Singularities of Birational Geometry via Arcs and Differential Operators

by

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ABSTRACT

We study singularities of algebraic varieties, in particular those arising in birational geometry, from several points of view. The first is that of arc schemes: arc schemes parametrize “infinitesimal curves” on a variety, and their geometry reflects properties of singularities. We show that morphisms of arc schemes (more precisely, of “local” arc schemes) can detect local isomorphisms of varieties. More precisely, we use the triviality of a certain ideal-closure operation to show that if a morphism induces an isomorphism of local arc schemes then it must be an isomorphism on local rings.

We then use arc schemes, in conjunction with the theory of determinantal rings, to verify the semicontinuity conjecture for the behavior of the minimal log discrepancy (a subtle invariant of singularities) in the case of determinantal varieties. In particular, we calculate the Nash ideal of a generic square determinantal variety, which then allows us to give an explicit formula for the minimal log discrepancies of pairs of determinantal varieties and determinantal subvarieties. This allows us to verify the semicontinuity conjecture for such pairs.

We then take another point of view, via the study of differential operators on singular rings. At least since [LS89], the question had been asked of whether one can characterize singularities of rings via certain properties of their rings of differential operators. In particular, one question is whether a ring with mild singularities is a simple module under the action of its ring of differential operators. While an answer in characteristic p had been provided by [Smi95], no answer had been forthcoming in characteristic 0. We provide a counterexample showing that the expected connection does not exist, through the study of the global geometry of Fano varieties. More specifically, we show that certain del Pezzo surfaces do not have big tangent bundles, and thus their homogeneous coordinate rings are not simple under the action of their rings of differential operators, despite having “mild” singularities.

CHAPTER I

Introduction

When studying the solutions of systems of polynomial equations (i.e., algebraic varieties), one encounters certain special points, at which the local structure of the set of solutions differs from the generic "smooth" behavior. Such points are called "singularities," and can be characterized in many ways; perhaps the most intuitive (at least over $\mathbb{C}$) is that a singular point is one at which the solution set fails to be a manifold, or equivalently where the inverse function theorem fails. Singular points are a complicated but unavoidable part of algebraic geometry: even if one is only concerned with smooth varieties, singular varieties arise as limits, intersections, or projections of smooth varieties. Understanding or classifying the types of singularities that occur is beyond our grasp; indeed, by considering affine cones over projective varieties, one sees that classifying all possible singularities in dimension $n+1$ necessitates classifying all projective varieties of dimension n. So, we frequently restrict our attention to certain special classes of "mild" singularities. There are more points of view than can be described here, but a few are particularly relevant in my work:

1. (Birational geometry) The minimal model program is a (still partially conjectural) program, which from a complex algebraic variety produces a "simplest" model, isomorphic to the original variety almost everywhere. In this simplification process, however, certain singularities are inevitably introduced. The singularities produced in this process (and its generalization to pairs consisting of a variety and a codimension-1 subvariety) form several classes of interest: terminal, canonical, Kawamata log terminal (or klt), log canonical, and more.

2. (Characteristic p) When working over a field k of characteristic $p > 0$, a k-algebra R has a natural map $F : R \to R$, the p-th power, or Frobenius map. The properties of F lead to several important classes of singularities: F-regular,

F-split/F-pure, F-rational, etc. Not only have these classes proved natural from the point of view of commutative algebra, but they have been shown to have deep connections with the classes arising from birational geometry

(3) (Differential operators) Given a k-algebra R, one can define the ring of k-linear differential operators on R, denoted $D_{R/k}$, which generalizes the Weyl algebra $\mathbb{C}\langle x_1, \ldots, x_n, \partial/\partial x_1, \ldots, \partial/\partial x_n\rangle$ of differential operators on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Properties of $D_{R/k}$ often reflect the singularities of R.

1.1 Singularities of birational geometry

Motivation for defining the singularities studied here comes in large part from higher-dimensional birational geometry, where these classes appear naturally when one attempts to find the "simplest" birational model of a variety.

We begin with a brief recollection of the surface case, which is described (for example) in [Har77, Chapter V].

Remark 1.1.1. Let X be a smooth projective surface. If $E \subset X$ is a smooth rational curve with $E^2 = -1$ (that is, E is a (-1)-curve), then Castelnuovo's contractibility theorem says that X is the blowup of another smooth projective surface X_1 at a point p, and E the exceptional divisor of this blowup. Now, one can look for smooth rational curves E_1 on X_1 with $E_1^2 = -1$, and continue in this manner blowing down (-1)-curves. Note that $\rho(X_1)$ (the rank of the Picard group of X modulo numerical equivalence) is $\rho(X) - 1$. Severi's theorem of the base states that $\rho(X)$ is finite, and thus the Picard rank strictly decreases with each contraction. Thus, the process terminates, and we obtain a smooth projective surface $\widetilde{X}$, birational to X, with no (-1)-curves.

Since birational morphisms of smooth projective surfaces can be factored into blow-ups of points, there can be no birational morphism $\widetilde{X} \to Y$ that is not an isomorphism. Thus, one can think of $\widetilde{X}$ as being a simplest, or "minimal", birational model of X. Note that in this case one does not have to leave the world of smooth projective varieties.

In higher dimensions, it is then natural to ask whether there is a similar method for producing a "simplest" birational representative of a given smooth projective variety.

Remark 1.1.2. Two related insights were necessary to make progress in higher dimensional algebraic geometry:

- When studying the possible birational modifications of a variety, one should examine the *curves* that are to be contracted. In the surface case, these curves are of course also divisors, but in higher dimensions one can better find contractibility criterion when studying curves rather than divisors.

- In the surface case, negativity of the self-intersection number allowed us to identify the contractible curves. In higher dimension, there will quite often be disjoint curves contracted to the same point, and so their intersection is not particularly illuminating.

  Instead, if X is a variety and C a curve on X, one should consider $K_X \cdot C$, the degree of the restriction of the canonical divisor to C. Concretely, if X is smooth one writes down a meromorphic volume form on X, and $K_X \cdot C$ is the degree of the restriction of this volume form to C. When X is a surface, the adjunction formula says that $K_X \cdot C + C^2 = 2g(C) - 2$, where $g(C)$ is the genus of C. If C is a smooth rational curve, so $g(C) = 0$, $C^2 = -1$ (and thus C is contractible to a smooth point) if and only if $K_X \cdot C = -1$.

The key idea of the minimal model program, then, is that in higher dimensions one should look for curves C such that $K_X \cdot C < 0$, and seek to contract these curves. If there are no curves C such that $K_X \cdot C < 0$, K_X is then a *nef* divisor (thought of as some "positivity" of K_X), and one can check that X has various nice "minimality" properties.

Remark 1.1.3 (Necessity of singularities)**.** In dimension ≥ 3, if one tries to find a "simplest" birational model by contracting curves on which the canonical divisor is negative, one inevitably encounters singularities. Indeed, it was precisely the realization that such singularities were unavoidable, and in many ways tractable, which allowed for many of the important developments of birational geometry of the last decades.

Perhaps the simplest example of the introduction of singularities is the following: let A be an abelian variety of dimension 3, and let $i : A \to A$ be the involution $p \mapsto -p$. By standard results on abelian varieties, i has exactly 64 fixed points. If X_0 is the quotient of A under the action of i, then, X_0 has exactly 64 isolated double points, each of which is (analytically) isomorphic to the cone over the degree-2 Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$. Let $X \to X_0$ be the resolution of singularities obtained by blowing up these 64 double points, with exceptional divisors $E_1, \ldots, E_{64}$, each isomorphic to $\mathbb{P}^2$. Then one can check that $\mathcal{O}_{E_i}(E_i) = \mathcal{O}_{\mathbb{P}^2}(-2)$. Adjunction states that

$$\mathcal{O}_X(K_X + E_i)|_{E_i} \cong \mathcal{O}_{E_i}(K_{E_i});$$

since $\mathcal{O}_{E_i}(K_{E_i}) = \mathcal{O}_{\mathbb{P}^2}(-3)$, we have that $\mathcal{O}_X(K_X)|_{E_i} = \mathcal{O}_{\mathbb{P}^2}(-1)$, so that K_X is not nef (as $K_X \cdot L = -1$ for any line $L \subset E_i$.).

On the other hand, K_{X_0} is nef: since the quotient morphism $\pi : A \to X_0$ is étale in codimension 1, we have $K_A = \pi^* K_{X_0}$. Since K_A is trivial, K_{X_0} is numerically trivial, thus nef. Thus, from the point of view of positivity of the canonical bundle, X_0 is a "more minimal" model than X, despite its singularities. Moreover, one can check that any attempt to contract the K_X-negative curves of X will result in a singular variety. Thus, if we truly want a nef canonical divisor, we must accept the presence of singularities.

The singularities that appear in the process of contracting K_X-negative curves on a smooth variety are called "terminal" singularities. They will be defined more systematically in what follows, but the original motivation for their definition arises exactly from their appearance in this process.

Remark 1.1.4 (Canonical singularities on surfaces)**.** There is a related class of singularities arising in the search for a representative of a birational equivalence class, which appears already in the study of surfaces, the *canonical* singularities. Their formal definition will appear later, but here we mention how they arise. If X is a smooth projective surface, one can consider the graded ring $R_X := \bigoplus H^0(X, mK_X)$. Because the plurigenera $H^0(X, mK_X)$ are birational invariants of smooth projective varieties, the ring R_X is unchanged by contracting (-1)-curves on X, and we may thus assume that X has no (-1)-curves. There are then two possibilities: $H^0(X, mK_X) = 0$ for all m, in which case it can be shown that X is either $\mathbb{P}^2$ or a projective bundle over a smooth curve, or $H^0(mK_X) \neq 0$ for some (hence infinitely many) m. In the latter case, we assume R_X is finitely generated (this is in fact automatic, because $\mathcal{O}_X(K_X)$ can be shown to be semiample). One can then consider the dimension of $\operatorname{Proj} R_X$, which may be 0, 1, or 2. When $\dim \operatorname{Proj} R_X = 2$, we set $X_{\text{can}} := \operatorname{Proj} R_X$ and call it the canonical model of X. Again, since the plurigenera (and more generally, R_X itself) are birational invariants of smooth projective varieties, X_{can} is a birational invariant of X.

When $\dim \operatorname{Proj} R_X = 2$, a priori one has only a birational map $X \dashrightarrow X_{\text{can}}$. However, if X contains no (-1)-curves, one can show that some multiple of $|mK_X|$ is in fact basepointfree, and thus defines a morphism $X \to X_{\text{can}}$. One can check that this morphism contracts precisely those curves C for which $K_X \cdot C = 0$, and that these are exactly the (-2)-curves, i.e., smooth rational curves C such that $C^2 = -2$. The resulting variety X_{can} will not be smooth, but will have isolated singularities. These are the so-called Du Val singularities, and can be described in several ways:

(1) They are exactly the ADE singularities, so called because they are classified by the Dynkin diagrams of types A, D, and E, and they can be given up to analytic isomorphism as a list of explicit hypersurface singularities.

(2) They are the rational double points.

(3) They are exactly the singularities occurring as quotients of $\mathbb{C}^2$ by a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$.

More generally, one can define canonical singularities as the singularities appearing on the "canonical models" of smooth projective varieties of general type. If X is a smooth projective variety, and the canonical ring $\bigoplus H^0(X, mK_X)$ is a finitely generated $\mathbb{C}$-algebra[1] and $\dim X = \dim \operatorname{Proj}\bigl(\bigoplus H^0(X, mK_X)\bigr)$, we set $X_{\mathrm{can}} := \operatorname{Proj}\bigl(\bigoplus H^0(X, mK_X)\bigr)$, and call it the canonical model of X. The singularities that appear on X_{can} are called canonical singularities. We will define them instead more systematically, but the description above provides their initial motivation and the use of the name itself.

Remark 1.1.5 (Motivation for pairs)**.** More generally, over the past several decades, the utility of studying *pairs* has become clear. In fact, many results are formulated and proved most naturally in this language, particularly those depending on induction. The technical definition appears in Chapter II, but the idea is as follows: One considers a variety X, which for simplicity we take to be $\mathbb{Q}$-factorial, and either an $\mathbb{R}$-divisor $\sum a_i D_i$, for $a_i \in \mathbb{R}_{>0}$ and the D_i prime divisors, or a formal sum $\sum a_i Y_i$, for Y_i closed subvarieties and $a_i \in \mathbb{R}_{>0}$. In the simplest case, when $D \subset X$ is a smooth divisor in a smooth variety, the adjunction formula relates K_D and K_X, by $\mathcal{O}_D(K_D) \cong O_X(K_X + D)|_D$. Thus, properties of K_D (e.g., ampleness, nefness, etc.) should be thought of as inherited not solely from X, but from the pair (X, D). It is often necessary to quantify the "singularities" of the pair (X, D): the formal definition for this imprecise formulation will be given shortly, but the idea is that we should study invariants of singularities that account for the singularities of X, the singularities of D, and their interaction all at once. For a more thorough introduction and motivation for the language of pairs, see [Kol97].

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[1] This is always the case when X is smooth, or merely klt, by deep results of [Bir+10, Corollary 1.1.2]

The above techniques and examples motivate the definition of several classes of singularities, defined by the behavior of the canonical class under resolutions of singularities. The definitions are a bit technical, and are given in Chapter II, but we recall the notions briefly here, and describe some related invariants and questions.

Let (X, D) be a pair consisting of a normal variety X and a divisor D such that $K_X + D$ is $\mathbb{Q}$-Cartier (that is, some multiple $m(K_X + D)$ is a Cartier divisor). Given a proper birational morphism $\pi : Y \to X$ with Y normal, one can write

$$K_Y + D_Y = \pi^*(K_X + D)$$

(precisely because of the hypotheses that $K_X + D$ is $\mathbb{Q}$-Cartier). For each divisor E on Y, we say that $1 - \mathrm{ord}_E(D_Y)$ is the *log discrepancy* of the divisor E_i with respect to (X, D), and write $a_E(X, D) := 1 - \mathrm{ord}_E(D_Y)$. One can check that for each divisor E over X this quantity depends only on the valuation ring $\mathcal{O}_{Y,E} \subset k(Y) = k(X)$, and not on the particular birational model Y on which E appears.

The idea is that one considers a log resolution $\pi : Y \to X$ of the pair (X, D). The log discrepancies of the exceptional divisors appearing on Y should be thought of as numeric invariants of the singularities of (X, D); the smaller the log discrepancies, the more singular the pair (X, D) is.

Remark 1.1.6. The word "log" arises in the following way: one motivation for considering pairs (X, D) is in studying a noncompact variety U. If X is a smooth compactification of U and $D = X - U$ a simple normal crossing boundary divisor, then properties of U are related to properties of the pair (X, D). One studies the logarithmic cotangent bundle $\Omega_X(\log D)$, so named because its local sections are differentials with "logarithmic poles" along D. If $D = V(x_1 \cdots x_r)$ in local coordinates, then $\Omega(\log D)$ is spanned by

$$\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n.$$

This is the motivation for the word "log" to describe resolutions of singularities producing simple normal crossing divisors, and invariants of singularities of such pairs.

Definition 1.1.7.

- (X, D) is called terminal if $a_E(X, D) > 1$ for all divisors E *exceptional* over X.

- (X, D) is called canonical if $a_E(X, D) \geq 1$ for all divisors E *exceptional* over X.

- (X, D) is called Kawamata log terminal (or klt) if $a_E(X, D) > 0$ for all divisors E over X.
- (X, D) is called log canonical if $a_E(X, D) \geq 0$ for all divisors E over X.

If we take $D = 0$, we refer simply to X being terminal, canonical, and so on. One can also replace a divisor D by a formal sum of subvarieties of higher codimension; we will consider both frameworks in Chapter II.

Remark 1.1.8. What if $a_E(X, D) < 0$ for some E? One can show that in this case by repeatedly blowing up we can obtain a proper birational morphism $Y_i \to X$ and a divisor E_i on Y_i with $a_{E_i}(X, D) \leq -i$; that is, if $a_E(X, D) < 0$ for some E, then the log discrepancies of (X, D) becomes arbitrarily negative.

Remark 1.1.9. In fact, if $a_E(X, D) \geq 0$ for all divisors E on a single log resolution $f : Y \to X$ of (X, D), then each of the above conditions can be checked only for divisors E on Y, rather than on all birational models.

Example 1.1.10 (Surfaces)**.** The above classification is interesting already in the case of surfaces with no boundary divisor. Let X be a surface. One can check that:

- If X has terminal singularities, then in fact X is smooth. (This corresponds to the fact that one can contract (-1)-curves on a variety and preserve smoothness.)
- If X has canonical singularities, then (as mentioned above) the singularities of X are exactly the Du Val singularities, or quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}(2; \mathbb{C})$.
- If X has klt singularities, then the singularities of X are quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{GL}(2; \mathbb{C})$.
- If X has log canonical singularities, then the singularities of X are quotients of simple elliptic singularities or smooth points by finite group actions.

Example 1.1.11. Consider $X = V(x^2 + y^2 + z^2)$, the cone in $\mathbb{A}^3$ over a smooth conic in $\mathbb{P}^2$. A single blowup at the singular point $\sigma_X : \widetilde{X} \to X$, the restriction of the blowup $\sigma : \mathrm{Bl}_p \mathbb{A}^3 \to \mathbb{A}^3$ to the strict transform $\widetilde{X}$ of X, resolves the singularity. The exceptional divisor E is isomorphic to $\mathbb{P}^1$. Using adjunction and the blowup formula, one can show that $a_E(X) = 0$, and thus X is canonical. For the full calculation, see Example 2.1.30.

Example 1.1.12. We may repeat the exact same calculation for $V(x^3+y^3+z^3) \subset \mathbb{A}^3$, the cone over the Fermat elliptic curve. Again, a single blowup resolves the singularity; now the exceptional divisor is a copy of the elliptic curve. Repeating the same calculation, we get that

$$K_{\widetilde{X}} - \sigma^* K_X = -E,$$

so that X is log canonical but not canonical.

Example 1.1.13 (Cone over a hypersurface)**.** In general, let f be a homogeneous polynomial of degree d defining a smooth hypersurface in $\mathbb{P}^{n-1}$. The cone $X = V(f) \subset \mathbb{A}^n$ then has an isolated singularity at the origin, and one can check that a single blowup at this cone point, say $\widetilde{X}$, resolves the singularity, with exceptional divisor E a copy of the original hypersurface. Using adjunction and the fact that

$$K_{\mathrm{Bl}_p \mathbb{A}^n} = \sigma^* K_{\mathbb{A}^n} + (n-1)E,$$

we obtain that $K_{\widetilde{X}/X} = (n-1-d)E$. Thus, X is terminal for $d \leq n-2$, canonical for $d \leq n-1$, and log canonical for $d \leq n$.

For a more general treatment of cones over varieties, and examples where non-integer discrepancies arise, see Example 2.1.31 and Example 2.1.32.

It is natural to define a single quantity capturing the "worst" possible behavior of the log discrepancies for a given pair (X, D). Because smaller log discrepancies correspond to worse singularities, it is natural to consider the smallest such number.

Definition 1.1.14. The minimal log discrepancy of the pair (X, D) along a subvariety W is defined to be $\inf\{a_E(X, D) : c_X(E) \subset W\}$, where $c_X(E)$ denotes the center of the valuation corresponding to E on X, or equivalently the image of E under any proper birational morphism $\pi : Y \to X$ on which E appears.

There are several important conjectures regarding the behavior of minimal log discrepancies, and all are still open in the general case:

Conjecture 1.1.15 ([Amb99; Sho88; Sho92])**.** *Let (X, D) be a pair consisting of a normal variety X such that $K_X + D$ is $\mathbb{Q}$-Cartier.*

- *(semicontinuity) The function $x \in X \mapsto \mathrm{mld}(x; X, D)$ is lower-semicontinuous.*
- *(ACC) Fix a dimension n and a set $\Gamma \subset [0, 1]$ satisfying the descending chain condition. The set $\{\mathrm{mld}(x; X, D) : \dim X \leq n, x \in X, \mathrm{coeff}\ D \subset \Gamma\}$ satisfies the*

ascending chain condition.[2]

- *(precise inversion of adjunction) Let H an effective Cartier divisor such that $H \not\subset \operatorname{Supp} D$. For every nonempty closed subset $W \subset H$, we have*

  $$\operatorname{mld}(W; X, D + H) = \operatorname{mld}(W; H, D|_H),$$

  where $D|_H$ is the restriction of the divisor D to H.

Partial results towards these have been proved using arc spaces, which is the first perspective on singularities this thesis will examine. In particular, [EM04], building on [EMY03], showed that the semicontinuity and precise inversion of adjunction conjectures hold for X a local complete intersection variety. The semicontinuity conjecture is also known if X has quotient singularities by [Nak16].

Remark 1.1.16. The above conjectures deal essentially with the local behavior of singularities and their invariants. However, they have ramifications for the global study of algebraic varieties: Shokurov [Sho04] showed that semicontinuity and the ascending chain condition imply termination of flips, which essentially says that an arbitrary sequence of elimination of K_X-negative curves C eventually terminates, and thus produces a minimal model (or a Mori fiber space). Thus, the behavior of minimal log discrepancies has been intensively studied, and is both of great importance and great subtlety.

1.2 Arc spaces

One powerful tool for understanding the above invariants of singularities is the notion of arc and jet schemes. If X is an algebraic variety over a field k, the ℓ-th jet scheme $J_\ell(X)$ is a moduli space for morphisms $\operatorname{Spec}(k[t]/t^{\ell+1}) \to X$. Such morphisms are called ℓ-jets, should be thought of as closed immersions into X of an ℓ-th order thickening of a point of a curve. Thus, for example, 1-jets on X are the same as tangent vectors on X (although one should be cautious with this intuition: the higher-order $J_\ell(X)$ will not be vector bundles for $\ell > 1$). For $\ell' > \ell$, there are natural "projection" morphisms $J_{\ell'}(X) \to J_\ell(X)$, induced by the natural truncation morphisms $\operatorname{Spec} k[t]/t^{\ell+1} \hookrightarrow \operatorname{Spec} k[t]/t^{\ell'+1}$. The inverse limit $J_\infty(X) := \varprojlim_\ell J_\ell(X)$ is called the arc scheme of X. The arc scheme is never of finite type (unless $\dim X = 0$).

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[2] We say a partially ordered set A satisfies the ascending chain condition (respectively, the descending chain condition) if there are is no infinite strictly increasing (respectively, strictly decreasing) sequence of elements of A.

Arc schemes were introduced by Nash [Nas95a], and used by Kontsevich [Kon95] to prove the birational invariance of the Hodge numbers of a Calabi–Yau variety:

Theorem 1.2.1. *Let X, Y be birationally equivalent complex Calabi–Yau varieties. Then $h^{p,q}(X) = h^{p,q}(Y)$.*

This was a generalization of the proof of Batyrev [Bat99] of the birational equivalence of the Betti numbers of birationally equivalent Calabi–Yau varieties. This earlier proof had used p-adic integration. Kontsevich's insight was to replace the mixed characteristic DVR $\mathbb{Z}_p$ by $\mathbb{C}[[t]]$, and instead of looking at the p-adic points $X(\mathbb{Z}_p)$ of a variety, he considers the $\mathbb{C}[[t]]$-valued points $X(\mathbb{C}[[t]])$. The arc space $J_\infty(X)$ is exactly the scheme parametrizing these points. Thus, the arc scheme serves as a "measure space" for what became known as motivic integration.

For the remainder of this section assume $k = \mathbb{C}$. Using this analogy with integration, there is a special class of subsets of the arc space, known as cylinders. By work of [ELM04], each divisorial valuation appearing on a resolution of singularities corresponds to an irreducible cylinders in the space of arcs, and the "codimension" (in a suitable sense) of each cylinder corresponds to the log discrepancy of the corresponding divisor. Thus, if one can describe or estimate the codimensions of these cylinders, one can obtain information about the singularities of the minimal model program. As a first example, the following theorem connects the singularities of X to properties of its jet schemes $J_\ell(X)$:

Theorem 1.2.2 ([EMY03])**.** *Let X be a normal local complete intersection variety. Then X is:*

(1) terminal if and only if $J_\ell(X)$ is normal for all ℓ.

(2) canonical if and only if $J_\ell(X)$ is irreducible for all ℓ.

(3) log canonical if and only if $J_\ell(X)$ is equidimensional for all ℓ.

Moreover, the results of [EMY03] showing that semicontinuity and precise inversion of adjunction hold in the local complete intersection setting were established using jet schemes, and to our knowledge no proof is known which avoids these methods.

Jet and arc schemes thus provide a powerful technical tool for studying the singularities of the minimal model program. At the same time, explicitly computing jet schemes is not easy, and it can be quite difficult to analyze the particular geometry of the jet schemes of a particular singularity. For example, even the jet schemes of the Du Val singularities are quite intricate; see for example [Nas95b; Plé08; PS12].

1.3 Differential operators

We turn now to another perspective on singularities, this time arising from the theory of differential operators. On a polynomial ring over the complex numbers $\mathbb{C}[x_1, \ldots, x_n]$ (or more generally on a smooth complex variety X), there is a well-developed theory of D-modules, or modules over the ring (or sheaf) of differential operators (see, e.g., [HTT08]). In the case of a polynomial ring over $\mathbb{C}$, the ring of differential operators is just the Weyl algebra $\mathbb{C}\langle x_1, \ldots, x_n, \partial/\partial x_1, \ldots, \partial/\partial x_n\rangle$.

For an arbitrary field k and a k-algebra R, one can define the ring of (k-linear) differential operators on R, denoted $D_{R/k}$. The formal definition will appear in Chapter II, but here we comment on a few key properties:

(1) $D_{R/k} = \bigcup_i D^i_{R/k}$ is naturally a (noncommutative) filtered ring, with $D^i_{R/k}$ called the differential operators of order $\leq i$.

(2) $D^0_{R/k} = R$, where $r \in R$ is thought of as the "multiplication-by-r" operator.

(3) $D^1_{R/k}$ is spanned by $D^0_{R/k}$ along with $\operatorname{Der}_k(R) = \operatorname{Hom}(\Omega_{R/k}, R)$, the k-linear derivations on R. When $\operatorname{char} k = 0$ and R is smooth over k, $D^1_{R/k}$ generates $D_{R/k}$, but this is not true generally; in fact, it is conjectured that when $\operatorname{char} k = 0$, $D^1_{R/k}$ can generate $D_{R/k}$ only when R is smooth over k [Nak61; MV73].

(4) $D_{R/k}$ is a finitely generated k-algebra when $\operatorname{char} k = 0$ and R is smooth over k; outside of this special case, $D_{R/k}$ will often fail to be finitely generated or Noetherian. As an example, if $R = \mathbb{F}_p[x_1, \ldots, x_n]$, then $D_{R/\mathbb{F}_p}$ is generated over R by the "divided power" operators $\frac{1}{\alpha_1! \cdots \alpha_n!}(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, and no finite subset of these generate over R.

Thus, when R is not smooth over k, many "nice" algebraic properties do not hold for $D_{R/k}$. Probably the earliest and most well-known example is that of the cone over an elliptic curve:

Example 1.3.1 ([BGG72])**.** Let $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$ be the affine cone over a smooth elliptic curve. Then D_R has no differential operators of negative degree, D_R is not a finitely generated $\mathbb{C}$-algebra, and is neither left- nor right-Noetherian. Since D_R has no differential operators of negative degree, the maximal homogeneous ideal (x, y, z) is a proper sub-D_R-module of R.

Thus, one expects that when R is singular, $D_{R/k}$ might be quite hard to describe explicitly. There are a variety of ways to use properties of D_R to describe the singularities of the ring R: one can consider Noetherianity of D_R, finite generation, generation by derivations, freeness of the R-module D_R^1, and more (see, for example, [LS89; Smi95; SV97; Ish87]). In particular, [LS89] posed the following questions:

(1) If $\operatorname{Spec} R$ has rational singularities, when is D_R simple? ([LS89, Question 0.13.1])

(2) When is R a simple D_R-module? ([LS89, Question 0.13.3])

If R is a regular k-algebra, then R is D_R-simple and D_R is a simple ring. These questions ask whether we can weaken this: do “mild” singularities guarantee similar properties?

There is a rather nice answer in positive characteristic: [Smi95, Theorem 2.2] showed that in characteristic p an F-pure ring R is a simple D_R-module if and only if R is strongly F-regular. Thus, one might expect a “mildly” singular ring R in characteristic 0 to be a simple D_R-module. If R is a simple D_R-module, we say that R is D-simple.

Remark 1.3.2. There are a few classes of singularities known to be D-simple in characteristic 0:

- If T is D_T-simple, and the inclusion of an T-submodule $R \hookrightarrow T$ splits as a map of R-modules, then R is D_R-simple [Smi95, Proposition 3.1].
- In particular, rings of invariants under finite group actions are D-simple, as are toric varieties and invariant subrings of polynomial rings under the action of classical algebraic groups (the latter is due originally to [LS89]).

These examples (which all have klt singularities), and analogies between strong F-regularity in characteristic p and klt singularities in characteristic 0, motivate the following more specific formulation:

Question 1.3.3 ([Hsi15, Question 5.1])**.** If R is a finitely generated Gorenstein $\mathbb{C}$-algebra such that $\operatorname{Spec} R$ has klt singularities, is R then a simple D_R-module?

Remark 1.3.4. One reason for considering the above question is the extension of the theory of holonomic D-modules and the Bernstein–Sato polynomial to the singular setting. For example, if $D = D_{\mathbb{C}[x_1,\ldots,x_n]/\mathbb{C}}$, then a D-module M must have $n \leq \dim M \leq 2n$ (where by dimension, we mean the dimension of the associated graded algebra). Recent work of [Mon+21b] examines the degree to which this generalizes to modules over the ring of differential operators of a singular ring, and shows that for this to hold one needs D-simplicity (and more!) to hold.

Similarly, [MHN17; Mon+21a] extends the theory of Bernstein–Sato functional equations and polynomials to the singular setting. However, not every ring admits a Bernstein–Sato polynomial. For example, the cone over an elliptic curve considered above does not, essentially for the same reason it is not D-simple: it has no differential operators of negative degree. Thus, the failure of D-simplicity is closely related to the existence (or lack) of Bernstein–Sato polynomials over singular rings.

Remark 1.3.5. The above question is also interesting through its converse: does a D-simple ring with log canonical singularities have klt singularities? This is interesting in part because of its connection to the conjectural relation between F-purity and log canonical singularities; see Remark 5.8.5.

1.4 Main results

In this thesis, we discuss three results concerning the above topics. After Chapter II, which consists of preliminary definitions and background results, each of the three results will occupy its own chapter.

1.4.1 Arc closures and the local isomorphism property

Chapter III will discuss the question of whether a morphism being a local isomorphism can be detected via the induced map of arc schemes. More precisely, we note that given a morphism $f : X \to Y$ of schemes, there is an induced morphism $f_\infty : J_\infty(X) \to J_\infty(Y)$, given by sending an arc $\operatorname{Spec} k[[t]] \to X$ to the composition $\operatorname{Spec} k[[t]] \to X \to Y$. Moreover, if $x \in X$ is the closed point of an arc γ, that is, the image of $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[[t]] \xrightarrow{\gamma} X$, then $f(x)$ is the closed point of $f_\infty(\gamma)$. If we let $J_\infty(X)_x$ and $J_\infty(Y)_{f(x)}$ be the set of all arcs with closed points x and $f(x)$, respectively, f_∞ induces a map

$$\bar{f}_\infty : J_\infty(X)_x \to J_\infty(Y)_{f(x)}.$$

[FEI18] stated the following question, due originally to Herwig Hauser:

Question 1.4.1 (Local isomorphism problem)**.** If $\bar{f}_\infty$ is an isomorphism, does f induce an isomorphism of local rings $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,f(y)}$?

Note that f induces an isomorphism of local rings, for example, if f is an open immersion, or more generally if and only if f induces an isomorphism on Zariski open neighborhoods. Thus, the morphism $\bar{f}_\infty$ clearly cannot determine whether f is a global isomorphism, and so the question can be thought of as asking how much local information the morphism $J_\infty(X)_x \to J_\infty(Y)_y$ carries.

[FEI18] recast this problem in purely algebraic terms, by defining a closure operation on ideals of a local ring, called the arc closure:

Theorem 1.4.2 ([FEI18])**.** *Let R be a local k-algebra. There exists an ideal closure operation $\mathfrak{a} \mapsto \mathfrak{a}^{\mathrm{ac}}$ on R; if $(0)^{\mathrm{ac}} = 0$, then a morphism $f : X \to \operatorname{Spec} R$ is a local isomorphism if and only if the induced map $\bar{f}_\infty$ is an isomorphism.*

The definition of the arc closure is given in Definition 3.3.1; the intuition is that an element $f \in R$ is in the arc closure of an ideal $\mathfrak{a}$ if the fiber of $J_\infty(V(f))$ over $\operatorname{Spec}(R/m)$ contains the fiber of $J_\infty(V(\mathfrak{a}))$ over $\operatorname{Spec}(R/m)$.

Put another way, the theorem states that if the zero ideal of R is equal to its arc-closure, then the local isomorphism problem has a positive answer for any map to $\operatorname{Spec} R$. Using this approach, we gave a positive answer to the local isomorphism problem, under very mild conditions (satisfied in essentially all areas of interest):

Theorem 1.4.3. *If (R, m, L) is a local k-algebra, and $k \hookrightarrow L$ is separable, then $(0)^{\mathrm{ac}} = 0$, and thus the local isomorphism problem has a positive solution.*

1.4.2 Arc schemes of determinantal ideals

Chapter IV is an application of jet-theoretic techniques discussed above to the specific case of generic square determinantal ideals. Determinantal varieties are almost never local complete intersections, and so the results of [EMY03] discussed above do not apply. However, they possess rich combinatorial structure. Previous work of [Doc13] used jet-theoretic methods to describe certain invariants of pairs of the form $(\mathbb{A}^{n^2}, D^k)$, where $D^k \subset \mathbb{A}^{n^2}$ is the subvariety of matrices of rank $\leq k$. However, these methods did not extend to the case of pairs $(D^k, \sum a_i D^{k-i})$ where the ambient variety is not smooth, but rather a singular determinantal variety.

In Chapter IV, we extend the jet-theoretic methods of [Doc13] to this case, obtaining the following results:

Theorem 1.4.4. *Consider the pair $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ (where the α_i may be zero).*

(1) $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ is log canonical at a matrix x_q of rank $q \leq k$ exactly when

$$\alpha_1 + \cdots + \alpha_j \leq m - k + (2j - 1)$$

for all $j = 1, \ldots, k - q$.

(2) In this case,

$$\operatorname{mld}\left(x_q; D^k, \sum_{i=1}^{k} \alpha_i D^{k-i}\right) = q(m-k) + km - \sum_{i=1}^{k-q} (k-q-i+1)\,\alpha_i.$$

In particular, we obtain:

Corollary 1.4.5 (Semicontinuity)**.** *If $\alpha_1, \dots, \alpha_k$ are nonnegative real numbers, the function $w \mapsto \operatorname{mld}\big(w; D^k, \sum_{i=1}^{k} \alpha_i D^{k-i}\big)$ is lower-semicontinuous on closed points.*

Remark 1.4.6. The key ingredient that allows us to describe the invariants of singularities from the codimension of certain cylinders in the space of arcs is our computation of the Nash ideal. This ideal (defined and discussed in Chapter II) is in general quite difficult to compute, and the difference between the Nash and Jacobian ideals reflects in some ways the failure of a variety to be a local complete intersection. Moreover, the Nash ideal defines the Nash blow-up, an important birational transformation of a variety. As far as we know, this result is one of the first nontrivial computations of this ideal. In order to perform this calculation, we make use of the combinatorial theory of determinantal rings, and in particular the existence and description of a straightening law on determinantal rings.

1.4.3 Differential operators on singular varieties

Chapter V is a study of differential operators on singular varieties, and in particular gives a negative answer to Question 1.3.3:

Theorem 1.4.7. *There are Gorenstein (graded) $\mathbb{C}$-algebras R with rational singularities such that $D_{R/k}$ contains no differential operators of negative degree, and thus such that R is not a simple $D_{R/k}$-module and $D_{R/k}$ is not simple. One example is $R = \mathbb{C}[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$.*

The proof is via the criterion given in [Hsi15] relating D-simplicity and bigness of the tangent bundle. Bigness of a vector bundle E is a measure of positivity, meaning essentially that the number of global sections of $\operatorname{Sym}^m E$ has maximal possible rate of growth as m increases. In particular, this theorem is a corollary of:

Theorem 1.4.8 (Theorem 5.5.2)**.** *Let X be a del Pezzo surface of degree 3, i.e., a smooth cubic surface. T_X is not big; in fact, $H^0(X, \operatorname{Sym}^m T_X) = 0$ for all m.*

Remark 1.4.9. This result is the first explicit example given of a Fano variety without big tangent bundle. (Note, though, that [HLS20] gave several additional examples shortly thereafter.) The study of the positivity of the tangent bundle has a rich history, and in general positivity of T_X imposes strong conditions on X: Mori's celebrated result [Mor79] proved a conjecture of Hartshorne that if T_X is ample then $X \cong \mathbb{P}^n$. Similarly, when X is Fano, nefness of T_X is conjectured in [CP91] to be equivalent to X being rational homogeneous. It has been long-known by experts (and an explicit statement and proof given in Chapter V) that if T_X is big, then X must be uniruled. Thus, it would be of great interest to understand which uniruled varieties, or even just which Fano varieties, have big tangent bundle.

CHAPTER II

Preliminaries

2.1 Notions of singularity

2.1.1 Weil and Cartier divisors

We recall briefly the notion of Weil and Cartier divisors from both a geometric and algebraic point of view.

Fix X a normal variety. By variety, we will mean a separated integral scheme of finite type over a field k; unless otherwise mentioned, k will be algebraically closed.

Definition 2.1.1. A Weil divisor D is a $\mathbb{Z}$-linear finite combination $\sum a_i D_i$ of codimension-1 irreducible subvarieties $D_i \subset X$. If $a_i > 0$ for all i, we say D is effective, and write $D \geq 0$. The set of Weil divisors carries a group structure given by the natural addition.

Since X is normal, and thus regular in codimension 1, the local ring $\mathcal{O}_{X,D}$ at the generic point of an irreducible subvariety D of codimension 1 is a discrete valuation ring. We write v_D for the corresponding valuation. If $f \in k(X)$ is a rational function on X, we have a corresponding Weil divisor $\operatorname{div}(f) = \sum_D v_D(f)D$. A Weil divisor D is called principal if it can be written as $D = \operatorname{div}(f)$ for some $f \in k(X)$. Two Weil divisors D, D' are linearly equivalent, which we write $D \sim D'$, if $D - D'$ is principal.

A Weil divisor D is called locally principal if there is an open cover $\{U_i\}$ of X such that the restriction $D|_{U_i}$ is principal for each i.

Definition 2.1.2. A Cartier divisor on X is a global section of $k(X)^*/\mathcal{O}_X^*$, where $k(X)^*$ is the sheaf of groups obtained by associating to each open affine $U = \operatorname{Spec} A$ of X the multiplicative group $\operatorname{Frac}(A) \smallsetminus \{0\}$ (by our integrality assumption on X, A is a domain and $k(X)^*$ is in fact the constant sheaf) and $\mathcal{O}_X^* \subset \mathcal{O}_X$ is the subsheaf of invertible sections. A Cartier divisor can thus be thought of as a collection $\{(U_i, f_i)\}$, with $\{U_i\}$ an open cover of X and $f_i \in k(X)^*$ such that $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$ for each i, j. The set of Cartier divisors carries a group structure, obtained by multiplication of elements of $H^0(X, k(X)^*/\mathcal{O}_X^*)$.

Remark 2.1.3. Since X is normal, a Cartier divisor gives rise to a corresponding Weil divisor, as follows: given $s \in H^0(X, k(X)^*/\mathcal{O}_X^*)$, the associated Weil divisor is then $\sum_D v_D(s)D$, where the sum is taken over the codimension-1 irreducible subvariety $D \subset X$. This map from Cartier divisors to Weil divisors is injective, and we will often identify Cartier divisors with their corresponding Weil divisor. A Weil divisor arises from a Cartier divisor (or is Cartier, for short) exactly when it is locally principal. We will say that two Cartier divisors are linearly equivalent if their corresponding Weil divisors are.

Remark 2.1.4. To a Weil divisor D, one can associate a subsheaf $\mathcal{O}_X(D)$ of $k(X)$, defined by $\mathcal{O}_X(D)(U) = \{f \in k(X) : \mathrm{div}_U(f) + D|_U \geq 0\}$. Two Weil divisors D, D' are linearly equivalent if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. A Weil divisor D is Cartier if and only if $\mathcal{O}_X(D)$ is an invertible sheaf. Since we assume X to be integral, *every* line bundle $\mathcal{L}$ can be written as $\mathcal{O}_X(D)$ for a Cartier divisor D on X; see [Har77, p. II.6].

One can show that if D_1, D_2 are Weil divisors then

$$\mathcal{O}_X(D_1 + D_2) = (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{**};$$

this holds because both sides are reflexive and they agree at all smooth points, which includes all codimension-1 points. (For background on reflexive modules and their properties, see, e.g., [Sch])

Remark 2.1.5. An effective Cartier divisor D corresponds to the locally principal ideal sheaf $\mathcal{O}_X(-D)$. On a variety with an ample line bundle, any Cartier divisor is a $\mathbb{Z}$-linear finite sum of effective Cartier divisors.

Remark 2.1.6. If $X = \operatorname{Spec} R$ is affine, a Cartier divisor corresponds to a projective R-module of rank 1 (and an effective Cartier divisor is simply a locally principal ideal $I \subset R$). A Weil divisor D corresponds to the reflexive rank-1 subsheaf $\mathcal{O}_X(D)$ inside $\operatorname{Frac} R$.

We now introduce $\mathbb{Q}$-divisors, a generalization of Weil divisors:

Definition 2.1.7. A $\mathbb{Q}$-divisor is a $\mathbb{Q}$-linear finite combination $\sum a_i D_i$ of codimension-1 irreducible subvarieties $D_i \subset X$. A $\mathbb{Q}$-divisor D is $\mathbb{Q}$-Cartier if mD is an integral Cartier divisor for some $m \in \mathbb{Z}$.

(Note that we will often refer to integral Weil divisors being $\mathbb{Q}$-Cartier as well.)

Example 2.1.8. Consider $V(x^2 - yz) \subset \mathbb{A}^3$, the cone over a smooth quadric in $\mathbb{P}^2$. Let $L = V(x, y) \subset X$ be a line through the vertex of the cone; one can check via the tangent space at the origin that L cannot be cut out by a single equation, so that L is not a Cartier divisor, but $2L$ is cut out by y, so $2L$ is Cartier and thus the integral Weil divisor L is a $\mathbb{Q}$-Cartier Weil divisor.

Remark 2.1.9 (Pulling back divisors)**.** In order to compare the behavior of divisors under birational morphisms, we need to be able to pull them back.

- It is immediate how to pull back a Cartier divisor on a variety X under any dominant morphism $\pi : \widetilde{X} \to X$: Given a Cartier divisor on X, thought of as $\{(U_i, f_i)\}$, we obtain a Cartier divisor $\{(f^{-1}(U_i), \pi^* f_i)\}$. To pull back an effective Cartier divisor, then, one simply pulls back the defining equations.

- Given *any* morphism $Y \to X$, one can pull back linear equivalence classes: if D is a Cartier divisor on X, then $\mathcal{O}_X(D)$ is a line bundle, which we pull back, obtaining $\pi^*\mathcal{O}_X(D)$. This then gives a linear equivalence class of Cartier divisors on $\widetilde{X}$.

- Finally, if $\pi : \widetilde{X} \to X$ is a dominant morphism, and D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, we can pull back D to a $\mathbb{Q}$-divisor on $\widetilde{X}$, by choosing m such that mD is Cartier, pulling mD back to $\widetilde{X}$, and dividing by m.

There is no obvious way to pull back Weil divisors under arbitrary morphisms though. We will often work with $\mathbb{Q}$-factorial varieties, where this issue does not arise:

Definition 2.1.10. A normal variety X is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier.

2.1.2 The canonical divisor

Now, we define a particular divisor on a variety X, under certain constraints on the singularities:

Definition 2.1.11. First, let X be a smooth variety of dimension n. Recall that in this case the cotangent sheaf $\Omega_{X/k}$ is locally free of rank n, so that $\omega_X := \bigwedge^n \Omega_{X/k}$ is a line bundle, the *canonical* bundle. We obtain a canonical divisor class via the correspondence between isomorphism classes of line bundles and linear equivalence classes of Cartier divisors. Although this gives us only a divisor class, we will frequently fix a particular choice of divisor K_X and refer it to *the* canonical divisor.

Now, let X be a normal variety. Recall that in this case Serre's criterion implies that the singular locus X_{sing} has codimension ≥ 2. Let $U = X \smallsetminus X_{\text{sing}}$ be the smooth locus, and fix a canonical divisor K_U as above. Since X_{sing} has codimension > 1, there is thus a unique Weil divisor K_X on X such that $K_X|_U = K_U$, called the canonical divisor of X. We write ω_X for the coherent sheaf $\mathcal{O}_X(K_X)$.

Equivalently:

(1) One can define $\omega_X = j_*(\omega_U)$, where $U \hookrightarrow X$ is the inclusion of the smooth locus. This is a reflexive rank-1 subsheaf of $k(X)$; take K_X to be a corresponding divisor.

(2) One can define ω_X by specifying its sections on each open set W as

$$\Gamma(W, \omega_X) = \{s \in \Omega_{k(X)} : s \text{ is regular on } W \cap X_{\text{sm}}\}.$$

(3) One may take $\omega_X = \left(\bigwedge^n \Omega_{X/k}\right)^{**}$.

Remark 2.1.12. For any normal domain R and finitely generated R-module M, one may check that M^{**} is reflexive, or equivalently torsionfree and S2. The latter, in particular, means that if $m \in M_P$ for all P of height 1 in R, then $m \in M$. Thus, the passage from $\bigwedge^n \Omega_{X/k}$ to $(\bigwedge^n \Omega_{X/k})^{**}$ kills any torsion and then adds all rational sections regular at all codimension-1 points.

As we have mentioned, we will often want to pull back divisors, and in particular the canonical divisor, under morphisms. The following definition is precisely what allows us to do so:

Definition 2.1.13. A variety X is $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier.

Note that a $\mathbb{Q}$-factorial variety is $\mathbb{Q}$-Gorenstein. We will encounter $\mathbb{Q}$-Gorenstein but not $\mathbb{Q}$-factorial varieties in Chapter IV.

Remark 2.1.14. Recall that on an open affine $U = \operatorname{Spec} R$ a Weil divisor D (or rather its linear equivalence class) corresponds to a rank-1 reflexive module M; to be $\mathbb{Q}$-Cartier, we must have that $((M)^{\otimes m})^{**}$ is projective (or, shrinking U, free) for some m. Locally, if P is a height-1 prime, the condition that the prime divisor $[V(P)]$ defined by P is $\mathbb{Q}$-Cartier is that some symbolic power $P^{(m)}$ is principal.

Remark 2.1.15. One may check that if $f : Y \to X$ is a proper birational morphism of normal varieties, then $f_*(K_Y)$ is a canonical divisor on X: normality allows us to reduce immediately to when X, Y are smooth, and we then can note that $f^*\omega_X \otimes \mathcal{O}_Y(E) \cong \omega_Y$, where E is effective and supported on $\operatorname{Exc}(f)$ (as argued in Example 2.1.17 below). This implies that for any choice of K_X, K_Y, we have that K_Y is linearly equivalent to $f^*K_X + E$, and thus that f_*K_Y is linearly equivalent to K_X, and thus a canonical divisor.

Given a proper birational morphism $f : Y \to X$ of normal varieties with X $\mathbb{Q}$-Gorenstein, we are interested in comparing the pullback of K_X with K_Y. The following definition is then natural:

Definition 2.1.16. If X is $\mathbb{Q}$-Gorenstein and $f : Y \to X$ is a proper birational morphism from a normal variety Y, we define the *relative canonical divisor* of $Y \to X$ by choosing a canonical divisor K_Y, taking $K_X = f_*(K_Y)$, and setting $K_{Y/X} := K_Y - f^*K_X$.

Although K_Y is well-defined only up to linear equivalence, the particular choice is of no importance when defining $K_{Y/X}$: one can check that $K_{Y/X}$ is independent of the choice of K_Y, and in fact $K_{Y/X}$ is the only Weil divisor supported on the f-exceptional locus and linearly equivalent to $K_Y - f^*K_X$ (for any choice of K_X, K_Y). Thus, $K_{Y/X}$ is an actual divisor, not just a linear equivalence class.

Example 2.1.17. If $f : Y \to X$ is a proper birational morphism of smooth varieties, given locally by some regular functions f_i on Y, the cotangent sequence

$$f^*\Omega_{X/k} \xrightarrow{(\partial f_i/\partial x_j)_{i,j}} \Omega_{Y/k} \to \Omega_{Y/X} \to 0$$

is left exact (the kernel of the leftmost map is torsion, since f is an isomorphism on an open subset, and $f^*\Omega_{X/k}$ is locally free, so the kernel is zero; see for example [Har77, p. II.8.19]). Thus, taking determinants we have that $K_{Y/X}$ is defined locally by the determinant of the Jacobian of the morphism.

As a concrete example, if $\sigma : \operatorname{Bl}_p \mathbb{A}^2 \to \mathbb{A}^2$ is the blowup of $\mathbb{A}^2$ at a point, then in one chart this map is given by $k[x, y] \to k[u, v]$, $x \mapsto uv$, $y \mapsto v$, so the Jacobian matrix is

$$\begin{pmatrix} v & 0 \\ u & 1 \end{pmatrix},$$

and the relative canonical divisor is cut out by v, i.e., the relative canonical divisor is just the exceptional divisor E of the blowup.

One can check this directly: a section of $\mathcal{O}_{\mathbb{A}^2}(K_{\mathbb{A}^2})$ is given by $dx \wedge dy$, which pulls back to $d(uv) \wedge d(v) = v\,du \wedge dv$, which is an element of $\mathcal{O}_{\operatorname{Bl}_p \mathbb{A}^2}(K_{\operatorname{Bl}_p \mathbb{A}^2} - E)$, and thus we have

$$K_{\mathrm{Bl}_p \mathbb{A}^2} \sim \sigma^*(K_{\mathbb{A}^2}) + E,$$

so we again see that the relative canonical divisor is E.

Remark 2.1.18. The same calculation shows that if we blow up $\mathbb{A}^n$ at a smooth subvariety Z of codimension r we get $K_{\mathrm{Bl}_Z \mathbb{A}^n/\mathbb{A}^n} = (r-1)E$, and in particular if $Z = \{p\}$ is a point we get $K_{\mathrm{Bl}_p \mathbb{A}^n/\mathbb{A}^n} = (n-1)E$.

The adjunction formula is an essential tool for calculating canonical divisors and arguing via induction on dimension; it can be stated in various levels of generality depending on the singularities involved (see, for example, [KM98; Kol13]), but the below formulation will suffice for our purposes:

Theorem 2.1.19 (Adjunction)**.** *If X is a normal $\mathbb{Q}$-Gorenstein variety, say with mK_X Cartier for some m, and H is a normal irreducible effective Cartier divisor, then $\mathcal{O}_H(mK_H) \cong \mathcal{O}_X(mK_X + mH)|_H$, and in particular K_H is also $\mathbb{Q}$-Cartier.*

2.1.3 Resolutions of singularities

In order to define many of the notions that follow, we will need resolution of singularities in characteristic 0. First, we define what it means for a divisor to have simple normal crossings.

Definition 2.1.20. If X is a *smooth* variety and $D = \sum a_i D_i$ a $\mathbb{Q}$-divisor, we say D has simple normal crossings (or that D is snc) if each D_i is smooth and the D_i intersect transversely, or equivalently if the D_i are smooth and at any point $x \in X$ there are algebraic coordinates $x_1, \dots, x_n$ such that the divisor $\sum D_i$ is defined by

$$x_{i_1} \cdots x_{i_k}$$

for some $i_1, \dots, i_k \in \{1, \dots, n\}$.

Definition 2.1.21. Let X be a variety.

- If $D \subset X$ is a $\mathbb{Q}$-divisor, a log resolution of the pair (X, D) is a proper birational morphism $\pi : \widetilde{X} \to X$, such that $\widetilde{X}$ is smooth and $\pi^* D \cup \mathrm{Exc}(\pi)$ is snc.
- If $Y = \sum a_i Y_i$ a $\mathbb{Q}$-linear sum of closed subschemes Y_i, a log resolution of the pair (X, Y) is a proper birational morphism $\pi : \widetilde{X} \to X$, such that $\widetilde{X}$ is smooth, $\mathcal{I}_{Y_i}\mathcal{O}_{\widetilde{X}}$ is an effective Cartier divisor $O_{\widetilde{X}}(-F_i)$ for each i, and $\sum F_i \cup \mathrm{Exc}(\pi)$ is snc.

Theorem 2.1.22 (Hironaka)**.** *Given a pair (X, D) consisting of a variety X and a $\mathbb{Q}$-divisor D, or a pair (X, Y) of a variety X and a $\mathbb{Q}$-linear sum of closed subschemes Y, there is a log resolution $\pi : \widetilde{X} \to X$ of the pair.*

2.1.4 Birational models and divisorial valuations

Let X be a normal variety. If $E \subset X$ is an irreducible divisor, then since X is regular in codimension 1, the local ring $\mathcal{O}_{X,E}$ is a discrete valuation ring of $k(X)$, and the corresponding valuation is just the order of vanishing along E. More generally, if $Y \to X$ is a birational morphism, with Y normal and $E \subset Y$ an irreducible divisor, then since $\mathcal{O}_{Y,E} \subset k(Y) = k(X)$, we have a discrete valuation on X. Moreover, one can check that this valuation depends only on $\mathcal{O}_{Y,E}$, and is thus independent of the particular birational model Y: that is, if $Y' \to Y$ is a proper birational morphism and $E' \subset Y'$ the strict transform of E, then $\mathcal{O}_{Y',E'} \cong \mathcal{O}_{Y,E}$, and thus they correspond to the same valuation on X.

Valuations of the above form are called divisorial valuations. If $f : Y \to X$ is a proper birational morphism and $E \subset Y$ an irreducible divisor, we call E a divisor over X; if f is not an isomorphism at the generic point of E, we call E an exceptional divisor over X. If $f : Y \to X$ is a proper birational morphism and E an irreducible divisor on Y, we write $c_X(E)$ for the closed subset $f(E)$; this is called the *center* of E. One can check via the valuative criterion for properness (see, e.g., [Har77, p. II.4.7]) that this depends only on the valuation ring of $k(X)$ corresponding to E and not on the particular normal birational model Y. Note also that when X is proper, every valuation of $k(X)$ has a center on X, again by the valuative criterion of properness.

2.1.5 Singularities of the minimal model program

Here we recall briefly the notion of log discrepancy and minimal log discrepancy. Our approach follows that of [EM09a], to which we refer for a comprehensive treatment. For this section, we will take X to be a normal $\mathbb{Q}$-Gorenstein variety over an algebraically closed field of characteristic 0; we let $Y := \sum_{i=1}^{s} a_i Y_i \geq 0$ be a formal $\mathbb{R}_{\geq 0}$-linear combination of proper closed subschemes Y_i. We refer to (X, Y) as a *pair*.

Definition 2.1.23. Let ord_E be a divisorial valuation of $k(X)$ with (nonempty) center $c_X(E)$ on X. The log discrepancy of E with respect to the pair (X, Y) is the real number

$$a_E(X, Y) := 1 + \operatorname{ord}_E(K_{X'/X}) - \sum a_i \operatorname{ord}_E(Y_i),$$

where $X' \to X$ is a birational morphism from a normal variety such that the center $c_{X'}(E)$ of ord_E on X' is a divisor. One can check that the quantity $a_E(X,Y)$ is independent of the choice of normal model $X' \to X$.

Remark 2.1.24. It is also common in the literature (e.g., in [KM98]) to consider the quantity $a_E(X,Y) - 1$ rather than $a_E(X,Y)$ for a divisor E with respect to a pair (X,Y). Often this is called just the "discrepancy" rather than "log discrepancy". This is of course largely a matter of convention, but the log discrepancy convention has some advantages, particularly if one wants to extend the definition of log discrepancies to arbitrary valuations. One extends the definition to quasimonomial valuations linearly, and so (for example) multiples of divisors with log discrepancy ≥ 0 will still have log discrepancy ≥ 0, but the same will not be true if we used the discrepancy rather than log discrepancy.

Definition 2.1.25. The minimal log discrepancy of the pair (X,Y) along a closed subset $W \subset X$, denoted $\mathrm{mld}(W;X,Y)$, is defined to be

$$\inf_E \{a_E(X,Y) : c_X(E) \subset W\},$$

where the infimum is taken over all irreducible divisors E (not necessarily exceptional) over X. If we consider a pair $(X,0)$, we will just write $\mathrm{mld}(W;X)$ for $\mathrm{mld}(W;X,0)$. If we take $W = X$, we write just $\mathrm{mld}(X,Y)$ for $\mathrm{mld}(X;X,Y)$. (If $\dim X = 1$ one must make the convention that if $\mathrm{mld}(W;X,Y) < 0$ then it is $-\infty$; this is automatic in higher dimension. We will not treat the 1-dimensional case at all in the following, so this issue will not arise.)

Definition 2.1.26. If X is a variety with K_X Cartier, we say X is:

(1) terminal if $\mathrm{mld}(X) > 1$.

(2) canonical if $\mathrm{mld}(X) \geq 1$.

If (X,Y) is a pair, we say (X,Y) is:

(1) klt if $\mathrm{mld}(X,Y) > 0$.

(2) log canonical if $\mathrm{mld}(X,Y) \geq 0$.

We say (X,Y) is klt (respectively, log canonical) along a closed subset $W \subset X$ if $(X|_U, Y|_U)$ is klt (respectively, log canonical) for some open neighborhood U of W in X.

Remark 2.1.27. One can define what it means for a pair (X, Y) to be terminal or canonical, but then one considers only divisors exceptional over X; we will not use this in what follows and so do not cover it here.

We will also require the notion of rational singularities:

Definition 2.1.28. If X is a normal variety (not necessarily $\mathbb{Q}$-Gorenstein), we say X has *rational singularities* if for some resolution of singularities $f : Y \to \operatorname{Spec} R$ we have $f_*\mathcal{O}_Y = \mathcal{O}_{\operatorname{Spec} R}$ and $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$.

A priori, the minimal log discrepancy seems impossible to calculate: how do we analyze all possible exceptional divisors appearing on smooth birational models of X? However, the following allows us to actually calculate these invariants from a single log resolution:

Lemma 2.1.29 ([EMY03, Proposition 1.4])**.** *Let $(X, Y = \sum d_i Y_i)$ be a pair, $W \subset X$ a closed subset, and $f : \widetilde{X} \to X$ a log resolution of (X, Y) such that additionally the preimage $f^{-1}(W)$ is a divisor and $f^{-1}(W) \cup f^{-1}(Y) \cup \operatorname{Exc}(f)$ is snc*[1]*. Write*

$$K_{\widetilde{X}/X} = \sum k_i E_i \qquad f^{-1}(Y) = \sum a_i E_i$$

for irreducible divisors $E_i \subset \widetilde{X}$.

(1) (X, Y) is log canonical along W if and only if $\min\{1 + k_i - a_i : c_X(E_i) \subset W\} \geq 0$.

(2) If (X, Y) is log canonical along W, then

$$\operatorname{mld}(W; X, Y) = \min\{1 + k_i - a_i : c_X(E_i) \subset W\}.$$

(We note that the precise formulation in [EMY03, Proposition 1.4] is useful for our purposes, but the result itself is much older and elementary.)

We will now examine several examples, which demonstrate the kind of techniques used in calculating discrepancies.

Example 2.1.30. Consider $X = V(x^2 + y^2 + z^2)$, the cone in $\mathbb{A}^3$ over a smooth conic in $\mathbb{P}^2$, which we treated briefly in Example 1.1.11. Here we give the details of the calculation. As mentioned, a blowup at the singular point $\sigma_X : \widetilde{X} \to X$, the restriction of the blowup $\sigma : \operatorname{Bl}_p \mathbb{A}^3 \to \mathbb{A}^3$ to the strict transform $\widetilde{X}$ of X, resolves the

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[1] or, if $W = X$, just that $f^{-1}(Y) \cup \operatorname{Exc}(f)$ is snc.

singularity. The exceptional divisor E is the conic in $\mathbb{P}^2$ defined by $x^2+y^2+z^2$, which is isomorphic to $\mathbb{P}^1$. We have morphisms

$$\begin{tikzcd}
\widetilde{X} = \operatorname{Bl}_p X \arrow[r, hook] \arrow[d, "\sigma_X"] & \operatorname{Bl}_p \mathbb{A}^3 \arrow[d, "\sigma"] \\
X \arrow[r, hook] & \mathbb{A}^3
\end{tikzcd}$$

To compute $K_{\widetilde{X}/X}$, we use adjunction for $X \subset \mathbb{A}^3$ and $\widetilde{X} \subset \operatorname{Bl}_p \mathbb{A}^3$, and the blowup formula for the blowup σ of $\mathbb{A}^3$ at a point:

$$\begin{aligned}
\mathcal{O}_X(K_X) &= \mathcal{O}_{\mathbb{A}^3}(K_{\mathbb{A}^3}+X)|_X, \\
\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) &= \mathcal{O}_{\operatorname{Bl}_p\mathbb{A}^3}(K_{\operatorname{Bl}_p\mathbb{A}^3}+\widetilde{X})|_{\widetilde{X}}, \\
K_{\operatorname{Bl}_p\mathbb{A}^3} &\sim \sigma^* K_{\mathbb{A}^3} + 2E_0.
\end{aligned}$$

From the second and third equations and the equality $(E_0)|_{\widetilde{X}} = E$, we have

$$\begin{aligned}
\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) &= \mathcal{O}_{\operatorname{Bl}_p\mathbb{A}^3}(K_{\operatorname{Bl}_p\mathbb{A}^3}+\widetilde{X})|_{\widetilde{X}} \\
&= \mathcal{O}_{\operatorname{Bl}_p\mathbb{A}^3}(\sigma^*K_{\mathbb{A}^3}+2E_0+\widetilde{X})|_{\widetilde{X}} \\
&= \mathcal{O}_{\widetilde{X}}\big(\sigma_X^*(K_{\mathbb{A}^3}|_X)+2E\big)\otimes\mathcal{O}_{\operatorname{Bl}_p\mathbb{A}^3}(\widetilde{X})|_{\widetilde{X}} \\
&= \mathcal{O}_{\widetilde{X}}(\sigma_X^*(K_X)+2E)\otimes\mathcal{O}_{\operatorname{Bl}_p\mathbb{A}^3}\big(-(\sigma^*(X)-\widetilde{X})\big)|_{\widetilde{X}}.
\end{aligned}$$

where the last equality follows from adjunction on X.

Because the defining equation for X has multiplicity 2 at the singular point, we have $\sigma^*(X)-\widetilde{X} = \operatorname{mult}_p(X)E = 2E_0$, so $(\sigma^*(X)-\widetilde{X})|_{\widetilde{X}} = 2E$. We obtain that

$$K_{\widetilde{X}} \sim \sigma_X^*(K_X) - 2E + 2E \sim \sigma_X^*(K_X),$$

and thus X is canonical.

Example 2.1.31. Let X_0 be the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. We know that $\mathcal{O}_{\mathbb{P}^5}(1)|_{X_0} = \mathcal{O}_{\mathbb{P}^2}(2)$, and $\omega_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$. Let $X \subset \mathbb{A}^6$ be the affine cone over X_0. $2K_X$ is clearly Cartier, so K_X is $\mathbb{Q}$-Cartier. Let $\sigma : Y \to X$ be the blowup at the cone point, giving us a resolution with exceptional divisor $E \cong \mathbb{P}^2$. Write $2K_Y \sim \sigma^*(2K_X) + aE$. Restricting this to E and using the adjunction $\mathcal{O}_Y(K_Y+E)|_E = \mathcal{O}_E(K_E)$ (and that $\sigma^*(\mathcal{O}_X(2K_X))|_E$ is trivial), we get

$$\mathcal{O}_E(2K_E) = \mathcal{O}_E((a+1)E)|_E.$$

Since $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^2}(-2)$ and $\mathcal{O}_E(K_E) = \omega_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$, we get that

$$\mathcal{O}_{\mathbb{P}^2}(-3) = \mathcal{O}_{\mathbb{P}^2}(-2(a+1))$$

so $a = 1/2$.

Thus the cone X is terminal but not smooth.

Example 2.1.32 (Cones)**.** The following calculation of the singularities of the cone over a variety will be used implicitly in Chapter V. Let $X_0 \subset \mathbb{P}^n$ be a smooth variety of positive dimension that is projectively normal (i.e., the homogeneous coordinate ring $k[x_0, \ldots, x_n]/I_{X_0}$ is normal), and let $X \subset \mathbb{A}^{n+1}$ be the cone over X_0. X thus has an isolated normal singularity at the origin.

One can give a criteria for X to be $\mathbb{Q}$-Gorenstein purely in terms of the embedding $X_0 \subset \mathbb{P}^n$. We claim first that $\bigoplus_m H^0(X_0, \omega_{X_0}(m))$ is the canonical module for X. Since X is affine, we have ω_X is the sheafification of $H^0(X, \omega_X)$, so it suffices to find the global sections of ω_X. Let $i : U = X \smallsetminus \{(0, \ldots, 0)\} \hookrightarrow X$ be the inclusion of the smooth locus. Since $X \smallsetminus U$ has codimension > 2 and ω_X is S2, we have that $H^0(X, \omega_X) = H^0(U, \omega_X|_U) = H^0(U, \omega_U)$. Then U is an $\mathbb{A}^1 \smallsetminus \{0\}$-bundle over X_0, which we call $\pi : U \to X_0$. One can then check that $\pi^*(\omega_{X_0}) = \omega_U$: to see this, note that π is a smooth map and thus taking the determinant of the cotangent sequence implies $\omega_U = \pi^*(\omega_{X_0}) \otimes \Omega_{U/X_0}$; one can then check that Ω_{U/X_0} is trivial.

Since $\pi^*(\omega_{X_0}) = \omega_U$, the projection formula says that

$$\pi_*(\pi^*(\omega_{X_0})) = \omega_{X_0} \otimes \pi_*\mathcal{O}_U = \omega_{X_0} \otimes \left(\bigoplus \mathcal{O}_{X_0}(i)\right),$$

where the second equality follows since $\pi_*\mathcal{O}_U = \bigoplus \mathcal{O}_{X_0}(i)$.

We then have that ω_X is equal to

$$H^0(U, \omega_U) = H^0(X_0, \pi^*\omega_X) = H^0(X_0, \pi_*\pi^*\omega_X) = \bigoplus H^0(X_0, \omega_{X_0}(i)).$$

Analogously, we have that

$$\mathcal{O}_X(mK_X) = (\omega_X^{\otimes m})^{**} = \bigoplus H^0(X_0, \omega_{X_0}^{\otimes m}(i))$$

(note that $\omega_X^{\otimes m}$ may not be S2, so we cannot repeat the argument for $m = 1$ for $\omega_X^{\otimes m}$, but its reflexification $(\omega_X^{\otimes m})^{**}$ is S2).

Since the coordinate ring $R = \bigoplus_i H^0(X, \mathcal{O}_X(i))$ of X is graded[2], a graded module

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[2] Note that this expression is the coordinate ring because X_0 is projectively normal

is locally free if and only if it is free, and thus $\mathcal{O}_X(mK_X)$ is locally free if and only if $(\omega_X^{\otimes m})^{**} \cong R(-a)$ for some a, which occurs if and only if $\omega_{X_0}^{\otimes m} = \mathcal{O}_{X_0}(mK_{X_0}) = \mathcal{O}_{X_0}(l) = \mathcal{O}_{\mathbb{P}^n}(l)|_{X_0}$ for some l, or (in terms of divisors) if and only if $mK_{X_0} \sim lH$, where H is the hyperplane class of the embedding.

To sum up, then, the cone over $X_0 \subset \mathbb{P}^n$ is $\mathbb{Q}$-Gorenstein exactly when some (nonzero) multiple of the canonical divisor on X_0 is a multiple (possibly zero) of the hyperplane section from $\mathbb{P}^n$.

Now, assume that $mK_{X_0} \sim lH$, and write $r = l/m$. One can check that if X_0 is a smooth variety in $\mathbb{P}^n$ and X the cone over X_0 in $\mathbb{A}^{n+1}$, blowing up X at the cone point gives a log resolution $Y \to X$ with exceptional divisor $E \cong X_0$; in fact, it is not hard to check that this blowup is the total space of the line bundle $\mathcal{O}_{X_0}(1)$.

It then follows that $K_{Y/X} = rE$, and thus X is

(1) terminal if and only if $r > 1$.

(2) canonical if and only if $r \geq 1$.

(3) Kawamata log terminal if and only if $r > 0$.

(4) log canonical if and only if $r \geq 0$.

There are several conjectures regarding minimal log discrepancies that are of great importance for the minimal model program. For the following discussion, we take (X, Y) to be a pair with X normal and $\mathbb{Q}$-Gorenstein and $Y = \sum a_i Y_i$ a $\mathbb{Q}$-linear sum of subschemes with coefficients $a_i > 0$. The following conjecture is a "precise" statement of inversion of adjunction, relating not just the inequalities of statements about being log terminal, log canonical, etc., but the actual minimal log discrepancy:

Conjecture 2.1.33 ([Sho92]). *Let H an effective Cartier divisor such that $H \not\subset \operatorname{Supp} Y$. For every nonempty proper closed subset $W \subset H$, we have*

$$\operatorname{mld}(W; X, Y + H) = \operatorname{mld}(W; H, Y|_H),$$

where $Y|_H = \sum a_i Y_i|_H$ is the restriction of Y to H.

The following ascending chain conjecture for minimal log discrepancies concerns the behavior of the numbers appearing as minimal log discrepancies of pairs of a fixed dimension with coefficients in a set satisfying the descending chain condition (e.g., rational numbers of the form $1 - 1/n$).

Conjecture 2.1.34 ([Sho88]). *Fix a dimension n and a set $\Gamma \subset [0,1]$ satisfying the descending chain condition. The set*

$$\{\operatorname{mld}(x; X, Y) : \dim X \leq n, x \in X, \operatorname{coeff} Y \subset \Gamma\}$$

(where X is taken to be normal and $\mathbb{Q}$-Gorenstein) satisfies the ascending chain condition.

Finally, the following conjecture concerns the behavior of the log discrepancies of divisors centered at a (closed) point as the point varies across the variety.

Conjecture 2.1.35 ([Amb99]). *Let (X, Y) be a pair. The function*

$$x \mapsto \operatorname{mld}(x; X, Y)$$

is lower-semicontinuous on the closed points of X.[3]

2.1.6 F-singularities

In Chapter V, we will need the following notions of characteristic-p singularities.

Definition 2.1.36. Let R be a ring of characteristic $p > 0$. We write F for the Frobenius $F : R \to R$, $r \mapsto r^p$. When R is reduced, we will also view F as an inclusion $R \hookrightarrow R^{1/p}$, where $R^{1/p}$ is the set of p-th roots of elements of R in some fixed algebraic closure of its total fraction field. (Note that F defines an isomorphism $R^{1/p} \cong R$.) We say R is F-finite if F is a finite ring map.

Remark 2.1.37. A finite field $\mathbb{F}_p$ is F-finite, and more generally so are perfect or algebraically closed fields. If R is F-finite and S is a finitely generated R-algebra, S is F-finite. Localizations and quotients of F-finite rings are F-finite. Thus, most rings of geometric interest will be F-finite. We will assume F-finiteness throughout the following, although we will try to explicitly mention it.

Definition 2.1.38. Let R be an essentially finite type k-algebra, with $[k : k^p] < \infty$ (so that R is F-finite). Assume that R is reduced. Then:

(1) R is strongly F-regular if for every $c \in R$, not a zerodivisor, there is an R-linear map $R^{1/p^e} \to R$ sending $c^{1/p^e} \mapsto 1$ for some $e \gg 0$.

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[3] That is, for each number t, there is an open neighborhood $U \subset X$ such that $\operatorname{mld}(x; X, Y) > t$ for any closed point $x \in U$.

(2) R is F-pure (or F-split) if there is an R-linear map $R^{1/p^e} \to R$ sending $1^{1/p^e} \mapsto 1$ for some $e \gg 0$.

Note that strongly F-regular implies F-pure (just by taking $c = 1$). Besides their intrinsic interest in commutative algebra (especially through connections to tight closure and invariants of singularities in characteristic p), these classes are characteristic-p analogues of klt and log canonical singularities, respectively.

Definition 2.1.39. Let R be a finitely generated $\mathbb{C}$-algebra of dimension d. There is a finitely generated $\mathbb{Z}$-algebra $A \subset \mathbb{C}$ such that R is defined over A, i.e., there is some A-algebra R_A and $R = R_A \otimes_A \mathbb{C}$. By the theorem on generic flatness, we may localize at element one element of A and thus assume that $A \to R_A$ is a flat extension, and thus that the fibers $R_p = R_A \otimes_A A_p/pA_p$ are of dimension d. We say that R is of F-regular type if X_p is strongly F-regular for p in an open dense subset of $\operatorname{Spec} A$, and R is of dense F-pure type if X_p is F-pure for p in an dense subset of $\operatorname{Spec} A$.

Theorem 2.1.40 ([Har98; HW02])**.** *Let R be a $\mathbb{Q}$-Gorenstein finitely generated $\mathbb{C}$-algebra. R has F-regular type if and only if R has klt singularities. Moreover, if R has dense F-pure type then R is log canonical.*

The following question was stated in [HW02], but was discussed previously by experts:

Conjecture 2.1.41 ([HW02, Problem 5.1.2])**.** *If R is log canonical, then it is of dense F-pure type.*

2.2 Arc schemes

2.2.1 The Jacobian and Nash ideals

We begin by reviewing two ideals, which are not themselves defined in terms of arc schemes, but whose existence and properties will be important in the study of cylinders of the arc scheme and their codimensions.

Definition 2.2.1. Let R be a finitely generated k-algebra of dimension d. The Jacobian ideal is the d-th Fitting ideal of the module of (k-linear) Kähler differentials $\Omega_{R/k}$. Equivalently, if one chooses a surjection $S \to R$, with S a polynomial ring over k in variables $x_1, \ldots, x_N$, and chooses generators $(f_1, \ldots, f_n)$ for the kernel of $R \to S$, then the Jacobian ideal is the ideal of $(N - d)$-th minors of the matrix $(\partial f_i / \partial x_j)$ of partial derivatives.

The description via Fitting ideals implies that on a k-variety X, the Jacobian ideal is canonically defined on each affine chart, independent of any choice of coordinates, and there is thus a Jacobian ideal sheaf, which we denote $\mathrm{Jac}_X \subset \mathcal{O}_X$. Note that the Jacobian ideal is cosupported on the singular locus of X (which is the same as the non-regular locus if k is algebraically closed or perfect).

Definition 2.2.2. We also need the notion of the Jacobian ideal of a proper birational morphism $f : \widetilde{X} \to X$ with $\widetilde{X}$ smooth, which we denote Jac_f. This is the ideal sheaf on $\widetilde{X}$ defined by 0-th Fitting ideal of $\Omega_{\widetilde{X}/X}$, or equivalently the ideal sheaf defined by the image of the morphism

$$f^*\Big(\bigwedge^n \Omega_X\Big) \to \bigwedge^n \Omega_{\widetilde{X}} = \omega_{\widetilde{X}},$$

where $\dim X = \dim \widetilde{X} = n$; since $\widetilde{X}$ is smooth, $\omega_{\widetilde{X}}$ is a line bundle, and thus the image of the left side defines an ideal of $\widetilde{X}$.

There is another ideal sheaf defined on a normal Gorenstein variety X, similar to but distinct from the Jacobian ideal, which plays an important role in the relation between jet spaces and discrepancies: the Nash ideal.

Recall that on a normal variety X of dimension d the canonical sheaf ω_X can be defined as $(\bigwedge^d \Omega_X)^{**}$, the reflexification of the d-th exterior power of the Kähler differentials. There is then in particular a natural map $\bigwedge^d \Omega_X \to (\bigwedge^d \Omega_X)^{**} = \omega_X$.

Definition 2.2.3. Let X be a normal Gorenstein variety of dimension d. Because X is Gorenstein, the image of the natural morphism

$$\bigwedge^d \Omega_X \to \Big(\bigwedge^d \Omega_X\Big)^{**} = \omega_X$$

is a coherent subsheaf of the *invertible* sheaf ω_X. This image then defines an ideal sheaf of $\mathcal{O}_X$ (obtained by tensoring the image by ω_X^{-1}); this ideal sheaf is called the Nash ideal sheaf of X, which we will denote by $J(X)$.

Note that the Nash ideal is cosupported on X_{sing}. If X is lci, then $J(X) = \mathrm{Jac}_X$, but in general they differ, as we will see in Chapter IV. See [EM09a, Section 9.2] for details on their relation.

Remark 2.2.4. By [SSU02, Section 2] and the references cited there, if $X = \operatorname{Spec} R$ for a finitely generated $\mathbb{N}$-graded k-algebra R with $R_0 = k$, then the morphism

$$\bigwedge^d \Omega_X \to \omega_X$$

is homogeneous. If X is Gorenstein as well, then we have $\omega_X \cong R(a)$ for some uniquely determined $a \in \mathbb{Z}$, and thus the Nash ideal will be homogeneous. For more on the canonical modules of graded rings, see [GW78, Chapter 2.1]

Remark 2.2.5. One can also define an analogue of Nash ideals for $\mathbb{Q}$-Gorenstein varieties: consider the image of $(\bigwedge^d \Omega_X)^{\otimes m} \to \omega_X^{\otimes m} \to (\omega_X^{\otimes m})^{**}$ for a positive integer m with $(\omega_X^{\otimes m})^{**}$ invertible; twisting by $\omega_X^{\otimes -m}$ we obtain an ideal sheaf, called the m-th Nash ideal of level m. We will not use this notion in the following.

2.2.2 Arc and jet schemes

We recall the definition of arc and jet schemes; for a comprehensive treatment see, e.g., [Voj13]. Let X be a scheme over k, and for each $\ell \in \mathbb{N}$ consider the functor from k-schemes to sets

$$T \mapsto \operatorname{Hom}\big(T \times_k \operatorname{Spec}(k[t]/t^{\ell+1}), X\big).$$

One can show that there is a k-scheme $J_\ell(X)$, the scheme of l-jets of X, representing this functor, i.e., such that

$$\operatorname{Hom}\big(T \times_k \operatorname{Spec}(k[t]/t^{\ell+1}), X\big) = \operatorname{Hom}(T, J_\ell(X));$$

in particular, k-points of $J_\ell(X)$ correspond to maps $\operatorname{Spec} k[t]/t^{\ell+1} \to X$. Moreover, if X is finite-type over k then so is $J_\ell(X)$.

The quotient maps

$$k[t]/t^{\ell+1} \to k[t]/t^{\ell'+1}$$

for $\ell' < \ell$ induce morphisms

$$\psi_{\ell,\ell'} \colon J_\ell(X) \to J_{\ell'}(X).$$

It follows by construction that these maps are affine, and thus the inverse limit over the system $\{J_\ell(X) \to J_{\ell'}(X) : \ell > \ell'\}$ exists in the category of k-schemes. We denote this limit by $J_\infty(X)$, and call it the arc scheme of X (note that $J_\infty(X)$ will not be of finite type over k in general). Since $J_\infty(X)$ is by construction an inverse limit, there are truncation maps $\psi_{\infty,\ell} \colon J_\infty(X) \to J_l(X)$ for any ℓ, arising from the quotient morphism

$$k[[t]] \to k[t]/t^{\ell+1}.$$

Remark 2.2.6. One can check that if $k \hookrightarrow L$ is a field extension then

$$\operatorname{Hom}(\operatorname{Spec}(L[[t]]), X) = \operatorname{Hom}(\operatorname{Spec}(L), J_\infty(X)).$$

In fact, by [Bha16] it is true (but highly nonelementary) that if X is quasicompact and quasiseparated over k and S is a k-algebra then

$$\operatorname{Hom}\bigl(\operatorname{Spec} S \times_k \operatorname{Spec}(k[[t]]), X\bigr) = \operatorname{Hom}(\operatorname{Spec} S, J_\infty(X)),$$

but we do not use this in the following.

In the following, we use ℓ to denote an element of $\mathbb{N} \cup \{\infty\}$, and write $k[[t]]/t^{\ell+1}$ to mean either $k[[t]]/t^{\ell+1} = k[t]/t^{\ell+1}$ when ℓ is finite or $k[[t]]$ when $\ell = \infty$.

For any ℓ we denote the truncation map $\psi_{\ell,0}\colon J_\ell(X) \to J_0(X) = X$ simply by ψ_ℓ; at the level of k-points, this just sends an arc $\operatorname{Spec} k[[t]]/t^{\ell+1} \to \operatorname{Spec} X$ to

$$\operatorname{Spec} k \to \operatorname{Spec} k[[t]]/t^{\ell+1} \to X,$$

i.e., to the image of the closed point of $\operatorname{Spec} k[[t]]/t^{\ell+1}$. For a point $x \in X$, not necessarily closed, we write $J_\ell(X)_x$ for $\psi_\ell^{-1}(x)$, the fiber over x.

Given a morphism $f\colon X \to Y$ of k-schemes, for any morphism

$$T \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}) \to X$$

we obtain a morphism

$$T \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}) \to X \to Y,$$

and by functoriality we obtain morphisms $f_\ell\colon J_\ell(X) \to J_\ell(Y)$ for all ℓ. Furthermore, it is clear that for $x \in X$ these morphisms restrict to morphisms $\bar{f}_\ell\colon J_\ell(X)_x \to J_\ell(Y)_{f(x)}$.

Remark 2.2.7. Let G be an algebraic group. Then, for any ℓ, by functoriality $J_\ell(G)$ is also an algebraic group (for example, the multiplication map $G \times G \to G$ gives rise to a morphism $J_\ell(G \times G) = J_\ell(G) \times J_\ell(G) \to J_\ell(G)$.) Similarly, if X is a variety, and G an algebraic group acting on X, then $J_\ell(G)$ acts on $J_\ell(X)$.

Now, we recall the construction of the arc and jet schemes of an affine scheme $\operatorname{Spec} R$ (the arc and jet schemes are obtained by simply gluing the construction over affine charts). Given a k-algebra R (not necessarily Noetherian or local), and $\ell \in \mathbb{N} \cup \{\infty\}$, we write R_ℓ for the ring defined as follows: take a surjection $k[x_\alpha]_{\alpha \in \mathcal{A}} \to R$, say with kernel $I = (f_\beta(x_\alpha))_{b \in \mathcal{B}}$. For each variable x_α, introduce variables $x_\alpha^{(i)}$ for $i = 0, \dots, \ell$. Define

$$R_\ell := \frac{k\big[x_\alpha^{(i)}\big]_{\alpha \in \mathcal{A}, i=0,\dots,\ell}}{\big(f_{\beta,i}(x_\alpha) : \beta \in \mathcal{B}, i = 0, \dots, \ell\big).}$$

where $f_{\beta,i}(x_\alpha)$ is the coefficient of t^i in the expansion of

$$f_\beta\big(x_\alpha^{(0)} + x_\alpha^{(1)} t + \cdots + x_\alpha^{(\ell)} t^\ell\big)$$

in $k[x_\alpha^{(i)}][[t]]/(t^{\ell+1})$ (when $\ell = \infty$, we mean simply $k[x_\alpha^{(i)}][[t]]$).

If $\ell < \infty$, for any k-scheme T we have

$$\operatorname{Hom}(T, \operatorname{Spec} R_\ell) = \operatorname{Hom}(T \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}), X),$$

so that $\operatorname{Spec} R_\ell$ is canonically isomorphic to $J_\ell(\operatorname{Spec} R)$; moreover, one can check that $\operatorname{Spec} R_\infty \cong J_\infty(\operatorname{Spec} R)$.

The projection maps $\psi_{\ell,\ell'}$ for $\ell > \ell'$ give ring maps $R_{\ell'} \to R_\ell$, and in particular an inclusion $R = R_0 \hookrightarrow R_\ell$ for any ℓ.

2.2.3 Cylinders in the space of arcs

For an in-depth treatment of this material, see [EM09a]; we will quickly survey the main notions. Fix an arbitrary finite-type k-scheme X.

Definition 2.2.8. A cylinder C in $J_\infty(X)$ is a set of the form $C = \psi_{\infty,\ell}^{-1}(S)$ for $S \subset J_\ell(X)$ a constructible subset.

Remark 2.2.9. Note that cylinders are closed under finite unions, finite intersections, and complements.

For a k-point $\gamma \in J_\infty(X)$, we write $\operatorname{ord}_\gamma(\mathfrak{a})$ for the value Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal sheaf. For a k-point $\gamma \in J_\infty(X)$, we write $\operatorname{ord}_\gamma(\mathfrak{a})$ for the value obtained by pulling back the ideal $\mathfrak{a}$ along $\gamma : \operatorname{Spec} k[[t]] \to X$ and applying the t-adic valuation (recall that if v is a valuation and I an ideal, $v(I) := \min\{v(f) : f \in I\}$).

Definition 2.2.10. We define the contact loci along $\mathfrak{a}$ as

$$\operatorname{Cont}^{\geq i}(\mathfrak{a}) = \{\gamma \in J_\infty(X) : \operatorname{ord}_\gamma(\mathfrak{a}) \geq i\} \text{ and } \operatorname{Cont}^{i}(\mathfrak{a}) = \{\gamma \in J_\infty(X) : \operatorname{ord}_\gamma(\mathfrak{a}) = i\}.$$

Note that these are cylinders in $J_\infty(X)$: we can write

$$\operatorname{Cont}^{\geq i}(\mathfrak{a}) = \psi_{\infty,i-1}^{-1}\big(J_{i-1}(\operatorname{Spec}(\mathcal{O}_X/\mathfrak{a}))\big),$$

where $J_{i-1}(\mathrm{Spec}(\mathcal{O}_X/\mathfrak{a})) \subset J_{i-1}(X)$ is the $(i-1)$-st jet scheme of the subscheme $\mathrm{Spec}(\mathcal{O}_X/\mathfrak{a})$, which is naturally a closed subscheme of $J_{i-1}(X)$. Since

$$\mathrm{Cont}^i(\mathfrak{a}) = \mathrm{Cont}^{\geq i}(\mathfrak{a}) \smallsetminus \mathrm{Cont}^{\geq i+1}(\mathfrak{a}),$$

it is a cylinder as well.

Given some subvarieties $Y_1, \ldots, Y_s$ and some s-tuple $\underline{w} = (w_1, \ldots, w_s) \in \mathbb{N}^s$, we write $\mathrm{Cont}^{\underline{w}}(Y) = \bigcap \mathrm{Cont}^{w_i}(Y_i)$; we refer to such intersections of contact loci as multicontact loci.

We now turn to the notion of codimension of a cylinder; for this, we specialize to the case where k is a field of characteristic 0, although much of this section can be adapted to any characteristic. Assume moreover that X is of pure dimension n over k.

The contact loci $\mathrm{Cont}^e(\mathrm{Jac}_X)$ along the Jacobian ideal are of particular importance in what follows. Given any cylinder C we will write $C^{(e)} := C \cap \mathrm{Cont}^e(\mathrm{Jac}_X)$.

Definition 2.2.11. Let C be a cylinder. If $C = \psi_{\infty,r}^{-1}(S) \subset \mathrm{Cont}^e(\mathrm{Jac}_X)$, then we define

$$\mathrm{codim}(C) := n(\ell+1) - \dim \psi_{\infty,\ell}(C)$$

for any $\ell \geq \max(e, r)$.

If C is an arbitrary cylinder in $J_\infty(X)$, we define

$$\mathrm{codim}(C) := \min_e \bigl(\mathrm{codim}(C^{(e)})\bigr).$$

Remark 2.2.12. Some comments on this definition are in order:

- By definition, we may write any cylinder as $\psi_{\infty,\ell}^{-1}(S)$ for some r and $S \subset J_\ell(X)$.
- The codimension is a nonnegative integer. This is not trivial; for details, see [EM09a, Section 5].
- The fact that for $C = \psi_{\infty,r}^{-1}(S) \subset \mathrm{Cont}^e(\mathrm{Jac}\, X)$ the quantity

  $$n(\ell+1) - \dim \psi_{\infty,\ell}(C)$$

  is independent of the choice of $\ell \geq \max(e, r)$ follows from the study of the truncation morphisms on the space of jets (see [EM09a, Theorem 4.1]).
- It is clear that $\mathrm{codim}(C_1 \cup C_2) = \min(\mathrm{codim}(C_1), \mathrm{codim}(C_2))$.

- When X is smooth, the codimension in the above sense of a cylinder C coincides with its topological codimension in the Zariski topology.

2.2.4 Jet schemes, birational morphisms, and minimal log discrepancies

Here we recall briefly the connection between arc schemes and minimal log discrepancy. Our approach follows that of [EM09a], to which we refer for a comprehensive treatment of this material. Recall that a pair (X, Y) consists o a normal $\mathbb{Q}$-Gorenstein variety X over an algebraically closed field of characteristic 0, and $Y := \sum_{i=1}^{s} a_i Y_i$ a formal $\mathbb{R}_{\geq 0}$-linear combination of proper closed subschemes Y_i.

A proper birational morphism will induce a set-theoretic bijection away from a "measure-0" subset of the space of arcs. More precisely, we have:

Theorem 2.2.13. *Let $f : \widetilde{X} \to X$ be a proper birational morphism of varieties. If $Z \subset X$ is the locus over which f is not an isomorphism, then f_∞ restricts to a bijection*

$$\widetilde{J_\infty}(X) \smallsetminus (f^{-1}(Z))_\infty \to J_\infty(X) \smallsetminus Z_\infty.$$

Proof. First, note that for any morphism $\operatorname{Spec} k[[t]] \to \widetilde{X}$ with image not contained in $f^{-1}(Z)$, the composition $\operatorname{Spec} k[[t]] \to \widetilde{X} \to X$ will have image not contained in Z. Thus, f_∞ restricts to a morphism $\widetilde{J_\infty}(X) \smallsetminus (f^{-1}(Z))_\infty \to J_\infty(X) \smallsetminus Z_\infty$.

Let $\gamma : \operatorname{Spec} k[[t]] \to X$ be an arc, with image not contained in Z. Since the image of the generic point $\operatorname{Spec} k((t)) \hookrightarrow \operatorname{Spec} k[[t]] \to X$ lands in $X \smallsetminus Z \cong \widetilde{X} \smallsetminus f^{-1}(Z)$, we have a map $\operatorname{Spec} k((t)) \to \widetilde{X}$ making the following diagram commute:

$$\begin{tikzcd}
\operatorname{Spec} k((t)) \arrow[r] \arrow[d, hook] & \widetilde{X} \arrow[d, "f"] \\
\operatorname{Spec} k[[t]] \arrow[r, "\gamma"] & X
\end{tikzcd}$$

The valuative criterion for the proper morphism γ then says that there is a unique morphism $\widetilde{\gamma} : \operatorname{Spec} k[[t]] \to \widetilde{X}$ making the diagram commute. Thus, we have that the morphism $\widetilde{J_\infty}(X) \smallsetminus (f^{-1}(Z))_\infty \to J_\infty(X) \smallsetminus Z_\infty$ is bijective. $\square$

While the morphism f_∞ is a set-theoretic bijection (on a "large" subset of the arc space at least), the codimension of a cylinder will change under f_∞. The connection between log discrepancies and jet schemes arises through the following birational transformation rule, which expresses how this codimension changes under proper birational morphisms:

Theorem 2.2.14 ([Kon95; DL99])**.** *Let X be a reduced equidimensional scheme, $f : \widetilde{X} \to X$ a proper birational morphism with $\widetilde{X}$ smooth, and let e, e' be nonnegative integers. Write*

$$C_{e,e'} := \mathrm{Cont}^{e}(\mathrm{Jac}_f) \cap f_\infty^{-1}(\mathrm{Cont}^{e'}(\mathrm{Jac}_X)) \subset J_\infty(\widetilde{X})$$

For $m \geq \max 2e, e+e'$, consider the map $f_m : J_m(\widetilde{X}) \to J_m(X)$, and write $\psi_m := \psi^{\widetilde{X}}_{\infty,m} : J_\infty(\widetilde{X}) \to J_m(\widetilde{X})$ for the truncation map.

- *$\psi_m(C_{e,e'})$ is the union of fibers of f_m, i.e., if $\gamma \in \psi_m(C_{e,e'})$ and $f_m(\gamma) = f_m(\gamma')$ for some γ' in $J_m(X)$, then $\gamma' \in \psi_m(C_{e,e'})$.*
- *The restriction $\psi_m(C_{e,e'}) \to f_m(\psi_m(C_{e,e'}))$ is a piecewise trivial $\mathbb{A}^e$-fibration.*

Remark 2.2.15. The statement of this might appear somewhat technical, but the essential content is that if one has a cylinder $D \subset J_\infty(\widetilde{X})$ that one can partition it up into $D \cap C_{e,e'}$ as e, e' varies, and that the truncation of each intersection $D \cap C_{e,e'}$ to a (high-enough) finite level m is a piecewise trivial $\mathbb{A}^e$-fibration over its image under f_m. At the level of the arc scheme, this says that the codimension of D itself changes under f_∞ based on the codimensions of each intersection $D \cap C_{e,e'}$ as e varies.

The relation between minimal log discrepancies and jet spaces is expressed through the following formula of Ein and Mustaţă; the proof proceeds by the use of the above birational transformation rule:

Theorem 2.2.16 ([EM09a, Theorem 7.4])**.** *Let (X, Y) be a pair and $W \subset X$ a proper closed subset. Then*

$$\begin{aligned}&\mathrm{mld}(W; X, Y)\\ &\qquad = \inf_{n,\underline{w}=(w_i)} \Big\{ \mathrm{codim}\big(\mathrm{Cont}^{\underline{w}}(Y) \cap \mathrm{Cont}^{n}(J(X)) \cap \mathrm{Cont}^{\geq 1}(W)\big) - n - \sum_i \alpha_i w_i \Big\}.\end{aligned}$$

2.2.5 Two lemmas on arc schemes

The following two lemmas will be used in Chapter IV. The first says that the contact loci of an ideal are unaffected by passing to the integral closure:

Lemma 2.2.17. *If X is a finite-type k-scheme, $\mathcal{J} \subset \mathcal{O}_X$ an ideal sheaf, and $\overline{\mathcal{J}}$ its integral closure, then $\mathrm{Cont}^{\geq i}(\mathcal{J}) = \mathrm{Cont}^{\geq i}(\overline{\mathcal{J}})$ and $\mathrm{Cont}^{i}(\mathcal{J}) = \mathrm{Cont}^{i}(\overline{\mathcal{J}})$.*

Proof. Clearly the first claim implies the second, since $\mathrm{Cont}^i(\mathcal{I}) = \mathrm{Cont}^{\geq i}(\mathcal{I}) \smallsetminus \mathrm{Cont}^{\geq i+1}(\mathcal{I})$ for any ideal sheaf $\mathcal{I}$. The claim is local on X, so let $X = \operatorname{Spec} R$ and $J \subset R$ be the ideal in question. A k-point $\gamma \in J_\infty(\operatorname{Spec} R)$ corresponds to a k-algebra homomorphism $\gamma^* : R \to k[[t]]$. Since $k[[t]]$ is a discrete valuation ring, we have that $\gamma^*(J) = \gamma^*(\overline{J})$ by the definition of integral closure. Thus, $\mathrm{ord}_\gamma(J) = \mathrm{ord}_\gamma(\overline{J})$ for any $\gamma \in J_\infty(X)$, and thus $\mathrm{Cont}^{\geq i}(J) = \mathrm{Cont}^{\geq i}(\overline{J})$. □

We introduce the following lemma to facilitate computation of codimensions of spaces of jets without having to calculate Jac_X or the contact loci along it explicitly:

Lemma 2.2.18. *Given any cylinder $C \subset J_\infty(X)$, not necessarily contained in some $\mathrm{Cont}^e(\mathrm{Jac}_X)$, we have*

$$\operatorname{codim}(C) = n(\ell + 1) - \dim \psi_{\infty,\ell}(C)$$

for $\ell \gg 0$.

Note that this does not give an explicit bound on how large we must take ℓ; in our applications here, the quantity

$$n(\ell + 1) - \dim \psi_{\infty,\ell}(C)$$

will be seen to be independent of ℓ for $\ell \gg 0$ directly.

The key ingredient in the proof of the lemma is the fact that $\lim_{e\to\infty} \operatorname{codim}(C^{(e)}) = \infty$; for a proof, see [EM09a, Proposition 5.11].

Proof. Take C to be a cylinder of codimension c. Using the fact that

$$\lim_{e->\infty} \operatorname{codim} \underbrace{C \cap \mathrm{Cont}^e \mathrm{Jac}_X}_{C^{(e)}} = \infty,$$

there is m such that for $m' > m$ we have $\operatorname{codim} C^{(m')} > c$. We can write

$$C = \underbrace{C^{(0)} \cup \cdots \cup C^{(m)}}_{C'} \cup \underbrace{\left(C \cap \mathrm{Cont}^{\geq m} \operatorname{Jac} X\right)}_{C''}.$$

Then C'' is a cylinder, and by definition of codimension it is clear $\operatorname{codim} C'' > c$, so that $\operatorname{codim} C = \operatorname{codim} C' = c$.

By the usual properties of dimension

$$\dim(\psi_{\infty,\ell}(C')) = \max_{e=0,\ldots,m} (\psi_{\infty,\ell}(C^{(e)})),$$

and so it is immediate that

$$\operatorname{codim}(C') = \min_{e=0,\ldots,m} \left(n(\ell+1) - \dim \psi_{\infty,\ell}(C^{(e)})\right) = n(\ell+1) - \dim \psi_{\infty,\ell}(C')$$

for $\ell \gg 0$.

Thus, all we need to show is that for $\ell \gg 0$,

$$n(\ell+1) - \dim \psi_{\infty,\ell}(C) = n(\ell+1) - \dim \psi_{\infty,\ell}(C'),$$

or equivalently that

$$\dim \psi_{\infty,\ell}(C') \geq \dim \psi_{\infty,\ell}(C'').$$

But this is immediate, because $\operatorname{codim} C'' < c = \operatorname{codim} C'$: for $\ell \gg 0$, we have

$$(\ell+1)n - \dim \psi_{\infty,\ell}(C') =: \operatorname{codim} C' > \operatorname{codim} C'' := (\ell+1)n - \dim \psi_{\infty,\ell}(C''),$$

and thus the desired inequality holds. $\square$

2.3 Differential operators

2.3.1 Definitions

Let A be a commutative ring, and R a commutative A-algebra. The (noncommutative) ring $D_{R/A}$ of A-linear differential operators on R is defined inductively as follows: let $D^0_{R/A} := \operatorname{Hom}_R(R, R) \cong R$ (thought of as multiplication by R), and

$$D^i_{R/A} := \{\varphi \in \operatorname{End}_A(R) : [\varphi, r] \in D^{i-1}_{R/A} \text{ for all } r \in R\}.$$

Then

$$D_{R/A} = \bigcup_i D^i_{R/A}.$$

We note $D_{R/A}$ is a subring of $\operatorname{Hom}_A(R, R)$ and thus R carries a canonical $D_{R/A}$-module structure.

Example 2.3.1. If R is the smooth A-algebra $A[x_1, \ldots, x_n]$, then $D_{R/A}$ is generated as an R-algebra by the "divided power partial derivatives"

$$\frac{1}{\alpha_1! \cdots \alpha_n!}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \cdots \circ \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

When $\mathbb{Q} \subset A$, then the coefficient $\frac{1}{\alpha_1! \cdots \alpha_n!}$ is irrelevant, and $D_{R/A}$ is generated as an

R-algebra by the partial derivatives $\partial/\partial x_i$. As a result, if k is a field containing $\mathbb{Q}$, then $D_{R/k}$ is Noetherian, finitely generated (in fact, generated over R by derivations), and a simple algebra [Smi86].

In characteristic p, however, we see already that even in the one-variable case, $D_{\mathbb{F}_p[x]/\mathbb{F}_p}$ is not finitely generated as an $\mathbb{F}_p$-algebra, and finite generation never holds in positive characteristic more generally (unless R is of dimension 0 over A).

Remark 2.3.2. If $R = k[x_1, \ldots, x_n]/I$ is the quotient of a polynomial ring, one can describe $D_{R/k}$ as a subquotient of $D_{S/k}$:

$$D_{R/k} = \frac{\{\delta \in D_{S/k} : \delta(I) \subset I\}}{ID_{S/k}}.$$

For a proof of this, see, e.g., [MR01, Theorem 5.13], though it goes back much further.

While this is more concrete than the above inductive description, it is very hard in practice to calculate the $\delta \in D_{S/k}$ preserving I (the *idealizer* of I); see [BJN19] for one such approach.

In all applications considered, we always take the base ring A to be a field k.

2.3.2 D-simplicity

Definition 2.3.3. An A-algebra R is called D-simple if R is a simple $D_{R/A}$-module, i.e., if the only proper $D_{R/A}$-submodule of R is the zero ideal.

Remark 2.3.4. One can also ask when $D_{R/A}$ is a simple algebra (i.e., there are no nonzero proper two-sided ideals). It is straightforward to verify that this forces R to be a simple $D_{R/A}$-module: If R were not a simple $D_{R/A}$-module, let $I \subset R$ be a nonzero proper $D_{R/A}$-submodule. Then R/I is a $D_{R/A}$-module with nonzero annihilator (since the annihilator includes multiplication by f for every $f \in I$), but the annihilator is proper (since it does not include 1), and thus $D_{R/A}$ cannot be simple. The converse is not true; see, e.g., [LS89, p. 0.13.3].

Remark 2.3.5. When $R = \bigoplus R_i$ is a graded A-algebra, $D_{R/A}$ is naturally graded as well: $(D_{R/A})_e$ consists of all differential operators $\delta \in D_{R/A}$ such that $\delta(R_i) \subset R_{i+e}$. This simple observation is key in our study of D-simplicity: if R is a graded A-algebra (not concentrated entirely in degree 0), and $D_{R/A}$ has no differential operators of negative degree, then $R_+ := \bigoplus_{i>0} R_i$ is a nonzero proper $D_{R/A}$-submodule of R, and thus R cannot be D-simple.

The question we will consider in Chapter V is when a k-algebra R is D-simple. In characteristic p, this has the following satisfying answer:

Theorem 2.3.6. *[Smi95, Theorem 2.2] Let (R, m) be an F-pure local ring essentially of finite type over an F-finite field k. Then R is D-simple if and only if it is strongly F-regular.*

Outside the context of characteristic p, much less is known. For example, if R is a D-simple ring, then:

- If R is reduced then R must be a domain.
- R is Cohen–Macaulay [Van91, Theorem 6.2.5].

As mentioned in Remark 1.3.2, few examples of D-simple rings in characteristic 0 are known, essentially all direct summands of regular rings.

CHAPTER III

Triviality of Arc Closures and the Local Isomorphism Problem

We give an answer in the "geometric" setting to a question of [FEI18] asking when local isomorphisms of k-schemes can be detected on the associated maps of local arc or jet schemes. In particular, we show that their ideal-closure operation $\mathfrak{a} \mapsto \mathfrak{a}^{\mathrm{ac}}$ (the arc-closure) on a local k-algebra $(R, \mathfrak{m}, L)$ is trivial when R is Noetherian and $k \hookrightarrow L$ is separable, and thus that such a germ $\operatorname{Spec} R$ has the (embedded) local isomorphism property.

3.1 Introduction

Let k be any field. Given a k-scheme X, morphisms $\operatorname{Spec} k[t]/t^{\ell+1} \to X$ (ℓ-jets) are parametrized by the ℓ-jet schemes $J_\ell(X)$, and morphisms $\operatorname{Spec} k[[t]] \to X$ (arcs) by the arc scheme $J_\infty(X)$. The arc and jet schemes encapsulate a great deal of information about X. They are central to the theory of motivic integration, which allowed Kontsevich to show the birational invariance of the Hodge numbers of Calabi–Yau varieties [Kon95] and since then has been applied to the study of various motivic invariants (see, e.g., [DL99; Loo00]). In particular, this has led to connections between singularities of the minimal model program and arc schemes (see [EM09b]). In a somewhat different direction, they are related further to singularities through the study of the Nash blow-up and Mather–Jacobian discrepancy (see [IR17; FD17b]).

There are morphisms $\psi_\infty \colon J_\infty(X) \to X$ and $\psi_\ell \colon J_\ell(X) \to X$, given by sending an arc $\operatorname{Spec} k[[t]] \to X$ to the image of the closed point of $\operatorname{Spec} k[[t]]$ in X, and likewise for ℓ-jets. Given a point x of X, one defines $J_\infty(X)_x := \psi_\infty^{-1}(x)$ and $J_\ell(X)_x := \psi_\ell^{-1}(x)$, the arcs or ℓ-jets based at $x \in X$. Given a morphism of k-schemes $f \colon X \to Y$, we obtain morphisms $f_\ell \colon J_\ell(X) \to J_\ell(Y)$ and $f_\infty \colon J_\infty(X) \to J_\infty(Y)$, defined on k-points by sending an arc $\operatorname{Spec} k[[t]] \to X$ on X to the arc $\operatorname{Spec} k[[t]] \to X \to Y$ on Y; these restrict to morphisms $\bar{f}_\ell \colon J_\ell(X)_x \to J_\ell(Y)_{f(x)}$ and $\bar{f}_\infty \colon J_\infty(X)_x \to J_\infty(Y)_{f(x)}$.

In [FEI18], de Fernex, Ein, and Ishii considered the question of how much local information about f is captured by the morphisms $\bar{f}_\ell \colon J_\ell(X)_x \to J_\ell(Y)_{f(x)}$ or $\bar{f}_\infty \colon J_\infty(X)_x \to J_\infty(Y)_{f(x)}$. More precisely, they asked the following question:

Question 3.1.1 (Local isomorphism problem)**.** If the morphisms $\bar{f}_\ell \colon J_\ell(X)_x \to J_\ell(Y)_{f(x)}$ are isomorphisms for all ℓ (including $\ell = \infty$), is f an isomorphism at x, i.e., does f induce an isomorphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$?

The question is local on X and Y, so we can restrict the setting to where X, Y are spectra of local rings and $x, y = f(x)$ are the closed points; we call such a pair (X, x) a germ.

The article [FEI18] also considers the following variant (and shows that this is equivalent to the original question when X is locally Noetherian):

Question 3.1.2 (Embedded local isomorphism problem)**.** If we assume furthermore that f is a closed embedding of germs, does the above question have a positive answer?

In order to understand the embedded version of the question, de Fernex, Ein, and Ishii introduce the arc closure, which is a closure operation $\mathfrak{a} \mapsto \mathfrak{a}^{\mathrm{ac}}$ on ideals of a local k-algebra R defined using the jet schemes of $\operatorname{Spec} R$. They then show that arc-closure of the zero ideal (the equality $(0) = (0)^{\mathrm{ac}}$) for a ring R is equivalent to a positive answer to the embedded local isomorphism problem for morphisms to $(\operatorname{Spec} R, \operatorname{Spec}(R/\mathfrak{m}))$. They furthermore give an example of a (non-Noetherian) k-algebra R in which the zero ideal is not arc-closed, and thus in which the embedded local isomorphism property does not hold, suggesting that some restrictions on R are necessary to ensure a positive answer.

In this chapter, we show that this closure operation is trivial for Noetherian local k-algebras $(R, \mathfrak{m}, L)$ for which $k \hookrightarrow L$ is separable, and thus that such germs have the embedded local isomorphism property:

Theorem 3.1.3. *If $(R, \mathfrak{m}, L)$ is a Noetherian local k-algebra with residue field L, $k \hookrightarrow L$ is separable, and $\mathfrak{a}$ is a proper ideal of R, then $\mathfrak{a}^{\mathrm{ac}} = \mathfrak{a}$.*

In particular, this holds when $\operatorname{char} k = 0$, when k is perfect, or when $L = k$, and thus holds in the cases of primary geometric interest.

Corollary 3.1.4. *Such germs have the embedded local isomorphism property.*

The strategy is relatively simple: We proceed by reducing first to showing that $\mathfrak{a}^{\mathrm{ac}} = \mathfrak{a}$ for $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal, and then by induction on the length of $R/\mathfrak{a}$ to the case where $R/\mathfrak{a}$ is a Gorenstein Artinian local k-algebra. At that point, we obtain an inclusion of R'-modules $R/\mathfrak{a} \hookrightarrow R'$ via the Matlis dual (*not* an inclusion of rings!) for a suitable *graded* Gorenstein Artinian local k-algebra R'. This step uses the Cohen structure theorem, and requires that $R/\mathfrak{a}$ has a coefficient field $L_0 \cong L$ containing k, which is where the assumption on separability of $k \hookrightarrow L$ comes in. This inclusion of modules necessitates the introduction and analysis of an arc-closure operation defined on submodules of modules; once a few elementary properties are shown, we use that the arc-closedness of the zero ideal of R', as shown in [FEI18, Theorem 5.8(a)], to conclude that the zero ideal of $R/\mathfrak{a}$ must be arc-closed as well.

The organization of the chapter is as follows: In Section 3.2 we recall our notation for arc and jet schemes, and describe the local isomorphism problem, and in Section 3.3 we recall the definition of arc and jet closures and their basic properties from [FEI18]. In Section 3.4 we generalize the definition of arc and jet closures to closures of submodules of modules and prove some elementary properties about these operations under module maps and restrictions of scalars along a ring quotient. Section 3.5 contains the core of our proof, and Section 3.6 has a few observations on further questions on the subject.

3.2 Arc and jet schemes

Recall that for any $\ell \in \mathbb{N} \cup \{\infty\}$ we have truncation maps $\psi_{\ell,0} \colon J_\ell(X) \to J_0(X) = X$, which we denote simply by ψ_ℓ; on k-points, this just sends an arc $\operatorname{Spec} k[[t]]/t^{\ell+1} \to \operatorname{Spec} X$ to

$$\operatorname{Spec} k \to \operatorname{Spec} k[[t]]/t^{\ell+1} \to X,$$

i.e., to the image of the closed point of $\operatorname{Spec} k[[t]]/t^{\ell+1}$. For a point $x \in X$, not necessarily closed, we write $J_\ell(X)_x$ for $\psi_\ell^{-1}(x)$, the fiber over x.

Given a morphism $f \colon X \to Y$ of k-schemes, for any morphism

$$T \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}) \to X$$

we obtain a morphism

$$T \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}) \to X \to Y,$$

and by functoriality we obtain morphisms $f_\ell\colon J_\ell(X) \to J_\ell(Y)$ for all ℓ. Furthermore, it is clear that for $x \in X$ these morphisms restrict to morphisms $\bar{f}_\ell\colon J_\ell(X)_x \to J_\ell(Y)_{f(x)}$.

We also recall fix some notation for the arc and jet schemes of an affine scheme $\operatorname{Spec} R$: Given a k-algebra R (not necessarily Noetherian or local), and $\ell \in \mathbb{N} \cup \{\infty\}$, we write R_ℓ for the coordinate ring of the jet scheme $J_\ell(\operatorname{Spec} R)$. If $\ell < \infty$, for any k-scheme T we have

$$\operatorname{Hom}(T, \operatorname{Spec} R_\ell) = \operatorname{Hom}(T \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}), X).$$

For an ideal I of R we write I_ℓ for the ideal of R_ℓ generated by $D_i(f)$ for $0 \leq i < \ell+1$ and $f \in I$, where D_i are the universal Hasse–Schmidt derivations $R \to R_\ell$. (For a full treatment of Hasse–Schmidt derivations, see [Voj13].)

Example 3.2.1. If $R = k[x_1, \dots, x_n]$ then

$$R_\ell = k\big[x_i^{(j)} : i = 1, \dots, n, 0 \leq j \leq \ell\big];$$

one can think of a point $(a_i^{(j)})$ of $\operatorname{Spec} R_\ell \cong \mathbb{A}^{n(\ell+1)}$ as parametrizing the arc

$$\operatorname{Spec} k[t]/t^{\ell+1} \to \mathbb{A}^n, \quad t \mapsto \left(\sum a_1^{(i)} t^i, \dots, \sum a_n^{(i)} t^i\right).$$

The Hasse–Schmidt derivations can be defined by setting

$$D_i(x_j) = x_j^{(i)}$$

and extending via the Leibniz rule $D_m(fg) = \sum_{i+j=m} D_i(f) D_j(g)$, so, e.g., if we write $x = x_1$, $y = x_2$, we have

$$D_2(xy) = D_2(x)D_0(y) + 2D_1(x)D_1(y) + D_0(x)D_2(y) = x^{(2)}y^{(0)} + 2x^{(1)}y^{(1)} + x^{(0)}y^{(2)}.$$

The projection maps $\psi_{\ell,\ell'}$ for $\ell > \ell'$ give ring maps $R_{\ell'} \to R_\ell$, and in particular an inclusion $R = R_0 \hookrightarrow R_\ell$ for any ℓ.

We now have the language to state the motivating questions of [FEI18]:

Question 3.2.2 (Local isomorphism problem)**.** Given a map $f\colon X \to Y$ and $x \in X$, if all the morphisms $\bar{f}_\ell\colon J_\ell(X)_x \to J_\ell(Y)_{f(x)}$ are isomorphisms (including $\ell = \infty$), is f a local isomorphism at x, i.e., does f induce an isomorphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$?

Question 3.2.3 (Embedded local isomorphism problem)**.** If we assume furthermore that f is a closed embedding, does the above question have a positive answer?

As remarked previously, these questions are local on source and target of the morphism, so we may assume X and Y are spectra of local rings with closed points x, y respectively; we refer to such a pair (X, x) or (Y, y) as a germ; if $(R, \mathfrak{m})$ is a local ring, we will refer simply to the germ $\operatorname{Spec} R$ when no confusion will occur. We say a germ (Y, y) has the local isomorphism property (respectively, the embedded local isomorphism property) if the local isomorphism problem (respectively, the embedded local isomorphism problem) has an affirmative answer for all maps of germs $(X, x) \to (Y, y)$.

Remark 3.2.4. As noted in [FEI18, Proposition 2.6, Lemma 2.7], if Y has the embedded local isomorphism property then the local isomorphism problem has an affirmative answer for maps $(X, x) \to (Y, y)$ with X Noetherian; thus, for most cases of geometric interest it suffices to consider just the embedded form of the problem.

3.3 Arc and jet closures

Now, say $(R, \mathfrak{m})$ is a local k-algebra, and write $\mathfrak{m}R_\ell$ for the expansion of $\mathfrak{m} \subset R$ to R_ℓ under the ring map $R \to R_\ell$. The following definition is key to the reduction in [FEI18] of the embedded local isomorphism problem to a ring-theoretic question:

Definition 3.3.1 ([FEI18])**.** For an ideal $\mathfrak{a}$ of R and $\ell < \infty$, define $\mathfrak{a}^{\ell-\mathrm{jc}}$, the ℓ-jet closure of $\mathfrak{a}$, as

$$\mathfrak{a}^{\ell-\mathrm{jc}} := \big(f \in R : (f)_\ell \subset \mathfrak{a}_\ell + \mathfrak{m}R_\ell\big),$$

and for $\ell = \infty$, define the arc closure of $\mathfrak{a}$ as

$$\mathfrak{a}^{\mathrm{ac}} = \big(f \in R : (f)_\infty \subset \mathfrak{a}_\infty + \mathfrak{m}R_\infty\big).$$

The ideal $\mathfrak{a}^{\ell-\mathrm{jc}}$ is the largest ideal of R whose higher differentials define the same closed subscheme in $J_\ell(\operatorname{Spec} R)_{\operatorname{Spec} R/\mathfrak{m}}$ (the fiber over the closed point of R) as that defined by the higher differentials of $\mathfrak{a}$.

Example 3.3.2. It is immediately seen that the $\mathfrak{a}^{\ell-\mathrm{jc}}$ are nontrivial closure operations; for example, by the Leibniz rule it is easily seen that if $f \in \mathfrak{m}^{\ell+1}$ then $D_\ell(f) \in \mathfrak{m}R_\ell$, so that $\mathfrak{a} + \mathfrak{m}^{\ell+1} \subset \mathfrak{a}^{\ell-\mathrm{jc}}$. For an example showing that this is in general a proper inclusion, see [FEI18, Example 3.11].

The following shows that we can compute these closures in the quotient ring $R/\mathfrak{a}$:

Lemma 3.3.3 ([FEI18, Lemma 3.2])**.** *Let* $\mathfrak{a} \subset \mathfrak{m}$*. For all* ℓ*, if* $\pi \colon R \to R/\mathfrak{a}$*, then*

$$\mathfrak{a}^{\ell-\mathrm{jc}} = \pi^{-1}((0_{R/\mathfrak{a}})^{\ell-\mathrm{jc}}),$$

and similarly

$$\mathfrak{a}^{\mathrm{ac}} = \pi^{-1}((0_{R/\mathfrak{a}})^{\mathrm{ac}}).$$

Thus it suffices to know how to compute the arc or ℓ-jet closure of the zero ideal, for which there is a nice interpretation in terms of the "universal" ℓ-jet: the identity morphism $\operatorname{Spec} R_\ell \to \operatorname{Spec} R_\ell$ corresponds to the "universal" ℓ-jet $(\operatorname{Spec} R_\ell) \times_k \operatorname{Spec}(k[t]/t^{\ell+1}) \to \operatorname{Spec} R$, given by the ring map

$$\mu_R \colon R \to R_\ell[t]/t^{\ell+1};$$

by composing with the quotient map

$$R_\ell[t]/t^{\ell+1} \to \big(R_\ell/\mathfrak{m}R_\ell\big)[t]/t^{\ell+1},$$

we obtain a map

$$\lambda_\ell \colon R \to \big(R_\ell/\mathfrak{m}R_\ell\big)[t]/t^{\ell+1}.$$

The following statement is now clear by definition:

Lemma 3.3.4 ([FEI18, Lemma 3.3])**.** $(0_R)^{\ell-\mathrm{jc}} = \ker \lambda_\ell$ *and* $(0_R)^{\mathrm{ac}} = \ker \lambda_\infty$*.*

Example 3.3.5. In the case $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ the universal ℓ-jet $R \to R_\ell[t]/t^{\ell+1}$ sends

$$x_i \mapsto x_i^{(0)} + x_i^{(1)}t + x_i^{(2)}t^2 + \cdots + x_i^{(\ell)}t^\ell.$$

The ideal $\mathfrak{m}R_\ell$ is just $\big(x_i^{(0)} : i = 1, \ldots, \ell\big)$; note that this is *not* the expansion of $\mathfrak{m}$ under the universal ℓ-jet.

The following result linking $\mathfrak{a}^{\ell-\mathrm{jc}}$ and $\mathfrak{a}^{\mathrm{ac}}$ is key to our proof below:

Proposition 3.3.6 ([FEI18, Proposition 3.12])**.** $\bigcap_{\ell \geq 0} \mathfrak{a}^{\ell-\mathrm{jc}} = \mathfrak{a}^{\mathrm{ac}}$*.*

The geometric interpretation following Definition 3.3.1 makes clear the motivation for this closure operation:

Proposition 3.3.7 ([FEI18, Proposition 5.1])**.** *Let* R *be a local* k*-algebra. The germ* $\operatorname{Spec} R$ *has the embedded local isomorphism property if and only if* $(0_R)^{\mathrm{ac}} = 0$*.*

Proof. This is essentially by definition: a closed embedding $X \to \operatorname{Spec} R$ corresponds to a quotient $R \to R/\mathfrak{a}$ for some $\mathfrak{a} \subset R$; this is an isomorphism of schemes if and only if $\mathfrak{a} = 0$, and it induces an isomorphism on the fibers of the jet schemes over the closed point if and only if $(\mathfrak{a})_\infty \subset \mathfrak{m}R_\infty$ if and only if $\mathfrak{a} \subset (0)^{\mathrm{ac}}$. ☐

Remark 3.3.8. It is observed in [FEI18] that the two preceding propositions imply that it is redundant in the statement of the embedded local isomorphism problem to ask for $\bar{f}_\ell$ to be an isomorphism for all $\ell \in \mathbb{N} \cup \{\infty\}$: $\bar{f}_\infty$ is an isomorphism if and only if $\bar{f}_\ell$ is an isomorphism for all finite ℓ.

Remark 3.3.9. In the non-Noetherian setting, Proposition 5.4 of [FEI18] provides an example of an ideal $\mathfrak{a}$ inside a power series ring in infinitely many variables such that $\mathfrak{a}^{\mathrm{ac}} \neq \mathfrak{a}$; this is proved via the observation that $\mathfrak{a}^{\ell-\mathrm{jc}} \supset \mathfrak{a} + \mathfrak{m}^{\ell+1}$, and then giving an explicit element contained in $\mathfrak{a} + \mathfrak{m}^{\ell+1}$ for all ℓ.

We remark here that this situation may in some sense be typical, at least for certain classes of non-Noetherian rings: if $(R, \mathfrak{m})$ is a non-Noetherian valuation ring, then one has $\mathfrak{m} = \mathfrak{m}^2 = \dots$ (see, for example, [HS06, Exercise 6.29]). Thus, for any ideal $\mathfrak{a} \subset \mathfrak{m}$, including the zero ideal, we have $\mathfrak{a} + \mathfrak{m}^\ell = \mathfrak{m}$ for all ℓ, and thus $\mathfrak{a}^{\mathrm{ac}} = \mathfrak{m}$.

The last result we need from [FEI18] is that says that a graded k-algebra has arc-closed zero ideal; we will write $R_{[i]}$ for the i-th graded piece of a graded ring R to avoid confusion with the jet schemes R_ℓ.

Theorem 3.3.10 ([FEI18, Theorem 5.8(a)])**.** *Let $(R, \mathfrak{m})$ be a local k-algebra with $\mathbb{N}$-grading such that $\mathfrak{m} = \bigoplus_{i \geq 1} R_{[i]}$. Then the zero ideal of R is arc-closed.*

Remark 3.3.11. The hypotheses do not demand that k be all of $R_{[0]}$ (which is the residue field of R); this is important in our application later.

We recall their proof here for ease of reference:

Proof. We construct an explicit arc using the data of the grading: define an arc $\rho\colon R \to R[[t]]$ by sending a homogeneous element $f \in R_{[i]}$ to ft^i. It is immediate that ρ is injective. By universality of the arc $R \to R_\infty[[t]]$ we get a map $\varphi\colon R_\infty \to R$, inducing a map $\widetilde{\varphi}\colon R_\infty[[t]] \to R[[t]]$ making the following diagram commute:

$$\begin{tikzcd}
 & R_\infty[[t]] \arrow[d, "\widetilde{\varphi}"] \\
R \arrow[ur, "\mu_R"] \arrow[r, hook, "\rho"'] & R[[t]]
\end{tikzcd}$$

Now, observe that for $f \in \mathfrak{m}$ we have

$$\rho(f) = \widetilde{\varphi}(\mu_R(f)) = \widetilde{\varphi}\big(d_0(f) + d_1(f)t + \cdots\big) = \varphi(d_0(f)) + \varphi(d_1(f))t + \cdots .$$

Since $\rho(f) \in tR[[t]]$, however, we must have that $\varphi(d_0(f)) = 0$ for all $f \in \mathfrak{m}$, and thus $\widetilde{\varphi}$ factors through $R_\infty/\mathfrak{m}R_\infty[[t]]$, yielding a commutative diagram

$$\begin{tikzcd}
& R_\infty[[t]] \arrow[r, two heads] \arrow[d, "\widetilde{\varphi}"] & (R_\infty/\mathfrak{m}R_\infty)[[t]] \arrow[dl, dashed] \\
R \arrow[ur, "\mu_R"] \arrow[r, hook, "\rho"'] & R[[t]] &
\end{tikzcd}$$

Thus, we must have that the composite map $\lambda_R \colon R \to R_\infty[[t]] \to (R_\infty/\mathfrak{m}R_\infty)[[t]]$ is injective since ρ is, and so $(0)^{\mathrm{ac}} = 0$. ☐

We also require the following persistence statement:

Lemma 3.3.12. *Arc closures of ideals are persistent under local ring homomorphisms; that is, if $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local rings and $\varphi \colon R \to S$ a local homomorphism, and $\mathfrak{a} \subset R$, then $\varphi(\mathfrak{a}^{\mathrm{ac}}) \subset (\varphi(\mathfrak{a})S)^{\mathrm{ac}}$.*

Proof. First, note that we have a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d, "\pi"] & S \arrow[d, "\pi'"] \\
R/\mathfrak{a} \arrow[r, "\widetilde{\varphi}"] & S/\mathfrak{a}S
\end{tikzcd}$$

By Lemma 3.3.3, elements $r \in \mathfrak{a}^{\mathrm{ac}}$ are precisely the elements of R such that $\pi(r)$ lies in the arc-closure of (0) in $R/\mathfrak{a}$; thus if we can show persistence for the map $\widetilde{\varphi}$ we have that $\pi'(\varphi(r)) = \widetilde{\varphi}(\pi(r))$ lies in the arc closure of 0 in $S/\mathfrak{a}S$, and thus applying Lemma 3.3.3 again we have $\varphi(r) \in (\varphi(\mathfrak{a})S)^{\mathrm{ac}}$. Thus, we may assume that $\mathfrak{a} = (0)$.

We have an $(S_\infty/\mathfrak{n}S_\infty)$-arc from R, i.e., the map $R \to S \to (S_\infty/\mathfrak{n}S_\infty)[[t]]$; by universality of the arc $R \to R_\infty[[t]]$ this induces a ring map $R_\infty \to S_\infty/\mathfrak{n}S_\infty$. Since φ is local, this descends to a ring map $R_\infty/\mathfrak{m}R_\infty \to S_\infty/\mathfrak{n}S_\infty$, and thus we have a ring map

$$(R_\infty/\mathfrak{m}R_\infty)[[t]] \to (S_\infty/\mathfrak{n}S_\infty)[[t]],$$

fitting into the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d, "\lambda_R"] & S \arrow[d, "\lambda_S"] \\
(R_\infty/\mathfrak{m}R_\infty)[[t]] \arrow[r] & (S_\infty/\mathfrak{n}S_\infty)[[t]]
\end{tikzcd}$$

Commutativity of this diagram then implies immediately that since $(0_R)^{\mathrm{ac}} = \ker \lambda_R$, we have $\lambda_S\big(\varphi((0_R)^{\mathrm{ac}})\big) = 0$, so that $\varphi((0_R)^{\mathrm{ac}}) \subset (0_S)^{\mathrm{ac}}$, yielding the result. $\square$

3.4 Arc closures of submodules

The key to our proof is to introduce the notion of arc-closure of an R-module:

Definition 3.4.1. For an R-module M, define

$$(0_M)^{\mathrm{ac}} = \ker\Big(M \xrightarrow{1_M \otimes \lambda_\infty} M \otimes_R \big(R_\infty/\mathfrak{m}R_\infty\big)[[t]]\Big).$$

For a submodule $N \subset M$, define $(N)^{\mathrm{ac}}_M = \pi_N^{-1}\big((0_{M/N})^{\mathrm{ac}}\big)$, where $\pi_N \colon M \to M/N$.

Lemma 3.4.2. *Arc closures of R-submodules are persistent under R-linear maps; that is, if $N \subset M$ is a submodule and $\varphi\colon M \to M'$ is an R-module map, then $\varphi((N)^{\mathrm{ac}}_M) \subset (\varphi(N))^{\mathrm{ac}}_{M'}$.*

Proof. Considering the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\pi"] & M' \arrow[d, "\pi'"] \\
M/N \arrow[r, "\bar{\varphi}"] & M'/\varphi(N)
\end{tikzcd}$$

we see that $m \in (N)^{\mathrm{ac}}_M$ exactly when $\pi(m) \in (0)^{\mathrm{ac}}_{M/N}$, and thus it suffices to show persistence under $\bar{\varphi}$ to obtain it for φ, i.e., it suffices to show persistence of arc closure of the zero submodule.

In this case, we have a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d] & M' \arrow[d] \\
M \otimes \big(R_\infty/\mathfrak{m}R_\infty\big)[t]/t^{\ell+1} \arrow[r] & M' \otimes \big(R_\infty/\mathfrak{m}R_\infty\big)[t]/t^{\ell+1}
\end{tikzcd}$$

Note that $m \in M$ lies in the arc closure of 0 exactly when it is in the kernel of the left vertical map; when this occurs, commutativity of the diagram immediately implies that $\varphi(m)$ is in the kernel of the right vertical map, so that $\varphi(m) \in (0)^{\mathrm{ac}}_{M'}$. $\square$

We also need a comparison for closures as R-modules versus R/I-modules:

Lemma 3.4.3. *Let R be a local ring and I an ideal. Let M be an R/I-module, $N \subset M$ an R/I-submodule. Then*

$$(N)^{\mathrm{ac}}_M \subset ({}_R N)^{\mathrm{ac}}_{{}_R M},$$

where the right side is the closure of N viewed as an R-submodule of the R-module M.

In fact, we will need this result only for the arc closure of 0 in R/I itself, but we present the proof in the general case:

Proof. It suffices to show this for $N = 0$, since the quotient map $M \to M/N$ is the same whether viewed as an R-module map or an R/I-module map. Writing

$$\begin{aligned} \lambda_R &\colon R \to (R_\infty/\mathfrak{m}R_\infty)[[t]], \\ \lambda_{R/I} &\colon R/I \to ((R/I)_\infty/\mathfrak{m}(R/I)_\infty)[[t]], \end{aligned}$$

we have a commutative diagram of ring maps

$$\begin{tikzcd}
 & R/I \otimes_R (R_\infty/\mathfrak{m}R_\infty)[[t]] \\
R/I \arrow[ru, "\mathrm{id}_{R/I} \otimes_R \lambda_R"] \arrow[r, "\lambda_{R/I}"'] & ((R/I)_\infty/\mathfrak{m}(R/I)_\infty)[[t]] \arrow[u, dashed]
\end{tikzcd}$$

where the right vertical side is induced by the universality of the arc $\lambda_{R/I}$. Tensoring over R with M, we obtain

$$\begin{tikzcd}
 & M \otimes_R R/I \otimes_R (R_\infty/\mathfrak{m}R_\infty)[[t]] \arrow[r, equal] & M \otimes_R (R_\infty/\mathfrak{m}R_\infty)[[t]] \\
M \arrow[ru, "\mathrm{id}_M \otimes_R \lambda_R"] \arrow[r, "\mathrm{id}_M \otimes_R \lambda_{R/I}"'] & M \otimes_R ((R/I)_\infty/\mathfrak{m}(R/I)_\infty)[[t]] \arrow[u, dashed] \arrow[r, equal] & M \otimes_{R/I} ((R/I)_\infty/\mathfrak{m}(R/I)_\infty)[[t]]
\end{tikzcd}$$

Thus we see that since

$$\mathrm{id}_M \otimes_R \lambda_R \colon M \to M \otimes_R (R_\infty/\mathfrak{m}R_\infty)[[t]]$$

factors through

$$\mathrm{id}_M \otimes_R \lambda_{R/I} = \mathrm{id}_M \otimes_{R/I} \lambda_{R/I} \colon M \to M \otimes_{R/I} ((R/I)_\infty/\mathfrak{m}(R/I)_\infty)[[t]],$$

we must have that

$$\underbrace{\ker\bigl(\mathrm{id}_M \otimes_{R/I} \lambda_{R/I}\bigr)}_{(N)^{\mathrm{ac}}_M} \subset \underbrace{\ker\bigl(\mathrm{id}_M \otimes_R \lambda_R\bigr)}_{({}_RN)^{\mathrm{ac}}_{{}_RM}},$$

yielding the result. ☐

3.5 The main result

Given a local k-algebra $(R, \mathfrak{m}, L)$ with residue field L, we say L is separable over k to mean that the field extension $k \subset R \to R/\mathfrak{m} \cong L$ is separable (not necessarily algebraic).

Theorem 3.5.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local* k*-algebra with residue field* L *separable over* k*, and* $\mathfrak{a}$ *a proper ideal of* R*. Then* $\mathfrak{a}^{\mathrm{ac}} = \mathfrak{a}$*.*

As stated in Section 3.1, the condition on separability of $k \hookrightarrow L$ is just to ensure that for a complete local k-algebra with residue field L we may choose a coefficient field containing k; this is sufficient but not necessary, as can be seen by taking L to be an inseparable extension of k and setting $R = L[[x]]$; $k \subset L$ is inseparable, but clearly R has a coefficient field containing k.

We note that the assumption on $k \hookrightarrow L$ is satisfied in particular when k has characteristic 0 or is perfect of positive characteristic, or when $k = L$, and thus in the primary case of geometric interest for the embedded local isomorphism question.

Proof of Theorem 3.5.1. The first step is to reduce to the case where $\mathfrak{a}$ is $\mathfrak{m}$-primary:

Lemma 3.5.2. *Let* $(R, \mathfrak{m})$ *be a local* k*-algebra. Then*

$$\mathfrak{a}^{\ell-\mathrm{jc}} = \bigcap_n (\mathfrak{a} + \mathfrak{m}^n)^{\ell-\mathrm{jc}}$$

for any ℓ*, and thus*

$$\mathfrak{a}^{\mathrm{ac}} = \bigcap_n (\mathfrak{a} + \mathfrak{m}^n)^{\mathrm{ac}}.$$

Proof. The second statement follows from the first due to the equality $\bigcap_\ell \mathfrak{a}^{\ell-\mathrm{jc}} = \mathfrak{a}^{\mathrm{ac}}$

(see Proposition 3.3.6), since then

$$\bigcap_n (\mathfrak{a}+\mathfrak{m}^n)^{\mathrm{ac}} = \bigcap_n \bigcap_\ell (\mathfrak{a}+\mathfrak{m}^n)^{\ell-\mathrm{jc}} = \bigcap_\ell \bigcap_n (\mathfrak{a}+\mathfrak{m}^n)^{\ell-\mathrm{jc}} = \bigcap_\ell \mathfrak{a}^{\ell-\mathrm{jc}} = \mathfrak{a}^{\mathrm{ac}}.$$

Fix ℓ. Clearly $\mathfrak{a}^{\ell-\mathrm{jc}} \subset (\mathfrak{a}+\mathfrak{m}^n)^{\ell-\mathrm{jc}}$ for all n by monotonicity of the closure operation. To see the other inclusion, note $(\mathfrak{a}+\mathfrak{m}^n)^{\ell-\mathrm{jc}} = \mathfrak{a}^{\ell-\mathrm{jc}} + (\mathfrak{m}^n)^{\ell-\mathrm{jc}}$. For $n > l$, though, the Leibniz rule says that $(\mathfrak{m}^n)^{\ell-\mathrm{jc}} \subset \mathfrak{m}R_n$, so that

$$\mathfrak{a}^{\ell-\mathrm{jc}} + \mathfrak{m}R_\ell = (\mathfrak{a}+\mathfrak{m}^n)^{\ell-\mathrm{jc}} + \mathfrak{m}R_\ell.$$

Thus $\mathfrak{a}^{\ell-\mathrm{jc}} = (\mathfrak{a}+\mathfrak{m}^n)^{\ell-\mathrm{jc}}$ for $n > \ell$, and the result follows. ☐

In the Noetherian case, then, to see that $\mathfrak{a}^{\mathrm{ac}} = \mathfrak{a}$ it suffices to show that $(\mathfrak{a}+\mathfrak{m}^n)^{\mathrm{ac}} = \mathfrak{a}+\mathfrak{m}^n$, since then

$$\mathfrak{a}^{\mathrm{ac}} = \bigcap_n (\mathfrak{a}+\mathfrak{m}^n)^{\mathrm{ac}} = \bigcap_n \mathfrak{a}+\mathfrak{m}^n = \mathfrak{a},$$

where the last equality follows by Krull's intersection theorem. Equivalently by Lemma 3.3.3, we must show that the zero ideal is closed in any Artinian local k-algebra. By induction, we may reduce further to the case of a Gorenstein Artinian local k-algebra:

Lemma 3.5.3. *If $(0_R)^{\mathrm{ac}} = 0_R$ for any Gorenstein Artinian local k-algebra R, the same is true for any Artinian local k-algebra.*

Proof. We induct on length(R). Say $f \in (0_R)^{\mathrm{ac}}$.

Case 1: Say there is $g \in \operatorname{Soc} R$ with $f \notin (g) = L \cdot g$, and consider the map $\pi \colon R \to R/(g)$. Then by persistence of arc-closure under ring maps (Lemma 3.3.12) we have that $\pi\big((0_R)^{\mathrm{ac}}\big) \subset (0_{R/(g)})^{\mathrm{ac}}$; since $g \in \operatorname{Soc} R$, though, we have length$(R/(g)) =$ length$(R) - 1$, and thus by induction we know $(0_{R/(g)})^{\mathrm{ac}} = 0_{R/(g)}$. But then $\pi(f) = 0$, so $f \in (g)$, contradicting our assumption, and thus $(0_R)^{\mathrm{ac}} = 0$.

Case 2: There is no such $g \in \operatorname{Soc} R$, in which case we must have that f itself generates the socle of R, and thus R must be Gorenstein. In this case though $f = 0$, by the assumption of the lemma. ☐

We are thus reduced to showing the zero ideal is arc-closed in a Gorenstein Artinian local k-algebra R with residue field L (since taking the quotient by an $\mathfrak{m}$-primary ideal did not change the residue field).

By our assumption that $k \hookrightarrow L$ is separable and R is an Artinian (hence complete) k-algebra with residue field L, there is a coefficient field $L_0 \cong L$ contained in R

containing k (see [Mat89, Theorem 28.3]). By the Cohen structure theorem, such a ring R can be written as the quotient of $S = L_0[[x_1, \ldots, x_n]]$ by an $(x_1, \ldots, x_n)$-primary ideal $I \subset S$, and the k-algebra structure on R is the same as the k-algebra structure on this quotient induced by the inclusion $k \hookrightarrow L_0$. From now on, we omit the subscript on L_0 and simply write L.

Since I is $(x_1, \ldots, x_n)$-primary, there exists N such that $\mathfrak{m}_N := (x_1^N, \ldots, x_n^N) \subset I$. Taking the surjection $S/\mathfrak{m}_N \to S/I$ and applying the Matlis duality functor $\mathrm{Hom}_S(-, E_S(L))$, where $E_S(L)$ is the injective hull of the residue field of S, we obtain an inclusion

$$\underbrace{\mathrm{Hom}_S(S/I, E_S(L))}_{E_{S/I}(L)} \hookrightarrow \underbrace{\mathrm{Hom}_S(S/\mathfrak{m}_N, E_S(L))}_{E_{S/\mathfrak{m}_N}(L)}.$$

Now, since S/I is assumed to be Gorenstein we have that $E_{S/I}(L)$ is isomorphic as an S-module to S/I; likewise for the complete intersection $S/\mathfrak{m}_N \cong E_{S/\mathfrak{m}_N}(L)$, so we have an inclusion of S-modules

$$S/I \hookrightarrow S/\mathfrak{m}_N;$$

note that this is in fact an inclusion of $S/\mathfrak{m}_N$-modules. Since $S/\mathfrak{m}_N$ is a graded local k-algebra, Theorem 5.8(a) of [FEI18] (appearing above as Theorem 3.3.10) implies that the zero ideal, viewed as a $S/\mathfrak{m}_N$-submodule is arc-closed. But via our comparison lemma (Lemma 3.4.3) we have that the arc-closure of $(0_{S/I})$ as an S/I-module is contained in the arc-closure of $(0_{S/I})$ as an $S/\mathfrak{m}_N$-module under the restriction of scalars along $S/(x_1^N, \ldots, x_n^M) \to S/I$. Thus it suffices to show that this latter arc-closure is the zero ideal; persistence of arc closure for the inclusion of $S/\mathfrak{m}_N$-modules $S/I \hookrightarrow S/\mathfrak{m}_N$ gives

$$(0_{S/I})^{\mathrm{ac}} \subset \left(0_{S/\mathfrak{m}_N}\right)^{\mathrm{ac}} = 0$$

(with both sides taken as $S/\mathfrak{m}_N$-modules) and thus $0_{S/I}^{\mathrm{ac}} = 0$. $\square$

Corollary 3.5.4. *Noetherian germs over perfect fields have the embedded local isomorphism property; likewise for local k-algebras with residue field k.*

3.6 Further questions

Despite the triviality of the arc-closures of ideals in this case, there are related questions:

Remark 3.6.1. There is another family of jet-theoretic closure operations appearing in [FEI18], the *jet support closures*, defined in terms of the reduced structure of the jet schemes. Explicitly, one can define a "reduced" universal ℓ-jet or arc via

$$\bar{\lambda}_\ell \colon R \to (R_\ell/\mathfrak{m}R_\ell)[t]/t^{\ell+1} \to (R_\ell/\mathfrak{m}R_\ell)_{\mathrm{red}}[t]/t^{\ell+1}$$

or

$$\bar{\lambda}_\infty \colon R \to (R_\infty/\mathfrak{m}R_\infty)_{\mathrm{red}}[[t]].$$

One then defines $(0)^{\ell-\mathrm{jsc}} = \ker \bar{\lambda}_\ell$, $(0)^{\mathrm{jsc}} = \bigcap \ker \bar{\lambda}_\ell$, and $(0)^{\mathrm{asc}} = \ker \bar{\lambda}_\infty$; one can then set $\mathfrak{a}^{\ell-\mathrm{jsc}} = \pi^{-1}((0_{R/\mathfrak{a}})^{\mathrm{jsc}})$ and likewise for a^{jsc} and a^{asc}. For any ideal $\mathfrak{a}$ there are inclusions

$$\mathfrak{a}^{\mathrm{ac}} \subset \mathfrak{a}^{\mathrm{jsc}} \subset \mathfrak{a}^{\mathrm{asc}}$$

and

$$\mathfrak{a}^{\mathrm{jsc}} \subset \bar{\mathfrak{a}},$$

where $\bar{\mathfrak{a}}$ is the integral closure of $\mathfrak{a}$. It is shown in [FEI18] that $\bar{\mathfrak{a}} = \mathfrak{a}^{\mathrm{jsc}}$ for ideals inside a regular ring R, but that in a nonregular ring (even for a complete intersection) we may have $\mathfrak{a}^{\mathrm{jsc}} \subsetneq \bar{\mathfrak{a}}$.

In contrast to the case for arc-closures, we note the inclusion $\mathfrak{a}^{\mathrm{jsc}} \subset \mathfrak{a}^{\mathrm{asc}}$ can in fact be proper: for example, if $R = k[x]/x^2$, then one can check explicitly that

$$R_\infty/\mathfrak{m}R_\infty = k[x_1, x_2, \dots]/(x_1^2, 2x_1x_2, 2x_1x_3 + 2x_2^2, \dots),$$

and the quotient by the nilradical is just k. Thus, the kernel of $R \to (R_\infty/\mathfrak{m}R_\infty)[[t]]$ is the maximal ideal (x), i.e., $(0)^{\mathrm{asc}} = (x)$. In contrast, one can check that $x \notin (0)^{\ell-\mathrm{jsc}}$ for any ℓ, and thus $(0)^{\mathrm{jsc}} \subsetneq (x) = (0)^{\mathrm{asc}}$.

This suggests that $\mathfrak{a}^{\mathrm{jsc}}$ may still be an interesting (and definitely nontrivial) closure operation in the Noetherian case, and provide a geometrically-motivated closure operation tighter than the integral closure in a nonregular ring.

Remark 3.6.2. In this chapter, we introduced arc-closures of submodules to show that arc-closures of ideals are trivial, but it is possible such arc-closures of submodules are nontrivial and interesting. In particular, base-change $\Omega_{R/k} \mapsto \Omega_{R/k} \otimes_R R_\infty[[t]]$ along the universal arc is used in [FD17a] as part of the description of the Kähler differentials of the arc scheme; thus, examination of the map $M \to M \otimes_R R_\infty[[t]]$ may have an interpretation in similar contexts.

Remark 3.6.3. For any Artinian k-algebra A, there is a scheme of A-jets $J_A(X)$, which represents the functor $T \mapsto \mathrm{Hom}(T \times_k A, X)$ on k-schemes. Given a k-algebra

R, $J_A(\operatorname{Spec} R)$ will be affine, say $\operatorname{Spec} R_A$; functoriality then gives a universal A-jet $\lambda_A \colon R \to R_A \otimes_k A \to R_A/\mathfrak{m}R_A \otimes_k A$. For more on this construction, see [Mus14]. Given a complete local ring $(C, \mathfrak{m})$, we can consider a family of Artinian rings $\{A_\lambda\}$ given by quotients of C by various $\mathfrak{m}$-primary ideals $\{I_\lambda\}$; one can then consider the ideal $\bigcap \ker \lambda_A$ of R, thought of as the $\{A_\lambda\}$-jet closure of (0_R), and ask if for some suitably chosen C and family of quotients A_λ we obtain an interesting closure operation in this way.

CHAPTER IV

Minimal Log Discrepancies of Determinantal Varieties via Jet Schemes

We compute the minimal log discrepancies of determinantal varieties of square matrices, and more generally of pairs $\left(D^k, \sum \alpha_i D^{k_i}\right)$ consisting of a determinantal variety (of square matrices) and an $\mathbb{R}$-linear sum of determinantal subvarieties. Our result implies the semicontinuity conjecture for minimal log discrepancies of such pairs. For these computations, we use the description of minimal log discrepancies via codimensions of cylinders in the space of jets; this necessitates the computations of an explicit generator for the canonical differential forms and the Nash ideal of determinantal varieties, which may be of independent interest.

4.1 Introduction

Let X be a normal $\mathbb{Q}$-Gorenstein complex algebraic variety and $Y = \sum q_i Y_i$ a formal $\mathbb{R}$-linear sum of subvarieties $Y_i \subset X$. The minimal log discrepancy $\operatorname{mld}(W; X, Y)$ is a measure of the singularities of the pair (X, Y) along a subvariety $W \subset X$, and its behavior, although subtle, is quite important for the minimal model program. In particular, one expects $\operatorname{mld}(x; X, Y)$ to be a lower-semicontinuous function of $x \in X$.

Semicontinuity is not known in general, but has been shown in the following situations:

- For varieties of dimension at most 3 and toric varieties of arbitrary dimension [Amb99].
- If the ambient variety is smooth or lci [EM04; EMY03].
- If X has only quotient singularities [Nak16].

The latter two results were both proved using jet schemes, and as far as we know no proofs are known which avoid the use of jet schemes.

In this chapter, we use jet schemes to compute minimal log discrepancies on determinantal varieties of square matrices, which fall outside the aforementioned cases (see the beginning of Section 4.2). Let $D^k \subset \mathbb{A}^{m^2}$ be the locus of $m \times m$-matrices of rank $\leq k$. We obtain the following description of the minimal log discrepancies of D^k:

Theorem 4.1.1. *If $w \in D^k$ is a matrix of rank exactly $q \leq k$, then*

$$\operatorname{mld}(w; D^k) = q(m-k) + km.$$

Moreover, we have

$$\operatorname{mld}(D^{k-1}; D^k) = m - k + 1.$$

Note that this recovers the fact that $D^k \subset \mathbb{A}^{m^2}$ has terminal singularities for any $k \leq m$.

Remark 4.1.2. We restrict our attention to the case of square matrices because it is the only setting in which D^k is $\mathbb{Q}$-Gorenstein (see Section 4.2).

More generally, we consider pairs of the form $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ for $\alpha_i \in \mathbb{R}$ (possibly zero). We compute when these pairs are log canonical, and moreover compute their minimal log discrepancies:

Theorem 4.1.3. *Consider the pair $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ (where the α_i may be zero).*

1. *$\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ is log canonical at a matrix x_q of rank $q \leq k$ exactly when*

   $$\alpha_1 + \cdots + \alpha_j \leq m - k + (2j-1)$$

   for all $j = 1, \ldots, k-q$.

2. *In this case,*

   $$\operatorname{mld}\left(x_q; D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right) = q(m-k) + km - \sum_{i=1}^{k-q} (k-q-i+1)\,\alpha_i.$$

3. *$\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ is log canonical along D^{k-j} (for $j > 0$) exactly when*

   $$\alpha_1 + \cdots + \alpha_j \leq m - k + (2j-1)$$

   for all $j = 1, \ldots, k$.

(4) In this case,

$$\operatorname{mld}\Big(D^{k-j}; D^k, \sum_{i=1}^{k} \alpha_i D^{k-i}\Big) = j(m-k+j) - \sum_{i=1}^{j} (j-i+1)\alpha_i$$

This immediately implies semicontinuity of the minimal log discrepancy for such pairs (when the coefficients are nonnegative):

Corollary 4.1.4 (Semicontinuity)**.** *If $\alpha_1, \ldots, \alpha_k$ are nonnegative real numbers, the function $w \mapsto \operatorname{mld}\big(w; D^k, \sum_{i=1}^{k} \alpha_i D^{k-i}\big)$ is lower-semicontinuous on closed points.*

Our work is by no means the first application of jet schemes to the calculation of invariants of determinantal varieties: Docampo [Doc13] uses jet schemes to compute the log canonical threshold of pairs $(\mathbb{A}^{m^2}, D^k)$, the irreducible components of the truncated jet schemes D^k_ℓ, and the topological zeta function of the D^k. Our application of jet schemes to the minimal log discrepancies of the determinantal varieties draws heavily from his methods there. Similarly, Johnson [Joh03] used explicit resolutions of singularities to calculate the multiplier ideals and log canonical thresholds of determinantal ideals in the ambient space $\mathbb{A}^{mn}$.

To calculate these minimal log discrepancies, we use the characterization of [EM09a] of minimal log discrepancies in terms of codimensions of various "multicontact" loci in the space of jets. To apply this characterization we need two main ingredients:

- Our computation of the Nash ideal of D^k (up to integral closure).
- Our calculation of the codimension of the $J_\infty(\mathrm{GL}_m \times \mathrm{GL}_m)$-orbits in the jet scheme $J_\infty(D^k)$.

The decomposition of the arc scheme $J_\infty(D^k)$ into orbits of the natural group action of $J_\infty(\mathrm{GL}_m \times \mathrm{GL}_m)$ is due to [Doc13], and our calculation of the codimension of these orbits in $J_\infty(D^k)$ is inspired by the methods of his chapter.

This chapter is organized as follows: We review some basic properties of determinantal rings in Section 4.2, as well as the straightening law on a determinantal ring. In Section 4.3 we describe the Nash ideal of a determinantal ring, and in Section 4.4 we actually compute minimal log discrepancies and prove the consequences noted above.

4.2 Determinantal rings

In this section we work over a field K of arbitrary characteristic. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates, and let $R := K[x_{ij}]$ be the polynomial ring on these indeterminates. For $k = 1, \dots, \min(m, n)$ we define the k-th determinantal ideal I_k to be the ideal generated by all $k \times k$ minors of (x_{ij}). We write $R_k = R/I_{k+1}$ for the corresponding quotient ring (note the difference in index here), so that R_k is the coordinate ring of the $m \times n$ matrices of rank $\leq k$; we write D^k for $\operatorname{Spec} R_k$. In what follows we will assume $k > 0$, since D^0 is just a point.

We record here some of the known properties of R_k:

- R_k has dimension $k(m+n-k)$, and thus I_{k+1} has codimension $mn-k(m+n-k)$.
- [HE71] R_k is normal. In fact, I_{k+1} is a prime ideal, and thus R_k is a Cohen–Macaulay domain.
- [BV88, Section 8] R_k is Gorenstein if and only if either $m = n$ or $k = \min(m, n)$ (note that in this last case $R_k = K$); R_k is $\mathbb{Q}$-Gorenstein if and only if it is Gorenstein.
- R_k is lci only when $k = 0$ or $k = \min(m, n)$: this follows easily by comparing the codimension of I_{k+1} and the number of $(k+1) \times (k+1)$ minors (which are homogeneous and thus by linear independence form a minimal generating set for I_{k+1}).
- The singular locus of $\operatorname{Spec} R_k$ is defined by I_k.

Since the (usual) notions of log discrepancies are specific to the $\mathbb{Q}$-Gorenstein case, after this section we will assume that $m = n$, i.e., we work with square matrices only.

4.2.1 The straightening law and an elementary consequence

We recall the straightening law on $R = K[x_{ij}]$ and $R_k = K[x_{ij}]/I_{k+1}$ from [CEP80], and then use it to prove an elementary proposition we will make use of later. This material will be used only for the calculation of the Nash ideal in Section 4.3.

Definition 4.2.1. A *Young diagram* σ corresponds to a nonincreasing sequence of integers $(\sigma_1, \dots, \sigma_t)$, and should be visualized as a set of left-justified rows of boxes of lengths $\sigma_1, \sigma_2, \dots$. We consider only Young diagrams with $\sigma_1 \leq m$. A *Young tableaux* T is a filling of a Young diagram σ with the integers $\{1, \dots, m\}$. We write $|T| = \sigma$ to indicate the underlying diagram has shape σ. The filling is *standard* if the filling is nondecreasing column-wise and strictly increasing row-wise. The *content* of a tableaux T is the function $\{1, \dots, m\} \to \mathbb{N}$ taking a number n to the number of times n appears in T. A *double tableaux* $(S|T)$ is a pair of Young tableaux with $|S| = |T|$; we say $(S|T)$ is standard if and only if S and T are both standard.

We partially order Young diagrams via the *dominance order*: $\sigma \leq \tau$ if and only if

$$\sum_{i=1}^{j} \sigma_i \leq \sum_{i=1}^{j} \tau_i$$

for all j.

We partially order Young tableaux as follows: given tableaux T, T' we say $T \leq T'$ when for any p, q the first p rows of T contain fewer integers $\leq q$ than the first p rows of T'. By [CEP80, Lemma 1.5], this refines the ordering on Young diagrams. We partially order the double tableaux by saying that $(S|T) \leq (S'|T')$ when $S \leq T$ and $S' \leq T'$.

To a double tableaux $(S|T)$ with the rows of S and T having no repeated entries, we can associate a monomial in the minors of (x_{ij}) as follows: for each row of S and T, say of length e, we view the entries in that row as the row and column indices specifying an $e \times e$ minor of (x_{ij}). We then multiply the resulting minor from each row to obtain a monomial in the minors, which we will write $x_{(S|T)}$ (this notation is nonstandard). When we write $x_{(S|T)}$, we will implicitly assume that S and T have no repeated entries in any row. We will refer to $x_{(S|T)}$ as a double tableaux, but note that the same monomial can arise from different double tableaux (i.e., any permutation of the rows gives the same monomial).

Example 4.2.2. Say $m = 3$. The double tableaux

$$(S|T) = \begin{array}{ccc} \boxed{3} & \boxed{1} & \boxed{2} \\ & \boxed{3} & \boxed{2} \\ & & \boxed{1} \end{array} \quad \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{2} & \\ \boxed{2} & & \end{array}$$

corresponds to the monomial

$$x_{(S|T)} = \det\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \cdot (x_{21}x_{32} - x_{22}x_{31}) \cdot x_{12}.$$

We will make use of the following *straightening law*; for context and a proof see [CEP80, Section 2]:

Theorem 4.2.3 (Straightening law)**.** *If $x_{(S|T)}$ is a double tableaux we can write*

$$x_{(S|T)} = \sum n_i x_{(S_i|T_i)}$$

with each $(S_i|T_i)$ *standard,* $n_i \in \mathbb{Z}$*,* $S_i \geq S$*,* $T_i \geq T$*, and with the content of each* $(S_i|T_i)$ *equal to that of* $(S|T)$*. Moreover, the double standard tableaux form a free* K*-basis for* $R = K[x_{ij}]$*.*

It is then a standard corollary (see, e.g., [Bae06, Proposition 1.0.2]) that R_k also has a straightening law, induced by the one on R. We will abuse notation and write $x_{(S|T)}$ for the image in R_k of the monomial $x_{(S|T)} \in R$; note that given a nonzero monomial $x_{(S|T)} \in R$, we have $x_{(S|T)} \neq 0$ in R_k exactly when no row of $|S| = |T|$ is of length $> k$. We say the image of $x_{(S|T)}$ in R_k is standard if $(S|T)$ is.

Corollary 4.2.4. *If* $x_{(S|T)}$ *is a nonzero double tableaux in* R_k *(so no row of* $|S| = |T|$ *has length* $> j$*) we can write*

$$x_{(S|T)} = \sum n_i x_{(S_i|T_i)}$$

with each $(S_i|T_i)$ *standard,* $n_i \in \mathbb{Z}$*,* $S_i \geq S$*,* $T_i \geq T$*, and with the content of each* $(S_i|T_i)$ *equal to that of* $(S|T)$*, and with no row of any* $|S_i| = |T_i|$ *of length* $> k$*. Moreover, the double standard tableaux with no row of length* $> k$ *form a free* K*-basis for* $R = K[x_{ij}]$*.*

We now establish an elementary consequence of the straightening law on R_k, which we will need for our calculation of the Nash ideal in Section 4.3. We write $S_k \subset R_k$ for the K-subalgebra generated by images of the $k \times k$ minors, and give S_k the grading induced by R_k (so S_k is generated in degree k). Let $\Delta \in S_k \subset R_k$ be the image of the $k \times k$ minor arising as the determinant of the first k rows and first k columns.

Proposition 4.2.5. *If* F *is a homogeneous element of* R_k *with* $\Delta \cdot F \in S_k$*, then* $F \in S_k$*.*

We'll set $G := \Delta \cdot F$. Since $G \in S_k$, we have that $k \mid \deg G$. Say $\deg G = k(d_0 + 1)$ for some d_0; note that $\deg F = kd_0$ then.

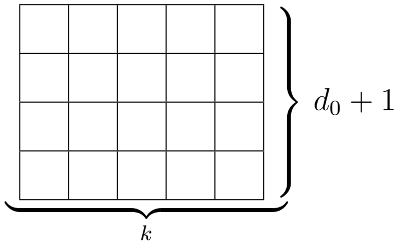
We prove the following lemma first:

Lemma 4.2.6. *Let* $G \in S_k$ *be of degree* $k(d_0 + 1)$*. If we expand* G *in the standard basis on* R_k*, say* $G = \sum \lambda_i x_{(S_i|T_i)}$*, then each* $(S_i|T_i)$ *has shape* $(k, \ldots, k)$ *(with* $d_0 + 1$ *entries).*

Proof. By assumption, $G \in S_k$ is a K-linear sum of monomials of shape

$$\underbrace{(k, k, \ldots, k)}_{d_0+1},$$

that is, corresponding to (double) Young diagrams of shape



It thus suffices to show the result for such monomials. The only issue is that they may not be *standard* monomials. If some monomial $x_{(S|T)}$ is not standard, we apply the straightening law (in R_k) to write

$$x_{(S|T)} = \sum \pm x_{(S_j|T_j)},$$

with $(S_j|T_j) \geq (S|T)$ having the same content (and thus the same degree). Let $\sigma = |S|$, $\sigma_j = |S_j|$. Note that for σ_j to dominate σ, it would have to have at least k entries in each row; however, if it had $k+1$ entries in any row it would be zero in R_k, and thus we must instead have $\sigma_j = \sigma$. ☐

Proof of Proposition 4.2.5. Expand F in the basis of standard monomials, say $F = \sum \lambda_i \, x_{(U_i|V_i)}$ with $\mu_i \in K$, $x_{(U_i|V_i)}$ standard of degree k with no row of any $|V_i|$ of length $> k$. The key observation is that each product of monomials

$$\Delta \cdot x_{(U_i|V_i)}$$

occurring in $\Delta \cdot F$ will again be standard. We take the standard-basis expansion of G, say $G = \sum \mu_i \, x_{(U_i|V_i)}$, as well, obtaining

$$\sum \lambda_i \, \Delta \cdot x_{(U_i|V_i)} = \Delta \cdot F = G = \sum \mu_i \, x_{(S_i|T_i)}.$$

Since by our preceding lemma the right side has all monomial terms of shape $|S_i| = (k, \ldots, k)$, the same must be true for the left side as well, i.e., each $\Delta \cdot x_{(U_i|V_i)}$ is of shape $(k, \ldots, k)$ (with $d_0 + 1$ entries). But this implies immediately that $x_{(U_i|V_i)}$ is of shape $(k, \ldots, k)$ (with d_0 entries) as well, and thus F is a degree-d_0 monomial in the $k \times k$ minors. ☐

4.2.2 $J_\infty(\mathrm{GL}_m \times \mathrm{GL}_m)$-orbits action on the jet spaces $J_\infty(D^k)$

For now, we specialize to the case where $\operatorname{char} K = 0$. We briefly recall here from [Doc13] the induced action of $\mathrm{GL}_m \times \mathrm{GL}_m$ on the jet spaces of determinantal varieties. Recall from Remark 2.2.7 that if an algebraic group G acts on a variety X, there is an induced action of the algebraic group $J_\ell(G)$ on $J_\ell(X)$, and likewise $J_\infty(G)$ on $J_\infty(X)$. One can think of jets on $\mathbb{A}^{m^2}$ as $m \times m$-matrices of power series, and jets on D^k as $m \times m$-matrices of power series whose $(k+1) \times (k+1)$ minors are zero, and the action of $J_\infty(G)$ on $J_\infty(D^k)$ is again by conjugation.

For the rest of the paper, we set $G := \mathrm{GL}_m \times \mathrm{GL}_m$. For each k, G acts on D^k by conjugation, so there is an induced action of $J_\infty(G)$ on $J_\infty(D^k)$ and $J_\ell(G)$ on $J_\ell(D^k)$ for all $\ell = 1, \dots, \infty$. We need one notion before we continue:

Definition 4.2.7. An extended partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of length m is a nonincreasing m-tuple of elements of $\mathbb{N} \cup \{\infty\}$.

The following gives an explicit description of the $J_\infty(G)$-orbits of D^m_∞, and of those which lie in $J_\infty(D^k)$:

Theorem 4.2.8 ([Doc13, Proposition 3.2])**.** *$J_\infty(G)$-orbits in D^m_∞ are in bijective correspondence with extended partitions of length m, under the correspondence sending $\lambda = (\lambda_1, \dots, \lambda_m)$ to the $J_\infty(G)$-orbit C_λ of the jet corresponding to the diagonal matrix*

$$\delta_\lambda := \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_m} \end{pmatrix}.$$

An orbit C_λ is contained in $J_\infty(D^k)$ if and only if $\lambda_1 = \cdots = \lambda_{m-k} = \infty$, and has finite codimension in $J_\infty(D^k)$ if and only if $\lambda_{m-k+1} < \infty$. More generally, $\operatorname{ord}_{\delta_\lambda}(I_k) = \lambda_{m-k+1} + \cdots + \lambda_m$.

Remark 4.2.9. For any $\ell \in \mathbb{N}$ and any extended partition $\lambda = (\lambda_1, \dots, \lambda_m)$ we write $\bar{\lambda}_\ell = (\bar{\lambda}_{1,\ell}, \dots, \bar{\lambda}_{m,\ell})$ for the partition defined by $\bar{\lambda}_{i,\ell} = \min(\ell, \lambda_i)$. We write $\delta_{\bar{\lambda},\ell}$ for the ℓ-jet corresponding to the matrix

$$\begin{pmatrix} t^{\bar{\lambda}_1} & & \\ & \ddots & \\ & & t^{\bar{\lambda}_m} \end{pmatrix}$$

and $C_{\bar{\lambda},\ell}$ for its orbit under the natural $J_\ell(\mathrm{GL}_m \times \mathrm{GL}_m)$-action. Note that compatibility of the truncation maps $\psi_{\infty,\ell}$ with the group action implies that $\psi_{\infty,\ell}(C_\lambda) = C_{\bar{\lambda},\ell}$.

4.3 The Nash ideal of a determinantal ring

For this section, there is no restriction on $\operatorname{char} K$. To apply Theorem 2.2.16 to the determinantal variety D^k we need to know $J(D^k)$, its Nash ideal; actually, by Lemma 2.2.17 it suffices to know $J(D^k)$ only up to integral closure. In this section, we show the following:

Theorem 4.3.1. *$J(D^k)$ has the same integral closure in R_k as I_k^{m-k}.*

In fact, we suspect that the equality $J(D^k) = I_k^{m-k}$ holds: we show below that $J(D^k) \subset I_k^{m-k}$, and the need to pass to integral closures would be avoided if one can show that this is an equality. It might be possible to prove this combinatorially by extending our approach below.

We begin by analyzing the relations on Ω_{D^k}:

Proposition 4.3.2. *If $\Delta = \Delta_{A,B}$ is a $(k+1) \times (k+1)$ minor, corresponding to a set A of $k+1$ rows and a set B of $k+1$ columns, then the image of Δ under the map*

$$d : k[x_{ij}] \to \Omega_{\mathbb{A}^{m^2}}$$

is

$$\sum_{(i,j)\in A\times B} \operatorname{sgn}(i,j) \cdot \Delta_{A\smallsetminus\{i\},B\smallsetminus\{j\}}\, dx_{ij},$$

where $\operatorname{sgn}(i,j)$ is 1 if the entry (i,j) lies on the first, third, etc. antidiagonal of the submatrix formed by the entries in the rows A and columns B, and is -1 if it lies on the second, fourth, etc. antidiagonal.

Proof. Without loss of generality we may assume $A = B = \{1, \dots, k+1\}$, so

$$\Delta = \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,k+1} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,k+1} \end{pmatrix}.$$

If we take the cofactor expansion along the top row, we get

$$\Delta = x_{1,1}\Delta_{[2,\dots,k+1|2,\dots,k+1]} - x_{1,2}\Delta_{[2,\dots,k+1|1,3,\dots,k+1} + \cdots + (-1)^{k+1}x_{1,k+1}\Delta_{[2,\dots,k+1|1,\dots,k]},$$

where we write $\Delta_{[i_1,\ldots,i_k|j_1,\ldots,j_k]}$ for the minor corresponding to rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$. Now, applying d, we see that we get

$$\begin{aligned} d\Delta = dx_{1,1} \cdot \Delta_{[2,\ldots,k+1|2,\ldots,k+1]} + \cdots + (-1)^{k+1} dx_{1,k+1} \cdot \Delta_{[2,\ldots,k+1|1,\ldots,k]}. \\ +x_{1,1} \cdot d\Delta_{[2,\ldots,k+1|2,\ldots,k+1]} - \cdots + (-1)^{k+1} x_{1,k+1} \cdot d\Delta_{[2,\ldots,k+1|1,\ldots,k]}. \end{aligned}$$

Note that none of the $k \times k$ minors appearing on the right side of the above formula involve $x_{1,1}$, so the only term where $dx_{1,1}$ can appear is in the term

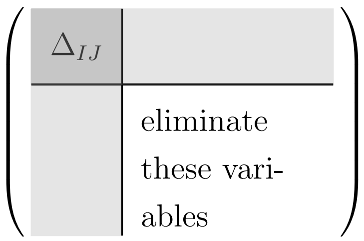
$$dx_{1,1} \cdot \Delta_{[2,\ldots,k+1|2,\ldots,k+1]}.$$

The same reasoning applies to the other $dx_{1,j}$, which then have coefficients

$$(-1)^{j+1} \Delta_{[2,\ldots,k+1|1,\ldots,j-1,j+1,\ldots,k+1]}.$$

Moreover, our choice of the top row to expand upon was arbitrary; repeating the same analysis for another row, we find the desired expression for the coefficients of the dx_{ij}. ☐

The smooth locus of D^k is covered by the open sets $D(\Delta_{IJ})$ where a $k \times k$ minor Δ_{IJ} does not vanish. In fact, as is well-known, if we invert Δ_{IJ}, we can use the cofactor expansion of a $(k+1) \times (k+1)$ minor involving Δ_{IJ} to eliminate the variables not occurring in the same row or column of Δ_{IJ}, obtaining that $D(\Delta_{IJ}) \cong \mathbb{A}^{k(2m-k)}$; thus certainly each $D(\Delta_{IJ})$ is contained in the smooth locus. Conversely, it is well-known that $D^k_{\text{sing}} = D^{k-1} = V(I_k)$ (see e.g., [BV88, Theorem 6.10]). We write $\mathcal{S}_{IJ}$ for the set $\{x_{ij} : i \in I \text{ or } j \in J\}$ of the $k(2m-k)$ variables occurring in the same row or column as Δ_{IJ}. The variables occurring in the gray region in the following diagram are exactly those contained in $\mathcal{S}_{IJ}$ (where the darker region denotes the minor Δ_{IJ} itself):



Thus, the variables in $\mathcal{S}_{IJ}$ give coordinates on $D(\Delta_{IJ}) \cong \mathbb{A}^{k(2m-k)}$, and thus on each $D(\Delta_{IJ})$ we have that

$$\Big(\bigwedge^{k(2m-k)} \Omega_{D^k}\Big)|_{D(\Delta_{IJ})} \cong \mathcal{O}_{D(\Delta_{IJ})} \cdot \Big\langle \bigwedge_{x_{ij} \in \mathcal{S}_{IJ}} dx_{ij} \Big\rangle.$$

(When we write the exterior product over some set of variables, if we do not specify we will implicitly mean that we consider the variables in lexicographic ordering on $\{1, \dots, m\} \times \{1, \dots, m\}$, i.e., from left to right over those appearing in the first row, then in the second, and so on.)

Thus, to give a $k(2m-k)$-form on the smooth locus of D^k (that is, a global canonical differential form), it suffices to define it on each $D(\Delta_{IJ})$ and demonstrate the compatibility of these definitions:

Proposition 4.3.3. *The rational $k(2m-k)$-form defined on $D(\Delta_{[1,\dots,k|1,\dots,k]})$ by*

$$\frac{1}{\Delta^{m-k}_{[1,\dots,k|1,\dots,k]}} \bigwedge_{x_{ij} \in \mathcal{S}_{[1,\dots,k|1,\dots,k]}} dx_{ij}$$

extends to a global canonical differential form $w \in H^0(D^k, \omega_{D^k}) = H^0(D^k, i_* \omega_{D^k_{\mathrm{sm}}})$, whose restriction to each $D(\Delta_{IJ})$ is

$$w|_{D(\Delta_{IJ})} = \pm \frac{1}{\Delta^{m-k}_{IJ}} \bigwedge_{x_{ij} \in \mathcal{S}_{IJ}} dx_{ij}.$$

Moreover, w generates ω_{D^k}.

The sign of the above expression for $w|_{D(\Delta_{IJ})}$ depends on the position of the columns and rows appearing in I and J relative to the entire matrix, but will be unimportant for our purposes.

Proof. It is clear that if w is indeed compatibly defined then it is a global generator of ω_{D^k}; this can be verified locally, and on each $D(\Delta_{IJ})$ it is immediate that w is a unit times a generator of $\omega|_{D(\Delta_{IJ})}$.

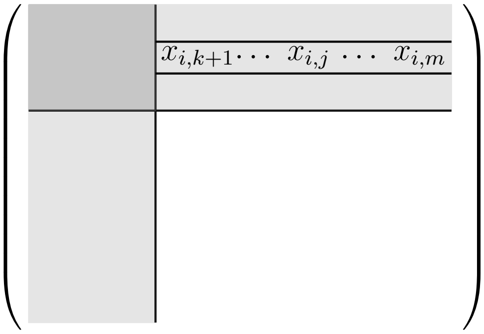
We thus just need to verify that the definitions on each $D(\Delta_{IJ})$ agree. Because D^k is irreducible, we may ignore the question of the sign: the rational $k(2m-k)$-form we defined on $D(\Delta_{[1,\dots,k|1,\dots,k]})$ will be defined on a dense open subset of each $D(\Delta_{IJ})$, and thus we just need to show that it extends to a regular $k(2m-k)$-form on $D(\Delta_{IJ})$ (which we will see will be of the form $\pm \frac{1}{\Delta^{m-k}_{IJ}} \bigwedge_{x_{ij} \in \mathcal{S}_{IJ}} dx_{ij}$). If it does, then this rational $k(2m-k)$-form defined on $D(\Delta_{[1,\dots,k|1,\dots,k]})$ extends to the entirety of each $D(\Delta_{IJ})$ and thus gives a regular $k(2m-k)$-form on D^k.

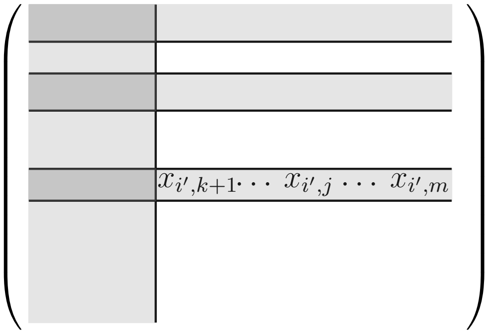
It suffices to show the definitions on $D(\Delta_{[1,\ldots,k|1,\ldots,k]})$ and $D(\Delta_{[1,\ldots,i-1,i+1,\ldots,k,i'|1,\ldots,k]})$ agree, i.e., that we can change one row; by symmetry we can then change one column as well, and by making one change at a time go from $D(\Delta_{[1,\ldots,k|1,\ldots,k]})$ to any $D(\Delta_{I',J'})$. So, fix $I = J = \{1, \ldots, k\}$ and $I' = \{1, \ldots, i-1, i+1, \ldots, k, i'\}$.

So, consider the rational $k(2m-k)$-forms

$$\bigwedge_{x_{ij} \in \mathcal{S}_{IJ}} dx_{ij} \quad \text{and} \quad \bigwedge_{x_{ij} \in \mathcal{S}_{I'J}} dx_{ij}.$$

The first involves the variables occurring in the shaded region on the left below, the second involves those occurring in the shaded region on the right (where the darker region in each denotes the minor Δ being localized at):





To go from $\bigwedge_{x_{ij} \in \mathcal{S}_{IJ}} dx_{ij}$ to $\bigwedge_{x_{ij} \in \mathcal{S}_{I'J}} dx_{ij}$ then, we need only replace the $m-k$ variables $x_{i,k+1}, \ldots, x_{i,m}$ by $x_{i',k+1}, \ldots, x_{i',m}$. For each $j = k+1, \ldots, m$, then, consider the $(k+1) \times (k+1)$ minor

$$\begin{pmatrix} x_{11} & \cdots & x_{1k} & x_{1j} \\ x_{21} & \cdots & x_{2k} & x_{2j} \\ \vdots & \ddots & \vdots & \vdots \\ x_{k1} & \cdots & x_{kk} & x_{kj} \\ x_{i'1} & \cdots & x_{i'k} & x_{i'j} \end{pmatrix}.$$

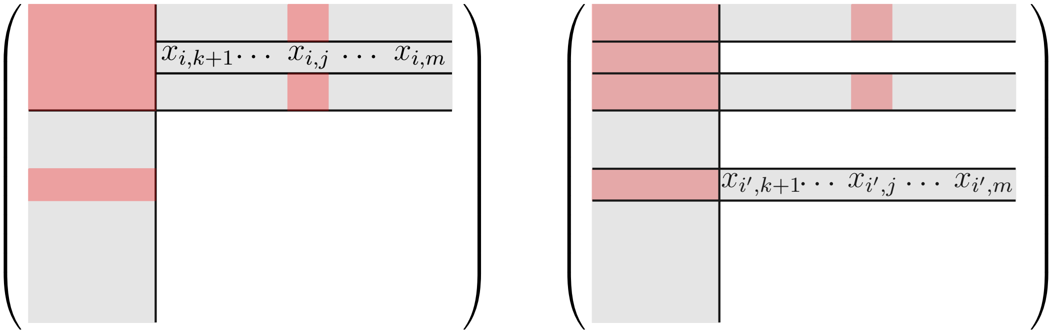
By Proposition 4.3.2, this yields the relation

$$\Delta_{[2,\ldots,k,i'|2,\ldots,k,j]}\, dx_{11} - \cdots + \Delta_{[1,\ldots,k|1,\ldots,k]}\, dx_{i'j} = 0 \tag{4.1}$$

on $\Omega^1_{D^k}$. Now, we take the exterior product of this relation with the $((k+1)^2-2)$-form

$$\Lambda_j := \bigwedge_{(p,q) \in \{1,\ldots,k,i'\} \times \{1,\ldots,k,j\} \smallsetminus \{(i,j),(i',j)\}} dx_{pq},$$

i.e., the product over all the indices appearing in the minor *except* dx_{ij} and $dx_{i'j}$. We have highlighted in darker gray below the variables in Λ_j, in relation to each of the shaded regions in question:



The only terms surviving on the left side of relation (4.1) then are then the wedge product with these missing indices, so we have that

$$\Lambda_j \wedge \left((-1)^{i+j}\Delta_{[1,\ldots,i-1,i+1,\ldots,k,i'|1,\ldots,k]}dx_{ij} + \Delta_{[1,\ldots,k|1,\ldots,k]}dx_{i'j}\right) = 0,$$

or equivalently

$$(-1)^{i+j+1}\underbrace{\Delta_{[1,\ldots,i-1,i+1,\ldots,k,i'|1,\ldots,k]}}_{\Delta_{I'J}}\cdot\Lambda_j \wedge dx_{ij} = \underbrace{\Delta_{[1,\ldots,k|1,\ldots,k]}}_{\Delta_{IJ}}\cdot\Lambda_j \wedge dx_{i'j}. \tag{4.2}$$

Note that the minors $\Delta_{I'J} = \Delta_{[1,\ldots,i-1,i+1,\ldots,k,i'|1,\ldots,k]}$ and $\Delta_{IJ} = \Delta_{[1,\ldots,k|1,\ldots,k]}$ appearing on each side are independent of the column j under consideration. We have switched one x_{ij} for $x_{i'j}$.

Since any Λ_j appears as a wedge factor of each of $\bigwedge_{x_{pq}\in\mathcal{S}_{IJ}} dx_{pq}$ and $\bigwedge_{x_{pq}\in\mathcal{S}_{I'J}} dx_{pq}$, we can use the above relation for each $j = m-k+1,\ldots,m$ to obtain

$$\frac{1}{\Delta_{IJ}^{m-k}}\bigwedge_{x_{pq}\in\mathcal{S}_{IJ}} dx_{pq} = \pm\frac{1}{\Delta_{I'J}^{m-k}}\bigwedge_{x_{pq}\in\mathcal{S}_{I'J}} dx_{pq}$$

(where the sign is determined by the $(m-k)$-fold product of $(-1)^{m+i}$ and the repeated use of skew-commutativity), giving the result. $\square$

We now prove Theorem 4.3.1 above, which states that the Nash ideal $J(D^k)$ and I_k^{m-k} have the same integral closure. The proof will occupy the rest of this section.

Proof. We have just seen that $\omega_{D^k} \cong \mathcal{O}_{D^k}\langle w\rangle$, with w the $k(2m-k)$-form we defined in Proposition 4.3.3. Since $\bigwedge^{k(2m-k)}\Omega_{D^k}$ is generated by the restriction of $k(2m-k)$-forms from $\mathbb{A}^{m^2}$, it suffices to consider how these forms restrict to D^k.

Lemma 4.3.4. $\{\Delta^{m-k} : \Delta \in I_k\} \subset J(D^k)$.

Proof. For any $k \times k$ minor $\Delta = \Delta_{IJ}$, consider the $k(2m-k)$-form $\rho := \bigwedge_{x_{ij} \in \mathcal{S}_{IJ}} dx_{ij}$. By definition, on $D(X_{IJ})$ we have $\rho = \Delta_{IJ}^{m-k} \cdot w$. Thus, we deduce that

$$\Delta_{IJ}^{m-k} \in J(D^k),$$

giving the lemma. □

Recalling that for arbitrary elements f_i of any ring R, $(f_1^d, \dots, f_m^d)$ and $(f_1, \dots f_m)^d$ have the same integral closure, we obtain:

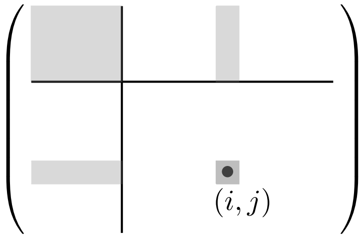
Corollary 4.3.5. *The integral closure of I_k^{m-k} is contained in the integral closure of $J(D^k)$.*

Now, we need the reverse inclusion, for which it suffices to show that $J(D^k)$ is contained in I_k^{m-k}.

Proposition 4.3.6. *Let $\partial = \bigwedge_{x_{ij} \in I, |I| = k(2m-k)} dx_{ij}$. Then the image of ∂ in ω_{D^k} is $F \cdot w$ for some $F \in I_k^{m-k}$; in fact, F is a degree-$(m-k)$ polynomial in the $k \times k$ minors.*

Proof. We think of the given set I as corresponding to a filling of the $m \times m$-matrix by $k(2m-k)$ entries. We want to use the relations of Corollary 4.3.2 to move the filled entries to those corresponding to some $\mathcal{S}_{IJ}$. For convenience's sake, we choose $I = J = \{1, \dots, k\}$; we write $\Delta = \Delta_{[1,\dots,k|1,\dots,k]}$. Let $(i,j) \in I$ be a "filled" entry with i, j both $\geq k+1$. That is, (i,j) lies in the "bad" region.

Consider the $(k+1) \times (k+1)$ minor formed by the first k rows and columns and the i-th row and j-th column; in the following diagram this minor is marked in gray:



All entries of this minor except the (i,j)-th entry lie in the "good" region corresponding to $\mathcal{S}_{IJ}$. The relation from Proposition 4.3.2 corresponding to this minor can be written as

$$\Delta_{[1,\ldots,k|1,\ldots,k]} \cdot dx_{ij} = - \sum_{(p,q)\neq(i,j)} (-1)^{p+q} \underbrace{\Delta_{[1,\ldots,p-1,p+1,\ldots,k,i|1,\ldots,q-1,q+1,\ldots,k,j]}}_{\Delta_{pq}} \cdot dx_{pq}.$$

The entries (p, q) appearing on the right side are all "good", so we can localize at $\Delta_{[1,\ldots,k|1,\ldots,k]}$ and use this equation to eliminate the "bad" entry dx_{ij} in the $k(2m-k)$-form ∂ in favor of good entries (and this creates no new "bad" entries). Note that the coefficients we pick up are all of the form Δ_{KL}/Δ.

The goal now is to show that F lies in I_k^{m-k}; in fact, we will show the stronger claim that it is a degree-$(m-k)$ polynomial in the $k \times k$ minors. We induce on the number of "bad" entries as follows: Note that when we eliminate dx_{ij} from the $k(2m-k)$-form ∂, we express ∂ as a linear combination (with coefficients of the form Δ_i/Δ) of $k(2m-k)$-forms ∂_i with fewer "bad" entries. When we rewrite each of *these* $k(2m-k)$-forms ∂_i as an element F_i times w, by induction we get

$$\partial_i = F_i \omega$$

for F_i a degree-$(m-k)$ polynomial in the $k \times k$ minors (and thus in I_k^{m-k}). Thus, we have

$$\Delta_{[1,\ldots,k|1,\ldots,k]} \cdot F = \sum \Delta_i F_i,$$

or, collecting the terms on the right-hand side,

$$\Delta_{[1,\ldots,k|1,\ldots,k]} \cdot F = G(\{\Delta_{pq}\}),$$

where $G(\{\Delta_{pq}\})$ is a degree-$(m-k+1)$ polynomial in the $k \times k$-minors (and thus in $S_k \subset R_k$).

This equality implies that F is homogeneous of degree $(m-k)k$; since $G(\{\Delta_{pq}\})$ is a degree-$(m-k+1)$ polynomial in the Δ_{IJ}, we can simply apply Proposition 4.2.5 to conclude that $F \in S_k$ (i.e., F is a degree-$(m-k)$ polynomial in the Δ_{IJ}), and thus $F \in I_k^{m-k}$. □

Having just shown that $J(D^k) \subset I_{m-k}^k$, we have that $J(D^k)$ and I_{m-k}^k have the same integral closure, concluding the proof of Theorem 4.3.1. □

4.4 Computing minimal log discrepancies

For the remainder of the chapter we work over a field of characteristic 0. Our aim is to compute minimal log discrepancies on determinantal varieties via the formula of Theorem 2.2.16. Specifically, we consider the case of a pair $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$, with $\alpha_i \in \mathbb{R}$ (possibly 0); our goal is to compute

$$\operatorname{mld}(w; D^k, \sum \alpha_i D^{k-i})$$

for w a closed point of D^k; by the same process, we also will compute

$$\operatorname{mld}(D^{k-j}; D^k, \sum \alpha_i D^{k-i})$$

for any j.

Via the $\mathrm{GL}_m \times \mathrm{GL}_m$-action on D^k we may assume that w is the point

$$x_q := \begin{array}{c} \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix} \\ \underbrace{\qquad\qquad}_{m-q} \; \underbrace{\qquad\qquad}_{q} \end{array}$$

for some $0 \leq q \leq k$.

Note that the multicontact loci

$$\operatorname{Cont}^i(J(D^k)) \cap \operatorname{Cont}^{w_1}(D^{k-1}) \cap \cdots \cap \operatorname{Cont}^{w_k}(D^0)$$

are $J_\infty(\mathrm{GL}_m \times \mathrm{GL}_m)$-invariant, so they are disjoint unions of $J_\infty(\mathrm{GL}_m \times \mathrm{GL}_m)$-orbits, say $\bigsqcup C_\lambda$. Thus, we have that the multicontact loci

$$\operatorname{Cont}^i(J(D^k)) \cap \operatorname{Cont}^{w_1}(D^{k-1}) \cap \cdots \cap \operatorname{Cont}^{w_k}(D^0) \cap \operatorname{Cont}^{\geq 1}(x_q)$$

appearing in the calculation of $\operatorname{mld}(x_q; X, Y)$ via Theorem 2.2.16 will decompose as

$$\bigsqcup (C_\lambda \cap \operatorname{Cont}^{\geq 1}(x_q)).$$

(Note that $\mathrm{Cont}^{\geq 1}(x_q)$ is *not* $J_\infty(\mathrm{GL}_m \times \mathrm{GL}_m)$-invariant, since x_q is not $\mathrm{GL}_m \times \mathrm{GL}_m$-invariant.)

We now need to do the following:

- Analyze which of the $C_\lambda \cap \mathrm{Cont}^{\geq 1}(x_q)$ appear in a given multicontact locus.
- Calculate the codimension of $C_\lambda \cap \mathrm{Cont}^{\geq 1}(x_q)$ in $J_\infty(D^k)$.

To answer the former, we have the following:

Proposition 4.4.1. *Fix $q \leq k$ and let $\lambda = (\lambda_1, \ldots, \lambda_m)$.*

(1) $C_\lambda \subset J_\infty(D^k)$ if and only if $\lambda_1 = \cdots = \lambda_{m-k} = \infty$.

(2) The codimension of C_λ in $J_\infty(D^k)$ is finite if and only if $\lambda_{m-k+1} < \infty$.

(3) $C_\lambda \cap \mathrm{Cont}^{\geq 1}(x_q) \neq \emptyset$ if and only if $\lambda_1, \ldots, \lambda_{m-q} > 0$ and $\lambda_{m-q+1} = \cdots = \lambda_m = 0$.

(4) $C_\lambda \subset \mathrm{Cont}^{w_i}(D^{k-i})$ if and only if $\lambda_{m-k-i+1} + \cdots + \lambda_m = w_i$.

(5) $C_\lambda \subset \mathrm{Cont}^i(J(D^k))$ if and only if $\lambda_{m-k+1} + \cdots + \lambda_m = i/(m-k)$.

Note that (5) implies in particular that $\mathrm{Cont}^i(J(D^k))$ is empty if $m-k$ does not divide i.

Proof. (1), (2), and (4) are just Propositions 3.2, 3.4, and 3.3 of [Doc13], respectively.

(3) follows by noting that the matrix

$$\delta_\lambda := \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_m} \end{pmatrix}$$

(which generates the $J_\infty(\mathrm{GL}_m \times \mathrm{GL}_m)$-orbit C_λ) is mapped to x_q under the map induced by the truncation $k[[t]] \to k$ if and only if the first $m-q$ entries are positive powers of t and the rest are $1 = t^0$.

Finally, to see (5), note that by Lemma 2.2.17 and Theorem 4.3.1 we have

$$\mathrm{Cont}^i(J(D^k)) = \mathrm{Cont}^i(I_k^{m-k});$$

since $\mathrm{ord}_\gamma(I_k^{m-k}) = (m-k)\,\mathrm{ord}_\gamma(I_k)$, we have immediately that $\mathrm{Cont}^i(I_k^{m-k})$ is empty if $m-k$ does not divide i, and is $\mathrm{Cont}^{i/(m-k)}(I_k)$ when it does; we can then apply part (4) to obtain the desired conclusion. ☐

Proposition 4.4.2. *(1) If the conditions in statements (1)–(2) of Proposition 4.4.1 hold (so that C_λ is in D^k_∞ and has finite codimension), then the codimension of C_λ in D^k_∞ is*

$$(2(m-k+1)-1)\lambda_{m-k+1}+\cdots+(2m-1)\lambda_m.$$

(2) If the conditions in statements (1)–(3) of Proposition 4.4.1 hold (so that $C_\lambda \cap \mathrm{Cont}^{\geq 1}(x_q)$ is in D^k_∞, nonempty, and has finite codimension), then the codimension of $C_\lambda \cap \mathrm{Cont}^{\geq 1}(x_q)$ in D^k_∞ is

$$q(2m-q)+(2(m-k+1)-1)\lambda_{m-k+1}+\cdots+(2m-1)\lambda_m.$$

Remark 4.4.3. Note that since $\lambda_{m-q+1}=\cdots=\lambda_m=0$ in part (2) of the theorem, we can just as well write the codimension of $C_\lambda \cap \mathrm{Cont}^{\geq 1}(x_q)$ in D^k_∞ as

$$q(2m-q)+(2(m-k+1)-1)\lambda_{m-k+1}+\cdots+(2(m-q)-1)\lambda_{m-q}.$$

In what follows, we will write G for $\mathrm{GL}_m \times \mathrm{GL}_m$ to lighten notation. Our proof of the proposition is exactly parallel to the proof of Proposition 5.3 of [Doc13].

Proof of Proposition 4.4.2. First, note that it suffices to prove (1), at which point (2) follows immediately: the $J_\infty(G)$-action on $J_\infty(D^k)$ and the G-action on D^k are compatible with the truncation morphisms $\psi_{\infty,\ell}$ and $\psi_{\ell,0}$, so we have a commutative diagram

$$\begin{tikzcd}
J_\infty(G) \times J_\infty(D^k) \arrow[r] \arrow[d] & J_\infty(D^k) \arrow[d] \\
G \times D^k \arrow[r] & D^k
\end{tikzcd}$$

Thus, we have that δ_ℓ lies over x_q if and only if $C_\lambda = J_\infty(G)\cdot\delta_\ell$ lies over $G\cdot x_q$, and the fibers $C_\lambda \to g\cdot x_q$ are constant for $g \in G$. But note that $G\cdot x_q$ is the matrices of rank exactly q, and thus $\dim(G\cdot x_q)=q(2m-q)$. Thus, if the codimension of C_λ is c, say, then we must have that $\mathrm{codim}(C_\lambda\cap\mathrm{Cont}^{\geq 1}) = \mathrm{codim}(C_\lambda)+q(2m-q)$, so that the formula in (1) implies (2).

By Proposition 2.2.18, it suffices to calculate $(\ell+1)\cdot\dim X - \dim(\psi_{\infty,\ell}(C_\lambda))$ for $\ell \gg 0$. As noted in Remark 4.2.9, the image of C_λ under $\psi_{\infty,\ell}$ is exactly $C_{\bar\lambda,\ell}$, where $(\bar\lambda)_i = \min(\lambda_i,\ell)$. We thus are led to calculating the dimensions of $C_{\bar\lambda,\ell}$ for $\ell \gg 0$. Choose $\ell > \lambda_{m-k+1}$ (by assumption $\lambda_{m-k+1} < \infty$). To know $\dim C_{\bar\lambda,\ell}$ it suffices to know the codimension of the stabilizer of $\delta_{\bar\lambda,\ell}$ in G_ℓ.

Consider the condition of an element

$$\left(\left(g_{ij} = \sum_{n=0}^{\ell} g_{ij}^n t^n\right)_{i,j}, \left(h_{ij} = \sum_{n=0}^{\ell} h_{ij}^n t^n\right)_{i,j}\right)$$

of G_ℓ stabilizing $\delta_{\bar{\lambda},\ell}$, which is the equality of matrices

$$\begin{pmatrix} 0 & \cdots & 0 & t^{\lambda_{m-k+1}} g_{1,m-k+1} & \cdots & t^{\lambda_m} g_{1,m} \\ 0 & \cdots & 0 & t^{\lambda_{m-k+1}} g_{2,m-k+1} & \cdots & t^{\lambda_m} g_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t^{\lambda_{m-k+1}} g_{m,m-k+1} & \cdots & t^{\lambda_m} g_{m,m} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ t^{\lambda_{m-k+1}} h_{m-k+1,1} & t^{\lambda_{m-k+1}} h_{m-k+1,2} & \cdots & t^{\lambda_{m-k+1}} h_{m-k+1,m} \\ \vdots & \ddots & \ddots & \vdots \\ t^{\lambda_m} h_{m,1} & t^{\lambda_m} h_{m,2} & \cdots & t^{\lambda_m} h_{m,m} \end{pmatrix}.$$

For $\max(i,j) < m-k+1$, equality of the (i,j)-th entries is trivial, since both entries are just 0. If $i < m-k+1$ but $j \geq m-k+1$, equality of the (i,j)-th entries gives the equation

$$t^{\lambda_j} g_{i,j} = 0,$$

i.e., that

$$t^{\lambda_j} g_{i,j}^0 + t^{\lambda_j+1} g_{i,j}^1 + \cdots + t^\ell g_{i,j}^{\ell-\lambda_j} = 0.$$

This gives $\ell - \lambda_j + 1$ equations $g_{i,j}^n = 0$ for $n = 0, \ldots, \ell - \lambda_j$. Likewise, if $j < m-k+1$ but $i \geq m-k+1$ we get $\ell - \lambda_i + 1$ equations $h_{i,j}^n = 0$ for $n = 0, \ldots, \ell - \lambda_i$.

For $\min(i,j) \geq m-k+1$, equality of the (i,j)-th entries gives the equation

$$t^{\lambda_j} g_{i,j} = t^{\lambda_i} h_{i,j}.$$

Say $i \leq j$, so $\lambda_i \geq \lambda_j$. Writing out the condition above, we have

$$t^{\lambda_j} g_{i,j}^0 + t^{\lambda_j+1} g_{i,j}^1 + \cdots + t^\ell g_{i,j}^{\ell-\lambda_j} = 0 + \cdots + 0 + t^{\lambda_i} h_{i,j}^0 + t^{\lambda_j+i} h_{i,j}^1 + \cdots + t^\ell h_{i,j}^{\ell-\lambda_i}.$$

This gives $\ell - \lambda_j + 1$ equations

(1) $g_{i,j}^n = 0$ for $n = 0, \ldots, \lambda_i - \lambda_j$.

(2) $g_{i,j}^n = h_{i,j}^{n-\lambda_i+\lambda_j}$ for $n = \lambda_i - \lambda_j + 1, \ldots, \ell - \lambda_j$.

For each of the $2k(m-k)+k^2$ indices (i,j) with $\max(i,j) \geq m-k+1$, we thus obtain

$$\ell + 1 - \min(\lambda_i, \lambda_j)$$

independent linear conditions. To see how many entries contribute a given $\ell + 1 - \lambda_i$ linear conditions, consider the filling of the matrix where the (i,j)-th entry with $\max(i,j) \geq m-k+1$ is filled with $\min(\lambda_i, \lambda_j)$:

$$\begin{pmatrix} & & & & \lambda_{m-k+1} & \lambda_{m-k+2} & \cdots & \lambda_m \\ & & & & \lambda_{m-k+1} & \lambda_{m-k+2} & \cdots & \lambda_m \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & \lambda_{m-k+1} & \lambda_{m-k+2} & \cdots & \lambda_m \\ \lambda_{m-k+1} & \lambda_{m-k+1} & \cdots & \lambda_{m-k+1} & \lambda_{m-k+1} & \lambda_{m-k+2} & \cdots & \lambda_m \\ \lambda_{m-k+2} & \lambda_{m-k+2} & \cdots & \lambda_{m-k+2} & \lambda_{m-k+2} & \lambda_{m-k+2} & \cdots & \lambda_m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_m & \lambda_m & \cdots & \lambda_m & \lambda_m & \lambda_m & \cdots & \lambda_m \end{pmatrix}.$$

We see that there are $2(m-k+1)-1$ entries with λ_{m-k+1}, $2(m-k+2)-1$ entries with λ_{m-k+1}, and so on, up to $2m-1$ entries with λ_m. This implies that the codimension of the stabilizer in G_ℓ is

$$(\ell+1)(2k(m-k)+k^2) - \big((2(m-k+1)-1)\lambda_{m-k+1} + \cdots + (2m-1)\lambda_m\big),$$

which is thus the dimension of $C_{\bar{\lambda},\ell}$.

Finally, this says that the codimension of C_λ in $J_\infty(D^k)$ is

$$k(2m-k)(\ell+1) - \\ \big((2k(m-k)+k^2)m^2(\ell+1) - (2(m-k+1)-1)\lambda_{m-k+1} + \cdots + (2m-1)\lambda_m\big),$$

or

$$(2(m-k+1)-1)\lambda_{m-k+1} + \cdots + (2m-1)\lambda_m,$$

giving the theorem. □

Theorem 4.4.4. *Consider the pair* $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ *(where the* α_i *may be zero).*

(1) $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ *is log canonical at a matrix* x_q *of rank* $q \leq k$ *exactly when*

$$\alpha_1 + \cdots + \alpha_j \leq m - k + (2j-1)$$

for all $j = 1, \ldots, k-q$.

(2) In this case,

$$\operatorname{mld}\left(x_q; D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right) = q(m-k) + km - \sum_{i=1}^{k-q} (k-q-i+1)\,\alpha_i.$$

(3) $\left(D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right)$ *is log canonical along* D^{k-j} *(for* $j > 0$*) exactly when*

$$\alpha_1 + \cdots + \alpha_j \leq m - k + (2j-1)$$

for all $j = 1, \ldots, k$.

(4) In this case,

$$\operatorname{mld}\left(D^{k-j}; D^k, \sum_{i=1}^k \alpha_i D^{k-i}\right) = j(m-k+j) - \sum_{i=1}^{j} (j-i+1)\alpha_i$$

Before proving the theorem, we mention a few corollaries:

Corollary 4.4.5 (Semicontinuity)**.** *If* $\alpha_1, \ldots, \alpha_k$ *are nonnegative real numbers, the function* $w \mapsto \operatorname{mld}\big(w; D^k, \sum_{i=1}^k \alpha_i D^{k-i}\big)$ *is lower-semicontinuous on closed points.*

Proof. The quantity

$$\operatorname{mld}\left(w; D^k, \sum \alpha_i D^{k-i}\right)$$

is constant on each locus of rank-q matrices, so we only need to check that it decreases when we go from q to $q-1$. Note that part (1) of the theorem guarantees that if $\operatorname{mld}\big(x_q; D^k, \sum \alpha_i D^{k-i}\big)$ is $-\infty$ then the same is true of $\operatorname{mld}\big(x_{q-1}; D^k, \sum \alpha_i D^{k-i}\big)$, so we may assume that both $\operatorname{mld}\big(x_q; D^k, \sum \alpha_i D^{k-i}\big)$ and $\operatorname{mld}\big(x_{q-1}; D^k, \sum \alpha_i D^{k-i}\big)$ are nonnegative, and thus we may apply the formula in part (2) of the theorem.

This formula implies that

$$\operatorname{mld}\left(x_q; D^k, \sum \alpha_i D^{k-i}\right) - \operatorname{mld}\left(x_{q-1}; D^k, \sum \alpha_i D^{k-i}\right) = (m-k) + \alpha_1 + \cdots + \alpha_{k-q+1} > 0,$$

yielding the result. ∎

Corollary 4.4.6. *Determinantal varieties (of square matrices) have terminal singularities.*

This follows easily from the fact determinantal varieties have a small resolution (see, e.g., [Har92, Example 16.18]), but this gives a proof avoiding the use of an explicit resolution. It also gives explicitly the log discrepancy along the singular locus.

Proof. We consider just the singularities of D^k, i.e., all α_i are 0. Since $D^m \cong \mathbb{A}^{m^2}$, we may assume $k < m$. Recall from Definition 2.1.26 it suffices to show that $\operatorname{mld}(D^{k-1}, D^k) > 1$. By part (3) of Theorem 4.4.4, this is $m - k + 1$, and this is > 1 except in the excluded case $k = m$. In particular, determinantal varieties of square matrices have terminal singularities. ☐

Now, we prove the theorem itself:

Proof of Theorem 4.4.4. We begin by proving parts (1) and (2): By Proposition 4.4.1, we can decompose the multicontact loci

$$\mathcal{C}_{n,w_1,\ldots,w_k} := \operatorname{Cont}^n(J(D^k)) \cap \operatorname{Cont}^{w_1}(D^{k-1}) \cap \cdots \cap \operatorname{Cont}^{w_k}(D^0) \cap \operatorname{Cont}^{\geq 1}(x_q)$$

as the disjoint union of

$$C_\lambda \cap \operatorname{Cont}^{\geq 1}(x_q),$$

with $\lambda = (\lambda_1, \ldots, \lambda_m)$ ranging over all m-tuples satisfying:

- $\lambda_1 = \cdots = \lambda_{m-k} = \infty$.
- $\lambda_{m-k+1} < \infty$.
- $\lambda_{m-q} > 0$ (and thus $\lambda_{m-k+1}, \ldots, \lambda_{m-q}$ are all > 0) and $\lambda_{m-q+1} = \cdots = \lambda_m = 0$.

Again by Proposition 4.4.1, it's immediate that a cylinder $C_\lambda \cap \operatorname{Cont}^{\geq 1}(x_q)$ will lie in

$$\operatorname{Cont}^{\lambda_{m-k}+\cdots+\lambda_{m-q}}(I_k) = \operatorname{Cont}^{(m-k)(\lambda_{m-k}+\cdots+\lambda_{m-q})}(J(D^k))$$

and in

$$\operatorname{Cont}^{\lambda_{m-k-j+1}+\cdots+\lambda_{m-q}}(D^{k-j})$$

for each i.

Equivalently, a given cylinder $C_\lambda \cap \operatorname{Cont}^{\geq 1}(x_q)$ is contained in $\mathcal{C}_{n,w_1,\ldots,w_k}$ for

$$n = (m - k)(\lambda_{m-k+1} + \cdots + \lambda_{m-q})$$

and

$$w_i = \lambda_{m-k-i+1} + \cdots + \lambda_{m-q}.$$

Finally, by part (2) of Proposition 4.4.2, we know that

$$\operatorname{codim}(C_\lambda \cap x_q) = q(2m-q) + (2(m-k+1)-1)\lambda_{m-k+1} + \cdots + (2(m-q)-1)\lambda_{m-q}.$$

The infimum in Theorem 2.2.16 can then be rewritten as

$$\begin{aligned} &q(2m-q) + (2(m-k+1)-1)\lambda_{m-k+1} \\ &\qquad + \cdots + (2(m-q)-1)\lambda_{m-q} - (m-k)(\lambda_{m-k+1} + \cdots + \lambda_{m-q}) \\ &\qquad -\alpha_1(\lambda_{m-k+1} + \cdots + \lambda_{m-q}) - \alpha_2(\lambda_{m-k+2} + \cdots + \lambda_{m-q}) - \cdots - \alpha_{k-q}(\lambda_{m-q}) \end{aligned}$$

over $\lambda_{m-k+1}, \ldots, \lambda_{m-q} > 0$.

Grouping terms by the λ_i, we can rewrite this quantity as

$$\begin{aligned} &q(2m-q) + \lambda_{m-k+1}(m-k+1-\alpha_1) + \lambda_{m-k+2}(m-k+3-(\alpha_1+\alpha_2)) \\ &\qquad + \cdots + \lambda_{m-q}(m-k+(2(k-q)-1)-(\alpha_1+\cdots+\alpha_{k-q})). \end{aligned}$$

Now, set

$$\begin{aligned} \beta_1 &= m-k+1-\alpha_1, \\ &\vdots \\ \beta_{k-q} &= m-k+(2(k-q)-1)-(\alpha_1+\cdots+\alpha_{k-q}), \end{aligned}$$

so β_i is the coefficient of λ_{m-k+i} in the above quantity. It is clear that if any β_i is negative then simply by taking $\lambda_{m-k+i} \gg 0$ we can make the quantity in question arbitrarily negative, and thus $(D^k, \sum \alpha_i D^{k-i})$ will not be log canonical, proving part (1) of the theorem.

If all β_i are nonnegative, then it is clear that the quantity

$$q(2m-q) + \lambda_{m-k+1}\beta_1 + \cdots + \lambda_{m-q}\beta_{k-q}$$

is minimized by taking $\lambda_{m-k+1} = \cdots = \lambda_{m-q} = 1$. Taking these values and simplifying, we see that the minimum value is

$$q(m-k) + km - \alpha_1(k-q) - \alpha_2(k-q-1) - \cdots - 2\alpha_{k-q-1} - \alpha_{k-q},$$

giving the claim in (2).

The proof of (3) and (4) follows in exactly the same fashion, except that one imposes the condition that $\lambda_{m-k+1}, \ldots, \lambda_{m-k+j} > 0$ instead of the conditions that $\lambda_{m-k+1}, \ldots, \lambda_{m-q} > 0$ and $\lambda_{m-q+1} = \cdots = \lambda_m = 0$, and uses the formula from part (1) of Proposition 4.4.2 instead of part (2). $\square$

CHAPTER V

Bigness of the Tangent Bundle of Del Pezzo Surfaces and D-Simplicity

We consider the question of simplicity of a ring R under the action of its ring of differential operators D_R. We give examples to show that even when R is Gorenstein and has rational singularities R need not be a simple D_R-module; for example, this is the case when R is the homogeneous coordinate ring of a smooth cubic surface. Our examples are homogeneous coordinate rings of smooth Fano varieties, and our proof proceeds by showing that the tangent bundle of such a variety need not be big. We also give a partial converse showing that when R is the homogeneous coordinate ring of a smooth projective variety X, embedded by some multiple of its canonical divisor, then simplicity of R as a D_R-module implies that X is Fano and thus R has rational singularities.

5.1 Introduction

Given a k-algebra R, let $D_{R/k}$ be the ring of k-linear differential operators on R. When $R = k[x_1, \ldots, x_n]$ (or when R is a smooth k-algebra), $D_{R/k}$ is well-studied and has several nice properties; however, when R is not a smooth k-algebra, $D_{R/k}$ is quite mysterious. For example, [BGG72] showed that if $R = \mathbb{C}[x,y,z]/(x^3 + y^3 + z^3)$, then $D_{R/\mathbb{C}}$ is not finitely generated over $\mathbb{C}$, not left- or right-Noetherian, and that R is not a simple $D_{R/\mathbb{C}}$-module.

We consider the following questions:

(1) [LS89, Question 0.13.1]: If $\operatorname{Spec} R$ has rational singularities, when is $D_{R/k}$ simple?

(2) [LS89, Question 0.13.3]: When is R a simple $D_{R/k}$-module?

In the setting of finite-type $\mathbb{C}$-algebras, [Hsi15, Question 5.1] asked whether in (2) above it is sufficient for R to have Gorenstein rational singularities. This criteria was motivated in part by [Smi95, Theorem 2.2], which showed that in characteristic p an F-pure ring R is a simple D_R-module if and only if R is strongly F-regular; thus, one might expect a "mildly" singular ring R in characteristic 0 to be a simple D_R-module.

In this chapter, we give a negative answer to Hsiao's question, illustrating the differing behavior of differential operators in characteristic p and characteristic 0.

Theorem 5.1.1. *There are Gorenstein (graded) $\mathbb{C}$-algebras R with rational singularities such that $D_{R/k}$ contains no differential operators of negative degree, and thus such that R is not a simple $D_{R/k}$-module and $D_{R/k}$ is not simple. One example is $R = \mathbb{C}[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$.*

We also show a partial converse: the necessity of the klt condition for D-simplicity in the special case where R is the homogeneous coordinate ring of a smooth projective variety embedded by a multiple of its canonical divisor:

Theorem (Theorem 5.7.4)**.** *Let R be a normal $\mathbb{Q}$-Gorenstein graded $\mathbb{C}$-algebra, generated in degree 1, with an isolated singularity. If R is a simple D_R-module, or merely admits a differential operator of negative degree, then R has klt singularities, and thus rational singularities.*

Both main theorems are proved by working with the smooth complex variety $X = \operatorname{Proj} R$, and using the following observation of [Hsi15]:

Theorem 5.1.2 ([Hsi15, Theorem 1.2])**.** *Let X be a smooth complex projective variety and L an ample line bundle. Set $R = S(X, L) = \bigoplus_m H^0(X, L^m)$. If R is a simple D_R-module then T_X is big.*

In Section 5.3 we discuss the notion of bigness of vector bundles; in the context of this theorem, bigness of T_X is equivalent to, for any $e > 0$, the existence of a nonzero global section of $H^0(\operatorname{Sym}^m T_X \otimes L^{-e})$ for some $m \gg 0$.

The singularities of rings of the form $R = S(X, L) = \bigoplus_m H^0(X, L^m)$ (i.e., rings that are section rings of polarized smooth complex projective varieties) translate to properties of the embedding $X \hookrightarrow \mathbb{P}^N$ determined by $H^0(X, L^m)$ for some m large enough. Recall from Example 2.1.32 that:

(1) R is always normal.

(2) R is Gorenstein if and only if $L = \mathcal{O}_X(aK_X)$ for some $a \in \mathbb{Z}$, and $\mathbb{Q}$-Gorenstein if and only if $L^{\otimes b} = \mathcal{O}_X(aK_X)$ for $a, b \in \mathbb{Z}$.

(3) R is klt if and only if it is $\mathbb{Q}$-Gorenstein and $a < 0$, i.e., if and only if it is $\mathbb{Q}$-Gorenstein and $-K_X$ is ample.

If R is klt then it has rational singularities, and the converse is true if R is Gorenstein. Thus, to give a counterexample to the sufficiency of Gorenstein rational singularities for D_R-simplicity, we find a variety X with $-K_X$ ample (i.e., X is Fano) and T_X not big. Thus, Theorem 5.1.1 will follow from

Theorem (Theorem 5.5.2)**.** *Let X be a del Pezzo surface of degree 3, i.e., a smooth cubic surface. Then T_X is not big; in fact, $H^0(X, \operatorname{Sym}^m T_X) = 0$ for all m.*

Remark 5.1.3. After the first version of these results was made public, the author realized that Theorem 5.5.2 also follows from [BD08, Theorem B]. The proof we give is distinct, and more direct but less general; see Section 5.6 for a discussion of their results.

Similarly, to show that R klt implies that R is D-simple, *if* $R = S(X, L)$ is the homogeneous coordinate ring of a smooth projective variety embedded by a power of its canonical divisor (i.e., such that either K_X or $-K_X$ is ample), we show that T_X big implies that $-K_X$ ample. In fact, we note the following statement (likely known to experts), which immediately implies our statement on necessity:

Proposition 5.1.4. *Let X be a smooth complex projective variety. If T_X is big, then X is uniruled.*

We begin by discussing differential operators and D-simplicity in Section 5.2. We then recall the definition and properties of big vector bundles in Section 5.3, and discuss the connection between D-simplicity and bigness of the tangent bundle in Section 5.4. In Section 5.5, we show that a Fano variety need not have big tangent bundle, by examining the tangent bundles of some del Pezzo surfaces. We show that if X is a del Pezzo surface of degree ≤ 4, then T_X is not big (and thus in particular the tangent bundle of a smooth cubic surface is not big). Section 5.6 discusses how the results discussed in Section 5.5 also follows from work of [BD08; DL19]. In Section 5.7, we prove Theorem 5.7.4 by showing that, if $R = S(X, L)$ for some smooth complex projective X with L very ample and a multiple of K_X, then D_R-simplicity of R forces X to be Fano and thus R to have klt (thus rational) singularities. Sections 5.8 and 5.9 contain no new results, but compare and contrast our results with known results in positive characteristic and characteristic 0 respectively.

Remark 5.1.5. Recently, there has been additional progress via work of [HLS20]: they completely classify which del Pezzo surfaces have big (or pseudoeffective) tangent bundle via different methods, and in particular recover the two examples we treat here (the del Pezzos of degrees 3 and 4). In addition, they are able to treat also the case of a hypersurface in $\mathbb{P}^n$, as well as certain del Pezzo threefolds. For further discussion, see Remark 5.9.5.

5.2 Differential operators and singularities

For the rest of the chapter, we consider a field k, most often $\mathbb{C}$, and a finitely generated k-algebra R. We will write simply D_R for $D_{R/k}$. Recall from Example 1.3.1 the following example:

Example 5.2.1 ([BGG72])**.** Let $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$ be the affine cone over a smooth elliptic curve. Then D_R has no differential operators of negative degree, D_R is not a finitely generated $\mathbb{C}$-algebra, and is neither left- nor right-Noetherian. We note here that since D_R has no differential operators of negative degree, the maximal homogeneous ideal (x, y, z) is a proper sub-D_R-module of R.

There are a variety of ways to use properties of D_R to describe the singularities of the ring R: one can consider Noetherianity of D_R, finite generation, generation by derivations, freeness of the R-module D^1_R, and more (see, for example, [LS89; Smi95; SV97; Ish87]). In particular, [LS89] posed the following questions:

(1) If $\operatorname{Spec} R$ has rational singularities, when is D_R simple? ([LS89, Question 0.13.1])

(2) When is R a simple D_R-module? ([LS89, Question 0.13.3])

Remark 5.2.2. We will also use the notion of *klt singularities* of R (by which we mean of $\operatorname{Spec} R$), which are of importance in the minimal model program. For the definition and properties of klt singularities, see Definition 2.1.26. (We do note that klt singularities are by definition $\mathbb{Q}$-Gorenstein.) We assemble here the only facts we will need in this chapter:

- If R has klt singularities then it has rational singularities [Elk81].
- If R is Gorenstein and has rational singularities then it has klt singularities (see, for example, [ST08, Proposition 3.1]).

- Let X be a smooth projective variety and L a very ample line bundle. Then $S(X,L) := \bigoplus H^0(X, L^{\otimes m})$ is $\mathbb{Q}$-Gorenstein if and only if $L^{\otimes b} \cong \mathcal{O}_X(aK_X)$ for some $a, b \in \mathbb{Z}$, and has klt singularities if and only if $-K_X$ is ample (see, for example, [Kol13, Lemma 3.1]).

The property in (2) above is called D-simplicity:

Definition 5.2.3. A k-algebra R is called D-simple if R is a simple D_R-module.

For clarity, we will sometimes say R is D_R-simple.

Remark 5.2.4. We note the following:

- If D_R is a simple ring, then R is D-simple (for a proof, see Remark 2.3.4).
- If R is an $\mathbb{N}$-graded ring with $R_0 = k$, and R is D-simple, then the graded ring D_R must contain differential operators of negative degree, i.e., $(D_R)_e \neq 0$ for some $e < 0$.

The examples of D-simple rings listed in Remark 1.3.2, as well as analogies between strong F-regularity in characteristic p and klt singularities in characteristic 0, motivate the following more specific formulation:

Question 5.2.5 ([Hsi15, Question 5.1])**.** If R is a finitely generated Gorenstein $\mathbb{C}$-algebra such that $\operatorname{Spec} R$ has rational singularities, is R then D-simple?

This proposes one potential solution to the question asked by [LS89] and considered in following work (e.g., [Smi95; SV97]) on what conditions beyond rational singularities ensure D-simplicity.

We note that since R is assumed to be Gorenstein it is equivalent to ask whether $\operatorname{Spec} R$ has klt singularities,

Remark 5.2.6. [LS89] gives an example of a ring with rational singularities which is not D-simple; the ring in question is obtained as the quotient of $\mathbb{C}[x,y,z]/(x^3+y^3+z^3)$ under a $\mathbb{Z}/3\mathbb{Z}$-action, which is a 2-dimensional normal isolated rational singularity. This example is why one must impose the Gorenstein condition in the phrasing of [Hsi15, Question 5.1].

Our Theorem 5.1.1 exhibits a klt hypersurface ring R which is not D-simple (and thus such that D_R is not a simple ring). This indicates that one must impose fairly strong conditions on R to obtain a sufficient condition for Question 0.13.1 of [LS89] on the simplicity of D_R.

5.3 Positivity of vector bundles

The rest of the chapter will use various notions of positivity for vector bundles. We recall some definitions and properties here, largely following [Laz04b, Chapter 6].

Let X be any variety and E a locally free sheaf of rank r on X. We write $\pi : \mathbb{P}E \to X$ for the projective bundle of 1-dimensional quotients of E. The variety $\mathbb{P}E$ carries a tautological line bundle $\mathcal{O}_{\mathbb{P}E}(1)$, such that $\pi_* \mathcal{O}_{\mathbb{P}E}(m) = \operatorname{Sym}^m E$ for $m \geq 0$.

Definition 5.3.1. The vector bundle E is said to be ample, nef or big if the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ is ample, nef, or big respectively.

Remark 5.3.2. There are conflicting conventions for defining bigness of vector bundles. The definition we take here is elsewhere called "L-big" (for "Lazarsfeld-big"). There is also the stronger notion of "V-big" (for "Viehwig-big"). These differ even in quite simple cases: for example, $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ is L-big but not V-big. For a detailed discussion of the different notions of positivity generally and bigness specifically, see [Bau+15; Jab07].

Remark 5.3.3. A line bundle L on a variety X of dimension N is big if and only if

$$\lim_{m \to \infty} \frac{h^0(X, L^{\otimes m})}{m^N} > 0.$$

We can give a similar characterization for bigness of vector bundles:

Say E is a rank-r vector bundle on a variety X of dimension n. Because $\pi_* \mathcal{O}_{\mathbb{P}E}(1) = \operatorname{Sym}^m E$, we have that

$$H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(m)) = H^0(X, \operatorname{Sym}^m E).$$

Since $\mathbb{P}E$ has dimension $n + r - 1$, we have that E is big if and only if

$$\lim_{m \to \infty} \frac{h^0(X, \operatorname{Sym}^m E)}{m^{n+r-1}} > 0.$$

The following characterization of bigness will be crucial in Section 5.4:

Proposition 5.3.4 ([Hsi15])**.** *Let L be an ample line bundle on X. The bundle E is big if and only if for all $e > 0$ there exists $m \geq 0$ such that $H^0(X, \operatorname{Sym}^m E \otimes L^{-e}) \neq 0$.*

We reproduce the proof of [Hsi15] here for convenience (note that just the "only if" implication is stated there, although the converse direction is straightforward).

Proof. Say $H^0(X, \operatorname{Sym}^m E \otimes L^{-e}) = H^0(\mathcal{O}_{\mathbb{P}E}(m) \otimes \pi^* L^{-e}) \neq 0$, i.e., $\mathcal{O}_{\mathbb{P}E}(m) \otimes \pi^* L^{-e}$ is effective (and thus so are all its positive tensor powers). Since $\mathcal{O}_{\mathbb{P}E}(1)$ is π-ample, we have that $\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L^j$ is ample for $j \gg 0$; choosing $j = Ne$ for $N \gg 0$, we have

$$\mathcal{O}_{\mathbb{P}E}(mN+1) = \underbrace{(\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L^{Ne})}_{\text{ample}} \otimes \underbrace{(\mathcal{O}_{\mathbb{P}E}(mN) \otimes L^{-Ne})}_{\text{effective}}$$

Thus, we have that $\mathcal{O}_{\mathbb{P}E}(mN+1)$ can be decomposed as the product of an ample line bundle and an effective line bundle, and is thus big; hence $\mathcal{O}_{\mathbb{P}E}(1)$ is big as well.

Conversely, if E is big, then Kodaira's lemma (Lemma 5.3.5 below) implies that for any e such that L^e is effective, there exists m such that $\mathcal{O}_{\mathbb{P}E}(m) \otimes \pi^* L^{-e}$ has a section for some $m \gg 0$. Then

$$0 \neq H^0(\mathbb{P}E, \mathcal{O}_{PE}(m) \otimes \pi^* L^{-e}) = H^0(X, \operatorname{Sym}^m E \otimes L^{-e}),$$

concluding the proof. $\square$

Lemma 5.3.5 (Kodaira's lemma)**.** *Let Y be a normal variety, A a big divisor and D an effective divisor. Then*

$$H^0(Y, \mathcal{O}_Y(mA - D)) \neq 0$$

for all m sufficiently large and divisible.

For a proof, see [Laz04a, Proposition 2.2.6].

Finally, we note here a fact we will use throughout:

Lemma 5.3.6. *Let k be a field and X a k-scheme, and let*

$$0 \to L \to E \to F \to 0$$

be a short exact sequence of vector bundles, with L a line bundle. Then for any $m > 0$ we have a short exact sequence

$$0 \to L \otimes \operatorname{Sym}^{m-1} E \to \operatorname{Sym}^m E \to \operatorname{Sym}^m F \to 0.$$

If k has characteristic 0, and we have a short exact sequence of vector bundles

$$0 \to E \to F \to L \to 0$$

with L again a line bundle, then for any $m > 0$ we have a short exact sequence

$$0 \to \operatorname{Sym}^m E \to \operatorname{Sym}^m F \to \operatorname{Sym}^{m-1} F \otimes L \to 0.$$

For the first fact see [Eis95, Proposition A2.2]; the second follows dualizing $0 \to E \to F \to L \to 0$, applying the first fact, and then dualizing again, and using the identifications $(\operatorname{Sym} E^\vee)^\vee \cong \operatorname{Sym}^m E$, which holds only in characteristic 0.

5.4 D-simplicity of section rings and bigness of the tangent bundle

In this section, we recall [Hsi15, Theorem 1.2]:

Theorem 5.4.1. *Let X be a smooth complex projective variety of dimension ≥ 2, let L be an ample line bundle on X, and let $R = S(X, L) := \bigoplus H^0(X, \mathcal{O}_X(mL))$ be the section ring of X with respect to L. If R has a differential operator of negative degree (e.g., if R is D_R-simple), then T_X is big.*

We recall the proof from [Hsi15] for the reader's convenience:

Proof. We first recall the connection between the tangent bundle of a smooth variety and the differential operators on its section ring from [Ish87].

Let X be a smooth projective variety and L an ample line bundle. We will assume for simplicity here that $R = S(X, L)$ is generated in degree 1. [Hsi15] and [Ish87] treat the general case; we note for our purposes here that we can also replace R by a Veronese subring while preserving the existence of a differential operator of negative degree (and thus reduce to the case here) and assume that $\operatorname{Proj} R$ is embedded in some $\mathbb{P}^n$ by $|L|$.

By Proposition 5.3.4, T_X is big if and only if for any $e < 0$ there exists $m > 0$ such that $H^0(X, \operatorname{Sym}^m T_X \otimes L^e) \neq 0$. We claim the vanishing $H^0(X, \operatorname{Sym}^m T_X \otimes L^e) = 0$ for all $e < 0$ and all m implies on the other hand that $D_e^m := (D_R^m)_e = 0$ for all $e < 0$ and all m, i.e., that R has no differential operators of negative degree. This then shows that R cannot be D-simple as the homogeneous maximal ideal will be a proper D_R-submodule.

Write D^m for the differential operators on R of order $\leq m$; write $D_l^m \subset D^m$ for the homogeneous differential operators of degree l. Let $\widehat{X} = \operatorname{Spec} R$ be the affine cone over X, which embeds naturally in $\mathbb{A}^{n+1} = \operatorname{Spec} k[x_0, \ldots, x_n]$, and let $U = \widehat{X} \smallsetminus \{\mathfrak{m}\}$ be the punctured cone, so $\pi : U \to X$ is an $\mathbb{A}^1$-bundle.

Let

$$I = \sum_{i=0}^{n+1} x_i \frac{\partial}{\partial x_i} \in D_0^1$$

be the Euler operator on R induced from that on $k[x_0, \ldots, x_n]$. Write $\mathcal{D}\mathrm{iff}^m$ for the sheaf of differential operators of order $\leq m$ on U. Note that by reflexivity of D^m we have that

$$D^m = H^0(U, \mathcal{D}\mathrm{iff}^m).$$

Thus I gives a global section of $\mathcal{D}\mathrm{iff}^1$, and let $\mathcal{D}\mathrm{iff}_e^m \subset \mathcal{D}\mathrm{iff}^m$ be the subsheaf of differential operators δ with $[I, \delta] = e\delta$. The global sections of $\mathcal{D}\mathrm{iff}_e^m$ are exactly those homogeneous differential operators δ on R such that for any homogeneous polynomial f we have

$$\deg \delta(f) - \deg f = e.$$

For any m, e, write $\Delta_e^m = \pi_*(\mathcal{D}\mathrm{iff}_e^m)$; note then that

$$H^0(X, \Delta_e^m) = H^0(U, \mathcal{D}\mathrm{iff}_e^m) = D_e^m.$$

One can then check:

1. $\Delta_e^m = \Delta_0^m \otimes L^e$ for any $m \geq 1$ and any $e \in \mathbb{Z}$.
2. Let $\sigma_1 = \Delta_0^1$ and let $\sigma_m = \Delta_0^m / \Delta_0^{m-1}$ for $m \geq 2$. Then $\sigma_m = \mathrm{Sym}^m \sigma_1$ and we have a short exact sequence

$$0 \to \mathcal{O}_X \to \sigma_1 \to T_X \to 0, \tag{5.1}$$

   and thus by Lemma 5.3.6 short exact sequences

$$0 \to \sigma_{m-1} \to \sigma_m \to \mathrm{Sym}^m T_X \to 0. \tag{5.2}$$

Now, let $e < 0$. Twisting (5.1) by L^e and taking global sections we get

$$0 \to H^0(X, L^e) \to H^0(X, \sigma_1 \otimes L^e) \to H^0(X, T_X \otimes L^e) \to H^1(X, L^e).$$

By Kodaira vanishing, $H^1(X, L^e) = 0$, while clearly $H^0(X, L^e) = 0$. Thus, if $H^0(X, T_X \otimes L^e) = 0$, then $H^0(X, \sigma_1 \otimes L^e) = 0$. Moreover, since by definition $\sigma_1 \otimes L^e = \pi_*(\mathcal{D}\mathrm{iff}_e^1)$, we have that

$$0 = H^0(X, \sigma_1 \otimes L^e) = H^0(U, \mathcal{D}\mathrm{iff}_e^1) = D_e^1$$

for $e < 0$. Thus, if $H^0(X, T_X \otimes L^e) = 0$ for $e < 0$, then R has no derivations of negative degree.

Now, we handle the higher order differential operators by induction on m. Again, let $e < 0$. Twisting (5.2) by L^e and taking global sections we get

$$0 \to H^0(X, \sigma_{m-1} \otimes L^e) \to H^0(X, \sigma_m \otimes L^e) \to H^0(X, \operatorname{Sym}^m T_X \otimes L^e)$$

Vanishing of $H^0(X, \operatorname{Sym}^m T_X \otimes L^e)$ for all m and all $e < 0$ will imply that $H^0(X, \sigma_m \otimes L^e) = H^0(X, \sigma_{m-1} \otimes L^e)$ for all m and all $e < 0$, and we have seen already that $H^0(X, \sigma_1 \otimes L^e) = 0$ for $e < 0$, and thus we obtain $H^0(X, \sigma_m \otimes L^e) = 0$ for all m and all $e < 0$.

By definition we have

$$0 \to \Delta_e^{m-1} \to \Delta_e^m \to \sigma_m \otimes L^e \to 0$$

and thus

$$0 \to \underbrace{H^0(X, \Delta_e^{m-1})}_{D_e^{m-1}} \to \underbrace{H^0(X, \Delta_e^m)}_{D_e^m} \to H^0(X, \sigma_m \otimes L^e).$$

Since the rightmost term vanishes, we know that $D_e^m = D_e^{m-1}$ for all $e < 0$, and we know already that $D_e^1 = 0$ for $e < 0$, and thus the result is shown. $\square$

5.5 The tangent bundle of degree-3 del Pezzo surfaces

In this section, we will treat the case of del Pezzo surfaces of degree 3, and show that their tangent bundles are not big. In the next section, we will treat those of degree 4. While our results for degree-4 del Pezzos actually imply the results for those of degree-3, the argument is simpler in the degree-3 case, and the statement actually slightly stronger. By [Hsi15, Corollary 1.3], toric del Pezzo surfaces (i.e., those of degree ≥ 6) have big tangent bundles, while combining the results of this section and the next implies those of degree ≤ 4 do not have big tangent bundles (see Corollary 5.6.3).

The del Pezzo surfaces of degrees 3 embed as surfaces in $\mathbb{P}^3$. If one attempts to use the resulting short exact sequences for their tangent bundles, however, one runs into difficulties. Instead, the key is to study the cotangent bundles, via the following elementary fact:

Lemma 5.5.1. *For any smooth surface Y, $T_Y \cong \Omega_Y(-K_Y)$.*

Proof. The nondegenerate pairing $\Omega_Y \times \Omega_Y \to \mathcal{O}_Y(K_Y)$ induces an isomorphism $(\Omega_Y)^\vee \cong \Omega_Y(-K_Y)$, and $(\Omega_Y)^\vee$ is simply T_Y. □

For the rest of this section we work over an arbitrary ground field of characteristic 0. We will prove:

Theorem 5.5.2. *Let X be a del Pezzo surface of degree 3, i.e., a smooth cubic surface. T_X is not big; in fact, $H^0(X, \operatorname{Sym}^m T_X) = 0$ for all m.*

This theorem immediately implies Theorem 5.1.1: Set $R = S(X, \mathcal{O}_X(1)) = \bigoplus_m H^0(X, \mathcal{O}_X(m))$. Combining Theorem 5.4.1 and Theorem 5.5.2, we have that R has no differential operators of negative degree, and thus R is not a simple D_R-module and D_R is itself not a simple ring. On the other hand, note that X is Fano (since by adjunction $\omega_X = \mathcal{O}_X(-1)$). Thus R has klt (thus also rational) singularities; since X is a hypersurface R is Gorenstein, and thus we have obtained the counterexample to Question 5.2.5 promised in Theorem 5.1.1. We note here that nothing in our results is specific to $\mathbb{C}[x, y, z, w]/(x^3+y^3+z^3+w^3)$, but applies also to $\mathbb{C}[x, y, z, w]/(F)$ for any homogeneous cubic F defining a smooth projective surface in $\mathbb{P}^3$.

Remark 5.5.3. Recent work of [HIM19] has considered positivity properties of the tangent bundle of del Pezzo surfaces, and in particular of T_X. The notions of positivity they examine are analytic in nature, and in particular the definitions of "big" for vector bundles they consider is *not* the same as the bigness of $\mathcal{O}_{\mathbb{P}T_X}(1)$. Thus, our result in Theorem 5.5.2 does not follow from their results, and the methods we use here are much more algebraic in nature.

Lemma 5.5.4. *Let $n \geq 3$ and $m \geq 1$. Then:*

(1) $H^0(\mathbb{P}^n, \operatorname{Sym}^m \Omega_{\mathbb{P}^n}(e)) = 0$ for $e < m+1$.

(2) $H^1(\mathbb{P}^n, \operatorname{Sym}^m \Omega_{\mathbb{P}^n}(e)) = 0$ for $e < m-1$.

(3) $H^i(\mathbb{P}^n, \operatorname{Sym}^m \Omega_{\mathbb{P}^n}(e)) = 0$ for $1 < i < n$ and any e.

Proof. The Euler sequence for $\Omega_{\mathbb{P}^n}$ is

$$0 \to \Omega_{P^3} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Since all terms are locally free and the rightmost term has rank 1, by Lemma 5.3.6 we have a short exact sequence of symmetric powers

$$0 \to \operatorname{Sym}^m \Omega_{\mathbb{P}^n} \to \underbrace{\operatorname{Sym}^m(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})}_{\bigoplus \mathcal{O}_{\mathbb{P}^n}(-m)} \to \underbrace{\operatorname{Sym}^{m-1}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})}_{\bigoplus \mathcal{O}_{\mathbb{P}^n}(-m+1)} \to 0$$

(the ranks are unimportant and we suppress them). Twisting by some $\mathcal{O}_{\mathbb{P}^n}(e)$ yields

$$0 \to \operatorname{Sym}^m \Omega_{\mathbb{P}^n}(e) \to \bigoplus \mathcal{O}_{\mathbb{P}^n}(-m+e) \to \bigoplus \mathcal{O}_{\mathbb{P}^n}(-m+e+1) \to 0.$$

Claim (1) of the lemma for $e < m$ follows (in any characteristic) by taking global sections, obtaining

$$0 \to H^0(\mathbb{P}^n, \operatorname{Sym}^m \Omega_{\mathbb{P}^n}(e)) \to H^0\big(\mathbb{P}^n, \bigoplus \mathcal{O}_{P^3}(-m+e)\big),$$

and noting that the right term vanishes for $-m+e<0$, but for $e=m$ we must take a slightly different approach, for which a slight change in notation will be helpful: Write $\mathbb{P}^n = \mathbb{P}(V)$ for a vector space V of dimension n. We twist the Euler sequence for the tangent bundle by $\mathcal{O}_{\mathbb{P}(V)}(1)$, obtaining

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \to T_{\mathbb{P}(V)}(-1) \to 0.$$

Taking symmetric powers we obtain

$$0 \to \underbrace{\operatorname{Sym}^{m-1}(V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)})(-1)}_{\operatorname{Sym}^{m-1}(V^\vee)\otimes\mathcal{O}_{\mathbb{P}(V)}(-1)} \to \underbrace{\operatorname{Sym}^m(V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)})}_{\operatorname{Sym}^m(V^\vee)\otimes\mathcal{O}_{\mathbb{P}(V)}} \to \operatorname{Sym}^m(T_{\mathbb{P}(V)}(-1)) \to 0.$$

Dualizing this sequence we have

$$0 \to \big(\operatorname{Sym}^m(T_{\mathbb{P}(V)}(1))\big)^\vee \to \operatorname{Sym}^m(V^\vee)^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \to \operatorname{Sym}^{m-1}(V^\vee)^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to 0.$$

Now, *using that we are in characteristic 0*, we know that

$$\big(\operatorname{Sym}^m(T_{\mathbb{P}(V)}(1))\big)^\vee \cong \operatorname{Sym}^m(\Omega_{\mathbb{P}(V)}(1)) = \operatorname{Sym}^m \Omega_{\mathbb{P}(V)}(m).$$

Thus, to see that $H^0(\mathbb{P}^n, \operatorname{Sym}^m \Omega_{\mathbb{P}(V)}(m)) = 0$, it suffices to show that the map

$$H^0(\mathbb{P}^n, \operatorname{Sym}^m(V^\vee)^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}) \to H^0\big(\mathbb{P}^n, \operatorname{Sym}^{m-1}(V^\vee)^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}(1)\big) \tag{5.3}$$

is injective. But this is just the canonical map of vector spaces

$$\operatorname{Sym}^m(V^\vee)^\vee \to \operatorname{Sym}^{m-1}(V^\vee)^\vee \otimes V.$$

which is dual to the canonical multiplication

$$\operatorname{Sym}^{m-1}(V^\vee) \otimes V^\vee \to \operatorname{Sym}^m(V^\vee),$$

which is obviously surjective, and thus (5.3) is injective and the $e = m$ case of claim (1) is shown.

For claim (2), the relevant terms of the long exact sequence are

$$H^0\big(\mathbb{P}^n, \bigoplus \mathcal{O}_{\mathbb{P}^n}(-m+e+1)\big) \to H^1(\mathbb{P}^n, \operatorname{Sym}^m \Omega_{\mathbb{P}^n}(e)) \to H^1\big(\mathbb{P}^n, \bigoplus \mathcal{O}_{\mathbb{P}^n}(-m+e)\big).$$

The right term vanishes always, while the left term is zero if $-m + e + 1 < 0$.

Finally, claim (3) follows by examining the terms

$$H^{i-1}\big(\mathbb{P}^n, \bigoplus \mathcal{O}_{\mathbb{P}^n}(-m+e+1)\big) \to H^i(\mathbb{P}^n, \operatorname{Sym}^m \Omega_{\mathbb{P}^n}(e)) \to H^i\big(\mathbb{P}^n, \bigoplus \mathcal{O}_{\mathbb{P}^n}(-m+e)\big)$$

and noting that the first and last terms vanish for any e and $1 < i < n$. □

Lemma 5.5.5. *Let $m \geq 1$. Then:*

(1) $H^0(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m)) = 0$.

(2) $H^1(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m-3)) = 0$.

Proof. We start with (1): Twisting the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$ by $\operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m)$, we have

$$0 \to \operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m-3) \to \operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m) \to \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m) \to 0.$$

Taking the long exact sequence in cohomology we get

$$H^0(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m)) \to H^0(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m)) \to H^1(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m-3)). \quad (5.4)$$

By Lemma 5.5.4 the first and last terms vanish, and thus $H^0(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m)) = 0$, as desired.

For (2), we twist $0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$ by $\operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m-3)$ and take the long exact sequence, yielding relevant terms

$$H^1(\mathbb{P}^3, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m-3)) \to H^1(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m-3)) \to H^2(\mathbb{P}^3, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}(m-6)).$$

Again, Lemma 5.5.4 says that the outer terms vanish and thus $H^1(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m-3)) = 0$. □

Proof of Theorem 5.5.2. By 5.5.1, we have that

$$\operatorname{Sym}^m(T_X) = \operatorname{Sym}^m(\Omega_X(1)) = \operatorname{Sym}^m(\Omega_X)(m),$$

and so it suffices to show that

$$H^0(X, \operatorname{Sym}^m(\Omega_X)(m)) = 0$$

for all m.

We have a presentation

$$0 \to \mathcal{O}_X(-3) \to \Omega_{\mathbb{P}^3}|_X \to \Omega_X \to 0$$

for Ω_X; taking symmetric powers and twisting by $\mathcal{O}_X(m)$, we have

$$0 \to \operatorname{Sym}^{m-1}(\Omega_{\mathbb{P}^3}|_X)(m-3) \to \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m) \to \operatorname{Sym}^m \Omega_X(m) \to 0.$$

Taking the long exact sequence in cohomology we have

$$H^0(X, \operatorname{Sym}^m \Omega_{\mathbb{P}^3}|_X(m)) \to H^0(X, \operatorname{Sym}^m \Omega_X(m)) \to H^1(X, \operatorname{Sym}^{m-1} \Omega_{\mathbb{P}^3}|_X(m-3)).$$

But Lemma 5.5.5 implies immediately that the outer terms vanish, and thus so does the middle term, proving the theorem. □

5.6 An alternate proof, and the case of degree-4 del Pezzos

In this section, we give an alternate proof of Theorem 5.5.2 as a corollary of [BD08]. We then use related work of [DL19] to treat the case of del Pezzos of degree 4.

We recall first:

Theorem 5.6.1 ([BD08, Theorem B]). *If $X \subset \mathbb{P}^n$ is a smooth hypersurface, then for any $m > 1$,*

$$H^0(X, \operatorname{Sym}^m \Omega_X(m)) \neq 0$$

if and only if X is a hyperquadric in $\mathbb{P}^n$.

Recall that if X is a smooth cubic surface in $\mathbb{P}^3$, then we have an isomorphism $T_X \cong \Omega_X(1)$, and thus applying the theorem we have immediately that

$$H^0(X, \operatorname{Sym}^m T_X) = H^0(X, \operatorname{Sym}^m \Omega_X(m)) = 0,$$

thus recovering Theorem 5.5.2.

We note that the proof of [BD08, Theorem B] involves a detailed study of the tangent map and the tangent 2-trisecant variety of the embedding $X \subset \mathbb{P}^n$, and thus the proof we gave in the preceding section is significantly more elementary, although correspondingly less general.

We now turn to the proof of the following theorem, using results of [DL19]:

Theorem 5.6.2. *Let X be a del Pezzo surface of degree 4. Then T_X is not big.*

Proof. Let $X \subset \mathbb{P}^4$ is the anticanonical embedding of X as a complete intersection of two quadrics in $\mathbb{P}^4$, say Q_1, Q_2. Let $f : \bigcup_{x \in X} T_x X \to \mathbb{P}^4$ be the tangent map of X, which associates to a point in a tangent plane to $x \in X$ the corresponding point of $\mathbb{P}^4$. Combining Corollaries 2.1 and 3.1 of [DL19], we have that if:

$$f \text{ is surjective with connected fibers} \tag{$*$}$$

then there is a graded isomorphism $\bigoplus H^0(X, \operatorname{Sym}^m \Omega_X(m)) = \operatorname{Sym}^\bullet H^0(X, \mathcal{I}_X(2)) = \mathbb{C}[Q_1, Q_2]$ (where $\deg Q_1 = \deg Q_2 = 2$). Assuming (1) and (2), then, we have that $H^0(X, \operatorname{Sym}^{2m} \Omega_X(2m))$ has as basis the set of degree-m monomials in the Q_i, and thus grows like m rather than m^3, and thus $\Omega_X(m)$ is not big. Since we already know that $T_X = \Omega_X(m)$ (using that X is a surface embedded by its anticanonical divisor), this implies that T_X is not big.

So, all that remains is to show that $(*)$ holds. First, note that $\bigcup_{x \in X} T_x X$ has dimension 4, so to obtain surjectivity of the tangent map f it suffices to check that it is generically finite. Since the tangent map is injective on each tangent plane $T_x X$, it suffices to check that a general point of $\mathbb{P}^4$ lies on only finitely many tangent planes to X. This follows immediately, however, from the fact that the Gauss map $\gamma : X \to \operatorname{Gr}(2, \mathbb{P}^4)$ associating to a point $x \in X$ the tangent hyperplane $T_x X \in \operatorname{Gr}(2, \mathbb{P}^4)$ is not just generically finite, but birational (see [Zak93, Corollary 2.8]).

Note that this implies that f itself is generically injective and dominant, and thus f is in fact birational. This immediately gives connectivity of the fibers $f^{-1}(p)$ for $p \in \mathbb{P}^4$: Since the tangent map f is a birational morphism onto the smooth variety $\mathbb{P}^4$, Zariski's main theorem implies that f has connected fibers, and thus the proof is complete. $\square$

Corollary 5.6.3. *Let X_i be a del Pezzo surface of degree i. Then T_{X_i} is not big for $i < 5$.*

Proof. First consider X_4, which is the blowup of $\mathbb{P}^2$ at five general points. $-K_{X_4}$ embeds X_4 as the intersection of two smooth quadrics in $\mathbb{P}^4$, which we have just seen does not have big tangent bundle. If $i < 4$, we can view X_i as the blowup of X_4 at $i - 4$ general points, say $\mu : X_i \to X_4$. We have an injection

$$T_{X_i} \hookrightarrow \mu^* T_{X_4};$$

taking the m-th symmetric power yields a morphism

$$\operatorname{Sym}^m T_{X_i} \hookrightarrow \operatorname{Sym}^m \mu^* T_{X_4} = \mu^* \operatorname{Sym}^m T_{X_4}.$$

This must be an injection, since $T_{X_i} \hookrightarrow \mu^* T_{X_4}$ is generically an isomorphism, so $\operatorname{Sym}^m T_{X_i} \to \operatorname{Sym}^m \mu^* T_{X_4}$ is generically an isomorphism well and thus has torsion kernel, but $\operatorname{Sym}^m T_{X_i}$ is locally free and thus cannot have a torsion subsheaf. Taking global sections and noting that $H^0(X_i, \mu^* \operatorname{Sym}^m T_{X_4}) = H^0(X_4, \operatorname{Sym}^m T_{X_4})$ since $\mu_* \mathcal{O}_{X_i} = \mathcal{O}_{X_4}$, we thus have a containment

$$H^0(X_i, \operatorname{Sym}^m T_{X_i}) \to H^0(X_4, \operatorname{Sym}^m T_{X_4}),$$

and thus $H^0(\operatorname{Sym}^m T_{X_i})$ cannot grow like m^3. □

Remark 5.6.4. In particular, as mentioned at the beginning of this section, once we know that T_{X_4} is not big, T_{X_3} cannot be big either; however, our result above actually shows that $H^0(X_3, \operatorname{Sym}^m T_{X_3}) = 0$ for all m, which does not follow from our treatment of T_{X_4}, as we saw above that $H^0(X_4, \operatorname{Sym}^2 T_{X_4})$ is 2-dimensional.

5.7 A partial converse

The preceding section showed that given a Fano variety X and an ample line bundle L, the section ring $S(X, L)$ may not be D-simple, even though it has only a Gorenstein rational singularity. Even though this is not true, however, one can formulate a partial converse (Theorem 5.7.4), which imposes conditions on the singularities of a D-simple ring.

Proposition 5.7.1. *Let X be a smooth complex projective variety. If T_X is big then X is uniruled.*

Proof. First, we recall the following theorem of Miyaoka:

Theorem 5.7.2 ([Miy87, Corollary 8.6])**.** *If a smooth complex projective variety X is not uniruled then Ω_X is generically nef, i.e., $\Omega_X|_C$ is nef for a general complete intersection curve $C \subset X$.*

Now, say X is not uniruled but T_X is big. Take L to be an ample line bundle on X and consider a nonzero global section $s \in H^0(X, \operatorname{Sym}^m T_X \otimes L^{-1})$. Choosing a general complete intersection curve $C \subset X$, which by generality will not lie in the zero locus of s, we obtain a nonzero global section $s|_C \in H^0(C, \operatorname{Sym}^m T_X|_C \otimes L|_C^{-1})$. We can view this nonzero global section equivalently as an injection

$$\mathcal{O}_C \hookrightarrow \operatorname{Sym}^m T_X|_C \otimes L|_C^{-1},$$

or as an injection

$$L|_C \hookrightarrow \operatorname{Sym}^m T_X|_C.$$

Moreover, we note that $(\operatorname{Sym}^m T_X|_C)^\vee = \operatorname{Sym}^m \Omega_X|_C$, and that since $\Omega_X|_C$ is nef so is $\operatorname{Sym}^m \Omega_X|_C$ (by [Laz04b, Theorem 6.2.12(iii)]).

Lemma 5.7.3. *If C is a smooth curve, L a line bundle on C of positive degree (thus ample), and E a vector bundle on C with $E^\vee$ nef, then there is no injection $L \hookrightarrow E$.*

Proof. Say we have $L \hookrightarrow E$. The cokernel $Q := E/L$ may not be torsionfree, but we may consider the surjection

$$E \to Q \to Q/\text{torsion}.$$

Since C is a smooth curve, $Q' := Q/\text{torsion}$ is locally free, and thus so is the kernel of the surjection

$$E \to Q'.$$

Call this locally free kernel L'; it is clear L' is a line bundle containing L, and thus $\deg L' \geq \deg L > 0$. The short exact sequence of locally free sheaves

$$0 \to L' \to E \to Q' \to 0$$

dualizes to

$$0 \to (Q')^\vee \to E^\vee \to (L')^\vee \to 0.$$

However, $E^\vee$ was supposed to be nef, yet the quotient $(L')^\vee$ is not, and we thus have a contradiction, so there can be no injection $L \hookrightarrow E$. $\square$

Applying this lemma with $E = \operatorname{Sym}^m T_X$ and $L = L|_C$, we obtain Theorem 5.7.1. □

The following theorem recovers and extends [BJN19, Corollary 4.49], which treated the Gorenstein case.

Theorem 5.7.4. *Let R be a normal $\mathbb{Q}$-Gorenstein graded $\mathbb{C}$-algebra, generated in degree 1, with an isolated singularity. If R is D-simple, then $\operatorname{Proj}(R)$ is Fano, and thus R has klt singularities, and thus rational singularities.*

Proof. Let $X = \operatorname{Proj} R$, with $L = \mathcal{O}_X(1)$ the corresponding ample line bundle. Thus, we have that $R = S(X, L)$ is the section ring of the smooth projective variety X under the projectively normal embedding defined by L. Since R is $\mathbb{Q}$-Gorenstein, we must have that $K_X \sim r \cdot L$ for $r \in \mathbb{Q}$. D-simplicity of R forces T_X to be big. Thus, applying Theorem 5.7.1, we have that X must be uniruled.

Since X is uniruled, we must have $H^0(X, mK_X) = 0$ for all $m > 0$. But for m sufficiently large and divisible we have $mK_X \sim a \cdot L$ for $a = mr \in \mathbb{Z}$, $|a| \gg 0$. Thus $H^0(X, aL) = 0$ for $a \gg 0$, and by ampleness of L we must have that $a < 0$, so $r < 0$ and $-K_X$ is ample. Thus X is Fano and embedded by a multiple of its canonical divisor, so $S(X, L)$ has klt singularities by [Kol13, Lemma 3.1]. □

5.8 Relationship to differential operators in characteristic p

As mentioned in Section 5.2, part of the motivation for the conjectural relationship between klt singularities and D-simplicity is the equivalence of D-simplicity and F-regularity for F-pure varieties, and the analogy between F-regularity in characteristic p and klt singularities in characteristic 0. In this section, we give a brief discussion of these analogies.

Remark 5.8.1. By [SS10, Theorem 5.1], a smooth (or klt) Fano variety X over $\mathbb{C}$ has globally F-regular type; this means that if one looks at various models X_p of X over finite fields $\mathbb{F}_p$, then X_p is globally F-regular for almost all p; this in turn is equivalent to the section ring $S(X_p, L_p)$ being strongly F-regular for any ample line bundle L_p on X_p. We avoid giving a formal definition here, but the following example is indicative of the general process, at least in the case where X can be defined over the subring $\mathbb{Z} \subset \mathbb{C}$:

Example 5.8.2. Let $X = \operatorname{Proj} \mathbb{C}[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$ be a smooth cubic surface. Then for each prime p, we have $X_p = \operatorname{Proj} \mathbb{F}_p[x, y, z, w]/(x^3+y^3+z^3+w^3)$; then we have a natural choice of section ring $S(X_p, L_p) = \mathbb{F}_p[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$. For $p \geq 5$, this is strongly F-regular (by Fedder's criteria [Fed83]). Thus, X has F-regular type.

Remark 5.8.3. For any $p \geq 5$, since $\mathbb{F}_p[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$ is strongly F-regular (and F-pure), it is D-simple, and in particular has differential operators of negative degree. However, our above results showed that $\mathbb{C}[x, y, z, w]/(x^3+y^3+z^3+w^3)$ is *not* D-simple, as it has no differential operators of negative degree. That is, there is no hope in general to "lift" differential operators of negative degree from characteristic p to characteristic 0. This offers another example of the phenomena discussed in [Smi95], where the ring $R_p = (\mathbb{Z}/p\mathbb{Z})[x, y, z]/(x^3 + y^3 + z^3)$ is shown to have a degree-0 differential operator for $p \equiv 1 \bmod 3$ (i.e., those p such that R_p is F-split) that does not arise as the image of a differential operator on $\mathbb{Z}[x, y, z]/(x^3 + y^3 + z^3)$. As is noted there, this shows that there are "more" differential operators in positive characteristic. The example of this chapter is further evidence for this heuristic: in positive characteristic $p \geq 5$, $\mathbb{F}_p[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$ has differential operators of arbitrarily negative degree, while $\mathbb{C}[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$ has no differential operators of negative degree.

Remark 5.8.4. Recent work of [BJN19] introduced an invariant $s(R)$ of a ring R, called the *differential signature*. One always has $0 \leq s(R) \leq 1$, and if $s(R) > 0$ then R is D_R-simple. We will not recall the definition of this invariant here, but want to note briefly that our results give an example of the contrasting behavior of $s(R)$ in positive characteristic and characteristic 0:

Let $R_p = \mathbb{F}_p[x, y, z, w]/(x^3+y^3+z^3+w^3)$ and $R = \mathbb{C}[x, y, z, w]/(x^3+z^3+z^3+w^3)$. Since R is not D_R-simple, the differential signature of R (over $\mathbb{C}$) must be zero. On the other hand, R_p is strongly F-regular for each p, so it has positive F-signature; moreover, one can calculate using [Shi18] the limit of the F-signatures as p goes to infinity to be 1/8. By [BJN19, Lemma 5.15], this bounds the limit of the differential signatures of R_p (over $\mathbb{Z}$) away from 0. Thus, one cannot expect to calculate the differential signature in characteristic 0 as a limit of differential signatures in characteristic p as $p \to \infty$. For further discussion on this question, see [BJN19, Section 5.3].

Remark 5.8.5. Although the main result of this chapter is that the characteristic-0 analogue of "strongly F-regular implies D-simple and F-pure" is false, another interesting connection to characteristic p arises from considering the potential characteristic-0 converse, that is, does a D-simple ring with log canonical singularities necessarily have klt singularities?

We note that this follows from the conjectural relation between F-purity and log canonical singularities, as follows: Let R be a D-simple $\mathbb{Q}$-Gorenstein essentially finite-type $\mathbb{C}$-algebra and assume that R has log canonical singularities. One can choose a finite-type $\mathbb{Z}$-algebra A and an essentially finite-type A-algebra R_A such that $R_A \otimes_{\mathbb{A}} \mathbb{C} = R$, and consider the reductions of R_A modulo the expansion of various maximal ideals of A. For simplicity, we assume that we can take A to be $\mathbb{Z}$, although the general case proceeds in the same way.

Conjecturally (see, e.g., [Tak13, Conjecture 2.4]), since R is log canonical there is a dense (but likely not open) subset of $\mathbb{Z}$ such that the reduction R_p is F-pure (i.e., R is of dense F-pure type) for p in this subset. By [SV97, Theorem 5.2.1], D-simplicity of R descends to D-simplicity of R_p for p in an *open* dense subset of $\mathbb{Z}$ as well. An open dense set and an arbitrary dense set intersect in a dense subset, and thus over a dense subset of $\mathbb{Z}$, R_p is F-pure and D-simple, and thus strongly F-regular by [Smi95, Theorem 2.2]. Thus, R is of (dense) strongly F-regular type. Theorem 3.3 of [HW02] then implies that R is klt (and thus also has rational singularities).

It would be interesting to have a proof that D-simple plus log canonical implies klt that does not rely on reduction to positive characteristic.

5.9 Big tangent bundles in characteristic 0

In this section, we briefly review what is known about bigness of the tangent bundle for smooth complex projective varieties; throughout, X will denote such a variety.

Remark 5.9.1. By [Wah83], if X is a smooth projective variety and L an ample line bundle, then $H^0(X, T_X \otimes L^{-1}) \neq 0$ forces X to be be projective space $\mathbb{P}^n$, and additionally $L = \mathcal{O}_{\mathbb{P}^n}(1)$ (except if $n = 1$ in which case L might be $\mathcal{O}_{\mathbb{P}^1}(2)$). That is, if $\dim X \geq 2$ and $R = S(X, L)$ has a derivation of negative degree, then X must be the projective n-space, and R just a polynomial ring.

It appears that less is known about the potential nonvanishing of global sections of the higher symmetric powers $H^0(X, \operatorname{Sym}^m T_X \otimes L^e)$. The following result is as much as is known to us:

Theorem 5.9.2 ([ADK08, Theorem 6.3]). *Let X be a smooth complex projective variety of Picard number 1 and L an ample line bundle. If $H^0(X, T_X^{\otimes m} \otimes L^{-m}) \neq 0$, then either $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ or Q is a quadric hypersurface and L is the restriction of the hyperplane class from the ambient projective space.*

Since in characteristic 0 we can embed $\operatorname{Sym}^m T_X \hookrightarrow T_X^{\otimes m}$, this implies that if X is as above, and not a projective space or a hyperquadric, then $H^0(X, \operatorname{Sym}^m T_X \otimes L^{-m}) = 0$. That is, for such X we can rule out differential operators on $R = S(X, L)$ of order m and degree $-m$. We do not know if one can extend this theorem to varieties with higher Picard number.

We emphasize that results of the above form are very specific to characteristic 0; for example, [Wah83] gives the example of the ring $R = (\mathbb{Z}/2\mathbb{Z})[x_0, x_1, x_2]/(x_0^2 + x_1 x_2)$; $\operatorname{Proj} R$ is a smooth quadric, but $\partial/\partial x_0$ is a differential operator on R of order 1 and degree -1.

Remark 5.9.3. Bigness of the tangent bundle of a smooth projective variety is known in the following cases:

- projective spaces.
- quadrics (of any dimension).
- varieties X admitting an ample line bundle L such that the section ring $S(X, L)$ is a split summand of a polynomial ring, and thus in particular:
- smooth toric varieties.
- Grassmannians and (partial) flag varieties.
- when T_X is nef (conjecturally, by [CP91] this is equivalent to X being rational homogeneous) and $\dim X \leq 3$.
- products of the above varieties.

Note that while many of these are Fano varieties, not all are (e.g., many toric varieties). However, by [Hsi15], if T_X is big *and nef* then X is Fano. Nefness of T_X is quite a restrictive condition though, and as already mentioned, it is conjectured in [CP91] to be equivalent to X being rational homogeneous (the quotient of a semisimple algebraic group by a parabolic subgroup).

We note here that other positivity properties of T_X are well-studied, and appear to be quite restrictive: Beyond the aforementioned conjecture on nefness, there is the celebrated result of Mori [Mor79] proving a conjecture of Hartshorne that if T_X is ample then $X \cong \mathbb{P}^n$. It is thus natural to ask about the following question:

Question 5.9.4. What conditions on a smooth complex variety X, beyond uniruledness, are imposed by bigness of the tangent bundle T_X?

Remark 5.9.5 (Recent work)**.** There has been additional progress recently towards this question through work of [HLS20]. There, the authors prove the following results:

(1) ([HLS20, Theorem 1.2]) Let X_i be a del Pezzo surface of degree i. Then T_X is big if and only if $i \geq 5$.

(2) ([HLS20, Theorem 1.4]) Let X be a hypersurface of degree d in $\mathbb{P}^n$ for $n \geq 3$. Then T_X is big if and only if $d = 2$.

We note that the missing case that (1) settles is that of the del Pezzo of degree 5, as those of degree 6 or higher are toric (and hence have big tangent bundle) and those degree 4 or lower are covered by the results here. However, their methods are able to treat all the non-toric del Pezzos in a uniform way, via the study of the dual variety of minimal rational tangents. The case of the del Pezzo of degree 5 illustrates the subtlety of the question of when a Fano variety has big tangent bundle: the del Pezzo of degree 5 is not toric, and in fact has finite automorphism group, and yet has big tangent bundle, while the del Pezzos of lower degree do not have big tangent bundle. (2) sheds further light on our question above, and indicates that one may expect bigness of T_X to be quite restrictive, as it implies that projective space and hyperquadrics are the only smooth hypersurfaces to have big tangent bundle.

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