Dressings and Asymptotic Symmetries in Quantum Field Theory

by

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List of Abbreviations

BCH Baker-Campbell-Hausdorff

BMS Bondi-van der Burg-Metzner-Sachs

BRST Becchi-Rouet-Stora-Tyutin

dS de Sitter spacetime

FK Faddeev-Kulish

IR infrared

JHEP Journal of High Energy Physics

LGT large gauge transformation

LHS left-hand side

NP Newman-Penrose

PRD Physical Review D

QED quantum electrodynamics

RHS right-hand side

Abstract

In quantum field theories with massless gauge bosons, the conventional formulation of scattering amplitudes in terms of momentum eigenstates lead to infrared divergences. To resolve this one should dress charged particles with an infinite number of low-energy (soft) gauge bosons. These dressings are closely related to the asymptotic symmetries of the theory. The aim of this dissertation is to investigate this relation in detail, and study the implications of dressings in a quantum field theoretical context in asymptotically flat and Schwarzschild spacetimes. In particular, the asymptotic symmetry of interest in gravity is the BMS supertranslation. First, we demonstrate in perturbative quantum gravity that the dressings facilitate charge conservation by carrying a definite supertranslation charge. We then use this property to derive dressed states from the charge conservation laws of asymptotic symmetries. We develop a unified quantum mechanical framework for the construction of dressings at the null infinity and the horizon of a Schwarzschild black hole, and show that the black hole soft hairs are planted on the horizon by dressed particles falling into the black hole. This framework is then extended to the magnetic parity soft charges of electromagnetism and gravity. It is shown that one can construct 't Hooft line dressings at the asymptotic boundaries, which are charged under the magnetic large gauge transformations and the dual supertranslations. Finally, we study the standard and dual supertranslation charges at the black hole horizon. We find central terms in their algebra in the presence of singularities in the parameter functions, which hints at further interesting soft structures yet to be revealed.

Chapter 1

Introduction

1.1 Dressings and the infrared

In quantum electrodynamics (QED), a charged particle may emit an infinite number of low-energy (or soft) photons with a finite total energy below the energy resolution of detectors. This gives rise to divergences in the S-matrix elements of real emission processes, which are referred to as infrared divergences. It was soon shown that the infrared-divergent contributions can be factored, order by order in the perturbation theory, and exponentiated [2]. The exponentiation of divergent factors implies that the corresponding S-matrix element vanishes as one removes the infrared cut-off. Since any charged particle may emit soft photons, the S-matrix elements for all processes in QED vanish. While this is problematic, it poses no problem in physical predictions. In practice, one sums up physically indistinguishable (inclusive) cross-sections, order by order in perturbation theory, which leads to cancellations of infrared-divergent factors. The infrared cutoff is replaced by the detector resolutions, and the inclusive cross-sections are well-defined even after one removes the infrared cut-offs. This method is referred to as the Bloch-Nordsieck resummation [3], and it is known to also work in perturbative quantum gravity and for appropriate processes in non-abelian gauge theories.

Although physically sensible predictions can be made using the Bloch-Nordsieck resummation, it is troubling that the S-matrix in gauge and gravitational theories in the standard Fock (momentum) basis is not well-defined. Thus, it is natural to ask if this is a feature deriving from the physical nature of the problem, or whether there exists a formalism in which the S-matrix can be well-defined. Chung [4] has shown that one can choose coherent states which contain not only charged particles but also an infinite set of photons, and therefore different from the set of conventional Fock states, superposed in such a way that the infrared-divergent factors cancel out, leaving the S-matrix elements well-defined. Soon afterwards, Kibble [5–7] introduced, based on the asymptotic behavior of charged particle Green's functions near the mass shell, a large non-separable space of asymptotic states in the basis of which the S-matrix can be well-defined as a unitary operator.

Building on their work, Faddeev and Kulish [8] developed a formalism which allows for the construction of appropriately defined S-matrix elements. This takes into account the fact that the early and late time dynamics in theories with massless gauge particles cannot be free. For example, in QED, they observed that there are terms in the interaction Hamiltonian that arise from the coupling of soft photons to creation or annihilation operators of a charged particle and give a non-vanishing contribution in the limit $t \to \pm\infty$. The proposed solution was to construct true asymptotic states which include multiple soft particle emissions to all orders in the coupling constant. Physically, these states describe dressing of the charged particles by soft photon clouds. The standard Dyson S-matrix between these asymptotic states is then free of infrared singularities. This Faddeev-Kulish method was extended to perturbative quantum gravity in [9]. It should be noted that the Faddeev-Kulish construction is valid in QED only for massive charged particles. In contrast, perturbative quantum gravity does not suffer from this restriction because of the cancellation of collinear divergences [10]. In this sense, the infrared behavior of gravity is simpler than that of QED.

Studies of the large time structure of gauge and gravitational theories have recently seen a resurgence following the original work of [11–13] on asymptotically flat spacetimes. In particular, an important realization has emerged that the infrared sector of these theories is governed by an infinite dimensional symmetry group generated by large gauge transformations (LGT) [14–17]. In perturbative gravity, for example, there are infinitely degenerate vacua which differ by the addition of soft gravitons and are related by spontaneously broken Bondi-van der Burg-Metzner-Sachs (BMS) symmetries. The infinite number of conservation laws associated with the BMS supertranslations forbids the transitions between equivalent vacua, and this is interpreted (for the case of QED see [18,19]) as the reason of the vanishing of the Fock space S-matrix elements for transitions involving soft gravitons. We have seen that the Faddeev-Kulish (FK) asymptotic states were introduced to precisely take care of this problem. Since the Dyson S-matrix is finite between the FK asymptotic states, the question that naturally arises is, what is the relation between the FK asymptotic operator which generates the asymptotic states and the BMS supertranslations? For the aforementioned interpretation to be valid, it must be that the FK dressings implicitly induce transitions between the degenerate vacua. This interpretation has been explicitly verified for the case of QED in [18] and [19]. Related discussions have appeared in [20–22], where the authors study the factorization of the hard and soft sectors in a scattering experiment, and the relation to BMS transformations, with a special emphasis on the application of their results to the black hole information paradox. Also, the relation between the soft and hard sectors was formulated using information-theoretic tools in [23], see also [24]. Since then, there has been an extensive analysis of asymptotic symmetries of gauge theories and soft theorems, see [25–53] for examples of some earlier work.

It is clear that there exists a close relationship between the Faddeev-Kulish dressings, the asymptotic symmetries and the infrared structures of quantum field theories. It is at this interface wherein the motivation of this dissertation lies. The Faddeev-Kulish dressings present a new perspective on the study of long-distance interactions in field theories. The asymptotic symmetries, while being classical, lead to rich structures in the quantum theory, and the dressings serve a natural tool to study such structures. Initially, there was hope that the BMS black hole soft hairs have the potential to resolve the black hole information paradox. But since then, there has been various works (see for example [54]) which show that BMS supertranslations do not have obvious connections to Hawking radiations. However, there are recent papers [55–57] that suggest soft near-horizon physics can have nontrivial bearings on the information paradox. We believe that the results presented in this dissertation is relevant to this active field of research. Let us summarize the contents of this dissertation below.

In chapter 2, we show that the ill-defined nature of scattering amplitudes in perturbative quantum gravity is tied to a transition between the degenerate vacua, and demonstrate how the dressings encapsulate the correct vacuum transition. Moreover, we show that the cancellation of infrared divergences in perturbative quantum gravity is closely related to the conservation law of BMS charge. This chapter is based on the paper *Asymptotic dynamics in perturbative quantum gravity and BMS supertranslations* with Ratindranath Akhoury and Uri Kol that has been published in JHEP [58].

In chapter 3, we observe that the dressed states of gravity in scattering amplitudes act essentially as BMS charge eigenstates. This allows us to re-derive the dressed states by introducing the BMS charge as a new quantum number. Using this, we prove a conjecture by Strominger and his collaborators [19] which asserts that BMS symmetry implies cancellation of infrared divergence. This chapter is based on the paper *BMS supertranslation symmetry implies Faddeev-Kulish amplitudes* with Ratindranath Akhoury, published in JHEP [59].

In chapter 4, we explore the possible implications of dressings on the black hole horizon. The degeneracy of vacuum in QED and gravity can be interpreted as the vacuum containing information encoded by the low-energy photons and gravitons, also referred to as the soft hair. Building on this, Hawking, Perry and Strominger [60, 61] proposed that the Schwarzschild black holes carry soft hair, and that this may bear non-trivial implications on the fate of information stored in evaporating black holes. They showed in the classical theory that BMS supertranslation (a subset of the BMS symmetry) leads to a non-trivial structure on the Schwarzschild horizon, exhibiting low-energy graviton degrees of freedom localized on the horizon. Along with some of the earlier work (for example [62]), this led to numerous investigations on the effects of non-trivial diffeomorphisms acting on black hole horizons [63–85,85–89,89,90]. This structure is actually a characteristic of horizon dressings: in a quantum field theory, particles falling into the horizon are accompanied by dressings, and these dressings show up as soft hairs on the horizon. Therefore, deriving the dressings yields a quantum-field-theoretical handle on the black hole soft hairs, thereby paving the way to studying microstates responsible for black hole entropy. In order to extend the dressing construction to curved spacetime, we adopt the work of Jakob and Stefanis [91] to express the dressing as the Wilson lines. We consider photons in a Rindler spacetime, which can be viewed as the near-horizon geometry of a Schwarzschild black hole. We obtain the photon dressings on the Rindler horizon, and noticed that the horizon has the low-energy photon degrees of freedom analogous to the results of Hawking, Perry and Strominger. Chapter 4 is based on the paper *Soft photon hair on Schwarzschild horizon from a Wilson line perspective* with Ratindranath Akhoury that has been published in JHEP [92].

In chapter 5, we demonstrate that asymptotic particles falling into the black hole leave behind a soft graviton hair on the horizon, by constructing gravitational dressings on the black hole horizon in the context of perturbative quantum gravity in a Schwarzschild background. This extends the result of chapter 4 to gravity, with the crucial difference that we work directly in a Schwarzschild background instead of a Rindler wedge. To this end, we quantize the metric perturbation as in [93,94]. We observe that the work of Jakob and Stefanis [91] extends to gravity in flat background without difficulty, and adopt Mandelstam's point of view [95] to construct Faddeev-Kulish dressings as gravitational Wilson lines along the geodesic of a massive, radially infalling scalar matter field. The gravitational Wilson line in curved background is taken to be a straightforward generalization of that in flat background, see [9] for instance. It is shown that the dressing thus constructed carries a definite soft supertranslation charge parametrized by the mass and energy of the matter particle being dressed, in accordance with the case of flat spacetime. More explicitly, the dressing operator for a radially infalling particle of mass m and energy E acting on a black hole state with no hair, creates a black hole with soft hair. This state is characterized by the energy E and mass m through the ratio m^2/E and by the spherical angles on the future boundary of the future horizon. This should be contrasted with the case of soft hair on the asymptotic infinities $\mathcal{I}^{\pm}$, where the corresponding state is labeled by a charge parametrized only by the momentum of the asymptotic particle (see section 5.1.2). Chapter 5 is based on the paper *Supertranslation hair of Schwarzschild black hole: a Wilson line perspective* with

Ratindranath Akhoury and Sandeep Pradhan, published in JHEP [96].

While the dressings constructed at the leading soft order is relevant for the infrared-divergent soft factor of scattering amplitudes, to study the infrared-finite part we need dressings that include corrections due to photons and gravitons that are in the subleading soft order. This is of great importance since it is relevant to physical predictions. The main obstacle to constructing such dressings is that the leading and subleading charges of the asymptotic symmetry do not commute, and therefore one cannot have simultaneous eigenstates of both. In chapter 6, we observe that by working to first order in the coupling constant, we can construct dressed states that behave as eigenstates of the leading and subleading charges in scattering amplitudes. Using these corrected dressed states, we show that that the infrared-finite parts of scattering amplitudes are in agreement with the cross-sections used in experiments. This is an important consistency check of the dressed-state formalism. Moreover, we show that there is no tree-level radiation of low-energy photons and gravitons. Chapter 6 is based on the paper *Subleading soft dressings of asymptotic states in QED and perturbative quantum gravity* with Ratindranath Akhoury, published in JHEP [97].

In the construction of Schwarzschild gravitational dressings in chapter 5, we observe that only the electric parity gravitons contributes to the horizon structure; the magnetic parities cancel out by themselves in a very non-trivial way. This issue is related to dual supertranslation [98–100], which has gained a lot of attention recently [101–115] as a “magnetic” dual of the BMS supertranslation. The electromagnetic duality of the vacuum Maxwell theory is broken in quantum electrodynamics with only electric sources. However, recent works [116, 117] have shown that this duality is regained, even in the absence of magnetic monopoles, for the asymptotic states which include the soft electric and magnetic modes. In a theory with electrically and magnetically charged particles, Strominger [32] has obtained the magnetic corrections to the usual soft photon theorems and explicitly constructed the charges which generate the magnetic large gauge transformations. A number of interesting questions, however, need to be further explored. Is the theory of electric and magnetic charges infrared finite? Are Strominger’s magnetic soft photon theorems [32] exact? In chapter 7, we aim to address these issues. Our approach is to construct the dressings which are charged under magnetic LGT in electrodynamics and under dual supertranslations in perturbative quantum gravity. First, working within the framework of a quantum field theory of magnetic and electric charges formulated by Blagojević and collaborators [118–120], we construct the asymptotic states of the magnetically charged particles by diagonalizing the asymptotic three point interaction potential of these with the photon. We emphasize that the field theory formulation of [118–120] is used only in the spirit of an effective field theory to determine the structure of the asymptotic three-point interaction. This construction at large times is non-perturbative, since the states can also be derived by other non-perturbative methods, such as writing Wilson line dressings or building eigenstates of the soft charge associated with the asymptotic symmetry. These methods give identical results (see [19, 59, 91]). Later in the chapter, we use only the second method for gravity. Having obtained the asymptotic states, we then show that the soft photon dressing associated to these states can be written as a 't Hooft line operator along the asymptotic trajectory of the magnetically charged particle. By direct computation, we demonstrate that the 't Hooft line dressings are charged under the magnetic LGT while neutral under electric LGT. This is in contrast to the dressings for electrons [8], which can be written as a Wilson line [91] and are charged under electric LGT while neutral under magnetic LGT. The construction of the 't Hooft line dressing parallels the treatment of electrically charged particles and this acts on the Fock states to create coherent states. The infrared finiteness of the quantum field theory of electric and magnetic charges is then manifest. The construction also makes clear that the leading magnetic dressings, just like their electric counterparts, are exact as was conjectured by Strominger in [32]. The 't Hooft line interpretation of the dressing allows us to extend the construction to perturbative quantum gravity. We construct gravitational 't Hooft line dressings that are charged under dual supertranslations but carry zero BMS supertranslation charge. Again, this is to be contrasted with gravitational Wilson line dressings, which are charged only under supertranslations. In gravity, we have no magnetic counterpart of the graviton coupling to the energy-momentum tensor. Thus, there are no particles that carry dual supertranslation charge, and hence no issue regarding infrared divergences. We study the algebra of dual supertranslation charges and the 't Hooft line dressings for smooth parameter functions on the sphere. Chapter 7 is based on the paper *Magnetic soft charges, dual supertranslations, and 't Hooft line dressings* with Ratindranath Akhoury, published in PRD [121].

Both the LGT and the BMS supertranslation charges are parametrized by a function on the sphere. This is true both for charges on null infinities and charges on the horizon. When this function is smooth, the algebra of standard and dual soft charges are known to be abelian. In electromagnetism, it has been shown by two independent groups [116, 117] that the algebra of charges bears a central term when the parameter function has singularities. In chapter 8, we extend this result to gravity on the Schwarzschild horizon. In the presence of a black hole, the future null infinity by itself does not form a Cauchy surface (in the absence of massive particles), and should be augmented by the future Schwarzschild horizon $\mathcal{H}^+$. Accordingly, the supertranslation charge obtains a contribution from the horizon. Using the first-order formalism of gravity to compute the horizon standard and dual supertranslation charges [108, 109], we derive that the horizon charge algebra exhibits a central term in the presence of poles in the parameter function. We discuss possible physical implications of such terms in the algebra, by drawing analogy to electromagnetism. Chapter 8 is based on an ongoing project with Ratindranath Akhoury and Malcolm Perry.

We end this introduction with a brief review of the two cornerstones of this dissertation: Faddeev-Kulish dressings of gravity and BMS supertranslation.

1.2 Dressings of perturbative quantum gravity

In this section, we review how the dressed states (and dressings) arise in perturbative quantum gravity around asymptotically flat spacetimes arise. The construction is very similar to the case of gauge theories [8]. The main reference is [9]. Throughout this section we work in the leading soft approximation, the eikonal limit, when the spin of the matter particles does not play a role. Thus, our results are equally valid for scalars, fermions, and gravitons; however, to be specific, we explicitly work with a massive scalar field φ coupled to gravity.

We expand around flat space $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and $\kappa^2 = 32\pi G$. In this expansion, $h_{\mu\nu}$ is the graviton field. We choose to work in the harmonic gauge (also known as the de Donder gauge)

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{1.1}$$

The Lagrangian of the theory assembles into the form

$$\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_h + \mathcal{L}_{\text{int}}, \tag{1.2}$$

where we have the free-field Lagrangians of the matter field φ and the graviton $h_{\mu\nu}$,

$$\mathcal{L}_\varphi = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 \tag{1.3}$$

$$\mathcal{L}_h = -\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \frac{1}{4}\partial_\alpha h\partial^\alpha h \tag{1.4}$$

as well as the leading-order interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \frac{\kappa}{2}\left[h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{h}{2}\left(\partial^\mu\varphi\partial_\mu\varphi + m^2\varphi^2\right)\right]. \tag{1.5}$$

Here $h = \eta^{\mu\nu}h_{\mu\nu} = h^\mu{}_\mu$ is the trace. It is sufficient to include only cubic interactions, since it has been shown in [9,10] that interactions at quadratic and higher order in the graviton field play no role in the infrared structure of the theory. One can read off the graviton propagator to be

$$\frac{1}{2}I^{\mu\nu\rho\sigma}\frac{i}{(-k^2+i\epsilon)} \tag{1.6}$$

where $I^{\mu\nu\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}$. The $h\varphi\varphi$ vertex rule, with scalar momentum p flowing in and p' flowing out is

$$\frac{i\kappa}{2}\left[p^\mu p'^\nu + p^\nu p'^\mu - \eta^{\mu\nu}\left(p\cdot p' + m^2\right)\right] \tag{1.7}$$

The fields $\varphi(x)$ and $h_{\mu\nu}(x)$ can be expanded in harmonic modes

$$\varphi(x) = \int \widetilde{d^3p}\,\left(b(p)e^{ip\cdot x} + b^\dagger(p)e^{-ip\cdot x}\right), \tag{1.8}$$

$$h_{\mu\nu}(x) = \int \widetilde{d^3k}\,\left(a_{\mu\nu}(k)e^{ik\cdot x} + a^\dagger_{\mu\nu}(k)e^{-ik\cdot x}\right), \tag{1.9}$$

where we have employed the shorthand notation

$$\widetilde{d^3p} = \frac{d^3p}{(2\pi)^3 2\omega_p}, \quad \widetilde{d^3k} = \frac{d^3k}{(2\pi)^3 2\omega_k}, \tag{1.10}$$

with $\omega_p = \sqrt{|\mathbf{p}|^2+m^2}$ and $\omega_k = |\mathbf{k}|$. With our choice of normalization (which is different from [9]), the creation and annihilation operators obey the commutation relations

$$\left[b(p), b^\dagger(p')\right] = (2\pi)^3(2\omega_p)\delta^3(\mathbf{p}-\mathbf{p}'), \tag{1.11}$$

$$\left[a_{\mu\nu}(k), a^\dagger_{\rho\sigma}(k')\right] = \frac{1}{2}I_{\mu\nu\rho\sigma}(2\pi)^3(2\omega_k)\delta^3(\mathbf{k}-\mathbf{k}'), \tag{1.12}$$

where $I^{\mu\nu\rho\sigma} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}$. We define the physical states as the subset of all the states in the Fock space that obey the gauge condition (1.1). The gauge condition (1.1) translates into the Gupta-Bleuler constraint on the Fock space

$$\left(k^\mu a_{\mu\nu}(k) - \frac{1}{2}k_\nu a^\mu_\mu(k)\right)|\Psi\rangle = 0 \quad \text{for all physical Fock states } |\Psi\rangle. \tag{1.13}$$

In the following we discuss the construction and properties of physical asymptotic states.

Now we consider the interaction potential, namely

$$V(t) = -\frac{\kappa}{2}\int d^3x\left[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{h}{2}\left(\partial^\mu\phi\partial_\mu\phi + m^2\phi^2\right)\right]. \tag{1.14}$$

Using the expansions for $h^{\mu\nu}$ and ϕ, we get

$$V(t) = -\frac{\kappa}{2}\int \widetilde{d^3p}\,\widetilde{d^3q}\,\widetilde{d^3k}\,\left(a^{\mu\nu}(k)e^{ik\cdot x} + a^{\dagger\mu\nu}(k)e^{-ik\cdot x}\right) \\ \times\left(ip_\mu b(p)e^{ip\cdot x} - ip_\mu b^\dagger(p)e^{-ip\cdot x}\right)\left(iq_\nu b(q)e^{iq\cdot x} - iq_\nu b^\dagger(q)e^{-iq\cdot x}\right). \tag{1.15}$$

The second term in the integrand of (1.14) does not contribute since $p^2 + m^2 = 0$. Due to the Riemann-Lebesgue lemma, for $|t| \to \infty$ this potential effectively becomes

$$V_{\text{as}}(t) = -\kappa\int \widetilde{d^3p}\,\widetilde{d^3k}\,\frac{p_\mu p_\nu}{2p^0}\rho(p)\left(a^{\mu\nu}(k)e^{i\frac{k\cdot p}{p^0}t} + a^{\dagger\mu\nu}(k)e^{-i\frac{k\cdot p}{p^0}t}\right) \tag{1.16}$$

where $\rho(p) \equiv b^\dagger(p)b(p)$. Then for $|t| \to \infty$ the asymptotic Hamiltonian is $H_{\text{as}} = H_0 + V_{\text{as}}$, with H_0 denoting the Hamiltonian for the free fields. The asymptotic time evolution operator solves

$$i\frac{d}{dt}U_{\text{as}}(t) = H_{\text{as}}(t)U_{\text{as}}(t) \tag{1.17}$$

To solve for U_{as}, let us define the function

$$Z(t) = e^{iH_0 t}U_{\text{as}}(t). \tag{1.18}$$

In terms of Z, the time evolution equation takes the form

$$i\frac{d}{dt}Z(t) = V_{\text{as}}(t)Z(t), \tag{1.19}$$

for which we obtain the solution

$$Z(t) = T\exp\left(-i\int^t V_{\text{as}}(\tau)d\tau\right) = e^{-R(t)}e^{i\Phi(t)}. \tag{1.20}$$

Here we have the two commuting operators $R(t)$ and $\Phi(t)$, defined as

$$R(t) = i\int^t V_{\text{as}}(\tau)d\tau, \tag{1.21}$$

$$\Phi(t) = \frac{i}{2}\int^t d\tau\int^\tau ds\,[V_{\text{as}}(\tau), V_{\text{as}}(s)]. \tag{1.22}$$

Plugging in our expression (1.16) for V_{as}, we obtain

$$R(t) = \frac{\kappa}{2} \int \widetilde{d^3p}\, \widetilde{d^3k}\, \rho(p) \frac{p^\mu p^\nu}{p \cdot k} \left(a^\dagger_{\mu\nu}(k) e^{-i\frac{p\cdot k}{\omega_p}t} - a_{\mu\nu}(k) e^{i\frac{p\cdot k}{\omega_p}t} \right), \tag{1.23}$$

$$\Phi(t) = -\frac{\kappa^2}{16(2\pi)^3} \int \widetilde{d^3p}\, \widetilde{d^3p'} \, : \rho(p)\rho(p') : \frac{2(p\cdot p')^2 - m^4}{\sqrt{(p\cdot p')^2 - m^4}} \int^t \frac{d\tau}{|\tau|}. \tag{1.24}$$

At large times $t \to \infty$, $e^{i\Phi(t)}$ is a divergent phase (that is related to Coulomb phase) and $e^{-R(t)}$ is an operator that carries an infinite number of soft gravitons. That the large-time interaction Hamiltonian induces an infinite number of soft graviton operators implies that the asymptotic state space $\mathcal{H}_{\text{as}}$ should be defined in terms of the space of Fock states $\mathcal{H}_{\text{F}}$ as

$$\mathcal{H}_{\text{as}} = e^{R(t)} \mathcal{H}_{\text{F}}, \tag{1.25}$$

where t, the asymptotic time, is taken to be very large. For QED [8] and for gravity [9], it was shown that this asymptotic space has a number of important properties including gauge invariance (linearized general coordinate invariance). However, recent works [14, 15] have clarified that the gauge invariance is only with respect to small gauge transformations. Large gauge transformations, or those that do not reduce to the identity at time-like and null infinity, are instead symmetries of the system. Among these, the ones relevant for us are the BMS supertranslations. The space of asymptotic states is divided into superselection sectors, each labeled by a BMS charge.

We note that there is a 3-graviton interaction vertex in perturbative quantum gravity, so the gravitons are dressed as well. Since the leading soft approximation is not sensitive to spin of the particle, the dressings for gravitons are the same as those for massless scalars.

1.3 BMS Symmetry

The group of BMS (named after Bondi, van der Burg, Metzner and Sachs) transformations form the asymptotic symmetry group of gravity in asymptotically flat spacetimes [11, 12]. Not only does it contain the Poincaré transformations, but it also includes a novel diffeomorphism called the supertranslation. Since supertranslations are intimately related to the gravitational dressings of perturbative quantum gravity, we explore in this section how supertranslation arises as the asymptotic symmetry of the asymptotically flat spacetime. The main reference for this section is [1].

Let us begin with the simplest example of an asymptotically flat metric, the Minkowski metric. An asymptotic boundary of this spacetime is the future null infinity $\mathcal{I}^+$, with the future and past boundaries $\mathcal{I}^+_+$ and $\mathcal{I}^+_-$ respectively. $\mathcal{I}^+$ is a three-dimensional manifold parametrized by the retarded time $u = t - r$ and the angular coordinates. To get to $\mathcal{I}^+$, one takes the limit $r \to \infty$ while keeping u fixed. In the retarded set of coordinates (u, r, x^A), where we use the capital Latin letter $A, B, \ldots$ to denote angular indices, the Minkowski metric takes the form

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B. \tag{1.26}$$

Here γ_{AB} is the metric on the unit sphere; for instance in the standard spherical coordinates $x^A = (\theta, \phi)$,

$$\gamma_{AB} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}. \tag{1.27}$$

Now, let us turn to the asymptotically flat spacetimes. What coordinates should we use to parametrize their asymptotic boundaries? Let us demand that we retain a nice set of coordinates (u, r, θ, ϕ) that we used for Minkowski spacetime. First, we want a constant-u surface to be a null hypersurface, that is, it is everywhere tangent to a light cone. This implies that the normal vector $\nabla_\mu u$ should be null, that is,

$$\nabla_\mu u \nabla^\mu u = g^{uu} = 0. \tag{1.28}$$

Next, we want the angular coordinates to be constant along any light ray. This implies that given a light ray defined by the tangent $\nabla^\mu u$, angular coordinates are orthogonal in the sense that

$$\nabla^\mu u \nabla_\mu x^A = g^{uA} = 0. \tag{1.29}$$

Notice that these two conditions that we impose on the coordinates fix three (out of ten) components of the inverse metric to zero:

$$g^{\mu\nu} = \begin{bmatrix} 0 & g^{ur} & 0 \\ g^{ur} & g^{rr} & g^{rA} \\ 0 & g^{rB} & g^{AB} \end{bmatrix}. \tag{1.30}$$

By direct computation, one can observe that this is enough to fix three components of the metric $g_{rr} = g_{rA} = 0$,

$$g_{\mu\nu} = \begin{bmatrix} g_{uu} & g_{ur} & g_{uA} \\ g_{ur} & 0 & 0 \\ g_{uA} & 0 & g_{AB} \end{bmatrix}, \tag{1.31}$$

as well as fix g_{AB} and g^{AB} to be inverses of each other.

Finally, we want r to represent the luminosity distance. In terms of the metric, this property translates to

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0. \tag{1.32}$$

Since (1.31) is to be an asymptotically flat metric, we want g_{AB} to reduce to $r^2\gamma_{AB}$ at large r, which implies the large-r expansion

$$\frac{g_{AB}}{r^2} = \gamma_{AB} + \frac{1}{r}C_{AB} + O(r^{-2}) \tag{1.33}$$

for some symmetric two-dimensional tensor C_{AB} that is a function of u and x^A. Using this expansion and the identity $\det M = \exp \operatorname{tr} \ln M$, one finds that the luminosity distance condition (1.32) reduces to tracelessness of C_{AB},

$$\gamma^{AB}C_{AB} = 0. \tag{1.34}$$

Metrics that satisfy $g_{rr} = g_{rA} = 0$ as well as (1.32) (or (1.34)) are said to be in the Bondi gauge. Einstein's equations require the metric in Bondi gauge to have the large-r expansion

$$\begin{aligned} ds^2 = &-du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B \\ &+ \frac{2m_B}{r}du^2 + rC_{AB}dx^A dx^B + U_A du dx^A + \cdots, \end{aligned} \tag{1.35}$$

where $m_B(u, x^A)$ is called the Bondi mass aspect, and $U_A(u, x^B)$ is a function that can be fixed in terms of C_{AB} by demanding some fall-off conditions on the Weyl tensor. These set of coordinates (u, r, x^A) are often referred to as the Bondi coordinates.

Now that we have the asymptotically flat metric on $\mathcal{I}^+$, let us see which diffeomorphisms ξ leave this structure invariant. Since we are interested in seeing something other than Poincaré transformations, let us exclude rotations and boosts. Recalling that rotation and boost generators are of the form $x_{[\mu}\partial_{\nu]}$, we restrict

$$\xi^u, \xi^r \sim O(1), \qquad \xi^A \sim O(r^{-1}). \tag{1.36}$$

This leaves us with translation-like transformations. Since we want to stay in the Bondi gauge, we demand that $\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{rA} = 0$ and $\gamma^{AB}\mathcal{L}_\xi g_{AB} = 0$. This is enough to fix the components of the vector field ξ at leading order in large r,

$$\xi^\mu \partial_\mu = f\partial_u + \frac{1}{2}D^2 f\partial_r - \frac{1}{r}D^A f\partial_A, \tag{1.37}$$

where f is a function on the unit sphere, D_A denotes covariant derivative on the unit sphere (and thus is compatible with γ_{AB}), $D^A = \gamma^{AB}D_B$, and $D^2 = D^A D_A$.

We have identified the translation-like diffeomorphisms ξ that retain the structure of asymptotically flat spacetimes, that are parametrized by a function f on the sphere. One finds that they are abelian at large r, in the sense that the Lie bracket of two such diffeomorphisms vanishes as $r \to \infty$. So, what are these transformations? It is instructive to write the vector field ξ in spherical coordinates,

$$\xi^\mu \partial_\mu = f\partial_u + \frac{1}{2}\mathbf{L}^2 f\partial_r - \frac{1}{r}\partial_\theta f\partial_\theta - \frac{1}{r\sin^2\theta}\partial_\phi f\partial_\phi. \tag{1.38}$$

Here we have used $D^2 = -\mathbf{L}^2$, where $\mathbf{L}^2$ is the angular momentum operator. Taking $f(x^A)$ to be the four lowest spherical harmonics (up to constant normalization)

$$Y_0^0 = 1, \qquad Y_1^0 = \cos\theta, \qquad Y_1^{\pm 1} = \sin\theta e^{\pm i\phi}, \tag{1.39}$$

we find that

$$\xi(Y_0^0) = T_0, \qquad \xi(Y_1^0) = -T_3, \qquad \xi(Y_1^{\pm 1}) = -(T_1 \pm iT_2), \tag{1.40}$$

where $T_0 = \partial_t$ and $T_i = \partial_i$ are the space and time generators respectively, with $t = u + r$. Therefore, the $l = 0, 1$ partial waves of $f(x^A)$ generate spacetime translation. Since $f(x^A)$ can be a generic function on the sphere, it also contains higher order partial waves,

$$f(x^A) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_l^m(x^A). \tag{1.41}$$

Therefore, the diffeomorphism generated by ξ can be thought of as a generalization of spacetime translations, and is referred to as *BMS supertranslation* in the literature. There is an infinite number of supertranslation generators, one for each spherical harmonic.

While our analysis was done on the future null infinity $\mathcal{I}^+$, one can follow a similar line of reasoning to obtain supertranslation on the past null infinity $\mathcal{I}^-$. These two supertranslations are closely related. The charge at $\mathcal{I}^+$ can be written as an integral over the past boundary $\mathcal{I}^+_-$,

$$Q^+_{f^+} = \frac{1}{4\pi G} \int_{\mathcal{I}^+_-} d\Omega\, f^+ m_B, \tag{1.42}$$

while at $\mathcal{I}^-$ it can be written as one over the future boundary $\mathcal{I}^-_+$,

$$Q^-_{f^-} = \frac{1}{4\pi G} \int_{\mathcal{I}^-_+} d\Omega\, f^- m_B. \tag{1.43}$$

For the scattering problem in general relativity to be well-defined, the charge must be conserved in the sense that

$$Q^+_{f^+} = Q^-_{f^-}, \qquad f^+ = f^-|_{\text{antipodal}}. \tag{1.44}$$

Here the two parameter functions f^+ and f^- are to be antipodally matched. This is usually implicit in most computations in the literature, and therefore one uses a single f for both null infinities. In perturbative quantum gravity, this conservation law becomes [14]

$$Q^+_f \mathcal{S} - \mathcal{S} Q^-_f = 0, \tag{1.45}$$

where $\mathcal{S}$ is the S-matrix operator. This conservation law plays a central role in the relation between dressings and supertranslation charges.

There exists an extension of the BMS symmetry called the BMS superrotations. Just as supertranslations are generalizations of translations, superrotations can be thought of as generalizing boosts and rotations. There are, however, complications to this generalization as superrotations are known to change the structure of the metric (1.35) (and hence the need for *extension* of BMS symmetry). We come back to superrotations in the context of subleading soft theorems in chapter 6.

There is an asymptotic symmetry transformation of gravity that is not a diffeomorphism, which goes under the name of dual supertranslation (see [98–100]). The relationship between standard and dual supertranslation is parallel to that between the LGTs of electric and magnetic charges in electromagnetism. We discuss dual supertranslations in chapters 7 and 8.

Chapter 2

BMS Supertranslations and Gravitational Dressings

2.1 Physical asymptotic states for gravity

Kulish and Faddeev have constructed physical asymptotic states in QED [8], following the works of Chung [4] and Kibble [5]. The physical asymptotic states (dressed states) were constructed by dressing the incoming and outgoing states with a coherent cloud of photons. This formalism has been used in [9] to construct asymptotic states in gravity, which involve coherent clouds of gravitons. In this chapter, we demonstrate that the these dressed states facilitate charge conservation by carrying a definite BMS supertranslation charge. We start with a brief overview of the work done in [9] and provide generalizations that are relevant to our discussion.

As reviewed in chapter 1, the Faddeev-Kulish formalism was used in [9] to construct an operator $e^{R(t)}$ that projects the Fock space $\mathcal{H}_{\mathrm{F}}$ into the space of asymptotic states $\mathcal{H}_{\mathrm{as}}$

$$e^{R(t)}\mathcal{H}_{\mathrm{F}} = \mathcal{H}_{\mathrm{as}}. \tag{2.1}$$

The anti-Hermitian operator $R(t)$ is given by

$$R(t) = \frac{\kappa}{2}\int \widetilde{d^3p}\,\widetilde{d^3k}\,\rho(p)\frac{p^\mu p^\nu}{p\cdot k}\left(a^\dagger_{\mu\nu}(k)e^{-i\frac{p\cdot k}{\omega_p}t} - a_{\mu\nu}(k)e^{i\frac{p\cdot k}{\omega_p}t}\right), \tag{2.2}$$

where t, the asymptotic time, is taken to be very large and $\rho(p) = b^\dagger(p)b(p)$ is the number operator of the scalar particle. For QED [8] and for gravity [9], it was shown that this asymptotic space has a number of important properties including gauge invariance (linearized general coordinate invariance). However, recent works [14, 15] have clarified that the gauge invariance is only with respect to small gauge transformations. Large gauge transformations, or those that do not reduce to the identity at time-like and null infinity, are instead symmetries of the system. Among these, the ones relevant for us are the BMS supertranslations. The space of asymptotic states is divided into superselection sectors, each labeled by a BMS charge which is explicitly constructed in section 2.4.

The operator $R(t)$, however, exhibits some properties which make it unfavorable to work with. For example, it does not preserve the Gupta-Bleuler condition (1.13):

$$\left[R(t), k^\mu a_{\mu\nu}(k) - \frac{1}{2}k_\nu a^\mu_\mu(k)\right] \neq 0. \tag{2.3}$$

To resolve this, we note that only the low energy behavior of $R(t)$ defines the space $\mathcal{H}_{\rm as}$, and introduce another operator R_f of the form:

$$R_f = \frac{\kappa}{2}\int \widetilde{d^3p}\,\widetilde{d^3k}\,\rho(p)\left(f^{\mu\nu*}(p,k)a^\dagger_{\mu\nu}(k) - f^{\mu\nu}(p,k)a_{\mu\nu}(k)\right), \tag{2.4}$$

which is characterized by an infrared function $f_{\mu\nu}(p,k)$. This function is different in different superselection sectors, but its form is restricted as we show now. One restriction comes from the fact that $e^{R(t)}$ and e^{R_f} must describe unitarily equivalent spaces, i.e.,

$$e^{R_f}\mathcal{H}_{\rm F} = e^{R(t)}\mathcal{H}_{\rm F} = \mathcal{H}_{\rm as}.$$

The constraints arising from these, which are discussed in [9] and in appendix A, make it convenient to write $f_{\mu\nu}(p,k)$ in a form analogous to that of QED in [8], which reads

$$f_{\mu\nu}(p,k) = \left[\frac{p_\mu p_\nu}{p\cdot k} + \frac{c_{\mu\nu}(p,k)}{\omega_k}\right]\phi(p,k) \tag{2.5}$$

with some function $c_{\mu\nu}(p,k)$, where $\phi(p,k)$ is a smooth function such that $\phi \to 1$ as $k \to 0$. This form is useful to work with, because it separates the two terms that play distinct roles: the first term is responsible for mapping Fock states into asymptotic states and, as we see later in this chapter, the second term parametrizes the superselection sector.

We note that since we have not restricted the form of $c_{\mu\nu}(p,k)$ yet, the expression (2.5) is still general. Depending on how we parametrize $c_{\mu\nu}(p,k)$, however, (2.5) might not be compatible with some possible forms of $f_{\mu\nu}(p,k)$. For example, if we demand that $c_{\mu\nu}(p,k)$ does not contain terms proportional to $p_\mu p_\nu$, then (2.5) becomes incompatible with certain parametrizations, such as the one used in [9]. However, since we use explicit parameterization only as an example, our results are valid in general.

With the new operator R_f, we demand that the physical asymptotic states be subject to the Gupta-Bleuler condition (1.13), which implies

$$\left[R_f, k^\mu a_{\mu\nu} - \frac{1}{2} k_\nu a^\mu_\mu\right] = 0, \tag{2.6}$$

or,

$$k^\mu f_{\mu\nu} = \frac{k^\mu c_{\mu\nu}}{\omega_k} + p_\nu = 0. \tag{2.7}$$

There are additional constraints on the function $f_{\mu\nu}$, or equivalently, on $c_{\mu\nu}$ arising again from the fact that $e^{R(t)}$ and e^{R_f} define unitarily equivalent spaces. In appendix A we show that, to leading order in k, these constraints are

$$c^*_{\mu\nu}(p,k) = c_{\mu\nu}(p,k) \tag{2.8}$$
$$c_{\mu\nu}(p,k) I^{\mu\nu\rho\sigma} c_{\rho\sigma}(p',k) = 0 \quad \text{for all } p \text{ and } p'. \tag{2.9}$$

Subleading corrections to (2.8)-(2.9) only rescales the operator e^{R_f} by a positive finite constant, and we could therefore absorb them in the normalization of the state.

With a $c_{\mu\nu}$ that satisfies (2.7)-(2.9), the graviton cloud operator e^{R_f} properly gives us the asymptotic states

$$|\Psi^{\text{as}}\rangle = e^{R_f} |\Psi\rangle , \tag{2.10}$$

where $|\Psi\rangle$ denotes Fock states of the matter fields. It is convenient to interpret e^{R_f} as an operator that dresses each scalar with its own cloud of gravitons. Indeed, this is seen most clearly by commuting e^{R_f} through the scalar operators using $[b(p), \rho(p')] = (2\pi)^3 (2\omega_p) \delta^3(\mathbf{p} - \mathbf{p}')b(p)$. In this way we obtain, for example

$$e^{R_f} b^\dagger(p_1) b^\dagger(p_2) |0\rangle = e^{R_f(p_1)} b^\dagger(p_1) e^{R_f(p_2)} b^\dagger(p_2) |0\rangle , \tag{2.11}$$

where

$$R_f(p) = \frac{\kappa}{2} \int \widetilde{d^3k} \left(f^{\mu\nu}(p,k) a^\dagger_{\mu\nu}(k) - f^{\mu\nu}(p,k) a_{\mu\nu}(k) \right) . \tag{2.12}$$

One can parameterize the c-matrix as the following to exclude terms proportional to $p_\mu p_\nu$,

$$c_{\mu\nu}(p,k) = a_1 q_{(\mu} p_{\nu)} + a_2 q_\mu q_\nu \ , \tag{2.13}$$

where $q(k)$ is some four-vector and a_1, a_2 are coefficients to be determined.[1] This parame-

[1] We have used the following notation for the symmetric combination $q_{(\mu} p_{\nu)} \equiv q_\mu p_\nu + q_\mu p_\nu$.

terization is similar to the one used in [18] for the case of QED. The gauge constraint (2.7) then fixes the coefficients to be

$$a_1 = -\frac{\omega_k}{k \cdot q} \quad \text{and} \quad a_2 = -\frac{k \cdot p}{k \cdot q} a_1 \ , \tag{2.14}$$

and therefore we have

$$c_{\mu\nu}(p,k) = \frac{\omega_k}{k \cdot q} \left[\frac{k \cdot p}{k \cdot q} q_\mu q_\nu - q_{(\mu} p_{\nu)} \right] . \tag{2.15}$$

The constraint (2.9) then reads

$$c_{\mu\nu}(p,k) I^{\mu\nu\rho\sigma} c_{\rho\sigma}(p',k) = \frac{\omega_k^2}{(k \cdot q)^2}\, q^2 \left[\frac{(k \cdot p)}{(k \cdot q)} q - p \right] \cdot \left[\frac{(k \cdot p')}{(k \cdot q)} q - p' \right] = 0, \tag{2.16}$$

and can be satisfied identically only if q is a null vector $q^2 = 0$. In addition, since rescaling q by a constant does not affect (2.15), we can assume that the time component of q is 1 without any loss of generality. As we see later, the null vector q parameterizes the space of superselection sectors, and the combination

$$c^{\mu\nu}(p,k) \epsilon^{\pm}_{\mu\nu}(k), \tag{2.17}$$

where $\epsilon^{\pm}_{\mu\nu}(k)$ are the transverse, traceless physical polarization tensors of graviton,

$$k^\mu \epsilon^{\pm}_{\mu\nu}(k) = 0 \quad \text{and} \quad \eta^{\mu\nu} \epsilon^{\pm}_{\mu\nu}(k) = 0, \tag{2.18}$$

is related to the conserved charge under BMS symmetry transformations. The BMS charge therefore characterizes the superselection sector. A similar conclusion was drawn for QED in [18]. In [9], the choice $c^{\mu\nu}(p,k)\epsilon^{\pm}_{\mu\nu}(k) = 0$ was made. This choice can be realized by (2.15) with $q^\mu = (1, -\hat{\mathbf{k}})$ and corresponds to a vanishing BMS charge.

The harmonic gauge condition (1.1) does not fix the gauge completely. BMS transformations parameterize the residual leftover gauge freedom [11, 12, 14, 15], which is given by

$$h_{\mu\nu} \ \rightarrow \ h_{\mu\nu} + \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu, \tag{2.19}$$

with the gauge parameter λ_μ satisfying the wave equation

$$\Box \lambda_\mu = 0, \tag{2.20}$$

We review this in section 2.2. Here it is worth noting that under this residual gauge freedom the infrared function $f_{\mu\nu}$ transforms as

$$f_{\mu\nu}(p,k) \;\to\; f_{\mu\nu}(p,k) + k_{(\mu}\lambda_{\nu)} - (k\cdot\lambda)\eta_{\mu\nu}, \tag{2.21}$$

with $k^2 = 0$, which is also implied by equation (2.7). It was shown in [9] that the action of e^{R_f} on a matter Fock state $|\Psi_{\rm in}\rangle$ is invariant under (2.21) for small gauge transformations. In the subsequent sections, we scrutinize how the BMS transformation, in particular the supertranslation, plays a role in the context of the asymptotic states.

We would now like to show that the S-matrix elements in the basis (2.10)

$$\langle\Psi^{\rm as}_{\rm out}|\,\mathcal{S}\,|\Psi^{\rm as}_{\rm in}\rangle = \langle\Psi_{\rm out}|\,e^{-R_f}\mathcal{S}e^{R_f}\,|\Psi_{\rm in}\rangle\,, \tag{2.22}$$

are free of IR divergence. This has been shown in [9] for a process between single-scalar asymptotic states using a specific choice of $c_{\mu\nu}$. In appendix B, we present a generalization of this result; we show that the divergences cancel for a process between asymptotic states with arbitrary number of scalar particles to all orders in the perturbative expansion, provided that the c-matrices satisfy, to leading order in the momentum k,

$$\sum_{j\in{\rm out}} c^{(\rm out)}_{\mu\nu}(p_j,k) = \sum_{i\in{\rm in}} c^{(\rm in)}_{\mu\nu}(p_i,k), \tag{2.23}$$

where “in" and “out" denote the set of incoming and outgoing scalar particles, respectively. We briefly describe these results now.

First, note that from the work of Weinberg [2] we know that the amplitude $\mathcal{M}$ of a process can be decomposed into

$$\mathcal{M} = \langle\Psi_{\rm out}|\,e^{-R_f}\mathcal{S}e^{R_f}\,|\Psi_{\rm in}\rangle = A_{\rm virt}\mathcal{M}', \tag{2.24}$$

where $A_{\rm virt}$ is the IR-divergent contribution of virtual gravitons and $\mathcal{M}'$ is the remainder of the amplitude. In (B.80) we show that

$$\mathcal{M}' = A_{\rm cloud}\widetilde{\mathcal{M}}, \tag{2.25}$$

where $\widetilde{\mathcal{M}}$ is the IR-finite part of the amplitude and $A_{\rm cloud}$ is the divergent factor coming from interactions that involve graviton clouds. The latter has the form

$$A_{\rm cloud} = (A_{\rm virt})^{-1}e^{-aC}, \tag{2.26}$$

where a is a positive constant, and

$$C \equiv \int \frac{d^3k}{\omega_k^3}\, c^{\rm tot}_{\mu\nu} I^{\mu\nu\rho\sigma} c^{\rm tot}_{\rho\sigma} \quad \text{with} \quad c^{\rm tot}_{\mu\nu} = \sum_{j\in \rm out} c_{\mu\nu}(p_j,k) - \sum_{i\in \rm in} c_{\mu\nu}(p_i,k). \tag{2.27}$$

The factor e^{-aC} derives solely from the interactions between graviton clouds. Since we have the same c-matrices for both the incoming and outgoing states, by (2.9) the integrand of (2.27) vanishes and $C = 0$. If we use different c-matrices, for instance $c_{\mu\nu}$ for incoming and $c'_{\mu\nu}$ for outgoing states, then (2.27) readily generalizes to the same expression for the integral with

$$c^{\rm tot}_{\mu\nu} = \sum_{j\in \rm out} c'_{\mu\nu}(p_j,k) - \sum_{i\in \rm in} c_{\mu\nu}(p_i,k). \tag{2.28}$$

If the condition (2.23) is not met, then C exhibits IR divergence and the amplitude vanishes. Therefore to obtain a non-zero amplitude, (2.23) must be satisfied and $C = 0$. It is worth noting that subleading corrections in the momentum k to equation (2.23) are finite and can therefore be absorbed in the normalization of the states. The amplitude thus becomes

$$\mathcal{M} = A_{\rm virt} A_{\rm cloud} \widetilde{\mathcal{M}} = A_{\rm virt} (A_{\rm virt})^{-1} e^{-aC} \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}, \tag{2.29}$$

which is IR finite.

2.2 The BMS group

2.2.1 Asymptotically flat spacetime

In this section we review the structure of asymptotically Minkowski geometry and BMS transformations. We follow closely the works of [14, 15].

Let us first define the retarded system of coordinates, which is related to the Cartesian system by

$$r^2 = x_1^2 + x_2^2 + x_3^2, \qquad u = t - r, \qquad z = \frac{x_1 + i x_2}{r + x_3}. \tag{2.30}$$

The inverse relations are given by

$$t = u + r\ , \qquad \mathbf{x} = r\hat{\mathbf{x}} = \frac{r}{1 + z\bar{z}} \left(z + \bar{z}, i(\bar{z} - z), 1 - z\bar{z}\right). \tag{2.31}$$

The flat Minkowski metric is then given by

$$\begin{aligned} ds_0^2 &= -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \\ &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \ , \end{aligned} \tag{2.32}$$

where

$$\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} \tag{2.33}$$

is the round metric on the unit S^2.

Asymptotically flat metrics have an expansion around future null infinity ($r = \infty$), whose leading order terms are given by

$$\begin{aligned} ds^2 &= ds_0^2 \\ &\quad + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 - 2U_z dudz - 2U_{\bar{z}}dud\bar{z} \\ &\quad + \ \dots \ , \end{aligned} \tag{2.34}$$

where

$$U_z = -\frac{1}{2}D^z C_{zz} \ , \qquad U_{\bar{z}} = -\frac{1}{2}D^{\bar{z}} C_{\bar{z}\bar{z}} \ , \tag{2.35}$$

and the dots denote higher order terms. The Bondi mass aspect m_B and the radiative data C_{zz}, $C_{\bar{z}\bar{z}}$ are functions of $(u, z, \bar{z})$. We also define the Bondi news by

$$N_{zz} \equiv \partial_u C_{zz} \ , \qquad N_{\bar{z}\bar{z}} \equiv \partial_u C_{\bar{z}\bar{z}} \ . \tag{2.36}$$

The $\mathcal{I}^+$ data m_B and C_{zz} are related by the constraint equation

$$\partial_u m_B = -\frac{1}{2}\partial_u \left[D^z U_z + D^{\bar{z}} U_{\bar{z}}\right] - T_{uu} \ , \tag{2.37}$$

where

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right] \tag{2.38}$$

is the total outgoing radiation energy flux. The first term of (2.38) is the gravitational contribution while T^M is the stress-energy tensor of the matter sector.

It is important to note that the metric in (2.34) is written in the Bondi gauge, which is convenient for the presentation of the asymptotic solution but is not compatible with the harmonic gauge. The transformation that relates the two gauges

$$h^H_{\mu\nu} = h^B_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{2.39}$$

obeys the following equation

$$\Box\xi_\mu = \frac{1}{2}\partial_\mu h^B - \partial^\nu h^B_{\mu\nu} \tag{2.40}$$

where the label H stands for Harmonic gauge and the label B stands for Bondi gauge. For a detailed discussion on the relation between the two gauges we refer the reader to references [36, 122, 123]. In the rest of the chapter, we work in the harmonic gauge.

2.2.2 BMS supertranslations

As discussed in the previous section, after fixing the harmonic gauge there is still a residual leftover gauge freedom given by (2.19)-(2.20). The gauge field λ_μ parameterizes the group of BMS transformations. At leading order it is given by [36]

$$\lambda^\mu\partial_\mu = f\partial_u + V^i\partial_i + \frac{1}{2}(D^iV_i)\,(u\partial_u - r\partial_r) + \dots\ , \tag{2.41}$$

where $i = 1, 2$ runs over the S^2 coordinates, and the dots stand for subleading terms. The function $f(z,\bar{z})$ is the transformation parameter of supertranslations, and the 2-vector $V^i(z,\bar{z})$ is the transformation parameter of superrotations. In this chapter, we are interested only in the supertranslations.

The retarded system of coordinates $(u, r, z, \bar{z})$ is useful to describe null infinity. However, in the following we study the action of BMS transformations on massive particles, which reach null infinity only asymptotically far in the future (at $\mathcal{I}^+_+$). For this purpose it is useful to adopt the hyperbolic system of coordinates defined as

$$\tau = \sqrt{t^2 - r^2} = \sqrt{u^2 + 2ur}\ , \qquad \rho = \frac{r}{\sqrt{t^2 - r^2}} = \frac{r}{\sqrt{u^2 + 2ur}}\ , \tag{2.42}$$

with the inverse relations given by

$$u = \tau\sqrt{1+\rho^2} - \rho\tau\ , \qquad r = \rho\tau\ . \tag{2.43}$$

In this system of coordinates, the world-line of a massive particle moving at a constant velocity is described by hypersurfaces of constant ρ. The Minkowski metric then takes the form

$$ds^2 = -d\tau^2 + \tau^2\left(\frac{d\rho^2}{1+\rho^2} + \rho^2\gamma_{z\bar{z}}dzd\bar{z}\right)\ . \tag{2.44}$$

An illustrative diagram of the causal structure of Minkowski spacetime in the hyperbolic coordinates is given in Strominger's lecture notes [1], which we reproduce in figure 2.1.

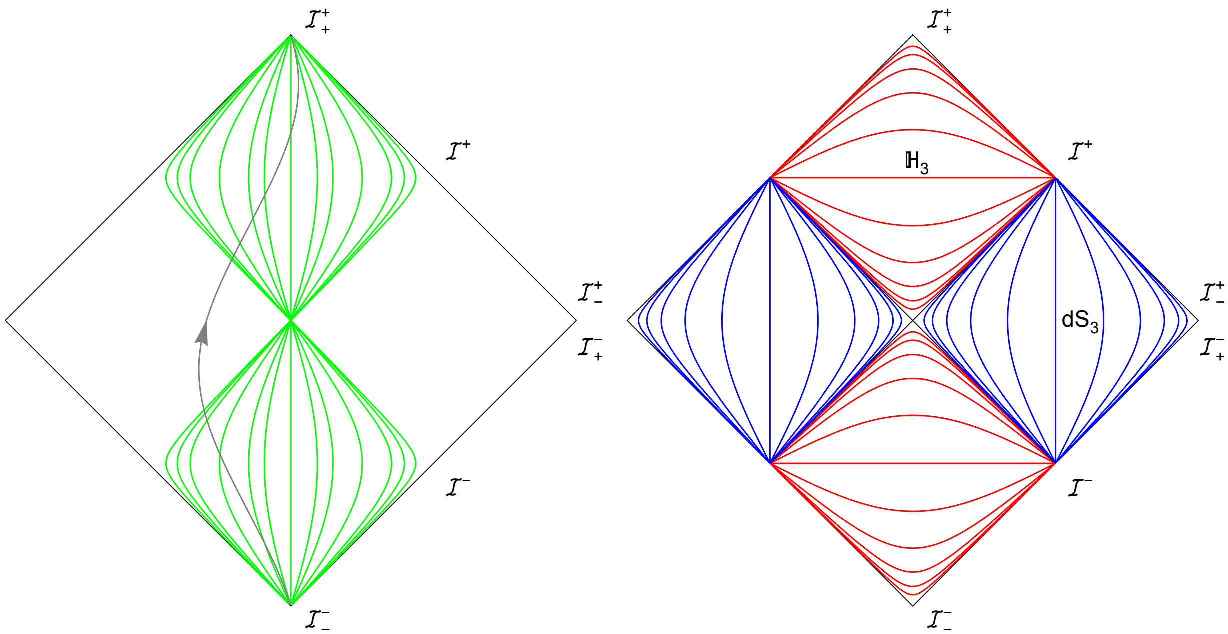


Figure 2.1: Diagrams illustrating the causal structure of Minkowski spacetime, reproduced from [1]. Left: the green lines describe hypersurfaces of constant ρ, and the grey line is the world-line of a massive particle moving at a constant velocity. Right: hyperbolic slicing of Minkowski spacetime. The slices correspond to constant τ hypersurfaces, where for $\tau^2 > 0$ the resulting surface is the hyperbolic space $\mathbb{H}_3$ and for $\tau^2 < 0$ it is dS_3.

It was shown in [36, 124] that at $\tau \to \infty$ the only non-vanishing component of λ_μ is λ_τ,

$$\lim_{\tau\to\infty} \lambda_\tau(\tau, \rho, z, \bar{z}) = \widetilde{\lambda}_\tau(\rho, z, \bar{z}) \ , \tag{2.45}$$

In appendix C we study the two solutions for $\widetilde{\lambda}_\tau(\rho, z, \bar{z})$. At time-like infinity they asymptote to

$$\lim_{\rho\to\infty} \widetilde{\lambda}_\tau(\rho, z, \bar{z}) = \alpha(z, \bar{z})\rho\,(1 + \dots) + \beta(z, \bar{z})\rho^{-3}\,(1 + \dots) \ , \tag{2.46}$$

where the dots denote subleading terms in $1/\rho$. The α-series is leading and do not vanish at time-like infinity $\rho \to \infty$. It is a large gauge transformation. The β-series is subleading and vanishes at time-like infinity. We also show that in terms of the radiative data, the α and β modes are given by

$$\begin{aligned} \alpha(z, \bar{z}) &= (\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z)_{\mathcal{I}^+_+} \ , \\ \beta(z, \bar{z}) &= i\,(\partial_z U_{\bar{z}} - \partial_{\bar{z}} U_z)_{\mathcal{I}^+_+} \ . \end{aligned} \tag{2.47}$$

The supertranslation charge (as well as the radiative data) is gauge-invariant to leading order in r [36, 122, 123]. At null infinity $\mathcal{I}^+$ it is given by

$$T(f) = \frac{1}{4\pi G} \int_{\mathcal{I}^+_-} d^2 z \gamma_{z\bar{z}} f(z,\bar{z}) m_B \ . \tag{2.48}$$

Using the constraint equation (2.37) we can write

$$T(f) = T_{\text{soft}}(f) + T_{\text{hard}}(f), \tag{2.49}$$

where we have the soft part, given by the boundary term

$$T_{\text{soft}}(f) = \frac{1}{8\pi G} \int d^2 z \left[\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z\right] f(z,\bar{z}), \tag{2.50}$$

and the hard part

$$T_{\text{hard}}(f) = \frac{1}{4\pi G} \int du d^2 z f(z,\bar{z}) \gamma_{z\bar{z}} T_{uu} \ . \tag{2.51}$$

The soft part of BMS supertranslations corresponds to the α-mode. The reason is that the graviton's zero-mode is precisely the pure gauge mode λ_μ. To isolate the BMS mode we therefore have to impose the following boundary conditions

$$\beta = i \left(\partial_z U_{\bar{z}} - \partial_{\bar{z}} U_z\right)_{\mathcal{I}^+_\pm} = i \left(D_z^2 C_{\bar{z}\bar{z}} - D_{\bar{z}}^2 C_{zz}\right)_{\mathcal{I}^+_\pm} = 0 \ . \tag{2.52}$$

After imposing these boundary conditions, the Bondi news and the radiative data transform as

$$\begin{aligned} \delta_f N_{zz} &= f \partial_u N_{zz} \\ \delta_f C_{zz} &= f \partial_u C_{zz} - 2 D_z^2 f \end{aligned} \tag{2.53}$$

under supertranslations, and the action of the BMS charge is described by the following Dirac (or Poisson) brackets

$$\begin{aligned} \{T(f), C_{zz}\} &= f \partial_u C_{zz} - 2 D_z^2 f \\ \{T(f), N_{zz}\} &= f \partial_u N_{zz} \ . \end{aligned} \tag{2.54}$$

Without imposing the boundary conditions (2.52) the result of the Dirac brackets would be different. Imposing different boundary conditions does not change the Dirac brackets, but fails to identify the BMS mode correctly (at leading order the β-mode does not contribute, but at subleading orders it does).

We would now like to express the generator of soft supertranslations in terms of the creation and annihilation operators. To do this, we use the results of [122, 123], where it was shown that to leading order in the asymptotic expansion, the soft supertranslations generator is gauge-invariant. Therefore, instead of performing the computation in the general Gupta-Bleuler quantization that we have been using so far, we can go further and fix the residual gauge freedom left after setting the harmonic gauge. That can be done by adopting the canonical quantization in terms of the physical, transverse-traceless, components of the graviton

$$a_{\mu\nu}(k) = \sum_{r=\pm} \epsilon^{r*}_{\mu\nu}(k) a_r(k), \tag{2.55}$$

where the momentum modes in the polarization basis obey the following commutation relations

$$\left[a_r(k), a_s^\dagger(k')\right] = \delta_{rs}(2\omega_k)(2\pi)^3\delta^3\left(\mathbf{k} - \mathbf{k}'\right). \tag{2.56}$$

The transverse-traceless components of the polarization tensor can be decomposed as follows

$$\epsilon^{\pm}_{\mu\nu}(k) = \epsilon^{\pm}_{\mu}(k)\epsilon^{\pm}_{\nu}(k), \tag{2.57}$$

and we further use the following concrete realization for them

$$\begin{aligned} \epsilon^{-\mu}(k) &= \frac{1}{\sqrt{2}}\left(z, 1, +i, -z\right), \\ \epsilon^{+\mu}(k) &= \frac{1}{\sqrt{2}}\left(\bar{z}, 1, -i, -\bar{z}\right). \end{aligned} \tag{2.58}$$

Using the physical components of the polarization tensor we completely fix the gauge and eliminate any ambiguities that could complicate the computation. As explained above, that does not change the result since, to leading order in the asymptotic expansion, the soft supertranslation generator is independent of the gauge choice [122, 123].

Using this and the plane wave expansion (for example see Appendix A of [18]), we write the radiative data as [15]

$$\begin{aligned} C_{zz}(u, z, \bar{z}) &= \kappa \lim_{r\to\infty} \frac{1}{r} h_{zz}(r, u, z, \bar{z}) \\ &= \kappa \lim_{r\to\infty} \frac{1}{r} \partial_z x^\mu \partial_z x^\nu h_{\mu\nu} \\ &= -\frac{i\kappa}{8\pi^2}\gamma_{z\bar{z}} \int_0^\infty d\omega_k \left[a_+(\omega_k \hat{\mathbf{x}}_z) e^{-i\omega_k u} - a_-^\dagger(\omega_k \hat{\mathbf{x}}_z) e^{i\omega_k u}\right]. \end{aligned} \tag{2.59}$$

where we have been using that the four-momentum of the graviton, being a massless excitation, can be expressed as

$$k^\mu = \frac{\omega_k}{1+z\bar{z}} \left(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\right). \tag{2.60}$$

The soft supertranslations generator (2.50) can then be written as[2]

$$\begin{aligned} T_{\text{soft}}(f) &= -\frac{1}{16\pi G}\int dud^2z \left[N_{\bar{z}}{}^{z} D_z^2 f + N_z{}^{\bar{z}} D_{\bar{z}}^2 f\right] \\ &= \lim_{\omega_k\to 0} \frac{\omega_k}{4\pi\kappa}\int d^2z \left[\left(a_+(\omega_k \hat{\mathbf{x}}_z) + a_-^\dagger(\omega_k \hat{\mathbf{x}}_z)\right) D_{\bar{z}}^2 f + \text{h.c.}\right]. \end{aligned} \tag{2.61}$$

In this form it is clear why T_{soft} is, indeed, described by soft gravitons.

2.3 Action of BMS supertranslation

In this section we study the action of BMS supertranslations on single-particle states, as well as on the vacuum state, using the expressions obtained in section 2.2.

2.3.1 Outgoing graviton

Using the expression (2.61) and the commutation relations (2.56), the action of the supertranslation generator on an outgoing soft graviton at future null infinity is given by

$$\left[T(f), a_{+/-}(\mathbf{k})\right] = \frac{8\pi^2}{\kappa}\frac{1}{\gamma_{z_k\bar{z}_k}}\delta(\omega_k) D_{z/\bar{z}}^2 f. \tag{2.62}$$

Since we take $a_{+/-}(\mathbf{k})$ to be soft, it is the soft part $T_{\text{soft}}(f)$ of $T(f)$ that contributes to (2.62), and hence the delta function on the right hand side.

2.3.2 Undressed massive particle

The action of BMS supertranslations on an undressed massive particle has been studied in detail by Campiglia and Laddha [36, 124, 125]. Here we briefly review this result. For simplicity we take the particle to be a scalar, but to leading order the result is the same for particles of any spin.

[2] We use that $\frac{1}{2\pi}\int du e^{-i\omega u} = \delta(\omega)$. Note that since the ω-integration is over half the real plane we have

$$\int_0^\infty d\omega\, \delta(\omega) f(\omega) = \frac{1}{2} f(0)$$

The retarded system of coordinates $(u, r, z, \bar{z})$ is useful to describe null infinity, and therefore more convenient when we discuss massless particles. However, massive particles reach null infinity only asymptotically (in the future), and to describe them it is more convenient to use the hyperbolic system of coordinates that we have introduced in subsection 2.2.2.

The canonically quantized massive scalar field is given by (1.8),

$$\varphi(x) = \int \widetilde{d^3p} \left[b(p) e^{ip\cdot x} + b^\dagger(p) e^{-ip\cdot x} \right]. \tag{2.63}$$

The creation and annihilation operators of the scalar particle obey the commutation relation (1.11),

$$\left[b(p), b^\dagger(p') \right] = (2\pi)^3 (2\omega_p) \delta^3(\mathbf{p} - \mathbf{p}\,'), \tag{2.64}$$

where $\omega_p^2 = |\mathbf{p}|^2 + m^2$. The phase factor is

$$x \cdot p = \tau \left(\rho\, \hat{\mathbf{x}} \cdot \mathbf{p} - \omega_p \sqrt{1 + \rho^2} \right). \tag{2.65}$$

At large τ the integral in (2.63) is dominated by a saddle point at $\mathbf{p} = m\rho\hat{\mathbf{x}}$,

$$\lim_{\tau\to\infty} \varphi(x) = \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} \left[b(m\rho\hat{\mathbf{x}}) e^{-im\tau} + b^\dagger(m\rho\hat{\mathbf{x}}) e^{im\tau} \right], \tag{2.66}$$

where the constant phase factors have been absorbed into the creation and annihilation operators. Asymptotically, the scalar field transforms under BMS supertranslations as

$$\delta_f \varphi = \widetilde{\lambda}_\tau(\rho, z, \bar{z}) \partial_\tau \varphi, \tag{2.67}$$

where $\widetilde{\lambda}_\tau$ is a function that parametrizes the transformation (see appendix C). The annihilation operator therefore transforms as

$$\delta_f b(p) = -im\widetilde{\lambda}_\tau(|\mathbf{p}|/m, z, \bar{z})\, b(p), \tag{2.68}$$

which is equivalent to the following commutation relation

$$\begin{aligned} [T(f), b(p)] &= -m\widetilde{\lambda}_\tau(|\mathbf{p}|/m, z, \bar{z}) b(p) \\ &= -b(p) \int \frac{d^2z}{4\pi} \sqrt{\gamma} \frac{m^4}{\left(\sqrt{m^2 + |\mathbf{p}|^2} - \mathbf{p}\cdot\hat{\mathbf{x}}_z\right)^3} f(z, \bar{z}). \end{aligned} \tag{2.69}$$

2.3.3 Vacuum

BMS supertranslations give rise to a freedom in the definition of the vacuum. We define the vacuum as the state that satisfies

$$a(\omega\hat{\mathbf{x}})\,|0\rangle = 0\;, \tag{2.70}$$

which applies, in particular, to a soft graviton annihilation operator. Alternatively, the state $T(f)\,|0\rangle = T_{\text{soft}}(f)\,|0\rangle$, for any function $f(z,\bar{z})$, could serve as the zero energy state (note that the hard part inside $T(f)$ annihilates the vacuum). Physically, this state differs from the original vacuum (2.70) by the addition of a soft graviton. In this section we show that these different choices are orthogonal to each another. More explicitly, we show that acting with the generator of BMS supertranslations on the original vacuum (2.70) creates a state which is orthogonal to any state constructed from the original vacuum

$$\langle 0|\,T(f)\hat{\Psi}^{\text{out}}e^{-R_f}\mathcal{S}e^{R_f}\hat{\Psi}^{\text{in}}\,|0\rangle = 0\;. \tag{2.71}$$

This implies that no physical process can transform the original vacuum into the new state generated by BMS supertranslations. This is one of our main results in this chapter.

We start by considering a scattering process with an emission of a single soft graviton. The amplitude for this process is given by

$$\begin{aligned}\mathcal{M}_{k,\,\text{soft}} &= \langle\text{out}|e^{-R_f}\mathcal{S}e^{R_f}|\text{in}\rangle\\ &= \langle\mathbf{k},r|\hat{\Psi}^{\text{out}}e^{-R_f}\mathcal{S}e^{R_f}\hat{\Psi}^{\text{in}}|0\rangle\;\;,\end{aligned} \tag{2.72}$$

where[3]

$$\langle\mathbf{k},r| = \langle 0|\,a_r(k) = \epsilon^r_{\rho\sigma}(k)\,\langle 0|\,a^{\rho\sigma}(k) \tag{2.73}$$

is the soft graviton state with polarization r. The scalar operators are given by

$$\hat{\Psi}^{\text{in}} \equiv \prod_{i\in\text{in}} b^\dagger(p_i) \qquad \text{and} \qquad \hat{\Psi}^{\text{out}} \equiv \prod_{j\in\text{out}} b(p_j), \tag{2.74}$$

where “in" and “out" denote the set of incoming and outgoing scalar particles, respectively. The soft graviton can connect to a diagram in three different ways:

1. Connect to an external scalar leg.
2. Connect to the graviton cloud $e^{\pm R_f}$ (or equivalently $e^{\pm R_f(p)}$).
3. Connect to an internal leg.

[3]We have used $\epsilon^t\cdot\epsilon^{-r} = \delta^{tr}$.

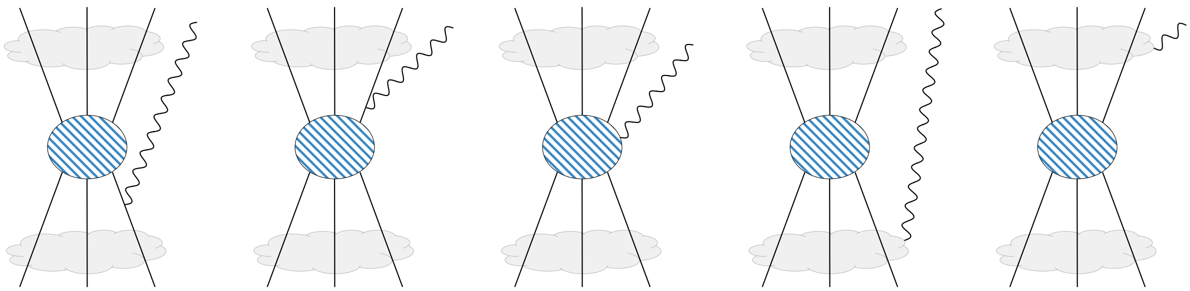


Figure 2.2: Different ways to connect an external soft graviton to a scattering amplitude. The first two diagrams on the left represent a soft graviton that is connected to an external leg. The last two diagrams on the right represent a soft graviton that is connected to the gravitons' cloud. The diagram in the middle represents a soft graviton that is connected to an internal leg.

These three options are depicted in figure 2.2. Contractions of the last type are IR-convergent, and therefore do not contribute to the amplitude at leading order. The cloud also dresses the soft graviton operator, but this dressing involves the scalar number operator $\rho(p) = b^\dagger(p)b(p)$ and vanishes by acting on the vacuum.

Consider the contraction of the first type. By virtue of the soft theorem [2], each such contraction contributes

$$\eta\,\kappa\,\frac{p_\mu p_\nu}{2p\cdot k}\epsilon^r_{\mu\nu}(k)\mathcal{M}\ , \tag{2.75}$$

where $\eta = +1$ for an outgoing state and $\eta = -1$ for an incoming state, and

$$\mathcal{M} \equiv \langle 0|\,\hat{\Psi}^{\rm out} e^{-R_f}\mathcal{S}e^{R_f}\hat{\Psi}^{\rm in}\,|0\rangle \tag{2.76}$$

is the amplitude without the soft graviton. Let us briefly review the derivation of (2.75). Using the commutation relation (1.12), we derive the momentum-space contraction rule to be

$$\begin{aligned}\langle \mathbf{k}, \overbrace{r|\,h}_{\mu\nu} &= \epsilon^{r,\rho\sigma}(k)\,\langle 0|\int \widetilde{d^3k'}\,\left[a_{\rho\sigma}(k), a^\dagger_{\mu\nu}(k')\right]\\ &= \frac{1}{2}\epsilon^{r,\rho\sigma}(k)I_{\rho\sigma\mu\nu}\,\langle 0|\\ &= \epsilon^r_{\mu\nu}(k)\,\langle 0|\,,\end{aligned} \tag{2.77}$$

where we used (2.18) in the last line. One may consider this to be the external "wavefunction" of a graviton with polarization r. Next, we observe that the insertion of a soft graviton to an external leg with momentum p adds a scalar propagator

$$\frac{-i}{(p \pm k)^2 + m^2} \quad \xrightarrow{k \to 0} \quad \mp \frac{i}{2p \cdot k} \tag{2.78}$$

and scalar-scalar-graviton vertex

$$\frac{i\kappa}{2} \left(p_\mu (p \pm k)_\nu + p_\nu (p \pm k)_\mu - \frac{1}{2} \eta_{\mu\nu} \left[p \cdot (p \pm k) + m^2 \right] \right) \quad \xrightarrow{k \to 0} \quad i\kappa p_\mu p_\nu \ , \tag{2.79}$$

where the upper (lower) sign is for an outgoing (incoming) state. Putting (2.77)-(2.79) together, we recover the result (2.75) in the soft limit.

Next, we study the soft gravitons' contractions of the second type, i.e. to the clouds of gravitons. The incoming and outgoing asymptotic states can be written in terms of single particle dressed states:

$$e^{R_f} \hat{\Psi}^{\text{in}} |0\rangle = \left[\prod_{i \in \text{in}} b^\dagger(p_i) e^{R_f(p_i)} \right] |0\rangle \tag{2.80}$$

and

$$\langle 0| \hat{\Psi}^{\text{out}} e^{-R_f} = \langle 0| \left[\prod_{j \in \text{out}} e^{-R_f(p_j)} b(p_j) \right] . \tag{2.81}$$

The contraction of the soft graviton with the cloud gives

$$\begin{aligned}
\langle \mathbf{k}, r| \, e^{\pm R_f(p)} &= \pm \langle \mathbf{k}, r| \, R_f(p) e^{\pm R_f(p)} \\
&= \pm \epsilon^{r,\rho\sigma}(k) \, \langle 0| \, \frac{\kappa}{2} \int \widetilde{d^3k'} \, f^{\mu\nu}(k', p) \left[a_{\rho\sigma}(k), a^\dagger_{\mu\nu}(k') \right] e^{\pm R_f(p)} \\
&= \pm \frac{\kappa}{4} f^{\mu\nu}(p, k) I_{\mu\nu\rho\sigma} \epsilon^{r,\rho\sigma}(k) \, \langle 0| \, e^{\pm R_f(p)} \\
&= \pm \frac{\kappa}{2} f^{\mu\nu}(p, k) \epsilon^r_{\mu\nu}(k) \, \langle 0| \, e^{\pm R_f(p)} \ ,
\end{aligned} \tag{2.82}$$

where the upper (lower) sign is for incoming (outgoing) particles.

Using these results, we obtain

$$\mathcal{M}_{k,\,\text{soft}} = \frac{\kappa}{2} \left[\left(\sum_{j \in \text{out}} \frac{p_j^\mu p_j^\nu}{p_j \cdot k} - \sum_{i \in \text{in}} \frac{p_i^\mu p_i^\nu}{p_i \cdot k} \right) - \left(\sum_{j \in \text{out}} f^{\mu\nu}(p_j, k) - \sum_{i \in \text{in}} f^{\mu\nu}(p_i, k) \right) \right] \epsilon^r_{\mu\nu}(k) \mathcal{M}, \tag{2.83}$$

where the first two sums come from soft factors (2.75), and the last two sums come from contractions with clouds (2.82). Multiplying by ω_k and taking the soft limit results in

$$\begin{aligned}\lim_{\omega_k\to 0}\omega_k\,\mathcal{M}_{k,\,\mathrm{soft}} &= \lim_{\omega_k\to 0}\omega_k \\ &\times\frac{\kappa}{2}\left[\sum_{j\in\mathrm{out}}\frac{p_j^\mu p_j^\nu}{p_j\cdot k}\Big(1-\phi(p_j,k)\Big)-\sum_{i\in\mathrm{in}}\frac{p_i^\mu p_i^\nu}{p_i\cdot k}\Big(1-\phi(p_i,k)\Big)\right. \\ &\left.-\frac{1}{\omega_k}\left(\sum_{j\in\mathrm{out}}c'^{\mu\nu}(p_j,k)\phi(p_j,k)-\sum_{i\in\mathrm{in}}c^{\mu\nu}(p_i,k)\phi(p_i,k)\right)\right]\epsilon^r_{\mu\nu}(k)\mathcal{M},\end{aligned} \tag{2.84}$$

where the c-matrix $c_{\mu\nu}$ ($c'_{\mu\nu}$) was used to construct the incoming (outgoing) state. Since $\phi(p,k)\to 1$ in this limit, each term in the second line of (2.84) having a factor of the form $(1-\phi)$ vanishes. Then, the right-hand side of (2.84) becomes

$$\lim_{\omega_k\to 0}-\frac{\kappa}{2}\left[\sum_{j\in\mathrm{out}}c'^{\mu\nu}(p_j,k)-\sum_{i\in\mathrm{in}}c^{\mu\nu}(p_i,k)\right]\epsilon^r_{\mu\nu}(k)\mathcal{M}. \tag{2.85}$$

With the parametrization (2.15), we observe that using the same q for incoming and outgoing states reduces this to

$$\lim_{\omega_k\to 0}-\frac{\kappa}{2}\left[\frac{\omega_k}{k\cdot q}\left(\frac{k\cdot p_{\mathrm{tot}}}{k\cdot q}q^\mu q^\nu-2q^\mu p^\nu_{\mathrm{tot}}\right)\right]\epsilon^r_{\mu\nu}(k)\mathcal{M}=0, \tag{2.86}$$

since

$$p_{\mathrm{tot}}\equiv\sum_{j\in\mathrm{out}}p_j-\sum_{i\in\mathrm{in}}p_i=0 \tag{2.87}$$

by energy-momentum conservation. For the general case where $c'_{\mu\nu}\neq c_{\mu\nu}$, we show at the end of appendix B that processes with non-zero amplitudes can only occur between states that satisfy

$$\sum_{j\in\mathrm{out}}c'^{\mu\nu}(p_j,k)=\sum_{i\in\mathrm{in}}c^{\mu\nu}(p_i,k). \tag{2.88}$$

Therefore, we conclude that

$$\lim_{\omega_k\to 0}\omega_k\mathcal{M}_{k,\,\mathrm{soft}}=0\ , \tag{2.89}$$

that is, $\mathcal{M}_{k,\,\mathrm{soft}}$ is not IR divergent. Since the creation operators annihilate the Bra vacuum, the action of the soft part of the BMS supertranslations (2.61) is given by

$$\langle 0|T(f)=\lim_{\omega_k\to 0}\frac{\omega_k}{4\pi\kappa}\int d^2z\,\langle 0|\left[a_-(\omega_k\hat{x})D_z^2f+a_+(\omega_k\hat{x})D_{\bar{z}}^2f\right]. \tag{2.90}$$

The soft limit of the amplitude, equation (2.89), together with (2.90), then implies the identity (2.71). Namely, the original vacuum state $|0\rangle$ and and the new state $T_{\text{soft}}(f)\,|0\rangle$ are orthogonal.

2.4 BMS supertranslation of asymptotic states

We are now in a position to compute BMS supertranslations of a physical asymptotic state. For simplicity we consider a single particle state with momentum p dressed with a graviton cloud. The action of the supertranslation generator on the physical asymptotic state can be decomposed into the following three pieces

$$\begin{aligned}\langle 0|\, e^{-R_f(p)}\, b(p)\, T(f) = \langle 0|\, T(f)\, e^{-R_f(p)}\, b(p) \\ + \langle 0| \left[e^{-R_f(p)}\,, T(f)\right] b(p) - \langle 0|\, e^{-R_f(p)}\, \left[T(f), b(p)\right].\end{aligned} \tag{2.91}$$

The first term in (2.91) is the action of BMS supertranslation on the vacuum. It vanishes when contracted with an incoming (ket) state, by the result of previous section. The second and third terms are the actions of BMS on the graviton cloud and on the massive particle, respectively.

Let us first compute the commutator of $R_f(p)$ and $T(f)$,

$$[R_f(p), T(f)] = \frac{\kappa}{2}\int \widetilde{d^3k}\, \left(f^{\mu\nu}(p,k)\epsilon^r_{\mu\nu}\left[a^\dagger_r(k), T(f)\right] - \text{h.c.}\right). \tag{2.92}$$

Using (2.62) we arrive at

$$\begin{aligned}[R_f(p), T(f)] &= -4\pi^2 \int \frac{\widetilde{d^3k}}{\gamma_{z_k\bar z_k}}\delta(\omega_k)\left[f^{\mu\nu}(p,k)\left(\epsilon^-_{\mu\nu}D_z^2 f + \epsilon^+_{\mu\nu}D_{\bar z}^2 f\right) + \text{h.c.}\right] \\ &= -\pi^2 \int \frac{d^2z}{(2\pi)^3}\omega_k\left[f^{\mu\nu}(p,k)\left(\epsilon^-_{\mu\nu}D_z^2 f + \epsilon^+_{\mu\nu}D_{\bar z}^2 f\right) + \text{h.c.}\right].\end{aligned} \tag{2.93}$$

Defining $\hat{k}_{z,\bar z} \equiv (1, \hat{\mathbf{k}}_{z,\bar z})$, the last expression takes the form

$$[R_f(p), T(f)] = -\pi^2 \int \frac{d^2z}{(2\pi)^3}\left(\frac{p^\mu p^\nu}{\hat{k}_{z,\bar z}\cdot p} + c^{\mu\nu}\right)\left[\left(\epsilon^-_{\mu\nu}(\hat{k}_{z,\bar z})D_z^2 f + \epsilon^+_{\mu\nu}(\hat{k}_{z,\bar z})D_{\bar z}^2 f\right) + \text{h.c.}\right], \tag{2.94}$$

where according to our convention the delta function yielded half the value of the integrand at 0. Since $\epsilon^{r*}_{\mu\nu} = \epsilon^{-r}_{\mu\nu}$, we arrive at

$$[R_f(p), T(f)] = -2\pi^2 \int \frac{d^2z}{(2\pi)^3}\left(\frac{p^\mu p^\nu}{\hat{k}_{z,\bar z}\cdot p} + c^{\mu\nu}\right)\left(\epsilon^-_{\mu\nu}(\hat{k}_{z,\bar z})D_z^2 f + \epsilon^+_{\mu\nu}(\hat{k}_{z,\bar z})D_{\bar z}^2 f\right). \tag{2.95}$$

Integrating by parts, we then have[4]

$$[R_f(p), T(f)] = -\int \frac{d^2z}{4\pi} \left[\partial_z \partial^{\bar{z}} \left(\gamma_{z\bar{z}} \frac{p^\mu p^\nu \epsilon^-_{\mu\nu}}{\hat{k}_{z,\bar{z}} \cdot p} \right) + \partial_{\bar{z}} \partial^z \left(\gamma_{z\bar{z}} \frac{p^\mu p^\nu \epsilon^+_{\mu\nu}}{\hat{k}_{z,\bar{z}} \cdot p} \right) + C(p, z, \bar{z}) \right] f \ , \tag{2.96}$$

where we have defined

$$C(p, z, \bar{z}) \equiv \partial_z \partial^{\bar{z}} (\gamma_{z\bar{z}} c^{\mu\nu} \epsilon^-_{\mu\nu}) + \partial_{\bar{z}} \partial^z (\gamma_{z\bar{z}} c^{\mu\nu} \epsilon^+_{\mu\nu}). \tag{2.97}$$

With $\epsilon^\pm_{\mu\nu} = \epsilon^\pm_\mu \epsilon^\pm_\nu$, we get

$$[R_f(p), T(f)] = -\int \frac{d^2z}{4\pi} \left[\partial_z \partial^{\bar{z}} \left(\gamma_{z\bar{z}} \frac{(p \cdot \epsilon^-)^2}{\hat{k}_{z,\bar{z}} \cdot p} \right) + \partial_{\bar{z}} \partial^z \left(\gamma_{z\bar{z}} \frac{(p \cdot \epsilon^+)^2}{\hat{k}_{z,\bar{z}} \cdot p} \right) + C(p, z, \bar{z}) \right] f \ . \tag{2.98}$$

An explicit calculation shows that

$$\partial_z \partial^{\bar{z}} \left(\gamma_{z\bar{z}} \frac{(p \cdot \epsilon^-)^2}{\hat{k}_{z,\bar{z}} \cdot p} \right) = \partial_{\bar{z}} \partial^z \left(\gamma_{z\bar{z}} \frac{(p \cdot \epsilon^+)^2}{\hat{k}_{z,\bar{z}} \cdot p} \right) = \frac{1}{2} \gamma_{z\bar{z}} \frac{p^4}{(p \cdot \hat{k}_{z,\bar{z}})^3} \ . \tag{2.99}$$

We therefore end up with

$$[R_f(p), T(f)] = -\int \frac{d^2z}{4\pi} \left[\gamma_{z\bar{z}} \frac{p^4}{(p \cdot \hat{k}_{z,\bar{z}})^3} + C(p, z, \bar{z}) \right] f \ . \tag{2.100}$$

The first contribution,

$$\frac{p^4}{(p \cdot \hat{k}_{z,\bar{z}})^3} = \frac{m^4}{\left(\mathbf{p} \cdot \hat{\mathbf{k}}_{z,\bar{z}} - \sqrt{m^2 + |\mathbf{p}|^2} \right)^3} \ , \tag{2.101}$$

is the Aichelburg-Sexl gravitational field of a massive particle [126]. This is the gravitational analogue of the Lienard-Wiechert electromagnetic radiation field of a moving charged particle.

We now see that the first term in the second line of (2.91) is equal to

$$\langle 0 | \, e^{-R_f(p)} \, [-R_f(p) \, , T(f)] \, b(p) \ , \tag{2.102}$$

[4]Derivatives with upper indices are defined as usual by $\partial^{\bar{z}} = \gamma^{\bar{z}z} \partial_z$ and $\partial^z = \gamma^{z\bar{z}} \partial_{\bar{z}}$.

since $[R_f(p)\ ,T(f)]$ is a c-number. We therefore have

$$\begin{aligned}\langle 0|\, e^{-R_f(p)}\, b(p)\, T(f) = \langle 0|\, T(f)\, e^{-R_f(p)}\, b(p) \\ - \langle 0|\, e^{-R_f(p)}\Big\{\, [R_f(p)\ ,T(f)]\, b(p) + [T(f), b(p)]\,\Big\}.\end{aligned} \tag{2.103}$$

From (2.69) and (2.100), we observe that the BMS of the bare particle and the momentum dependent part of the BMS of the graviton cloud exactly cancel each other[5]. Finally, the outgoing BMS charge between the two physical asymptotic states is then given by

$$\frac{\langle 0|\, \hat{\Psi}^{\rm out}_{\rm as} T_{\rm soft} \mathcal{S} \hat{\Psi}^{\rm in}_{\rm as}\, |0\rangle}{\langle 0|\, \hat{\Psi}^{\rm out}_{\rm as} \mathcal{S} \hat{\Psi}^{\rm in}_{\rm as}\, |0\rangle} = -\sum_{j\in \rm out} \int \frac{d^2z}{4\pi} C(p_j, z, \bar{z}) f(z, \bar{z}). \tag{2.104}$$

Similarly, one can also construct the BMS charge of an incoming physical asymptotic state. The BMS charge (2.104) parameterizes the asymptotic state and is conserved as long as BMS supertranslation is a symmetry of the system, in line with the discussion in section 2.1.

To better understand the meaning of the BMS charge and the implications of the BMS symmetry, we end this section by looking at a BMS eigenstate defined as

$$\langle \Omega_\Lambda|\, T(f) \equiv \int \frac{d^2z}{4\pi} \Lambda(z, \bar{z}) f(z, \bar{z})\, \langle \Omega_\Lambda|\,, \tag{2.105}$$

and which is related to the vacuum by

$$\langle 0| = \int \mathcal{D}[\Lambda] e^{-\frac{1}{2}\Lambda^2}\, \langle \Omega_\Lambda|\,, \qquad \text{where} \qquad \Lambda^2 = \int \frac{d^2z}{4\pi} \Lambda^2(z, \bar{z})\ . \tag{2.106}$$

in a similar fashion to the case of QED [18]. The asymptotic states built from these eigenstates are also eigenstates of BMS transformations

$$\langle \Omega_\Lambda|\, e^{-R_f(p)} b(p) T(f) = \langle \Omega_\Lambda|\, e^{-R_f(p)} b(p) \int \frac{d^2z}{4\pi} \left[\Lambda(z, \bar{z}) - C(p, z, \bar{z})\right] f(z, \bar{z}), \tag{2.107}$$

and similarly their BMS charge is given by

$$\frac{\langle \Omega_\Lambda|\, \hat{\Psi}^{\rm out}_{\rm as} T_{\rm soft} \mathcal{S} \hat{\Psi}^{\rm in}_{\rm as}\, |\Omega_\Lambda\rangle}{\langle \Omega_\Lambda|\, \hat{\Psi}^{\rm out}_{\rm as} \mathcal{S} \hat{\Psi}^{\rm in}_{\rm as}\, |\Omega_\Lambda\rangle} = \int \frac{d^2z}{4\pi} \left[\Lambda(z, \bar{z}) - \sum_{j\in \rm out} C(p_j, z, \bar{z})\right] f(z, \bar{z}) \tag{2.108}$$

The state $|\Omega_\Lambda\rangle$ belongs to a superselection sector which is characterized by its BMS charge. We can now study the transition amplitude between two different BMS eigenstates by com-

[5]Note that $\sqrt{\gamma} = \gamma_{z\bar{z}}$.

puting the expectation value of the following commutator

$$\begin{aligned}
&\langle\Omega_{\Lambda_1}|\hat{\Psi}_{\rm as}^{\rm out}[T(f),\mathcal{S}]\hat{\Psi}_{\rm as}^{\rm in}|\Omega_{\Lambda_2}\rangle \\
&\quad=\int\frac{d^2z}{4\pi}\left[\Lambda_1(z,\bar{z})-\Lambda_2(z,\bar{z})\right]f(z,\bar{z})\,\langle\Omega_{\Lambda_1}|\,\hat{\Psi}_{\rm as}^{\rm out}\mathcal{S}\hat{\Psi}_{\rm as}^{\rm in}|\Omega_{\Lambda_2}\rangle\,,
\end{aligned}\tag{2.109}$$

where we used (2.88) to remove the terms involving $C(z,\bar{z})$. The left hand side of equation (2.109) is the difference between the total incoming and outgoing BMS charges, which is zero by the conservation law of the symmetry. The right hand side vanishes when either

$$\Lambda_1=\Lambda_2 \tag{2.110}$$

or

$$\langle\Omega_{\Lambda_1}|\,\hat{\Psi}_{\rm as}^{\rm out}\mathcal{S}\hat{\Psi}_{\rm as}^{\rm in}|\Omega_{\Lambda_2}\rangle=0 \qquad \text{for} \qquad \Lambda_1\neq\Lambda_2 \tag{2.111}$$

We therefore conclude that BMS symmetry implies that the amplitude for transition between different superselection sectors is zero, once the contribution of the FK clouds is taken into account.

2.5 Summary

In this chapter, we have studied the effect of BMS supertranslations on physical asymptotic states in perturbative quantum gravity. These states were constructed in [9] using the method of Kulish and Faddeev for QED [8] by dressing the Fock states with a cloud of soft gravitons. BMS supertranslations, in turn, give rise to a freedom in the definition of the vacuum. By acting with the BMS generator on the vacuum one generates a different state which could equally serve as the zero energy state. Therefore there exists a family of states generated by the action of BMS supertranslations on the vacuum. This is a continuous family (or a moduli) which is parameterized by the BMS transformation parameter.

We end with a summary of the main results. First, we have shown that all the states in this family are orthogonal to each other once we take into account the contribution of the FK clouds, see equation (2.71). In other words, the amplitude for transition between any two states in this family is zero for any physical process. Second, we have computed the BMS charge of a physical asymptotic state, see equation (2.107). The BMS charge is conserved if BMS supertranslation is a symmetry of the system, see equations (2.110) and (2.111). It characterizes the superselection sector to which the state belongs and the conservation law implies that there is no transition between different superselection sectors.

Chapter 3

BMS Supertranslation Symmetry Implies Faddeev-Kulish Amplitudes

3.1 BMS charge and eigenstates

In this chapter, we show that the dressed states of gravity in scattering amplitudes act essentially as BMS charge eigenstates. We then use this property to re-derive the dressed states by introducing the BMS charge as a new quantum number.

In order to establish notation and to make connections with earlier work, we begin with a review of BMS symmetry and the conserved charges. As is customary, we employ the retarded coordinates $(u, r, z, \bar{z})$, defined in terms of the Cartesian coordinates (t, x_1, x_2, x_3) as

$$u = t - r, \qquad r^2 = x_1^2 + x_2^2 + x_3^2, \qquad z = \frac{x_1 + ix_2}{r + x_3}. \tag{3.1}$$

Here u is the retarded time and z is the complex coordinate on the unit 2-sphere with the metric $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$. Then in the Bondi gauge [11, 12], the asymptotically flat metric has the expansion [14]

$$\begin{aligned} ds^2 = &- du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^zC_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} \\ &+ \cdots , \end{aligned} \tag{3.2}$$

where m_B is the Bondi mass aspect and D^z, $D^{\bar{z}}$ are the 2-sphere covariant derivatives. The gravitational radiation is characterized by the Bondi news tensor $N_{zz} = \partial_u C_{zz}$.

The BMS supertranslation charge for a 2-sphere function $f = f(w, \bar{w})$ is then

$$Q(f) = Q_S(f) + Q_H(f), \tag{3.3}$$

where, explicit expressions for the soft part Q_S and the hard part Q_H are given in [1, 36]. We are interested in these expressions at the leading terms in the large-r expansion which are known to be gauge-invariant [122].

The action of the hard charge Q_H on a Fock state of N massive particles can be expressed as [36]

$$Q_H \left|\mathbf{p}_1, \ldots, \mathbf{p}_N\right\rangle = \sum_{i=1}^{N} \widetilde{f}(p_i) \left|\mathbf{p}_1, \ldots, \mathbf{p}_N\right\rangle, \tag{3.4}$$

where $p_i^\mu = (E_k, \mathbf{p}_i)$, and

$$\widetilde{f}(p) = -\frac{1}{2\pi} \int d^2w \, \frac{(\epsilon^+(w, \bar{w}) \cdot p)^2}{p \cdot \hat{x}_w} D_{\bar{w}}^2 f(w, \bar{w}). \tag{3.5}$$

Here $\hat{x}_w^\mu = (1, \hat{\mathbf{x}}_w)$ with the unit vector $\hat{\mathbf{x}}_w$ pointing in the direction $(w, \bar{w})$, and the polarization vectors have components

$$\epsilon^{-\mu}(z, \bar{z}) = \frac{1}{\sqrt{2}}(z, 1, i, -z) \quad \text{and} \quad \epsilon^{+\mu}(z, \bar{z}) = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}). \tag{3.6}$$

The action of the soft charge Q_S on the same state is [14]

$$Q_S \left|\mathbf{p}_1, \ldots, \mathbf{p}_N\right\rangle = -\frac{1}{8\pi G} \int du \, d^2w \, \gamma_{w\bar{w}} N^{\bar{w}\bar{w}} D_{\bar{w}}^2 f \left|\mathbf{p}_1, \ldots, \mathbf{p}_N\right\rangle. \tag{3.7}$$

Conservation of BMS supertranslation charges imply,

$$\langle \mathrm{out}| \left[Q(f), \mathcal{S}\right] |\mathrm{in}\rangle = 0, \tag{3.8}$$

which should hold for all functions $f(w, \bar{w})$. In particular, let us choose

$$f(w, \bar{w}) = \frac{(1 + w\bar{w})(\bar{w} - \bar{z})}{(1 + z\bar{z})(w - z)}, \tag{3.9}$$

such that [36]

$$D_{\bar{w}}^2 f(w, \bar{w}) = 2\pi \delta^2(w - z). \tag{3.10}$$

With this choice, the conservation law (3.8) reads

$$\frac{\gamma_{z\bar{z}}}{4G}\int_{-\infty}^{\infty} du\,\langle\text{out}|(N^{\bar{z}\bar{z}}\mathcal{S}-\mathcal{S}N^{\bar{z}\bar{z}})|\text{in}\rangle = -\sum_i \eta_i \frac{(p_i\cdot\epsilon^+(z,\bar{z}))^2}{p_i\cdot\hat{x}_z}\langle\text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{3.11}$$

where the sum on the RHS runs over all external particles and $\eta_i = +1$ (-1) if i is an outgoing (incoming) particle. Let us define the operator

$$N(z,\bar{z}) \equiv \gamma_{z\bar{z}}\int_{-\infty}^{\infty} du\, N^{\bar{z}\bar{z}} = \gamma^{z\bar{z}}\int_{-\infty}^{\infty} du\, N_{zz}. \tag{3.12}$$

Then (3.11) becomes

$$\langle\text{out}|(N(z,\bar{z})\mathcal{S}-\mathcal{S}N(z,\bar{z}))|\text{in}\rangle = -\frac{\kappa^2}{8\pi}\sum_i \eta_i \frac{(p_i\cdot\epsilon^+(z,\bar{z}))^2}{p_i\cdot\hat{x}_z}\langle\text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{3.13}$$

where $\kappa = \sqrt{32\pi G}$. If the in- and out-states are eigenstates of $N(z,\bar{z})$ such that

$$\langle\text{out}|\, N(z,\bar{z}) = N_{\text{out}}\,\langle\text{out}| \qquad \text{and} \qquad N(z,\bar{z})\,|\text{in}\rangle = N_{\text{in}}\,|\text{in}\rangle\,, \tag{3.14}$$

then we obtain

$$(N_{\text{out}}-N_{\text{in}})\,\langle\text{out}|\mathcal{S}|\text{in}\rangle = \Omega^{\text{soft}}\,\langle\text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{3.15}$$

with a soft factor that is analogous to that of [19]:

$$\Omega^{\text{soft}} = -\frac{\kappa^2}{8\pi}\sum_i \eta_i \frac{p_i^\mu p_i^\nu}{p_i\cdot\hat{x}_z}\epsilon^+_{\mu\nu}. \tag{3.16}$$

To see what the eigenstates look like, we first note that $N(z,\bar{z})$ can be expressed in terms of the graviton creation and annihilation operators as [15]

$$N(z,\bar{z}) = -\frac{\kappa}{8\pi}\lim_{\omega\to 0}\left[\omega a_+(\omega\hat{x}_z) + \omega a_-^\dagger(\omega\hat{x}_z)\right]. \tag{3.17}$$

This suggests that the eigenstate should take some form of a coherent graviton state. Next, consider the following state

$$|N\rangle = \exp\left\{\int \widetilde{d^3k}\, N^{\mu\nu}(k)\left[a^\dagger_{\mu\nu}(k) - a_{\mu\nu}(k)\right]\right\}|0\rangle\,, \tag{3.18}$$

where $\widetilde{d^3k} = \frac{d^3k}{(2\pi)^3(2\omega_k)}$ is the Lorentz-invariant measure,

$$a^\dagger_{\mu\nu}(k) = \sum_r \epsilon^r_{\mu\nu}(k) a^{r\dagger}(k), \qquad a_{\mu\nu}(k) = \sum_r \epsilon^{r*}_{\mu\nu}(k) a^r(k), \tag{3.19}$$

$N^{\mu\nu}$ is an arbitrary symmetric tensor and the sum runs over all polarizations, including the unphysical ones. We next show that if the symmetric tensor $N^{\mu\nu}(k)$ has soft poles, then the above state is an eigenstate of both $\lim \omega a_+$ and $\lim \omega a^\dagger_-$. Indeed,

$$\lim_{\omega\to 0} \omega a_+(\omega \hat{x}_z) |N\rangle = \lim_{\omega\to 0} \omega \left[a_+(\omega\hat{x}_z), \int \widetilde{d^3k}\, N^{\mu\nu}(k) \left(a^\dagger_{\mu\nu}(k) - a_{\mu\nu}(k) \right) \right] |N\rangle \tag{3.20}$$

$$= \lim_{\omega\to 0} \frac{\omega}{2} N_{\mu\nu}(\omega\hat{x}_z) I^{\mu\nu\rho\sigma} \epsilon^+_{\rho\sigma}(z,\bar{z}) |N\rangle \tag{3.21}$$

$$= \lim_{\omega\to 0} \omega N^{\mu\nu}(\omega\hat{x}_z) \epsilon^+_{\mu\nu}(z,\bar{z}) |N\rangle . \tag{3.22}$$

Thus we see that the eigenvalue is non-zero only if $N^{\mu\nu}$ has poles for soft momenta. Similarly,

$$\lim_{\omega\to 0} \omega a^\dagger_-(\omega\hat{x}_z) |N\rangle = \lim_{\omega\to 0} \omega N^{\mu\nu}(\omega\hat{x}_z) \epsilon^+_{\mu\nu}(z,\bar{z}) |N\rangle . \tag{3.23}$$

It should be noted that in (3.23), the term with the creation operator acting on the vacuum vanishes upon taking the soft limit $\omega \to 0$. From this we can immediately see that $|N\rangle$ is an eigenstate of $N(z,\bar{z})$, i.e.,

$$N(z,\bar{z}) |N\rangle = -\frac{\kappa}{4\pi} \left(\lim_{\omega\to 0} \omega N^{\mu\nu} \epsilon^+_{\mu\nu} \right) |N\rangle . \tag{3.24}$$

In particular, the Fock vacuum $|0\rangle$, which corresponds to $N^{\mu\nu} = 0$, is itself an eigenstate with eigenvalue 0. Later for convenience, when considering S matrix elements, we put $N^{\mu\nu} = 0$ for the incoming state, which amounts to assuming that the incoming state is a Fock state. This does not entail a loss of generality because as can be seen from (3.15), it is only the difference $N^{\mu\nu}_{\text{out}} - N^{\mu\nu}_{\text{in}}$ that matters. Similarly, the bra state

$$\langle N| = \langle 0| \exp\left[-\int \widetilde{d^3k}\, N^{\mu\nu} \left(a^\dagger_{\mu\nu} - a_{\mu\nu} \right) \right] \tag{3.25}$$

is an eigenstate of $N(z,\bar{z})$:

$$\langle N| N(z,\bar{z}) = -\frac{\kappa}{4\pi} \langle N| \left(\lim_{\omega\to 0} \omega N^{\mu\nu} \epsilon^+_{\mu\nu} \right) . \tag{3.26}$$

We want to treat these eigenstates as alternative vacuums, so we restrict the momentum integrals to run over only the soft momenta. With these choices, $|N\rangle$ remains an eigenstate with any number of hard particle operators acting on it.

The conservation law $N_{\text{out}} - N_{\text{in}} = \Omega^{\text{soft}}$ implied by (3.15) is then

$$\lim_{\omega\to 0} \omega \Big[N^{\mu\nu}_{\text{out}}(\omega \hat{x}_z) - N^{\mu\nu}_{\text{in}}(\omega \hat{x}_z) \Big] \epsilon^{+}_{\mu\nu}(z,\bar{z}) = \frac{\kappa}{2} \sum_i \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot \hat{x}_z} \epsilon^{+}_{\mu\nu}(z,\bar{z}). \tag{3.27}$$

As shown above, the leading soft terms in $N^{\mu\nu}$ are the only ones contributing to the eigenvalue, which therefore satisfy

$$N^{\mu\nu}_{\text{out}}(k) - N^{\mu\nu}_{\text{in}}(k) = \frac{\kappa}{2} \sum_i \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot k}, \tag{3.28}$$

where we have put $k = \omega \hat{x}_z$. We should emphasize that either this conservation law is satisfied or the amplitude $\langle \text{out}|\mathcal{S}|\text{in}\rangle$ vanishes. This implies that if the initial state is built on the Fock vacuum $|0\rangle$, i.e.

$$|\text{in}\rangle = \prod_{i\in\text{in}} b^\dagger(p_i) \, |0\rangle \,, \tag{3.29}$$

where $b^\dagger$ is the creation operator of hard massive particles, then this state does not scatter into any state built on the same vacuum $|0\rangle$, since in that case $N_{\text{out}} = N_{\text{in}} = 0$, thereby violating the conservation law $N_{\text{out}} - N_{\text{in}} = \Omega^{\text{soft}}$. Instead, scattering must take place into states built on the vacuum $|N_{\text{out}}\rangle$ with

$$N^{\mu\nu}_{\text{out}} = \frac{\kappa}{2} \sum_i \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot k}. \tag{3.30}$$

Such states therefore have the form, (see Eq. (3.18))

$$\langle \text{out}| = \langle 0| \left[\prod_{j\in\text{out}} b(p_j) \right] \exp\left[-\frac{\kappa}{2} \sum_i \eta_i \int \widetilde{d^3k} \, \frac{p_i^\mu p_i^\nu}{p_i \cdot k} (a^\dagger_{\mu\nu} - a_{\mu\nu}) \right]. \tag{3.31}$$

The scattering amplitude now can be written in the form:

$$\langle \text{out}| \, \mathcal{S} \, |\text{in}\rangle = \langle \Psi_{\text{out}}| \exp\left[-\frac{\kappa}{2} \sum_i \eta_i \int \widetilde{d^3k} \, \frac{p_i^\mu p_i^\nu}{p_i \cdot k} (a^\dagger_{\mu\nu} - a_{\mu\nu}) \right] \mathcal{S} \, |\Psi_{\text{in}}\rangle \,, \tag{3.32}$$

where Ψ_{out}, Ψ_{in} denote the usual Fock states for the hard particles. The form of (3.32) is reminiscent of the Faddeev-Kulish amplitudes. In the following two sections, we spell out this equivalence more precisely. It turns out that any amplitude that obeys the conservation law (3.28), an example being (3.32), is equal to the Faddeev-Kulish amplitude and is therefore IR-finite.

Before moving on, it is worth noting that the Faddeev-Kulish states of gravity have zero supertranslation charge for the specific choice made in (3.9) (but not for a general function $f(z,\bar{z})$, as shown in [18, 58]). This is analogous to the case of QED [19], where the corresponding Faddeev-Kulish states have zero large gauge charge for $f(z,\bar{z}) = \frac{1}{z-w}$. Therefore, Faddeev-Kulish states trivially conserve this large gauge charge, and this appears to have been a motivation for the conjecture in [19], that the conservation of large gauge charge leads to infrared-finiteness. The authors provide an argument based on crossing symmetry that, just as Faddeev-Kulish amplitudes are infrared-finite, other states with non-zero charge should lead to infrared-finite S-matrix elements as well. We prove this conjecture in the next section.

3.2 Relation to Faddeev-Kulish amplitudes

As a first step in establishing this equality, we demonstrate a crucial feature of the Faddeev-Kulish amplitudes which, although technical, has important physical consequences. Since a Faddeev-Kulish amplitude is constructed by dressing each external particle with its cloud of soft gravitons, an amplitude with n incoming and n' outgoing particles necessarily has n clouds on the right of the scattering operator $\mathcal{S}$, and n' clouds on the left. Although the clouds commute with each other, it was not clear how things change if, for example, one moves a cloud dressing an incoming particle (therefore sitting on the right of $\mathcal{S}$) to the left of $\mathcal{S}$. In this connection, based on the conservation of supertranslation charge and the crossing symmetry, the authors of [19] conjectured that such amplitudes exhibit the same cancellation of IR divergences. In this section, we explicitly show that the clouds "weakly commute" with $\mathcal{S}$, in the sense that in an $\mathcal{S}$ matrix element, any incoming cloud can be moved to the outgoing state without affecting the amplitude, and vice versa. This result proves the aforementioned conjecture, since it follows that the amplitudes considered in [19] are equal to the Faddeev-Kulish amplitude. Then in the next section, we use this to show that any amplitude that conserves supertranslation charge, for example (3.32), is equal to the Faddeev-Kulish amplitude with the same external particle configuration. This establishes the notion that Faddeev-Kulish amplitudes naturally arise from the charge conservation of asymptotic symmetries.

In order to relate (3.32) to the Faddeev-Kulish amplitude, let us denote its left hand side by $\mathcal{M}$,

$$\mathcal{M} = \langle \Psi_{\text{out}}| \exp\left[-\frac{\kappa}{2}\sum_i \eta_i \int \widetilde{d^3k}\, \frac{p_i^\mu p_i^\nu}{p_i \cdot k}(a^\dagger_{\mu\nu} - a_{\mu\nu})\right] \mathcal{S}\, |\Psi_{\text{in}}\rangle\,. \tag{3.33}$$

Next, consider another amplitude $\mathcal{M}_c$, given by

$$\mathcal{M}_c = \langle \Psi_{\text{out}} | \exp \left\{ -\frac{\kappa}{2} \sum_i \eta_i \int \widetilde{d^3k} \left[\frac{p_i^\mu p_i^\nu}{p_i \cdot k} + \frac{c^{\mu\nu}(p_i, k)}{\omega_k} \right] (a^\dagger_{\mu\nu} - a_{\mu\nu}) \right\} \mathcal{S} | \Psi_{\text{in}} \rangle \tag{3.34}$$

$$= \langle \Psi_{\text{out}} | \exp \left[-\sum_i \eta_i R_f(p_i) \right] \mathcal{S} | \Psi_{\text{in}} \rangle , \tag{3.35}$$

where we inserted a term proportional to $c^{\mu\nu}/\omega_k$ to the argument of the exponential. Here $c^{\mu\nu}(p, k)$ is the tensor of [9] that parametrizes the asymptotic space, and

$$R_f(p_i) = \frac{\kappa}{2} \int \widetilde{d^3k} \left[\frac{p_i^\mu p_i^\nu}{p_i \cdot k} + \frac{c^{\mu\nu}(p_i, k)}{\omega_k} \right] (a^\dagger_{\mu\nu} - a_{\mu\nu}) \tag{3.36}$$

is the anti-Hermitian operator appearing in the construction of Faddeev-Kulish state [9] with $\phi = 1$. In contrast to $\mathcal{M}$ and $\mathcal{M}_c$, the IR-finite Faddeev-Kulish amplitude $\mathcal{M}_{\text{FK}}$ is given by

$$\mathcal{M}_{\text{FK}} = \langle \Psi_{\text{out}} | \exp \left[-\sum_{i \in \text{out}} R_f(p_i) \right] \mathcal{S} \exp \left[\sum_{i \in \text{in}} R_f(p_i) \right] | \Psi_{\text{in}} \rangle . \tag{3.37}$$

We aim to establish $\mathcal{M}_{FK} = \mathcal{M}_c = \mathcal{M}$.

3.2.1 Moving the graviton clouds

Let us start by considering the simplest case, i.e., the Faddeev-Kulish amplitude for single-particle external states to leading order in the interaction. We follow the shorthand notations used in [58]:

$$P_{\mu\nu}(p, k) = \frac{\kappa}{2} \left(\frac{p_\mu p_\nu}{p \cdot k} \right), \qquad C_{\mu\nu}(p, k) = \frac{\kappa}{2} \frac{c_{\mu\nu}(p, k)}{\omega_k}, \tag{3.38}$$

and $S_{\mu\nu}(p, k) = P_{\mu\nu}(p, k) + C_{\mu\nu}(p, k)$. These allow us to write, (see [58] for details)

$$\mathcal{M}_{\text{FK}} = \langle 0 | b(p_f) e^{-S_f \cdot (a^\dagger - a)} \mathcal{S} e^{S_i \cdot (a^\dagger - a)} b^\dagger(p_i) | 0 \rangle , \tag{3.39}$$

where $S_f^{\mu\nu} \equiv S^{\mu\nu}(p_f, k)$ and $S_i^{\mu\nu} \equiv S^{\mu\nu}(p_i, k)$. The subscript FK is written to emphasize that this is a Faddeev-Kulish amplitude. In what follows we employ the following notation,

$$S \cdot (a^\dagger - a) \equiv \int \widetilde{d^3k}\, S^{\mu\nu} (a^\dagger_{\mu\nu} - a_{\mu\nu}), \tag{3.40}$$

and

$$S_f \cdot I \cdot S_i \equiv \int \widetilde{d^3k}\, S_f^{\mu\nu} I_{\mu\nu\rho\sigma} S_i^{\rho\sigma}, \tag{3.41}$$

where

$$I^{\mu\nu\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}. \tag{3.42}$$

Up to the one loop order, this amplitude is

$$\mathcal{M}_{\text{FK}} = \langle 0|\, b(p_f) \left(1 + S_f \cdot a - \frac{1}{4} S_f \cdot I \cdot S_f\right) \mathcal{S} \left(1 + S_i \cdot a^\dagger - \frac{1}{4} S_i \cdot I \cdot S_i\right) b^\dagger(p_i)\, |0\rangle\,. \tag{3.43}$$

Working out the infrared divergences (see [58] for details), we see that they factor out and cancel as

$$\Bigg(1 - \underbrace{\frac{1}{4} P \cdot I \cdot P}_{\text{virtual}} + \underbrace{\frac{1}{2} S \cdot I \cdot P}_{\text{interacting}} - \underbrace{\frac{1}{4} S \cdot I \cdot S}_{\text{cloud-to-cloud}}\Bigg) \langle p_f|\mathcal{S}|p_i\rangle = \langle p_f|\mathcal{S}|p_i\rangle\,, \tag{3.44}$$

where $P = P_f - P_i$ and $S = S_f - S_i$. Note that the various infrared divergent contributions are indicated in braces. These are (1) corrections due to virtual graviton exchange, (2) the interacting graviton corrections arising from gravitons connecting the Faddeev-Kulish clouds to external legs, and finally (2) corrections due to cloud-to-cloud graviton exchanges. These have been discussed in detail in appendix B of [58].

Now let us see what happens if we put all the clouds in the outgoing state. We denote this amplitude as,

$$\mathcal{M}_c = \langle 0|\, b(p_f) e^{-S_f \cdot (a^\dagger - a)} e^{S_i \cdot (a^\dagger - a)} \mathcal{S} b^\dagger(p_i)\, |0\rangle\,. \tag{3.45}$$

Let us consider the various infrared divergent contributions in this case. The virtual graviton contribution remains unchanged. For the interacting gravitons, it used to be that the graviton contractions with a cloud gives the factor

$$\frac{1}{2} \int \widetilde{d^3k}\, S^{\mu\nu} I_{\mu\nu\rho\sigma}, \tag{3.46}$$

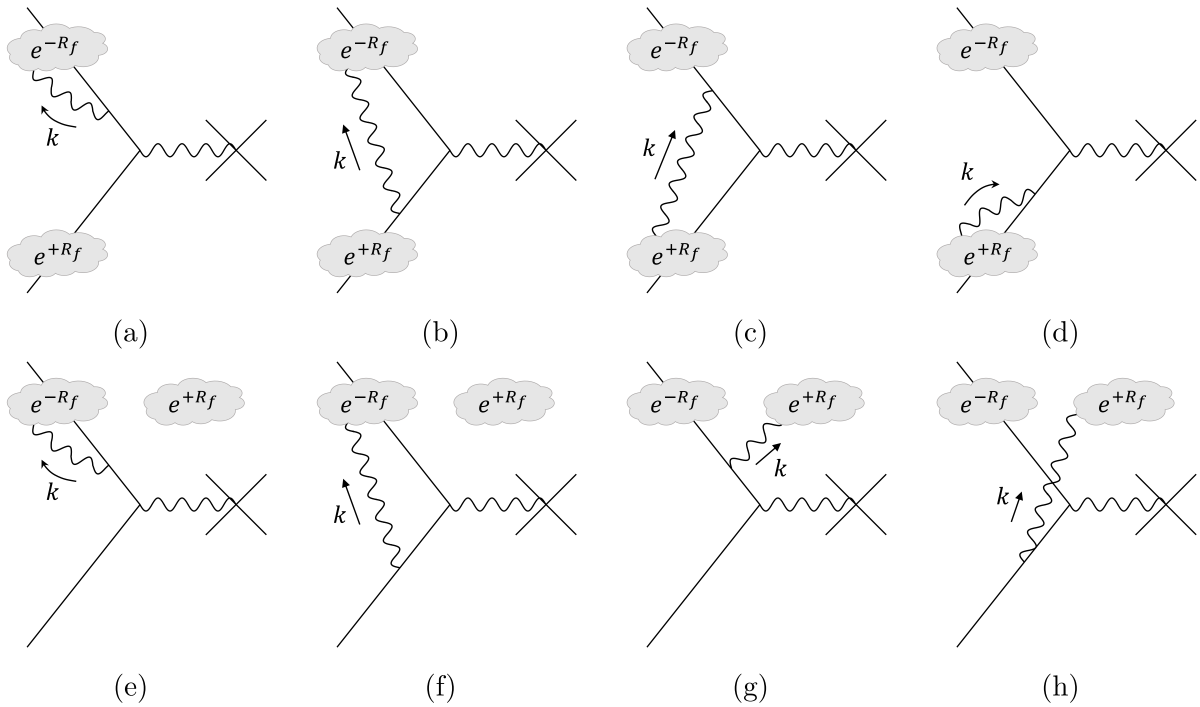


Figure 3.1: Diagrams (a)-(d) represent processes with Faddeev-Kulish asymptotic states. Diagrams (e)-(h) represent the same processes with the incoming cloud moved to the outgoing state. Notice the "wrong" sign $+R_f$ compared to a normal outgoing cloud with $-R_f$.

and depending on whether it was an incoming or an outgoing cloud, the contraction became

$$+\frac{\eta}{2}\int \widetilde{d^3k}\, S_f^{\mu\nu} I_{\mu\nu\rho\sigma} P^{\rho\sigma} \quad \text{for outgoing cloud (Figures 3.1(a),(b)), and} \tag{3.47}$$

$$-\frac{\eta}{2}\int \widetilde{d^3k}\, S_i^{\mu\nu} I_{\mu\nu\rho\sigma} P^{\rho\sigma} \quad \text{for incoming cloud (Figures 3.1(c),(d)),} \tag{3.48}$$

due to the difference in the sign of soft factor for absorption and emission. Figures 3.1(a) and 3.1(c) have $\eta = +1$, while 3.1(b) and 3.1(d) have $\eta = -1$. But now, we have two clouds that are in the outgoing state, so the graviton contraction gives the factor

$$+\frac{1}{2}\int \widetilde{d^3k}\, S_f^{\mu\nu} I_{\mu\nu\rho\sigma} \quad \text{for the } p_f \text{ cloud, and} \tag{3.49}$$

$$-\frac{1}{2}\int \widetilde{d^3k}\, S_i^{\mu\nu} I_{\mu\nu\rho\sigma} \quad \text{for the } p_i \text{ cloud,} \tag{3.50}$$

due to the difference in the signs of R_f. Since both are outgoing clouds, we have the same sign for the soft factor,

$$+\frac{\eta}{2}\int \widetilde{d^3k}\, S_f^{\mu\nu} I_{\mu\nu\rho\sigma} P^{\rho\sigma} \quad \text{for the } p_f \text{ cloud (Figures 3.1(e),(f)), and} \tag{3.51}$$

$$-\frac{\eta}{2}\int \widetilde{d^3k}\, S_i^{\mu\nu} I_{\mu\nu\rho\sigma} P^{\rho\sigma} \quad \text{for the } p_i \text{ cloud (Figures 3.1(g),(h)),} \tag{3.52}$$

where Figures 3.1(e) and 3.1(g) have $\eta = +1$, while 3.1(f) and 3.1(h) have $\eta = -1$. One can see that the results stay the same, meaning that contributions of interacting gravitons are unaltered. It remains to check the cloud-to-cloud contributions, but since these arise from contractions between operators in the clouds, they do not depend on which side of $\mathcal{S}$ the cloud is located and therefore are unchanged. We have thus shown that the infrared divergent part of the single-particle, leading order amplitudes $\mathcal{M}_c$ and $\mathcal{M}_{\rm FK}$ remains unchanged upon shifting the cloud around, i.e. between the in and out states.

Next, we generalize this result to the most general case of multiple external particles and all loop orders. Again, we begin by considering the individual contributions, i.e., virtual, interacting, and cloud-to-cloud gravitons. The virtual graviton contribution is unchanged from the one given in [58]. For the interacting gravitons, consider the amplitude of a diagram with N (N') absorbed (emitted) interacting gravitons,

$$(-1)^N \left[\prod_{r=1}^{N+N'} \frac{1}{2}\int \widetilde{d^3k_r}\, S_{\mu\nu}(p_r, k_r) I^{\mu\nu\rho_r\sigma_r}\right] \mathcal{J}_{\rho_1\sigma_1\rho_2\sigma_2\cdots\rho_{N+N'}\sigma_{N+N'}}, \tag{3.53}$$

where p_r is the momentum of the external particle that exchanges graviton r, and $\mathcal{J}$ is a complicated tensor whose detailed form is given in equation (B.58) of [58]. Taking the k-th incoming cloud and moving it to the outgoing state has the two following effects (see Figure 3.2):

1. The factor $(-1)^N$, which comes from the signs in the soft factors, becomes $(-1)^{N-n_k}$, where n_k is the number of interacting gravitons connected to the k-th (previously) incoming cloud. This is because these gravitons used to be absorbed but are now emitted.

2. The following factor in (3.53),

$$\left[\prod_{r=1}^{N+N'} \frac{1}{2}\int \widetilde{d^3k_r}\, S_{\mu\nu}(p_r, k_r) I^{\mu\nu\rho_r\sigma_r}\right], \tag{3.54}$$

came from contractions of gravitons with the clouds. For the Faddeev-Kulish amplitude, where all clouds are in the proper locations, each cloud gives the same factor

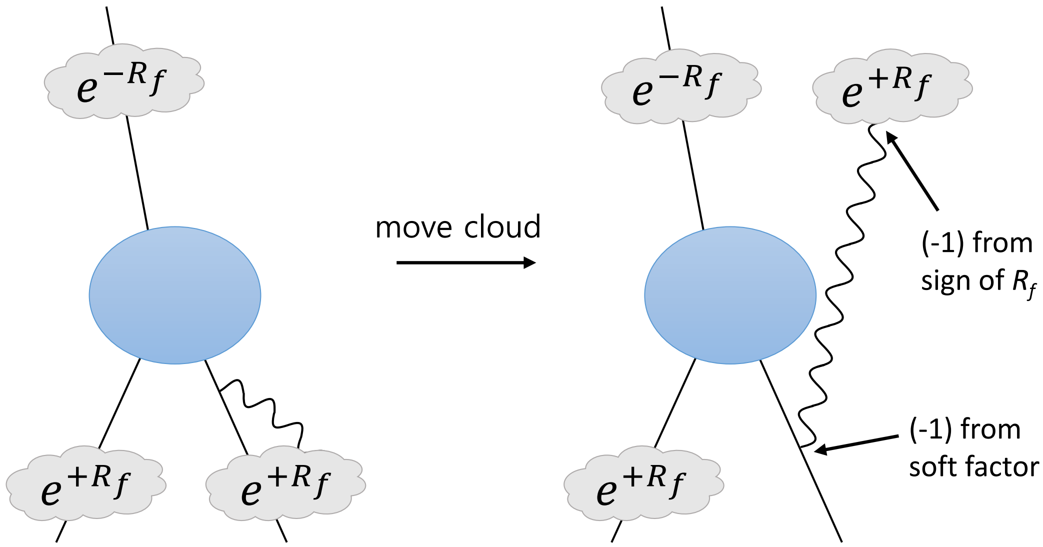


Figure 3.2: An example of an incoming cloud being moved to the out-state. Each boson connecting this cloud to an external propagator obtains two factors of (-1), one from the soft factor and the other from the "wrong" sign of R_f. These two factors cancel, and thus the overall amplitude is unaffected by such a change.

$\frac{1}{2}\int \widetilde{d^3k}\, S^{\mu\nu} I_{\mu\nu\rho\sigma}$ upon contraction. But now that the k-th incoming cloud is sitting in the outgoing state with a wrong sign (incoming and outgoing clouds have different signs $e^{\pm R_f}$), only this cloud gives an additional factor of -1. The above factor changes to

$$\left[(-1)^{n_k} \prod_{r=1}^{N+N'} \frac{1}{2} \int \widetilde{d^3k_r}\, S_{\mu\nu}(p_r, k_r) I^{\mu\nu\rho_r\sigma_r}\right]. \tag{3.55}$$

It follows that we obtain two factors $(-1)^{-n_k}$ and $(-1)^{n_k}$, which cancel each other and the overall contribution remains unchanged. It remains to consider the cloud-to-cloud gravitons. There are three types: out-to-out, in-to-in, and the disconnected. The contributions of l disconnected gravitons factored out as

$$l! \left[\frac{1}{2} S^{\text{out}} \cdot I \cdot S^{\text{in}}\right]^l, \tag{3.56}$$

but with the k-th incoming cloud moved to the out-state (as an outgoing cloud with the wrong sign), this is adjusted to

$$l! \left[\frac{1}{2}(S^{\text{out}} - S^k) \cdot I \cdot (S^{\text{in}} - S^k)\right]^l, \tag{3.57}$$

which eventually exponentiates to

$$\exp\left\{\frac{1}{2}(S^{\rm out} - S^k)\cdot I\cdot(S^{\rm in} - S^k)\right\}. \tag{3.58}$$

The in-to-in and out-to-out contributions change from

$$\exp\left\{-\frac{1}{4}S^{\rm out}\cdot I\cdot S^{\rm out} - \frac{1}{4}S^{\rm in}\cdot I\cdot S^{\rm in}\right\} \tag{3.59}$$

to

$$\exp\left\{-\frac{1}{4}(S^{\rm out} - S^k)\cdot I\cdot(S^{\rm out} - S^k) - \frac{1}{4}(S^{\rm in} - S^k)\cdot I\cdot(S^{\rm in} - S^k)\right\}. \tag{3.60}$$

Putting (3.58) and (3.60) together, we obtain

$$\exp\left\{-\frac{1}{4}(S^{\rm out} - S^{\rm in})\cdot I\cdot(S^{\rm out} - S^{\rm in})\right\}, \tag{3.61}$$

which is the same factor that was obtained without moving the cloud, and thus the cloud-to-cloud contribution also remains unaltered.

It follows that we can write

$$\begin{aligned}
&\langle\Psi_{\rm out}|\left[\prod_{j\in{\rm out}} e^{-S_j\cdot(a^\dagger - a)}\right]\mathcal{S}\left[\prod_{i\in{\rm in}} e^{S_i\cdot(a^\dagger - a)}\right]|\Psi_{\rm in}\rangle \\
&= \langle\Psi_{\rm out}|\left[\prod_{j\in{\rm out}} e^{-S_j\cdot(a^\dagger - a)}\right]\left[\prod_{i\in{\rm in}} e^{S_i\cdot(a^\dagger - a)}\right]\mathcal{S}\,|\Psi_{\rm in}\rangle && (3.62)\\
&= \langle\Psi_{\rm out}|\,\mathcal{S}\left[\prod_{j\in{\rm out}} e^{-S_j\cdot(a^\dagger - a)}\right]\left[\prod_{i\in{\rm in}} e^{S_i\cdot(a^\dagger - a)}\right]|\Psi_{\rm in}\rangle\,, && (3.63)
\end{aligned}$$

and so on. Therefore, we conclude that the Faddeev-Kulish amplitude does not change under a shift of the cloud from one side of the scattering operator to the other.

3.2.2 Equality of the amplitudes

From (3.37) and (3.35), it is clear that the only difference between $\mathcal{M}_c$ and $\mathcal{M}_{\rm FK}$ is in the location of the clouds; the incoming cloud, which should be dressing the incoming state, is located in the out-state. We have seen in the previous subsection that in an amplitude the clouds can freely be commuted through the scattering operator. This implies that the amplitude $\mathcal{M}_c$, which has all the clouds in the outgoing state, is actually equal to the

Faddeev-Kulish amplitude, i.e.

$$\mathcal{M}_c = \mathcal{M}_{\text{FK}}. \tag{3.64}$$

Now let us consider the original amplitude $\mathcal{M}$ of (3.35) that emerged from the conservation of supertranslation charge. This is a special case of $\mathcal{M}_c$, in the sense that putting $c^{\mu\nu} = 0$ in $\mathcal{M}_c$ recovers $\mathcal{M}$. Thus, $\mathcal{M}$ is equal to the Faddeev-Kulish amplitude constructed using the $R(t)$ operator of [9] instead of R_f. Since states constructed with $R(t)$ and R_f are related by a unitary transformation, this implies that $\mathcal{M} = \mathcal{M}_c$. We can see this more directly by noting that amplitudes constructed with $c^{\mu\nu} = 0$ are related to those with non-zero $c^{\mu\nu}$ by the following relation [58]

$$\mathcal{M}_c = \exp\left[-\frac{\kappa^2}{4}\sum_{n,m}\eta_n\eta_m\int\frac{\widetilde{d^3k}}{\omega_{\mathbf{k}}^2}c_{\mu\nu}(p_n,k)I^{\mu\nu\rho\sigma}c_{\rho\sigma}(p_m,k)\right]\mathcal{M} = \mathcal{M}, \tag{3.65}$$

where each sum runs over the whole set of external particles. The summand vanishes term by term, due to one of the constraints that $c^{\mu\nu}$ has to satisfy. Therefore

$$\mathcal{M} = \mathcal{M}_c = \mathcal{M}_{\text{FK}}, \tag{3.66}$$

and the amplitude $\mathcal{M}$ of (3.35) is the IR-finite Faddeev-Kulish amplitude.

3.3 Soft gravitons and decoherence of momentum configurations of hard matter particles

In this section we reconsider the problem of the decoherence of momentum superpositions of hard matter particles due to low energy soft gravitons that was discussed in [23]. The same conclusions were reached in [24] using a different approach. In [23] the usual Bloch-Nordsieck mechanism was introduced to cancel the infrared divergences in order to obtain finite density matrices, and the asymptotic symmetries discussed in section 3.1 do not play any role. The question we address in this section is how a consistent application of the results of the previous sections might change the conclusions of [23]. This problem also has potential implications in the recent debate regarding the proposal of Hawking, Perry and Strominger [60], concerning new charges on black holes and their consequences for the information paradox. We refer interested readers to the relevant literature [18,20,21,60,61].

First, we briefly outline the logic of [23]. Consider an "in" Fock state $|\alpha\rangle_{\text{in}}$ at time

$t = -\infty$, which is related to the "out" Fock state at $t = \infty$ by the S matrix:

$$\begin{aligned} |\alpha\rangle \quad \rightarrow \quad |\alpha\rangle_{\text{in}} &= \mathcal{S}\,|\alpha\rangle_{\text{out}} &(3.67)\\ &= \left(\sum_{\beta b} |\beta b\rangle\,\langle\beta b|\right)\mathcal{S}\,|\alpha\rangle_{\text{out}} &(3.68)\\ &= \sum_{\beta b} S_{\beta b,\alpha}\,|\beta b\rangle_{\text{out}}\,, &(3.69)\end{aligned}$$

where, $S_{\beta b,\alpha} \equiv \langle\beta b|\mathcal{S}|\alpha\rangle$, and β (b) stands for the set of hard (soft) particles. We drop subscripts on the kets which, unless specified, are the asymptotic out-states. Then the authors construct a reduced density matrix by tracing out the external soft bosons $|b\rangle$:

$$\rho = \sum_{\beta\beta' b} S_{\beta b,\alpha} S^{*}_{\beta' b,\alpha}\,|\beta\rangle\,\langle\beta'|\,. \tag{3.70}$$

By factoring out the divergences from the sum,

$$\begin{aligned}\sum_b S_{\beta b,\alpha} S^{*}_{\beta' b,\alpha} &= S_{\beta,\alpha} S^{*}_{\beta',\alpha}\underbrace{\left(\frac{E}{\lambda}\right)^{\widetilde{A}_{\beta\beta',\alpha}}\left(\frac{E}{\lambda}\right)^{\widetilde{B}_{\beta\beta',\alpha}} f\left(\frac{E}{E_T},\widetilde{A}_{\beta\beta',\alpha}\right) f\left(\frac{E}{E_T},\widetilde{B}_{\beta\beta',\alpha}\right)}_{\text{real soft bosons}}\\ &= S^{\Lambda}_{\beta,\alpha} S^{\Lambda *}_{\beta',\alpha}\underbrace{\left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2+A_{\beta',\alpha}/2}\left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2+B_{\beta',\alpha}/2}}_{\text{virtual bosons}}\\ &\quad\times\underbrace{\left(\frac{E}{\lambda}\right)^{\widetilde{A}_{\beta\beta',\alpha}}\left(\frac{E}{\lambda}\right)^{\widetilde{B}_{\beta\beta',\alpha}} f\left(\frac{E}{E_T},\widetilde{A}_{\beta\beta',\alpha}\right) f\left(\frac{E}{E_T},\widetilde{B}_{\beta\beta',\alpha}\right)}_{\text{real soft bosons}},\end{aligned}$$

and by considering the limit as the IR cut-off λ is removed, the authors of [23] observed the decoherence of momentum configurations of hard particles or conversely, the strong correlations between the hard and soft particles. We refer to [23] for details of the notations and derivations of this equation. However, note that it is essential in this approach to sum over the soft bosons because otherwise the infrared divergences do not cancel.

We now show that this conclusion implicitly assumes that the vacuum is unique and before the cancellation of IR divergences for the inclusive process, one is dealing with S matrix elements which vanish as the cut-off is removed. We have seen that conservation of BMS charge, namely

$$(N_{\text{out}} - N_{\text{in}})\,\langle\text{out}|\mathcal{S}|\text{in}\rangle = \Omega^{\text{soft}}\,\langle\text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{3.71}$$

dictates that scattering processes starting from a state built on the Fock vacuum $|0\rangle$ evolves only into states that are built on the coherent vacuum

$$\exp\left[\int_{\text{soft}} \widetilde{d^3k}\, N^{\mu\nu}_{\text{out}}(a^{\dagger}_{\mu\nu} - a_{\mu\nu})\right] |0\rangle \,, \tag{3.72}$$

where,

$$N^{\mu\nu}_{\text{out}} = \frac{\kappa}{2}\sum_i \eta_i \frac{p_i^{\mu} p_i^{\nu}}{p_i \cdot k}, \tag{3.73}$$

with the sum running over all external particles[1]. Therefore, if we started with a state $|\alpha\rangle$ built on $|0\rangle$, then the outgoing state cannot be just $|\beta b\rangle$, which is a Fock state built on $|0\rangle$; all S-matrix elements between such states vanish. Instead, $|\alpha\rangle$ scatters into states accompanied by a coherent cloud,

$$|\beta; N_{\text{out}}\rangle = |\beta\rangle \exp\left[\int_{\text{soft}} \widetilde{d^3k}\, N^{\mu\nu}_{\text{out}}(a^{\dagger}_{\mu\nu} - a_{\mu\nu})\right], \tag{3.74}$$

with N_{out} dependent on the sets of external hard momenta α and β. We therefore should consider,

$$|\alpha\rangle_{\text{in}} = \sum_{\beta} S^{\text{FK}}_{\beta,\alpha} |\beta; N_{\text{out}}\rangle \,, \tag{3.75}$$

where we have written,

$$S^{\text{FK}}_{\beta,\alpha} \equiv \langle\beta| \exp\left[-\int_{\text{soft}} \widetilde{d^3k}\, N^{\mu\nu}_{\text{out}}(a^{\dagger}_{\mu\nu} - a_{\mu\nu})\right] \mathcal{S} |\alpha\rangle \,. \tag{3.76}$$

The states $|\alpha\rangle$ and $|\beta\rangle$ are just the conventional Fock states. We have seen earlier that the right hand side is exactly equivalent to the amplitude constructed using the Faddeev-Kulish asymptotic states, i.e.,

$$\langle\beta| e^{-R_f} \mathcal{S} e^{R_f} |\alpha\rangle \,, \tag{3.77}$$

and hence the the left hand side has the superscript FK on the S matrix element. Now the density matrix becomes

$$\sum_{\beta\beta'} S^{\text{FK}}_{\beta,\alpha} S^{\text{FK}*}_{\beta',\alpha} |\beta; N_{\text{out}}\rangle \langle\beta'; N'_{\text{out}}| \,. \tag{3.78}$$

[1]Note that here the in and out labels refer to incoming or outgoing particles. The Fock states are all in the "out" basis.

The amplitudes $S^{\mathrm{FK}}_{\beta,\alpha}$ do not have infrared divergences coming from the virtual bosons. In the "virtual bosons" part of (3.71), the λ-dependent part is exactly canceled by interactions involving the clouds, as seen in [58]. Thus, in this framework there is no longer the decoherence that was observed in [23].

To sum up, due to the conservation of BMS charge, any conventional Fock state $|\alpha\rangle$ evolves not into another Fock state $|\beta b\rangle$, but instead into a coherent state $|\beta; N_{\text{out}}\rangle$. If the starting state is a coherent state, then the end state is just be another coherent state, and the BMS charge conservation guarantees that the amplitudes $S^{\mathrm{FK}}_{\beta,\alpha}$ coincide with the infrared-finite Faddeev-Kulish amplitudes. We reiterate, that the presence of the coherent boson cloud cancels all the problematic dependence on the infrared cut-off λ, and therefore one is no longer mathematically forced to sum over the soft particles in order to obtain well-defined density matrix elements.

It is noteworthy that although the density matrix elements (3.78) are now well-defined, depending on what kind of measurement is being carried out, one may still construct a reduced density matrix by summing over the soft particles. Would the decoherence of the momentum configurations of the hard matter particles return in this case? This analysis has recently been carried out in [127]. We next reanalyze this within the framework introduced in the previous sections of this chapter.

The $\beta\beta'$-component of the reduced density matrix is

$$\rho_{\beta\beta'} = \sum_b S^{\mathrm{FK}}_{\beta,\alpha} S^{\mathrm{FK}*}_{\beta',\alpha} \langle b|N_{\text{out}}\rangle \langle N'_{\text{out}}|b\rangle \tag{3.79}$$

$$= S^{\mathrm{FK}}_{\beta,\alpha} S^{\mathrm{FK}*}_{\beta',\alpha} \langle N'_{\text{out}}| \left(\sum_b |b\rangle\langle b|\right) |N_{\text{out}}\rangle \tag{3.80}$$

$$= S^{\mathrm{FK}}_{\beta,\alpha} S^{\mathrm{FK}*}_{\beta',\alpha} \langle N'_{\text{out}}|N_{\text{out}}\rangle . \tag{3.81}$$

By normal-ordering the graviton operators, we obtain

$$\langle N'_{\text{out}}|N_{\text{out}}\rangle = \langle 0| \exp\left\{\frac{\kappa}{2}\int_{\text{soft}} \widetilde{d^3k} \left(N^{\mu\nu}_{\text{out}} - N'^{\mu\nu}_{\text{out}}\right)\left(a^\dagger_{\mu\nu} - a_{\mu\nu}\right)\right\}|0\rangle \tag{3.82}$$

$$= \exp\left\{-\frac{\kappa^2}{16}\int_{\text{soft}} \widetilde{d^3k} \left(N^{\mu\nu}_{\text{out}} - N'^{\mu\nu}_{\text{out}}\right) I_{\mu\nu\rho\sigma} \left(N^{\rho\sigma}_{\text{out}} - N'^{\rho\sigma}_{\text{out}}\right)\right\}, \tag{3.83}$$

where we can write

$$N^{\mu\nu}_{\text{out}} - N'^{\mu\nu}_{\text{out}} = \sum_{p\in\beta} \frac{p^\mu p^\nu}{p\cdot k} - \sum_{p\in\beta'} \frac{p^\mu p^\nu}{p\cdot k}. \tag{3.84}$$

Therefore, if $\beta \neq \beta'$ then the integral in (3.83) is infrared-divergent and the expression (3.83) vanishes. This implies that the off-diagonal elements of the reduced density matrix is zero and the decoherence of momentum configurations of the hard particles reappears.

Does this conclusion change if we include external states with soft gravitons? The density matrix with external soft gravitons is

$$\sum_{\beta\beta' bb'} S^{\mathrm{FK}}_{\beta b,\alpha} S^{\mathrm{FK}*}_{\beta' b',\alpha} \left|\beta b; N_{\mathrm{out}}\right\rangle \left\langle \beta' b'; N'_{\mathrm{out}}\right| , \tag{3.85}$$

and the reduced density matrix, after tracing out the soft particles, becomes

$$\rho_{\beta\beta'} = \sum_{b''} \sum_{bb'} S^{\mathrm{FK}}_{\beta b,\alpha} S^{\mathrm{FK}*}_{\beta' b',\alpha} \left\langle b'' | b; N_{\mathrm{out}}\right\rangle \left\langle b'; N'_{\mathrm{out}} | b''\right\rangle \tag{3.86}$$

$$= \sum_{bb'} S^{\mathrm{FK}}_{\beta b,\alpha} S^{\mathrm{FK}*}_{\beta' b',\alpha} \left\langle b'; N'_{\mathrm{out}} | b; N_{\mathrm{out}}\right\rangle . \tag{3.87}$$

Let us employ a notation similar to that of [127]:

$$W(\beta) = \exp\left\{ \frac{\kappa}{2} \int_{\mathrm{soft}} \widetilde{d^3k} \sum_{p\in\beta} \frac{p^\mu p^\nu}{p\cdot k} (a^\dagger_{\mu\nu} - a_{\mu\nu}) \right\} , \tag{3.88}$$

$$W^\dagger(\beta') = \exp\left\{ -\frac{\kappa}{2} \int_{\mathrm{soft}} \widetilde{d^3k} \sum_{p\in\beta'} \frac{p^\mu p^\nu}{p\cdot k} (a^\dagger_{\mu\nu} - a_{\mu\nu}) \right\} , \tag{3.89}$$

such that $|b; N_{\mathrm{out}}\rangle = W(\beta)\,|b\rangle$. Then, the reduced density matrix element is

$$\rho_{\beta\beta'} = \sum_{bb'} S^{\mathrm{FK}}_{\beta b,\alpha} S^{\mathrm{FK}*}_{\beta' b',\alpha} \left\langle b'\right| W^\dagger(\beta') W(\beta) \left| b\right\rangle . \tag{3.90}$$

Let us see what we can say about $\langle b'|W^\dagger(\beta')W(\beta)|b\rangle$. Let m and n be the particle number of b' and b, respectively. Then,

$$\langle b'|W^\dagger(\beta')W(\beta)|b\rangle = \langle 0|\, a_{\ell'_1}(k'_1)\cdots a_{\ell'_m}(k'_m) W^\dagger(\beta')W(\beta) a^\dagger_{\ell_1}(k_1)\cdots a^\dagger_{\ell_n}(k_n)\, |0\rangle , \tag{3.91}$$

where ℓ'_i and k'_i (ℓ_i and k_i) are the polarization and momentum of the i-th graviton in b' (b). Let us use the shorthand

$$W^2 \equiv W^\dagger(\beta')W(\beta), \tag{3.92}$$

and observe that since

$$a_\ell(k) = \epsilon_\ell^{\mu\nu}(k) a_{\mu\nu}(k), \tag{3.93}$$

$$a_\ell^\dagger(k) = \epsilon_\ell^{\mu\nu *}(k) a_{\mu\nu}^\dagger(k), \tag{3.94}$$

we have the commutators

$$\left[W^2, a_\ell^\dagger(k)\right] = -\frac{\kappa}{2} \int_{\text{soft}} \widetilde{d^3k'} \sum_{p\in\beta,\beta'} \eta_p \frac{p^\mu p^\nu}{p\cdot k'} \left[a_{\mu\nu}^\dagger(k') - a_{\mu\nu}(k'), a_\ell^\dagger(k)\right] W^2 \tag{3.95}$$

$$= +\frac{\kappa}{2} \sum_{p\in\beta,\beta'} \eta_p \frac{p^\mu p^\nu}{p\cdot k} \epsilon_{\mu\nu}^{\ell *}(k) W^2, \tag{3.96}$$

$$\left[a_\ell(k), W^2\right] = -\frac{\kappa}{2} \int_{\text{soft}} \widetilde{d^3k'} \sum_{p\in\beta,\beta'} \eta_p \frac{p^\mu p^\nu}{p\cdot k'} \left[a_\ell(k), a_{\mu\nu}^\dagger(k') - a_{\mu\nu}(k')\right] W^2 \tag{3.97}$$

$$= -\frac{\kappa}{2} \sum_{p\in\beta,\beta'} \eta_p \frac{p^\mu p^\nu}{p\cdot k} \epsilon_{\mu\nu}^{\ell}(k) W^2, \tag{3.98}$$

where $\eta_p = +1$ if $p \in \beta'$ and $\eta_p = -1$ if $p \in \beta$. Using this, we can commute the left-most creation operator $a_{\ell_1}^\dagger(k_1)$ to the left side of W^2 to obtain

$$\begin{aligned} \langle b'|W^2|b\rangle = &\left[\frac{\kappa}{2} \sum_{p\in\beta,\beta'} \eta_p \frac{p^\mu p^\nu}{p\cdot k_1} \epsilon_{\mu\nu}^{\ell_1 *}(k_1)\right] \langle 0|\, a_{\ell_1'}(k_1') \cdots a_{\ell_m'}(k_m') W^2 a_{\ell_2}^\dagger(k_2) \cdots a_{\ell_n}^\dagger(k_n)\, |0\rangle \\ &+ \langle 0|\, a_{\ell_1'}(k_1') \cdots a_{\ell_m'}(k_m') a_{\ell_1}^\dagger(k_1) W^2 a_{\ell_2}^\dagger(k_2) \cdots a_{\ell_n}^\dagger(k_n)\, |0\rangle . \end{aligned} \tag{3.99}$$

However, we next show that the contribution from the second term in the parentheses is vanishingly small. To see this, one may consider commuting $a_{\ell_1}^\dagger(k_1)$ all the way to the left, aiming to act it on the vacuum. This creates one term for each annihilation operator which has a factor of the following form,

$$\begin{aligned} &\int_{\text{soft}} \widetilde{d^3k_j'}\, \widetilde{d^3k_1}\, S_{\beta b,\alpha}^{\text{FK}} S_{\beta' b',\alpha}^{\text{FK}*} \left[a_{\ell_j'}(k_j'), a_{\ell_1}^\dagger(k_1)\right] \\ &= \delta_{\ell_j',\ell_1} \int d\Omega_j' d\Omega_1 \delta^2(\Omega_j' - \Omega_1) S_{\beta b,\alpha}^{\text{FK}} S_{\beta' b',\alpha}^{\text{FK}*} \int_{\text{soft}} \frac{|\mathbf{k}_j'|^2 d|\mathbf{k}_j'|}{(2\pi)^3 |\mathbf{k}_j'|}, \end{aligned} \tag{3.100}$$

where we have separated out the radial parts from the momentum integrals. The radial integrals can be computed separately, because the Faddeev-Kulish amplitudes $S_{\beta b,\alpha}^{\text{FK}}$ and $S_{\beta' b',\alpha}^{\text{FK}*}$ are $O(|\mathbf{k}|^0)$ in each soft momentum $\mathbf{k}$, which can be seen in figure 3.3 for a single outgoing soft graviton; the first two diagrams cancel the last two diagrams, and only the one in the middle, which is infrared-finite, contribute. If the momentum integral was over the whole momentum space, then the last integral in (3.100) diverges. But since it is only over

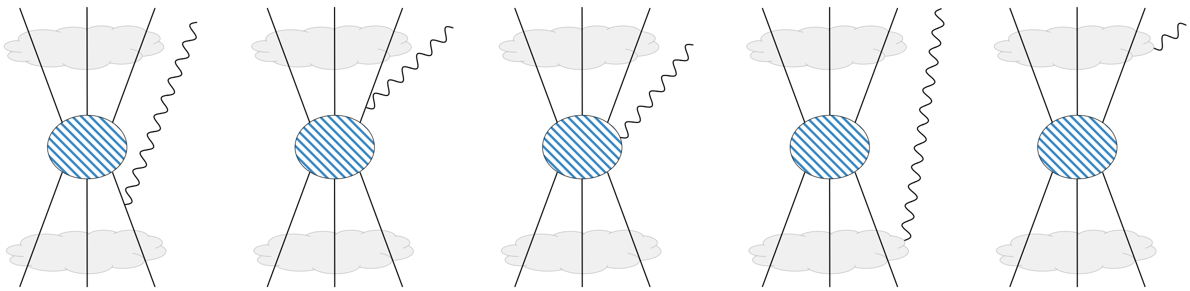


Figure 3.3: Diagrams contributing to an amplitude with external soft boson. The first two diagrams cancel the last two diagrams, and only the diagram in the middle remains, which is of zeroth order in the soft momentum.

the soft momentum space, it has a vanishingly small value (proportional to some momentum cutoff squared, ω_c^2, where we think of $\omega_c \to 0$) and therefore the expression vanishes. Thus, we have

$$\langle b'|W^2|b\rangle = \left[\frac{\kappa}{2}\sum_{p\in\beta,\beta'}\eta_p\frac{p^\mu p^\nu}{p\cdot k_1}\epsilon^{\ell_1 *}_{\mu\nu}(k_1)\right]\langle 0|\, a_{\ell'_1}(k'_1)\cdots a_{\ell'_m}(k'_m)W^2 a^\dagger_{\ell_2}(k_2)\cdots a^\dagger_{\ell_n}(k_n)\,|0\rangle\,. \tag{3.101}$$

Each creation operator gives a factor analogous to that in the square brackets, so we may write

$$\langle b'|W^2|b\rangle = \prod_{i=1}^{n}\left[\frac{\kappa}{2}\sum_{p\in\beta,\beta'}\eta_p\frac{p^\mu p^\nu}{p\cdot k_i}\epsilon^{\ell_i *}_{\mu\nu}(k_i)\right]\langle 0|\, a_{\ell'_1}(k'_1)\cdots a_{\ell'_m}(k'_m)W^2\,|0\rangle\,. \tag{3.102}$$

We can perform a similar process for the annihilation operators, where this time the factors have an additional minus sign, and this yields

$$\langle b'|W^2|b\rangle = \prod_{i=1}^{n}\left[\frac{\kappa}{2}\sum_{p\in\beta,\beta'}\eta_p\frac{p^\mu p^\nu}{p\cdot k_i}\epsilon^{\ell_i *}_{\mu\nu}(k_i)\right]\prod_{j=1}^{m}\left[-\frac{\kappa}{2}\sum_{p\in\beta,\beta'}\eta_p\frac{p^\mu p^\nu}{p\cdot k'_j}\epsilon^{\ell_j}_{\mu\nu}(k'_j)\right]\langle 0|\,W^2\,|0\rangle\,. \tag{3.103}$$

This explicitly shows that each term in the sum of (3.90) contains a product of infrared-finite integrals as well as the vacuum expectation value $\langle 0|W^2|0\rangle = \langle N'_{\text{out}}|N_{\text{out}}\rangle$, but we have seen that this value vanishes for the off-diagonal elements $\beta \neq \beta'$. Therefore, the reduced density matrix still exhibits a complete decoherence of the hard particle momentum configurations.

We conclude this section with a discussion of the two formulations of the density matrix: the one using the Bloch-Nordsieck mechanism and the one using dressed states. It is straightforward to see that only the off-diagonal elements of the reduced density matrix are different, whereas, the diagonal element which is essentially the Bloch-Nordsieck cross section is the same in the two approaches. Indeed, the cross section of the process $\alpha \to \beta b$ is given (up to a factor) by the absolute square of the amplitude:

$$\Gamma_{\beta b,\alpha} = S_{\beta b,\alpha} S^*_{\beta b,\alpha}. \tag{3.104}$$

These cross sections exhibit two types of infrared divergence, one arising from the real soft bosons and the other from the virtual bosons. The Bloch-Nordsieck method of dealing with these divergences is to sum over all unobservable soft bosons,

$$\Gamma_{\beta,\alpha} = \sum_b \Gamma_{\beta b,\alpha} = \sum_b S_{\beta b,\alpha} S^*_{\beta b,\alpha}, \tag{3.105}$$

and performing this sum results in the exponentiation of the soft factors of real bosons, which then cancels the divergence due to virtual bosons. It is clear that every diagonal element of the reduced density matrix in (3.70) is a Bloch-Nordsieck cross section:

$$\rho_{\beta\beta} = \langle \beta | \rho | \beta \rangle = \sum_b S_{\beta b,\alpha} S^*_{\beta b,\alpha}. \tag{3.106}$$

Thus, only the off-diagonal elements are affected (see Eq. (3.90)). The practical use of the Bloch-Nordsieck mechanism for obtaining IR finite cross sections does not require any modifications.

3.4 Summary

In this chapter, we have demonstrated that graviton cloud operators weakly commute with the scattering operator (see equations (3.62) and (3.63)), and used this to show that scattering amplitudes which conserve BMS supertranslation charge are equivalent to the Faddeev-Kulish amplitudes, see equation (3.66). Since Faddeev-Kulish amplitudes are free of infrared divergence, this proves the conjecture in [19], which proposes that conservation of asymptotic charge leads to infrared finite scattering amplitudes. The contents of this chapter ties up some loose ends on the relation between the Faddeev-Kulish formalism, asymptotic symmetry and infrared divergences.

We have also applied our formalism to the intriguing problem considered in [23] where it was found that tracing out soft degrees of freedom leads to the decoherence of hard particle momenta, whether or not one employs the Faddeev-Kulish states [127]. In contrast to their work, we have constructed the corresponding reduced density matrices conserving the BMS supertranslation charge at all stages and arrived at a similar conclusion, see equation (3.103).

Chapter 4

Soft Photon Hair on Schwarzschild Horizon from a Wilson Line Perspective

4.1 Wilson lines and soft charge in Minkowski spacetime

In this chapter, we extend the construction of photon dressing to the future horizon of Rindler spacetime. Then, we demonstrate that their structure is reminiscent of that exhibited by soft gravitons on the Schwarzschild horizon, which was derived earlier by Hawking, Perry and Strominger [60, 61].

We start by reviewing topics in flat Minkowski spacetime that are crucial in our subsequent analysis of Rindler and Schwarzschild horizons. In the first subsection, selected materials from [128] and [91] are used to show that the Faddeev-Kulish dressings of asymptotic states [8,9] are in fact Wilson lines along a specific time-like path. Then in the second subsection, we explore the connections between Faddeev-Kulish dressings, Wilson line punctures, edge states and surface charges associated with the asymptotic symmetry transformation developed in [19].

4.1.1 Equivalence of Wilson lines and Faddeev-Kulish dressings

A gauge-invariant formulation of QED using path-dependent variables dates back to Mandelstam's work [128]. In this formulation, the conventional matter fields are dressed by Wilson lines extending out to infinity. It is known [91, 129] that taking the path in each Wilson line to be the time-like path of an asymptotic particle yields the Faddeev-Kulish dressings [8]. In this section we briefly review this connection.

Under a gauge transformation, the gauge and matter fields transform as

$$\varphi(x) \to e^{-ie\Lambda(x)}\varphi(x), \tag{4.1}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x). \tag{4.2}$$

Mandelstam introduces a non-local path dependent variable which is the matter field dressed with a Wilson line, i.e.,

$$\Psi(x \,|\, \Gamma) = \mathbb{P}\exp\left\{ie\int_\Gamma^x d\xi^\mu A_\mu(\xi)\right\}\varphi(x), \tag{4.3}$$

along the path Γ, with the path-ordering operator $\mathbb{P}$. We show that this path-dependent dressing describes the Faddeev-Kulish dressing for a particular time-like, straight line path relevant for asymptotic field. In the Lorenz gauge, the equation of motion of the gauge field with source J_μ is

$$\Box A_\mu(x) = J_\mu(x). \tag{4.4}$$

Therefore one may use the retarded Green's function $G_{\rm ret}$ to decompose

$$A_\mu(x) = A_\mu^{\rm in}(x) + \int d^4z\, G_{\rm ret}(x-z)J_\mu(z), \tag{4.5}$$

where the $G_{\rm ret}$ solves $\Box G_{\rm ret}(x) = \delta^{(4)}(x)$, and the incoming asymptotic field $A_\mu^{\rm in}$ is the homogeneous solution satisfying $\Box A_\mu^{\rm in}(x) = 0$. Then, the dressing in (4.3) can be written as

$$\mathbb{P}\exp\left\{ie\int_\Gamma^x d\xi^\mu A_\mu(\xi)\right\} = \mathbb{T}\exp\left\{ie\int_\Gamma^x d\xi^\mu A_\mu^{\rm in}(\xi) + ie\int_\Gamma^x d\xi^\mu \int d^4z\, G_{\rm ret}(\xi-z)J_\mu(z)\right\} \tag{4.6}$$

$$= \exp\left\{ie\int_\Gamma^x d\xi^\mu A_\mu^{\rm in}(\xi)\right\} \times (\text{phase factors}), \tag{4.7}$$

where $\mathbb{P}$ is replaced by the time-ordering operator $\mathbb{T}$ since Γ is time-like. The path-ordering has been removed at the price of gaining an infinite c-number phase, which is related to the Coulomb phase and thus is not of interest for us. We anticipate letting $x^0 \to -\infty$ for an asymptotic incoming particle. With this limit in mind, we may assume Γ to be the trajectory of a free particle described by a constant four-velocity v^μ, and parametrize $\xi^\mu = x^\mu + \tau v^\mu$.

Then,

$$\exp\left\{ie\int_\Gamma^x d\xi^\mu A_\mu^{\rm in}(\xi)\right\} = \exp\left\{ie\int_{-\infty}^0 d\tau \frac{d\xi^\mu}{d\tau} A_\mu^{\rm in}(\xi)\right\} \tag{4.8}$$

$$= \exp\left\{iev^\mu \int_{-\infty}^0 d\tau A_\mu^{\rm in}(x+v\tau)\right\}. \tag{4.9}$$

For the in field, we have the standard asymptotic mode expansion

$$A_\mu^{\rm in}(x) = \int \widetilde{d^3k}\,\left[a_\mu(\mathbf{k})e^{ik\cdot x} + a_\mu^\dagger(\mathbf{k})e^{-ik\cdot x}\right], \tag{4.10}$$

where $\widetilde{d^3k} = \frac{d^3k}{(2\pi)^3(2\omega)}$ with $\omega = |\mathbf{k}|$ is the Lorentz-invariant measure, and

$$a_\mu(\mathbf{k}) = \sum_{\ell=\pm}\epsilon_\mu^{\ell *}(\mathbf{k})a_\ell(\mathbf{k}), \quad a_\mu^\dagger(\mathbf{k}) = \sum_{\ell=\pm}\epsilon_\mu^{\ell}(\mathbf{k})a_\ell^\dagger(\mathbf{k}), \tag{4.11}$$

with the polarization tensor $\epsilon_\mu^\ell(\mathbf{k})$. The creation and annihilation operators satisfy the standard commutation relations

$$\left[a_\ell(\mathbf{k}), a_{\ell'}^\dagger(\mathbf{k}')\right] = \delta_{\ell\ell'}(2\pi)^3(2\omega)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{4.12}$$

Using the mode expansion, we may write

$$v^\mu\int_{-\infty}^0 d\tau A_\mu^{\rm in}(x+v\tau) = v^\mu\int_{-\infty}^0 d\tau\int\widetilde{d^3k}\,\left[a_\mu(\mathbf{k})e^{ik\cdot(x+v\tau)} + \text{h.c.}\right] \tag{4.13}$$

$$= -i\int\widetilde{d^3k}\,\frac{p^\mu}{p\cdot k}\left[a_\mu(\mathbf{k})e^{ik\cdot x} - \text{h.c.}\right], \tag{4.14}$$

where $p^\mu = mv^\mu$, and we have used the boundary condition [8]

$$\int_{-\infty}^0 d\tau\, e^{ik\cdot v\tau} = \frac{1}{ik\cdot v}. \tag{4.15}$$

Therefore, we obtain

$$\exp\left\{ie\int_\Gamma^x d\xi^\mu A_\mu^{\rm in}(\xi)\right\} = \exp\left\{-e\int\widetilde{d^3k}\,\frac{p^\mu}{p\cdot k}\left(a_\mu^\dagger(\mathbf{k})e^{-ik\cdot x} - a_\mu(\mathbf{k})e^{ik\cdot x}\right)\right\}. \tag{4.16}$$

Now, recall that we anticipate $x^0\to-\infty$. Under this limit, non-vanishing contribution to the integral comes only from $k\to 0$ by virtue of the Riemann-Lebesgue lemma. Following the construction of [8], we implement this by replacing $e^{\pm ik\cdot x}$ with a scalar function $\phi(p,k)$ having support in a small neighborhood of $k=0$ and satisfying $\phi\to 1$ as $k\to 0$. Then, we

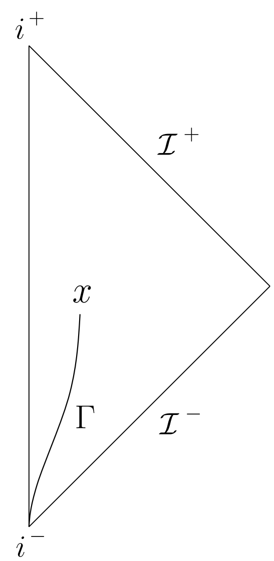


Figure 4.1: A Penrose diagram of the Minkowski spacetime, where $\mathcal{I}^+$ ($\mathcal{I}^-$) represents the future (past) null infinity and i^+ (i^-) represents the future (past) time-like infinity. The spacetime point x is the position of a massive dressed particle, and the time-like path Γ extends from i^- to x. The time-like infinities $i^\pm$ are 3-dimensional hyperbolic spaces $\mathbb{H}_3$ each parametrized by a 3-vector, see for example [1].

may write,

$$\exp\left\{ie\int_\Gamma^x d\xi^\mu A_\mu^{\text{in}}(\xi)\right\} = W(\mathbf{p}), \tag{4.17}$$

where $W(\mathbf{p})$ is, up to a unitary transformation, the Faddeev-Kulish operator (or dressing) of an asymptotic incoming particle of momentum $\mathbf{p}$,

$$W(\mathbf{p}) = \exp\left\{-e\int \widetilde{d^3k}\,\frac{p^\mu}{p\cdot k}\phi(p,k)\left(a_\mu^\dagger(\mathbf{k}) - a_\mu(\mathbf{k})\right)\right\}. \tag{4.18}$$

Equation (4.17) shows that a Wilson line along the trajectory of an asymptotic particle corresponds to the Faddeev-Kulish dressing.

4.1.2 Faddeev-Kulish dressings and the soft charge

In the previous subsection, we have seen that Faddeev-Kulish dressings are essentially Wilson lines. Let us consider a Wilson line along a time-like curve Γ of constant momentum p ending at a point x, as in figure 4.1. The Wilson line stretches all the way down to the past time-like infinity i^-, where the asymptotic phase space is the hyperbolic space $\mathbb{H}_3$. If this Wilson line is dressing an asymptotic massive charged particle, as is the case under our consideration, we can assume that the limit $x^0 \to -\infty$ is being taken. In this picture, one can see that the Faddeev-Kulish dressing can essentially be viewed as a Wilson line puncture on i^-, the asymptotic boundary of Minkowski spacetime. The term “puncture" henceforth is be used to denote a Wilson line along a time-like path piercing the spacetime boundary of our interest.

Let us put this intuitive description on a more formal ground. The Minkowski spacetime has the following metric in terms of the Cartesian coordinates

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2. \tag{4.19}$$

We introduce the advanced set of coordinates $(v, r, z, \bar{z})$, which is related to the Cartesian coordinates by

$$v = t + r, \qquad r^2 = x_1^2 + x_2^2 + x_3^2, \qquad z = \frac{x_1 + ix_2}{r + x_3}. \tag{4.20}$$

The Minkowski metric can then be written as

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \tag{4.21}$$

where $\gamma_{z\bar{z}} = 2/(1 + z\bar{z})^2$ is the unit 2-sphere metric. In terms of these coordinates, the momentum measure is $d^3k = \omega^2 d\omega\gamma_{z\bar{z}}d^2z$, so we may write the asymptotic gauge field as

$$A_\mu^{\text{in}}(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left[a_\mu(\mathbf{k})e^{ik\cdot x} + a_\mu^\dagger(\mathbf{k})e^{-ik\cdot x}\right] \tag{4.22}$$

$$= \frac{1}{16\pi^3}\int \omega d\omega\gamma_{z\bar{z}}d^2z\left[a_\mu(\omega\hat{\mathbf{k}})e^{ik\cdot x} + a_\mu^\dagger(\omega\hat{\mathbf{k}})e^{-ik\cdot x}\right]. \tag{4.23}$$

Here $\hat{\mathbf{k}}$ is a unit 3-vector that points in the direction defined by $(z, \bar{z})$, such that $\mathbf{k} = \omega\hat{\mathbf{k}}$. Let us employ the usual polarization tensors

$$\epsilon^{+\mu} = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \quad \epsilon^{-\mu} = \frac{1}{\sqrt{2}}(z, 1, i, -z), \tag{4.24}$$

the plane-wave expansion,

$$e^{ik\cdot x} = 4\pi e^{-i\omega t}\sum_{\ell=0}^{\infty} i^\ell j_\ell(\omega r)\sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{k}})Y_{\ell m}^*(\hat{\mathbf{x}}) \tag{4.25}$$

$$= \frac{2\pi i}{\omega r}e^{-i\omega v}\delta^{(2)}(\hat{\mathbf{k}} + \hat{\mathbf{x}}) + O\left(r^{-2}\right), \tag{4.26}$$

as well as the relation $A_z = \partial_z(x^\mu A_\mu)$ to obtain the non-vanishing components [19]:

$$A_z(v, z, \bar{z}) = \lim_{r\to\infty} A_z(v, r, z, \bar{z}) \tag{4.27}$$
$$= \frac{i}{8\pi^2}\sqrt{\gamma_{z\bar{z}}}\int d\omega \left(a_-^\dagger(-\omega\hat{\mathbf{x}})e^{i\omega v} - a_+(-\omega\hat{\mathbf{x}})e^{-i\omega v}\right). \tag{4.28}$$

Here the expression $a_\pm(-\omega\hat{\mathbf{x}})$ should be understood as a particle operator with $\omega > 0$ with momentum in the direction $-\hat{\mathbf{x}}$. The minus sign comes from the fact that a massless particle moving in the direction $(z, \bar{z})$ is mapped to its antipodal point in the past infinity.

Observe in (4.28) that taking the limit $v \to -\infty$ forces the integral to get contributions only from the zero-modes $\omega = 0$ by virtue of the Riemann-Lebesgue lemma. We may use the method of [8, 9] to implement this explicitly, by introducing an infrared scalar function $\phi(\omega)$ that has support only in a small neighborhood of $\omega = 0$ and satisfies $\phi(0) = 1$:

$$A_z(z, \bar{z}) = \frac{i}{8\pi^2}\sqrt{\gamma_{z\bar{z}}}\int dw \left(a_-^\dagger(-\omega\hat{\mathbf{x}}) - a_+(-\omega\hat{\mathbf{x}})\right)\phi(\omega). \tag{4.29}$$

A Faddeev-Kulish dressing $W(\mathbf{p})$ of a particle with momentum $\mathbf{p}$ can be written in terms of these boundary modes.

$$W(\mathbf{p}) = \exp\left\{\frac{ie}{2\pi}\int d^2z\sqrt{\gamma_{z\bar{z}}}\left[\left(\frac{p\cdot\epsilon^-}{p\cdot\hat{k}}\right)A_z(z, \bar{z}) + \left(\frac{p\cdot\epsilon^+}{p\cdot\hat{k}}\right)A_{\bar{z}}(z, \bar{z})\right]\right\}, \tag{4.30}$$

where $\hat{k}^\mu = (1, \hat{\mathbf{k}}) = (1, -\hat{\mathbf{x}})$.

In terms of the language used in [130], the edge modes $A_z(z, \bar{z})$ and $A_{\bar{z}}(z, \bar{z})$ are the zero modes that exponentiate to the Wilson line sourced and localized at the boundary. In this reference which deals with the case of Rindler space, for each edge mode annihilation operator $a_{\mathbf{k}}$, there is a conjugate variable $q_{\mathbf{k}}$ such that

$$[a_{\mathbf{k}}, q_{-\mathbf{k}'}] = i\delta_{\mathbf{k}\mathbf{k}'}, \tag{4.31}$$

The eigenspace of this conjugate variable is more natural (compared to the eigenspace of $a_{\mathbf{k}}$) in the sense that it diagonalizes the boundary Hamiltonian of the Rindler space. We see below that even in flat Minkowski space, the vacua of Faddeev-Kulish states define an eigenspace analogous to that of $q_{\mathbf{k}}$ in a manner which was previously discussed in [16].

For this purpose, consider a function $\varepsilon(z, \bar{z})$ on the 2-sphere. The conserved charge Q_ε associated with this function can be written as [16]

$$Q_\varepsilon = Q_\varepsilon^{\text{soft}} + Q_\varepsilon^{\text{hard}}, \tag{4.32}$$

where the hard charge $Q_\varepsilon^{\text{hard}}$ contains charged matter current and hence commutes with the boundary field $A_z(z,\bar{z})$, and the soft charge $Q_\varepsilon^{\text{soft}}$ is given by

$$Q_\varepsilon^{\text{soft}} = -2\int d^2z\, \partial_{\bar{z}}\varepsilon(z,\bar{z})N_z(z,\bar{z}) = -2\int d^2z\, \partial_z\varepsilon(z,\bar{z})N_{\bar{z}}(z,\bar{z}). \tag{4.33}$$

Here the operator $N_z(z,\bar{z})$ is defined as

$$N_z(z,\bar{z}) = \int_{-\infty}^{\infty} dv\, \partial_v A_z(z,\bar{z}), \tag{4.34}$$

and contains only zero-energy photon operators, as one can see from the expression

$$N_z(z,\bar{z}) = -\frac{1}{4\pi}\sqrt{\gamma_{z\bar{z}}}\int_0^\infty d\omega\, \omega\, \delta(\omega)\left(a_-^\dagger(-\omega\hat{\mathbf{x}}) + a_+(-\omega\hat{\mathbf{x}})\right), \tag{4.35}$$

obtained by using the mode expansion (4.29) and the integral representation

$$\int_{-\infty}^{\infty} d\omega\, e^{\pm i\omega v} = 2\pi\delta(\omega). \tag{4.36}$$

From (4.29) and (4.35), we obtain by direct calculation

$$[A_z(z,\bar{z}), N_{\bar{w}}(w,\bar{w})] = \frac{i}{2}\sqrt{\gamma_{z\bar{z}}\gamma_{w\bar{w}}}\int d\omega\, d\omega'\, \omega\phi(\omega)\omega'\delta(\omega')\delta^{(3)}(\omega\hat{\mathbf{x}}_z - \omega'\hat{\mathbf{x}}_w), \tag{4.37}$$

where we used the commutation relation (4.12). Now, note that we may write

$$\delta^{(3)}(\omega\hat{\mathbf{x}}_z - \omega'\hat{\mathbf{x}}_w) = \frac{1}{\omega^2\gamma_{z\bar{z}}}\delta(\omega-\omega')\delta^{(2)}(z-w). \tag{4.38}$$

Therefore, with the convention

$$\int_0^\infty d\omega\, \delta(\omega)f(\omega) = \frac{1}{2}f(0), \tag{4.39}$$

of delta functions acting on the boundary of the integration domain, we obtain the commutation relation

$$[A_z(z,\bar{z}), N_{\bar{w}}(w,\bar{w})] = \frac{i}{2}\delta^{(2)}(z-w). \tag{4.40}$$

It follows from the expression (4.33) that

$$[Q_\varepsilon, A_z(z,\bar{z})] = i\partial_z\varepsilon(z,\bar{z}). \tag{4.41}$$

It is worth noting here that Q_ε may be replaced by $Q_\varepsilon^{\text{soft}}$ since $Q_\varepsilon^{\text{hard}}$ commutes with $A_z(z,\bar{z})$.

There is an immediate consequence of (4.41) and (4.30), which has been pointed out in [19]. Consider a vacuum $|0\rangle$ such that $Q_\varepsilon |0\rangle = 0$ and dress it with the Faddeev-Kulish operator to construct a state $W(\mathbf{p}) |0\rangle$. The expression (4.30) shows that $W(\mathbf{p})$ involves only the boundary gauge fields and thus only the zero-mode photon operators, qualifying $W(\mathbf{p}) |0\rangle$ as a vacuum. The two vacua $|0\rangle$ and $W(\mathbf{p}) |0\rangle$ are distinct, since the latter carries soft charge,

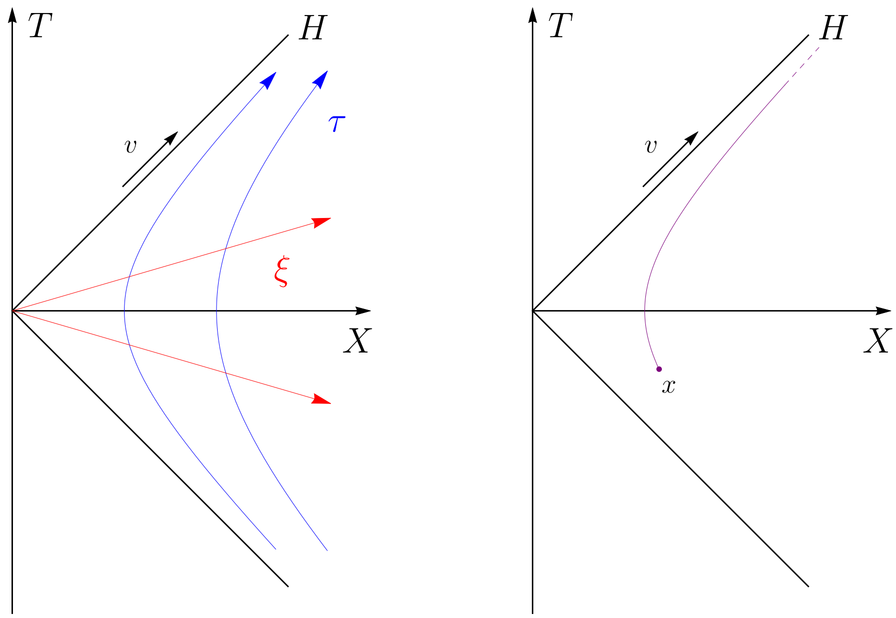
$$Q_\varepsilon W(\mathbf{p}) |0\rangle = -\frac{e}{2\pi} \int d^2 z \sqrt{\gamma_{z\bar{z}}} \left\{ \left(\frac{p \cdot \epsilon^-}{p \cdot \hat{k}} \right) \partial_z \varepsilon(z, \bar{z}) + \left(\frac{p \cdot \epsilon^+}{p \cdot \hat{k}} \right) \partial_{\bar{z}} \varepsilon(z, \bar{z}) \right\} W(\mathbf{p}) |0\rangle \,, \quad (4.42)$$

as implied by (4.41). Since there are infinitely many dressings $W(\mathbf{p})$, there exists an infinite number of degenerate vacua, each characterized by its soft charge. The selection rule arising from the charge conservation manifests itself as the infrared divergence of scattering amplitudes, and the asymptotic states of Faddeev and Kulish are the eigenstates of the conserved charge Q_ε. This has been investigated for the flat spacetime both in QED [19] and in perturbative gravity [59].

In summary, we have seen that the flat-space Faddeev-Kulish dressings can be written as Wilson line punctures on the asymptotic boundary of Minkowski spacetime. The massless gauge field has non-vanishing components at the asymptotic boundary, and the Wilson line can be written as a linear combination of these fields. On the other hand, the soft charge of the large gauge symmetry is a linear combination of a variable which is canonically conjugate to the boundary gauge field. As a consequence, each dressing carries a definite soft charge, parametrized by a 3-momentum $\mathbf{p}$. The dressings carry zero energy, and therefore can be used to generate a Hilbert space consisting of an infinite number of distinct vacua. This space is referred to as the edge Hilbert space in some literature, see for example [130]. The soft charge of large gauge transformation is a good quantum number to label the states. In the next section, we extend this work to the Rindler spacetime and the future Rindler horizon, aiming to draw results consistent with the analysis made in [130].

4.2 Soft photon hair on future Rindler horizon

In the previous section, we have seen that the Faddeev-Kulish dressings of asymptotic states are Wilson lines along a time-like path at the future/past time-like infinity. From this, along with the previous works exploring the connection between large gauge symmetry and Faddeev-Kulish dressings [18, 19, 58], it follows that the set of degenerate vacua carrying soft charge of large gauge transformation is obtained by dressing the vacuum with Wilson lines. Then it is only natural to expect that, in spacetimes exhibiting an event horizon, Wilson lines piercing the horizon along a time-like path are the dressings carrying soft hair on the



(a) (b)

Figure 4.2: A depiction of the Rindler spacetime. The Rindler coordinates (τ, ξ) are related to the Minkowski coordinates (X, T) by $T = \frac{1}{a}e^{a\xi}\sinh(a\tau)$ and $X = \frac{1}{a}e^{a\xi}\cosh(a\tau)$. At the future Rindler horizon H, one has $\xi = -\infty$ and $\tau = \infty$. H is parametrized by the advanced time $v = \tau + \xi$, along with the coordinates $\mathbf{x}_\perp$ which are omitted in the diagram. (a) The constant-ξ curves (marked blue) are parametrized by τ, while the constant-τ curves (marked red) are parametrized by ξ. (b) The purple curve illustrates a Wilson line along a time-like trajectory of a massive particle, starting at a point x and extending into H.

horizon. In this section, we show that this expectation indeed holds in the Rindler spacetime. We begin by reviewing a canonical quantization scheme of gauge fields developed in [131]. This is then be used to demonstrate that radial (time-like) Wilson lines in the vicinity of the future horizon are the analogue Faddeev-Kulish operators that dress the Fock vacuum to create degenerate vacua carrying soft charge.

Following the notation of [131], we use lower case Latin letters such as i, j to denote the spatial components of tensors and capital Latin letters such as I, J to denote the perpendicular components $\mathbf{x}_\perp$ (that is, x^2 and x^3).

4.2.1 Review of transverse gauge fields in Rindler spacetime

Here we present a brief review the quantization scheme developed in [131]. This quantization in Weyl gauge especially proves to be useful because it involves only the physical, transverse gauge fields. The methods introduced here are relevant to the Schwarzschild case as well.

The Rindler metric takes the following form,

$$ds^2 = e^{2a\xi}(-d\tau^2 + d\xi^2) + d\mathbf{x}_\perp^2. \tag{4.43}$$

We sometimes write $(x^0, x^1, x^2, x^3) = (\tau, \xi, \mathbf{x}_\perp)$. The quantization is carried out in the Weyl gauge,

$$A_0(\tau, \xi, \mathbf{x}_\perp) = 0. \tag{4.44}$$

The Lagrangian density of the gauge field A_μ coupled to an external current j^μ is

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu\right), \tag{4.45}$$

where the field strength tensor $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.46}$$

The equations of motion are then given by

$$\partial_\mu \sqrt{-g} g^{\mu\rho} g^{\nu\sigma}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) = \sqrt{-g} j^\nu, \tag{4.47}$$

and the conjugate momenta are

$$\Pi^i = -\sqrt{-g} g^{00} \partial_0 A^i. \tag{4.48}$$

With the metric of the form (4.43), the equations of motion reduces to

$$\partial_0 \sqrt{-g} g^{00} \partial_0 A^i + \partial_j \sqrt{-g} g^{jk} g^{il}(\partial_k A_l - \partial_l A_k) = \sqrt{-g} j^i, \tag{4.49}$$

and the Gauss Law ($\nu = 0$ in (4.47)),

$$\partial_i \Pi^i = \sqrt{-g} j^0, \tag{4.50}$$

is no longer part of the equations of motion; it becomes a constraint on the conjugate momentum. Canonical quantization is achieved by postulating the equal-time commutation relation

$$\left[\Pi^i(\tau, \xi, \mathbf{x}_\perp), A_j(\tau, \xi', \mathbf{x}'_\perp)\right] = \frac{1}{i}\delta^i_j \delta(\xi - \xi')\delta^{(2)}(\mathbf{x} - \mathbf{x}'). \tag{4.51}$$

Define the transverse projection operator as

$$P^i{}_j = \delta^i_j - \partial^i \frac{1}{\Delta} \partial_j, \tag{4.52}$$

where Δ is given by

$$\Delta = \partial_i \partial^i = \partial_\xi e^{-2a\xi} \partial_\xi + \nabla_\perp^2, \tag{4.53}$$

with $\nabla_\perp^2 = \partial_2^2 + \partial_3^2$. The projection operator satisfies the following identities,

$$P^i{}_j P^j{}_k = P^i{}_k, \qquad \partial_i P^i{}_j = 0, \qquad P^i{}_j \partial^j = 0. \tag{4.54}$$

Using this, one can define the transverse components of the gauge field and the conjugate momentum which we denote with a hat,

$$\hat{A}^i = P^i{}_j A^i, \qquad \hat{\Pi}^i = P^i{}_j \Pi^j. \tag{4.55}$$

A nice property of these transverse projections are that by a proper choice of gauge-fixing, the Hamiltonian of the gauge field can be formulated completely in terms of the transverse fields (4.55), plus a c-number contribution describing the effect of the external current – therefore, no unphysical degrees of freedom need to be carried around. We do not delve into the details of how the dynamics can be written down in terms of the transverse fields; we refer the interested readers to [131].

The equal-time commutation relation of the transverse fields can be obtained by transverse projection of the canonical relation (4.51). To this end, it is convenient to first consider a massless scalar field φ in Rindler spacetime, because, as we later see, the radial gauge fields satisfy the same equation of motion. The free-field equation of motion is

$$\left(\partial_\tau^2 - \Delta_s \right) \varphi = 0, \tag{4.56}$$

where the scalar Laplacian Δ_s is given by

$$\Delta_s = \partial_\xi^2 + e^{2a\xi} \nabla_\perp^2. \tag{4.57}$$

This can be solved with the ansatz

$$\varphi = e^{-i\omega\tau} e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} k_{i\frac{w}{a}}(z), \tag{4.58}$$

where we defined

$$z = \frac{k_\perp}{a} e^{a\xi}, \tag{4.59}$$

with $k_\perp = |\mathbf{k}_\perp|$. Here $k_{i\frac{w}{a}}$ is the appropriately normalized MacDonald function,

$$k_{i\frac{\omega}{a}}(z) = \frac{1}{\pi}\sqrt{\frac{2\omega}{a}\sinh\frac{\pi\omega}{a}}\, K_{i\frac{\omega}{a}}(z), \tag{4.60}$$

which forms a complete orthonormal set. In particular, it satisfies the completeness relation

$$\int_0^\infty d\omega\, k_{i\frac{\omega}{a}}(z) k_{i\frac{\omega}{a}}(z') = az\delta(z-z') = \delta(\xi - \xi'). \tag{4.61}$$

Among the components of the transverse projection operator (4.52), the one that is be relevant for our purposes is $i = j = 1$. Using the completeness relations and noting that $P^1{}_1$ can be written in terms of the scalar Laplacian Δ_s as[1]

$$P^1{}_1 = \frac{1}{\Delta_s} e^{2a\xi}\nabla_\perp^2, \tag{4.62}$$

one obtains the commutation relation

$$\begin{aligned} \left[\hat{\Pi}^1(\tau,\xi,\mathbf{x}_\perp), \hat{A}_1(\tau,\xi',\mathbf{x}'_\perp)\right] &= -ig_{11}P^1{}_1 g^{11}\delta(\xi-\xi')\delta^{(2)}(\mathbf{x}_\perp - \mathbf{x}'_\perp) && (4.63)\\ &= -i\int \frac{d^2k_\perp}{(2\pi)^2} z^2 e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp - \mathbf{x}'_\perp)} \int_0^\infty d\omega \frac{a^2}{\omega^2} k_{i\frac{\omega}{a}}(z) k_{i\frac{\omega}{a}}(z'). && (4.64) \end{aligned}$$

Now, using the properties (4.54) of the transverse projection operator, one can show that the equation of motion of the transverse field $\hat{A}_\mu$ with no external current is

$$\sqrt{-g}g^{00}\partial_0^2\hat{A}^i + \partial_j\sqrt{-g}g^{jk}g^{il}(\partial_k\hat{A}_l - \partial_l\hat{A}_k) = 0, \tag{4.65}$$

which can be written as

$$\partial_\tau^2\hat{A}^1 - \Delta_s\hat{A}^1 = 0, \tag{4.66}$$

$$\partial_\tau^2\hat{A}^I - \Delta_s\hat{A}^I + 2ae^{2a\xi}\partial_I\hat{A}^1 = 0, \tag{4.67}$$

where $I = 2, 3$. The commutation relation (4.64) and the equation of motion (4.66) can be used to write down the normal mode expansion of the transverse gauge field $\hat{A}_1$ and its

[1] This can be seen by showing $\Delta_s P^1{}_1 = e^{2a\xi}\nabla_\perp^2$.

conjugate momentum $\hat{\Pi}^1$:

$$\hat{A}_1(\tau,\xi,\mathbf{x}_\perp) = \int \frac{d\omega}{\sqrt{2\omega}}\frac{d^2k_\perp}{2\pi}\frac{a^2}{\omega k_\perp}\left[a_1(\omega,\mathbf{k}_\perp)e^{-i\omega\tau}e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} + \text{h.c.}\right] z^2 k_{i\frac{\omega}{a}}(z), \tag{4.68}$$

$$\hat{\Pi}^1(\tau,\xi,\mathbf{x}_\perp) = -i\int \frac{d\omega}{\sqrt{2\omega}}\frac{d^2k_\perp}{2\pi}k_\perp\left[a_1(\omega,\mathbf{k}_\perp)e^{-i\omega\tau}e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} - \text{h.c.}\right] k_{i\frac{\omega}{a}}(z). \tag{4.69}$$

Requiring that the creation/annihilation operators satisfy the standard commutation relation

$$[a_1(\omega,\mathbf{k}_\perp), a_1^\dagger(\omega',\mathbf{k}'_\perp)] = \delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp), \tag{4.70}$$

one can readily check that the transverse fields (4.68), (4.69) satisfy the relation (4.64). The remaining components $\hat{A}_I$ can also be obtained by writing down a relation similar to (4.62) and using the equations of motion (4.67). The fields $\hat{A}_I$ has been worked out in [131], the result of which we state here for later reference:

$$\begin{aligned}\hat{A}_I(\tau,\xi,\mathbf{x}_\perp) = \int & \frac{d\omega}{\sqrt{2\omega}}\frac{d^2k_\perp}{2\pi} \\ & \times\left[\left\{\epsilon_I(\mathbf{k}_\perp)a_2(\omega,\mathbf{k}_\perp) + i\frac{ak_I}{\omega k_\perp}a_1(\omega,\mathbf{k}_\perp)z\frac{d}{dz}\right\}k_{i\frac{\omega}{a}}(z)e^{-i\omega\tau}e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} + \text{h.c.}\right],\end{aligned} \tag{4.71}$$

where k_I denotes the I-th component of $\mathbf{k}_\perp$, and the second pair of creation/annihilation operators satisfy the commutation relations

$$[a_2(\omega,\mathbf{k}_\perp), a_2^\dagger(\omega',\mathbf{k}'_\perp)] = \delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp), \tag{4.72}$$

$$[a_1(\omega,\mathbf{k}_\perp), a_2^\dagger(\omega',\mathbf{k}'_\perp)] = 0, \tag{4.73}$$

and the polarization vector $\epsilon_I(\mathbf{k}_\perp)$ is transverse to $\mathbf{k}_\perp$,

$$k_I\epsilon^I(\mathbf{k}_\perp) = 0. \tag{4.74}$$

4.2.2 Wilson lines, edge modes and surface charges on the horizon

In order to investigate the behavior of the transverse fields near the future Rindler horizon, we note that the horizon is parametrized by the advanced time $v = \tau + \xi$ as well as the 2-dimensional plane coordinates $\mathbf{x}_\perp$. In terms of the advanced coordinates $(v,\xi,\mathbf{x}_\perp)$, the

Rindler metric is

$$ds^2 = e^{2a\xi}(-dv^2 + 2dvd\xi) + d\mathbf{x}_\perp^2, \tag{4.75}$$

and the transverse gauge field (4.68) is

$$\hat{A}_1(v, \xi, \mathbf{x}_\perp) = \int \frac{d\omega}{\sqrt{2\omega}} \frac{d^2k_\perp}{2\pi} \frac{a^2}{\omega k_\perp} \left[a_1(\omega, \mathbf{k}_\perp)e^{-i\omega v + i\omega\xi} e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} + \text{h.c.}\right] z^2 k_{i\frac{\omega}{a}}(z). \tag{4.76}$$

Notice the factors $e^{\pm i\omega\xi}$ in the integrand. As we approach the horizon $\xi \to -\infty$, by virtue of the Riemann-Lebesgue lemma only the leading soft modes contribute to the integral. Similar to the construction of [8,9], we can explicitly implement this by replacing $e^{\pm i\omega\xi}$ with a scalar function $\phi(\omega)$, which satisfies $\phi(0) = 1$ and has support only in a small neighborhood of $\omega = 0$. With this and the asymptotic form

$$k_{i\frac{\omega}{a}}(z) \sim \frac{\omega}{a}\sqrt{\frac{2}{\pi}} K_0(z) \quad \text{as } z \to 0, \tag{4.77}$$

we obtain

$$\hat{A}_1(v, \xi, \mathbf{x}_\perp) \sim \int \frac{d\omega}{\sqrt{\pi\omega}} \frac{d^2k_\perp}{2\pi} \frac{a}{k_\perp} \phi(\omega) \left[a_1(\omega, \mathbf{k}_\perp)e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} + \text{h.c.}\right] z^2 K_0(z) \quad \text{as } \xi \to -\infty. \tag{4.78}$$

Now, let us consider the exponent in the Wilson line, i.e., the line integral

$$\mathcal{A}(x) = \int_\Gamma dx^\mu \hat{A}_\mu(x), \tag{4.79}$$

where Γ is a time-like path in the vicinity of the horizon. In the following we evaluate this assuming that the gauge fields satisfy sourceless, quasi-free equations of motion, i.e., we do not consider its interactions with any currents, classical or otherwise. As we have seen in section 4.1.1, the current for the straight line asymptotic path is essentially classical. Use of the Yang-Feldman equation (4.5) in the evaluation of the line integral implies that the interaction terms with the external current gives additional c-number terms in the expression for the Wilson line, which are related to the Coulomb phase. These were not relevant for the analysis of the soft hair for that case and we think it is safe to assume this to be the case here as well. From the metric (4.43), one can see that $d\mathbf{x}_\perp^2 = 0$ along a time-like geodesic as

$\xi \to -\infty$. Thus we may write

$$\mathcal{A}(\mathbf{x}_\perp) = \int_\Gamma d\xi \hat{A}_1(v, \xi, \mathbf{x}_\perp) \tag{4.80}$$

$$= -\int \frac{d\omega}{\sqrt{\pi\omega}} \frac{d^2k_\perp}{2\pi} \frac{1}{k_\perp} \phi(\omega) \left[a_1(\omega, \mathbf{k}_\perp) e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} + \text{h.c.} \right], \tag{4.81}$$

where we used a boundary condition analogous to (4.15),

$$\int_\Gamma dz\, z K_0(z) = -z K_1(z), \tag{4.82}$$

and took the limit corresponding to $\xi \to -\infty$,

$$\lim_{z \to 0} z K_1(z) = 1. \tag{4.83}$$

Drawing analogy from Minkowski spacetime, we expect the Wilson line $\exp\{ie\mathcal{A}(x)\}$ to serve as the Faddeev-Kulish dressings in Rindler spacetime. To see this, consider any function $\varepsilon(\mathbf{x}_\perp)$ on the 2-dimensional plane. The conserved charge Q_ε associated with this function is then [1]

$$Q_\varepsilon = Q_\varepsilon^{\text{soft}} + Q_\varepsilon^{\text{hard}}, \tag{4.84}$$

where the soft and hard charges are given as

$$Q_\varepsilon^{\text{soft}} = \int_H \mathrm{d}\varepsilon \wedge *F, \qquad Q_\varepsilon^{\text{hard}} = \int_H \varepsilon * j. \tag{4.85}$$

Here H denotes the future Rindler horizon, j is the charged matter current, and $*F$ is the dual field strength tensor

$$(*F)_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{4.86}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the antisymmetric Levi-Civita tensor with $\epsilon_{0123} = \sqrt{-g} = e^{2a\xi}$. Since the operator $\mathcal{A}(x)$ involves only the soft photon modes (cf. (4.81)), it commutes with the hard charge $Q_\varepsilon^{\text{hard}}$ and we may thus focus our attention on the soft charge $Q_\varepsilon^{\text{soft}}$. The horizon is parametrized by $(v, \mathbf{x}_\perp)$ and $\partial_v \varepsilon(\mathbf{x}_\perp) = 0$, which implies that the relevant components of the dual tensor $*F$ are, up to some magnetic fields that vanish at the future horizon,

$$(*F)_{02} = -\partial_v \hat{A}^3, \qquad (*F)_{03} = \partial_v \hat{A}^2. \tag{4.87}$$

We therefore have

$$Q_\varepsilon^{\rm soft} = -\lim_{\xi\to-\infty}\int_{-\infty}^{\infty} dv \int d^2\mathbf{x}_\perp \partial_I \varepsilon(\mathbf{x}_\perp)\partial_v \hat{A}^I(v,\xi,\mathbf{x}_\perp) \tag{4.88}$$

$$= \int d^2\mathbf{x}_\perp \varepsilon(\mathbf{x}_\perp) N(\mathbf{x}_\perp), \tag{4.89}$$

where, in the last line we defined

$$N(\mathbf{x}_\perp) = \lim_{\xi\to-\infty}\int_{-\infty}^{\infty} dv\, \partial_v \partial_I \hat{A}^I(v,\xi,\mathbf{x}_\perp), \tag{4.90}$$

after an integration by parts. The transverse property (4.54) of the projection operator implies that $\partial_i \hat{A}^i = 0$, or equivalently $\partial_I \hat{A}^I = -\partial_\xi \hat{A}^1 = -az\partial_z \hat{A}^1$. Thus with the integral representation (4.36) of the Dirac delta function, we obtain

$$N(\mathbf{x}_\perp) = \lim_{z\to 0} i \int \frac{d\omega}{\sqrt{2\omega}} d^2k_\perp k_\perp \delta(\omega)\left[a_1(\omega,\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} - \text{h.c.}\right] az\frac{d}{dz}k_{i\frac{\omega}{a}}(z). \tag{4.91}$$

$$= -i\int \frac{d\omega}{\sqrt{\pi\omega}} d^2k_\perp k_\perp \omega\delta(\omega)\left[a_1(\omega,\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} - \text{h.c.}\right], \tag{4.92}$$

where in the second line we used the asymptotic form (4.77), along with the relation

$$\frac{d}{dz}K_0(z) = -K_1(z), \tag{4.93}$$

and then took the limit $z\to 0$ using (4.83). With the commutation relation (4.70), we obtain

$$\begin{aligned}
[N(\mathbf{x}_\perp), \mathcal{A}(\mathbf{x}'_\perp)] &= i\int \frac{d\omega d\omega'}{\pi\sqrt{\omega\omega'}}\frac{d^2k_\perp d^2k'_\perp}{2\pi}\frac{k_\perp}{k'_\perp}\omega\delta(\omega)\phi(\omega') \\
&\qquad \times\left[a_1(\omega,\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} - \text{h.c.},\ a_1(\omega',\mathbf{k}'_\perp)e^{i\mathbf{k}'_\perp\cdot\mathbf{x}'_\perp} + \text{h.c.}\right] &(4.94)\\
&= i\int \frac{d\omega d\omega'}{\pi\sqrt{\omega\omega'}}\frac{d^2k_\perp d^2k'_\perp}{2\pi}\omega\delta(\omega)\phi(\omega') \\
&\qquad \times \delta(\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\left[e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp)} + e^{-i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp)}\right] &(4.95)\\
&= 2i\delta^{(2)}(\mathbf{x}_\perp - \mathbf{x}'_\perp)\int_0^\infty d\omega\,\delta(\omega) &(4.96)\\
&= i\delta^{(2)}(\mathbf{x}_\perp - \mathbf{x}'_\perp), &(4.97)
\end{aligned}$$

where we used the following convention of delta functions,

$$\int_0^\infty d\omega\,\delta(\omega)f(\omega) = \frac{1}{2}f(0). \tag{4.98}$$

Equation (4.97) shows that $N(\mathbf{x}_\perp)$ and $\mathcal{A}(\mathbf{x}_\perp)$ are canonically conjugate variables, and therefore $\mathcal{A}(\mathbf{x}_\perp)$ satisfies the commutator

$$[Q_\varepsilon, \mathcal{A}(\mathbf{x}_\perp)] = i\varepsilon(\mathbf{x}_\perp). \tag{4.99}$$

This has immediate consequence. Consider the following state,

$$|q, \mathbf{x}_\perp\rangle = e^{iq\mathcal{A}(\mathbf{x}_\perp)} |0\rangle , \tag{4.100}$$

where we choose $|0\rangle$ to be the vacuum satisfying $Q_\varepsilon |0\rangle = 0$. Since $\mathcal{A}(\mathbf{x}_\perp)$ only involves zero-energy photon operators, this state carries zero energy and is therefore a vacuum. However, it follows from (4.99) that

$$Q_\varepsilon |q, \mathbf{x}_\perp\rangle = Q_\varepsilon^{\text{soft}} |q, \mathbf{x}_\perp\rangle = -q\varepsilon(\mathbf{x}_\perp) |q, \mathbf{x}_\perp\rangle . \tag{4.101}$$

which implies that this is a degenerate vacuum carrying soft charge. This is analogous to the case of flat space [19, 59], where the set of degenerate vacua is obtained by dressing the vacuum with the Faddeev-Kulish operators. We are thus led to the conclusion that the time-like Wilson lines near the horizon are the Faddeev-Kulish dressings of Rindler spacetime.

We end this section with an instructive derivation of the boundary values of gauge fields to see how the charge Q_ε acts on them. Let us define the boundary fields

$$\hat{A}_i^H = \lim_{\xi \to -\infty} \hat{A}_i(v, \xi, \mathbf{x}_\perp). \tag{4.102}$$

From (4.78) we can see that $\hat{A}_1^H = 0$ since

$$\lim_{z \to 0} z^2 K_0(z) = 0. \tag{4.103}$$

In the advanced coordinates the remaining components (4.71) can be written as

$$\hat{A}_I(v, \xi, \mathbf{x}_\perp) = \int \frac{d\omega}{\sqrt{2\omega}} \frac{d^2k_\perp}{2\pi} \left[\left\{ \epsilon_I a_2 + i \frac{ak_I}{\omega k_\perp} a_1 z \frac{d}{dz} \right\} k_{i\frac{\omega}{a}}(z) e^{-i\omega v + i\omega\xi} e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} + \text{h.c.} \right]. \tag{4.104}$$

The first term in the curly brackets involving the polarization tensor ϵ_I is proportional to the expression

$$e^{i\omega\xi} k_{i\frac{\omega}{a}}(z) = \frac{\omega}{a} \sqrt{\frac{2}{\pi}} K_0(z) + O(\omega^2) \to 0 \quad \text{as } \omega \to 0, \tag{4.105}$$

so it retains no zero-energy modes at the horizon. However, the second term survives, yielding the non-zero components

$$\hat{A}_I^H(\mathbf{x}_\perp) = -i \int \frac{d\omega}{\sqrt{\pi\omega}} \frac{d^2k_\perp}{2\pi} \frac{k_I}{k_\perp} \phi(\omega) \left[a_1(\omega, \mathbf{k}_\perp) e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} - \text{h.c.} \right]. \tag{4.106}$$

The expression (4.81) tells us that we can write these fields in terms of $\mathcal{A}(\mathbf{x}_\perp)$ as

$$\hat{A}_I^H(\mathbf{x}_\perp) = \partial_I \mathcal{A}(\mathbf{x}_\perp), \tag{4.107}$$

from which we obtain the commutation relation

$$\left[Q_\varepsilon, \hat{A}_I^H(\mathbf{x}_\perp) \right] = i\partial_I \varepsilon(\mathbf{x}_\perp). \tag{4.108}$$

This is reminiscent of the action of charge on boundary fields in Minkowski spacetime [16],

$$[Q_\varepsilon, A_z(u, z, \bar{z})] = i\partial_z \varepsilon(z, \bar{z}), \tag{4.109}$$

and shows that Q_ε correctly generates the boundary degrees of freedom (large gauge transformations), which in our case are the fields at the Rindler horizon.

In summary, we have shown that, similar to the Minkowski spacetime, the Wilson line puncture on the future Rindler horizon carries a definite soft horizon charge. This identifies the puncture as the Faddeev-Kulish dressing of Rindler spacetime, which can be used to generate the edge Hilbert space. The edge Hilbert space consists of an infinite number of degenerate vacua, where each state is labeled by its soft horizon charge. This result is consistent with the analysis made in [130] with regards to the edge states in the Lorenz gauge. In the next section, we apply similar methods to extend our analysis to the Schwarzschild spacetime and its horizon.

4.3 Soft photon hair on Schwarzschild horizon

In this section, we investigate the soft photon hair directly on the Schwarzschild horizon using the quantization method of [131]. The Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.110}$$

where $M = GM_0$, with G the Newton's constant and M_0 the mass of the black hole. From the lessons learned in Rindler spacetime, we know that it is only the near-horizon physics that plays a role in the analysis: it is expected that the Wilson lines along a time-like path in the vicinity of the horizon are again the analogue of the Faddeev-Kulish dressings, the building blocks of the edge Hilbert space. This motivates us to restrict our attention to the near-horizon region of Schwarzschild, by writing

$$\rho = r - 2M. \tag{4.111}$$

The leading terms in the small-ρ expansion of (4.110) yields the near-horizon metric,

$$ds^2 = -\frac{\rho}{2M}dt^2 + \frac{2M}{\rho}d\rho^2 + 4M^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.112}$$

Then, let us define new coordinates ξ and y as

$$\xi = 2M\ln\left(\frac{\rho}{2M}\right), \qquad y = \cos\theta, \tag{4.113}$$

in terms of which the metric reads

$$ds^2 = e^{2a\xi}\left(-dt^2 + d\xi^2\right) + 4M^2\left[\frac{dy^2}{1-y^2} + (1-y^2)d\phi^2\right], \tag{4.114}$$

where $a = \frac{1}{4M}$ is the surface gravity of the black hole. We use Ω to denote the spherical coordinates (y, ϕ) collectively. The (t, ξ) space resembles the Rindler spacetime, but the remaining space is a 2-sphere, not a 2-plane. There are two reasons for choosing the coordinate $y = \cos\theta$ over the conventional θ: One is that this achieves $\partial_i(\sqrt{-g}g^{00}) = 0$ which simplifies a lot of calculations [131], and the other is that it is difficult to obtain a simple operator relation such as (4.62) if we use θ.

In the following subsections, we work with the metric (4.114). We begin by deriving the mode expansion of transverse gauge fields. These are used to show that the near-horizon, time-like Wilson lines are the Faddeev-Kulish dressings that build the edge Hilbert space.

4.3.1 Transverse gauge fields

We work in the Weyl gauge as before,

$$A_0(t, \xi, \Omega) = 0. \tag{4.115}$$

With the metric (4.114), the transverse projection operator is defined as,

$$P^i{}_j = \delta^i_j - \partial^i \frac{1}{\Delta} \partial_j, \tag{4.116}$$

where the operator $\Delta = \partial_i \partial^i$ is now given by

$$\Delta = \partial_\xi e^{-2a\xi} \partial_\xi - \frac{\mathbf{L}^2}{4M^2}. \tag{4.117}$$

Here $\mathbf{L}^2$ is the angular momentum squared operator,

$$\mathbf{L}^2 = -\partial_y (1 - y^2) \partial_y - \frac{\partial_\phi^2}{1 - y^2}, \tag{4.118}$$

whose eigenfunctions are the spherical harmonics $Y_{\ell m}$,

$$\mathbf{L}^2 Y_{\ell m}(y, \phi) = \ell(\ell + 1) Y_{\ell m}(y, \phi). \tag{4.119}$$

In order to find the mode expansion of transverse gauge fields, as for the Rindler case, let us first look at the case of free scalar field φ, whose equation of motion has the form

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \varphi = 0, \tag{4.120}$$

or equivalently,

$$\left(\partial_t^2 - \Delta_s\right) \varphi = 0, \tag{4.121}$$

where Δ_s is the scalar Laplacian,

$$\Delta_s = \partial_\xi^2 - \frac{e^{2a\xi}}{4M^2} \mathbf{L}^2. \tag{4.122}$$

From symmetry, solutions take the form

$$\varphi(t, \xi, y, \phi) = e^{-i\omega t} Y_{\ell m}(y, \phi) R(\xi) \tag{4.123}$$

with some function R of ξ. Then, the equation of motion (4.121) reduces to an equation for R, which reads

$$\frac{d^2 R}{d\xi^2} + \left[\omega^2 - \frac{e^{2a\xi}}{4M^2} \ell(\ell + 1)\right] R = 0. \tag{4.124}$$

Now, define a new variable

$$z = 2\sqrt{\ell(\ell+1)}e^{a\xi}, \tag{4.125}$$

where we exclude the $\ell = 0$ mode. We see later that $\ell = 0$ is associated to the total electric charge and hence is not of our interest. In terms of z, equation (4.124) becomes

$$z^2 \frac{d^2 R}{dz^2} + z\frac{dR}{dz} + \left(\frac{\omega^2}{a^2} - z^2\right) R = 0. \tag{4.126}$$

This is the same modified Bessel equation that we saw in Rindler space, whose solutions are the properly normalized MacDonald functions

$$R = k_{i\frac{\omega}{a}}(z) = \frac{1}{\pi}\sqrt{\frac{2\omega}{a}\sinh\left(\frac{\pi\omega}{a}\right)}K_{i\frac{\omega}{a}}(z). \tag{4.127}$$

Now we're in the right place to obtain the commutation relation of the transverse fields. From the free Maxwell Lagrangian density,

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right), \tag{4.128}$$

we obtain the momentum density Π^i conjugate to the gauge field to be

$$\Pi^i = -\sqrt{-g}g^{00}\partial_0 A^i = 4M^2\partial_0 A^i. \tag{4.129}$$

Then we quantize the fields by imposing the equal-time commutation relation,

$$\left[\Pi^i(t,\xi,y,\phi), A_j(t,\xi',y',\phi')\right] = \frac{1}{i}\delta^i_j\delta(\xi-\xi')\delta(y-y')\delta(\phi-\phi'). \tag{4.130}$$

Commutation relation between transverse components of the fields is obtained by applying transverse projection onto (4.130). The projection operator relevant to our analysis is ${P^1}_1$. Observing that

$$\Delta_s\left(1 - \partial^1\frac{1}{\Delta}\partial_1\right) = -\frac{e^{2a\xi}}{4M^2}\mathbf{L}^2, \tag{4.131}$$

we find that the projection operator can be written as

$${P^1}_1 = -\frac{1}{\Delta_s}\frac{e^{2a\xi}}{4M^2}\mathbf{L}^2. \tag{4.132}$$

Then, we obtain

$$\left[\hat{\Pi}^1(t,\mathbf{x}'),\hat{A}_1(t,\mathbf{x})\right]=\frac{1}{i}g_{11}P^1{}_1g^{11}\delta(\xi-\xi')\delta(y-y')\delta(\phi-\phi') \tag{4.133}$$

$$=i\frac{e^{2a\xi}}{4M^2}\frac{1}{\Delta_s}\mathbf{L}^2\delta(\xi-\xi')\delta(y-y')\delta(\phi-\phi'), \tag{4.134}$$

which, making use of the completeness relations

$$\int_0^\infty d\omega\, k_{i\frac{\omega}{a}}(z)k_{i\frac{\omega}{a}}(z')=\delta(\xi-\xi'), \tag{4.135}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}(y,\phi)Y^*_{\ell m}(y',\phi')=\delta(y-y')\delta(\phi-\phi'), \tag{4.136}$$

can be written as

$$\left[\hat{\Pi}^1(t,\mathbf{x}'),\hat{A}_1(t,\mathbf{x})\right]=i\sum_{\ell m}z^2Y_{\ell m}(y,\phi)Y^*_{\ell m}(y',\phi')\int_0^\infty d\omega\frac{a^2}{\omega^2}k_{i\frac{\omega}{a}}(z)k_{i\frac{\omega}{a}}(z'). \tag{4.137}$$

Now that we have the commutation relation, let us consider the equation of motion of free gauge field in Weyl gauge, which reads

$$\partial_0\sqrt{-g}g^{00}\partial_0A^i+\partial_j\sqrt{-g}g^{jk}g^{il}(\partial_kA_l-\partial_lA_k)=0. \tag{4.138}$$

Noting that $\partial_iA_j-\partial_jA_i=\partial_i\hat{A}_j-\partial_j\hat{A}_i$, one may write the equations of motion of the transverse fields as

$$\sqrt{-g}g^{00}\partial_0^2\hat{A}^i+\partial_j\sqrt{-g}g^{jk}g^{il}(\partial_k\hat{A}_l-\partial_l\hat{A}_k)=0, \tag{4.139}$$

or written out explicitly,

$$-\partial_0^2\hat{A}^1+\Delta_s\hat{A}^1=0, \tag{4.140}$$

$$-\partial_0^2\hat{A}^y+\Delta_s\hat{A}^y+\frac{e^{2a\xi}}{4M^2}\left[2y\partial_\phi\hat{A}^\phi-2a\left(1-y^2\right)\partial_y\hat{A}^1+2y\partial_y\hat{A}^y\right]=0, \tag{4.141}$$

$$-\partial_0^2\hat{A}^\phi+\Delta_s\hat{A}^\phi-\frac{e^{2a\xi}}{4M^2}\left[\partial_y\left(2y\hat{A}^\phi\right)+\frac{2a\partial_\phi\hat{A}^1}{1-y^2}+\frac{2y\partial_\phi\hat{A}^y}{(1-y^2)^2}\right]=0. \tag{4.142}$$

Using the equal-time commutation relation (4.137) and the equation of motion (4.140), we obtain the ξ-component of the transverse fields to be

$$\hat{A}^1(t,\xi,\Omega) = \sum_{\ell m} 4\sqrt{\ell(\ell+1)} \int \frac{d\omega}{\sqrt{2\omega}} \frac{a^2}{\omega} \left[a_{\ell m}(\omega) e^{-i\omega t} Y_{\ell m}(\Omega) + \text{h.c.}\right] k_{i\frac{\omega}{a}}(z), \tag{4.143}$$

$$\hat{A}_1(t,\xi,\Omega) = \sum_{\ell m} \frac{1}{\sqrt{\ell(\ell+1)}} \int \frac{d\omega}{\sqrt{2\omega}} \frac{a^2}{\omega} \left[a_{\ell m}(\omega) e^{-i\omega t} Y_{\ell m}(\Omega) + \text{h.c.}\right] z^2 k_{i\frac{\omega}{a}}(z). \tag{4.144}$$

Due to the property $\partial_i(\sqrt{-g} g^{00}) = 0$ of our metric (4.114), the transverse conjugate momentum is simply

$$\hat{\Pi}^1(t,\xi,\Omega) = 4M^2 \partial_0 \hat{A}^1(t,\xi,\Omega) \tag{4.145}$$

$$= -i \sum_{\ell m} \sqrt{\ell(\ell+1)} \int \frac{d\omega}{\sqrt{2\omega}} \left[a_{\ell m}(\omega) e^{-i\omega t} Y_{\ell m}(\Omega) - \text{h.c.}\right] k_{i\frac{\omega}{a}}(z). \tag{4.146}$$

One can readily check that by postulating the standard commutation relation

$$\left[a_{\ell m}(\omega), a^\dagger_{\ell' m'}(\omega')\right] = \delta_{\ell\ell'} \delta_{mm'} \delta(\omega - \omega'), \tag{4.147}$$

the transverse fields (4.144) and (4.146) satisfy (4.137).

4.3.2 Wilson lines and edge degrees of freedom

Since the horizon is parametrized by the 2-sphere coordinates (y,ϕ) and the advanced time $v = t + \xi$, let us change coordinates to (v,ξ,y,ϕ). In terms of these coordinates, the radial field (4.144) is simply obtained by replacing t with $v - \xi$,

$$\hat{A}_1(v,\xi,\Omega) = \sum_{\ell m} \frac{1}{\sqrt{\ell(\ell+1)}} \int \frac{d\omega}{\sqrt{2\omega}} \frac{a^2}{\omega} \left[a_{\ell m}(\omega) e^{-i\omega v + i\omega\xi} Y_{\ell m}(\Omega) + \text{h.c.}\right] z^2 k_{i\frac{\omega}{a}}(z). \tag{4.148}$$

In the vicinity of the horizon $\xi \to -\infty$, only the leading soft modes contribute to the integral, due to the factor $e^{\pm i\omega\xi}$ and the Riemann-Lebesgue lemma. As in [8, 9], we implement this with a scalar function $\phi(\omega)$ that has support in a small neighborhood of $\omega = 0$ and satisfies $\phi(0) = 1$:

$$\hat{A}_1(v,\xi,\Omega) = \sum_{\ell m} \frac{1}{\sqrt{\ell(\ell+1)}} \int \frac{d\omega}{\sqrt{2\omega}} \frac{a^2}{\omega} \phi(\omega) \left[a_{\ell m}(\omega) Y_{\ell m}(\Omega) + \text{h.c.}\right] z^2 k_{i\frac{\omega}{a}}(z). \tag{4.149}$$

Let us consider the line integral

$$\mathcal{A}(x) = \int_\Gamma^x dz^\mu \hat{A}_\mu(z), \tag{4.150}$$

where Γ is a time-like path in the vicinity of the horizon. We are again treating the gauge fields to satisfy sourceless quasi-free equations of motion, for the same reason discussed in section 4.2.2. From the metric (4.114), one can observe that $dy, d\phi \to 0$ along Γ as $\xi \to -\infty$, which allows us to write

$$\mathcal{A}(\Omega) = \int_\Gamma d\xi \hat{A}_1(v, \xi, \Omega) \tag{4.151}$$

$$= -\sum_{\ell m} \frac{1}{\sqrt{\ell(\ell+1)}} \int \frac{d\omega}{\sqrt{\pi\omega}} \phi(\omega) \left[a_{\ell m}(\omega) Y_{\ell m}(\Omega) + \text{h.c.}\right] \tag{4.152}$$

where we used the asymptotic form (4.77) of the MacDonald function and the boundary condition (4.82).

As in the case of Rindler spacetime, we want to show that the Wilson line $\exp\{ie\mathcal{A}(x)\}$ is the Faddeev-Kulish dressing that implements soft hair on the Schwarzschild horizon. To this end, let us consider a 2-sphere function $\varepsilon(\Omega)$. The conserved charge Q_ε of QED associated with $\varepsilon(\Omega)$ is [1]

$$Q_\varepsilon = Q_\varepsilon^{\text{soft}} + Q_\varepsilon^{\text{hard}}, \tag{4.153}$$

where we have

$$Q_\varepsilon^{\text{soft}} = \int_H \mathrm{d}\varepsilon \wedge *F, \qquad Q_\varepsilon^{\text{hard}} = \int_H \varepsilon * j, \tag{4.154}$$

with the Schwarzschild horizon H and the charged matter current j. Equation (4.152) shows that $\mathcal{A}(\Omega)$ only involves zero-energy photon operators, which implies that it commutes with the hard charge $Q_\varepsilon^{\text{hard}}$. To obtain an explicit expression for the soft charge $Q_\varepsilon^{\text{soft}}$, we note that the horizon H is parametrized by v, y, and ϕ. Thus the two relevant components of the dual field tensor are, up to some magnetic fields that vanish at H,

$$(*F)_{vy} = -4M^2 \partial_v \hat{A}^\phi, \qquad (*F)_{v\phi} = 4M^2 \partial_v \hat{A}^y, \tag{4.155}$$

using which we may write the soft charge $Q_\varepsilon^{\text{soft}}$ as

$$Q_\varepsilon^{\text{soft}} = -4M^2 \int_{-\infty}^{\infty} dv \int_{-1}^{1} dy \int_0^{2\pi} d\phi \left\{\partial_y \varepsilon(y,\phi) \partial_v \hat{A}^y + \partial_\phi \varepsilon(y,\phi) \partial_v \hat{A}^\phi\right\}. \tag{4.156}$$

After a partial integration, we may write

$$Q_\varepsilon^{\text{soft}} = \int d\Omega\, \varepsilon(\Omega) N(\Omega), \tag{4.157}$$

where we defined the operator $N(\Omega)$ as

$$N(\Omega) = 4M^2 \int_{-\infty}^{\infty} dv\, \partial_v (\partial_y \hat{A}^y + \partial_\phi \hat{A}^\phi). \tag{4.158}$$

Using the property $\partial_i \hat{A}^i = 0$ of transverse fields and the definition $a = 1/4M$, we obtain

$$N(\Omega) = -\frac{1}{4a^2} \int_{-\infty}^{\infty} dv\, \partial_v \partial_1 \hat{A}^1. \tag{4.159}$$

Now, we can substitute the mode expansion (4.143) and use (4.77), (4.93) as well as the integral representation (4.36) to write $N(\Omega)$ in terms of the zero-mode photon operators,

$$N(\Omega) = -4i \sum_{\ell m} \sqrt{\ell(\ell+1)} \int \frac{d\omega}{\sqrt{\pi\omega}} \omega \delta(\omega) \left[a_{\ell m}(\omega) Y_{\ell m}(\Omega) - \text{h.c.} \right], \tag{4.160}$$

where we also took the limit (4.83) since the gauge field is evaluated at the horizon. Now that we have expressions (4.152) and (4.160), one can see by direct calculation that $N(\Omega)$ and $\mathcal{A}(\Omega)$ are conjugate variables, up to a constant,

$$\begin{aligned}
[N(\Omega), \mathcal{A}(\Omega')] = 2i \sum_{\ell m} \sum_{\ell' m'} \sqrt{\frac{\ell(\ell+1)}{\ell'(\ell'+1)}} \int \frac{d\omega d\omega'}{\sqrt{\omega\omega'}} \omega \delta(\omega) \phi(\omega') \\
\times \left[a_{\ell m}(\omega) Y_{\ell m}(\Omega) - \text{h.c.}\,,\, a_{\ell' m'}(\omega') Y_{\ell' m'}(\Omega') + \text{h.c.} \right] \qquad (4.161) \\
= i \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega') \qquad (4.162) \\
= i\delta(y - y')\delta(\phi - \phi') - \frac{i}{4\pi}, \qquad (4.163)
\end{aligned}$$

where we used the convention (4.98) of delta function and the completeness relation of spherical harmonics (4.136). If we expand the gauge parameter $\varepsilon(\Omega)$ in spherical harmonics, the $\ell = 0$ mode is associated to the conservation of total electric charge [1], which is not of our interest. Thus we want to restrict our attention to the case where

$$\int d\Omega\, \varepsilon(\Omega) = 0. \tag{4.164}$$

Then it follows from (4.157) that the following commutation relation is satisfied,

$$[Q_\varepsilon, \mathcal{A}(\Omega)] = i\varepsilon(\Omega). \tag{4.165}$$

The implication of this is that the Wilson lines $e^{ie\mathcal{A}}$ are indeed the Faddeev-Kulish dressings corresponding to the soft hair residing at the Schwarzschild horizon. To illustrate this point, let $|M\rangle$ denote a state describing a Schwarzschild black hole with no soft hair. Since Schwarzschild black holes carry no electromagnetic charge, $Q_\varepsilon |M\rangle = 0$. Let us construct another state by dressing $|M\rangle$ with a Wilson line,

$$|M, (q, \Omega)\rangle = e^{iq\mathcal{A}(\Omega)} |M\rangle . \tag{4.166}$$

From (4.152) we can see that the operator $\mathcal{A}(\Omega)$ only involves soft photons; the dressing carries no additional energy, angular momentum, or electromagnetic charge. Unlike $|M\rangle$, however, this new state carries soft hair on the horizon,

$$Q_\varepsilon |M, (q, \Omega)\rangle = \left[Q_\varepsilon, e^{iq\mathcal{A}(\Omega)}\right] |M\rangle = -q\varepsilon(\Omega) |M, (q, \Omega)\rangle . \tag{4.167}$$

This implies that there exists an infinite number of such degenerate states, each labeled by its soft charge configuration. Given a quantum black hole state with soft hair, one can shift its soft charge using a Wilson line operator.

We end the section by analyzing the action of Q_ε on the boundary gauge fields, given by

$$\hat{A}^H_y(y, \phi) \equiv \lim_{\xi \to -\infty} \hat{A}_y(v, \xi, y, \phi), \tag{4.168}$$

$$\hat{A}^H_\phi(y, \phi) \equiv \lim_{\xi \to -\infty} \hat{A}_\phi(v, \xi, y, \phi). \tag{4.169}$$

Since these fields are purely large-gauge, they can be obtained indirectly via the relations $\hat{A}^H_y(y, \phi) = \partial_y \mathcal{A}(y, \phi)$ and $\hat{A}^H_\phi(y, \phi) = \partial_\phi \mathcal{A}(y, \phi)$.[2] Under a large gauge transformation $\delta \hat{A}_i = \partial_i \varepsilon$, the commutation relation (4.165) implies

$$\left[Q_\varepsilon, \hat{A}^H_y(y, \phi)\right] = i\partial_y \varepsilon(y, \phi) = i\delta \hat{A}^H_y(y, \phi), \tag{4.170}$$

$$\left[Q_\varepsilon, \hat{A}^H_\phi(y, \phi)\right] = i\partial_\phi \varepsilon(y, \phi) = i\delta \hat{A}^H_\phi(y, \phi). \tag{4.171}$$

Therefore, we conclude that the conserved charge Q_ε correctly generates the boundary de-

[2]This can be explicitly shown by obtaining a mode expansion of the transverse fields using the equations of motion (4.141) and (4.142) and then taking the limit $\xi \to -\infty$. As an example, a derivation for $\hat{A}^H_y$ is done in appendix D.

grees of freedom.

To summarize, we have identified the Wilson line punctures on the Schwarzschild horizon as the Faddeev-Kulish dressings that carry definite soft horizon charge. Similar to the case of Minkowski and Rindler spacetimes, these dressings can be used to generate the edge Hilbert space consisting of an infinite number of states, each of which is labeled by its soft horizon charge. In this case, the bulk state is a quantum state labeled solely by the mass of the Schwarzschild black hole. The existence of the edge Hilbert state implies that this bulk state is degenerate, and thus a new quantum number, e.g. the soft horizon charge, should be introduced to correctly identify the state. This is consistent with the Hawking-Perry-Strominger analysis, which claims that the Schwarzschild black holes carry soft hairs [60, 61].

4.4 Summary

In this chapter, we have applied the Weyl-gauge quantization scheme of transverse photon fields developed in [131] to show that for the QED in Rindler and Schwarzschild backgrounds, the Wilson line punctures on the horizon are objects that correspond to the Faddeev-Kulish dressings (see equation (4.81)). By computing the commutation relation between Wilson lines and the soft charge, we have shown that each dressing carries a definite soft horizon charge (see equations (4.101) and (4.167)). The dressings can be used as building blocks to generate the so-called edge Hilbert space (as opposed to the bulk Hilbert space), that consists of an infinite number of degenerate states each of which is labeled by its charge. This shows that the Wilson line dressing is an effective tool to study the soft hair at both infinity and the horizon. Moreover, this approach provides for a systematic way to construct the edge Hilbert space which has applications in studies of entanglement entropy of gauge fields [132].

We have provided a straightforward quantum-mechanical calculation that demonstrates the existence of soft charges localized on the Rindler and Schwarzschild horizons, supports the claim that Schwarzschild black holes carry soft hair [60, 61], and also bridges the gap between the Hawking-Perry-Strominger analysis and the Wilson line formulation [133] of Rindler edge states. Moreover, our calculations show explicitly that the limit of gauge fields at the horizon only involve static photons, see for example equation (4.152). This suggests that similar results are expected in curved spacetimes exhibiting an infinite red-shift surface, for example the cosmological horizon of a de Sitter space.

Chapter 5

Supertranslation Hair of Schwarzschild Black Hole: A Wilson Line Perspective

5.1 Review of gravitational dressings at infinity in flat spacetime

In this chapter, we extend the results of the previous chapter to construct graviton dressings on the future Schwarzschild horizon. We then use the dressed states to demonstrate that asymptotic particles falling into the black hole leave behind a soft graviton hair on the horizon.

To this end, we start by briefly reviewing the gravitational FK dressings in flat spacetime [9] and its Wilson line representation [91, 95]. We also review how the dressings carry a definite supertranslation charge [58, 59]. These results are central to our construction of dressings on the Schwarzschild horizon.

5.1.1 Dressing as a Wilson line

Mandelstam [95] formulated a method for quantizing gravity using path-dependent but coordinate-independent variables. This involves quantizing of the curvature tensor field directly in a path-dependent way instead of the standard quantization of the metric tensor field (gauge field). Consider the interaction between a scalar field and gravitational field.

Let us consider a small perturbation with respect to the flat spacetime,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \tag{5.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the flat metric and $\kappa^2 = 32\pi G$ with Newton's constant G. The prescription of writing a path-dependent variable $A(x, P)$ in terms of a coordinate-dependent variable $a(x)$ is given in [95], which reads

$$A(x,P) = a(x) + \frac{i\kappa}{4}\int_P^x dz^\lambda \left\{\frac{\partial h_{\mu\lambda}(z)}{\partial z^\nu} - \frac{\partial h_{\nu\lambda}(z)}{\partial z^\mu}\right\}[J^{\mu\nu}(z), a(x)] - \frac{\kappa}{2}\int_P^x dz^\lambda h_{\mu\lambda}(z)\frac{\partial a(x)}{\partial x_\mu}, \tag{5.2}$$

to first order in κ. Here $J_{\mu\nu}(z)$ is the angular momentum operator about z in the $\mu\nu$-plane. Let us consider the case where the variables are scalar fields of mass m, and write $A(x, P) = \Phi(x, P)$ and $a(x) = \phi(x)$. Since our focus is on supertranslation charges, and the angular momentum term is sub-leading, we arrive at the following expression for the leading order:

$$\Phi(x,P) = \phi(x) - \frac{\kappa}{2}\int_P^x dz^\lambda h_{\mu\lambda}(z)\frac{\partial\phi(x)}{\partial x_\mu} \tag{5.3}$$

$$= \left\{1 - \frac{i\kappa}{2}\int_P^x dz^\lambda h_{\mu\lambda}(z)\,(-i\partial^\mu)\right\}\phi(x) \tag{5.4}$$

$$= W_1(x,P)\phi(x), \tag{5.5}$$

where the operator $W_1(x, P)$ is defined as

$$W_1(x,P) \equiv 1 - \frac{i\kappa}{2}\int_P^x dz^\lambda h_{\mu\lambda}(z)\,(-i\partial^\mu)\,. \tag{5.6}$$

This can be interpreted as an operator that dresses the scalar field. When the scalar field is quantized, we have the standard expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left\{a(\mathbf{p})e^{ip\cdot x} + a^\dagger(\mathbf{p})e^{-ip\cdot x}\right\}, \tag{5.7}$$

where $E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$ and the creation/annihilation operators a, $a^\dagger$ satisfy the commutation relation

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = (2\pi)^3(2E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{5.8}$$

The expansion of $\phi(x)$ shows that dressing each scalar field with the operator $W_1(x, P)$ is essentially equivalent to dressing each operator $a(\mathbf{p})$ with $W_1(\mathbf{p}; x, P)$ defined as

$$W_1(\mathbf{p}; x, P) = 1 - \frac{i\kappa}{2} \int_P^x dz^\lambda h_{\mu\lambda}(z) p^\mu, \tag{5.9}$$

where $p^\mu = (E_p, \mathbf{p})$, and each operator $a^\dagger(\mathbf{p})$ with $W_1^\dagger(\mathbf{p}; x, P)$. Notice that this is the first-order approximation of the Wilson line $\mathcal{W}(\mathbf{p}; x, P)$, defined as

$$W(\mathbf{p}; x, P) = \mathcal{P} \exp\left\{-\frac{i\kappa}{2} \int_P^x dz^\lambda h_{\mu\lambda}(z) p^\mu\right\} \tag{5.10}$$

with path-ordering $\mathcal{P}$. Taking the path P to be a trajectory of a free particle with velocity $v = p/m$ allows us to parametrize $z = x + v\tau$ and write

$$W(\mathbf{p}; x) = \mathcal{P} \exp\left\{-\frac{i\kappa}{2} \int_{-\infty}^0 d\tau \, p^\mu v^\nu h_{\mu\nu}(x + v\tau)\right\}. \tag{5.11}$$

Notice the similarity between this Wilson line operator defined in gravity with that defined in QED [91, 92]. Next we consider the metric perturbations.

The Einstein's equations $G_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}$ in terms of the perturbation reads [134]

$$O_{\mu\nu\rho\sigma} h^{\rho\sigma} = \frac{\kappa}{2} T_{\mu\nu}, \tag{5.12}$$

where

$$O^{\mu\nu}{}_{\rho\sigma} = \left(\frac{1}{2}\delta^\mu_\rho \delta^\nu_\sigma + \frac{1}{2}\delta^\nu_\rho \delta^\mu_\sigma - \eta^{\mu\nu}\eta_{\rho\sigma}\right) \Box - \frac{1}{2}\left(\delta^\mu_\rho \partial^\nu \partial_\sigma + \delta^\mu_\sigma \partial^\nu \partial_\rho + (\mu \leftrightarrow \nu)\right) + \eta_{\rho\sigma}\partial^\mu \partial^\nu + \eta^{\mu\nu}\partial_\rho \partial_\sigma. \tag{5.13}$$

The operator $O_{\mu\nu\rho\sigma}$ is not invertible, so we must fix the gauge using the harmonic gauge condition

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h = 0, \tag{5.14}$$

after which we obtain the retarded Green's function [134]

$$G^{\text{ret}}_{\mu\nu\rho\sigma}(x) = \frac{1}{2} I_{\mu\nu\rho\sigma} \frac{1}{4\pi|\mathbf{x}|} \delta(|\mathbf{x}| - x^0), \tag{5.15}$$

where $I_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}$, satisfying

$$O_{\mu\nu}{}^{\rho\sigma} G^{\text{ret}}_{\rho\sigma\alpha\beta}(x-y) = \frac{1}{2}\mathbf{1}_{\mu\nu\alpha\beta}\delta^{(4)}(x-y), \qquad G^{\text{ret}}_{\mu\nu\rho\sigma}(x-y) = 0 \quad \text{if } x^0 < y^0. \tag{5.16}$$

Here the "identity" tensor $\mathbf{1}_{\mu\nu\alpha\beta}$ is defined as

$$\mathbf{1}_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}). \tag{5.17}$$

Using the retarded Green's function, we may express metric perturbations in terms of free fields as

$$h_{\mu\nu}(x) = h^{\text{in}}_{\mu\nu}(x) + 2\int d^4y\, G^{\text{ret}}_{\mu\nu\rho\sigma}(x-y)T^{\rho\sigma}(y), \tag{5.18}$$

where $h^{\text{in}}_{\mu\nu}$ is the free field satisfying $O^{\mu\nu\rho\sigma}h_{\rho\sigma} = 0$, and $T^{\mu\nu}$ is the energy-momentum tensor of the scalar field which is given by (see eq. (4.2) in [9])

$$T^{rs}(y) = \kappa \int d^3\mathbf{p}\rho(\mathbf{p})\frac{p^r p^s}{2p_0}\delta^{(3)}\left(\mathbf{y} - \frac{\mathbf{p}t}{p_0}\right), \tag{5.19}$$

where $\rho(\mathbf{p})$ is the (unintegrated) number operator $a^\dagger(\mathbf{p})a(\mathbf{p})$. It follows that the Wilson line $W(\mathbf{p};x)$ can be written in terms of free field and the energy-momentum tensor as

$$\begin{aligned} W(\mathbf{p};x) = \mathcal{P}\exp&\left\{-\frac{i\kappa}{2}\int_{-\infty}^{0} d\tau v^\mu p^\nu h^{\text{in}}_{\mu\nu}(x+v\tau)\right\} && (5.20)\\ &\times\exp\left\{-i\kappa\int_{-\infty}^{0} d\tau v^\mu p^\nu \int d^4y\, G^{\text{ret}}_{\mu\nu\rho\sigma}(x+v\tau-y)T^{\rho\sigma}(y)\right\} && (5.21)\end{aligned}$$

Next let us recall the Wick's ordering theorem, given by

$$T\exp\left[-i\int dt\, H_I(t)\right] = \exp\left[-i\int H_I(t)\right]\exp\left[-\frac{i}{2}\int dt\int ds\,\theta(t-s)[H_I(t), H_I(s)]\right], \tag{5.22}$$

where θ is the step function. Using this, we can express the Wilson line operator as [91]

$$\begin{aligned} W(\mathbf{p};x) &= \exp\left\{-\frac{i\kappa}{2}\int_{-\infty}^{0} d\tau\, v^\mu p^\nu h^{\text{in}}_{\mu\nu}(x+v\tau)\right\} \times (\text{phases}) && (5.23)\\ &= \widetilde{W}(\mathbf{p};x) \times (\text{phases}). && (5.24)\end{aligned}$$

We do not concern ourselves with the phases, and focus on the first factor,

$$\widetilde{W}(\mathbf{p};x) \equiv \exp\left\{-\frac{i\kappa}{2}\int_{-\infty}^{0} d\tau\, v^\mu p^\nu h^{\text{in}}_{\mu\nu}(x+v\tau)\right\}. \tag{5.25}$$

Using the standard mode expansion of the asymptotic in-field,

$$h_{\mu\nu}^{\rm in}(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left[a_{\mu\nu}(\mathbf{k})e^{ik\cdot x} + a_{\mu\nu}^\dagger(\mathbf{k})e^{-ik\cdot x}\right], \tag{5.26}$$

where $\omega \equiv |\mathbf{k}|$, we obtain

$$\int_{-\infty}^{0} d\tau\, v^\mu p^\nu h_{\mu\nu}^{\rm in}(x+v\tau) = \int_{-\infty}^{0} d\tau\, v^\mu p^\nu \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left[a_{\mu\nu}(\mathbf{k})e^{ik\cdot(x+v\tau)} + a_{\mu\nu}^\dagger(\mathbf{k})e^{-ik\cdot(x+v\tau)}\right] \tag{5.27}$$

$$= i\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\frac{p^\mu p^\nu}{p\cdot k}\left[a_{\mu\nu}^\dagger(\mathbf{k})e^{-ik\cdot x} - a_{\mu\nu}(\mathbf{k})e^{ik\cdot x}\right], \tag{5.28}$$

where we used $p = mv$ and the boundary condition

$$\int_{-\infty}^{0} d\tau\, e^{ik\cdot v\tau} = \frac{1}{ik\cdot v}. \tag{5.29}$$

Recall that x is the position of the scalar field which is dressed by the Wilson line. The dressing of a scalar field at the past time-like infinity can be obtained by taking the limit $x^0 \to -\infty$. Due to the factors $e^{\pm ik\cdot x}$, by the Riemann-Lebesgue lemma only the leading soft particles contribute to the integral. Following [8], we implement this using an infrared function $\phi(\omega)$ that has support in a small neighborhood of $\omega = 0$ and $\phi(0) = 1$,

$$W(\mathbf{p}) = \lim_{x^0\to-\infty} \widetilde{W}(\mathbf{p};x) \tag{5.30}$$

$$= \exp\left\{\frac{\kappa}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\frac{p^\mu p^\nu}{p\cdot k}\phi(\omega)\left[a_{\mu\nu}^\dagger(\mathbf{k}) - a_{\mu\nu}(\mathbf{k})\right]\right\}, \tag{5.31}$$

which is, up to a unitary transformation, identified with the Faddeev-Kulish dressing of gravity [9]. The dressings may be interpreted as Wilson line punctures [92, 130, 133] on the spacetime boundary.

5.1.2 Wilson line punctures and boundary charges

In this subsection, let us review how the dressing (5.31) carries BMS supertranslation charge [58, 59]. We work with the asymptotically flat metric, and use the Bondi coordinates $(v, r, z, \bar{z})$ [15],

$$\begin{aligned} ds^2 &= -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &\quad + \frac{2m_B}{r}dv^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 - 2V_z dvdz - 2V_{\bar{z}}dvd\bar{z} + \cdots, \end{aligned} \tag{5.32}$$

where $\gamma_{z\bar{z}} = 2/(1+z\bar{z})^2$ is the 2-sphere metric, m_B is the Bondi mass aspect, and $V_z = \frac{1}{2}D^z C_{zz}$ with D_z the covariant derivative on the 2-sphere. The first line corresponds to the flat metric.

We start by using (5.1) to write the radiative data C_{zz} as (see for example [15])

$$\begin{aligned}
C_{zz}(v, z, \bar{z}) &= \kappa \lim_{r\to\infty} \frac{1}{r} h_{zz}(r, v, z, \bar{z}) && (5.33)\\
&= \kappa \lim_{r\to\infty} \frac{1}{r} \partial_z x^\mu \partial_z x^\nu h_{\mu\nu} && (5.34)\\
&= -\frac{i\kappa\gamma_{z\bar{z}}}{8\pi^2} \int_0^\infty d\omega \left[a_+(-\omega\hat{\mathbf{x}}_z)e^{-i\omega v} - a_-^\dagger(-\omega\hat{\mathbf{x}}_z)e^{i\omega v}\right]. && (5.35)
\end{aligned}$$

Taking the limit $v \to -\infty$, we obtain

$$\begin{aligned}
C_{zz}(z, \bar{z}) &\equiv \lim_{v\to-\infty} C_{zz}(v, z, \bar{z}) && (5.36)\\
&= \frac{i\kappa\gamma_{z\bar{z}}}{8\pi^2} \int_0^\infty d\omega \left[a_-^\dagger(-\omega\hat{\mathbf{x}}_z) - a_+(-\omega\hat{\mathbf{x}}_z)\right] \phi(\omega), && (5.37)
\end{aligned}$$

where we have again used the Riemann-Lebesgue lemma and the infrared function $\phi(\omega)$. Now rewrite the dressing (5.31) as,

$$\begin{aligned}
W(\mathbf{p}) &= \exp\left[\frac{\kappa}{2}\int \frac{d\omega d^2z\gamma_{z\bar{z}}}{16\pi^3}\frac{p^\mu p^\nu}{p\cdot\hat{k}}\phi(\omega)\left(a_{\mu\nu}^\dagger(\mathbf{k}) - a_{\mu\nu}(\mathbf{k})\right)\right] && (5.38)\\
&= \exp\left[\frac{\kappa}{2}\int \frac{d^2z\gamma_{z\bar{z}}}{16\pi^3}\left\{\frac{(p\cdot\epsilon^-)^2}{p\cdot\hat{k}}\int d\omega\,\phi(\omega)\left(a_-^\dagger(-\omega\hat{\mathbf{x}}) - a_+(-\omega\hat{\mathbf{x}})\right)\right.\right. && (5.39)\\
&\qquad\left.\left. + \frac{(p\cdot\epsilon^+)^2}{p\cdot\hat{k}}\int d\omega\,\phi(\omega)\left(a_+^\dagger(-\omega\hat{\mathbf{x}}) - a_-(-\omega\hat{\mathbf{x}})\right)\right\}\right], && (5.40)
\end{aligned}$$

where $\hat{k}^\mu = (1, \hat{\mathbf{k}}) = (1, -\hat{\mathbf{x}})$ and we have used

$$a_{\mu\nu}(\mathbf{k}) = \sum_{r=\pm} \epsilon_{\mu\nu}^{r*}(\mathbf{k}) a_r(\mathbf{k}),$$

with graviton polarization tensors $\epsilon_{\mu\nu}^\pm(\mathbf{k}) = \epsilon_\mu^\pm(\mathbf{k})\epsilon_\nu^\pm(\mathbf{k})$, defined as $\epsilon^{-\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}[z, 1, +i, -z]$ and $\epsilon^{+\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}[\bar{z}, 1, -i, -\bar{z}]$. In this form (5.40), we can see that the dressing may be written in terms of $C_{zz}(z, \bar{z})$ as

$$W(\mathbf{p}) = \exp\left[-\frac{i}{4\pi}\int d^2z \left\{\frac{(p\cdot\epsilon^-)^2}{p\cdot\hat{k}} C_{zz}(z, \bar{z}) + \frac{(p\cdot\epsilon^+)^2}{p\cdot\hat{k}} C_{\bar{z}\bar{z}}(z, \bar{z})\right\}\right]. \qquad (5.41)$$

This form is convenient in computing the dressing's supertranslation charge.

Next we consider the soft BMS supertranslation charge $Q_f^{\mathcal{I}^-}$, which is given as [15]

$$Q_f^{\mathcal{I}^-} = \frac{4}{\kappa^2}\int dvd^2z\gamma^{z\bar{z}}D_z^2 f(z,\bar{z})\partial_v C_{zz} = \frac{4}{\kappa^2}\int dvd^2z\gamma^{z\bar{z}}D_{\bar{z}}^2 f(z,\bar{z})\partial_v C_{\bar{z}\bar{z}}, \tag{5.42}$$

where $f(z,\bar{z})$ is the 2-sphere function that parametrizes the transformation. Let us define the operator

$$\begin{aligned} M_{zz}(z,\bar{z}) &= \frac{4}{\kappa^2}\gamma^{z\bar{z}}\int_{-\infty}^{\infty} dv\,\partial_v C_{zz}(v,z,\bar{z}) & (5.43)\\ &= -\frac{1}{\kappa\pi}\int_0^{\infty} d\omega\,\omega\delta(\omega)\left[a_+(-\omega\hat{\mathbf{x}}_z) + a_-^\dagger(-\omega\hat{\mathbf{x}}_z)\right], & (5.44)\end{aligned}$$

where we have used the mode expansion (5.35) and the integral representation

$$\pm 2\pi i\omega\delta(\omega) = \int_{-\infty}^{\infty} dv\,\partial_v\, e^{\pm i\omega v}. \tag{5.45}$$

Then, we may write $Q_f^{\mathcal{I}^-}$ as

$$Q_f^{\mathcal{I}^-} = \int d^2z\, M_{zz}D_z^2 f(z,\bar{z}) = \int d^2z\, M_{\bar{z}\bar{z}}D_{\bar{z}}^2 f(z,\bar{z}). \tag{5.46}$$

Using (5.37), one can see by direct calculation that the following commutation relation is satisfied:

$$\begin{aligned} [M_{zz}(z,\bar{z}), C_{\bar{w}\bar{w}}(w,\bar{w})] &= \frac{i\gamma_{w\bar{w}}}{4\pi^3}\int_0^{\infty} d\omega\,\omega\delta(\omega)\frac{(2\pi)^3(2\omega)}{\omega^2\gamma_{z\bar{z}}}\delta^{(2)}(w-z) & (5.47)\\ &= 4i\delta^{(2)}(w-z)\int_0^{\infty} d\omega\,\delta(\omega) & (5.48)\\ &= 2i\delta^{(2)}(w-z), & (5.49)\end{aligned}$$

where in the last line we used the convention

$$\int_0^{\infty} d\omega\, f(\omega)\delta(\omega) = \frac{1}{2}f(0). \tag{5.50}$$

This commutation relation shows that the operator $\frac{1}{2}M_{zz}$ is the canonical conjugate variable of the radiative mode C_{zz}. It follows from (5.46) that

$$[Q_f^{\mathcal{I}^-}, C_{zz}] = 2iD_z^2 f(z,\bar{z}), \tag{5.51}$$

which is the correct commutator between the soft supertranslation charge and the radiative mode C_{zz} [15] in the past infinity.

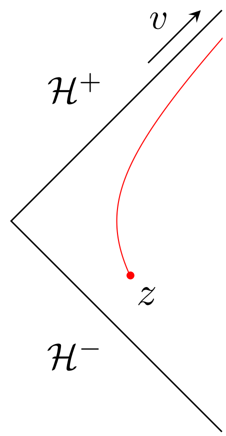


Figure 5.1: Depiction of a gravitational Wilson line in the Schwarzschild background (marked red in the figure). The line extends from the spacetime point $z = (t_0, r_0, \Omega_0)$ at which the field being dressed is located. For the dressing of an asymptotic massive particle falling into the black hole, we take the Wilson line to be along the particle's geodesic and take the limit $r_0 \to 2M$. We refer to this as the Wilson line puncture.

From (5.41) and (5.51), we obtain the commutator

$$\left[Q_f^{\mathcal{I}}, W(\mathbf{p})\right] = Q_f(\mathbf{p}) W(\mathbf{p}), \tag{5.52}$$

where $Q_f(\mathbf{p})$ is given in terms of the 2-sphere function f and the momentum $\mathbf{p}$,

$$Q_f(\mathbf{p}) = \frac{1}{2\pi} \int d^2 z \left\{ \frac{(p \cdot \epsilon^-)^2}{p \cdot \hat{k}} D_z^2 f + \frac{(p \cdot \epsilon^+)^2}{p \cdot \hat{k}} D_{\bar{z}}^2 f \right\}. \tag{5.53}$$

The commutator (5.52) shows that the FK dressing $W(\mathbf{p})$ carries a definite supertranslation charge $Q_f(\mathbf{p})$.

5.2 Gravitational dressing on the Schwarzschild horizon

In this section, we apply the methods we reviewed in section 5.1 to construct the dressing of an asymptotic massive scalar field that falls into the Schwarzschild black hole. Following Mandelstam's approach [95], we assume that the scalar field is made coordinate-invariant by a gravitational Wilson line dressing, which now follows a time-like trajectory into the black hole. After quantizing the graviton, we observe that the dressing comprises only zero-frequency graviton excitations. It is then shown in section 5.4 that the dressing we construct carries a definite horizon supertranslation charge of Hawking, Perry and Strominger [60, 61].

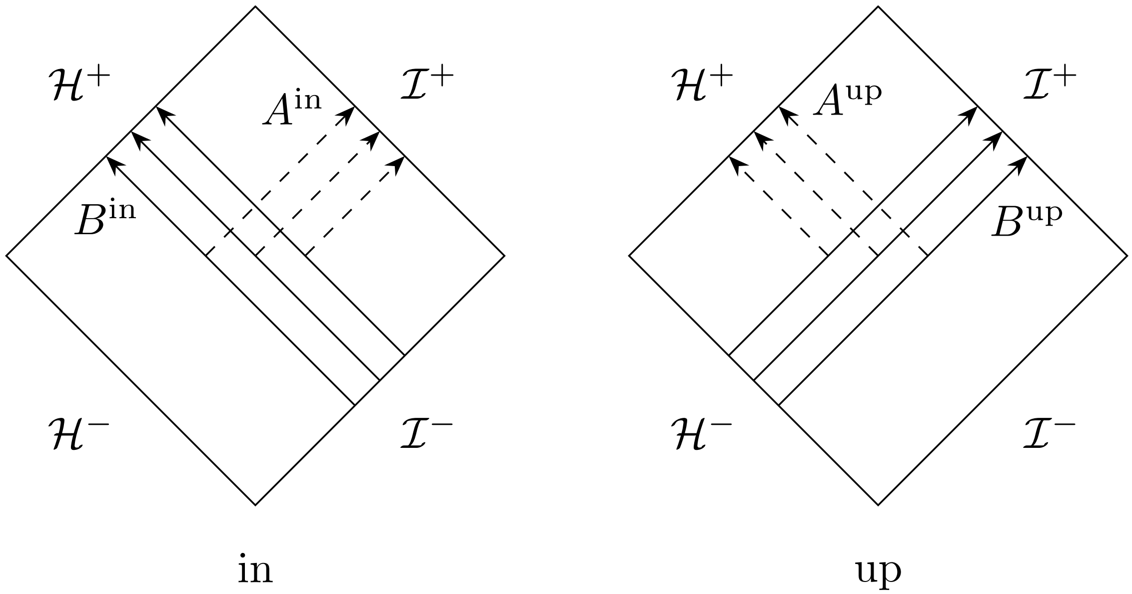


Figure 5.2: Representations of the "in" and "up" modes in the Penrose diagram of the exterior of a Schwarzschild black hole. The in-modes consist purely of traveling waves incoming from the past null infinity $\mathcal{I}^-$ and therefore vanish on the past horizon $\mathcal{H}^-$. The up-modes consist purely of traveling waves incoming from $\mathcal{H}^-$ and therefore vanish on $\mathcal{I}^-$. For each mode Λ, ${}_sR^{\Lambda}_{l\omega}e^{-i\omega t}$ is the incoming partial wave, $\left|A^{\Lambda}\right|^2$ is the reflection coefficient and $\left|B^{\Lambda}\right|^2$ is the transmission coefficient.

To this end, let us first consider the dressing of a particle of mass m at a spacetime point $z = (t_0, r_0, \theta_0, \phi_0)$. Drawing analogy from section 5.1, we write its dressing as the gravitational Wilson line,

$$\exp(W) \equiv \exp\left\{-\frac{i}{2}m\kappa\int_{\Gamma}^{z} dx^{\mu}h_{\mu\nu}(x)\frac{dx^{\nu}}{d\tau}\right\}, \tag{5.54}$$

along a radial geodesic Γ of a massive particle of mass m, extending from z to the future horizon $\mathcal{H}^+$, see figure 5.1. As usual, we employ the boundary condition [8, 92] that the contribution to W comes from only the upper bound, z, of the integral. To obtain the dressing of an asymptotic particle on $\mathcal{H}^+$, we evaluate the integral under the limit $r_0 \to 2M$. In this case, the entirety of the geodesic Γ lies in the vicinity of $\mathcal{H}^+$. Since this Wilson line acts like a puncture on the future boundary $\mathcal{H}^+_+$ of the horizon, we refer to this as the Wilson line puncture, following [92, 133].

We now employ the graviton quantization of Candelas et. al. [93] (see appendix F for details), where the graviton field $h_{\mu\nu}(x)$ has the mode expansion

$$h_{\mu\nu}(x) = \sum_{\Lambda} \sum_{lmP} \int_0^\infty d\omega \left[a^\Lambda_{lmP}(\omega) h^\Lambda_{\mu\nu}(l,m,\omega,P;x) + \text{h.c.} \right]. \tag{5.55}$$

The mode functions $h^\Lambda_{\mu\nu}(l,m,\omega,P;x)$ and their complex conjugates form a complete orthonormal set with $l \geq 2$, $|m| \leq l$, $P = \pm 1$, and $\Lambda \in \{\text{in}, \text{up}\}$. Here $P = +1$ (-1) is referred to as the electric (magnetic) parity. Modes with $\Lambda = \text{in}$ (up) are referred to as the in-modes (up-modes); it denotes the boundary conditions satisfied by the modes (see Fig. 5.2). Throughout this chapter, it is tacitly assumed that the sum over l, m and P span $l \geq 2$, $|m| \leq l$ and $P = \pm 1$:

$$\sum_{lmP} (\cdots) \equiv \sum_{l\geq 2} \sum_{m=-l}^{l} \sum_{P=\pm 1} (\cdots), \tag{5.56}$$

unless explicitly stated otherwise. The graviton is quantized by promoting $a^\Lambda_{lmP}(\omega)$ and $a^{\Lambda\dagger}_{lmP}(\omega)$ to operators satisfying the canonical commutation relation (F.24).

Coming back to the dressing (5.54), let us first consider the contribution to W coming from the up-modes.[1] First, we separate the graviton field (5.55) into two parts,

$$h_{\mu\nu}(x) = h^{\text{in}}_{\mu\nu}(x) + h^{\text{up}}_{\mu\nu}(x), \tag{5.57}$$

$$h^\Lambda_{\mu\nu}(x) \equiv \sum_{lmP} \int_0^\infty d\omega \left[a^\Lambda_{lmP}(\omega) h^\Lambda_{\mu\nu}(l,m,\omega,P;x) + \text{h.c.} \right], \quad \Lambda \in \{\text{in}, \text{up}\}. \tag{5.58}$$

Then, we may write (5.54) as

$$W = -\frac{i}{2} m\kappa \int_\Gamma^z dx^\mu h^{\text{up}}_{\mu\nu}(x) \frac{dx^\nu}{d\tau} + (\text{in-mode contribution}). \tag{5.59}$$

Let E be the total energy of the particle at infinity. This fixes the geodesic Γ, along which we have

$$\frac{dt}{d\tau} = \frac{E}{mV}, \qquad \frac{dr}{d\tau} = -\left(\frac{E^2}{m^2} - V\right)^{1/2}, \qquad \frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = 0. \tag{5.60}$$

To simplify the calculations, we move to the ingoing Eddington-Finkelstein coordinates

[1] From figure 5.2, we see that the in-modes are incoming waves from $\mathcal{I}^-$, which are known to have vanishing contribution to the horizon supertranslation charge (see [61] for a discussion). For this reason, we expect the dressing to receive vanishing contribution from the in-modes as well, and later in this section we see that this is indeed the case.

(v, r, θ, ϕ), where we have $h^{\rm up}_{vr}(x) = 0$ and $h^{\rm up}_{rr}(x) = 0$. Then (5.59) simplifies to

$$W = -\frac{i}{2} m\kappa \int^{v_0} dv\, h^{\rm up}_{vv}(x) \frac{dv}{d\tau} + (\text{in-mode contribution}) \tag{5.61}$$

$$= -\frac{im^2\kappa}{4E} \int^{v_0} dv\, h^{\rm up}_{vv}(x) + (\text{in-mode contribution}), \tag{5.62}$$

where in the last equation we used

$$\frac{dv}{d\tau} = \frac{dt}{d\tau} + \frac{dr_*}{dr}\frac{dr}{d\tau} = \frac{E}{mV}\left[1 - \left(1 - \frac{m^2 V}{E^2}\right)^{1/2}\right] = \frac{m}{2E} + \mathcal{O}(r - 2M), \tag{5.63}$$

and discarded subleading terms in the expansion, since for an asymptotic particle Γ lies entirely in the vicinity of $\mathcal{H}^+$. The component $h^{\rm up}_{vv}(x)$ is a linear combination of the modes $h^{\rm up}_{vv}(l, m, \omega, P; x)$, whose explicit form may be read off from (F.9),

$$h^{\rm up}_{vv}(l, m, \omega, P; x) = N^{\rm up} \left\{\Upsilon_{vv}\, {}_{-2}Y_{lm}(\theta, \phi) + P\Upsilon^*_{vv}\, {}_{+2}Y_{lm}(\theta, \phi)\right\}\, {}_{+2}R^{\rm up}_{l\omega}(r) e^{-i\omega t}, \tag{5.64}$$

where $N^{\rm up}$ is a normalization constant, Υ_{vv} is a second-order differential operator defined as (F.11), ${}_{\pm 2}Y_{lm}$ are $s = \pm 2$ spin-weighted spherical harmonics, and ${}_{+2}R^{\rm up}_{l\omega}$ is a radial function satisfying the boundary condition for up-modes; see appendix F for details. Now, observe that (G.9) can be used to write the spin-weighted spherical harmonics in terms of ordinary spherical harmonics,

$$\Upsilon_{\mu\nu}\, {}_{-2}Y_{lm}(\theta, \phi) = -\frac{r^2 V^2}{8} \eth\eth\, {}_{-2}Y_{lm}(\theta, \phi) = -\frac{r^2 V^2}{8} \sqrt{\frac{(l+2)!}{(l-2)!}}\, Y_{lm}(\theta, \phi), \tag{5.65}$$

$$\Upsilon^*_{\mu\nu}\, {}_{+2}Y_{lm}(\theta, \phi) = -\frac{r^2 V^2}{8} \bar{\eth}\bar{\eth}\, {}_{+2}Y_{lm}(\theta, \phi) = -\frac{r^2 V^2}{8} \sqrt{\frac{(l+2)!}{(l-2)!}}\, Y_{lm}(\theta, \phi). \tag{5.66}$$

Thus we may write (5.64) as

$$h^{\rm up}_{vv}(l, m, \omega, P; x) = -N^{\rm up}(1 + P)\frac{r^2 V^2}{8} \sqrt{\frac{(l+2)!}{(l-2)!}}\, Y_{lm}(\theta, \phi)\, {}_{+2}R^{\rm up}_{l\omega}(r) e^{-i\omega t}. \tag{5.67}$$

One immediately sees that $h^{\rm up}_{vv}(x)$ does not have $P = -1$ contribution,

$$h^{\rm up}_{vv}(l, m, \omega, P = -1; x) = 0. \tag{5.68}$$

We also see in section 5.3 that supertranslation horizon charge only receives contribution from $P = 1$ modes. This is reminiscent of the gravitational memory at the asymptotic infinities (see for example [135–137]). Since Γ is near $\mathcal{H}^+$, we can replace the radial function ${}_{+2}R^{\rm up}_{l\omega}(r)$ by its asymptotic form (F.15) for $r \to 2M$,

$$
{}_{+2}R^{\rm up}_{l\omega}(r) \sim \frac{A^{\rm up}_{l\omega}}{(2M)^4 V^2} e^{-i\omega r_*} \qquad \text{near } \mathcal{H}^+, \tag{5.69}
$$

which leads to

$$
h^{\rm up}_{vv}(l,m,\omega,P;x) \sim -\frac{N^{\rm up} A^{\rm up}_{l\omega}}{8(2M)^2}(1+P)\sqrt{\frac{(l+2)!}{(l-2)!}}\, Y_{lm}(\theta,\phi) e^{-i\omega v} \qquad \text{near } \mathcal{H}^+. \tag{5.70}
$$

Expanding (5.62) into modes and substituting the above expression yields

$$
\begin{aligned}
W = &-\frac{im^2\kappa}{4E}\sum_{lmP}\int_0^\infty d\omega \int^{v_0} dv\, [a^{\rm up}_{lmP}(\omega) h^{\rm up}_{vv}(l,m,\omega,P;x) + \text{h.c.}] \\
&+ (\text{in-mode contribution}) \qquad (5.71)\\
= &\frac{im^2\kappa}{4E}\sum_{lm}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^\infty d\omega \left[a^{\rm up}_{lm,P=1}(\omega)\frac{N^{\rm up} A^{\rm up}_{l\omega}}{4(2M)^2}\, Y_{lm}(\theta,\phi)\frac{e^{-i\omega v_0}}{(-i\omega)} + \text{h.c.}\right] \\
&+ (\text{in-mode contribution}), \qquad (5.72)
\end{aligned}
$$

where we have used a boundary condition analogous to that used in [8, 92] to evaluate

$$
\int^{v_0} dv\, e^{-i\omega v} = \frac{e^{-i\omega v_0}}{(-i\omega)}. \tag{5.73}
$$

Recall that the line integral was along a time-like geodesic Γ of a particle with total energy E at infinity, which implies that as $r_0 \to 2M$, the advanced time v_0 diverges to infinity. Thus in this limit, the presence of $e^{\pm i\omega v_0}$ in the integrand removes all contributions except those from $\omega = 0$ by virtue of the Riemann-Lebesgue lemma. Following the previous approaches [8,9,92], we explicitly implement this by replacing $e^{\pm i\omega v_0}$ with an infrared function $\phi(\omega)$, which we define to have support only in a small neighborhood of $\omega = 0$ and satisfy $\phi(0) = 1$. This yields

$$
\begin{aligned}
W = &\frac{im^2\kappa}{4E}\sum_{lm}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^\infty d\omega\phi(\omega) \left[a^{\rm up}_{lm,P=1}(\omega)\frac{N^{\rm up} A^{\rm up}_{l\omega}}{(-4i\omega)(2M)^2}\, Y_{lm}(\theta,\phi) + \text{h.c.}\right] \\
&+ (\text{in-mode contribution}). \qquad (5.74)
\end{aligned}
$$

Due to the function $\phi(\omega)$, only the leading soft term in the integrand contributes to the integral. From (F.21) and (F.17), we have the soft expansion

$$N^{\text{up}} A^{\text{up}}_{l\omega} = (2M)^2 \frac{(-4i\omega)}{\sqrt{\pi\omega}} \left[\frac{(l-2)!}{(l+2)!} + \mathcal{O}(\omega) \right]. \tag{5.75}$$

We can substitute this to (5.74) to obtain

$$\begin{aligned} W = \frac{im^2\kappa}{4E} \sum_{lm} \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^\infty \frac{d\omega}{\sqrt{\pi\omega}} \phi(\omega) \left[a^{\text{up}}_{lm,P=1}(\omega)\, Y_{lm}(\theta,\phi) + \text{h.c.} \right] \\ + (\text{in-mode contribution}). \end{aligned} \tag{5.76}$$

Now, let us turn our attention to last term in (5.76), the contribution from the in-modes. One could imagine carrying out a similar set of steps, after which one would obtain an expression analogous to the first term in (5.74), where the integrand is proportional to $\phi(\omega) N^{\text{in}} B^{\text{in}}_{l\omega}$. From (F.20) and (F.16), we have the expansion

$$N^{\text{in}} B^{\text{in}}_{l\omega} = -\frac{(-4iM\omega)^{l+1}}{2(2M)^3\sqrt{\pi\omega}} \left[\frac{l!(l-2)!(l+2)!}{2(2l+1)!(2l)!} + \mathcal{O}(\omega) \right], \tag{5.77}$$

which, when compared to (5.74), contains far more factors of ω since $l \geq 2$. This leads to the in-mode contribution being sub-leading soft in comparison to that of up-modes, and therefore negligible in comparison due to the presence of $\phi(\omega)$. There is a subtlety here that is worth mentioning: unlike the up-modes, the in-modes are in the ingoing radiation gauge $h_{(a)(1)} = 0$, $g^{\mu\nu} h_{\mu\nu} = 0$, which is not compatible with the Bondi gauge. However, that the in-modes are sub-leading soft to the up-modes on $\mathcal{H}^+$ is still true in Bondi gauge. To see this, we note that the radial function ${}_{-2}R^{\text{in}}_{l\omega}(r)$ derives its origin from the contribution of each in-mode to the Weyl scalar Ψ_4 [93, 94],

$$\delta\Psi_4 \left[h^{\text{in}}(l,m,\omega,P;x) \right] = -\frac{N^{\text{in}}}{8r^4} \left[\frac{(l+2)!}{(l-2)!} + 12iM\omega P \right] {}_{-2}R^{\text{in}}_{l\omega}(r)\, {}_{-2}Y_{lm}(\theta,\phi) e^{-i\omega t}. \tag{5.78}$$

Since $\delta\Psi_4$ is a gauge-invariant quantity, the Bondi gauge expression of the in-mode contribution to W also includes the radial dependence ${}_{-2}R^{\text{in}}_{l\omega}(r)$, which includes the factor ω^l near $\mathcal{H}^+$, rendering the in-mode contribution sub-dominant in comparison to that of the up-modes. Therefore, one obtains the final expression

$$\exp(W) = \exp\left[\frac{im^2\kappa}{4E} \sum_{lm} \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^\infty \frac{d\omega}{\sqrt{\pi\omega}} \phi(\omega) \left[a^{\text{up}}_{lm,P=1}(\omega)\, Y_{lm}(\theta,\phi) + \text{h.c.} \right] \right] \tag{5.79}$$

of the gravitational Wilson line, dressing an asymptotic particle of mass m and total energy E falling into the black hole.

We have seen that the in-modes are sub-leading soft to up-modes on $\mathcal{H}^+$ and vanish by choice of boundary conditions on $\mathcal{H}^-$. Due to this observation, we can restrict our attention to the up-modes when dealing with supertranslation (and zero-modes) on both horizons.

5.3 Supertranslation charge and horizon fields

Classical analysis of the black hole horizon [61] suggests that there exist horizon degrees of freedom of the form h_{AB} that live on $\mathcal{H}^\pm$. Our goal in this section is to derive an expression of the horizon supertranslation charge and horizon fields in terms of the graviton Fock space operators. From the previous work with regards to Maxwell fields [92], it is reasonable to expect that such fields can be obtained as an appropriate limit of the bulk fields (5.55). However, one fails to do so directly on $\mathcal{H}^+$, since ${}_{+2}R^{\mathrm{up}}_{l\omega}$ blows up as one approaches $\mathcal{H}^+$. This is perhaps due to the fact that the bulk fields are Klein-Gordon normalized on $\mathcal{H}^- \cup \mathcal{I}^-$. The limit, on the other hand, is well defined on $\mathcal{H}^-$, which leads us to take an alternate approach: we derive the horizon fields on the past horizon $\mathcal{H}^-$ first, and use time-inversion symmetry of Schwarzschild spacetime to obtain the corresponding fields on the future horizon $\mathcal{H}^+$.

5.3.1 Past horizon

Let $\mathcal{H}^- \cup \Sigma^-$ be a Cauchy surface in the past (for instance, in the absence of massive particles $\Sigma^- = \mathcal{I}^-$). The linearized supertranslation charge Q_f^- on this surface can be decomposed as

$$Q_f^- = Q_f^{\mathcal{H}^-} + Q_f^{\Sigma^-}. \tag{5.80}$$

We loosely refer to $Q_f^{\mathcal{H}^-}$ as the supertranslation charge on $\mathcal{H}^-$.

To obtain an expression for $Q_f^{\mathcal{H}^-}$, we move to the outgoing Eddington-Finkelstein coordinates (u, r, Ω), where $u = t - r_*$ is the retarded time. The Schwarzschild metric in these coordinates reads

$$ds^2 = -V du^2 - 2du dr + r^2 \gamma_{AB} dx^A dx^B, \qquad V \equiv 1 - \frac{2M}{r}, \tag{5.81}$$

with the 2-sphere metric γ_{AB}. The Bondi gauge conditions read [61]

$$h_{rr} = h_{rA} = \gamma^{AB} h_{AB} = 0. \tag{5.82}$$

We want to find infinitesimal diffeomorphisms $\delta x^\mu = \xi^\mu$ that preserve the Bondi gauge conditions as well as the standard falloffs at large r [61]. Gauge conditions put the following constraints on ξ^μ,

$$\frac{1}{2}\mathcal{L}_\xi g_{rr} = \partial_r \xi^u = 0, \tag{5.83}$$

$$\mathcal{L}_\xi g_{Ar} = \partial_r \xi^A - \frac{1}{r^2} D^A f = 0, \tag{5.84}$$

$$\frac{1}{2}\gamma^{AB}\mathcal{L}_\xi g_{AB} = D_A \xi^A + \frac{2}{r}\xi^r = 0. \tag{5.85}$$

We restrict our attention to supertranslations by choosing $\xi^u = f$ such that $\partial_u f = 0$. Then (5.83) leads to $f = f(\Omega)$, and (5.84) with falloff condition on ξ^A implies $\xi^A = -\frac{1}{r} D^A f$. Substituting this into (5.85), one obtains $\xi^r = \frac{1}{2} D^2 f$. Therefore we obtain

$$\xi^\alpha \partial_\alpha = f\partial_u + \frac{1}{2} D^2 f \partial_r - \frac{1}{r} D^A f \partial_A. \tag{5.86}$$

In order to exclude ordinary spacetime translations which are not of our interest, we restrict the angular function $f(\Omega)$ to contain only partial waves with $l \geq 2$.

The supertranslation charge associated with the diffeomorphism ξ on $\mathcal{H}^-$ reads (see appendix H for a derivation)

$$Q_f^{\mathcal{H}^-} = \frac{1}{\kappa M} \int_{\mathcal{H}^-} d\Omega\, du\, f(\Omega) D^A D^B \partial_u h^-_{AB}(u, \Omega), \tag{5.87}$$

where $h^-_{AB}(u,\Omega)$ are the horizon fields related to the supertranslation fields on $\mathcal{H}^+$ obtained in [61]; the "$-$" superscript emphasizes that these fields are defined on $\mathcal{H}^-$. The u-integral and u-derivative introduce a delta function $\delta(u)$ into the mode expansion of h^-_{AB}, which implies that it is only the zero-energy modes that are relevant for horizon supertranslation.

We can obtain the horizon fields h^-_{AB} by taking the quantized graviton field $h_{\mu\nu}(x)$ and taking the limit to $\mathcal{H}^-$. The boundary conditions are such that the in-modes vanish on $\mathcal{H}^-$, so it suffices to consider the up-modes. Although the up-modes are in the outgoing radiation gauge, the angular components h^{up}_{AB} already satisfy the Bondi gauge condition $\gamma^{AB} h_{AB} = 0$ and therefore is expected to retain their functional form on $\mathcal{H}^-$ under a gauge transformation to Bondi gauge. Recalling from (F.15) that

$${}_{+2}R^{\text{up}}_{l\omega}(r) e^{-i\omega t} \sim e^{-i\omega u} \qquad \text{near } \mathcal{H}^-, \tag{5.88}$$

and observing from (F.11) that,

$$\Upsilon_{AB} = -r^4 e_{(3)A} e_{(3)B} \left(\Delta + 5\mu - 2\gamma\right)\left(\Delta + \mu - 4\gamma\right) \tag{5.89}$$

$$= -r^4 e_{(3)A} e_{(3)B} \left(\partial_u - \frac{V}{2}\partial_r - \frac{5V}{2r} - \frac{M}{r^2}\right)\left(\partial_u - \frac{V}{2}\partial_r - \frac{V}{2r} - \frac{2M}{r^2}\right), \tag{5.90}$$

one obtains from (F.9) the asymptotic form

$$h^{\text{up}}_{AB}(l, m, \omega, P; x) \sim -\frac{1}{2}(2M)^2 N^{\text{up}} H_{AB}(P; \Omega) e^{-i\omega u} + \cdots \qquad \text{near } \mathcal{H}^-, \tag{5.91}$$

where "$\cdots$" contains terms with additional factors of ω, which are omitted since we're ultimately interested in leading soft modes. Here we have defined

$$H_{AB}(P; \Omega) \equiv \left[e_{(3)A} e_{(3)B}\ {}_{-2}Y_{lm}(\Omega) + P e^*_{(3)A} e^*_{(3)B}\ {}_{+2}Y_{lm}(\Omega)\right]\Big|_{r=2M}. \tag{5.92}$$

A tedious but straightforward computation shows that (see I for a derivation)

$$D^A D^B H_{AB}(P = -1; \Omega) = 0, \tag{5.93}$$

which implies

$$D^A D^B h^{\text{up}}_{AB}(l, m, \omega, P = -1; x) = 0, \tag{5.94}$$

that is, the magnetic parity modes $P = -1$ do not contribute to the supertranslation charge (5.87). Again, this is similar to the fact that gravitational memory at the asymptotic infinities receive contribution only from the electric parity modes; see [135–137]. For $P = 1$, it can be shown that

$$H_{AB}(P = 1; \Omega) = (2M)^2 \sqrt{\frac{(l-2)!}{(l+2)!}} \left(2D_A D_B - \gamma_{AB} D^2\right) Y_{lm}(\Omega). \tag{5.95}$$

Substituting the expressions to (5.91), keeping only the relevant leading soft contribution and plugging the modes into the expansion (5.55) yields

$$h^-_{AB}(u, \Omega) = -\frac{(2M)^4}{2}\left(2D_A D_B - \gamma_{AB} D^2\right) \times \sum_{lm} \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^\infty d\omega \left[N^{\text{up}} a^{\text{up}}_{lm,P=1}(\omega) Y_{lm}(\Omega) e^{-i\omega u} + \text{h.c.}\right]. \tag{5.96}$$

Since $D^2 Y_{lm}(\Omega) = -l(l+1)Y_{lm}(\Omega)$, and

$$D^A D^B (2D_A D_B - \gamma_{AB} D^2) Y_{lm}(\Omega) = D^2(D^2+2)Y_{lm}(\Omega) = \frac{(l+2)!}{(l-2)!} Y_{lm}(\Omega), \tag{5.97}$$

we immediately obtain

$$\begin{aligned} D^A D^B \partial_u h^-_{AB}(u,\Omega) = -\frac{(2M)^4}{2} \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}} \\ \times \int_0^\infty d\omega (-i\omega) \left[N^{\mathrm{up}} a^{\mathrm{up}}_{lm,P=1}(\omega) Y_{lm}(\Omega) e^{-i\omega u} - \mathrm{h.c.} \right]. \end{aligned} \tag{5.98}$$

Let us define the operator

$$N^-(\Omega) \equiv \frac{1}{\kappa M} \int_{-\infty}^{\infty} du\, D^A D^B \partial_u h^-_{AB}(u,\Omega). \tag{5.99}$$

Using (5.98) and the integral representation

$$\int_{-\infty}^{\infty} du\, e^{\pm i\omega u} = 2\pi \delta(\omega) \tag{5.100}$$

of the delta function, we obtain the expression

$$N^-(\Omega) = \frac{2i\pi(2M)^3}{\kappa} \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}} \int_0^\infty d\omega\, \omega\, \delta(\omega) \left[N^{\mathrm{up}} a^{\mathrm{up}}_{lm,P=1}(\omega) Y_{lm}(\Omega) - \mathrm{h.c.} \right]. \tag{5.101}$$

The supertranslation charge (5.87) can be written in terms of the operator $N^-(\Omega)$ as

$$Q^{\mathcal{H}^-}_f = \int d\Omega\, f(\Omega) N^-(\Omega). \tag{5.102}$$

The presence of the delta function $\delta(\omega)$ clearly shows that only the zero-energy gravitons contribute to the supertranslation charge.

Now, let us take the horizon field $h^-_{AB}(u,\Omega)$ and take the limit $u \to \infty$, which brings the field to the future infinity $\mathcal{H}^-_+$ of the past horizon. Due to the factors $e^{\pm i\omega u}$ in the integrand, by the the Riemann-Lebesgue lemma only the zero-energy modes have non-vanishing contributions to the integral. We can implement this by introducing the infrared function $\phi(\omega)$ in place of $e^{\pm i\omega u}$, which vanishes outside of a small neighborhood of $\omega = 0$ and satisfies

$\phi(0) = 1$. Using this trick, we may define

$$h^-_{AB}(\Omega) \equiv \lim_{u\to\infty} h^-_{AB}(u,\Omega) \tag{5.103}$$

$$= -2M\left(2D_A D_B - \gamma_{AB}D^2\right)\mathcal{A}^-(\Omega) \tag{5.104}$$

where we introduced the scalar field

$$\mathcal{A}^-(\Omega) = \frac{1}{2}(2M)^3 \sum_{lm} \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^\infty d\omega\, \phi(\omega) \left[N^{\rm up} a^{\rm up}_{lm,P=1}(\omega) Y_{lm}(\Omega) + \text{h.c.}\right], \tag{5.105}$$

with the infrared function $\phi(\omega)$. The operators $N^-(\Omega)$ and $\kappa\mathcal{A}^-(\Omega)$ satisfy the commutation relation[2]

$$\begin{aligned}\left[N^-(\Omega'), \kappa\mathcal{A}^-(\Omega)\right] &= i\pi(2M)^6 \sum_{lm}\sum_{l'm'} \sqrt{\frac{(l'+2)!}{(l'-2)!}} \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^\infty d\omega' d\omega\, \omega'\, \delta(\omega')\phi(\omega) \\ &\quad \times \left[N^{\rm up'} a^{\rm up}_{l'm',P=1}(\omega') Y_{l'm'}(\Omega') - \text{h.c.}, N^{\rm up} a^{\rm up}_{lm,P=1}(\omega) Y_{lm}(\Omega) + \text{h.c.}\right]\end{aligned} \tag{5.107}$$

$$= i \sum_{l\geq 2}\sum_m Y_{lm}(\Omega') Y^*_{lm}(\Omega) \tag{5.108}$$

$$= i\delta^{(2)}(\Omega - \Omega') + (l = 0, 1 \text{ terms}). \tag{5.109}$$

Since $f(\Omega)$ does not contain partial waves with $l = 0, 1$, equations (5.102), (5.104) and (5.109) lead to the commutator

$$\left[Q^{\mathcal{H}^-}_f, \kappa h^-_{AB}(\Omega)\right] = -2iM(2D_A D_B - \gamma_{AB}D^2) f(\Omega), \tag{5.110}$$

which is the anticipated quantum action of supertranslation on $\mathcal{H}^-$, reflecting the Lie derivative

$$\mathcal{L}_\xi g_{AB}|_{\mathcal{H}^-} = -2M(2D_A D_B - \gamma_{AB}D^2) f(\Omega) \tag{5.111}$$

of the metric.

[2] In deriving (5.109), we used a crossing relation similar to that used in the original quantization [93] to avoid dealing with factors of 1/2 coming from delta functions sitting on the boundary of integration domains; see appendix J. This is similar to using conventions such as

$$\int_0^\infty d\omega f(\omega)\delta(\omega) = \frac{1}{2} f(0), \tag{5.106}$$

which were used in, for example, [18, 58, 92].

5.3.2 Future horizon

Now that we have derived the supertranslation charge and the horizon fields on $\mathcal{H}^-$, we use the time-reversal symmetry of the Schwarzschild spacetime to obtain analogous results on $\mathcal{H}^+$. The appropriate choice of coordinates is the ingoing Eddington-Finkelstein coordinates (v, t, Ω) with advanced time $v = t + r_*$, in which the Schwarzschild metric reads

$$ds^2 = -V dv^2 + 2dvdr + r^2 \gamma_{AB} dx^A dx^B. \tag{5.112}$$

Since there are the horizon degrees of freedom $h^-_{AB}(u, \Omega)$ on $\mathcal{H}^-$, one should obtain their counterparts $h^+_{AB}(v, \Omega)$ on $\mathcal{H}^+$ by taking $t \to -t$, or equivalently $u \to -v$. Applying this to (5.96), we obtain the future horizon field to be

$$\begin{aligned} h^+_{AB}(v, \Omega) &= h^-_{AB}(-v, \Omega) && (5.113) \\ &= -\frac{(2M)^4}{2} \left(2D_A D_B - \gamma_{AB} D^2\right) \\ &\quad \times \sum_{lm} \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^\infty d\omega \left[N^{\mathrm{up}} a^{\mathrm{up}}_{lm,P=1}(\omega) Y_{lm}(\Omega) e^{i\omega v} + \mathrm{h.c.}\right]. && (5.114) \end{aligned}$$

From [61], we know that the vector field which generates supertranslation on $\mathcal{H}^+$ is

$$\zeta^\alpha \partial_\alpha = f\partial_v - \frac{1}{2} D^2 f \partial_r + \frac{1}{r} D^A f \partial_A, \tag{5.115}$$

and that the associated supertranslation charge on $\mathcal{H}^+$ is

$$Q^{\mathcal{H}^+}_f = \frac{1}{\kappa M} \int_{\mathcal{H}^+} d\Omega\, dv\, f(\Omega) D^A D^B \partial_v h^+_{AB}(v, \Omega). \tag{5.116}$$

Let us define the operator

$$N^+(\Omega) \equiv \frac{1}{\kappa M} \int_{-\infty}^{\infty} dv\, D^A D^B \partial_v h^+_{AB}(v, \Omega), \tag{5.117}$$

in terms of which the charge $Q^{\mathcal{H}^+}_f$ has the simple form

$$Q^{\mathcal{H}^+}_f = \int d\Omega\, f(\Omega) N^+(\Omega). \tag{5.118}$$

Similar to the derivation of (5.101), we plug in the mode expansion (5.114) and use (5.97) to obtain

$$N^{+}(\Omega) = -\frac{2i\pi(2M)^3}{\kappa}\sum_{lm}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^\infty d\omega\,\omega\,\delta(\omega)\left[N^{\mathrm{up}}a^{\mathrm{up}}_{lm,P=1}(\omega)Y_{lm}(\Omega)-\mathrm{h.c.}\right], \quad (5.119)$$

which, as expected, only involves zero-energy gravitons.

As in the case of $\mathcal{H}^-$, we can obtain the zero-modes $h^{+}_{AB}(\Omega)$ by taking $h^{+}_{AB}(v,\Omega)$ to the past boundary $\mathcal{H}^{+}_{-}$ of the future horizon. From (5.114), we have

$$\begin{aligned} h^{+}_{AB}(\Omega) &\equiv \lim_{v\to-\infty} h^{+}_{AB}(v,\Omega) && (5.120)\\ &= 2M(2D_AD_B-\gamma_{AB}D^2)\mathcal{A}^{+}(\Omega), && (5.121)\end{aligned}$$

where $\mathcal{A}^{+}(\Omega)$ is the scalar field defined as

$$\begin{aligned}\mathcal{A}^{+}(\Omega) &= -\frac{1}{2}(2M)^3\sum_{lm}\sqrt{\frac{(l-2)!}{(l+2)!}}\int_0^\infty d\omega\,\phi(\omega)\left[N^{\mathrm{up}}a^{\mathrm{up}}_{lm,P=1}(\omega)Y_{lm}(\Omega)+\mathrm{h.c.}\right] && (5.122)\\ &= -\frac{1}{2}\sum_{lm}\sqrt{\frac{(l-2)!}{(l+2)!}}\int_0^\infty \frac{d\omega}{\sqrt{\pi\omega}}\phi(\omega)\left[a^{\mathrm{up}}_{lm,P=1}(\omega)Y_{lm}(\Omega)+\mathrm{h.c.}\right]. && (5.123)\end{aligned}$$

In the second equality we used (F.21). The field $h^{+}_{AB}(\Omega)$ are, up to a factor of κ, the supertranslation zero modes $\delta_f g_{AB}$ obtained in [61]. The operators $N^{+}(\Omega)$ and $\kappa\mathcal{A}^{+}(\Omega)$ satisfy a commutation relation similar to (5.109),

$$\left[N^{+}(\Omega'),\kappa\mathcal{A}^{+}(\Omega)\right] = i\delta^{(2)}(\Omega-\Omega') + (l=0,1\text{ terms}), \quad (5.124)$$

from which we obtain

$$\begin{aligned}\left[Q^{\mathcal{H}^+}_f,\kappa\mathcal{A}^{+}(\Omega)\right] &= if(\Omega), && (5.125)\\ \left[Q^{\mathcal{H}^+}_f,\kappa h^{+}_{AB}(\Omega)\right] &= 2iM(2D_AD_B-\gamma_{AB}D^2)f(\Omega), && (5.126)\end{aligned}$$

which is the anticipated quantum action of supertranslation on the metric perturbation, correctly reflecting the classical result [61] on $\mathcal{H}^+$,

$$\mathcal{L}_\zeta g_{AB}|_{\mathcal{H}^+} = 2M(2D_AD_B-\gamma_{AB}D^2)f. \quad (5.127)$$

Equation (5.126) shows that the supertranslation zero modes κh^{+}_{AB}, written as a linear combination of zero-frequency electric-parity up-mode gravitons, are the symplectic partners of the linearized horizon charge $Q_f^{\mathcal{H}^+}$ that enlarge the horizon phase space, as anticipated from [61].

5.3.3 Comments

We notice that the structures of the horizon fields and zero-modes on $\mathcal{H}^\pm$ are very similar to those of the past/future null infinities $\mathcal{I}^\pm$ that are extensively studied in the literature. The commutator (5.125) and its counterpart on $\mathcal{H}^-$ suggest that the two scalar fields $\mathcal{A}^\pm(\Omega)$ are the analogs of the Goldstone boson modes on $\mathcal{I}^\pm$ for asymptotically flat spacetimes [16]. Recall that we obtained the horizon fields on $\mathcal{H}^+$ from those on $\mathcal{H}^-$ via time-inversion symmetry of Schwarzschild spacetime. This was used to derive (5.105) and (5.122), from which one obtains

$$\mathcal{A}^-(\Omega) = -\mathcal{A}^+(\Omega). \tag{5.128}$$

This is reminiscent of the antipodal matching conditions of $\mathcal{I}^\pm$ for Christodoulou-Klainerman spaces [13, 14, 16].

Also, we note that the relations (5.104) and (5.121) suggest $\mathcal{A}^\pm$ to be related to the horizon analogs of the "scalar memory" T introduced in [137] at $\mathcal{I}^\pm$ for asymptotically flat spacetimes. Recall that magnetic parity modes dropped out in the construction of the dressing and the charge, which is reminiscent of the situation of gravitational memory at infinities.

5.4 Gravitational dressing implants supertranslation charge

In section 5.2 we obtained the dressing $\exp(W)$ for a particle of mass m with energy E that falls into the black hole. In this section, we show that this dressing carries a definite horizon supertranslation charge.

Comparing the expression (5.79) for $\exp(W)$ with the expression (5.123) for $\mathcal{A}^+$, one immediately recognizes that the exponent W is proportional to the operator $\mathcal{A}^+$,

$$W = W(m, E, \Omega) = -\frac{im^2}{2E}\kappa\mathcal{A}^+(\Omega). \tag{5.129}$$

This with (5.125) implies the commutation relation

$$\left[Q_f^{\mathcal{H}^+}, e^{W(m,E,\Omega)}\right] = \frac{m^2}{2E} f(\Omega) e^{W(m,E,\Omega)}. \tag{5.130}$$

Recall from (5.79) that the dressing $\exp(W)$ is written purely in terms of zero-energy gravitons and therefore carries no energy. Given a Schwarzschild black hole state $|M_0, 0\rangle$ of mass M_0 with zero soft supertranslation charge, i.e.

$$Q_f^{\mathcal{H}^+} |M_0, 0\rangle = 0, \tag{5.131}$$

one can use the dressings to obtain other Schwarzschild black hole states carrying non-zero soft supertranslation charge,

$$|M_0, (m, E, \Omega)\rangle \equiv e^{W(m,E,\Omega)} |M_0, 0\rangle\,, \tag{5.132}$$

$$Q_f^{\mathcal{H}^+} |M_0, (m, E, \Omega)\rangle = \frac{m^2}{2E} f(\Omega) |M_0, (m, E, \Omega)\rangle\,. \tag{5.133}$$

Therefore, our derivation of the gravitational dressing provides an example in the quantum theory of the classical construction of the supertranslation hair in [61].

5.5 Summary

In this chapter, we have explicitly shown how to construct soft supertranslation hair on the horizon of Schwarzschild black holes within a quantum field theoretical framework. The essential ingredient was the construction of dressed states by attaching Wilson lines to the infalling scalar particles, see equation (5.79). We have observed the horizon graviton supertranslation modes proposed by Hawking, Perry and Strominger [60, 61] in the quantum theory, see equation (5.126). We have then used the dressings to show that infalling particles implant supertranslation charge on the black hole, see equations (5.132) and (5.133).

This perspective works for dressed states implanting hair both at $\mathcal{I}^\pm$ and at the horizon. Our quantization procedure implies that a crucial component in our construction of soft charges and Faddeev-Kulish dressings is the existence of an infinite red-shift surface. At this surface the Killing vector ∂_u associated with the time-translation symmetry of the background spacetime becomes null. A massive particle approaching this surface only makes contact asymptotically at $u = \infty$ and at this point its dressing carrying the soft charge only contains soft gravitons ($\omega = 0$). This can be seen by expanding the dressing in terms of plane waves $e^{\pm i\omega u}$; as $u \to \infty$ only soft modes contribute by virtue of the Riemann-Lebesgue lemma. Thus, we can conclusively confirm that there is structure at the horizon and not just any null surface. However, there is evidence that this particular example of supertranslation hair does not appear to have relevance to the black hole information paradox. Evidence that Hawking radiation is not modified by soft hair implanted by supertranslating shock waves comes from [54], at least using the mechanisms analyzed therein. Their results complement those of [47] obtained from the perspective of dressing states with soft hair where the authors showed that the spectrum of Hawking radiation (without backreaction) emitted in the Schwarzschild background is unchanged after including the dressing of asymptotic states with soft form factors. See also [138, 139] for arguments against the role of soft hair carrying black hole information. The relevance of soft hair to black hole entropy is still an open question.

Chapter 6

Subleading Soft Dressings of Asymptotic States in QED and Perturbative Quantum Gravity

6.1 Dressings in perturbative quantum gravity

In this chapter, we construct dressed states at the subleading soft order as eigenstates of the leading and subleading charges, by working to first order in the coupling constant. We then show that that the infrared-finite parts of scattering amplitudes are in agreement with the cross-sections used in experiments. To this end, we start with a brief review of the relevant asymptotic symmetry of gravity: BMS superrotation.

6.1.1 Review of superrotation in asymptotically flat spacetime

We start by establishing our notation regarding asymptotically flat spacetimes and reviewing the materials associated with the superrotation on $\mathcal{I}^{\pm}$. We follow the construction of [140] closely. For the sake of simplicity we assume that all matter particles are massless scalars; for particles with spin see appendix K.1.2.

Metric and mode expansions

In Bondi coordinates, the metric for an asymptotically flat spacetime near the future null infinity $\mathcal{I}^+$ reads [140]

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &\quad + \frac{2m_B^+}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{uz}dudz + 2g_{u\bar{z}}dud\bar{z} + \cdots , \end{aligned} \tag{6.1}$$

where the first line corresponds to the flat metric. Here $z = e^{i\phi}\tan\frac{\theta}{2}$ is the stereographic coordinate, $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ is the metric on the 2-sphere and m_B^+ is the Bondi mass aspect. The uz-component of the metric has the expansion

$$g_{uz} = \frac{1}{2}D^zC_{zz} + \frac{1}{6r}C_{zz}D_zC^{zz} + \frac{2}{3r}N_z^+ + \cdots , \tag{6.2}$$

where D_z is the covariant derivative on S^2 and N_z^+ is the angular momentum aspect. In general, m_B^+, N_z^+ and C_{zz} are functions of u, z and $\bar{z}$.

Let us define the graviton field $h_{\mu\nu}$ through

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \qquad \kappa^2 = 32\pi G, \tag{6.3}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Near $\mathcal{I}^+$, the graviton field can be approximated by the on-shell mode expansion

$$h^{\mathrm{out}}_{\mu\nu}(x) = \sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left[\epsilon^{s*}_{\mu\nu}(\mathbf{k})a^{\mathrm{out}}_s(\mathbf{k})e^{ik\cdot x} + \epsilon^{s}_{\mu\nu}(\mathbf{k})a^{\mathrm{out}\dagger}_s(\mathbf{k})e^{-ik\cdot x}\right], \tag{6.4}$$

where $\omega_k = k^0 = |\mathbf{k}|$, and $\epsilon^{\pm}_{\mu\nu} = \epsilon^{\pm}_{\mu}\epsilon^{\pm}_{\nu}$ are the spin-2 polarization tensors. By parametrizing the graviton momentum k^μ by $(\omega_k, z, \bar{z})$,

$$k^\mu = \frac{\omega_k}{1+z\bar{z}}\Big(1+z\bar{z}, \bar{z}+z, i(\bar{z}-z), 1-z\bar{z}\Big), \tag{6.5}$$

we may write the polarization tensors as

$$\epsilon^{+\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \qquad \epsilon^{-\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}(z, 1, i, -z). \tag{6.6}$$

The out-operators satisfy the standard commutation relation,

$$\left[a^{\mathrm{out}}_s(\mathbf{k}), a^{\mathrm{out}\dagger}_r(\mathbf{k}')\right] = \delta_{sr}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{6.7}$$

Near the past null infinity $\mathcal{I}^-$, the asymptotically flat metric reads

$$
\begin{aligned}
ds^2 = &-dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\
&+ \frac{2m_B^-}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{vz}dvdz + 2g_{v\bar{z}}dvd\bar{z} + \cdots\,,
\end{aligned} \tag{6.8}
$$

where

$$g_{vz} = -\frac{1}{2}D^zD_{zz} - \frac{1}{6r}D_{zz}D_zD^{zz} - \frac{2}{3r}N_z^- + \cdots\,. \tag{6.9}$$

We have the mode expansion for the incoming graviton field

$$h_{\mu\nu}^{\text{in}}(x) = \sum_{s=\pm}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left[\epsilon_{\mu\nu}^{s*}(\mathbf{k})a_s^{\text{in}}(\mathbf{k})e^{ik\cdot x} + \epsilon_{\mu\nu}^{s}(\mathbf{k})a_s^{\text{in}\dagger}(\mathbf{k})e^{-ik\cdot x}\right], \tag{6.10}$$

where the in-operators satisfy

$$\left[a_s^{\text{in}}(\mathbf{k}), a_r^{\text{in}\dagger}(\mathbf{k}')\right] = \delta_{sr}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{6.11}$$

Superrotation charge

Superrotation near $\mathcal{I}^\pm$ are generated by the following vector fields respectively [140],

$$\xi^+(Y) = \left(1+\frac{u}{2r}\right)Y^z\partial_z - \frac{u}{2r}D^{\bar{z}}D_zY^z\partial_{\bar{z}} - \frac{(r+u)}{2}D_zY^z\partial_r + \frac{u}{2}D_zY^z\partial_u + \text{c.c.}, \tag{6.12}$$

$$\xi^-(Y) = \left(1-\frac{v}{2r}\right)Y^z\partial_z + \frac{v}{2r}D^{\bar{z}}D_zY^z\partial_{\bar{z}} - \frac{(r-v)}{2}D_zY^z\partial_r + \frac{v}{2}D_zY^z\partial_v + \text{c.c.}, \tag{6.13}$$

parametrized by the same vector field Y^z on the sphere. We drop the constraint that Y^z is a conformal Killing vector, following [35]. The conserved charges associated with superrotation have the expressions [1]

$$Q_Y^\pm = \frac{4}{\kappa^2}\int_{\mathcal{I}_\mp^\pm} d^2z\left(Y_{\bar{z}}N_z^\pm + Y_zN_{\bar{z}}^\pm\right), \tag{6.14}$$

where $\mathcal{I}_-^+$ ($\mathcal{I}_+^-$) is the past (future) boundary of the future (past) null infinity. The charges can be decomposed into soft and hard parts,

$$Q_Y^\pm = Q_S^\pm + Q_H^\pm, \tag{6.15}$$

where the soft charges are given by

$$Q_S^+ = -\frac{2}{\kappa^2}\int_{\mathcal{I}^+} du\, d^2z\, \gamma^{z\bar{z}} D_z^3 Y^z u N_{\bar{z}\bar{z}} + \text{h.c.} \tag{6.16}$$

$$= -\frac{i}{4\pi\kappa}\lim_{\omega\to 0}(1+\omega\partial_\omega)\int d^2z\, D_z^3 Y^z \left[a_-^{\text{out}}(\omega\mathbf{x}_z) - a_+^{\text{out}\dagger}(\omega\mathbf{x}_z)\right] + \text{h.c.}, \tag{6.17}$$

$$Q_S^- = \frac{2}{\kappa^2}\int_{\mathcal{I}^-} dv\, d^2z\, \gamma^{z\bar{z}} D_z^3 Y^z v M_{\bar{z}\bar{z}} + \text{h.c.} \tag{6.18}$$

$$= \frac{i}{4\pi\kappa}\lim_{\omega\to 0}(1+\omega\partial_\omega)\int d^2z\, D_z^3 Y^z \left[a_-^{\text{in}}(\omega\mathbf{x}_z) - a_+^{\text{in}\dagger}(\omega\mathbf{x}_z)\right] + \text{h.c.}, \tag{6.19}$$

with $N_{zz} = \partial_u C_{zz}$ and $M_{zz} = \partial_v D_{zz}$; we refer to [140] for details. Here $\mathbf{x}_z$ denotes a unit 3-vector whose direction is given by $(z, \bar{z})$,

$$\mathbf{x}_z = \frac{1}{1+z\bar{z}}\Big(\bar{z}+z, i(\bar{z}-z), 1-z\bar{z}\Big). \tag{6.20}$$

The hard charges $Q_H^\pm$ are defined by their actions on the Fock states,

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_n|\, Q_H^+ = i\sum_{i=1}^n \left(Y^z(z_i)\partial_{z_i} - \frac{E_i}{2} D_z Y^z(z_i)\partial_{E_i} + (z\to\bar{z})\right)\langle \mathbf{p}_1, \ldots, \mathbf{p}_n|\,, \tag{6.21}$$

$$Q_H^- \,|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle = -i\sum_{i=1}^n \left(Y^z(z_i)\partial_{z_i} - \frac{E_i}{2} D_z Y^z(z_i)\partial_{E_i} + (z\to\bar{z})\right)|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle\,. \tag{6.22}$$

where the momentum of the massless scalar p_i is written as

$$p_i^\mu = \frac{E_i}{1+z_i\bar{z}_i}\Big(1+z_i\bar{z}_i, \bar{z}_i+z_i, i(\bar{z}_i-z_i), 1-z_i\bar{z}_i\Big). \tag{6.23}$$

6.1.2 Distinction between in and out operators

Notice that in (6.17) and (6.19) we follow the notation of [14] and others to distinguish between the "out" operators on $\mathcal{I}^+$ and the "in" operators on $\mathcal{I}^-$. They are related by the boundary condition $N_{\bar{z}\bar{z}}|_{\mathcal{I}_-^+} = -M_{\bar{z}\bar{z}}|_{\mathcal{I}_+^-}$ such that the subleading soft contribution to the S-matrix element from insertions of $a_s^{\text{in}\dagger}$ in the incoming state and $-a_{-s}^{\text{out}}$ in the outgoing state are equivalent, see [140] for a discussion. An alternative approach, which was taken in [59], is to make this relation explicit by taking in (6.19) and (6.26),

$$a_s^{\text{out}}(\omega\mathbf{x}_z) \to a_s(\omega\mathbf{x}_z), \qquad a_s^{\text{in}}(\omega\mathbf{x}_z) \to -a_s(\omega\mathbf{x}_z), \tag{6.24}$$

such that $Q_S^+ = (Q_S^-)^\dagger = Q_S^-$, and

$$\left[a_r(\mathbf{k}), a_s^\dagger(\mathbf{k}')\right] = \delta_{rs}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{6.25}$$

Then we can remove the superscript in $Q_S^\pm$ and write

$$Q_S = \frac{-i}{4\pi\kappa}\lim_{\omega\to 0}(1+\omega\partial_\omega)\int d^2z\, D_z^3 Y^z\left[a_-(\omega\mathbf{x}_z) - a_+^\dagger(\omega\mathbf{x}_z)\right] + \text{h.c.}. \tag{6.26}$$

Either approach yields the same result; with the distinction intact, one just has to be cautious when contracting an "out" and an "in" operator. We employ the latter convention (6.24) and use (6.26) for it removes some complications in the amplitude computation.

6.1.3 Subleading soft dressing

Let us consider the scattering from some incoming state $|\text{in}\rangle$ to some outgoing state $\langle\text{out}|$ – the states can be either dressed states or Fock states. Superrotation symmetry states that the associated charge must be conserved in a scattering process, i.e.

$$\langle\text{out}|\left(Q_Y^+\mathcal{S} - \mathcal{S}Q_Y^-\right)|\text{in}\rangle = 0, \tag{6.27}$$

where $\mathcal{S}$ is the scattering matrix. This can be written as

$$\langle\text{out}|[Q_S(Y),\mathcal{S}]|\text{in}\rangle = -i\sum_i\left(Y^z(z_i)\partial_{z_i} - \frac{E_i}{2}D_zY^z(z_i)\partial_{E_i} + (z\to\bar{z})\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{6.28}$$

where we used (6.21) and (6.22).

Let us choose the vector field

$$Y = Y_g \equiv \frac{(z-w)^2}{(\bar{z}-\bar{w})}\partial_z, \tag{6.29}$$

for which (6.28) becomes [35, 140] (see appendix K.1 for a derivation)

$$\begin{aligned}\langle\text{out}|[Q_S(Y_g),\mathcal{S}]|\text{in}\rangle &= -\sum_i\frac{p_i^\mu k_\lambda\epsilon^-_{\mu\nu}(\omega\mathbf{x}_z)}{p_i\cdot k}\left(p_i^\lambda\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\lambda}}\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle & (6.30)\\ &= -iS_g^{(1)-}\langle\text{out}|\mathcal{S}|\text{in}\rangle\,, & (6.31)\end{aligned}$$

where $k^\mu \equiv (\omega, \omega \mathbf{x}_z)$, and $S_g^{(1)-}$ is the subleading soft factor for negative-helicity graviton,

$$S_g^{(1)-} = -i \sum_i \eta_i \frac{p_i^\mu k_\lambda J_i^{\lambda\nu}}{p_i \cdot k} \epsilon_{\mu\nu}^-(\omega \mathbf{x}_z), \tag{6.32}$$

with $\eta_i = +1$ for incoming particles and $\eta_i = -1$ for outgoing particles.[1] Using the identity $D_z^3 Y_g^z = 4\pi\delta^{(2)}(z-w)$, we may write the soft charge as

$$Q_S(Y_g) = -\frac{i}{\kappa} \lim_{\omega\to 0}(1+\omega\partial_\omega)\left[a_-(\omega\mathbf{x}_z) - a_+^\dagger(\omega\mathbf{x}_z)\right]. \tag{6.33}$$

We now claim that under certain circumstances stated below, we may write for a ket vacuum $|0\rangle$,

$$Q_S\,|0\rangle \approx 0. \tag{6.34}$$

Strictly speaking, the subleading soft charge does not annihilate the vacuum state (and hence the symbol $\approx$), but rather creates a state containing a soft graviton. In section 6.3 we show that no state may scatter to such a state and vice versa in the dressed state formalism. Therefore, Q_S may be taken to annihilate $|0\rangle$ insofar as scattering processes are concerned.

As was done in [59] for the leading soft dressings, we aim to construct the subleading soft dressed state by using superrotation charge conservation. Since $Q_S(Y_g)$ is of the form $a - a^\dagger$, we want to consider a coherent state of the form

$$\exp\left\{\frac{i\kappa}{2}\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k} N_{\text{in}}^{\mu\nu}\left[\epsilon_{\mu\nu}^{s*}(\mathbf{k})a_s(\mathbf{k}) + \epsilon_{\mu\nu}^{s}(\mathbf{k})a_s^\dagger(\mathbf{k})\right]\right\}|\mathbf{p}_1,\cdots,\mathbf{p}_m\rangle, \tag{6.35}$$

where $N_{\text{in}}^{\mu\nu}$ is a tensor whose components are to be determined by charge conservation, $\phi(\omega_k)$ is an infrared function [8,9] that has support only in a small neighborhood of $\omega_k = 0$ satisfying $\phi(0) = 1$. Here $|\mathbf{p}_1,\cdots,\mathbf{p}_m\rangle$ is an m-particle Fock state,

$$|\mathbf{p}_1,\cdots,\mathbf{p}_m\rangle = \prod_{i=1}^m b^\dagger(\mathbf{p}_i)\,|0\rangle, \tag{6.36}$$

where $b^\dagger(\mathbf{p})$ is the creation operator of the scalar field. However, since we are using the tree-level subleading soft theorem, we can only construct a dressing that can be trusted to

[1] In this definition of $S_g^{(1)-}$ we deviate from the convention used in [140]. The signs η_i derive from the different momentum-space representations of the action of angular momentum on bras and kets. Alternatively, one could define $J_i^{\lambda\nu}$ as in [140], for which case the statement of angular momentum conservation becomes $\sum_i J_i = 0$. We adopt (6.32), in order for angular momentum conservation to take the more natural form $\sum_{i\in\text{in}} J_i = \sum_{j\in\text{out}} J_j$. The two approaches are equivalent.

order κ in the exponent. In this spirit, let us define the incoming state as the linearized version of (6.35),

$$|\text{in}\rangle = \left\{1 + \frac{i\kappa}{2}\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}N_{\text{in}}^{\mu\nu}\left[\epsilon_{\mu\nu}^{s*}(\mathbf{k})a_s(\mathbf{k}) + \epsilon_{\mu\nu}^{s}(\mathbf{k})a_s^\dagger(\mathbf{k})\right]\right\}|\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle\,. \quad (6.37)$$

By direct computation,

$$\begin{aligned} Q_S(Y_g)\,|\text{in}\rangle &= \frac{1}{2}\lim_{\omega\to 0}(1+\omega\partial_\omega)\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}N_{\text{in}}^{\mu\nu} \\ &\quad\times\left[a_-(\omega\mathbf{x}_z) - a_+^\dagger(\omega\mathbf{x}_z), \epsilon_{\mu\nu}^{s*}(\mathbf{k})a_s(\mathbf{k}) + \epsilon_{\mu\nu}^{s}(\mathbf{k})a_s^\dagger(\mathbf{k})\right]|\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle \quad (6.38) \\ &= \frac{1}{2}\lim_{\omega\to 0}(1+\omega\partial_\omega)\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}N_{\text{in}}^{\mu\nu} \\ &\quad\times (2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\omega\mathbf{x}_z)\left(\epsilon_{\mu\nu}^{s}(\mathbf{k})\delta_{s,-} + \epsilon_{\mu\nu}^{s*}(\mathbf{k})\delta_{s,+}\right)|\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle \quad (6.39) \\ &= \lim_{\omega\to 0}(1+\omega\partial_\omega)N_{\text{in}}\cdot\epsilon^-(\omega\mathbf{x}_z)\,|\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle\,, \quad (6.40) \end{aligned}$$

where in the last line we used the notation $N_{\text{in}}\cdot\epsilon^- \equiv N_{\text{in}}^{\mu\nu}\epsilon_{\mu\nu}^-$.

Similarly, we can construct a bra state,

$$\langle\text{out}| \equiv \langle\mathbf{p}_{m+1}, \cdots, \mathbf{p}_{m+n}|\left(1 - \frac{i\kappa}{2}\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}N_{\text{out}}^{\mu\nu}\left[\epsilon_{\mu\nu}^{s*}(\mathbf{k})a_s(\mathbf{k}) + \epsilon_{\mu\nu}^{s}(\mathbf{k})a_s^\dagger(\mathbf{k})\right]\right), \quad (6.41)$$

such that

$$\begin{aligned} \langle\text{out}|\,Q_S(Y_g) &= -\frac{1}{2}\lim_{\omega\to 0}(1+\omega\partial_\omega)\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}\langle\mathbf{p}_{m+1}, \cdots, \mathbf{p}_{m+n}|\,N_{\text{out}}^{\mu\nu} \\ &\quad\times\left[\epsilon_{\mu\nu}^{s*}(\mathbf{k})a_s(\mathbf{k}) + \epsilon_{\mu\nu}^{s}(\mathbf{k})a_s^\dagger(\mathbf{k}), a_-(\omega\mathbf{x}_z) - a_+^\dagger(\omega\mathbf{x}_z)\right] \quad (6.42) \\ &= \frac{1}{2}\lim_{\omega\to 0}(1+\omega\partial_\omega)\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\omega\mathbf{x}_z) \\ &\quad\times\left(\epsilon_{\mu\nu}^{s*}(\mathbf{k})\delta_{s,+} + \epsilon_{\mu\nu}^{s}(\mathbf{k})\delta_{s,-}\right)\langle\mathbf{p}_{m+1}, \cdots, \mathbf{p}_{m+n}|\,N_{\text{out}}^{\mu\nu} \quad (6.43) \\ &= \lim_{\omega\to 0}(1+\omega\partial_\omega)\,\langle\mathbf{p}_{m+1}, \cdots, \mathbf{p}_{m+n}|\,N_{\text{out}}\cdot\epsilon^-(\omega\mathbf{x}_z). \quad (6.44) \end{aligned}$$

With these states, we may write

$$\begin{aligned}\langle \text{out}|\,[Q_S(Y_g),\mathcal{S}]\,|\text{in}\rangle &= \lim_{\omega\to 0}(1+\omega\partial_\omega)\Big[\,\langle \mathbf{p}_{m+1},\cdots,\mathbf{p}_{m+n}|(N_{\text{out}}\cdot\epsilon^-)\mathcal{S}|\text{in}\rangle \\ &\qquad - \langle \text{out}|\mathcal{S}(N_{\text{in}}\cdot\epsilon^-)|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\,\Big] \qquad (6.45)\\ &= \lim_{\omega\to 0}(1+\omega\partial_\omega)\Big[(N_{\text{out}}\cdot\epsilon^-)\,\langle \mathbf{p}_{m+1},\cdots,\mathbf{p}_{m+n}|\mathcal{S}|\text{in}\rangle \\ &\qquad - (N_{\text{in}}\cdot\epsilon^-)\,\langle \text{out}|\mathcal{S}|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\,\Big], \qquad (6.46)\end{aligned}$$

where in the second equality we employ a convenient abuse of notation to write $N\cdot\epsilon^-$ both as an operator and as its action on the amplitude in the momentum-basis. In section 6.4 we see that, due to the presence of $\phi(\omega_k)$, adding or removing subleading dressings do not change the value of the amplitude, that is,

$$\langle \mathbf{p}_{m+1},\cdots,\mathbf{p}_{m+n}|\mathcal{S}|\text{in}\rangle = \langle \text{out}|\mathcal{S}|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle = \langle \text{out}|\mathcal{S}|\text{in}\rangle\,. \qquad (6.47)$$

This shows that the non-commutativity of subleading charges and the nonexistence of simultaneous eigenstates do not cause difficulties to S-matrix calculations at this order. Using (6.47), one may write (6.46) as

$$\langle \text{out}|\,[Q_S(Y_g),\mathcal{S}]\,|\text{in}\rangle = \lim_{\omega\to 0}(1+\omega\partial_\omega)\,(N^{\mu\nu}_{\text{out}}-N^{\mu\nu}_{\text{in}})\,\epsilon^-_{\mu\nu}\,\langle \text{out}|\mathcal{S}|\text{in}\rangle\,. \qquad (6.48)$$

Thus, the superrotation charge conservation (6.31) reads

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\,(N^{\mu\nu}_{\text{out}}-N^{\mu\nu}_{\text{in}})\,\epsilon^-_{\mu\nu}\,\langle \text{out}|\mathcal{S}|\text{in}\rangle = -iS^{(1)-}_g\,\langle \text{out}|\mathcal{S}|\text{in}\rangle\,, \qquad (6.49)$$

which, with (6.32), implies for $\langle \text{out}|\mathcal{S}|\text{in}\rangle \neq 0$,

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\,(N^{\mu\nu}_{\text{out}}-N^{\mu\nu}_{\text{in}})\,\epsilon^-_{\mu\nu} = -\sum_{i=1}^{m+n}\eta_i\frac{(p_i)^\mu k_\lambda (J_i)^{\lambda\nu}}{p_i\cdot k}\epsilon^-_{\mu\nu}. \qquad (6.50)$$

One can derive a similar relation associated with ϵ^+ by choosing $Y=(\bar{z}-\bar{w})^2(z-w)^{-1}\partial_{\bar{z}}$. A natural split for the dressings is

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)N^{\mu\nu}_{\text{in}} = -\sum_{i=1}^{m}\frac{(p_i)^\mu k_\lambda (J_i)^{\lambda\nu}}{p_i\cdot k}, \qquad (6.51)$$

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)N^{\mu\nu}_{\text{out}} = -\sum_{j=m+1}^{m+n}\frac{(p_j)^\mu k_\lambda (J_j)^{\lambda\nu}}{p_j\cdot k}. \qquad (6.52)$$

If we treat supertranslation (associated with simple poles) separately as in [59], we may assume that $N^{\mu\nu}_{\text{in,out}}$ do not possess poles. Then it follows that

$$N^{\mu\nu}_{\text{in}} = -\sum_{i=1}^{m} \frac{(p_i)^\mu k_\lambda (J_i)^{\lambda\nu}}{p_i \cdot k}, \qquad N^{\mu\nu}_{\text{out}} = -\sum_{j=m+1}^{m+n} \frac{(p_j)^\mu k_\lambda (J_j)^{\lambda\nu}}{p_j \cdot k}, \tag{6.53}$$

which, substituted into (6.37), yields the subleading soft Faddeev-Kulish dressings. Put together with the leading soft gravitational Faddeev-Kulish dressings [9, 58, 59], we denote the dressed asymptotic state with double brackets as

$$\|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle\!\rangle = W_g(\mathbf{p}_1, \cdots, \mathbf{p}_n) \left|\mathbf{p}_1, \cdots, \mathbf{p}_n\right\rangle, \tag{6.54}$$

where $W_g(\mathbf{p}_1, \cdots, \mathbf{p}_n)$ is the gravitational n-particle dressing, which to the subleading order in soft momentum expansion and leading order in κ is given by[2]

$$\begin{aligned} W_g = \exp&\left\{ \frac{\kappa}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu}{p_i \cdot k} (a^\dagger_{\mu\nu} - a_{\mu\nu}) \right\} \\ &\times \left(1 - \frac{\kappa}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \sum_{i=1}^{n} \frac{p_i^\mu k_\rho J_i^{\rho\nu}}{p_i \cdot k} i (a^\dagger_{\mu\nu} + a_{\mu\nu}) + \mathcal{O}(\kappa^2) \right). \end{aligned} \tag{6.55}$$

Keeping in mind that only order κ terms can be trusted, (6.55) can be conveniently written as[3]

$$W_g = \exp\left[\frac{\kappa}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \sum_{i=1}^{n} \frac{p_i^\mu}{p_i \cdot k} \left\{ \left(p_i^\nu - ik_\rho J_i^{\rho\nu}\right) a^\dagger_{\mu\nu} - \left(p_i^\nu + ik_\rho J_i^{\rho\nu}\right) a_{\mu\nu} \right\} + \mathcal{O}(\kappa^2) \right]. \tag{6.56}$$

Here we employed the notation $a_{\mu\nu}(\mathbf{k}) = \sum_s \epsilon^{s*}_{\mu\nu}(\mathbf{k}) a_s(\mathbf{k})$, where s spans all polarizations. This includes unphysical polarizations, since the projection to physical polarizations in (6.32) is a consequence of our choice (6.29) of Y; superrotation charge should be conserved for a generic vector field. Unphysical polarizations are also required (at the leading soft order) for canceling out infrared divergence, see [4, 9, 58] for example. The expression (6.56) expanded to first order in κ agrees with the gravitational Wilson line dressing of Mandelstam [95]. Thus to this order in the coupling, one observes the equivalence between Wilson lines and FK dressings as discussed in [91, 92].

[2]In the original constructions of FK dressings [8, 9], a tensor $c^{\mu\nu}$ (four-vector c^μ in QED) was introduced in the dressing to account for gauge invariance. It has recently been argued [141] that gauge invariance can be achieved without such terms, so we do not consider them here.

[3]Non-commutativity of p and J may be ignored at this order.

For explicit calculations, it is convenient to define the infrared function as

$$\phi(\omega) = \begin{cases} 1 & \text{if } \lambda < \omega < \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{6.57}$$

where λ is the infrared cutoff which we take to be zero at the very end of the calculation, and Λ is a very small energy scale below which particles are considered to be soft. With this definition, the dressing (6.56) becomes

$$W_g = \exp\left[\frac{\kappa}{2}\int_{\lambda<\omega_k<\Lambda} \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\sum_{i=1}^{n}\frac{p_i^\mu}{p_i \cdot k}\left\{(p_i^\nu - ik_\rho J_i^{\rho\nu})\, a^\dagger_{\mu\nu} - (p_i^\nu + ik_\rho J_i^{\rho\nu})\, a_{\mu\nu}\right\} + \mathcal{O}(\kappa^2)\right]. \tag{6.58}$$

The dressing W_g acting on an n-particle Fock state is to be understood as an n-particle dressing with the corresponding momenta of the hard particles, unless explicitly stated otherwise. The dressed states automatically implement conservation of supertranslation charge [58] and superrotation charge, as shown above.

An important point concerning the validity of (6.58) should be emphasized here. In any scattering process there are contributions from real emissions and from virtual diagrams. The applicability of the subleading soft graviton and soft photon theorems to a $2 \to 2$ scattering process has been studied in [142]. There it was shown that the subleading soft photon theorem correctly reproduces the scattering amplitude to subleading order both for real and virtual photon emissions. However, for the case of soft gravitons, the subleading soft graviton theorem correctly reproduces the real external emissions, but there is a violation of the theorem for virtual gravitons. Thus, for the choice of vector field in (6.29), we expect our dressing to be correct for the case of real emissions which we discuss in section 6.3. For scattering involving virtual gravitons we would need to generalize (6.58).

6.2 Dressing in QED

We present here the construction of subleading soft dressing in QED, which is fairly parallel to the case of gravity in section 6.1. In this section we mostly follow the notation and conventions of [27].

6.2.1 Mode expansions and conserved charges

The photon field near $\mathcal{I}^+$ becomes nearly free and can be approximated by

$$A_\mu^{\text{out}}(x) = e \sum_{\alpha=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[\epsilon_\mu^{\alpha*}(\mathbf{k}) a_\alpha^{\text{out}}(\mathbf{k}) e^{ik\cdot x} + \epsilon_\mu^{\alpha}(\mathbf{k}) a_\alpha^{\text{out}\dagger}(\mathbf{k}) e^{-ik\cdot x}\right], \tag{6.59}$$

where the out-operators satisfy the commutator

$$\left[a_\alpha^{\text{out}}(\mathbf{k}), a_\beta^{\text{out}\dagger}(\mathbf{k}')\right] = \delta_{\alpha\beta}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{6.60}$$

Likewise, near $\mathcal{I}^-$ we have the incoming photon field,

$$A_\mu^{\text{in}}(x) = e \sum_{\alpha=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[\epsilon_\mu^{\alpha*}(\mathbf{k}) a_\alpha^{\text{in}}(\mathbf{k}) e^{ik\cdot x} + \epsilon_\mu^{\alpha}(\mathbf{k}) a_\alpha^{\text{in}\dagger}(\mathbf{k}) e^{-ik\cdot x}\right], \tag{6.61}$$

with the standard commutator

$$\left[a_\alpha^{\text{in}}(\mathbf{k}), a_\beta^{\text{in}\dagger}(\mathbf{k}')\right] = \delta_{\alpha\beta}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{6.62}$$

The QED analog of superrotation is the asymptotic symmetry on $\mathcal{I}^\pm$ associated with Low's theorem. In terms of the mode expansions (6.59) and (6.61), the corresponding conserved charges are[4] [27]

$$\mathcal{Q}_Y^\pm = \mathcal{Q}_S^\pm(Y) + \mathcal{Q}_H^\pm(Y), \tag{6.63}$$

where the soft parts are given by

$$\mathcal{Q}_S^+(Y) = -\frac{i}{4\pi e} \lim_{\omega\to 0}(1+\omega\partial_\omega) \int d^2z\, D_z^2 Y^z \frac{\sqrt{2}}{1+z\bar{z}} \left[a_-^{\text{out}}(\omega\mathbf{x}_z) - a_+^{\text{out}\dagger}(\omega\mathbf{x}_z)\right] + \text{h.c.}, \tag{6.64}$$

$$\mathcal{Q}_S^-(Y) = \frac{i}{4\pi e} \lim_{\omega\to 0}(1+\omega\partial_\omega) \int d^2z\, D_z^2 Y^z \frac{\sqrt{2}}{1+z\bar{z}} \left[a_-^{\text{in}}(\omega\mathbf{x}_z) - a_+^{\text{in}\dagger}(\omega\mathbf{x}_z)\right] + \text{h.c.}. \tag{6.65}$$

Here $\mathbf{x}_z$ is a unit 3-vector whose direction is given by $(z,\bar{z})$; its Cartesian components are given in (6.20). The hard parts are defined by their actions on the Fock states,

$$\langle \mathbf{p}_1,\ldots,\mathbf{p}_n|\, \mathcal{Q}_H^+(Y) = -i\sum_{i=1}^n Q_i \left(D_A Y^A(z_i)\partial_{E_i} - \frac{1}{E_i}\mathcal{L}_{Y(z_i)}\right) \langle \mathbf{p}_1,\ldots,\mathbf{p}_n|, \tag{6.66}$$

$$\mathcal{Q}_H^-(Y)\, |\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle = i\sum_{i=1}^n Q_i \left(D_A Y^A(z_i)\partial_{E_i} - \frac{1}{E_i}\mathcal{L}_{Y(z_i)}\right) |\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle, \tag{6.67}$$

[4]We use calligraphic font $\mathcal{Q}$ for the QED charge to minimize notational overlap with gravity.

where $A \in \{z, \bar{z}\}$, Q_i is the electric charge of the i-th particle, and $\mathcal{L}_Y$ is the Lie derivative on S^2, see appendix K.2 for details. Scattering processes conserve $\mathcal{Q}_Y$, which implies that

$$\langle \text{out}|(\mathcal{Q}_Y^+ \mathcal{S} - \mathcal{S}\mathcal{Q}_Y^-)|\text{in}\rangle = 0, \tag{6.68}$$

for any asymptotic states $|\text{in}\rangle$ and $\langle\text{out}|$.

As was the case in gravity, the contribution to the subleading soft matrix element from a soft photon insertion $a_\alpha^{\text{in}\dagger}$ in the incoming state is equivalent to the contribution from an insertion $-a_{-\beta}^{\text{out}}$ in the outgoing state. Therefore, we can follow the procedure of section 6.1.2 and define

$$\mathcal{Q}_S(Y) \equiv -\frac{i}{4\pi e} \lim_{\omega\to 0}(1+\omega\partial_\omega) \int d^2z\, D_z^2 Y^z \frac{\sqrt{2}}{1+z\bar{z}} \left[a_-(\omega \mathbf{x}_z) - a_+^\dagger(\omega \mathbf{x}_z)\right] + \text{h.c.}, \tag{6.69}$$

where the soft operators are related by $\mathcal{Q}_S^+ = \mathcal{Q}_S = \left(\mathcal{Q}_S^-\right)^\dagger = \mathcal{Q}_S^-$, and

$$\left[a_\alpha(\mathbf{k}), a_\beta^\dagger(\mathbf{k}')\right] = \delta_{\alpha\beta}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{6.70}$$

We emphasize that this procedure is done to avoid defining separate rules for contractions between operators on $\mathcal{I}^+$ and $\mathcal{I}^-$; one may obtain the same result with the distinction intact.

6.2.2 Subleading soft dressing

Now we construct the subleading soft Faddeev-Kulish dressing in QED to leading order in the coupling constant e as linearized coherent states that respect $\mathcal{Q}_Y$ charge conservation (6.68).

Using (6.63)-(6.67), (6.68) can be written as

$$\langle \text{out}|\left[\mathcal{Q}_S(Y), \mathcal{S}\right]|\text{in}\rangle = i\sum_i Q_i \left(D_A Y^A(z_i)\partial_{E_i} - \frac{1}{E_i}\mathcal{L}_{Y(z_i)}\right)\langle \text{out}|\mathcal{S}|\text{in}\rangle\,. \tag{6.71}$$

Let us choose

$$Y = Y_e \equiv \frac{(z-w)(1+z\bar{z})}{(\bar{z}-\bar{w})}\partial_z. \tag{6.72}$$

Then, (6.71) takes the form [27] (see appendix K.2 for details)

$$\langle \text{out}|[\mathcal{Q}_S(Y_e), \mathcal{S}]|\text{in}\rangle = -\frac{\sqrt{2}i}{e} S_e^{(1)-}\langle \text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{6.73}$$

where $k^\mu \equiv (\omega, \omega \mathbf{x}_z)$, and $S_e^{(1)-}$ is the subleading soft factor for negative-helicity photon,

$$S_e^{(1)-} = -ie \sum_i \eta_i Q_i \frac{k_\lambda J_i^{\lambda\nu}}{p_i \cdot k} \epsilon_\nu^-(\omega \mathbf{x}_z), \tag{6.74}$$

with $\eta_i = +1$ (-1) for outgoing (incoming) particle (by introducing η_i we deviate from the convention of [27], see footnote 1). An analogous expression involving ϵ^+ can be derived by choosing $Y = (\bar{z} - \bar{w})(1 + z\bar{z})(z - w)^{-1}\partial_{\bar{z}}$ instead. Using the identity $D_z^2 Y_e^z = 2\pi(1 + z\bar{z})\delta^{(2)}(z - w)$, one obtains the following expression for the soft charge,

$$\mathcal{Q}_S(Y_e) = -\frac{i}{\sqrt{2}e} \lim_{\omega \to 0} (1 + \omega \partial_\omega) \left[a_-(\omega \mathbf{x}_z) - a_+^\dagger(\omega \mathbf{x}_z) \right]. \tag{6.75}$$

Let us begin by considering a vacuum $|0\rangle$ such that,

$$\mathcal{Q}_S(Y_e) |0\rangle \approx 0. \tag{6.76}$$

As noted in section 6.1.3, formally the subleading soft charge does not annihilate the vacuum, but rather adds to it a soft photon. As is shown in section 6.3, in scattering processes such a state completely factors out, and therefore in S-matrix computations one may act as if Q_S annihilates the vacuum (hence the symbol $\approx$).

Now we consider states which are dressed to first order in e that take the form

$$|\text{in}\rangle = \left(1 + ie \sum_{\alpha = \pm} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \mathcal{N}_{\text{in}}^\mu \left[\epsilon_\mu^{\alpha *}(\mathbf{k}) a_\alpha(\mathbf{k}) + \epsilon_\mu^\alpha(\mathbf{k}) a_\alpha^\dagger(\mathbf{k}) \right] \right) |\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle, \tag{6.77}$$

$$\langle \text{out}| = \langle \mathbf{p}_{m+1}, \cdots, \mathbf{p}_{m+n}| \left(1 - ie \sum_{\alpha = \pm} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \mathcal{N}_{\text{out}}^\mu \left[\epsilon_\mu^{\alpha *}(\mathbf{k}) a_\alpha(\mathbf{k}) + \epsilon_\mu^\alpha(\mathbf{k}) a_\alpha^\dagger(\mathbf{k}) \right] \right), \tag{6.78}$$

where $\mathcal{N}_{\text{in,out}}^\mu$ are operators to be determined, and $\phi(\omega_k)$ is the infrared function that restricts the momentum integrals to soft modes. Then,

$$\begin{aligned}
\mathcal{Q}_S(Y_e) |\text{in}\rangle &= \frac{1}{\sqrt{2}} \lim_{\omega \to 0} (1 + \omega \partial_\omega) \sum_{\alpha = \pm} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \mathcal{N}_{\text{in}}^\mu \\
&\quad \times \left[a_-(\omega \mathbf{x}_z) - a_+^\dagger(\omega \mathbf{x}_z), \epsilon_\mu^{\alpha *}(\mathbf{k}) a_\alpha(\mathbf{k}) + \epsilon_\mu^\alpha(\mathbf{k}) a_\alpha^\dagger(\mathbf{k}) \right] |\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle &(6.79) \\
&= \frac{1}{\sqrt{2}} \lim_{\omega \to 0} (1 + \omega \partial_\omega) \sum_{\alpha = \pm} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \mathcal{N}_{\text{in}}^\mu \\
&\quad \times (2\pi)^3 (2\omega_k) \delta^{(3)}(\mathbf{k} - \omega \mathbf{x}_z) \left(\epsilon_\mu^{\alpha *}(\mathbf{k}) \delta_{\alpha,+} + \epsilon_\mu^\alpha(\mathbf{k}) \delta_{\alpha,-} \right) |\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle &(6.80) \\
&= \sqrt{2} \lim_{\omega \to 0} (1 + \omega \partial_\omega) \mathcal{N}_{\text{in}} \cdot \epsilon^-(\omega \mathbf{x}_z) |\mathbf{p}_1, \cdots, \mathbf{p}_m\rangle, &(6.81)
\end{aligned}$$

and

$$\langle\text{out}|\,\mathcal{Q}_S(Y_e) = -\frac{1}{\sqrt{2}}\lim_{\omega\to 0}(1+\omega\partial_\omega)\sum_{\alpha=\pm}\int\frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}\langle\mathbf{p}_{m+1},\cdots,\mathbf{p}_{m+n}|\,\mathcal{N}^\mu_{\text{out}}$$
$$\times\left[\epsilon^{\alpha*}_\mu(\mathbf{k})a_\alpha(\mathbf{k})+\epsilon^\alpha_\mu(\mathbf{k})a^\dagger_\alpha(\mathbf{k}),a_-(\omega\mathbf{x}_z)-a^\dagger_+(\omega\mathbf{x}_z)\right] \tag{6.82}$$
$$=\sqrt{2}\lim_{\omega\to 0}(1+\omega\partial_\omega)\,\langle\mathbf{p}_{m+1},\cdots,\mathbf{p}_{m+n}|\,\mathcal{N}_{\text{out}}\cdot\epsilon^-(\omega\mathbf{x}_z). \tag{6.83}$$

Along the same line of reasoning as in (6.48), this leads to

$$\langle\text{out}|\,[\mathcal{Q}_S(Y_e),\mathcal{S}]\,|\text{in}\rangle = \sqrt{2}\lim_{\omega\to 0}(1+\omega\partial_\omega)\,(\mathcal{N}^\mu_{\text{out}}-\mathcal{N}^\mu_{\text{in}})\,\epsilon^-_\mu\,\langle\text{out}|\mathcal{S}|\text{in}\rangle\,. \tag{6.84}$$

Assume that the simple poles (associated with large gauge symmetry) have been treated separately, as in [19]. Then $\lim_{\omega\to 0}(1+\omega\partial_\omega)\mathcal{N}_{\text{in,out}}\cdot\epsilon^- = \mathcal{N}_{\text{in,out}}\cdot\epsilon^-$, and the equation of charge conservation (6.73) becomes

$$(\mathcal{N}^\mu_{\text{out}}-\mathcal{N}^\mu_{\text{in}})\,\epsilon^-_\mu(\omega\mathbf{x}_z) = -\sum_{i=1}^{m+n}\eta_i Q_i\frac{k_\rho J_i^{\rho\mu}}{p_i\cdot k}\epsilon^-_\mu(\omega\mathbf{x}_z), \tag{6.85}$$

if the matrix element $\langle\text{out}|\mathcal{S}|\text{in}\rangle$ is to not vanish. A natural splitting for the dressing is

$$\mathcal{N}^\mu_{\text{in}} = -\sum_{i=1}^{m}Q_i\frac{k_\rho J_i^{\rho\mu}}{p_i\cdot k}. \qquad \mathcal{N}^\mu_{\text{out}} = -\sum_{i=m+1}^{m+n}Q_i\frac{k_\rho J_i^{\rho\mu}}{p_i\cdot k}, \tag{6.86}$$

Combining this with the known leading soft dressing, one deduces the dressed asymptotic state for QED to subleading order in the soft expansion and to first order in e to be

$$\|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\!\rangle = W_e\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\,, \tag{6.87}$$

where the dressing W_e is

$$W_e = \exp\left\{e\int\frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}\sum_{i=1}^{n}\frac{Q_i p_i^\mu}{p_i\cdot k}(a^\dagger_\mu - a_\mu)\right\}$$
$$\times\left(1-e\int\frac{d^3k}{(2\pi)^3}\frac{\phi(\omega_k)}{2\omega_k}\sum_{i=1}^{n}Q_i\frac{k_\rho J_i^{\rho\mu}}{p_i\cdot k}i(a^\dagger_\mu+a_\mu)+\mathcal{O}(e^2)\right), \tag{6.88}$$

which, keeping in mind that only terms to first order in e may be trusted, can be conveniently put as

$$W_e = \exp\left[e \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \sum_{i=1}^{n} \frac{Q_i}{p_i \cdot k} \left\{ \left(p_i^\mu - ik_\nu J_i^{\nu\mu}\right) a_\mu^\dagger - \left(p_i^\mu + ik_\nu J_i^{\nu\mu}\right) a_\mu \right\} + \mathcal{O}(e^2)\right]. \tag{6.89}$$

The term $\mathcal{O}(e^2)$ emphasizes that the subleading dressing is valid only to order e. The photon operator is defined as $a_\mu(\mathbf{k}) = \sum_\alpha \epsilon_\mu^{\alpha*}(\mathbf{k}) a_\alpha(\mathbf{k})$, where α spans all polarizations including unphysical ones, since the projection to physical polarizations is due to our choice (6.72) of Y; the charge should be conserved for a generic vector field. The unphysical polarizations are also required (at the leading soft order) to cancel out Weinberg's infrared-divergent factor [2], see [4, 9, 58] for example.

With the explicit implementation (6.57) of $\phi(\omega_k)$, we obtain

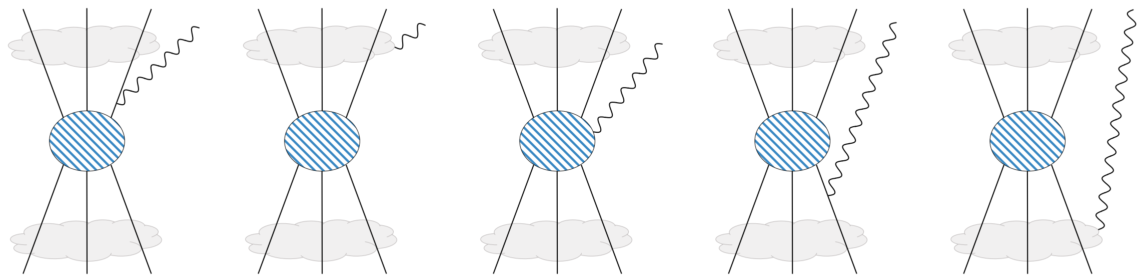
$$W_e = \exp\left[e \int_{\lambda<\omega_k<\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_{i=1}^{n} \frac{Q_i}{p_i \cdot k} \left\{ \left(p_i^\mu - ik_\nu J_i^{\nu\mu}\right) a_\mu^\dagger - \left(p_i^\mu + ik_\nu J_i^{\nu\mu}\right) a_\mu \right\} + \mathcal{O}(e^2)\right], \tag{6.90}$$

where λ is the infrared cutoff and Λ is the separation scale below which we consider particles to be soft. Notice that the structure is very similar to the gravitational dressing (6.58). One can obtain the QED dressing from gravity by the replacement $\frac{\kappa}{2}(p_i^\mu \epsilon_{\mu\nu}^s) \to eQ_i\epsilon_\nu^s$ for each particle. The comments about non-commutativity of subleading and leading soft charges with all its complications discussed towards the end of section (6.1.3) also apply here. In the subsequent sections, we work with the gravitational dressing with the understanding that same results can be shown for QED with minimal modifications.

6.3 External soft gravitons and photons

With the leading soft Faddeev-Kulish states, it is known that adding an external soft graviton does not induce infrared divergence; the divergent soft factors from the dressings cancel those from external legs [18, 58]. Now that we have constructed dressings to subleading order in the soft expansion, we are in a position to investigate what happens to the $\mathcal{O}(\omega^0)$ subleading soft factors. Although we work with gravitons, the derivation for QED is very similar and the final result is also valid for external soft photons. Every step in the calculation can be changed to the corresponding expression for QED by replacing $\frac{\kappa}{2}(p_i^\mu \epsilon_{\mu\nu}^s)$ with $eQ_i\epsilon_\nu^s$ for each hard particle.

As in (6.57), let Λ be the soft energy scale, below which particles are considered to be



(a) (b) (c) (d) (e)

Figure 6.1: Different contributions to the emission amplitude of a soft graviton.

soft, and let λ be the infrared cutoff which we take to be zero at the end of calculations. Let us consider a scattering amplitude from the dressed m-particle state $\|\mathrm{in}\rangle\!\rangle$ to the dressed n-particle state $\|\mathrm{out}\rangle\!\rangle$, with a soft graviton insertion of polarization s and momentum $k^\mu = (\omega, \omega\mathbf{x})$ where ω is soft,

$$\mathcal{M} \equiv \mathcal{M}(k, s; \{p\}) \equiv \langle \mathrm{out}|\, a_s(\omega\mathbf{x}) W_g^\dagger \mathcal{S} W_g \,|\mathrm{in}\rangle\,, \qquad (\lambda < \omega < \Lambda). \tag{6.91}$$

The dressed amplitude $\mathcal{M}$ has the small-ω expansion

$$\mathcal{M}(k, s; \{p\}) = \frac{1}{\omega}\mathcal{M}^{(-1)} + \mathcal{M}^{(0)} + \mathcal{O}(\omega), \tag{6.92}$$

where each $\mathcal{M}^{(n)}$ is independent of ω. The different contributions to $\mathcal{M}(k, s; \{p\})$ are illustrated in figure 6.1.

It is known that the first term involving the infrared-divergent amplitude $\mathcal{M}^{(-1)}$ vanishes [58]. To see this, note that $\mathcal{M}^{(-1)}$ receives contribution from diagrams 6.1(a), 6.1(b), 6.1(d) and 6.1(e). Using the notation $\eta_i = +1$ (-1) if i is outgoing (incoming), we may write

$$\frac{1}{\omega}\mathcal{M}^{(-1)} = \frac{\kappa}{2}\Bigg[\underbrace{\sum_{i=1}^{m+n} \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot k}}_{\text{6.1(a) and 6.1(d)}} - \underbrace{\sum_{i=1}^{m+n} \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot k}\phi(\omega)}_{\text{6.1(b) and 6.1(e)}}\Bigg]\epsilon^s_{\mu\nu}(\mathbf{k})\overline{\mathcal{M}} = 0, \tag{6.93}$$

where $\overline{\mathcal{M}} \equiv \langle\!\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle\!\rangle = \langle \mathrm{out}|W_g^\dagger \mathcal{S} W_g|\mathrm{in}\rangle$ is the dressed amplitude without graviton insertion. In the second equation we used (6.57) to write $\phi(\omega) = 1$. The first sum in the square brackets comes from graviton emission from external legs (figures 6.1(a) and 6.1(d)); the second sum comes from graviton emission from dressings (figures 6.1(b) and 6.1(e)).

We can determine $\mathcal{M}^{(0)}$ by collecting the $\mathcal{O}(\omega^0)$ terms in the amplitude. To do so, let us first decompose the gravitational dressing W_g into leading and subleading parts,

$$W_g = W_g^{(0)} W_g^{(1)}, \tag{6.94}$$

$$W_g^{(0)} = \exp\left\{ \frac{\kappa}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \sum_i \frac{p_i^\mu p_i^\nu}{p_i \cdot k} (a_{\mu\nu}^\dagger - a_{\mu\nu}) \right\}, \tag{6.95}$$

$$W_g^{(1)} = 1 - \frac{\kappa}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\phi(\omega_k)}{2\omega_k} \sum_i \frac{p_i^\mu k_\rho J_i^{\rho\nu}}{p_i \cdot k} i(a_{\mu\nu}^\dagger + a_{\mu\nu}). \tag{6.96}$$

Then, we obtain the commutators

$$\left[a_s(\omega\mathbf{x}), W_g^\dagger(\mathbf{p}_{m+1}, \cdots, \mathbf{p}_{m+n})\right] = -\frac{\kappa}{2} \sum_{i=m+1}^{m+n} \left(\frac{p_i^\mu p_i^\nu}{p_i \cdot k} W_g^\dagger - i \frac{p_i^\mu k_\lambda J_i^{\lambda\nu}}{p_i \cdot k} W_g^{(0)\dagger} \right) \phi(\omega) \epsilon_{\mu\nu}^s, \tag{6.97}$$

$$\left[a_s(\omega\mathbf{x}), W_g(\mathbf{p}_1, \cdots, \mathbf{p}_m)\right] = \frac{\kappa}{2} \sum_{i=1}^{m} \left(W_g \frac{p_i^\mu p_i^\nu}{p_i \cdot k} - i W_g^{(0)} \frac{p_i^\mu k_\lambda J_i^{\lambda\nu}}{p_i \cdot k} \right) \phi(\omega) \epsilon_{\mu\nu}^s. \tag{6.98}$$

The first and second terms in the summands correspond respectively to the leading and subleading soft contributions from figures 6.1(b) and 6.1(e). The second terms contribute to $\mathcal{M}^{(0)}$, along with emissions from internal propagators (figure 6.1(c)) and external legs (figures 6.1(a) and 6.1(d)). There is one subtlety here – the second terms in the summands are missing the subleading dressing factors $W_g^{(1)}$. However, as we see in section 6.4, insertion of such factors only add $\mathcal{O}(\Lambda)$ corrections to the amplitude, which is negligible by the definition of the soft energy scale Λ. Therefore, within the amplitudes one may replace $W_g^{(0)}$ of (6.97) and (6.98) with W_g and write

$$\mathcal{M}^{(0)} = \frac{\kappa}{2} \Bigg[\underbrace{-i \sum_{i=1}^{m+n} \eta_i \frac{p_i^\mu k_\lambda J_i^{\lambda\nu}}{p_i \cdot k}}_{\text{6.1(a), 6.1(c) and 6.1(d)}} + \underbrace{i \sum_{i=1}^{m+n} \eta_i \frac{p_i^\mu k_\lambda J_i^{\lambda\nu}}{p_i \cdot k} \phi(\omega)}_{\text{6.1(b) and 6.1(e)}} \Bigg] \epsilon_{\mu\nu}^s(\mathbf{k}) \overline{\mathcal{M}} = \mathcal{O}(\Lambda), \tag{6.99}$$

since $\phi(\omega) = 1$ for $\omega < \Lambda$. We remind the reader that the sign η_i in the subleading soft factor comes from the different momentum-space representations of the action of $J_i^{\mu\nu}$ on bras and kets:

$$\langle \mathbf{p} | J^{\mu\nu} = -i \left(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu} \right) \langle \mathbf{p} |, \tag{6.100}$$

$$J^{\mu\nu} | \mathbf{p} \rangle = i \left(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu} \right) | \mathbf{p} \rangle, \tag{6.101}$$

which may differ from some conventions in the literature, see footnote 1.

Collecting the results (6.93) and (6.99), equation (6.92) becomes

$$\mathcal{M}(k, s; \{p\}) = \underbrace{\frac{1}{\omega}\mathcal{M}^{(-1)}}_{=0} + \underbrace{\mathcal{M}^{(0)}}_{=\mathcal{O}(\Lambda)} + \mathcal{O}(\omega) = \mathcal{O}(\Lambda). \tag{6.102}$$

At this point one may remove the infrared regulator $\lambda \to 0$, and conclude that the soft emission amplitude is negligible since ω is by definition less than the soft energy scale Λ, which in turn is by definition much less than any energy scales of our interest. As the emission amplitude vanishes in the soft limit, the state containing a zero-energy graviton can be treated as null as far as scattering processes are concerned:

$$\lim_{\omega\to 0} a_s^\dagger(\omega \mathbf{x})\,|0\rangle \approx 0. \tag{6.103}$$

In summary, the use of leading and subleading Faddeev-Kulish states do not allow absorption and emission of on-shell soft gravitons at tree level.

6.4 Equivalence of Faddeev-Kulish amplitudes and traditional amplitudes

In this section, we show that the Faddeev-Kulish amplitude is equivalent to the infrared-finite part of traditional amplitudes constructed using Fock states, up to power-law type corrections in the soft energy scale Λ which is negligible by definition. Keeping both leading and subleading terms to first order in κ in the exponent of the dressing function W_g, we explicitly show that this equivalence is up to order Λ for radiation-less amplitudes in the case of scattering of a scalar from an external potential. In reference [58] only the infrared-finiteness was shown, keeping just the leading order term in the exponent of the dressing. Again, although we only derive the result explicitly for gravity, the derivation for QED is similar and the result also holds for QED amplitudes.

For simplicity we consider a $1 \to 1$ gravitational potential scattering between the dressed states $\|\mathbf{p}_i\rangle\!\rangle$ and $\langle\!\langle \mathbf{p}_f\|$ at one-loop order. Let us define the shorthand notation

$$P_i^{\mu\nu} \equiv \frac{\kappa}{2}\left(\frac{p_i^\mu p_i^\nu}{p_i \cdot k}\right), \quad Q_i^{\mu\nu} \equiv \frac{\kappa}{2}\left(-i\frac{p_i^\mu k_\rho J_i^{\rho\nu}}{p_i \cdot k}\right), \tag{6.104}$$

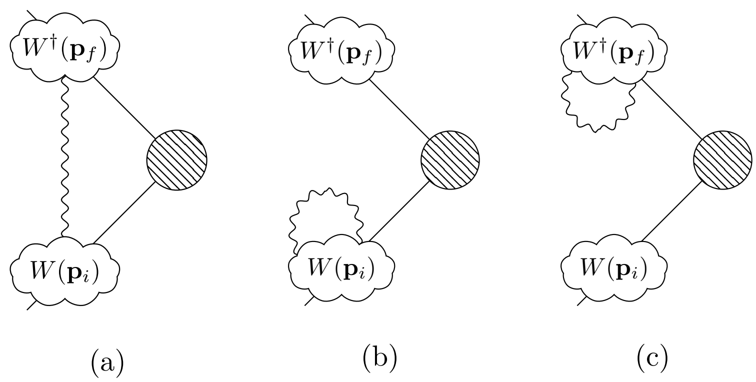


Figure 6.2: We consider the simple case of $1 \to 1$ gravitational potential scattering, where the incoming and outgoing momenta are $\mathbf{p}_i$ and $\mathbf{p}_f$, respectively. The figures illustrate different contributions to the FK amplitude of this process. Blob represents the internal diagram, including the gravitational potential.

and similarly $P_f^{\mu\nu}$ and $Q_f^{\mu\nu}$ corresponding to p_f. We also use the notation

$$\int \widetilde{d^3k} \equiv \int_{\lambda<\omega_k<\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k}, \tag{6.105}$$

where λ is the infrared cutoff and Λ is the soft energy scale. We can use these and (6.54) to write the dressed states as

$$\langle\!\langle \mathbf{p}_f \| = \langle \mathbf{p}_f | \, W_g^\dagger(\mathbf{p}_f) = \langle \mathbf{p}_f | \left(1 - \int \widetilde{d^3k}\, Q_f a\right) \exp\left\{-\int \widetilde{d^3k}\, P_f(a^\dagger - a)\right\}, \tag{6.106}$$

$$\| \mathbf{p}_i \rangle\!\rangle = W_g(\mathbf{p}_i)\, |\mathbf{p}_i\rangle = \exp\left\{\int \widetilde{d^3k}\, P_i(a^\dagger - a)\right\} \left(1 + \int \widetilde{d^3k}\, Q_i a^\dagger\right) |\mathbf{p}_i\rangle\,, \tag{6.107}$$

where concatenation implies contraction, for example,

$$P_i(a^\dagger - a) \equiv P_i^{\mu\nu}(a^\dagger_{\mu\nu} - a_{\mu\nu}). \tag{6.108}$$

First consider the contribution to the matrix element due to graviton exchange between dressings. There are two self-interactions of the dressings, each coming from $W_g(\mathbf{p}_i)$ and $W_g^\dagger(\mathbf{p}_f)$ (figures 6.2(b) and 6.2(c)), and one cross-interaction between $W_g(\mathbf{p}_i)$ and $W_g^\dagger(\mathbf{p}_f)$ (figure 6.2(a)). Using the Baker-Campbell-Hausdorff (BCH) formula,

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \qquad \text{if } [A,[A,B]] = [B,[A,B]] = 0, \tag{6.109}$$

we may write,

$$\exp\left\{-\int \widetilde{d^3k}\, P_f(a^\dagger - a)\right\} = \exp\left\{-\int \widetilde{d^3k}\, P_f a^\dagger\right\}\exp\left\{\int \widetilde{d^3k}\, P_f a\right\}\exp\left\{-\frac{1}{4}\int \widetilde{d^3k}\, P_f I P_f\right\} \tag{6.110}$$

and similar for the incoming dressing, where we used (see [9, 134] for example)

$$\left[a_{\mu\nu}(\mathbf{k}), a^\dagger_{\rho\sigma}(\mathbf{k}')\right] = \frac{1}{2} I_{\mu\nu\rho\sigma}(2\pi)^3(2\omega_k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{6.111}$$

$$I_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}, \tag{6.112}$$

and employed the notation $P_f I P_f \equiv P_f^{\mu\nu} I_{\mu\nu\rho\sigma} P_f^{\rho\sigma}$. Then to first order in κ,[5]

$$\langle\!\langle \mathbf{p}_f \| = \langle \mathbf{p}_f | \left(1 - \int \widetilde{d^3k}\, Q_f a + \int \widetilde{d^3k}\, P_f a - \frac{1}{4}\int \widetilde{d^3k}\, P_f I P_f\right), \tag{6.113}$$

$$\| \mathbf{p}_i \rangle\!\rangle = \left(\int \widetilde{d^3k}\, P_i a^\dagger + \int \widetilde{d^3k}\, Q_i a^\dagger - \frac{1}{4}\int \widetilde{d^3k}\, P_i I P_i\right) | \mathbf{p}_i \rangle. \tag{6.114}$$

The last terms on the RHS of (6.113) and (6.114) are the self-interaction contributions of the dressings to the matrix element,

$$-\frac{1}{4}\int \widetilde{d^3k}\, (P_f I P_f + P_i I P_i) \langle \mathbf{p}_f | \mathcal{S} | \mathbf{p}_i \rangle. \tag{6.115}$$

The cross-interaction between the two dressings (figure 6.2(a)) come from the contraction between the graviton operators of (6.113) and (6.114), which introduces the term

$$\frac{1}{2}\int \widetilde{d^3k}\, (P_f I P_i - Q_f I P_i + Q_i I P_f) \langle \mathbf{p}_f | \mathcal{S} | \mathbf{p}_i \rangle. \tag{6.116}$$

Now, let us consider the remaining contributions, namely the ones due to a dressing exchanging gravitons with either an external leg or an internal propagator, see figure 6.3. At the leading soft (divergent) order, graviton exchange with an internal propagator does not contribute, while each exchange with an external leg (figures 6.3(a)-(d)) induces a soft factor $\pm\eta P^{\mu\nu}$, where the $+$ ($-$) sign corresponds to emission (absorption) of the graviton, and $\eta = 1$ (-1) if the external leg is outgoing (incoming). At the next order, graviton exchanges with the internal diagram (figures 6.3(e) and 6.3(f)) induce one subleading soft factor $\eta Q^{\mu\nu} + \widetilde{Q}^{\mu\nu}$ for each external leg, where $\widetilde{Q}^{\mu\nu} = \mathcal{O}(\omega^0)$ is a subleading soft factor due to the graviton being off-shell; see [142] for example. Since there are two dressings and two external legs, at

[5] While we write the dressings to first order in κ, we keep the κ^2-order infrared-divergent term $\int \widetilde{d^3k}\, P_f I P_f$ since it is needed to cancel infrared divergence [9].

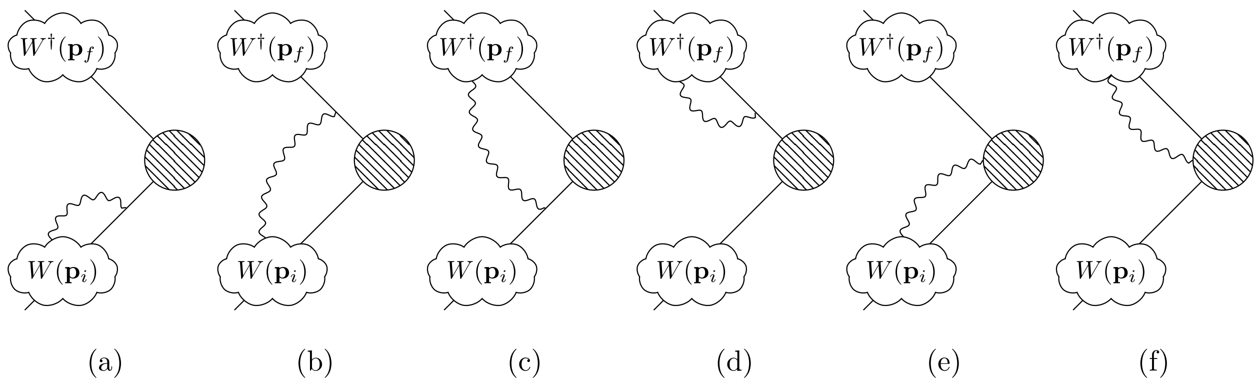


Figure 6.3: Contributions to the amplitude due to graviton exchange between a dressing and either an external leg or the internal diagram.

one-loop level this introduces eight terms, four from $W^\dagger(\mathbf{p}_f)$ which correspond to diagrams 6.3(c), 6.3(d) and 6.3(f),

$$\frac{1}{2}\int \widetilde{d^3k}\left[(P_f - Q_f)IP_f + (Q_f + \widetilde{Q}_f)IP_f - (P_f - Q_f)IP_i + (-Q_i + \widetilde{Q}_i)IP_f\right]\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle\,, \tag{6.117}$$

and four from $W(\mathbf{p}_i)$ corresponding to figures 6.3(a), 6.3(b) and 6.3(e),

$$\frac{1}{2}\int \widetilde{d^3k}\left[-(P_i + Q_i)IP_f + (Q_f + \widetilde{Q}_f)IP_i + (P_i + Q_i)IP_i + (-Q_i + \widetilde{Q}_i)IP_i\right]\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle\,. \tag{6.118}$$

Expressions (6.115), (6.116), (6.117) and (6.118) comprise the full contribution of the graviton interaction to the matrix element that involves Faddeev-Kulish dressings at one-loop. The net contribution to subleading order in soft momentum is

$$\frac{1}{4}\int \widetilde{d^3k}\left[(P_f - P_i)I(P_f - P_i) - 2(Q_i - \widetilde{Q}_i - \widetilde{Q}_f)IP_f + 2(Q_f + \widetilde{Q}_i + \widetilde{Q}_f)IP_i\right]\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle\,. \tag{6.119}$$

Now, the traditional amplitude $\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle$ is infrared divergent, owing to the presence of soft virtual graviton loops. The divergence can be factored out, such that at one loop [2,58],

$$\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle = \left(1 - \frac{1}{4}\int \widetilde{d^3k}\,(P_f - P_i)I(P_f - P_i)\right)\overline{\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle}, \tag{6.120}$$

where $\overline{\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle}$ is the infrared-finite matrix element where all virtual graviton loop momenta below the soft energy scale Λ has been removed. In figure 6.4, only diagrams 6.4(a)-(c)

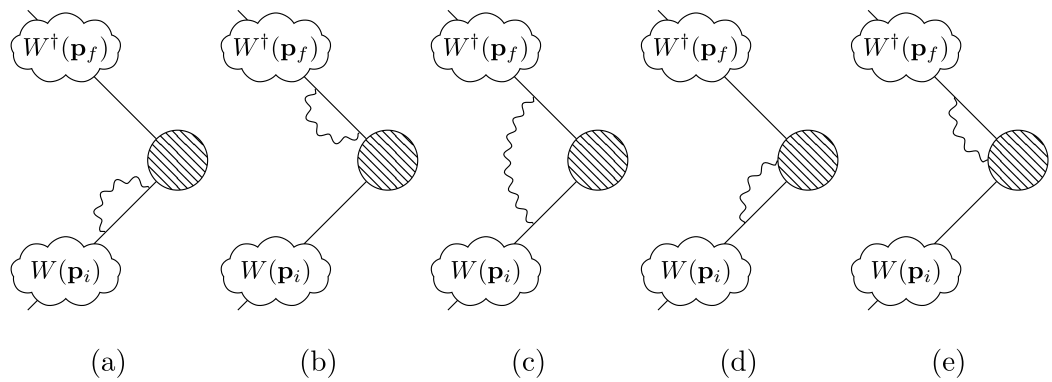


Figure 6.4: Graviton exchanges that do not involve dressings. The soft graviton loops exponentiate and factor out [2].

contribute to this term. We do not concern ourselves with subleading corrections to this factor coming from diagrams 6.4(a)-(e), since such corrections do not alter our conclusion.

Putting (6.119) and (6.120) together, at one-loop we observe that

$$\langle\!\langle \mathbf{p}_f \| \mathcal{S} \| \mathbf{p}_i \rangle\!\rangle = \left(1 + \frac{1}{2}\int \widetilde{d^3k}\ \left[(Q_f + \widetilde{Q}_i + \widetilde{Q}_f)IP_i - (Q_i - \widetilde{Q}_i - \widetilde{Q}_f)IP_f\right]\right)\overline{\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle}. \tag{6.121}$$

Recall that the integral spans only the soft sector $\lambda < \omega_k < \Lambda$. Since the integrand is of order $\mathcal{O}(\omega_k^0)$, after removing the infrared regulator λ the second term on the RHS of (6.121) becomes

$$\frac{1}{2}\int\limits_{0<\omega_k<\Lambda} \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left[(Q_f + \widetilde{Q}_i + \widetilde{Q}_f)IP_i - (Q_i - \widetilde{Q}_i - \widetilde{Q}_f)IP_f\right]\overline{\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle} = \mathcal{O}(\Lambda), \tag{6.122}$$

By definition, Λ is the soft scale and is therefore negligible compared to all other energy scales of significance. Therefore,

$$\langle\!\langle \mathbf{p}_f \| \mathcal{S} \| \mathbf{p}_i \rangle\!\rangle = \overline{\langle \mathbf{p}_f|\mathcal{S}|\mathbf{p}_i\rangle}. \tag{6.123}$$

That is, the Faddeev-Kulish amplitude dressed to subleading soft order is equivalent to the infrared-finite traditional matrix element. This extends the results of [9, 58] where the analyses were done only at the level of infrared-divergent terms.

Now, recall that our construction of dressings does not account for loop-corrections to the subleading soft theorems. As one can see from (6.31) and (6.73), such corrections introduce $\ln\omega$ terms to the integrand in the exponent of the dressings. Then, from (6.122) we observe that the corresponding corrections to dressed amplitudes involve infrared-finite integrals of the form

$$\lim_{\lambda\to 0}\int_{\lambda}^{\Lambda} d\omega_k \,\ln\omega_k = \mathcal{O}(\Lambda\ln\Lambda), \tag{6.124}$$

which becomes arbitrarily small as we decrease Λ, and therefore is negligible compared to other energy scales of our interest. It follows that the dressed amplitudes remain unaffected by loop corrections to the subleading soft theorem.

One should check that the equivalence (6.123) reproduces the inclusive cross section obtained by the Bloch-Nordsieck method [3], which sums over emissions of real gravitons with energies below the detector resolution $E_{\rm res}$. It was shown in [2] that the cross section for a transition $\alpha\to\beta$ accompanied by any number of real gravitons with total energy below $E_{\rm res}$ can be written as

$$\begin{aligned}\Gamma_{\alpha\to\beta}(\leq E_{\rm res}) = \frac{1}{\pi}\sum_{N=0}^{\infty}\int_{\lambda}^{E_{\rm res}} d\omega_1\cdots\int_{\lambda}^{E_{\rm res}} d\omega_N\int_{-\infty}^{\infty} d\sigma\frac{\sin(E_{\rm res}\sigma)}{\sigma}\\ \times\exp\left\{i\sigma\sum_{i=1}^{N}\omega_i\right\}\Gamma_{\alpha\to\beta}(\omega_1,\cdots,\omega_N),\end{aligned} \tag{6.125}$$

where $\Gamma_{\alpha\to\beta}(\omega_1,\cdots,\omega_N)$ denotes the cross section for emission of N gravitons with energies $\omega_1,\cdots,\omega_N$. The cross sections are now the norm-squared of the dressed amplitudes. In the previous section, we have shown in (6.102) that scattering amplitudes, and therefore cross sections, for processes that emit/absorb real gravitons with energy below the soft scale Λ are negligible. This has the effect of replacing the infrared cutoff λ in (6.125) with the soft energy scale Λ, which results in [2]

$$\Gamma_{\alpha\to\beta}(\leq E_{\rm res}) = \left(\frac{E_{\rm res}}{\Lambda}\right)^{B} b(B)\Gamma_{\alpha\to\beta}, \tag{6.126}$$

where $b(x) = 1-\frac{1}{12}\pi^2x^2+\cdots$ and

$$B = \frac{\kappa^2}{64\pi^2}\sum_{ij}\eta_i\eta_j\frac{m_im_j(1+\beta_{ij}^2)}{\beta_{ij}(1-\beta_{ij}^2)^{1/2}}\ln\left(\frac{1+\beta_{ij}}{1-\beta_{ij}}\right), \qquad \beta_{ij}^2\equiv 1-\frac{m_i^2m_j^2}{(p_i\cdot p_j)^2}. \tag{6.127}$$

$\Gamma_{\alpha\to\beta}$ is the cross section for $\alpha\to\beta$ without the undetectable real gravitons. In the original construction of [2] with Fock states, one factors out the soft loop contribution from $\Gamma_{\alpha\to\beta}$. In our construction with the dressed amplitude (6.123), we have no soft graviton loops (they have been canceled on account of the dressings) and thus may write

$$\Gamma_{\alpha\to\beta} = \Gamma^0_{\alpha\to\beta}, \tag{6.128}$$

where $\Gamma^0_{\alpha\to\beta}$ is the cross section computed excluding virtual graviton loop momenta below the soft scale Λ. Therefore,

$$\Gamma_{\alpha\to\beta}(\leq E_{\rm res}) = \left(\frac{E_{\rm res}}{\Lambda}\right)^B b(B)\Gamma^0_{\alpha\to\beta}, \tag{6.129}$$

which agrees with the inclusive cross section computed in [2] using Fock states via the Bloch-Nordsieck method.

6.5 Summary

In this chapter, using ideas similar to the one presented in [59], at leading order in the coupling we constructed the Faddeev-Kulish dressing for gravity and QED to subleading order in the soft energy expansion, see equations (6.58) and (6.90). We have shown that the dressed amplitudes are equivalent to the infrared-finite part of the traditional matrix elements, up to negligible power-law type corrections of the soft energy scale, see equation (6.102). We have also shown that, to first-order in the coupling constant, the FK state formalism does not allow soft radiation of photon and graviton, see equation (6.129). This supports the proposition that for the FK states soft particles carry information about the hard particles and vice versa [23, 127].

Chapter 7

Dual Soft Charges and ’t Hooft Line Dressings

7.1 Quantum field theory of electric and magnetic charges

In this chapter, we extend our formalism of the previous chapters to asymptotic symmetries of magnetic parity. We construct dressings charged under magnetic LGT in electrodynamics and under dual supertranslations in perturbative quantum gravity.

Let us begin with a brief review of the one-potential Lagrangian formulation of the quantum field theory of electric and magnetic charges by Blagojević and collaborators [118, 119]. This theory has been shown in [118,119] to be equivalent to two of the more well-known quantum field theories of electric and magnetic charges, namely the Hamiltonian formulation of Schwinger [143] and the Lagrangian formulation of Zwanziger [144]. Our interest in this theory is not to calculate amplitudes but to understand the structure of the three-point interactions at large times. This is all that is needed to construct the dressings. Once the dressings are constructed, the soft theorems and the coherent states follow straightforwardly.

The formulation is based on the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left[\gamma^\mu(i\partial_\mu - eA_\mu) - m_\psi\right]\psi + \bar{\chi}(\gamma^\mu i\partial_\mu - m_\chi)\chi, \tag{7.1}$$

where ψ and $\bar{\psi}$ (χ and $\bar{\chi}$) are the fermionic fields describing an electrically (magnetically) charged spin-1/2 particle, and the field strength tensor, following Dirac [145], is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}, \tag{7.2}$$

with A_μ the photon field and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric tensor. Here $G^{\rho\sigma}$ is a functional of χ and $\bar{\chi}$, defined as

$$G_{\mu\nu}(x) = \int d^4y\, h_\mu(x-y) j_\nu^g(y), \tag{7.3}$$

where $j_\nu^g = g\bar{\chi}\gamma_\nu\chi$ is the magnetic current, g is the magnetic charge, and $h_\mu(x)$ is any c-number function satisfying $\partial_\mu h^\mu(x) = -\delta(x)$. A convenient form of h_μ can be given in terms of an arbitrary real vector n^μ,

$$h_\mu(x) = -n_\mu(n\cdot\partial)^{-1}(x), \tag{7.4}$$

which comes in handy for concrete calculations. A coordinate space representation of $(n\cdot\partial)^{-1}(x)$ in terms of step functions can be found in [118]; we do not write it here since we'll mostly be working in momentum space.

This Lagrangian yields the following Maxwell's equations,

$$\partial^\mu F_{\mu\nu} = j_\nu^e, \tag{7.5}$$

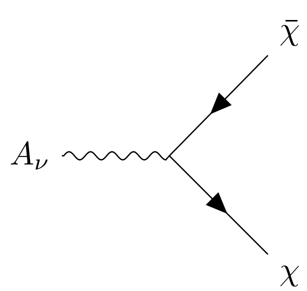
$$\partial^\mu(*F)_{\mu\nu} = j_\nu^g, \tag{7.6}$$

where $j_\nu^e = e\bar{\psi}\gamma_\nu\psi$ ($j_\nu^g = g\bar{\chi}\gamma_\nu\chi$) is the conserved electric (magnetic) current, and

$$(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{7.7}$$

is the tensor dual of $F_{\mu\nu}$.

One can read off the momentum space Feynman rules from the Lagrangian. Of interest to us is the 3-point vertex; all momenta are flowing into the vertex:



$$= -ig\epsilon_{\mu\nu\rho\sigma}k^\mu\frac{n^\rho\gamma^\sigma}{n\cdot k} \equiv -igA_{\nu\sigma}\gamma^\sigma, \tag{7.8}$$

Here we defined $A_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\frac{n^\rho k^\sigma}{n\cdot k}$. The divergence appearing when $n\cdot k = 0$ is spurious, see [146]. Other standard Feynman rules such as the standard propagators and the electron-photon vertex have been omitted and can be found in [146]. It is worth noting that while the Lagrangian is non-local in coordinate space, the Feynman rules are local in momentum space.

7.2 ’t Hooft line dressing

In this section, we construct the infrared-finite dressed state of an asymptotic magnetically charged particle, along the same lines as Faddeev and Kulish [8,9], and show that the dressing can be written as a ’t Hooft line dressing.

7.2.1 Faddeev-Kulish construction

In order to construct the dressed state, we write out the Lagrangian (7.1) as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu) - j_e^\mu A_\mu - \frac{1}{2}n^2 j_g^\mu \left(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}\right)(n\cdot\partial)^{-2} j_g^\nu \\ &- \epsilon^{\mu\nu\rho\sigma}(n\cdot\partial)^{-1}\partial_\mu A_\nu n_\rho j^g_\sigma + \bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi + \bar{\chi}(i\partial\!\!\!/ - m_\chi)\chi,\end{aligned} \tag{7.9}$$

where we recall that n^μ is an arbitrary 4-vector, $j_e^\mu = e\bar{\psi}\gamma^\mu\psi$ and $j_g^\mu = g\bar{\chi}\gamma^\mu\chi$. We observe that the normal-ordered interaction potential relevant to the χ-photon scattering is

$$V_{A\bar{\chi}\chi}(t) = g\epsilon^{\mu\nu\rho\sigma}\int d^3x\,(n\cdot\partial)^{-1} : \partial_\mu A_\nu n_\rho \bar{\chi}\gamma_\sigma\chi : . \tag{7.10}$$

The photon and χ fields have the standard mode expansions

$$A_\mu(x) = \int \widetilde{d^3k}\,\left(\epsilon_\mu^{\alpha *}(\mathbf{k})a_\alpha(\mathbf{k})e^{ik\cdot x} + \epsilon_\mu^\alpha(\mathbf{k})a_\alpha^\dagger(\mathbf{k})e^{-ik\cdot x}\right), \tag{7.11}$$

$$\chi(x) = \int \widetilde{d^3p}\,\left(b_s(\mathbf{p})u^s(\mathbf{p})e^{ip\cdot x} + d_s^\dagger(\mathbf{p})v^s(\mathbf{p})e^{-ip\cdot x}\right), \tag{7.12}$$

$$\bar{\chi}(x) = \int \widetilde{d^3p}\,\left(d_s(\mathbf{p})\bar{v}^s(\mathbf{p})e^{ip\cdot x} + b_s^\dagger(\mathbf{p})\bar{u}^s(\mathbf{p})e^{-ip\cdot x}\right), \tag{7.13}$$

where we employ the usual notation for the Lorentz invariant measures,

$$\widetilde{d^3k} \equiv \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}, \qquad \widetilde{d^3p} \equiv \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}, \tag{7.14}$$

$\epsilon_\mu^\alpha(\mathbf{k})$ is the photon polarization vector, $u^s(\mathbf{p})$ ($v^s(\mathbf{p})$) is the fermion (anti-fermion) spinor amplitudes, $\omega = |\mathbf{k}|$, $E_p = \sqrt{\mathbf{p}^2 + m_\chi^2}$, and the creation and annihilation operators satisfy the appropriate commutation/anti-commutation relations

$$\left[a_\alpha(\mathbf{k}), a^\dagger_\beta(\mathbf{k}')\right] = \delta_{\alpha\beta}(2\pi)^3(2\omega)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{7.15}$$

$$\left\{b_r(\mathbf{p}), b^\dagger_s(\mathbf{q})\right\} = \delta_{rs}(2\pi)^3(2E_p)\delta^{(3)}(\mathbf{p}-\mathbf{q}), \tag{7.16}$$

$$\left\{d_r(\mathbf{p}), d^\dagger_s(\mathbf{q})\right\} = \delta_{rs}(2\pi)^3(2E_p)\delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{7.17}$$

Plugging in the expansions to (7.10) and taking the large-time limit $|t| \to \infty$, we arrive at the asymptotic potential

$$V^{\mathrm{as}}_{A\bar{\chi}\chi}(t) = g\epsilon^{\mu\nu\rho\sigma}\int \widetilde{d^3k}\,\widetilde{d^3p}\,\frac{p_\nu}{E_p}\frac{n_\rho k_\sigma}{n\cdot k}\rho(\mathbf{p})\left(\epsilon^\alpha_\mu(\mathbf{k})a^\dagger_\alpha(\mathbf{k})e^{-i\frac{p\cdot k}{E_p}t} + \epsilon^{\alpha *}_\mu(\mathbf{k})a_\alpha(\mathbf{k})e^{i\frac{p\cdot k}{E_p}t}\right), \tag{7.18}$$

where $\rho(\mathbf{p}) = \sum_s\left(b^\dagger_s(\mathbf{p})b_s(\mathbf{p}) - d^\dagger_s(\mathbf{p})d_s(\mathbf{p})\right)$ is the number density operator of the magnetically charged particle. We note that this asymptotic form of the 3-point interaction is all we need from the original theory, (7.1), to construct dressings, coherent states and soft theorems. One could argue the form of this interaction from symmetry arguments in the spirit of an effective field theory. Following the same line of arguments as in the original Faddeev-Kulish construction [8], one obtains from this potential the asymptotic state of the magnetically charged particles,

$$\|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle\!\rangle = e^{\widetilde{R}}\left|\mathbf{p}_1, \ldots, \mathbf{p}_n\right\rangle, \tag{7.19}$$

with the dressing

$$\begin{aligned} e^{\widetilde{R}} &= \lim_{t\to\infty}\exp\left(i\int^t V^{\mathrm{as}}_{A\bar{\chi}\chi}(\tau)d\tau\right) &(7.20)\\ &= \exp\left\{-g\epsilon^{\mu\nu\rho\sigma}\int \widetilde{d^3k}\,\widetilde{d^3p}\,\frac{p_\nu}{p\cdot k}\frac{n_\rho k_\sigma}{n\cdot k}\rho(\mathbf{p})\phi(\omega)\left(\epsilon^\alpha_\mu(\mathbf{k})a^\dagger_\alpha(\mathbf{k}) - \epsilon^{\alpha *}_\mu(\mathbf{k})a_\alpha(\mathbf{k})\right)\right\} &(7.21)\\ &= \exp\left\{-g\int \widetilde{d^3k}\,\widetilde{d^3p}\,\frac{A^{\mu\nu}p_\nu}{p\cdot k}\rho(\mathbf{p})\phi(\omega)\left(\epsilon^\alpha_\mu(\mathbf{k})a^\dagger_\alpha(\mathbf{k}) - \epsilon^{\alpha *}_\mu(\mathbf{k})a_\alpha(\mathbf{k})\right)\right\}, &(7.22)\end{aligned}$$

where $A_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\frac{n^\rho k^\sigma}{n\cdot k}$. Note that the factors $\exp(\pm i(p\cdot k)t/E_p)$ in $V^{\mathrm{as}}_{A\bar{\chi}\chi}(t)$ at large t suppress $\omega > 0$ contributions to the integral by the Riemann-Lebesgue lemma, so we have replaced such factors with an infrared function $\phi(\omega)$ that only has support in a small neighborhood of $\omega = 0$ and satisfies $\phi(0) = 1$ to reflect this; see [8, 9] for instance. Due to the number density operator $\rho(\mathbf{p})$, the dressing decomposes into a product of single particle dressings

$e^{\widetilde{R}} = \prod_i e^{\widetilde{R}(p_i)}$, where

$$e^{\widetilde{R}(p)} = \exp\left\{ -g \int \widetilde{d^3k}\, \frac{A^{\mu\nu} p_\nu}{p \cdot k} \phi(\omega) \left(\epsilon^{\alpha}_{\mu}(\mathbf{k}) a^{\dagger}_{\alpha}(\mathbf{k}) - \epsilon^{\alpha *}_{\mu}(\mathbf{k}) a_{\alpha}(\mathbf{k}) \right) \right\}. \tag{7.23}$$

For physical states Ψ, the Gupta-Bleuler condition demands $k^\mu a_\mu(\mathbf{k}) |\Psi\rangle = 0$, where $a_\mu(\mathbf{k}) = \epsilon^{\alpha *}_{\mu}(\mathbf{k}) a_\alpha(\mathbf{k})$. The consistency of the dressing (7.23) with this condition boils down to the commutator

$$0 = \left[\frac{A^{\mu\nu} p_\nu}{p \cdot k} \left(a^{\dagger}_{\mu}(\mathbf{k}) - a_\mu(\mathbf{k}) \right), k'^{\rho} a_\rho(\mathbf{k}') \right] \tag{7.24}$$

$$= \frac{k^\mu A_{\mu\nu} p^\nu}{p \cdot k} (2\pi)^3 (2\omega) \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{7.25}$$

which is is automatically satisfied since $k^\mu A_{\mu\nu} = 0$ for any choice of n by antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$. Therefore, the dressing commutes with the Gupta-Bleuler condition. In the original construction of dressings in QED [8], Faddeev and Kulish introduced a vector c^μ into the dressing to make it compatible with gauge fixing; here we do not need such a treatment. It was recently shown by Hirai and Sugishita [141] that a careful BRST analysis removes the need for c^μ even in the Faddeev-Kulish construction. An analogous BRST analysis of the theory including magnetically charged particles is left for future investigation.

As the construction of the dressing (7.22) was fairly parallel to that of [8], it is natural to expect that it resolves the infrared divergence of the theory (7.1). To see this, let us consider a generalization of (7.1) to a theory containing dyons, which is fairly straightforward (see for instance [120]. In the generalized theory, dyons are dressed with the magnetic dressing (7.22) as well as the original Faddeev-Kulish dressing of QED [8], where the latter takes the form

$$e^{R} = \exp\left\{ -e \int \widetilde{d^3p}\, \widetilde{d^3k}\, \rho(\mathbf{p}) \phi(\omega) \frac{p^\mu}{p \cdot k} \left(a^{\dagger}_{\mu}(\mathbf{k}) - a_\mu(\mathbf{k}) \right) \right\}. \tag{7.26}$$

Since $\left[R, \widetilde{R}\right] = 0$, the dressing in a dyonic generalization of (7.1) takes the form

$$\exp\left\{ - \int \widetilde{d^3p}\, \widetilde{d^3k}\, \rho(\mathbf{p}) \phi(\omega) \left(e\eta_{\mu\nu} + g A_{\mu\nu} \right) \frac{p^\nu}{p \cdot k} \left(a^{\mu\dagger}(\mathbf{k}) - a^\mu(\mathbf{k}) \right) \right\}, \tag{7.27}$$

where $\rho(\mathbf{p})$ is now a number density operator of the dyon field. When acted on a single dyon state, this dressing reproduces the coherent state of Antunović and Senjanović [147] that was shown to resolve the infrared divergences, as expected. Thus, the dressings we have constructed ensure the infrared finiteness of any theory whose asymptotic three-point interaction is given by (7.18).

In the next subsection, we see that this dressing can be written as a 't Hooft line dressing along the timelike trajectory of an asymptotic particle with momentum p.

7.2.2 Dressing as 't Hooft line

We now re-derive the magnetic Faddeev-Kulish dressing (7.23) by considering a 't Hooft line operator. This is to be contrasted with the usual (electric) Faddeev-Kulish dressing [8, 92] having a Wilson line representation.

Consider a 't Hooft operator $\exp\left(ig\int_S *F\right)$ associated with a simple connected 2-dimensional surface S with boundary loop C. If we can write $*F = d\widetilde{A}$ for a vector field $\widetilde{A}$ in the absence of electrons, by Stoke's theorem, the 't Hooft operator becomes $\exp\left(ig\oint_C \widetilde{A}\right)$. Then, we can consider the field χ dressed with a 't Hooft line,

$$\exp\left(ig\int_\infty^x d\xi^\mu \widetilde{A}_\mu(\xi)\right)\chi(x). \tag{7.28}$$

For an asymptotic particle, this is possible and we see that the dressing of the asymptotic particle agrees with (7.23). We are interested in large time dynamics, so in what follows below we consider free equations of motion with $j_g = j_e = 0$.

To this end, we write the Minkowski metric in the retarded time coordinates,

$$ds^2 = -du^2 + 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \tag{7.29}$$

where $u = t - r$ is the retarded time suitable for describing the future null infinity $\mathcal{I}^+$, $z = e^{i\phi}\cot(\theta/2)$ is the complex angular coordinate, and $\gamma_{z\bar{z}} = 2/(1+z\bar{z})^2$ is the unit 2-sphere metric. The radiative mode of photon on $\mathcal{I}^+$ can be expanded as (7.11), where we sum over the two physical polarizations $\alpha = \pm$. In terms of the complex angular coordinates $(z_k, \bar{z}_k)$, the (t, x, y, z) components of the photon momentum k becomes

$$k^\mu = \frac{\omega}{1+z_k\bar{z}_k}\left(1+z_k\bar{z}_k, \bar{z}_k+z_k, i(\bar{z}_k - z_k), 1-z_k\bar{z}_k\right), \tag{7.30}$$

where $\omega = |\mathbf{k}|$. Then the two transverse polarization vectors can be defined as

$$\epsilon^{+\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}(\bar{z}_k, 1, -i, -\bar{z}_k), \qquad \epsilon^{-\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}(z_k, 1, i, -z_k). \tag{7.31}$$

The definition of $*F$ (7.7) and $*F = d\widetilde{A}$ implies

$$\partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu = \varepsilon_{\mu\nu}{}^{\rho\sigma} \partial_\rho A_\sigma. \tag{7.32}$$

The vector field $\widetilde{A}_\mu$ is essentially the photon field with a different polarization vector [32], so let us write the ansatz

$$\widetilde{A}_\mu = \int \widetilde{d^3k} \left(\widetilde{\epsilon}_\mu^{\alpha *}(\mathbf{k}) a_\alpha(\mathbf{k}) e^{ik\cdot x} + \widetilde{\epsilon}_\mu^{\alpha}(\mathbf{k}) a_\alpha^\dagger(\mathbf{k}) e^{-ik\cdot x} \right), \tag{7.33}$$

for some polarization vectors $\widetilde{\epsilon}_\mu^\alpha$. Then equation (7.32) boils down to

$$k_\mu \widetilde{\epsilon}_\nu^{\alpha *}(\mathbf{k}) - k_\nu \widetilde{\epsilon}_\mu^{\alpha *}(\mathbf{k}) = \varepsilon_{\mu\nu}{}^{\rho\sigma} k_\rho \epsilon_\sigma^{\alpha *}. \tag{7.34}$$

Now, let us make the choice

$$\widetilde{\epsilon}_\mu^\pm(\mathbf{k}) = -A_{\mu\nu} \epsilon_\pm^\nu(\mathbf{k}), \qquad A_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma} \frac{n^\rho k^\sigma}{n \cdot k}, \tag{7.35}$$

where $n^\mu \neq k^\mu$ is an arbitrary non-zero 4-vector of our choice. It is straightforward to check that this constitutes an infinite number of solutions for (7.34). An illuminating choice is $n^\mu = (1, 0, 0, -1)$, for which we obtain

$$A_{\mu\nu} = \begin{bmatrix} 0 & -\frac{i}{2}(\bar{z}_k - z_k) & \frac{1}{2}(\bar{z}_k + z_k) & 0 \\ \frac{i}{2}(\bar{z}_k - z_k) & 0 & -1 & \frac{i}{2}(\bar{z}_k - z_k) \\ -\frac{1}{2}(\bar{z}_k + z_k) & 1 & 0 & -\frac{1}{2}(\bar{z}_k + z_k) \\ 0 & -\frac{i}{2}(\bar{z}_k - z_k) & \frac{1}{2}(\bar{z}_k + z_k) & 0 \end{bmatrix}, \tag{7.36}$$

and accordingly, the polarization vectors become

$$\widetilde{\epsilon}^{+\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}(-i\bar{z}_k, -i, 1, i\bar{z}_k) = -i\epsilon^{+\mu}(\mathbf{k}), \tag{7.37}$$

$$\widetilde{\epsilon}^{-\mu}(\mathbf{k}) = \frac{1}{\sqrt{2}}(iz_k, i, -1, -iz_k) = i\epsilon^{-\mu}(\mathbf{k}). \tag{7.38}$$

One can see that this $\widetilde{\epsilon}^\pm$ can essentially be obtained by a $\frac{\pi}{2}$-rotation of $\epsilon^\pm$ in the complex plane, reflecting the electromagnetic duality $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$.

Having obtained a solution for $\widetilde{A}$, we now use the methods of Wilson line dressing construction [91, 92, 96] to derive the 't Hooft line dressing $\widetilde{W}(p)$. We first write the dressing

(7.28) along a path Γ at the asymptotic future,

$$\widetilde{W} = \exp\left\{ig \int_\Gamma d\xi^\mu \widetilde{A}_\mu(\xi)\right\} \tag{7.39}$$

Now we assert that Γ is the straight line geodesic of an asymptotic particle with momentum p, for which we may parametrize $\xi^\mu = \xi_0^\mu + \frac{p^\mu}{m_\chi}\tau$. Then, using (7.33) and (7.35),

$$\widetilde{W}(p) = \exp\left\{ig \int^t d\tau \, \frac{p^\mu}{m_\chi} \widetilde{A}_\mu\left(\xi_0 + \frac{p}{m_\chi}\tau\right)\right\} \tag{7.40}$$

$$= \exp\left\{-g \int \widetilde{d^3k}\, \frac{A^{\mu\nu} p_\nu}{p \cdot k} \left(\epsilon_\mu^\alpha(\mathbf{k}) a_\alpha^\dagger(\mathbf{k}) e^{ik\cdot(\xi_0 + \frac{p}{m_\chi}t)} - \epsilon_\mu^{\alpha*}(\mathbf{k}) a_\alpha(\mathbf{k}) e^{ik\cdot(\xi_0 + \frac{p}{m_\chi}t)}\right)\right\}, \tag{7.41}$$

where we used the boundary condition $\int^t d\tau e^{i\frac{k\cdot p}{m_\chi}\tau} = \frac{m_\chi}{ik\cdot p} e^{i\frac{k\cdot p}{m_\chi}t}$, see [8] for a discussion. For an asymptotic particle t diverges to infinity, so by virtue of the Riemann-Lebesgue lemma non-zero frequency modes do not contribute to the integral. To reflect this, we write

$$\widetilde{W}(p) = \exp\left\{-g \int \widetilde{d^3k}\, \frac{p^\mu}{p \cdot k} \phi(\omega) \left(\widetilde{\epsilon}_\mu^\alpha a_\alpha^\dagger(\mathbf{k}) - \widetilde{\epsilon}_\mu^{\alpha*} a_\alpha(\mathbf{k})\right)\right\}, \tag{7.42}$$

where $\phi(\omega)$ is any smooth infrared function [8, 9] that satisfies $\phi(0) = 1$ and has support only in a small neighborhood of $\omega = 0$.

One can immediately see that this is the dressing $e^{\widetilde{R}(p)}$ we obtained in (7.23) from the asymptotic $A\bar{\chi}\chi$ interaction potential,

$$\widetilde{W}(p) = e^{\widetilde{R}(p)}. \tag{7.43}$$

Therefore the magnetic Faddeev-Kulish dressing may be written as a 't Hooft line dressing. This is analogous to the electric counterpart associated with Wilson line dressing [94]. These results further justify the choice of an effective large-time interaction of the form (7.18), irrespective of the validity of the full theory (7.1).

7.2.3 Soft LGT charges

We now show that the 't Hooft line dressing (7.42) carries a definite soft magnetic LGT charge. The soft part of the magnetic LGT charge on $\mathcal{I}^+$ has the expression [32] (up to different normalization)

$$\widetilde{Q}_\varepsilon^+ = i \int d^2z \left(\partial_z \varepsilon(z, \bar{z}) F_{\bar{z}}^+ - \partial_{\bar{z}} \varepsilon(z, \bar{z}) F_z^+\right), \tag{7.44}$$

where $\varepsilon(z, \bar{z})$ is a 2-sphere function parametrizing the LGT, which we assume does not introduce poles or branch cuts. The soft photon operator F_z^+ is defined as

$$F_z^+ = \int_{-\infty}^{\infty} du\, F_{uz}^{(0)} = \lim_{r\to\infty} \int_{-\infty}^{\infty} du\, \partial_u A_z(u, r, z, \bar{z}). \tag{7.45}$$

The field A_z in the large r limit takes the form (see for example Appendix A of [18])

$$\lim_{r\to\infty} A_z(u, r, z, \bar{z}) = -\frac{i}{8\pi^2}\sqrt{\gamma_{z\bar{z}}} \int_0^{\infty} d\omega \left(a_+(\omega \mathbf{x}_z) e^{-i\omega u} - a_-^{\dagger}(\omega \mathbf{x}_z) e^{i\omega u}\right) \tag{7.46}$$

where $\mathbf{x}_z$ is the unit 3-vector corresponding to the direction $(z, \bar{z})$, with components

$$\mathbf{x}_z = \left(\frac{\bar{z}+z}{1+z\bar{z}}, \frac{i(\bar{z}-z)}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right). \tag{7.47}$$

Substituting (7.46) into (7.44) and using $\int_{-\infty}^{\infty} du\, \partial_u e^{\pm i\omega u} = \pm 2\pi i \omega \delta(\omega)$, we obtain

$$\widetilde{Q}_\varepsilon^+ = \frac{i}{4\pi} \int d\omega d^2 z \sqrt{\gamma_{z\bar{z}}}\, \omega\delta(\omega) \left[\partial_{\bar{z}}\varepsilon(z, \bar{z}) \left(a_-^{\dagger}(\omega \mathbf{x}_z) + a_+(\omega \mathbf{x}_z)\right) - \text{h.c.}\right], \tag{7.48}$$

where the presence of the delta function shows that only zero-frequency photon operators contribute (hence the soft charge).

Using this expression and the canonical commutation relation (7.15), one can directly compute the commutator of the charge and the operator $\widetilde{R}(p)$ of (7.43), i.e.,

$$\left[\widetilde{Q}_\varepsilon^+, \widetilde{R}(p)\right] = -\frac{ig}{4\pi} \int d^2 z \sqrt{\gamma_{z\bar{z}}} \frac{A^{\mu\nu} p_\nu}{p \cdot \hat{k}} \left(\epsilon_\mu^+(\mathbf{x}_z) \partial_{\bar{z}}\varepsilon - \epsilon_\mu^-(\mathbf{x}_z) \partial_z \varepsilon\right), \tag{7.49}$$

where $k^\mu = \omega \hat{k}^\mu$, and we have used the convention [18, 58]

$$\int_0^{\infty} d\omega\, \delta(\omega) f(\omega) = \frac{1}{2} f(0). \tag{7.50}$$

By partial integration, this can be put in the form

$$\left[\widetilde{Q}_\varepsilon^+, \widetilde{R}(p)\right] = \frac{ig}{4\pi} \int d^2 z\, \varepsilon(z, \bar{z}) \left[\partial_{\bar{z}} \left(\sqrt{\gamma_{z\bar{z}}} \frac{\epsilon^+ \cdot A \cdot p}{p \cdot \hat{k}}\right) - \partial_z \left(\sqrt{\gamma_{z\bar{z}}} \frac{\epsilon^- \cdot A \cdot p}{p \cdot \hat{k}}\right)\right] \tag{7.51}$$

$$= -\frac{g}{2\pi} \int d^2 z\, \varepsilon(z, \bar{z})\ \operatorname{Im} \left[\partial_{\bar{z}} \left(\sqrt{\gamma_{z\bar{z}}} \frac{\epsilon^+ \cdot A \cdot p}{p \cdot \hat{k}}\right)\right], \tag{7.52}$$

where we used the notation $\epsilon^+ \cdot A \cdot p = \epsilon^+_\mu A^{\mu\nu} p_\nu$. Using the identity,

$$\partial_{\bar{z}} \left(\sqrt{\gamma_{z\bar{z}}} \, \frac{\epsilon^+ \cdot A \cdot p}{p \cdot \hat{k}} \right) = -\frac{i}{2} \gamma_{z\bar{z}} \left(\frac{n^2}{(n \cdot \hat{k})^2} - \frac{p^2}{(p \cdot \hat{k})^2} \right), \tag{7.53}$$

which shows that the expression of (7.52) in square brackets is purely imaginary for all real vectors n^μ, we obtain the commutator

$$\left[\widetilde{Q}^+_\varepsilon, \widetilde{R}(p) \right] = \frac{g}{4\pi} \int d^2 z \, \gamma_{z\bar{z}} \, \varepsilon(z, \bar{z}) \left(\frac{m_\chi^2}{(p \cdot \hat{k})^2} + \frac{n^2}{(n \cdot \hat{k})^2} \right), \tag{7.54}$$

where we have used $p^2 + m_\chi^2 = 0$. It follows that the 't Hooft line dressing $\widetilde{W}(p) = \exp \widetilde{R}(p)$ carries a definite soft magnetic LGT charge parametrized by the momentum p,

$$\left[\widetilde{Q}^+_\varepsilon, \widetilde{W}(p) \right] = \frac{g}{4\pi} \int d^2 z \, \gamma_{z\bar{z}} \, \varepsilon(z, \bar{z}) \left(\frac{m_\chi^2}{(p \cdot \hat{k})^2} + \frac{n^2}{(n \cdot \hat{k})^2} \right) \widetilde{W}(p). \tag{7.55}$$

It is worth noting that while the charge eigenvalue in (7.55) has an n-dependent term, this term does not interfere with magnetic charge conservation since it takes the form $g \times (\text{const})$, and $\sum g_{\text{in}} = \sum g_{\text{out}}$ for a scattering process. This term acts as a constant shift in the soft magnetic LGT charge of the state and is unmeasurable.

On the other hand, the 't Hooft line dressing does not carry soft electric LGT charge. To see this, we note that the soft electric charge takes the form [32],

$$\begin{aligned} Q^+_\varepsilon &= -\int d^2 z \left(\partial_z \varepsilon(z, \bar{z}) F^+_{\bar{z}} + \partial_{\bar{z}} \varepsilon(z, \bar{z}) F^+_z \right) & (7.56) \\ &= \frac{1}{4\pi} \int d\omega d^2 z \sqrt{\gamma_{z\bar{z}}} \, \omega \delta(\omega) \left[\partial_{\bar{z}} \varepsilon(z, \bar{z}) \left(a^\dagger_-(\omega \mathbf{x}_z) + a_+(\omega \mathbf{x}_z) \right) + \text{h.c.} \right]. & (7.57) \end{aligned}$$

A computation similar to the magnetic case shows that the dressing's electric LGT charge involves the real part of the expression (7.53),

$$\left[Q^+_\varepsilon, \widetilde{W}(p) \right] = \frac{g}{2\pi} \int d^2 z \, \varepsilon(z, \bar{z}) \;\; \text{Re} \left[\partial_{\bar{z}} \left(\sqrt{\gamma_{z\bar{z}}} \, \frac{\epsilon^+ \cdot A \cdot p}{p \cdot \hat{k}} \right) \right] \widetilde{W}(p) = 0 \tag{7.58}$$

which vanishes identically.

This is to be contrasted with the electric dressing of QED (Wilson line dressing) $W(p)$, which takes the form [8]

$$W(p) = \exp \left\{ -e \int \widetilde{d^3 k} \, \frac{p^\mu}{p \cdot k} \phi(\omega) \left(\epsilon^\alpha_\mu a^\dagger_\alpha(\mathbf{k}) - \epsilon^{\alpha *}_\mu a_\alpha(\mathbf{k}) \right) \right\}. \tag{7.59}$$

One can show the identity,

$$\partial_{\bar{z}}\left(\sqrt{\gamma_{z\bar{z}}}\,\frac{p\cdot\epsilon^{+}}{p\cdot\hat{k}}\right)=-\frac{1}{2}\gamma_{z\bar{z}}\frac{p^{2}}{(p\cdot\hat{k})^{2}}, \tag{7.60}$$

to obtain the following results,

$$\begin{aligned}
\left[Q^{+}_{\varepsilon},W(p)\right] &= \frac{e}{2\pi}\int d^{2}z\,\varepsilon(z,\bar{z})\ \mathrm{Re}\left[\partial_{\bar{z}}\left(\sqrt{\gamma_{z\bar{z}}}\,\frac{p\cdot\epsilon^{+}}{p\cdot\hat{k}}\right)\right]W(p) && (7.61)\\
&= \frac{e}{4\pi}\int d^{2}z\,\gamma_{z\bar{z}}\,\varepsilon(z,\bar{z})\frac{m_{\psi}^{2}}{(p\cdot\hat{k})^{2}}W(p), && (7.62)\\
\left[\widetilde{Q}^{+}_{\varepsilon},W(p)\right] &= -\frac{e}{2\pi}\int d^{2}z\,\varepsilon(z,\bar{z})\ \mathrm{Im}\left[\partial_{\bar{z}}\left(\sqrt{\gamma_{z\bar{z}}}\,\frac{p\cdot\epsilon^{+}}{p\cdot\hat{k}}\right)\right]W(p) && (7.63)\\
&= 0. && (7.64)
\end{aligned}$$

Here we have used $p^2 + m_\psi^2 = 0$, with m_ψ the mass of the electrically charged field ψ. Therefore, the Faddeev-Kulish dressing of QED carries only carries a non-zero electric LGT charge. The duality between the electric and the magnetic charges and their dressings is now manifest.

Although we have constructed the dressings in a theory that has magnetically charged particles, we know from [117] that one can retain the electromagnetic duality on the boundary even without bulk degrees of freedom carrying magnetic charge. Accordingly, we can consider the operators that can be obtained by replacing the momentum p^μ in the dressing (7.42) with a constant vector C^μ:

$$\widetilde{W}(C)=\exp\left\{-g\int\widetilde{d^{3}k}\,\frac{C^{\mu}}{C\cdot k}\phi(\omega)\left(\widetilde{\epsilon}^{\alpha}_{\mu}a^{\dagger}_{\alpha}(\mathbf{k})-\widetilde{\epsilon}^{\alpha*}_{\mu}a_{\alpha}(\mathbf{k})\right)\right\}. \tag{7.65}$$

From our construction, we can see that these are 't Hooft line operators along a straight line path at the future null infinity $\mathcal{I}^+$, whose direction is given by C^μ. The vector C^μ can be understood as a parameter that encodes the way soft magnetic charge is distributed over the sphere. The 't Hooft line operators are charged under magnetic LGT and neutral under electric LGT, with a constant charge:

$$\begin{aligned}
\left[\widetilde{Q}^{+}_{\varepsilon},\widetilde{W}(C)\right] &= \frac{g}{4\pi}\int d^{2}z\,\gamma_{z\bar{z}}\,\varepsilon(z,\bar{z})\left(\frac{n^{2}}{(n\cdot\hat{k})^{2}}-\frac{C^{2}}{(C\cdot\hat{k})^{2}}\right)\widetilde{W}(C), && (7.66)\\
\left[Q^{+}_{\varepsilon},\widetilde{W}(C)\right] &= 0. && (7.67)
\end{aligned}$$

Such operators can be used to translate a vacuum to another vacuum carrying a different soft magnetic LGT charge.

7.2.4 Remarks on holomorphic potential and soft theorem

We have seen that there is an infinite number of choices for the dual field $\widetilde{A}$, parametrized by the 4-vector n^μ. The relation between $\widetilde{A}_\mu$ and A_μ can be obtained from the relation between polarization vectors (7.35),

$$\widetilde{A}_\mu = -\epsilon_{\mu\nu\rho\sigma}(n\cdot\partial)^{-1}n^\rho\partial^\sigma A^\nu. \tag{7.68}$$

While this involves unpleasant differential operators, we can expect a more desirable relation between the holomorphic/antiholomorphic potentials at $r\to\infty$, since in that limit the angular position $(z,\bar{z})$ is identified with the angular direction of outgoing momentum $(z_k,\bar{z}_k)$. By a derivation analogous to (7.46), we can obtain

$$\begin{aligned}\widetilde{A}_\mu(u,z,\bar{z}) &\equiv \lim_{r\to\infty}\widetilde{A}_\mu(u,r,z,\bar{z}) &(7.69)\\ &= -\frac{i}{8\pi^2 r}\int_0^\infty d\omega\left(\widetilde{\epsilon}^{\alpha *}_\mu(\mathbf{x}_z)a_\alpha(\omega\mathbf{x}_z)e^{-i\omega u}-\widetilde{\epsilon}^{\alpha}_\mu(\mathbf{x}_z)a^\dagger_\alpha(\omega\mathbf{x}_z)e^{i\omega u}\right). &(7.70)\end{aligned}$$

It is straightforward to show that while the dual polarization vectors $\widetilde{\epsilon}^{\pm}_\mu = -A_{\mu\nu}\epsilon^{\pm\nu}$ are complicated functions of $n^\mu=(n^0,n^1,n^2,n^3)$ with components

$$\widetilde{\epsilon}^{+0} = \frac{i\left(n^0\left(\bar{z}^2-1\right)+in^2\left(\bar{z}^2+1\right)+2n^3\bar{z}\right)}{\sqrt{2}(n^0z\bar{z}+n^0-n^1z-n^1\bar{z}+in^2z-in^2\bar{z}+n^3(z\bar{z}-1))} \tag{7.71}$$

$$\widetilde{\epsilon}^{+1} = \frac{i\left(n^0\left(\bar{z}^2-1\right)+2in^2\bar{z}+n^3\bar{z}^2+n^3\right)}{\sqrt{2}(n^0z\bar{z}+n^0-n^1z-n^1\bar{z}+in^2z-in^2\bar{z}+n^3(z\bar{z}-1))} \tag{7.72}$$

$$\widetilde{\epsilon}^{+2} = \frac{-n^0\left(\bar{z}^2+1\right)+2n^1\bar{z}-n^3\left(\bar{z}^2-1\right)}{\sqrt{2}(n^0z\bar{z}+n^0-n^1z-n^1\bar{z}+in^2z-in^2\bar{z}+n^3(z\bar{z}-1))} \tag{7.73}$$

$$\widetilde{\epsilon}^{+3} = \frac{2in^0\bar{z}-in^1\left(\bar{z}^2+1\right)+n^2\left(\bar{z}^2-1\right)}{\sqrt{2}(n^0z\bar{z}+n^0-n^1z-n^1\bar{z}+in^2z-in^2\bar{z}+n^3(z\bar{z}-1))}, \tag{7.74}$$

and $\widetilde{\epsilon}^{-\mu}=(\widetilde{\epsilon}^{+\mu})^*$, their $z/\bar{z}$ components $\widetilde{\epsilon}^+_z=\widetilde{\epsilon}^-_{\bar{z}}=0$, $\widetilde{\epsilon}^-_z=-\widetilde{\epsilon}^+_{\bar{z}}=ir\sqrt{\gamma_{z\bar{z}}}$ do not depend on n^μ. They satisfy $\widetilde{\epsilon}^+_{z/\bar{z}}=-i\epsilon^+_{z/\bar{z}}$, $\widetilde{\epsilon}^-_{z/\bar{z}}=i\epsilon^-_{z/\bar{z}}$, which implies the relation

$$\widetilde{A}_z(u,z,\bar{z}) = iA_z(u,z,\bar{z}), \tag{7.75}$$

$$\widetilde{A}_{\bar{z}}(u,z,\bar{z}) = -iA_{\bar{z}}(u,z,\bar{z}), \tag{7.76}$$

in agreement with the identification made in [32].[1] Therefore for any n, we arrive at the same complexified transformations that act on the holomorphic potential A_z,

$$(\delta_\varepsilon + i\widetilde{\delta}_\varepsilon)A_z = 2\partial_z\varepsilon, \tag{7.77}$$

$$(\delta_\varepsilon + i\widetilde{\delta}_\varepsilon)\widetilde{A}_z = 0. \tag{7.78}$$

As expected, the unphysical freedom to choose n^μ does not affect the results.

Also, at the end of section 7.2.1 we have seen that, in a dyonic generalization of the theory, the dressing takes the form (7.27). This can written using (7.35) as

$$\exp\left\{-\int \widetilde{d^3p}\,\widetilde{d^3k}\,\rho(\mathbf{p})\phi(\omega)\left(\frac{p\cdot(e\epsilon^\alpha + g\widetilde{\epsilon}^\alpha)}{p\cdot k}a^\dagger_\alpha(\mathbf{k}) - \text{h.c.}\right)\right\}. \tag{7.79}$$

This expression is closely related to the soft theorems and ward identities, since one can use the latter to reconstruct dressed states as LGT charge eigenstates [19,59,97] (see also [18,58]). In particular, the coefficient of the creation operator $a^\dagger_\alpha(\mathbf{k})$ is tied to the soft factor. In [32], it was conjectured that the leading soft factor

$$\sum_{j\in\text{out}}\frac{p_j\cdot(e_j\epsilon^\alpha + g_j\widetilde{\epsilon}^\alpha)}{p_j\cdot k} - \sum_{i\in\text{out}}\frac{p_i\cdot(e_i\epsilon^\alpha + g_i\widetilde{\epsilon}^\alpha)}{p_i\cdot k} \tag{7.80}$$

is exact for all abelian gauge theories. We emphasize that this result, like the dressing we have constructed, depends only on the asymptotic form of the three-point interaction, (7.18). The full content of the theory (7.1) is not needed.

7.3 Gravitational ’t Hooft line dressing

In this section, we consider perturbative quantum gravity in asymptotically flat spacetimes and investigate how the construction of ’t Hooft line dressings can be extended to the gravitational context. Since we do not have a Lagrangian field theory to guide us, we proceed by analogy with electromagnetism.

[1] The factors $\frac{4\pi}{e^2}$ in [32] are due to the different normalizations of the gauge fields.

7.3.1 Preliminaries

Consider the asymptotically flat metric on $\mathcal{I}^+$ in Bondi coordinates [1]

$$\begin{aligned} ds^2 = &-du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dz^2d\bar{z}^2 \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 - 2U_z dudz - 2U_{\bar{z}}dud\bar{z} + \cdots , \end{aligned} \tag{7.81}$$

where the first line is the Minkowski metric, $m_B(u, z, \bar{z})$ is the Bondi mass aspect, $C_{zz}(u, z, \bar{z})$ is the radiative mode and $U_z = \frac{1}{2}D^z C_{zz}$. Here D_z denotes covariant derivative on the 2-sphere (i.e. with respect to $\gamma_{z\bar{z}}$).

Let us define the graviton field $h_{\mu\nu}(x)$ to be

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \qquad \kappa^2 = 32\pi G. \tag{7.82}$$

This relates the radiative gravitons on $\mathcal{I}^+$ to the metric (7.81). These gravitons have the mode expansion

$$h_{\mu\nu}(x) = \int \widetilde{d^3k} \left(\epsilon^{\alpha *}_{\mu\nu}(\mathbf{k}) a_\alpha(\mathbf{k}) e^{ik\cdot x} + \epsilon^{\alpha}_{\mu\nu}(\mathbf{k}) a^\dagger_\alpha(\mathbf{k}) e^{-ik\cdot x} \right), \tag{7.83}$$

where we demand the canonical commutation relation

$$\left[a_\alpha(\mathbf{k}), a^\dagger_\beta(\mathbf{k}')\right] = \delta_{\alpha\beta}(2\pi)^3(2\omega)\delta^{(2)}(\mathbf{k} - \mathbf{k}'). \tag{7.84}$$

We are implicitly summing over the physical polarizations $\alpha = \pm$. Parametrizing graviton momentum k^μ as in (7.30), the graviton polarization vector $\epsilon^{\pm}_{\mu\nu}(\mathbf{k})$ can be written in terms of the two photon polarization vectors (7.31) as $\epsilon^{\pm}_{\mu\nu}(\mathbf{k}) = \epsilon^{\pm}_{\mu}(\mathbf{k})\epsilon^{\pm}_{\nu}(\mathbf{k})$.

7.3.2 Construction of dressing

In general relativity, we do not have the magnetic counterpart to the local source term $h_{\mu\nu}T^{\mu\nu}$, so we do not have bulk degrees of freedom that carry magnetic BMS charge. However, from [117], we know that the electromagnetic duality can still be retained on the boundary in this case. Following this perspective, our goal in this section is to obtain generic operators charged under dual supertranslations. We achieve this by analogy with electromagnetism: we first construct gravitational 't Hooft line dressings of particles, and then replace the particle momentum with a constant vector as in (7.65).

The gravitational Wilson line along a curve Γ takes the form [96]

$$\exp\left(-im_0\frac{\kappa}{2}\int_\Gamma dx^\mu h_{\mu\nu}\frac{dx^\nu}{d\tau}\right), \tag{7.85}$$

where m_0 is a parameter with dimension of mass that is possibly different from the particle mass m. This is basically the Wilson line of QED with the replacement $eA_\mu \to -m(\kappa/2)h_{\mu\nu}dx^\nu/d\tau$. For an asymptotic particle the trajectory is a straight line, so $mdx^\nu/d\tau = p^\nu$ is a constant vector. This implies that the gravitational 't Hooft line dressing can be obtained in the same way as in QED, by the replacement $\epsilon_{\mu\nu} \to \widetilde{\epsilon}_{\mu\nu}$, where

$$\widetilde{\epsilon}^{\pm}_{\mu\nu}(\mathbf{k}) = \widetilde{\epsilon}^{\pm}_{\mu}(\mathbf{k})\epsilon^{\pm}_{\nu}(\mathbf{k}). \tag{7.86}$$

Here $\widetilde{\epsilon}^{\pm}(\mathbf{k})$ is given by (7.35). Analogous to the case of QED, by taking Γ to be the trajectory of an asymptotic particle with momentum p, the expression (7.85) becomes the gravitational Wilson line dressing $W_g(p)$,

$$W_g(p) = \exp\left\{\frac{\kappa}{2}\int \widetilde{d^3k}\,\frac{p^\mu p^\nu}{p\cdot k}\phi(\omega)\left(\epsilon^{\alpha}_{\mu\nu}(\mathbf{k})a^\dagger_\alpha(\mathbf{k}) - \epsilon^{\alpha*}_{\mu\nu}(\mathbf{k})a_\alpha(\mathbf{k})\right)\right\}, \tag{7.87}$$

where $\phi(\omega)$ is the infrared function. We obtain the gravitational 't Hooft line $\widetilde{W}_g(p) = \exp\widetilde{R}_g(p)$ dressing by the replacement $\epsilon_{\mu\nu} \to \widetilde{\epsilon}_{\mu\nu}$, which yields

$$\widetilde{W}_g(p) = \exp\left\{-\frac{\kappa m_0}{2m}\int \widetilde{d^3k}\,\phi(\omega)\left(\frac{(p\cdot A\cdot\epsilon^\alpha)(p\cdot\epsilon^\alpha)}{p\cdot k}a^\dagger_\alpha(\mathbf{k}) - \text{h.c.}\right)\right\}, \tag{7.88}$$

where we denote $p\cdot A\cdot\epsilon^\alpha = p_\mu A^{\mu\nu}\epsilon^\alpha_\nu$.

One can think of the two dressings

$$W_g(p) \qquad \text{and} \qquad \widetilde{W}_g(p) \tag{7.89}$$

to be realizing the analog of electromagnetic duality of Freidel and Pranzetti [117] *on the boundary*. Unlike electromagnetism, where we have the bulk duality between $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$, here in gravity this duality is not realized in the bulk. Therefore, we must make a departure from the notion of "dressing" magnetically charged particles. In this sense, we define p not as some particle momentum but rather as a general 4-vector, which we later see parametrizes how the dual supertranslation charge of $\widetilde{W}_g$ is distributed over the sphere.

Unlike the case of photons and Lorentz gauge, in gravity we need more work to show that the dressed states exist in de Donder gauge. The physical states Ψ of the theory are the ones that satisfy the Gupta-Bleuler condition implementing the de Donder gauge,

$$\left(k^\mu a_{\mu\nu}(\mathbf{k}) - \frac{1}{2}k_\nu a^\mu{}_\mu(\mathbf{k})\right)|\Psi\rangle\,. \tag{7.90}$$

It is straightforward to see that the commutator,

$$\left[\frac{A^{\alpha\rho}p_\rho p^\beta}{p\cdot k}\left(a^\dagger_{\alpha\beta}(\mathbf{k}) - a_{\alpha\beta}(\mathbf{k})\right), k'^\mu a_{\mu\nu}(\mathbf{k}') - \frac{1}{2}k'_\nu a^\mu{}_\mu(\mathbf{k}')\right] = \frac{1}{2}p^\mu A_{\mu\nu}(2\pi)^3(2\omega)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{7.91}$$

does not vanish, which implies that the dressing in its current form is incompatible with de Donder gauge. Therefore, just as for the Wilson line dressing [9, 58], we must introduce a symmetric tensor $\widetilde{c}_{\mu\nu}(p,k)$ to fix this. Let us re-define the dressing to be

$$\widetilde{W}_g(p) = \exp\left\{\frac{\kappa m_0}{2m}\int \widetilde{d^3k}\,\phi(\omega)\left(\frac{A^{\mu\rho}p_\rho p^\nu}{p\cdot k} + \frac{1}{\omega}\widetilde{c}^{\mu\nu}(p,k)\right)\left(a^\dagger_{\mu\nu}(\mathbf{k}) - a_{\mu\nu}(\mathbf{k})\right)\right\}. \tag{7.92}$$

It was shown in [58] that, in order for this correction to just be a unitary transformation of (7.88) and not introduce additional singularities, we require the condition

$$\widetilde{c}_{\mu\nu}(p',k)I^{\mu\nu\rho\sigma}\widetilde{c}_{\rho\sigma}(p,k) = O(k) \qquad \text{for all } p, p', \tag{7.93}$$

where $I_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}$. For the states to be well-defined under the Gupta-Bleuler condition, we require

$$0 = \left[\left(\frac{A^{\alpha\rho}p_\rho p^\beta}{p\cdot k} + \frac{\widetilde{c}^{\alpha\beta}}{\omega}\right)\left(a^\dagger_{\alpha\beta}(\mathbf{k}) - a_{\alpha\beta}(\mathbf{k})\right), k'^\mu a_{\mu\nu}(\mathbf{k}') - \frac{1}{2}k'_\nu a^\mu{}_\mu(\mathbf{k}')\right] \tag{7.94}$$

$$= \left(p^\mu A_{\mu\nu} - \frac{2k^\mu \widetilde{c}_{\mu\nu}}{\omega}\right)\frac{1}{2}(2\pi)^3(2\omega)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{7.95}$$

which translates to the condition

$$k^\mu \widetilde{c}_{\mu\nu} = \frac{\omega}{2}p^\mu A_{\mu\nu}. \tag{7.96}$$

The two conditions (7.93) and (7.96) are exactly the ones that we encounter for the Wilson line dressing [58] with the replacement

$$p_\nu \;\to\; \widetilde{p}_\nu \equiv -\frac{1}{2}p^\mu A_{\mu\nu}. \tag{7.97}$$

This implies that the solution is exactly that of [58] with the same replacement:

$$\widetilde{c}_{\mu\nu}(p,k) = \frac{\omega}{q\cdot k}\left(\frac{\widetilde{p}\cdot k}{q\cdot k}q_\mu q_\nu - q_\mu \widetilde{p}_\nu - q_\nu \widetilde{p}_\mu\right) \tag{7.98}$$

$$= \frac{\omega}{2(q\cdot k)}\left(p^\alpha A_{\alpha\nu}q_\mu + p^\alpha A_{\alpha\mu}q_\nu\right), \tag{7.99}$$

where q is any null vector; a suitable choice is $q^\mu = (1, -\mathbf{k})$, which preserves rotational invariance. One can readily check that the two consistency conditions (7.93) and (7.96) are satisfied. Therefore, by introducing $\widetilde{c}_{\mu\nu}$ we have a well-defined dressing that preserves the de Donder gauge condition.

Recently it has been shown by Hirai and Sugishita [141] that a careful BRST analysis can remove the necessity of objects such as $\widetilde{c}_{\mu\nu}$, at least in QED. We henceforth assume that this can be done here as well, since from [58] we understand that all $\widetilde{c}_{\mu\nu}$ does is shift the soft charges by some function of p; the important part was the existence of a solution $\widetilde{c}_{\mu\nu}$ to the conditions (7.93) and (7.96), which we have already shown. We leave the proof along the lines of [141] for future work.

7.3.3 Dual supertranslation charge

We now see that the gravitational 't Hooft lines are charged under dual supertranslations. A dual supertranslation is parametrized by a 2-sphere function $f(z,\bar{z})$, which we assume does not introduce poles or branch cuts. Its charge has the expression [101]

$$M_f = \frac{2i}{\kappa^2}\int_{\mathcal{I}^+_-} d^2z\gamma^{z\bar{z}} f(z,\bar{z})\left(D_{\bar{z}}^2 C_{zz} - D_z^2 C_{\bar{z}\bar{z}}\right). \tag{7.100}$$

In general relativity, there is no source term like $h_{\mu\nu}T^{\mu\nu}$ for the magnetic case. This implies that we do not have a hard charge. Thus under a partial integration, we have no contribution from the future boundary $\mathcal{I}^+_+$ and one can cast (7.100) into a total derivative over $\mathcal{I}^+$,

$$M_f = -\frac{2i}{\kappa^2}\int_{\mathcal{I}^+} dud^2z\gamma^{z\bar{z}}\left(D_{\bar{z}}^2 f\partial_u C_{zz} - D_z^2 f\partial_u C_{\bar{z}\bar{z}}\right), \tag{7.101}$$

where we also integrated by parts twice on the sphere. Using the mode expansion of h_{zz}, the radiative mode C_{zz} can be written as [58]

$$C_{zz}(u,z,\bar{z}) = \kappa \lim_{r\to\infty}\frac{1}{r}h_{zz}(u,r,z,\bar{z}) \tag{7.102}$$

$$= \frac{i\kappa}{8\pi^2}\gamma_{z\bar{z}}\int_0^\infty d\omega\left(a_-^\dagger(\omega\mathbf{x}_z)e^{i\omega u} - a_+(\omega\mathbf{x}_z)e^{-i\omega u}\right). \tag{7.103}$$

Using the identity $\int_{-\infty}^{\infty} du \partial_u e^{\pm i\omega u} = \pm 2\pi i \omega \delta(\omega)$, we may write

$$M_f = \frac{i}{2\pi\kappa} \int d\omega d^2z \, \omega\delta(\omega) \left[D_{\bar{z}}^2 f \left(a_-^\dagger(\omega \mathbf{x}_z) + a_+(\omega \mathbf{x}_z) \right) - \text{h.c.} \right]. \tag{7.104}$$

Writing the gravitational 't Hooft line dressing as $\widetilde{W}_g(p) = \exp \widetilde{R}_g(p)$ and using the canonical commutation relation (7.84), one can obtain

$$\left[M_f, \widetilde{R}_g(p) \right] = \frac{im_0}{4\pi m} \int d^2z \left(D_{\bar{z}}^2 f \frac{(\epsilon^+ \cdot A \cdot p)(p \cdot \epsilon^+)}{p \cdot \hat{k}} - D_z^2 f \frac{(\epsilon^- \cdot A \cdot p)(p \cdot \epsilon^-)}{p \cdot \hat{k}} \right). \tag{7.105}$$

Noting that $\Gamma_{zz}^z = \frac{-2\bar{z}}{1+z\bar{z}} = \gamma^{z\bar{z}} \partial_z \gamma_{z\bar{z}}$ and integrating by parts, we can write this in the form

$$\begin{aligned} \left[M_f, \widetilde{R}_g(p) \right] &= \frac{im_0}{4\pi m} \int d^2z \, f(z,\bar{z}) \left[\partial_{\bar{z}} \partial^z \left(\gamma_{z\bar{z}} \frac{(\epsilon^+ \cdot A \cdot p)(p \cdot \epsilon^+)}{p \cdot \hat{k}} \right) - \text{c.c.} \right] &(7.106)\\ &= -\frac{m_0}{2\pi m} \int d^2z \, f(z,\bar{z}) \ \text{Im} \left[\partial_{\bar{z}} \partial^z \left(\gamma_{z\bar{z}} \frac{(\epsilon^+ \cdot A \cdot p)(p \cdot \epsilon^+)}{p \cdot \hat{k}} \right) \right], &(7.107) \end{aligned}$$

where we denote $\partial^z = \gamma^{z\bar{z}} \partial_{\bar{z}}$. One can show that

$$\partial_{\bar{z}} \partial^z \left(\gamma_{z\bar{z}} \frac{(\epsilon^+ \cdot A \cdot p)(p \cdot \epsilon^+)}{p \cdot \hat{k}} \right) = -\frac{i}{2} \gamma_{z\bar{z}} \left(\frac{m^4}{(p \cdot \hat{k})^3} - \frac{n^2}{(n \cdot \hat{k})^3} B(p,n,z) \right), \tag{7.108}$$

where

$$\begin{aligned} B(p,n,z) = &(n^1 + in^2)(p^1 - ip^2) - (n^0 - n^3)(p^0 + p^3) \\ &- (n^0 + n^3)(p^1 - ip^2)z + (n^1 - in^2)(p^0 + p^3)z. \end{aligned} \tag{7.109}$$

Unlike the case of electromagnetism, one can see that an obscure choice of n can make the expression in square brackets of (7.107) contain both imaginary and real components. A sufficient condition to prevent this is to choose n to be null; an example is $n^\mu = (1,0,0,-1)$ which, as we saw in section 7.2.2, corresponds to a $\frac{\pi}{2}$-rotation of $\epsilon^\pm$ in the complex plane. Then, we have the commutator

$$\begin{aligned} \left[M_f, \widetilde{W}_g(p) \right] &= \left[M_f, e^{\widetilde{R}_g(p)} \right] &(7.110)\\ &= \left(\frac{m_0}{4\pi m} \int d^2z \, \gamma_{z\bar{z}} \, f(z,\bar{z}) \frac{p^4}{(p \cdot \hat{k})^3} \right) \widetilde{W}_g(p), &(7.111) \end{aligned}$$

which implies that the gravitational 't Hooft line dressing (7.88) carries dual supertranslation charge.

On the other hand, it does not carry supertranslation charge. To see this, we first recall that the soft supertranslation charge T_f has the form [58, 148]

$$T_f = \frac{1}{2\pi\kappa}\int d\omega d^2z\,\omega\delta(\omega)\left[D_{\bar{z}}^2 f\left(a_-^\dagger(\omega\mathbf{x}_z) + a_+(\omega\mathbf{x}_z)\right) + \text{h.c.}\right]. \tag{7.112}$$

Then, by a similar analysis, one obtains,

$$\left[M_f, \widetilde{W}_g(p)\right] = \frac{1}{2\pi}\int d^2z\, f(z,\bar{z})\ \operatorname{Re}\left[\partial_{\bar{z}}\partial^z\left(\gamma_{z\bar{z}}\,\frac{(\epsilon^+\cdot A\cdot p)(p\cdot\epsilon^+)}{p\cdot\hat{k}}\right)\right]\widetilde{W}_g(p) = 0, \tag{7.113}$$

which proves the statement.

It is instructive to contrast this with the gravitational Faddeev-Kulish dressing (gravitational Wilson line dressing), which takes the form [9]

$$W_g(p) = \exp\left\{\frac{\kappa}{2}\int\widetilde{d^3k}\,\phi(\omega)\frac{p^\mu p^\nu}{p\cdot k}\left(\epsilon_{\mu\nu}^{\alpha}(\mathbf{k})a_\alpha^\dagger(\mathbf{k}) - \text{h.c.}\right)\right\}. \tag{7.114}$$

The following identity plays a role:

$$\partial_{\bar{z}}\partial^z\left(\gamma_{z\bar{z}}\,\frac{(p\cdot\epsilon^+)^2}{p\cdot\hat{k}}\right) = \frac{1}{2}\gamma_{z\bar{z}}\,\frac{m^4}{(p\cdot\hat{k})^3}. \tag{7.115}$$

Use of this identity immediately gives:

$$\begin{aligned}
[T_f, W_g(p)] &= \frac{1}{2\pi}\int d^2z\, f(z,\bar{z})\ \operatorname{Re}\left[\partial_{\bar{z}}\partial^z\left(\gamma_{z\bar{z}}\,\frac{(p\cdot\epsilon^+)^2}{p\cdot\hat{k}}\right)\right]W_g(p) && (7.116)\\
&= \left(\frac{1}{4\pi}\int d^2z\,\gamma_{z\bar{z}}\,f(z,\bar{z})\frac{m^4}{(p\cdot\hat{k})^3}\right)W_g(p), && (7.117)\\
[M_f, W_g(p)] &= -\frac{1}{2\pi}\int d^2z\, f(z,\bar{z})\ \operatorname{Im}\left[\partial_{\bar{z}}\partial^z\left(\gamma_{z\bar{z}}\,\frac{(p\cdot\epsilon^+)^2}{p\cdot\hat{k}}\right)\right]W_g(p) && (7.118)\\
&= 0, && (7.119)
\end{aligned}$$

implying that the gravitational Wilson line dressing only carries a definite supertranslation charge.

Since we do not have a "magnetic" counterpart to the source term $h_{\mu\nu}T^{\mu\nu}$ in general relativity, we have no bulk degrees of freedom carrying dual supertranslation charge. Therefore, instead of the dressing (7.88) of "magnetically" charged particles, we generalize them to generic operators that are charged under dual supertranslation. These operators can be obtained by replacing p^μ with a constant vector C^μ as in (7.65) and absorb the dimensionless factor m_0/m in C^μ:

$$\widetilde{W}_g(C) = \exp\left\{-\frac{\kappa}{2}\int \widetilde{d^3k}\,\phi(\omega)\left(\frac{(C\cdot A\cdot\epsilon^\alpha)(C\cdot\epsilon^\alpha)}{C\cdot k}a^\dagger_\alpha(\mathbf{k}) - \text{h.c.}\right)\right\}. \tag{7.120}$$

From our construction, we see that these are 't Hooft line operators along a straight line geodesic at $\mathcal{I}^+$ whose direction is given by C^μ. The vector C^μ parametrizes how the soft dual supertranslation charge is distributed over the 2-sphere. Choosing n to be null, we see that

$$\left[M_f, \widetilde{W}_g(C)\right] = \left(\frac{1}{4\pi}\int d^2z\,\gamma_{z\bar{z}}\,f(z,\bar{z})\frac{C^4}{(C\cdot\hat{k})^3}\right)\widetilde{W}_g(C), \tag{7.121}$$

$$\left[T_f, \widetilde{W}_g(C)\right] = 0. \tag{7.122}$$

These 't Hooft line operators are charged under dual supertranslation and neutral under supertranslation, and we can use them to translate a vacuum to another vacuum carrying a different dual supertranslation charge.

7.4 Summary

In this chapter, we have constructed the asymptotic states by following the original method of Faddeev and Kulish, using a quantum field theory of electric and magnetic charges by Blagojević and collaborators [118, 119]. We have shown that the magnetic dressings can be expressed as a 't Hooft line operator (see equation (7.43)), and that they are charged under magnetic large gauge transformations (see equation (7.66)). The 't Hooft line interpretation allowed us to formulate a gravitational 't Hooft line operator, which is charged under dual supertranslations while carrying zero supertranslation charge (see equations (7.121) and (7.122)).

Throughout the chapter, we have assumed the 2-sphere function parameters $\varepsilon(z,\bar{z})$ and $f(z,\bar{z})$ to be smooth. In the next chapter, we observe that relaxing this condition brings about interesting results.

Chapter 8

Standard and Dual BMS Charges on the Schwarzschild Horizon

8.1 Horizon supertranslation in Bondi gauge

In this chapter, we study the standard and dual supertranslation charges at the future Schwarzschild horizon. We demonstrate that their algebra exhibits central terms in the presence of singularities in the parameter function. We briefly discuss its possible implications on the structure of the black hole horizon.

We begin with a review of BMS transformations on the Schwarzschild horizon. We work in the Bondi gauge,

$$g_{rr} = g_{rA} = 0, \qquad \partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0. \tag{8.1}$$

In the ingoing Eddington-Finkelstein coordinates, the Schwarzschild metric is given by

$$ds^2 = -\Lambda dv^2 + 2dvdr + r^2\gamma_{AB}d\Theta^A d\Theta^B, \qquad \Lambda \equiv 1 - \frac{2M}{r}, \tag{8.2}$$

where γ_{AB} is the metric on the unit 2-sphere. A diffeomorphism ξ on Schwarzschild that preserves these conditions should satisfy

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{rA} = 0, \qquad \gamma^{AB}\mathcal{L}_\xi g_{AB} = 0. \tag{8.3}$$

Such diffeomorphisms can be parametrized as [61]

$$\xi = X\partial_v - \frac{1}{2}\left(rD_AX^A + D^2X\right)\partial_r + \left(X^A + \frac{1}{r}D^AX\right)\partial_A, \tag{8.4}$$

where $X^A = X^A(v, \Theta)$ is an arbitrary vector field and $X = X(v, \Theta)$ is an arbitrary scalar field on the future horizon $\mathcal{H}^+$. Here D_A denotes covariant derivative on the unit 2-sphere, $D^A = \gamma^{AB} D_B$, and $D^2 \equiv D^A D_A = \gamma^{AB} D_A D_B$. A supertranslation is given by

$$X = f(\Theta), \qquad X^A = 0, \tag{8.5}$$

where f is a smooth function on the 2-sphere. In the later sections, we relax the smoothness condition to allow f to have poles. A superrotation is given by

$$X = \frac{v}{2} D_A Y^A, \qquad X^A = Y^A(\Theta), \tag{8.6}$$

where Y^A is a smooth vector field on the 2-sphere.

Since supertranslations and superrotations are metric-dependent, the diffeomorphisms (8.4) do not form a closed algebra under the Lie bracket of vector fields

$$[\xi_1, \xi_2]^a = \xi_1^b \partial_b \xi_2^a - \xi_2^b \partial_b \xi_1^a. \tag{8.7}$$

In performing two consecutive diffeomorphisms, the first one ruins the metric for the second. To take this into account, we need a *modified* Lie bracket [149] of vector fields,

$$[\xi_1, \xi_2]_M^a = [\xi_1, \xi_2]^a - \delta_{\xi_1} \xi_2^a + \delta_{\xi_2} \xi_1^a, \tag{8.8}$$

where $\delta_{\xi_1} \xi_2^a$ denotes the change in the vector component ξ_2^a induced by the diffeomorphism ξ_1. Supertranslations and superrotations form a closed algebra under the modified bracket. Given a pair of vector fields ξ_i $(i = 1, 2)$ that generate supertranslation f_i and superrotation Y_i, one can show that

$$[\xi_1, \xi_2]_M = \xi_{[1,2]}, \tag{8.9}$$

where $\xi_{[1,2]}$ is another vector field that generates supertranslation $\hat{f}$ and superrotation $\hat{Y}$ given by

$$\hat{f} = \frac{1}{2} f_1 D_A Y_2^A - \frac{1}{2} f_2 D_A Y_1^A + Y_1^A D_A f_2 - Y_2^A D_A f_1, \tag{8.10}$$

$$\hat{Y}^A = Y_1^B D_B Y_2^A - Y_2^B D_B Y_1^A. \tag{8.11}$$

A derivation is given in appendix L. We note that this is the same as the BMS_4 algebra at null infinity [149].

8.2 Horizon charges

Following [61], let us define $\mathcal{X}^+$ to be the hypersurface extending from the future boundary $\mathcal{H}^+_+$ of $\mathcal{H}^+$ to the past boundary $\mathcal{I}^+_-$ of $\mathcal{I}^+$, such that

$$\Sigma = \mathcal{H}^+ \cup \mathcal{X}^+ \tag{8.12}$$

forms a Cauchy surface. Then, a charge Q^Σ associated with Σ breaks into two parts,

$$Q^\Sigma = Q^{\mathcal{H}^+} + Q^{\mathcal{X}^+}. \tag{8.13}$$

In [108, 109], the authors provide a formula for the (possibly non-integrable) variation of electric and magnetic charges associated with a vector field ξ,

$$\not{\delta} Q^\Sigma_E = \frac{1}{16\pi}\epsilon_{\alpha\beta\gamma\delta}\int_{\partial\Sigma}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta, \tag{8.14}$$

$$\not{\delta} Q^\Sigma_M = \frac{i}{8\pi}\int_{\partial\Sigma}(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta, \tag{8.15}$$

where $\partial\Sigma$ denotes the boundary of a Cauchy surface Σ. Each of these break into two contributions $\not{\delta} Q^{\mathcal{H}^+}$ and $\not{\delta} Q^{\mathcal{X}^+}$, where the horizon contributions take the form

$$\not{\delta} Q^{\mathcal{H}^+}_E = \frac{1}{16\pi}\epsilon_{\alpha\beta\gamma\delta}\int_{\partial\mathcal{H}^+}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta, \tag{8.16}$$

$$\not{\delta} Q^{\mathcal{H}^+}_M = \frac{i}{8\pi}\int_{\partial\mathcal{H}^+}(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta. \tag{8.17}$$

Throughout this chapter, we take the viewpoint that the black hole ultimately evaporates, and thus there is no future boundary $\mathcal{H}^+_+$ of the horizon. This means that we take $\partial\mathcal{H}^+ = \mathcal{H}^+_-$, and ignore all contributions of $\mathcal{H}^+_+$ to the integral.[1]

Expressions for the horizon contributions in Bondi coordinates are derived in appendix M. Taking ξ to be the supertranslation vector field

$$\xi = f\partial_v - \frac{1}{2}D^2 f\partial_r + \frac{1}{r}D^A f\partial_A, \tag{8.18}$$

we obtain the horizon supertranslation charge $\not{\delta} Q^{\mathcal{H}^+}_f$ from (M.50) and the dual supertrans-

[1]For eternal black holes, one should add boundary degrees of freedom on $\mathcal{H}^+_+$ such that they cancel the contribution of $\mathcal{H}^+_+$ to the integral, since $\mathcal{H}^+_+$ is not a genuine part of the boundary $\partial\Sigma$. See [116,117,150–152] for a discussion of electromagnetism on $\mathcal{I}^+$.

lation charge $\oint\widetilde{Q}_f^{\mathcal{H}^+}$ from (M.77) to be

$$\oint Q_f^{\mathcal{H}^+} = \frac{M}{8\pi}\int_{\partial\mathcal{H}^+} d^2\Theta\sqrt{\gamma}\bigg[D^A\left(\frac{f}{M}h_{vA} + (D_Af)h_{vr}\right) \\ - (D^Af)\partial_r h_{vA} + 2fh_{vv} + (D^2f)h_{vr}\bigg], \tag{8.19}$$

$$\oint\widetilde{Q}_f^{\mathcal{H}^+} = \frac{-i}{32\pi M}\int_{\partial\mathcal{H}^+} d^2\Theta\sqrt{\gamma}(D^Bf)\epsilon_A{}^C D^A h_{BC}. \tag{8.20}$$

We use ϵ^{AB} to denote the alternating tensor on the unit 2-sphere, with $\epsilon^{\theta\phi} = \frac{1}{\sin\theta}$ and $\epsilon_A{}^C = \gamma_{AB}\epsilon^{BC}$.

For smooth functions everywhere, we can discard total derivatives in the integrand, and the supertranslation charge is in exact agreement with that of [61]. After residual gauge fixing and using a combination of the constraints on $\mathcal{H}^+$, the supertranslation charge then simplifies to the expression

$$\oint Q_f^{\mathcal{H}^+} = \frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2\Theta\sqrt{\gamma}f(\Theta)D^AD^B\sigma_{AB}, \tag{8.21}$$

where $\sigma_{AB} = \frac{1}{2}\partial_v h_{AB}$ is the conjugate momentum of h_{AB}. The phase space of the horizon $\mathcal{H}^+$ has the Dirac bracket [61],

$$\{\sigma_{AB}(v,\Omega), h_{CD}(v',\Omega')\} = 32\pi M^2\gamma_{ABCD}\delta(v-v')\delta(\Omega-\Omega'), \tag{8.22}$$

where $\gamma_{ABCD} \equiv \gamma_{AC}\gamma_{BC} + \gamma_{AD}\gamma_{BC} - \gamma_{AB}\gamma_{CD}$ is proportional to the DeWitt metric [153].

Since we can integrate by parts freely without having to worry about boundary terms, we can move all covariant derivatives to act on f. As such, we define the integrable horizon supertranslation charge $\delta Q_f^{\mathcal{H}^+}$ and dual supertranslation charge $\delta\widetilde{Q}_f^{\mathcal{H}^+}$ as

$$\delta Q_f^{\mathcal{H}^+} \equiv \frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2\Theta\sqrt{\gamma}\,(D^BD^Af)\sigma_{AB}, \tag{8.23}$$

$$\delta\widetilde{Q}_f^{\mathcal{H}^+} \equiv \frac{-i}{32\pi M}\int_{\mathcal{H}^+_-} d^2\Theta\sqrt{\gamma}\,(D^BD^Af)\epsilon_A{}^C h_{BC}. \tag{8.24}$$

Notice that in this form, the dual supertranslation charge is related to supertranslation charge by the twisting procedure $h_{AB} \to \epsilon_A{}^C h_{CB}$ proposed by [99, 100].

When we have smooth functions everywhere, $\oint Q_f^{\mathcal{H}^+} = \delta Q_f^{\mathcal{H}^+}$ and $\oint\widetilde{Q}_f^{\mathcal{H}^+} = \delta\widetilde{Q}_f^{\mathcal{H}^+}$, i.e. the charges are integrable. In the next section, we see that allowing f to have isolated simple poles leads $\oint Q_f^{\mathcal{H}^+}$ to acquire additional pieces.

8.3 Supertranslation charge with poles on the complex plane

Let us employ the complex stereographic coordinates $(z, \bar{z})$, defined as

$$z = e^{i\phi} \tan \frac{\theta}{2}, \qquad \bar{z} = e^{-i\phi} \tan \frac{\theta}{2}, \tag{8.25}$$

where θ and ϕ are the standard spherical coordinates on a unit sphere. The unit sphere metric in these coordinates is $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$, $\gamma_{zz} = \gamma_{\bar{z}\bar{z}} = 0$. The integration measure is

$$d^2\Theta\sqrt{\gamma} = d^2z\sqrt{\gamma}, \qquad d^2z \equiv idz \wedge d\bar{z}, \qquad \sqrt{\gamma} = \gamma_{z\bar{z}}. \tag{8.26}$$

We have organized the notation such that d^2z is real. The volume form is $i\sqrt{\gamma}dz \wedge d\bar{z}$, so the alternating tensor has components $\epsilon_{z\bar{z}} = i\sqrt{\gamma}$. The non-vanishing Christoffel symbols are ${}^{(2)}\Gamma^{z}_{zz} = \frac{-2\bar{z}}{1+z\bar{z}}$ and ${}^{(2)}\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \frac{-2z}{1+z\bar{z}}$.

Let us compute the supertranslation charge $\oint Q_f^{\mathcal{H}^+}$ when $f(z, \bar{z})$ has a pole at some complex coordinate w, that is, $f = \frac{1}{z-w}$. After fully fixing the residual gauge freedom as in [61], on $\mathcal{H}^+$ we have

$$\begin{aligned} h_{vv} &= h_{vA} = 0, \\ h_{vr} &= \frac{1}{4M^2}[D^2 - 1]^{-1} D^B D^C h_{BC}, \\ \partial_r h_{vA} &= -\frac{1}{4M^2} D_A [D^2 - 1]^{-1} D^B D^C h_{BC} + \frac{1}{4M^2} D^B h_{AB}, \end{aligned} \tag{8.27}$$

and the supertranslation charge (8.19) takes the form

$$\oint Q_f^{\mathcal{H}^+} = \frac{M}{8\pi} \int_{\partial \mathcal{H}^+} d^2z\sqrt{\gamma} \left(-(D^A f) \frac{1}{4M^2} D^B h_{AB} + 2D_A(D^A f h_{vr}) \right). \tag{8.28}$$

We have used (8.27) for $D_A h_{vr} + \partial_r h_{vA}$. Let us take a look at the total derivative term $D_A(D^A f h_{vr})$. For $f = \frac{1}{z-w}$ we have,

$$\int d^2z\sqrt{\gamma}\, D^A(D_A f h_{vr}) = i \int dz \wedge d\bar{z}\, (\partial_{\bar{z}}(h_{vr}\partial_z f) + \partial_z(h_{vr}\partial_{\bar{z}} f)) \tag{8.29}$$

$$= -i \oint_w dz\, h_{vr}\partial_z f + i \oint_w d\bar{z}\, h_{vr}\partial_{\bar{z}} f \tag{8.30}$$

$$= -2\pi \partial_z h_{vr}\Big|_{z=w}. \tag{8.31}$$

The second term on the r.h.s. of the second line vanishes because $f = \frac{1}{z-w}$ satisfies the identity[2]

$$\partial_{\bar{z}} f = 2\pi\delta^2(z-w), \tag{8.32}$$

and the contour of $\oint_w d\bar{z}$ is a circle around w that does not cross the delta function singularity. The first term in the second line has $\partial_z f = \frac{-1}{(z-w)^2}$, so it contributes a residue proportional to $\partial_z h_{vr}$ evaluated at w, as shown in the third line (8.31). Thus, plugging in the expression (8.27) for h_{vr} we have

$$\not{\delta} Q_f^{\mathcal{H}^+} = -\frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2 z\sqrt{\gamma}\,(D^A f)D^B\sigma_{AB} - \frac{1}{4M}\int_{-\infty}^{\infty} dv D_z[D^2-1]^{-1}D^B D^A\sigma_{AB}\bigg|_{z=w}. \tag{8.33}$$

Partial integrating the first term, we obtain

$$\not{\delta} Q_f^{\mathcal{H}^+} = \frac{1}{16\pi M}\int_{\mathcal{H}^+} dv\, d^2 z\sqrt{\gamma}\,(D^B D^A f)\sigma_{AB} - \frac{1}{4M}\int_{-\infty}^{\infty} dv D_z[D^2-1]^{-1}D^B D^A\sigma_{AB}\bigg|_{z=w}, \tag{8.34}$$

where one can see that the total derivative due to the partial integration vanishes,

$$\int d^2 z\sqrt{\gamma}\, D^B(\sigma_{AB}D^A f) = \int d^2 z\left(\partial_{\bar{z}}(\sigma_{zz}D^z f) + \partial_z(\sigma_{\bar{z}\bar{z}}D^{\bar{z}} f)\right) \tag{8.35}$$

$$= -i\oint_w dz\gamma^{z\bar{z}}\sigma_{zz}\partial_{\bar{z}} f + i\oint_w d\bar{z}\gamma^{z\bar{z}}\sigma_{\bar{z}\bar{z}}\partial_z f \tag{8.36}$$

$$= 0, \tag{8.37}$$

since the $\partial_{\bar{z}} f = 2\pi\delta^2(z-w)$ and the contour of $\oint_w dz$ is a circle around w that does not cross the delta function singularity, and $\sigma_{\bar{z}\bar{z}}\partial_z f = -\frac{1}{2}(z-w)^{-2}\partial_v h_{\bar{z}\bar{z}}$ does not have poles in $\bar{z}$.

We recognize the first term in (8.34) to be the integrable supertranslation charge $\delta Q_f^{\mathcal{H}^+}$ (8.23). Thus, we find that a pole in f leads $\delta Q_f^{\mathcal{H}^+}$ to acquire a non-integrable part $\mathcal{N}_f^{\mathcal{H}^+}$,

$$\not{\delta} Q_f^{\mathcal{H}^+} = \delta Q_f^{\mathcal{H}^+} + \mathcal{N}_f^{\mathcal{H}^+}, \tag{8.38}$$

where $\delta Q_f^{\mathcal{H}^+}$ is given in (8.23), and

$$\mathcal{N}_f^{\mathcal{H}^+} = -\frac{1}{4M}\int_{-\infty}^{\infty} dv D_z[D^2-1]^{-1}D^B D^A\sigma_{AB}\bigg|_{z=w}. \tag{8.39}$$

[2]Note that we normalize $\delta^2(z-w)$ as a real density, so $1 = \int d^2 z\,\delta^2(z-w) = \int \epsilon\frac{1}{\sqrt{\gamma}}\delta^2(z-w)$, where $\epsilon = d^2 z\sqrt{\gamma}$ is the volume form on the unit sphere.

This splitting into integrable and non-integrable parts is not unique (see for instance [109]). Our choice is justified by the following points:

1. $\delta Q_f^{\mathcal{H}^+}$ is the horizon supertranslation charge in the absence of poles in f, and
2. $\mathcal{N}_f^{\mathcal{H}^+}$ has zero Dirac bracket with both $\delta Q_g^{\mathcal{H}^+}$ and $\delta\widetilde{Q}_g^{\mathcal{H}^+}$, so it carries no degrees of freedom.

We have seen point 1 at the end of section 8.2. We demonstrate point 2 in appendix N.

8.4 Dirac bracket between charges

In this section, we use this bracket to compute the bracket $\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\}$, where $f = \frac{1}{z-w}$ and g we assume to be smooth. This bracket can probe the central term of the algebra of the full charges, since the full charges have the expansions,

$$Q_f^{\mathcal{H}^+} = Q_f^{(h=0)} + \delta Q_f^{\mathcal{H}^+} + O(h^2), \tag{8.40}$$

$$\widetilde{Q}_g^{\mathcal{H}^+} = \widetilde{Q}_g^{(h=0)} + \delta\widetilde{Q}_g^{\mathcal{H}^+} + O(h^2), \tag{8.41}$$

where $Q_f^{(h=0)}$ and $\widetilde{Q}_g^{(h=0)}$ are the constant charges of the background metric and hence do not carry degrees of freedom, so this to

$$\{Q_f^{\mathcal{H}^+}, \widetilde{Q}_g^{\mathcal{H}^+}\} = \underbrace{\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\}}_{\text{constant}} + O(h). \tag{8.42}$$

Therefore, the constant term corresponds to the central charge of the charge algebra.

Now let us compute $\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\}$, with $f = \frac{1}{z-w}$ and g smooth. Using the expressions (8.23) and (8.24) and applying the bracket (8.22), we obtain

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\right\} = \frac{-i}{2(16\pi M)^2}\left\{\int_{\mathcal{H}^+} dv\, d^2z\sqrt{\gamma}\,(D^B D^A f)\sigma_{AB}, \int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\,(D^E D^C g)\epsilon_E{}^D h_{DC}\right\} \tag{8.43}$$

$$= \frac{-i}{16\pi}\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\,(D^B D^A f)(D^E D^C g)\epsilon_E{}^D \gamma_{ABDC}. \tag{8.44}$$

We can manipulate D^B to write this as

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta \widetilde{Q}_g^{\mathcal{H}^+}\right\} = \frac{-i}{16\pi}\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\Bigg(D^B\left((D^Af)(D^ED^Cg)\epsilon_E{}^D\gamma_{ABDC}\right) - (D^Af)(D^BD^ED^Cg)\epsilon_E{}^D\gamma_{ABDC}\Bigg). \tag{8.45}$$

Plugging in the expressions for $\epsilon_A{}^B$ and γ_{ABCD}, we can see that the first term is zero,

$$\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\, D^B\left((D^Af)(D^ED^Cg)\epsilon_E{}^D\gamma_{ABDC}\right) = -2\oint_w dz\,(\partial_{\bar{z}}f)D^{\bar{z}}D^{\bar{z}}g\gamma_{z\bar{z}} - 2\oint_w d\bar{z}\,(\partial_z f)D^zD^zg\gamma_{z\bar{z}} \tag{8.46}$$

$$= 0. \tag{8.47}$$

The $\oint_w dz$ integral vanishes since its contour is a circle around w and does not meet the singularity of the delta function $\partial_{\bar{z}}f = 2\pi\delta^2(z-w)$, and the $\oint_w d\bar{z}$ integral vanishes since $(\partial_z f)D^zD^zg\gamma_{z\bar{z}}$ does not have a pole in $\bar{z}$. Therefore, we obtain

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta \widetilde{Q}_g^{\mathcal{H}^+}\right\} = \frac{i}{8\pi}\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\,{\gamma_{z\bar{z}}}^2\left((D^zf)(D^zD^{\bar{z}}D^{\bar{z}}g)\epsilon_{\bar{z}}{}^{\bar{z}} + (D^{\bar{z}}f)(D^{\bar{z}}D^zD^zg)\epsilon_z{}^z\right) \tag{8.48}$$

$$= \frac{1}{8\pi}\int_{\mathcal{H}^+_-} d^2z\left((\partial_{\bar{z}}f)D^zD_z^2g - (\partial_z f)D^{\bar{z}}D_{\bar{z}}^2g\right). \tag{8.49}$$

$$= \frac{1}{8\pi}\int_{\mathcal{H}^+_-} d^2z\gamma^{z\bar{z}}\Bigg((\partial_{\bar{z}}f)[D_{\bar{z}}, D_z]D_zg + (\partial_{\bar{z}}f)D_zD_{\bar{z}}D_zg - (\partial_z f)[D_z, D_{\bar{z}}]D_{\bar{z}}g - (\partial_z f)D_{\bar{z}}D_zD_{\bar{z}}g\Bigg). \tag{8.50}$$

In the second line, we have used $\sqrt{\gamma} = \gamma_{z\bar{z}}$ and $\epsilon_z{}^z = -\epsilon_{\bar{z}}{}^{\bar{z}} = i$. The commutators are $[D_{\bar{z}}, D_z]D_zg = \gamma_{z\bar{z}}D_zg$ and $[D_z, D_{\bar{z}}]D_{\bar{z}}g = \gamma_{z\bar{z}}D_{\bar{z}}g$, which one can check by direct computation using ${}^{(2)}\Gamma^z_{zz} = \frac{-2\bar{z}}{1+z\bar{z}}$ and ${}^{(2)}\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \frac{-2z}{1+z\bar{z}}$. Thus we have

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta \widetilde{Q}_g^{\mathcal{H}^+}\right\} = \frac{1}{8\pi}\int d^2z\left((\partial_{\bar{z}}f)D_zg - (\partial_z f)D_{\bar{z}}g + (\partial_{\bar{z}}f)D_zD_{\bar{z}}D^{\bar{z}}g - (\partial_z f)D_{\bar{z}}D_zD^zg\right). \tag{8.51}$$

For the last two terms in the parentheses, we have used $\gamma^{z\bar{z}}$ to purposely raise the index of the first derivative acting on g. This allows us to write the third covariant derivatives acting on g as regular partial derivatives,

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\right\} = \frac{1}{8\pi}\int d^2z \left((\partial_{\bar{z}}f)D_z g - (\partial_z f)D_{\bar{z}}g + (\partial_{\bar{z}}f)\partial_z D_{\bar{z}}D^{\bar{z}}g - (\partial_z f)\partial_{\bar{z}}D_z D^z g\right). \tag{8.52}$$

Now we partial integrate all $\partial_A f$'s inside the parentheses. Only the boundary terms survive, since partial derivatives commute and

$$D_{\bar{z}}D^{\bar{z}}g - D_z D^z g = \gamma^{z\bar{z}}(\partial_{\bar{z}}\partial_z g - \partial_z\partial_{\bar{z}})g = 0. \tag{8.53}$$

Therefore, we have via Stokes' theorem,

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\right\} = \frac{1}{8\pi}\int d^2z \left(\partial_{\bar{z}}(fD_z g) - \partial_z(fD_{\bar{z}}g) + \partial_{\bar{z}}(f\partial_z D_{\bar{z}}D^{\bar{z}}g) - \partial_z(f\partial_{\bar{z}}D_z D^z g)\right) \tag{8.54}$$

$$= -\frac{i}{8\pi}\oint_w \left(dz\frac{(D_z g + \partial_z D_{\bar{z}}D^{\bar{z}}g)}{z-w} + d\bar{z}\frac{(D_{\bar{z}}g + \partial_{\bar{z}}D_z D^z g)}{z-w}\right). \tag{8.55}$$

The $\oint_w d\bar{z}$ integral vanishes due to the absence of $\bar{z}$-poles. Observe that we can use the identity

$$[D_{\bar{z}}, D_z]D_z g = \gamma_{z\bar{z}}D_z g \tag{8.56}$$

to simplify

$$D_z g + \partial_z D_{\bar{z}}D^{\bar{z}}g = D_z g + D_z D_{\bar{z}}D^{\bar{z}}g \tag{8.57}$$

$$= D_z g + \gamma^{z\bar{z}}D_z D_{\bar{z}}D_z g \tag{8.58}$$

$$= D_z g + \gamma^{z\bar{z}}[D_z, D_{\bar{z}}]D_z g + \gamma^{z\bar{z}}D_{\bar{z}}D_z D_z g \tag{8.59}$$

$$= D^z D_z D_z g. \tag{8.60}$$

and write

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\right\} = -\frac{i}{8\pi}\oint_w dz\frac{D^z D_z D_z g}{z-w} \tag{8.61}$$

The $\oint_w dz$ integrals contribute a residue to the bracket so long as $D^z D_z^2 g$ does not contain a factor of $(z-w)$. If it does, it vanishes when evaluated at w. Either way, we obtain

$$\left\{\delta Q_f^{\mathcal{H}^+}, \delta \widetilde{Q}_g^{\mathcal{H}^+}\right\} = \frac{1}{4} D^z D_z D_z g\bigg|_{z=w}. \tag{8.62}$$

8.5 Another way of computing central term

Here, we obtain the central term of the previous section using a different method.

We start from our expression (8.24) for the integrable variation $\delta\widetilde{Q}_f^{\mathcal{H}^+}$ of dual supertranslation charge, which reads

$$\delta\widetilde{Q}_g^{\mathcal{H}^+} = \frac{-i}{32\pi M}\int_{\mathcal{H}^+_-} d^2z\sqrt{\gamma}\,(D^B D^A g)\epsilon_A{}^C h_{BC}, \tag{8.63}$$

and invoke equation (3.1) in the paper [149],

$$\{Q_f^{\mathcal{H}^+}, \widetilde{Q}_g^{\mathcal{H}^+}\} = \delta_f \widetilde{Q}_g^{\mathcal{H}^+}, \tag{8.64}$$

where $\delta_f\widetilde{Q}_g$ denotes taking the expression (8.63) for $\delta\widetilde{Q}_g^{\mathcal{H}^+}$ and replacing h_{AB} with the diffeomorphism mode

$$h_{BC} \;\to\; 2M(2D_B D_C f - \gamma_{BC} D^2 f). \tag{8.65}$$

This leads to the expression

$$\{Q_f^{\mathcal{H}^+}, \widetilde{Q}_g^{\mathcal{H}^+}\} = \frac{-i}{16\pi}\int d^2z\sqrt{\gamma}\,(D^B D^A g)\epsilon_A{}^C(2D_B D_C f - \gamma_{BC} D^2 f) \tag{8.66}$$

$$= \frac{-i}{8\pi}\int d^2z\sqrt{\gamma}\,(D^B D^A g)\epsilon_A{}^C D_B D_C f + \frac{i}{16\pi}\int d^2z\sqrt{\gamma}\,(D^B D^A g)\epsilon_{AB} D^2 f. \tag{8.67}$$

The second term on the r.h.s. is zero, since $D^B D^A g$ is symmetric while ϵ_{AB} is antisymmetric. So we have just the first term,

$$\{Q_f^{\mathcal{H}^+}, \widetilde{Q}_g^{\mathcal{H}^+}\} = -\frac{i}{8\pi}\int d^2z\sqrt{\gamma}\,(D^B D^A g)\epsilon_A{}^C D_B D_C f \tag{8.68}$$

which we manipulate D_B to write

$$\{Q_f^{\mathcal{H}^+}, \widetilde{Q}_g^{\mathcal{H}^+}\} = -\frac{i}{8\pi}\left((1)+(2)\right), \tag{8.69}$$

$$(1) \equiv \int d^2z\sqrt{\gamma}D_B\left((D^BD^Ag)\epsilon_A{}^C D_C f\right), \tag{8.70}$$

$$(2) \equiv -\int d^2z\sqrt{\gamma}(D^2D^Ag)\epsilon_A{}^C D_C f. \tag{8.71}$$

Let's compute (1) first. This is of the form

$$\int d^2z\sqrt{\gamma}D_BV^B = i\int dz\wedge d\bar{z}\,\gamma_{z\bar{z}}(D_zV^z + D_{\bar{z}}V^{\bar{z}}) \tag{8.72}$$

$$= i\int dz\wedge d\bar{z}\,(\partial_zV_{\bar{z}} + \partial_{\bar{z}}V_z) \tag{8.73}$$

$$= i\oint_w d\bar{z}\,V_{\bar{z}} - i\oint_w dz\,V_z, \tag{8.74}$$

where in the first equation we have used $d^2z = idz\wedge d\bar{z}$ and $\sqrt{\gamma} = \gamma_{z\bar{z}}$, and in the second equation we have used the fact the only non-vanishing Christoffel symbols are Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ to write $D_zV_{\bar{z}} = \partial_zV_{\bar{z}}$ and $D_{\bar{z}}V_z = \partial_{\bar{z}}V_z$. The third equation is Stokes' theorem. This implies that (1) can be written as

$$(1) = i\oint_w d\bar{z}(D_{\bar{z}}D^Ag)\epsilon_A{}^C D_C f - i\oint dz(D_zD^Ag)\epsilon_A{}^C D_C f. \tag{8.75}$$

Everything is smooth except for $f = \frac{1}{z-w}$, so the first term with $\oint d\bar{z}$ never sees a pole in $\bar{z}$ and therefore vanishes. Writing out the second term while noting that the only non-vanishing components of $\epsilon_A{}^B$ are $\epsilon_z{}^z = -\epsilon_{\bar{z}}{}^{\bar{z}} = i$, we obtain

$$(1) = \oint dz(D_zD^zg)\partial_z f - \oint dz(D_zD^{\bar{z}}g)\partial_{\bar{z}} f, \tag{8.76}$$

where we have written $D_Cf = \partial_Cf$. The second term vanishes since it has $\partial_{\bar{z}}f = 2\pi\delta^2(z-w)$ and the contour never crosses w. We can partial integrate the first term and obtain via residue theorem using $f = \frac{1}{z-w}$,

$$(1) = -\oint dz(\partial_zD_zD^zg)f \tag{8.77}$$

$$= -\oint dz\frac{\partial_zD_zD^zg}{z-w} \tag{8.78}$$

$$= -2\pi i\partial_zD_zD^zg|_{z=w}. \tag{8.79}$$

Now we turn to (2) in (8.69), which reads

$$(2) = -\int d^2z\sqrt{\gamma}(D^2D^Ag)\epsilon_A{}^C D_C f \tag{8.80}$$

$$= -\int d^2z\sqrt{\gamma}D_C\left((D^2D^Ag)\epsilon_A{}^C f\right) + \int d^2z\sqrt{\gamma}(D_CD^2D^Ag)\epsilon_A{}^C f. \tag{8.81}$$

One can quickly see that the second term vanishes,

$$\epsilon_A{}^C D_CD^2D^Ag = \epsilon^{AC}D_CD^2D_Ag \tag{8.82}$$

$$= \epsilon^{AC}D_C[D^2, D_A]g + \epsilon^{AC}D_CD_AD^2g \tag{8.83}$$

$$= \epsilon^{AC}D_CD_Ag + \epsilon^{AC}D_CD_AD^2g \tag{8.84}$$

$$= 0, \tag{8.85}$$

since both D_CD_Ag and $D_CD_AD^2g$ are symmetric in A and C. Here we have used $[D^2, D_A]g = D_Ag$. So we are left with just

$$(2) = -\int d^2z\sqrt{\gamma}D_C\left((D^2D^Ag)\epsilon_A{}^C f\right), \tag{8.86}$$

which again is of the form (8.74), so we can write

$$(2) = -i\oint_w d\bar{z}(D^2D^zg)\epsilon_{z\bar{z}}f + i\oint_w dz(D^2D^{\bar{z}}g)\epsilon_{\bar{z}z}f \tag{8.87}$$

$$= \oint_w d\bar{z}\frac{(D^2D_{\bar{z}}g)}{z-w} + \oint_w dz\frac{(D^2D_zg)}{z-w}. \tag{8.88}$$

We have explicitly wrote out $f = \frac{1}{z-w}$, and used $\epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = i\gamma_{z\bar{z}}$. The first term is zero since there are no poles in $\bar{z}$, and the second term yields the residue at $z = w$,

$$(2) = 2\pi D^2D_zg|_{z=w}. \tag{8.89}$$

Collecting the results (8.79) and (8.89) and plugging them into (8.69), we obtain

$$\{Q_f^{\mathcal{H}^+}, \widetilde{Q}_g^{\mathcal{H}^+}\} = -\frac{i}{8\pi}\left((1) + (2)\right) \tag{8.90}$$

$$= \frac{1}{4}\left(-\partial_zD_zD^zg + D^2D_zg\right)\Big|_{z=w}. \tag{8.91}$$

We can manipulate the indices to simplify

$$-\partial_z D_z D^z g + D^2 D_z g = -\partial_z D^z D_z g + D^2 D_z g \tag{8.92}$$

$$= -D_z D^z D_z g + D_z D^z D_z g + D_{\bar{z}} D^{\bar{z}} D_z g \tag{8.93}$$

$$= D_{\bar{z}} D^{\bar{z}} D_z g \tag{8.94}$$

$$= D^z D_z D_z g, \tag{8.95}$$

where in the second line we have used $\partial_z D^z D_z g = D_z D_{\bar{z}} D^{\bar{z}} g = D_z D^z D_z g$. This finally leads to

$$\{Q_f^{\mathcal{H}^+}, \widetilde{Q}_g^{\mathcal{H}^+}\} = \frac{1}{4} D^z D_z D_z g|_{z=w}. \tag{8.96}$$

This is in agreement with our earlier result (8.62) for the infinitesimal bracket $\{\delta Q_f^{\mathcal{H}^+}, \delta\widetilde{Q}_g^{\mathcal{H}^+}\}$.

What is the meaning of this central term? To answer this question, we take a look at the simpler case of the electromagnetic charges of large gauge transformation, and see that a similar problem arises there.

8.6 Electromagnetism

Now we review the electromagnetic duality on the Schwarzschild horizon, which is parallel to the case of the future null infinity $\mathcal{I}^+$ since both $\mathcal{H}^+$ and $\mathcal{I}^+$ are null hypersurfaces. We refer the reader to [116, 117] for electromagnetism on $\mathcal{I}^+$.

Like the BMS charges, the electromagnetic charges split into the $\mathcal{H}^+$ and $\mathcal{X}^+$ contributions (8.13). Horizon contributions of the (soft) electric and magnetic charges are given by

$$\mathcal{Q}_\lambda^{\mathcal{H}^+} = \int_{\mathcal{H}^+} d\alpha \wedge *F, \tag{8.97}$$

$$\widetilde{\mathcal{Q}}_\lambda^{\mathcal{H}^+} = \int_{\mathcal{H}^+} d\alpha \wedge F, \tag{8.98}$$

where $\lambda(\Theta)$ is a function on the sphere. We use the curly letter $\mathcal{Q}$ to distinguish these charges from the diffeomorphism charges.

In the complex coordinates (8.25), we can write

$$\mathcal{Q}_\lambda^{\mathcal{H}^+} = -i \int_{\mathcal{H}^+} dv\, d^2z\, (\partial_{\bar{z}}\lambda(*F)_{vz} - \partial_z\lambda(*F)_{v\bar{z}}) \tag{8.99}$$

$$= - \int_{\mathcal{H}^+} dv\, d^2z\, (F_{vz}\partial_{\bar{z}}\lambda + F_{v\bar{z}}\partial_z\lambda)\,, \tag{8.100}$$

$$\widetilde{\mathcal{Q}}_\lambda^{\mathcal{H}^+} = -i \int_{\mathcal{H}^+} dv\, d^2z\, (F_{vz}\partial_{\bar{z}}\lambda - F_{v\bar{z}}\partial_z\lambda)\,. \tag{8.101}$$

In the temporal gauge $A_v = 0$, we have $F_{vz} = \partial_v A_z$ and

$$\mathcal{Q}_\lambda^{\mathcal{H}^+} = \int_{\mathcal{H}^+_-} d^2z\, (A_z\partial_{\bar{z}}\lambda + A_{\bar{z}}\partial_z\lambda)\,, \tag{8.102}$$

$$\widetilde{\mathcal{Q}}_\lambda^{\mathcal{H}^+} = -i \int_{\mathcal{H}^+} dv\, d^2z\, (F_{vz}\partial_{\bar{z}}\lambda - F_{v\bar{z}}\partial_z\lambda)\,. \tag{8.103}$$

The Poisson bracket has the form

$$\{A_z(v, z, \bar{z}), F_{v\bar{z}}(v', z', \bar{z}')\} = \delta(v - v')\delta^2(z - z'), \tag{8.104}$$

using which we obtain

$$\{\mathcal{Q}_\lambda^{\mathcal{H}^+}, \widetilde{\mathcal{Q}}_\sigma^{\mathcal{H}^+}\} = \int_{\mathcal{H}^+_-} d^2z \sqrt{\gamma}\, \epsilon^{AB}\partial_A\lambda\partial_B\sigma \tag{8.105}$$

$$= \int_{S^2} d\lambda \wedge d\sigma. \tag{8.106}$$

For λ with poles in z, this gives rise to a central term in the algebra, just as in the case of gravity.

To get rid of the central term in the algebra, one may imagine that there exists a boundary theory on $\mathcal{H}^+$ such that the anomalous term is canceled. For this purpose, let us consider a $U(1) \times U(1)$ Chern-Simons theory with two independent 1-form fields a and $\tilde{a}$ on the null surface Σ,

$$S = \frac{k}{4\pi} \int_\Sigma a \wedge d\tilde{a}. \tag{8.107}$$

Under an electric large gauge transformation a and $\tilde{a}$ transform as

$$a \;\to\; a + d\phi, \tag{8.108}$$

$$\tilde{a} \;\to\; \tilde{a}, \tag{8.109}$$

and under a magnetic large gauge transformation they transform as

$$a \;\to\; a, \tag{8.110}$$
$$\widetilde{a} \;\to\; \widetilde{a} + d\widetilde{\phi}. \tag{8.111}$$

From the action we find the equations of motion to be $da = 0$ and $d\widetilde{a} = 0$. Variation of the action yields

$$\begin{aligned}\delta S &= \frac{k}{4\pi}\int_{\Sigma}(\delta a \wedge d\widetilde{a} + a \wedge d\delta\widetilde{a}) &(8.112)\\ &= \frac{k}{4\pi}\int_{\Sigma}(\delta a \wedge d\widetilde{a} - da \wedge \delta\widetilde{a}) + \frac{k}{4\pi}\int_{\partial\Sigma} a \wedge \delta\widetilde{a}, &(8.113)\end{aligned}$$

from which we obtain the symplectic potential to be

$$\theta(a, \widetilde{a}, \delta a, \delta\widetilde{a}) = \frac{k}{4\pi} a \wedge \delta\widetilde{a}. \tag{8.114}$$

Accordingly, the symplectic current density is

$$\omega(a, \widetilde{a}, \delta_1 a, \delta_1\widetilde{a}, \delta_2 a, \delta_2\widetilde{a}) = \frac{k}{4\pi}\left(\delta_1 a \wedge \delta_2\widetilde{a} - \delta_2 a \wedge \delta_1\widetilde{a}\right). \tag{8.115}$$

Since there are two types of LGT's, we have two integrable charge variations. One is the electric charge,

$$\begin{aligned}\delta\mathcal{Q}_{\phi} &= \int_{\partial\Sigma}\omega(a, \widetilde{a}, \delta a, \delta\widetilde{a}, d\phi, 0) &(8.116)\\ &= -\frac{k}{4\pi}\int_{\partial\Sigma} d\phi \wedge \delta\widetilde{a}, &(8.117)\end{aligned}$$

the other is the magnetic charge,

$$\begin{aligned}\delta\widetilde{\mathcal{Q}}_{\phi} &= \int_{\partial\Sigma}\omega(a, \widetilde{a}, \delta a, \delta\widetilde{a}, 0, d\phi) &(8.118)\\ &= \frac{k}{4\pi}\int_{\partial\Sigma} \delta a \wedge d\phi. &(8.119)\end{aligned}$$

We can compute the algebra using either one of the variations,

$$\{\mathcal{Q}_{\phi}, \widetilde{\mathcal{Q}}_{\varphi}\} = \delta_{\phi}\widetilde{\mathcal{Q}}_{\varphi} = -\delta_{\varphi}\mathcal{Q}_{\phi}, \tag{8.120}$$

and one can see that we get the same answer for both cases,

$$\{\mathcal{Q}_\phi, \tilde{\mathcal{Q}}_\varphi\} = -\frac{k}{4\pi}\int_{\partial\Sigma} d\phi \wedge d\varphi. \tag{8.121}$$

The electric-electric and magnetic-magnetic brackets vanish regardless of the presence of poles,

$$\{\mathcal{Q}_\phi, \mathcal{Q}_\varphi\} = 0, \tag{8.122}$$

$$\{\tilde{\mathcal{Q}}_\phi, \tilde{\mathcal{Q}}_\varphi\} = 0. \tag{8.123}$$

Therefore, one finds the algebra to be exactly parallel to that of standard and dual LGT charges on the horizon. The algebra (8.121), (8.122) and (8.123) tells us that putting a $U(1) \times U(1)$ Chern-Simons theory with the proper choice of the level k on the horizon, we can get rid of the central term in the standard and dual LGT algebra.

8.7 Summary and Remarks

In this chapter, we have used the formalism of Godazgar, Godazgar and Perry [108, 109] to compute the standard and dual supertranslation charges on the future Schwarzschild horizon, see equations (8.23) and (8.24). We have demonstrated that having poles in the supertranslation parameter function leads to the algebra exhibiting a central term, proportional to triple derivative of the dual supertranslation parameter (see equations (8.62) and (8.96)), which is reminiscent of the case of near-horizon diffeomorphism algebra of rotating black holes [89, 154].

What does this imply for the standard and dual BMS algebra on the horizon? A similar central term is observed in the algebra of LGTs in electromagnetism, and we have seen that putting a $U(1) \times U(1)$ Chern-Simons theory may remove this term, see equations (8.106) and (8.121). Therefore, it is plausible that the central term of the supertranslation algebra hints at the existence of a gravitational Chern-Simons theory living on the horizon. What this gravitational theory should be is still not clear. We leave this for future investigation.

Appendix A

Convergence Constraints of Graviton Dressings

In this appendix, we briefly discuss some constraints that the gravitational dressing at null infinity must satisfy.

Starting from the interaction term, one can show [9] that the graviton cloud operator is of the form $e^{R(t)}$, where

$$R(t) = \frac{\kappa}{2} \int \widetilde{d^3p}\, \widetilde{d^3k}\, \rho(p) \frac{p^\mu p^\nu}{p \cdot k} \left(a^\dagger_{\mu\nu}(k) e^{-i\frac{p\cdot k}{\omega_p} t} - a_{\mu\nu}(k) e^{i\frac{p\cdot k}{\omega_p} t} \right). \tag{A.1}$$

We used the shorthand notation (1.10), and $\rho(p) = b^\dagger(p) b(p)$ is the number operator of the scalar particle. $e^{R(t)}$ maps the Fock space $\mathcal{H}_{\mathrm{F}}$ to the Faddeev-Kulish asymptotic space $\mathcal{H}_{\mathrm{as}}$, i.e.

$$e^{R(t)} \mathcal{H}_{\mathrm{F}} = \mathcal{H}_{\mathrm{as}} \ . \tag{A.2}$$

An operator of the form e^{R_f}, where R_f is given by

$$R_f = \frac{\kappa}{2} \int \widetilde{d^3p}\, \widetilde{d^3k}\, \rho(p) \left(f^{\mu\nu *} a^\dagger_{\mu\nu} - f^{\mu\nu} a_{\mu\nu} \right), \tag{A.3}$$

can be constructed such that e^{R_f} also yields the Faddeev-Kulish asymptotic space:

$$\mathcal{H}_{\mathrm{as}} = e^{R_f} \mathcal{H}_{\mathrm{F}} \ . \tag{A.4}$$

We wish to identify the constraints on $f^{\mu\nu}$ that allows the operator e^{R_f} to have this property.

To this end, let us use the Baker-Campbell-Hausdorff (BCH) formula to decompose $e^{R(t)}$ as

$$e^{R(t)} = e^{R_f} e^{R(t)-R_f} e^{-\frac{1}{2}[R_f, R(t)]}. \tag{A.5}$$

Demanding that $e^{R(t)-R_f}$ and $e^{-\frac{1}{2}[R_f,R(t)]}$ be unitary operators in the Fock space yields the desired property,

$$\mathcal{H}_{\text{as}} = e^{R(t)}\mathcal{H}_{\text{F}} = e^{R_f} e^{R(t)-R_f} e^{-\frac{1}{2}[R_f, R(t)]}\mathcal{H}_{\text{F}} = e^{R_f}\mathcal{H}_{\text{F}} \ . \tag{A.6}$$

Let us start with $e^{-[R_f,R(t)]/2}$. The definitions (A.1) and (A.3) tell us that both R_f and $R(t)$ are anti-Hermitian. Since the commutator of two anti-Hermitian operators is itself anti-Hermitian, $e^{-[R_f,R(t)]/2}$ is a unitary operator (up to normalization) as long as the commutator converges. By direct calculation, we obtain

$$\begin{aligned}[R_f, R(t)] =& \frac{\kappa^2}{8}\int \widetilde{d^3p_1}\,\widetilde{d^3p_2}\,\widetilde{d^3k}\,\rho(p_1)\rho(p_2) \\ &\times I_{\mu\nu\rho\sigma}\Big[f^{\mu\nu *}(p_1,k)\mathcal{P}^{\rho\sigma}(p_2,k) - f^{\mu\nu}(p_1,k)\mathcal{P}^{\rho\sigma *}(p_2,k)\Big],\end{aligned} \tag{A.7}$$

where

$$\mathcal{P}^{\mu\nu}(p,k) \equiv \frac{p^\mu p^\nu}{p\cdot k} e^{i\frac{p\cdot k}{\omega_p}t}. \tag{A.8}$$

This commutator involves the k-integral

$$\int \frac{d^3k}{\omega_k}\frac{p_2^\rho p_2^\sigma}{p_2\cdot k}\phi(p_1,k)\left\{\left(\frac{p_1^\mu p_1^\nu}{p_1\cdot k} + \frac{c^{\mu\nu *}}{\omega_k}\right)e^{i\frac{p_2\cdot k}{\omega_{p_2}}t} - \left(\frac{p_1^\mu p_1^\nu}{p_1\cdot k} + \frac{c^{\mu\nu}}{\omega_k}\right)e^{-i\frac{p_2\cdot k}{\omega_{p_2}}t}\right\}, \tag{A.9}$$

which has IR divergence if the leading term of $c_{\mu\nu}$ in k has non-zero imaginary part. Therefore, the unitarity of $e^{-\frac{1}{2}[R_f,R(t)]}$ demands

$$c^*_{\mu\nu}(p,k) - c_{\mu\nu}(p,k) = O(k). \tag{A.10}$$

The subleading terms of $c_{\mu\nu}$ does not contribute to the commutator (A.7); the asymptotic time t is taken to be very large, i.e. $|t| \to \infty$, and by virtue of the Riemann-Lebesgue lemma, the only contribution comes from small k.

Next, we consider $e^{R(t)-R_f}$. Using the BCH formula to write this in a normal-ordered form, we obtain

$$
\begin{aligned}
e^{R(t)-R_f} = \exp & \left\{ -\frac{\kappa^2}{16} \int \widetilde{d^3p_1}\, \widetilde{d^3p_2}\, \widetilde{d^3k}\, \rho(p_1)\rho(p_2) \left(\mathcal{P}_1^{\mu\nu *} - f_1^{\mu\nu *}\right) I_{\mu\nu\rho\sigma} \left(\mathcal{P}_2^{\rho\sigma} - f_2^{\rho\sigma}\right) \right\} \\
& \times \exp \left\{ \frac{\kappa}{2} \int \widetilde{d^3p}\, \widetilde{d^3k}\, \rho(p) \left(\mathcal{P}^{\mu\nu *} - f^{\mu\nu *}\right) a^\dagger_{\mu\nu} \right\} \\
& \times \exp \left\{ -\frac{\kappa}{2} \int \widetilde{d^3p}\, \widetilde{d^3k}\, \rho(p) \left(\mathcal{P}^{\mu\nu} - f^{\mu\nu}\right) a_{\mu\nu} \right\},
\end{aligned}
\tag{A.11}
$$

where $\mathcal{P}_i^{\mu\nu} \equiv \mathcal{P}^{\mu\nu}(p_i, k)$ and $f_i^{\mu\nu} \equiv f^{\mu\nu}(p_i, k)$ for $i = 1, 2$. The first exponential involves an integrand of the form

$$
\frac{1}{\omega_k} \left(\mathcal{P}_1^{\mu\nu *} - f_1^{\mu\nu *}\right) I_{\mu\nu\rho\sigma} \left(\mathcal{P}_2^{\rho\sigma} - f_2^{\rho\sigma}\right) \quad \xrightarrow{k\to 0} \quad \frac{1}{\omega_k^3} c^{\mu\nu *}(p_1, k) I_{\mu\nu\rho\sigma} c^{\rho\sigma}(p_2, k). \tag{A.12}
$$

Due to the gauge constraint (2.7), the leading term of $c_{\mu\nu}$ cannot cancel any poles $1/\omega_k$ in this limit, meaning that the integral exhibits IR divergence unless the leading term of (A.12) in k vanishes. Using (A.10) to write $c^{\mu\nu *} = c^{\mu\nu} + O(k)$ in (A.12), we find that the following constraint,

$$
c^{\mu\nu}(p_1, k) I_{\mu\nu\rho\sigma} c^{\rho\sigma}(p_2, k) = O(k) \quad \text{for all } p_1 \text{ and } p_2, \tag{A.13}
$$

is sufficient for $e^{R(t)-R_f}$ to form a unitary operator. Notice that when this is satisfied, the last two exponentials (A.11) form unitary operators in the Fock space as well.

The subleading $O(k)$ terms in (A.13), which also include the subleading terms in (A.10) due to rewriting $c^{\mu\nu *} = c^{\mu\nu} + O(k)$, give a finite value to the k-integral in (A.11). These terms therefore only contribute to the normalization of the states and can be ignored.

Appendix B

Cancellation of Infrared Divergence

By constructing asymptotic states analogous to that of Faddeev and Kulish [8], IR divergence in gravity was shown [9] to cancel to all loop orders for single-particle asymptotic states by making a convenient choice of $c_{\mu\nu}$, i.e. $c^{\mu\nu}(p,k)\epsilon^{\pm}_{\mu\nu}(k) = 0$. Here we generalize this to multi-particle asymptotic states using a general $c_{\mu\nu}$ that is only subject to the basic constraints (2.7)-(2.9). We set $\phi(p,k) = 1$ without any loss of generality since this only changes the overall normalization of the states.

The equations involved turn out to be cumbersome, so let us begin by laying down some shorthand notations. We remind the reader that the dressed creation and annihilation operators of the scalar particle take the form

$$e^{R_f(p)}b^\dagger(p) = \exp\left[\frac{\kappa}{2}\int \widetilde{d^3k}\ \left(f_{\mu\nu}(p,k)a^{\dagger\mu\nu}(k) - f_{\mu\nu}(p,k)a^{\mu\nu}(k)\right)\right] b^\dagger(p) \tag{B.1}$$

$$e^{-R_f(p)}b(p) = \exp\left[-\frac{\kappa}{2}\int \widetilde{d^3k}\ \left(f_{\mu\nu}(p,k)a^{\dagger\mu\nu}(k) - f_{\mu\nu}(p,k)a^{\mu\nu}(k)\right)\right] b(p)\ . \tag{B.2}$$

The dressings $e^{\pm R_f(p)}$ commute with the undressed operators b, $b^\dagger$. If we define

$$S_{\mu\nu}(p,k) = \frac{\kappa}{2}f_{\mu\nu}(p,k) \tag{B.3}$$

$$P_{\mu\nu}(p,k) = \frac{\kappa}{2}\left(\frac{p_\mu p_\nu}{p\cdot k}\right) \tag{B.4}$$

$$C_{\mu\nu}(p,k) = \frac{\kappa}{2}\frac{c_{\mu\nu}(p,k)}{\omega_k} \tag{B.5}$$

so that $S_{\mu\nu} = P_{\mu\nu} + C_{\mu\nu}$, we have

$$e^{R_f(p)} b^\dagger(p) = \exp\left[\int \widetilde{d^3k} \left(S_{\mu\nu}(p,k)a^{\dagger\mu\nu}(k) - S_{\mu\nu}(p,k)a^{\mu\nu}(k)\right)\right] b^\dagger(p) \tag{B.6}$$

$$e^{-R_f(p)} b(p) = \exp\left[-\int \widetilde{d^3k} \left(S_{\mu\nu}(p,k)a^{\dagger\mu\nu}(k) - S_{\mu\nu}(p,k)a^{\mu\nu}(k)\right)\right] b(p). \tag{B.7}$$

We use the superscript "in" ("out") to denote the quantity summed over all incoming (outgoing) scalar particles. The superscript "tot" denotes the difference between "out" and "in". For example,

$$S^{\text{in}}_{\mu\nu}(k) = \sum_{i\in\text{in}} S_{\mu\nu}(p_i,k), \quad S^{\text{out}}_{\mu\nu}(k) = \sum_{i\in\text{out}} S_{\mu\nu}(p_i,k), \quad S^{\text{tot}}_{\mu\nu} = S^{\text{out}}_{\mu\nu} - S^{\text{in}}_{\mu\nu}, \tag{B.8}$$

$$P^{\text{in}}_{\mu\nu}(k) = \sum_{i\in\text{in}} P_{\mu\nu}(p_i,k), \quad P^{\text{out}}_{\mu\nu}(k) = \sum_{i\in\text{out}} P_{\mu\nu}(p_i,k), \quad P^{\text{tot}}_{\mu\nu} = P^{\text{out}}_{\mu\nu} - P^{\text{in}}_{\mu\nu}, \tag{B.9}$$

$$C^{\text{in}}_{\mu\nu}(k) = \sum_{i\in\text{in}} C_{\mu\nu}(p_i,k), \quad C^{\text{out}}_{\mu\nu}(k) = \sum_{i\in\text{out}} C_{\mu\nu}(p_i,k), \quad C^{\text{tot}}_{\mu\nu} = C^{\text{out}}_{\mu\nu} - C^{\text{in}}_{\mu\nu}. \tag{B.10}$$

We sometimes write

$$S^n_{\mu\nu} \equiv S_{\mu\nu}(p_n,k) \tag{B.11}$$

in contexts where the graviton momentum k is unambiguous.

B.1 Sources of infrared divergence

Listed below are the possible sources of IR divergence:

1. Virtual gravitons. It is well known that only the virtual gravitons connecting two external legs produce IR divergence, and that their contribution exponentiates [2]. This contribution takes the form [9]

$$\exp\left[-\frac{\kappa^2}{128\pi^3}\sum_{n,m}\int \frac{d^3k}{\omega_k}\frac{\eta_n\eta_m\left[(p_n\cdot p_m)^2 - (1/2)p_n^2p_m^2\right]}{(p_n\cdot k)(p_m\cdot k)}\right], \tag{B.12}$$

   where each sum runs over the external particles. $\eta = +1$ for an outgoing particle, and $\eta = -1$ for an incoming particle.

2. Real gravitons. External soft gravitons are another source of IR divergence [2]. In this section the external states involve gravitons only in the form of Faddeev-Kulish clouds.

3. Interacting gravitons. We reserve the term "interacting" to denote the gravitons that

connect a Faddeev-Kulish cloud to either an external or an internal leg. We follow the procedure analogous to the work of Chung [4] to factor out the IR divergence from this type of contribution.

4. Cloud-to-cloud gravitons. These gravitons propagate from one cloud to another. We can further group these into two types:

   (a) "Disconnected" gravitons. We use this term to denote gravitons that connect the cloud of an incoming particle with the cloud of an outgoing particle.

   (b) In-to-in/out-to-out gravitons. In-to-in (out-to-out) gravitons connect two incoming (outgoing) clouds. Note that the graviton can be emitted and absorbed by the same cloud, see figures B.3(b) and B.3(c).

B.2 Single-particle external states, cancellation to one loop

We start with the case of single-scalar in, single-scalar out, and show that the divergent factors cancel to second-order in the interaction. In the next subsection we see how this generalizes to multiple-scalar in, multiple-scalar out, and show the cancellation to all orders of interaction.

Consider the single-scalar asymptotic in-state

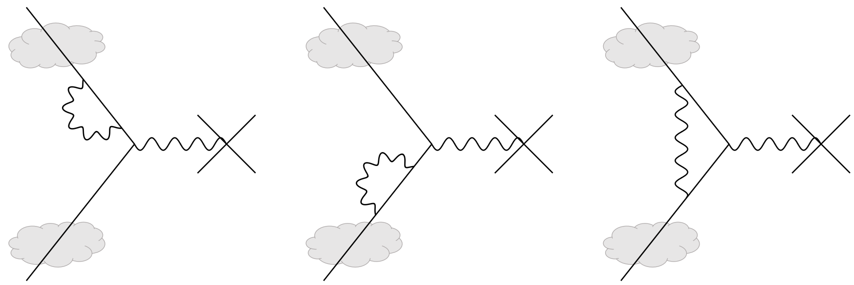
$$
\begin{aligned}
|\mathrm{i}\rangle &= e^{R_f(p_i)} b^\dagger(p_i)\,|0\rangle && \text{(B.13)}\\
&= \exp\left[\int \widetilde{d^3k}\,\left(S^i_{\mu\nu} a^{\dagger\mu\nu} - S^i_{\mu\nu} a^{\mu\nu}\right)\right] b^\dagger(p_i)\,|0\rangle\,. && \text{(B.14)}
\end{aligned}
$$

The commutator

$$
\left[\left(\int \widetilde{d^3k}\, S^i_{\mu\nu} a^{\dagger\mu\nu}\right), \left(-\int \widetilde{d^3k}\, S^i_{\mu\nu} a^{\mu\nu}\right)\right] = \frac{1}{2}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma} \tag{B.15}
$$

is a c-number, so we can use the BCH formula $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ to write

$$
\begin{aligned}
&\exp\left[\int \widetilde{d^3k}\,\left(S^i_{\mu\nu} a^{\dagger\mu\nu} - S^i_{\mu\nu} a^{\mu\nu}\right)\right]\\
&= \exp\left(\int \widetilde{d^3k}\, S^i_{\mu\nu} a^{\dagger\mu\nu}\right) \exp\left(-\int \widetilde{d^3k}\, S^i_{\mu\nu} a^{\mu\nu}\right) \exp\left(-\frac{1}{4}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma}\right).
\end{aligned} \tag{B.16}
$$



(a) (b) (c)

Figure B.1: Contributions of a virtual graviton.

Therefore, the in-state may be written as

$$|\mathrm{i}\rangle = \exp\left(-\frac{1}{4}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma}\right) \exp\left(\int \widetilde{d^3k}\, S^i_{\mu\nu} a^{\dagger\mu\nu}\right) b^\dagger(p_i)\,|0\rangle\,, \tag{B.17}$$

since $a^{\mu\nu}$ commutes with $b^\dagger$ and annihilates the vacuum. To the lowest order, this is

$$|\mathrm{i}\rangle = \left(1 - \frac{1}{4}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma} + \int \widetilde{d^3k}\, S^i_{\mu\nu} a^{\dagger\mu\nu}\right) b^\dagger(p_i)\,|0\rangle\,. \tag{B.18}$$

Similarly, we may write the asymptotic out-state as

$$\langle \mathrm{f}| = \langle 0|\, b(p_f) e^{-R_f(p_f)} \tag{B.19}$$

$$= \langle 0|\, b(p_f) \exp\left[-\int \widetilde{d^3k}\, \left(S^f_{\mu\nu} a^{\dagger\mu\nu} - S^f_{\mu\nu} a^{\mu\nu}\right)\right] \tag{B.20}$$

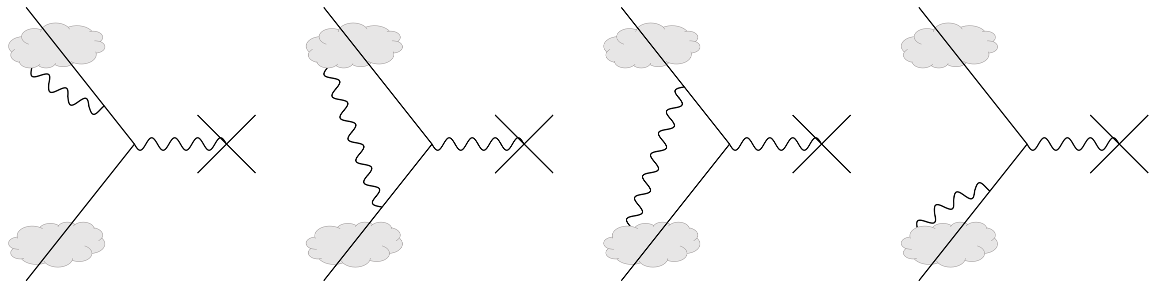
$$= \langle 0|\, b(p_f) \exp\left(\int \widetilde{d^3k}\, S^f_{\mu\nu} a^{\mu\nu}\right) \exp\left(-\frac{1}{4}\int \widetilde{d^3k}\, S^f_{\mu\nu} I^{\mu\nu\rho\sigma} S^f_{\rho\sigma}\right), \tag{B.21}$$

or to the lowest order,

$$\langle \mathrm{f}| = \langle 0|\, b(p_f) \left(1 - \frac{1}{4}\int \widetilde{d^3k}\, S^f_{\mu\nu} I^{\mu\nu\rho\sigma} S^f_{\rho\sigma} + \int \widetilde{d^3k}\, S^f_{\mu\nu} a^{\mu\nu}\right). \tag{B.22}$$

We now demonstrate that an amplitude of the form $\langle \mathrm{f}|\mathcal{S}|\mathrm{i}\rangle$ is free of IR divergence.

Let us begin with the contribution of virtual gravitons. Diagrams that fall into this category are given in figure B.1. (B.12) sums up these contributions, which in our case of



(a) (b) (c) (d)

Figure B.2: Contributions of interacting gravitons.

single particle external states can be written as

$$\exp\left[-\kappa^2 \sum_{n,m} \int \frac{d^3k}{128\pi^3\omega_k} \frac{\eta_n\eta_m \left[(p_n \cdot p_m)^2 - (1/2)p_n^2 p_m^2\right]}{(p_n \cdot k)(p_m \cdot k)}\right] \tag{B.23}$$

$$\approx 1 - \kappa^2 \sum_{n,m} \int \frac{d^3k}{128\pi^3\omega_k} \frac{\eta_n\eta_m \left[(p_n \cdot p_m)^2 - (1/2)p_n^2 p_m^2\right]}{(p_n \cdot k)(p_m \cdot k)} \tag{B.24}$$

$$= 1 - \frac{\kappa^2}{128\pi^3} \int \frac{d^3k}{\omega_k} \left[\frac{p_f^4}{2(p_f \cdot k)^2} + \frac{p_i^4}{2(p_i \cdot k)^2} - 2\left(\frac{(p_f \cdot p_i)^2 - \frac{1}{2}p_f^2 p_i^2}{(p_f \cdot k)(p_i \cdot k)}\right)\right]. \tag{B.25}$$

Thus we find the contribution $A^{(1)}_{\text{virt}}$ of virtual gravitons to be

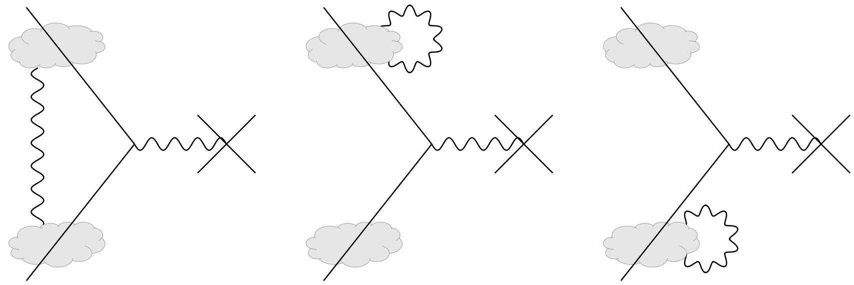
$$A^{(1)}_{\text{virt}} = -\frac{\kappa^2}{128\pi^3} \int \frac{d^3k}{\omega_k} \left[\frac{p_f^4}{2(p_f \cdot k)^2} + \frac{p_i^4}{2(p_i \cdot k)^2} - 2\left(\frac{(p_f \cdot p_i)^2 - \frac{1}{2}p_f^2 p_i^2}{(p_f \cdot k)(p_i \cdot k)}\right)\right], \tag{B.26}$$

where the superscript (1) emphasizes that this is the leading term in the interaction.

Next, we consider the contributions of interacting gravitons. There are four diagrams that are IR-divergent, which are shown in figure B.2. Contribution from figure B.2(a) yields a factor of

$$\int \widetilde{d^3k}\, S^f_{\mu\nu} \frac{1}{2} I^{\mu\nu\rho\sigma} \left(\frac{-i}{2p^f \cdot k}\right) \left(i\kappa p^f_\rho p^f_\sigma\right) = \frac{1}{2} \int \widetilde{d^3k}\, S^f_{\mu\nu} I^{\mu\nu\rho\sigma} P^f_{\rho\sigma}\ , \tag{B.27}$$

where $-i/2p^f \cdot k$ is the propagator, $i\kappa p^f_\rho p^f_\sigma$ comes from the vertex rule, and the rest comes from the contraction of an outgoing cloud and $h^{\rho\sigma}(x)$. Similarly, diagrams (b), (c) and (d)



(a) (b) (c)

Figure B.3: Contributions of cloud-to-cloud gravitons.

contribute the following factors respectively:

$$-\frac{1}{2}\int \widetilde{d^3k}\, S^f_{\mu\nu} I^{\mu\nu\rho\sigma} P^i_{\rho\sigma} \tag{B.28}$$

$$-\frac{1}{2}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} P^f_{\rho\sigma} \tag{B.29}$$

$$\frac{1}{2}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} P^i_{\rho\sigma} \tag{B.30}$$

The net contribution of interacting graviton is the sum of (B.27)-(B.30), which reads

$$\frac{1}{2}\int \widetilde{d^3k}\, \left(S^f_{\mu\nu} - S^i_{\mu\nu}\right) I^{\mu\nu\rho\sigma} \left(P^f_{\rho\sigma} - P^i_{\rho\sigma}\right) = \frac{1}{2}\int \widetilde{d^3k}\, S^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma}\ . \tag{B.31}$$

The last contribution comes from the cloud-to-cloud gravitons. There are three diagrams that correspond to this category, shown in figure B.3. Figure B.3(a) shows the “disconnected" graviton line. Recalling that the initial and final states are

$$|\text{i}\rangle = \left(1 - \frac{1}{4}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma} + \int \widetilde{d^3k}\, S^i_{\mu\nu} a^{\dagger\mu\nu}\right) b^\dagger(p_i)\, |0\rangle \tag{B.32}$$

$$\langle \text{f}| = \langle 0|\, b(p_f) \left(1 - \frac{1}{4}\int \widetilde{d^3k}\, S^f_{\mu\nu} I^{\mu\nu\rho\sigma} S^f_{\rho\sigma} + \int \widetilde{d^3k}\, S^f_{\mu\nu} a^{\mu\nu}\right), \tag{B.33}$$

we can see that the disconnected line corresponds to the contraction of the last terms of (B.32) and (B.33),

$$\int \widetilde{d^3k}\, \widetilde{d^3k'}\, S_{\mu\nu}(p_f, k) S_{\rho\sigma}(p_i, k')\, \langle 0|a^{\mu\nu}(k) a^{\dagger\rho\sigma}(k')|0\rangle \tag{B.34}$$

$$= \frac{1}{2}\int \widetilde{d^3k}\, S^f_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma} \tag{B.35}$$

$$= \frac{1}{4}\int \widetilde{d^3k}\, S^f_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma} + \frac{1}{4}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} S^f_{\rho\sigma}\ , \tag{B.36}$$

where in the last equation we used the symmetry of $I^{\mu\nu\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}$ under

$(\mu\nu) \leftrightarrow (\rho\sigma)$. Figures B.3(b) and B.3(c) are the out-to-out and in-to-in graviton lines, respectively. These contribute a factor coming from the second terms of (B.33) and (B.32),

$$-\frac{1}{4}\int \widetilde{d^3k}\,\left(S^f_{\mu\nu} I^{\mu\nu\rho\sigma} S^f_{\rho\sigma} + S^i_{\mu\nu} I^{\mu\nu\rho\sigma} S^i_{\rho\sigma}\right), \tag{B.37}$$

which, combined with (B.36), form the cloud-to-cloud contribution

$$-\frac{1}{4}\int \widetilde{d^3k}\,\left(S^f_{\mu\nu} - S^i_{\mu\nu}\right) I^{\mu\nu\rho\sigma} \left(S^f_{\rho\sigma} - S^i_{\rho\sigma}\right) = -\frac{1}{4}\int \widetilde{d^3k}\, S^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\text{tot}}_{\rho\sigma}\ . \tag{B.38}$$

The leading contribution $A^{(1)}_{\text{cloud}}$ involving the clouds can therefore be written as the sum of (B.31) and (B.38):

$$A^{(1)}_{\text{cloud}} = -\frac{1}{4}\int \widetilde{d^3k}\, S^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\text{tot}}_{\rho\sigma} + \frac{1}{2}\int \widetilde{d^3k}\, S^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma}\ . \tag{B.39}$$

Noting that $S^{\text{tot}}_{\mu\nu} = P^{\text{tot}}_{\mu\nu} + C^{\text{tot}}_{\mu\nu}$, we write

$$\begin{aligned}
A^{(1)}_{\text{cloud}} &= \frac{1}{4}\int \widetilde{d^3k}\, I^{\mu\nu\rho\sigma}\Big[-\left(P^{\text{tot}}_{\mu\nu} + C^{\text{tot}}_{\mu\nu}\right)\left(P^{\text{tot}}_{\rho\sigma} + C^{\text{tot}}_{\rho\sigma}\right) + 2\left(P^{\text{tot}}_{\mu\nu} + C^{\text{tot}}_{\mu\nu}\right) P^{\text{tot}}_{\rho\sigma}\Big] && \text{(B.40)}\\
&= \frac{1}{4}\int \widetilde{d^3k}\, P^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma} - \frac{1}{4}\int \widetilde{d^3k}\, C^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} C^{\text{tot}}_{\rho\sigma}. && \text{(B.41)}
\end{aligned}$$

The second term involving the integral

$$\int \frac{d^3k}{\omega_k^3}\left(c^f_{\mu\nu} - c^i_{\mu\nu}\right) I^{\mu\nu\rho\sigma}\left(c^f_{\rho\sigma} - c^i_{\rho\sigma}\right), \tag{B.42}$$

derives solely from the interactions between graviton clouds. Note that in this case of single-particle states, we cannot use different $c_{\mu\nu}$ for the incoming and outgoing particles, since that renders the integral (B.42) divergent. This point becomes more clear when we study the case of multi-particle states in the next subsection. This term thus vanishes due to the convergence constraint (2.9). Then we are left with

$$\begin{aligned}
A^{(1)}_{\text{cloud}} &= \frac{1}{4}\int \widetilde{d^3k}\, P^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma} && \text{(B.43)}\\
&= \frac{\kappa^2}{16}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(\frac{p^f_\mu p^f_\nu}{p^f\cdot k} - \frac{p^i_\mu p^i_\nu}{p^i\cdot k}\right) I^{\mu\nu\rho\sigma}\left(\frac{p^f_\rho p^f_\sigma}{p^f\cdot k} - \frac{p^i_\rho p^i_\sigma}{p^i\cdot k}\right) && \text{(B.44)}\\
&= \frac{\kappa^2}{128\pi^3}\int \frac{d^3k}{\omega_k}\left[\frac{p_f^4}{2(p_f\cdot k)^2} + \frac{p_i^4}{2(p_i\cdot k)^2} - 2\left(\frac{(p_f\cdot p_i)^2 - \frac{1}{2}p_f^2 p_i^2}{(p_f\cdot k)(p_i\cdot k)}\right)\right]. && \text{(B.45)}
\end{aligned}$$

This is precisely $A^{(1)}_{\text{virt}}$ with the opposite sign, and therefore cancels the contribution of the virtual gravitons.

B.3 Multi-particle external states, cancellation to all orders

To all loop orders, the contribution A_{virt} of soft gravitons in loops is given by (B.12), which reads

$$A_{\text{virt}} = \exp\left[-\frac{\kappa^2}{128\pi^3}\sum_{n,m}\int \frac{d^3k}{\omega_k}\frac{\eta_n\eta_m\left[(p_n\cdot p_m)^2-(1/2)p_n^2p_m^2\right]}{(p_n\cdot k)(p_m\cdot k)}\right] \tag{B.46}$$

$$= \exp\left(-\frac{1}{4}\sum_{n,m}\eta_n\eta_m\int \widetilde{d^3k}\, P^n_{\mu\nu}I^{\mu\nu\rho\sigma}P^m_{\rho\sigma}\right), \tag{B.47}$$

where the summation indices n and m run over all external particles.

Next we compute the interacting gravitons' contribution. To this end, let us first examine how the insertion of a soft graviton affects the amplitude of a diagram, following the procedure analogous to that of Chung [4] for QED. Suppose we have a diagram with amplitude $M^{(0)}$ that does not contain any soft gravitons. Inserting a soft graviton $a_{\mu_1\nu_1}(k_1)$ or $a^\dagger_{\mu_1\nu_1}(k_1)$ gives us a new amplitude

$$M^{(1)}_{\mu_1\nu_1}(k_1) = \pm P^{\text{tot}}_{\mu_1\nu_1}(k)M^{(0)} + \widetilde{\xi}_{\mu_1\nu_1}(k_1), \tag{B.48}$$

where the net soft factor $P^{\text{tot}}_{\mu_1\nu_1}(k)$ comes from attaching the graviton to the external legs, and $\widetilde{\xi}_{\mu_1\nu_1}(k_1)$ comes from attaching it to the body of the diagram and does not contain IR divergence in k_1. The $+$ $(-)$ sign corresponds to emission (absorption) of the graviton. The Lorentz indices μ_1 and ν_1 eventually contract with the clouds $\int \widetilde{d^3k}\, S_{\rho\sigma}\frac{1}{2}I^{\rho\sigma\mu\nu}$, but we leave them free for now. We can see that an amplitude $M^{(n)}_{\mu_1\nu_1\cdots\mu_n\nu_n}$ with n real soft gravitons may be written as

$$\begin{aligned} M^{(n)}_{\mu_1\nu_1\cdots\mu_n\nu_n}(k_1,\cdots,k_n) = \pm P^{\text{tot}}_{\mu_n\nu_n}(k_n)M^{(n-1)}_{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(k_1,\cdots,k_{n-1}) \\ + \widetilde{\xi}_{\mu_1\nu_1\cdots\mu_n\nu_n}(k_1,\cdots,k_{n-1};k_n), \end{aligned} \tag{B.49}$$

where $\widetilde{\xi}_{\mu_1\nu_1\cdots\mu_n\nu_n}(k_1,\cdots,k_{n-1};k_n)$ does not contain IR divergence in k_n. We know from [155] that such equation can be unwound as a sum over all permutations of the gravitons, in this case represented by the labels (μ,ν,k)'s:

$$M^{(n)}_{\mu_1\nu_1\cdots\mu_n\nu_n}(k_1,\cdots,k_n)=\sum_{s=0}^{n}\sum_{\substack{\text{Perm}\\(\mu,\nu,k)}}\frac{(-1)^m}{s!(n-s)!}\left[\prod_{i=1}^{s}P^{\text{tot}}_{\mu_i\nu_i}(k_i)\right]\xi_{\mu_{s+1}\nu_{s+1}\cdots\mu_n\nu_n}(k_{s+1},\cdots,k_n),\tag{B.50}$$

where m is the number of absorbed gravitons and ξ's are some IR-convergent functions symmetric in the gravitons, or equivalently in the labels (μ,ν,k)'s.

We now examine the amplitude of a diagram with N (N') interacting soft gravitons that connect to the clouds of incoming (outgoing) scalars. This puts $n=N+N'$, so let us write

$$M^{(N+N')}_{\mu_1\nu_1\cdots\mu_{N+N'}\nu_{N+N'}}(k_1,\cdots,k_{N+N'})=(-1)^N\sum_{s=0}^{N+N'}\sum_{\substack{\text{Perm}\\(\mu,\nu,k)}}\frac{M^{(N+N',s)}_{\mu_1\nu_1\cdots\mu_{N+N'}\nu_{N+N'}}(k_1,\cdots,k_{N+N'})}{s!(N+N'-s)!}\tag{B.51}$$

with the restricted amplitude defined by

$$M^{(n,s)}_{\mu_1\nu_1\cdots\mu_n\nu_n}(k_1,\cdots,k_n)\equiv\left[\prod_{i=1}^{s}P^{\text{tot}}_{\mu_i\nu_i}(k_i)\right]\xi_{\mu_{s+1}\nu_{s+1}\cdots\mu_n\nu_n}(k_{s+1},\cdots,k_n),\tag{B.52}$$

representing the sum of all diagrams where *the first* s gravitons connect to external legs and the rest to internal legs. One such diagram is shown in figure B.4. The product $\prod_i P^{\text{tot}}_{\mu_i\nu_i}$ is the IR-divergent factor due to the gravitons (red in the figure) connecting to external legs. The function $\xi_{\mu_{s+1}\nu_{s+1}\cdots}$ is the contribution of the remaining gravitons (blue in the figure) connecting to internal legs. One can see that $M^{(n,s)}_{\mu_1\nu_1\cdots}$ is symmetric in the in the first s and the last $N+N'-s$ labels (μ,ν,k). The expression

$$\sum_{\substack{\text{Perm}\\(\mu,\nu,k)}}\frac{1}{s!(N+N'-s)!}M^{(N+N',s)}_{\mu_1\nu_1\cdots\mu_{N+N'}\nu_{N+N'}}(k_1,\cdots,k_{N+N'})\tag{B.53}$$

hence represents the sum of all diagrams that have $N+N'$ interacting gravitons where any s of them are connected to the external legs. Since $M^{(N+N')}_{\mu_1\nu_1\cdots}$ sums over these diagrams for all $0\le s\le N+N'$, apart from the factor $(-1)^N$, it represents the amplitude (with loose ends) of a process involving $N+N'$ interacting gravitons.

Now we connect the loose ends to the graviton clouds. Let us restrict our attention to a specific configuration, where the ith (jth) incoming (outgoing) cloud has N_i (N'_j) interacting gravitons connected to it, so that $\sum_{i\in\text{in}}N_i=N$ and $\sum_{j\in\text{out}}N'_j=N'$. Later we sum over all possible configurations. As we saw in the case of a single-particle state, connecting a

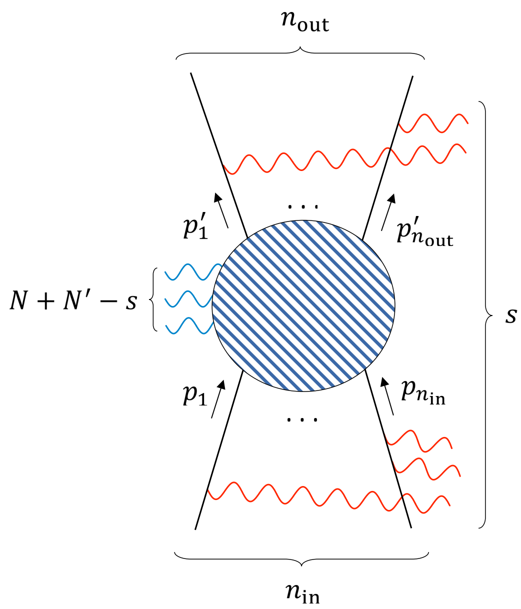


Figure B.4: A diagram with n_{in} incoming, n_{out} outgoing scalar particles, and $N + N'$ interacting gravitons. The loose ends of graviton lines connect to the clouds, which are not drawn here. There are s gravitons (colored red) connected to external legs, each contributing an IR-divergent factor $\pm P_{\mu\nu}$. The remaining $N + N' - s$ gravitons (colored blue) connect to the internal legs and constitute the IR-convergent part ξ.

graviton to the cloud of an external particle having momentum p amounts to contracting with an expression of the form

$$\frac{1}{2}\int \widetilde{d^3k}\, S_{\mu\nu}(p,k) I^{\mu\nu\rho\sigma}. \tag{B.54}$$

The Lorentz indices ρ and σ contract with the indices of the corresponding loose end. After connecting all loose ends, we have $N + N'$ copies of these integrals with varying momenta p, their Lorentz indices contracted with $M^{(N+N')}_{\mu_1\nu_1\cdots}$. The order in which the gravitons are connected is irrelevant as long as we have the same configuration ($\{N_i\}$, $\{N'_j\}$), because in $M^{(N+N')}_{\mu_1\nu_1\cdots}$ we are summing over all permutations of the loose ends. Since order does not matter, let us simply connect the first N_1 gravitons $(\mu_1, \nu_1, k_1), \cdots, (\mu_{N_1}, \nu_{N_1}, k_{N_1})$ to the first incoming cloud, the next N_2 gravitons to the second cloud, and so on. By the time we exhaust all of the incoming clouds, we would have connected N gravitons, leaving us with N' loose ends. Then, we repeat this procedure for the outgoing clouds - connect the first N'_1 among the N' leftover gravitons to the first outgoing cloud, etc. For notational simplicity, let us define the sequence

$$(a_i)_{i=1}^{N+N'} = \Bigg(\overbrace{\underbrace{p_1, \cdots, p_1}_{N_1}, \underbrace{p_2, \cdots, p_2}_{N_2}, \cdots, \underbrace{p_{n_{\rm in}}, \cdots, p_{n_{\rm in}}}_{N_{n_{\rm in}}}}^{N}, \overbrace{\underbrace{p'_1, \cdots, p'_1}_{N'_1}, \cdots, \underbrace{p'_{n_{\rm out}}, \cdots, p'_{n_{\rm out}}}_{N'_{n_{\rm out}}}}^{N'} \Bigg). \tag{B.55}$$

Using this, we can connect all loose ends by writing

$$\left[\prod_{r=1}^{N+N'} \frac{1}{2} \int \widetilde{d^3k_r}\, S^{a_r}_{\mu\nu} I^{\mu\nu\rho_r\sigma_r} \right] M^{(N+N')}_{\rho_1\sigma_1\cdots\rho_{N+N'}\sigma_{N+N'}}(k_1, \cdots, k_{N+N'}), \tag{B.56}$$

where $S^{a_r}_{\mu\nu} \equiv S_{\mu\nu}(a_r, k)$. Writing out the expression for $M^{(N+N')}_{\rho_1\sigma_1\cdots}$, we obtain

$$\begin{aligned} &(-1)^N \sum_{s=0}^{N+N'} \left[\prod_{r=1}^{N+N'} \frac{1}{2} \int \widetilde{d^3k_r}\, S^{a_r}_{\mu\nu} I^{\mu\nu\rho_r\sigma_r} \right] \\ &\quad \times \sum_{\substack{\text{Perm} \\ (\rho,\sigma,k)}} \left[\frac{1}{s!} \prod_{i=1}^{s} P^{\rm tot}_{\rho_i\sigma_i}(k_i) \right] \frac{\xi_{\rho_{s+1}\sigma_{s+1}\cdots\rho_{N+N'}\sigma_{N+N'}}(k_{s+1}, \cdots, k_{N+N'})}{(N+N'-s)!}. \end{aligned} \tag{B.57}$$

The summand of $\sum_s$ is the sum of all diagrams where s of the $N+N'$ interacting gravitons are being connected to external legs. For a given s, let us say there are s_i (s'_i) gravitons connecting the ith (jth) incoming (outgoing) cloud to external legs, so that $\sum_{i\in\rm in} s_i + \sum_{j\in\rm out} s'_j = s$. Then, instead of summing over the total number s, we can sum over each of the numbers s_i an s'_i. This yields

$$\begin{aligned} &(-1)^N \left[\prod_{i\in\rm in} \sum_{s_i=0}^{N_i} \right] \left[\prod_{j\in\rm out} \sum_{s'_j=0}^{N'_j} \right] \left[\prod_{r=1}^{N+N'} \frac{1}{2} \int \widetilde{d^3k_r}\, S^{a_r}_{\mu\nu} I^{\mu\nu\rho_r\sigma_r} \right] \\ &\quad \times \sum_{\substack{\text{Perm} \\ (\rho,\sigma,k)}} \left[\frac{1}{s!} \prod_{i=1}^{s} P^{\rm tot}_{\rho_i\sigma_i}(k_i) \right] \frac{\xi_{\rho_{s+1}\sigma_{s+1}\cdots\rho_{N+N'}\sigma_{N+N'}}(k_{s+1}, \cdots, k_{N+N'})}{(N+N'-s)!}, \end{aligned} \tag{B.58}$$

where s is now defined as $s = \sum_{i\in\rm in} s_i + \sum_{j\in\rm out} s'_j$. Among the $N+N'$ copies of $\frac{1}{2}\int \widetilde{d^3k}\, S_{\mu\nu} I^{\mu\nu\rho\sigma}$, s copies contract with $\prod P_{\rho\sigma}$ (corresponding to external legs) and form the IR-divergent factor; the remaining $N+N'-s$ copies contract with $\xi_{\rho\sigma\cdots}$ and end up in the IR-convergent part. For the ith incoming cloud, there are N_i copies of $\frac{1}{2}\int \widetilde{d^3k}\, S_{\mu\nu} I^{\mu\nu\rho\sigma}$ but only s_i copies of $\prod P^{\rm tot}_{\rho\sigma}$, which indicates that we get $\binom{N_i}{s_i}$ identical contractions. By the same token, the jth outgoing cloud has $\binom{N'_j}{s'_j}$ identical contractions. Therefore, contracting the indices and distributing $(-1)^N$ yields

$$\left[\prod_{i\in\text{in}}\sum_{s_i=0}^{N_i}\binom{N_i}{s_i}\left(-\frac{1}{2}\int\widetilde{d^3k}\,S^i_{\mu\nu}I^{\mu\nu\rho\sigma}P^{\text{tot}}_{\rho\sigma}\right)^{s_i}\right]\left[\prod_{j\in\text{out}}\sum_{s'_j=0}^{N'_j}\binom{N'_j}{s'_j}\left(\frac{1}{2}\int\widetilde{d^3k}\,S^j_{\mu\nu}I^{\mu\nu\rho\sigma}P^{\text{tot}}_{\rho\sigma}\right)^{s'_j}\right]$$
$$\times\widetilde{\mathcal{M}}'_{N_1-s_1,\cdots,N_{n_\text{in}}-s_{n_\text{in}},N'_1-s'_1,\cdots,N'_{n_\text{out}}-s'_{n_\text{out}}}\,, \tag{B.59}$$

where $\widetilde{\mathcal{M}}'_{N_1-s_1,\cdots}$ is the IR-convergent part of the amplitude (from the contractions with ξ), given by

$$\begin{aligned}\widetilde{\mathcal{M}}'_{N_1-s_1,\cdots,N_{n_\text{in}}-s_{n_\text{in}},N'_1-s'_1,\cdots,N'_{n_\text{out}}-s'_{n_\text{out}}} &= (-1)^{\sum_{i\in\text{in}}(N_i-s_i)}\\ \times\left[\prod_{r=1}^{N+N'-s}\frac{1}{2}\int\widetilde{d^3k_r}\,S^{a'_r}_{\mu\nu}I^{\mu\nu\rho_r\sigma_r}\right]&\xi_{\rho_1\sigma_1\cdots\rho_{N+N'-s}\sigma_{N+N'-s}}(k_1,\cdots,k_{N+N'-s}).\end{aligned} \tag{B.60}$$

Here we used a sequence a' similar to (B.55) to simplify the notation:

$$\left(a'_i\right)_{i=1}^{N+N'-s}\equiv\Bigg(\overbrace{\underbrace{p_1,\cdots,p_1}_{N_1-s_1},\cdots,\underbrace{p_{n_\text{in}},\cdots,p_{n_\text{in}}}_{N_{n_\text{in}}-s_{n_\text{in}}}}^{N-\sum_i s_i},\overbrace{\underbrace{p'_1,\cdots,p'_1}_{N'_1-s'_1},\cdots,\underbrace{p'_{n_\text{out}},\cdots,p'_{n_\text{out}}}_{N'_{n_\text{out}}-s'_{n_\text{out}}}}^{N'-\sum_j s'_j}\Bigg). \tag{B.61}$$

The first line of (B.59) is the IR-divergent contribution of the configuration $(\{N_i\},\{N'_j\})$ factored out of the amplitude.

There is a combinatorial factor that accompanies (B.59), and to compute this we have to take into account other types of contributing gravitons. A cloud has three types of gravitons attached to it: the interacting gravitons, the disconnected gravitons, and the in-to-in/out-to-out gravitons. The in-to-in and out-to-out gravitons are treated separately later, so for the moment let us assume that there are only the first two types. Let l_i (l'_j) denote the number of disconnected gravitons attached to a cloud of an incoming (outgoing) scalar. l is the total number of disconnected graviton lines, so that $\sum_i l_i = \sum_j l'_j = l$. A cloud with N_i interacting and l_i disconnected graviton lines attached to it involves $N_i + l_i$ graviton creation/annihilation operators, which means it comes from the $(N_i + l_i)$-th term in the Taylor expansion of $e^{\pm R_f(p)}$. This term is accompanied by the factor $1/(N_i+l_i)!$. Since there are $\binom{N_i+l_i}{N_i} = (N_i + l_i)!/l_i!N_i!$ ways to group these into interacting/disconnected gravitons, this cloud has a net factor of $1/l_i!N_i!$. This applies to every incoming and outgoing cloud, and therefore the configuration $(\{N_i, l_i\}, \{N'_j, l'_j\})$ has a net combinatorial factor of

$$\left[\prod_{i\in\text{in}} \frac{1}{l_i! N_i!}\right]\left[\prod_{j\in\text{out}} \frac{1}{l'_j! N'_j!}\right]. \tag{B.62}$$

Multiplying this with (B.59) yields

$$\left[\prod_{i\in\text{in}} \frac{1}{l_i!} \sum_{s_i=0}^{N_i} \frac{1}{s_i!}\left(-\frac{1}{2}\int \widetilde{d^3k}\, S^i_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma}\right)^{s_i}\right]\left[\prod_{j\in\text{out}} \frac{1}{l'_j!} \sum_{s'_j=0}^{N'_j} \frac{1}{s'_j!}\left(\frac{1}{2}\int \widetilde{d^3k}\, S^j_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma}\right)^{s'_j}\right]$$
$$\times \widetilde{\mathcal{M}}_{N_1-s_1,\cdots,N_{n_\text{in}}-s_{n_\text{in}},N'_1-s'_1,\cdots,N'_{n_\text{out}}-s'_{n_\text{out}}}, \tag{B.63}$$

where

$$\widetilde{\mathcal{M}}_{j_1,j_2,\cdots,j_{n_\text{in}},j'_1,j'_2,\cdots,j'_{n_\text{out}}} \equiv \frac{\widetilde{\mathcal{M}}'_{j_1,j_2,\cdots,j_{n_\text{in}},j'_1,j'_2,\cdots,j'_{n_\text{out}}}}{j_1!j_2!\cdots j_{n_\text{in}}!j'_1!j'_2!\cdots j'_{n_\text{out}}!} \tag{B.64}$$

is the the rescaled finite amplitude.

We also have the contribution from the disconnected gravitons. A graviton line connecting the ith incoming cloud to the jth outgoing cloud contributes a factor

$$\frac{1}{2}\int \widetilde{d^3k}\, S_{\mu\nu}(p'_j, k) I^{\mu\nu\rho\sigma} S_{\rho\sigma}(p_i, k). \tag{B.65}$$

Summing over all possible disconnected lines therefore contributes the factor

$$l!\left[\frac{1}{2}\sum_{\substack{n\in\text{out}\\ m\in\text{in}}} \int \widetilde{d^3k}\, S^n_{\mu\nu} I^{\mu\nu\rho\sigma} S^m_{\rho\sigma}\right]^l = l!\left[\frac{1}{2}\int \widetilde{d^3k}\, S^{\text{out}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\text{in}}_{\rho\sigma}\right]^l, \tag{B.66}$$

where $l!$ is the number of ways we can pair l incoming gravitons with l outgoing gravitons.

The product of (B.63) and (B.66) form the contribution of a single configuration $(\{N_i, l_i\}, \{N'_j, l'_j\})$. Taking these two expressions and summing over all N_i, N'_j, l_i, l'_j, and l gives us the amplitude

$$
\begin{aligned}
&\sum_{l=0}^{\infty} \sum_{\sum l_i = l} \sum_{\sum l'_j = l} \sum_{\{N_i\}} \sum_{\{N'_j\}} l! \left[\frac{1}{2}\int \widetilde{d^3k}\, S^{\rm out}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\rm in}_{\rho\sigma}\right]^l \\
&\times \left[\prod_{i\in{\rm in}} \frac{1}{l_i!} \sum_{s_i=0}^{N_i} \frac{1}{s_i!}\left(-\frac{1}{2}\int \widetilde{d^3k}\, S^{i}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\rm tot}_{\rho\sigma}\right)^{s_i}\right] \\
&\times \left[\prod_{j\in{\rm out}} \frac{1}{l'_j!} \sum_{s'_j=0}^{N'_j} \frac{1}{s'_j!}\left(\frac{1}{2}\int \widetilde{d^3k}\, S^{j}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\rm tot}_{\rho\sigma}\right)^{s'_j}\right] \widetilde{\mathcal{M}}_{N_1-s_1,\cdots,N_{n_{\rm in}}-s_{n_{\rm in}},N'_1-s'_1,\cdots,N'_{n_{\rm out}}-s'_{n_{\rm out}}}\,.
\end{aligned}
\tag{B.67}
$$

Let us use the identity

$$
\sum_{\sum l_i = l} \frac{l!}{l_1! l_2! \cdots l_{n_{\rm in}}!} = 1 \tag{B.68}
$$

to eliminate the l_i's and l'_j's, and rearrange the sums

$$
\sum_{N_i=0}^{\infty} \sum_{s_i=0}^{N_i} \quad\to\quad \sum_{s_i=0}^{\infty} \sum_{N_i=s_i}^{\infty} \quad\to\quad \sum_{s_i=0}^{\infty} \sum_{m_i=0}^{\infty} \tag{B.69}
$$

with $m_i \equiv N_i - s_i$, after which (B.67) becomes

$$
\begin{aligned}
&\sum_{l=0}^{\infty} \frac{1}{l!} \left[\frac{1}{2}\int \widetilde{d^3k}\, S^{\rm out}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\rm in}_{\rho\sigma}\right]^l \left[\prod_{i\in{\rm in}} \sum_{s_i=0}^{\infty} \sum_{m_i=0}^{\infty} \frac{1}{s_i!}\left(-\frac{1}{2}\int \widetilde{d^3k}\, S^{i}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\rm tot}_{\rho\sigma}\right)^{s_i}\right] \\
&\qquad \times \left[\prod_{j\in{\rm out}} \sum_{s'_j=0}^{\infty} \sum_{m'_j=0}^{\infty} \frac{1}{s'_j!}\left(\frac{1}{2}\int \widetilde{d^3k}\, S^{j}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\rm tot}_{\rho\sigma}\right)^{s'_j}\right] \widetilde{\mathcal{M}}_{m_1,\cdots,m_{n_{\rm in}},m'_1,\cdots,m'_{n_{\rm out}}}\,.
\end{aligned}
\tag{B.70}
$$

The divergent factors exponentiate, leaving us with

$$
\exp\left(\frac{1}{2}\int \widetilde{d^3k}\, S^{\rm out}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\rm in}_{\rho\sigma} + \frac{1}{2}\int \widetilde{d^3k}\, S^{\rm tot}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\rm tot}_{\rho\sigma}\right) \widetilde{\mathcal{M}}\ , \tag{B.71}
$$

where the leftover, IR-finite part $\widetilde{\mathcal{M}}$ of the amplitude is given by

$$
\widetilde{\mathcal{M}} \equiv \left[\prod_{i\in{\rm in}} \sum_{m_i=0}^{\infty}\right] \left[\prod_{j\in{\rm out}} \sum_{m'_j=0}^{\infty}\right] \widetilde{\mathcal{M}}_{m_1,\cdots,m_{n_{\rm in}},m'_1,\cdots,m'_{n_{\rm out}}}\ . \tag{B.72}
$$

Now we consider the contribution of the in-to-in and out-to-out gravitons. These contributions manifest themselves in the form of normalization of the in- and out-states. In the single-particle case, we used the BCH formula to discard the annihilation operators. We should be more careful in doing so when dealing with the general case of multi-particle state. Consider for example the following two-particle state:

$$|\mathrm{i}\rangle = e^{R_f(p_1)} b^\dagger(p_1) e^{R_f(p_2)} b^\dagger(p_2) |0\rangle \tag{B.73}$$

$$\begin{aligned} &= \exp\left(\int \widetilde{d^3k}\, S^1_{\mu\nu} a^{\dagger\mu\nu}\right) \exp\left(-\int \widetilde{d^3k}\, S^1_{\mu\nu} a^{\mu\nu}\right) \exp\left(-\frac{1}{4}\int \widetilde{d^3k}\, S^1_{\mu\nu} I^{\mu\nu\rho\sigma} S^1_{\rho\sigma}\right) \\ &\times \exp\left(\int \widetilde{d^3k}\, S^2_{\mu\nu} a^{\dagger\mu\nu}\right) \exp\left(-\frac{1}{4}\int \widetilde{d^3k}\, S^2_{\mu\nu} I^{\mu\nu\rho\sigma} S^2_{\rho\sigma}\right) b^\dagger(p_1) b^\dagger(p_2) |0\rangle\,. \end{aligned} \tag{B.74}$$

We wish to eliminate $\exp\left(-\int \widetilde{d^3k}\, S^1_{\mu\nu} a^{\mu\nu}\right)$ in (B.74), by commuting it all the way to the vacuum; but it does not commute with $\exp\left(\int \widetilde{d^3k}\, S^2_{\mu\nu} a^{\dagger\mu\nu}\right)$, and thus this procedure induces an extra factor. Since

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} = e^{B+A} e^{\frac{1}{2}[A,B]} = e^B e^A e^{[A,B]} \tag{B.75}$$

for $[A,B] \in \mathbb{C}$, the extra factor is

$$\exp\left\{-\int \widetilde{d^3k}\, \widetilde{d^3k'}\, S_{\mu\nu}(p_1,k) S_{\rho\sigma}(p_2,k') \left[a^{\mu\nu}(k), a^{\dagger\rho\sigma}(k')\right]\right\} \tag{B.76}$$

$$= \exp\left\{-\frac{1}{2}\int \widetilde{d^3k}\, S_{\mu\nu}(p_1,k) I^{\mu\nu\rho\sigma} S_{\rho\sigma}(p_2,k)\right\} \tag{B.77}$$

$$= \exp\left\{-\frac{1}{4}\int \widetilde{d^3k}\, \left(S^1_{\mu\nu} I^{\mu\nu\rho\sigma} S^2_{\rho\sigma} + S^2_{\mu\nu} I^{\mu\nu\rho\sigma} S^1_{\rho\sigma}\right)\right\}, \tag{B.78}$$

where in the last line we used the symmetry of $I^{\mu\nu\rho\sigma}$ to write the expression in a symmetric fashion. For a multi-particle in-state, we get a factor of this form for each unordered pair of incoming scalars. With a similar line of reasoning for a multi-particle out-state, the total contribution of the in-to-in and out-to-out gravitons results in a factor of

$$\exp\left\{-\frac{1}{4}\int \widetilde{d^3k}\, \left(S^{\mathrm{in}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\mathrm{in}}_{\rho\sigma} + S^{\mathrm{out}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\mathrm{out}}_{\rho\sigma}\right)\right\}. \tag{B.79}$$

Multiplying (B.79) with the divergent factor in (B.71) gives us the expression for the net divergent factor A_{cloud} due to the amplitude interactions involving the clouds.

$$\begin{aligned} A_{\text{cloud}} &= \exp\Bigg\{ -\frac{1}{4}\int \widetilde{d^3k}\,\left(S^{\text{in}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\text{in}}_{\rho\sigma} + S^{\text{out}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\text{out}}_{\rho\sigma}\right) \\ &\qquad + \frac{1}{2}\int \widetilde{d^3k}\, S^{\text{out}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\text{in}}_{\rho\sigma} + \frac{1}{2}\int \widetilde{d^3k}\, S^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma} \Bigg\} \end{aligned} \tag{B.80}$$

$$= \exp\left(-\frac{1}{4}\int \widetilde{d^3k}\, S^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} S^{\text{tot}}_{\rho\sigma} + \frac{1}{2}\int \widetilde{d^3k}\, S^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma}\right). \tag{B.81}$$

Since $S^{\text{tot}}_{\mu\nu}(k) = P^{\text{tot}}_{\mu\nu}(k) + C^{\text{tot}}_{\mu\nu}(k)$,

$$A_{\text{cloud}} = \exp\left\{\frac{1}{4}\int \widetilde{d^3k}\left[-\left(P^{\text{tot}}_{\mu\nu} + C^{\text{tot}}_{\mu\nu}\right) I^{\mu\nu\rho\sigma}\left(P^{\text{tot}}_{\rho\sigma} + C^{\text{tot}}_{\rho\sigma}\right) + 2\left(P^{\text{tot}}_{\mu\nu} + C^{\text{tot}}_{\mu\nu}\right) I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma}\right]\right\} \tag{B.82}$$

$$= \exp\left\{\frac{1}{4}\int \widetilde{d^3k}\,\left(P^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} P^{\text{tot}}_{\rho\sigma} - C^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} C^{\text{tot}}_{\rho\sigma}\right)\right\} \tag{B.83}$$

$$= \exp\left(\frac{1}{4}\sum_{n,m}\eta_n\eta_m \int \widetilde{d^3k}\, P^m_{\mu\nu} I^{\mu\nu\rho\sigma} P^n_{\rho\sigma}\right)\exp\left(-\frac{1}{4}\int \widetilde{d^3k}\, C^{\text{tot}}_{\mu\nu} I^{\mu\nu\rho\sigma} C^{\text{tot}}_{\rho\sigma}\right) \tag{B.84}$$

The first exponential of (B.84) is the inverse of A_{virt}, so let us write

$$A_{\text{cloud}} = (A_{\text{virt}})^{-1}\exp(-aC) \tag{B.85}$$

where $a = \kappa^2/256\pi^3$ and

$$C \equiv \int \frac{d^3k}{\omega_k^3}\left(\sum_{j\in\text{out}} c^j_{\mu\nu} - \sum_{i\in\text{in}} c^i_{\mu\nu}\right) I^{\mu\nu\rho\sigma}\left(\sum_{j\in\text{out}} c^j_{\rho\sigma} - \sum_{i\in\text{in}} c^i_{\rho\sigma}\right). \tag{B.86}$$

The factor e^{-aC} derives solely from the interactions between graviton clouds. It only contributes to the normalization of states, and we can use (2.9) to set $C = 0$. Therefore, A_{cloud} exactly cancels the divergent factor A_{virt}, proving the cancellation of IR divergence to all orders.

Lastly, let us consider the general case where we use different dressings for the incoming and outgoing state. Then, the expression (B.86) readily generalizes to

$$C \equiv \int \frac{d^3k}{\omega_k^3} c^{\text{tot}}_{\mu\nu}(k) I^{\mu\nu\rho\sigma} c^{\text{tot}}_{\rho\sigma}(k), \quad \text{with} \quad c^{\text{tot}}_{\mu\nu}(k) \equiv \sum_{j\in\text{out}} c'_{\mu\nu}(p_j,k) - \sum_{i\in\text{in}} c_{\mu\nu}(p_i,k). \tag{B.87}$$

If $c^{\text{tot}}_{\mu\nu}(k)$ does not vanish as $k \to 0$, then C diverges and $e^{-aC} \to 0$, forcing the amplitude to be zero. Non-zero amplitudes are therefore only allowed between asymptotic states whose c-matrices satisfy

$$\sum_{j \in \text{out}} c'_{\mu\nu}(p_j, k) = \sum_{i \in \text{in}} c_{\mu\nu}(p_i, k), \tag{B.88}$$

up to subleading corrections of order $O(k)$.

Appendix C

BMS Modes at Null Infinity

In this appendix we review and extend on the work of [36, 124, 125] on the solutions of the pure gauge mode λ_μ. The wave equation (2.20) in the hyperbolic system of coordinates takes the form

$$\partial_\nu \partial^\nu \lambda_\mu = \left(\frac{\triangle_\rho}{\tau^2} - \partial_\tau^2 \right) \lambda_\mu = 0 \ , \tag{C.1}$$

where

$$\triangle_\rho = (1+\rho^2)\partial_\rho^2 + \frac{1}{\rho}(2+3\rho^2)\partial_\rho + \frac{1}{\rho^2}(1+z\bar{z})^2 \partial_z \partial_{\bar{z}} \ . \tag{C.2}$$

At $\tau \to \infty$ the only non-vanishing component of λ_μ is λ_τ,

$$\lim_{\tau \to \infty} \lambda_\tau(\tau, \rho, z, \bar{z}) = \widetilde{\lambda}_\tau(\rho, z, \bar{z}) \ , \tag{C.3}$$

the asymptotic form of which obeys the following equation

$$\triangle_\rho \widetilde{\lambda}_\tau = n(n-2)\widetilde{\lambda}_\tau \ , \tag{C.4}$$

where $n = 3$ in our case (for a $U(1)$ gauge symmetry, $n = 2$). $\widetilde{\lambda}_\tau(\rho, z, \bar{z})$ can be written in terms of the Green's function

$$\widetilde{\lambda}_\tau(\rho, z, \bar{z}) = \int d^2\omega \, G(\rho, z, \bar{z}; \omega, \bar{\omega}) f(\omega, \bar{\omega}) \ . \tag{C.5}$$

The Green's function obeys

$$\triangle_\rho G = n(n-2)G \tag{C.6}$$

$$\lim_{\rho \to \infty} \rho^{2-n} G(\rho, z, \bar{z}; \omega, \bar{\omega}) = \delta^2(z - \omega) \ . \tag{C.7}$$

The two solutions to equation (C.6) are given by

$$G(\rho, z, \bar{z}; \omega, \bar{\omega}) = \alpha f^{(n)}(\rho, z, \bar{z}; \omega, \bar{\omega}) + \beta f^{(2-n)}(\rho, z, \bar{z}; \omega, \bar{\omega}) \tag{C.8}$$

where

$$f^{(n)}(\rho, z, \bar{z}; \omega, \bar{\omega}) = \frac{n-1}{2^{n-1}} \frac{\sqrt{\gamma}}{2\pi} \left(\sqrt{1+\rho^2} - \rho\, \hat{x}_z \cdot \hat{x}_\omega \right)^{-n} . \tag{C.9}$$

The asymptotic of the function $f^{(n)}$ is

$$\lim_{\rho\to\infty} f^{(n)}(\rho, z, \bar{z}; \omega, \bar{\omega}) \sim \begin{cases} \rho^{-n}, & \hat{x}_z \neq \hat{x}_\omega \\ \rho^{+n}, & \hat{x}_z = \hat{x}_\omega \end{cases} \tag{C.10}$$

and its integral over S^2 asymptotes to

$$\lim_{\rho\to\infty} \rho^{2-n} \int d^2\omega\, f^{(n)}(\rho, z, \bar{z}; \omega, \bar{\omega}) = 1 . \tag{C.11}$$

The solution for the gauge mode therefore asymptotes to

$$\lim_{\rho\to\infty} \widetilde{\lambda}_\tau(\rho, z, \bar{z}) = \alpha(z, \bar{z})\, \rho^{n-2} \left(1 + \dots\right) + \beta(z, \bar{z})\, \rho^{-n} \left(1 + \dots\right) , \tag{C.12}$$

where the dots stand for subleading terms in $1/\rho$. The α-series is leading and do not vanish at time-like infinity $\rho \to \infty$. It is a large gauge transformation. The β-series is subleading and vanishes at time-like infinity.

We would now like to express the α and β modes in terms of the radiative data C_{zz} and $C_{\bar{z}\bar{z}}$. To do this we should study the solutions to equation (2.40) for the gauge mode ξ_μ. At leading order, only the τ-component is non-vanishing and its solution can be written in terms of the Green's function

$$\xi_\tau = \int d^2\omega\, G(\rho, z, \bar{z}; \omega, \bar{\omega}) \left(\frac{1}{2} \partial_\tau h^B - \partial^\nu h^B_{\tau\nu} \right) . \tag{C.13}$$

Plugging the solution for the Green's function, and the asymptotic form of the metric in the Bondi gauge, we get for the α-mode

$$\lim_{\rho\to\infty} \xi^\alpha_\tau = \rho^{n-2} \left(\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z \right)_{\mathcal{I}^+_+} . \tag{C.14}$$

By comparing to (C.12), we conclude that

$$\alpha = \left(\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z \right)_{\mathcal{I}^+_+} . \tag{C.15}$$

To solve for the subleading β-mode in a similar way we have to study subleading corrections to ξ^α. Here, we do not solve this problem explicitly, but instead give a heuristic explanation based on properties of 2D conformal field theories. On S^2 the leading α-mode is a left mover, while the subleading β-mode is a right mover. This implies that the β-mode is orthogonal to (C.15) and is therefore given by

$$\beta = i\left(\partial_z U_{\bar{z}} - \partial_{\bar{z}} U_z\right)_{\mathcal{I}^+_+} \ . \tag{C.16}$$

The factor of i is required to make β real. We leave the explicit analysis and further exploration of this direction to future work. See [156] for a related work.

Appendix D

Horizon Gauge Field in Rindler Spacetime

In this appendix we show explicitly that the horizon field defined by

$$\hat{A}_y^H(\Omega) = \lim_{\xi \to -\infty} \hat{A}_y(v, \xi, \Omega), \tag{D.1}$$

satisfies the relation $\hat{A}_y^H = \partial_y \mathcal{A}(\Omega)$. To this end, we want to obtain a mode expansion for $\hat{A}^y(t, \xi, \Omega)$. Let us take the equation of motion (4.141) and use $\partial_i \hat{A}^i = 0$ to write

$$-\partial_0^2 \hat{A}^y + \Delta_s \hat{A}^y - \frac{e^{2a\xi}}{4M^2} \left[2y\partial_1 \hat{A}^1 - 2a \left(1 - y^2 \right) \partial_y \hat{A}^1 \right] = 0. \tag{D.2}$$

Now, consider the following ansatz

$$\hat{A}^y = \hat{a}^y + (1 - y^2)\partial_y \frac{1}{\mathbf{L}^2} \partial_1 \hat{A}^1, \tag{D.3}$$

for some field $\hat{a}^y$. Substituting (D.3) into (D.2) yields the equation of motion of $\hat{a}^y$, which is essentially that of a free scalar field,

$$\partial_0^2 \hat{a}^y - \Delta_s \hat{a}^y = 0. \tag{D.4}$$

Using equations (D.2) and (D.4) along with the mode expansion of $\hat{A}^1$ yields the following mode expansion for $\hat{A}^y$,

$$\hat{A}^y(t,\xi,\Omega) = \sum_{\ell m}\int \frac{d\omega}{\sqrt{2\omega}}\left\{\left[(\cdots) + \frac{4a^3}{\omega}\frac{a_{\ell m}(\omega)}{\sqrt{\ell(\ell+1)}}(1-y^2)\partial_y Y_{\ell m}(\Omega) z\frac{d}{dz}k_{i\frac{\omega}{a}}(z)\right]e^{-i\omega t} + \text{h.c.}\right\}, \tag{D.5}$$

where the omitted terms in the parentheses $(\cdots)$ correspond to the mode expansion for the field $\hat{a}^y$. The covariant component $\hat{A}_y$ in terms of the advanced time coordinates (v,ξ,Ω) is therefore

$$\hat{A}_y(v,\xi,\Omega) = \sum_{\ell m}\int \frac{d\omega}{\sqrt{2\omega}}\left\{\left[(\cdots) + \frac{a}{\omega}\frac{a_{\ell m}(\omega)}{\sqrt{\ell(\ell+1)}}\partial_y Y_{\ell m}(\Omega) z\frac{d}{dz}k_{i\frac{\omega}{a}}(z)\right]e^{-i\omega v + i\omega\xi} + \text{h.c.}\right\}. \tag{D.6}$$

We can obtain the horizon gauge field $\hat{A}_y^H$ by taking the limit $\xi \to -\infty$. Since $\hat{a}^y$ satisfies the free scalar field equation (D.4), the terms in $(\cdots)$ are proportional to $k_{i\frac{\omega}{a}}(z) = O(\omega)$ and therefore vanish at the horizon due to the relation (4.105). Hence,

$$\hat{A}_y^H(\Omega) = \lim_{\xi\to-\infty}\hat{A}_y(v,\xi,\Omega) \tag{D.7}$$

$$= -\sum_{\ell m}\int \frac{d\omega}{\sqrt{\pi\omega}}\phi(\omega)\left\{\frac{a_{\ell m}(\omega)}{\sqrt{\ell(\ell+1)}}\partial_y Y_{\ell m}(\Omega) + \text{h.c.}\right\}. \tag{D.8}$$

From (4.152), we can immediately obtain

$$\hat{A}_y^H(\Omega) = \partial_y \mathcal{A}(\Omega), \tag{D.9}$$

which proves the claim.

Appendix E

Newman-Penrose Formalism in Schwarzschild Spacetime

In this appendix, we reproduce some relevant details of the Newman-Penrose (NP) formalism used in the quantization of [93, 94]. We use $(a), (b), \ldots$ to denote tetrad indices in contrast to the tensor indices $\mu, \nu, \ldots$.

The components of the Kinnersley tetrad ${e_{(a)}}^{\mu}$ are, in the Schwarzschild coordinates (t, r, θ, ϕ),

$$ {e_{(1)}}^{\mu} = \left(\frac{1}{V}, 1, 0, 0\right), \qquad {e_{(2)}}^{\mu} = \left(\frac{1}{2}, -\frac{V}{2}, 0, 0\right), \tag{E.1} $$

$$ {e_{(3)}}^{\mu} = {e_{(4)}}^{\mu *} = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right). \tag{E.2} $$

These satisfy the orthonormality condition

$$ g_{\mu\nu}{e_{(a)}}^{\mu}{e_{(b)}}^{\nu} = \eta_{(a)(b)}, \tag{E.3} $$

where $g_{\mu\nu}$ is the Schwarzschild metric and $\eta_{(a)(b)}$ is a constant symmetric matrix,

$$ \eta_{(a)(b)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{E.4} $$

Tetrad indices $(a), (b), \ldots$ are lowered/raised by $\eta_{(a)(b)}$ and its inverse $\eta^{(a)(b)}$. In terms of the

Kinnersley tetrad, only one of the five independent Weyl scalars is non-zero,

$$\Psi_0 \equiv -C_{(1)(3)(1)(3)} = 0, \tag{E.5}$$

$$\Psi_1 \equiv -C_{(1)(2)(1)(3)} = 0, \tag{E.6}$$

$$\Psi_2 \equiv -C_{(1)(3)(4)(2)} = \frac{M}{r^3}, \tag{E.7}$$

$$\Psi_3 \equiv -C_{(1)(2)(4)(2)} = 0, \tag{E.8}$$

$$\Psi_4 \equiv -C_{(2)(4)(2)(4)} = 0. \tag{E.9}$$

Here we used the notation $C_{(a)(b)(c)(d)} = C_{\mu\nu\rho\sigma}{e_{(a)}}^{\mu}{e_{(b)}}^{\nu}{e_{(c)}}^{\rho}{e_{(d)}}^{\sigma}$, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor of the Schwarzschild spacetime. Furthermore, among the spin coefficients

$$\gamma_{(a)(b)(c)} \equiv \nabla_\mu {e_{(a)}}^{\nu} e_{(b)\nu} {e_{(c)}}^{\mu}, \tag{E.10}$$

all but the following vanish:

$$\rho \equiv \gamma_{(3)(1)(4)} = -\frac{1}{r}, \tag{E.11}$$

$$\mu \equiv \gamma_{(2)(4)(3)} = -\frac{V}{2r}, \tag{E.12}$$

$$\gamma \equiv \frac{1}{2}\Big(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}\Big) = \frac{M}{2r^2}, \tag{E.13}$$

$$\alpha \equiv \frac{1}{2}\Big(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}\Big) = -\frac{\cot\theta}{2\sqrt{2}r}, \tag{E.14}$$

$$\beta \equiv \frac{1}{2}\Big(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}\Big) = \frac{\cot\theta}{2\sqrt{2}r}. \tag{E.15}$$

We also make use of the tetrad operators (cf. [157]),

$${e_{(a)}}^{\mu}\partial_\mu = (D, \Delta, \delta, \delta^*), \tag{E.16}$$

which, in the Schwarzschild coordinates (t, r, θ, ϕ), can be written out explicitly as

$$D = \frac{1}{V}\partial_t + \partial_r, \qquad \Delta = \frac{1}{2}\partial_t - \frac{V}{2}\partial_r, \qquad \delta = \frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right). \tag{E.17}$$

Appendix F

Quantizing Metric Perturbations of Schwarzschild

In this appendix, we review the quantization of the linear perturbations of the Schwarzschild metric studied in [93, 94], which takes advantage of the simplifications given by the NP formalism. Among the many conventions of the NP formalism, we follow those given in Appendix A of [94]. The quantization is done in two particular gauges called the ingoing and outgoing radiation gauge – since we work in the Bondi gauge, it is worth noting that the outgoing radiation gauge satisfies the Bondi gauge conditions in the advance time coordinates.

For a Schwarzschild black hole of mass M_{bh}, the spacetime is described by the metric $g_{\mu\nu}(x)$ with the line element

$$ds^2 = -V dt^2 + \frac{dr^2}{V} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad V \equiv 1 - \frac{2M}{r}, \tag{F.1}$$

in the usual coordinates (t, r, θ, ϕ), where $2M = 2GM_{\text{bh}}$ is the Schwarzschild radius. The appropriate choice of tetrad that reflects the symmetries of the Schwarzschild spacetime is the Kinnersley tetrad (see appendix E), defined as

$$e_{(1)}{}^{\mu} = \left(\frac{1}{V}, 1, 0, 0\right), \tag{F.2}$$

$$e_{(2)}{}^{\mu} = \left(\frac{1}{2}, -\frac{V}{2}, 0, 0\right), \tag{F.3}$$

$$e_{(3)}{}^{\mu} = e_{(4)}{}^{\mu *} = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right). \tag{F.4}$$

In terms of the Kinnersley tetrads, all spin coefficients vanish except

$$\rho = -\frac{1}{r}, \quad \mu = -\frac{V}{2r^2}, \quad \gamma = \frac{M}{2r^2}, \quad -\alpha = \beta = \frac{\cot\theta}{2\sqrt{2}r}. \tag{F.5}$$

We consider the perturbed metric $g'_{\mu\nu}$ around the Schwarzschild background $g_{\mu\nu}$,

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \kappa h_{\mu\nu}(x), \tag{F.6}$$

where $\kappa^2 = 32\pi G$. The complete set of modes is

$$\left\{h^{\Lambda}_{\mu\nu}(l,m,\omega,P;x), h^{\Lambda *}_{\mu\nu}(l,m,\omega,P;x)\right\}_{\Lambda,l,m,\omega,P}, \tag{F.7}$$

where $l \geq 2$, $|m| \leq l$, and $\Lambda \in \{\text{in}, \text{up}\}$ indicates the boundary condition satisfied by the mode (see Fig. 5.2). Each mode has a definite parity, labeled by $P = \pm 1$. In the literature $P = +1$ and $P = -1$ are referred to as the electric and magnetic parities, respectively (see for example [158]). The $\Lambda = \text{in}$ modes (henceforth the in-modes) have the form

$$h^{\text{in}}_{\mu\nu}(l,m,\omega,P;x) = N^{\text{in}}\left\{\Theta_{\mu\nu}\, {}_{+2}Y_{lm}(\theta,\phi) + P\Theta^*_{\mu\nu}\, {}_{-2}Y_{lm}(\theta,\phi)\right\}\, {}_{-2}R^{\text{in}}_{l\omega}(r)e^{-i\omega t} \tag{F.8}$$

in the ingoing radiation gauge $h_{\mu\nu}e_{(1)}{}^{\nu} = 0$, $g^{\mu\nu}h_{\mu\nu} = 0$. The $\Lambda = \text{up}$ modes (henceforth the up-modes) have the form

$$h^{\text{up}}_{\mu\nu}(l,m,\omega,P;x) = N^{\text{up}}\left\{\Upsilon_{\mu\nu}\, {}_{-2}Y_{lm}(\theta,\phi) + P\Upsilon^*_{\mu\nu}\, {}_{+2}Y_{lm}(\theta,\phi)\right\}\, {}_{+2}R^{\text{up}}_{l\omega}(r)e^{-i\omega t} \tag{F.9}$$

in the outgoing radiation gauge $h_{\mu\nu}e_{(2)}{}^{\nu} = 0$, $g^{\mu\nu}h_{\mu\nu} = 0$. Here N^{Λ} are the normalization constants that are independent of the spacetime point x, and $\Theta_{\mu\nu}$, $\Upsilon_{\mu\nu}$ are second-order differential operators defined as[1]

$$\begin{aligned}
\Theta_{\mu\nu} &= -e_{(1)\mu}e_{(1)\nu}(\delta^* - 2\alpha)(\delta^* - 4\alpha) - e_{(4)\mu}e_{(4)\nu}(D - \rho)(D + 3\rho) \\
&\quad + \frac{1}{2}\left(e_{(1)\mu}e_{(4)\nu} + e_{(4)\mu}e_{(1)\nu}\right)\left[D(\delta^* - 4\alpha) + (\delta^* - 4\alpha)(D + 3\rho)\right], \qquad \text{(F.10)} \\
\Upsilon_{\mu\nu} &= r^4\Big\{ - e_{(2)\mu}e_{(2)\nu}(\delta - 2\alpha)(\delta - 4\alpha) - e_{(3)\mu}e_{(3)\nu}(\Delta + 5\mu - 2\gamma)(\Delta + \mu - 4\gamma) \\
&\quad + \frac{1}{2}\left(e_{(2)\mu}e_{(3)\nu} + e_{(3)\mu}e_{(2)\nu}\right)\left[(\delta - 4\alpha)(\Delta + \mu - 4\gamma) + (\Delta + 4\mu - 4\gamma)(\delta - 4\alpha)\right]\Big\},
\end{aligned}$$
$$\text{(F.11)}$$

[1]The authors of [159] argue that there is a typo in the expression for $\Upsilon_{\mu\nu}$ in the literature. The corrections proposed therein do not affect our results.

where D, Δ and δ are differential operators defined by the relation [157]

$$e_{(a)}{}^{\mu}\partial_{\mu} \equiv (D, \Delta, \delta, \delta^*). \tag{F.12}$$

The angular functions ${}_sY_{lm}(\theta, \phi)$ are spin-weighted spherical harmonics, whose relevant properties are spelled out in appendix G. The radial functions ${}_{-2}R^{\text{in}}_{l\omega}(r)$ and ${}_{+2}R^{\text{up}}_{l\omega}(r)$ are solutions to the ordinary differential equation

$$\begin{aligned}\Bigg[\frac{1}{[r(r-2M)]^s}\frac{d}{dr}\left([r(r-2M)]^{s+1}\frac{d}{dr}\right)& \\ +\frac{\omega^2r^4+2is\omega r^2(r-3M)}{r(r-2M)}&-(l-s)(l+s+1)\Bigg]{}_sR_{l\omega}(r)=0,\end{aligned} \tag{F.13}$$

with $s=-2$ and $s=+2$ respectively, subject to the boundary conditions

$${}_{-2}R^{\text{in}}_{l\omega}(r) \sim \begin{cases} B^{\text{in}}_{l\omega}(4M^2V)^2e^{-i\omega r_*} & \text{as } r\to 2M, \\ r^{-1}e^{-i\omega r_*}+A^{\text{in}}_{l\omega}r^3e^{+i\omega r_*} & \text{as } r\to\infty, \end{cases} \tag{F.14}$$

$${}_{+2}R^{\text{up}}_{l\omega}(r) \sim \begin{cases} A^{\text{up}}_{l\omega}(4M^2V)^{-2}e^{-i\omega r_*}+e^{+i\omega r_*} & \text{as } r\to 2M, \\ B^{\text{up}}_{l\omega}r^{-5}e^{+i\omega r_*} & \text{as } r\to\infty. \end{cases} \tag{F.15}$$

Here $r_* = r + 2M\ln(r/2M-1)$ is the tortoise coordinate, and $A^{\Lambda}_{l\omega}$, $B^{\Lambda}_{l\omega}$ are the reflection and transmission amplitudes respectively; see Fig. 5.2. In particular, we note that $B^{\text{in}}_{l\omega}$ and $A^{\text{up}}_{l\omega}$ have the small-ω expansions

$$B^{\text{in}}_{l\omega} = \frac{1}{(2M)^5}\frac{l!(l-2)!(l+2)!}{2(2l+1)!(2l)!}(-4iM\omega)^{l+3}+\mathcal{O}(\omega^{l+4}), \tag{F.16}$$

$$A^{\text{up}}_{l\omega} = 2(2M)^4\frac{(l-2)!}{(l+2)!}(-4iM\omega)+\mathcal{O}(\omega^2), \tag{F.17}$$

which prove to be useful later. The normalization constants N^{Λ} are fixed by the orthonormality condition

$$\langle h^{\Lambda}(l,m,\omega,P;x), h^{\Lambda'}(l',m',\omega',P';x)\rangle = \delta_{\Lambda\Lambda'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\delta_{PP'}. \tag{F.18}$$

The Klein-Gordon inner product $\langle\cdot,\cdot\rangle$ between two symmetric tensor fields $\psi_{\alpha\beta}$ and $\phi_{\alpha\beta}$ is defined as

$$\langle\psi,\phi\rangle = \frac{i}{2}\int_S d\Sigma^\mu\left(\psi^{\alpha\beta*}\nabla_\mu\bar{\phi}_{\alpha\beta} - \phi^{\alpha\beta}\nabla_\mu\bar{\psi}^*_{\alpha\beta} + 2\bar{\phi}_{\alpha\mu}\nabla_\beta\bar{\psi}^{\alpha\beta*} - 2\bar{\psi}_{\alpha\mu}\nabla_\beta\bar{\phi}^{\alpha\beta}\right), \tag{F.19}$$

where $\bar{\psi}_{\alpha\beta}$, $\bar{\phi}_{\alpha\beta}$ are the trace-free parts of $\psi_{\alpha\beta}$, $\phi_{\alpha\beta}$, respectively, and S is some Cauchy surface; see [93] or [160] for the construction of the inner product. Taking S to be $\mathcal{H}^-\cup\mathcal{I}^-$, the normalization constants become[2]

$$\left|N^{\mathrm{in}}\right|^{-2} = 64\pi\omega^5, \tag{F.20}$$

$$\left|N^{\mathrm{up}}\right|^{-2} = (2M)^6\pi\omega(1+4M^2\omega^2)(1+16M^2\omega^2). \tag{F.21}$$

Having obtained the complete set of orthonormal modes, we can write the linear perturbation of the Schwarzschild background as the expansion

$$h_{\mu\nu}(x) = \sum_\Lambda\sum_{lmP}\int_0^\infty d\omega\left[a^\Lambda_{lmP}(\omega)h^\Lambda_{\mu\nu}(l,m,\omega,P;x) + \mathrm{h.c.}\right] \tag{F.22}$$

and quantize the field by promoting $a^\Lambda_{lmP}(\omega)$ and $a^{\Lambda\dagger}_{lmP}(\omega)$ to operators that satisfy the commutation relations

$$\left[a^\Lambda_{lmP}(\omega), a^{\Lambda'\dagger}_{l'm'P'}(\omega')\right] = \delta_{\Lambda\Lambda'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\delta_{PP'}, \tag{F.23}$$

$$\left[a^\Lambda_{lmP}(\omega), a^{\Lambda'}_{l'm'P'}(\omega')\right] = 0 = \left[a^{\Lambda\dagger}_{lmP}(\omega), a^{\Lambda'\dagger}_{l'm'P'}(\omega')\right]. \tag{F.24}$$

A peculiar feature of this method of quantization is that the two modes ("in" and "up") are in different gauges. This does not cause problems for us, because the in-modes are, by definition, the linearized gravity waves sent in from $\mathcal{I}^-$, and it is known that these waves carry zero supertranslation charge; see [61], or [161] for a recent account. One can also observe this directly from the soft expansion (F.16) of the black hole absorption amplitude $B^{\mathrm{in}}_{l\omega}$, which is proportional to ω^{l+3}. This point becomes relevant in section 5.3 when we compute the contribution of the up-modes to the supertranslation charge.

It is noteworthy that the up-modes, which are in the outgoing radiation gauge, also satisfy the Bondi gauge conditions in the ingoing Eddington-Finkelstein coordinates (v,r,θ,ϕ), where $v = t + r_*$. To see this, we note that in these coordinates,

$$e_{(2)v} = -\frac{V}{2}, \qquad e_{(2)r} = 0, \tag{F.25}$$

[2]Note that the normalization is different from [94] since we expand $g'_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}$ and quantize $h_{\mu\nu}$ in order to give the graviton field a mass dimension.

which, when substituted into (F.9), implies that

$$h^{\mathrm{up}}_{vr}(l, m, \omega, P; x) = 0, \tag{F.26}$$

$$h^{\mathrm{up}}_{rr}(l, m, \omega, P; x) = 0, \tag{F.27}$$

$$h^{\mathrm{up}}_{rA}(l, m, \omega, P; x) = 0. \tag{F.28}$$

Furthermore, the orthonormality (E.3) implies that

$$\gamma^{AB}\Upsilon_{AB} \propto \gamma^{AB} e_{(3)A} e_{(3)B} = r^2 g^{\mu\nu} e_{(3)\mu} e_{(3)\nu} = 0, \tag{F.29}$$

from which we obtain

$$\gamma^{AB} h^{\mathrm{up}}_{AB}(l, m, \omega, P; x) = 0. \tag{F.30}$$

Equations (F.27), (F.28) and (F.30) are the Bondi gauge conditions in these coordinates [61], so as far as we're in the ingoing Eddington-Finkelstein coordinates, we can safely pretend that the up-modes are quantized in the Bondi gauge.

Appendix G

Spin-Weighted Spherical Harmonics

In this appendix, we review the relevant definition and properties of the spin-weighted spherical harmonics. For more details we refer the reader to [162, 163].

The spin-weighted spherical harmonics ${}_sY_{lm}(\theta, \phi)$ are defined for integers $|s| \leq l$ by the equations

$$ {}_0Y_{lm}(\theta, \phi) = Y_{lm}(\theta, \phi), \tag{G.1} $$

$$ {}_{s+1}Y_{lm}(\theta, \phi) = [(l-s)(l+s+1)]^{-1/2}\, \eth\, {}_sY_{lm}(\theta, \phi), \tag{G.2} $$

$$ {}_{s-1}Y_{lm}(\theta, \phi) = -\,[(l-s)(l+s+1)]^{-1/2}\, \bar{\eth}\, {}_sY_{lm}(\theta, \phi), \tag{G.3} $$

where $Y_{lm}(\theta, \phi)$ are the ordinary spherical harmonics,

$$ Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta) e^{im\phi}, \tag{G.4} $$

and $\eth$,$\bar{\eth}$ are operators which act on a function η of spin-weight s as

$$ \eth\eta = -\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - s\cot\theta\right)\eta, \tag{G.5} $$

$$ \bar{\eth}\eta = -\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + s\cot\theta\right)\eta. \tag{G.6} $$

The spin-weighted spherical harmonics ${}_sY_{lm}$ form a complete orthonormal set for any function of θ and ϕ with spin-weight s:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, {}_sY_{lm}(\theta,\phi) {}_sY^*_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}, \tag{G.7}$$

$$\sum_{l=|s|}^{\infty} \sum_{m=-l}^{l} {}_sY_{lm}(\theta,\phi) {}_sY^*_{lm}(\theta',\phi') = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi'). \tag{G.8}$$

G.1 Spin-2 spherical harmonics

In the NP formulation of Schwarzschild spacetime reviewed in appendix F, the operators ð, $\bar{\text{ð}}$ can be expressed in terms of the tetrad operator δ (E.16) and the spin coefficient α (F.5):

$$\begin{aligned} \delta + 2s\alpha &= \frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi - s\cot\theta\right) = -\frac{\eth}{\sqrt{2}r}, \\ \delta^* - 2s\alpha &= \frac{1}{\sqrt{2}r}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + s\cot\theta\right) = -\frac{\bar{\eth}}{\sqrt{2}r}. \end{aligned} \tag{G.9}$$

These combinations appear for example in the definitions of the differential operators $\Theta_{\mu\nu}$ (F.10) and $\Upsilon_{\mu\nu}$ (F.11).

There is a relation between ${}_{\pm2}Y_{lm}(\theta,\phi)$ and a 2-sphere tensor of the form

$$(2D_AD_B - \gamma_{AB}D^2)Y_{lm}(\theta,\phi), \tag{G.10}$$

which we show below. Each spin-2 spherical harmonics can be written as two spin operators acting on the ordinary spherical harmonics,

$${}_{-2}Y_{lm} = \sqrt{\frac{(l-2)!}{(l+2)!}}\,\bar{\eth}\bar{\eth}Y_{lm} \tag{G.11}$$

$$= \sqrt{\frac{(l-2)!}{(l+2)!}}\left(\partial_\theta^2 + \frac{2i\cos\theta}{\sin^2\theta}\partial_\phi - \frac{2i}{\sin\theta}\partial_\theta\partial_\phi - \frac{1}{\sin^2\theta}\partial_\phi^2 - \cot\theta\partial_\theta\right)Y_{lm}, \tag{G.12}$$

$${}_{+2}Y_{lm} = \sqrt{\frac{(l-2)!}{(l+2)!}}\,\eth\eth Y_{lm} \tag{G.13}$$

$$= \sqrt{\frac{(l-2)!}{(l+2)!}}\left(\partial_\theta^2 - \frac{2i\cos\theta}{\sin^2\theta}\partial_\phi + \frac{2i}{\sin\theta}\partial_\theta\partial_\phi - \frac{1}{\sin^2\theta}\partial_\phi^2 - \cot\theta\partial_\theta\right)Y_{lm}. \tag{G.14}$$

Consider the following linear combinations,

$$
\begin{aligned}
{}_{-2}Y_{lm} + {}_{+2}Y_{lm} &= 2\sqrt{\frac{(l-2)!}{(l+2)!}}\left(\partial_\theta^2 - \frac{1}{\sin^2\theta}\partial_\phi^2 - \cot\theta\partial_\theta\right)Y_{lm}, \\
{}_{-2}Y_{lm} - {}_{+2}Y_{lm} &= 2\sqrt{\frac{(l-2)!}{(l+2)!}}\left(\frac{2i\cos\theta}{\sin^2\theta}\partial_\phi - \frac{2i}{\sin\theta}\partial_\theta\partial_\phi\right)Y_{lm}.
\end{aligned}
\tag{G.15}
$$

The components and the trace of the 2-sphere tensor $D_A D_B Y_{lm}$ are

$$D_\theta D_\theta Y_{lm} = \partial_\theta^2 Y_{lm}, \tag{G.16}$$

$$D_\theta D_\phi Y_{lm} = (\partial_\theta\partial_\phi - \cot\theta\partial_\phi)\,Y_{lm} = D_\phi D_\theta Y_{lm}, \tag{G.17}$$

$$D_\phi D_\phi Y_{lm} = \left(\partial_\phi^2 + \sin\theta\cos\theta\partial_\theta\right)Y_{lm}, \tag{G.18}$$

$$D^2 Y_{lm} = \gamma^{AB} D_A D_B Y_{lm} = \left(\partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2 + \cot\theta\partial_\theta\right)Y_{lm}. \tag{G.19}$$

We can use these to write

$$(2D_\theta D_\theta - \gamma_{\theta\theta}D^2)Y_{lm} = \left(\partial_\theta^2 - \frac{1}{\sin^2\theta}\partial_\phi^2 - \cot\theta\partial_\theta\right)Y_{lm}, \tag{G.20}$$

$$(2D_\theta D_\phi - \gamma_{\theta\phi}D^2)Y_{lm} = 2D_\theta D_\phi Y_{lm} = i\sin\theta\left(\frac{2i\cos\theta}{\sin^2\theta}\partial_\phi - \frac{2i}{\sin\theta}\partial_\theta\partial_\phi\right)Y_{lm}, \tag{G.21}$$

$$(2D_\phi D_\phi - \gamma_{\phi\phi}D^2)Y_{lm} = -\sin^2\theta\left(\partial_\theta^2 - \frac{1}{\sin^2\theta}\partial_\phi^2 - \cot\theta\partial_\theta\right)Y_{lm}. \tag{G.22}$$

Comparison with (G.15) yields the relations

$${}_{-2}Y_{lm} + {}_{+2}Y_{lm} = 2\sqrt{\frac{(l-2)!}{(l+2)!}}\,(2D_\theta D_\theta - \gamma_{\theta\theta}D^2)Y_{lm} \tag{G.23}$$

$$= -\frac{2}{\sin^2\theta}\sqrt{\frac{(l-2)!}{(l+2)!}}\,(2D_\phi D_\phi - \gamma_{\phi\phi}D^2)Y_{lm}, \tag{G.24}$$

$${}_{-2}Y_{lm} - {}_{+2}Y_{lm} = -\frac{2i}{\sin\theta}\sqrt{\frac{(l-2)!}{(l+2)!}}\,(2D_\theta D_\phi - \gamma_{\theta\phi}D^2)Y_{lm}. \tag{G.25}$$

Appendix H

Supertranslation Charge on the Past Schwarzschild Horizon

In this appendix, we follow a line of computation similar to that of [61] to derive the supertranslation charge on the *past* Schwarzschild horizon $\mathcal{H}^-$.

To this end, we move to the outgoing Eddington-Finkelstein coordinates (u, r, Ω), where $u = t - r_*$. The Schwarzschild metric in these coordinates reads

$$ds^2 = -Vdu^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B, \qquad V \equiv 1 - \frac{2M}{r}, \tag{H.1}$$

with the 2-sphere metric γ_{AB}. The non-zero inverse metric components are given by

$$g^{ur} = -1, \quad g^{rr} = V, \quad g^{AB} = \frac{1}{r^2}\gamma^{AB}. \tag{H.2}$$

The Bondi gauge conditions read [61]

$$h_{rr} = h_{rA} = \gamma^{AB}h_{AB} = 0. \tag{H.3}$$

We want to find infinitesimal diffeomorphisms $\delta x^\mu = \xi^\mu$ that respect the Bondi gauge conditions as well as the falloffs at large r. Bondi gauge conditions put the following constraints on ξ^μ,

$$\nabla_r \xi_r = \partial_r \xi^u = 0, \tag{H.4}$$

$$\nabla_A \xi_r + \nabla_r \xi_A = \partial_r \xi^A - \frac{1}{r^2}D^A f = 0, \tag{H.5}$$

$$\gamma^{AB}\nabla_A \xi_B = D_A \xi^A + \frac{2}{r}\xi^r = 0. \tag{H.6}$$

We restrict our attention to supertranslations by choosing $\xi^u = f$ such that $\partial_u f = 0$. Then (H.4) leads to $f = f(\Omega)$, and (H.5) with falloff condition on ξ^A implies $\xi^A = -\frac{1}{r}D^A f$. Substituting this into (H.6), one obtains $\xi^r = \frac{1}{2}D^2 f$. Therefore we obtain

$$\xi^\alpha \partial_\alpha = f\partial_u + \frac{1}{2}D^2 f \partial_r - \frac{1}{r}D^A f \partial_A. \tag{H.7}$$

The supertranslation charge associated with the boundary $\partial\Sigma$ of a Cauchy surface Σ and diffeomorphism ξ reads

$$Q_\xi^\Sigma = -\frac{2}{\kappa^2}\int_{\partial\Sigma} *F, \tag{H.8}$$

where F is a two-form with components

$$F_{\mu\nu} = \frac{1}{2}(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu)h + (\nabla_\mu h^\alpha{}_\nu - \nabla_\nu h^\alpha{}_\mu)\xi_\alpha + (\nabla_\alpha \xi_\mu h^\alpha{}_\nu - \nabla_\alpha \xi_\nu h^\alpha{}_\mu) \tag{H.9}$$

$$+ (\nabla_\alpha h^\alpha{}_\mu \xi_\nu - \nabla_\alpha h^\alpha{}_\nu \xi_\mu) + (\xi_\mu \nabla_\nu h - \xi_\nu \nabla_\mu h). \tag{H.10}$$

When $\partial\Sigma$ is the boundary of the past horizon $\mathcal{H}^-$, we may write

$$Q_\xi^{\mathcal{H}^-} = -\frac{2(2M)^2}{\kappa^2}\int d\Omega\, F_{ru}\Big|_{\mathcal{H}^-_-}^{\mathcal{H}^-_+}, \tag{H.11}$$

where $d\Omega = \sin\theta d\theta d\phi$. The relevant component reads

$$\begin{aligned} F_{ru} = {} & \frac{\kappa}{2}(\nabla_r \xi_u - \nabla_u \xi_r)h + \kappa(\nabla_r h_{\alpha u} - \nabla_u h_{\alpha r})\xi^\alpha + \kappa(\nabla_\alpha \xi_r h^\alpha{}_u - \nabla_\alpha \xi_u h^\alpha{}_r) \\ & + \kappa(\nabla_\alpha h^\alpha{}_r \xi_u - \nabla_\alpha h^\alpha{}_u \xi_r) + \kappa(\xi_r \nabla_u h - \xi_u \nabla_r h). \end{aligned} \tag{H.12}$$

In the Bondi gauge $h = g^{\mu\nu}h_{\mu\nu} = -2h_{ru}$, so at $r = 2M$ the terms are given by

$$\frac{1}{2}(\nabla_r\xi_u - \nabla_u\xi_r)h = \frac{f}{2M}h_{ru}, \tag{H.13}$$

$$(\nabla_r h_{\alpha u} - \nabla_u h_{\alpha r})\xi^\alpha = f(\partial_r h_{uu} - \partial_u h_{ur} - \frac{1}{2M}h_{ru}) + D^A f(\frac{1}{4M^2}h_{Au} - \frac{1}{2M}\partial_r h_{Au}) + \frac{1}{2}D^2 f\partial_r h_{ru} \tag{H.14}$$

$$\nabla_\alpha\xi_r h^\alpha{}_u - \nabla_\alpha\xi_u h^\alpha{}_r = -\frac{f}{2M}h_{ur}, \tag{H.15}$$

$$\nabla_\alpha h^\alpha{}_r\xi_u - \nabla_\alpha h^\alpha{}_u\xi_r = f\left(-\partial_u h_{ru} - \partial_r h_{uu} + \frac{1}{2M}h_{ru} + \frac{1}{4M^2}D^A h_{Au} - \frac{1}{M}h_{uu}\right) + \frac{1}{2}D^2 f\partial_r h_{ur} + \frac{1}{2M}D^2 f h_{ur} \tag{H.16}$$

$$\xi_r\nabla_u h - \xi_u\nabla_r h = -D^2 f\partial_r h_{ru} + 2f\partial_u h_{ru}. \tag{H.17}$$

Substituting these into F_{ru} yields, up to a total derivative on S^2,

$$F_{ru} = -\frac{\kappa}{2M}D^A f\partial_r h_{Au} - \frac{\kappa f}{M}h_{uu} + \frac{\kappa}{2M}D^2 f h_{ur}, \tag{H.18}$$

which leads to the following expression for the charge on $\mathcal{H}^-$,

$$Q_f^{\mathcal{H}^-} = -\frac{2(2M)^2}{\kappa}\int d\Omega\left(-\frac{1}{2M}D^A f\partial_r h_{Au} - \frac{f}{M}h_{uu} + \frac{1}{2M}D^2 f h_{ur}\right)\Bigg|^{\mathcal{H}^-_+}_{\mathcal{H}^-_-} \tag{H.19}$$

$$= \frac{4M}{\kappa}\int d\Omega\, du\, f(\Omega)\partial_u\left(-D^A\partial_r h_{Au} + 2h_{uu} - D^2 h_{ur}\right). \tag{H.20}$$

Given the linearly perturbed metric

$$g'_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}, \tag{H.21}$$

the perturbed Ricci tensor is given by [164]

$$R'_{\mu\nu} = R_{\mu\nu} - \frac{\kappa}{2}\left(\nabla_\nu\nabla_\mu h - \nabla_\rho\nabla_\mu h_\nu{}^\rho - \nabla_\rho\nabla_\nu h_\mu{}^\rho + \nabla_\rho\nabla^\rho h_{\mu\nu}\right) + \mathcal{O}(\kappa^2). \tag{H.22}$$

We take the background $g_{\mu\nu}$ to be the Schwarzschild metric, and use the obvious notation that quantities with primes are computed from the perturbed metric $g'_{\mu\nu}$ and those without are computed from $g_{\mu\nu}$. We keep everything only up to linear order in $h_{\mu\nu}$, and indices are raised/lowered using $g_{\mu\nu}$. The non-trivial constraints on $\mathcal{H}^-$ are,

$$G'_{uu} = \frac{\kappa^2}{4} T'_{uu} = 0, \tag{H.23}$$

$$G'_{uA} = \frac{\kappa^2}{4} T'_{uA} = 0, \tag{H.24}$$

where $G'_{\mu\nu} = R'_{\mu\nu} - \frac{1}{2} g'_{\mu\nu} R'$ is the perturbed Einstein tensor. The perturbed energy-momentum tensor $T'_{\mu\nu}$ vanishes since we keep terms only up to linear order in $h_{\mu\nu}$. At $r = 2M$ we have

$$-\frac{1}{2}\nabla_u \nabla_u h = \partial_u^2 h_{ru} + \frac{1}{4M}\partial_u h_{ru}, \tag{H.25}$$

$$-\frac{1}{2}\nabla_\rho \nabla^\rho h_{uu} = \partial_r \partial_u h_{uu} + \frac{1}{2M}\partial_u h_{uu} + \frac{1}{4M}\partial_r h_{uu} - \frac{1}{8M^2} D^2 h_{uu} - \frac{1}{2M}\partial_u h_{ru} - \frac{1}{8M^2} h_{ru}, \tag{H.26}$$

$$\begin{aligned}\nabla_\rho \nabla_u h_u{}^\rho = {} & -\partial_u^2 h_{ur} - \partial_r \partial_u h_{uu} + \frac{1}{4M^2}\partial_u D^A h_{uA} + \frac{1}{4M}\partial_u h_{ru} - \frac{1}{M}\partial_u h_{uu} - \frac{1}{4M}\partial_r h_{uu} \\ & + \frac{1}{8M^2} h_{ru} + \frac{1}{16M^3} D^A h_{uA}.\end{aligned} \tag{H.27}$$

The constraint $G'_{uu} = 0$ thus reduces to

$$0 = \frac{4M^2}{\kappa} G'_{uu} = 4M^2 \left(-\frac{1}{2}\nabla_u \nabla_u h - \frac{1}{2}\nabla_\rho \nabla^\rho h_{uu} + \nabla_\rho \nabla_u h_u{}^\rho \right) \tag{H.28}$$

$$= \partial_u (D^A h_{uA} - 2M h_{uu}) + \frac{1}{4M} D^A h_{uA} - \frac{1}{2} D^2 h_{uu}. \tag{H.29}$$

We can use the following expressions at $r = 2M$,

$$\begin{aligned}\nabla_\rho \nabla_A h_u{}^\rho = {} & -\partial_r D_A h_{uu} - \partial_u D_A h_{ur} + \frac{1}{2M} D_A h_{ru} - \frac{1}{M} D_A h_{uu} - \frac{3}{8M^2} h_{uA} \\ & + \frac{1}{4M^2} D^B D_A h_{uB},\end{aligned} \tag{H.30}$$

$$\nabla_\rho \nabla_u h_A{}^\rho = -\partial_u \partial_r h_{Au} - \frac{1}{M}\partial_u h_{uA} - \frac{1}{8M^2} h_{uA} + \frac{1}{4M^2}\partial_u D^B h_{AB}, \tag{H.31}$$

$$-\nabla_u \nabla_A h = 2\partial_u D_A h_{ur}, \tag{H.32}$$

$$-\nabla_\rho \nabla^\rho h_{Au} = 2\partial_u \partial_r h_{Au} - \frac{1}{4M^2} D^2 h_{Au} + \frac{1}{M} D_A h_{uu}, \tag{H.33}$$

to write the constraint $G'_{Au} = R'_{Au} = 0$ as

$$0 = \frac{2}{\kappa} G'_{Au} = \nabla_\rho \nabla_A h_u{}^\rho + \nabla_\rho \nabla_u h_A{}^\rho - \nabla_u \nabla_A h - \nabla_\rho \nabla^\rho h_{Au}. \tag{H.34}$$

$$\begin{aligned} &= \partial_u \left(D_A h_{ur} + \partial_r h_{Au} - \frac{1}{M} h_{uA} + \frac{1}{4M^2} D^B h_{AB} \right) - D_A \partial_r h_{uu} + \frac{1}{2M} D_A h_{ru} \\ &\quad + \frac{1}{4M^2} D_A D^B h_{uB} - \frac{1}{4M^2} D^2 h_{Au} - \frac{1}{4M^2} h_{uA} \end{aligned} \tag{H.35}$$

Taking the linear combination $0 = \frac{4M}{\kappa} G'_{uu} + \frac{2}{\kappa} D^A G'_{Au}$, we obtain the equation

$$\begin{aligned} \partial_u \left(-D^A \partial_r h_{Au} + 2h_{uu} - D^2 h_{ur} \right) = \frac{1}{4M^2} D^A D^B \partial_u h_{AB} - \frac{1}{2M} D^2 h_{uu} \\ - D^2 \partial_r h_{uu} + \frac{1}{2M} D^2 h_{ru} - \frac{1}{4M^2} D^A h_{uA}. \end{aligned} \tag{H.36}$$

Substituting this back into (H.20) yields the following expression for the horizon charge,

$$\begin{aligned} Q_f^{\mathcal{H}^-} = \frac{1}{\kappa M} \int d\Omega \, du \, f(\Omega) \Bigg(D^A D^B \partial_u h_{AB} - D^A h_{uA} \\ - 2M D^2 h_{uu} - 4M^2 D^2 \partial_r h_{uu} + 2M D^2 h_{ru} \Bigg). \end{aligned} \tag{H.37}$$

By performing a gauge fixing analogous to [61], the expression reduces to

$$Q_f^{\mathcal{H}^-} = \frac{1}{\kappa M} \int d\Omega \, du \, f(\Omega) D^A D^B \partial_u h_{AB}. \tag{H.38}$$

Appendix I

Magnetic Parity Gravitons on the Past Schwarzschild Horizon

In this appendix, we show that the magnetic parity gravitons do not contribute to the supertranslation charge (5.87) on $\mathcal{H}^-$. By the choice of boundary conditions, the in-modes vanish on $\mathcal{H}^-$, so it suffices to consider the up-modes.

Recall from (5.91) that $D^A D^B h^{\mathrm{up}}_{AB}$ is proportional to the following quantity,

$$D^A D^B H_{AB} = D_\theta D_\theta H_{\theta\theta} + \frac{1}{\sin^2\theta} D_\theta D_\phi H_{\theta\phi} + \frac{1}{\sin^2\theta} D_\phi D_\theta H_{\phi\theta} + \frac{1}{\sin^4\theta} D_\phi D_\phi H_{\phi\phi}, \tag{I.1}$$

where the tensor H_{AB} is defined in (5.92). Let us restrict our attention to the magnetic parity $P = -1$, for which the components of H_{AB} reads

$$H_{\theta\theta}(P=-1;\Omega) = 2M^2\left({}_{-2}Y_{lm} - {}_{+2}Y_{lm}\right), \tag{I.2}$$

$$H_{\theta\phi}(P=-1;\Omega) = 2M^2 i \sin\theta\left({}_{-2}Y_{lm} + {}_{+2}Y_{lm}\right) \tag{I.3}$$

$$H_{\phi\phi}(P=-1;\Omega) = -2M^2\sin^2\theta\left({}_{-2}Y_{lm} - {}_{+2}Y_{lm}\right), \tag{I.4}$$

where we have, from appendix G.1,

$${}_{-2}Y_{lm} + {}_{+2}Y_{lm} = 2\sqrt{\frac{(l-2)!}{(l+2)!}}\left(\partial_\theta^2 - \frac{1}{\sin^2\theta}\partial_\phi^2 - \cot\theta\,\partial_\theta\right) Y_{lm}, \tag{I.5}$$

$${}_{-2}Y_{lm} - {}_{+2}Y_{lm} = 2\sqrt{\frac{(l-2)!}{(l+2)!}}\left(\frac{2i\cos\theta}{\sin^2\theta}\partial_\phi - \frac{2i}{\sin\theta}\partial_\theta\partial_\phi\right) Y_{lm}. \tag{I.6}$$

With some algebra, one can show that

$$D_\theta D_\theta H_{\theta\theta}(-1;\Omega) = \frac{8iM^2}{\sin\theta}\sqrt{\frac{(l-2)!}{(l+2)!}}\Bigg[\left(\cot^3\theta + \frac{5\cot\theta}{\sin^2\theta}\right)\partial_\phi$$
$$- \partial_\theta^3\partial_\phi + 3\left(1-\frac{2}{\sin^2\theta}\right)\partial_\theta\partial_\phi + 3\cot\theta\partial_\theta^2\partial_\phi\Bigg]Y_{lm}(\Omega), \tag{I.7}$$

$$D_\theta D_\phi H_{\theta\phi}(-1;\Omega) = \frac{-4iM^2}{\sin\theta}\sqrt{\frac{(l-2)!}{(l+2)!}}\Bigg[\frac{11\cos\theta+\cos 3\theta}{\sin\theta}\partial_\phi - 3\cot\theta\partial_\phi^3 - \sin^2\theta\partial_\theta^3\partial_\phi$$
$$+ \partial_\theta\partial_\phi^3 - \frac{1}{2}(19+9\cos 2\theta)\partial_\theta\partial_\phi + 3\sin 2\theta\partial_\theta^2\partial_\phi\Bigg]Y_{lm}(\Omega), \tag{I.8}$$

and

$$D_\phi D_\theta H_{\theta\phi}(-1;\Omega) = -4iM^2\sin\theta\sqrt{\frac{(l-2)!}{(l+2)!}}\Bigg[13\partial_\theta\partial_\phi + \frac{10\cos\theta+2\cos 3\theta}{\sin^3\theta}\partial_\phi - \frac{3\cot\theta}{\sin^2\theta}\partial_\phi^3$$
$$- \partial_\theta^3\partial_\phi - \frac{14}{\sin^2\theta}\partial_\theta\partial_\phi + \frac{1}{\sin^2\theta}\partial_\theta\partial_\phi^3 + 6\cot\theta\partial_\theta^2\partial_\phi\Bigg]Y_{lm}(\Omega), \tag{I.9}$$

$$D_\phi D_\phi H_{\phi\phi}(-1;\Omega) = 8iM^2\sqrt{\frac{(l-2)!}{(l+2)!}}\Bigg[\sin\theta\left(-8\cos^2\theta\partial_\theta\partial_\phi + \partial_\theta\partial_\phi^3 + 3\cos\theta\sin\theta\partial_\theta^2\partial_\phi\right)$$
$$+ (\cos\theta + 5\cos^3\theta)\partial_\phi - 3\cos\theta\partial_\phi^3\Bigg]Y_{lm}(\Omega). \tag{I.10}$$

Substituting (I.7)-(I.10) into (I.1) yields

$$D^A D^B H_{AB}(-1;\Omega) = 0. \tag{I.11}$$

Therefore, with (5.91) we conclude

$$D^A D^B \partial_u h^{\mathrm{up}}_{AB}(l,m,\omega,P=-1;x) = 0 \qquad \text{on } \mathcal{H}^-, \tag{I.12}$$

which shows that the magnetic parity modes $P=-1$ do not contribute to the supertranslation charge. This is similar to the situation of gravitational memory at the infinities of asymptotically flat spacetimes, see [135–137] for relevant discussions of the gravitational memory effects.

Appendix J

Unfolding the Energy Integrals

In this appendix, we briefly discuss the treatment of energy integrals that arise in the graviton mode expansions in chapter 5.

In order to avoid dealing with expressions of the form $\int_0^\infty d\omega\,\delta(\omega)$, let us expand

$$h_{AB}(x) = \sum_{\Lambda} \sum_{l,m,P} \int_{-\infty}^{\infty} d\omega\, a^{\Lambda}_{lmP}(\omega) h^{\Lambda}_{AB}(l,m,\omega,P;x), \tag{J.1}$$

where $\Lambda \in \{\text{in}, \text{up}\}$, and a crossing relation analogous to that used in [93] is used to write

$$h^{\Lambda}_{AB}(l,m,\omega,P;x) = \left[h^{\Lambda}_{AB}(l,m,-\omega,P;x)\right]^* \qquad \text{for } \omega < 0. \tag{J.2}$$

The commutator between operators $a^{\Lambda}_{lmP}(\omega)$ becomes

$$\left[a^{\Lambda}_{lmP}(\omega), a^{\Lambda'}_{l'm'P'}(\omega')\right] = \delta_{\Lambda\Lambda'}\delta_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega+\omega'). \tag{J.3}$$

Recall that only the up-modes and the electric parity $P = 1$ contribute to the supertranslation charge. From (5.91) and (5.95), the asymptotic expression

$$h^{\text{up}}_{AB}(l,m,\omega,P=1;x) \sim -M\sqrt{\frac{(l-2)!}{\pi\omega(l+2)!}}(2D_AD_B - \gamma_{AB}D^2)Y_{lm}(\Omega)e^{-i\omega u} \tag{J.4}$$

holds near $\mathcal{H}^-$ for $\omega > 0$. We omit the subleading soft modes for they are irrelevant for this discussion. The crossing relation (J.2) implies that the above expression can be extended as

$$h^{\text{up}}_{AB}(l,m,\omega,P=1;x) \sim -M\sqrt{\frac{(l-2)!}{\pi|\omega|(l+2)!}}(2D_AD_B - \gamma_{AB}D^2)\widetilde{Y}_{lm}(u,\Omega)e^{-i\omega u} \tag{J.5}$$

near $\mathcal{H}^-$, to apply both to positive and negative ω, where we defined

$$\widetilde{Y}_{lm}(\omega,\Omega) \equiv \begin{cases} Y_{lm}(\Omega) & \text{for } \omega > 0, \\ Y^*_{lm}(\Omega) & \text{for } \omega < 0. \end{cases} \tag{J.6}$$

Using (J.5) and following an analogous set of steps to derive (5.101), we now obtain an alternate expression for N^-, which reads

$$N^-(\Omega) = \frac{2i}{\kappa}\sum_{lm}\int_{-\infty}^{\infty} d\omega\, a^{\text{up}}_{lm,P=1}(\omega)\delta(\omega)\sqrt{\frac{\pi|\omega|(l+2)!}{(l-2)!}}\widetilde{Y}_{lm}(\omega,\Omega). \tag{J.7}$$

Similarly, the zero-modes and the scalar fields are given by

$$h^-_{AB}(\Omega) = -2M(2D_AD_B - \gamma_{AB}D^2)\mathcal{A}^-(\Omega), \tag{J.8}$$

$$\mathcal{A}^-(\Omega) = \frac{1}{2}\sum_{lm}\int_{-\infty}^{\infty} d\omega\, \phi(\omega)a^{\text{up}}_{lm,P=1}(\omega)\sqrt{\frac{(l-2)!}{\pi|\omega|(l+2)!}}\widetilde{Y}_{lm}(\omega,\Omega). \tag{J.9}$$

Then by direct calculation,

$$\begin{aligned}
[N^-(\Omega), \kappa\mathcal{A}^-(\Omega')] &= i\sum_{lm}\sum_{l'm'}\int_{-\infty}^{\infty} d\omega d\omega'\phi(\omega')\delta(\omega)\sqrt{\frac{|\omega|(l+2)!(l'-2)!}{|\omega'|(l-2)!(l'+2)!}} \\
&\quad\times\left[a^{\text{up}}_{lm,P=1}(\omega)\widetilde{Y}_{lm}(\omega,\Omega), a^{\text{up}}_{lm,P=1}(\omega')\widetilde{Y}_{l'm'}(\omega',\Omega')\right] && \text{(J.10)} \\
&= i\sum_{lm}\int_{-\infty}^{\infty} d\omega d\omega'\phi(\omega')\delta(\omega+\omega')\delta(\omega)\widetilde{Y}_{lm}(\omega,\Omega)\widetilde{Y}_{lm}(\omega,\Omega'). && \text{(J.11)}
\end{aligned}$$

Since the delta function $\delta(\omega+\omega')$ is non-zero only when $\omega\omega' < 0$, let us keep the Hermitian combination[1] at $\omega = 0$,

$$\widetilde{Y}_{lm}(\omega,\Omega)\widetilde{Y}_{lm}(\omega,\Omega')\Big|_{\omega=0} = \frac{1}{2}\left(Y_{lm}(\Omega)Y^*_{lm}(\Omega') + Y^*_{lm}(\Omega)Y_{lm}(\Omega')\right). \tag{J.12}$$

This leads to the commutator

$$\begin{aligned}
[N(\Omega), \kappa\mathcal{A}^-(\Omega')] &= i\sum_{l=2}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\Omega)Y^*_{lm}(\Omega') && \text{(J.13)} \\
&= i\delta^{(2)}(\Omega-\Omega') + (l = 0, 1 \text{ terms}), && \text{(J.14)}
\end{aligned}$$

where we used the completeness relation of spherical harmonics.

[1] This is similar to the construction of zero-modes as a Hermitian combination of $\omega > 0$ and $\omega < 0$ modes in [16].

Appendix K

Subleading Soft Factors and Spin Angular Momenta

In this appendix, we review the steps presented in [27, 140] of deriving the subleading soft factors from the action of hard charge on matter particles. We treat gravity with massless scalars first, and then examine how the presence of spin affects the result. This is then used to derive analogous results for QED.

K.1 Gravity

K.1.1 Massless scalars

From the actions of soft and hard superrotation charges [140], we have

$$\langle \text{out}|[Q_H, \mathcal{S}]|\text{in}\rangle = i\sum_i \left(Y^z(z_i)\partial_{z_i} - \frac{E_i}{2} D_z Y^z(z_i)\partial_{E_i} + (z \to \bar{z}) \right) \langle \text{out}|\mathcal{S}|\text{in}\rangle \,, \tag{K.1}$$

$$\langle \text{out}|[Q_S, \mathcal{S}]|\text{in}\rangle = -\frac{i}{2\pi\kappa} \lim_{\omega\to 0}(1+\omega\partial_\omega) \int d^2z D_z^3 Y^z \langle \text{out}|a_-(\omega \mathbf{x}_z)\mathcal{S}|\text{in}\rangle + \text{h.c.}. \tag{K.2}$$

Here $\mathbf{x}_z$ denotes the unit 3-vector pointing in the direction defined by $(z, \bar{z})$,

$$\mathbf{x}_z = \left(\frac{\bar{z}+z}{1+z\bar{z}}, \frac{i(\bar{z}-z)}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}} \right). \tag{K.3}$$

Superrotation is a symmetry of the S-matrix, which implies that the superrotation charge

$$Q(Y) = Q_S(Y) + Q_H(Y), \tag{K.4}$$

is conserved in scattering process, that is, $\langle \text{out}|[Q(Y), \mathcal{S}]|\text{in}\rangle = 0$. Equivalently,

$$\langle \text{out}|[Q_S, \mathcal{S}]|\text{in}\rangle = - \langle \text{out}|[Q_H, \mathcal{S}]|\text{in}\rangle \,. \tag{K.5}$$

Using (K.1) and (K.2), this can be written as

$$\begin{aligned} &\frac{i}{2\pi\kappa} \lim_{\omega \to 0} (1 + \omega \partial_\omega) \int d^2 z D_z^3 Y^z \langle \text{out}|a_-(\omega \mathbf{x}_z)\mathcal{S}|\text{in}\rangle + \text{h.c.} \\ &\qquad = i \sum_i \left(Y^z(z_i)\partial_{z_i} - \frac{E_i}{2} D_z Y^z(z_i) \partial_{E_i} + (z \to \bar{z}) \right) \langle \text{out}|\mathcal{S}|\text{in}\rangle \,. \end{aligned} \tag{K.6}$$

For the vector field Y^z, let us choose

$$Y = \frac{(z - w)^2}{(\bar{z} - \bar{w})} \partial_z, \tag{K.7}$$

which satisfies $D_z^3 Y^z = 4\pi \delta^{(2)}(z - w)$, as well as

$$D_z Y^z(z_i) = \frac{2(w - z_i)(1 + w\bar{z}_i)}{(\bar{w} - \bar{z}_i)(1 + z_i \bar{z}_i)}. \tag{K.8}$$

Then (K.6) becomes

$$\begin{aligned} &\lim_{\omega \to 0} (1 + \omega \partial_\omega) \langle \text{out}|a_-(\omega \mathbf{x}_w)\mathcal{S}|\text{in}\rangle \\ &\qquad = \frac{\kappa}{2} \sum_i \left(-\frac{(w - z)^2}{(\bar{w} - \bar{z})} \partial_{z_i} - \frac{E_i (w - z_i)(1 + w\bar{z}_i)}{(\bar{w} - \bar{z}_i)(1 + z_i \bar{z}_i)} \partial_{E_i} \right) \langle \text{out}|\mathcal{S}|\text{in}\rangle \,. \end{aligned} \tag{K.9}$$

We now show that the factor in the parentheses on the RHS is the subleading soft factor,

$$S_g^{(1)-} = -i \frac{\kappa}{2} \sum_i \eta_i \frac{p_i^\mu k_\lambda J_i^{\lambda\nu}}{p_i \cdot k} \epsilon^-_{\mu\nu}(\mathbf{k}). \tag{K.10}$$

Since all hard particles are assumed to be scalars, the angular momentum consists of only the orbital part,

$$\eta_i (J_i)_{\mu\nu} = \eta_i (L_i)_{\mu\nu} \equiv -i \left(p_{i\mu} \frac{\partial}{\partial p_i^\nu} - p_{i\nu} \frac{\partial}{\partial p_i^\mu} \right), \tag{K.11}$$

where $\eta_i = +1$ (-1) if i-th particle is outgoing (incoming). Our definition of $J_{\mu\nu}$ is different from that of [140] by a sign, see footnote 1. Now let us parametrize

$$\begin{aligned} p^\mu &= \frac{E}{1+z\bar{z}}\left(1+z\bar{z}, \bar{z}+z, i(\bar{z}-z), 1-z\bar{z}\right), \\ k^\mu &= \frac{\omega_k}{1+w\bar{w}}\left(1+w\bar{w}, \bar{w}+w, i(\bar{w}-w), 1-w\bar{w}\right), \\ \epsilon^{-\mu} &= \frac{1}{\sqrt{2}}(w, 1, i, -w), \end{aligned} \tag{K.12}$$

and $\epsilon^{-\mu\nu} = \epsilon^{-\mu}\epsilon^{-\nu}$. The quantities $(E, z, \bar{z})$ are related to p^μ by

$$E = \sqrt{(p^x)^2 + (p^y)^2 + (p^z)^2}, \qquad z = \frac{p^x + ip^y}{p^t + p^z}, \qquad \bar{z} = \frac{p^x - ip^y}{p^t + p^z}. \tag{K.13}$$

Let us write (K.10) out as

$$S_g^{(1)-} = -(p\cdot\epsilon^-)\left(\epsilon^{-\mu}\frac{\partial}{\partial p^\mu} - \frac{(p\cdot\epsilon^-)}{(p\cdot k)}k^\mu\frac{\partial}{\partial p^\mu}\right). \tag{K.14}$$

One can show using (K.12) that

$$p\cdot\epsilon^- = \frac{-\sqrt{2}E(w-z)}{(1+z\bar{z})}, \tag{K.15}$$

$$p\cdot k = -\frac{2\omega_k E(w-z)(\bar{w}-\bar{z})}{(1+w\bar{w})(1+z\bar{z})}. \tag{K.16}$$

Now, noting that

$$\frac{\partial}{\partial p^\mu} = \frac{\partial E}{\partial p^\mu}\partial_E + \frac{\partial z}{\partial p^\mu}\partial_z + \frac{\partial \bar{z}}{\partial p^\mu}\partial_{\bar{z}}, \tag{K.17}$$

we obtain

$$\frac{\partial}{\partial p^t} = -\frac{z(1+z\bar{z})}{2E}\partial_z - \frac{\bar{z}(1+z\bar{z})}{2E}\partial_{\bar{z}}, \tag{K.18}$$

$$\frac{\partial}{\partial p^x} = \frac{(\bar{z}+z)}{(1+z\bar{z})}\partial_E + \frac{(1+z\bar{z})}{2E}\partial_z + \frac{(1+z\bar{z})}{2E}\partial_{\bar{z}}, \tag{K.19}$$

$$\frac{\partial}{\partial p^y} = \frac{i(\bar{z}-z)}{(1+z\bar{z})}\partial_E + \frac{i(1+z\bar{z})}{2E}\partial_z - \frac{i(1+z\bar{z})}{2E}\partial_{\bar{z}}, \tag{K.20}$$

$$\frac{\partial}{\partial p^z} = \frac{(1-z\bar{z})}{(1+z\bar{z})}\partial_E - \frac{z(1+z\bar{z})}{2E}\partial_z - \frac{\bar{z}(1+z\bar{z})}{2E}\partial_{\bar{z}}. \tag{K.21}$$

Using these with (K.15) and (K.16), we may write (K.14) as

$$S_g^{(1)-} = \frac{\kappa}{2}\sum_i \left(-\frac{E_i(w-z_i)(1+w\bar{z}_i)}{(\bar{w}-\bar{z}_i)(1+z_i\bar{z}_i)}\partial_{E_i} - \frac{(w-z_i)^2}{(\bar{w}-\bar{z}_i)}\partial_{z_i} \right). \tag{K.22}$$

This is exactly the expression appearing on the RHS of (K.9), which was to be shown.

K.1.2 Spin correction

Now suppose the hard particles have non-zero spin. The angular momentum $J^{\mu\nu}$ appearing in the subleading soft factor now contains the spin contribution $S^{\mu\nu}$,

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}. \tag{K.23}$$

Let us define helicity in terms of the Pauli-Lubansky pseudovector,

$$hp_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}p^\sigma = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}S^{\nu\rho}p^\sigma, \tag{K.24}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor with $\epsilon_{0123} = 1$, and in the second equation the orbital part drops out due to antisymmetry in p. In this basis, the spin angular momentum has components [165]

$$S_{\mu\nu} = \frac{h}{E}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & p^z & -p^y \\ 0 & -p^z & 0 & p^x \\ 0 & p^y & -p^x & 0 \end{pmatrix} = \frac{h}{1+z\bar{z}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1-z\bar{z} & -i(\bar{z}-z) \\ 0 & -(1-z\bar{z}) & 0 & \bar{z}+z \\ 0 & i(\bar{z}-z) & -(\bar{z}+z) & 0 \end{pmatrix}, \tag{K.25}$$

where in the second equation we used the parametrization (K.12).

For particles with spin, the action of the hard charge is [140]

$$\langle \text{out}|[Q_H, \mathcal{S}]|\text{in}\rangle = i\sum_i \left(\mathcal{L}_Y - \frac{E_i}{2}D_A Y^A(z_i)\partial_{E_i} \right) \langle \text{out}|\mathcal{S}|\text{in}\rangle, \tag{K.26}$$

where $\mathcal{L}_Y$ is the Lie derivative on the 2-sphere with respect to Y,

$$\mathcal{L}_Y = Y^A\partial_A + \frac{i}{2}D_A Y_B S^{AB}, \qquad A, B \in \{z, \bar{z}\}, \tag{K.27}$$

where S_{AB} is the pullback of (K.25) to the 2-sphere. By coordinate transformation from $\hat{x}^\mu$

to $(z, \bar{z})$, one finds that

$$S_{\bar{z}z} = \frac{\partial \hat{x}^\mu}{\partial \bar{z}} \frac{\partial \hat{x}^\nu}{\partial z} S_{\mu\nu} = -\frac{2ih}{(1 + z\bar{z})^2} = -ih\gamma_{z\bar{z}}, \tag{K.28}$$

and similarly $S_{z\bar{z}} = ih\gamma_{z\bar{z}}$, $S_{zz} = S_{\bar{z}\bar{z}} = 0$. Thus,

$$\mathcal{L}_Y = Y^z \partial_{\bar{z}} + \frac{h}{2} D_z Y^z + Y^{\bar{z}} \partial_{\bar{z}} - \frac{h}{2} D_{\bar{z}} Y^{\bar{z}}, \tag{K.29}$$

and (K.26) can be written as

$$\begin{aligned} \langle \text{out} | [Q_H, \mathcal{S}] | \text{in} \rangle = i \sum_i \left[Y^z(z_i) \partial_{z_i} - \frac{E_i}{2} D_z Y^z(z_i) \partial_{E_i} + \frac{h_i}{2} D_z Y^z(z_i) + (z \to \bar{z}, h \to -h) \right] \\ \times \langle \text{out} | \mathcal{S} | \text{in} \rangle \,. \end{aligned} \tag{K.30}$$

In the previous section, we saw that with the choice $Y = (z - w)^2 (\bar{z} - \bar{w})^{-1} \partial_z$, the first two terms in the square brackets correspond to the subleading soft factor coming from the orbital angular momentum $L^{\mu\nu}$. The third term then must correspond to the factor coming from spin angular momentum $S^{\mu\nu}$, which is

$$-i \sum_i \frac{p_i^\mu k_\lambda S_i^{\lambda\nu}}{p_i \cdot k} \epsilon^-_{\mu\nu}. \tag{K.31}$$

We have already computed $p_i \cdot \epsilon^-$ and $p_i \cdot k$ in (K.15) and (K.16). Using (K.25) and (K.12), one can directly show that

$$k^\lambda S_{\lambda\nu} \epsilon^{-\nu} = \frac{\sqrt{2} i h_i \omega_k (w - z_i)(1 + w\bar{z}_i)}{(1 + w\bar{w})(1 + z_i \bar{z}_i)}. \tag{K.32}$$

Combining the results, we obtain

$$-i \frac{p_i^\mu k_\lambda S_i^{\lambda\nu}}{p_i \cdot k} \epsilon^-_{\mu\nu} = \frac{h_i (w - z_i)(1 + w\bar{z}_i)}{(\bar{w} - \bar{z}_i)(1 + z_i \bar{z}_i)}. \tag{K.33}$$

But from (K.8) we observe that this can be written as

$$-i \frac{p_i^\mu k_\lambda S_i^{\lambda\nu}}{p_i \cdot k} \epsilon^-_{\mu\nu} = \frac{h_i}{2} D_z Y^z(z_i), \tag{K.34}$$

which is exactly the third term in the square brackets of (K.30), showing that the formalism extends to particles with spin.

K.2 QED

The results of appendix K.1.2 is directly relevant to QED since charged particles have spin. The construction is very similar – we start with charged scalars and move on to spin corrections.

K.2.1 Massless scalars

From the action of hard and soft charges for charged scalars, we have [27]

$$\langle \text{out}|[\mathcal{Q}_H,\mathcal{S}]|\text{in}\rangle = i\sum_i Q_i\left(\frac{1}{E_i}Y^z(z_i)\partial_{z_i} - D_zY^z(z_i)\partial_{E_i} + (z\to\bar{z})\right)\langle \text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{K.35}$$

$$\langle \text{out}|[\mathcal{Q}_S,\mathcal{S}]|\text{in}\rangle = -\frac{i}{4\pi e}\lim_{\omega\to 0}(1+\omega\partial_\omega)\int d^2z D_z^2Y^z\frac{\sqrt{2}}{1+z\bar{z}}\langle \text{out}|a_-(\omega\mathbf{x}_z)\mathcal{S}|\text{in}\rangle + \text{h.c.}. \tag{K.36}$$

Now let us choose the vector field

$$Y = \frac{(z-w)(1+z\bar{z})}{(\bar{z}-\bar{w})}\partial_z, \tag{K.37}$$

which satisfies $D_z^2Y^z = 2\pi(1+z\bar{z})\delta^{(2)}(z-w)$ and

$$D_zY^z = -\frac{(1+w\bar{z})}{(\bar{w}-\bar{z})}. \tag{K.38}$$

This leads the charge conservation $\langle \text{out}|[\mathcal{Q}_S,\mathcal{S}]|\text{in}\rangle = -\langle \text{out}|[\mathcal{Q}_H,\mathcal{S}]|\text{in}\rangle$ to be written as

$$\begin{aligned}&\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle \text{out}|a_-(\omega\mathbf{x}_z)\mathcal{S}|\text{in}\rangle\\ &\qquad = \frac{e}{\sqrt{2}}\sum_i Q_i\left(\frac{(w-z_i)(1+z_i\bar{z}_i)}{E_i(\bar{w}-\bar{z}_i)}\partial_{z_i} + \frac{(1+w\bar{z}_i)}{(\bar{w}-\bar{z}_i)}\partial_{E_i}\right)\langle \text{out}|\mathcal{S}|\text{in}\rangle\,.\end{aligned} \tag{K.39}$$

We now show that the factor on the RHS is the subleading soft factor for scalars,

$$S_e^{(1)-} = -ie\sum_i \eta_i Q_i\frac{k_\lambda L_i^{\lambda\nu}}{p_i\cdot k}\epsilon_\nu^-. \tag{K.40}$$

Comparing this to gravity (K.10), we observe that this is just $S_g^{(1)-}$ times $2eQ_i[\kappa(p_i\cdot\epsilon^-)]^{-1}$. From (K.15) and (K.22), we thus obtain

$$S_e^{(1)-} = \frac{e}{\sqrt{2}}\sum_i Q_i\left(\frac{(w-z_i)(1+z_i\bar{z}_i)}{E_i(\bar{w}-\bar{z}_i)}\partial_{z_i} + \frac{(1+w\bar{z}_i)}{(\bar{w}-\bar{z}_i)}\partial_{E_i}\right), \tag{K.41}$$

which exactly agrees with the RHS of (K.39).

K.2.2 Spin correction

In the presence of spin, the subleading factor (K.41) gains an additional spin contribution, which can be obtained using the results of appendix K.1.2. We shall show that this contribution is exactly the spin corrected term in the action of hard charge obtained by replacing $Y^z\partial_z$ with $\mathcal{L}_Y$ in (K.35).

Just as the orbital angular momentum piece of $S_e^{(1)-}$, multiplying $eQ_i(p_i\cdot\epsilon^-)^{-1}$ to (K.33) should give us the spin angular momentum part, i.e.

$$-ieQ_i\frac{k_\lambda S_i^{\lambda\nu}}{p_i\cdot k}\epsilon_\mu^- = -\frac{eQ_ih_i}{\sqrt{2}E_i}\frac{(1+w\bar{z}_i)}{(\bar{w}-\bar{z}_i)}. \tag{K.42}$$

Thus the full subleading soft factor becomes

$$S_e^{(1)-} = \frac{e}{\sqrt{2}}\sum_i Q_i\left(\frac{(1+w\bar{z}_i)}{(\bar{w}-\bar{z}_i)}\partial_{E_i} + \frac{(w-z_i)(1+z_i\bar{z}_i)}{E_i(\bar{w}-\bar{z}_i)}\partial_{z_i} - h_i\frac{(1+w\bar{z}_i)}{E_i(\bar{w}-\bar{z}_i)}\right). \tag{K.43}$$

The spin-corrected action of hard charge is obtained by replacing $Y^z\partial_z$ with $\mathcal{L}_Y$ in (K.35),

$$\langle\text{out}|[\mathcal{Q}_H,\mathcal{S}]|\text{in}\rangle = i\sum_i Q_i\left(\frac{1}{E_i}\mathcal{L}_Y - D_zY^z(z_i)\partial_{E_i} + (z\to\bar{z})\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle \tag{K.44}$$

$$= i\sum_i Q_i\left(\frac{1}{E_i}Y^z(z_i)\partial_{z_i} + \frac{h}{E_i}D_zY^z(z_i) - D_zY^z(z_i)\partial_{E_i} + (z\to\bar{z}, h\to -h)\right) \times \langle\text{out}|\mathcal{S}|\text{in}\rangle\,. \tag{K.45}$$

Accordingly, the charge conservation (K.39) becomes

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle\text{out}|a_-(\omega\mathbf{x}_z)\mathcal{S}|\text{in}\rangle = \frac{e}{\sqrt{2}}\sum_i Q_i\left(\frac{(w-z_i)(1+z_i\bar{z}_i)}{E_i(\bar{w}-\bar{z}_i)}\partial_{z_i} + \frac{(1+w\bar{z}_i)}{(\bar{w}-\bar{z}_i)}\partial_{E_i} - h_i\frac{(1+w\bar{z}_i)}{E_i(\bar{w}-\bar{z}_i)}\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle\,. \tag{K.46}$$

The new term is exactly the spin contribution (K.42) to the subleading soft factor $S_e^{(1)-}$.

Appendix L

Modified Lie Bracket

In this appendix, we discuss the modified Lie bracket algebra of BMS transformations on the future Schwarzschild horizon.

The vector field ξ that generates supertranslation $f(\Theta)$ and superrotation $Y(\Theta)$ is

$$\xi = \left(f + \frac{v}{2}\psi\right)\partial_v - \frac{1}{2}\left(D^2(f + \frac{v}{2}\psi) + r\psi\right)\partial_r + \left(\frac{1}{r}D^A(f + \frac{v}{2}\psi) + Y^A\right)\partial_A, \tag{L.1}$$

where $\psi \equiv D_A Y^A$. Let us ease the notation by defining

$$F(v, \Theta) \equiv f(\Theta) + \frac{v}{2}\psi(\Theta), \tag{L.2}$$

such that $\partial_v F = \frac{1}{2}\psi$. Then,

$$\xi = F\partial_v - \frac{1}{2}D^2 F\partial_r + \frac{1}{r}D^A F\partial_A - \frac{r}{2}\psi\partial_r + Y^A\partial_A, \tag{L.3}$$

$$\xi_v = -\Lambda F - \frac{1}{2}D^2 F - \frac{r}{2}\psi, \qquad \xi_r = F, \qquad \xi_A = rD_A F + r^2 Y_A, \tag{L.4}$$

where $Y_A = \gamma_{AB}Y^B$. In this form, ξ is like a supertranslation F (which now is v-dependent) with "corrections" $-\frac{r}{2}\psi\partial_r + Y^A\partial_A$. Since we are only interested in terms linear in ξ, we can compute the contributions of F and the remainders separately.

Using the non-vanishing Christoffel symbols

$$\bar{\Gamma}^v_{vv} = \frac{M}{r^2}, \qquad \bar{\Gamma}^v_{AB} = -r\gamma_{AB}, \tag{L.5}$$

$$\bar{\Gamma}^r_{vv} = \frac{M\Lambda}{r^2}, \qquad \bar{\Gamma}^r_{vr} = -\frac{M}{r^2}, \qquad \bar{\Gamma}^r_{AB} = -r\Lambda\gamma_{AB}, \tag{L.6}$$

$$\bar{\Gamma}^A_{rB} = \frac{1}{r}\delta^A_B, \qquad \bar{\Gamma}^A_{BC} = {}^{(2)}\Gamma^A_{BC}, \tag{L.7}$$

where we use $\bar{\Gamma}$ to denote Christoffel symbols of unperturbed Schwarzschild spacetime, we compute $\delta\bar{g}_{ab} \equiv \mathcal{L}_\xi \bar{g}_{ab}$ to be

$$\delta\bar{g}_{vv} = \frac{M}{r^2} D^2 F - \psi + \frac{3M}{r}\psi - \frac{1}{2} D^2 \psi, \tag{L.8}$$

$$\delta\bar{g}_{vr} = 0, \tag{L.9}$$

$$\delta\bar{g}_{vA} = -D_A \left(\Lambda F + \frac{1}{2} D^2 F \right), \tag{L.10}$$

$$\delta\bar{g}_{AB} = 2r D_A D_B F - r\gamma_{AB} D^2 F + r^2 \left(D_A Y_B + D_B Y_A - \gamma_{AB}\psi \right). \tag{L.11}$$

The perturbed metric is hence

$$\begin{aligned} ds^2 = &- \left(\Lambda - \frac{M}{r^2} D^2 F + \psi - \frac{3M}{r}\psi + \frac{1}{2} D^2\psi \right) dv^2 + 2dvdr - D_A \left(2\Lambda F + D^2 F \right) dv d\Theta^A \\ &+ \left[r^2 \gamma_{AB} + 2r D_A D_B F - r\gamma_{AB} D^2 F + r^2 \left(D_A Y_B + D_B Y_A - \gamma_{AB}\psi \right) \right] d\Theta^A d\Theta^B. \end{aligned} \tag{L.12}$$

Using this and the relation

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} + \frac{1}{2} \bar{g}^{ad} \left(\bar{\nabla}_b \delta\bar{g}_{dc} + \bar{\nabla}_c \delta\bar{g}_{db} - \bar{\nabla}_d \delta\bar{g}_{bc} \right) + O(\delta\bar{g})^2, \tag{L.13}$$

let us compute some perturbed Christoffel symbols to linear order in ξ,

$$\Gamma^v_{rr} = \Gamma^r_{rr} = \Gamma^A_{rr} = 0, \tag{L.14}$$

$$\Gamma^v_{rA} = 0, \tag{L.15}$$

$$\Gamma^r_{rA} = \frac{1}{r} D_A F - \frac{3M}{r^2} D_A F + \frac{1}{2r} D_A D^2 F, \tag{L.16}$$

$$\Gamma^B_{rA} = \frac{1}{r}\delta^B_A - \frac{1}{2r^2} \left(2D^B D_A F - \delta^B_A D^2 F \right), \tag{L.17}$$

which turn out to be exactly the same as the components of supertranslated metric with just $f \to F$, and

$$\gamma^{AB} \Gamma^v_{AB} = -2r, \tag{L.18}$$

$$\gamma^{AB} \Gamma^r_{AB} = -2r\Lambda - D^2 F + \frac{4M}{r} D^2 F - 2r\psi + 6M\psi - rD^2\psi - \frac{1}{2} D^2 D^2 F, \tag{L.19}$$

$$\gamma^{AB} \Gamma^C_{AB} = \gamma^{AB\,(2)}\Gamma^C_{AB} + \frac{4M}{r^2} D^C F + Y^C + D^2 Y^C. \tag{L.20}$$

Using the above symbols, we can write

$$\nabla_r \zeta_r = \partial_r \zeta_r, \tag{L.21}$$

$$\begin{aligned}\nabla_r \zeta_A + \nabla_A \zeta_r = \partial_r \zeta_A + D_A \zeta_r - \zeta_r \left(\frac{2}{r} D_A F - \frac{6M}{r^2} D_A F + \frac{1}{r} D_A D^2 F\right) \\ - \frac{2}{r}\zeta_A + \frac{1}{r^2}\zeta_B \left(2D^B D_A F - \delta_A^B D^2 F\right),\end{aligned} \tag{L.22}$$

$$\begin{aligned}\gamma^{AB}\nabla_A \zeta_B = D^A \zeta_A + \zeta_r \left(D^2 F - \frac{4M}{r} D^2 F + 2r\psi - 6M\psi + rD^2\psi + \frac{1}{2} D^2 D^2 F\right) \\ + 2r\zeta_v + 2r\Lambda\zeta_r - \zeta_C \left(\frac{4M}{r^2} D^C F + Y^C + D^2 Y^C\right).\end{aligned} \tag{L.23}$$

Now, let us raise the indices using the perturbed metric,

$$\zeta_v = -\Lambda\zeta^v + \zeta^v \left(\frac{M}{r^2} D^2 F - \psi + \frac{3M}{r}\psi - \frac{1}{2} D^2\psi\right) + \zeta^r - \zeta^A D_A \left(\Lambda F + \frac{1}{2} D^2 F\right), \tag{L.24}$$

$$\zeta_r = \zeta^v, \tag{L.25}$$

$$\begin{aligned}\zeta_A = -\zeta^v D_A \left(\Lambda F + \frac{1}{2} D^2 F\right) + r^2 \gamma_{AB} \zeta^B \\ + \zeta^B \left(2r D_A D_B F - r\gamma_{AB} D^2 F + r^2 \left(D_A Y_B + D_B Y_A - \gamma_{AB}\psi\right)\right),\end{aligned} \tag{L.26}$$

and expand around the Schwarzschild supertranslation + superrotation vector field parametrized by $g(\Theta)$ and $Z^A(\Theta)$, employing the shorthand $\phi \equiv D_A Z^A$ and $G \equiv g + \frac{v}{2}\phi$,

$$\zeta^v = G + \delta\zeta^v, \qquad \zeta^r = -\frac{1}{2} D^2 G - \frac{r}{2}\phi + \delta\zeta^r, \qquad \zeta^A = \frac{1}{r} D^A G + Z^A + \delta\zeta^A, \tag{L.27}$$

which to first order in the perturbation gives

$$\begin{aligned}\zeta_v = -\Lambda\delta\zeta^v + \delta\zeta^r + G\left(-\Lambda + \frac{M}{r^2} D^2 F - \psi + \frac{3M}{r}\psi - \frac{1}{2} D^2\psi\right) - \frac{1}{2} D^2 G - \frac{r}{2}\phi \\ - \left(\frac{1}{r} D^A G + Z^A\right)\left(\Lambda D_A F + \frac{1}{2} D_A D^2 F\right),\end{aligned} \tag{L.28}$$

$$\zeta_r = G + \delta\zeta^v, \tag{L.29}$$

$$\begin{aligned}\zeta_A = -G\left(\Lambda D_A F + \frac{1}{2} D_A D^2 F\right) + r D_A G + r^2 Z_A + r^2 \gamma_{AB} \delta\zeta^B \\ + \left(\frac{1}{r} D^B G + Z^B\right)\left(2r D_A D_B F - r\gamma_{AB} D^2 F + r^2 \left(D_A Y_B + D_B Y_A - \gamma_{AB}\psi\right)\right).\end{aligned} \tag{L.30}$$

Plugging these back in and demanding that $\nabla_r \zeta_r = \nabla_A \zeta_r + \nabla_r \zeta_A = \gamma^{AB} \nabla_A \zeta_B = 0$, we obtain

$$
\begin{aligned}
0 &= \partial_r \delta\zeta^v, && \text{(L.31)}\\
0 &= r^2 \gamma_{AB} \partial_r \delta\zeta^B + D_A \delta\zeta^v - \frac{2}{r}(D^B G) D_A D_B F + \frac{1}{r}(D_A G) D^2 F - (D^B G)(D_A Y_B + D_B Y_A) \\
&\quad + (D_A G)\psi, && \text{(L.32)}\\
0 &= -\Lambda (D^A G) D_A F + 2(D^A D^B G) D_A D_B F - (D^2 G) D^2 F \\
&\quad + \frac{r^2}{2}(D^A Z^B + D^B Z^A)(D_A Y_B + D_B Y_A) - r^2 \phi\psi - r(D^2 G)\psi - r(D^2 F)\phi + 2r\delta\zeta^r \\
&\quad + 2r(D^A D^B G) D_A Y_B - \frac{1}{2}(D^A G) D_A D^2 F + 2r(D^A Z^B) D_A D_B F + r^2 D_A \delta\zeta^A. && \text{(L.33)}
\end{aligned}
$$

Solving for $\delta\zeta$, we obtain

$$
\begin{aligned}
\delta\zeta^v &= 0, && \text{(L.34)}\\
\delta\zeta^r &= \frac{1}{2r}\left(\Lambda (D^A G) D_A F - (D^A D^B G) D_A D_B F + \frac{1}{2}(D^2 G) D^2 F\right) - r D^{(A} Z^{B)} D_{(A} Y_{B)} + \frac{r}{2}\phi\psi \\
&\quad + \frac{1}{2r}(D^A G) D^2 D_A F - (D^A Z^B) D_A D_B F + \frac{1}{2}(D^2 F)\phi + \frac{1}{2}(D_B G) D^2 Y^B + \frac{1}{2}(D_B G) Y^B, \\
& && \text{(L.35)}\\
\delta\zeta^A &= -\frac{1}{r^2}(D^B G) D^A D_B F + \frac{1}{2r^2}(D^A G) D^2 F - \frac{1}{r}(D_B G)(D^A Y^B + D^B Y^A) + \frac{1}{r}(D^A G)\psi. \\
& && \text{(L.36)}
\end{aligned}
$$

These are the changes in ζ^a due to the transformation ξ^a. To emphasize this point, we change the notation to $\delta\zeta^a \to \delta_\xi \zeta^a$. The changes in ξ^a due to ζ^a can be obtained by exchanging $\xi \leftrightarrow \zeta$, and we denote this as $\delta_\zeta \xi^a$.

The regular Lie bracket $[\xi, \zeta]^a = \xi^b \partial_b \zeta^a - \zeta^b \partial_b \xi^a$ of two vector fields can be computed straightforwardly from (L.1),

$$[\xi,\zeta]^v = \frac{1}{2}F\phi - \frac{1}{2}G\psi + Y^A D_A G - Z^A D_A F, \tag{L.37}$$

$$\begin{aligned}[\xi,\zeta]^r = &-\frac{1}{4}FD^2\phi + \frac{1}{4}GD^2\psi + \frac{1}{4}\phi D^2 F - \frac{1}{4}\psi D^2 G - \frac{1}{2r}(D^A F)D_A D^2 G + \frac{1}{2r}(D^A G)D_A D^2 F \\ &- \frac{1}{2}(D^A F)D_A\phi + \frac{1}{2}(D^A G)D_A\psi - \frac{1}{2}Y^A D_A D^2 G + \frac{1}{2}Z^A D_A D^2 F \\ &- \frac{r}{2}Y^A D_A\phi + \frac{r}{2}Z^A D_A\psi,\end{aligned} \tag{L.38}$$

$$\begin{aligned}[\xi,\zeta]^A = &\frac{1}{2r}FD^A\phi - \frac{1}{2r}GD^A\psi + \frac{1}{2r^2}(D^2F)D^A G - \frac{1}{2r^2}(D^2 G)D^A F + \frac{1}{2r}\psi D^A G - \frac{1}{2r}\phi D^A F \\ &+ \frac{1}{r^2}(D^B F)D_B D^A G - \frac{1}{r^2}(D^B G)D_B D^A F + \frac{1}{r}Y^B D_B D^A G - \frac{1}{r}Z^B D_B D^A F \\ &+ \frac{1}{r}(D^B F)D_B Z^A - \frac{1}{r}(D^B G)D_B Y^A + Y^B D_B Z^A - Z^B D_B Y^A.\end{aligned} \tag{L.39}$$

Now, we define the modified bracket by correcting this by $\delta_\xi \zeta^a$ and $\delta_\zeta \xi^a$,

$$[\xi,\zeta]^a_M = [\xi,\zeta]^a - \delta_\xi\zeta^a + \delta_\zeta\xi^a. \tag{L.40}$$

Using the expressions for $\delta_\xi\zeta^a$ that we have computed earlier, we obtain

$$[\xi,\zeta]^v_M = \frac{1}{2}F\phi + Y^A D_A G - (\xi \leftrightarrow \zeta), \tag{L.41}$$

$$\begin{aligned}[\xi,\zeta]^r_M = &-\frac{1}{4}FD^2\phi - \frac{1}{4}(D^2F)\phi - \frac{1}{2}(D^A F)D_A\phi - \frac{1}{2}Y^A D^2 D_A G - (D^A Y^B)D_A D_B G \\ &- \frac{1}{2}(D_B G)D^2 Y^B - \frac{r}{2}Y^A D_A\phi - (\xi \leftrightarrow \zeta),\end{aligned} \tag{L.42}$$

$$\begin{aligned}[\xi,\zeta]^A_M = &\frac{1}{2r}FD^A\phi - \frac{1}{2r}\psi D^A G + \frac{1}{r}Y^B D_B D^A G + Y^B D_B Z^A + \frac{1}{r}(D_B G)D^A Y^B \\ &- (\xi \leftrightarrow \zeta).\end{aligned} \tag{L.43}$$

The v-component can be reorganized as

$$[\xi,\zeta]^v = \frac{1}{2}f\phi - \frac{1}{2}g\psi + Y^A D_A g - Z^A D_A f + \frac{v}{2}D_A\left(Y^B D_B Z^A - Z^B D_B Y^A\right). \tag{L.44}$$

Let us define

$$\hat{f} = \frac{1}{2}f\phi - \frac{1}{2}g\psi + Y^A D_A g - Z^A D_A f, \tag{L.45}$$

$$\hat{Y}^A = Y^B D_B Z^A - Z^B D_B Y^A. \tag{L.46}$$

Then, define $\hat{\psi} \equiv D_A \hat{Y}^A$, and take $\hat{F} \equiv \hat{f} + \frac{v}{2}\hat{\psi}$ so that we have $[\xi, \zeta]^v = \hat{F}$,

$$\hat{F} = \frac{1}{2} F\phi + Y^A D_A G - (\xi \leftrightarrow \zeta). \tag{L.47}$$

With this definition, observe that we have exactly the modified bracket components

$$\begin{aligned} -\frac{1}{2} D^2 \hat{F} - \frac{r}{2}\hat{\psi} &= -\frac{1}{4} F D^2 \phi - \frac{1}{4}(D^2 F)\phi - \frac{1}{2}(D^A F) D_A \phi - \frac{1}{2} Y^A D^2 D_A G - (D^A Y^B) D_A D_B G \\ &\quad - \frac{1}{2}(D_A G) D^2 Y^A - \frac{r}{2} Y^A D_A \phi - (\xi \leftrightarrow \zeta) &\text{(L.48)} \\ &= [\xi, \zeta]^r_M, &\text{(L.49)} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{r} D^A \hat{F} + \hat{Y}^A &= \frac{1}{2r} F D^A \phi - \frac{1}{2r}\psi D^A G + \frac{1}{r} Y^B D_B D^A G + Y^B D_B Z^A + \frac{1}{r}(D_B G) D^A Y^B \\ &\quad - (\xi \leftrightarrow \zeta) &\text{(L.50)} \\ &= [\xi, \zeta]^A_M. &\text{(L.51)} \end{aligned}$$

This implies that

$$[\xi, \zeta]_M = \left(\hat{f} + \frac{v}{2}\hat{\psi}\right)\partial_v - \frac{1}{2}\left(D^2\left(\hat{f} + \frac{v}{2}\hat{\psi}\right) + r\hat{\psi}\right)\partial_r + \left(\frac{1}{r} D^A \left(\hat{f} + \frac{v}{2}\hat{\psi}\right) + \hat{Y}^A\right)\partial_A. \tag{L.52}$$

Comparing the RHS to the expression (L.1), we can see that it is another supertranslation $\hat{f}$ together with superrotation $\hat{Y}^A$.

We conclude that given two pairs $(f_1, Y_1), (f_2, Y_2)$ of supertranslation and superrotation, the modified bracket has the algebra

$$[(f_1, Y_1), (f_2, Y_2)]_M = (\hat{f}, \hat{Y}), \tag{L.53}$$

with the product being another supertranslation + superrotation parametrized by

$$\begin{aligned} \hat{f} &= \frac{1}{2} f_1 D_A Y_2^A - \frac{1}{2} f_2 D_A Y_1^A + Y_1^A D_A f_2 - Y_2^A D_A f_1, &\text{(L.54)} \\ \hat{Y}^A &= Y_1^B D_B Y_2^A - Y_2^B D_B Y_1^A. &\text{(L.55)} \end{aligned}$$

Appendix M

Standard and Dual Supertranslation Charges on the Schwarzschild Horizon

In this appendix, we give a derivation of the supertranslation and dual supertranslation charges using the formula of [108, 109],

$$\not{\delta} Q_E^{\mathcal{H}^+} = \frac{1}{16\pi}\epsilon_{\alpha\beta\gamma\delta}\int_{\partial\mathcal{H}^+}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta, \tag{M.1}$$

$$\not{\delta} Q_M^{\mathcal{H}^+} = \frac{i}{8\pi}\int_{\partial\mathcal{H}^+}(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta. \tag{M.2}$$

Here $\omega_{\alpha\beta}$ is the (torsion-free) spin connection 1-form, and $\delta\omega$ is the change in ω induced by the variation $\delta g_{ab} = h_{ab}$ of the metric.

In order to incorporate the variation of the metric, we parametrize a generic metric in Bondi gauge by

$$g_{ab} = \begin{pmatrix} V + W_A W^A & U & W_B \\ U & 0 & 0 \\ W_A & 0 & g_{AB} \end{pmatrix}, \tag{M.3}$$

where V, U, W_A are real functions of v, r, Θ^A. The inverse metric is

$$g^{ab} = \begin{pmatrix} 0 & U^{-1} & 0 \\ U^{-1} & -VU^{-2} & -U^{-1}W^B \\ 0 & -U^{-1}W^A & g^{AB} \end{pmatrix}, \tag{M.4}$$

where g^{AB} is the inverse of the two-dimensional metric g_{AB}, and $W^A = g^{AB}W_B$ (not $\gamma^{AB}W_B$). Since this metric may deviate from that of Schwarzschild, the two-dimensional curved indices

$A, B, C, \ldots$ in this section (and only in this section) are lowered and raised using g_{AB} and g^{AB} instead of the unit 2-sphere metric γ_{AB}.

We employ the following set of vielbein $E^\alpha = E^\alpha{}_a dx^a$,

$$E^1 = \frac{V}{2}dv + U dr, \tag{M.5}$$

$$E^2 = -dv, \tag{M.6}$$

$$E^3 = W_A \mu^A dv + \mu_A d\Theta^A, \tag{M.7}$$

$$E^4 = W_A \bar{\mu}^A dv + \bar{\mu}_A d\Theta^A, \tag{M.8}$$

where μ_A, $\bar{\mu}_A$ are complex functions of v, r, Θ^A, and $\mu^A = g^{AB}\mu_B$, $\bar{\mu}^A = g^{AB}\bar{\mu}_B$ (bar denotes complex conjugation, so $\bar{\mu}_A$ is the complex conjugate of μ_A and hence $E^3 = \overline{E^4}$). They satisfy the conditions

$$\mu_A \bar{\mu}_B + \mu_B \bar{\mu}_A = g_{AB}, \tag{M.9}$$

$$\mu^A \bar{\mu}_A = 1, \tag{M.10}$$

$$\mu^A \mu_A = \bar{\mu}^A \bar{\mu}_A = 0. \tag{M.11}$$

The tangent space metric and its inverse are

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{M.12}$$

and the inverse vielbeins $E_\alpha = E_\alpha{}^a \partial_a$ are

$$E_1 = U^{-1}\partial_r, \tag{M.13}$$

$$E_2 = -\partial_v + \frac{V}{2U}\partial_r + W^A \partial_A, \tag{M.14}$$

$$E_3 = \bar{\mu}^A \partial_A, \tag{M.15}$$

$$E_4 = \mu^A \partial_A. \tag{M.16}$$

One can readily check that

$$E_\alpha{}^a E^\alpha{}_b = \delta^a{}_b, \qquad E_\alpha{}^a E^\beta{}_a = \delta_\alpha{}^\beta, \tag{M.17}$$

$$E^\alpha{}_a E^\beta{}_b \eta_{\alpha\beta} = g_{ab}, \qquad E_\alpha{}^a E_\beta{}^b \eta^{\alpha\beta} = g^{ab}. \tag{M.18}$$

The spin connection 1-form $\omega_{\alpha\beta}$ is defined as

$$dE^\alpha = -\omega^\alpha{}_\beta \wedge E^\beta = \frac{1}{2}c^\alpha{}_{\beta\gamma}E^\beta \wedge E^\gamma, \tag{M.19}$$

$$\omega_{\alpha\beta} = \frac{1}{2}(c_{\alpha\beta\gamma} - c_{\beta\alpha\gamma} - c_{\gamma\alpha\beta})E^\gamma, \tag{M.20}$$

where $c_{\alpha\beta\gamma}$ are the anholonomy coefficients. The coefficients are given by

$$c_{1\beta\gamma} = 0, \tag{M.21}$$

$$c_{212} = \frac{1}{U}\left(\frac{1}{2}V' - \dot{U} + W^A\partial_A U\right), \tag{M.22}$$

$$c_{213} = \frac{\bar{\mu}^A\partial_A U}{U}, \tag{M.23}$$

$$c_{223} = -\frac{1}{2}\bar{\mu}^A\partial_A V + \frac{\bar{\mu}^A\partial_A U}{2U}V, \tag{M.24}$$

$$c_{234} = 0, \tag{M.25}$$

$$c_{312} = -\frac{W^{A\prime}\bar{\mu}_A}{U}, \tag{M.26}$$

$$c_{313} = \frac{\bar{\mu}^A\bar{\mu}'_A}{U}, \tag{M.27}$$

$$c_{314} = \frac{\mu^A\bar{\mu}'_A}{U}, \tag{M.28}$$

$$c_{323} = \bar{\mu}^A\partial_A(W\cdot\bar{\mu}) - \bar{\mu}^A\dot{\bar{\mu}}_A + \frac{\bar{\mu}^A\bar{\mu}'_A}{2U}V + W^A\bar{\mu}^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A), \tag{M.29}$$

$$c_{324} = \mu^A\partial_A(W\cdot\bar{\mu}) - \mu^A\dot{\bar{\mu}}_A + \frac{\mu^A\bar{\mu}'_A}{2U}V + W^A\mu^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A), \tag{M.30}$$

$$c_{334} = \left(\mu^A\bar{\mu}^B - \bar{\mu}^A\mu^B\right)\partial_B\bar{\mu}_A. \tag{M.31}$$

The remaining coefficients can be obtained using the antisymmetry $c_{\alpha\beta\gamma} = -c_{\alpha\gamma\beta}$ and the fact $E^3 = \overline{E^4}$ implies switching indices $3 \leftrightarrow 4$ corresponds to complex conjugation, for instance $c_{213} = \overline{c_{214}}$ and $c_{434} = \overline{c_{343}} = -\overline{c_{334}}$. Using this to compute $\omega_{\alpha\beta}$, we obtain

$$\begin{aligned}
\omega_{12} &= \frac{1}{U}\left(-\frac{1}{2}V' + \dot{U} - W^A\partial_A U\right)E^2 \\
&\quad + \frac{1}{2U}\left(-\bar{\mu}^A\partial_A U + W^{A\prime}\bar{\mu}_A\right)E^3 + \frac{1}{2U}\left(-\mu^A\partial_A U + W^{A\prime}\mu_A\right)E^4, && \text{(M.32)} \\
\omega_{13} &= \frac{1}{2U}\left(W^{A\prime}\bar{\mu}_A - \bar{\mu}^A\partial_A U\right)E^2 - \frac{\bar{\mu}^A\bar{\mu}'_A}{U}E^3 - \frac{1}{2U}\left(\mu^A\bar{\mu}'_A + \bar{\mu}^A\mu'_A\right)E^4, && \text{(M.33)} \\
\omega_{23} &= \frac{1}{2U}\left(-\bar{\mu}^A\partial_A U - W^{A\prime}\bar{\mu}_A\right)E^1 + \frac{1}{2}\left(\bar{\mu}^A\partial_A V - \frac{\bar{\mu}^A\partial_A U}{U}V\right)E^2 \\
&\quad - \left(\bar{\mu}^A\partial_A(W\cdot\bar{\mu}) - \bar{\mu}^A\dot{\bar{\mu}}_A + \frac{\bar{\mu}^A\bar{\mu}'_A}{2U}V + W^A\bar{\mu}^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A)\right)E^3 \\
&\quad - \frac{1}{2}\left(\mu^A\partial_A(W\cdot\bar{\mu}) - \mu^A\dot{\bar{\mu}}_A + \frac{\mu^A\bar{\mu}'_A}{2U}V + W^A\mu^B(\partial_A\bar{\mu}_B - \partial_B\bar{\mu}_A) + \text{c.c.}\right)E^4, && \text{(M.34)} \\
\omega_{34} &= \frac{1}{2U}\left(-\mu^A\bar{\mu}'_A + \bar{\mu}^A\mu'_A\right)E^1 \\
&\quad + \frac{1}{2}\left(\bar{\mu}^A\partial_A(W\cdot\mu) - \bar{\mu}^A\dot{\mu}_A + \frac{\bar{\mu}^A\mu'_A}{2U}V + W^A\bar{\mu}^B(\partial_A\mu_B - \partial_B\mu_A) - \text{c.c.}\right)E^2 \\
&\quad - \left(\mu^A\bar{\mu}^B - \bar{\mu}^A\mu^B\right)\partial_B\bar{\mu}_A E^3 - \left(\mu^A\bar{\mu}^B - \bar{\mu}^A\mu^B\right)\partial_B\mu_A E^4. && \text{(M.35)}
\end{aligned}$$

We keep in mind that $E^3 = \overline{E^4}$. The remaining components can be obtained by antisymmetry and complex conjugation, for instance $\omega_{42} = \overline{\omega_{32}} = -\overline{\omega_{23}}$.

M.1 Supertranslation charge

The conserved electric charge involves the differential form

$$\frac{1}{16\pi}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta. \tag{M.36}$$

We are interested in integrating

$$\begin{aligned}
\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta}\wedge E^\delta &= \epsilon_{1234}i_\xi E^3\delta\omega^{12}\wedge E^4 + \epsilon_{1324}i_\xi E^2\delta\omega^{13}\wedge E^4 + \epsilon_{2314}i_\xi E^1\delta\omega^{23}\wedge E^4 \\
&\quad + \epsilon_{1243}i_\xi E^4\delta\omega^{12}\wedge E^3 + \epsilon_{1423}i_\xi E^2\delta\omega^{14}\wedge E^3 + \epsilon_{2413}i_\xi E^1\delta\omega^{24}\wedge E^3 \\
&\quad + \cdots && \text{(M.37)}
\end{aligned}$$

over S^2. Observe that the alternating tensor $\epsilon_{\alpha\beta\gamma\delta}$ is purely imaginary,

$$\overline{\epsilon_{1234}} = \epsilon_{1243} = -\epsilon_{1234}. \tag{M.38}$$

By explicit computation, one finds that

$$\epsilon_{1234} = E_1{}^a E_2{}^b E_3{}^c E_4{}^d \epsilon_{abcd} = -i, \tag{M.39}$$

where ϵ_{abcd} is the alternating tensor in the curved coordinates with $\epsilon_{vr\theta\phi} = \sqrt{-\det g} = Ur^2 \sin\theta$. Using this and rearranging the indices, we obtain

$$\begin{aligned}\frac{i}{2}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta} \wedge E^\delta = & -i_\xi E^3 \delta\omega_{12} \wedge E^4 + i_\xi E^2 \delta\omega_{24} \wedge E^4 - i_\xi E^1 \delta\omega_{14} \wedge E^4 \\ & + i_\xi E^4 \delta\omega_{12} \wedge E^3 - i_\xi E^2 \delta\omega_{23} \wedge E^3 + i_\xi E^1 \delta\omega_{13} \wedge E^3 \\ & + \cdots . \end{aligned} \tag{M.40}$$

Let us look at this one by one. We are interested only in coefficients of $E^3 \wedge E^4$. The first and fourth terms combine to yield

$$-i_\xi E^3 \delta\omega_{12} \wedge E^4 + i_\xi E^4 \delta\omega_{12} \wedge E^3 = \frac{1}{2}\xi^A \left(\partial_A h_{vr} + \frac{2}{r}h_{vA} - \partial_r h_{vA}\right) E^3 \wedge E^4 + \cdots . \tag{M.41}$$

For the second term we have

$$i_\xi E^2 \delta\omega_{24} \wedge E^4 = \frac{\xi^v}{2}\left(\frac{1}{r}h_{vv} + \partial_A h^A{}_v + h^A{}_v \left(\bar{\mu}^B \partial_A \mu_B + \mu^B \partial_A \bar{\mu}_B\right)\right) E^3 \wedge E^4 + \cdots , \tag{M.42}$$

where we have used $\delta(\bar{\mu}^A \dot{\mu}_A) = \delta(\mu^A \dot{\bar{\mu}}_A) = 0$. It turns out that

$$\partial_A h^A{}_v + h^A{}_v \left(\bar{\mu}^B \partial_A \mu_B + \mu^B \partial_A \bar{\mu}_B\right) = g^{AB} D_A h_{vB} = \frac{1}{r^2}\gamma^{AB} D_A h_{vB}, \tag{M.43}$$

where D_A denotes covariant derivative on the unit 2-sphere (that is, compatible with γ_{AB}, not g_{AB}). Thus, we can write

$$i_\xi E^2 \delta\omega_{24} \wedge E^4 = \frac{\xi^v}{2}\left(\frac{1}{r}h_{vv} + \frac{1}{r^2}\gamma^{AB} D_A h_{vB}\right) E^3 \wedge E^4 + \cdots . \tag{M.44}$$

The coefficient of $E^3 \wedge E^4$ is real, i.e. its complex conjugate is the same,

$$i_\xi E^2 \delta\omega_{24} \wedge E^4 - i_\xi E^2 \delta\omega_{23} \wedge E^3 = \xi^v \left(\frac{1}{r}h_{vv} + \frac{1}{r^2}\gamma^{AB} D_A h_{vB}\right) E^3 \wedge E^4 + \cdots . \tag{M.45}$$

We also have

$$-i_\xi E^1 \delta\omega_{14} \wedge E^4 = \xi^r \delta\left[\frac{1}{2U}\left(\bar{\mu}^A \mu'_A + \mu^A \bar{\mu}'_A\right)\right] E^3 \wedge E^4 + \frac{\xi^r}{2}\left(\bar{\mu}^A \mu'_A + \mu^A \bar{\mu}'_A\right) \delta E^3 \wedge E^4 + \cdots , \tag{M.46}$$

$$i_\xi E^1 \delta\omega_{13} \wedge E^3 = \xi^r \delta\left[\frac{1}{2U}\left(\mu^A \bar{\mu}'_A + \bar{\mu}^A \mu'_A\right)\right] E^3 \wedge E^4 + \frac{\xi^r}{2}\left(\mu^A \bar{\mu}'_A + \bar{\mu}^A \mu'_A\right) E^3 \wedge \delta E^4 + \cdots . \tag{M.47}$$

Together we have

$$-i_\xi E^1 \delta\omega_{14} \wedge E^4 + i_\xi E^1 \delta\omega_{13} \wedge E^3 = -\frac{2\xi^r}{r} h_{vr} E^3 \wedge E^4 + \frac{\xi^r}{r} \delta(E^3 \wedge E^4) + \cdots , \tag{M.48}$$

where we have used $\delta(\mu^A \bar{\mu}'_A) = \delta(\bar{\mu}^A \mu'_A) = 0$. With $\delta r = 0$, we also have $\delta(E^3 \wedge E^4) = 0$ due to the Bondi gauge condition $\gamma^{AB} h_{AB} = 0$.

Collecting the results, we obtain

$$\begin{aligned}\frac{i}{2}\epsilon_{\alpha\beta\gamma\delta}(i_\xi E^\gamma)\delta\omega^{\alpha\beta} \wedge E^\delta = \Bigg[&\frac{1}{2}\xi^A\left(\partial_A h_{vr} + \frac{2}{r} h_{vA} - \partial_r h_{vA}\right) + \xi^v\left(\frac{1}{r} h_{vv} + \frac{1}{r^2}\gamma^{AB} D_A h_{vB}\right) \\ &- \frac{2\xi^r}{r} h_{vr}\Bigg] E^3 \wedge E^4 + \cdots . \end{aligned} \tag{M.49}$$

Plugging this into (M.1), we obtain the electric diffeomorphism charge associated with vector field ξ on the Schwarzschild horizon $r = 2M$ to be

$$\begin{aligned}\not{\delta} Q_E^{\mathcal{H}^+} = \frac{M^2}{4\pi}\int d^2\Theta \sqrt{\gamma}\Bigg[&\xi^A\left(\partial_A h_{vr} + \frac{1}{M} h_{vA} - \partial_r h_{vA}\right) \\ &+ \frac{1}{M}\xi^v\left(h_{vv} + \frac{1}{2M}\gamma^{AB} D_A h_{vB}\right) - \frac{2\xi^r}{M} h_{vr}\Bigg]. \end{aligned} \tag{M.50}$$

M.2 Dual supertranslation charge

The magnetic diffeomorphism charge associated with a vector field ξ takes the form

$$\not{\delta} Q_M^{\mathcal{H}^+} = \frac{i}{8\pi}\int_{\partial\mathcal{H}^+} (i_\xi E^\alpha)\delta\omega_{\alpha\beta} \wedge E^\beta. \tag{M.51}$$

For this we need to compute the $d\Theta^A \wedge d\Theta^B$ component of the two-form

$$(i_\xi E^\alpha)\delta\omega_{\alpha\beta} \wedge E^\beta. \tag{M.52}$$

Since only E^3 and E^4 carry $d\Theta^A$ components, the only part of the expression relevant to the S^2 integral is

$$\begin{aligned}(i_\xi E^\alpha)\delta\omega_{\alpha\beta}\wedge E^\beta = (i_\xi E^1)(\delta\omega_{13}\wedge E^3 + \delta\omega_{14}\wedge E^4) + (i_\xi E^2)(\delta\omega_{23}\wedge E^3 + \delta\omega_{24}\wedge E^4)\\ + (i_\xi E^3)\delta\omega_{34}\wedge E^4 + (i_\xi E^4)\delta\omega_{43}\wedge E^3 + \cdots\,,\end{aligned} \tag{M.53}$$

where $\cdots$ contains all the irrelevant components such as $dv\wedge d\Theta^A$. Using the expression (M.34) for the spin connection, we can write

$$\begin{aligned}(\delta\omega_{13}\wedge E^3 + \delta\omega_{14}\wedge E^4)|_{d\Theta^A\wedge d\Theta^B} = -\delta\left[\frac{1}{2U}\left(\bar\mu^A\mu'_A + \mu^A\bar\mu'_A\right)\right](E^3\wedge E^4 + E^4\wedge E^3)\\ - \frac{1}{2}\left(\bar\mu^A\mu'_A + \mu^A\bar\mu'_A\right)(\delta E^3\wedge E^4 + \delta E^4\wedge E^3)\\ - (\bar\mu^A\bar\mu'_A)\delta E^3\wedge E^3 - (\mu^A\mu'_A)\delta E^4\wedge E^4.\end{aligned} \tag{M.54}$$

The first line on the RHS is clearly zero since $E^3\wedge E^4 + E^4\wedge E^3 = 0$. The third line is also zero since

$$\bar\mu^A\bar\mu'_A = \frac{1}{r}\bar\mu^A\bar\mu_A = 0, \qquad \mu^A\mu'_A = \frac{1}{r}\mu^A\mu_A = 0. \tag{M.55}$$

In the second line, we have

$$\delta E^3\wedge E^4 + \delta E^4\wedge E^3 = (\delta\mu_A\bar\mu_B + \delta\bar\mu_A\mu_B)d\Theta^A\wedge d\Theta^B. \tag{M.56}$$

One can show that the expression in parentheses on the RHS is $\frac{1}{2}h_{AB}$ and is therefore symmetric,

$$h_{AB} = \delta(\mu_A\bar\mu_B + \bar\mu_A\mu_B) = 2(\delta\mu_A\bar\mu_B + \delta\bar\mu_A\mu_B), \tag{M.57}$$

which implies $\delta E^3\wedge E^4 + \delta E^4\wedge E^3 = 0$. Therefore we have

$$(\delta\omega_{13}\wedge E^3 + \delta\omega_{14}\wedge E^4)|_{d\Theta^A\wedge d\Theta^B} = 0. \tag{M.58}$$

The expression for $\delta\omega_{23}\wedge E^3 + \delta\omega_{24}\wedge E^4$ is similar but with just more complicated coefficients. To see this, first observe that the E^3 and E^4 components of ω_{23} and ω_{24} have the form

$$\omega_{23} = \cdots - AE^3 - BE^4, \qquad \omega_{24} = \cdots - BE^3 - \bar AE^4, \tag{M.59}$$

where A is complex and B is real,

$$A = \bar{\mu}^A \partial_A (W \cdot \bar{\mu}) - \bar{\mu}^A \dot{\bar{\mu}}_A + \frac{\bar{\mu}^A \bar{\mu}'_A}{2U}(V - W^2) + W^A \bar{\mu}^B (\partial_A \bar{\mu}_B - \partial_B \bar{\mu}_A), \tag{M.60}$$

$$B = \frac{1}{2}\left(\mu^A \partial_A (W \cdot \bar{\mu}) - \mu^A \dot{\bar{\mu}}_A + \frac{\mu^A \bar{\mu}'_A}{2U}(V - W^2) + W^A \mu^B (\partial_A \bar{\mu}_B - \partial_B \bar{\mu}_A) + \text{c.c.}\right). \tag{M.61}$$

Note that $A = B = 0$ on Schwarzschild; it is only the variations δA and δB that do not necessarily vanish. Thus, we have

$$\begin{aligned}(\delta\omega_{23} \wedge E^3 + \delta\omega_{24} \wedge E^4)|_{d\Theta^A \wedge d\Theta^B} &= -(\delta B)(E^3 \wedge E^4 + E^4 \wedge E^3) - B(\delta E^3 \wedge E^4 + \delta E^4 \wedge E^3) \\ &\quad - A\delta E^3 \wedge E^3 - \bar{A}\delta E^4 \wedge E^4 \qquad \text{(M.62)} \\ &= 0, \qquad \text{(M.63)}\end{aligned}$$

where the second line vanishes since $A = B = 0$, and the first line vanishes due to $E^3 \wedge E^4 + E^4 \wedge E^3 = 0$.

At this point we are left with the two terms,

$$(i_\xi E^3)\delta\omega_{34} \wedge E^4 + (i_\xi E^4)\delta\omega_{43} \wedge E^3. \tag{M.64}$$

We first note that the E^3 and E^4 components of $\omega_{34} = -\omega_{43}$ can be written compactly using $\mu^A \bar{\mu}^B - \bar{\mu}^A \mu^B = i\epsilon^{AB}$ as

$$\omega_{34} = \cdots + i\epsilon^{AB}\left(\partial_A \bar{\mu}_B E^3 + \partial_A \mu_B E^4\right). \tag{M.65}$$

The variation $\delta\epsilon^{AB}$ is proportional to the trace $\gamma^{AB} h_{AB}$ and therefore vanishes in Bondi gauge. Therefore if we vary ω_{34}, the variation only acts on the expression inside the parentheses,

$$\begin{aligned}\delta\omega_{34} &= \cdots + i\epsilon^{AB}\delta\left(\partial_A \bar{\mu}_B E^3 + \partial_A \mu_B E^4\right) \qquad \text{(M.66)} \\ &= \cdots + i\epsilon^{AB}\left(\partial_A \delta\bar{\mu}_B E^3 + \partial_A \delta\mu_B E^4 + \partial_A \bar{\mu}_B \delta E^3 + \partial_A \mu_B \delta E^4\right). \qquad \text{(M.67)}\end{aligned}$$

Plugging this in and using $i_\xi E^3 = \xi^A \mu_A$ and $i_\xi E^4 = \xi^A \bar{\mu}_A$, we obtain

$$\begin{aligned}(i_\xi E^3)\delta\omega_{34} \wedge E^4 + (i_\xi E^4)\delta\omega_{43} \wedge E^3 &= i\epsilon^{AB}\xi^C \mu_C \left(\partial_A \delta\bar{\mu}_B E^3 + \partial_A \bar{\mu}_B \delta E^3 + \partial_A \mu_B \delta E^4\right) \wedge E^4 \\ &\quad - i\epsilon^{AB}\xi^C \bar{\mu}_C \left(\partial_A \delta\mu_B E^4 + \partial_A \bar{\mu}_B \delta E^3 + \partial_A \mu_B \delta E^4\right) \wedge E^3 \qquad \text{(M.68)} \\ &= \xi^C X_C, \qquad \text{(M.69)}\end{aligned}$$

where X_C takes the form

$$\begin{aligned}X_C &= i\epsilon^{AB}\mu_C \left(\partial_A \delta\bar{\mu}_B E^3 + \partial_A \bar{\mu}_B \delta E^3 + \partial_A \mu_B \delta E^4\right) \wedge E^4 \\ &\quad - i\epsilon^{AB}\bar{\mu}_C \left(\partial_A \delta\mu_B E^4 + \partial_A \bar{\mu}_B \delta E^3 + \partial_A \mu_B \delta E^4\right) \wedge E^3 \qquad \text{(M.70)} \\ &= i\epsilon^{AB}\Big[\mu_C(\partial_A \delta\bar{\mu}_B)\mu_D\bar{\mu}_E + \mu_C(\partial_A\bar{\mu}_B)\delta\mu_D\bar{\mu}_E + \mu_C(\partial_A \mu_B)\delta\bar{\mu}_D\bar{\mu}_E \\ &\quad + \bar{\mu}_C(\partial_A\delta\mu_B)\mu_D\bar{\mu}_E + \bar{\mu}_C(\partial_A\bar{\mu}_B)\mu_D\delta\mu_E + \bar{\mu}_C(\partial_A\mu_B)\mu_D\delta\bar{\mu}_E\Big] d\Theta^D \wedge d\Theta^E. \qquad \text{(M.71)}\end{aligned}$$

One finds that this expression is

$$\begin{aligned}X_C &= \frac{1}{2}\left(\partial_\theta \frac{h_{\phi\theta}}{\sin\theta} + \frac{2\cos\theta}{\sin^2\theta} h_{\phi\theta} - \frac{\partial_\phi h_{\theta\theta}}{\sin\theta}, \sin\theta \partial_\theta \frac{h_{\phi\phi}}{\sin^2\theta} + \frac{2\cos\theta}{\sin^2\theta} h_{\phi\phi} - \partial_\phi \frac{h_{\theta\phi}}{\sin\theta}\right) d\Omega \qquad \text{(M.72)} \\ &= -\frac{r^2}{2}\epsilon^{AB} D_A h_{BC} d\Omega, \qquad \text{(M.73)}\end{aligned}$$

where $d\Omega = \sin\theta d\theta \wedge d\phi$, and D_A denotes the unit 2-sphere covariant derivative compatible with γ_{AB}. Notice that ϵ^{AB} here is the Levi-Civita tensor for the metric g_{AB}, which contains the r^2 factor. If we write $\bar{\epsilon}^{AB}$ for the Levi-Civita tensor corresponding to the S^2 metric γ_{AB}, we have the relation $\bar{\epsilon}^{AB} = r^2 \epsilon^{AB}$ and

$$X_C = -\frac{1}{2}\bar{\epsilon}^{AB} D_A h_{BC} d\Omega. \qquad \text{(M.74)}$$

Collecting the results, we obtain the magnetic diffeomorphism charge associated with a vector field ξ to be

$$\begin{aligned}\not{\delta} Q_M^{\mathcal{H}^+} &= \frac{i}{8\pi}\int_{\partial\mathcal{H}^+} (i_\xi E^\alpha)\delta\omega_{\alpha\beta} \wedge E^\beta \qquad \text{(M.75)} \\ &= \frac{i}{8\pi}\int_{\partial\mathcal{H}^+} \xi^C X_C \qquad \text{(M.76)} \\ &= -\frac{i}{16\pi}\int_{\partial\mathcal{H}^+} d^2\Theta \sqrt{\gamma}\, \xi^C \bar{\epsilon}^{AB} D_A h_{BC}. \qquad \text{(M.77)}\end{aligned}$$

Appendix N

Non-Integrable Part of Horizon Supertranslation Charge

In this appendix, we demonstrate that the non-integrable part $\mathcal{N}_f^{\mathcal{H}^+}$ of the horizon supertranslation charge exhibits vanishing Dirac bracket with other horizon supertranslation charges.

We can re-write $\mathcal{N}_f^{\mathcal{H}^+}$ in terms of the delta function $\partial_{\bar{z}} f = 2\pi\delta^2(z-w)$. Doing so and taking note that the covariant derivative D_z is acting on a scalar and is therefore a plain partial derivative, we obtain

$$\mathcal{N}_f^{\mathcal{H}^+} = -\frac{1}{8\pi M}\int_{\mathcal{H}^+} dv\, d^2z\, (\partial_{\bar{z}} f)\partial_z [D^2-1]^{-1} D^B D^A \sigma_{AB} \tag{N.1}$$

Partial integration in the second term by $\bar{z}$ yields

$$\mathcal{N}_f^{\mathcal{H}^+} = \frac{1}{8\pi M}\int_{\mathcal{H}^+} dv\, d^2z\, (\partial_z\partial_{\bar{z}} f)[D^2-1]^{-1} D^B D^A \sigma_{AB}. \tag{N.2}$$

The boundary term arising from this vanishes, since $\partial_{\bar{z}} f = 2\pi\delta^2(z-w)$ and the contour does not cross w. To treat $[D^2-1]^{-1}$ explicitly, let us consider its Green's function $\Delta(z,z')$ of D^2-1,[1]

$$(D^2-1)\Delta(z,z') = \frac{1}{\gamma_{z\bar{z}}}\delta^2(z-z'), \tag{N.3}$$

[1] The Green's function depends on both $(z,\bar{z})$ and $(z',\bar{z}')$, so we should have written $\Delta(z,\bar{z},z',\bar{z}')$ to be precise. We use the shorthand $\Delta(z,z')$ for notational brevity.

which is derived in appendix N.1 to be,

$$\Delta(z,z') = \frac{1}{4\sin(\pi\lambda)} P_\lambda(-\mathbf{n}_z \cdot \mathbf{n}_{z'}), \tag{N.4}$$

where $\lambda = \frac{1}{2}(-1 + i\sqrt{3})$, P_λ is the Legendre function, and

$$\mathbf{n}_z = \left(\frac{z+\bar{z}}{1+z\bar{z}}, \frac{i(\bar{z}-z)}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}} \right) \tag{N.5}$$

is the Cartesian coordinates of a unit vector on the sphere characterized by $(z,\bar{z})$. The quantity $\mathbf{n}_z \cdot \mathbf{n}_{z'}$ reduces to $\cos\theta$ when $(z',\bar{z}')$ is set to the north pole, as it should. Using Δ, we can write (8.39) as

$$\mathcal{N}_f^{\mathcal{H}^+} = \frac{1}{8\pi M} \int_{\mathcal{H}^+} dv\, d^2z\, (\partial_z \partial_{\bar{z}} f) \int d^2z' \sqrt{\gamma'}\, \Delta(z,z') D^{B'} D^{A'} \sigma_{A'B'}. \tag{N.6}$$

In the second term on the r.h.s., let us partial integrate the two covariant derivatives on $\sigma_{A'B'}$ to Δ. This gives rise to two boundary terms, but one can use (N.4) to show that they vanish, see appendix N.2 for details,

$$\mathcal{N}_f^{\mathcal{H}^+} = \frac{1}{8\pi M} \int_{\mathcal{H}^+} d^2z\, (\partial_z \partial_{\bar{z}} f) \int d^2z' \sqrt{\gamma'}\, (D^{A'} D^{B'} \Delta(z,z')) \sigma_{A'B'}. \tag{N.7}$$

First, let us compute the Dirac bracket $\{\mathcal{N}_f^{\mathcal{H}^+}, \delta Q_g^{\mathcal{H}^+}\}$. This is zero, since it is proportional to

$$\begin{aligned} &\left\{ \int_{\mathcal{H}^+} dv\, d^2z (\partial_z \partial_{\bar{z}} f) \int d^2z' \sqrt{\gamma'} (D^{A'} D^{B'} \Delta(z,z')) \sigma_{A'B'}, \int_{\mathcal{H}^+} dv\, d^2z'' \sqrt{\gamma''} (D^{E''} D^{C''} g) \sigma_{D''C''} \right\} \\ &= 0. \end{aligned} \tag{N.8}$$

Next, we compute $\{\mathcal{N}_f^{\mathcal{H}^+}, \delta \widetilde{Q}_g^{\mathcal{H}^+}\}$. It is proportional to the quantity

$$\begin{aligned} &\left\{ \int_{\mathcal{H}^+} dv\, d^2z (\partial_z \partial_{\bar{z}} f) \int d^2z' \sqrt{\gamma'} (D^{A'} D^{B'} \Delta(z,z')) \sigma_{A'B'}, \int_{\mathcal{H}^+_-} d^2z'' \sqrt{\gamma''} (D^{E''} D^{C''} g) \epsilon_{E''}{}^{D''} h_{D''C''} \right\} \\ &= 32\pi M^2 \int_{\mathcal{H}^+_-} d^2z\, (\partial_z \partial_{\bar{z}} f) \int d^2z' \sqrt{\gamma'} (D^{A'} D^{B'} \Delta(z,z')) (D^{E'} D^{C'} g) \epsilon_{E'}{}^{D'} \gamma_{A'B'D'C'}, \end{aligned} \tag{N.9}$$

where we have used (8.22), with $\gamma_{ABCD} = \gamma_{AC}\gamma_{BD} + \gamma_{AD}\gamma_{BC} - \gamma_{AB}\gamma_{CD}$. Partial integrating the two covariant derivatives on Δ to g while noting that $D_A \epsilon_{BC} = 0$ and $D_A \gamma_{BCDE} = 0$, we obtain

$$\left\{\int_{\mathcal{H}^+} dv\, d^2z(\partial_z\partial_{\bar{z}}f)\int d^2z'\sqrt{\gamma'}(D^{A'}D^{B'}\Delta(z,z'))\sigma_{A'B'},\int_{\mathcal{H}^+_-} d^2z''\sqrt{\gamma''}(D^{E''}D^{C''}g)\epsilon_{E''}{}^{D''}h_{D''C''}\right\}$$

$$= 32\pi M^2\int d^2z\,(\partial_z\partial_{\bar{z}}f)\int d^2z'\sqrt{\gamma'}\Delta(z,z')(D^{B'}D^{A'}D^{E'}D^{C'}g)\epsilon_{E'}{}^{D'}\gamma_{A'B'D'C'} \tag{N.10}$$

$$= 64\pi i M^2\int d^2z\,(\partial_z\partial_{\bar{z}}f)\int d^2z'\sqrt{\gamma'}\Delta(z,z')(\gamma^{z'\bar{z}'})^2[D^2_{z'},D^2_{\bar{z}'}]g. \tag{N.11}$$

The boundary term arising from the partial integration is similar to that discussed in appendix N.2 and vanish for the same reason.[2] In the second equation, we have used the fact that the only non-vanishing components of $\epsilon_A{}^B$ and γ_{ABCD} are $\epsilon_z{}^z = -\epsilon_{\bar{z}}{}^{\bar{z}} = i$ and $\gamma_{zz\bar{z}\bar{z}} = \gamma_{\bar{z}\bar{z}zz} = \frac{8}{(1+z\bar{z})^2} = 2\gamma_{z\bar{z}}{}^2$ respectively. One can readily check that $[D^2_z, D^2_{\bar{z}}]g = 0$.

Therefore, we conclude that $\mathcal{N}^{\mathcal{H}^+}_f$ has zero bracket with both charges,

$$\{\mathcal{N}^{\mathcal{H}^+}_f, \delta Q^{\mathcal{H}^+}_g\} = 0, \tag{N.13}$$

$$\{\mathcal{N}^{\mathcal{H}^+}_f, \delta \widetilde{Q}^{\mathcal{H}^+}_g\} = 0. \tag{N.14}$$

N.1 Green's function for $D^2 - 1$

Let us derive the Green's function for $D^2 - 1$ on the unit sphere. We want a solution to

$$(D^2-1)\Delta(\Omega,\Omega') = \delta(\Omega-\Omega') \equiv \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi'), \tag{N.15}$$

where Ω and Ω' represent points on the unit sphere, and the differential operator acts on Ω. Due to spherical symmetry, the Green's function only depends on the geodesic distance between Ω and Ω'. Without any loss of generality, we can assign the coordinates on the sphere such that Ω' sits at the north pole. Then, the geodesic distance between Ω and Ω' is given by θ. By spherical symmetry, this solution must be the same as when Ω' is not necessarily at the north pole but instead $\phi = \phi'$, in which case the geodesic distance is

[2]The boundary term arising from the partial integration is proportional to the expression

$$\int d^2z'\sqrt{\gamma'}\Big[D^{A'}\left(D^{B'}\Delta(z,z')D^{E'}D^{C'}g\right) - D^{B'}\left(\Delta(z,z')D^{A'}D^{E'}D^{C'}g\right)\Big]\epsilon_{E'}{}^{D'}\gamma_{A'B'D'C'}$$

$$= 2i\oint_z dz'\gamma^{z'\bar{z}'}\left((D^2_z g)\,\partial_{\bar{z}'}\Delta(z,z') - \Delta(z,z')D_{\bar{z}'}D^2_z g\right) + 2i\oint_z d\bar{z}'\gamma^{z'\bar{z}'}\left((D^2_{\bar{z}'}g)\,\partial_{z'}\Delta(z,z') - \Delta(z,z')D_{z'}D_{\bar{z}'^2}g\right). \tag{N.12}$$

It is shown in appendix N.1 that $\Delta \sim \frac{1}{4}\log|z-z'|^2$ as $z \to z'$, so the above expression vanishes due to lack of appropriate poles.

$|\theta - \theta'|$. Thus, we solve the following equation first,

$$(D^2 - 1)\Delta(|\theta - \theta'|) = \frac{1}{2\pi \sin\theta}\delta(\theta - \theta'), \tag{N.16}$$

and restore the ϕ-dependence later. The operator D^2 in spherical coordinates reads

$$D^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{N.17}$$

so by changing variables to $t = \cos\theta$, we can write (N.16) as

$$\left(\frac{d}{dt}(1-t^2)\frac{d}{dt} - 1\right)\Delta(t, t') = \frac{1}{2\pi}\delta(t - t'). \tag{N.18}$$

We can obtain the Green's function Δ by solving this equation for $t < t'$ and $t > t'$, and then stitching the two solutions together at $t = t'$.

The differential equation (N.18) states that a second-order differential operator acting on Δ yields a delta function. This implies that Δ is continuous at $t = t'$; otherwise the discontinuity can locally be written in terms of the Heaviside step function, and $\frac{d^2}{dt^2}$ acting on it yields a derivative of the delta function, which is not present in (N.18). So, we have

$$\lim_{\epsilon \to 0^+} \Delta(t' - \epsilon, t') = \lim_{\epsilon \to 0^+} \Delta(t' + \epsilon, t'). \tag{N.19}$$

On the other hand, $\frac{d\Delta}{dt}$ is discontinuous, which can be seen by integrating (N.18) around an infinitesimal region around $t = t'$,

$$\lim_{\epsilon \to 0^+}(1 - t'^2)\left(\left.\frac{d\Delta}{dt}\right|_{t=t'+\epsilon} - \left.\frac{d\Delta}{dt}\right|_{t=t'-\epsilon}\right) = \frac{1}{2\pi}. \tag{N.20}$$

With the stitching conditions (N.19) and (N.20) in mind, let us solve (N.18) for $t \neq t'$. Equation (N.18) for $t \neq t'$ takes the form of a Legendre equation,

$$\left(\frac{d}{dt}(1-t^2)\frac{d}{dt} + \lambda(\lambda+1)\right)\Delta(t, t') = 0, \tag{N.21}$$

with $\lambda = \frac{-1 \pm i\sqrt{3}}{2}$ (such that $\lambda(\lambda+1) = -1$). Being a second-order ordinary differential equation, this has two linearly independent solutions, the Legendre functions $P_\lambda(t)$ and $Q_\lambda(t)$ of the first and second kind. When $\lambda = n$ where n is an integer, $P_n(t)$ and $Q_n(t)$ become the Legendre polynomials of the first and second kind. Legendre polynomials have a definite parity, so for instance $P_n(t)$ and $P_n(-t) = (-1)^n P_n(t)$ are not linearly independent.

However, for non-integer λ, $P_\lambda(t)$ is linearly independent to $P_\lambda(-t)$, (equations 8.2.3 and 8.3.1 of [166])

$$P_\lambda(-t) = \cos(\lambda\pi) P_\lambda(t) - \frac{2}{\pi}\sin(\pi\lambda) Q_\lambda(t). \tag{N.22}$$

This relation implies that for non-integer λ, we can use $\{P_\lambda(t), P_\lambda(-t)\}$ instead of $\{P_\lambda(t), Q_\lambda(t)\}$ as the basis of solutions to (N.21). Thus, we can write

$$\Delta(t,t') = \begin{cases} a_1 P_\lambda(t) + a_2 P_\lambda(-t) & \text{for } t < t', \\ b_1 P_\lambda(t) + b_2 P_\lambda(-t) & \text{for } t > t', \end{cases} \tag{N.23}$$

where a_1, a_2, b_1 and b_2 are functions of t' only. We demand that the Green's function $\Delta(t,t')$ is well-defined everywhere but $t = t'$. Taking note that $P_\lambda(1) = 1$ and $P_\lambda(-1) = \infty$, one can see that this fixes $a_1 = b_2 = 0$,

$$\Delta(t,t') = \begin{cases} a_2 P_\lambda(-t) & \text{for } t < t', \\ b_1 P_\lambda(t) & \text{for } t > t'. \end{cases} \tag{N.24}$$

The remaining coefficients a_2 and b_1 are fixed by the stitching conditions (N.19) and (N.20), which read

$$a_2 P_\lambda(-t') = b_1 P_\lambda(t'), \tag{N.25}$$

$$b_1 P'_\lambda(t') + a_2 P'_\lambda(-t') = \frac{1}{2\pi(1-t'^2)}. \tag{N.26}$$

These can equivalently be written as

$$\begin{pmatrix} P_\lambda(-t') & -P_\lambda(t') \\ P'_\lambda(-t') & P'_\lambda(t') \end{pmatrix} \begin{pmatrix} a_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2\pi(1-t'^2)} \end{pmatrix}. \tag{N.27}$$

Solving for a_2 and b_1, we obtain

$$\begin{pmatrix} a_2 \\ b_1 \end{pmatrix} = \frac{1}{(P_\lambda(-t')P'_\lambda(t') + P_\lambda(t')P'_\lambda(-t'))} \begin{pmatrix} P'_\lambda(t') & P_\lambda(t') \\ -P'_\lambda(-t') & P_\lambda(-t') \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2\pi(1-t'^2)} \end{pmatrix} \tag{N.28}$$

$$= \frac{-1}{2\pi(1-t'^2)\mathcal{W}\{P_\lambda(t), P_\lambda(-t)\}|_{t=t'}} \begin{pmatrix} P_\lambda(t') \\ P_\lambda(-t') \end{pmatrix}, \tag{N.29}$$

where $\mathcal{W}\{\cdot,\cdot\}|_{t=t'}$ is the Wronskian,

$$\mathcal{W}\{P_\lambda(t), P_\lambda(-t)\} = \begin{vmatrix} P_\lambda(t) & P_\lambda(-t) \\ \frac{d}{dt}P_\lambda(t) & \frac{d}{dt}P_\lambda(-t) \end{vmatrix} = \begin{vmatrix} P_\lambda(t) & P_\lambda(-t) \\ P'_\lambda(t) & -P'_\lambda(-t) \end{vmatrix} \tag{N.30}$$

$$= -\left(P_\lambda(t)P'_\lambda(-t) + P_\lambda(-t)P'_\lambda(t)\right), \tag{N.31}$$

evaluated at $t = t'$. To compute the Wronskian of $P_\lambda(t)$ and $P_\lambda(-t)$, we first note that the Wronskian of $P_\lambda(t)$ and $Q_\lambda(t)$ is (equation 8.1.9 of [166])

$$\mathcal{W}\{P_\lambda(t), Q_\lambda(t)\} = \frac{1}{1-t^2}. \tag{N.32}$$

Then, we use the relation (N.22) to obtain

$$\mathcal{W}\{P_\lambda(t), P_\lambda(-t)\} = \cos(\lambda\pi)\mathcal{W}\{P_\lambda(t), P_\lambda(t)\} - \frac{2}{\pi}\sin(\pi\lambda)\mathcal{W}\{P_\lambda(t), Q_\lambda(t)\} \tag{N.33}$$

$$= \frac{-2\sin(\pi\lambda)}{\pi(1-t^2)}, \tag{N.34}$$

since $\mathcal{W}\{P_\lambda(t), P_\lambda(t)\} = 0$. This with (N.29) implies that a_2 and b_1 are

$$\begin{pmatrix} a_2 \\ b_1 \end{pmatrix} = \frac{1}{4\sin(\pi\lambda)} \begin{pmatrix} P_\lambda(t') \\ P_\lambda(-t') \end{pmatrix}. \tag{N.35}$$

Plugging these into (N.24), we obtain the Green's function

$$\Delta(t,t') = \frac{1}{4\sin(\pi\lambda)} \begin{cases} P_\lambda(t')P_\lambda(-t) & \text{for } t < t', \\ P_\lambda(-t')P_\lambda(t) & \text{for } t > t'. \end{cases} \tag{N.36}$$

Putting Ω' back at the north pole (and hence $\theta' = 0$ and $t' = 1$) and recalling that $\lambda = \frac{-1+i\sqrt{3}}{2}$, we obtain

$$\Delta(\theta) = \frac{1}{4\sin(\pi\lambda)} P_{\frac{-1+i\sqrt{3}}{2}}(-\cos\theta). \tag{N.37}$$

So, this is the Green's function when Ω' is the north pole. For a generic point Ω' on the sphere, spherical symmetry demands that Δ only depend on the geodesic distance γ between Ω and Ω', which is given as

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'), \tag{N.38}$$

and we have

$$\Delta(\Omega,\Omega') = \frac{1}{4\sin(\pi\lambda)} P_{\frac{-1+i\sqrt{3}}{2}}(-\cos\gamma), \tag{N.39}$$

as a solution to the equation (N.15). We note that it does not matter which of the two orders $\lambda = \frac{-1\pm i\sqrt{3}}{2}$ we choose, since $P_\lambda(t) = P_{\lambda^*}(t)$; we have just chosen plus sign for definiteness.

N.2 Treatment of boundary term

In this section, we show that the boundary terms arising from partial integrating the r.h.s. of (N.6) vanish.

One can see that this partial integration involves

$$\begin{aligned}\int d^2z'\sqrt{\gamma'}\Delta(z,z')D^{B'}D^{A'}\sigma_{A'B'} = &\int d^2z'\sqrt{\gamma'}D^{B'}\left(\Delta(z,z')D^{A'}\sigma_{A'B'}\right)\\ &-\int d^2z'\sqrt{\gamma'}D^{A'}\left(\sigma_{A'B'}D^{B'}\Delta(z,z')\right)\\ &+\int d^2z'\sqrt{\gamma'}\left(D^{A'}D^{B'}\Delta(z,z')\right)\sigma_{A'B'},\end{aligned} \tag{N.40}$$

so the boundary term arising from this procedure is proportional to the quantity

$$\begin{aligned}&\int d^2z'\sqrt{\gamma'}D^{B'}\left(\Delta(z,z')D^{A'}\sigma_{A'B'}\right) - \int d^2z'\sqrt{\gamma'}D^{A'}\left(\sigma_{A'B'}D^{B'}\Delta(z,z')\right)\\ &= \int d^2z'\left[\partial_{\bar z'}\left(\Delta(z,z')D^{z'}\sigma_{z'z'}\right) + \partial_{z'}\left(\Delta(z,z')D^{\bar z'}\sigma_{\bar z'\bar z'}\right)\right]\\ &\quad - \int d^2z'\left[\partial_{\bar z'}\left(\sigma_{z'z'}D^{z'}\Delta(z,z')\right) + \partial_{z'}\left(\sigma_{\bar z'\bar z'}D^{\bar z'}\Delta(z,z')\right)\right] \qquad \text{(N.41)}\\ &= -i\oint_z dz'\gamma^{z'\bar z'}\left(\Delta(z,z')\partial_{\bar z'}\sigma_{z'z'} - \sigma_{z'z'}\partial_{\bar z'}\Delta(z,z')\right)\\ &\quad + i\oint_z d\bar z'\gamma^{z'\bar z'}\left(\Delta(z,z')\partial_{z'}\sigma_{\bar z'\bar z'} - \sigma_{\bar z'\bar z'}\partial_{z'}\Delta(z,z')\right), \qquad \text{(N.42)}\end{aligned}$$

where in the last line we have used Stokes' theorem. This vanishes if (a) Δ and $\partial_{\bar z'}\Delta$ do not have z'-poles at $z' = z$ and (b) Δ and $\partial_{z'}\Delta$ do not have $\bar z'$-poles at $z' = z$.

To show that both (a) and (b) are true, we start from the Green's function $\Delta(z,z')$ given in (N.4). For the moment, let us put $z', \bar z' = 0$ (the north pole) and restore them later. This gives

$$\Delta(z,0) = \frac{1}{4\sin(\lambda\pi)} P_\lambda\left(\frac{z\bar z - 1}{z\bar z + 1}\right). \tag{N.43}$$

Only the asymptotic behavior of $\Delta(z,0)$ near $z, \bar z = 0$ is relevant for the boundary contribution (N.42), and for this we need the asymptotic behavior of $P_\lambda(t)$ near $t = -1$. This can be derived via the asymptotic behaviors of $P_\lambda(t)$ and $Q_\lambda(t)$ near $t = 1$, which read [167]

$$P_\lambda(t) \sim 1, \tag{N.44}$$

$$Q_\lambda(t) \sim \frac{1}{2}\ln\left(\frac{2}{1-t}\right), \qquad \text{as } t \to 1, \tag{N.45}$$

and using the relation (N.22), which yields

$$P_\lambda(t) \sim \frac{1}{\pi}\sin(\pi\lambda)\ln(1+t), \qquad \text{as } t \to -1. \tag{N.46}$$

Applying this to the Green's function (N.43) with $t = (z\bar{z}-1)/(z\bar{z}+1)$, we obtain

$$\Delta(z,0) \sim \frac{1}{4}\ln(z\bar{z}), \qquad \text{as } z,\bar{z} \to 0. \tag{N.47}$$

Restoring the reference point z', the asymptotic form of the Green's function near $z = z'$ is[3]

$$\Delta(z,z') \sim \frac{1}{4}\ln|z-z'|^2, \qquad \text{as } (z,\bar{z}) \to (z',\bar{z}'). \tag{N.50}$$

One immediately sees that Δ has a logarithmic singularity at $z = z'$ and therefore has no poles there. Also, $\partial_{z'}\Delta = \frac{1}{4(z'-z)}$ has no $\bar{z}'$-pole at $z' = z$, and $\partial_{\bar{z}'}\Delta = \frac{1}{4(\bar{z}'-\bar{z})}$ has no z'-pole at $z = z'$. Therefore, the boundary term (N.42) receives no residues and vanishes.

[3]One can also derive this without putting $z' = 0$ in the first place. To do so, one notes that $\cos\gamma$ in (N.38) for generic z and z' can be obtained by taking the dot product of two vectors of the form (N.5), and that it satisfies

$$1-\cos\gamma = 1 - \mathbf{n}_z \cdot \mathbf{n}_{z'} = \frac{2(z'-z)(\bar{z}'-\bar{z})}{(1+z\bar{z})(1+z'\bar{z}')}. \tag{N.48}$$

Then, taking $z = z' + re^{i\phi}$ and expanding around $r = 0$ leads to

$$1-\cos\gamma = \frac{2r^2}{(1+z'\bar{z}')^2} + O(r^3), \tag{N.49}$$

which plugged into (N.46) for $P_\lambda(-\cos\gamma)$ and then into (N.4) leads to $\Delta(z,z') \sim \frac{1}{4}\ln r^2 = \frac{1}{4}\ln(z-z')(\bar{z}-\bar{z}')$ for $r \to 0$, in agreement with (N.50).

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