \in \mathrm{BSR}(\mathfrak{a})$, and choose a sequence (ν_e) with $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ such that α is the p-adic limit of ν_e. By part (i), for every $e \geq 0$ there is some $i_e \in \{0, 1, \dots, k\}$ such that $\nu_e - i_e \in \nu_{\mathfrak{b}}^{\bullet}(p^e)$. We conclude there is some $i \in \{0, 1, \dots, k\}$ and a subsequence (ν_{e_j}) such that $\nu_{e_j} - i \in \nu_{\mathfrak{b}}^{\bullet}(p^{e_j})$. The p-adic limit of this subsequence is $\alpha - i$ which, by Theorem IV.17, is a Bernstein-Sato root of $\mathfrak{b}$. Therefore, $\alpha \in \mathrm{BSR}(\mathfrak{b}) + i$. The other statement follows similarly. □

Example V.6. Given an ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ and a reduction $\mathfrak{b} \subseteq \mathfrak{a}$, the Bernstein-Sato polynomials of $\mathfrak{a}$ and $\mathfrak{b}$ may differ. For example, in the ring $\mathbb{C}[x, y]$ we may consider the ideal $\mathfrak{a} = (x^3, x^2y, xy^2, y^3)$ and its reduction $\mathfrak{b} = (x^3, y^3)$ [HS06, Prop. 1.4.6], which has reduction number 1. Using computational software [LT], we find:

$$\begin{aligned} b_{\mathfrak{a}}(s) &= (s+1)(s+\frac{2}{3})(s+\frac{4}{3}) \\ b_{\mathfrak{b}}(s) &= (s+1)(s+\frac{2}{3})(s+\frac{4}{3})(s+\frac{5}{3})(s+2). \end{aligned}$$

In Chapter VI we will see that, for monomial ideals like $\mathfrak{a}$ and $\mathfrak{b}$ above, the formation of Bernstein-Sato roots is compatible with mod-p reduction (see Theorem VI.6 for a precise statement). This tells us that if for all primes p large enough, the Bernstein-Sato roots of the ideals $\mathfrak{a} = (x^3, x^2y, xy^2, y^3) \subseteq \mathbb{F}_p[x, y]$ and $\mathfrak{b} = (x^3, y^3) \subseteq \mathbb{F}_p[x, y]$ are given by

$$\begin{aligned} \mathrm{BSR}(\mathfrak{a}) &= \{-1, -2/3, -4/3\} \\ \mathrm{BSR}(\mathfrak{b}) &= \{-1, -2/3, -4/3, -5/3, -2\}. \end{aligned}$$

In particular, $\mathrm{BSR}(\mathfrak{a}) \neq \mathrm{BSR}(\mathfrak{b})$. Note the agreement with Lemma V.5.

V.4: Rationality and negativity

In characteristic zero, Kashiwara used resolution of singularities to show that the roots of the Bernstein-Sato polynomial of a principal ideal are rational and negative [Kas77]; this result was later extended to arbitrary ideals by Budur, Mustaţă and Saito [BMS06a]. In this section we prove an analogous statement in characteristic p.

We base our approach for proving the rationality of Bernstein-Sato roots in the strategy used for F-jumping numbers. In order to show that the F-jumping numbers of an r-generated ideal $\mathfrak{a}$ are rational, one starts by first showing that they form a discrete set and then the rationality is forced by the fact that F-jumping numbers come with some "dynamics"; namely, if α is an F-jumping number of $\mathfrak{a}$ then so are $p\alpha$ and $\alpha - \lfloor \alpha/r \rfloor$ [BMS08, Prop. 3.4].

We already know that an ideal has finitely many Bernstein-Sato roots (this is the analogue of "discreteness") and our next goal is to find some dynamics on the set of Bernstein-Sato roots. We will work in our usual setting: R is a regular F-finite ring of characteristic p and $\mathfrak{a} \subseteq R$ is an ideal.

Lemma V.7. *Suppose $\mathfrak{a}$ can be generated by r elements. If $n \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ then there is some $i \in \{0, 1, \ldots, r(p-1)\}$ such that $pn + i \in \nu_{\mathfrak{a}}^{\bullet}(p^{e+1})$.*

Proof. Consider the following chain of ideals:

$$\mathfrak{a}^{np} \supseteq \mathfrak{a}^{n[p]} \supseteq \mathfrak{a}^{(n+1)[p]} \supseteq \mathfrak{a}^{(n+1)p+(r-1)(p-1)} = \mathfrak{a}^{np+r(p-1)+1}$$

(for the last inclusion, see Lemma II.5 (iii)). We apply the operators $\mathcal{C}_R^{(e+1)}$ to the above chain and, by recalling that $\mathcal{C}_R^{(e)} \cdot \mathfrak{b} = \mathcal{C}_R^{(e+1)} \cdot \mathfrak{b}^{[p]}$ for any $\mathfrak{b} \subseteq R$ (see Corollary II.48), we arrive that the chain

$$\mathcal{C}_R^{(e+1)} \cdot \mathfrak{a}^{np} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{n+1} \supseteq \mathcal{C}_R^{(e+1)} \cdot \mathfrak{a}^{np+r(p-1)+1}.$$

Since $n \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ the two ideals in the middle differ, and we conclude that the two outer ideals must also differ. The result follows (see Corollary IV.13). ☐

Lemma V.8. *Suppose $\mathfrak{a}$ can be generated by r elements. If $\alpha \in \mathbb{Z}_p$ is a Bernstein-Sato root of $\mathfrak{a}$ then there is some $i \in \{0, 1, \ldots, r(p-1)\}$ such that $p\alpha + i$ is also a Bernstein-Sato root of $\mathfrak{a}$.*

Proof. We use the alternative characterization of Bernstein-Sato roots given in Theorem IV.17. Pick a sequence $(\nu_e)_{e=0}^{\infty}$ such that $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ and such that α is the p-adic limit of ν_e. By Lemma V.7, for every e there is some $i_e \in \{0, 1, \ldots, r(p-1)\}$ such that $p\nu_e + i_e \in \nu_{\mathfrak{a}}^{\bullet}(p^{e+1})$. We conclude there is some $i \in \{0, 1, \ldots, r(p-1)\}$ and a subsequence (ν_{e_j}) such that $p\nu_{e_j} + i \in \nu_{\mathfrak{a}}^{\bullet}(p^{e_j+1})$. The p-adic limit of this sequence is $p\alpha + i$, and we conclude that $p\alpha + i$ is a Bernstein-Sato root. ☐

Lemma V.9. *Let $(R, \mathfrak{m}, k)$ be a regular F-finite local ring and consider the extension $(S, \mathfrak{n}, L)$ given by*

$$(S, \mathfrak{n}, L) = (R[x]_{\mathfrak{m}R[x]}, \mathfrak{m}R[x]_{\mathfrak{m}R[x]}, k(x)).$$

Then:

(i) *The extension* $(S, \mathfrak{n}, L)$ *is faithfully flat, regular, F-finite and local.*

(ii) *For every ideal* $\mathfrak{a} \subseteq R$ *we have* $\ell(\mathfrak{a}) = \ell(\mathfrak{a}S)$*, where* ℓ *denotes analytic spread.*

(iii) *For every integer* $e \geq 0$ *and every ideal* $\mathfrak{a} \subseteq R$ *we have* $\nu^\bullet_{\mathfrak{a}}(p^e) = \nu^\bullet_{\mathfrak{a}S}(p^e)$*.*

(iv) *We have* $\mathrm{BSR}(\mathfrak{a}) = \mathrm{BSR}(\mathfrak{a}S)$ *for every ideal* $\mathfrak{a} \subseteq R$*.*

Proof. Statement (i) is a standard fact, and for (ii) we refer to [HS06, Lemma 8.4.2]. For (iii) observe that, by Proposition II.58, we have $D^{(e)}_{R[x]} \cdot (\mathfrak{b}R[x]) = (D^{(e)}_R \cdot \mathfrak{b})R[x]$ for any ideal $\mathfrak{b} \subseteq R$ and therefore $D^{(e)}_S \cdot (\mathfrak{b}S) = (D^{(e)}_R \cdot \mathfrak{b})S$. The claim on ν-invariants then follows by faithful flatness. Part (iv) follows from (iii) and the alternative characterization of Bernstein-Sato roots given in Theorem IV.17. □

Theorem V.10. *Let* R *be a regular F-finite ring of characteristic* p *and* $\mathfrak{a} \subseteq R$ *be an ideal and let* ℓ *denote* $\ell = \max\{\ell(\mathfrak{a}R_{\mathfrak{m}}) \mid \mathfrak{m} \in \mathrm{Max}(R)\}$*. Then the Bernstein-Sato roots of* $\mathfrak{a}$ *are rational and lie in the interval* $[-\ell, 0]$*.*

Proof. We begin with the rationality. Recall that $\mathrm{BSR}(\mathfrak{a})$ denotes the set of Bernstein-Sato roots of $\mathfrak{a}$, and let $\widetilde{\mathrm{BSR}}(\mathfrak{a}) \subseteq \mathbb{Z}_p/\mathbb{Z}$ be its image under the quotient map $\mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z}$; in other words, $\widetilde{\mathrm{BSR}}(\mathfrak{a}) = \{\alpha + \mathbb{Z} \mid \alpha \in \mathrm{BSR}(\mathfrak{a})\}$. Note that, by Lemma V.8, $\widetilde{\mathrm{BSR}}(\mathfrak{a})$ is closed under multiplication by p.

Let $\alpha \in \mathbb{Z}_p$ be a Bernstein-Sato root. Since $\widetilde{\mathrm{BSR}}(\mathfrak{a})$ is a finite set, there exist some $n < m$ such that $p^n\alpha \equiv p^m\alpha \mod \mathbb{Z}$. We conclude that there exists some $c \in \mathbb{Z}$ such that $p^n\alpha = p^m\alpha + c$, and since p^n must divide c, we have $c = p^n c'$ for some $c' \in \mathbb{Z}$. It follows that $\alpha = c'/(p^{m-n} - 1) \in \mathbb{Z}_{(p)}$ as required.

Next we claim that if α is a Bernstein-Sato root we must have $\alpha \leq 0$. To see this, let $\alpha \in \mathrm{BSR}(\mathfrak{a})$ be the largest Bernstein-Sato root and suppose that $\alpha > 0$. Then, by Lemma V.8, we could find another root $p\alpha + i > \alpha$, giving a contradiction.

We now show that all Bernstein-Sato roots are no smaller than $-\ell$. By Proposition V.2 we can restrict to the local case, and by Lemma V.9, we may assume that the residue field of R is infinite. In this situation we can find a reduction $\mathfrak{b}$ of $\mathfrak{a}$ that is generated by ℓ elements, and by Lemma V.5 it suffices to show that all Bernstein-Sato roots of $\mathfrak{b}$ are no smaller than $-\ell$.

Let $\alpha \in \mathrm{BSR}(\mathfrak{b})$ be the smallest Bernstein-Sato root of $\mathfrak{b}$ and suppose that $\alpha < -\ell$ for a contradiction. By Lemma V.8 there is some $i \in \{0, 1, \dots, \ell(p-1)\}$ such that $p\alpha + i \in \mathrm{BSR}(\mathfrak{b})$, and we observe that

$$p\alpha + i \leq p\alpha + \ell(p-1) < p\alpha - \alpha(p-1) = \alpha,$$

thus giving a contradiction. $\square$

Recall that the roots of the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$ of a nonzero ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ are strictly negative. Since we have only shown that the Bernstein-Sato roots are nonpositive in positive characteristic, the question of whether zero can be a Bernstein-Sato root arises. It is easy to find a trivial example: for the zero ideal we have $\mathrm{BSR}((0)) = \{0\}$.

A reasonable hope is that zero can never arise as the Bernstein-Sato root of a nonzero ideal, but this is not quite correct if $\mathrm{Spec}(R)$ has multiple components. Because of the local nature of Bernstein-Sato roots (see Proposition V.2), if an ideal $\mathfrak{a} \subseteq R$ is zero on an irreducible component of $\mathrm{Spec}(R)$ then we will still have $0 \in \mathrm{BSR}(\mathfrak{a})$. We will show that, as long as $\mathfrak{a}$ is nonzero on every component of $\mathrm{Spec}(R)$ then zero cannot be a Bernstein-Sato root of $\mathfrak{a}$.

Definition V.11. We say that an ideal $\mathfrak{a} \subseteq R$ is *regular* if it contains a nonzerodivisor.

In particular, if R is a domain then $\mathfrak{a}$ is regular precisely when it is nonzero. In full generality, since R is regular it is a product of domains $R = R_1 \times R_2 \times \cdots \times R_k$, and thus $\mathrm{Spec}(R)$ is the disjoint union of the $\mathrm{Spec}\, R_i$. The ideal $\mathfrak{a}$ is then regular precisely when its restriction to every $\mathrm{Spec}(R_i)$ is nonzero.

Lemma V.12. *Let $\mathfrak{a} \subseteq R$ be a regular ideal. Then there is some d large enough so that $\mathcal{C}_R^{(d)} \cdot \mathfrak{a} = R$.*

Proof. The construction of $\mathcal{C}_R^{(d)} \cdot \mathfrak{a}$ is compatible with localization, so we may restrict to the case where $R = (R, \mathfrak{m})$ is local. Pick a nonzero element $g \in \mathfrak{a}$. By Krull's interchapter theorem there is some d large enough so that $g \notin \mathfrak{m}^{[p^d]}$ which, by Proposition II.44, is equivalent to $\mathcal{C}_R^{(d)} \cdot g \not\subseteq \mathfrak{m}$. We conclude that $R = \mathcal{C}_R^{(d)} \cdot g \subseteq \mathcal{C}_R^{(d)} \cdot \mathfrak{a}$, which proves the result. $\square$

Note that, by Proposition II.44, we also conclude that $D_R^{(d)} \cdot \mathfrak{a} = R$ and therefore $D_R \cdot \mathfrak{a} = R$.

The reader who is familiar with the concept of strongly F-regular rings will note that Lemma V.12 shows that a regular ring is strongly F-regular.

Lemma V.13. *Let $\mathfrak{a} \subseteq R$ be a regular ideal and $s \geq 0$ be an integer. Then there is some e large enough so that*

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{sp^e} = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{sp^e+1}.$$

This lemma follows easily from a stronger result of Blickle, Mustaţă and Smith [BMS08, Prop. 2.14]. In order to keep the discussion self-contained, we provide a proof by adapting their technique to our particular situation.

Proof. Note that, for every integer $e \geq 0$, we have

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{sp^e+1} = \mathcal{C}_R^{(e+1)} \cdot \mathfrak{a}^{(sp^e+1)[p]} \subseteq \mathcal{C}_R^{(e+1)} \cdot \mathfrak{a}^{sp^{e+1}+p} \subseteq \mathcal{C}_R^{(e+1)} \cdot \mathfrak{a}^{sp^{e+1}+1},$$

and therefore we get an increasing chain of ideals

$$\mathcal{C}_R^{(0)} \cdot \mathfrak{a}^{s+1} \subseteq \mathcal{C}_R^{(1)} \cdot \mathfrak{a}^{sp+1} \subseteq \cdots \subseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{sp^e+1} \subseteq \cdots$$

which, by noetherianity, stabilizes to some ideal I. We want to show that $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{sp^e} \subseteq I$ for large enough e or, equivalently, that $\mathfrak{a}^{sp^e} \subseteq I^{[p^e]}$ (see Corollary II.46). By Lemma V.12 there is some d large enough, some $g \in \mathfrak{a}$ and some $\varphi \in \mathcal{C}_R^{(d)}$ such that $\varphi(g) = 1$.

Let $f \in \mathfrak{a}^{sp^e}$ for e large. Note that

$$f^{p^d} g \in \mathfrak{a}^{sp^{e+d}+1} \subseteq D_R^{(e+d)} \cdot \mathfrak{a}^{sp^{e+d}+1} = I^{[p^{e+d}]} = I^{[p^e][p^d]},$$

where the penultimate equality follows from Corollary II.46. We conclude that

$$f = \varphi(f^{p^d} g) \in \mathcal{C}_R^{(d)} \cdot I^{[p^e][p^d]} = I^{[p^e]}$$

as required. □

Proposition V.14. *Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be a regular ideal. Then $0 \notin \mathrm{BSR}(\mathfrak{a})$.*

Proof. Let $r \geq 1$ be such that $\mathfrak{a}$ can be generated by r elements, and suppose that $0 \in \mathrm{BSR}(\mathfrak{a})$. Let $(\nu_e) \subseteq \mathbb{Z}_{\geq 0}$ be a sequence with $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$, $\nu_e \equiv 0 \mod p^e\mathbb{Z}_p$ and $0 \leq \nu_e < rp^e$ (see Remark IV.18). For every e there is some $s \in \{0, 1, \ldots, r-1\}$ such that $\nu_e = sp^e$, and therefore $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{sp^e} \neq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{sp^e+1}$. This contradicts Lemma V.13. □

V.5: Bernstein-Sato roots and F-jumping numbers

Let $\mathfrak{a}$ be an ideal in a domain R that is finitely generated over $\mathbb{C}$. There is a subtle relationship between the roots of the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$ of $\mathfrak{a}$ and the jumping numbers of the multiplier ideal of $\mathfrak{a}$: the log-canonical threshold $\mathrm{lct}(\mathfrak{a})$ of $\mathfrak{a}$ is the smallest root of $b_{\mathfrak{a}}(-s)$, and every jumping number in the interval $[\mathrm{lct}(\mathfrak{a}), \mathrm{lct}(\mathfrak{a})+1)$ is also a root of $b_{\mathfrak{a}}(-s)$ [BMS06a] [ELSV04].

This suggests that, when $\mathfrak{a}$ is an ideal in a regular F-finite ring R of characteristic p, there should be a relationship between the Bernstein-Sato roots of $\mathfrak{a}$ and the F-jumping numbers of $\mathfrak{a}$. This guess is further encouraged by the fact that we can obtain the Bernstein-Sato roots of $\mathfrak{a}$ from the sets $\nu^\bullet_{\mathfrak{a}}(p^e)$ of ν-invariants of $\mathfrak{a}$ (Theorem IV.17), and the fact that the ν-invariants of $\mathfrak{a}$ are also intimately linked to the F-jumping numbers of $\mathfrak{a}$ (in fact, they were first introduced by Mustaţă, Takagi and Watanabe in order to study these latter invariants [MTW05]).

We begin to explore this connection in the case where $\mathfrak{a}$ is principal, say $\mathfrak{a} = (f)$. In this case, the relationship is very strong: a theorem of Bitoun states that the Bernstein-Sato roots of f are (up to a sign) the F-jumping numbers of f that lie in $\mathbb{Z}_{(p)}$. Our first goal is to give a new proof of Bitoun's result by using the alternative characterization of Bernstein-Sato roots (Theorem IV.17), together with a result of Mustaţă, Takagi and Watanabe.

Lemma V.15. *Let $\alpha \in \mathbb{Z}_{(p)}$ be negative, and let $\alpha = \alpha_0 + p\alpha_1 + p^2\alpha_2 + \cdots$ be the p-adic expansion of α. Choose an integer $d > 0$ such that $\alpha(p^d - 1) \in \mathbb{Z}$. For all integers $e \geq 0$ large enough we have*

$$\alpha_0 + p\alpha_1 + \cdots + p^{ed-1}\alpha_{ed-1} = \alpha + p^{ed}(\lfloor \alpha \rfloor + 1 - \alpha).$$

Proof. Note that $\alpha_0 + p\alpha_1 + \cdots + p^{ed-1}\alpha_{ed-1}$ is the unique integer n with $0 \leq n < p^{ed}$ and $n \equiv \alpha \mod p^{ed}\mathbb{Z}_p$. The right hand side can be written as $-\alpha(p^{ed} - 1) + p^{ed}(\lfloor\alpha\rfloor + 1)$, and is therefore an integer. We also have $\alpha + p^{ed}(\lfloor\alpha\rfloor + 1 - \alpha) \equiv \alpha \mod p^{ed}\mathbb{Z}_p$. Since $0 < \lfloor\alpha\rfloor + 1 - \alpha \leq 1$, and since $\alpha < 0$ by assumption, we conclude that $0 \leq \alpha + p^{ed}(\lfloor\alpha\rfloor + 1 - \alpha) < p^{ed}$ for all e large enough. □

Lemma V.16 ([MTW05, Prop. 1.9])**.** *Suppose that λ is an F-jumping number of f, and let $J = \tau(f^\lambda)$. For all integers $e \geq 0$ we have $\nu^J_f(p^e) = \lceil \lambda p^e \rceil - 1$.*

Proof. We have

$$\begin{aligned}
\nu^J_f(p^e) &= \max\{n \geq 0 \mid (f^n) \not\subseteq \tau(f^\lambda)^{[p^e]}\} \\
&= \max\{n \geq 0 \mid \mathcal{C}^{(e)}_R \cdot f^n \not\subseteq \tau(f^\lambda)\} && \text{(Prop. II.44)} \\
&= \max\{n \geq 0 \mid \tau(f^{n/p^e}) \not\subseteq \tau(f^\lambda)\} && \text{(Prop. II.51)} \\
&= \max\{n \geq 0 \mid n/p^e < \lambda\} \\
&= \lceil \lambda p^e \rceil - 1.
\end{aligned}$$

□

Theorem V.17 ([Bit18])**.** *Let R be a regular F-finite ring and $f \in R$ be a nonzerodivisor. Then*

$$\mathrm{BSR}(f) = -\big(\mathrm{FJN}(f) \cap \mathbb{Z}_{(p)} \cap (0,1]\big).$$

Proof. Suppose that $\alpha \in \mathbb{Z}_p$ is a Bernstein-Sato root of f. By Theorem V.10 and Proposition V.14, we know that α is rational and lies in the interval $[-1, 0)$; our goal is to show that $-\alpha$ is an F-jumping number of f. Pick a sequence $(\nu_e) \subseteq \mathbb{Z}_{\geq 0}$ with $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ such that $\nu_e \equiv \alpha \mod p^e\mathbb{Z}_p$ and $0 \leq \nu_e < p^e$ (see Remark IV.18). It follows that ν_e is the e-th truncation of the p-adic expansion of α. If we fix an integer $d > 0$ so that $\alpha(p^d - 1) \in \mathbb{Z}$ then, by Lemma V.15, we have $\nu_{ed} = -\alpha(p^{ed} - 1)$ for all integers $e \geq 0$. Since $\nu_{ed} \in \nu_{\mathfrak{a}}^{\bullet}(p^{ed})$, we have

$$\tau(f^{-\alpha(p^{ed}-1)/p^{ed}}) = \mathcal{C}_R^{(ed)} \cdot \mathfrak{a}^{-\alpha(p^{ed}-1)} \neq \mathcal{C}_R^{(ed)} \cdot \mathfrak{a}^{-\alpha(p^{ed}-1)+1} = \tau(f^{(-\alpha(p^{ed}-1)+1)/p^{ed}})$$

(see Proposition II.51). We conclude that for every integer $e \geq 0$ there is an F-jumping number in the interval

$$\left(\frac{-\alpha(p^{ed}-1)}{p^{ed}}, \frac{-\alpha(p^{ed}-1)+1}{p^{ed}}\right] = \left(-\alpha - \frac{-\alpha}{p^{ed}}, -\alpha + \frac{1+\alpha}{p^{ed}}\right].$$

Since the sequence $-\alpha - (-\alpha)/p^{ed}$ increases to $-\alpha$ and the sequence $-\alpha + (1+\alpha)/p^{ed}$ decreases to $-\alpha$, we conclude that $-\alpha$ is an F-jumping number of f.

Suppose now that $\lambda \in (0, 1] \cap \mathbb{Z}_{(p)}$ is an F-jumping number of f, and we will show that $-\lambda$ is a Bernstein-Sato root of f. Pick some $d > 0$ large enough so that $\lambda(p^d - 1) \in \mathbb{Z}$. By Lemma V.16 we know that $\nu_{ed} := \lceil \lambda p^{ed} \rceil - 1$ is in $\nu_f^{\bullet}(p^{ed})$. We have $0 \leq 1 - \lambda < 1$, and note that $\lambda p^{ed} + (1 - \lambda) = \lambda(p^{ed} - 1) + 1$ is an integer. We conclude that $\nu_{ed} = \lambda(p^{ed} - 1)$ and, since the p-adic limit of this sequence is $-\lambda$, we conclude that $-\lambda$ is a Bernstein-Sato root of f from Theorem IV.17. ∎

We now explore the case where the ideal $\mathfrak{a}$ is not necessarily principal.

Lemma V.18. *Suppose that $\mathfrak{a}$ can be generated by r elements and let $\lambda > 0$ be an F-jumping number of $\mathfrak{a}$. Then for every e large enough there is some $k \in \{1, 2, \dots, r\}$ such that $\lceil \lambda p^e \rceil - k \in \nu_{\mathfrak{a}}^{\bullet}(p^i)$.*

Proof. We claim that for all integers $e, i \geq 0$ we have inclusions

$$\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil - r} \supseteq \mathcal{C}_R^{(e+i)} \cdot \mathfrak{a}^{(\lceil \lambda p^e \rceil - 1)p^i} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}.$$

To prove the first inclusion, we use Lemma II.5 to observe that we have a chain

$$\mathfrak{a}^{(\lceil \lambda p^e \rceil - r)[p^i]} \supseteq \mathfrak{a}^{(\lceil \lambda p^e \rceil - r)p^i + (r-1)(p^i - 1)} = \mathfrak{a}^{(\lceil \lambda p^e \rceil - 1)p^i + r - 1} \supseteq \mathfrak{a}^{(\lceil \lambda p^e \rceil - 1)p^i},$$

to which we apply $\mathcal{C}_R^{(e+i)}$; the claimed inclusion then follows by Corollary II.46. The second inclusion follows by observing

$$\mathfrak{a}^{(\lceil \lambda p^e - 1\rceil - 1)p^i} \supseteq \mathfrak{a}^{\lceil \lambda p^e \rceil p^i} \supseteq \mathfrak{a}^{\lceil \lambda p^e \rceil [p^i]},$$

and once again applying $\mathcal{C}_R^{(e+i)}$ and using Corollary II.46.

We now prove the statement of the lemma by contradiction. Suppose that there is some e large enough so that $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} = \tau(\mathfrak{a}^\lambda)$, and that $\lceil \lambda p^e \rceil - k \notin \nu_{\mathfrak{a}}^\bullet(p^e)$ for all $k \in \{1, 2, \dots, r\}$. Then we have $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil - r} = \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} = \tau(\mathfrak{a}^\lambda)$. The claim then entails that $\mathcal{C}_R^{(e+i)} \cdot \mathfrak{a}^{(\lceil \lambda p^e \rceil - 1)p^i} = \tau(\mathfrak{a}^\lambda)$ for all $i \geq 0$ and, by picking i large enough, we obtain $\tau(\mathfrak{a}^{(\lceil \lambda p^e \rceil - 1)/p^e}) = \tau(\mathfrak{a}^\lambda)$. Since $(\lceil \lambda p^e \rceil - 1)/p^e < \lambda$, we conclude that λ is not an F-jumping number. $\square$

Lemma V.19. *Let $s \geq 0$ be a stable exponent for $\mathfrak{a}$ and $e \geq 0$ be an integer. For every $n \in \nu_{\mathfrak{a}}^\bullet(p^{e+s})$ there is an F-jumping number of $\mathfrak{a}$ in the interval*

$$\left(\frac{\lfloor n/p^s \rfloor}{p^e}, \frac{\lceil (n+1)/p^s \rceil}{p^e} \right].$$

Proof. Consider the following chain of ideals

$$\mathcal{C}_R^{(e+s)} \cdot \mathfrak{a}^{\lfloor n/p^s \rfloor p^s} \supseteq \mathcal{C}_R^{(e+s)} \cdot \mathfrak{a}^n \supseteq \mathcal{C}_R^{(e+s)} \cdot \mathfrak{a}^{n+1} \supseteq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\lceil (n+1)/p^s \rceil p^s}.$$

The two middle ideals differ, and hence so do the outer ideals. Since s is stable, the ideal on the left is $\tau(\mathfrak{a}^{\lfloor n/p^s \rfloor / p^e})$ and the ideal on the right is $\tau(\mathfrak{a}^{\lceil (n+1)/p^s \rceil / p^e})$, which gives the result. $\square$

Theorem V.20. *Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be a regular ideal generated by r elements.*

(i) Suppose that α is a Bernstein-Sato root of $\mathfrak{a}$. Then there is some $m \in \{\lfloor \alpha \rfloor + 1, \lfloor \alpha \rfloor + 2, \dots, \lfloor \alpha \rfloor + r\}$ such that $m - \alpha$ is an F-jumping number of $\mathfrak{a}$.

(ii) Suppose that $\lambda \in \mathbb{Z}_{(p)}$ is an F-jumping number of $\mathfrak{a}$. Then there is some $m \in \{\lceil \lambda \rceil - r, \lceil \lambda \rceil - r + 1, \dots, \lceil \lambda \rceil - 1\}$ such that $m - \lambda$ is a Bernstein-Sato root of $\mathfrak{a}$.

Proof. Let α be a Bernstein-Sato root of $\mathfrak{a}$, and let $\alpha = \alpha_0 + p\alpha_1 + p^2\alpha_2 + \cdots$ be its p-adic expansion. By Theorem V.10 and Proposition V.14 we know that α is in $\mathbb{Z}_{(p)}$ and that it is negative. Pick an integer $d > 0$ such that $\alpha(p^d - 1) \in \mathbb{Z}$ and, by replacing it with a big multiple if necessary, assume that d is a stable exponent for $\mathfrak{a}$.

Let $(\nu_e) \subseteq \mathbb{Z}_{\geq 0}$ be a sequence with $\nu_e \in \nu_{\mathfrak{a}}^\bullet(p^e)$ and such that $\nu_e \equiv \alpha \mod p^e\mathbb{Z}_p$ and $0 \leq \nu_e < rp^e$ (see Remark IV.18). For every integer $e \geq 0$ there is some $k_e \in \{0, 1, \dots, r-1\}$

such that

$$\begin{aligned}\nu_{(e+1)d} &= \alpha_0 + p\alpha_1 + \cdots + p^{(e+1)d-1}\alpha_{(e+1)d-1} + p^{(e+1)d}k_e \\ &= \alpha + p^{(e+1)d}(\lfloor \alpha \rfloor + 1 + k_e - \alpha) \\ &= \alpha + p^{(e+1)d}(m_e - \alpha)\end{aligned}$$

where $m_e = \lfloor \alpha \rfloor + 1 + k_e \in \{\lfloor \alpha \rfloor + 1, \ldots, \lfloor \alpha \rfloor + r\}$. We conclude that there is some $m \in \{\lfloor \alpha \rfloor + 1, \ldots, \lfloor \alpha \rfloor + r\}$ and an increasing sequence (e_i) such that $\nu_{(e_i+1)d} = \alpha + p^{(e_i+1)d}(m - \alpha)$. By Lemma V.19, for every i there is an F-jumping number in the interval

$$\left(k + \frac{\lfloor -\alpha(p^{(e_i+1)d} - 1)/p^d \rfloor}{p^{e_i d}}, k + \frac{\lceil (-\alpha(p^{(e_i+1)d} - 1) + 1)/p^d \rceil}{p^{e_i d}}\right].$$

As i grows, both endpoints of the interval converge to $k - \alpha$, and thus $k - \alpha$ is an F-jumping number of $\mathfrak{a}$.

Suppose now that $\lambda \in \mathbb{Z}_{(p)}$ is an F-jumping number of $\mathfrak{a}$, and fix an integer $d > 0$ such that $\lambda(p^d - 1) \in \mathbb{Z}$. By Lemma V.18, for every e there is some $k_e \in \{1, 2, \ldots, r\}$ such that

$$\nu_{ed} := \lceil \lambda p^{ed} \rceil - k_e \in \nu_{\mathfrak{a}}^{\bullet}(p^{ed}).$$

Note that $\lceil \lambda p^e \rceil = \lceil \lambda(p^{ed} - 1) + \lambda \rceil = \lambda(p^{ed} - 1) + \lceil \lambda \rceil$, and thus $\nu_{ed} = \lambda(p^{ed} - 1) + m_e$ where $m_e := \lceil \lambda \rceil - k_e \in \{\lceil \lambda \rceil - 1, \ldots, \lceil \lambda \rceil - r\}$. We conclude that there is some $m \in \{\lceil \lambda \rceil - 1, \ldots, \lceil \lambda \rceil - r\}$ and an increasing sequence (e_i) such that $\nu_{e_i d} = \lambda(p^{e_i d} - 1) + m$. The p-adic limit of this sequence is $-\lambda + m$, and thus $-\lambda + m$ is a Bernstein-Sato root of $\mathfrak{a}$. $\square$

Corollary V.21. *We have an equality* $\mathrm{BSR}(\mathfrak{a}) + \mathbb{Z} = -\big(\mathrm{FJN}(\mathfrak{a}) \cap \mathbb{Z}_{(p)}\big) + \mathbb{Z}$ *of subsets of* $\mathbb{Z}_{(p)}$.

CHAPTER VI
Behavior under Mod-p Reduction

VI.1: General results

Let $A = \mathbb{Z}[a^{-1}]$ where $a > 0$ is an integer, and consider an ideal $\mathfrak{a} \subseteq A[x_1, \dots, x_n]$. We let $\mathfrak{a}_{\mathbb{C}}$ denote the expansion of $\mathfrak{a}$ to $\mathbb{C}[x_1, \dots, x_n]$ and, given a prime number p that does not divide a, we denote by $\mathfrak{a}_p$ the image of $\mathfrak{a}$ in $\mathbb{F}_p[x_1, \dots, x_n]$ ($\mathfrak{a}_p$ is called the mod-p reduction of $\mathfrak{a}$). The study of how the singularities of $\mathfrak{a}_{\mathbb{C}}$ reflect the singularities of $\mathfrak{a}_p$ for various p, and viceversa, constitutes a deep and interesting field of study. In this chapter we pursue this philosophy and study how the Bernstein-Sato polynomial of $\mathfrak{a}_{\mathbb{C}}$ and the Bernstein-Sato roots of $\mathfrak{a}_p$ are related to one another.

Let us set up some notation. The Bernstein-Sato polynomial of $\mathfrak{a}_{\mathbb{C}}$ will be denoted by $b_{\mathfrak{a}}(s)$ and, given a prime number p, an integer $e \geq 0$ and a proper ideal $J \subseteq \mathbb{Z}[x_1, \dots, x_n]$ that contains $\mathfrak{a}$ in its radical, we will simply write $\nu_{\mathfrak{a}}^J(p^e)$ for the invariant $\nu_{\mathfrak{a}}^J(p^e) = \nu_{\mathfrak{a}_p}^{J_p}(p^e)$ and, similarly, $\nu_{\mathfrak{a}}^{\bullet}(p^e)$ will denote the set of ν-invariants for $\mathfrak{a}_p$ of level e. We fix generators $\mathfrak{a} = (f_1, \dots, f_r)$ for the ideal $\mathfrak{a}$ and, for $B = A, \mathbb{C}$ or $\mathbb{F}_p$ (where $p \not| a$), we let

$$H_B := H^r_{(f_1 - t_1, \dots, f_r - t_r)} B[x_1, \dots, x_n][t_1, \dots, t_r],$$

realized via the Čech complex on the given generators; in the case $B = \mathbb{F}_p$, we will denote $H_{\mathbb{F}_p}$ by H_p, and similar usage of the subscript p will be used in the notation introduced below.

We let $\delta_B \in H_B$ denote the class of the element $(f_1 - t_1)^{-1} \cdots (f_r - t_r)^{-1}$. Note that H_A is free over A, and that there are isomorphisms $\mathbb{C} \otimes_A H_A \cong H_{\mathbb{C}}$ and $\mathbb{F}_p \otimes_A H_A \cong H_p$ which identify $1 \otimes \delta_A$ with $\delta_{\mathbb{C}}$ and δ_p respectively.

We let D_B denote the ring of B-linear differential operators on $B[x_1, \dots, x_n][t_1, \dots, t_r]$, which we equip with the $\mathbb{Z}$-grading for which $\deg x_i = 0$ and $\deg t_i = 1$. The subring of D_B consisting of homogeneous elements of degree zero will be denoted $(D_B)_0$. We also let $\{V^i D_B\}_{i \in \mathbb{Z}}$ denote the V-filtration on D_B with respect to the ideal $(t_1, \dots, t_r)$ (see Definition

II.59).

Recall that, since H_B is a local cohomology module of the D_B-module $B[x_1, \dots, x_n][t_1, \dots, t_n]$, it itself has a natural D_B-module structure (see Section II.3). The module N_B is then defined as

$$N_B = \frac{V^0 D_B \cdot \delta_B}{V^1 D_B \cdot \delta_B} = \frac{(D_B)_0 \cdot \delta_B}{(D_B)_0 \cdot \mathfrak{a}\delta_B}$$

(for the last equality, see the proof of Lemma IV.3). Recall that when $B = \mathbb{C}$ (resp. $B = \mathbb{F}_p$) the module N_B is the one used to define Bernstein-Sato polynomials (resp. Bernstein-Sato roots).

We begin with the following result of Mustaţă, Takagi and Watanabe.

Proposition VI.1 ([MTW05, Prop. 3.11]). *Let $\mathfrak{a} \subseteq A[x_1, \dots, x_n]$ be an ideal. For all p large enough and all $\nu \in \nu^\bullet_{\mathfrak{a}}(p^1)$ we have $b_{\mathfrak{a}}(\nu) \equiv 0 \mod p$.*

Recall that we have $\nu^\bullet_{\mathfrak{a}}(p^e) \subseteq \nu^\bullet_{\mathfrak{a}}(p^1)$ for all $e \geq 1$ (Corollary IV.14), and therefore we can also conclude that $b_{\mathfrak{a}}(\nu) \equiv 0 \mod p$ for all $\nu \in \nu^\bullet_{\mathfrak{a}}(p^e)$.

Proof. The polynomial $b_{\mathfrak{a}}(s)$ is the minimal polynomial of the operator $s = -\sum_{i=1}^r \partial_{t_i} t_i$ acting on the module $N_{\mathbb{C}} = (D_{\mathbb{C}})_0 \cdot \delta_{\mathbb{C}} / (D_{\mathbb{C}})_0 \cdot \mathfrak{a}\delta_{\mathbb{C}}$. We conclude that there exist some differential operators $\xi_i \in (D_{\mathbb{C}})_0$ such that we have the following equality in $H_{\mathbb{C}}$:

$$b_{\mathfrak{a}}(s) \cdot \delta_{\mathbb{C}} = \sum_{i=1}^r \xi_i \cdot f_i \delta_{\mathbb{C}}.$$

By [BMS06b, Prop. 2.1], the operators ξ_i have $\mathbb{Q}$-coefficients, so after further localizing A we may assume that this equation holds in H_A. Now let p be large enough so that p does not divide a and such that all the ξ_i have order $\leq p-1$. For each i, let $\xi_{i,p}$ denote the mod-p reduction of ξ_i (Corollary II.24). Since $\xi_{i,p}$ has order $\leq p-1$, we have $\xi_{i,p} \in (D_p)_0 \cap D_p^{(1)}$ (Proposition II.33). By reducing the above equation mod-p, we observe that $b_{\mathfrak{a}}(s) \cdot N_p^1 = 0$.

Suppose now that $\nu \in \nu^\bullet_{\mathfrak{a}}(p^1)$. By Proposition IV.15, we know that $(N_p^1)_\nu \neq 0$, and note that s acts by ν in the module $(N_p^1)_\nu$ (see Remark III.4). We conclude that $0 = b_{\mathfrak{a}}(s) \cdot (N_p^1)_\nu = b_{\mathfrak{a}}(\nu)(N_p^1)_\nu$ and therefore $b_{\mathfrak{a}}(\nu) = 0$ in $\mathbb{F}_p$. $\square$

This suggests the following method for finding roots of the Bernstein-Sato polynomial.

Corollary VI.2 ([MTW05, Rmk. 3.13]). *Let $J \subseteq A[x_1, \dots, x_n]$ be a proper ideal which contains $\mathfrak{a}$ in its radical, and let $e \geq 0$ be an integer. Suppose that there exist some integer M and a polynomial $P(t) \in \mathbb{Q}[t]$ such that $\nu^{J_p}_{\mathfrak{a}_p}(p^e) = P(p^e)$ whenever $p^e \equiv 1 \mod M$. Then $P(0)$ is a root of $b_{\mathfrak{a}}(s)$.*

Proof. By Dirichlet's theorem, there are infinitely many primes p with $p \equiv 1 \mod M$. Therefore, $b_{\mathfrak{a}}(P(0)) \equiv b_{\mathfrak{a}}(P(p^e)) \equiv 0 \mod p$ for infinitely many primes p, and thus $b_{\mathfrak{a}}(P(0)) = 0$. □

One can find such integer M and polynomial $P(t)$ in concrete examples (e.g. see Example VI.7 below, and [MTW05, §4]).

Proposition VI.1 also allows us to prove the following result.

Theorem VI.3. *Let $\mathfrak{a} \subseteq A[x_1, \dots, x_n]$ be an ideal. Suppose that $\alpha \in \mathbb{Q}$ is such that $\alpha \in \mathrm{BSR}(\mathfrak{a}_p)$ for infinitely many p. Then α is a root of $b_{\mathfrak{a}}(s)$.*

Proof. Let p be large enough so that it does not divide any of the denominators of the roots of $b_{\mathfrak{a}}(s)$. Then $b_{\mathfrak{a}}(s)$ defines a continuous function $b_{\mathfrak{a}} : \mathbb{Z}_p \to \mathbb{F}_p$, and in fact we have $b_{\mathfrak{a}} \in C^1(\mathbb{Z}_p, \mathbb{F}_p)$.

If p is such that $\alpha \in \mathrm{BSR}(\mathfrak{a}_p)$ then, by Theorem IV.17, there is some ν-invariant $\nu \in \nu_{\mathfrak{a}}^{\bullet}(p^1)$ such that $\nu \equiv \alpha \mod p$, and therefore $b_{\mathfrak{a}}(\alpha) \equiv b_{\mathfrak{a}}(\nu) \equiv 0 \mod p$ (the last congruence coming from Proposition VI.1). Since there are infinitely many such p by assumption, the result follows. □

VI.2: The ν-invariants of monomial ideals

Let A be a commutative ring and fix a nonzero monomial ideal $\mathfrak{a} \subseteq A[x_1, \dots, x_n]$. In this setting whenever J is a also a monomial ideal we can define the invariant $\nu_{\mathfrak{a}}^J(s)$ without passing to prime characteristic. First of all, given a monomial ideal $J \subseteq A[x_1, \dots, x_n]$ and a positive integer q (not necessarily a prime power) we define an ideal $J^{[q]}$ of $A[x_1, \dots, x_n]$ as follows:

$$J^{[q]} = (\mu^q : \mu \in J \text{ a monomial}).$$

If J is a monomial ideal containing $\mathfrak{a}$ in its radical we define

$$\nu_{\mathfrak{a}}^J(q) = \max\{n \geq 0 \mid \mathfrak{a}^n \not\subseteq J^{[q]}\}.$$

Observe that both of these notions recover the usual ones when A has prime characteristic p and q is a p-th power. When J is generated by powers of variables, the following description of $\nu_{\mathfrak{a}}^J(q)$ — taken from [BMS06a] — is useful for computations.

Remark VI.4 ([BMS06b])**.** For $j = 1, \dots, r$ pick monomials $\mu_j = \prod_{i=1}^n x_i^{c_{ij}} \in A[x_1, \dots, x_n]$, let $\mathfrak{a} = (\mu_1, \dots, \mu_r)$ be the ideal they generate and, for $i = 1, \dots, n$ consider the linear form $\ell_i(t) = \sum_{j=1}^r c_{ij} t_j$ on $\mathbb{Z}^r$. Given an integer $s \geq 0$, the ideal $\mathfrak{a}^s$ is generated by monomials

$x_1^{\ell_1(\underline{u})} \cdots x_n^{\ell(\underline{u})}$, where $\underline{u} \in (\mathbb{Z}_{\geq 0})^r$ is such that $|\underline{u}| = s$ (recall the notation $|\underline{u}| = u_1 + \cdots + u_r$). If $J \subseteq A[x_1, \ldots, x_n]$ is a monomial ideal of the form $J = (x_1^{a_1}, \cdots, x_n^{a_n})$, we have $\mathfrak{a}^s \not\subseteq J^{[q]}$ if and only if there is some $\underline{u}$ with $|\underline{u}| = s$ such that $\ell_i(\underline{u}) < a_i q$ for all $i = 1, 2, \ldots, n$. We conclude that

$$\nu_{\mathfrak{a}}^J(q) = \max\{|\underline{u}| \mid \ell_i(\underline{u}) \leq a_i q - 1 \ \forall i = 1, \ldots, n\}.$$

We now state two theorems from [BMS06b], which roughly say that the method suggested by Corollary VI.2 for finding the roots of the Bernstein-Sato polynomial works for monomial ideals. While the behavior illustrated below has been shown to also hold for some examples of hypersurfaces [MTW05, Chapter 4], monomial ideals exhibit remarkable behavior in two ways: in order to recover all the roots it suffices to take $p^e \equiv 1 \mod M$ large and for J to be a monomial ideal.

We state the theorems in a slightly weaker form which suffices for our purposes.

Theorem VI.5 ([BMS06b, Thm. 4.1])**.** *If $\mathfrak{a} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ is a nonzero monomial ideal then there is a positive integer M with the following property: if J is a monomial ideal whose radical contains $\mathfrak{a}$ then there are rational numbers $\beta > 0$ and η such that $\nu_{\mathfrak{a}}^J(q) = \beta q + \eta$ for all q large enough with $q \equiv 1 \mod M$.*

Observe that, by Corollary VI.2, the rational number η in Theorem VI.5 will be a root of $b_{\mathfrak{a}}(s)$.

Theorem VI.6 ([BMS06b, Thm. 4.9])**.** *Let $\mathfrak{a} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be a nonzero monomial ideal and α be a root of $b_{\mathfrak{a}}(s)$. Then there is a monomial ideal J together with a rational number β and a positive integer M such that $\nu_{\mathfrak{a}}^J(q) = \beta q + \alpha$ for all q large enough with $q \equiv 1 \mod M$.*

Let us illustrate these results with an example.

Example VI.7. Consider the ideal $\mathfrak{a} = (x_1^2 x_2 x_3, x_1 x_2^2 x_3, x_1 x_2 x_3^2)$ (see [BMS06b, Ex. 5.2]), and let $J = (x_1^3, x_2^3, x_3^3)$. We claim that for all integers $q > 0$ with $q \equiv 1 \mod 4$ we have $\nu_{\mathfrak{a}}^J(q) = \frac{9}{4}q - \frac{5}{4}$. Indeed, Remark VI.4 tells us that $\nu_{\mathfrak{a}}^J(q)$ is the maximum over $u_1 + u_2 + u_3$ such that

$$\begin{aligned} 2u_1 + u_2 + u_3 &\leq 3q - 1 \\ u_1 + 2u_2 + u_3 &\leq 3q - 1 \\ u_2 + u_2 + 2u_3 &\leq 3q - 1. \end{aligned}$$

Adding the three inequalities gives $u + u_2 + u_3 \leq \lfloor (9q-3)/4 \rfloor = (9q-5)/4$. Equality is proven by taking $u_1 = u_2 = (3q-3)/4$ and $u_3 = (3q+1)/4$. For this example, we have $b_{\mathfrak{a}}(s) = (s + \frac{3}{4})(s + \frac{5}{4})(s + \frac{6}{4})(s+1)^3$ and, indeed, we observe that $-5/4$ is a root of $b_{\mathfrak{a}}(s)$.

VI.3: Reduction mod-p for monomial ideals

In this chapter, we use the results of Budur, Mustaţă and Saito given above to show that mod-p reduction "works" for Bernstein-Sato roots of monomial ideals (see Theorem VI.11 below for a precise statement). We begin with two preliminary results.

Lemma VI.8. *Let $\mathfrak{a} \subseteq \mathbb{F}_p[x_1, \dots, x_n]$ be a monomial ideal and $e, m \geq 0$ be integers. If $\mathfrak{a}$ is generated in degrees $\leq D$ then $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^m$ is a monomial ideal generated in degrees $\leq \lfloor Dm/p^e \rfloor$.*

Proof. Follows from the description of $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^m$ given in proposition II.47. $\square$

Lemma VI.9. *Let A be a commutative ring and consider the polynomial ring $R = A[x_1, \dots, x_n]$. Given integers $b_1, \dots, b_n \geq 0$, consider the monomial $\mu = x_1^{b_1} \cdots x_n^{b_n}$ and the ideal $J = (x_1^{b_1+1}, \dots, x_n^{b_n+1})$. For all monomial ideals $I \subseteq R$, $\mu \in I$ if and only if $I \not\subseteq J$.*

Proof. The ($\Rightarrow$) direction is clear, since $\mu \notin J$. For ($\Leftarrow$), suppose $I \not\subseteq J$. This means that there exists some monomial $x_1^{a_1} \cdots x_n^{a_n}$ in I with $a_i \leq b_i$ for all i. By multiplying it with the appropriate monomial, we conclude $\mu \in I$. $\square$

One may think of the following result as a characteristic p analogue of Theorem VI.6.

Proposition VI.10. *Let $\mathfrak{a} \subseteq \mathbb{F}_p[x_1, \dots, x_n]$ be a nonzero monomial ideal, let α be a Bernstein-Sato root of $\mathfrak{a}$ and let $d > 0$ be an integer such that $\alpha(p^d - 1) \in \mathbb{Z}$. Then there is a monomial ideal J whose radical contains $\mathfrak{a}$, a rational number β and a sequence $e_i \nearrow \infty$ of positive integers such that*

$$\nu_{\mathfrak{a}}^J(p^{e_i d}) = \beta p^{e_i d} + \alpha.$$

Proof. We know that $\alpha \in \mathbb{Z}_{(p)}$ and that $\alpha < 0$ by Theorem V.10 and Proposition V.14. We pick some integer $d > 0$ such that $\alpha(p^d - 1) \in \mathbb{Z}$ and some integers r, D such that $\mathfrak{a}$ is generated by r monomials of degrees $\leq D$. Let (ν_e) be a sequence with $\nu_e \in \nu_{\mathfrak{a}}^\bullet(p^e)$, $\nu_e \equiv \alpha \mod p^e \mathbb{Z}_p$ and $0 \leq \nu_e < rp^e$ (see Remark IV.18).

For every e there is some $s_{ed} \in \{0, 1, \dots, r-1\}$ such that

$$\begin{aligned} \nu_{ed} &= \alpha_0 + p\alpha_1 + \cdots + p^{ed-1}\alpha_{ed-1} + p^{ed} s_{ed} \\ &= \alpha + (\lfloor \alpha \rfloor - \alpha + 1 + s_{ed}) p^{ed} \end{aligned}$$

(we use Lemma V.15 in the second equality).

The set $\{0, 1, \dots, r-1\}$ of possible values for s_{ed} is finite, so there is some constant $s \in \{0, 1, \dots, r-1\}$ and a subsequence $(\nu_{e_i d})$ such that $\nu_{e_i d} = \alpha + (\lfloor \alpha \rfloor - \alpha + s) p^{e_i d}$. Since $\nu_{e_i d} \in \nu_{\mathfrak{a}}^\bullet(p^{e_i d})$, we have

$$\mathcal{C}_R^{(e_i d)} \cdot \mathfrak{a}^{\nu_{e_i d}} \neq \mathcal{C}_R^{(e_i d)} \cdot \mathfrak{a}^{\nu_{e_i d}+1}$$

for every i. These are two monomial ideals which, by Lemma VI.8, are generated in degrees $\leq \lfloor D(\nu_{e_i d+1})/p^{e_i d} \rfloor \leq rD$. We conclude that, for every i, there is some monomial $\mu_{e_i d} \in \mathbb{F}_p[x_1, \dots, x_n]$ of degree $\leq rD$ such that $\mu_{e_i d} \in \mathcal{C}_R^{(e_i d)} \cdot \mathfrak{a}^{\nu_{e_i d}}$ and $\mu_{e_i d} \notin \mathcal{C}_R^{(e_i d)} \cdot \mathfrak{a}^{\nu_{e_i d}+1}$. There are only finitely many monomials of degree $\leq rD$, so we may pass to a further subsequence to assume that the monomials $\mu = \mu_{e_i d}$ are independent of i.

Suppose that $\mu = x_1^{b_1} \cdots x_n^{b_n}$, and consider the ideal $J = (x_1^{b_1+1}, \dots, x_n^{b_n+1})$. By Lemma VI.9 we have $\nu_{\mathfrak{a}}^J(p^{e_i d}) = \nu_{e_i d}$ and therefore $\nu_{\mathfrak{a}}^J(p^{e_i d}) = \alpha + (\lfloor \alpha \rfloor - \alpha + 1 + s)p^{e_i d}$. □

Theorem VI.11. *Let $\mathfrak{a} \subseteq \mathbb{Z}[x_1, \dots, x_n]$ be a monomial ideal. Then, for all primes p large enough, the set of Bernstein-Sato roots of $\mathfrak{a}_p$ coincides with the set of roots of $b_{\mathfrak{a}}(s)$.*

Proof. First, let α be a root of the $b_{\mathfrak{a}}(s)$. By Theorem VI.6 we may find a monomial ideal $J \subseteq \mathbb{Z}[x_1, \dots, x_n]$, a rational number $\beta \in \mathbb{Q}$ and an integer M such that $\nu_{\mathfrak{a}}^J(q) = \beta q + \alpha$ whenever q is large enough and $q \equiv 1 \mod M$. Observe that, by replacing M with a big multiple, M can be chosen independently of α, and we may also assume that $M\beta \in \mathbb{Z}$. Let p be a prime number that does not divide M and such that $\alpha \in \mathbb{Z}_{(p)}$. Then there exists some d such that $p^d \equiv 1 \mod M$ and therefore $\nu_{\mathfrak{a}}^J(p^{ed}) = \beta p^{ed} + \alpha$ for all $e > 0$. Since the p-adic limit of the sequence $(\beta p^{ed} + \alpha)_{e=0}^{\infty}$ is α, Theorem IV.17 implies that α is a Bernstein-Sato root of $\mathfrak{a}_p$.

We now prove the other containment. We let M be a number satisfying the conclusion of Theorem VI.5 for the ideal $\mathfrak{a}$, and pick p large enough so that it does not divide M. Suppose then that α is a Bernstein-Sato root of $\mathfrak{a}_p$, and we will show that α is a root of $b_{\mathfrak{a}}$.

We know that α is in $\mathbb{Z}_{(p)}$ and thus we may find some $d > 0$ such that $\alpha(p^d - 1) \in \mathbb{Z}$. By replacing d with a multiple, we may also assume that $p^d \equiv 1 \mod M$. By Proposition VI.10 we can find some monomial ideal J containing $\mathfrak{a}$ in its radical, a rational number β and a sequence $e_i \nearrow \infty$ such that $\nu_{\mathfrak{a}}^J(p^{e_i d}) = \beta p^{e_i d} + \alpha$. On the other hand, Theorem VI.6 says that there are some rational numbers β' and η such that $\nu_{\mathfrak{a}}^J(q) = \beta' q + \eta$ for all $q \equiv 1 \mod M$ large enough. We conclude that $\beta' = \beta$ and $\eta = \alpha$ and, by Corollary VI.2, α is a root of $b_{\mathfrak{a}}(s)$. □

Recall that, in positive characteristic, the analytic spread provides a lower bound on the Bernstein-Sato roots of an ideal (Theorem V.10). In characteristic zero the analogous result is false in general[1], but we can use Theorem VI.6 to show that such a bound holds for monomial ideals.

[1]There are principal ideals $(f) \subseteq \mathbb{C}[x_1, \dots, x_n]$ whose Bernstein-Sato polynomial $b_f(s)$ has roots $\lambda < -1$; for example $f = x^2 + y^3$ and $\lambda = -7/6$.

Corollary VI.12. *Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring over $\mathbb{C}$, $\mathfrak{m} \subseteq R$ be the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$, and $\mathfrak{a} \subseteq R$ be a monomial ideal. For every root λ of the Bernstein-Sato polynomial of $\mathfrak{a}$ we have $-\ell(\mathfrak{a}R_{\mathfrak{m}}) \leq \lambda$.*

Proof. Let $\mathfrak{a}_{\mathbb{Z}} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be the corresponding monomial ideal over $\mathbb{Z}$; more precisely, $\mathfrak{a}_{\mathbb{Z}}$ is generated by those monomials μ for which $\mu \in \mathfrak{a}$. Given a ring B we let $R_B = B[x_1, \ldots, x_n]$ be the polynomial ring over B, $\mathfrak{m}_B \subseteq R_B$ be the ideal $\mathfrak{m}_B = (x_1, \ldots, x_n)$, and $\mathfrak{a}_B \subseteq R_B$ be the expansion of $\mathfrak{a}_{\mathbb{Z}}$. We let $\mathcal{F}_B$ be the graded ring

$$\mathcal{F}_B = \frac{R_B}{\mathfrak{m}_B} \oplus \frac{\mathfrak{a}_B}{\mathfrak{m}_B \mathfrak{a}_B} \oplus \frac{\mathfrak{a}_B^2}{\mathfrak{m}_B \mathfrak{a}_B^2} \oplus \cdots$$

and ℓ_B be its Krull dimension. Note that when B is a field $\mathfrak{m}_B$ is a maximal ideal and ℓ_B is the analytic spread of $\mathfrak{a}_B(R_B)_{\mathfrak{m}_B}$ (see Section V.3). Given a prime number p we let R_p denote $R_p = R_{\mathbb{F}_p}$, and we similarly define $\mathfrak{m}_p$, $\mathfrak{a}_p$, $\mathcal{F}_p$ and ℓ_p.

We claim that we have $\ell_{\mathbb{C}} = \ell_p$ for all primes p large enough[2]. Indeed, if $\mathfrak{b} \subseteq R_{\mathbb{Z}}$ is a monomial ideal then $\mathfrak{b} = \bigoplus_{\mu} \mathbb{Z}\mu$ where the sum ranges over all monomials μ contained in $\mathfrak{b}$. In particular, $\mathfrak{b}$ is free over $\mathbb{Z}$ and the natural map $B \otimes_{\mathbb{Z}} \mathfrak{b} \to \mathfrak{b}_B$ is an isomorphism (where $\mathfrak{b}_B$ is the expansion of $\mathfrak{b}$ to R_B). We conclude that there is a ring isomorphism $\mathcal{F}_B \cong B \otimes_{\mathbb{Z}} \mathcal{F}_{\mathbb{Z}}$ for every ring B. Therefore, the generic fibre of the morphism $\operatorname{Spec} \mathbb{F}_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is $\operatorname{Spec} \mathcal{F}_{\mathbb{Q}}$, and that the fibre over a prime p is $\operatorname{Spec} \mathcal{F}_p$. It follows that $\ell_{\mathbb{Q}} = \ell_p$ for all p large enough [GW10, Prop. 10.95] and, since $\mathcal{F}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} \mathcal{F}_{\mathbb{Q}}$, we also have $\ell_{\mathbb{Q}} = \ell_{\mathbb{C}}$ [GW10, Prop. 5.38].

By Theorem VI.11, and the claim above, there is some p large enough so that λ is a Bernstein-Sato root of $\mathfrak{a}_p$ and such that $\ell_p = \ell_{\mathbb{C}}$. Since $\mathfrak{a}_p$ is homogeneous, λ is a Bernstein-Sato root of $\mathfrak{a}_p(R_p)_{\mathfrak{m}_p}$ (Proposition V.4), and by Theorem V.10 we conclude that $-\ell_p \leq \lambda$. ☐

VI.4: Examples of monomial ideals in small characteristics

To finish we would like to illustrate the behavior in small characteristics by computing some examples. Let us remark that both of the examples below exhibit the following behavior: the Bernstein-Sato roots of $\mathfrak{a}_p$ are always roots of $b_{\mathfrak{a}}(s)$ and, moreover, they are precisely the roots that lie in $\mathbb{Z}_{(p)}$. We do not know any example where this is not the case. To make these computations we will use the description of the ν-invariants given in Remark VI.4, the description of Bernstein-Sato roots given in Theorem IV.17, and the following result.

[2] In fact, a result of Bivià-Ausina, the analytic spread of a monomial ideal depends only on the Newton polygon of $\mathfrak{a}$, and is therefore independent of the ground field [BA03] (see also [Sin07, Cor. 4.10]). We thus have $\ell_{\mathbb{C}} = \ell_p$ for all primes p.

Lemma VI.13. *Let $\mathfrak{a} \subseteq \mathbb{F}_p[x_1, \ldots, x_n]$ be a monomial ideal. For all $\nu \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$ there is a monomial ideal J of the form $J = (x_1^{a_1}, \ldots, x_n^{a_n})$ such that $\nu = \nu_{\mathfrak{a}}^J(p^e)$.*

Proof. We have $\mathcal{C}_R^{(e)} \cdot \mathfrak{a}^\nu \neq \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\nu+1}$. Since both of these are monomial ideals, there is a monomial $\mu = x_1^{b_1} \cdots x_n^{b_n}$ such that $\mu \in \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^\nu$ and $\mu \notin \mathcal{C}_R^{(e)} \cdot \mathfrak{a}^{\nu+1}$. By Lemma VI.9, we get $\nu = \nu_{\mathfrak{a}}^J(p^e)$ where $J = (x_1^{b_1+1}, \ldots, x_n^{b_n+1})$. □

Example VI.14. Consider the ideal $\mathfrak{a} = (x_1^2, x_2^3)$. In this case, using computational software [LT], we find:

$$b_{\mathfrak{a}}(s) = (s + \frac{5}{6})(s + \frac{7}{6})(s + \frac{4}{3})(s + \frac{3}{2})(s + \frac{5}{3})(s + 2).$$

We compute the ν-invariants of $\mathfrak{a}$ and, by Lemma VI.13, it suffices to compute all $\nu_{\mathfrak{a}}^J(p^e)$ for $J = (x_1^{a_1}, x_2^{a_2})$. For such a J we have, by Remark VI.4,

$$\begin{aligned} \nu_{\mathfrak{a}}^J(p^e) &= \max\{u_1 + u_2 \mid 2u_1 \leq a_1 p^e - 1, 3u_2 \leq a_2 p^e - 1\} \\ &= \left\lfloor \frac{a_1 p^e - 1}{2} \right\rfloor + \left\lfloor \frac{a_2 p^e - 1}{3} \right\rfloor \end{aligned}$$

and therefore

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) = \left\{ \left\lfloor \frac{a_1 p^e - 1}{2} \right\rfloor + \left\lfloor \frac{a_2 p^e - 1}{3} \right\rfloor \;\middle|\; a_1, a_2 \in \mathbb{Z}_{>0} \right\}.$$

We begin by noting that the following hold for all $a_1, a_2 \in \mathbb{Z}_{>0}$, where we always take $c_1, c_2 \in \mathbb{Z}_{>0}$:

- If $p^e \equiv 0 \mod 2$ then

$$\left\lfloor \frac{a_1 p^e - 1}{2} \right\rfloor = \frac{1}{2} a_1 p^e - 1.$$

- If $p^e \equiv 1 \mod 2$ then

$$\left\lfloor \frac{a_1 p^e - 1}{2} \right\rfloor = \begin{cases} c_1 p^e - 1 \text{ if } a_1 = 2c_1 \\ (c_1 - \frac{1}{2})p^e - \frac{1}{2} \text{ if } a_1 = 2c_1 - 1. \end{cases}$$

- If $p^e \equiv 0 \mod 3$ then

$$\left\lfloor \frac{a_2 p^e - 1}{3} \right\rfloor = \frac{1}{3} a_2 p^e - 1.$$

- If $p^e \equiv 1 \mod 3$ then

$$\left\lfloor \frac{a_2p^e - 1}{3} \right\rfloor = \begin{cases} c_2p^e - 1 \text{ if } a_2 = 3c_2 \\ (c_2 - \frac{1}{3})p^e - \frac{2}{3} \text{ if } a_2 = 3c_2 - 1 \\ (c_2 - \frac{2}{3})p^e - \frac{1}{3} \text{ if } a_2 = 3c_2 - 2. \end{cases}$$

(i) Suppose that $p = 2$ and that e is even, so that $p^e \equiv 0 \mod 2$ and $p^e \equiv 1 \mod 3$. We have:

$$\nu_{\mathfrak{a}}^{\bullet}(2^e) = \left\{a_1 2^{e-1} + c_2\ 2^e - 2\right\}_{a_1,c_2=1}^{\infty} \cup \left\{a_1 2^{e-1} + (c_2 - \frac{1}{3})2^e - \frac{5}{3}\right\}_{a_1,c_2=1}^{\infty}$$
$$\cup \left\{a_1 2^{e-1} + (c_2 - \frac{2}{3})2^e - \frac{4}{3}\right\}_{a_1,c_2=1}^{\infty}$$

and therefore $\mathrm{BSR}(\mathfrak{a}) = \{-4/3, -5/3, -2\}$.

(ii) When $p = 3$ and $e \geq 1$ we have $p^e \equiv 1 \mod 2$ and $p^e \equiv 0 \mod 3$, and therefore

$$\nu_{\mathfrak{a}}^{\bullet}(3^e) = \left\{c_1\ 3^e + a_2 3^{e-1} - 2\ \right\}_{c_1,a_2=1}^{\infty} \cup \left\{(c_1 - \frac{1}{2})3^e + a_2 3^{e-1} - \frac{3}{2}\right\}_{c_1,a_2=1}^{\infty},$$

which gives $\mathrm{BSR}(\mathfrak{a}) = \{-3/2, -2\}$.

(iii) Suppose that $p \geq 5$ and that e is such that $p^e \equiv 1 \mod 6$. Then

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) = \left\{(c_1 + c_2)p^e - 2\right\}_{c_1,c_2=1}^{\infty} \cup \left\{(c_1 + c_2 - \frac{1}{2})p^e - \frac{3}{2}\right\}_{c_1,c_2=1}^{\infty}$$
$$\cup \left\{(c_1 + c_2 - \frac{1}{3})p^e - \frac{5}{3}\right\}_{c_1,c_2=1}^{\infty} \cup \left\{(c_1 + c_2 - \frac{5}{6})p^e - \frac{7}{6}\right\}_{c_1,c_2=1}^{\infty}$$
$$\cup \left\{(c_1 + c_2 - \frac{5}{3})p^e - \frac{4}{3}\right\}_{c_1,c_2=1}^{\infty} \cup \left\{(c_1 + c_2 - \frac{7}{6})p^e - \frac{5}{6}\right\}_{c_1,c_2=1}^{\infty},$$

and therefore $\mathrm{BSR}(\mathfrak{a}) = \{-5/6, -7/6, -4/3, -3/2, -5/3, -2\}$, in agreement with Theorem VI.11.

Example VI.15. Let $\mathfrak{a} = (x_2x_3, x_1x_3, x_1x_2)$. By again using [LT] we find that

$$b_{\mathfrak{a}} = (s + 2)^2(s + \frac{3}{2}).$$

In this case we have $\ell_1(t_1,t_2,t_3)=t_2+t_3$, $\ell_2(t_1,t_2,t_3)=t_1+t_3$ and $\ell_3(t_1,t_2,t_3)=t_1+t_2$. For $J=(x_1^{a_1},x_2^{a_2},x_3^{a_3})$, Remark VI.4 tells us that $\nu_{\mathfrak{a}}^J(p^e)$ is the largest $u_1+u_2+u_3$ where

$$\begin{aligned} u_2+u_3 &\le a_1p^e-1\\ u_1+u_3 &\le a_2p^e-1\\ u_1+u_2 &\le a_3p^3-1. \end{aligned}$$

By adding the first two inequalities, we observe that $\nu_{\mathfrak{a}}^J(p^e)\le(a_1+a_2)p^e-2$, and we obtain similar inequalities by considering the other two pairs. By adding all three inequalities we also see that $\nu_{\mathfrak{a}}^J(p^e)\le\lfloor((a_1+a_2+a_3)p^e-3)/2\rfloor$, and thus

$$\nu_{\mathfrak{a}}^J(p^e)\le\min\left\{(a_1+a_2)p^e-2,\ (a_1+a_3)p^e-2,\ (a_2+a_3)p^e-2,\ \left\lfloor\frac{(a_1+a_2+a_3)p^e-3}{2}\right\rfloor\right\}.$$

We claim that this is an equality, and let us first prove it in the case where the minimum is $(a_1+a_2)p^e-2$. In this case have $(a_1+a_2)p^e-2\le((a_1+a_2+a_3)p^e-3)/2$, which yields $(a_1+a_2)p^e\le a_3p^e+1$, and the claim follows by taking $u_1=a_2p^e-1$, $u_2=a_1p^e-1$ and $u_3=0$. The cases where the minimum is $(a_1+a_3)p^e-2$ or $(a_2+a_3)p^e-2$ follow by symmetry.

We now consider the case where the minimum is $\lfloor((a_1+a_2+a_3)p^e-3)/2\rfloor$, in which case we have $((a_1+a_2+a_3)p^e-3)/2\le(a_1+a_2)p^e-2$, which yields $a_3p^e+1\le(a_1+a_2)p^e$; similarly, we obtain $a_2p^e+1\le(a_1+a_3)p^e$ and $a_1p^e+1\le(a_2+a_3)p^e$. When $(a_1+a_2+a_3)p^e-3$ is divisible by 2, the claim follows by considering $u_1=((-a_1+a_2+a_3)p^e-1)/2$, $u_2=((a_1-a_2+a_3)p^e-1)/2$ and $u_3=((a_1+a_2-a_3)p^e-1)/2$; when $(a_1+a_2+a_3)p^e-3$ is not divisible by 2 we can take $u_1=((-a_1+a_2+a_3)p^e-2)/2$, $u_2=((a_1-a_2+a_3)p^e-2)/2$ and $u_3=(a_1+a_2-a_3)p^e/2$.

(i) Suppose that $p=2$. By taking $a_1=a_2=a_3=1$ we see that $(3p^e-3)/2\in\nu_{\mathfrak{a}}^\bullet(p^e)$ for all e large , and therefore $-2\in\mathrm{BSR}(\mathfrak{a})$. On the other hand, observe that $\lfloor((a_1+a_2+a_3)p^e-3)/2\rfloor=(a_1+a_2+a_3)2^{e-1}-2$, and thus every $\nu\in\nu_{\mathfrak{a}}^\bullet(p^e)$ has the form $\nu=k2^{e-1}-2$ for some integer k. We conclude that $\mathrm{BSR}(\mathfrak{a})=\{-2\}$.

(ii) Suppose that $p>2$. By taking $a_1=a_2=1$ and $a_3=10^{10}$, we observe that $2p^e-2\in\nu_{\mathfrak{a}}^\bullet(p^e)$, giving $-2\in\mathrm{BSR}(\mathfrak{a})$. By taking $a_1=a_2=a_3=1$, we get $(3p^e-3)/2\in\nu_{\mathfrak{a}}^\bullet(p^e)$ for e large, giving $-3/2\in\mathrm{BSR}(\mathfrak{a})$. On the other hand, all $\nu\in\nu_{\mathfrak{a}}^\bullet(p^e)$ have the form $\nu=kp^e-2$ or $\nu=(kp^e-3)/2$ for some integer k, and therefore $\mathrm{BSR}(\mathfrak{a})=\{-3/2,-2\}$. Once again, we observe the agreement with Theorem VI.11

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