Some Results on Tori in p-adic Groups

by

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For my parents

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ABSTRACT

The rational conjugacy classes of tori play an important role in the representation theory of reductive p-adic groups defined over a finite extension k of the p-adic numbers. In an upcoming paper, Adler and DeBacker give a parameterization of the rational conjugacy classes of embeddings of maximal tori using Bruhat-Tits theory, and our primary goal will be to move towards an analogous parameterization of maximal θ-split tori, which play a prominent role in the theory of p-adic symmetric spaces. In particular, we will parameterize the rational conjugacy classes of maximal θ-split tori in finite groups of Lie type and in groups defined over the maximal unramified extension K of k. We will then use Bruhat-Tits theory to parameterize the θ-split tori which split over K and then determine which of these tori can emerge as the maximal K-split subtorus of a maximal θ-split torus. We will also provide a similar parameterization of a class of unramified tori which we will call unramified θ-perfect tori. These tori will play an important role in future work where we will use them to determine how the conjugacy classes of θ-split tori over K split into rational conjugacy classes. Finally, in the case of symplectic groups, we will compare DeBacker's parameterization of maximal unramified tori to another parameterization due to Waldspurger.

CHAPTER I

Introduction

Suppose $\mathbf{G}$ is a connected reductive algebraic group defined over a finite extension k of the p-adic numbers $\mathbb{Q}_p$, with $p \neq 2$, and suppose that θ is an involution (i.e. an order 2 automorphism) of $\mathbf{G}$ defined over k. If $\mathbf{G}^\theta$ denotes the points in $\mathbf{G}$ fixed by θ and $(\mathbf{G}^\theta)^\circ$ denotes its identity component, then we let $\mathbf{H}$ be a k-subgroup of $\mathbf{G}$ so that $(\mathbf{G}^\theta)^\circ \subseteq \mathbf{H} \subseteq \mathbf{G}^\theta$. We then have that $\mathbf{H}^\circ$, the identity component of $\mathbf{H}$, is reductive by [36], and we say that the quotient $\mathbf{G}/\mathbf{H}$ is a p-adic symmetric space. The study of p-adic symmetric spaces has played a key role in several important results concerning the representations theory of p-adic groups and the Langlands program. For example, in [15], the authors use symmetric space methods to determine when two data in Yu's construction of supercuspidal representations [38, 13] correspond to equivalent supercuspidal representations, and symmetric space methods also show up in the study of the relative trace formulas [19].

If $\mathbf{G}'$ is a connected reductive algebraic group defined over k, then perhaps the simplest example of a p-adic symmetric space emerges from defining $\mathbf{G}$ to be the direct product $\mathbf{G}' \times \mathbf{G}'$ and by letting θ be the map which sends $(g_1, g_2) \in \mathbf{G}' \times \mathbf{G}'$ to (g_2, g_1). Then $\mathbf{H} = \{(g,g) | g \in \mathbf{G}'\}$, and we can identify the p-adic symmetric space $\mathbf{G}/\mathbf{H}$ with $\mathbf{G}'$ via the map $\mathbf{G} \to \mathbf{G}'$ which sends (g_1, g_2) to $(g_1)^{-1} g_2$.

Many results about a reductive p-adic group $\mathbf{G}'$ have natural analogues for p-adic symmetric spaces that specialize to the original result when we consider $\mathbf{G} = \mathbf{G}' \times \mathbf{G}'$. (See, for example, [3, 12, 21, 22, 31].) In this thesis, our primary goal is to study an analogue of maximal k-tori associated to p-adic symmetric spaces. In particular, we say that a k-torus $\mathbf{S}$ in $\mathbf{G}$ is θ-split if $\theta(s) = s^{-1}$ for all $s \in \mathbf{S}$. Such tori correspond to Cartan subalgebras in the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ which lie in the -1-eigenspace of the differential $d\theta$, and they have filled a role analogous to that of maximal k-tori in most existing results on p-adic symmetric spaces [3, 12, 21, 23, 24]. More specifically, we want to move towards a parameterization of the $\mathbf{H}(k)$-conjugacy classes of the the maximal θ-split k-tori, with the hope that it will advance the understanding of the structure of p-adic symmetric spaces and facilitate the development of symmetric space analogues of results in the representation theory of p-adic groups in which maximal k-tori play a prominent role (e.g. [20]).

In moving towards this goal, we will attempt to model the approach of Jeff Adler and Stephen DeBacker in their work on parameterizing the $\mathbf{G}(k)$-conjugacy classes of embeddings of maximal k-tori in $\mathbf{G}$ [10, 1, 2]. In their work, they begin in [10, 1] by using Bruhat-Tits theory to parameterize the tori which split over the maximal unramified extension $K \leq \overline{k}$ of k, where $\overline{k}$ denotes a fixed algebraic closure of k, and by determining which of these tori emerge as the K-split component of a maximal k-torus in $\mathbf{G}$. They then determine in [2] how the $\mathbf{G}(K)$-conjugacy class of a K-minisotropic maximal k-torus in $\mathbf{G}$ splits into $\mathbf{G}(k)$-conjugacy classes, and so by applying this splitting to the K-minisotropic k-tori contained in the centralizers of the unramified tori arising in [1], they are able to arrive at a parameterization of the $\mathbf{G}(k)$-conjugacy classes of all maximal k-tori.

In Chapter 2, we will introduce some relevant notation and recall some important facts about p-adic symmetric spaces. Then, in Chapter 3, we will begin working towards our parameterization by solving the analogous problem in a finite group of Lie type G defined over the residue field $\mathfrak{f}$ of k. As we will be particularly interested in the case where G is the reductive quotient associated to a point in the Bruhat-Tits building of $\mathbf{G}(K)$ which is fixed under the action induced by θ, we will again use θ to denote a $\mathfrak{f}$-involution of G, and we will parameterize the θ-split $\mathfrak{f}$-tori of G by relating them to twisted conjugacy classes in a finite group called the extended little Weyl group. The proof of this parameterization closely mirrors the classical result found in, for example, [8] and essentially follows from Lang's theorem applied to the connected reductive group H°.

From there, assuming that $\mathbf{G}$ satisfies a tameness condition (in particular, the p does not divide the order of the Weyl group), we will determine in Chapter 4 the $\mathbf{H}(K)$-conjugacy classes of maximal θ-split K-tori. We will again relate the conjugacy classes to twisted conjugacy classes in the extended little Weyl group, but we will be forced this time to rely on an argument using Galois cohomology. We will also provide a parameterization of the $\mathbf{G}(K)$-conjugacy classes of the tame twisted Levi K-subgroups in $\mathbf{G}$ by combining two proofs in the work of Adler and DeBacker. While this result will be independent of the theory of symmetric spaces a priori, it will prove useful in showing how the $\mathbf{H}(K)$-conjugacy class of a maximal θ-split torus splits into $\mathbf{H}(k)$-conjugacy classes and is also independently interesting as a step towards a parameterization of the $\mathbf{G}(k)$-conjugacy classes of twisted Levi k-subgroups, which, for example, play a prominent role in Yu's construction of supercuspidal representations [38, 13].

We will consider two types of unramified tori associated to p-adic symmetric spaces, the first being the unramified θ-split tori. A parameterization of the $\mathbf{H}(k)$-conjugacy classes of maximal unramified θ-split k-tori is given in [28]. Here, in Chapter 5 we will extend Portilla's results by defining and parameterizing an analogue for θ-split tori of the unramified tori appearing in [1]. We will show that the unramified component of a maximal θ-split k-torus will be a torus of this form, and assuming a conjecture (Conjecture 5.3.4), we can then show that all of these tori will arise as the unramified component of some maximal θ-split k-torus.

These tori and their centralizers, which are examples of unramified twisted Levi subgroups, are expected to play a significant role in the theory of p-adic symmetric spaces. In particular, the unramified twisted Levi subgroups arising from these tori are associated to θ-split parabolic subgroups, a type of parabolic subgroup of $\mathbf{G}$ which plays a prominent role in the symmetric space analogue of supercuspidal representations of $\mathbf{G}$. (See [21], for example.)

Thus if one hopes to prove a symmetric space analogue of Yu's construction of supercuspidal representations [38, 13] or another result involving twisted Levi subgroups, then one would expect the unramified twisted Levi subgroups arising from our unramified θ-split tori to fill the role normally played by unramified twisted Levi subgroups.

However, in order to demonstrate how the $\mathbf{H}(K)$-conjugacy of a maximal θ-split k-torus splits into $\mathbf{H}(k)$-conjugacy classes, we will need to consider another class of unramified tori. In Chapter 6, we will consider unramified θ-perfect tori, which are the unramified tori in $\mathbf{G}$ (in the sense of [1]) whose associated unramified twisted Levi subgroup contains a maximal θ-split k-torus of $\mathbf{G}$. While these tori are not expected to be particularly important in the theory of p-adic symmetric spaces, they will play a prominent role in the author's upcoming work on ramified θ-split tori. In particular, given a maximal θ-split k-torus $\mathbf{S}$ of $\mathbf{G}$, we will need to construct a θ-stable unramified twisted Levi subgroup $\mathbf{M}$ so that $C_{\mathbf{M}}(\mathbf{S})$ is a K-minisotropic k-torus. Upon constructing such a Levi subgroup, we will then be able to associate a point in the Bruhat-Tits building of $\mathbf{M}(k)$ to $\mathbf{S}$, and using this point in the building, we will be able to show how the $\mathbf{H}(K)$-conjugacy class of $\mathbf{S}$ breaks into $\mathbf{H}(k)$-conjugacy classes. The twisted Levi subgroups $\mathbf{M}$ arising in this way will all correspond θ-perfect tori, and so the results in Chapter 6 will play a prominent role in our analysis.

The methods in Chapters 5 and 6 will rely heavily on Bruhat-Tits theory. In particular, we will show that all of the unramified tori we need to consider emerge from lifts of tori in the reductive quotients associated to facets in the Bruhat-Tits building of $\mathbf{G}(k)$ which intersect the Bruhat-Tits building of $\mathbf{H}(k)$.

Finally, in Chapter 7 we will review the parameterization of maximal unramified tori in [10], and we will provide a comparison with an analogous parameterization for symplectic groups in [37]. The work and notation in this chapter will be completely independent of the rest of the thesis.

CHAPTER II

Definitions and Notation

In this chapter, we will recall basic definitions and known results about p-adic groups and p-adic symmetric spaces. We will also introduce notation which we will use for all but the very last chapter.

2.1 Fields, Groups, and Tori

Let k be a finite extension of the p-adic numbers $\mathbb{Q}_p$ for $p \neq 2$ with nontrivial discrete valuation ν.[1] Let $\overline{k}$ denote an algebraic closure of k, and let $K \leq \overline{k}$ be the maximal unramified extension of k in $\overline{k}$. Then the valuation ν extends uniquely to K, and we will also use ν to denote the valuation of the extension. We let $\mathfrak{o}_k$ (resp. $\mathfrak{o}_K$) denote the ring of integers of k (resp. K), and we fix a uniformizer ϖ for k and thus K. We let $\mathfrak{f} := \mathfrak{o}_k / < \varpi >$ and $\mathfrak{F} := \mathfrak{o}_K / < \varpi >$ denote the residue fields of k and K respectively. Then $\mathfrak{F}$ is an algebraic closure of $\mathfrak{f}$. We can and do identify $\Gamma = \mathrm{Gal}(K/k)$ with $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$, and we fix a topological generator Fr for Γ.

If $\mathcal{G}$ is a group and $x, y \in \mathcal{G}$, then we may use both $\mathrm{Int}(x)(y)$ and ${}^x y$ to denote xyx^{-1}.

If $\mathbf{G}$ is a reductive algebraic group defined over k, then we use $\mathrm{Lie}(\mathbf{G})$ or $\mathfrak{g}$ to denote its Lie algebra. We will identify $\mathbf{G}$ with its $\overline{k}$-rational points, and we will also use G instead

[1] It is quite likely that most if not all of the results in this thesis apply to the more general setting of non-archimedean local fields whose characteristic and residue characteristic are both not 2. However, the author has not checked this carefully.

of $\mathbf{G}(K)$ to denote the group of K-points of an algebraic K-group. We use $\mathbf{G}^\circ$ to denote the identity component of $\mathbf{G}$, and if τ is a k-automorphism of $\mathbf{G}$, then we write $\mathbf{G}^\tau$ for the points in $\mathbf{G}$ fixed by τ. Given a subset $\mathbf{A}$ of $\mathbf{G}$, we will use $N_{\mathbf{G}}(\mathbf{A})$ and $C_{\mathbf{G}}(\mathbf{A})$ to denote the normalizer and centralizer of $\mathbf{A}$ in $\mathbf{G}$ respectively.

When we refer to a maximal torus, we mean a maximal torus in $\mathbf{G}$ unless otherwise specified. If $\mathbf{S}$ is a k-torus in $\mathbf{G}$ and E is an extension of k, we will use $\mathbf{S}^E$ to denote the unique maximal k-torus which is contained in $\mathbf{S}$ and splits over E.

Throughout this thesis, we will frequently consider maximal k-tori in $\mathbf{G}$ which are K-split. We call such a torus a maximal unramified torus. Let $\mathbf{A}$ be a maximal unramified torus in $\mathbf{G}$ that contains a maximal k-split torus of $\mathbf{G}$. Such a torus exists and is unique up to G^{Fr}-conjugacy [29]. We denote by $\Phi = \Phi(\mathbf{G}, \mathbf{A})$ the root system of $\mathbf{G}$ with respect to $\mathbf{A}$ and by $W = W(\mathbf{G}, \mathbf{A})$ the Weyl group $N_{\mathbf{G}}(\mathbf{A})/C_{\mathbf{G}}(\mathbf{A})$. We denote by $\Psi = \Psi(\mathbf{G}, \mathbf{A}, \nu)$ the set of affine roots of $\mathbf{G}$ with respect to $\mathbf{A}$ and ν, and for $\psi \in \Psi$, we let $\dot{\psi} \in \Phi$ denote the gradient of ψ.

2.2 Involutions and p-adic Symmetric Spaces

We fix a connected reductive algebraic group $\mathbf{G}$ defined over k, and we fix an involution θ which is defined over k. If we let $\mathbf{H}$ be a linear algebraic k-group with $(\mathbf{G}^\theta)^\circ \subset \mathbf{H} \subset \mathbf{G}^\theta$, then we have that $\mathbf{H}^\circ$ is reductive by [30, 36], and $\mathbf{G}/\mathbf{H}$ defines a p-adic symmetric space.

We say that a torus $\mathbf{S}$ is θ-split if $\theta(s) = s^{-1}$ for all $s \in \mathbf{S}$. If k' is an extension of k, then we say that a k-torus is (θ, k')-split if it is both θ-split and k'-split. We call a maximal (θ, K)-split k-torus a maximal unramified θ-split torus. By [18], the condition that $\mathbf{S}$ is a maximal θ-split k'-torus is equivalent to the condition that $\mathbf{S}$ is a maximal θ-split torus that is defined over k'. Throughout the thesis, we will generally reserve $\mathbf{T}$ for maximal k-tori in $\mathbf{G}$ and $\mathbf{S}$ for maximal θ-split k-tori. The one noticable exception to this will be in Section 4.1, where $\mathbf{S}$ will play the role of a maximal K-torus in $\mathbf{G}$.

If $\mathbf{T}$ is a θ-stable torus, then again by [18], we have a decomposition $\mathbf{T} = \mathbf{T}^+\mathbf{T}^-$, where

$\mathbf{T}^+$ denotes the maximal θ-fixed torus in $\mathbf{T}$ and $\mathbf{T}^-$ is a θ-split torus equalling the identity component of the connected algebraic group $\{t \in \mathbf{T} | \theta(t) = t^{-1}\}$. The product is an *almost direct product* in the sense that the map from $\mathbf{T}^+ \times \mathbf{T}^-$ to $\mathbf{T}$ which sends (t_1, t_2) to $(t_1)^{-1} t_2$ is an isogeny whose kernel $\mathbf{T}^+ \cap \mathbf{T}^-$ is a finite 2-group. Since θ is defined over k, we have that if $\mathbf{T}$ is defined over k, then both $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined over k as well (although an element in $\mathbf{T}(k)$ cannot always be written as the product of elements of $\mathbf{T}^+(k)$ and $\mathbf{T}^-(k)$).

There are several important results about p-adic symmetric spaces that will play an important role in our analysis of maximal θ-split k-tori. First, we note that by Vust, all maximal θ-split tori are $(\mathbf{G}^\theta)^\circ$-conjugate (and hence $\mathbf{H}$-conjugate). It is not generally the case that two maximal (θ, k)-split tori are $\mathbf{H}(k)$-conjugate, but by [18], they are $\mathbf{G}(k)$-conjugate, meaning that all such tori have the same k-rank.

The difference between $\mathbf{G}(k)$-conjugacy and $\mathbf{H}(k)$-conjugacy accounts for most of the difficulty in determining the $\mathbf{H}(k)$-conjugacy classes of maximal θ-split k-tori. This difference generally stems from the fact that if $\mathbf{G}$ is not simply connected, then we will generally not have that $\mathbf{G}^\theta$ is connected. However, a result of Vust [36] often allows us to circumvent this issue. In particular, if $\mathbf{A}$ is any maximal θ-split torus in $\mathbf{G}$ and $\mathbf{C} := \mathbf{A} \cap \mathbf{G}^\theta$, then we have a decomposition $\mathbf{G}^\theta = \mathbf{C} \cdot (\mathbf{G}^\theta)^\circ$. As all of the elements of $\mathbf{A} \cap \mathbf{G}^\theta$ necessarily have order two, we see that the component group of $\mathbf{G}^\theta$ is an abelian 2-group.

A powerful result of Helminck and Wang [18] illuminates the structure of the centralizers of θ-split k-tori. Recall that $\mathbf{G}$ can be written as an almost direct product $\mathbf{Z_G}\mathbf{G}_1\mathbf{G}_2$, where $\mathbf{Z_G}$ is contained in the center of $\mathbf{G}$, $\mathbf{G}_1$ is k-anisotropic (i.e. has k-rank 0), and $\mathbf{G}_2$ is k-isotropic (i.e. has k-rank greater than 0). Then if we let $\mathbf{A}$ be a maximal (θ, k)-split k-torus in $\mathbf{G}$ and let $\mathbf{Z}$, $\mathbf{L}_1$, and $\mathbf{L}_2$ denote the central, anisotropic, and isotropic factors of $\mathbf{L} = C_{\mathbf{G}}(\mathbf{A})$ over k respectively, then Helminck and Wang show that

- $\mathbf{A}$ is the unique maximal (θ, k)-split torus of $C_{\mathbf{G}}(\mathbf{A})$.

- $\mathbf{L}_2 \subset \mathbf{H}$.

- If $\mathbf{A}_0$ is any maximal k-split torus of $C_{\mathbf{G}}(\mathbf{A})$, then $\mathbf{A}_0$ is θ-stable and $\mathbf{Z}\mathbf{L}_1 \subset C_{\mathbf{G}}(\mathbf{A}_0)$.

Applying this result when $k = \overline{k}$, we see that the centralizer of any maximal θ-split torus has its derived subgroup contained in $(\mathbf{G}^\theta)^\circ$. As another consequence, one shows that if $\mathbf{A}$ is a maximal (θ, k)-split k-torus of $\mathbf{G}$, $\mathbf{A}_0 \supset \mathbf{A}$ is a maximal k-split k-torus in $\mathbf{G}$, and $\mathbf{S} \supset \mathbf{A}$ is a maximal θ-split k-torus, then $\mathbf{A}_0$ and $\mathbf{S}$ commute. In particular, there is a maximal torus $\mathbf{T}$ in $C_{\mathbf{G}}(\mathbf{A})$ which contains both a maximal k-split torus in $\mathbf{G}$ and a maximal θ-split k-torus in $\mathbf{G}$.

We also note that there is an analogue of the Weyl group for maximal θ-split k-tori of $\mathbf{G}$. In particular, if $\mathbf{S}$ is a maximal θ-split k-torus in $\mathbf{G}$, then we define the *little Weyl group* of $\mathbf{S}$ in $\mathbf{G}$, which we will denote by $W_\theta = W_\theta(\mathbf{S}, G)$, to be $N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S})$. By [32], every element of the little Weyl group has a representative in $\mathbf{H}^\circ$ so that we may instead define it as $N_{\mathbf{H}^\circ}(\mathbf{S})/C_{\mathbf{H}^\circ}(\mathbf{S})$. The little Weyl group is in fact the Weyl group of the reduced root system for the roots of $\mathbf{S}$ in $\mathbf{G}$, and if $\mathbf{T}$ is a maximal k-torus containing $\mathbf{S}$, then the little Weyl group can be realized as the quotient of the elements of $W(\mathbf{T}, \mathbf{G})$ which normalize $\mathbf{S}$ by the elements of $W(\mathbf{T}, \mathbf{G})$ which centralize $\mathbf{S}$. Since the derived subgroup of $C_{\mathbf{G}}(\mathbf{S})$ is contained in $\mathbf{H}^\circ$, we also have that the image of any element in $N_{\mathbf{H}^\circ}(\mathbf{S})$ in the little Weyl group has a representative in $N_{\mathbf{H}^\circ}(\mathbf{T})$. As $C_{\mathbf{H}^\circ}(\mathbf{S})$ is frequently disconnected, we will generally have to work with variants of this object in our parameterizations.

2.3 The Bruhat-Tits Building and Symmetric Spaces

We let $\mathcal{B}(G)$ denote the (enlarged) Bruhat-Tits building of G, and we note $\mathcal{B}(G)^\Gamma$ (which we also sometimes write as $\mathcal{B}(G)^{\mathrm{Fr}}$) is the (enlarged) Bruhat-Tits building of $\mathbf{G}(k)$. Similarly, we let $\mathcal{B}(H)$ denote the (enlarged) Bruhat-Tits building of H°, from which we then have that $\mathcal{B}(H)^\Gamma$ (or $\mathcal{B}(H)^{\mathrm{Fr}}$) is the building of $\mathbf{H}^\circ(k)$.

If $\mathbf{T}$ (resp. $\mathbf{T}'$) is a maximal K-split torus in $\mathbf{G}$ (resp. $\mathbf{H}$), then we will let $\mathcal{A}(\mathbf{T})$ (resp. $\mathcal{A}(\mathbf{T}')$) denote the associated apartment in $\mathcal{B}(G)$ (resp. $\mathcal{B}(H)$). If $\mathcal{A}$ is an apartment in one of our various buildings and Ω is a subset of $\mathcal{A}$, then we let $A(\mathcal{A}, \Omega)$ denote the smallest affine subspace of $\mathcal{A}$ containing Ω.

For $x \in \mathcal{B}(G)$, we let G_x and G_x^+ denote the parahoric subgroup associated to x and its pro-unipotent radical. Recall that both G_x and G_x^+ only depend on the facet F in $\mathcal{B}(G)$ containing x so that we may write G_F and G_F^+ for G_x and G_x^+ respectively. If F is Γ-stable, then the quotient G_F/G_F^+ is the group of $\mathfrak{F}$-points of a connected, reductive group G_F defined over $\mathfrak{f}$. If $x \in \mathcal{B}(G)^\Gamma$, then we have that Γ acts on G_x and G_x^+, and the quotient of their Γ-fixed points coincides with the group of $\mathfrak{f}$-rational points of the connected reductive group G_x defined over $\mathfrak{f}$. We also have that $\mathsf{G}_x(\mathfrak{f}) = \mathsf{G}_x(\mathfrak{F})^\Gamma$. We can define these notions analogously for H.

If $\mathbf{A}$ is a maximal unramified torus in $\mathbf{G}$ containing a maximal k-split torus of $\mathbf{G}$ and $\mathcal{A}(\mathbf{A})$ is the apartment in $\mathcal{B}(G)$ corresponding to $\mathbf{A}$, then for a facet F in $\mathcal{A}(\mathbf{A})^{\mathrm{Fr}}$, we have that the image of $A \cap G_F$ in G_F, which we call A_F, is a maximally $\mathfrak{f}$-split maximal $\mathfrak{f}$-torus in G_F. In other words, A_F is a maximal $\mathfrak{f}$-torus in G_F which contains a maximal $\mathfrak{f}$-split torus of G_F. If we denote by W_F the Weyl group $N_{\mathsf{G}_F}(\mathsf{A}_F)/\mathsf{A}_F$, then we may identify W_F with a subgroup of W. We let Ψ_F denote the set of affine roots of $\mathbf{A}$ that vanish on F, and we let Φ_F denote the corresponding set of gradients. We then let ${}_F\mathbf{M}$ denote the Levi k-subgroup which contains $\mathbf{A}$ and corresponds to Φ_F, and we recall that ${}_F\mathsf{M}_F = \mathsf{G}_F$. For a facet F in $\mathcal{A}(A)$, we let $\tilde{W}$ denote the affine Weyl group $N_G(\mathbf{A})/(C_G(\mathbf{A}) \cap G_F)$. Again we may define all of these objects analogously for a maximal unramified torus in $\mathbf{H}$.

The action of θ on G and $\mathbf{G}(k)$ induces an involution on $\mathcal{B}(G)$ and $\mathcal{B}(G)^{\mathrm{Fr}}$ which we will also denote by θ. Prasad and Yu [30] show that we can identify $\mathcal{B}(H)$ with $\mathcal{B}(G)^\theta$, and thus we can also identify $\mathcal{B}(H)^{\mathrm{Fr}}$ with $(\mathcal{B}(G)^{\mathrm{Fr}})^\theta$. The facets in $\mathcal{B}(H)$ are too large for parameterizing maximal θ-split k-tori. Thus we will be particularly concerned with θ-facets, which are defined by Portilla in [27, 28] to be a nonempty subset $F \subset \mathcal{B}(H)$ which is the set of θ-fixed points of some facet F' in $\mathcal{B}(G)$.

In studying θ-facets, it will be important to understand how the parahoric subgroup G_x

of G corresponding to a point x in $\mathcal{B}(G)^\theta$ relates to the corresponding parahoric subgroup H_x of H. Upon restricting to the respective pro-unipotent radicals, we have that $G_x^+ \cap H = H_x^+$ by [27]. However, a similar result does not hold for the parahoric subgroups, and we only have that $(\mathsf{G}_x^\theta)^\circ = \mathsf{H}_x$, again by [27].

To see why $G_x \cap H$ does not necessarily equal H_x, we look at the example of $\mathbf{G} = \mathrm{PGL}_2$, where we define θ to be the inner automorphism $\mathrm{Int}(m)$, where m is the image of the element

$$\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$$

in PGL_2. Then H° is a maximal tamely ramified elliptic torus, and so the building of H is a point x. In particular, x is the midpoint of an alcove lying in the apartment of the diagonal torus. However, the diagonal torus is θ-split, and so the reductive quotient G_x is a θ-split torus. But a θ-split torus has finitely many θ-fixed points (in particular, the order two elements), and so the θ-fixed points are disconnected. Thus we see that G_x^θ is not H_x, which is the trivial group. By Hakim-Murnaghan, the θ-cohomology of G_x^+ is trivial, and so each element of G_x^θ has a representative in $G_x \cap H$, meaning we have the desired counterexample.

Even so, with Portilla's result on the pro-unipotent radicals, he is still able to prove that the points in a given θ-facet all lie in the same facet in the building of H. Thus every facet in $\mathcal{B}(H)$ can be written as the union of θ-facets, and we have that $H_x = H_y$ for all points x and y in a given θ-facet F, allowing us to write H_F instead of H_x or H_y.

CHAPTER III

Tori over the Residue Field

The goal of this section is to parameterize conjugacy classes of θ-split tori in finite groups of Lie type. A less general version of this result was attempted in [28]. Let G be a connected, reductive group defined over a finite field $\mathfrak{f}$ with characteristic $p \neq 2$. Let θ be an involution of G defined over $\mathfrak{f}$, and let $(\mathsf{G}^\theta)^\circ \subseteq \mathsf{H} \subseteq \mathsf{G}^\theta$. Let S be a maximal θ-split $\mathfrak{f}$-torus in G, an $\mathfrak{f}$-torus so that $\theta(s) = s^{-1}$ for all $s \in \mathsf{S}$. Recall that the little Weyl group of S is defined to be $W_\theta = N_{\mathsf{G}}(\mathsf{S})/C_{\mathsf{G}}(\mathsf{S})$, and recall that every element of the little Weyl group has a representative in H° so that we may instead define it as $N_{\mathsf{H}^\circ}(\mathsf{S})/C_{\mathsf{H}^\circ}(\mathsf{S})$. (See [32, 24].) Define the extended little Weyl group by $W_{\theta,c} = N_{\mathsf{H}}(\mathsf{S})/(C_{\mathsf{H}}(\mathsf{S}))^\circ$, and we consider the subgroup $W'_{\theta,c} := N_{\mathsf{H}^\circ}(\mathsf{S})/(C_{\mathsf{H}}(\mathsf{S}))^\circ$ of $W_{\theta,c}$.

Let $\mathcal{S}_1$ denote the set of $\mathsf{H}(\mathfrak{f})$-conjugacy classes of maximal θ-split $\mathfrak{f}$-tori. Define a relation $\sim$ on $W'_{\theta,c}$ by saying $w \sim w'$ if there is an x in $W_{\theta,c}$ so that $w = xw'\mathrm{Fr}(x^{-1})$. One checks that this defines an equivalence relation on $W_{\theta,c}$

Recall from [36] that if S' is another maximal θ-split $\mathfrak{f}$-torus in G, then there is an element h in H° so that $\mathsf{S}' = {}^h\mathsf{S}$. Modeling Carter's parameterization of conjugacy classes of maximal tori in a finite group of Lie type in [8], we use this result to show that there is a bijective correspondence between $\mathcal{S}_1$ and $W'_{\theta,c}/\sim$.

Proposition 3.0.1. The map $\sigma : \mathcal{S}_1 \to W'_{\theta,c}/\sim_F$ given by ${}^h\mathsf{S} \mapsto h^{-1}\mathrm{Fr}(h)$ is a well-defined bijection.

Proof. If ${}^h\mathsf{S}$ is a maximal θ-split $\mathfrak{f}$-torus in G, then we have

$$ {}^h\mathsf{S} = \mathrm{Fr}({}^h\mathsf{S}) = {}^{\mathrm{Fr}(h)}\mathrm{Fr}(\mathsf{S}) = {}^{\mathrm{Fr}(h)}\mathsf{S} $$

so that $h^{-1}\mathrm{Fr}(h)$ is in $N_{\mathsf{H}^\circ}(\mathsf{S})$.

Now suppose that S_1 and S_2 are maximal θ-split $\mathfrak{f}$-tori for which there is some h in $\mathsf{H}(\mathfrak{f})$ such that $\mathsf{S}_1 = {}^h\mathsf{S}_2$. Suppose $\mathsf{S}_1 = {}^{h_1}\mathsf{S}$ and $\mathsf{S}_2 = {}^{h_2}\mathsf{S}$ for some h_1 and h_2 in H°. We have that $h_1^{-1}hh_2$ is an element of $N_{\mathsf{H}}(\mathsf{S})$. Then we see that

$$ (h_1^{-1}hh_2)(h_2^{-1}\mathrm{Fr}(h_2))\mathrm{Fr}((h_1^{-1}hh_2)^{-1}) = h_1^{-1}\mathrm{Fr}(h_1), $$

showing that the image of $h_2^{-1}\mathrm{Fr}(h_2)$ in $W'_{\theta,c}$ is Frobenius conjugate to the image of $h_1^{-1}\mathrm{Fr}(h_1)$ by the image of $(h_1^{-1}hh_2)$ in $W_{\theta,c}$. Hence our map is well-defined.

We now need to show that the map is injective. Suppose that we have elements h_1 and h_2 in H° for which ${}^{h_1}\mathsf{S}$ and ${}^{h_2}\mathsf{S}$ determine the same equivalence class in $W'_{\theta,c}$. Letting π denote the projection map from $N_{\mathsf{H}}(S)$ to $W_{\theta,c}$, we have that there is some element n in $N_{\mathsf{H}}(\mathsf{S})$ so that

$$ \pi(h_2^{-1}\mathrm{Fr}(h_2)) = \pi(nh_1^{-1}\mathrm{Fr}(h_1)\mathrm{Fr}(n^{-1})). $$

Rearranging terms, we have that

$$ t := nh_1^{-1}\mathrm{Fr}(h_1)\mathrm{Fr}(n^{-1})\mathrm{Fr}(h_2^{-1})h_2 $$

is an element of $(C_{\mathsf{H}}(\mathsf{S}))^\circ$. Conjugating by h_2, we then have that

$$ {}^{h_2}t = h_2nh_1^{-1}\mathrm{Fr}(h_1)\mathrm{Fr}(n^{-1})\mathrm{Fr}(h_2^{-1}) $$

is an element of $(C_{\mathsf{H}}({}^{h_2}\mathsf{S}))^\circ$. Applying Lang-Steinberg, we can find an element u in $(C_{\mathsf{H}}(\mathsf{S}))^\circ$

so that $({}^{h_2}u)^{-1}\mathrm{Fr}({}^{h_2}u) = {}^{h_2}t$. Rearranging terms, we then find that

$$h_2unh_1^{-1} = \mathrm{Fr}(h_2unh_1^{-1}),$$

so that $h_2unh_1^{-1}$ is in $\mathsf{H}(\mathfrak{f})$. But we know that un is an element of $N_{\mathsf{H}}(\mathsf{S})$, so we have that

$${}^{h_2unh_1^{-1}}({}^{h_1}S) = {}^{h_2}S,$$

meaning that the tori are rationally conjugate and that we have injectivity.

Finally, for surjectivity, note that if w is in $W'_{\theta,c}$, we can find an element $n \in N_{\mathsf{H}^\circ}(\mathsf{S})$ so that $\pi(n) = w$. Then applying Lang-Steinberg, we can find an element $h \in \mathsf{H}^\circ$ so that $h^{-1}\mathrm{Fr}(h) = n$, and so we have surjectivity, completing the proof. $\square$

Note that we can analogously prove that the $\mathsf{G}(\mathfrak{f})$-conjugacy classes of maximal θ-split $\mathfrak{f}$-tori are parameterized by Frobenius conjugacy classes in the little Weyl group W_θ. The component group $C_{\mathsf{H}^\circ}(\mathsf{S})/(C_{\mathsf{H}^\circ}(\mathsf{S}))^\circ$ thus controls the difference between the $\mathsf{H}(\mathfrak{f})$-conjugacy classes and $\mathsf{G}(\mathfrak{f})$-conjugacy classes. We will perform a similar type of measurement when parameterizing $H(k)$-conjugacy classes of K-minisotropic maximal θ-split k-tori in the p-adic setting.

CHAPTER IV

Results over the Maximal Unramified Extension

We let $\mathbf{G}$ be a connected reductive group defined over a p-adic field k with $p \neq 2$. We assume that $\mathbf{G}$ is tame, i.e. that p does not divide the order of the Weyl group. Let θ be an involution of $\mathbf{G}$ defined over k, and let $\mathbf{H}$ be a subgroup so that $(\mathbf{G}^\theta)^\circ \subseteq \mathbf{H} \subseteq \mathbf{G}^\theta$. We will use $\mathbf{G}(k), G, \mathbf{G}$ for the k, K, and $\overline{k}$ points of $\mathbf{G}$, where K is the maximal unramified extension of k in $\overline{k}$, and we will use similar notation for $\mathbf{H}$ and other subgroups of $\mathbf{G}$. We will parameterize tame twisted Levi K-subgroups up to G-conjugacy. Then we will parameterize θ-split tori up to H-conjugacy, and we will then discuss which tame twisted Levi K-subgroups arise as the centralizer of a θ-split torus. Finally, we will parameterize the embeddings of maximal θ-split tori up to H-conjugacy as well as the conjugacy classes of θ-regular semisimple elements lying in a fixed maximal θ-split torus $\mathbf{S}$.

4.1 Tame Twisted Levis over K

We suppose $\mathbf{G}$ is a connected reductive group defined over K, and we fix a Borel K-subgroup $\mathbf{B}$ of $\mathbf{G}$. We also fix a maximal K-torus $\mathbf{S}$ of $\mathbf{B}$, and we let $\Delta_{\mathbf{G}} = \Delta(\mathbf{G}, \mathbf{B}, \mathbf{S})$ be the corresponding set of simple roots. We denote the Weyl group of $\mathbf{S}$ in $\mathbf{G}$ by $W_{\mathbf{G}} = W(\mathbf{G}, \mathbf{S})$. For $\pi \subset \Delta_{\mathbf{G}}$, we let $W_{\mathbf{G},\pi}$ denote the corresponding parabolic subgroup of $W_{\mathbf{G}}$. Suppose the residue characteristic of K does not divide the order of the Weyl group, and let σ be a topological generator of $\mathrm{Gal}(E/K)$, where E is a tame extension of K over which all maximal

K-tori of G split.

Recall that we call a subgroup $\mathbf{L}$ of $\mathbf{G}$ a twisted Levi K-subgroup if it is defined over K and if there exists a parabolic E-subgroup $\mathbf{P}$ of $\mathbf{G}$ so that $\mathbf{L}$ is a Levi component of $\mathbf{P}$. We seek to study the set $\mathcal{L}$ of twisted Levi K-subgroups up to G-conjugacy, modeling the work in [1] for quasi-finite fields and the parameterization in [2] of maximal K-tori up to G-conjugacy. As in [1], we define a *Levi torus* to be a K-torus $\mathbf{T}$ in $\mathbf{G}$ that is equal to the identity component of the center of $C_{\mathbf{G}}(\mathbf{T})$, and if $\mathbf{L}$ is a twisted Levi K-subgroup, then we let $\mathbf{T}_{\mathbf{L}}$ be the corresponding Levi torus, i.e. $\mathbf{T}_{\mathbf{L}}$ is the identity component of the center of $\mathbf{L}$. Since $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T}_{\mathbf{L}})$, understanding $\mathcal{L}$ up to G-conjugacy is equivalent to understanding the set of Levi tori up to G-conjugacy.

Let $\mathcal{L}/\sim_G$ be the set of G-conjugacy classes in $\mathcal{L}$. We will relate $\mathcal{L}/\sim_G$ to the set I_G consisting of pairs (π, w) where $\pi \subset \Delta_{\mathbf{G}}$ and $w \in W_{\mathbf{G}}$ such that $\sigma(\pi) = w\pi$. For two pairs (π', w') and (π, w) in I_G, we will say that $(\pi', w') \sim (\pi, w)$ if there is an element u in $W_{\mathbf{G}}$ for which

- $\pi = u\pi'$
- $w = \sigma(u)w'u^{-1}$.

One may check that this defines an equivalence relation on I_G.

Proposition 4.1.1. There is a natural bijective correspondence between $I_G/\sim$ and $\mathcal{L}/\sim_G$.

Proof. We begin by defining a map $\phi : I_G \to \mathcal{L}/\sim_G$. To do so, suppose we have a pair (π, w) in I_G. Looking at the proof of surjectivity in the parameterization of maximal K-tori up to G-conjugacy in [2], we can find $g \in \mathbf{G}(E)$ so that $n_g := \sigma(g^{-1})(g)$ lies in $N_{\mathbf{G}(E)}(\mathbf{S})$ and has image w in $W_{\mathbf{G}}$. Let $\mathbf{S}_\pi = (\cap_{\alpha\in\pi}\ker(\alpha))^\circ$ and $\mathbf{M}_\pi = C_{\mathbf{G}}(\mathbf{S}_\pi)$. Because $\sigma(\pi) = w\pi$, we have

$$\sigma({}^{g}\mathbf{M}_\pi) = {}^{\sigma(g)}\sigma(\mathbf{M}_\pi) = {}^{gn_g^{-1}}(\mathbf{M}_{\sigma(\pi)}) = {}^{g}({}^{n_g^{-1}}(\mathbf{M}_{w\pi})) = {}^{g}\mathbf{M}_\pi,$$

so that ${}^{g}\mathbf{M}_\pi$ is a twisted Levi K-subgroup of $\mathbf{G}$ with $\mathbf{T}_{{}^{g}\mathbf{M}_\pi} = {}^{g}\mathbf{T}_{\mathbf{M}_\pi} = {}^{g}\mathbf{S}_\pi$.

We need to show that this construction is independent of our choice of g. In other words, we need to show that a different choice results in a twisted Levi K-subgroup that is G-conjugate to ${}^{g}\mathbf{M}_{\pi}$. Choose $g_i \in \mathbf{G}(E)$ $(i = 1, 2)$ so that $\sigma(g_i^{-1})g_i$ has image w in $W_{\mathbf{G}}$. Then there is some s in $\mathbf{S}(E)$ so that $\sigma(g_1^{-1})g_1 = \sigma(g_2^{-1})g_2 s$. Fix an element X of $\mathrm{Lie}({}^{g_1}\mathbf{T}_{\mathbf{M}_\pi})(K)$ so that $C_{\mathbf{G}}(X) = {}^{g_1}\mathbf{M}_{\pi}$. (Such an X exists. The elements of $\mathrm{Lie}({}^{g_1}\mathbf{T}_{\mathbf{M}_\pi})$ that do not vanish on any root of $(g_1 \cdot \Delta_G) \setminus (g_1 \cdot \pi)$ is non-empty and open in $\mathrm{Lie}({}^{g_1}\mathbf{T}_{\mathbf{M}_\pi})$, and tori are unirational. Thus there is an element of $\mathrm{Lie}({}^{g_1}\mathbf{T}_{\mathbf{M}_\pi})(K)$ which does not vanish on any root of $(g_1 \cdot \Delta_G) \setminus (g_1 \cdot \pi)$ and so that $C_{\mathbf{G}}(X) = C_{\mathbf{G}}({}^{g_1}\mathbf{T}_{\mathbf{M}_\pi}) = {}^{g_1}\mathbf{M}_{\pi}$.) Then we have

$$\sigma({}^{g_2 g_1^{-1}}X) = {}^{g_2 s g_1^{-1}}X = {}^{g_2 g_1^{-1}}X.$$

Thus ${}^{g_2 g_1^{-1}}X$ lies in $\mathrm{Lie}({}^{g_2}\mathbf{T}_{M_\pi})(K)$. However, since the Galois cohomology $\mathrm{H}^1(K, C_{\mathbf{G}}(X))$ is trivial by [33], we have that X and ${}^{g_2 g_1^{-1}}X$ are G-conjugate, and so $C_{\mathbf{G}}(X) = {}^{g_1}\mathbf{M}_{\pi}$ and $C_{\mathbf{G}}({}^{g_2 g_1^{-1}}X) = {}^{g_2}\mathbf{M}_{\pi}$ are also G-conjugate so that ϕ is well-defined.

We claim that ϕ descends to a map from $I_G/\sim$ to $\mathcal{L}/\sim_G$, which we will also call ϕ. Suppose $(\pi, w) \sim (\pi', w')$ with $w' = \sigma(u)wu^{-1}$, and choose $g \in G(E)$ and $n' \in N_{\mathbf{G}(E)}(\mathbf{S})$ so that $\sigma(g^{-1})g$ has image w in W and n' has image u in W. Then $g' = g(n')^{-1}$ is an element of $\mathbf{G}(E)$ so that $\sigma(g'^{-1})g'$ has image w' in W. We then see that ${}^{g'}\mathbf{M}_{\pi'} = {}^{g}\mathbf{M}_{\pi}$ so that $\phi(\pi, w) = \phi(\pi', w')$.

We now need to show that the map ϕ is injective. To see this, suppose we have (π, w) and (π', w') in I_G so that $\phi(\pi, w) = \phi(\pi', w')$. Choose g and g' in $\mathbf{G}(E)$ so that images of $n_g = \sigma(g)^{-1}g$ and $n_{g'} = \sigma(g')^{-1}g'$ in $W_{\mathbf{G}}$ are w and w' respectively. Replacing g with kg for some k in G, we may assume without loss of generality that ${}^{g}\mathbf{M}_{\pi} = {}^{g'}\mathbf{M}_{\pi'}$ and hence that ${}^{g}\mathbf{S}$ and ${}^{g'}\mathbf{S}$ are maximal E-split K-tori in ${}^{g}\mathbf{M}_{\pi}$. Consequently, there is an element $m' = {}^{g}m$ for m in $\mathbf{M}_{\pi}(E)$ so that ${}^{m'g'}\mathbf{S} = {}^{g}\mathbf{S}$ and ${}^{m'g'}(\mathbf{B} \cap \mathbf{M}_{\pi}) = {}^{g}(\mathbf{B} \cap \mathbf{M}_{\pi})$. Since we also have

$${}^{\sigma(m')}({}^{g'}\mathbf{S}) = {}^{\sigma(m')}\sigma({}^{g'}\mathbf{S}) = \sigma({}^{m'g'}\mathbf{S}) = {}^{m'g'}\mathbf{S},$$

we may conclude that $m'\sigma(m')^{-1}$ is an element of $N_{{}^g\mathbf{M}_\pi(E)}({}^g\mathbf{S})$, which then implies that $g^{-1}(m'\sigma(m')^{-1})^{-1}g = n_g^{-1}\sigma(m)n_g m^{-1}$ is an element of $N_{\mathbf{M}_\pi(E)}(\mathbf{S})$. Now set $n = mg^{-1}g' = g^{-1}m'g'$, which we note is an element of $N_{\mathbf{G}(E)}(\mathbf{S})$ so that $n\pi' = \pi$ (because ${}^{m'g'}(\mathbf{B}\cap\mathbf{M}_\pi) = {}^g(\mathbf{B}\cap\mathbf{M}_\pi)$). We have

$$\sigma(n)(n_{g'})n^{-1} = \sigma(mg^{-1}g')\sigma(g')^{-1}g'(mg^{-1}g')^{-1} = \sigma(m)(n_g)m^{-1} = n_g(n_g^{-1}\sigma(m)(n_g)m^{-1}),$$

and so looking at images in $W_\mathbf{G}$, we have that $\sigma(\overline{n})w'\overline{n}^{-1}$ is in the coset $wW_{\mathbf{G},\pi}$, where $\overline{n}$ denotes the image of n in $W_\mathbf{G}$. Choose x in $W_{\mathbf{G},\pi}$ so that $\sigma(\overline{n})w'\overline{n}^{-1} = wx$, and note that

$$x\pi = w^{-1}\sigma(\overline{n})w'\overline{n}^{-1}\pi = w^{-1}\sigma(\overline{n})w'\pi' = w^{-1}\sigma(\overline{n})\sigma(\pi') = w^{-1}\sigma(\pi) = \pi.$$

But the action of $W_{\mathbf{G},\pi}$ on the set of bases for the root system spanned by π is simply transitive, and so we have $x = 1$ and $(\pi, w) \sim (\pi', w')$ as desired.

It remains to show that our map is surjective. Suppose that $\mathbf{L}$ is in $\mathcal{L}$. Let $\mathbf{T_L}$ be the connected component of the center of $\mathbf{L}$. Choose a Borel K-subgroup $\mathbf{B_L}$ in $\mathbf{L}$ and a maximal K-torus $\mathbf{S_L}$ in $\mathbf{B_L}$. Denote by $\Delta_\mathbf{L} = \Delta(\mathbf{L}, \mathbf{B_L}, \mathbf{S_L})$ the corresponding set of simple roots, and choose g in $\mathbf{G}(E)$ so that $\mathbf{S_L} = {}^g\mathbf{S}$ and $\mathbf{B_L} \leq {}^g\mathbf{B}$. Define $\pi_\mathbf{L} = g^{-1}\cdot\Delta_\mathbf{L}$, and note that $\pi_\mathbf{L} \subset \Delta_\mathbf{G}$. Let $w_\mathbf{L}$ denote the image of $\sigma(g^{-1})g$ in $W_\mathbf{G}$ and put $\mathbf{S}_{\pi_\mathbf{L}} = (\cap_{\alpha\in\pi_\mathbf{L}}\ker(\alpha))^\circ \leq \mathbf{S}$. Then we have

- $\mathbf{T_L} = {}^g\mathbf{S}_{\pi_\mathbf{L}}$ and
- $\sigma(\pi_\mathbf{L}) = w_\mathbf{L}\pi_\mathbf{L}$

Thus we have constructed a pair $(\pi_\mathbf{L}, w_\mathbf{L})$ in I_G whose image under ϕ is the G-conjugacy class of $\mathbf{L}$.

□

4.2 Maximal θ-split Tori over K

We now seek to parameterize the maximal θ-split K-tori up to H-conjugacy. We are forced to take an approach using Galois cohomology.

Fix a maximal θ-split K-torus $\mathbf{S}$, and let X be the set of all maximal θ-split tori in $\mathbf{G}$. Let $W_\theta = N_{\mathbf{H}}(\mathbf{S})/C_{\mathbf{H}}(\mathbf{S})$ and $W_{\theta,c} = N_{\mathbf{H}}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^\circ$ be the little Weyl group and connected little Weyl group of $\mathbf{S}$ respectively. Consider the exact sequence

$$0 \to N_{\mathbf{H}}(\mathbf{S}) \to \mathbf{H} \to X \to 0.$$

Taking the Galois cohomology sequence over K, we have

$$0 \to N_H(\mathbf{S}) \to H \to X(K) \to \mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S})) \to \mathrm{H}^1(K, \mathbf{H}).$$

Then the set of H-conjugacy classes is parameterized by the kernel of the map $\mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S})) \to \mathrm{H}^1(K, \mathbf{H})$. In order to compute this kernel, we will begin by computing $\mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S}))$.

We slightly revise our definition of σ from the previous section to let it be a topological generator for the Galois group of the maximal tame extension K_{tame} of K in the maximal separable extension K_{sep} in $\overline{k}$.

Lemma 4.2.1. Assume that p does not divide the order of the Weyl group of $\mathbf{G}$. Then $\mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S}))$ is in bijection with the set of σ-conjugacy classes in $W_{\theta,c}$.

Proof. First consider the exact sequence

$$0 \to (C_{\mathbf{H}}(\mathbf{S}))^\circ \to N_{\mathbf{H}}(\mathbf{S}) \to W_{\theta,c} \to 0.$$

Then by [33], the Galois cohomology sequence gives us a bijection between $\mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S}))$ and $\mathrm{H}^1(K, W_{\theta,c})$, and so it suffices to compute the cohomology $\mathrm{H}^1(K, W_{\theta,c})$.

To do so, first note that any prime dividing the order of W_θ also divides the order of W.

This is because if $\mathbf{T}$ is a maximal torus in $\mathbf{G}$ containing $\mathbf{S}$ and n is in $N_{\mathbf{H}}(\mathbf{S})$, then $\mathbf{T}$ and ${}^n\mathbf{T}$ are $C_{\mathbf{G}}(\mathbf{S})$-conjugate. Since the derived group of $C_{\mathbf{G}}(\mathbf{S})$ is θ-fixed by [18], they are actually $C_{\mathbf{H}^\circ}(\mathbf{S})$-conjugate, meaning that n is equivalent in W_θ to an element of $N_{\mathbf{H}^\circ}(\mathbf{T})$ and thus that W_θ is isomorphic to a quotient of a subgroup of W (since W_θ is defined by modding $N_{\mathbf{H}}(\mathbf{T})$ by a subgroup containing $\mathbf{T}$).

Now $W_{\theta,c}$ is an extension of W_c by a 2-group, since by [36], the component group of $C_{\mathbf{H}}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^\circ$ is a finite abelian 2-group. Since 2 clearly divides the order of W, we have that each prime dividing the order of $W_{\theta,c}$ divides the order of W as well.

Now since the residue characteristic p does not divide the order of W by assumption, we see that there can be no non-trivial cocycles from $\mathrm{Gal}(K_{sep}/K_{tame})$ to $W_{\theta,c}$, as the former is a pro-p group and the order of the latter is not divisible by p by our tameness assumption.

Thus it suffices to consider $\mathrm{H}^1(\mathrm{Gal}(K_{tame}/K), W_{\theta,c})$. Note that $\mathrm{Gal}(K_{tame}/K)$ is isomorphic to the product of the groups $\mathbb{Z}_l$ for $l \neq p$ and is topologically generated by σ. Thus a cocycle is determined entirely by the image of σ, and since $W_{\theta,c}$ is finite, we can form a cocycle by mapping σ to any element of $W_{\theta,c}$ (see e.g. [11, 2.1.2]). Then two cocycles are cohomologous if and only if the respective images of σ in $W_{\theta,c}$ are σ-conjugate in $W_{\theta,c}$, meaning we are done.

□

Consequently, if we have that $\mathbf{H}$ is connected (e.g. if $\mathbf{G}$ is simply connected by e.g. [30]), or if $\mathbf{H}$-conjugacy is equivalent to $\mathbf{H}^\circ$-conjugacy (e.g. if $\mathbf{G}$ splits over K or, more generally, if $\mathbf{G}$ contains a maximal θ-split K-torus which splits over K since by Vust, the component group of $\mathbf{H}$ has representatives in the order two elements of some fixed maximal θ-split K-torus, which are all K-rational if the maximal θ-split torus splits over K), then we have our parameterization since $\mathrm{H}^1(K, \mathbf{H}^\circ)$ vanishes by Steinberg's theorem ([33]). To deal with the case of a general $\mathbf{G}$ and $\mathbf{H}$, first consider the map of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N_{\mathbf{H}^\circ}(\mathbf{S}) & \longrightarrow & \mathbf{H}^\circ & \longrightarrow & X & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & N_{\mathbf{H}}(\mathbf{S}) & \longrightarrow & \mathbf{H} & \longrightarrow & X & \longrightarrow & 1
\end{array}$$

where the vertical maps are given by the obvious inclusions. We then obtain a commutative diagram of the Galois cohomology sequences:

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & N_{H^\circ}(\mathbf{S}) & \longrightarrow & H^\circ & \longrightarrow & X(K) & \longrightarrow & \mathrm{H}^1(K, N_{\mathbf{H}^\circ}(\mathbf{S})) & \longrightarrow & \mathrm{H}^1(K, \mathbf{H}^\circ) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & N_{H}(\mathbf{S}) & \longrightarrow & H & \longrightarrow & X(K) & \longrightarrow & \mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S})) & \longrightarrow & \mathrm{H}^1(K, \mathbf{H})
\end{array}$$

Now suppose that w is in the kernel of the map $\mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S})) \to \mathrm{H}^1(K, \mathbf{H})$. Then it is in the image of the map $X(K) \to \mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S}))$, and so by the commutativity of the above diagram, w is also the image of an element w' in $\mathrm{H}^1(K, N_{\mathbf{H}^\circ}(\mathbf{S}))$ under the vertical map $\mathrm{H}^1(K, N_{\mathbf{H}^\circ}(\mathbf{S})) \to \mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S}))$. On the other hand, since $\mathrm{H}^1(K, \mathbf{H}^\circ)$ is trivial, the commutativity of our diagram tells us that every element of $\mathrm{H}^1(K, N_{\mathbf{H}^\circ}(\mathbf{S}))$ maps to an element in the kernel of the map $\mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S})) \to \mathrm{H}^1(K, \mathbf{H})$. Thus we see that the kernel of the map $\mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S})) \to \mathrm{H}^1(K, \mathbf{H})$ is precisely the image of the map $\mathrm{H}^1(K, N_{\mathbf{H}^\circ}(\mathbf{S})) \to \mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S}))$. By the proof of our lemma, we know that $\mathrm{H}^1(K, N_{\mathbf{H}^\circ}(\mathbf{S}))$ is the set of σ-conjugacy classes in $W'_{\theta,c} := N_{\mathbf{H}^\circ}(\mathbf{S})/(C_{\mathbf{H}^\circ}(\mathbf{S}))^\circ$, and the latter is the set of σ-conjugacy classes in $W_{\theta,c}$.[1] Thus we have

Proposition 4.2.2. The H-conjugacy classes of maximal θ-split K-tori are in bijection with the the elements of $W'_{\theta,c}$, modulo σ-conjugacy by the elements of $W_{\theta,c}$.

Note that by replacing $\mathbf{H}$ and $\mathbf{H}^\circ$ with $\mathbf{G}$ in the proof of the previous proposition allows us to show that the G-conjugacy classes of maximal θ-split K-tori are in bijection with the σ-conjugacy classes in the little Weyl group W_θ. Note that by [32] every element in the little Weyl group has a representative in $\mathbf{H}^\circ$, and by looking at its image in the extended little

[1]Note that we could avoid using commutative diagrams in this argument by noting that all the maximal θ-split K-tori are $\mathbf{H}^\circ$-conjugate by [36], meaning that we can just directly show that every cocycle in the kernel is cohomologous to a cocycle in $\mathbf{H}^\circ$.

Weyl group $W_{\theta,c}$, we see from our proof that there is an element $h \in \mathbf{H}^\circ$ so that $h^{-1}\sigma(h)$ has the same image in $W_{\theta,c}$ and hence in W_θ as well. Thus we see that every G-conjugacy class contains a maximal θ-split K-torus, and we have that the difference between the σ-conjugacy classes in W_θ and those in $W_{\theta,c}$ measures the difference between G-conjugacy and H-conjugacy of maximal θ-split K-tori in $\mathbf{G}$.

We now ask which of the G-conjugacy classes of twisted Levi K-subgroups from Proposition 4.1 have a representative which emerges as the centralizer of a θ-split torus in $\mathbf{G}$. In other words, by [18], we want to know which twisted Levis occur in the Levi decomposition of a θ-split parabolic subgroup over $\overline{k}$, i.e. a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ so that $\theta(\mathbf{P}) \cap \mathbf{P}$ is a Levi subgroup. First, note that we may choose $\mathbf{T}$ to be a maximal K-torus which is both θ-stable and contains a maximal θ-split torus $\mathbf{S}$ [18]. Fix a θ-basis Δ_θ for the roots of $\mathbf{T}$ in $\mathbf{G}$, i.e. a simple system for the roots of $\mathbf{T}$ in $\mathbf{G}$ as in [17] so that every simple root is either fixed by θ or sent to a negative root by θ, and let Δ_θ^+ be the roots in Δ_θ fixed by θ. Then by [17], every θ-split parabolic subgroup is $\mathbf{H}^\circ$-conjugate to one of the form $\mathbf{P}_\pi$ for π a subset of Δ_θ containing Δ_θ^+ whose span is θ-stable. Thus we see that any pair (π, w) in $I_{\mathbf{G}}$ corresponding to the Levi of a θ-split parabolic must satisfy

1. π is a subset of Δ_θ containing Δ_θ^+ whose span is θ-stable and

2. w is the image of an element in $N_{\mathbf{H}^\circ}(\mathbf{T}) \subseteq N_{N_{\mathbf{G}}(\mathbf{S})}(\mathbf{T})$.

The previous paragraph is enough for what we ultimately want to do, but one may also ask the question of whether all pairs (π, w) satisfying the two conditions above emerge as the Levi of a θ-split parabolic subgroup over $\overline{k}$. Given a pair $(\pi, w) \in I_{\mathbf{G}}$ so that π is a subset of Δ_θ containing Δ_θ^+ whose span is θ-stable and that w is the image of an element n in $N_{\mathbf{H}^\circ}(\mathbf{S}) \subseteq N_{N_{\mathbf{G}}(\mathbf{S})}(\mathbf{T})$, choose $g \in G(E)$ so that $\sigma(g^{-1})g = n$. Then the proof of our parameterization of the H-conjugacy classes of maximal θ-split K-tori tells us that there is also an element $h \in \mathbf{H}^\circ(E)$ so that $h^{-1}\sigma(h)$ has the same image in the extended little Weyl group $W_{\theta,c}$ as n and hence also in the little Weyl group W_θ. Since they have the same image in W_θ, the parameterization of the G-conjugacy classes of maximal θ-split K-tori then tells us that we may multiply g by an element of $g' \in G$ so that ${}^h\mathbf{S} = {}^g\mathbf{S}$ is a maximal θ-split K-torus in $\mathbf{G}$. But by [17], we know that the centralizer in $\mathbf{G}$ of ${}^h(\mathbf{T}_\pi \cap \mathbf{S}) = {}^g(\mathbf{T}_\pi \cap \mathbf{S})$ equals the centralizer in $\mathbf{G}$ of ${}^g\mathbf{T}_\pi$. In particular, $C_\mathbf{G}({}^h(\mathbf{T}_\pi \cap \mathbf{S}))$ is the Levi component of a θ-split parabolic subgroup over $\overline{k}$ of type π and also a twisted Levi subgroup corresponding to the pair (π, w) in our parameterization of twisted Levi K-subgroups.

4.3 θ-regular Semisimple Elements over K

Recall that $\mathbf{S}$ is a fixed maximal θ-split K-torus in $\mathbf{G}$. Let s be a strongly θ-regular semisimple element in S, i.e. an element so that $C_\mathbf{G}(s) = C_\mathbf{G}(\mathbf{S})$. We seek to parameterize the H-orbits in ${}^\mathbf{H}s \cap G$. Note that since all maximal θ-split K-tori are $\mathbf{H}(K_{tame})$-conjugate and since the little Weyl group has representatives in $\mathbf{H}^\circ(K_{tame})$, we have that all such elements are conjugate over $\mathbf{H}(K_{tame})$ so that it is again enough to consider the Galois action of σ. We will again use a cohomological argument.

Let X' be the set of all elements of the form ${}^hs$ for h in $\mathbf{H}$, and consider the exact sequence

$$0 \to C_\mathbf{H}(\mathbf{S}) \to \mathbf{H} \to X' \to 0.$$

Taking the Galois cohomology sequence over K, we have

$$0 \to C_H(\mathbf{S}) \to H \to X'(K) \to \mathrm{H}^1(K, C_\mathbf{H}(\mathbf{S})) \to \mathrm{H}^1(K, \mathbf{H}).$$

Then the set of H-conjugacy classes is parameterized by the kernel of the map $\mathrm{H}^1(K, C_\mathbf{H}(\mathbf{S})) \to \mathrm{H}^1(K, \mathbf{H})$.

Lemma 4.3.1. $\mathrm{H}^1(K, C_\mathbf{H}(\mathbf{S}))$ is in bijection with the set of σ-conjugacy classes in $C_\mathbf{H}(\mathbf{S})/(C_\mathbf{H}(\mathbf{S}))^\circ$.

The proof is nearly identical to that of Lemma 4.2.1 in the previous section, replacing $N_\mathbf{H}(\mathbf{S})$ with $C_\mathbf{H}(\mathbf{S})$. Note that $C_\mathbf{H}(\mathbf{S})/(C_\mathbf{H}(\mathbf{S}))^\circ$ is a finite abelian 2-group by [36], and since the residue characteristic p is not equal to 2 by assumption, we see that $\mathrm{Gal}(K_{sep}/K_{tame})$ acts trivially on $C_{\mathbf{H}}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^{\circ}$ so that it does in fact carry an action of σ. To complete the classification, we can look at the analogous commutative diagrams to the ones in the previous section and perform an identical argument to find

Proposition 4.3.2. The H-conjugacy classes of elements in $({}^{\mathbf{H}}s \cap G)$, where s is strongly θ-regular semisimple element in G, is in bijection with the elements of $C_{\mathbf{H}^{\circ}}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^{\circ}$, modulo σ-conjugacy by $C_{\mathbf{H}}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^{\circ}$.

It is also of interest to know which of these conjugacy classes intersect $\mathbf{S}$. To compute this, we need to compute the kernel of the map $\mathrm{H}^1(K, C_{\mathbf{H}}(\mathbf{S})) \to \mathrm{H}^1(K, N_{\mathbf{H}}(\mathbf{S}))$. Combining our lemmas from this section and the previous, we see that the kernel is the set of σ-conjugacy classes in $C_{\mathbf{H}}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^{\circ}$ which are σ-conjugate to the trivial class in the connected little Weyl group $W_{\theta,c}$ of $\mathbf{S}$.

CHAPTER V

Unramified θ-split Tori

We now attempt to adapt the parameterization of unramified twisted Levi subgroups and unramified tori in [1] to the setting of θ-split tori. We carry over notation from the previous section, including the choice of G, H, etc. as the K points, with $\mathbf{G}(k), \mathbf{H}(k)$, etc. denoting the k-points. Recall that a θ-stable k-torus $\mathbf{S}$ is the almost direct product of subtori $\mathbf{S}^+ \cdot \mathbf{S}^-$, where $\mathbf{S}^+ \subseteq \mathbf{G}^\theta$ is a k-torus and $\mathbf{S}^-$ is a θ-split k-torus.

5.1 θ-Perfect Tori, Roots, and θ-split Parabolics

We say that a maximal k-torus $\mathbf{T}$ in $\mathbf{G}$ is *θ-perfect* if it is θ-stable and contains both a maximally k-split, maximal K-split k-torus and a maximal θ-split k-torus which is both maximally (θ, k)-split and maximally (θ, K)-split. In other words, we have

- $\mathbf{T}$ is a maximal k-torus,
- $\mathbf{T}$ contains a maximal K-split k-torus $\mathbf{T}_K$,
- $\mathbf{T}_K$ contains a maximal k-split k-torus $\mathbf{T}_k$,
- $\mathbf{T}$ contains a maximal θ-split k-torus $\mathbf{S}$,
- $\mathbf{S}$ contains a maximal (θ, K)-split k-torus $\mathbf{S}_K$,
- $\mathbf{S}$ contains a maximal (θ, k)-split k-torus $\mathbf{S}_k$.

We first show that such a torus exists. Begin with a maximal (θ, k)-split torus $\mathbf{S}_k$, i.e. a k-torus which is both θ-split and split over k and maximal among such tori. By a result of Helminck in [16], if we choose any maximal k-split torus $\mathbf{T}'_k$ containing $\mathbf{S}_k$ and any maximal θ-split k-torus $\mathbf{S}'$ containing $\mathbf{S}_k$, then $\mathbf{S}'$ and $\mathbf{T}'_k$ commute so that there is a maximal k-torus $\mathbf{T}'$ containing both. Then to construct a θ-perfect torus, we first choose a maximally (θ, K)-split maximal θ-split k-torus $\mathbf{S}$ containing $\mathbf{S}_k$, and let $\mathbf{S}_K$ denote the maximal (θ,K)-split k-torus in $\mathbf{S}$. We also choose a maximal k-split torus $\mathbf{T}_k$ containing $\mathbf{S}_k$. Then since $\mathbf{S}$ and $\mathbf{T}_k$ commute by Helminck's result, $\mathbf{S}_K$ and $\mathbf{T}_k$ also commute, meaning we can choose a maximal K-split k-torus $\mathbf{T}_K$ containing both $\mathbf{S}_K$ and $\mathbf{T}_k$. Then applying Helminck's result again, this time over K, we have that $\mathbf{S}$ and $\mathbf{T}_K$ commute, so there is a maximal k-torus $\mathbf{T}$ containing both $\mathbf{S}$ and $\mathbf{T}_K$. We then have that $\mathbf{T}$ is a θ-perfect torus.

We next show that a θ-perfect k-torus $\mathbf{T}$ containing a maximal k-split torus $\mathbf{T}_k$ and a maximal θ-split k-torus $\mathbf{S}$ is unique up to $(\mathbf{H}N_{\mathbf{G}}(\mathbf{T}))(k)$-conjugacy. To see this, first note that $\mathbf{T}$ is unique up to $\mathbf{G}(k)$-conjugacy by [29]. Now note that $\mathbf{S}$ is unique up to $\mathbf{H}$-conjugacy by [36], and again using [29], $\mathbf{T}$ is unique up to $C_{\mathbf{G}(k)}(\mathbf{S})$-conjugacy. But by [18], the derived subgroup of $C_{\mathbf{G}(k)}(\mathbf{S})$ is contained in $\mathbf{H}^\circ$, meaning that $\mathbf{T}$ is also unique up to $\mathbf{H}$-conjugacy. Thus since $\mathbf{T}$ is unique up to $\mathbf{G}(k)$-conjugacy and $\mathbf{H}$-conjugacy, we have that $\mathbf{T}$ is also unique up to $(\mathbf{H}N_{\mathbf{G}}(\mathbf{T}))(k)$-conjugacy, and so we have our claim.

Henceforth, we fix a θ-perfect torus $\mathbf{T}$ containing a maximal k-split $\mathbf{T}_k$, a maximal θ-split k-torus $\mathbf{S}$, and a maximal (θ, k)-split torus $\mathbf{S}_k$. We will now choose a special simple system for the roots of $\mathbf{T}$. In particular, by [17], there is a *θ-basis* for the roots of $\mathbf{T}$ in $\mathbf{G}$, which we will denote by Δ, so that Δ is a basis for the roots of $\mathbf{T}$ in $\mathbf{G}$ and for all α in Δ, we have either

- $\theta(\alpha) = \alpha$ or
- $\theta(\alpha)$ is a negative root with respect to the basis defined by Δ.

We let Δ_+ denote the simple roots satisfying the former condition and let Δ_- denote the simple roots satisfying the latter. Again by Helminck [17], the restrictions of the roots in Δ to the maximal θ-split torus $\mathbf{S}$ contained in $\mathbf{T}$ gives a simple system for the roots of $\mathbf{S}$ in $\mathbf{G}$, and the subsets of the corresponding simple system for the roots of $\mathbf{S}$ are in bijection with subsets π of Δ so that

1. $\Delta_+ \subset \pi$
2. the subsystem spanned by π is θ-stable.

We call such a subset *θ-admissible.* Note that since the action of Fr on $\mathbf{T}$ commutes with the action of θ by assumption, we have that Fr acts on the set of θ-admissible subsets of the θ-bases for the roots of $\mathbf{T}$. The little Weyl group $W_\theta := N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S})$ can be identified with the Weyl group of the roots of $\mathbf{S}$ in $\mathbf{G}$ and acts simply transitively on the set of θ-bases for the roots of $\mathbf{T}$ in $\mathbf{G}$ [17]. We let $W_{\theta,c} := N_{\mathbf{H}}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^\circ$ be the connected little Weyl group of $\mathbf{S}$ in $\mathbf{G}$, and we consider the subgroup $W'_{\theta,c} := N_{\mathbf{H}^\circ}(\mathbf{S})/(C_{\mathbf{H}}(\mathbf{S}))^\circ$ of $W_{\theta,c}$. Given a θ-admissible subset π of Δ, we let $W_{\theta,c}(\pi)$ and $W'_{\theta,c}(\pi)$ denote the respective subgroups of elements whose image in W_θ under the obvious projection lies in the parabolic subgroup corresponding to π. Note that through this projection, all of these groups act on the roots of $\mathbf{S}$.

Recall that a θ-split parabolic K-subgroup is a parabolic K-subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $\theta(\mathbf{P}) \cap \mathbf{P}$ is a Levi subgroup of $\mathbf{P}$. In other words, θ sends $\mathbf{P}$ to an opposite parabolic subgroup. By [18], a minimal θ-split parabolic K-subgroup has an associated Levi subgroup of the form $C_{\mathbf{G}}(\mathbf{S}_1)$ for $\mathbf{S}_1$ a maximal (θ, K)-split torus. Again by [17], the minimal θ-split K-parabolic subgroup $\mathbf{P}$ containing the Borel subgroup $\mathbf{B}$ corresponding to Δ has associated Levi subgroup $C_{\mathbf{G}}(\mathbf{S})$, and the root system of $C_{\mathbf{G}}(\mathbf{S})$ has simple system Δ_+. More generally, the θ-split parabolics containing $\mathbf{B}$ are in bijection with θ-admissible subsets π of Δ. The θ-split parabolic subgroup corresponding to a θ-admissible subset $\pi \subseteq \Delta$ has a Levi subgroup of the form $C_{\mathbf{G}}(\mathbf{S}_\pi)$, where $\mathbf{S}_\pi := \bigcap_{\alpha\in\pi} (\ker(\alpha|_{\mathbf{S}}))^\circ$. We call such a Levi subgroup a θ-split Levi subgroup. Given such a subset π, we let $W_{\theta,c(\pi)} := N_{\mathbf{H}}(\mathbf{S})/((C_{\mathbf{H}}(\mathbf{S}_\pi))^\circ \cap C_{\mathbf{H}}(\mathbf{S}))$

and $W'_{\theta,c(\pi)} := N_{\mathbf{H}^\circ}(\mathbf{S})/((C_{\mathbf{H}}(\mathbf{S}_\pi))^\circ \cap C_{\mathbf{H}^\circ}(\mathbf{S}))$. These groups also act on the roots of $\mathbf{S}$ by projecting onto the little Weyl group.

Our primary goal for this section is to parameterize the unramified θ-split twisted Levi subgroups up to $\mathbf{H}(k)$-conjugacy.

5.2 θ-split Twisted Levi Subgroups for Finite Groups of Lie Type

We first look at the analogous question over the residue field $\mathfrak{f}$. We carry over all notation from the previous subsection, except we use G, H, etc. for groups over $\mathfrak{f}$. We fix a θ-stable maximal $\mathfrak{f}$-torus T of G containing a maximal θ-split $\mathfrak{f}$-torus S of G as well as a maximal $\mathfrak{f}$-torus in $\mathbf{G}$ and a maximal (θ, k)-split $\mathfrak{f}$-torus in $\mathbf{G}$ (in other words, a *θ-perfect* $\mathfrak{f}$-torus $\mathbf{T}$ in G, which exists by [18] as in the case of groups $\mathbf{G}$ defined over k). We also fix a θ-basis Δ of T. We say that a reductive subgroup L of G is called a *θ-split twisted Levi $\mathfrak{f}$-subgroup* of G if L is defined over $\mathfrak{f}$ and there exists a θ-split parabolic $\mathfrak{F}$-subgroup of G for which L is the associated Levi factor. We let $\mathcal{L}_\theta$ denote the set of θ-split twisted Levi $\mathfrak{f}$-subgroups of $\mathbf{G}$.

Every θ-split twisted Levi $\mathfrak{f}$-subgroup L of $\mathbf{G}$ can naturally be associated to a θ-split $\mathfrak{f}$-torus. In particular, we let S_L denote the maximal θ-split subtorus of the connected component of the center of L, which is θ-stable. We call an $\mathfrak{f}$-torus S in G that is equal to the θ-split component of the center of $C_\mathsf{G}(\mathsf{S})$ a θ-split Levi torus. Then for a θ-split twisted Levi $\mathfrak{f}$-subgroup of G, we have that $\mathsf{L} = C_\mathsf{G}(\mathsf{S}_\mathsf{L})$, and so we have a bijective correspondence between the set of θ-split Levi tori in G and the set of θ-split twisted Levi $\mathfrak{f}$-subgroups in G. Thus understanding $\mathcal{L}_\theta$ up to $\mathsf{H}(\mathfrak{f})$-conjugacy is equivalent to understanding the set of θ-split Levi tori in G up $\mathsf{H}(\mathfrak{f})$-conjugacy.

Let $\mathcal{L}_\theta / \sim_\mathsf{H}$ denote the set of $\mathsf{H}(\mathfrak{f})$-conjugacy classes in $\mathcal{L}_\theta$. Let I_H denote the set of pairs (π, w) where π is a θ-admissible subset of Δ and $w \in W'_{\theta,c(\pi)} = N_{\mathsf{H}^\circ}(\mathsf{S})/((C_\mathsf{H}(\mathsf{S}_\pi))^\circ \cap C_\mathsf{H}(\mathsf{S}))$ so that $w\pi = \mathrm{Fr}(\pi)$. For (π', w') and (π, w) in I_H, we write $(\pi', w') \sim (\pi, w)$ if there exists an element n' in $N_{\mathbf{H}}(\mathbf{S})$ so that $\theta = n'\theta'$ and $w = \mathrm{Fr}(n')w'(n')^{-1}$. One can check that this gives a well-defined equivalence relation on the set I_H.

Lemma 5.2.1. There is a natural bijective correspondence between $I_{\mathsf{H}}/\sim$ and $\mathcal{L}_\theta/\sim_{\mathsf{H}}$.

Proof. We begin by defining a map $\varphi : I_{\mathsf{H}} \to \mathcal{L}_\theta/\sim_{\mathsf{H}}$. Suppose that we have a pair $(\pi, w) \in I_{\mathsf{H}}$. Then by Lang-Steinberg for H°, we can choose an element $h \in \mathsf{H}^\circ$ so that the image of $\mathrm{Fr}(h^{-1})h$ in $W'_{\theta,c(\pi)}$ is w. Set $n_h = \mathrm{Fr}(h^{-1})h \in N_{\mathsf{H}^\circ}(\mathsf{S})$. Let $\mathsf{S}_\pi := \bigcap_{\alpha\in\pi} (\ker(\alpha|_{\mathsf{S}}))^\circ$ and $\mathsf{M}_\pi = C_{\mathsf{G}}(\mathsf{S}_\pi)$. Since $\mathrm{Fr}(\pi) = w\pi$, we have

$$\mathrm{Fr}({}^h\mathsf{M}_\pi) = {}^{\mathrm{Fr}(h)}\mathrm{Fr}(\mathsf{M}_\pi) = {}^{hn_h^{-1}}(\mathsf{M}_{\mathrm{Fr}(\pi)}) = {}^h({}^{n_h^{-1}}(\mathsf{M}_{w\pi})) = {}^h\mathsf{M}_\pi.$$

Thus we have that ${}^h\mathsf{M}_\pi$ is a θ-split twisted Levi $\mathfrak{f}$-subgroup of G.

We need to show that a different choice of h results in a θ-split twisted Levi $\mathfrak{f}$-subgroup which is $\mathsf{H}(\mathfrak{f})$-conjugate to ${}^h\mathsf{M}_\pi$. Suppose $h' \in \mathsf{H}^\circ$ is chosen so that $n_{h'} := \mathrm{Fr}(h')^{-1}h'$ also has image w in $W'_{\theta,c(\pi)}$. Then we can choose $s \in (C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ \cap C_{\mathsf{H}^\circ}(\mathsf{S})$ so that $n_{h'} = n_h s$. Then we have $\mathrm{Fr}(h'h^{-1})^{-1}h'h^{-1} = {}^h s \in {}^h((C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ \cap C_{\mathsf{H}^\circ}(\mathsf{S}))$, and applying Lang-Steinberg to $(C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ$, we can find an element $s' \in {}^h((C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ) \subseteq {}^h\mathsf{M}_\pi$ so that $\mathrm{Fr}(h'h^{-1})^{-1}h'h^{-1} = \mathrm{Fr}(s')^{-1}s'$. Thus $s'h(h')^{-1} = \mathrm{Fr}(s'h(h')^{-1})$, meaning that $s'h(h')^{-1} \in \mathsf{H}(\mathfrak{f})$, and we have

$${}^h\mathsf{M}_\pi = {}^{s'h}\mathsf{M}_\pi = {}^{(s'h(h')^{-1})h'}\mathsf{M}_\pi.$$

Thus we have that ${}^{h'}\mathsf{M}_\pi$ is $\mathsf{H}(\mathfrak{f})$-conjugate to ${}^h\mathsf{M}_\pi$, and so φ is well-defined.

We now show that φ descends to an injective map from $I_{\mathsf{H}}/\sim$ to $\mathcal{L}_\theta/\sim_{\mathsf{H}}$, which we shall also call φ. Suppose (π, w) and (π', w') are in I_{H}, and choose h and h' in H° so that the images of n_h and n'_h in $W'_{\theta,c(\pi)}$ are w and w' respectively. If $\varphi(\pi, w) = \varphi(\pi', w')$, then there is a $k \in \mathsf{H}(\mathfrak{f})$ so that ${}^h\mathsf{M}_\pi = {}^{kh'}\mathsf{M}_{\pi'}$. Replacing h' by kh', we may assume without loss of generality that ${}^h\mathsf{M}_\pi = {}^{h'}\mathsf{M}_{\pi'}$. We then have that both ${}^h\mathsf{S}$ and ${}^{h'}\mathsf{S}$ are maximal θ-split $\mathfrak{f}$-tori in ${}^h\mathsf{M}_\pi$, and so there exists some $m' = {}^h m$ with m in $(C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ$ so that ${}^{m'h'}\mathsf{S} = {}^h\mathsf{S}$ and ${}^{m'h'}(\mathsf{P} \cap \mathsf{M}_{\pi'}) = {}^h(\mathsf{P} \cap \mathsf{M}_\pi)$, where P is the minimal θ-split parabolic subgroup corresponding to our θ-basis for the roots of T. Since we also have that ${}^{\mathrm{Fr}(m')}({}^{h'}\mathsf{S}) = {}^h\mathsf{S}$ and since the two tori are defined over $\mathfrak{f}$, we may conclude that $m'\mathrm{Fr}(m')^{-1} \in N_{{}^h(C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ}({}^h\mathsf{S})$, which implies that $n_h^{-1}\mathrm{Fr}(m)n_h m^{-1} \in N_{(C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ}(\mathsf{S})$. Then set $n = mh^{-1}h' \in N_{\mathsf{H}}(\mathsf{S})$ and note that $n\pi' = \pi$. We have

$$\mathrm{Fr}(n)(n_{h'})n^{-1} = \mathrm{Fr}(mh^{-1}h')\mathrm{Fr}(h')^{-1}h'(mh^{-1}h')^{-1} = \mathrm{Fr}(m)(n_h)m^{-1} = n_h(n_h^{-1}\mathrm{Fr}(m)(n_h)m^{-1}).$$

Looking at images in the little Weyl group W_θ of $\mathbf{S}$, we see that the image of $\mathrm{Fr}(n)(w')n^{-1}$ is equal to the image of w times an element of the parabolic subgroup $W_\theta(\pi)$ of the little Weyl group corresponding to the θ-admissible subset π of Δ, an element which we call x and which is equal to the image of $n_h^{-1}\mathrm{Fr}(m)(n_h)m^{-1}$ in W_θ. Note that

$$x\pi = w^{-1}\mathrm{Fr}(n)w'n^{-1}\pi = w^{-1}\mathrm{Fr}(n)w'\pi' = w^{-1}\mathrm{Fr}(n)\mathrm{Fr}(\pi') = w^{-1}\mathrm{Fr}(\pi) = \pi.$$

But since the action of $W_\theta(\pi)$ on the set of θ-bases for the root system spanned by π is simply transitive, we must have that $x = 1$ in W_θ, so that $n_h^{-1}\mathrm{Fr}(m)(n_h)m^{-1}$ lies in $C_{\mathsf{H}}(\mathsf{S})$. Thus $n_h^{-1}\mathrm{Fr}(m)(n_h)m^{-1}$ lies in $(C_{\mathsf{H}}(\mathsf{S}_\pi))^\circ \cap C_{\mathsf{H}}(\mathsf{S})$, and so we have that its image in $W_{\theta,c(\pi)}$ is trivial. Consequently, we have that $(\pi, w) \sim (\pi', w')$ so that the map is injective as claimed.

Finally, we show that φ is surjective. Suppose $\mathsf{L} \in \mathcal{L}_\theta$. Let A_{L} denote the maximal θ-split subtorus of the connected component of the center of L. Choose a minimal θ-split parabolic $\mathfrak{f}$-subgroup P_{L} in L and a maximal $\mathfrak{f}$-torus T_{L} in P_{L} containing a maximal θ-split $\mathfrak{f}$-torus S_{L}. Denote by Δ_{L} the corresponding θ-basis for the roots of T_{L}. Choose $h \in \mathsf{H}^\circ$ so that $\mathsf{S}_{\mathsf{L}} = {}^h\mathsf{S}$ and $\mathsf{P}_{\mathsf{L}} \leq {}^h\mathsf{P}$. Define $\pi_{\mathsf{L}} = h^{-1} \cdot \Delta_{\mathsf{L}}$, and note that $\pi_{\mathsf{L}} \subset \Delta_\theta$. Let w_{L} denote the image of $\mathrm{Fr}(h^{-1})h$ in $W_{\theta,c(\pi)}$ and put $\mathsf{S}_{\pi_{\mathsf{L}}} = (\bigcap_{\alpha\in\pi_{\mathsf{L}}} (\ker(\alpha|_{\mathsf{S}}))^\circ \leq \mathsf{S}$. Then we have that $\mathsf{A}_{\mathsf{L}} = {}^h\mathsf{S}_{\pi_{\mathsf{L}}}$ and $\mathrm{Fr}(\pi_{\mathsf{L}}) = w_{\mathsf{L}}\pi_{\mathsf{L}}$, which shows that $\varphi(\pi_L, w_L)$ gives the $\mathsf{H}(\mathfrak{f})$-conjugacy class of L.

$\square$

5.3 More on θ-split Tori and θ-split Levi Subgroups

We fix a Galois extension E of k. (Note that E here is not necessarily the extension E from chapter 4, which was defined to be a tame extension of K over which all K-tori split.) Then we call a subgroup $\mathbf{M}$ of $\mathbf{G}$ a *θ-split Levi (E, k)-subgroup* if it is a k-subgroup so that $\mathbf{M} = \mathbf{P} \cap \theta(\mathbf{P})$ for some θ-split parabolic E-subgroup $\mathbf{P}$ of $\mathbf{G}$ (which then implies that $\mathbf{M}$ is a Levi (E, k)-subgroup in the sense of [1]). If E is a tame Galois extension of k and M is the group of K-rational points of a Levi (E, k)-subgroup of G, then we can and do identify $\mathcal{B}(M)$ with a subset of $\mathcal{B}(G)$, noting that there is no canonical way to do this but that all such identifications have the same image. Given a θ-split k-torus $\mathbf{S}$ of $\mathbf{G}$, we let $\mathbf{S}^E$ denote the maximal E-split subtorus in $\mathbf{G}$, and given a θ-stable maximal k-torus $\mathbf{T}$ of $\mathbf{G}$, we let $\mathbf{T}^-$ denote the maximal θ-split subtorus of $\mathbf{T}$.

Lemma 5.3.1. If $\mathbf{M}$ is a θ-split Levi (E, k)-subgroup and $\mathbf{Z}_{\mathbf{M}}^E$ is defined to be the maximal (θ, E)-split torus in the center of $\mathbf{M}$, then $\mathbf{Z}_{\mathbf{M}}^E$ is defined over k and $\mathbf{M} = C_{\mathbf{G}}(\mathbf{Z}_{\mathbf{M}}^E)$.

Proof. Since $\mathbf{M}$ is defined over k, the center of $\mathbf{M}$ is also defined over k. Thus by the uniqueness of $\mathbf{Z}_M^E$, we have that it is also defined over k.

Now since $\mathbf{M}$ is a θ-split Levi (E, k)-subgroup, we have that there is θ-split parabolic E-subgroup $\mathbf{P}$ of $\mathbf{G}$ so that $\mathbf{M}$ is equal to $\mathbf{P} \cap \theta(\mathbf{P})$. By [17], there is a (θ, E)-split torus $\mathbf{S}$ of $\mathbf{G}$ so that $\mathbf{M} = \mathrm{C}_G(\mathbf{S})$. Since $\mathbf{S}$ is in the center of $\mathbf{M}$ and is (θ, E)-split, we must have that $\mathbf{S}$ is contained in $\mathbf{Z}_{\mathbf{M}}^E$. Thus we have that $\mathbf{M} \le C_{\mathbf{G}}(\mathbf{Z}_{\mathbf{M}}^E) \le C_{\mathbf{G}}(\mathbf{S}) = \mathbf{M}$, and so we are done. □

Corollary 5.3.2. Suppose $\mathbf{M}$ is a θ-split Levi (E, k)-subgroup whose center has θ-split component $\mathbf{Z}_{\mathbf{M}}$. Then if $\mathbf{C}$ is a k-subgroup of $\mathbf{G}$ which lies between $\mathbf{Z}_{\mathbf{M}}^E$ and $\mathbf{Z}_{\mathbf{M}}$, then $\mathbf{M} = C_{\mathbf{G}}(\mathbf{C})$.

Proof. Since $\mathbf{Z}_{\mathbf{M}}^E \le \mathbf{C} \le \mathbf{Z}_M$, we have $\mathbf{M} \le C_{\mathbf{G}}(\mathbf{Z}_{\mathbf{M}}) \le C_{\mathbf{G}}(\mathbf{C}) \le C_{\mathbf{G}}(\mathbf{Z}_{\mathbf{M}}^E) = \mathbf{M}$. □

Lemma 5.3.3. If $\mathbf{T}$ is a maximal θ-split k-torus in $\mathbf{G}$, then $C_{\mathbf{G}}(\mathbf{T}^E)$ is the unique θ-split Levi (E, k)-subgroup in $\mathbf{G}$ that is minimal among Levi (E, k)-subgroups that contain $\mathbf{T}$.

Proof. Let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T}^E)$. Then we must first show that $\mathbf{M}$ is a θ-split Levi (E,k)-subgroup in $\mathbf{G}$. Since $\mathbf{T}^E$ is a θ-split torus, $\mathbf{M}$ is a θ-split Levi subgroup by [17], and since $\mathbf{T}^E$ is the unique maximal E-split torus in $\mathbf{T}$, it is defined over k. Thus $\mathbf{M}$ is defined over k as well. Now if $\mathbf{Z}^E_{\mathbf{M}}$ denotes the maximal (θ, E)-split torus in the center of $\mathbf{M}$, then we must have that $\mathbf{T}^E \leq \mathbf{Z}^E_{\mathbf{M}}$. Furthermore, since $\mathbf{T}$ is a maximal θ-split k-torus in $\mathbf{M}$, we also have $\mathbf{Z}^E_{\mathbf{M}} \leq \mathbf{T}$. But then by the maximality of $\mathbf{T}^E$ in $\mathbf{T}$, we conclude that $\mathbf{Z}^E_{\mathbf{M}} \leq \mathbf{T}^E$. This then implies that $\mathbf{T}^E = \mathbf{Z}^E_{\mathbf{M}}$, and so $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T}^E) = C_{\mathbf{G}}(\mathbf{Z}^E_{\mathbf{M}})$.

Now let $\mathbf{T}'$ be a maximal (θ, E)-split torus in $\mathbf{G}$ that contains $\mathbf{T}^E$, and let $\mathbf{P}'$ be a minimal θ-split parabolic E-subgroup so that $\mathbf{P}' \cap \theta(\mathbf{P}') = C_{\mathbf{G}}(\mathbf{T}')$. (Such a parabolic exists by [17].) Then since $\mathbf{T}' \leq \mathbf{M}$, we have that the subgroup $\mathbf{MP}'$ of $\mathbf{G}$ is a θ-split parabolic E-subgroup of $\mathbf{G}$ for which $\mathbf{M}$ is the associated θ-split Levi E-subgroup.

We now show that $\mathbf{M}$ is the unique minimal θ-split Levi (E,k)-subgroup in $\mathbf{G}$ that contains $\mathbf{T}$. Suppose that $\mathbf{M}'$ is another Levi (E,k)-subgroup that contains $\mathbf{T}$, and let $\mathbf{Z}^E_{\mathbf{M}'}$ denote the maximal (θ, E)-split torus in the center of $\mathbf{M}'$. Then by the previous lemma, we have that $\mathbf{M}' = C_{\mathbf{G}}(\mathbf{Z}^E_{\mathbf{M}'})$, and since $\mathbf{T} \leq \mathbf{M}'$, we have that $\mathbf{Z}^E_{\mathbf{M}'} \leq \mathbf{T}$. Thus $\mathbf{Z}^E_{\mathbf{M}'} \leq \mathbf{T}^E = \mathbf{Z}^E_{\mathbf{M}}$, and so $\mathbf{M} \leq \mathbf{M}'$. □

We now adopt the following language when E is the maximal unramified extension K of k. First, we say that a subgroup $\mathbf{L}$ of $\mathbf{G}$ is an *unramified θ-split twisted Levi subgroup* provided that $\mathbf{L}$ is a θ-split Levi (K,k)-subgroup of $\mathbf{G}$. In addition, we say that a k-torus $\mathbf{S}$ is an *unramified θ-split torus* in $\mathbf{G}$ provided that $\mathbf{S}$ is the (θ, K)-split component of the center of an unramified θ-split twisted Levi subgroup in $\mathbf{G}$.

Note that by our lemmas, we know that if $\mathbf{L}$ is an unramified θ-split twisted Levi subgroup in $\mathbf{G}$ and $\mathbf{S}$ is the (θ, K)-split component of the center of $\mathbf{L}$, then $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$. In addition, since two θ-split Levi (K,k)-subgroups are $\mathbf{H}(k)$-conjugate if and only if the (θ, K)-split components of their centers are $\mathbf{H}(k)$-conjugate, we have that a parameterization of the $\mathbf{H}(k)$-conjugacy classes of unramified tori also gives a parameterization of the the $\mathbf{H}(k)$-conjugacy classes of θ-split Levi (K,k)-subgroups.

5.3.1 A Conjecture and a Consequence

We state the following as a conjecture. It likely follows from applying the analogous result in [1] to the reductive subgroup constructed in [23], but this needs to be checked more carefully. Note that this conjecture is not used outside of this subsection.

Conjecture 5.3.4. If $\mathbf{G}$ is a connected reductive k-group and θ is an involution defined over k, then $\mathbf{G}$ contains a K-minisotropic maximal θ-split k-torus.

Assuming the conjecture is true, we can prove the following:

Lemma 5.3.5. Suppose $\mathfrak{f}$ is finite. Then a θ-split torus $\mathbf{T}$ in $\mathbf{G}$ is an unramified θ-split torus if and only if there exists a maximal θ-split k-torus $\mathbf{T}'$ in $\mathbf{G}$ such that $\mathbf{T}$ is the maximal K-split subtorus of $\mathbf{T}'$.

Proof. Suppose $\mathbf{T}$ is the (θ, K)-split component of the center of a θ-split Levi (K, k)-subgroup $\mathbf{L}$. Then by the conjecture, there is a K-minisotropic maximal θ-split k-torus $\mathbf{T}'$ in $\mathbf{L}$. Then $\mathbf{T}$ is the maximal K-split subtorus of $\mathbf{T}'$.

Now suppose there exists a maximal θ-split k-torus $\mathbf{T}'$ in $\mathbf{G}$ for which $\mathbf{T}$ is the maximal K-split subtorus of $\mathbf{T}'$. Then since $\mathbf{T}'$ is defined over k, $\mathbf{T}$ is as well. Let $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T})$. Then by our previous lemma, we have that $\mathbf{L}$ is the unique minimal θ-split Levi (K, k)-subgroup containing $\mathbf{T}'$. We also have that $\mathbf{T}$ is contained in $\mathbf{Z}_{\mathbf{L}}$, the θ-split component of the center of $\mathbf{L}$. Thus $\mathbf{T} = \mathbf{T}^K \leq \mathbf{Z}_{\mathbf{L}}^K$. But then since $\mathbf{T}'$ contains $\mathbf{Z}_{\mathbf{L}}$ and $\mathbf{T}$ is the maximal (θ, K)-split subtorus of $\mathbf{T}'$, we conclude that $\mathbf{Z}_{\mathbf{L}}^K$ is contained in $\mathbf{T}$. Hence $\mathbf{T} = \mathbf{Z}_{\mathbf{L}}^K$, and so $\mathbf{T}$ is an unramified torus in $\mathbf{G}$. $\square$

Regardless of whether the conjecture holds, note that we have shown that the maximal (θ, K)-split subtorus of a maximal θ-split k-torus in $\mathbf{G}$ must be an unramified θ-split torus.

5.4 θ-split (K,k)-tori

Adjusting our notation from the previous section, we let $\mathbf{Z} = \mathbf{Z_G}$ denote the center of $\mathbf{G}$, and we let $\mathbf{Z}^-$ and $\mathbf{Z}^{K-}$ denote its θ-split component and (θ, K)-split component respectively. We call a torus in $\mathbf{G}$ a θ-split (K,k)-torus in $\mathbf{G}$ if it is a (θ, K)-split k-torus that contains $\mathbf{Z}^{K-}$. If we let $\mathcal{T}_K$ denote the set of θ-split (K,K)-tori in $\mathbf{G}$, then $\mathcal{T}_K$ carries a natural action of $\mathrm{Gal}(K/k)$, and we denote the set of points in $\mathcal{T}_K$ fixed by $\mathrm{Gal}(K/k)$ by $\mathcal{T}_k$. To ease notation, we will also call the K-rational points S of a θ-split (K,k)-torus $\mathbf{S}$ a θ-split (K,k)-torus.

Our goal for this subsection is to parameterize the $\mathbf{H}(k)$-conjugacy classes in $\mathcal{T}_k$. To begin, we introduce indexing sets modeling those in [28] and [1]. For a θ-facet F in $\mathcal{B}(G)$ (i.e. a non-empty subset of points in $\mathcal{B}(H) = \mathcal{B}(G)^\theta$ which equals the set of θ-fixed points of some facet F' of $\mathcal{B}(G)$), we let Z_F^- denote the group corresponding to the image of $G_F \cap \mathbf{Z}^{K-}(K)$ in G_F/G_F^+, where we write G_F for the parahoric of the facet F' in $\mathcal{B}(G)$ containing F. Now consider the indexing set

$$J := \{(F, \mathsf{S}) | F \text{ is a } \theta\text{-facet in } \mathcal{B}(G) \text{ and } \mathsf{S} \text{ is a } \theta\text{-split torus in } \mathsf{G}_F \text{ which contains } \mathsf{Z}_F^-\}.$$

Definition 5.4.1. We say that a (θ, K)-split torus $\mathbf{S}$ in $\mathcal{T}_K$ is a lift of $(F, \mathsf{S}) \in J$ provided that we have

1. $F \subset \mathcal{B}(C_G(S))$
2. the image of $S \cap G_F$ in $\mathsf{G}_F = G_F/G_F^+$ is S.

Now suppose that $(F, \mathsf{S}) \in J$, and let $\Gamma := \mathrm{Gal}(K/k)$. Note that if $\Gamma(F) = F$, then G_F is defined over the residue field $\mathfrak{f}$ of k. In this situation, it makes sense to consider $\Gamma(\mathsf{S})$, and so we define J^Γ to be the set of pairs (F, S) in J so that both F and S are Γ-stable.

Next, we say that a pair $(F, \mathsf{S}) \in J^\Gamma$ is maximal if whenever a θ-facet F_1 in $\mathcal{B}(G)$ is both Γ-stable and contains F in its closure, then S belongs to the $\mathfrak{f}$-parabolic subgroup G_{F_1}/G_F^+ of

G_F if and only if $F = F_1$. We let $J^\Gamma_{\max}$ denote the subset of maximal pairs in J^Γ.

5.4.1 Lifts of Tori over $\mathfrak{f}$

Suppose $(F, \mathsf{S}) \in J^\Gamma$. Our first goal for this subsection is to show that there is an element of $\mathcal{T}_k$ that lifts (F, S) and to show that any two such lifts are conjugate by an element of $(H_F^+)^\Gamma$. (Recall from [28] that F is contained in a unique facet F_2 in $\mathcal{B}(H)$. Thus we may use H_F and H_F^+ to denote the subgroups of H associated to F_2.) We will then show that all elements of $\mathcal{T}_k$ arise in this way.

First, recall that we defined a θ-perfect $\mathfrak{f}$-torus in a group defined over $\mathfrak{f}$ to be a θ-stable maximal $\mathfrak{f}$-torus which contains a maximal $\mathfrak{f}$-split torus, a maximal θ-split $\mathfrak{f}$-torus, and a maximal (θ, k)-split $\mathfrak{f}$-torus.

Lemma 5.4.2. Set $\mathsf{M} = C_{\mathsf{G}_F}(\mathsf{S})$, and let T denote a θ-perfect $\mathfrak{f}$-torus in M. Then there is a θ-stable maximal unramified torus $\mathbf{T}$ in $\mathbf{G}$ which lifts (F, T). Moreover, for all such $\mathbf{T}$ lifting (F, T) there exists a unique lift $\mathbf{S} \in \mathcal{T}_k$ of (F, S) with the property that $\mathbf{S} \leq \mathbf{T}$.

Proof. Such an unramified torus $\mathbf{T}$ exists by [28]. Now note that $\mathrm{X}_*(\mathsf{T}) = \mathrm{X}_*(\mathbf{T})$ as Γ-modules and as θ-modules, and so we can choose a subtorus $\mathbf{S}$ of $\mathbf{T}$ corresponding to the image of $\mathrm{X}_*(\mathsf{S})$ under the map $\mathrm{X}_*(\mathsf{S}) \hookrightarrow \mathrm{X}_*(\mathsf{T}) = \mathrm{X}_*(\mathbf{T})$. Then $\mathbf{S} \in \mathcal{T}_k$, and since $T \leq C_G(S)$, we have $F \subseteq \mathcal{B}(T) \subseteq \mathcal{B}(C_G(S))$, giving us that S is a lift of (F, S) as required.

Now if $\mathbf{S}' \in \mathcal{T}_k$ is another lift of (F, S) that lies in $\mathbf{T}$, then $\mathrm{X}_*(\mathbf{S}') = \mathrm{X}_*(\mathbf{S}) = \mathrm{X}_*(\mathsf{S})$ in $\mathrm{X}_*(\mathbf{T})$, and so $\mathbf{S}' = \mathbf{S}$. $\square$

Corollary 5.4.3. If $\mathbf{S}, \mathbf{S}' \in \mathcal{T}_k$ both lift (F, S), then there exists an element $h \in (H_F^+)^\Gamma$ so that ${}^h\mathbf{S} = \mathbf{S}'$

Proof. We reuse the notation from the proof of the preceding lemma.

Set $\mathbf{M}' = C_{\mathbf{G}}(\mathbf{S}')$. Then note that $F \subset \mathcal{B}(M')$ by the definition of a lift, and we have that the image of $M' \cap G_F$ in G_F is $\mathsf{M} = C_{\mathsf{G}_F}(\mathsf{S})$. Now let $\mathbf{T}' \leq \mathbf{M}'$ be a θ-stable lift of (F, T). Since $\mathbf{S}'$ is in the center of $\mathbf{M}'$, we have $\mathbf{S}' \leq \mathbf{T}'$, and since $\mathbf{S}'$ (resp. $\mathbf{T}'$) is a K-split torus lifting (F, S) (resp. (F, T)), we conclude from the preceding lemma that $\mathbf{S}'$ is the unique lift of (F, S) in $\mathbf{T}'$. Then by [28], there is an h in $(H_F^+)^\Gamma$ so that ${}^h\mathbf{T} = \mathbf{T}'$. Our result then follows from uniqueness in the preceding lemma. $\square$

Thanks to the preceding lemma and corollary, we can define an action of $\mathbf{H}(k)$ on $J^\Gamma_{\max}$. Suppose $h \in \mathbf{H}(k)$ and $(F, \mathsf{S}) \in J^\Gamma_{\max}$. Then if $\mathbf{S}$ is a lift of (F, S), let ${}^h\mathsf{S}$ denote the image of ${}^hS \cap G_{hF}$ in G_{hF} and set $h(F, \mathsf{S}) := (hF, {}^h\mathsf{S}) \in J^\Gamma_{\max}$.

We now want to move in the opposite direction, showing that every element of $\mathcal{T}_k$ arises as a lift from a pair in $J^\Gamma_{\max}$.

Lemma 5.4.4. For all $\mathbf{S} \in \mathcal{T}_k$ there exists $(F, \mathsf{S}) \in J^\Gamma_{\max}$ so that $\mathbf{S}$ lifts (F, S).

Before proving the lemma, given a reductive k-subgroup $\mathbf{C}$ of $\mathbf{G}$ having the same K-rank as $\mathbf{G}$, we define a (θ, C)-facet to be a non-empty subset of $\mathcal{B}(G)^\theta$ which equals the set of θ-fixed points of some C-facet in $\mathcal{B}(C) \subseteq \mathcal{B}(G)$.

Proof. Fix $\mathbf{S} \in \mathcal{T}_k$, and let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{S})$. Note that $\mathbf{M}$ is a Levi (K, k)-subgroup of $\mathbf{G}$.

Choose a Γ-stable (θ, M)-facet F' in $\mathcal{B}(M)$ so that F' has maximal dimension among θ-facets in $\mathcal{B}(M)$. Then since F' can be written as the disjoint union of (θ, G)-facets in $\mathcal{B}(G)$, we may choose a Γ-stable (θ, G)-facet F in $\mathcal{B}(H)$ so that $F \subset F'$ and $\dim(F^\Gamma) \geq \dim(\tilde{F}^\Gamma)$ for all Γ-stable (θ, G)-facets $\tilde{F}$ in $\overline{F}'$. In fact, $\dim(F^\Gamma) \geq \dim(\tilde{F}^\Gamma)$ for all Γ-stable (θ, G)-facets $\tilde{F}$ in $\mathcal{B}(M)$. This is because every such facet is $(M^\theta)^\Gamma$-conjugate to an element in the closure of an alcove of $\mathcal{B}(M^\theta) = \mathcal{B}(M)^\theta$ [30] containing F', and by [5, 9.2.5], the Γ-fixed points of all (θ, G)-facets lying in the closure of this alcove which do not lie in the closure of another (θ, G)-facet have the same dimension, which will be that of F^Γ.

Now let S be the $\mathfrak{f}$-torus in G_F corresponding to the image of $S \cap G_F$ in G_F. Then, by our construction, we have that $\mathbf{S}$ is a lift of the pair (F, S). It remains to show that $(F, \mathsf{S}) \in J^\Gamma_{\max}$. Suppose that $F'' \subset \mathcal{B}(G)$ is a Γ-stable (θ, G)-facet with $F \subset \overline{F}''$ and $F \neq F''$. Then if S belongs to the proper parabolic $\mathfrak{f}$-subgroup $G_{F''}/G_F^+$ of $\mathsf{G}_F = G_F/G_F^+$, then we have that $S \cap G_{\mathsf{F}} = S \cap G_{\mathsf{F}''}$ fixes F'' and $(F'', \mathsf{S}') \in J^\Gamma$ where S' is the θ-split $\mathfrak{f}$-torus in $\mathsf{G}_{F''}$

corresponding to the image of $S \cap G_{F''}$ in $\mathsf{G}_{F''}$. By [9, 4.4.2], we then have that F'' is in $\mathcal{B}(M)$, but the dimension of F''^{Γ} is strictly larger than than of F^{Γ}, contradicting the previous paragraph. □

5.4.2 An Equivalence Relation

Thanks to the results of the previous subsection, we have a well-defined surjective map φ from $J^{\Gamma}_{\max}$ to the set of H^{Γ}-conjugacy classes in $\mathcal{T}_k$. In this subsection, we introduce an equivalence relation $\sim$ on $J^{\Gamma}_{\max}$ so that φ descends to a bijection.

Suppose $\mathcal{A}$ is an apartment in $\mathcal{B}(H)^{\Gamma}$ so that $\mathcal{A} = \mathcal{A}(\mathbf{S}, k)$ for some maximal k-split torus $\mathbf{S} \leq \mathbf{H}$. Note that any two such apartments are conjugate by an element $h \in H^{\Gamma}$.

If $\Omega \subset \mathcal{A}$, then we denote the smallest affine subspace of $\mathcal{A}$ that contains Ω by $A(\mathcal{A}, \Omega)$. If F_1, F_2 are two (θ, G^{Γ})-facets in $\mathcal{A}$ for which $\emptyset \neq A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$, then we say that F_1 and F_2 are equivalent. Note that if F_i' denotes the G^{Γ}-facet containing F_i for $i = 1, 2$, and $\mathcal{A}'$ is an apartment of $\mathcal{B}(G)^{\Gamma}$ containing F_1', F_2', then by [27] $A(\mathcal{A}, F_1) = A(\mathcal{A}, F_2)$ if and only if $A(\mathcal{A}', F_1') = A(\mathcal{A}', F_2')$. Thus there is a natural identification of G_{F_1} with G_{F_2} as in [10], and we write $\mathsf{G}_{F_1} \approx \mathsf{G}_{F_2}$.

Now suppose $(F_i, \mathsf{S}_i) \in J^{\Gamma}$. Then we write $(F_1, \mathsf{S}_1) \sim (F_2, \mathsf{S}_2)$ if there exists an element $h \in H^{\Gamma}$ and an apartment $\mathcal{A}$ in $\mathcal{B}(H)^{\Gamma}$ so that

1. $\emptyset \neq A(\mathcal{A}, F_1^{\Gamma}) = A(\mathcal{A}, hF_2^{\Gamma})$
2. $\mathsf{S}_1 \approx {}^{h}\mathsf{S}_2$ in $\mathsf{G}_{F_1} \approx \mathsf{G}_{hF_2}$.

Lemma 5.4.5. The relation $\sim$ is an equivalence relation on $J^{\Gamma}_{\max}$.

Proof. The proof is nearly identical to the one in [28] or [10]. □

5.4.3 A Bijective Correspondence

Suppose $\mathbf{S} \in \mathcal{T}_k$ is a lift of $(F, \mathsf{S}) \in J^{\Gamma}_{\max}$. Let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{S})$, and note that by our definition of a lift (Definition 5.4.1), we have that $F \subset \mathcal{B}(M)$.

Lemma 5.4.6. Let C denote a Γ-stable (θ, M)-facet in $\mathcal{B}(M)$ that contains F in its closure. Then C^Γ is a maximal (θ, M^Γ)-facet in $\mathcal{B}(M)^\Gamma$, and F^Γ is an open subset of C^Γ.

Proof. It will be enough to show that F^Γ is a maximal (θ, G^Γ)-facet in $\mathcal{B}(M)^\Gamma$. Now choose a (θ, G^Γ)-facet $D \subset \mathcal{B}(M)^\Gamma$ so that $F^\Gamma \subset \overline{D}$. If $F^\Gamma \neq D$, then as $\mathbf{S}$ is in the center of $\mathbf{M}$, the image of $S \cap G_F = S \cap G_D$ in G_F/G_F^+ belongs to the parabolic $\mathfrak{f}$-subgroup G_D/G_F^+, contradicting that $(F, \mathsf{S}) \in J^\Gamma_{\max}$. $\square$

Lemma 5.4.7. Suppose $(F_i, \mathsf{S}_i) \in J^\Gamma_{\max}$ with lifts $\mathbf{S}_i \in \mathcal{T}_k$. Then if there exists $h \in H^\Gamma$ so that ${}^h\mathbf{S}_1 = \mathbf{S}_2$, then $(F_1, \mathsf{S}_1) \sim (F_2, \mathsf{S}_2)$.

Proof. Replacing (F_1, S_1) with $(hF_1, {}^h\mathsf{S}_1)$, we may and do assume that $\mathbf{S} := \mathbf{S}_1 = \mathbf{S}_2$. Now set $\mathbf{M} = C_{\mathbf{G}}(\mathbf{S})$. Then since $\mathbf{S}$ is a lift of (F_i, S_i), we know from the definition of a lift that $F_i \subset \mathcal{B}(M)$. Let C_i denote the (θ, M)-facet in $\mathcal{B}(M)$ to which F_i belongs. By the preceding lemma, we have that C_i^Γ is a maximal (θ, M^Γ)-facet in $\mathcal{B}(M^\Gamma)$, and so in particular, C_i^Γ must lie in an alcove $\tilde{C}_i^\Gamma$ of $\mathcal{B}(M^\Gamma)^\theta$ for some M-facet $\tilde{C}_i$ in $\mathcal{B}(M)^\theta$. Thus there exists an $m \in M^\Gamma \cap H$ so that $m\tilde{C}_1 = \tilde{C}_2$. Replacing (F_1, S_1) by $(mF_1, {}^m\mathsf{S}_1)$, then since F_1^Γ and F_2^Γ are open in C_1^Γ and C_2^Γ, and hence also open in $\tilde{C}_1^\Gamma = \tilde{C}_2^\Gamma$, for any apartment $\mathcal{A}$ in $\mathcal{B}(M^\Gamma)^\theta \subset \mathcal{B}(H)^\Gamma$ containing $\tilde{C}_1^\Gamma$ we have $\emptyset \neq A(\mathcal{A}, F_1^\Gamma) = A(\mathcal{A}, F_2^\Gamma)$. Then since ${}^m\mathbf{S} = \mathbf{S}$, we see that $(F_1, \mathsf{S}_1) \sim (F_2, \mathsf{S}_2)$. $\square$

Corollary 5.4.8. There exists a bijection between $J^\Gamma_{\max}/\sim$ and the set of H^Γ-conjugacy classes in $\mathcal{T}_k$.

Proof. The only thing we still need to check is that if $(F_1, \mathsf{S}_1), (F_2, \mathsf{S}_2) \in J^\Gamma_{\max}$ with $(F_1, \mathsf{S}_1) \sim (F_2, \mathsf{S}_2)$, then they have lifts that are H^Γ-conjugate. Suppose we have such (F_1, S_1) and (F_2, S_2) with $(F_1, \mathsf{S}_1) \sim (F_2, \mathsf{S}_2)$. Then there is some element $h \in H^\Gamma$ and an apartment $\mathcal{A}$ in $\mathcal{B}(H)^\Gamma$ so that

- $\emptyset \neq A(\mathcal{A}, F_1^\Gamma) = A(\mathcal{A}, hF_2^\Gamma)$
- $\mathsf{S}_1 \approx {}^h\mathsf{S}_2$ in $\mathsf{G}_{F_1} \approx \mathsf{G}_{hF_2}$

We may and do assume that h is the identity and that $\mathcal{A} \subset \mathcal{A}'(A)^\Gamma$, where $\mathcal{A}'(A)$ is an apartment in $\mathcal{B}(G)$ for some maximal k-split torus $\mathbf{A}$ of $\mathbf{G}$.

Let $\mathbf{M}_i$ denote the Levi (k,k)-subgroup of $\mathbf{G}$ corresponding to the G-facet F_i' containing the (θ, G)-facet F_i. Since $A(\mathcal{A}', (F_1')^\Gamma) = A(\mathcal{A}', (F_2')^\Gamma)$ by our earlier remark, we have that $\mathbf{M}_1 = \mathbf{M}_2$, so we can set $\mathbf{M} = \mathbf{M}_1$. By construction, the image of $M \cap G_{F_i}$ in G_{F_i} is G_{F_i} itself.

Now since $\mathsf{S}_1 \approx \mathsf{S}_2$ in $\mathsf{G}_{F_1} \approx \mathsf{G}_{F_2}$, we can find a θ-stable (K,k)-torus $\mathbf{T}$ so that the image of $T \cap M_{F_1} \cap M_{F_2}$ in $\mathsf{M}_{F_i} = \mathsf{G}_{F_i}$ is a θ-perfect $\mathfrak{f}$-torus in $C_{\mathsf{G}_{F_i}}(\mathsf{S}_i)$. But by Lemma 5.4.2, there is exactly one lift $\mathbf{S}$ of (F_i, S_i) in $\mathbf{T}$. Then the image of $S \cap M$ in $\mathsf{M}_{F_i} = \mathsf{G}_{F_i}$ is S_i, and so the proof is complete. □

5.4.4 θ-split (K,k)-tori and Extendable Levi (k,k)-subgroups

If $\mathbf{M}'$ is a Levi (k,k)-subgroup of $\mathbf{G}$, then we let $(\mathbf{M}')$ denote the H^Γ-conjugacy class of $\mathbf{M}'$. For two conjugacy classes $(\mathbf{M}_1)$ and $(\mathbf{M}_2)$, we write $(\mathbf{M}_1) \leq (\mathbf{M}_2)$ if there exists $\mathbf{L}_i \in (\mathbf{M}_i)$ so that $\mathbf{L}_1 \leq \mathbf{L}_2$. If a Levi (k,k)-subgroup $\mathbf{M}$ is θ-stable, we say that it is *extendable* if $\mathbf{M} = C_{\mathbf{G}}((\mathbf{T}^+)^k)$ for some k-torus $\mathbf{T}^+$ so that $(\mathbf{T}^+)^k \leq \mathbf{H}$. As the minimal θ-stable parabolic k-subgroups in $\mathbf{G}$ have a Levi component equal to the centralizer of a maximal k-split torus in $\mathbf{H}$ by [18], the extendable Levi (k,k)-subgroups should correspond to the Levi subgroups arising as Levi component of a θ-stable parabolic k-subgroup, hence the term extendable.

Given a θ-facet F in $\mathcal{B}(H)$, we let ${}_F\mathbf{M}$ denote the Levi k-subgroup of $\mathbf{G}$ associated to the facet F' in $\mathcal{B}(G)$ containing F.

Lemma 5.4.9. Fix $(F, \mathsf{S}) \in J^\Gamma_{\max}$, and let $\mathbf{S} \in \mathcal{T}_k$ be a lift of (F, S). Then there exists a θ-stable $(H_F^+)^\Gamma$-conjugate $\mathbf{M}'$ of ${}_F\mathbf{M}$ so that $\mathbf{S} \leq \mathbf{M}'$ and so that every extendable Levi (k,k)-subgroup $\mathbf{M}''$ containing $\mathbf{S}$, satisfies $(\mathbf{M}') \leq (\mathbf{M}'')$.

Proof. Let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{S})$. Then from the previous section, we have that F^Γ is a maximal (θ, G^Γ)-facet in $\mathcal{B}(M)^\Gamma$ and hence contained in an $(M \cap H)^\Gamma$-alcove. Choose a θ-perfect k-torus T in $C_{\mathsf{G}_F}(\mathsf{S})$. Then there is a θ-stable lift $\mathbf{T}$ of T containing $\mathbf{S}$, and if C denotes the $\mathfrak{f}$-split component of the center of G_F, there is a unique k-torus $\mathbf{C}_F$ which lifts $\mathsf{C}\mathsf{Z}_F$ and which is contained in both $\mathbf{T}$ and the center of $\mathbf{M}$ by [1]. We know then that $\mathbf{M}' = C_{\mathbf{G}}(\mathbf{C}_F)$ is θ-stable since $\mathbf{C}_F$ is θ-stable by uniqueness, and our previous result tells us that it is $(H_F^+)^\Gamma$-conjugate to ${}_F\mathbf{M}$ since the central tori are θ-stable and have the same image in G_F. Note that $\mathbf{M}'$ contains $\mathbf{S}$.

Now suppose that $\mathbf{M}'' = C_{\mathbf{G}}((\mathbf{A}^+)^k)$ is an extendable θ-stable Levi (k,k)-subgroup which contains $\mathbf{S}$, where $\mathbf{A}^+$ is a k-torus in $\mathbf{H}$. (Recall that $(\mathbf{A}^+)^k$ denotes the maximal k-split torus contained in $\mathbf{A}^+$.) Then since $(\mathbf{A}^+)^k$ commutes with $\mathbf{S}$, we have that $(\mathbf{A}^+)^k \leq \mathbf{M}$. Choose a maximally k-split maximal (K,k)-torus $\mathbf{T}'$ in $\mathbf{M} \cap \mathbf{H}$ that contains $(\mathbf{A}^+)^k$. Then since F^Γ is contained in an $(M \cap H)^\Gamma$-alcove, after replacing $\mathbf{T}'$ and $\mathbf{M}''$ with a $(\mathbf{M} \cap \mathbf{H})(k)$-conjugate we may assume that $F^\Gamma \subset \mathcal{B}(T')^\Gamma \subset \mathcal{B}(M^\theta)^\Gamma$. Since F^Γ is a maximal (θ, G^Γ)-facet in $\mathcal{B}(M)^\theta$, we see that the image of $(\mathbf{T}')^k$ in G_F is contained in C. After potentially replacing $\mathbf{T}'$ by an $(H_F^+)^\Gamma$-conjugate, we then have that $(\mathbf{A}^+)^k \leq (\mathbf{T}')^k \leq \mathbf{C}_F$. We then have that $\mathbf{M}' \leq \mathbf{M}''$, and so we are done.

□

5.5 A Parameterization of Unramified θ-split Tori

In this section we seek to parameterize the unramified θ-split tori, those θ-split k-tori $\mathbf{S}$ in $\mathbf{G}$ for which $\mathbf{S}$ is the (θ, K)-split component of the the center of $C_{\mathbf{G}}(\mathbf{S})$. In the case of maximal unramified θ-split tori, we have a parameterization by Portilla. In this case, there is a bijective correspondence between the set of H^Γ-conjugacy classes of maximal unramified θ-split tori in $\mathbf{G}$ and equivalence classes of pairs (F, S), where F is a θ-facet in $\mathcal{B}(G^\theta)^{\mathrm{Fr}}$ and S is a maximal θ-elliptic θ-stable $\mathfrak{f}$-torus in G_F, where we recall from [28] that a θ-stable torus in said to be θ-elliptic if $(\mathsf{T}^+)^{\mathfrak{f}} = (\mathsf{Z}^+_{\mathsf{G}_F})^{\mathfrak{f}}$. (Note that there is another notion of a θ-elliptic $\mathfrak{f}$-torus appearing in the work of Murnaghan, where it is defined to be a $\mathfrak{f}$-torus T so that $(\mathsf{T}^-)^{\mathfrak{f}} = (\mathsf{Z}^-_{\mathsf{G}_F})^{\mathfrak{f}}$.)

A general unramified θ-split torus will arise as a lift of one of the pairs (F, S) from the previous section, where F is a θ-facet in $\mathcal{B}(G^\theta)^{\mathrm{Fr}}$ and S is a θ-split $\mathfrak{f}$-torus in G_F. However, not all of the θ-split (K, k)-tori from the previous section are unramified θ-split tori, and so we need to refine this parameterization.

We fix an H-alcove $\mathcal{C}$ in $\mathcal{B}(H)$ lying in the apartment of a maximally k-split maximal unramified k-torus $\mathbf{A}$ of $\mathbf{H}$, which in turn lies inside of a maximally k-split maximal unramified k-torus $\mathbf{A}'$ of $C_{\mathbf{G}}(\mathbf{A})$. Then $\mathcal{C}^\Gamma$ is a union of θ-facets in $\mathcal{B}(G^\theta)^\Gamma$, and for each such θ-facet F such that a pair (F, A_F) arises in our parameterization from the previous section, we fix a maximally k-split maximal θ-split $\mathfrak{f}$-torus A_F in G_F and a θ-perfect torus A'_F containing A_F so that if for two θ-facets F and F', there is an $h \in H^\Gamma$ so that F and ${}^hF'$ are strongly associated, then A_F is identified with $\mathsf{A}_{F'}$ and A'_F with $\mathsf{A}'_{F'}$ under the identification of G_F with $\mathsf{G}_{F'}$.

By taking lifts of each of the A_F above we find a family of θ-split tori $\mathbf{A}_F$ and maximally k-split θ-stable maximal unramified tori $\mathbf{A}'_F$ in $\mathbf{G}$ containing them. By [17], the set of roots in $\mathbf{G}$ of each of the $\mathbf{A}'_F$ has a θ-basis, which we denote by Δ_F.

5.5.1 An Indexing Set over $\mathfrak{f}$

Given $\mathbf{A}'_F$ as above, we set $\Pi(\mathbf{G}, F)$ to be the set of all θ-admissible subsets of θ-bases of the roots of $\mathbf{A}'_F$ in $\mathbf{G}$. Recall that we may and do identify $\Pi(\mathbf{G}, F)$ with the subsets of simple systems for the roots of $\mathbf{A}_F$ in $\mathbf{G}$. Recall that we identified Fr with a topological generator for Γ. Set

$$I_{\theta,F} = \{(\pi, w) | \pi \in \Pi(\mathbf{G}, F), w \in W_{\theta,c(\pi),F}, \text{ and } \mathrm{Fr}(\Phi_\pi) = w\Phi_\pi\},$$

where Φ_π denotes the root system spanned by π and $W_{\theta,c(\pi),F} = N_{H^\circ}(\mathbf{A}_F)/((C_{H^\circ}((\mathbf{A}_F)_\pi))^\circ \cap C_{H^\circ}(\mathbf{A}_F))$.

Modeling [1], if F is θ-facet contained in the facet F' in $\mathcal{B}(G)$, we let $\Phi(F)$ denote the set of gradients of the affine roots of $\mathbf{A}'$ in $\mathbf{G}$ whose restriction to F' is constant, and we let $\mathbf{A}'(F) = \left(\bigcap_{\alpha\in\Phi(F)} \ker(\alpha)\right)^\circ$. Recall that ${}_F\mathbf{M} = C_{\mathbf{G}}(\mathbf{A}'(F))$ and that the image of $A'(F) \cap G_F$

in G_F is the group of $\mathfrak{F}$-points of the $\mathfrak{f}$-split component of the center of $\mathsf{G}_F \cong {}_F\mathsf{M}_F$.

For a θ-admissible subset π of roots of $\mathbf{A}'_F$, we define

$$(W_{\theta,c(\pi)})_F := N_{\mathsf{H}_F}(\mathsf{A}_F)/((C_{\mathsf{H}_F}((\mathsf{A}_F)_\pi))^\circ \cap C_{\mathsf{H}_F}(\mathsf{A}_F)),$$

where $(\mathsf{A}_F)_\pi$ is the image of $(A_F)_\pi$ in G_F. Note that we can and do identify $(W_{\theta,c(\pi)})_F$ with a subgroup of $W_{\theta,c(\pi),F}$.

We now define

$$I(F) := \{(\pi, w) \in I_{\theta,F} | w \in (W_{\theta,c(\pi)})_F \leq W_{\theta,c(\pi),F}\}.$$

For (θ', w') and (θ, w) in $I(F)$, we write $(\theta', w') \sim_F (\theta, w)$ if there exists $n \in N_{H_F}(A_F)$ so that $\Phi_{\pi'} = n\Phi_\pi$ and $\mathrm{Fr}(n)wn^{-1} \in w'((W_{\theta,c(\pi')})_F \cap W_{\theta,c(\pi')}(\pi'))$, where $W_{\theta,c(\pi')}(\pi')$ denotes the subgroup of $W_{\theta,c(\pi')}$ whose image in the little Weyl group under the natural projection lies in the parabolic subgroup corresponding to π'. One checks that $\sim_F$ defines an equivalence relation.

We will say that $(\theta, w) \in I(F)$ is F-elliptic provided that for all θ-facets F' in $\mathcal{C}$ so that $F \subseteq \overline{F}'$, for all $(\theta', w') \in I(F)$ with $(\theta', w') \sim_F (\theta, w)$, and for all $h_{F'} \in H_F^\Gamma$ so that $(\mathbf{A}_F)_{\theta'} \subseteq {}^{h_{F'}}\mathbf{A}_{F'}$, we have that ${}^{h_{F'}^{-1}}w'$ does not have a representative in $H_{F'}$. We set $I^e(F)$ to be the set of pairs in $I(F)$ which are F-elliptic.

Lemma 5.5.1. Suppose $(\pi, w) \in I(F)$. Then we can choose $h \in H_F$ so that the image of $n = \mathrm{Fr}(h)^{-1}h \in N_{H_F}(\mathbf{A}_F)$ in $(W_{\theta,c(\pi)})_F$ is w.

Proof. Choose $\overline{h} \in \mathsf{H}_F$ so that the image of $\mathrm{Fr}(\overline{h})^{-1}\overline{h}$ in $(W_{\theta,c(\pi)})_F$ is w, which we can do by Lang's theorem applied to H_F. Note that $\mathsf{S} = {}^{\overline{h}}\mathsf{A}_F$ is a maximal θ-split $\mathfrak{f}$-torus in G_F by our choice of A_F. Now let $\mathbf{S}$ be a lift of (F, S). Then since $\mathbf{S}$ is a maximal unramified θ-split torus with $F \subset \mathcal{A}(S)$, there exists an element $x \in H_F$ so that ${}^x\mathbf{A}_F = \mathbf{S}$. Let $\overline{x}$ denote the image of x in H_F. Then since $\mathsf{S} = {}^{\overline{x}}\mathsf{A}_F$, the image of $\mathrm{Fr}(\overline{x})^{-1}\overline{x}$ in $(W_{\theta,c(\pi)})_F$ is of the form $\mathrm{Fr}(w')^{-1}ww'$

for some w' in $(W_{\theta,c(\pi)})_F$. Then let $n' \in N_{H_F}(\mathsf{A}_F)$ be a lift of w', and set $h = xn'$. □

5.5.2 Relevant θ-split Tori over $\mathfrak{f}$

Suppose $(F, \mathsf{S}) \in J^{\mathrm{Fr}}$, and let $\mathbf{S}$ be a lift of (F, S). Then we will say that S is relevant in G_F provided that $\mathbf{S}$ is the (θ, K)-split component of the center of $C_{\mathbf{G}}(\mathbf{S})$. Let $\mathcal{R}(F)$ denote the set of relevant θ-split tori in G_F. Fix $\iota = (\pi, w) \in I(F)$. Then thanks to Lemma 5.5.1, we can fix $h \in H_F$ so that the image of $n = \mathrm{Fr}(h)^{-1}h \in N_{H_F}(\mathbf{A}_F)$ in $(W_{\theta,c(\pi)})_F$ is w. Let $\overline{h}$ denote the image of h in H_F, and let

$$(\mathbf{A}_F)_\pi = \left(\bigcap_{\alpha\in\pi} \ker(\alpha)|_{\mathbf{A}_F}\right)^\circ \leq \mathbf{A}_F.$$

Set $\mathsf{S}_\iota = {}^{\overline{h}}(\mathsf{A}_F)_\pi$ and $\mathbf{S}_\iota = {}^{h}(\mathbf{A}_F)_\pi$. Then $\mathbf{S}_\iota$ is a lift of (F, S_ι). Set $\mathsf{L}_\iota = C_{\mathsf{G}_F}(\mathsf{S}_\iota)$ and $\mathbf{L}_\iota = C_G(\mathbf{S}_\iota)$. Then note that $\Phi_\pi = {}^{h^{-1}}\Phi(\mathbf{L}_\iota, {}^{h}\mathbf{A}'_F)$, and note that since $\mathbf{S}_\iota$ is the (θ, K)-split component of the center of $\mathbf{L}_\iota$, S_ι is relevant.

Lemma 5.5.2. The map that sends $\iota \in I(F)$ to the H_F^Γ-conjugacy class of S_ι is well-defined.

Proof. We first show that the the H_F^Γ-conjugacy class of S_ι is independent of the choice of h above. Suppose $h' \in H_F$ so that the image of $\mathrm{Fr}(h')^{-1}h' \in N_{H_F}(\mathbf{A}_F)$ in $(W_{\theta,c(\pi)})_F$ is also w and let $\overline{h}'$ denote the image of h' in H_F. Let $\mathsf{S}'_\iota = {}^{\overline{h}'}(\mathsf{A}_F)_\pi$ and $\mathbf{S}'_\iota = {}^{h'}(\mathbf{A}_F)_\pi$. Then $\mathbf{S}'_\iota$ is a lift of (F, S'_ι), and since $\mathrm{Fr}(h')^{-1}h'$ and $\mathrm{Fr}(h)^{-1}h$ have image w in $(W_{\theta,c(\pi)})_F$, there exists $s' \in (C_{H_F}((\mathsf{A}_F)_\pi))^\circ \cap C_{H_F}(\mathsf{A}_F)$ so that $\mathrm{Fr}(h')^{-1}h's' = \mathrm{Fr}(h)^{-1}h$. Let $x = h'h^{-1} \in H_F$. Then for all $t \in S_\iota$ we have

$$\mathrm{Fr}({}^{x}t) = {}^{\mathrm{Fr}(h')\mathrm{Fr}(h^{-1})}\mathrm{Fr}(t) = {}^{h'h^{-1}}({}^{hs'}\mathrm{Fr}(t)) = {}^{x}\mathrm{Fr}(t).$$

Hence $\mathrm{Int}(x)$ and $\mathrm{Int}(\mathrm{Fr}(x))$ both carry S_ι^{Fr} to $S_\iota'^{\mathrm{Fr}}$, hence they carry $\mathbf{S}_\iota$ to $\mathbf{S}'_\iota$. Moreover, $\mathrm{Fr}(x)^{-1}x \in (C_{H_F}(\mathbf{S}_\iota))^\circ$. Then since we know that $\mathrm{H}^1(\mathrm{Fr}, (C_{\mathsf{H}_F}(\mathsf{S}_\iota)^\circ) = 1$, we know there exists $l \in (C_{H_F}(\mathbf{S}_\iota))^\circ$ so that $\mathrm{Fr}(x)^{-1}x = \mathrm{Fr}(l)^{-1}l$ modulo H_F^+. Thus $\overline{y}$, the image of xl^{-1} in

H_F, belongs to $\mathsf{H}_F^{\mathrm{Fr}}$, and we have that that S_ι and S'_ι are H_F^Γ-conjugate by $\overline{y}$.

If $\mathbf{S}'_\iota$ is H^Γ-conjugate to $\mathbf{S}_\iota$, say by $h_1 \in H^\Gamma$, then $\mathbf{S}'_\iota = {}^{h_1 h}(\mathbf{A}_F)_\pi$. Then since $\mathrm{Fr}(h_1 h)^{-1} h_1 h = \mathrm{Fr}(h)^{-1} h$, the map is independent of our choice of lift, and so we have a well-defined map from $I(F)$ to the set of $\mathsf{H}_F^{\mathrm{Fr}}$-conjugacy classes in $\mathcal{R}(F)$.

□

Lemma 5.5.3. The map that sends $\iota \in I(F)$ to the $\mathsf{H}_F^{\mathrm{Fr}}$-conjugacy class of S_ι descends to a bijective map from $I(F)/\sim_F$ to the set of $\mathsf{H}_F^{\mathrm{Fr}}$-conjugacy classes in $\mathcal{R}(F)$.

Proof. We first show that the map is injective. Suppose $\iota_i = (\pi_i, w_i) \in I(F)$ and $h_i \in H_F$ so that image of $\mathrm{Fr}(h_i)^{-1} h_i$ in $(W_{\theta,c(\pi)})_F$ is w_i. Set $\mathbf{S}_i = {}^{h_i}(\mathbf{A}_F)_{\pi_i}$ and $\mathsf{S}_i = {}^{\overline{h}_i}(\mathsf{A}_F)_{\pi_i}$, where $\overline{h}_i$ is the image of h_i in H_F. Note that $\mathbf{S}_i$ is a lift of (F, S_i). Now suppose there exists $\overline{h} \in \mathsf{H}_F^{\mathrm{Fr}}$ so that $\mathsf{S}_1 = {}^{\overline{h}}\mathsf{S}_2$. Then by the previous section, there exists a lift $h \in H_F^{\mathrm{Fr}}$ of $\overline{h}$ for which $\mathbf{S}_1 = {}^{h}\mathbf{S}_2$. Without loss of generality replace h_2 by hh_2 so that $\mathbf{S}_1 = \mathbf{S}_2$ and $\mathsf{S}_1 = \mathsf{S}_2$. Let $\mathsf{L}_1 = C_{\mathsf{G}_F}(\mathsf{S}_1)$, and let $\mathbf{L}_1 = C_{\mathbf{G}}(\mathbf{S}_1)$. Then there exists $\bar{l} \in (\mathsf{L}_1 \cap \mathsf{H}_F)^\circ$ for which ${}^{\overline{h}_2}\mathsf{A}_F = {}^{\overline{lh_1}}\mathsf{A}_F$. Then there is a lift $l \in (L_1 \cap H_F)$ of $\bar{l}$ so that ${}^{h_2}\mathbf{A}_F = {}^{lh_1}\mathbf{A}_F$. Choose $m \in N_H(\mathbf{A}_F)$ for which $lh_1 = h_2 m$, and note that $m = h_2^{-1} l h_1 \in H_F$. Let $\mathbf{M}_{\pi_i} = C_{\mathbf{G}}((\mathbf{A}_F)_{\pi_i})$. Then we have

$$
\begin{aligned}
\Phi_{\pi_1} &= \Phi(\mathbf{M}_{\pi_1}, \mathbf{A}_F) \\
&= h_1^{-1}\Phi(\mathbf{L}_1, {}^{h_1}\mathbf{A}_F) = h_1^{-1} l^{-1}\Phi(\mathbf{L}_1, {}^{lh_1}\mathbf{A}_F) = h_1^{-1} l^{-1}\Phi(\mathbf{L}_1, {}^{h_2}\mathbf{A}_F) \\
&= h_1^{-1} l^{-1} h_2 \Phi(\mathbf{M}_{\pi_2}, \mathbf{A}_F) = m^{-1}\Phi(\mathbf{M}_{\pi_2}, \mathbf{A}_F) = \Phi(\mathbf{M}_{m^{-1}\pi_2}, \mathbf{A}_F) \\
&= m^{-1}\Phi_{\pi_2}
\end{aligned}
$$

so that $m\Phi_{\pi_1} = \Phi_{\pi_2}$.

Since the image of ${}^{h^{-1}}(\mathrm{Fr}(l)^{-1} l) \in N_{H_F}(\mathbf{A}_F)$ in $W_{\theta,c(\pi_1)}$ belongs to the parabolic subgroup $W_{\theta,c(\pi_1)}(\pi_1)$, we then have that

$$\mathrm{Fr}(m)^{-1} w_2 m = \mathrm{Fr}(lh_1)^{-1} lh_1((C_{H_F}((\mathbf{A}_F)_\pi))^\circ \cap C_{H_F}(\mathbf{A}_F))$$

$$= w_1 h_1^{-1}(\mathrm{Fr}(l)^{-1}l)h_1((C_{H_F}((\mathbf{A}_F)_\pi))^\circ \cap C_{H_F}(\mathbf{A}_F)) \in w_1 W_{\theta,c(\pi_1)}(\pi_1).$$

Since the representatives are all in H_F, we have that $\mathrm{Fr}(m)^{-1}w_2 m$ is in $(W_{\theta,c(\pi)})_F$, and so we conclude that $\iota_1 \sim_F \iota_2$.

We now show that the map is surjective. Suppose $\mathsf{T} \le \mathsf{G}_F$ belongs to $\mathcal{R}(F)$. Let T' be a maximal θ-stable $\mathfrak{f}$-torus in G_F that contains T and has the largest possible $(\theta, \mathfrak{f})$-split rank among θ-stable tori in G_F that contain T. Then T' contains the center of G_F and there exist lifts $\mathbf{T}$ of (F, T) and $\mathbf{T}'$ of (F, T') such that $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T})$ is a θ-split Levi (K, k)-subgroup, $\mathbf{T}$ is the (θ, K)-split component of the center of $\mathbf{L}$, and $\mathbf{T} \le \mathbf{T}' \le \mathbf{L}$. Let $\mathbf{B_L} \le \mathbf{L}$ be a Borel K-subgroup of $\mathbf{L}$ contained in a minimal θ-split parabolic K-subgroup of $\mathbf{L}$ which contains $\mathbf{T}'$. Since $\mathbf{T}'$ is a lift of (F, T'), there is a $h \in H_F$ so that ${}^h\mathbf{A}'_F = \mathbf{T}'$. Let $\pi = h^{-1}\Delta(\mathbf{L}, \mathbf{B_L}, \mathbf{T}') \in \Pi(\mathbf{G}, F)$. Let w denote the image of $\mathrm{Fr}(h)^{-1}h$ in $(W_{\theta,c(\pi)})_F$. Then the pair (π, w) belongs to $I(F)$ and corresponds to T. $\square$

5.5.3 Parameterizing H^{Fr}-Conjugacy Classes of Unramified θ-split Tori in G

Define

$$I_{\mathrm{un}} = \{(F, \pi, w) : F \subseteq \mathcal{A}(A)^{\mathrm{Fr}} \text{ is a } (G^{\mathrm{Fr}}, \theta)\text{-facet and } (\pi, w) \in I(F)\}$$

and let $\mathcal{U}$ denote the set of H^{Fr}-conjugacy classes of unramified θ-split tori in $\mathbf{G}$. Then by the previous subsections, we can define a function $j : I_{\mathrm{un}} \to \mathcal{U}$ as follows. For $(F, \pi, w) \in I_{\mathrm{un}}$, let $\mathsf{S} \in \mathcal{R}(F)$ be a relevant torus associated to (π, w) and let $j(F, \pi, w)$ be the H^{Fr}-conjugacy class of any lift of (F, S).

For $(F', \pi', w'), (F, \pi, w) \in I_{\mathrm{un}}$ we write $(F', \pi', w') \sim (F, \pi, w)$ provided that there exists an element $n \in (\tilde{W}(\mathbf{H}, A))^{\mathrm{Fr}}$, where $\tilde{W}(\mathbf{H}, A)$ is defined to be the affine Weyl group $N_H(\mathbf{A})/(C_H(\mathbf{A}) \cap H_F)$ of $\mathbf{A}$ in H, for which $A(\mathcal{A}(A)^{\mathrm{Fr}}, F') = A(\mathcal{A}(A)^{\mathrm{Fr}}, nF)$ and with the identifications of $\mathsf{G}_{F'} = \mathsf{G}_{nF}$ and $X^*(\mathsf{A}_{F'}) = X^*(\mathsf{A}_{nF}) = X^*(\mathbf{A}_F)$ arising in this way, we have that $(\pi', w') \sim_{F'} (n\pi, {}^n w)$ in $I(F') = I(nF)$. One check that this defines an equivalence relation on I_{un}.

We say that $(F,\pi,w) \in I_{\mathrm{un}}$ is elliptic provided that $(\pi,w) \in I^e(F)$, and we set I^e_{un} to be the set of elliptic triples (F,π,w) in I_{un}.

Theorem 5.5.4. The map j induces a bijection from $I^e_{\mathrm{un}}/\sim$ to $\mathcal{U}$.

Proof. We first show that j is surjective. Let $\mathbf{S}$ be an unramified θ-split torus in $\mathbf{G}$. Let $\mathbf{L}$ denote the centralizer of $\mathbf{S}$ and let F be a maximal $(\theta, G^{\mathrm{Fr}})$-facet in $\mathcal{B}(L^\theta)^{\mathrm{Fr}} \subseteq \mathcal{B}(G^\theta)^{\mathrm{Fr}}$. Choose a maximally (θ,K)-split maximal unramified torus $\mathbf{S}'$ of $\mathbf{L}$ that contains $\mathbf{S}$ so that $F \subset \mathcal{B}(S')$. Also fix a minimal θ-split parabolic K-subgroup $\mathbf{P_L}$ of $\mathbf{L}$ that contains $\mathbf{S}'$ and a Borel K-subgroup $\mathbf{B_L}$ contained in $\mathbf{P_L}$. Choose $h \in H^{\mathrm{Fr}}$ so that $hF \subset \mathcal{A}(A)^{\mathrm{Fr}}$. Then after replacing $\mathbf{S}$ with ${}^h\mathbf{S}$, we may assume that $F \subset \mathcal{A}(A)^{\mathrm{Fr}}$.

Let S denote the θ-split $\mathfrak{f}$-torus in G_F whose group of $\mathfrak{f}$-rational points coincides with the image of $S \cap G_F$ in G_F. Then there exists an $h \in H_F$ so that $\mathbf{S}' = {}^h\mathbf{A}'_F$. Let $\pi = h^{-1}\Delta(\mathbf{L},\mathbf{B_L},\mathbf{S}') \in \Pi(\mathbf{G},F)$, and let w denote the image of $\mathrm{Fr}(h)^{-1}h$ in $(W_{\theta,c(\pi)})_F$. Then note that $\mathbf{S}$ belongs to $j(F,\pi,w)$.

To conclude our proof of surjectivity, we need to show that the triple (F,π,w) is elliptic. If it is not elliptic, then there exists $(\pi',w') \in I(F)$ with $(\pi,w) \sim_F (\pi',w')$, an element $h_{F'} \in H_F$, and a (θ,G^{Fr})-facet F' with $F \subset \overline{F}$ so that ${}^{h_{F'}^{-1}}w'$ has a representative in $H_{F'}$ and $(\mathbf{A}_F)_{\pi'} \subseteq {}^{h_{F'}}\mathbf{A}_{F'}$. Then there exists $h \in \tilde{H} := {}^{h_{F'}}H_{F'} \subset H_F$ so that $\mathrm{Fr}(h)^{-1}h$ lies in $N_{\tilde{H}}(\mathbf{A}_F)$ and has image w' in $(W_{\theta,c(\pi)})_F$. Note that $hh_{F'}F'$ is a facet in $\mathcal{B}({}^{hh_{F'}}A'_{F'})$, and since $(\pi,w) \sim_F (\pi',w')$, from the preceding lemma we have that ${}^{hh_{F'}}(\mathbf{A}_{F'})_{h_{F'}^{-1}\pi'} = {}^h(\mathbf{A}_F)_{\pi'} = {}^x\mathbf{S}$ for some $x \in H^\Gamma$. Note that then ${}^{x^{-1}hh_{F'}}\mathbf{A}'_{F'} \leq \mathbf{L}$, and so $x^{-1}hh_{F'}F' \subset \mathcal{B}({}^{x^{-1}hh_{F'}}\mathbf{A}'_{F'})^{\mathrm{Fr}} \subseteq \mathcal{B}(L)^{\mathrm{Fr}}$, contradicting the maximality of F.

It remains to show that if (F_i,π_i,w_i) for $i \in \{1,2\}$ are two elements of I^θ_{un} with $j(F_1,\pi_1,w_1) = j(F_2,\pi_2,w_2)$, then $(F_1,\pi_1,w_1) \sim (F_2,\pi_2,w_2)$. Choose $\mathsf{S}_i \in \mathcal{R}(F_i)$ corresponding to $(\pi_i,w_i) \in I^e(F_i)$ and let $\mathbf{S}_i$ be a lift of (F_i,S_i). Note that $(F_i,\mathsf{S}_i) \in J^\Gamma_{\max}$. Since $j(F_1,\pi_1,w_1) = j(F_2,\pi_2,w_2)$, we conclude that $\mathbf{S}_1$ is H^{Fr}-conjugate to $\mathbf{S}_2$. Thus we know that there exists $h \in H^{\mathrm{Fr}}$ and an apartment $\mathcal{A}'$ in $\mathcal{B}(G^\theta)^{\mathrm{Fr}}$ so that $\emptyset \neq A(\mathcal{A}',F_1) = A(\mathcal{A}',hF_2)$

and $\mathsf{S}_1 = {}^h\mathsf{S}_2$ in $\mathsf{G}_{F_1} = \mathsf{G}_{hF_2}$ by Corollary 5.4.8. After conjugating by an element of $H_{F_1}^{\mathrm{Fr}}$, we may assume that $\mathcal{A}' = \mathcal{A}(A)$. By the affine Bruhat decomposition for $\mathbf{H}$, we may choose $n \in N_{\mathbf{H}(k)}(\mathbf{A})$ so that $n^{-1}h \in H_{F_2}^{\mathrm{Fr}}$. Then there exists $x \in \mathsf{H}_{F_2}^{\mathrm{Fr}}$ such that after replacing S_2 by ${}^x\mathsf{S}_2$ we may assume $A(\mathcal{A}(A)^{\mathrm{Fr}}, F_1) = A(\mathcal{A}(A)^{\mathrm{Fr}}, nF_2)$ and $\mathsf{S}_1 = {}^n\mathsf{S}_2$ in $\mathsf{G}_{F_1} = \mathsf{G}_{nF_2}$. Identifying n with its image in the affine Weyl group $\tilde{W}(H, A)$ of A in H, we then have that $(\pi_1, w_1) \sim_F (n\pi_2, {}^n w_2)$ in $I(F_1) = I(nF_2)$.

□

CHAPTER VI

Unramified θ-perfect Tori

While the previous section provides a serviceable analogue of the results in [1], working with unramified θ-split tori alone does not allow us to show how the $\mathbf{H}(K)$-conjugacy class of a maximal θ-split k-torus splits into $\mathbf{H}(k)$-conjugacy classes. In this section, we will provide a very similar parameterization of the $\mathbf{H}(k)$-conjugacy classes of a class of θ-stable tori called *unramified θ-perfect tori*, which are unramified tori in the sense of [1] which are contained in a H-conjugate of a θ-perfect torus in the sense of the previous section. Using these tori, we will be able to show how an $\mathbf{H}(K)$-conjugacy class splits into $\mathbf{H}(k)$-conjugacy classes in a future paper.

6.1 Results for Finite Groups of Lie Type

In this subsection, we will look at the analogous problem for finite groups of Lie type. Recall that a θ-stable maximal $\mathfrak{f}$-torus in $\mathbf{G}$ is called θ-perfect if it contains a maximal $\mathfrak{f}$-torus, a maximal θ-split $\mathfrak{f}$-torus, and a maximal (θ, k)-split $\mathfrak{f}$-torus. We say that a reductive subgroup L of G is a *θ-perfect twisted Levi $\mathfrak{f}$-subgroup* of G if

1. L is defined over $\mathfrak{f}$,

2. there exists a parabolic $\mathfrak{F}$-subgroup of G for which L is the associated Levi factor,

3. and the center of L is a θ-stable subtorus of an H-conjugate of a θ-perfect torus of G (or, equivalently, a θ-stable maximal $\mathfrak{f}$-torus containing a maximal θ-split $\mathfrak{f}$-torus.

We let $\mathcal{L}_{per}$ denote the set of θ-perfect twisted Levi $\mathfrak{f}$-subgroups of G.

If S is a θ-stable subtorus of an H-conjugate of a θ-perfect torus of G, then we say that it is a θ-perfect Levi torus if it equals the connected component of the center of $C_{\mathsf{G}}(\mathsf{S})$. Now if L is a θ-perfect twisted Levi $\mathfrak{f}$-subgroup, let S_{L} denote the the connected component of the center of L, which by definition is contained in a θ-perfect torus of G. Then we have $\mathsf{L} = C_{\mathsf{G}}(\mathsf{S}_{\mathsf{L}})$, and this gives a bijective correspondence between the set of θ-perfect Levi tori in G and the set of θ-perfect twisted Levi $\mathfrak{f}$-subgroups in G. This means that understanding $\mathcal{L}_{per}$ up to $\mathsf{H}(\mathfrak{f})$-conjugacy is equivalent to understanding the set of θ-perfect Levi tori in G up to $\mathsf{H}(\mathfrak{f})$-conjugacy.

Now let $\mathcal{L}_{per}/\sim_{\mathsf{H}}$ denote the set of $\mathsf{H}(\mathfrak{f})$-conjugacy classes in $\mathcal{L}_{per}$. Fix a θ-perfect $\mathfrak{f}$-torus T, and let Δ denote a θ-basis for the roots of $\mathbf{T}$ as in the previous section. Then if π is a subset of Δ whose span is θ-stable, we let $W'_{\theta,c(\pi)} = N_{\mathsf{H}^\circ}(\mathsf{T})/((C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ \cap C_{\mathsf{H}^\circ}(\mathsf{T}))$. Note the $W'_{\theta,c(\pi)}$ acts on the set of such π by considering the action which arises from projecting to the usual Weyl group of T. Let I_{per} denote the set of pairs (π, w), where π is a subset of a fixed θ-basis Δ for the roots of $\mathbf{T}$ and $w \in W'_{\theta,c(\pi)}$ so that

1. the span of π is θ-stable and
2. $w\pi = \mathrm{Fr}(\pi)$.

For (π', w') and (π, w) in I_{per}, we write $(\pi', w') \sim (\pi, w)$ if there exists an element n' in $N_{\mathsf{H}}(\mathsf{T})$ so that $\theta = n'\theta'$ and $w = \mathrm{Fr}(n')w'(n')^{-1}$. One checks that this gives a well-defined equivalence relation on the set I_{per}.

Lemma 6.1.1. There is a natural bijective correspondence between $I_{per}/\sim$ and $\mathcal{L}_{per}/\sim_{\mathsf{H}}$.

Proof. We begin by defining a map $\varphi : I_{per} \to \mathcal{L}_{per}/\sim_{\mathsf{H}}$. To do this, first suppose that we have a pair $(\pi, w) \in I_{per}$. Then applying Lang-Steinberg to H°, we can choose an element

$h \in \mathsf{H}^\circ$ so that the image of $\mathrm{Fr}(h^{-1})h$ in $W'_{\theta,c(\pi)}$ is w. Then we set $n_h = \mathrm{Fr}(h^{-1})h \in N_{\mathsf{H}^\circ}(\mathsf{T})$, and we let $\mathsf{T}_\pi := (\bigcap_{\alpha\in\pi}(\ker(\alpha)))^\circ$. Let $\mathsf{M}_\pi = C_{\mathsf{G}}(\mathsf{T}_\pi)$. Then since $\mathrm{Fr}(\pi) = w\pi$, we have

$$\mathrm{Fr}({}^h\mathsf{M}_\pi) = {}^{\mathrm{Fr}(h)}\mathrm{Fr}(\mathsf{M}_\pi) = {}^{hn_h^{-1}}(\mathsf{M}_{\mathrm{Fr}(\pi)}) = {}^h({}^{n_h^{-1}}(\mathsf{M}_{w\pi})) = {}^h\mathsf{M}_\pi.$$

Furthermore, since the span of π is θ-stable, we have that the center ${}^h\mathsf{T}_\pi$ of ${}^h\mathsf{M}_\pi$ is a θ-stable subtorus of a θ-perfect torus of G. Thus ${}^h\mathsf{M}_\pi$ is the center of a θ-perfect twisted Levi $\mathfrak{f}$-subgroup.

In order to show that we have a well-defined map, we need to show that a different choice of h results in a θ-perfect twisted Levi $\mathfrak{f}$-subgroup which is $\mathsf{H}(\mathfrak{f})$-conjugate to ${}^h\mathsf{M}_\pi$. Suppose that $h' \in \mathsf{H}^\circ$ is chosen so that $n_{h'} := \mathrm{Fr}(h')^{-1}h'$ also has image w in $W'_{\theta,c(\pi)}$. Then we can choose $s \in (C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ \cap C_{\mathsf{H}^\circ}(\mathsf{T})$ so that $n_{h'} = n_h s$. Then we have $\mathrm{Fr}(h'h^{-1})^{-1}h'h^{-1} = {}^hs \in {}^h((C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ \cap C_{\mathsf{H}^\circ}(\mathsf{T}))$, and applying Lang-Steinberg to ${}^h((C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ)$, we can find an element $s' \in^h ((C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ) \subseteq {}^h\mathsf{M}_\pi$ so that

$$\mathrm{Fr}(h'h^{-1})^{-1}h'h^{-1} = \mathrm{Fr}(s')^{-1}s'.$$

Thus $s'h(h')^{-1} = \mathrm{Fr}(s'h(h')^{-1})$, meaning that $s'h(h')^{-1} \in \mathsf{H}(\mathfrak{f})$, and we have

$${}^h\mathsf{M}_\pi = {}^{s'h}\mathsf{M}_\pi = {}^{(s'h(h')^{-1})h'}\mathsf{M}_\pi.$$

Thus we have that ${}^{h'}\mathsf{M}_\pi$ is $\mathsf{H}(\mathfrak{f})$-conjugate to ${}^h\mathsf{M}_\pi$, and so ϕ is well-defined.

We now show that φ descends to an injective map from $I_{per}/\sim$ to $\mathcal{L}_{per}/\sim_{\mathsf{H}}$, which we will also call φ. To do this, suppose that (π, w) and (π', w') are in I_{per}, and choose h and h' in H° so that the images of n_h and $n_{h'}$ in $W'_{\theta,c(\pi)}$ are w and w' respectively. If $\varphi(\pi, w) = \varphi(\pi', w')$, then we know that there is $k \in \mathsf{H}(\mathfrak{f})$ so that ${}^h\mathsf{M}_\pi = {}^{kh'}\mathsf{M}_{\pi'}$. Replacing h' with kh', we may then assume without loss of generality that ${}^h\mathsf{M}_\pi = {}^{h'}\mathsf{M}_{\pi'}$. We then have that both ${}^h\mathsf{T}$ and ${}^{h'}\mathsf{T}$ are maximal θ-perfect $\mathfrak{f}$-tori in ${}^h\mathsf{M}_\pi$, and so there exists some $m' = {}^hm$

with m in $(C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ$ so that ${}^{m'h'}\mathsf{T} = {}^h\mathsf{T}$ and ${}^{m'h'}(\mathsf{B} \cap \mathsf{M}_{\pi'}) = {}^h(\mathsf{B} \cap \mathsf{M}_\pi)$, where B is the Borel subgroup of G corresponding to our fixed θ-basis Δ for the roots of T. Since ${}^{h'}\mathsf{T}$ and ${}^h\mathsf{T}$ are both defined over $\mathfrak{f}$, we also see that ${}^{\mathrm{Fr}(m')}({}^{h'}\mathsf{T}) = {}^h\mathsf{T}$, and so we can conclude that $m'\mathrm{Fr}(m')^{-1} \in N_{{}^h(C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ}({}^h\mathsf{T})$. This then implies that $n_h^{-1}\mathrm{Fr}(m)n_h m^{-1} \in N_{(C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ}(\mathsf{T})$. We now set $n = mh^{-1}h' \in N_{\mathsf{H}}(\mathsf{T})$ and note that $n\pi' = \pi$. We also have

$$\mathrm{Fr}(n)(n_{h'})n^{-1} = \mathrm{Fr}(mh^{-1}h')\mathrm{Fr}(h')^{-1}h'(mh^{-1}h')^{-1} = \mathrm{Fr}(m)(n_h)m^{-1} = n_h(n_h^{-1}\mathrm{Fr}(m)(n_h)m^{-1}).$$

Looking at images in the Weyl group W of T in G, we see that the image of $\mathrm{Fr}(n)(w')n^{-1}$ is equal to the image of w times an element of the parabolic subgroup W_π of W associated to π, which we call x and equals the image of $n_h^{-1}\mathrm{Fr}(m)n_h m^{-1}$ in W. Note that

$$x\pi = w^{-1}\mathrm{Fr}(n)w'n^{-1}\pi = w^{-1}\mathrm{Fr}(n)w'\pi' = w^{-1}\mathrm{Fr}(n)\mathrm{Fr}(\pi') = w^{-1}\mathrm{Fr}(\pi) = \pi.$$

But since the action of W_π on the set of simple systems for the root system spanned by π is simply transitive, we must have that $x = 1$ in W, meaning that $n_h^{-1}\mathrm{Fr}(m)n_h m^{-1}$ lies in $C_{\mathsf{H}}(\mathsf{T})$. Thus we have shown that $n_h^{-1}\mathrm{Fr}(m)(n_h)m^{-1}$ lies in $(C_{\mathsf{H}}(\mathsf{T}_\pi))^\circ \cap C_{\mathsf{H}}(\mathsf{T})$, and so its image in $W'_{\theta,c(\pi)}$ is trivial. Consequently, we have shown that $(\pi, w) \sim (\pi', w')$ so that the map is injective as claimed.

Finally, we show that the map ϕ is surjective. To do so, suppose that $\mathsf{L} \in \mathcal{L}_{per}$, and let A_{L} denote the connected component of the center of L. Choose a θ-perfect torus T_{L} in L, and let B_{L} be a Borel $\mathfrak{f}$-subgroup in L containing T_{L} and corresponding to a θ-basis Δ_{L} for the roots of T_{L}. Now choose an $h \in \mathsf{H}^\circ$ so that $\mathsf{T}_{\mathsf{L}} = {}^h\mathsf{T}$. and $\mathsf{B}_{\mathsf{L}} \leq {}^h\mathsf{B}$. Define $\pi_{\mathsf{L}} = h^{-1} \cdot \Delta_{\mathsf{L}}$, and note that $\pi_{\mathsf{L}} \subset \Delta$. Let w_{L} denote the image of $\mathrm{Fr}(h^{-1})h$ in $W'_{\theta,c(\pi)}$ and put $\mathsf{T}_{\pi_{\mathsf{L}}} = (\bigcap_{\alpha \in \pi_{\mathsf{L}}} (\ker(\alpha)))^\circ \leq \mathsf{T}$. Then we have that $\mathsf{A}_{\mathsf{L}} = {}^h\mathsf{T}_{\pi_{\mathsf{L}}}$ and $\mathrm{Fr}(\pi_{\mathsf{L}}) = w_{\mathsf{L}}\pi_{\mathsf{L}}$, which shows that $\varphi(\pi_{\mathsf{L}}, w_{\mathsf{L}})$ gives the $\mathsf{H}(\mathfrak{f})$-conjugacy class of L. $\square$

6.2 Some Notation and a Counterexample

Fix a Galois extension E of k. We call a subgroup $\mathbf{M}$ of $\mathbf{G}$ a *θ-perfect Levi (E,k)-subgroup* if it is a θ-stable k-subgroup so that $\mathbf{M}$ is the Levi factor of a parabolic E-subgroup of $\mathbf{G}$ and if $\mathbf{M}$ contains an $\mathbf{H}$-conjugate of a θ-perfect torus of $\mathbf{G}$. The latter condition is equivalent to the identity component of the center of $\mathbf{M}$ being contained in an $\mathbf{H}$-conjugate of a θ-perfect torus of $\mathbf{G}$. (Note that the $\mathbf{H}$-conjugates of a θ-perfect torus of $\mathbf{G}$ are precisely the θ-stable maximal tori in $\mathbf{G}$ which contain a maximal θ-split torus. This is because all maximal θ-split tori are $\mathbf{H}$-conjugate and because the derived subgroup of the centralizer of a maximal θ-split torus is in $\mathbf{H}^\circ$ by [18].) Recall that if E is a tame Galois extension of k and M is the group of K-rational points of a Levi (E,k)-subgroup of G, then we can and do identify $\mathcal{B}(M)$ with a subset of $\mathcal{B}(G)$, noting that there is no canonical way to do this but that all such identifications have the same image. As in Section 5.3, given a k-torus $\mathbf{S}$ of $\mathbf{G}$, we let $\mathbf{S}^E$ denote the maximal E-split subtorus in $\mathbf{G}$.

When E is the maximal unramified extension K of k, we adopt the following language. First, we say that a subgroup $\mathbf{L}$ of $\mathbf{G}$ is an *unramified θ-perfect twisted Levi subgroup* provided that $\mathbf{L}$ is a θ-perfect Levi (K,k)-subgroup of $\mathbf{G}$. In addition, we say that a k-torus $\mathbf{S}$ is an unramified θ-perfect torus in $\mathbf{G}$ provided that $\mathbf{S}$ is the maximal K-split subtorus of the identity component of the center of an unramified θ-perfect twisted Levi subgroup of $\mathbf{G}$.

Note that a θ-perfect Levi (E,k)-subgroup is also a Levi (E,k)-subgroup in the sense of [1], and so we can carry over results from section 3 there. In particular, we know that if $\mathbf{L}$ is an unramified θ-perfect twisted Levi subgroup in $\mathbf{G}$ and $\mathbf{S}$ is the maximal K-split subtorus of the identity component of the center of $\mathbf{L}$, then we know that $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$. This gives a bijective correspondence between the unramified θ-perfect twisted Levi subgroups of $\mathbf{G}$ and the unramfied θ-perfect tori in $\mathbf{G}$, and we have that two unramified θ-perfect twisted Levi subgroups are $\mathbf{H}(k)$-conjugate if and only if the corresponding unramified θ-perfect tori are $\mathbf{H}(k)$-conjugate. Consequently, parameterizing the $\mathbf{H}(k)$-conjugacy classes of unramified θ-perfect twisted Levi subgroups in $\mathbf{G}$ is equivalent to parameterizing the $\mathbf{H}(k)$-conjugacy calsses of unramified θ-perfect tori in $\mathbf{G}$.

It is natural to ask whether the conjecture from Section 5.3 and the lemma following it have analogues for θ-perfect tori. In particular, one asks:

1. Does $\mathbf{G}$ contain a $\mathbf{H}$-conjugate of a θ-perfect torus which is K-minisotropic?

2. Is a K-split torus $\mathbf{T}$ contained in an $\mathbf{H}$-conjugate of a θ-perfect torus unramified if and only if there exists an $\mathbf{H}$-conjugate $\mathbf{T}'$ of a θ-perfect k-torus so that $\mathbf{T}$ is the maximal K-split subtorus of $\mathbf{T}'$.

The answer to both of these questions is no, as the following example illustrates. Let $\mathbf{G} = \mathrm{SL}_3$ defined over, as always, a finite extension k of the p-adic numbers $\mathbb{Q}_p$ with $p \neq 2$, and let θ be the involution given by conjugating by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Then the $\mathbf{H}(k)$-conjugacy classes of θ-split k-tori with respect to this involution can be directly computed. In particular, for each $x \in \{1, \epsilon, \varpi, \epsilon\varpi\}$, where we fix $\epsilon \in \mathfrak{o}^\times \setminus (\mathfrak{o}^\times)^2$, there is an $\mathbf{H}(k)$-conjugacy class which has a representative torus T_x consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & xb & a \end{pmatrix}.$$

The centralizers of these θ-split tori consist of matrices having the form

$$\begin{pmatrix} c & 0 & 0 \\ 0 & a & b \\ 0 & xb & a \end{pmatrix},$$

and as these centralizers are tori, we have that they are representatives for the $\mathbf{H}(k)$-conjugacy classes of the maximal k-tori which are $\mathbf{H}$-conjugate to a θ-perfect torus. (The θ-perfect tori correspond to the class where $x = 1$, and the other $\mathbf{H}(k)$-conjugacy class of maximal unramifed θ-perfect tori is the class where $x = \epsilon$.) But notice that the intersection of each representative with the diagonal torus in SL_3 gives us a k-split torus of rank 1, and so none of the $\mathbf{H}$-conjugates of a θ-perfect torus are K-minisotropic. Furthermore, the trivial torus is K-split and clearly contained in a θ-perfect torus, yet it is not the maximal K-split subtorus of any $\mathbf{H}$-conjugate of a θ-perfect torus, as each such torus contains a k-split torus of rank 1. Thus we have shown that the answer to both of our questions is no.

6.3 θ-perfect (K, k)-tori

We call a torus in $\mathbf{G}$ a θ-perfect (K, k)-torus if it is a θ-stable K-split k-torus which contains $\mathbf{Z}_{\mathbf{G}}^K := \mathbf{Z}^K$ and is contained in a $\mathbf{H}$-conjugate of a θ-perfect torus. We let $\mathcal{T}_{K,per}$ denote the set of θ-perfect (K, K)-tori in $\mathbf{G}$, then we have that $\mathcal{T}_{K,per}$ carries a natural action of $\mathrm{Gal}(K/k)$, and we denote the set of points in $\mathcal{T}_{K,per}$ fixed by $\mathrm{Gal}(K/k)$ by $\mathcal{T}_{k,per}$. To ease notation, we will also call the K-rational points T of a θ-perfect (K, k)-torus $\mathbf{T}$ a θ-perfect (K, k)-torus.

Our goal for this subsection is to parameterize the $\mathbf{H}(k)$-conjugacy classes in $\mathcal{T}_{k,per}$. To begin, as in the previous section we introduce our indexing sets. For a θ-facet F in $\mathcal{B}(G)$, we let Z_F denote the group corresponding to the image of $(G_F \cap Z^K)$ in G_F/G_F^+, where as in the previous section we write G_F for the parahoric of the facet F' in $\mathcal{B}(G)$ containing F. Now we define the indexing set J_{per} to be the set of pairs (F, T) so that F is a θ-facet in $\mathcal{B}(G)$ and T is a subtorus of an H_F-conjugate of a θ-perfect torus in G_F containing Z_F.

We say that a (K, K)-torus $\mathbf{T}$ (in the sense of [1]) is a lift of (F, T) provided that we have

1. $\mathbf{T}$ is θ-stable
2. $F \subset \mathcal{B}(C_G(T))$

3. the image of $T \cap G_F$ in $\mathsf{G}_F = G_F/G_F^+$ is T.

Now suppose that $(F, \mathsf{T}) \in J_{per}$, and let $\Gamma :=$ Gal (K/k). If $\Gamma(F) = F$, then G_F is defined over the residue field $\mathfrak{f}$ of k, and so it makes sense to consider $\Gamma(\mathsf{T})$. We then define J_{per}^{Γ} to be the set of pairs (F, T) in J_{per} so that both F and T are Γ-stable.

Next, we say that a pair (F, T) in J_{per}^{Γ} is maximal if whenever a θ-facet F_1 in $\mathcal{B}(G)$ is Γ-stable and contains F in its closure, then T belongs to the $\mathfrak{f}$-parabolic subgroup G_{F_1}/G_F^+ of G_F if and only if $F = F_1$. We let $J_{m,per}^{\Gamma}$ denote the subset of maximal pairs in J_{per}^{Γ}.

6.3.1 Lifts of Tori over $\mathfrak{f}$

Suppose that $(F, \mathsf{T}) \in J_{per}^{\Gamma}$. Our first goal for this subsection is to show that there is a (K, k)-torus which lifts (F, T) and to show that any two such lifts are conjugate by an element of $(H_F^+)^{\Gamma}$. We will then show that all elements of $\mathcal{T}_{k,per}$ arise in this way.

The proofs here are almost identical to the ones in 5.4.1 from the previous chapter, the main difference being the tori T which show up in the pairs (F, T) in J_{per}^{Γ}. In particular, a torus T in G_F may not be θ-split. For example, letting $\mathbf{G}$ be SL_3 as in our example from 6.2, one may take a special vertex x in the apartment of the diagonal torus, and then take the torus T in G_x consisting of diagonal matrices of the form $\mathrm{diag}(c, a, a)$. Then T is contained in a θ-perfect torus in G_x, meaning that $(x, \mathsf{T}) \in J_{per}^{\Gamma}$, but T is certainly not θ-split. On the other hand, the torus S in a pair $(F, \mathsf{S}) \in J^{\Gamma}$ from the previous section may not contain all of Z_F. For example, if one lets $\mathbf{G}$ be the direct product of our SL_3 example from 6.2 with a non-trivial k-split torus $\mathbf{T}_k$ on which θ is defined to act trivially, then the maximal θ-split tori in the product $\mathrm{SL}_3 \times \mathbf{T}_k$ are the product of maximal θ-split tori in SL_3 with the trivial subtorus and hence do not contain the center of the direct product. Consequently, the pairs (F, S) in J^{Γ} associated to the direct product $\mathbf{G}$ will not contain the center of the associated reductive quotient G_F and hence will not be in J_{per}^{Γ}.

Lemma 6.3.1. Set $\mathsf{M} = C_{\mathsf{G}_F}(\mathsf{T})$, and let T' denote a θ-perfect $\mathfrak{f}$-torus in M. Then there is a θ-stable maximal unramified torus $\mathbf{T}'$ in $\mathbf{G}$ which lifts (F, T'). Moreover, for all such $\mathbf{T}'$

lifting (F, T') there exists a unique lift $\mathbf{T} \in \mathcal{T}_{k,per}$ of (F, T) with the property that $\mathbf{T} \leq \mathbf{T}'$.

Proof. We know that such an unramified torus $\mathbf{T}'$ exists by [28]. We have that $\mathrm{X}_*(\mathsf{T}') = X_*(\mathbf{T}')$ both as Γ-modules and as θ-modules, and so we can choose a subtorus $\mathbf{T}$ of $\mathbf{T}'$ corresponding to the image of $\mathrm{X}_*(\mathbf{T})$ under the map $\mathrm{X}_*(\mathsf{T}) \hookrightarrow \mathrm{X}_*(\mathbf{T}') = \mathrm{X}_*(\mathbf{T}')$. Then we have that $\mathbf{T}$ is a (K, k)-torus, and since we have $T' \leq C_G(T)$ by construction, we have that $F \subseteq \mathcal{B}(T') \subseteq \mathcal{B}(C_G(T))$, which shows that T is a lift of (F, T) as required.

Now if $\mathbf{T}_2 \in \mathcal{T}_{k,per}$ is another lift of (F, T) that lies in $\mathbf{T}'$, then $\mathrm{X}_*(\mathbf{T}_2) = \mathrm{X}_*(\mathbf{T}) = \mathrm{X}_*(\mathsf{T})$ in $\mathrm{X}_*(\mathbf{T}')$, and so $\mathbf{T} = \mathbf{T}_2$. □

Corollary 6.3.2. If $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}_{k,per}$ both lift (F, T), then there exists an element $h \in (H_F^+)^\Gamma$ so that ${}^h\mathsf{T}_1 = \mathsf{T}_2$.

Proof. Set $\mathbf{M}_i = C_{\mathbf{G}}(\mathbf{T}_i)$. Then note that $F \subseteq \mathcal{B}(M_i)$ by the definition of a lift, and we know that the image of $M_i \cap G_F$ in G_F is $\mathsf{M} = C_{\mathsf{G}_F}(\mathsf{T})$. Now let $\mathbf{T}_i' \leq \mathbf{M}'$ be a θ-stable lift of (F, T'), where as in Lemma 6.3.1, T' is a θ-perfect $\mathfrak{f}$-torus in M. Then since $\mathbf{T}_i$ is in the center of $\mathbf{M}_i$, we have that $\mathbf{T}_i \leq \mathbf{T}_i'$, and since $\mathbf{T}_i$ (resp. $\mathbf{T}_i'$) is a K-split torus lifting (F, T) (resp. (F, T')), we know from the previous lemma that $\mathbf{T}_i$ is the unique lift of (F, T) in $\mathbf{T}_i'$. Then by [28], we know that there is an h in $(H_F^+)^\Gamma$ so that ${}^h\mathbf{T}_1' = \mathbf{T}_2'$. Our result then follows from uniqueness in Lemma 6.3.1. □

The preceding lemma and corollary allow us to define an action of H^Γ on $J_{m,per}^\Gamma$. Suppose that $h \in H^\Gamma$ and $(F, \mathsf{T}) \in J_{m,per}^\Gamma$. Then if $\mathbf{T}$ is a lift of (F, T), let ${}^h\mathsf{T}$ denote the image of ${}^hT \cap G_{hF}$ in G_{hF} and set $h(F, \mathsf{T}) := (hF, {}^h\mathsf{T}) \in J_{m,per}^\Gamma$.

We now move in the opposite direction, showing that every element of $\mathcal{T}_{k,per}$ arises as a lift of a pair in $J_{m,per}^\Gamma$. Recall from Section 5.4 that given a θ-stable reductive k-subgroup $\mathbf{C}$ of $\mathbf{G}$ having the same K-rank as $\mathbf{G}$, we define a (θ, C)-facet to be a non-empty subset of $\mathcal{B}(G)^\theta$ which equals the set of θ-fixed points of some C-facet in $\mathcal{B}(C) \subseteq \mathcal{B}(G)$.

Lemma 6.3.3. For all $\mathbf{T} \in \mathcal{T}_{k,per}$ there exists $(F, \mathsf{T}) \in J_{m,per}^\Gamma$ so that $\mathbf{T}$ lifts (F, T).

Proof. Fix $\mathbf{T} \in \mathcal{T}_{k,per}$, and let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T})$. Note that $\mathbf{M}$ is a θ-stable Levi (K,k)-subgroup of $\mathbf{G}$.

Choose a Γ-stable (θ, M)-facet F' in $\mathcal{B}(M)$ so that F' has maximal dimension among θ-facets in $\mathcal{B}(M)$. Then since F' can be written as the disjoint union of (θ, G)-facets in $\mathcal{B}(G)$, we are able to choose a Γ-stable (θ, G)-facet F in $\mathcal{B}(H)$ so that $F \subset F'$ and $\dim(F^\Gamma) \geq \dim(\tilde{F}^\Gamma)$ for all Γ-stable (θ, G)-facets $\tilde{F}$ in $\overline{F}'$. As in the analogous proof for θ-split (K,k)-tori in Section 5.4, we actually have that $\dim(F^\Gamma) \geq \dim(\tilde{F}^\Gamma)$ for all Γ-stable (θ, G)-facets $\tilde{F}$ in $\mathcal{B}(M)$.

Now let T be the $\mathfrak{f}$-torus in G_F corresponding to the image of $T \cap G_F$ in G_F. Then, by construction, we have that $\mathbf{T}$ is a lift of the pair (F, T). It remains to show that $(F, \mathsf{T}) \in J^\Gamma_{m,per}$. Suppose that $F'' \subset \mathcal{B}(G)$ is a Γ-stable (θ, G)-facet with $F \subset \overline{F}''$ and $F \neq F''$. Then if T lies in the proper parabolic $\mathfrak{f}$-subgroup $G_{F''}/G_F^+$ of $\mathsf{G}_F = G_F/G_F^+$, then we have that $S \cap G_F = S \cap G_{F''}$ fixes F'' and $(F'', \tilde{\mathsf{T}}) \in J^\Gamma_{per}$, where $\tilde{\mathsf{T}}$ is the $\mathfrak{f}$-torus in $\mathsf{G}_{F''}$ corresponding to the image of $S \cap G_{F''}$ in $\mathsf{G}_{F''}$. By [9, 4.4.2], we then have that F'' is in $\mathcal{B}(M)$. However, the dimension of $(F'')^\Gamma$ is strictly larger than that of F^Γ, contradicting our choice of F. $\square$

6.3.2 An Equivalence Relation

The previous subsection gives us a well-defined surjective map φ from $J^\Gamma_{m,per}$ to the set of H^Γ-conjugacy classes in $\mathcal{T}_{k,per}$. In this subsection, we will define an equivalence relation $\sim$ on $J^\Gamma_{m,per}$ so that φ will descend to a bijection.

Suppose $(F_i, \mathsf{T}_i) \in J^\Gamma_{per}$. Then we write $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$ if there exists an element $h \in H^\Gamma$ and an apartment $\mathcal{A}$ in $\mathcal{B}(H)^\Gamma$ so that

1. $\emptyset \neq A(\mathcal{A}, F_1^\Gamma) = A(\mathcal{A}, hF_2^\Gamma)$

2. $\mathsf{T}_1 \approx {}^h\mathsf{T}_2$ in $\mathsf{G}_{F_1} \approx \mathsf{G}_{hF_2}$.

Lemma 6.3.4. The relation $\sim$ is an equivalence relation on $J^\Gamma_{m,per}$.

Proof. The proof is nearly identical to the one in [28] or [10]. $\square$

6.3.3 Reducing $J^{\Gamma}_{m,per}$

Suppose that $\mathbf{T} \in \mathcal{T}_{k,per}$ is a lift of $(F, \mathsf{T}) \in J^{\Gamma}_{m,per}$. Let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T})$, and note that by our definition of a lift, we have that $F \subset \mathcal{B}(M)$.

Lemma 6.3.5. Let C denote a Γ-stable (θ, M)-facet in $\mathcal{B}(M)$ that contains F in its closure. Then C^{Γ} is a maximal (θ, M^{Γ})-facet in $\mathcal{B}(M)^{\Gamma}$, and F^{Γ} is an open subset of C^{Γ}.

Proof. It will be enough to show that F^{Γ} is a maximal (θ, G^{Γ})-facet in $\mathcal{B}(M)^{\Gamma}$. Now choose a (θ, G^{Γ})-facet $D \subset \mathcal{B}(M)^{\Gamma}$ so that $F^{\Gamma} \subset \overline{D}$. If $F^{\Gamma} \neq D$, then since $\mathbf{T}$ is contained in the center of $\mathbf{M}$, we have that the image of $T \cap G_F = T \cap G_D$ in G_F/G_F^+ belongs to the parabolic $\mathfrak{f}$-subgroup G_D/G_F^+, contradicting that $(F, \mathsf{T}) \in J^{\Gamma}_{m,per}$. □

In contrast to the case of θ-split (K, k)-tori, not every lift of a pair (F, T) in J^{Γ}_{per} is contained in a θ-perfect (K, k) torus in $\mathbf{G}$. We can see this by looking at an alcove in $\mathcal{B}(H)^{\Gamma}$ in the SL_3 example from 6.2. Thus we want to determine a condition for a lift of (F, T) to lie in $\mathcal{T}_{k,per}$.

First, note that since any two lifts of (F, T) are $(H_F^+)^{\Gamma}$-conjugate, it suffices to check whether a single lift lies in $\mathcal{T}_{k,per}$. With this in mind, we have the following lemma:

Lemma 6.3.6. Let d be the rank of a maximal (θ, K)-split k-torus in $\mathbf{G}$. Let $\mathbf{T}$ be a lift of a pair $(F, \mathsf{T}) \in J^{\Gamma}_{per}$. Then $\mathbf{T}$ lies in $\mathcal{T}_{k,per}$ if and only if there exists a θ-facet F_1 so that

- F_1 is Γ-stable
- F contains F_1 in its closure
- the rank of a maximal θ-split $\mathfrak{f}$-torus in G_{F_1} is d
- the image of $T \cap G_F$ in $G_F/G_{F_1}^+$ is contained in a H_{F_1}-conjugate of a θ-perfect torus in G_{F_1}.

Proof. First suppose there exists a θ-facet F_1 as in the statement of the lemma. Then choose a θ-perfect torus T' in G_{F_1} containing the image of $T \cap G_F$ in $G_F/G_{F_1}^+$. Then a lift $\mathbf{T}'$ of T'

will contain a maximal θ-split k-torus in $\mathbf{G}$, and so $\mathbf{T}'$ is an $\mathbf{H}$-conjugate of a θ-perfect torus in $\mathbf{G}$. After further conjugating, by an element of $(H_F^+)^\Gamma$, we may assume that $\mathbf{T}'$ contains $\mathbf{T}$, and so $\mathbf{T}$ is in $\mathcal{T}_{k,per}$ as desired.

Now suppose that $\mathbf{T} \in \mathcal{T}_{k,per}$ is a lift of (F, T), and choose a θ-perfect torus $\mathbf{T}''$ in $C_{\mathbf{G}}(\mathbf{T})$, and let $\mathbf{T}'$ be its maximal K-split subtorus. Then since $\mathbf{T} \in \mathcal{T}_{k,per}$, we have that $\mathbf{T}''$ is an $\mathbf{H}$-conjugate of a subtorus of a θ-perfect torus, and so $\mathbf{T}' \in \mathcal{T}_{k,per}$ as well. Thus by the previous lemma, there is a pair $(F', \mathsf{T}') \in J_{m,per}^\Gamma$ such that $\mathbf{T}'$ lifts (F', T'). Then F' is an open subset of a $(C_G(\mathbf{T}) \cap H)^\Gamma$-alcove, and so after replacing $\mathbf{T}'$ and (F', T') with a $(C_G(\mathbf{T}) \cap H)(k)$-conjugate, we may assume that F is contained in the closure of the $(M \cap H)^\Gamma$-alcove containing F'. But all maximal (θ, G^Γ)-facets contained in the alcove will be open and thus strongly associated. Thus replacing F' with a strongly associated (θ, G^Γ)-facet F_1 containing F in its closure, we may assume that F lies in the closure of $F' = F_1$. The other three conditions on F_1 are satisfied by construction, and so we are done. □

We let $J_{m,per}^{\Gamma\prime}$ denote the pairs in $J_{m,per}^\Gamma$ which have a lift $\mathbf{T}$ satisfying the conditions of the previous lemma.

6.3.4 A Bijective Correspondence

Upon restricting to $J_{m,per}^{\Gamma\prime}$ and applying our equivalence relation to it, we will finally obtain our bijective correspondence.

Lemma 6.3.7. Suppose that $(F_i, \mathsf{T}_i) \in J_{m,per}^{\Gamma\prime}$ with lifts $\mathbf{T}_i \in \mathcal{T}_{k,per}$. Then if there exists $h \in H^\Gamma$ so that ${}^h\mathbf{T}_1 = \mathbf{T}_2$, then $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$.

Proof. Replacing (F_1, T_1) with $(hF_1, {}^h\mathsf{T}_1)$, we may and do assume that $\mathbf{T} := \mathbf{T}_1 = \mathbf{T}_2$. Now set $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T})$. Then since $\mathbf{T}$ is a lift of (F_i, T_i), we must have that $F_i \subset \mathcal{B}(M)$. Let C_i denote the (θ, M)-facet in $\mathcal{B}(M)$ to which F_i belongs. Then by Lemma 6.6, we have that C_i^Γ is a maximal (θ, M^Γ)-facet in $\mathcal{B}(\mathbf{M}^\Gamma)$, and so in particular, we have that C_i^Γ must lie in an alcove $\tilde{C}_i^\Gamma$ of $\mathcal{B}(\mathbf{M}^\Gamma)^\theta$ for some M-facet $\tilde{C}_i$ in $\mathcal{B}(M)^\theta$. Then there exists an $m \in M^\Gamma \cap H$

so that $m\tilde{C}_1 = \tilde{C}_2$. Replacing (F_1, T_1) by $(mF_1, {}^m\mathsf{T}_1)$, then since F_1^Γ and F_2^Γ are open in C_1^Γ and C_2^Γ, and hence also open in $\tilde{C}_1^\Gamma = \tilde{C}_2^\Gamma$, we have that for any apartment $\mathcal{A}$ in $\mathcal{B}(M^\Gamma)^\theta \subset \mathcal{B}(H)^\Gamma$ containing $\tilde{C}_1^\Gamma$, $\emptyset \neq A(\mathcal{A}, F_1^\Gamma) = A(\mathcal{A}, F_2^\Gamma)$. Then since ${}^m\mathbf{T} = \mathbf{T}$, we see that $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$. □

Corollary 6.3.8. There exists a bijection between $J_{m,per}^{\Gamma'}/\sim$ and the set of H^Γ-conjugacy classes in $\mathcal{T}_{k,per}$.

Proof. It remains to check that if $(F_1, \mathsf{T}_1), (F_2, \mathsf{T}_2) \in J_{m,per}^{\Gamma'}/\sim$ with $(F_1, \mathsf{T}_1) \sim (F_2, \mathsf{T}_2)$, then they have lifts that are H^Γ-conjugate. To do this, fix such (F_1, T_1) and (F_2, T_2). Then there is some element $h \in H^\Gamma$ and an apartment $\mathcal{A}$ in $\mathcal{B}(H)^\Gamma$ so that

- $\emptyset \neq A(\mathcal{A}, F_1^\Gamma) = A(\mathcal{A}, hF_2^\Gamma)$
- $\mathsf{T}_1 \approx {}^h\mathsf{T}_2$ in $\mathsf{G}_{F_1} \approx \mathsf{G}_{hF_2}$.

We may and do assume that $\mathcal{A} \subset \mathcal{A}'(A)^\Gamma$, where $\mathcal{A}'(A)$ is an apartment in $\mathcal{B}(G)$ for some maximal k-split torus $\mathbf{A}$ of $\mathbf{G}$, and we may also assume that h is the identity element.

Let $\mathbf{M}_i$ denote the Levi (k,k)-subgroup of $\mathbf{G}$ corresponding to the G-facet F_i' containing the (θ, G)-facet F_i. Then since $A(\mathcal{A}', (F_1')^\Gamma) = A(\mathcal{A}', (F_2')^\Gamma)$ by [28], we have that $\mathbf{M} := \mathbf{M}_1 = \mathbf{M}_2$. By construction, we have that the image of $M \cap G_{F_i}$ in G_{F_i} is G_{F_i} itself.

Now since $\mathsf{T}_1 \approx \mathsf{T}_2$ in $\mathsf{G}_{F_1} \approx \mathsf{G}_{F_2}$, we can find a θ-stable (K,k)-torus $\mathbf{T}'$ so that the image of $T \cap M_{F_1} \cap M_{F_2}$ in $\mathsf{M}_{F_i} = \mathsf{G}_{F_i}$ is a θ-perfect $\mathfrak{f}$-torus in $C_{\mathsf{G}_{F_i}}(\mathsf{T}_i)$. But we know that there is exactly one lift $\mathbf{T}$ of (F_i, T_i) in $\mathbf{T}'$, meaning that the image of $T \cap M$ in $\mathsf{M}_{F_i} = \mathsf{G}_{F_i}$ is T_i and that the proof is complete. □

6.3.5 θ-perfect (K,k)-tori and Extendable Levi (k,k)-subgroups

Recall that if a Levi (k,k)-subgroup $\mathbf{M}$ is θ-stable, we say that it is extendable if $\mathbf{M} = C_{\mathbf{G}}((\mathbf{T}^+)^k)$ for some k-torus $\mathbf{T}^+$ so that $(\mathbf{T}^+)^k \leq \mathbf{H}$. Fixing a pair $(F, \mathsf{T}) \in J_{m,per}^{\Gamma'}$ and a lift $\mathbf{T} \in \mathcal{T}_{k,per}$ of (F, T), we are able to prove the following lemma:

Lemma 6.3.9. There exists a θ-stable $(H_F^+)^\Gamma$-conjugate $\mathbf{M}'$ of ${}_F\mathbf{M}$ so that $\mathbf{T} \leq \mathbf{M}'$ and so that every extendable Levi (k,k)-subgroup $\mathbf{M}''$ containing $\mathbf{T}$ satisfies $(\mathbf{M}') \leq (\mathbf{M}'')$.

Proof. First let $\mathbf{M} = C_{\mathbf{G}}(\mathbf{T})$. Then from the previous section, we have that F^Γ is a maximal (θ, G^Γ)-facet in $\mathcal{B}(M)^\Gamma$ and hence contained in an $(M \cap H)^\Gamma$-alcove. Choose a θ-perfect k-torus T' in $C_{\mathsf{G}_F}(\mathsf{T})$. Then there is a θ-stable lift $\mathbf{T}'$ of T' containing $\mathbf{T}$, and if C denotes the $\mathfrak{f}$-split component of the center of G_F, then there is a unique lift $\mathbf{C}_F$ of $\mathsf{C}\mathsf{Z}_F$ contained in $\mathbf{T}$ by [1]. We must have that $\mathbf{M}' = C_{\mathbf{G}}(\mathbf{C}_F)$ is θ-stable by uniqueness and the fact that $\mathsf{C}\mathsf{Z}_F$ is θ-stable. Since any two lifts of T' are $(H_F^+)^\Gamma$-conjugate, we must have that $\mathbf{M}'$ is $(H_F^+)^\Gamma$-conjugate to ${}_F\mathbf{M}$, and $\mathbf{M}'$ clearly contains $\mathbf{T}$.

Now suppose that $\mathbf{T}_1^+$ is a k-torus in $\mathbf{H}$ so that $\mathbf{M}'' = C_{\mathbf{G}}((\mathbf{T}_1^+)^k)$ is an extendable θ-stable Levi (k,k)-subgroup which also contains $\mathbf{T}$. Then since $(\mathbf{T}_1^+)^k$ commutes with $\mathbf{T}$, we have that $(\mathbf{T}_1^+)^k \leq \mathbf{M}$. Now choose a maximally k-split maximal (K,k)-torus $\mathbf{T}_1'$ in $\mathbf{M} \cap \mathbf{H}$ that contains $(\mathbf{T}_1^+)^k$. Then since F^Γ is contained in an $(M \cap H)^\Gamma$-alcove, we may replace $\mathbf{T}_1'$ and $\mathbf{M}''$ with a $(\mathbf{M} \cap \mathbf{H})(k)$-conjugate in order to obtain $F^\Gamma \subset \mathcal{B}(T_1') \subset \mathcal{B}(M^\theta)^\Gamma$. Then since F^Γ is a maximal (θ, G^Γ)-facet in $\mathcal{B}(M)^\theta$, we must then have that the image of $((\mathbf{T}_1')^k)(K) \cap G_F$ in G_F is contained in C. After potentially replacing $\mathbf{T}_1'$ by an $(H_F^+)^\Gamma$-conjugate, we then have that $(\mathbf{T}_1^+)^k \leq (\mathbf{T}_1')^k \leq \mathbf{C}_F$. We then have that $\mathbf{M}' \leq \mathbf{M}''$, and so we are done. $\square$

6.4 A Parameterization of Unramified θ-perfect Tori

In this section, we seek to parameterize the unramified θ-perfect tori, the k-tori $\mathbf{T}$ in $\mathbf{G}$ for which $C_{\mathbf{G}}(\mathbf{T})$ contains a $\mathbf{H}$-conjugate of a θ-perfect torus and for which $\mathbf{T}$ is the K-split component of the center of $C_{\mathbf{G}}(\mathbf{T})$.

The unramified θ-perfect tori will arise as a lift of one of the pairs (F, T) from Section 6.3, where F is a θ-facet in $\mathcal{B}(G^\theta)^\Gamma$ and T is a subtorus of an H_F-conjugate of a θ-perfect torus in G_F. However, not all of the θ-perfect (K,k)-tori are unramified θ-perfect tori, and so we need to refine the parameterization.

We first fix an H-alcove $\mathcal{C}$ in $\mathcal{B}(H)$ lying in the apartment of a maximally k-split maximal unramified torus $\mathbf{A}$ of $\mathbf{H}$, which in turn lies inside of a fixed maximally k-split maximal unramified torus $\mathbf{A}'$ of $C_{\mathbf{G}}(\mathbf{A})$. Then $\mathcal{C}^\Gamma$ is a union of θ-facets in $\mathcal{B}(G^\theta)^\Gamma$, and for each such θ-facet F such that a pair (F, T) arises in our parameterization of θ-perfect (K,k)-tori from Section 6.3, we fix a θ-perfect torus A'_F in G_F. We choose these tori A'_F so that if for two θ-facet F and F' there is an $h \in H^\Gamma$ such that F and ${}^hF'$ are strongly associated, then A'_F is identified with $\mathsf{A}'_{F'}$ under the identification of G_F with $\mathsf{G}_{F'}$. For each such θ-perfect torus A'_F, we let A_F denote the maximal θ-split subtorus of A'_F.

By taking lifts of each of the A'_F above we find a family of θ-stable maximal unramified tori $\mathbf{A}'_F$ in $\mathbf{G}$. Note that not all of the $\mathbf{A}'_F$ are θ-perfect tori in $\mathbf{G}$, but we have a criterion for checking whether this is the case from 6.3.3.

6.4.1 An Indexing Set over $\mathfrak{f}$

Given $\mathbf{A}'_F$ as above, we set $\Pi'(\mathbf{G}, F)$ to be the set of all subsets π_F of bases of the roots of $\mathbf{A}'_F$ in $\mathbf{G}$ so that the span of π_F is θ-stable. Set

$$I'_{\theta,F} = \{(\pi, w) | \pi \in \Pi'(G,F), w \in W'_{\theta,c(\pi),F}, \text{ and } \mathrm{Fr}(\Phi_\pi) = w\Phi_\pi),$$

where Φ_π denotes the root system spanned by π and

$$W'_{\theta,c(\pi),F} = N_{H^\circ}(\mathbf{A}'_F)/((C_{H^\circ}((\mathbf{A}'_F)_\pi))^\circ \cap C_{H^\circ}(\mathbf{A}'_F)).$$

Modeling [1] and our parameterization of unramified θ-split tori, if F is a θ-facet contained in the facet F' in $\mathcal{B}(G)$, we let $\Phi(F)$ denote the set of gradients of the affine roots of $\mathbf{A}'$ in $\mathbf{G}$ whose restriction to F' is constant, and we let $\mathbf{A}'(F) = \left(\cap_{\alpha\in\Phi(F)}\ker(\alpha)\right)^\circ$. Recall that ${}_F\mathbf{M} = C_{\mathbf{G}}(\mathbf{A}'(F))$ and that the image of $A'(F) \cap G_F$ in G_F is the group of $\mathfrak{F}$-points of the $\mathfrak{f}$-split component of the center of $\mathsf{G}_F \cong {}_F\mathsf{M}_F$.

For a set of roots of $\pi \in \Pi'(\mathbf{G}, F)$ of the torus $\mathbf{A}'_F$, we define

$$(W'_{\theta,c(\pi)})_F := N_{\mathsf{H}_F}(\mathsf{A}_F)/((C_{\mathsf{H}_F}((\mathsf{A}_F)_\pi))^\circ \cap C_{\mathsf{H}_F}(\mathsf{A}_F)),$$

where $(\mathsf{A}_F)_\pi$ is the image of $(A_F)_\pi \cap G_F$ in G_F. Note that we can and do identify $(W'_{\theta,c(\pi)})_F$ with a subgroup of $W'_{\theta,c(\pi),F}$.

We now define

$$I'(F) := \{(\pi, w) \in I'_{\theta,F} | w \in (W_{\theta,c(\pi)})_F \leq W_{\theta,c(\pi),F}\}.$$

For (θ', w') and (θ, w) in $I'(F)$, we write $(\theta', w') \sim_F (\theta, w)$ if there exists an element $n \in N_{H_F}(\mathbf{A}'_F)$ so that $\Phi_{\pi'} = n\Phi_\pi$ and $\mathrm{Fr}(n)wn^{-1} \in w'((W'_{\theta,c(\pi')})_F \cap W'_{\theta,c(\pi'),F}(\pi'))$, where $W'_{\theta,c(\pi'),F}(\pi')$ denotes the subgroup of $W_{\theta,c(\pi')}$ whose image in the Weyl group of $\mathbf{A}_F$ under the natural projection lies in the parabolic subgroup corresponding to π'. One checks that $\sim_F$ defines an equivalence relation on $I'(F)$.

We will say that $(\theta, w) \in I'(F)$ is F-elliptic provided that for all θ-facets F' in our H-alcove $\mathcal{C}$ so that $F \subseteq \overline{F}'$, for all $(\theta', w') \in I'(F)$ with $(\theta', w') \sim_F (\theta, w)$, and for all $h_{F'} \in H_F^\Gamma$ so that $(\mathbf{A}'_F)_{\theta'} \subseteq {}^{h_{F'}^{-1}}\mathbf{A}'_{F'}$, we have that ${}^{h_{F'}}w'$ does not have a representative in $H_{F'}$. We set $I'_e(F)$ to be the set of pairs in $I'(F)$ which are F-elliptic.

Lemma 6.4.1. Suppose $(\pi, w) \in I'(F)$. Then we can choose $h \in H_F$ so that the image of $n = \mathrm{Fr}(h)^{-1}h \in N_{H_F}(\mathbf{A}'_F)$ in $(W'_{\theta,c(\pi)})_F$ is w.

Proof. Choose $\overline{h} \in \mathsf{H}_F$ so that the image of $\mathrm{Fr}(\overline{h})^{-1}\overline{h})$ in $(W'_{\theta,c(\pi)})_F$ is w, which we can do by applying Lang's theorem to H_F. Note that $\mathsf{T} = {}^{\overline{h}}\mathsf{A}'_F$ is by construction an H_F-conjugate of a maximal θ-perfect $\mathfrak{f}$-torus in G_F. Now let $\mathbf{T}$ be a lift of (F, T). Then there exists an element $x \in H_F$ so that ${}^x\mathbf{A}'_F = \mathbf{T}$. Now let $\overline{x}$ denote the image of x in H_F. Then since $\mathsf{T} = {}^{\overline{x}}\mathsf{A}'_F$, the image of $\mathrm{Fr}(\overline{x}^{-1})\overline{x}$ in $(W'_{\theta,c(\pi)})_F$ is of the form $\mathrm{Fr}(w')^{-1}ww'$ for some w' in $(W'_{\theta,c(\pi)})_F$. Then let $n' \in N_{H_F}(\mathbf{A}'_F)$ be a lift of w', and set $h = xn'$. $\square$

6.4.2 Relevant θ-perfect Tori over $\mathfrak{f}$

Suppose $(F, \mathsf{T}) \in J^{\Gamma}_{per}$, and let $\mathbf{T}$ be a lift of (F, T). Then we will say that T is relevant in G_F provided that $\mathbf{T}$ is the K-split component of the center of $C_{\mathbf{G}}(\mathbf{T})$. Let $\mathfrak{R}'(F)$ denote the set of relevant θ-perfect tori in G_F, and fix $\iota = (\pi, w) \in I'(F)$. Then thanks to Lemma 6.11, we can fix $h \in H_F$ so that the image of $n = \mathrm{Fr}(h)^{-1}h \in N_{H_F}(\mathbf{A}'_F)$ in $(W'_{\theta,c(\pi)})_F$ is w. Let $\overline{h}$ denote the image of h in H_F, and let

$$(\mathbf{A}'_F)_\pi = \left(\bigcap_{\alpha \in \pi} \ker(\alpha)|_{\mathbf{A}'_F} \right)^{\circ} \leq \mathbf{A}'_F.$$

Set $\mathsf{T}_\iota = {}^{\overline{h}}(\mathsf{A}'_F)_\pi$ and $\mathbf{T}_\iota = {}^{h}(\mathbf{A}'_F)$. Then we have that $\mathbf{T}_\iota$ is a lift of (F, T_ι). We now set $\mathsf{L}_\iota = C_{\mathsf{G}_F}(\mathsf{T}_\iota)$ and $\mathbf{L}_\iota = C_G(\mathbf{T}_\iota)$. Then note that $\Phi_\pi = {}^{h^{-1}}\Phi(\mathbf{L}_\iota, {}^{h}\mathbf{A}'_F)$, and note that since $\mathbf{T}_\iota$ is the K-split component of the center of $\mathbf{L}_\iota$, we have that T_ι is relevant.

Lemma 6.4.2. The map that sends $\iota \in I'(F)$ to the $(\mathsf{H}_F)^{\Gamma}$-conjugacy class of T_ι is well-defined.

Proof. We first show that the $(\mathsf{H}_F)^{\Gamma}$-conjugacy class of T_i is independent of the choice of h above. Suppose $h' \in H_F$ so that image of $\mathrm{Fr}(h')^{-1}h' \in N_{H_F}(\mathbf{A}'_F)$ in $(W'_{\theta,c(\pi)})_F$ is also w and let $\overline{h}'$ denote the image of h' in H_F. Let $\mathsf{T}'_\iota = {}^{\overline{h}'}(\mathsf{A}'_F)_\pi$ and $\mathbf{T}'_\iota = {}^{h'}(\mathbf{A}'_F)_\pi$. Then $\mathbf{T}'_\iota$ is a lift of (F, T'_ι), and since $\mathrm{Fr}(h')^{-1}h'$ and $\mathrm{Fr}(h)^{-1}h$ both have image w in $(W_{\theta,c(\pi)})_F$, there exists $s' \in (C_{\mathsf{H}_F}((\mathsf{A}'_F)_\pi))^{\circ} \cap C_{\mathsf{H}_F}(\mathsf{A}'_F)$ so that $\mathrm{Fr}(h')^{-1}h's' = \mathrm{Fr}(h)^{-1}h$. Let $x = h'h^{-1} \in H_F$. Then for all $t \in \mathsf{T}_\iota$ we have

$$\mathrm{Fr}({}^{x}t) = {}^{\mathrm{Fr}(h')\mathrm{Fr}(h^{-1})}\mathrm{Fr}(t) = {}^{h'h^{-1}}({}^{hs'}\mathrm{Fr}(t)) = {}^{x}\mathrm{Fr}(t).$$

Hence we see that $\mathrm{Int}(x)$ and $\mathrm{Int}(\mathrm{Fr}(x))$ both carry T_ι^{Fr} to $T_\iota'^{\mathrm{Fr}}$, and so they carry $\mathbf{T}_\iota$ to $\mathbf{T}'_\iota$. In addition, we have that $\mathrm{Fr}(x)^{-1}x \in (C_{H_F}(\mathbf{T}_\iota))^{\circ}$. Then because we know that $\mathrm{H}^1(\mathrm{Fr}, (C_{\mathsf{H}_F}(\mathsf{T}_\iota)^{\circ}) = 1$, we know that there exists $l \in (C_{H_F}(\mathbf{T}_\iota))^{\circ}$ so that $\mathrm{Fr}(x)^{-1}x = \mathrm{Fr}(l)^{-1}l$ modulo H_F^+. Thus $\overline{y}$, the image of xl^{-1} in H_F, belongs to $\mathsf{H}_F^{\mathrm{Fr}}$, and we have that T_ι and T'_ι

are $\mathsf{H}_F^{\mathrm{Fr}}$-conjugate by $\overline{y}$.

Now if $\mathbf{T}'_\iota$ is H^Γ-conjugate to $\mathbf{T}_\iota$, say by $h_1 \in H^\Gamma$, then $\mathbf{T}'_\iota = {}^{h_1 h}(\mathbf{A}'_F)_\pi$. Then since $\mathrm{Fr}(h_1 h)^{-1} h_1 h = \mathrm{Fr}(h)^{-1} h$, the map is independent of our choice of lift, and so we have a well-defined map from $I'(F)$ to the set of $\mathsf{H}_F^{\mathrm{Fr}}$-conjugacy classes in $\mathcal{R}(F)$. $\square$

Lemma 6.4.3. The map that sends $\iota \in I'(F)$ to the $\mathsf{H}_F^{\mathrm{Fr}}$-conjugacy class of T_ι descends to a bijective map from $I'(F)/\sim_F$ to the set of $\mathsf{H}_F^{\mathrm{Fr}}$-conjugacy classes in $\mathcal{R}(F)$.

Proof. We first show that the map is injective. Suppose $\iota_i = (\pi_i, w_i) \in I'(F)$ and $h_i \in H_F$ so that the image of $\mathrm{Fr}(h_i)^{-1} h_i$ in $(W'_{\theta,c(\pi_i)})_F$ is w_i. Set $\mathbf{T}_i = {}^{h_i}(\mathbf{A}'_F)_{\pi_i}$ and $\mathsf{T}_i = {}^{\overline{h_i}}(\mathsf{A}_F)_{\pi_i}$, where $\overline{h}_i$ is the image of h_i in H_F. Note that $\mathbf{T}_i$ is a lift of (F, T_i). Now suppose that there exists $\overline{h} \in \mathsf{H}_F^{\mathrm{Fr}}$ so that $\mathsf{T}_1 = {}^{\overline{h}}\mathsf{T}_2$. Then we know there exists a lift $h \in H_F^{\mathrm{Fr}}$ of $\overline{h}$ for which $\mathbf{T}_1 = {}^{h}\mathbf{T}_2$. Without loss of generality, we may replace h_2 by hh_2 so that $\mathbf{T}_1 = \mathbf{T}_2$ and $\mathsf{T}_1 = \mathsf{T}_2$. Let $\mathsf{L}_1 = C_{\mathsf{G}_F}(\mathsf{T}_1)$, and let $\mathbf{L}_1 = C_G(\mathbf{T}_1)$. Then there exists $\bar{l} \in (\mathsf{L}_1 \cap \mathsf{H}_F)^\circ$ for which ${}^{\overline{h_2}}\mathsf{A}'_F = {}^{\overline{lh_1}}\mathsf{A}'_F$. Choose $m \in N_H(\mathbf{A}'_F)$ for which $lh_1 = h_2 m$, and note that $m = h_2^{-1} l h_1 \in H_F$. Let $\mathbf{M}_{\pi_i} = C_{\mathbf{G}}((\mathbf{A}'_F)_{\pi_i})$. Then we have

$$\begin{aligned}
\Phi_{\pi_1} &= \Phi(\mathbf{M}_{\pi_1}, \mathbf{A}'_F) \\
&= h_1^{-1}\Phi(\mathbf{L}_1, {}^{h_1}\mathbf{A}'_F) = h_1^{-1} l^{-1} \Phi(\mathbf{L}_1, {}^{lh_1}\mathbf{A}'_F) = h_1^{-1} l^{-1} \Phi(\mathbf{L}_1, {}^{h_2}\mathbf{A}'_F) \\
&= h_1^{-1} l^{-1} h_2 \Phi(\mathbf{M}_{\pi_2}, \mathbf{A}'_F) = m^{-1}\Phi(\mathbf{M}_{\pi_2}, \mathbf{A}'_F) = \Phi(\mathbf{M}_{m^{-1}\pi_2}, \mathbf{A}'_F) \\
&= m^{-1}\Phi_{\pi_2}
\end{aligned}$$

so that $m\Phi_{\pi_1} = \Phi_{\pi_2}$.

Since the image of ${}^{h^{-1}}(\mathrm{Fr}(l)^{-1} l) \in N_{H_F}(\mathbf{A}'_F)$ in $W'_{\theta,c(\pi_1)}$ belongs to the parabolic subgroup $W'_{\theta,c(\pi_1)}(\pi_1)$, we then have that

$$\begin{aligned}
\mathrm{Fr}(m)^{-1} w_2 m &= \mathrm{Fr}(lh_1)^{-1} l h_1 ((C_{H_F}((\mathbf{A}'_F)_\pi))^\circ \cap C_{H_F}(\mathbf{A}'_F)) \\
&= w_1 h_1^{-1}(\mathrm{Fr}(l)^{-1} l) h_1 ((C_{H_F}((\mathbf{A}'_F)_\pi))^\circ \cap C_{H_F}(\mathbf{A}'_F)) \in w_1 W'_{\theta,c(\pi_1)}(\pi_1).
\end{aligned}$$

Since the representatives are all in H_F, we have that $\mathrm{Fr}(m)^{-1}w_2m$ is in $(W'_{\theta,c(\pi)})_F$, and so we conclude that $\iota_1 \sim_F \iota_2$.

We now show that the map is surjective. Suppose $\mathsf{T} \leq \mathsf{G}_F$ belongs to $\mathcal{R}(F)$. Let T' be a maximal θ-stable $\mathfrak{f}$-torus in G_F which contains T and has the largest possible $(\theta, \mathfrak{f})$-split rank among θ-stable tori in G_F that contain T. Then T contains the center of G_F and there exists lifts $\mathbf{T}$ of (F, T) and $\mathbf{T}'$ of (F, T') such that $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T})$ is a θ-perfect Levi (K, k)-subgroup, $\mathbf{T}$ is the K-split component of the center of $\mathbf{L}$, and $\mathbf{T} \leq \mathbf{T}' \leq \mathbf{L}$. Let $\mathbf{B_L} \leq \mathbf{L}$ be a Borel K-subgroup of $\mathbf{L}$. Then since $\mathbf{T}'$ is a lift of (F, T'), there is an $h \in H_F$ so that ${}^h\mathbf{A}'_F = \mathbf{T}'$. Let $\pi = h^{-1}\Delta(\mathbf{L}, \mathbf{B_L}, \mathbf{T}') \in \Pi'(\mathbf{G}, F)$. Let w denote the image of $\mathrm{Fr}(h)^{-1}h$ in $(W'_{\theta,c(\pi)})_F$. Then the pair (π, w) belongs to $I(F)$ and corresponds to T. $\square$

6.4.3 Parameterizing H^Γ-Conjugacy Classes of Unramified θ-perfect tori in G

Note that not all of the tori in $\mathcal{R}(F)$ from the previous subsection lift to a θ-perfect (K, k)-torus. However, we have a condition in Lemma 6.3.6 which allows us to check whether this is the case. Given a Γ-stable θ-facet F_1 which is contained in the closure of F, we can check whether or not G_{F_1} contains a maximal θ-split $\mathfrak{f}$-torus which lifts to a maximal (θ, K)-split k-torus in $\mathbf{G}$ by looking at the θ-bases of the roots of our fixed maximal unramified torus $\mathbf{A}'_{F_1}$ associated to F_1.

In particular, if Δ is a θ-basis for the roots of $\mathbf{A}'_{F_1}$, Φ^- is the set of negative roots of $\mathbf{A}'_{F_1}$ with respect to Δ, and θ^* is the automorphism of the Dynkin diagram of Δ induced by θ, then we can partition Δ into $\Delta^+ = \{\alpha \in \Delta | \theta(\alpha) = \alpha\}$, $\Delta_1^- = \{\beta \in \Delta | \theta(\beta) \in \Phi^-, \theta^*(\beta) = \beta\}$, and $\{\Delta_2^- = \{\gamma \in \Delta | \theta(\beta) \in \Phi^-, \theta^*(\beta) \neq \beta\}$. Then by [17], the reduced root system of the maximal θ-split subtorus $\mathbf{A}_{F_1}$ of $\mathbf{A}'_{F_1}$ has rank $|\Delta_1^-| + \frac{|\Delta_2^-|}{2}$. Thus if this number equals the rank of a maximal (θ, K)-split k-torus of $\mathbf{G}$, then $\mathbf{A}_{F_1}$ must be a maximal (θ, K)-split k-torus, meaning that A_{F_1} satisfies the third bullet point of Lemma 6.3.6.

For the fourth condition of Lemma 6.3.6, note that the torus in $\mathcal{R}(F)$ corresponding to a pair $(\theta, w) \in I'(F)$ satisfies the condition if and only if $(\mathbf{A}'_F)_\theta$ also satisfies the condition, significantly reducing the number of tori we need to check. However, there does not appear to be a unique minimal θ, as we can conjugate $(\mathbf{A}'_F)_\theta$ by elements in $N_{H_F}(\mathbf{A}'_F)$. Furthermore, there does not appear to be a condition on θ significantly simpler than the fourth condition of Lemma 6.3.6.

We let $I(F)_{per}$ denote the set of pairs $(\pi, w) \in I'(F)$ so that the conjugacy class in $\mathcal{R}(F)$ corresponding to the equivalence class of (π, w) under our map from the previous section consists of tori which lift to θ-perfect (K, k)-tori. Now define

$$I_{un,per} := \{(F, \pi, w) : F \subseteq \mathcal{A}(A)^{\mathrm{Fr}} \text{ is a } (\theta, G^{Fr})\text{-facet and } (\pi, w) \in I(F)_{per}\}$$

and let $\mathcal{U}_{per}$ denote the set of H^Γ-conjugacy classes of unramified θ-perfect tori in $\mathbf{G}$. Then by the previous subsections, we can define a function $j : I_{un,per} \to \mathcal{U}_{per}$ as follows. For $(F, \pi, w) \in I_{un,per}$, we let $\mathsf{T} \in \mathcal{R}(F)$ denote a relevant torus associated to (π, w) and let $j(F, \pi, w)$ denote the H^Γ-conjugacy class of any lift of (F, T) which is a θ-perfect (K, k)-torus.

For $(F', \pi', w'), (F, \pi, w) \in I_{un,per}$ we write $(F', \pi', w') \sim (F, \pi, w)$ provided there exists an element $n \in (\tilde{W}(\mathbf{H}, A))^{\mathrm{Fr}}$, where we recall that $\tilde{W}(\mathbf{H}, A)$ is defined to be the affine Weyl group $N_H(\mathbf{A})/(C_H(\mathbf{A}) \cap H_F)$ of A in $\mathbf{H}$, for which $A(\mathcal{A}(A)^{\mathrm{Fr}}, F) = A(\mathcal{A}(A)^{\mathrm{Fr}}, nF)$ and, with the identifications of $\mathsf{G}_{F'} = \mathsf{G}_{nF}$ and $X^*(\mathsf{A}'_{F'}) = X^*(\mathsf{A}'_{nF}) = X^*(\mathbf{A}'_{nF})$ arising in this way, we have that $(\pi', w') \sim_{F'} (n\pi, {}^n w)$ in $I'(F') = I'(nF)$. One checks that this defines an equivalence relation on $I_{un,per}$.

We say that $(F, \pi, w) \in I_{un,per}$ is elliptic provided that $(\pi, w) \in I'_e(F)$, and we set $I^e_{un,per}$ to be the set of elliptic triples (F, π, w) in $I_{un,per}$.

Theorem 6.4.4. The map j induces a bijection from $I^e_{un,per}/\sim$ to $\mathcal{U}$.

Proof. We first show that j is surjective. Let $\mathbf{T}$ be an unramified θ-perfect torus in $\mathbf{G}$. Let $\mathbf{L}$ denote the centralizer of $\mathbf{T}$, and let F be a maximal $(\theta, G^{\mathrm{Fr}})$-facet in $\mathcal{B}(\mathbf{L}^\theta)^{\mathrm{Fr}} \subseteq \mathcal{B}(G^\theta)^{\mathrm{Fr}}$. Choose a maximally (θ, K)-split maximal unramified torus $\mathbf{T}'$ of $\mathbf{L}$ that contains $\mathbf{T}$ and so that $F \subset \mathcal{B}(\mathbf{T}')$. Also fix a Borel K-subgroup $\mathbf{B_L}$ of $\mathbf{L}$ that contains $\mathbf{T}'$. Choose $h \in H^\Gamma$ so that $hF \subset \mathcal{A}(A)^{\mathrm{Fr}}$. Then after replacing $\mathbf{T}$ with ${}^h\mathbf{T}$, we may assume that $F \subset \mathcal{A}(A)^{\mathrm{Fr}}$.

Let T denote the θ-split $\mathfrak{f}$-torus in G_F whose group of $\mathfrak{f}$-rational points coincides with the image of $T \cap G_F$ in G_F. Then there exists an $h \in H_F$ so that $\mathbf{T}' = {}^h\mathbf{A}'_F$. Then let $\pi = h^{-1}\Delta(\mathbf{L}, \mathbf{B}_\mathbf{L}, \mathbf{T}') \in \Pi'(\mathbf{G}, F)$, and let w denote the image of $\mathrm{Fr}(h)^{-1}h$ in $(W'_{\theta,c(\pi)})_F$. Then note that $\mathbf{T}$ belongs to $j(F, \pi, w)$.

To conclude our proof of surjectivity, we need to show that the triple (F, π, w) we have constructed is elliptic. If this is not the case, then there exists $(\pi', w') \in I'(F)$ with $(\pi, w) \sim_F (\pi', w')$, an element $h_{F'} \in H_F^\Gamma$, and a $(\theta, G^{\mathrm{Fr}})$-facet F' with $F \subset \overline{F}'$ so that ${}^{h_{F'}^{-1}}w'$ has a representative in $H_{F'}$ and $(\mathbf{A}'_F)_{\pi'} \subseteq {}^{h_{F'}}\mathbf{A}'_{F'}$. Then there exists $h \in {}^{h_{F'}}H_{F'} \subset H_F$ so that $\mathrm{Fr}(h)^{-1}h$ lies in $N_{{}^{h_{F'}}H_{F'}}(\mathbf{A}'_F)$ and has image w' in $(W'_{\theta,c(\pi)})_F$. Note that $hh_{F'}F'$ is a facet in $\mathcal{B}({}^{hh_{F'}}\mathbf{A}'_{F'})$, and since $(\pi, w) \sim_F (\pi', w')$, from Lemma 6.13 we have that ${}^{hh_{F'}}(\mathbf{A}'_{F'})_{h_{F'}^{-1}\pi'} = {}^h(\mathbf{A}'_F)_{\pi'} = {}^x\mathbf{T}$ for some $x \in H(k)$. Note that then ${}^{x^{-1}hh_{F'}}\mathbf{A}'_{F'} \leq \mathbf{L}$, and so $x^{-1}hh_{F'}F' \subset \mathcal{B}({}^{x^{-1}hh_{F'}}\mathbf{A}'_{F'})^{\mathrm{Fr}} \subseteq \mathcal{B}(\mathbf{L})^{\mathrm{Fr}}$, contradicting the maximality of F.

It remains to show that if (F_i, π_i, w_i) for $i \in \{1, 2\}$ are two elements of $I^e_{un,per}$ with $j(F_1, \pi_1, w_1) = j(F_2, \pi_2, w_2)$, then $(F_1, \pi_1, w_1) \sim (F_2, \pi_2, w_2)$. Choose $\mathsf{T}_i \in \mathcal{R}(F_i)$ corresponding to $(\pi, w_i) \in I'_e(F_i)$, and let $\mathbf{T}_i$ be a lift of (F_i, T_i). Note that $(F_i, \mathsf{T}_i) \in J^{\Gamma'}_{m,per}$. Then since $j(F_1, \pi_1, w_1) = j(F_2, \pi_2, w_2)$, we know that $\mathbf{T}_1$ is H^Γ-conjugate to $\mathbf{T}_2$. Thus we know that there exists $h \in H^\Gamma$ and an apartment $\mathcal{A}'$ in $\mathcal{B}(G^\theta)^{\mathrm{Fr}}$ so that $\emptyset \neq A(\mathcal{A}', F_1) = A(\mathcal{A}', hF_2)$ and $\mathsf{T}_1 = {}^h\mathsf{T}_2$ in $\mathsf{G}_{F_1} = \mathsf{G}_{hF_2}$. After conjugating by an element of $H^\Gamma_{F_1}$, we may assume that $\mathcal{A}' = \mathcal{A}(A)$. Then by the affine Bruhat decomposition for $\mathbf{H}$, we may choose $n \in N_{H(k)}(\mathbf{A})$ so that $n^{-1}h \in H^{\mathrm{Fr}}_{F_2}$. Then there exists $x \in \mathsf{H}^{\mathrm{Fr}}_{F_2}$ such that after replacing T_2 with ${}^x\mathsf{T}_2$ we may assume that $A(\mathcal{A}(A)^{\mathrm{Fr}}, F_1) = A(\mathcal{A}(A)^{\mathrm{Fr}}, nF_2)$ and $\mathsf{T}_1 = {}^n\mathsf{T}_2$ in $\mathsf{G}_{F_1} = \mathsf{G}_{nF_2}$. Identifying n with its image in $\tilde{W}(\mathbf{H}, A)$, we then have that $(\pi_1, w_1) \sim_F (n\pi_2, {}^nw_2)$ in $I'(F_1) = I'(nF_2)$. □

6.5 Unramified θ-split Tori versus Unramified θ-perfect Tori: An Example

At first glance, the concept of an unramified θ-split torus is quite similar to that of an unramified θ-perfect torus, and one may notice that our parameterizations and proofs for these two classes of tori are nearly identical. Here, we seek to briefly demonstrate the difference between the two parameterizations through an example, and we will attempt to motivate why each of these concepts is worthwhile.

We return to the SL_3 example from Section 6.2 earlier in the chapter. In particular, we let θ be the involution given by conjugation by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $\mathbf{H}$ is the $(2,1)$-Levi subgroup of SL_3, and the diagonal torus in SL_3 is a maximal k-split torus in $\mathbf{H}$. Then the torus $\mathbf{S}$ consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}$$

is a maximal θ-split k-torus, and a θ-perfect torus $\mathbf{T}$ containing $\mathbf{S}$ consists of matrices of the form

$$\begin{pmatrix} c & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}.$$

Then one computes that $N_{\mathbf{H}}(\mathbf{T})/C_{\mathbf{H}}(\mathbf{T}) = N_{\mathbf{H}}(\mathbf{S})/C_{\mathbf{H}}(\mathbf{S})$ is a subgroup of order two whose non-trivial element w has representative

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(Note that it is not generally the case that the set of Weyl group elements having representatives in $\mathbf{H}$ equals the little Weyl group, as an analogous example in SL_4 shows.)

The roots of $\mathbf{T}$ act on an element

$$t = \begin{pmatrix} c & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix}$$

as follows: The root α sends t to $c(a+b)^{-1}$, the root β sends t to $(a+b)(a-b)^{-1}$. We then define $-\alpha, -\beta$, and $\pm(\alpha+\beta)$ in the obvious way. We check that θ acts on the roots by sending β to $-\beta$ and by permuting α and $\alpha+\beta$. One then checks that $\Delta = \{\alpha+\beta, -\alpha\}$ gives a θ-basis for the roots of $\mathbf{T}$ and that $-\Delta$ is the only other θ-basis. The only θ-admissible subsets of Δ are Δ itself and the empty set. The only other subsets of simple roots which span a θ-stable root space are the sets $\{\beta\}$ and $\{-\beta\}$. Note that for each of these subsets π, one can check that the θ-fixed points of the centralizer of $(\mathbf{T})_\pi$ is connected so that there is no need to distinguish between the little Weyl group (resp., the Weyl group elements having representatives in $\mathbf{H}$) and the variants of the Weyl group showing up in our parameterizations.

With all of this in mind, we can give the data corresponding to the conjugacy classes of unramified θ-split tori and unramified θ-perfect tori. It will suffice to work in a fixed alcove of the apartment of the diagonal torus of SL_2. Then if we let β' be the root of the diagonal torus which sends the element

$$t' = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}.$$

to yz^{-1}. Then all the maximal unramified θ-split tori are lifts of tori in the facet intersecting the hyperplane $H_{\tilde{\beta}}$, where $\tilde{\beta}$ denotes the gradient of β'. In particular, we have the torus $\mathbf{S}$, which corresponds to the equivalence class of the triple $(H_{\tilde{\beta}}, \emptyset, \mathrm{Id})$ and the torus $\mathbf{S}_\epsilon$, where we fix $\epsilon \in \mathfrak{o}^\times \setminus (\mathfrak{o}^\times)^2$, which has representatives of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & \epsilon b & a \end{pmatrix}$$

and corresponds to the equivalence class of the triple $(H_{\tilde{\beta}}, \emptyset, w)$. Since the maximal unramified θ-split tori have rank 1, the only other possible unramified θ-split torus is the trivial torus, which lies in the interior F of our fixed alcove in the apartment of the diagonal torus and corresponds to the triple $(F, \Delta', \mathrm{Id})$, where Δ' is some set of simple roots for the roots of the diagonal torus.

Similarly, we see that the maximal unramified θ-perfect tori are lifts of tori in the facet $H_{\tilde{\beta}}$. In particular, we have the torus $\mathbf{T}$, which corresponds to the equivalence class of the triple $(H_{\tilde{\beta}}, \emptyset, \mathrm{Id})$, and the torus $\mathbf{T}_\epsilon$, which has representatives of the form

$$\begin{pmatrix} c & 0 & 0 \\ 0 & a & b \\ 0 & \epsilon b & a \end{pmatrix}$$

and corresponds to the equivalence class of the triple $(H_{\tilde{\beta}}, \emptyset, w)$. In this case, the maximal unramified θ-perfect tori have rank 2, and so we have other non-trivial unramified θ-perfect tori. However, as it turns out the only such torus is the subtorus of the diagonal torus having representatives of the form

$$\begin{pmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},$$

which corresponds to the triple (F, β', Id). Finally, the trivial torus again corresponds to the triple $(F, \Delta', \mathrm{Id})$.

In the general setting, even though unramified θ-split tori are not unramified θ-perfect tori in general, we can associate to each unramified θ-split torus $\mathbf{S}$ an unramified θ-perfect torus $\mathbf{T}$ by taking the identity component of the center of $C_{\mathbf{G}}(\mathbf{S})$. (In other words, by taking the identity component of the unramified θ-split twisted Levi subgroup corresponding to $\mathbf{S}$.) If $\mathbf{S}$ corresponds to a triple of the form (F, π, w), then the corresponding unramified θ-perfect torus will correspond to a torus of the form (F, π, w') where w' has image w upon restricting from the unramified θ-perfect torus $\mathbf{A}'_F$ associated to F to its θ-split component $\mathbf{A}_F$. Thus we could have first parameterized θ-perfect tori first and then identified the unramified θ-split twisted Levi subgroups and their corresponding unramified θ-split tori by restricting to triples (F, π, w') where π is θ-admissible. However, this approach is more unwieldy in practice, as the various lifts of elements in $(W_{\theta,c(\pi)})_F$ to $(W'_{\theta,c(\pi)})_F$ will typically enlarge the size of the various equivalence classes of triples and hence the data we need to sort through. Furthermore, as we discuss below, unramified θ-split tori are more likely to play a prominent role in the theory of p-adic symmetric spaces outside of the parameterization of general θ-split k-tori.

Finally, we point out that both of these classes of unramified tori should play an important role in the theory of p-adic symmetric spaces. In [21], the authors study a symmetric space analogue of supercuspidal representations called relatively cuspidal representations, and they show that each irreducible H^Γ-distinguished representation of G^Γ can be embedded in some $i_{P^\Gamma}^{G^\Gamma}(\sigma)$, where P is a θ-split parabolic k-subgroup of G and σ is a relatively cuspidal representation of the θ-split Levi k-subgroup corresponding to P. This suggests that the θ-split twisted Levi k-subgroups are likely to be the proper analogue of twisted Levi k-

subgroups for proving symmetric space analogues of results where twisted Levi k-subgroups of G play a prominent role (for example, a potential analogue of Yu's construction of supercuspidal representations of G in [38, 13] for relatively cuspidal representations). Thus one would like to understand the $\mathbf{H}(k)$-conjugacy classes of all θ-split twisted Levi k-subgroups, and understanding the unramified θ-split twisted Levi subgroups, which correspond to the unramified θ-split tori, is an important first step in that parameterization.

On the other hand, our motivation for parameterizing unramified θ-perfect tori comes entirely from its role in showing how the H-conjugacy class of a maximal θ-split k-torus breaks up into $\mathbf{H}(k)$-conjugacy classes. In order to do this, one will need to construct from a representative $\mathbf{S}$ in our H-conjugacy class an unramified twisted Levi k-subgroup $\mathbf{L}$ containing $\mathbf{S}$ so that the centralizer of $\mathbf{S}$ in $\mathbf{L}$ is a K-minisotropic maximal k-torus $\mathbf{T}$. As it turns out, the unramified twisted Levi k-subgroups arising in this way are all unramified θ-perfect twisted Levi subgroups, and so expanding our parameterization of unramified θ-split twisted Levi subgroups to unramified θ-perfect twisted Levi subgroups is essential for parameterizing maximal θ-split k-tori.

For example, in the SL_3 example, to show how the H-conjugacy class of the maximal θ-split k-torus $\mathbf{S}_1$ having representatives of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & \varpi b & a \end{pmatrix}$$

breaks into $\mathbf{H}(k)$-conjugacy classes, we need to work within the $(1,2)$-Levi k-subgroup which is an unramified θ-perfect twisted Levi k-subgroup corresponding to the unramified θ-perfect torus having representatives of the form

$$\begin{pmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},$$

which is not an unramified θ-split torus.

CHAPTER VII

Maximal Unramified Tori in Symplectic Groups

If $\mathbf{G}$ is a reductive algebraic group over a p-adic field F with residue field $\mathbb{F}_q$, an unramified torus is the group of F-rational points of a F-torus in $\mathbf{G}$ that splits over an unramified extension of F. In [10], the author gives a parameterization of $\mathbf{G}(F)$-conjugacy classes of maximal unramified tori using Bruhat-Tits theory. In particular, modulo an equivalence, the conjugacy classes are parameterized by pairs (F, T) where F is a facet in the F-points of the Bruhat-Tits building of $\mathbf{G}$ and T is an elliptic $\mathbb{F}_q$-torus in the reductive quotient of $\mathbf{G}$ at the facet F. Given such a pair, T can be lifted to a maximal unramified torus in $\mathbf{G}$, and this gives a representative for the conjugacy class associated to the pair. The elliptic $\mathbb{F}_q$-tori in the reductive quotient are in turn parameterized by the elliptic Frobenius conjugacy classes in the Weyl group of the reductive quotient.

On the other hand, for symplectic, orthogonal, and unramified unitary groups, Waldspurger in [37] gives a parameterization of the maximal unramified tori in terms of triples of partitions. For each part of one of these partitions, he defines an F-algebra whose structure is determined by the partition, and he also constructs an F-endomorphism of the algebra. Taking the sum of these F-algebras, he obtains a symplectic F-vector space, and the sum of the F-endomorphisms determines a regular semisimple element for a torus in the associated conjugacy class.

The goal of this chapter is to provide a comparison of the two parameterizations in the case of the symplectic group, similar to the work in [26] and [4] for nilpotent orbits. In particular, given a triple of partitions (μ_0, μ', μ'') in the Waldspurger parameterization, we will associate a facet in the Bruhat-Tits building and an elliptic conjugacy class in the Weyl group of the reductive quotient which determine the same conjugacy class of tori as (μ_0, μ', μ''). We will also construct an inverse map, showing that the two indexing sets are in bijective correspondence.

The facet corrresponding to a triple (μ_0, μ', μ'') will correspond to a subdiagram of the extended Dynkin diagram of type C_n. This subdiagram will be a union of two subdiagrams of type C_k, which will be determined by the partitions μ' and μ'', and subdiagrams of type A_j, which will be determined by the partition μ_0.

Each element in the Weyl group of the reductive quotient attached to this facet is a product of signed k_i-cycles for natural numbers $k_i < n$. We say that a signed k_i-cycle σ is even if $\sigma^{k_i} = \mathrm{Id}$ and odd if $\sigma^{k_i} = -\mathrm{Id}$. The conjugacy class of an element is determined by its cycle-type, and the parts of the partitions μ_0, μ', and μ'' give us the cycle-type of the associated element w. In particular, each part x_i of μ_0 give us an even x_j-cycle while each part y_j of μ' or μ'' give us an odd y_j-cycle.

The chapter is organized as follows. In Section 2, we introduce some of the general notation needed for our result, and we recall some of the structure of the symplectic group Sp_{2n}. In Section 3, we discuss the Bruhat-Tits building of Sp_{2n}, and we discuss the DeBacker parameterization in more detail. In Section 4, we discuss Waldspurger's parameterization for Sp_{2n}. In Section 5, we present and prove the main result, discussing each partition in Waldspurger's parameterization in a separate subsection before putting everything together. Finally, in Section 6 we discuss an inverse to our construction, showing how one can move from a pair (F, w) to the associated triple of partitions in the Waldspurger construction.

7.1 General Notation and the Group $\mathbf{Sp}_{2n}$

Let F be a finite extension of $\mathbb{Q}_p$. Let $\mathfrak{o}$ be the ring of integers of F and $\mathfrak{p}$ the maximal ideal in $\mathfrak{o}$. Let ϖ be a uniformizer of F. Let q be the order of the residue field of F, which we denote by $\mathbb{F}_q$. We let Fr denote a topological generator of the absolute Galois group of $\mathbb{F}_q$. We can and do identify Fr with a topological generator of $\mathrm{Gal}(F_{\mathrm{un}}/F)$, where $F_{\mathrm{un}} \subseteq \overline{F}$ is the maximal unramified extension of F.

If $\mathbf{C}$ is an algebraic group defined over F, we will by abuse of notation also use $\mathbf{C}$ (bold) to denote the $\overline{F}$-points, and we will use C (not bold) to denote the group of F-points.

Let V be a vector space over F of even dimension $2n$ for $p \geq 6n+1$,[1] and fix an ordered basis $B = \{e_1, \ldots, e_n\}$ of V. Let q_V be the non-degenerate anti-symmetric bilinear form of V whose matrix with respect to B is the anti-diagonal matrix

$$\begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$$

Let $\mathbf{G}$ be the symplectic group Sp_{2n} preserving q_V. Then $\mathbf{G}$ is a connected reductive group defined over F. We will use $\mathfrak{g}$ to denote the vector space of F-rational points of the Lie algebra of $\mathbf{G}$, which we may identify with a subalgebra of the F-endomorphisms of V.

If $\mathbf{S}$ denotes the diagonal torus in $\mathbf{G}$, which of course is an F-split maximal torus in $\mathbf{G}$, we will let $\mathrm{diag}(t_1, \ldots, t_n)$ denote the matrix in $\mathbf{S}$ whose (i,i) entry is t_i for $i \leq n$ and t_{2n+1-i}^{-1} for $i > n$. Let $W = N_{\mathbf{G}}(\mathbf{S})/\mathbf{S}$ be the Weyl group of $\mathbf{S}$ in $\mathbf{G}$, and we will let Φ denote the set of roots of $\mathbf{S}$ in $\mathbf{G}$. We fix the simple system $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}, \beta\}$, where α_j is the root that

[1] This is Waldspurger's assumption on p in [37]. In our proofs, we require $q > 2n$ when choosing the generators a_i in our sections 5.1 - 5.3. We also want $p \neq 2$, else regular semisimple elements may not exist in the residue field.

takes $\mathrm{diag}(t_1, \dots, t_n)$ to $t_j t_{j+1}^{-1}$ for $1 \le j \le n-1$ and β is the root which takes $\mathrm{diag}(t_1, \dots, t_n)$ to t_n^2. Then $e := 2\alpha_1 + \cdots + 2\alpha_{n-1} + \beta$ is the highest root with respect to this simple system, and it takes $\mathrm{diag}(t_1, \dots, t_n)$ to t_1^2.

We conclude this section by recalling the root space decomposition of $\mathfrak{g}$. We let $E_{i,j}$ denote the elementary matrix having 1 in the (i, j) entry and 0 elsewhere. Then we have

- For $1 \le i \ne j \le n$, the matrix $E_{i,j} - E_{2n+1-j,2n+1-i}$ spans the root space of the root sending $\mathrm{diag}(t_1, \dots, t_n)$ to $t_i t_j^{-1}$.
- For $1 \le i < j \le n$, $E_{i,2n+1-j} + E_{j,2n+1-i}$ spans the root space of the root sending $\mathrm{diag}(t_1, \dots, t_n)$ to $t_i t_j$, while $E_{2n+1-j,i} + E_{2n+1-i,j}$ spans the root space of the root sending $\mathrm{diag}(t_1, \dots, t_n)$ to $(t_i t_j)^{-1}$.
- For $1 \le i \le n$, $E_{i,2n+1-i}$ spans the root space of the root sending $\mathrm{diag}(t_1, \dots, t_n)$ to t_i^2, while $E_{2n+1-i,i}$ spans the root space of the root sending $\mathrm{diag}(t_1, \dots, t_n)$ to t_i^{-2}.

Notice that if $X \in \mathfrak{g}$ has the block form $\begin{pmatrix} X^{1,1} & X^{1,2} \\ X^{2,1} & X^{2,2} \end{pmatrix}$, where each block is an $n \times n$ matrix, then the root spaces of the roots in the subsystem spanned by $\alpha_1, \dots, \alpha_{n-1}$ are contained in $X^{1,1}$ and $X^{2,2}$, while every root space lying in $X^{1,2}$ or $X^{2,1}$ must be associated to a root so that the coordinate of β with respect to the simple system Δ is non-zero.

7.2 The DeBacker Parameterization for G

Let $\mathcal{B}(G) = \mathcal{B}(\mathbf{G}, F)$ be the Bruhat-Tits building of G, which we identify with the Fr-fixed points of $\mathcal{B}(\mathbf{G}, F_{\mathrm{un}})$. We let $\mathcal{A} = \mathcal{A}(S)$ denote the apartment of the diagonal torus S in $\mathcal{B}(G)$. Within the apartment, we fix a fundamental alcove $\mathcal{C}$ whose walls are determined by the hyperplanes of $n+1$ affine roots so that the set of their gradients is precisely $\Delta \cup \{e\}$. By a slight abuse of notation, we denote these hyperplanes by $H_{\alpha_1}, \dots, H_{\alpha_{n-1}}, H_\beta, H_e$.

The facets in our fundamental alcove $\mathcal{C}$ can be identified with proper subdiagrams of the extended Dynkin diagram of $\mathbf{G}$. In particular, given such a subdiagram Γ, the corresponding facet is the one vanishing at the hyperplanes H_γ, where $\gamma \in \{e, \alpha_1, \ldots, \alpha_{n-1}, \beta\}$ is a root so that the corresponding vertex in the extended Dynkin diagram occurs in Γ. We adopt the following notation for the facets in the fundamental alcove:

Definition 7.2.1. If $a, b \geq 0$ and $x_1, \ldots, x_t > 0$ are integers so that $a + b + \sum_{i=1}^{t} x_i = n$, the facet $\langle a | x_1, \ldots, x_t | b \rangle$ is defined as follows. Let

$$H_e^a = \begin{cases} \mathcal{A} & a = 0 \\ H_e & a \neq 0 \end{cases},$$

and define H_β^b analogously. Then $\langle a | x_1, \ldots, x_t | b \rangle$ is the facet in $\mathcal{C}$ lying on

$$\begin{aligned} H_e^a \cap H_{\alpha_1} \cap \cdots \cap H_{\alpha_{a-1}} \cap \\ H_{\alpha_{a+1}} \cap \cdots \cap H_{\alpha_{a+x_1-1}} \cap \\ \vdots \\ H_{\alpha_{a+x_1+\cdots+x_{t-1}+1}} \cap \cdots \cap H_{\alpha_{a+x_1+\cdots+x_t-1}} \cap \\ H_{\alpha_{a+x_1+\cdots+x_t+1}} \cap \cdots \cap H_{\alpha_{n-1}} \cap H_\beta^b, \end{aligned}$$

where by convention we ignore $H_{\alpha_{a+x_1+\cdots+x_{i-1}+1}} \cap H_{\alpha_{a+x_1+\cdots+x_i-1}}$ if $x_i = 1$.

Note that the special vertex in $\mathcal{C}$ lying on the hyperplanes $H_{\alpha_1}, \ldots, H_{\alpha_n-1}, H_\beta$ is the facet $\langle 0 |\, | n \rangle$, while the other special vertex in $\mathcal{C}$, which lies on H_e instead of H_β, is the facet $\langle n |\, | 0 \rangle$

Example: Let $\mathbf{G} = \mathrm{Sp}_4$. In Figure 7.1 below, we label the facets in the alcove $\mathcal{C}$ determined by the simple roots $\Delta = \{\alpha, \beta\}$. There are three vertices and three edges in addition to the interior of the alcove.

If $x \in \mathcal{B}(G)$, we let G_x denote the parahoric subgroup of G attached to x, and we let G_x^+ denote the prounipotent radical of G_x. Note that both G_x and G_x^+ depend only on the facet F to which x belongs, so we may write G_F and G_F^+. For a facet F in $\mathcal{B}(G)$, the quotient

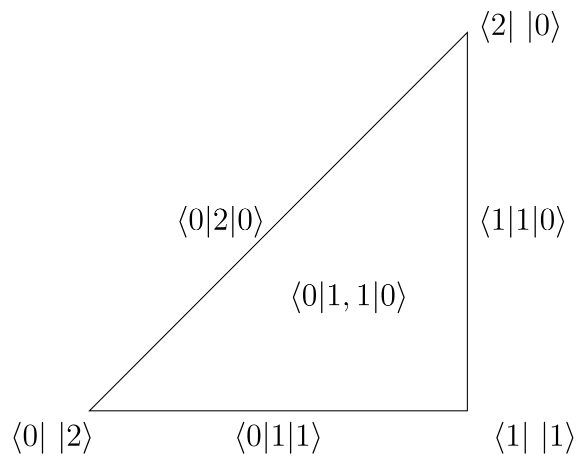

Figure 7.1: A labeling of the facets for Sp_4

$\mathsf{G}_\mathsf{F} := G_\mathsf{F}/G_\mathsf{F}^+$ is the group of $\mathbb{F}_q$-rational points of a reductive $\mathbb{F}_q$-group. The root system of this reductive group is the subdiagram of the extended Dynkin diagram associated to the facet. So in particular, the reductive quotient at the facet $\langle a|x_1,\ldots,x_t|b\rangle$ has root system of type

$$C_a \times A_{x_1-1} \times \cdots \times A_{x_t-1} \times C_b,$$

where again by convention we ignore terms with subscript 0.

We now discuss the parameterization in [10] for $\mathbf{G}$. Let $\mathcal{C}^S$ denote the set of G-conjugacy classes of maximal unramified tori. We will relate $\mathcal{C}^S$ to the set I^m of pairs (F,T), where F is a facet in $\mathcal{B}(G)$ and T is a $\mathbb{F}_q$-minisotropic maximal torus in G_F. Given such a pair, DeBacker shows that T can be lifted to a maximal unramified F-torus T, and he shows that a representative of each G-conjugacy class in $\mathcal{C}^S$ arises in this way. To attain a bijection, one needs to define an equivalence relation on I^m.

Given a facet F in an apartment $\mathcal{A}'$ of $\mathcal{B}(G)$, let $A(\mathcal{A}',\mathsf{F})$ be the affine subspace of $\mathcal{A}'$ spanned by F. Then if $A(\mathcal{A}',\mathsf{F}) = A(\mathcal{A}',\mathsf{F}')$ for two facets $\mathsf{F},\mathsf{F}' \subseteq \mathcal{A}'$, we can identify the reductive quotients G_F and $\mathsf{G}_{\mathsf{F}'}$ (see [9, Lemma 3.5.1]). This identification then allows us to define an equivalence relation by saying $(\mathsf{F},\mathsf{T}) \sim (\mathsf{F}',\mathsf{T}')$ provided there is an apartment $\mathcal{A}'$ in $\mathcal{B}(G)$ and an element $g \in G$ so that

- $\emptyset \neq A(\mathcal{A}', \mathsf{F}) = A(\mathcal{A}', g\mathsf{F}')$ and

- T is identified with ${}^g\mathsf{T}'$ by the identification of G_F with $\mathsf{G}_{g\mathsf{F}'}$.

Then this equivalence relation gives us a bijection

$$I^m/\sim \;\; \to \;\; \mathcal{C}^S.$$

Modeling upcoming work in [1], we can refine this parameterization. First, note that since $\mathcal{C}$ is a fundamental alcove, every facet F in $\mathcal{B}(G)$ is conjugate to at least one facet in $\mathcal{C}$. Thus we may restrict to pairs (F, T) with $\mathsf{F} \subseteq \mathcal{C}$. Additionally, by [10, Lemma 4.2.1] or [8], the maximal $\mathbb{F}_q$-tori in the reductive quotient G_F are parameterized by the conjugacy classes of the Weyl group W_F associated to the image of $S \cap G_\mathsf{F}$ in G_F, which we can and do the identify with a subgroup of the Weyl group W of S in G. Futhermore, the $\mathbb{F}_q$-minisotropic tori in the reductive quotient correspond to the elliptic conjugacy classes. Thus we can refine our parameterization to look at equivalence classes of pairs (F, w), where F is a facet in $\mathcal{C}$ and w is an elliptic element in the Weyl group W_F.

The elements in the Weyl group of G can be identified with signed permutations of $\{1, \ldots, n\}$. By ignoring the sign changes, each element τ of W determines a permutation τ' of $\{1, \ldots, n\}$, and this permutation can be written as a product of disjoint cycles. If j is in a cycle of length k in the cycle decomposition of τ', then $\tau^k(j) = \pm j$. If $\tau^k(j) = j$, then we say that the cycle is even, and if $\tau^k(j) = -j$, then we say that the cycle is odd. Mimicking the notation in [7], we write C_k for an odd cycle of length k and A_{k-1} for an even cycle of length k. In this notation, using Carter's classification of conjugacy classes in the Weyl group found in [7], given a facet F in $\mathcal{C}$ of type $\langle a | x_1, \ldots, x_t | b \rangle$, the elliptic elements in W_F are of the form

$$(C_{a_1} \times \cdots \times C_{a_k}) \times A_{x_1 - 1} \times \cdots \times A_{x_t - 1} \times (C_{b_1} \times \cdots \times C_{b_j}),$$

where $\sum_{i=1}^{k} a_i = a$ and $\sum_{i=1}^{j} b_i = b$.

For example, in Sp_{18}, the facet $\langle 2|2,1,3|1\rangle$ is the facet vanishing on the hyperplanes $H_e, H_{\alpha_1}, H_{\alpha_3}, H_{\alpha_6}$, H_{α_7}, and H_β. It has root system of type $C_2 \times A_1 \times A_2 \times C_1$, and the elliptic Weyl group elements are of type $C_2 \times A_1 \times A_2 \times C_1$ and $C_1 \times C_1 \times A_1 \times A_2 \times C_1$ in Carter's notation.

Example: The conjugacy classes of maximal unramified tori in Sp_4 are parameterized as in Figure 7.2 below. In this example, none of the facets in the alcove $\mathcal{C}$ determined by the simple roots $\Delta = \{\alpha, \beta\}$ are equivalent, and there are nine pairs (F, w) up to equivalence.

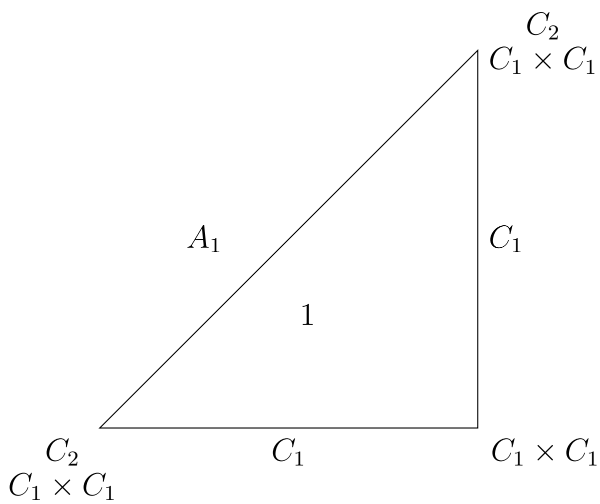

Figure 7.2: A labeling of the pairs (F, w) for Sp_4

7.3 The Waldspurger Parameterization for G

We will first discuss Waldspurger's parameterization of conjugacy classes of regular semisimple elements in $\mathfrak{g}$, which will then allow us to discuss his parameterization of the conjugacy classes of maximal unramified tori.

7.3.1 Regular Semisimple Elements in G

We let $\mathfrak{g}_{reg}$ denote the set of regular semisimple elements of $\mathfrak{g}$. We are going to recall the description of the G-conjugacy classes in $\mathfrak{g}_{reg}$ given in [37]. To start, we will consider the following objects:

1. A finite set I
2. For all $i \in I$, a finite extension $F_i^\#$ of F and a $F_i^\#$-algebra F_i which is 2-dimensional over $F_i^\#$.
3. For all $i \in I$, elements a_i and c_i in $F_i^\times$.

For all $i \in I$, we let τ_i be the unique non-trivial automorphism of F_i over $F_i^\#$. We assume that our choices above satisfy

(a) For all i, a_i generates F_i over F.

(b) For all $i, j \in I$ with $i \neq j$, there does not exist an F-linear isomorphism from F_i to F_j which maps a_i to a_j

(c) For all $i \in I$, $\tau_i(a_i) = -a_i$ and $\tau_i(c_i) = -c_i$

(d) $2n = \sum_{i \in I}[F_i : F]$

Thus the choice of a_i determines F_i and $F_i^\#$. Write $W = \bigoplus_{i \in I} F_i$, and define a symplectic form q_W on W by

$$q_W(\sum_{i \in I} w_i, \sum_{i \in I} w_i') = \sum_{i \in I}[F_i : F]^{-1}\mathrm{trace}_{F_i/F}(\tau_i(w_i)w_i'c_i).$$

We let X_W be the element of $\mathrm{End}_F(W)$ defined by $X_W(\sum_{i \in I} w_i) = \sum_{i \in I} a_i w_i$. Then if we fix an isomorphism from (W, q_W) onto (V, q_V), the element X_W identifies an element $X \in \mathfrak{g}$ which is regular and semisimple. The orbit does not depend on the choice of isomorphism, and if we call it $\mathcal{O}(I, (a_i), (c_i))$, then all orbits in $\mathfrak{g}_{reg}$ are of this form.

Now for all $i \in I$, let $\mathrm{sgn}_{F_i/F_i^\#}$ be the quadratic character of $F_i^{\#\times}$ associated to the algebra F_i. Let I^* be the set of $i \in I$ so that F_i is a field, i.e. so that $\mathrm{sgn}_{F_i/F_i^\#}$ is non-trivial. Then for two families $(I, (a_i), (c_i))$ and $(I', (a_i'), (c_i'))$ satisfying the above conditions, we have that the corresponding orbits $\mathcal{O}(I, (a_i), (c_i))$ and $\mathcal{O}(I', (a_i'), (c_i'))$ are equal if and only if

1. There is a bijection $\phi : I \to I'$.

2. For all $i \in I$, there is an F-linear isomorphism $\sigma_i : F'_{\phi(i)} \to F_i$ so that

    i. For all $i \in I$, $\sigma_i(a'_{\phi(i)}) = a_i$

    ii. For all $i \in I$, $\mathrm{sgn}_{F_i/F_i^{\#}}(c_i \sigma_i(c'_{\phi(i)})^{-1}) = 1$

We also have that the two orbits are in the same stable conjugacy class, i.e. there are regular semisimple elements of $\mathfrak{g}$ in the respective orbits which are conjugate by an element of $\mathbf{G}$, if maps ϕ and σ_i satisfying all but condition ii. above exist. A stable class $\mathcal{O}^{\mathrm{st}}(I, (a_i), (c_i))$ thus splits into exactly $(\mathbb{Z}/2\mathbb{Z})^{I^*}$ G-conjugacy classes.

7.3.2 Unramified Tori in G

Given a partition $\lambda = (x_1, \ldots, x_k)$, let $\mathrm{S}(\lambda) := \sum_{i=1}^{k} x_i$ denote the sum of the parts of λ. We always order the parts of λ so that $x_1 \geq \cdots \geq x_k$, and we do not allow parts of λ to be zero. Let $\theta_{\max}(V)$ be the set of triples of partitions (μ_0, μ', μ'') so that $S(\mu_0) + S(\mu') + S(\mu'') = n$. Then we have a bijection $\mathcal{C}^S \to \theta_{\max}(V)$ given as follows:

Let T be a maximal unramified torus, and fix a regular semisimple element X in $\mathfrak{t}$, the Lie algebra of T in $\mathfrak{g}$. Choose $(I, (a_i), (c_i))$ so that X is in the orbit $\mathcal{O}(I, (a_i), (c_i))$. For an integer $m \geq 1$, let $F^{(m)}$ be the unique unramified extension of F of degree m. Since T is unramified, for all $i \in I$, there exists an integer $m(i)$ so that $F_i^{\#} = F^{(m(i))}$. Let I', resp. I'', be the set of $i \in I^*$ so that the valuation $v_{F_i}(c_i)$ of c_i in F_i is even, resp. odd. We define the triple (μ_0, μ', μ'') by setting μ', resp. μ'', to be the the partition which has the same parts as the family $(m(i))_{i \in I'}$, resp. $(m(i))_{i \in I''}$. Finally, we define μ_0 to be the partition having the same non-zero terms as the family $(m(i))_{i \in I \setminus I^*}$. Thus we have defined an element of $\theta_{\max}(V)$ from a given maximal unramified torus T. According to [37], it does not depend on the choice of X in T, and ${}^gX$ defines the same element of $\theta_{\max}(V)$ for all $g \in G$. Thus the construction gives a bijection

$$\mathcal{C}^s \to \theta_{\max}(V)$$

.

7.4 Main Result

We are now ready to compare the two parameterizations.

Theorem 7.4.1. Consider $(\mu_0, \mu', \mu'') \in \theta_{\max}(V)$, with $\mu_0 = (x_1, \dots, x_r)$, $\mu' = (y_1, \dots, y_s)$, $\mu'' = (z_1, \dots, z_t)$. Then we have that the corresponding G-conjugacy class in $\mathcal{C}^S$ corresponds to the equivalence class of (F, w) in the DeBacker parameterization, where F is the facet $\langle \mathrm{S}(\mu'') | x_1, \dots, x_r | \mathrm{S}(\mu') \rangle$ and w is in the conjugacy class of type

$$(C_{z_1} \times \cdots \times C_{z_t}) \times A_{x_1 - 1} \times \cdots \times A_{x_r - 1} \times (C_{y_1} \times \cdots \times C_{y_s}).$$

In particular, the tori corresponding to the special vertex of $\mathcal{C}$ vanishing at the hyperplanes $H_{\alpha_1}, \dots, H_{\alpha_{n-1}}$ and H_β are those corresponding to triples of the form $(\emptyset, \mu', \emptyset)$. Similarly, the tori vanishing corresponding to the special vertex of $\mathcal{C}$ vanishing at the hyperplanes $H_{\alpha_1}, \dots, H_{\alpha_{n-1}}$ and H_e are those corresponding to triples of the form $(\emptyset, \emptyset, \mu'')$. The diagonal torus S corresponds to the triple $((1, \dots, 1), \emptyset, \emptyset)$.

Example: In Figure 7.3 below, for each pair (F, w) in Sp_4, we give the corresponding triple in the Waldspurger paramaterization.

We will prove this theorem throughout the rest of this section. We will deal with each of the partitions in a separate subsection before putting everything together at the end. For each partition, we will carefully construct the necessary field extensions, algebras, and generators a_i. We will then use our construction to produce an ordered symplectic basis and determine the structure of the matrix of the multiplication map with respect to our choice of ordered basis. We will pay particularly close attention to:

1. The location of the non-zero entries of the matrix of multiplication by a_i, particularly in our analysis of μ_0. We need to choose our basis of W so that X lies in the parahoric

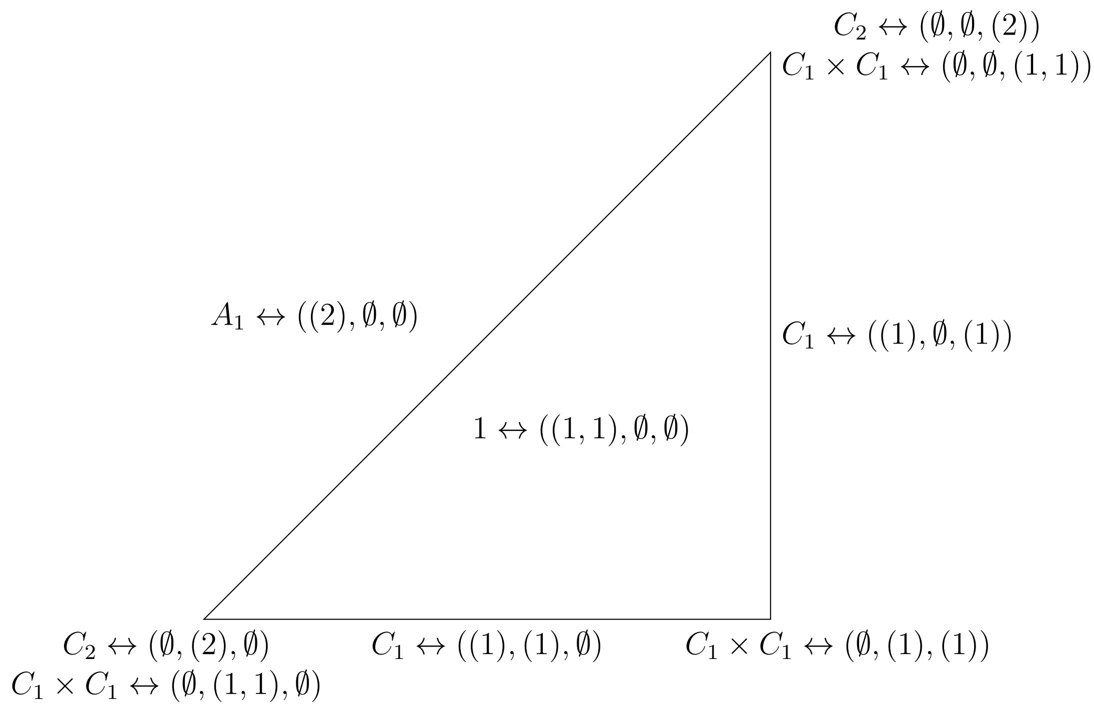

Figure 7.3: The Theorem for Sp_4

of our proposed facet F and descends to a regular semisimple element in the reductive quotient. In particular, we will choose our ordered basis so that all of the non-zero entries of X will lie on either the diagonal or in the root space of a root α so that $\mathsf{F} \subseteq H_\alpha$.

2. The way in which Fr acts on the eigenvalues of the matrix of multiplication by a_i. When we put everything together, we will use this information to determine the conjugacy class in W associated to our torus.

7.4.1 The Partition μ_0

We begin with a lemma:

Lemma 7.4.2. The degree m extension $\mathbb{F}_q^m$ of $\mathbb{F}_q$ contains an element η so that $\mathbb{F}_q^m = \mathbb{F}_q(\eta^2)$. In other words, η^2 is a primitive element of the extension $\mathbb{F}_q^m/\mathbb{F}_q$.

Proof. We will treat the cases of $p = 2$ and $p \neq 2$ separately. First, suppose $p \neq 2$, and choose an element $\eta \in \mathbb{F}_q^m$ so that η is a cyclic generator of the multiplicative group of $\mathbb{F}_q^m$. Then η has order $q^m - 1$, and since $p \neq 2$, η^2 has order $(q^m-1)/2 = \dfrac{q-1}{2}(1+q+\cdots+q^{m-1})$. If η^2 is not a primitive element of the extension $\mathbb{F}_q^m/\mathbb{F}_q$, then η^2 must belong to some proper subextension, and so the order of η^2 is less or equal $q^k - 1$ for some $k < m$. But

$$q^k - 1 < q^k < 1 + q + \cdots + q^{m-1} < \frac{q-1}{2}(1+q+\cdots+q^{m-1})$$

since $q > p > 2$, and so η^2 is the desired primitive element of the extension $\mathbb{F}_q^m/\mathbb{F}_q$.

On the other hand, if $p = 2$, then since 2 is then coprime to $q^m - 1$, η^2 is another cyclic generator of the multiplicative group of $\mathbb{F}_q^m$ for any cyclic generator η of $\mathbb{F}_q^m$, and so we are done. $\square$

For the part x_1 of the partition μ_0, we may thus assume that $F^{(x_1)} = F(\eta^2)$ for some $\eta \in \mathfrak{o}_{F^{(x_1)}}^{\times}$. Let $f(x)$ be the minimal polynomial of η^2 over F so that $F^{(x_1)} = F[x]/(f(x))$. Note that $F^{(x_1)} = F(\eta)$ as well, and so the minimal polynomial $g(x)$ of η over F also has degree x_1, which we will use shortly. Then we define the algebra $F_{x_1} := F^{(x_1)}[y]/(y^2 - \eta^2) \cong F[y]/(f(y^2))$. Then F_{x_1} is 2-dimensional over $F^{(x_1)}$ and is not a field, meaning that it is the algebra corresponding to x_1 in the Waldspurger construction. We have that the non-trivial $F^{(x_1)}$-automorphism τ sends y to $-y$.

We set $c_1 = y$ so that our symplectic form on F_{x_1}, viewed as an F-algebra, is given by the pairing $\langle v_1, v_2 \rangle = \frac{1}{2x_1}\mathrm{trace}_{F_{x_1}/F}(\tau(v_1)v_2 y)$ for $v_1, v_2 \in F_{x_1}$. We set $a_1 = y$ so that our F-endomorphism X_{x_1} on F_{x_1} is given by multiplication by y.

We will now determine the eigenvalues of our multiplication map X_{x_1}, and we will determine how Fr acts on them. Again viewing F_{x_1} as an F-vector space, we begin by fixing the ordered basis $\{1, y, \eta^2, \eta^2 y, \eta^4, \ldots, \eta^{2(x_1-1)}, \eta^{2(x_1-1)}y\}$. Then the matrix of X_{x_1} with respect to this basis is in rational canonical form, and so the characteristic polynomial of X_{x_1} is $f(x^2)$. We then claim that $f(x^2) = \pm g(x)g(-x)$. To see this, first note that η and $-\eta$ cannot both be roots of $g(x)$. If they were, we would have $\eta = \mathrm{Fr}^k(-\eta)$ for some integer $k < x_1$, as every unramified extension of F is cyclic. But then we would have that $\eta^2 = \mathrm{Fr}^k(\eta^2)$, implying that the extension $\mathbb{F}_q(\eta^2)$ has degree less than x_1 and hence contradicting our choice of η. Thus the minimal polynomial of $-\eta$, which is $\pm g(-x)$, is not equal to $g(x)$, and so in particular $g(x)$ and $g(-x)$ have distinct roots. But η and $-\eta$ are both roots of the monic polynomial $f(x^2)$, and so we must have that both $g(x)$ and $\pm g(-x)$ divide $f(x^2)$. Since the degrees match and the roots are distinct, we have our claim.

With this equality, we have that Fr permutes the roots of $g(x)$, which are elements of $\mathfrak{o}^\times_{F^{(x_1)}}$ having distinct images in the residue field, and the roots of $g(-x)$, which are the negatives of the roots of $g(x)$, cyclically, and so Fr acts on the roots of $f(x^2)$, which are the eigenvalues of X_{x_1}, via two x_1-cycles.

We will now begin working towards a symplectic basis. To begin, we reorder our basis as

$$\{1, \eta^2, \eta^4, \dots, \eta^{2(x_1-1)}, y, \eta^2 y, \dots, \eta^{2(x_1-1)} y\}.$$

Then with respect to the basis, X_{x_1} is a block matrix of the form

$$\begin{pmatrix} 0 & A \\ \mathrm{Id} & 0 \end{pmatrix}$$

where each block is a $x_1 \times x_1$ matrix with non-zero entries in $\mathfrak{o}$. However, if we consider $F(\eta^2) = F^{(x_1)}$ as an F-vector space and let Y be the linear transformation given by multiplication by η^2, we notice that A is also the matrix of Y with respect to the ordered basis $B = \{1, \eta^2, \eta^4, \dots, \eta^{2(x_1-1)}\}$. Thus the matrix A has a natural square root $\sqrt{A}$ obtained by taking the matrix of the linear transformation on $F(\eta^2)$ given by multiplication by η with respect to the basis B. We also see that $\sqrt{A}$ is invertible, with inverse given by multiplication by η^{-1} in $F(\eta^2)$. Note that $\sqrt{A}$ and $\sqrt{A}^{-1}$ both have entries in $\mathfrak{o}$.

Now let T be the block matrix

$$\begin{pmatrix} \sqrt{A} & -\sqrt{A} \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

and let B_T be the transformed ordered basis $B_T = \{T \cdot 1, T \cdot \eta^2, \ldots, T \cdot \eta^{2(x_1-1)}, T \cdot y, \ldots, T \cdot \eta^{2(x_1-1)}y\}$. Then the matrix of X_{x_1} with respect to B_T is given by

$$\begin{pmatrix} \frac{1}{2}\sqrt{A}^{-1} & \frac{1}{2}\,\mathrm{Id} \\ -\frac{1}{2}\sqrt{A}^{-1} & \frac{1}{2}\,\mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & A \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{A} & -\sqrt{A} \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \sqrt{A} & 0 \\ 0 & -\sqrt{A} \end{pmatrix}$$

From here, we finally construct a symplectic basis. First, we claim that $\langle T \cdot \eta^{2i}, T \cdot \eta^{2j} \rangle = 0 = \langle T \cdot \eta^{2i}y, T \cdot \eta^{2j}y \rangle$ for all i, j in $\{0, \ldots, x_1 - 1\}$. To see this, note that

$$T \cdot \eta^{2i} = \eta \cdot \eta^{2i} + \eta^{2i}y = \eta^{2i+1} + \eta^{2i}y$$

since $\sqrt{A}$ acts on $F^{(x_1)}$ by multiplication by η by construction and

$$T \cdot \eta^{2i}y = -\eta \cdot \eta^{2i} + \eta^{2i}y = -\eta^{2i+1} + \eta^{2i}y.$$

Thus

$$\begin{aligned} \langle T \cdot \eta^{2i}, T \cdot \eta^{2j} \rangle = \langle \eta^{2i+1} + \eta^{2i}y, \eta^{2j+1} + \eta^{2j}y \rangle &= \frac{1}{2x_1}\mathrm{trace}_{F_{x_1}/F}((\eta^{2i+1} - \eta^{2i}y)(\eta^{2j+1} + \eta^{2j}y)y) \\ &= \frac{1}{2x_1}\mathrm{trace}_{F_{x_1}/F}(y\eta^{2i+2j+2} + \eta^{2i+2j+3} - \eta^{2i+2j+3} - y\eta^{2i+2j+2}) \\ &= 0. \end{aligned}$$

The computation for $\langle T \cdot \eta^{2i}, T \cdot \eta^{2j} \rangle$ is nearly identical.

Thus we can decompose the F-vector space F_{x_1} into two Lagrangian subspaces, x_1-dimensional subspaces $F_{x_1}^+ = \mathrm{span}_F\{T \cdot 1, T \cdot \eta^2, \ldots, T \cdot \eta^{2(x_1-1)}\}$ and $F_{x_1}^- = \mathrm{span}_F\{T \cdot y, \ldots, T \cdot \eta^{2(x_1-1)}y\}$ so that $\langle x, y \rangle = 0$ for all $x, y \in F_{x_1}^+$, resp. $F_{x_1}^-$. Then for $v' \in F_{x_1}^-$, the map which sends $v \in F_{x_1}^+$ to $\langle v, v' \rangle$ gives a linear functional on $F_{x_1}^+$. Since the symplectic form is non-degenerate, we obtain an isomorphism $F_{x_1}^- \xrightarrow{\sim} (F_{x_1}^+)^*$, where $(F_{x_1}^+)^*$ denotes the dual space of $F_{x_1}^+$. We can then construct a dual basis $\{\gamma_0^*, \ldots, \gamma_{x_1-1}^*\}$ of $(F_{x_1}^+)^*$ so that $\gamma_i^*(T \cdot \eta^{2j}) =$

$\begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$, and letting B' be the ordered basis $\{T \cdot 1, T \cdot \eta^2, \ldots, T \cdot \eta^{2(x_1-1)}, \gamma_{x_1-1}, \ldots, \gamma_0\}$, where γ_i is the inverse image of γ_i^* under our isomorphism $F_{x_1}^- \xrightarrow{\sim} (F_{x_1}^+)^*$, we have obtained our symplectic basis.

Note that the change of basis matrix from B_T to B' has the block-diagonal form $\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & M \end{pmatrix}$ for some $x_1 \times x_1$ matrix M, and so the matrix of X_{x_1} with respect to B' also has block diagonal form

$$\begin{pmatrix} \sqrt{A} & 0 \\ 0 & N \end{pmatrix}$$

where N is the $x_1 \times x_1$ matrix given by conjugating $-\sqrt{A}^{\mathsf{T}}$ by the matrix $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. Thus we note that the matrix of X_{x_1} with respect to the ordered basis B' still has entries in $\mathfrak{o}$.

For future use, we let $\gamma_{j+1}^{x_1,+} := T \cdot \beta^{2j}$ and $\gamma_{j+1}^{x_1,-} := \gamma_j$ for $j \in \{0, \ldots, x_1 - 1\}$ so that $B' = \{\gamma_1^{x_1,+}, \ldots, \gamma_{x_1}^{x_1,+}, \gamma_{x_1}^{x_1,-}, \ldots, \gamma_1^{x_1,-}\}$. We also let

$$\begin{pmatrix} M_{x_1}^{1,1} & 0 \\ 0 & M_{x_1}^{2,2} \end{pmatrix}$$

denote the matrix of X_{x_1} with respect to this ordered basis.

Remark: Note that we could have constructed the symplectic basis before applying the transformation T to our original basis. However, the resulting matrix of X_{x_1} would not have the block-diagonal form we have just obtained, which is essential to ensuring our resulting element will be a lift of a regular semisimple element in the reductive quotient of our eventual facet F. In particular, we want the non-zero entries of this piece to only show up in the root spaces in the subsystem spanned by the simple roots α_i.

Thus we are done with the entry x_1 in μ_0. For the other $x_j \in \mu_0$, we can repeat this process. The only difference is if $x_{j_1} = x_{j_2}$ for $j_1 \neq j_2$, we need to rescale our choice of a_{j_2} by a representative of a non-trivial class in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times \cap R_{2x_{j_1}})$, where $R_{2x_{j_1}}$ denotes the group of $2x_{j_1}$th roots of unity which belong to $\mathbb{F}_q^\times$. The resulting matrix of $X_{x_{j_2}}$ and its eigenvalues are then scaled by the same element so that the entries of $X_{x_{j_2}}$ still lie in $\mathfrak{o}$ and the eigenvalues of $X_{x_{j_2}}$ still lie in $\mathfrak{o}_{F^{(x_1)}}^\times$. Since $q > 2n$, we can scale so that for all $x_{j_m} = x_{j_1}$, each of the a_{j_m} are multiplied by representatives of different classes. We need to do this so that condition (b) at the start of Section 4.1 is satisfied and so that the the eigenvalues of all the matrices $X_{x_{j_m}}$ are distinct elements of $\mathfrak{o}_{F_{\mathrm{un}}}^\times$ with distinct images in $\overline{\mathbb{F}}_q^\times$.

7.4.2 The Partition μ'

We choose a lift $\delta \in \mathfrak{o}_{F^{(y_1)}}^\times$ of an element of the degree y_1 extension $\mathbb{F}_q^{y_1}$ of $\mathbb{F}_q$ which generates the cyclic group $(\mathbb{F}_q^{y_1})^\times$. Then we have that the degree y_1 unramified extension $F^{(y_1)}$ is equal to $F[\delta]$. Furthermore, the image of δ cannot have a square root in $(\mathbb{F}_q^{y_1})^\times$ since $(\mathbb{F}_q^{y_1})^\times$ has even order, and so the degree 2 unramified extension F_{y_1} of $F^{(y_1)}$ is generated by $\delta^{1/2}$ for some fixed square root of δ. We let τ denote the non-trivial element of the Galois group of the extension $F_{y_1}/F^{(y_1)}$, so that $\tau(\delta^{1/2}) = -\delta^{1/2}$.

We choose $a_{y_1} = c_{y_1} = \delta^{1/2}$ in the Waldspurger construction. Thus our F-endomorphism X_{y_1} of F_{y_1} is given by multiplication by $\delta^{1/2}$, and our symplectic form on F_{y_1} is given by the pairing $\langle v_1, v_2\rangle = \frac{1}{2y_1}\mathrm{trace}_{F_{y_1}/F}(\tau(v_1)v_2\delta^{1/2})$ for $v_1, v_2 \in F_{y_1}$. Let $f(x)$ be the minimal polynomial of δ over F. Then $f(x^2)$ is the minimal polynomial of $\delta^{1/2}$ over x, and so it also the characteristic polynomial of X_{y_1}. Since $f(x^2)$ is the minimal polynomial of an unramified extension of F, we know that the eigenvalues of X_{y_1} are in $\mathfrak{o}_{F^{(y_1)}}^\times$ and have distinct images in $\mathbb{F}_q^{y_1}$. We also know that Fr permutes the $2y_1$ roots of $f(x^2)$ cyclically. Furthermore, for all roots $\gamma^{1/2}$ of $f(x^2)$, we must have that $\mathrm{Fr}^{y_1}(\gamma^{1/2}) = -\gamma^{1/2}$ since Fr permutes the y_1 roots of $f(x)$ cyclically, meaning that Fr acts as an odd y_1-cycle on the roots of $f(x^2)$.

Before constructing our symplectic basis, we begin with the ordered basis

$$\{1, \delta, \ldots, \delta^{y_1-1}, \delta^{1/2}, \delta^{3/2}, \ldots, \delta^{y_1-1/2}\}.$$

Then the matrix of our multiplication map with respect to this basis has the block form

$$\begin{pmatrix} 0 & A \\ \mathrm{Id} & 0 \end{pmatrix}$$

where each block is a $y_1 \times y_1$ matrix. If we let

$$F_{y_1}^+ = \mathrm{span}_F\{1, \delta, \delta^2, \ldots, \delta^{y_1-1}\} \qquad \text{and} \qquad F_{y_1}^- = \mathrm{span}_F\{\delta^{1/2}, \delta^{3/2}, \ldots, \delta^{y_1-1/2}\},$$

then note that any two elements of $F_{y_1}^+$ (resp. $F_{y_1}^-$) are mutually orthogonal with respect to our symplectic form. For v' in $F_{y_1}^-$, the map which sends $v \in F_{y_1}^+$ to $\langle v, v' \rangle$ gives a linear functional on $F_{y_1}^+$, and since the symplectic form is non-degenerate, we have an isomorphism $F_{y_1}^- \xrightarrow{\sim} (F_{y_1}^+)^*$, where $(F_{y_1}^+)^*$ denotes the dual basis of $F_{y_1}^+$. Constructing a dual basis $\{\gamma_0^*, \ldots, \gamma_{y_1-1}^*\}$ of $(F_{y_1}^+)^*$ so that $\gamma_i^*(\delta^j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$, we let $B' = \{1, \ldots, \delta^{y_1-1}, \gamma_{y_1-1}, \ldots, \gamma_0\}$, where γ_i is the inverse image of γ_i^* under our isomorphism $F_{y_1}^- \xrightarrow{\sim} (F_{y_1}^+)^*$. For future use, we let $\gamma_{j+1}^{y_1,+} = \delta^j$ and $\gamma_{j+1}^{y_1,-} = \gamma_j$ for all $j \in \{0, \ldots, y_1 - 1\}$ so that $B' = \{\gamma_1^{y_1,+}, \ldots, \gamma_{y_1}^{y_1,+}, \gamma_{y_1}^{y_1,-}, \ldots, \gamma_1^{y_1,-}\}$.

Note that the change of basis matrix from our initial basis to the ordered basis B' has the block-diagonal form $\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & M \end{pmatrix}$ for some $y_1 \times y_1$ matrix M, and so the matrix of our multiplication map with respect to B' has the block form

$$\begin{pmatrix} 0 & M_{y_1}^{1,2} \\ M_{y_1}^{2,1} & 0 \end{pmatrix}.$$

This is a matrix with entries in $\mathfrak{o}$.

For the other $y_j \in \mu'$, we can repeat this process. The only change we need to make is that if $y_{j_1} = y_{j_2}$, for $j_1 \neq j_2$, we need to rescale our choice of $a_{y_{j_2}}$ by a representative of a non-trivial class in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times \cap R_{2y_{j_1}})$, where $R_{2y_{j_1}}$ denotes the group of $2y_{j_1}$ roots of unity. The resulting matrix of $X_{y_{j_2}}$ and its eigenvalues are then scaled by the same element of $\mathbb{F}_q^\times$ so that the entries of $X_{y_{j_2}}$ still lie in $\mathfrak{o}$ and the eigenvalues of $X_{y_{j_2}}$ still lie in $\mathfrak{o}_{F(y_1)}^\times$. Since $q > 2n$, we can scale so that for all $y_{j_m} = y_{j_1}$, all of the a_{j_m} are multiplied by representatives of different classes. As with μ_0, we do this to ensure condition (b) at the start of section 4.1 is satisfied.

7.4.3 The Partition μ''

For $z_i \in \mu''$, we choose δ as in the previous section so that $F^{(z_i)} = F(\delta)$ and $F_{z_i} = F(\delta^{1/2})$. The primary difference in the construction from the previous section is that we set $c_{z_i} = \varpi\delta^{1/2}$, so that the symplectic form is given by $\langle v_1, v_2\rangle = \frac{1}{2z_i}\mathrm{tr}_{F_{z_i}/F}(\tau(v_1)v_2\varpi\delta^{1/2})$ for $v_1, v_2 \in F_{z_i}$. We take $a_{z_i} = d_{z_i}\cdot\delta^{1/2}$, where d_{z_i} is an element of $\mathbb{F}_q$ chosen so that for all $z_i = z_j$, d_{z_i} and d_{z_j} are distinct representatives of $\mathbb{F}_q^\times/(\mathbb{F}_q^\times \cap R_{2z_i})$, where $R_{2z_{j_1}}$ denotes multiplicative group of $2z_i$th roots of unity in $\mathbb{F}_q^\times$, and so that for all $z_i = y_k$ for $y_k \in \mu'$, a_{y_k} was not scaled by an element in the same class as d_{z_i} in the previous section. Then our F-endomorphism is multiplication by $d_{z_i}\cdot\delta^{1/2}$. We take our initial basis to be $B = \{1, \delta, \ldots, \delta^{z_i-1}, \frac{\delta^{1/2}}{\varpi}, \frac{\delta^{3/2}}{\varpi}, \ldots, \frac{\delta^{z_i-1/2}}{\varpi}\}$. We then construct the change of basis matrix that we would have constructed for the basis $\{1, \delta, \ldots, \delta^{z_i-1}, \delta^{1/2}, \delta^{3/2}, \ldots, \delta^{z_i-1/2}\}$ in the μ' case, and apply it to B to obtain an ordered basis $B' = \{\gamma_1^{z_i,+}, \ldots, \gamma_{z_i}^{z_i,+}, \gamma_{z_i}^{z_i,-}, \ldots, \gamma_1^{z_i,-}\}$. Then we again have a symplectic basis, and the matrix of the multiplication map X_{z_i} with respect to B' has the block form

$$\begin{pmatrix} 0 & M_{z_i}^{1,2} \\ M_{z_i}^{2,1} & 0 \end{pmatrix}.$$

The $z_i \times z_i$ matrix $M_{z_i}^{1,2}$ (resp. $M_{z_i}^{2,1}$) has non-zero entries in $\mathfrak{p}^{-1}$ (resp. $\mathfrak{p}$), and we again see that the eigenvalues of X_{z_i} are elements of $\mathfrak{o}_{F^{(z_i)}}^\times$ with distinct images in $\mathbb{F}_q^{z_i}$.

7.4.4 Completing the Proof

We are now ready to put everything together. Let

$$W = \bigoplus_{x_i \in \mu_0} F_{x_i} \oplus \bigoplus_{y_j \in \mu'} F_{y_j} \oplus \bigoplus_{z_k \in \mu''} F_{z_k}.$$

Then W is the space associated to the triple (μ_0, μ', μ'') in the Waldspurger construction. The symplectic form and F-endomorphism determining our regular semisimple element are the respective sums of the symplectic forms and F-endomorphisms of each subspace, as described in section 4.

For $x_i \in \mu_0$, set $B^+_{x_i} = \{\gamma_1^{x_i,+}, \dots, \gamma_{x_i}^{x_i,+}\}$ and $B^-_{x_i} = \{\gamma_{x_i}^{x_i,-}, \dots, \gamma_1^{x_i,-}\}$. Define $B^{\pm}_{y_j}$ and $B^{\pm}_{z_k}$ analogously. Fix the ordered basis

$$B_W = B^+_{z_1} \cup \cdots \cup B^+_{z_t} \cup B^+_{x_1} \cup \cdots \cup B^+_{x_r} \cup B^+_{y_1} \cup \cdots \cup B^+_{y_s} \cup B^-_{y_s} \cup \cdots \cup B^-_{y_1} \cup B^-_{x_r} \cup \cdots \cup B^-_{x_1} \cup B^-_{z_t} \cup \cdots \cup B^-_{z_1}.$$

By our work in the previous subsections, we know that the matrix of our symplectic form on W has the form

$$\begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$$

with respect to the ordered basis B_W so that we have an obvious isomorphism from W into V. With respect to B_W, we have that our F-endomorphism (which again is a regular semisimple element of our unramified torus), has the block form

$$\begin{pmatrix} M^{1,1} & M^{1,2} \\ M^{2,1} & M^{2,2} \end{pmatrix},$$

where each block has size $n \times n$. These blocks $M^{i,j}$ are in turn block matrices. The matrix $M^{1,1}$ is block diagonal of the form

$$\begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & M^{1,1}_{x_1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & M^{1,1}_{x_r} & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix},$$

where the blocks have size $z_1 \times z_1, \ldots, z_t \times z_t, x_1 \times x_1, \ldots, x_r \times x_r, y_1 \times y_1, \ldots, y_s \times y_s$ respectively. Similarly, the block $M^{2,2}$ is block diagonal of the form

$$\begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & M^{2,2}_{x_r} & & & & & \\ & & & & \ddots & & & & \\ & & & & & M^{2,2}_{x_1} & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix},$$

where the listed blocks have size $y_s \times y_s, \ldots, y_1 \times y_1, x_r \times x_r, \ldots, x_1 \times x_1, z_t \times z_t, \ldots, z_1 \times z_1$. The matrices $M^{1,2}$ and $M^{2,1}$ are block anti-diagonal. $M^{1,2}$ has the form

$$\begin{pmatrix} &&&&&&&& M^{1,2}_{z_1} \\ &&&&&&& \iddots & \\ &&&&&& M^{1,2}_{z_t} && \\ &&&&& 0 &&& \\ &&&& \iddots &&&& \\ &&& 0 &&&&& \\ && M^{1,2}_{y_1} &&&&&& \\ & \iddots &&&&&&& \\ M^{1,2}_{y_s} &&&&&&&& \end{pmatrix},$$

where the listed blocks have size, from top to bottom, $z_1 \times z_1, \ldots, z_t \times z_t, x_1 \times x_1, \ldots, x_r \times x_r, y_1 \times y_1, \ldots, y_s \times y_s$, and $M^{2,1}$ has the form

$$\begin{pmatrix} &&&&&&&& M^{2,1}_{y_s} \\ &&&&&&& \iddots & \\ &&&&&& M^{2,1}_{y_1} && \\ &&&&& 0 &&& \\ &&&& \iddots &&&& \\ &&& 0 &&&&& \\ && M^{2,1}_{z_t} &&&&&& \\ & \iddots &&&&&&& \\ M^{2,1}_{z_1} &&&&&&&& \end{pmatrix},$$

where the listed blocks have size, from top to bottom, $y_s \times y_s, \ldots, y_1 \times y_1, x_r \times x_r, \ldots, x_1 \times x_1, z_t \times z_t, \ldots, z_1 \times z_1$. Henceforth, we identify our F-endomorphism of W with the matrix we have constructed above, and we will write X_W for both.

Thus X_W is the regular semisimple element for our torus arising from the Waldspurger construction. We now determine a pair (F, w) giving our torus in the DeBacker parameterization. We begin with w. To compute w, we first note that there is a diagonal matrix s in the

Lie algebra of S and an element $g \in \mathbf{G}(F_{\mathrm{un}})$ so that $X_w = {}^g s$. Then since X_W is F-rational, we have that $n = g^{-1}\mathrm{Fr}(g)$ is an element of $N_G(S)$ sending $\mathrm{Fr}(s)$ to s, and we know from [1] that w is the image of this element in the Weyl group. But by linear algebra, note that the columns of g must be an eigenbasis for the endomorphism X_W. Since X_W is a regular semisimple element in the Lie algebra of Sp_{2n} its eigenvalues are distinct, and so $g^{-1}\mathrm{Fr}(g)$ is a permutation matrix, where the cycle type of the permutation is determined by how Fr permutes the eigenvalues of X_W. But by our previous analysis, each $x_i \in \mu_0$ gives us an even cycle of type A_{x_i-1} in the Carter notation, while each $y_j \in \mu'$ and $z_k \in \mu''$ gives us an odd cycle of type C_{y_j} or C_{z_k} respectively. Thus the Weyl group element w associated to X_W is of type

$$(C_{z_1} \times \cdots \times C_{z_t}) \times A_{x_1-1} \times \cdots \times A_{x_r-1} \times (C_{y_1} \times \cdots \times C_{y_s})$$

in the Carter notation.

To conclude the proof, we need to show that the image of X_W in $\mathrm{Lie}(G_{\mathsf{F}})/\mathrm{Lie}(G_{\mathsf{F}}^{+})$ is still a regular semisimple element, where F denotes the facet in the fundamental alcove $\mathcal{C}$ of type $\langle \mathrm{S}(\mu'')|x_1, \ldots, x_t|\mathrm{S}(\mu')\rangle$. To do this, we claim that it suffices to check

- The eigenvalues of X_W lie in $\mathfrak{o}_{F_{\mathrm{un}}}^{\times}$ and are distinct in the reductive quotient.

- Every non-zero entry in the root space of a root α in the subsystem Φ' spanned by $\{-e, \alpha_1, \ldots, \alpha_{\mathrm{S}(\mu'')-1}\}$ lies in $\mathfrak{o}$ if the coefficient of $-e$ in α is 0 with respect to the simple system $\{-e, \alpha_1, \ldots, \alpha_{\mathrm{S}(\mu'')-1}\}$ of Φ', lies in $\mathfrak{p}$ if the coefficient of $-e$ is positive, or lies in $\mathfrak{p}^{-1}$ if the coefficient of $-e$ is negative.

- Every non-zero entry in the root space of a root α in the subsystem spanned by the simple roots α_j corresponding to the vertices in the subdiagram of the extended Dynkin diagram associated to a part $x_i \in \mu_0$ lies in $\mathfrak{o}$.

- Every non-zero entry in the root space of a root α in the subsystem spanned by the simple roots β and $\alpha_{n-\mathrm{S}(\mu')+1}, \ldots, \alpha_{n-1}$ is in $\mathfrak{o}$.

- Every non-zero diagonal entry is in $\mathfrak{o}$.

- Every other entry is zero.

The last five conditions ensure that the element X_W lies in the Lie algebra of the parahoric at our facet, and the first ensures that the image X_W of X_W in the associated reductive quotient is a regular semisimple element of Lie(G_{F}). Note that the centralizer of X_W, call it T, in the quotient $G_{\mathsf{F}}/G_{\mathsf{F}}^+$ is a maximal $\mathbb{F}_q$-torus that corresponds to the image of $C_G(X_W) \cap G_{\mathsf{F}}$ in $G_{\mathsf{F}}/G_{\mathsf{F}}^+$. By [10], the maximal unramified torus $C_G(X_W)$ is a lift of T.

We now check that X_W does in fact satisfy the six conditions. The first condition follows from our choice of the elements a_i in the preceding sections for each partition. In particular, we showed that the eigenvalues associated to each part are distinct elements of $\mathfrak{o}_{F_{\mathrm{un}}}^{\times}$, and since they are the roots of the minimal polynomial of an unramified extension, they have distinct image in the residue field. Furthermore, by scaling at the end of each subsection whenever one of our partitions contains parts that are equal, we ensured that eigenvalues of X_W in the direct sum are also distinct elements of $\mathfrak{o}_{F_{\mathrm{un}}}^{\times}$ which still have distinct image in $\overline{\mathbb{F}}_q^{\times}$.

We can see that all of the remaining conditions hold from the block form of our matrix X_W. In the first $\mathrm{S}(\mu'')$ columns and last $\mathrm{S}(\mu'')$ columns of X_W, all non-zero entries are in $M^{2,1}$ and $M^{1,2}$ respectively. In particularly, they lie in a block of the form $M_{z_i}^{2,1}$ or $M_{z_i}^{1,2}$. The entries in the blocks $M_{z_i}^{2,1}$ correspond to root spaces of roots in the subsystem spanned by $\{-e, \alpha_1, \ldots, \alpha_{\mathrm{S}(\mu'')-1}\}$ so that the coefficient of $-e$ is positive. By our construction in the previous section, we know that the non-zero entries of $M_{z_i}^{2,1}$ lie in $\mathfrak{p}$. Similarly, the entries in the blocks $M_{z_i}^{1,2}$ correspond to root spaces of roots so that the coefficient of $-e$ is negative, and again by our construction in the previous section, we know that the non-zero entries of $B_{z_i}^{1,2}$ lie in $\mathfrak{p}^{-1}$.

For $x_i \in \mu_0$, we consider the columns $S(\mu'')+x_1+\cdots+x_{i-1}+1, \ldots, S(\mu'')+x_1+\cdots+x_{i-1}+x_i$ and $2n-(S(\mu'')+x_1+\cdots+x_i)+1, \ldots, 2n-(S(\mu'')+x_1+\cdots+x_{i-1})$. Then the only non-zero entries are in the matrices $M^{1,1}_{x_i}$ and $M^{2,2}_{x_i}$. The non-zero entries of these matrices are in $\mathfrak{o}$. They contain the diagonal entries of X_W lying in the columns, and they also contain the root spaces of the roots in the span of $\alpha_{S(\mu'')+x_1+\cdots+x_{i-1}+1}, \ldots, \alpha_{S(\mu'')+x_1+\ldots x_i-1}$, which are the simple roots corresponding to the vertices in the subdiagram of the extended Dynkin diagram associated to x_i.

Finally, in the middle $2\cdot$ $\mathrm{S}(\mu')$ columns of X_W, all non-zero entries are in the blocks of the form $M^{2,1}_{y_i}$ or $M^{1,2}_{y_i}$. But the non-zero entries of these blocks are in $\mathfrak{o}$, and the entries of the blocks are root spaces of roots in the subsystem spanned by β and $\alpha_{n-\mathrm{S}(\mu')}, \ldots, \alpha_{n-1}$.

Consequently, X_W defines the torus associated to the pair (F, w), and so we are done.

7.5 Inverse Map

The construction in the previous section produces one pair (F, w) in the equivalence class associated to our maximal unramified torus. It is natural to ask whether we can obtain other pairs (F', w') in our equivalence class so that we may construct an inverse map. To answer this, note that our construction only produces facets of the form $\langle a|x_1, \ldots, x_r|b\rangle$ for $x_1 \geq \cdots \geq x_r$. However, in our construction of W in the previous section, note that if σ is a permutation of the parts of μ_0, then we can permute the sub-bases $B^+_{x_i}$ by σ when choosing our ordered basis B_W, as long as we also permute the $B^-_{x_i}$ analogously. Then with respect to this permuted ordered basis, the matrix of X_W will still be of type w, and it will define an element in the same G-conjugacy class. However, now it will be regular semisimple in the reductive quotient of the facet of type $\langle \mathrm{S}(\mu'')|\sigma(x_1), \ldots, \sigma(x_r)|\mathrm{S}(\mu')\rangle$.

Consequently, given a pair (F, w), where F is of the form $\langle a|x_1, \ldots, x_r|b\rangle$ and $w \in W_{\mathsf{F}}$ is elliptic of type $(C_{a_1} \times \cdots \times C_{a_t}) \times A_{x_1-1} \times \cdots \times A_{x_r-1} \times (C_{b_1} \times \cdots \times C_{b_s})$, then we can define μ_0 to be the unique partition containing the parts $x_1, \ldots, x_r$, μ' to be the partition containing the parts $b_1, \ldots, b_s$, and μ'' to be the partition containing the parts $a_1, \ldots, a_t$. Then the F-conjugacy class associated to the triple (μ_0, μ', μ'') is the F-conjugacy class associated to (F, w).

BIBLIOGRAPHY

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[1] J. Adler and S. DeBacker, *Parameterizing Unramified Tori*, Preprint.

[2] J. Adler and S. DeBacker, Untitled Preprint on Maximal Tori in p-adic Groups.

[3] Y. Benoist and H. Oh, *Polar decomposition for p-adic symmetric spaces*, Internat. Math. Res. Notices. 251 21pp. (2011).

[4] T. Bernstein, J. Ma, M. Nevins, and J. W. Yap, *Nilpotent Orbits of Orthogonal Groups over p-adic Fields, and the DeBacker Parametrization*, Algebr. Represent. Theory 23 (2020), no. 5, 2033–2058.

[5] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. I. Données radicielles valuées*, Inst. Hautes Études Sci. Publ. Math. 41 (1972), 5–251

[6] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. 60 (1984), 197–376.

[7] R. Carter, *Conjugacy classes in the Weyl group*, Compositio Math. 25 (1972), 1–59.

[8] R. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, Wiley, Chichester, 1993 [1985].

[9] S. DeBacker, *Parametrizing nilpotent orbits via Bruhat-Tits theory*, Ann. of Math. (2) 156 (2002), no. 1, 295–332.

[10] S. DeBacker, *Parameterizing Conjugacy Classes of Maximal Unramified Tori via Bruhat–Tits Theory*, Michigan Math. J. 54 (2006), no. 1, 157–178.

[11] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) 169 (2009), no. 3, 795–901.

[12] P. Delorme and V. Secherre, *An analogue of the Cartan decomposition for p-adic symmetric spaces of split p-adic reductive groups*, Pacific J. Math. 251 1–21 (2011)

[13] J. Fintzen, *On the construction of tame supercuspidal representations,* Preprint, arXiv:1908.09819.

[14] T. Haines and M. Rapoport, *Appendix: On parahoric subgroups*, Advances in Mathematics, Volume 219, Issue 1, 2008, 188-198.

[15] J. Hakim and F. Murnaghan, *Distinguished tame supercuspidal representations*, Int. Math. Res. Pap. IMRP, Art. ID rpn005(2):166pp., 2008.

[16] A. Helminck, *On the classification of k-involutions*, Adv. Math. 153 (2000), no. 1, 1–117.

[17] A. Helminck and G. Helminck, *A class of parabolic k-subgroups associated with symmetric k-varieties*, Trans. Amer. Math. Soc. 350 (1998), no. 11, 4669–4691.

[18] A. Helminck and S. Wang, *On rationality properties of involutions of reductive groups* Adv. Math. 99 (1993), no. 1, 26–96.

[19] H. Jacquet, K. Lai, and S. Rallis, *A trace formula for symmetric spaces*, Duke Math. J. 70 (1993), no. 2, 305–372.

[20] T. Kaletha, *Regular supercuspidal representations*, J. Amer. Math. Soc. 32 (2019), no. 4, 1071–1170.

[21] S. Kato and K. Takano, *Subrepresentation theorem for p-adic symmetric spaces* Int. Math. Res. Not. IMRN 2008, no. 11, Art. ID rnn028, 40 pp.

[22] S. Kato and K. Takano, *Square integrability of representations on p-adic symmetric spaces* J. Funct. Anal. 258 (2010), no. 5, 1427–1451.

[23] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math., 93:753–809, 1971.

[24] P. Levy, *Involutions of reductive Lie algebras in positive characteristic*, Adv. Math. 210 (2007), no. 2, 505–559.

[25] L. Morris, *Some tamely ramified supercuspidal representations of symplectic groups*, Proc. London Math. Soc. (3) 63 (1991), no. 3, 519–551.

[26] M. Nevins, *On nilpotent orbits of SL_n and Sp_{2n} over a local non-Archimedean field*, Algebr. Represent. Theory 14 (2011), no. 1, 161–190.

[27] R. Portilla, *Finite order automorphisms and a parametrization of nilpotent orbits in p-adic Lie algebras*, Amer. J. Math. 136 (2014), no. 3, 551–598.

[28] R. Portilla, *Maximal unramified tori in symmetric spaces*, Preprint.

[29] G. Prasad, *A new approach to unramified descent in Bruhat-Tits theory*, Amer. J. Math. 142 (2020), no. 1, 215–253.

[30] G. Prasad and J. K. Yu, *On finite group actions on reductive groups and buildings*, Invent. Math. 147 (2002), no. 3, 545–560.

[31] C. Rader and S. Rallis, *Spherical characters on p-adic symmetric spaces*, Amer. J. Math. 118 91–178 (1996).

[32] R. Richardson, *Orbits, invariants, and representations associated to involutions of reductive groups* Invent. Math., 66(2):287–312, 1982.

[33] J. P. Serre, *Galois Cohomology*, Translated from the French by Patrick Ion and revised by the author. Corrected reprint of the 1997 English edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002. x+210 pp. ISBN: 3-540-42192-0

[34] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations, and L-functions (A. Borel, W. Casselman, eds.), Proc. Sympos. Pure Math., 33, pp. 29–69, Amer. Math. Soc., Providence, RI, 1979.

[35] E. B. Vinberg, *The Weyl group of a graded Lie algebra* Izv. Akad. Nauk SSSR Ser. Mat., 1976.

[36] T. Vust, *Opérationde groupes réductifs dans un type de cones presque homogènes*, Bull. Soc. Math. France 102 (1974), 317–333.

[37] J. L. Waldspurger, *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés*, Astérisque 269 (2001).

[38] J. K. Yu, *Construction of tame supercuspidal representations* J. Amer. Math. Soc., 50(2):263–286, 2002.