Anosov Representations, Strongly Convex Cocompact Groups and Eigenvalue Gaps

by

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Dedicated to Anthi and the memory of Yannis

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ABSTRACT

Anosov representations were introduced by Labourie for fundamental groups of closed negatively curved Riemannian manifolds in his study of the Hitchin component and further generalized by Guichard-Wienhard for more general Gromov hyperbolic groups. Anosov representations of hyperbolic groups form a rich and structurally stable class of discrete subgroups of real reductive Lie groups and are recognized as a higher rank analogue of classical convex cocompact representations of hyperbolic groups into simple Lie groups of real rank 1. In this thesis, we obtain characterizations of Anosov representations in the spirit of the work of Guéritaud-Guichard-Kassel-Wienhard and Kapovich-Leeb-Porti in terms of equivariant limit maps, the Cartan property, the uniform gap summation property and weak uniform eigenvalue gaps. As an application, we obtain a characterization of strongly convex cocompact subgroups of the projective linear group $\mathsf{PGL}(d,\mathbb{R})$. We also compute the Hölder exponent of the Anosov limit maps of an Anosov representation in terms of the Cartan and Lyapunov projection of the image of the representation. Finally, we also provide a complete characterization of the domain groups of Borel Anosov representations into the projective linear group $\mathsf{PGL}(4q+2,\mathbb{R})$ for every q greater or equal than 1.

CHAPTER I

Introduction & Statement of Results

In this chapter, we provide a brief introduction to Gromov hyperbolic spaces and groups, a definition of Anosov representations into special linear groups along with some classes of examples and also state our main results.

1.1 Gromov hyperbolic groups

Gromov hyperbolic spaces were introduced by Gromov during the 80's in his seminal work [Gro87] and since then they play a fundamental role in the study of the geometry and topology of metric spaces with hyperbolicity. Let us provide a definition attributed to Rips in terms of slim triangles. Given a geodesic metric space (X, d), a geodesic triangle defined by three points $p, q, r \in X$ (and geodesics between the vertices $[pq], [qr], [rp]$), is called *δ-thin* if the union of the δ-neighbourhoods of any two of the geodesics $[pq], [qr], [rp]$ contains the third one. A geodesic metric space (X, d) is called *Gromov hyperbolic* if there exists $\delta \geq 0$ such that every geodesic triangle on X is δ-thin. Gromov hyperbolic spaces can be thought as a generalization of the class of metric spaces of strictly negative curvature. The fundamental examples of Gromov hyperbolic spaces include Hadamards manifolds of sectional curvature at most $-\kappa < 0$ (i.e. complete simply connected Riemannian manifolds (M, g) of sectional curvature at most $-\kappa < 0$). An important property of Gromov hyperbolicity is that it is invariant under quasi-isometry (see [Gro87], [BH13], [CDP06]): a geodesic metric space (Y, d) which has the same large scale geometry as a Gromov hyperbolic space has to be Gromov hyperbolic.

Given a finitely generated group Γ and a finite generating subset S of Γ, we say that Γ is *word hyperbolic* if its Cayley graph equipped with the word metric d_S induced from S is a *Gromov hyperbolic space*. Moreover, since Gromov hyperbolicity is invariant under quasi-isometries, the definition does not depend on the choice of the generating subset S of Γ. It is an immediate consequence of the Svarc-Milnor lemma that a group Γ is word hyperbolic if and only if it admits a proper discontinuous and cocompact action by isometries on a Gromov hyperbolic metric space.

The free group F_k of rank k, $k \geq 1$, is word hyperbolic since its Cayley graph, equipped with the word metric defined with respect to a set of free generators, is a tree and hence 0-hyperbolic. The fundamental group $\pi_1(\Sigma)$ of a closed orientable hyperbolic surface Σ is also word hyperbolic, since $\pi_1(\Sigma)$ admits a proper discontinuous and cocompact action by isometries on the upper half plane $\mathbb{H}^2_{\mathbb{R}} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped with the Poincare metric

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

of constant negative curvature -1.

Another large class of examples arises from fundamental groups of certain mapping tori of closed hyperbolic surfaces. Thurston, proved in [Thu98] (see also [Ota96]) that given a closed orientable hyperbolic surface Σ and an orientation preserving pseudo-Anosov diffeomorphism $\phi : \Sigma \to \Sigma$, the mapping torus $M_\phi = \Sigma \times [0,1]/\{(s,0) \sim (\phi(s),1)\}$ admits a hyperbolic structure. The fundamental group of M_ϕ, $\pi_1(M_\phi)$, is word hyperbolic since it admits a discrete, faithful and cocompact representation into $\mathsf{PSL}(2,\mathbb{C})$, the group of orientation preserving isometries of the 3-dimensional real hyperbolic space $\mathbb{H}^3_{\mathbb{R}} = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$ equipped with the Riemannian metric

$$(ds)^2 = \frac{(dx)^2 + (dy)^2 + (dz)^2}{z^2}$$

of constant negative curvature -1. More generally, uniform lattices (i.e. discrete and co-compact subgroups) of a simple Lie group G of real rank 1 (e.g. $G = \mathsf{SO}(n,1)$, $n \geq 1$) are word hyperbolic.

The Bestvina-Feighn combination theorem [BF+92] provides examples constructed as amalgamated free products and HNN extensions of word hyperbolic groups along certain subgroups. More precisely, given two word hyperbolic groups Γ_1 and Γ_2 and a quasiconvex malnormal subgroup Γ_0 of both Γ_1 and Γ_2, the amalgamated free product $\Gamma_1 *_{\Gamma_0} \Gamma_2$ is word hyperbolic. Other examples include certain mapping tori of free groups (see [BF+92, page 85]) and certain small cancellation groups.

The class of word hyperbolic groups generalizes, in many aspects, the class of fundamental groups of closed negatively curved Riemannian manifolds. Associated to every word hyperbolic group Γ there exists a finite dimensional and contractible simplicial complex, called the *Rips complex*, on which Γ acts properly discontinuously and cocompactly (see [BH13] for more details). Word hyperbolic groups are finitely presented and satisfy several remarkable algorithmic properties. For example, the word and conjugacy problems are solvable in the class of word hyperbolic groups (see for example [BH13, III. Γ.2]). However, besides the remarkable properties that word hyperbolic groups enjoy, there exist examples with features completely different from those of uniform lattices in linear rank 1 Lie groups. For instance, there are examples which fail to admit faithful representations into any matrix group (see the constructions of Kapovich [Kap05] and Canary-Stover-T. [CST19]). Moreover, there are examples of word hyperbolic groups capturing some of the pathologies occuring in non-hyperbolic finitely presented groups. The Rips construction [Rip82] and its generalizations (see [Wis03]) exhibit such examples.

1.2 Anosov representations of hyperbolic groups and a definition into special linear groups

Higher Teichmüller theory is the study of spaces of discrete and faithful representations of finitely generated groups into real Lie groups by using tools from several areas of Mathematics such as differential geometry, Lie theory, dynamical systems, geometric group theory and many more. The prototypical example of a collection of (classes of) discrete faithful representations is the Teichmüller space $\mathcal{T}(S)$, where S is a closed orientable hyperbolic surface of genus greater or equal than 2. The space $\mathcal{T}(S)$ can be identified with the set of discrete and faithful representations of the fundamental group $\pi_1(S)$ into $\mathsf{PSL}(2,\mathbb{R})$, up to conjugation by elements of the projective linear group $\mathsf{PGL}(2,\mathbb{R})$. Teichmüller space, and its quotient moduli space, play a central role in diverse fields including algebraic geometry, complex analysis, low-dimensional topology, dynamics and geometric group theory. Higher Teichmüller spaces can be thought as the higher rank analogue of the space $\mathcal{T}(S)$, i.e. a collection of connected components of the space of representations of a finitely generated infinite group Γ into a non-compact linear semisimple Lie group G.

An important class of discrete faithful representations into linear semisimple Lie groups are *Anosov representations*. Anosov representations of word hyperbolic groups form a rich and structurally stable class of discrete subgroups of real semisimple Lie groups with special dynamical and geometric properties, playing central role in the study of Higher Teichmuüler spaces. They were introduced by Labourie [Lab06] for fundamental groups of closed negatively curved Riemannian manifolds in his work on the Hitchin component. Guichard-Wienhard later extended Labourie's dynamical definition for more general word hyperbolic groups in [GW12] by using the Gromov geodesic flow space associated to a word hyperbolic group (see [Gro87], [Cha94], [Min05]). In several cases, Anosov representations of a word hyperbolic group Γ form an entire connected component of the representation variety $\mathrm{Hom}(\Gamma, G)$. For example, this is the case for Hitchin representations see [Lab06] and Benoist representations (see [Ben05]).

Labourie's dynamical definition of Anosov representations is inspired by the definition of an Anosov flow on a compact C^∞-manifold [Ano67]. The prototypical example of an Anosov flow is the geodesic flow $\{g_t\}_{t\in\mathbb{R}}$ on the unit tangent bundle T^1M of a closed Riemannian manifold M of negative sectional curvature established by Anosov in [Ano67]. More precisely, let us fix a Riemannian metric $g = ||\cdot||$ on the compact manifold $N := T^1M$. Then, there exist $\{g_t\}_{t\in\mathbb{R}}$-invariant sub-bundles E^u, E^0 and E^s of the tangent bundle TN with the following properties:

(i) E^0 is one dimensional and spanned by vector field on TN, $\hat{m} \mapsto \frac{d}{dt}g_t(\hat{m})\big|_{t=0}$, $\hat{m} \in N$.

(ii) $TN = E^u \oplus E^0 \oplus E^s$.

(iii) The geodesic flow on E^u (resp. E^s) is uniformly dilating (resp. uniformly contracting). In other words, there exist $C, c > 0$ such that for every $\hat{m} \in N$, $t \geq 0$, $v^+ \in E^u_{\hat{m}}$ and $v^- \in E^s_{\hat{m}}$:

$$\begin{aligned}
\Big|\Big|dg_{-t}(u^+)\Big|\Big|_{g_{-t}(\hat{m})} &\leq Ce^{-ct}\Big|\Big|u^+\Big|\Big|_{\hat{m}} \\
\Big|\Big|dg_{t}(u^-)\Big|\Big|_{g_{t}(\hat{m})} &\leq Ce^{-ct}\Big|\Big|u^-\Big|\Big|_{\hat{m}}
\end{aligned}$$

The Gromov geodesic flow $(\hat{\Gamma}, \varphi_t)$ associated to a word hyperbolic group Γ, introduced by Gromov [Gro87] (see also [Cha94] and [Min05] for other contructions), is a metric space on which Γ acts properly discontinuously and cocompactly by isometries and has similar properties as the unit tangent bundle of the universal cover of a closed negatively curved Riemannian manifold. For every Anosov representation $\rho : \Gamma \to G$ into a semisimple Lie group G, there exist explicit vector bundles $E^{\pm}_\rho$ over the flow space $(\hat{\Gamma}, \varphi_t)$, obtained from $\hat{\Gamma}$, ρ and the *Anosov limit maps* of ρ with the following properties (see [Lab06], [GW12], or subsection 2.5 for the precise definition):

(i) there exists a lift of the geodesic flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on $\hat{\Gamma}$ on the bundles E^+_ρ and E^-_ρ.

(ii) the geodesic flow on E^+_ρ (resp. E^-_ρ) is uniformly dilating (resp. uniformly contracting).

Now let us provide an alternative definiton of Anosov representations into the special linear groups $\mathsf{PSL}(d,\mathbb{R})$ and $\mathsf{SL}(d,\mathbb{R})$, in terms of eigenvalue and singular value gaps established by the work of [KLP18], [BPS16] and [KP20]. For an introduction to Anosov representations we also refer the reader to Canary's lecture notes [Can20].

Let us fix $d \geq 2$. For an element $g \in \mathsf{SL}(d,\mathbb{R})$, let $\lambda_1(g) \geq \lambda_2(g) \geq ... \geq \lambda_d(g)$ denote the logarithms of the moduli of the eigenvalues of g in non-increasing order (counting multiplicity). We denote by $\mu_1(g) \geq \mu_2(g) \geq ... \geq \mu_d(g)$ the logarithms of the singular values of g (defined by the relation $\mu_i(g) = \frac{1}{2}\lambda_i(gg^t)$) in non-increasing order. The definition of an Anosov representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ depends on the choice of a pair of opposite parabolic subgroups (P^+, P^-) of

$\mathsf{SL}(d,\mathbb{R})$. Up to conjugation, every pair of opposite parabolic subgroups (P^+,P^-) arises as the stabilizer of a k-plane and a complementary $(d-k)$-plane for some $1\leq k\leq \frac{d}{2}$. In this case, we say that ρ is *P_k-Anosov.*

Definition 1.2.1. ([KLP18], [BPS16], [KP20]) *Let Γ be a finitely generated group and $d_S:\Gamma\times\Gamma\to\mathbb{N}$ be a left invariant word metric on Γ induced by a finite generating subset S of Γ. For $\gamma\in\Gamma$ we set $|\gamma|_\Gamma=d(\gamma,e)$ and let $|\gamma|_{\Gamma,\infty}=\lim_{n\to\infty}\frac{|\gamma^n|_\Gamma}{n}$ be the stable translation length of γ. For $d\geq 2$, $1\leq k\leq\frac{d}{2}$ and a representation $\rho:\Gamma\to\mathsf{SL}(d,\mathbb{R})$ the following are equivalent:*

(i) *ρ is P_k-Anosov.*

(ii) *There exist constants $C,a>0$ with the property:*

$$\mu_k(\rho(\gamma))-\mu_{k+1}(\rho(\gamma))\geq a|\gamma|_\Gamma-C\quad\forall\gamma\in\Gamma.$$

(iii) *Γ is word hyperbolic and there exists $c>0$ with the property:*

$$\lambda_k(\rho(\gamma))-\lambda_{k+1}(\rho(\gamma))\geq c|\gamma|_{\Gamma,\infty}\quad\forall\gamma\in\Gamma.$$

The equivalence (i)$\Leftrightarrow$(ii) has been established by Kapovich-Leeb-Porti [KLP18] and Bochi-Potrie-Sambarino [BPS16]. The equivalence (iii)$\Leftrightarrow$(ii) was established by Kassel-Potrie [KP20] answering a question of Bochi-Potrie-Sambarino in [BPS16].

In general, discrete representations of word hyperbolic groups into semisimple Lie groups are not very well understood. Even restricting to the case of Anosov representations, there are several open questions concerning word hyperbolic groups admitting Anosov representations. Some restrictions on groups admitting projective Anosov representations into $\mathsf{SL}(3,\mathbb{R})$ and $\mathsf{SL}(4,\mathbb{R})$ have been estabilshed in [CT20]. However, there is no known classification of which word hyperbolic groups are P_1-Anosov into $\mathsf{SL}(d,\mathbb{R})$ for any $d\geq 5$. Moreover, as of now, there is no known example of a linear word hyperbolic group which fails to admit Anosov representations into any linear semisimple Lie group (see for example [Kas18, page 24]). One hope for the construction of such examples is to consider amalgamated free products involving super-rigid lattices in the rank 1 Lie group $\mathsf{Sp}(d,1)$, $d\geq 2$, following the point of view of the constructions in [CST19].

During the last decade several results have been established for Anosov representations including various characterizations completely different from Labourie's original dynamical definition [Gué+17], [KLP17], [KLP18], [KLP14], constructions of domains of discontinuity associated to Anosov representations [GW12], constructions of Riemannian metrics on higher Teichmüller spaces containing Anosov representations [Bri+15], analogues of the collar lemma for Hitchin and maximal representations (see [LZ17] and [BP17] respectively), generalizations of the notion of Anosov representation for relatively hyperbolic groups [KL18], [Zhu19] and many more.

Recently, the work of Danciger-Guéritaud-Kassel [DGK17] and Zimmer [Zim17] offers a connection of Anosov representations with real projective structures. More precisely, their work shows that, up to composing an Anosov representation with an explicit Lie group homomorphism, the image of the composition acts convex cocompactly on a properly convex domain of some real projective space. In particular, Anosov representations of word hyperbolic groups can be thought as a generalization of convex cocompact representations into simple rank 1 Lie groups.

1.3 Examples of Anosov representations

Let us now provide some examples of Anosov representations into $\mathsf{SL}(d,\mathbb{R})$. A linear representation $\rho:\Gamma\to\mathsf{SL}(d,\mathbb{R})$ is called *Borel Anosov* if ρ is P_k-Anosov for every $1\leq k\leq \frac{d}{2}$.

(i) *Convex cocompact subgroups of rank 1 Lie groups.* Let G be the full isometry group of the locally symmetric hyperbolic space $\mathbb{H}^m_{\mathbf{k}}$, where $m\geq 2$ and $\mathbf{k}=\mathbb{R},\mathbb{C}$ or the ring of Hamiltonian quaternions $\mathcal{H}$ (i.e. G is locally isomorphic to $\mathsf{SO}(m,1),\mathsf{SU}(m,1)$ or $\mathsf{Sp}(m,1)$). Let $x_0\in\mathbb{H}^m_{\mathbf{k}}$ be a fixed basepoint. The visual boundary $\partial_\infty\mathbb{H}^m_{\mathbf{k}}$ of the Gromov hyperbolic space $\mathbb{H}^m_{\mathbf{k}}$ is by definition the equivalence classes of geodesic rays starting at x_0, where two geodesics $\sigma_1,\sigma_2:[0,\infty)\to\mathbb{H}^m_{\mathbf{k}}$ are equivalent if there exists $M>0$ such that $d_{\mathbb{H}^m_{\mathbf{k}}}(\sigma_1(t),\sigma_2(t))\leq M$ for every $t\geq 0$. The visual boundary $\partial_\infty\mathbb{H}^m_{\mathbf{k}}$ is topologically the sphere of dimension $[\mathbf{k}:\mathbb{R}]m-1$. The topology on $\partial_\infty\mathbb{H}^m_{\mathbf{k}}$ is induced by a visual metric (see Section 2.2).

Given a subgroup Γ of G, the *limit set* Λ_Γ of Γ in $\mathbb{H}^m_{\mathbf{k}}$ is the set of accumulation points of orbits of Γ in the visual boundary $\partial_\infty\mathbb{H}^m_{\mathbf{k}}$. The subgroup $\Gamma\subset G$ is called *convex cocompact* if Γ is a discrete subgroup of G and acts cocompactly on the convex hull $\mathcal{C}$ of its limit set Λ_Γ into the space $\mathbb{H}^m_{\mathbf{k}}$. It is an immediate consequence of the Svarc-Milnor lemma (see [BH13]) that Γ is a word hyperbolic group and the inclusion $\Gamma\hookrightarrow G$ is a *quasi-isometric embedding*, i.e., after fixing a word metric d_S on Γ, there exist $C,c>0$ such that

$$d_{\mathbb{H}^m_{\mathbf{k}}}(\gamma x_0,x_0)\geq c|\gamma|_\Gamma - C \quad \forall\gamma\in\Gamma$$

and $|\gamma|_\Gamma := d_S(\gamma,e)$. The inclusion representation $\Gamma\hookrightarrow G$ is Anosov. We also remark that additional examples arise from convex cocompact subgroups of the isometry group of the octonionic hyperbolic plane $\mathbb{H}^2_{\mathbb{O}}$ equipped with the Killing metric.

In general, given a representation $\rho:\Gamma\to G$ into a linear simple Lie group G of real rank 1, the following are equivalent (for instance see [GW12, Theorem 1.8]).

(i) ρ is Anosov.

(ii) The kernel $\ker(\rho)$ is finite and $\rho(\Gamma)$ is a convex cocompact subgroup of G.

(iii) ρ is a quasi-isometric embedding, i.e there exist $A, a > 0$ such that

$$a|\gamma|_\Gamma + A \geq d_{\mathbb{H}^m_{\mathbf{k}}}(\rho(\gamma)x_0, x_0) \geq \frac{1}{a}|\gamma|_\Gamma - A$$

for every $\gamma \in \Gamma$.

(ii) *Johnson-Millson deformations.* We describe here the type of deformations defined in [JM87]. Let Γ be a torsion-free and cocompact lattice (i.e. discrete with compact quotient) of the Lie group $\mathsf{SO}(d,1)$. Suppose that Γ contains a subgroup Γ_0 which is a cocompact lattice of $\mathsf{SO}(d-1,1) \subset \mathsf{SO}(d,1)$ and $\Gamma_0\backslash\mathbb{H}^{d-1}_{\mathbb{R}} \hookrightarrow \Gamma\backslash\mathbb{H}^d_{\mathbb{R}}$ is a totally geodesic embedding. The van Kampen theorem implies that the fundamental group of the compact hyperbolic d-manifold $\Gamma\backslash\mathbb{H}^d_{\mathbb{R}}$, Γ, splits as an amalgamated free product $A *_{\Gamma_0} B$ in the non-separating case, or as an HNN extension $C*_{\Gamma_0=\phi(\Gamma_0)} = \langle C, s \mid s\gamma s^{-1} = \phi(\gamma),\ \gamma \in \Gamma_0\rangle$ for some monomorphism $\phi : \Gamma_0 \hookrightarrow C$ in the separating case. Let ρ denote the inclusion of Γ in $\mathsf{SO}(d,1)$. Let us also fix an one parameter subgroup $\{c_t\}_{t\in\mathbb{R}}$ in $\mathsf{SL}(d+1,\mathbb{R})$ centralizing Γ_0 and consider the following family of deformations $\{\rho_t : \Gamma \to \mathsf{SL}(d+1,\mathbb{R})\}_{t\in\mathbb{R}}$ of ρ, defined as follows

$$\rho_t(\gamma) = \begin{cases} \rho(\gamma),\ \gamma \in A \\ c_t\rho(\gamma)c_t^{-1},\ \gamma \in B \end{cases} \text{ and } \rho_t(\gamma) = \begin{cases} \rho(s)c_t,\ \gamma = s \\ \rho(\gamma),\ \gamma \in C \end{cases}$$

in the non-separating and separating case respectively. The representation ρ is P_1-Anosov, so for small enough values of t, the representation ρ_t remains P_1-Anosov by the stability of Anosov representations (see [Lab06] and [GW12, Theorem 5.12]). In fact, Benoist's theorem in [Ben05] shows that ρ_t is P_1-Anosov for every $t \in \mathbb{R}$.

The previous examples are P_1-Anosov. It is also possible to produce P_k-Anosov examples for $k \neq 1$. For example, let $\Gamma \hookrightarrow \mathsf{SL}(2,\mathbb{C})$ be a torsion-free uniform lattice containing a separating totally geodesic surface Σ and, up to conjugation, we may assume that $\pi_1(\Sigma)$ is a convex cocompact subgroup of $\mathsf{SL}(2,\mathbb{R})$. Note that Γ splits as an amalgamated free product $\Gamma = A *_{\pi_1(\Sigma)} B$. Now consider the Lie group homomorphism $\tau_2 : \mathsf{SL}(2,\mathbb{C}) \hookrightarrow \mathsf{SL}(4,\mathbb{R})$ defined as follows:

$$\tau_2(g) = \begin{bmatrix} \mathrm{Re}(g) & -\mathrm{Im}(g) \\ \mathrm{Im}(g) & \mathrm{Re}(g) \end{bmatrix},\ g \in \mathsf{SL}(2,\mathbb{C}).$$

Observe that the one parameter subgroup $\{c_s\}_{s\in\mathbb{R}}$, $c_s = \mathrm{diag}(e^s, e^s, e^{-s}, e^{-s})$, centralizes $\tau_2(\mathsf{SL}(2,\mathbb{R}))$. The Johnson-Millson deformations $\{\rho_q : \Gamma \to \mathsf{SL}(4,\mathbb{R})\}_{q\in\mathbb{R}}$ of $\tau_2 : \Gamma \to \mathsf{SL}(4,\mathbb{R})$ defined as

$$\rho_q(\gamma) = \begin{cases} \tau_2(\gamma),\ \gamma \in A \\ c_q\tau_2(\gamma)c_q^{-1},\ \gamma \in B \end{cases}$$

are Zariski dense and P_2-Anosov in $\mathsf{SL}(4,\mathbb{R})$ as soon as $0 < |q| < \varepsilon$ and $\varepsilon > 0$ is small enough.

(iii) *Hitchin representations.* For $d \geq 3$, let $\mathrm{i}_d : \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ be the unique, up to conjugation, irreducible representation. For a closed orientable hyperbolic surface S, a representation $\rho : \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ is called *Fuchsian*, if $\rho = \mathrm{i}_d \circ j$ for some discrete faithful representation $j : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$. A continuous deformation of a Fuchsian representation of $\pi_1(S)$ is called a *Hitchin representation.* The Hitchin component(s) $\mathcal{H}_d(S)$ is (are) by definition the set of Hitchin representations of $\pi_1(S)$ into $\mathsf{PSL}(d,\mathbb{R})$ up to conjugation by elements of $\mathsf{PSL}(d,\mathbb{R})$. Hitchin proved in [Hit92] that each connected component of $\mathcal{H}_d(S)$ is a real analytic manifold diffeomorphic to $\mathbb{R}^{(d^2-1)|\chi(S)|}$.

By using Definition 2.5(ii) and the fact that j is convex cocompact in $\mathsf{PSL}(2,\mathbb{R})$, we may deduce that $\mathrm{i}_d \circ \rho$ is Borel Anosov. Labourie in [Lab06] showed that every Hitchin representation is Borel Anosov and established strong transversality properties for their Anosov limit maps. In particular, it follows by Labourie's work that $\mathcal{H}_d(S)$ is an example of a higher Teichmüller space.

(iv) *Barbot type representations.* Let $j_0 : \mathsf{SL}(2,\mathbb{R}) \hookrightarrow \mathsf{SL}(3,\mathbb{R})$ be the reducible embedding defined as $j_0(g) = \mathrm{diag}(g, 1)$, $g \in \mathsf{SL}(2,\mathbb{R})$. Barbot proved in [Bar10] that for every convex cocompact representation $\rho : \pi_1(S) \to \mathsf{SL}(2,\mathbb{R})$ the composition $j_0 \circ \rho$ is P_1-Anosov. By using Definition 1.2.1 (iii), we may check that the product $(\mathrm{i}_{2k+1} \circ \rho) \times (\mathrm{i}_{2m} \circ \rho) : \pi_1(S) \to \mathsf{SL}(2k+2m+1,\mathbb{R})$ is Borel Anosov for every $k, m \in \mathbb{N}$. Moreover, by stability, nearby deformations of $(\mathrm{i}_{2k+1} \circ \rho) \times (\mathrm{i}_{2m} \circ \rho)$ are also Borel Anosov.

We would like to remark that the two known classes of torsion-free word hyperbolic groups which are known to admit Borel Anosov representations are free groups and surface groups. To us, the only know examples of Borel Anosov representations of surface groups are either Hitchin representations or Barbot type representations.

(v) *Benoist representations.* Another large class of Anosov representations arises from strictly convex real projective structures on closed manifolds. A domain Ω of the real projective space $\mathbb{P}(\mathbb{R}^d)$ is called *properly convex* if it is a bounded and convex domain contained in an affine chart of $\mathbb{P}(\mathbb{R}^d)$ (i.e. the complement of a projective $(d-1)$-plane) and Ω is called *strictly convex* if additionally its boundary $\partial\Omega$ does not contain projective line segments. Benoist in [Ben04] proved that every discrete subgroup Γ of $\mathsf{SL}(d,\mathbb{R})$, $d\geq 3$, acting cocompactly on a strictly convex domain Ω of the real projective space $\mathbb{P}(\mathbb{R}^d)$ is word hyperbolic and the Hilbert geodesic flow on $\Gamma\backslash T^1\Omega$ is Anosov [Ben04]. The inclusion $\Gamma\hookrightarrow\mathsf{SL}(d,\mathbb{R})$ is called a *Benoist representation* and is P_1-Anosov (see [GW12, Prop. 6.1]). However, for every $2\leq k\leq\frac{d}{2}$, a Benoist representation into $\mathsf{SL}(d,\mathbb{R})$ is not P_k-Anosov [CT20, Corollary 1.4]. Moreover, Benoist proved in [Ben05] that the set of Benoist representations in the connected component $\mathcal{C}_i$ of i in $\mathrm{Hom}(\Gamma,\mathsf{SL}(d,\mathbb{R}))$ is closed. Benoist's closedness result, combined with Koszul's openess theorem [Kos68], implies that $\mathcal{C}_i$ contains entirely Benoist representations. In particular, $\mathcal{C}_i$ is a higher Teichmüller space.

1.4 Statement of the results

In this thesis, we present the main results obtained by the author in [Tso20a], [Tso20b] and a characterization of Benoist representations established by Richard Canary and the author in [CT20] in terms of limit maps.

First, we provide characterizations of Anosov representations in the spirit of the characterizations of Guéritaud-Guichard-Kassel-Wienhard [Gué+17] and Kapovich-Leeb-Porti [KLP17], [KLP14], in terms of the existence of limit maps, the Cartan property and the Lyapunov and Cartan projection. We use our main result in order to obtain characterizations of strongly convex cocompact subgroups of the projective linear group $\mathsf{PGL}(d,\mathbb{R})$. We also study the relation between weak Property U, introduced by Kassel-Potrie in [KP20], and the uniform gap summation property introduced in [Gué+17]. In particular, we provide conditions for a linear representation of a finitely generated group with weak uniform gaps in eigenvalues to be Anosov. We also introduce a Gromov product associated to the linear forms on the Cartan projection of a representation. For a representation satisfying the uniform gap summation property of [Gué+17] we compare the Gromov product on the Cartan projection of its image with the usual Gromov product on the domain group. We also compute the Hölder exponents of the limit maps of an Anosov representation in terms of the Cartan and Lyapunov projection of the image. Furthermore, by using a result of Benoist in [Ben00], we provide a complete characterization of the domain groups of Borel Anosov representations into the projective linear group $\mathsf{PSL}(4q+2,\mathbb{R})$, $q\geq 2$, answering in the affirmative a question of Andrés Sambarino.

For some background on Lie theory and word hyperbolic groups, we refer the reader to Section II. We briefly provide some notation here in order to state our main results. Let Γ be an infinite word hyperbolic group, G be a linear, non-compact semisimple Lie group with finitely many connected components and fix K a maximal compact subgroup of G. We also fix a Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$, $\overline{\mathfrak{a}}^+$ a closed Weyl chamber of $\mathfrak{a}$, a Cartan decomposition $G=K\exp(\overline{\mathfrak{a}}^+)K$ and consider the Cartan projection $\mu : G \to \overline{\mathfrak{a}}^+$. Given a linear form $\alpha \in \mathfrak{a}^*$, we set $\alpha(H) = \langle \alpha, H\rangle$. Every subset $\theta \subset \Delta$ of simple restricted roots of G defines a pair of opposite parabolic subgroups P_θ^+ and P_θ^-, well defined up to conjugation.

Let $\rho : \Gamma \to G$ be a representation and $\xi^+ : \partial_\infty\Gamma \to G/P_\theta^+$ and $\xi^- : \partial_\infty\Gamma \to G/P_\theta^-$ be two ρ-equivariant maps. The maps ξ^+ and ξ^- are called *transverse* if for any two distinct points $x^+, x^- \in \partial_\infty\Gamma$ there exists $g \in G$ such that $\xi^+(x^+) = gP_\theta^+$ and $\xi^-(x^-) = gP_\theta^-$. The representation ρ is called P_θ-divergent if for any infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ and $\alpha \in \theta$, the sequence $(\langle \alpha, \mu(\rho(\gamma_n))\rangle)_{n\in\mathbb{N}}$ is unbounded. The map $\xi^+ : \partial_\infty\Gamma \to G/P_\theta^+$ satisfies the *Cartan property* if for any sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ which converges to a point $x \in \partial_\infty\Gamma$ in the Gromov boundary $\partial_\infty\Gamma$ we have $\xi^+(x) = \lim_n k_{\rho(\gamma_n)}P_\theta^+$, where $\rho(\gamma_n) = k_{\rho(\gamma_n)} \exp(\mu(\rho(\gamma_n))k'_{\rho(\gamma_n)}$ is written in the Cartan decomposition of G. The limit maps of an Anosov representation (see sub-section 2.5 for the definition) satisfy the Cartan property, see [Gué+17, Theorem 1.3 (4) & 5.3 (4)]. We discuss the Cartan property in more detail in Section 3.2, where we prove (see Corollary 3.2.5) that for a Zariski dense representation $\rho : \Gamma \to G$ a (necessarily unique) continuous ρ-equivariant map ξ has to satisfy the Cartan property.

Our first characterization of Anosov representations is based on the existence of a pair of transverse limit maps, where one of the limit maps satisfies the Cartan property:

Theorem 1.4.1. *Let Γ be a word hyperbolic group, G a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G and $\rho : \Gamma \to G$ a representation. Then ρ is P_θ-Anosov if and only if the following conditions hold:*

(i) *ρ is P_θ-divergent.*

(ii) *There exists a pair of continuous, ρ-equivariant transverse maps*

$$\xi^+ : \partial_\infty\Gamma \to G/P_\theta^+ \quad \text{and} \quad \xi^- : \partial_\infty\Gamma \to G/P_\theta^-$$

and the map ξ^+ satisfies the Cartan property.

We explain how Theorem 1.4.1 is related with [KLP14, Theorem 1.7], [KLP17, Theorem 5.47] and [Gué+17, Theorem 1.3] in sub-section 1.4.6. From Theorem 1.4.1 we deduce the following characterization of Anosov representations entirely from the Cartan projection of the image of the representation.

Corollary 1.4.2. *Let Γ be an infinite word hyperbolic group, G a real semisimple Lie group with Cartan projection $\mu : G \to \overline{\mathfrak{a}}^+$, $\theta \subset \Delta$ a subset of simple restricted roots of G, $\{\omega_\alpha\}_{\alpha\in\theta}$ the* *associated set of fundamental weights and $\rho : \Gamma \to G$ a representation. We fix $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ a left invariant word metric on Γ. The representation ρ is P_θ-Anosov if and only if the following conditions are simultaneously satisfied:*

(i) *There exist $C, c > 1$ such that*

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \geqslant c \log |\gamma|_\Gamma - C$$

for every $\gamma \in \Gamma$ and $\alpha \in \theta$.

(ii) *There exist $A, a > 0$ such that*

$$\langle \omega_\alpha, 2\mu(\rho(\gamma)) - \mu(\rho(\gamma^2)) \rangle \leqslant A \cdot (2|\gamma|_\Gamma - |\gamma^2|_\Gamma) + a$$

for every $\gamma \in \Gamma$ and $\alpha \in \theta$.

For a group Γ, we denote by Γ_∞ the set of infinite order elements of Γ. For two linear representations $\rho_1 : \Gamma \to \mathsf{SL}(m, \mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{SL}(d, \mathbb{R})$, where ρ_2 is P_1-Anosov, we define

$$\upsilon_-(\rho_1, \rho_2) := \inf_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho_1(\gamma))}{\lambda_1(\rho_2(\gamma))} \quad \text{and} \quad \upsilon_+(\rho_1, \rho_2) := \sup_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho_1(\gamma))}{\lambda_1(\rho_2(\gamma))}$$

By using Definition 1.2.1 (iii), we may see that there exists $a > 0$ such that $\lambda_1(\rho_2(\gamma)) \geq a|\gamma|_\infty$ for every $\gamma \in \Gamma$ and hence the previous two quantities are well defined. As an application of Theorem 1.4.1 we obtain the following approximation result in Section 3.8 which refines the density result of Benoist obtained in [Ben+00].

Corollary 1.4.3. *Let Γ be a word hyperbolic group and $\rho_1 : \Gamma \to \mathsf{SL}(m, \mathbb{R})$, $\rho_2 : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ be two representations. Suppose that ρ_2 is P_1-Anosov and ρ_1 satisfies one of the following conditions:*
(i) *ρ_1 is P_1-Anosov.*
(ii) *$\rho_1(\Gamma)$ is contained in a semisimple P_1-proximal Lie subgroup of $\mathsf{SL}(m, \mathbb{R})$ of real rank 1.*

Then for any $\delta > 0$ and $p, q \in \mathbb{N}$ with $\upsilon_-(\rho_1, \rho_2) \leqslant \frac{p}{q} \leqslant \upsilon_+(\rho_1, \rho_2)$, there exists an infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of Γ such that

$$\left| \frac{p}{q} - \frac{\mu_1(\rho_1(\gamma_n))}{\mu_1(\rho_2(\gamma_n))} \right| \leqslant \frac{\delta}{q} \cdot \frac{\log |\gamma_n|_\Gamma}{|\gamma_n|_\Gamma}$$

for every $n \in \mathbb{N}$.

Now let $\rho : \Gamma \to G$ be a Zariski dense representation which admits a pair of ρ-equivariant limit maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta^+$ and $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$. In [GW12, Theorem 5.11], Guichard-Wienhard proved that ρ is P_θ-Anosov if and only if ξ^+ and ξ^- are compatible and transverse. By using Theorem 1.4.1 and Corollary 3.2.5 we obtain a generalization of [GW12, Theorem 5.11]. For a quasi-convex subgroup H of Γ (see Definitions 2.2.6 (ii)) we denote by $\iota_H : \partial_\infty H \hookrightarrow \partial_\infty \Gamma$ the Cannon-Thurston map extending the inclusion $H \hookrightarrow \Gamma$.

Theorem 1.4.4. *Let Γ be a word hyperbolic group, H a quasiconvex subgroup of Γ, G a semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G and $\rho : \Gamma \to G$ a Zariski dense representation. Suppose that ρ admits continuous ρ-equivariant maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta^+$ and $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$. Then $\rho|_H$ is P_θ-Anosov if and only if the maps $\xi^+ \circ \iota_H$ and $\xi^- \circ \iota_H$ are transverse.*

In Theorem 1.4.1 we do not assume that the image $\rho(\Gamma)$ contains a proximal element in $G/P_\theta^\pm$ or that the pair of maps (ξ^+, ξ^-) is compatible at some point $x \in \partial_\infty \Gamma$, i.e. $\mathrm{Stab}_G(\xi^+(x)) \cap \mathrm{Stab}_G(\xi^-(x))$ is a parabolic subgroup of G. Under the assumption that both maps (ξ^+, ξ^-) satisfy the Cartan property, Theorem 1.4.1 also follows from [KLP14, Theorem 1.7].

1.4.1 Strongly convex cocompact groups

Classical examples of Anosov representations arise from convex cocompact subgroups of real rank 1 simple Lie groups and their nearby deformations (e.g. Johnson-Millson deformations) into higher rank Lie groups. In general, Anosov representations into higher rank semisimple Lie groups (e.g. $\mathsf{SL}(d, \mathbb{R})$, $d \geq 3$) have many connections with real projective geometry and geometric structures. Let us fix an integer $d \geqslant 3$. Let Γ be a discrete subgroup of $\mathsf{PGL}(d, \mathbb{R})$ which preserves a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$. The *full orbital limit set* $\Lambda_\Omega(\Gamma)$ of Γ in Ω is the set of accumulation points of all Γ-orbits in $\partial\Omega$ (see [DGK17, Definition 1.10]). The group Γ acts *convex cocompactly on* Ω if the convex hull of $\Lambda_\Omega(\Gamma)$ in Ω is non-empty and has compact quotient by Γ (see [DGK17, Definition 1.11]). The group Γ is called *strongly convex cocompact in* $\mathbb{P}(\mathbb{R}^d)$ if it acts convex cocompactly on some properly convex domain Ω with strictly convex and C^1-boundary. The work of Danciger-Guéritaud-Kassel [DGK17] and Zimmer [Zim17] shows that Anosov representations can be realized as convex cocompact actions on properly convex domains of real projective spaces. More precisely, suppose that $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ is a P_1-Anosov representation with Anosov limit map $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$ and suppose that $\rho(\Gamma)$ preserves a properly convex domain of $\mathbb{P}(\mathbb{R}^d)$. Then there exists a $\rho(\Gamma)$-invariant properly convex domain Ω on which $\rho(\Gamma)$ acts convex cocompactly. The proximal limit $\Lambda_{\rho(\Gamma)}^{\mathbb{P}}$ (see sub-section 2.1.2) of $\rho(\Gamma)$ in $\mathbb{P}(\mathbb{R}^d)$ is exactly $\xi(\partial_\infty \Gamma) \subset \partial\Omega$, and $\rho(\Gamma)$ acts cocompactly on the convex hull of $\xi(\partial_\infty \Gamma)$ in Ω.

The following result of Danciger-Guéritaud-Kassel [DGK17] offers a connection between Anosov representations and strongly convex cocompact actions:

Theorem 1.4.5. ([DGK17, Theorem 1.4]) *Let Γ be an infinite discrete subgroup of $\mathsf{PGL}(d,\mathbb{R})$ which preserves a properly convex domain of $\mathbb{P}(\mathbb{R}^d)$. Then Γ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ if and only if Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ is P_1-Anosov.*

For a properly convex domain Ω let d_Ω be the Hilbert metric defined on Ω. By using Theorem 1.4.1 we prove the following geometric characterization of strongly convex cocompact subgroups of $\mathsf{PGL}(d,\mathbb{R})$ which are semisimple, i.e. their Zariski closure in $\mathsf{PGL}(d,\mathbb{R})$ is a reductive Lie group.

Theorem 1.4.6. *Let Γ be a finitely generated subgroup of $\mathsf{PGL}(d,\mathbb{R})$. Suppose that Γ preserves a strictly convex domain of $\mathbb{P}(\mathbb{R}^d)$ with C^1-boundary and the natural inclusion $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ is semisimple. Then the following conditions are equivalent:*

(i) Γ *is strongly convex cocompact in* $\mathbb{P}(\mathbb{R}^d)$.

(ii) $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ *is a quasi-isometric embedding,* Γ *preserves a properly convex domain* Ω *of* $\mathbb{P}(\mathbb{R}^d)$ *and there exists a* Γ*-invariant closed convex subset* $\mathcal{C}$ *of* Ω *such that* $(\mathcal{C}, d_\Omega)$ *is Gromov hyperbolic.*

The previous theorem generalizes the well known fact that a subgroup Γ of $\mathsf{PO}(n,1)$, $n \geq 2$, is convex cocompact if and only if its is quasi-isometrically embedded in $\mathsf{PO}(n,1)$. We remark that the assumption that the inclusion $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ is a quasi-isometric embedding cannot be dropped.

For a torsion-free word hyperbolic group Γ, $\mathrm{cd}(\Gamma)$ denotes the cohomological dimension of Γ. Note that given a convex a representation $\rho : \Gamma \to \mathsf{GL}(d,\mathbb{R})$ whose image acts convex cocompactly on a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$ is a Benoist representation if and only if $\mathrm{cd}(\Gamma) = d-1$. We also obtain the following characterization of P_1-Anosov representations into $\mathsf{GL}(d,\mathbb{R})$ whose domain group is of cohomolical dimension at least $d-1$.

Theorem 1.4.7. ([CT20, Theorem 1.5 & 1.7]). *Let Γ be a torsion free word hyperbolic group of cohomological dimension at least $d-1 \geq 3$ and suppose that $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a representation. The following conditions are equivalent:*

(i) ρ *is a Benoist representation.*

(ii) ρ *is* P_1*-Anosov.*

(iii) *There exists a non-constant continuous* ρ*-equivariant map* $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$.

1.4.2 Gromov products.

We also give the following definition of a Gromov product on $G\times G$ which we use in the proofs of our previous characterization (see Lemma 3.7.1).

Definition 1.4.8. *Let G be a real semisimple Lie group, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ and let $\mu : G \to \overline{\mathfrak{a}}^+$ be the Cartan projection. For every linear form $\varphi \in \mathfrak{a}^*$ the map $(\cdot)_\varphi : G\times G \to \mathbb{R}$ is called the Gromov product relative to φ and is defined as follows: for $g, h \in G$*

$$(g\cdot h)_\varphi := \frac{1}{4}\Big\langle \varphi, \mu(g)+\mu(g^{-1})+\mu(h)+\mu(h^{-1})-\mu(g^{-1}h)-\mu(h^{-1}g)\Big\rangle$$

We prove that for every P_θ-Anosov representation $\rho : \Gamma \to G$, the restriction of the Gromov product on $\rho(\Gamma)\times\rho(\Gamma)$, with respect to a fundamental weight ω_α for $\alpha\in\theta$, is comparable with the usual Gromov product on $\Gamma\times\Gamma$. We fix $|\cdot|_\Gamma : \Gamma\to\mathbb{N}$ a left invariant word metric on Γ and for $\gamma\in\Gamma$ and recall that $|\gamma|_{\Gamma,\infty} = \lim_n \frac{|\gamma^n|_\Gamma}{n}$ denotes the stable translation length of γ.

Proposition 1.4.9. *Let G be a real semisimple Lie group, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ and let $\mu : G\to\overline{\mathfrak{a}}^+$ and $\lambda : G\to\overline{\mathfrak{a}}^+$ be the Cartan and Lyapunov projections respectively. We fix $\theta\subset\Delta$ a subset of simple restricted roots of G and $\{\omega_\alpha\}_{\alpha\in\theta}$ the associated set of fundamental weights. Suppose that Γ is a word hyperbolic group and $\rho:\Gamma\to G$ is a P_θ-Anosov representation. There exist constants $C, c>0$ such that*

(i) $\frac{1}{C}(\gamma\cdot\delta)_e - c \leqslant \big(\rho(\gamma)\cdot\rho(\delta)\big)_{\omega_\alpha} \leqslant C(\gamma\cdot\delta)_e + c$ *for every $\alpha\in\theta$ and $\gamma,\delta\in\Gamma$.*

(ii) *If $\theta=\Delta$, then $\frac{1}{C}(\gamma\cdot\delta)_e - c \leqslant \big(\rho(\gamma)\cdot\rho(\delta)\big)_{\alpha} \leqslant C(\gamma\cdot\delta)_e + c$ for every $\alpha\in\Sigma^+$ and $\gamma,\delta\in\Gamma$.*

(iii) $\frac{1}{C}\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big) - c \leqslant \langle\omega_\alpha, \mu(\rho(\gamma))-\lambda(\rho(\gamma))\rangle \leqslant C\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)+c$ *for every $\alpha\in\theta$ and $\gamma\in\Gamma$.*

We remark that in the case where $\omega_\alpha=\varepsilon_1$, statement (i) of the previous proposition is not enough to guarantee that ρ is a P_1-Anosov representation (see Example 3.10.3). However, ρ is P_1-Anosov if we additionally assume that $\rho(\Gamma)$ preserves a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$ with strictly convex and C^1-boundary (see Proposition 3.7.1). Proposition 1.4.9 is proved as follows: by [Gué+17, Proposition 1.8] we may replace ρ with its semisimplification, then we compare the Gromov product relative to the fundamental weight $\{\omega_\alpha\}_{\alpha\in\theta}$ with the Gromov product with respect to the Hilbert metric d_Ω for some properly convex domain and then use Theorem 1.4.5.

1.4.3 Eigenvalue gaps and the uniform gap summation property

Kassel-Potrie introduced the following definition in [KP20]:

Definition 1.4.10. *Let Γ be a finitely generated group, $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ be a representation and fix $1 \leqslant i \leqslant d-1$. The representation ρ has a weak uniform i-gap in eigenvalues if there exists $c > 0$ such that*

$$\lambda_i(\rho(\gamma)) - \lambda_{i+1}(\rho(\gamma)) \geqslant c|\gamma|_{\Gamma,\infty}$$

for every $\gamma \in \Gamma$.

Guéritaud-Guichard-Kassel-Wienhard in [Gué+17, Theorem 1.7 (c)] proved that if Γ is word hyperbolic, ρ has a uniform i-gap into eigenvalues and admits a pair of continuous, ρ-equivariant, dynamics preserving and transverse maps $\xi^+ : \partial_\infty\Gamma \to \mathsf{Gr}_i(\mathbb{R}^d)$ and $\xi^- : \partial_\infty\Gamma \to \mathsf{Gr}_{d-i}(\mathbb{R}^d)$, then ρ is P_i-Anosov. Kassel-Potrie proved (see [KP20, Proposition 4.12]) that if Γ satisfies weak Property U (see Definition 3.5.1) and ρ has a weak uniform i-gap in eigenvalues, then ρ has a strong i-gap in singular values: there exist $L, \ell > 0$ such that $\mu_i(\rho(\gamma)) - \mu_{i+1}(\rho(\gamma)) \geqslant \ell|\gamma|_\Gamma - L$ for every $\gamma \in \Gamma$. The work of Kapovich-Leeb-Porti [KLP18] and Bochi-Potrie-Sambarino [BPS16] then shows that Γ is word hyperbolic and ρ is P_i-Anosov.

A function $f : \mathbb{N} \to \mathbb{R}^+$ satisfying the following two conditions:

(i) $\sum_{n=1}^{\infty} f(n) < +\infty$.

(ii) there exists $m > 0$ such that $f(k+1) \leqslant f(k) \leqslant mf(k+1)$ for every $k \in \mathbb{N}$,

is called a *Floyd function*. For the statement of our next theorem, we need the following definition (see also [Gué+17, Definition 5.2] and Definition 3.2.6 for a more general definition).

Definition 1.4.11. *Let Γ be a finitely generated group, $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ be a representation. We say that ρ satisfies the uniform gap summation property if there exists $C > 0$, a Floyd function f (e.g. $f(x) = x^{-1-\kappa}$, $\kappa > 0$) and $1 \leq k \leq \frac{d}{2}$ such that*

$$\mu_k(\rho(\gamma)) - \mu_{k+1}(\rho(\gamma)) \geqslant -\log\left(f(|\gamma|_\Gamma)\right) - C \quad \forall \gamma \in \Gamma.$$

The following theorem, motivated by [KP20, Question 4.9], provides further conditions under which a representation $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ with a weak i-uniform gap in eigenvalues is P_i-Anosov.

Theorem 1.4.12. *Let Γ be a non-virtually cyclic finitely generated group and $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ be a left invariant word metric on Γ. Suppose that $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ is a representation which has a weak uniform i-gap in eigenvalues for some $1 \leqslant i \leqslant d-1$. Then the following are equivalent:*

(i) Γ *is word hyperbolic and ρ is P_i-Anosov.*

(ii) *There exists a Floyd function $f : \mathbb{N} \to \mathbb{R}^+$ such that the Floyd boundary $\partial_f\Gamma$ of Γ contains at least three points.*

(iii) Γ *admits a representation* $\rho_1 : \Gamma \to \mathsf{GL}(m, \mathbb{R})$ *satisfying the uniform gap summation property.*

(iv) Γ *admits a semisimple representation* $\rho_2 : \Gamma \to \mathsf{GL}(p, \mathbb{R})$ *with the property*

$$\lim_{|\gamma|_\Gamma \to \infty} \frac{\mu_1(\rho_2(\gamma)) - \mu_p(\rho_2(\gamma))}{\log |\gamma|_\Gamma} = +\infty$$

We prove that conditions (ii), (iii) and (iv) imply that Γ satisfies weak Property U, so (i) follows by [KP20, Proposition 1.2]. In particular, in Theorem 3.5.3 we prove that every finitely generated group with non-trivial Floyd boundary has to satisfy weak Property U.

1.4.4 Hölder exponents

Sambarino in [Sam16] used the Hölder exponent of the Anosov limit maps in order to provide upper bounds for the entropy of a P_1-Anosov representation of the fundamental group of a hyperbolic manifold. Let (X_1, d_1) and (X_2, d_2) be two metric spaces and $f : (X_1, d_1) \to (X_2, d_2)$ be a Hölder continuous map. The Hölder exponent $a_f(d_1, d_2)$ of f is defined as the supremum among all numbers $\alpha > 0$ such that there exists $C > 0$ with $d_2(f(x), f(y)) \leqslant C \cdot d_1(x, y)^\alpha$ for every $x, y \in X_1$. We give a computation of the Hölder exponent of the Anosov limit map ξ of a (not necessarily semisimple) P_1-Anosov representation of a word hyperbolic group in the following theorem:

Theorem 1.4.13. *Let (X, d) be a Gromov hyperbolic space and let Γ be a word hyperbolic group acting properly discontinuously and cocompactly on X by isometries. We fix $x_0 \in X$ and $a > 0$ such that there exists a visual metric d_a on $\partial_\infty X$ with $d_a(x, y) \asymp a^{-(x \cdot y)_{x_0}}$ for $x, y \in \partial_\infty X$. Suppose that $q \geqslant 2$ and $\rho : \Gamma \to \mathsf{SL}(q, \mathbb{R})$ is a P_1-Anosov representation whose Anosov limit map $\xi : (\partial_\infty X, d_a) \to (\mathbb{P}(\mathbb{R}^q), d_\mathbb{P})$ is spanning. Then*

$$\alpha_\xi(d_a, d_\mathbb{P}) = \frac{1}{\log a} \cdot \sup_{n \geqslant 1} \inf_{|\gamma|_X \geqslant n} \frac{\mu_1(\rho(\gamma)) - \mu_2(\rho(\gamma))}{|\gamma|_X}$$

where $|\gamma|_X = d(\gamma x_0, x_0)$.

In Theorem 1.4.13, $d_\mathbb{P}$ denotes the angle metric defined as $d_\mathbb{P}([u], [v]) = \big|\sin \measuredangle(u, v)\big|$ for $u, v \in \mathbb{R}^d$ unit vectors. Moreover, in the case where ρ is irreducible we may replace the singular values with eigenvalues in the previous formula (see Corollary 3.9.2).

Now let us fix the visual metric on the real hyperbolic space $\mathbb{H}^d_\mathbb{R}$ defined as follows: $d_{\mathrm{v}}(x, y) \asymp e^{-(x \cdot y)}$ for $x, y \in \partial_\infty \mathbb{H}^d$ (see [BH13, Chapter III.H.3]). We remark that every Anosov subgroup Γ of

$\mathsf{PO}(d,1)$ is quasi-isometrically embedded and acts convex cocompactly on the real hyperbolic space $\mathbb{H}^d_{\mathbb{R}}$. As a corollary of the previous theorem we have:

Corollary 1.4.14. *Let $d \geqslant 2$ and Γ be a convex cocompact subgroup of $\mathsf{PO}(d,1)$ with limit set $\Lambda_\Gamma \subset \partial_\infty \mathbb{H}^d$. Suppose that $q \geqslant 2$ and $\rho : \Gamma \to \mathsf{PGL}(q,\mathbb{R})$ is an irreducible P_1-Anosov representation with Anosov limit map $\xi : (\Lambda_\Gamma, d_{\mathrm{v}}) \to (\mathbb{P}(\mathbb{R}^q), d_{\mathbb{P}})$. Then we have*

$$\alpha_\xi(d_{\mathrm{v}}, d_{\mathbb{P}}) = \inf_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma))}{\ell_{\mathbb{H}^d}(\gamma)}$$

where $\ell_{\mathbb{H}^d}(\gamma)$ is the translation length of γ and $\Gamma_\infty \subset \Gamma$ denotes the set of all infinite order elements of Γ. Moreover, if Γ is a cocompact lattice in $\mathsf{PO}(d,1)$, then for every $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$\lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)) \leqslant (1+\varepsilon) \cdot \ell_{\mathbb{H}^d}(\gamma)$$

1.4.5 Borel Anosov representations in even dimensions

Let us recall that a representation $\rho : \Gamma \to \mathsf{GL}(d,\mathbb{R})$ is called *Borel Anosov* if ρ is P_k-Anosov for every $1 \leq k \leq \frac{d}{2}$. We address the following question of Andrés Sambarino and provide a positive answer when $d = 4q+2$ and $q \in \mathbb{N}$.

Sambarino's Question: *Suppose that Γ is a torsion free word hyperbolic group which admits a Borel Anosov representation into $\mathsf{SL}(d,\mathbb{R})$. Is Γ necessarily free or a surface group?*

The only known examples of Borel Anosov representations are constructed from representations of free groups or surface groups (i.e. the fundamental group of a closed surface of negative Euler characteristic). Hitchin representations are the only known examples of Borel Anosov representations of surface groups in even dimensions. In odd dimensions, Barbot type representations and their nearby deformations are the only known examples except from Hitchin representations (see sub-section 1.3 (iii)).

A positive answer to Sambarino's question was given in [CT20] for $d = 3$ or 4. By using results of Benoist in [Ben00] and [Ben05], we prove that a torsion free word hyperbolic group admitting a P_{2q+1}-Anosov representation into $\mathsf{GL}(4q+2,\mathbb{R})$ has to be either free or a surface group. Moreover, by using Wilton's result [Wil18] on the existence of quasiconvex surface groups or rigid subgroups in one ended-word hyperbolic groups and a theorem of Kapovich-Leeb-Porti in [KLP18] (see also [KLP14, Theorem 6]), we prove the following stronger statement obtained in [Tso20b]:

Theorem 1.4.15. *Let Γ be a word hyperbolic group and $\rho : \Gamma \to \mathsf{GL}(4q+2, \mathbb{R})$ a representation. Suppose that there exists a continuous, ρ-equivariant dynamics preserving map $\xi : \partial_\infty \Gamma \to \mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. Then Γ is virtually free or virtually a surface group.*

The group Γ is virtually free (resp. a surface group) if it contains a finite-index subgroup which is free (resp. a surface group). The map ξ is called dynamics preserving whenever $\gamma \in \Gamma$ is an infinite order element, $\rho(\gamma)$ is P_k-proximal and $\xi(\gamma^+)$ is its attracting fixed point in $\mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. An analogue of Theorem 1.4.15 does not hold in dimensions which are multiples of 4, see Section 4.3.

Corollary 1.4.16. *Let G_{4q+2} be either $\mathsf{GL}(4q+2, \mathbb{R})$ or $\mathsf{PGL}(4q+2, \mathbb{R})$. If Γ is a word hyperbolic group and $\rho : \Gamma \to \mathsf{G}_{4q+2}$ is a P_{2q+1}-Anosov representation, then Γ is virtually free or virtually a surface group.*

A torsion-free word hyperbolic group Γ is called *rigid* if Γ does not admit a non-trivial splitting over a cyclic subgroup. Let $\tau_k^+ : \mathsf{Gr}_k(\mathbb{R}^d) \to \mathbb{P}(\wedge^k \mathbb{R}^d)$ be the Plücker embedding (see subsection 2.1.2). By using the connectedness properties of the boundary of a rigid hyperbolic group with the methods of the proof of Theorem 1.4.15 we have:

Corollary 1.4.17. *Let Γ be a torsion free rigid word hyperbolic group and $\rho : \Gamma \to \mathsf{GL}(4q+2, \mathbb{R})$ be a representation. Suppose there exists a continuous ρ-equivariant map $\xi : \partial_\infty \Gamma \to \mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. Then the map ξ is nowhere dynamics preserving and $\tau_{2q+1}^+ \circ \xi$ is not spanning.*

The map ξ is called *nowhere dynamics preserving* if for every infinite order element $\gamma \in \Gamma$ the restriction of ξ on $\{\gamma^-, \gamma^+\}$ is not dynamics preserving.

1.4.6 Historical remarks

We first explain how Theorem 1.4.1 is related with the equivalence (3) $\Leftrightarrow$ (5) in [KLP14, Theorem 1.7], see also [KLP17, Theorem 5.47]. A subgroup Γ of a real reductive Lie group G is called τ_{mod}-asymptotically embedded [KLP14, Definition 6.12], if it is τ_{mod}-regular, τ_{mod}-antipodal, word hyperbolic and there exists a Γ-equivariant homeomorphism $\alpha : \partial_\infty \Gamma \to \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. Here τ_{mod} corresponds to the choice of a subset of simple restricted roots $\eta \subset \Delta$ of G, τ_{mod}-antipodal means that the map α is transverse to itself i.e. for $x \neq y$ the pair $(\alpha(x), \alpha(y))$ is transverse and τ_{mod}-regular corresponds to P_η-divergence. Theorem 1.4.1 follows from [KLP14, Theorem 1.7] in the case both maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta^+$ and $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$ are assumed to satisfy the Cartan property (see Definition 3.2.1). In this case, we obtain a ρ-equivariant embedding $\xi : \partial_\infty \Gamma \to G/P$ with $P = P_\theta^+ \cap P_{\theta^\star}^+$. Here, $^\star : \Delta \to \Delta$ denotes the opposition involution and $\theta^\star = \{\alpha^\star : \alpha \in \theta\}$. Note that the pair of maps (ξ^+, ξ^-) is compatible and transverse, hence ξ is injective. The map ξ satisfies the Cartan property, maps onto the τ_{mod}-limit set $\Lambda_{\tau_{\mathrm{mod}}}(\rho(\Gamma))$ and $\rho(\Gamma)$ is τ_{mod}-asymptotically embedded.

We also remark that Theorem 1.3 of Guéritaud-Guichard-Kassel-Wienhard (see [Gué+17, Theorem 1.3, (1)$\Leftrightarrow$ (2)]) implies that a representation $\rho : \Gamma \to G$ is P_θ-Anosov if and only if ρ is P_θ-divergent and admits a pair of continuous, ρ-equivariant, dynamics preserving and transverse maps $\xi^{\pm} : \partial_\infty \Gamma \to G/P_\theta^{\pm}$. If we assume that ρ is semisimple, the argument of the proof of Corollary 3.2.5 shows that ξ^+ satisfies the Cartan property and ρ has to be Anosov by Theorem 1.4.1.

In case (i) of Corollary 1.4.3, by [Gué+17, Proposition 1.8], we may replace both ρ_1 and ρ_2 with their semisimplifications ρ_1^{ss} and ρ_2^{ss}. In this case the inequality also follows from Benoist's main result in [Ben+00].

The upper bound of Corollary 3.9.2 (i), $\alpha_{\eta^+}(d_a, d_{\mathbb{P}}) \leqslant \frac{1}{\log a} \inf_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma))}{|\gamma|_{X,\infty}}$, has been established by Sambarino in [Sam16, Lemma 6.8]. We prove the other bound by using Theorem 1.4.13 and Benoist's approximation result for the Cartan projection by the Lyapunov projection (see [Ben97] and [Gué+17, Theorem 4.12]).

CHAPTER II

Background

In this chapter, we recall definitions from Lie theory, review several facts for hyperbolic groups and the Floyd boundary and provide the dynamical definition of Anosov representations from [Lab06] and [GW12] along with some of their main properties and also discuss several facts for semisimple representations. We will mainly follow the notation from §2 of [Gué+17]. For more background on reductive Lie groups we refer the reader to [Kna02].

2.1 Lie theory

We will always consider G to be a semisimple Lie subgroup of $\mathsf{SL}(m, \mathbb{R})$ for some $m \geqslant 2$, of non-compact type which has finitely many connected components. The Zariski topology on G is the subspace topology induced from real algebraic subsets of $\mathsf{SL}(m, \mathbb{R})$. The group G has finite-index into a Zariski closed real semsimple subgroup of $\mathsf{SL}(m, \mathbb{R})$. We denote by $\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g})$ the adjoint representation and by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ its derivative. The Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is the bilinear form $B(X, Y) = \mathrm{tr}\big(\mathrm{ad}_X \mathrm{ad}_Y\big)$ and is non-degenerate as soon as $\mathfrak{g}$ is semisimple.

2.1.1 Root space decomposition, Cartan subspaces and parabolic subgroups.

We fix a maximal compact subgroup K of G, unique up to conjugation, a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathrm{Lie}(K)$, $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to B and the Cartan subspace $\mathfrak{a}$ which is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. The *real rank* of G is the dimension of $\mathfrak{a}$ as a real vector space. For a linear form $\beta \in \mathfrak{a}^*$ we use the notation $\langle \beta, H \rangle = \beta(H)$ for $H \in \mathfrak{a}$. The transformations $\mathrm{ad}_H : \mathfrak{g} \to \mathfrak{g}$, $H \in \mathfrak{a}$ are diagonizable and mutually commute. Thus we obtain a decomposition of $\mathfrak{g}$ into the common eigenspaces of $\mathrm{ad}_H, H \in \mathfrak{a}$ called the restricted root decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}_H(X) = \langle \alpha, H\rangle X \ \forall H \in \mathfrak{a}\}$ and $\Sigma = \{\alpha \in \mathfrak{a}^* : \mathfrak{g}_\alpha \neq 0\}$ is the set of restricted roots of G. We fix $H_0 \in \mathfrak{a}$ with $\alpha(H_0) \neq 0$ for every $\alpha \in \Sigma$. We denote by $\Sigma^+ = \{\alpha \in \Sigma : \langle \alpha, H_0\rangle > 0\}$ the set of positive roots and fix $\Delta \subset \Sigma^+$ the set of simple positive roots.

Note also that there exists a unique vector $H_\beta \in \mathfrak{a}$ such that $\langle \beta, H\rangle = B(H, H_\beta)$, since $B|_{\mathfrak{a}\times\mathfrak{a}}$ is a positive inner product. For two elements $\alpha, \beta \in \mathfrak{a}^*$ we set $(\alpha, \beta) = B(H_{\alpha,}, H_\beta)$. For a simple restricted root $\alpha \in \Delta$, we consider the associated *fundamental weight* $\omega_\alpha \in \mathfrak{a}^*$ satisfying

$$2\frac{(\omega_\alpha, \beta)}{(\beta, \beta)} = \delta_{\alpha\beta} \quad \forall \beta \in \Delta$$

For every $\theta \subset \Delta$, $\Sigma_\theta = \Sigma \cap \left(\sum_{\alpha\in\theta} \mathbb{Z}\alpha\right)$ denotes the set of all roots in Σ which are linear combinations of elements of θ. We consider the parabolic Lie algebras

$$\mathfrak{p}_\theta^\pm = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^\pm\cup\Sigma_{\Delta-\theta}} \mathfrak{g}_\alpha$$

and denote by $P_\theta^\pm = \{g \in G : \mathrm{Ad}(g)\mathfrak{p}_\theta^\pm = \mathfrak{p}_\theta^\pm\}$ to be the normalizer of $\mathfrak{p}_\theta^\pm$ in G. A subgroup P of G is called *parabolic* if it is the normalizer in G of some parabolic sub-algebra of $\mathfrak{g}$. Two parabolic subgroups P^+ and P^- of G are called *opposite* if there $\theta \subset \Delta$ and $g \in G$ such that $P^+ = gP_\theta^+ g^{-1}$ and $P^- = gP_\theta^- g^{-1}$.
Let $\overline{\mathfrak{a}}^+ := \bigcap_{\alpha\in\Sigma^+} \{H \in \mathfrak{a} : \langle \alpha, H\rangle \geqslant 0\}$. There exists a decomposition $G = K \exp(\overline{\mathfrak{a}}^+)K$ called the *Cartan decomposition* where each element $g \in G$ is written as $g = k_g \exp(\mu(g))k'_g$, $k_g, k'_g \in K$ and $\mu(g)$ denotes the Cartan projection of g. The *Lyapunov projection* $\lambda : G \to \overline{\mathfrak{a}}^+$ is defined as

$$\lambda(g) = \lim_{n\to\infty} \frac{\mu(g^n)}{n}$$

Example 2.1. *The case of* $G = \mathsf{SL}(d, \mathbb{R})$. The unique (up to conjugation) maximal compact subgroup of G is the special orthogonal group $\mathsf{SO}(d) = \{g \in \mathsf{SL}(d, \mathbb{R}) : gg^t = I_d\}$. A Cartan subspace for $\mathfrak{g}$ is the subspace $\mathfrak{a} = \mathrm{diag}_0(d)$ of all diagonal matrices with zero trace. Let $\varepsilon_i \in \mathfrak{a}^*$ be the projection to the (i, i)-entry. The closed dominant Weyl chamber of $\mathfrak{a}$ is $\overline{\mathfrak{a}}^+ := \{\mathrm{diag}(a_1, ..., a_d) : a_1 \geqslant ... \geqslant a_d,\ \sum_{i=1}^d a_i = 0\}$ and we have the Cartan decomposition $\mathsf{SL}(d, \mathbb{R}) = \mathsf{SO}(d) \exp(\overline{\mathfrak{a}}^+)\mathsf{SO}(d)$. The restricted root decomposition is $\mathfrak{sl}(d, \mathbb{R}) = \mathfrak{a} \oplus \bigoplus_{i\neq j} \mathbb{R}E_{ij}$, where E_{ij} denotes the $d \times d$ elementary matrix with 1 at the (i,j) entry and 0 everywhere else. The set of restricted roots is $\{\varepsilon_i-\varepsilon_j : i\neq j\}$ and of simple positive roots is $\{\varepsilon_i-\varepsilon_{i+1} : i=1,...,d-1\}$. For each $1\leqslant i\leqslant d-1$ the associated fundamental weight is $\omega_{\varepsilon_i-\varepsilon_{i+1}}=\sum_{k=1}^{i}\varepsilon_k$. For an element $g\in\mathsf{SL}(d,\mathbb{R})$ we denote by $\lambda_i(g)$ (resp. $\mu_i(g)$) the logarithm of the modulus of the i-th eigenvalue (resp. singular value) of g. We recall that the $\mu_i(g)=\frac{1}{2}\lambda_i(g^tg)$. The Cartan and Lyapunov projections of $g\in\mathsf{SL}(d,\mathbb{R})$ are given by the diagonal matrices:

$$\mu(g)=\mathrm{diag}\big(\mu_1(g),...,\mu_d(g)\big)\ \text{ and }\ \lambda(g)=\mathrm{diag}\big(\log\lambda_1(g),...,\lambda_d(g)\big).$$

2.1.2 Proximality.

An element $g\in G$ is called P_θ*-proximal* if $\langle\alpha,\lambda(\rho(\gamma))\rangle>0$ for all $\alpha\in\theta$. Equivalently, g has two fixed points $x_g^+\in G/P_\theta^+$ and $V_g^-\in G/P_\theta^-$ such that the pair (x_g^+,V_g^-) is transverse and for every $x\in G/P_\theta^+$ transverse to V_g^-, we have $\lim_n g^nx=x_g^+$. The element g is called P_θ*-biproximal* if g and g^{-1} are both P_θ-proximal. In this case, we denote by x_g^- the attracting fixed point of g^{-1} in G/P_θ^-.

Let $d\geqslant 2$ and $e_1,...,e_d$ be the canonical basis of $\mathbb{R}^d$. We denote by $(e_1,...,e_d)$ the canonical basis of $\mathbb{R}^d$ and set $e_j^\perp=\oplus_{j\neq i}\mathbb{R}e_j$. The group $K=\mathsf{SO}(d)$ is the unique, up to conjugation, maximal compact subgroup of $\mathsf{SL}(d,\mathbb{R})$. For $1\leqslant k\leqslant\frac{d}{2}$, we denote by P_k^+ the stabilizer of the plane $\langle e_1,..,e_k\rangle$ and by P_k^- the stabilizer of the complementary $(d-k)$-plane $\langle e_{k+1},...,e_d\rangle$. The Grassmannian of k-planes, $\mathsf{Gr}_k(\mathbb{R}^d)$ is identified with the quotient manifold $\mathsf{SL}(d,\mathbb{R})/P_k^+$. Similarly $\mathsf{Gr}_{d-k}(\mathbb{R}^d)$ is identified with $\mathsf{GL}(d,\mathbb{R})/P_k^-$. A pair of planes $(V^+,V^-)\in\mathsf{Gr}_k(\mathbb{R}^d)\times\mathsf{Gr}_{d-k}(\mathbb{R}^d)$ is *transverse* if there exists $h\in\mathsf{SL}(d,\mathbb{R})$ such that $V^+=h\langle e_1,...,e_k\rangle$ and $V^-=h\langle e_{k+1},..,e_d\rangle$. An element $g\in\mathsf{SL}(d,\mathbb{R})$ is called P_k*-proximal* if $\lambda_k(g)>\lambda_{k+1}(g)$. Equivalently, g has two fixed points $x_g^+\in\mathsf{Gr}_k(\mathbb{R}^d)$ and $V_g^-\in\mathsf{Gr}_{d-k}(\mathbb{R}^d)$ such that the pair (x_g^+,V_g^-) is transverse and for every k-plane V_0 transverse to V_g^- we have $\lim_n g^nV_0=x_g^+$. The element g is called P_k*-biproximal* if g and g^{-1} are P_k-proximal. We denote by x_g^- the attracting fixed point of g^{-1} in $\mathsf{Gr}_k(\mathbb{R}^d)$. For $k=1$, a P_1-proximal element $g\in\mathsf{GL}(d,\mathbb{R})$ in $\mathbb{P}(\mathbb{R}^d)$ has a unique eigenvalue, $\ell_1(g)$, of maximum modulus with multiplicity exactly one. The matrix g is called P_1*-positively proximal* if $\ell_1(g)>0$.

The Plücker embeddings $\tau_k^+:\mathsf{Gr}_k(\mathbb{R}^d)\to\mathbb{P}(\wedge^k\mathbb{R}^d)$ and $\tau_k^-:\mathsf{Gr}_{d-k}(\mathbb{R}^d)\to\mathsf{Gr}_{d_k-1}(\wedge^k\mathbb{R}^d)$, $d_k=\binom{d}{k}$, are defined as follows

$$\tau_k^+(gP_k)=[ge_1\wedge...\wedge ge_k]\quad\text{and}\quad\tau_k^-(gP_k^-)=[(\wedge^kg)W_k]$$

where $W_k = \langle e_{i_1} \wedge ... \wedge e_{i_k} : \{i_1, ..., i_k\} \neq \{1, ..., k\}\rangle$. The maps τ_k^+ and τ_k^- define embeddings of $\mathsf{Gr}_k(\mathbb{R}^d)$ and $\mathsf{Gr}_{d-k}(\mathbb{R}^d)$ into $\mathbb{P}(\wedge^k\mathbb{R}^d)$ and $\mathsf{Gr}_{d_k-1}(\wedge^k\mathbb{R}^d)$ respectively. An element $g \in \mathsf{SL}(d,\mathbb{R})$ is P_k-proximal if and only if $\tau_k^+(g)$ is P_1-proximal (see also [Gué+17, Proposition 3.3] for more details).

Limit sets. For a subgroup Γ of G, the *P_θ-proximal limit set* $\Lambda_\Gamma^{G/P_\theta^+}$ of Γ in G/P_θ^+ is defined to be the closure of the attracting fixed points of P_θ-proximal elements of Γ in G/P_θ^+. In the case where $G = \mathsf{SL}(d,\mathbb{R})$ and Γ is an irreducible subgroup containing a P_1-proximal element, the action of Γ on $\Lambda_\Gamma^{\mathbb{P}}$ in $\mathbb{P}(\mathbb{R}^d)$ is minimal (i.e. for every $x \in \Lambda_\Gamma^{\mathbb{P}}$, $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ is a dense subset of $\Lambda_\Gamma^{\mathbb{P}}$) (see [Ben00, Lemma 2.5]).

A linear representation $\rho : \Gamma \to \mathsf{GL}(d,\mathbb{R})$ is *irreducible* if $\rho(\Gamma)$ does not preserve a proper subspace of $\mathbb{R}^d$. The representation ρ is called *strongly irreducible* if for every finite-index subgroup H of Γ the restriction $\rho|_H$ is irreducible.

Let G be a semisimple linear Lie group. A representation $\tau : G \to \mathsf{GL}(d,\mathbb{R})$ is called *proximal* if $\tau(G)$ contains a P_1-proximal element. For an irreducible and proximal representation τ we denote by χ_τ the highest weight of τ. The functional $\chi_\tau \in \mathfrak{a}^*$ is of the form $\chi_\tau = \sum_{\alpha\in\Delta} n_\alpha\omega_\alpha$ and the representation τ is called *θ-compatible* if $\theta = \{\alpha \in \Delta : n_\alpha > 0\}$.

The *restricted Weyl group* of $\mathfrak{a}$ in $\mathfrak{g}$ is the group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$ (resp. $Z_K(\mathfrak{a})$) is the normalizer (resp. centralizer) of $\mathfrak{a}$ in K. The group W is a finite group, acts simply transitively on the set of Weyl chambers of $\mathfrak{a}$ and contains a unique order two element $wZ_K(\mathfrak{a}) \in W$ such that $\mathrm{Ad}(w)\overline{\mathfrak{a}}^+ = -\overline{\mathfrak{a}}^+$. The element $w \in K$ defines an involution $^\star : \Delta \to \Delta$ on the set of simple restricted roots Δ as follows: if $\alpha \in \Delta$ then $\alpha^\star \in \Delta$ is the unique root with $\alpha^\star(H) = -\alpha(\mathrm{Ad}(w)H)$ for every $H \in \mathfrak{a}$. By the definition of * we have $\langle \alpha, \mu(g)\rangle = \langle \alpha^\star, \mu(g^{-1})\rangle$ for every $\alpha \in \Delta$ and $g \in G$. Now let $\theta \subset \Delta$ be a subset of simple restricted roots of G determining the pair of opposite parabolic subgroups P_θ^+ and P_θ^-. The homogeneous spaces G/P_θ^+ and G/P_θ^- admit K-invariant metrics. We first fix an irreducible linear θ-proximal representation $\tau : G \to \mathsf{GL}(d,\mathbb{R})$ such that $P_\theta^+ = \mathrm{Stab}_G(\mathbb{R}e_1)$, $P_\theta^- = \mathrm{Stab}_G(e_1^\perp)$ and $\tau(K) \subset \mathsf{O}(d)$ (see [Gué+17, Proposition 3.3]). The metrics d_{G/P_θ^+} and d_{G/P_θ^-} are defined as follows: for $g_1, g_2 \in \mathsf{O}(d)$ we have

$$\begin{aligned} d_{G/P_\theta^+}(g_1P_\theta^+, g_2P_\theta^+) &:= d_{\mathbb{P}}(\tau(g_1)P_1^+, \tau(g_2)P_1^+) \\ d_{G/P_\theta^-}(g_1P_\theta^-, g_2P_\theta^-) &:= d_{\mathbb{P}}(\tau(g_1)^*P_1^+, \tau(g_2)^*P_1^+) \end{aligned}$$

where $d_{\mathbb{P}}$ is the angle metric on $\mathbb{P}(\mathbb{R}^d)$ with

$$d_{\mathbb{P}}(g_1P_1^+, g_2P_1^+) = \left|\sin\measuredangle(g_1e_1, g_2e_1)\right| = \sqrt{1 - \frac{\langle g_1e_1, g_2e_1\rangle^2}{||g_1e_1||^2 \cdot ||g_2e_1||^2}}$$

and $\tau(g_i)^* = \tau(g_i)^{-t}$ for $i = 1, 2$.

Let $g \in G$ such that $\langle \alpha, \mu(g) \rangle > 0$ (resp. $\langle \alpha, \mu(g^{-1}) \rangle > 0$) for every $\alpha \in \theta$. If we write $g = k_g \exp(\mu(g)) k_g'$ in the Cartan decomposition of G, the coset $k_g P_\theta^+$ (resp. $k_g w P_\theta^-$) is independent of the choice of the element $k_g \in K$ (see [Gué+17] and the references therein). Hence we define

$$\Xi_\theta^+(g) = k_g P_\theta^+ \qquad (\text{resp. } \ \Xi_\theta^-(g) = k_g w P_\theta^-)$$

2.2 Gromov hyperbolic spaces.

In this section, we provide some background on Gromov hyperbolic spaces. We also refer the reader to [BH13], [CDP06], [Gro87], [KB02] and [Väi05] for a detailed discussion and more background on Gromov hyperbolic spaces and their boundaries. By default, all metric spaces we consider will be assumed to be locally compact and proper.

Let (X, d) be a metric space. A curve $\sigma : [0, a] \to X$, $a > 0$ will be called a *geodesic* if $d(\sigma(t), \sigma(s)) = |t - s|$ for every $0 \leq t \leq a$. The space (X, d) is called *geodesic* if for every two points $x, y \in X$ there exists a geodesic connecting x with y.

Let (X, d) be proper geodesic metric and $x_0 \in X$ a fixed basepoint. For an isometry $\gamma : X \to X$ we define $|\gamma|_X = d(\gamma x_0, x_0)$. The *translation length* and the *stable translation length* of the isometry γ are:

$$\ell_X(\gamma) = \inf_{x \in X} d(\gamma x, x) \;\; \text{and} \;\; |\gamma|_{X,\infty} = \lim_{n \to \infty} \frac{|\gamma^n|_X}{n}$$

respectively.

Remark: Note that the stable transaltion length of γ, $|\gamma|_{X\infty}$, is well defined and $\lim_n \frac{|\gamma^n|_X}{n} = \inf_{n \geq 1} \frac{|\gamma^n|_X}{n}$ (see for example [BH13, p. 230]).

Let Γ be a finitely generated group and S be a finite generating subset of Γ. The left invariant word metric d_S on the Cayley graph C_Γ is defined as follows: for $g, h \in \Gamma$ we set

$$d_S(g, h) = \inf \left\{ k : g^{-1}h = s_1 \cdots s_k, s_i \in S \cup S^{-1} \right\}$$

For an element $\gamma \in \Gamma$ we set $|\gamma|_\Gamma = d_S(\gamma, e)$ and $|\gamma|_{\Gamma,\infty} = \lim_n \frac{d_S(\gamma^n, e)}{n}$ denotes the stable translation length of γ.

2.2.1 Quasi-Isometries

Definitions 2.2.1. *A map $f : (X, d) \to (Y, d')$ is called a (K, c)-quasi-isometric embedding if*

$$\frac{1}{K}d(x, y) - c \leq d'(f(x), f(y)) \leq Kd(x, y) + c$$

for every $x, y \in X$. The map f is called a quasi-isometry if:

(i) *f is a (K, c)-quasi-isometric embedding for some $K, c > 0$.*

(ii) *f is coarsely onto, i.e. there exists $M > 0$ such that for every $y \in Y$, there exists $x \in X$ with $d(y, f(x)) \leq M$.*

A (K, ε)-quasi-isometric embedding $\sigma : [0, R] \to X$, $R \in [0, \infty]$, will be called a (K, ε)-*quasi geodesic*. The following proposition is a fundamental observation in geometric group theory known as the Svarc-Milnor lemma (see for example [BH13]).

Proposition 2.2.2. (Svarc-Milnor lemma) *Let Γ be a group acting properly discontinuously and cocompactly by isometries on a proper geodesic metric space (X, d). Then:*

(i) *Γ is finitely generated.*

(ii) *If S is a finitely generated subset of Γ and $x_0 \in X$, the orbit map $x_0 \mapsto \gamma x_0$ is a quasi-isometry between (Γ, d_S) and (X, d).*

2.2.2 Gromov hyperbolicity

The *Gromov product* on a metric space (X, d) with respect to $x_0 \in X$ is defined as follows

$$(x \cdot y)_{x_0} := \frac{1}{2}\Big(d(x, x_0) + d(y, x_0) - d(x, y)\Big)$$

The triangle inequality shows that $(x \cdot y)_{x_0} \leqslant \mathrm{dist}_X(x_0, [x, y])$ for every geodesic $[x, y] \subset X$ joining x and y.

Definitions 2.2.3. *The metric space (X, d) is called Gromov hyperbolic if there exists $\delta \geqslant 0$ with the following property: for every $x, y, z \in X$*

$$(x \cdot y)_{x_0} \geqslant \min\left\{(x \cdot z)_{x_0}, (z \cdot y)_{x_0}\right\} - \delta$$

An infinite sequence of elements $(x_n)_{n\in\mathbb{N}}$ *in* X *is called a Gromov sequence if* $\lim_{n,m}(x_n \cdot x_m)_{x_0}$ *exists and is infinite. Two Gromov sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are equivalent if* $\underline{\lim}_n(x_n \cdot y_n)_{x_0} = +\infty$. *The Gromov boundary of* X, *denoted* $\partial_\infty X$, *is the set of equivalence classes of Gromov sequences.*

Given a Gromov hyperbolic space (X, d), the Gromov product with respect to $x_0 \in X$ extends also to $\partial_\infty X \times \partial_\infty X$ as follows: for $x, y \in \partial_\infty X$ define

$$(x \cdot y)_{x_0} := \sup\Big\{ \underline{\lim}_{m,n\to\infty}(x_m \cdot y_n)_{x_0} : \lim_{m\to\infty} x_m = x \text{ and } \lim_{n\to\infty} y_n = y\Big\}$$

The Gromov boundary $\partial_\infty X$ is metrizable (see [Gro87] and [BH13, Proposition 3.2]): there exist constants $C, a > 1$ and a *visual metric* $d_a : \partial_\infty X \times \partial_\infty X \to \mathbb{R}_+$ such that for every $x, y \in \partial_\infty X$:

$$\frac{1}{C}a^{-(x\cdot y)_{x_0}} \leqslant d_a(x, y) \leqslant Ca^{-(x\cdot y)_{x_0}}$$

For more details on boundaries of Gromov hyperbolic spaces we refer the reader to [KB02].

Given a subset A of X and $M > 0$, the M-neighbourhood of A in X is the set $B_M(A) := \cup_{x\in A}\{y \in X : d(x, y) \leq M\}$. One of the key properties of Gromov hyperbolic spaces is that they satisfy the following stability property for quasi-geodesics known as the fellow traveler property (see for example [BH13, Theorem 1.7, Chapter III.H.1]):

Theorem 2.2.4. *Suppose that* (X, d) *is a geodesic metric space which is* δ*-hyperbolic. For every* $K, \varepsilon > 0$ *there exists* $M = M(\delta, \varepsilon, K) > 0$ *with the following property: for every two* (K, ε)*-quasi-geodesics* $\alpha_1 : I_1 \to X$ *and* $\alpha_2 : I_2 \to X$ *with the same endpoints then* $\alpha_1(I_1) \subset B_M\big(\alpha_2(I_2)\big)$ *and* $\alpha_2(I_2) \subset B_M\big(\alpha_1(I_1)\big)$.

An immediate consequence of the fellow traveller property is the fact that Gromov hyperbolicity is invariant under quasi-isometries (see for example [BH13], [KB02]):

Proposition 2.2.5. *Let* (X, d) *and* (Y, d') *be two proper geodesic spaces and suppose that* $f : (X, d) \to (Y, d')$ *is a quasi-isometry.*

(i) (X, d) *is Gromov hyperbolic if and only if* (Y, d') *is.*

(ii) *Suppose that* (X, d) *and* (Y, d') *are Gromov hyperbolic and fix two visual metrics* d_a *and* d_b, $a, b > 0$, *on* $\partial_\infty X$ *and* $\partial_\infty Y$. *The map* f *induces the homeomorphism* $\partial f : (\partial_\infty, d_a) \to (\partial_\infty Y, d_b)$ *defined as follows*

$$\partial f([(x_n)_n]) = [(f(x_n))_n]$$

Moreover, tha map ∂f is bi-Hölder continuous, i.e. there exist $C, \alpha, \beta > 0$ such that

$$\frac{1}{C} d_a(x,y)^{\beta} \leq d_b(\partial f(x), \partial f(y)) \leq C d_a(x,y)^{\alpha}$$

for every $x, y, \in \partial_\infty X$.

Definitions 2.2.6. *Let Γ be a finitely generated group, S a finite generating subset of Γ and let C_Γ be the Cayley graph of Γ with respect to S.*

(i) *The group Γ is word hyperbolic if C_Γ equipped with the word metric d_S is a Gromov hyperbolic space. The Gromov boundary of Γ, $\partial_\infty \Gamma$, is by definition the Gromov boundary of the space (Γ, d_S).*
(ii) *Let H be a finitely generated subgroup of a word hyperbolic group Γ. The subgroup H is called quasiconvex in Γ if there exists $L > 0$ with the following property: for every $g, h \in H$, every geodesic $[g,h]$ in (C_Γ, d_S) is contained in the L-neighbourhood of H in C_Γ (with respect to d_S).*

Remarks: (i) Given a word hyperbolic group Γ, the definition of the Gromov boundary $\partial_\infty \Gamma$ is independent of the choice of the generating subset S of Γ. Note that for every generating subset S' of Γ, the spaces (Γ, d_S) and $(\Gamma, d_{S'})$ are quasi-isometric, hence a sequence $(\gamma_n)_{n\in\mathbb{N}}$ is a Gromov sequence with respect to d_S if and only if it is a Gromov sequence with respect to $d_{S'}$.

(ii) Theorem 3.5.4 implies that the definition of a quasi-convex subgroup H of a word hyperbolic group Γ does not depend on the choice of the generating subset S of Γ. In addition, H is a word hyperbolic group and the inclusion $\iota : H \hookrightarrow \Gamma$ is a quasi-isometric embedding.

For a word hyperbolic group Γ and $\gamma \in \Gamma$ an infinite order element, the map $\mathbb{Z} \to \Gamma$, $n \mapsto \gamma^n$, is a quasi-isometric embedding and hence $|\gamma|_{\Gamma,\infty} > 0$. Moreover, γ has exactly two fixed points γ^+ and γ^- in $\partial_\infty \Gamma$, represented by the sequences:

$$\gamma^+ = \big[(\gamma^n)_{n\in\mathbb{N}}\big] \ \text{ and } \ \gamma^- = \big[(\gamma^{-n})_{n\in\mathbb{N}}\big]$$

called the attracting and repelling fixed points of γ respectively.

We shall use the following inequality relating the stable translation length and the word length of an element of a group Γ acting isometrically, properly discontinously and cocompactly on a Gromov hyperbolic space.

Lemma 2.2.7. *Let (X,d) be a Gromov hyperbolic space and $x_0 \in X$. Suppose that Γ is a group acting properly discontinously and cocompactly by isometries on X. There exists $C > 0$ depending* *only on X such that for every infinite order element $\gamma \in \Gamma$ we have*

$$\left|\left(\gamma^{+} \cdot \gamma^{-1} x_0\right)_{x_0} - \frac{1}{2}\left(|\gamma|_X - |\gamma|_{X,\infty}\right)\right| \leqslant C$$

where $\gamma^{+} = \lim_n \gamma^n x_0$.

Proof. Since (X, d) is Gromov hyperbolic, we may choose $\delta > 0$ such that

$$(x \cdot y)_{x_0} \geqslant \min\left\{(x \cdot z)_{x_0}, (z \cdot y)_{x_0}\right\} - \delta$$

for every $x, y, z \in X \cup \partial_\infty X$. Observe that since the orbit map $x_0 \mapsto \delta x_0$, $\delta \in \Gamma$, is a quasi-isometry, $(\gamma^n x_0)_{n \in \mathbb{N}}$ is a Gromov sequence and $\lim_n \gamma^n x_0$ is well defined. Since $|\gamma|_{X,\infty} = \lim_n \frac{|\gamma^n|_X}{n}$, it is not hard to check that

$$\varliminf_{n \to \infty} \left(|\gamma^{n+1}|_X - |\gamma^n|_X\right) \leqslant |\gamma|_{X,\infty} \leqslant \varlimsup_{n \to \infty} \left(|\gamma^{n+1}|_X - |\gamma^n|_X\right)$$

Let $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ be a sequence such that $\varlimsup_n \left(|\gamma^{n+1}|_X - |\gamma^n|_X\right) = \lim_n \left(|\gamma^{k_n+1}|_X - |\gamma^{k_n}|_X\right)$. Note that for large $n \in \mathbb{N}$ we have $\left(\gamma^{+} \cdot \gamma^{k_n} x_0\right)_{x_0} > \left(\gamma^{+} \cdot \gamma^{-1} x_0\right)_{x_0}$, so

$$\begin{aligned}
\left(\gamma^{+} \cdot \gamma^{-1} x_0\right)_{x_0} &\leqslant \lim_{n \to \infty} \left(\gamma^{k_n} x_0 \cdot \gamma^{-1} x_0\right)_{x_0} + \delta \\
&= \frac{1}{2} \lim_{n \to \infty} \left(|\gamma|_X + \left|\gamma^{k_n}\right|_X - \left|\gamma^{k_n+1}\right|_X\right) + \delta \\
&\leqslant \frac{1}{2}\left(|\gamma|_X - |\gamma|_{X,\infty}\right) + \delta
\end{aligned}$$

Similarly, let $(m_n)_{n \in \mathbb{N}}$ be a sequence with $\varliminf_n \left(|\gamma^{n+1}|_X - |\gamma^n|_X\right) = \lim_n \left(|\gamma^{m_n+1}|_X - |\gamma^{m_n}|_X\right)$. Then

$$\begin{aligned}
\left(\gamma^{+} \cdot \gamma^{-1} x_0\right)_{x_0} &\geqslant \lim_{n \to \infty} \left(\gamma^{m_n} x_0 \cdot \gamma^{-1} x_0\right)_{x_0} - \delta \\
&= \frac{1}{2} \lim_{n \to \infty} \left(|\gamma|_X + \left|\gamma^{m_n}\right|_X - \left|\gamma^{m_n+1}\right|_X\right) - \delta \\
&\geqslant \frac{1}{2}\left(|\gamma|_X - |\gamma|_{X,\infty}\right) - \delta
\end{aligned}$$

The inequality follows. □

2.3 The Floyd boundary.

Let Γ be a finitely generated group, S a finite generating subset of Γ defining the left invariant metric d_S and let $|\gamma|_\Gamma = d_S(\gamma, e)$ for $\gamma \in \Gamma$. A function $f : \mathbb{N} \to \mathbb{R}^+$ satisfying the following two conditions:

(i) $\sum_{n=1}^{\infty} f(n) < +\infty$.

(ii) there exists $m > 0$ such that $f(k+1) \leqslant f(k) \leqslant mf(k+1)$ for every $k \in \mathbb{N}$,

is called a *Floyd function*. There exists a metric d_f on Γ defined as follows (see [Flo80]): for two adjacent vertices $g, h \in \Gamma$ the distance is $d_f(g,h) = f\big(\max\{|g|_\Gamma, |h|_\Gamma\}\big)$. The length of a finite path $\mathbf{p}$ defined by the sequence of adjacent vertices $\mathbf{p} = \{x_0, x_1, ..., x_k\}$ is $L_f(\mathbf{p}) = \sum_{i=0}^{k-1} d_f(x_i, x_{i+1})$. For two arbitrary vertices $g, h \in \Gamma$ their distance is defined as:

$$d_f(g,h) = \inf \Big\{ L_f(\mathbf{p}) : \mathbf{p} \text{ is a path from } g \text{ to } h \Big\}$$

It is easy to verify that d_f defines a metric on Γ and let $\overline{\Gamma}$ be the the metric completion of Γ with respect to d_f. Every two elements $x, y \in \overline{\Gamma}$ are represented by Cauchy sequences $(\gamma_n)_{n\in\mathbb{N}}, (\delta_n)_{n\in\mathbb{N}}$ with respect to d_f and their distance is $d_f(x,y) = \lim_n d_f(\gamma_n, \delta_n)$. The *Floyd boundary of* Γ *with respect to* f is defined to be the complement $\partial_f\Gamma := \overline{\Gamma} - \Gamma$ equipped with the metric d_f. The Floyd boundary $\partial_f\Gamma$ is called *non-trivial* if it contains at least three points. For every infinite order element $\gamma \in \Gamma$ the limit $\lim_n \gamma^{\pm n}$ exists (see for example [Kar03, Proposition 4]) and is denoted by $\gamma^{\pm}$.

If Γ is a word hyperbolic group, there exists $k > 0$ such that the Floyd boundary of Γ with respect to $f(x) = e^{-kx}$ is the Gromov boundary of Γ equipped with a visual metric (see [Gro87]). For more details and properties of the Floyd boundary we refer the reader to [Flo80] and [Kar03].

2.4 Flow spaces for hyperbolic groups.

Flow spaces for hyperbolic groups were introduced by Gromov in [Gro87] and further developed by Mineyev [Min05] and Champetier [Cha94]. Associated to any word hyperbolic group Γ there exists a metric space $\big(\hat{\Gamma}, d_{\hat{\Gamma}}\big)$ equipped with an $\mathbb{R}$-action $\{\varphi_t\}_{t\in\mathbb{R}}$ called the *geodesic flow* of Γ having the following properties:

(a) The action of Γ on $\hat{\Gamma}$ commutes with the action of the geodesic flow.

(b) The group Γ acts properly discontinuously and cocompactly by isometries on the flow space $\hat{\Gamma}$.

(c) There exist $C, c > 0$ such that for every $\hat{m} \in \hat{\Gamma}$, the map $t \mapsto \varphi_t(\hat{m})$ is a (C, c)-quasi-isometric embedding $(\mathbb{R}, d_{\mathbb{E}}) \to (\hat{\Gamma}, d_{\hat{\Gamma}})$.

It follows by property (a) that the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ descends to a well defined flow on the quotient $\Gamma\backslash\hat{\Gamma}$ which we also denote by $\{\varphi_t\}_{t\in\mathbb{R}}$. Moreover, property (c) guarantees that the map $(\tau^+, \tau^-) : \hat{\Gamma} \to \partial_\infty\Gamma \times \partial_\infty\Gamma - \{(x, x) \mid x \in \partial_\infty\Gamma\}$

$$(\tau^+, \tau^-)(\hat{m}) = \Big(\lim_{t\to\infty} \varphi_t(\hat{m}),\ \lim_{t\to\infty} \varphi_{-t}(\hat{m})\Big)$$

is well defined, continuous and equivariant with respect to the action of Γ on $\hat{\Gamma}$.

For example, if $\Gamma = \pi_1(M)$, where (M, g) is a closed negatively curved Riemannian manifold, a flow space for Γ satisfying the previous conditions is the unit tangent bundle $T^1\widetilde{M}$ equipped with the usual geodesic flow. Let Γ be a torsion free, discrete subgroup of $\mathsf{PGL}(d, \mathbb{R})$ acting properly discontinuously and cocompactly on a strictly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$. By Benoist's theorem [Ben04, Théorème 1] the group Γ is word hyperbolic. A flow space for Γ is the manifold $\Gamma\backslash T^1\Omega$ equipped with the Hilbert geodesic flow.

2.5 Anosov representations

Let Γ be a word hyperbolic group, $\rho : \Gamma \to G$ a representation and fix $\theta \subset \Delta$ a subset of simple restricted roots of G. We denote by $L_\theta = P_\theta^+ \cap P_\theta^-$ the common Levi subgroup. There exists a G-equivariant map $G/L_\theta \to G/P_\theta^+ \times G/P_\theta^-$ mapping the coset gL_θ to the pair $(gP_\theta^-, gP_\theta^-)$. The tangent space of G/L_θ at $(gP_\theta^+, gP_\theta^-)$ splits as the direct sum $T_{gP_\theta^+}G/P_\theta^+ \oplus T_{gP_\theta^-}G/P_\theta^-$ and hence we obtain a G-equivariant splitting of the tangent bundle $T(G/L_\theta) = \mathcal{E} \oplus \mathcal{E}^-$. We consider the quotient spaces:

$$\mathcal{X}_\rho = \Gamma\backslash\Big(\hat{\Gamma} \times G/L_\theta\Big) \quad \text{and} \quad \mathcal{E}_\rho^\pm = \Gamma\backslash\Big(\hat{\Gamma} \times \mathcal{E}^\pm\Big)$$

where the action of $\gamma \in \Gamma$ on $T(G/L_\theta)$ is given by the differential of the left translation $L_{\rho(\gamma)} : G/L_\theta \to G/L_\theta$. Let $\pi : \mathcal{X}_\rho \to \Gamma\backslash\hat{\Gamma}$ and $\pi_\pm : \mathcal{E}_\rho^\pm \to \mathcal{X}_\rho$ be the natural projections. the maps $\pi_\pm$ define vector bundles over the space $\mathcal{X}_\rho$ where the fiber over the point $[\hat{m}, (gP_\theta^+, gP_\theta^-)]_\Gamma$ is identified with the vector space $T_{gP_\theta^\pm}G/P_\theta^\pm$.

Given a flow space $(\hat{\Gamma}, \{\varphi_t\}_{t\in\mathbb{R}})$ associated to Γ, there is a natural flow, which we continue to denote by $\{\varphi_t\}_{t\in\mathbb{R}}$, on the quotients $\mathcal{E}_\rho^\pm$ and $\mathcal{X}_\rho$ defined as follows:

$$\varphi_t([\hat{m}, gL_\theta]_\Gamma) = [\varphi_t(\hat{m}), (gP_\theta^+, gP_\theta^-)]_\Gamma \ \text{ and } \ \varphi_t([\hat{m}, gL_\theta]_\Gamma) = [\varphi_t(\hat{m}), u]_\Gamma$$

for every $\hat{m} \in \hat{\Gamma}$, $g \in G$ and $u \in \mathcal{E}^{\pm} \subset T(G/L_\theta)$.

Now we are ready to provide the dynamical definition of an Anosov representation introduced by Labourie in [Lab06] for fundamental groups of negatively curved closed Riemannian manifolds, extended by Guichard-Wienhard in [GW12] for more general word hyperbolic groups. We also refer the reader to Canary's lecture notes [Can20] for more background on Anosov representations.

For the defintion, we shall need the following definition.

Definitions 2.5.1. *Let $\theta \subset \Delta$ be a subset of simple restricted roots of G, Γ be a word hyperbolic group and $\rho : \Gamma \to G$ a representation. For a coset $gP_\theta^{\pm}$, the stabilizer $\mathrm{Stab}_G(gP_\theta^{\pm})$ is the parabolic subgroup $gP_\theta^{\pm}g^{-1}$ of G. Suppose that $\xi^+ : \partial_\infty\Gamma \to G/P_\theta^+$ and $\xi^- : \partial_\infty\Gamma \to G/P_\theta^-$ are two continuous ρ-equivariant maps.*

(i) *The maps ξ^+ and ξ^- are called compatible if for every $x \in \partial_\infty\Gamma$ the intersection $\mathrm{Stab}_G(\xi^+(x)) \cap \mathrm{Stab}_G(\xi^-(x))$ is a parabolic subgroup of G.*

(ii) *The map ξ^+ (resp. ξ^-) is dynamics preserving if for every infinite order element $\gamma \in \Gamma$, $\rho(\gamma)$ is proximal in G/P_θ^+ (resp. G/P_θ^-) and $\xi^+(\gamma^+)$ (resp. $\xi^-(\gamma^+)$) is the attracting fixed point of $\rho(\gamma)$ in G/P_θ^+ (resp. G/P_θ^-).*

(iii) *Two maps ξ^+ and ξ^- are called transverse if for for every $x, y \in \partial_\infty\Gamma$ with $x \neq y$, there exists $h \in G$ such that $(\xi^-(x), \xi^-(y)) = (hP_\theta^+, hP_\theta^-)$.*

Definition 2.5.2. ([GW12], [Lab06]) *Let Γ be a word hyperbolic group and fix $\theta \subset \Delta$ a subset of restricted roots of G. A representation $\rho : \Gamma \to G$ is called P_θ-Anosov if:*

(i) *There exists a section $\sigma : \Gamma\backslash\hat{\Gamma} \to \mathcal{X}_\rho$ flat along the flow lines.*

(ii) *The lift of the geodesic flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on the pullback bundle $\sigma_*\mathcal{E}^+$ (resp. $\sigma_*\mathcal{E}^-$) is dilating (resp. contracting).*

The previous definition is equivalent to the existence of a pair of continuous ρ-equivariant transverse maps $\xi^+ : \partial_\infty\Gamma \to G/P_\theta^+$ and $\xi^- : \partial_\infty\Gamma \to G/P_\theta^-$ defining the flat section $\sigma : \mathcal{X}_\rho \to G/L_\theta$

$$\sigma([\hat{m}]_\Gamma) = \left[\hat{m}, \xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m}))\right]_\Gamma \ \ \forall\ \hat{m} \in \hat{\Gamma}$$

and a continuous equivariant family of norms $(||\cdot||_x)_{x\in\Gamma\backslash\hat{\Gamma}}$ with the property:
There exist $C, a > 0$ such that for every $x = [\hat{m}]_\Gamma$, $t \geqslant 0$, $v \in T_{\xi^+(\tau^+(\hat{m}))}G/P_\theta^+$ and $u \in$

$T_{\xi^-(\tau^-(\hat{m}))}G/P_\theta^-$:

$$\Big|\Big|\varphi_{-t}(X_v^+)\Big|\Big|_{\varphi_{-t}(x)} \leqslant Ce^{-at}\Big|\Big|X_v^+\Big|\Big|_x$$
$$\Big|\Big|\varphi_{t}(X_u^-)\Big|\Big|_{\varphi_{t}(x)} \leqslant Ce^{-at}\Big|\Big|X_u^-\Big|\Big|_x$$

where X_v^+ and X_u^- denote copies of the vectors v and u in the fibers $\pi_+^{-1}(x)$ and $\pi_-^{-1}(x)$ respectively.

We summarize here some of the main properties of Anosov representations obtained by Labourie [Lab06] and Guichard-Wienhard [GW12]. A represention $\rho : \Gamma \to G$ is called a *quasi-isometric embedding* if there exist $C, K > 0$ such that

$$\frac{1}{C}|\gamma|_\Gamma - K \leqslant \Big|\Big|\mu(\rho(\gamma))\Big|\Big| \leqslant C|\gamma|_\Gamma + K$$

for every $\gamma \in \Gamma$.

Theorem 2.5.3. ([GW12], [Lab06]) *Let Γ be a word hyperbolic group and $\theta \subset \Delta$ a subset of simple restricted roots of G. Suppose that $\rho : \Gamma \to G$ is a P_θ-Anosov representation. Then*

(i) *The kernel of ρ is finite and $\rho(\Gamma)$ is a discrete subgroup of G.*

(ii) *There exist $C, c > 0$ such that $\langle \alpha, \mu(\rho(\gamma))\rangle \geqslant C|\gamma|_\Gamma - c$ for every $\gamma \in \Gamma$ and $\alpha \in \theta$. In particular, ρ is a quasi-isometric embedding.*

(iii) *There exists a pair of compatible continuous ρ-equivariant maps*

$$\xi^+ : \partial_\infty\Gamma \to G/P_\theta^+ \quad \text{and} \quad \xi^- : \partial_\infty\Gamma \to G/P_\theta^-$$

which are dynamics preserving and transverse. Moreover $\xi^\pm(\partial_\infty\Gamma)$ identifies with the $P_\theta^\pm$-proximal limit set of $\rho(\Gamma)$ is $G/P_\theta^\pm$.

(iv) *The set of P_θ-Anosov representations of Γ in G is an open subset of* $\mathrm{Hom}(\Gamma, G)$ *and the map assigning a P_θ-Anosov representation to its Anosov limit maps is continuous.*

Remark: Let G be a higher rank semisimple Lie group and $\theta \subset \Delta$ be a subset of simple restricted roots of G. A quasi-isometric embedding $\rho : \Gamma \to G$ might fail to be P_θ-Anosov (see Example 3.10.1). Moreover, if $G = \mathsf{SL}(d, \mathbb{R})$, $d \geq 4$, for every $1 \leq i \leq \frac{d}{2}$, ρ might even fail to be in the closure of P_i-Anosov representations of Γ into G (see [Tso20c]).

The following result is the content of [GW12, Proposition 4.3], [Gué+17, Propostion 3.6] and [Gué+17, Lemma 3.7] and is used to reduce statements for P_θ-Anosov representations to statements for P_1-Anosov representations.

Proposition 2.5.4. ([Gué+17], [GW12]) *Let G a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G. There exists $d = d(G, \theta)$ and an irreducible θ-proximal representation $\tau : G \to \mathsf{GL}(d, \mathbb{R})$ such that $\tau(P_\theta^+)$ and $\tau(P_\theta^-)$ stabilize the line $[e_1]$ and the hyperplane $e_1^\perp = \langle e_1, ..., e_{d-1}\rangle$ respectively, so there exist continuous and τ-equivariant embeddings*

$$\iota^+ : G/P_\theta^+ \hookrightarrow \mathbb{P}(\mathbb{R}^d) \quad \text{and} \quad \iota^- : G/P_\theta^- \hookrightarrow \mathsf{Gr}_{d-1}(\mathbb{R}^d)$$

Let $\mathbb{R}^d = V^{\chi_\tau} \oplus V^{\chi_1} \oplus ... \oplus V^{\chi_k}$ be a decomposition of $\mathbb{R}^d$ into weight spaces, where $V^{\chi_i} := \{v \in \mathbb{R}^d : d\tau(H)v = \chi_i(H)v \ \forall H \in \mathfrak{a}\}$ and $\chi_i \in \mathfrak{a}^*$. The following properties hold:

(i) *A representation $\rho : \Gamma \to G$ is P_θ-Anosov if and only if $\tau \circ \rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ is P_1-Anosov. The pair of Anosov limit maps of $\tau \circ \rho$ is $(\iota^+ \circ \xi^+, \iota^- \circ \xi^-)$, where (ξ^+, ξ^-) is the pair of the limit maps of ρ.*

(ii) *For each $1 \leqslant i \leqslant k$, $\chi_\tau - \chi_i = \sum_{\beta \in \Sigma_\theta^+} n_{\beta,i}\beta$ where $n_{\beta,i} \in \mathbb{N}$ and $\Sigma_\theta^+ := \Sigma^+ - \mathrm{span}(\Delta - \theta)$.*

(iii) *For each $\alpha \in \theta$ there exists $1 \leqslant i \leqslant k$ such that $\chi_i = \chi_\tau - \alpha$.*

(iv) $\min\limits_{\alpha \in \theta} \langle \alpha, \mu(g)\rangle = \mu_1(\tau(g)) - \mu_2(\tau(g))$ *and* $\min\limits_{\alpha \in \theta} \langle \alpha, \lambda(g)\rangle = \lambda_1(\tau(g)) - \lambda_2(\tau(g))$ *for every $g \in G$.*

We need the following estimates which help us verify, in several cases, the Cartan property (see Section 3.2) of limit maps into the homogeneous spaces G/P_θ^+ and G/P_θ^-. The second part of the following proposition has been established in [BPS16, Lemma A4] and [Gué+17, Lemma 5.8], but for completeness we give a short proof.

Proposition 2.5.5. *Let G be a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G and $\tau : G \to \mathsf{GL}(d, \mathbb{R})$ an irreducible, θ-proximal representation such that $\tau(P_\theta^+)$ stabilizes the line $[e_1]$ in $\mathbb{R}^d$. Let χ_τ be the highest weight of τ and $g, r \in G$.*

(i) *If g is P_θ-proximal in G/P_θ^+ with attracting fixed point x_g^+, then*

$$d_{G/P_\theta^+}\Big(x_g^+, \Xi_\theta^+(g)\Big) \leqslant \exp\Big(- \min_{\alpha \in \theta} \langle \alpha, \mu(g)\rangle + \langle \chi_\tau, \mu(g) - \lambda(g)\rangle\Big)$$

(ii)

$$d_{G/P_\theta^+}\Big(\Xi_\theta^+(gr), \Xi_\theta^+(g)\Big) \leqslant C_{d,r} \exp\Big(- \min_{\alpha \in \theta} \langle \alpha, \mu(g)\rangle\Big)$$

where $C_{d,r} = \sigma_1(\tau(r))\sigma_1(\tau(r^{-1}))\sqrt{d-1}$.

Proof. By the definition of the metric d_{G/P_θ^+} and Proposition 2.5.4 we may assume that $G = \mathsf{SL}(d,\mathbb{R})$, $\theta = \{\varepsilon_1 - \varepsilon_2\}$ and $G/P_\theta^+ = \mathbb{P}(\mathbb{R}^d)$.

(i) Since g is P_1-proximal there exist $h \in \mathsf{GL}(d,\mathbb{R})$, $A_g \in \mathsf{GL}(d-1,\mathbb{R})$ and $k_g, k_g' \in \mathsf{O}(d)$ such that

$$g = h \begin{bmatrix} \ell_1'(g) & 0 \\ 0 & A_g \end{bmatrix} h^{-1} = k_g \exp(\mu(g)) k_g' \quad \text{and} \quad |\ell_1'(g)| = \ell_1(g).$$

We can write $\Xi_1^+(g) = k_g P_1^+$ and $x_g^+ = hP_1^+ = w_1 P_1^+$ for some $w_1 \in \mathsf{O}(d)$. Note that

$$h \begin{bmatrix} \ell_1'(g) & 0 \\ 0 & A_g \end{bmatrix} h^{-1} = w_1 \begin{bmatrix} \ell_1'(g) & * \\ 0 & * \end{bmatrix} w_1^{-1}$$

hence $k_g^{-1} w_1 \begin{bmatrix} \ell_1'(g) & * \\ 0 & * \end{bmatrix} = \exp(\mu(g)) k_g' w_1$ and $\left|\left\langle k_g^{-1} w_1 e_1, e_i \right\rangle\right| = \frac{\sigma_i(g)}{\ell_1(g)} \left|\left\langle k_g' w_1 e_1, e_i \right\rangle\right|$ for $2 \leqslant i \leqslant d$. Therefore,

$$d_{\mathbb{P}}(x_g^+, \Xi_1^+(g))^2 = 1 - \left\langle k_g^{-1} w_1 e_1, e_1 \right\rangle^2 = \sum_{i=2}^{d} \frac{\sigma_i(g)^2}{\ell_1(g)^2} \left\langle k_g' w_1 e_1, e_i \right\rangle^2 \leqslant \frac{\sigma_2(g)^2}{\ell_1(g)^2}$$

(ii) We have $k_{gr} \exp(\mu(gr)) k_{gr}' = k_g \exp(\mu(g)) k_g' r$ and in particular

$$\left\langle k_g^{-1} k_{gr} e_1, e_i \right\rangle = \frac{\sigma_i(g)}{\sigma_1(gr)} \left\langle k_g' r (k_{gr}')^{-1} e_1, e_i \right\rangle$$

for every $2 \leqslant i \leqslant d$. Note that since $\sigma_1(gr) \geqslant \frac{\sigma_1(g)}{\sigma_1(r^{-1})}$ and $\left|\left\langle k_g' r (k_{gr}')^{-1} e_1, e_i \right\rangle\right| \leqslant \sigma_1(r)$, we have

$$\left|\left\langle k_g^{-1} k_{gr} e_1, e_i \right\rangle\right| \leqslant \frac{\sigma_i(g)}{\sigma_1(g)} \sigma_1(r)\sigma_1(r^{-1})$$

Finally,

$$d_{\mathbb{P}}\Big(\Xi_1^+(gr), \Xi_1^+(g)\Big)^2 = \sum_{i=2}^{d} \langle k_g^{-1} k_{gr} e_1, e_i \rangle^2 = \sum_{i=2}^{d} \frac{\sigma_i(g)^2}{\sigma_1(gr)^2} \left\langle k_g' r (k_{gr}')^{-1} e_1, e_i \right\rangle^2 \leqslant C_{d,r}^2 \frac{\sigma_2(g)^2}{\sigma_1(g)^2}$$

□

2.6 Semisimple representations.

Let Γ be a finitely generated group and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ be a representation. The representation ρ is called *semisimple* if the Zariski closure $H := \overline{\rho(\Gamma)}^{\text{Zar}}$ of $\rho(\Gamma)$ in $\mathsf{SL}(d, \mathbb{R})$ is a real reductive Lie group. Equivalently ρ is semisimple if one of the following conditions are satisfied:

(i) If $\mathfrak{h} := \text{Lie}(H)$ denotes the Lie algebra of H, $[\mathfrak{h}, \mathfrak{h}]$ is a semisimple Lie algbera and $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}]$, where $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$.

(ii) There exists a decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_k$ and irreducible representations $\psi_i : H \to \mathsf{GL}(V_i)$, $i = 1, ..., k$, such that the inclusion $\mathrm{i} : H \hookrightarrow \mathsf{SL}(d, \mathbb{R})$ decomposes as the direct sum $\mathrm{i} = \psi_1 \oplus \cdots \oplus \psi_k$.

Let us fix an Euclidean norm $|| \cdot ||$ on the Cartan subspace $\mathfrak{a} \subset \mathfrak{g}$. The following result was established by Benoist using a result of Abels-Margulis-Soifer [AMS95] and allows one to control the Lyapunov projection of a semisimple representation in terms of its Cartan projection. We refer to [Gué+17, Theorem 4.12] for a proof.

Theorem 2.6.1. ([Ben97]) *Let G be a real reductive Lie group, Γ be a finitely generated group and $\rho_i : \Gamma \to G$, $1 \leqslant i \leqslant s$ be semisimple representations. Then there exists $C > 0$ and a finite subset F of Γ such that for every $\gamma \in \Gamma$ there exists $f \in F$ with the property:*

$$\max_{1 \leqslant i \leqslant s} \Big|\Big| \lambda(\rho_i(\gamma f)) - \mu(\rho_i(\gamma)) \Big|\Big| \leqslant C$$

Let $\mathcal{H}$ be a real algebraic subgroup of $\mathsf{GL}(d, \mathbb{R})$. There exists a unique connected normal unipotent subgroup of $\mathcal{H}$, $R_u(\mathcal{H})$, called the *unipotent radical of* $\mathcal{H}$ and a semisimple subgroup $\mathcal{L}$ (isomorphic to the quotient $\mathcal{H}/R_u(\mathcal{H})$) such that $\mathcal{H} = R_u(\mathcal{H})\mathcal{L}$.

Let $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ be a representation. Guéritaud-Guichard-Kassel-Wienhard in [Gué+17] observe that from ρ one may define the *semisimplification* ρ^{ss} as follows: let H be the Zariski closure of $\rho(\Gamma)$, $R_u(H)$ be the unipotent radical of H and fix L a semisimple Lie subgroup of h with $H = R_u(H)L$. Let $\pi : H \to L$ be the projection onto L. The representation $\rho^{ss} := \rho \circ \pi$ is a semisimple representation into $\mathsf{SL}(d, \mathbb{R})$. We shall use several times the following result for the semisimplification ρ^{ss} established in [Gué+17].

Proposition 2.6.2. ([Gué+17, Proposition 1.8]) *Let Γ be a finitely generated group, G a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G and $\rho : \Gamma \to G$ a representation with semisimplification $\rho^{ss} : \Gamma \to G$. Then, $\lambda(\rho(\gamma)) = \lambda(\rho^{ss}(\gamma))$ for every $\gamma \in \Gamma$ and ρ is P_θ-Anosov if and only if ρ^{ss} is P_θ-Anosov.*

2.7 Convex cocompact groups

Let us recall that a subset Ω of the projective space $\mathbb{P}(\mathbb{R}^d)$ is *properly convex* if it is contained in an affine chart of $\mathbb{P}(\mathbb{R}^d)$ in which Ω is bounded and convex. The domain Ω is called *strictly convex* if it is properly convex and $\partial\Omega$ does not contain projective line segments. Suppose that Ω is bounded and convex in some affine chart A. We fix an Euclidean metric $d_{\mathbb{E}}$ on A. We denote by d_Ω the Hilbert metric on Ω defined as follows

$$d_\Omega(x,y) = \frac{1}{2}\log\frac{d_{\mathbb{E}}(y,a)\cdot d_{\mathbb{E}}(x,b)}{d_{\mathbb{E}}(a,x)\cdot d_{\mathbb{E}}(y,b)}$$

where a,b are the intersection points of the projective line $[x,y]$ with $\partial\Omega$, x is between a and y, and y is between x and b. The group

$$\mathrm{Aut}(\Omega) = \Big\{g \in \mathsf{PGL}(d,\mathbb{R}) : g\Omega = \Omega\Big\}$$

is a Lie subgroup of $\mathsf{PGL}(d,\mathbb{R})$ and acts by isometries for the Hilbert metric d_Ω. Any discrete subgroup of $\mathrm{Aut}(\Omega)$ acts properly discontinuously on Ω.

We shall use the following estimate obtained by Danciger-Guéritaud-Kassel in [DGK17] which implies that convex cocompact subgroups of $G = \mathsf{PGL}(d,\mathbb{R})$ are quasi-isometrically embedded in G.

Proposition 2.7.1. ([DGK17, Proposition 10.1]) *Let Ω be a properly convex domain of $\mathbb{P}(\mathbb{R}^d)$. For any $x_0 \in \Omega$, there exists $\kappa > 0$ such that for any $g \in \mathrm{Aut}(\Omega)$,*

$$\mu_1(g) - \mu_d(g) \geqslant 2d_\Omega(gx_0, x_0) - \kappa$$

Let Γ be a subgroup of $\mathsf{PGL}(d,\mathbb{R})$ preserving a properly convex domain Ω. Recall that the definition of the Gromov product with respect to a linear form on $\mathfrak{a}^*$ was given in Definition 3.6.1. By using the previous proposition we can control the Gromov product with respect to $\varepsilon_1 \in \mathfrak{a}^*$ as follows:

Lemma 2.7.2. *Let Γ be a subgroup of $\mathsf{PGL}(d,\mathbb{R})$ which preserves a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$. Suppose that the natural inclusion of $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ is semisimple. Then for every $x_0 \in \Omega$ there exists $C > 0$ such that*

$$\Big|(\mu_1(\gamma) - \mu_d(\gamma)) - 2d_\Omega(\gamma x_0, x_0)\Big| \leqslant C \quad \text{and} \quad \Big|(\gamma\cdot\delta)_{\varepsilon_1} - (\gamma x_0 \cdot \delta x_0)_{x_0}\Big| \leqslant C$$

for every $\gamma, \delta \in \Gamma$ of infinite order.

Proof. By Theorem 2.6.1 there exists a finite subset F of Γ and $M>0$ such that for every $\gamma\in\Gamma$ there exists $f\in F$ such that $\lambda_1(\gamma f)-\lambda_d(\gamma f)\geqslant \mu_1(\gamma)-\mu_d(\gamma)-M$. The translation length of an isometry $g\in \mathrm{Aut}(\Omega)$ is exactly $\frac{1}{2}(\lambda_1(g)-\lambda_d(g))$, see [CLT15, Proposition 2.1]. In particular, if $\gamma\in\Gamma$ and $f\in F$ are as previously we have that

$$\begin{aligned}
2d_\Omega(\gamma x_0,x_0) &\geqslant 2d_\Omega(\gamma f x_0,x_0)-2\sup_{f\in F} d_\Omega(fx_0,x_0)\\
&\geqslant \lambda_1(\gamma f)-\lambda_d(\gamma f)-2\sup_{f\in F} d_\Omega(fx_0,x_0)\\
&\geqslant \mu_1(\gamma f)-\mu_d(\gamma f)-M-2\sup_{f\in F} d_\Omega(fx_0,x_0)\\
&\geqslant \mu_1(\gamma)-\mu_d(\gamma)-(\mu_1(f)-\mu_d(f))-2\sup_{f\in F} d_\Omega(fx_0,x_0)
\end{aligned}$$

Then, by Proposition 2.7.1, we obtain a uniform constant $L>0$ such that

$$\Big|(\mu_1(\gamma)-\mu_d(\gamma))-2d_\Omega(\gamma x_0,x_0)\Big|\leqslant L$$

for every $\gamma\in\Gamma$. The conclusion follows. □

Definitions 2.7.3. ([DGK17]) *Let Γ be an infinite discrete subgroup of $\mathsf{PGL}(d,\mathbb{R})$ and suppose that Γ preserves a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$. Let $\Lambda_\Omega(\Gamma)$ be the set of accumulation points of all Γ-orbits in $\partial\Omega$. The group Γ acts convex cocompactly on Ω if the convex hull of $\Lambda_\Omega(\Gamma)$ in Ω is non-empty and acted on cocompactly by Γ. The group Γ is called strongly convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ if Γ acts convex cocompactly on some strictly convex domain Ω with C^1-boundary.*

The following lemma follows immediately from [DGK17, Theorem 1.4] and [Zim17, Theorem 1.27] and is used to replace an arbitrary P_1-Anosov representation with a convex cocompact one.

Lemma 2.7.4. ([DGK17],[Zim17]) *Let V_d be the vector space of $d\times d$-symmetric matrices and $S_d:\mathsf{GL}(d,\mathbb{R})\to\mathsf{GL}(V_d)$ be the representation defined as follows $S_d(g)X=gXg^t$ for $X\in V_d$. For every P_1-Anosov representation $\rho:\Gamma\to\mathsf{GL}(d,\mathbb{R})$, the representation $S_d\circ\rho$ is P_1-Anosov and $S_d(\rho(\Gamma))$ is a strongly convex cocompact subgroup of $\mathsf{GL}(V_d)$.*

We will also need the following lemma which allows us to control the Cartan projection of an Anosov representation ρ in terms of the Cartan projection of a semisimplification ρ^{ss} of ρ. This follows by the techniques of Guichard-Wienhard in §5 of [GW], showing that Anosov representations have strong proximality properties. Given two representations $\rho_1:\Gamma\to\mathsf{GL}(n,\mathbb{R})$

and $\rho_2 : \Gamma \to \mathsf{GL}(m, \mathbb{R})$, we say that ρ_1 *uniformly dominates* ρ_2 is there exists $\delta > 0$ with the property

$$(1-\delta)\lambda_1(\rho_1(\gamma)) \geqslant \lambda_1(\rho_2(\gamma))$$

for every $\gamma \in \Gamma$.

Lemma 2.7.5. *Let Γ be a word hyperbolic group, G a real semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G. Suppose $\psi : \Gamma \to G$ is a P_θ-Anosov representation with semisimplification $\psi^{ss} : \Gamma \to G$. Then there exists a constant $C > 0$ such that*

$$\Big|\langle \omega_\alpha, \mu(\psi(\gamma)) - \mu(\psi^{ss}(\gamma))\rangle\Big| \leqslant C$$

for every $\gamma \in \Gamma$ and $\alpha \in \theta$.

Proof. By Proposition 2.5.4, we may compose ψ with an irreducible representation $\tau : G \to \mathsf{GL}(n, \mathbb{R})$ such that $\rho := \tau \circ \psi$ is P_1-Anosov. We remark that $\rho^{ss} = \tau \circ \psi^{ss}$. Note also $\mu_1(S_n(\rho(\gamma))) = 2\mu(\rho(\gamma))$ for every $\gamma \in \Gamma$, where S_n is defined as in Lemma 2.7.4. Therefore, by Lemma 2.7.4, we may further assume that $\rho(\Gamma)$ preserves a properly convex domain Ω in $\mathbb{P}(\mathbb{R}^n)$.

Note that if ρ is irreducible, then its Zariski closure is a reductive Lie group and the conclusion is immediate. Therefore we continue by assuming that ρ is reducible. Notice that the dual representation $\rho^*(\gamma) = \rho(\gamma^{-1})^t$ for $\gamma \in \Gamma$, also preserves a properly convex domain Ω' of $\mathbb{P}(\mathbb{R}^n)$. Then, up to conjugating ρ and considering possibly the dual of this conjugate, we may assume that $\rho(\Gamma)$ still preserves a properly convex domain Ω_0 of $\mathbb{P}(\mathbb{R}^n)$ and there exists a decomposition $\mathbb{R}^n = V_1 \oplus ... \oplus V_\ell$ such that

$$\rho = \begin{bmatrix} \rho_1 & * & * & * \\ 0 & \rho_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \rho_\ell \end{bmatrix} \quad \text{and} \quad \rho^{ss} = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \rho_\ell \end{bmatrix}$$

where $\rho_i : \Gamma \to \mathsf{GL}(V_i)$ are irreducible representations, ρ_1 is P_1-Anosov and uniformly dominates ρ_i for $2 \leqslant i \leqslant \ell$. In particular, ρ_1 is the restriction of ρ^{ss} on the vector subspace $\langle \xi_{\rho^{ss}}(\partial_\infty \Gamma)\rangle$. We conclude that for every $\gamma \in \Gamma$, $\mu_1(\rho(\gamma)) \geq \mu_1(\rho^{ss}(\gamma))$ so $\langle \omega_\alpha, \mu(\psi(\gamma))\rangle \geq \langle \omega_\alpha, \mu(\psi^{ss}(\gamma))\rangle$.

Now we prove that there exists $C > 0$ such that $\langle \omega_\alpha, \mu(\psi(\gamma) - \mu(\psi^{ss}(\gamma))\rangle \leqslant C$ for every $\gamma \in \Gamma$.

By using induction, it is enough to consider the case when $\ell = 2$ and

$$\rho(\gamma) = \begin{bmatrix} \rho_1(\gamma) & u(\gamma) \\ 0 & \rho_2(\gamma) \end{bmatrix}, \ \gamma \in \Gamma$$

for some matrix valued function $u : \Gamma \to \mathrm{Hom}(V_2, V_1)$. The group $\rho_1(\Gamma)$ preserves the properly convex domain $\Omega_0 \cap \mathbb{P}(V_1)$ of $\mathbb{P}(V_1)$. By [DGK17], [Zim17], there exists a closed $\rho_1(\Gamma)$-invariant properly convex domain $\Omega_1 \subset \mathbb{P}(V_1)$ and a $\rho_1(\Gamma)$-invariant closed convex subset $\mathcal{C} \subset \Omega_1$ such that $\rho_1(\Gamma)\backslash\mathcal{C}$ is compact. We fix a basepoint $x_0 \in \mathcal{C}$ such that every point of $\mathcal{C}$ is within d_{Ω_1}-distance M from the orbit $\rho_1(\Gamma) \cdot x_0$. Let $g \in \Gamma$ and consider $x_0, x_1, ..., x_k \in [x_0, gx_0]$ such that $\frac{1}{2} \leqslant d_\Omega(x_i, x_{i+1}) \leqslant 1$. For every i, there exists $g_i \in \Gamma$ such that $d_\Omega(\rho_1(g_i)x_0, x_0) \leqslant M$ and let $h_i = g_i^{-1}g_{i+1}$, $0 \leqslant i \leqslant k-1$, where $g_0 = e$ and $g_k = g$. A straightforward computation shows that

$$u(g) = u(h_0 \cdots h_{k-1}) = \sum_{i=1}^{k-1} \rho_2(h_{i+1} \cdots h_{k-1})^t \cdot \rho_1(h_0 \cdots h_{i-1}) \cdot u(h_i)$$

By Theorem 2.6.1, there exists a finite subset F of Γ and $C_1 > 0$, such that for every $\gamma \in \Gamma$ there exists $f \in F$ with $||\lambda(\rho_i(\gamma f)) - \mu(\rho_i(\gamma))|| \leqslant C_1$ for $i = 1, 2$. Since ρ_1 is semisimple, P_1-Anosov and uniformly dominates ρ_2, we can find constants $A, E, a, b, \varepsilon > 0$ such that for every $\gamma \in \Gamma$ we have:

$$b\sigma_1(\rho_1(\gamma))^{1-\varepsilon} \geqslant \sigma_1(\rho_2(\gamma)), \quad \sigma_1(\rho_1(\gamma)) \geqslant Ae^{ad_{\Omega_1}(\rho_1(\gamma)x_0, x_0)}$$

$$\text{and} \quad \Big|\big(\mu_1(\rho_1(\gamma)) - \mu_{d_1}(\rho_1(\gamma))\big) - 2d_{\Omega_1}(\gamma x_0, x_0)\Big| \leqslant E.$$

For $0 \leq i \leq k-1$ we set $w_i := h_0 \cdots h_i$. The triangle inequality shows $|d_{\Omega_1}(\rho_1(w_i)x_0, x_0) - d_{\Omega_1}(x_i, x_0)| \leqslant M$ and $|d_{\Omega_1}(\rho_1(w_i)x_0, gx_0) - d_{\Omega_1}(x_i, \rho_1(g)x_0)| \leqslant M$. There exists $R > 0$ independent of g, such that $h_i \in \Gamma$ lie in a metric ball of radius $R > 0$ of Γ and hence there exists $C_R > 0$ independent of g such that:

$$\begin{aligned}
||u(g)|| &\leqslant C_R \sum_{i=0}^{k-1} \sigma_1\big(\rho_2(h_{i+1} \cdots h_k)\big) \cdot \sigma_1\big(\rho_1(h_1 \cdots h_i)\big) \leqslant bC_R \sum_{i=0}^{k-1} \frac{\sigma_1(\rho_1(w_i^{-1}g))\sigma_1(\rho_1(w_i))}{\sigma_1(\rho_1(w_i^{-1}g))^{\varepsilon}} \\
&= bC_R \sum_{i=0}^{k-1} \frac{1}{\sigma_1(\rho_1(g^{-1}w_i))\sigma_1(\rho_1(w_i^{-1}))} \cdot \frac{\sigma_1}{\sigma_{d_1}}(\rho_1(w_i^{-1}g)) \cdot \frac{\sigma_1}{\sigma_{d_1}}(\rho_1(w_i)) \cdot \frac{1}{\sigma_1(\rho_1(w_i^{-1}g))^{\varepsilon}} \\
&\leqslant bC_R \sum_{i=0}^{k-1} \frac{1}{\sigma_1(\rho_1(g^{-1}))} \cdot e^{2d_{\Omega_1}(\rho_1(w_i^{-1}g)x_0, x_0)+E} \cdot e^{2d_{\Omega_1}(\rho_1(w_i)x_0, x_0)+E} \cdot \big(A^{-\varepsilon}e^{-a\varepsilon|w_i^{-1}g|_\Gamma}\big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{bC_R e^{2E}}{\sigma_1(\rho_1(g^{-1}))A^\varepsilon} \sum_{i=0}^{k-1} e^{2d_\Omega(\rho(w_i x_0),\rho(g)x_0)+2d_\Omega(\rho(x_i)x_0,x_0)} e^{-a\varepsilon d_\Omega(\rho_1(w_i^{-1}g)x_0,x_0)} \\
&\leqslant \frac{bC_R e^{2E+2M+2Ma\varepsilon}}{A^\varepsilon \sigma_1(\rho_1(g^{-1}))} e^{2d_{\Omega_1}(\rho_1(g)x_0,x_0)} \Big(\sum_{i=0}^{k-1} e^{-\varepsilon a(k-i)} \Big) \leqslant \frac{bC_R e^{2E+2M+2Ma\varepsilon}}{A^\varepsilon (1-e^{-a\varepsilon})} \sigma_1(\rho_1(g))
\end{aligned}
$$

It follows that there exists $L > 0$ depending only on ρ such that $\sigma_1(\rho^{ss}(g)) \leqslant \sigma_1(\rho(g)) \leqslant L\sigma_1(\rho^{ss}(g))$ for every $g \in \Gamma$. In particular, for the highest weight χ_τ we obtain $L' > 0$ such that

$$
\Big| \big\langle \chi_\tau, \mu(\psi(g)) - \mu(\psi^{ss}(g)) \big\rangle \Big| \leqslant L'
$$

for every $g \in \Gamma$. Since τ is θ-compatible the conclusion follows. $\square$

CHAPTER III

Characterizations of Anosov Representations

In this chapter, we prove our main characterizations of Anosov representations and Benoist representations in terms of the existence of equivariant limit maps, the Cartan property and the existsence of a weak uniform gap in eigenvalues. As an application we obtain a characterization of strongly convex cocompact subgroups of $\mathsf{PGL}(d,\mathbb{R})$. We also provide a computation of the Hölder exponent of the Anosov limit maps of a semisimple P_θ-Anosov rperesentation in terms of the Cartan and Lyapunov projection.

3.1 The contraction property

Let Γ be a word hyperbolic group, $(\hat{\Gamma},\varphi_t)$ a flow space on which Γ acts properly discontinuously and cocompactly and $F\subset\hat{\Gamma}$ a compact fundamental domain for the action of Γ on $\hat{\Gamma}$. Let also $\rho:\Gamma\rightarrow\mathsf{GL}(d,\mathbb{R})$ be a representation which admits a pair of transverse ρ-equivariant maps $\xi^+:\partial_\infty\Gamma\rightarrow\mathbb{P}(\mathbb{R}^d)$ and $\xi^-:\partial_\infty\Gamma\rightarrow\mathsf{Gr}_{d-1}(\mathbb{R}^d)$ defining the flat section $\sigma:\Gamma\backslash\hat{\Gamma}\rightarrow\mathcal{X}_\rho$ of the fiber bundle $\pi:\mathcal{X}_\rho\rightarrow\Gamma\backslash\hat{\Gamma}$. We fix an equivariant family of norms $\left(||\cdot||_x\right)_{x\in\Gamma\backslash\hat{\Gamma}}$ on the fibers of the bundle $\pi_\pm:\mathcal{E}_\rho^\pm\rightarrow\Gamma\backslash\hat{\Gamma}$. For a given point $\hat{m}\in\hat{\Gamma}$ we fix an element $h\in G$ so that $\xi^+(\tau^+(\hat{m}))=hP_1^+$ and $\xi^-(\tau^-(\hat{m}))=hP_1^-$ and denote by $L_h:G\rightarrow G$ the left translation by $h\in G$. Then we have

$$T_{hP_1^+}\mathbb{P}(\mathbb{R}^d)=\Big\{dL_hd\pi^+(X):X\in\bigoplus_{i=2}^d\mathbb{R}E_{i1}\Big\}$$
$$T_{hP_1^-}\mathsf{Gr}_{d-1}(\mathbb{R}^d)=\Big\{dL_hd\pi^-(X):X\in\bigoplus_{i=2}^d\mathbb{R}E_{1i}\Big\}$$

For $u \in \{0\} \times \mathbb{R}^{d-1}$, we denote by X_u^+ and X_u^- the tangent vectors

$$X_u^+ = \left[\hat{m}, \xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m})), dL_h d\pi^+ \left(\begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix}\right)\right]_\Gamma$$
$$X_u^- = \left[\hat{m}, \xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m})), dL_h d\pi^- \left(\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}\right)\right]_\Gamma$$

in the fibers of the bundles $\sigma_* \mathcal{E}^\pm \to \Gamma \backslash \hat{\Gamma}$ over $x = [\hat{m}]_\Gamma$ and π^+, π^- are the projections from $\mathsf{SL}(d,\mathbb{R})$ to $\mathbb{P}(\mathbb{R}^d)$ and $\mathsf{Gr}_{d-1}(\mathbb{R}^d)$ respectively.

The following lemma shows that when the geodesic flow on $\sigma_*\mathcal{E}^-$ is weakly contracting, then the geodesic flow on $\sigma_*\mathcal{E}^+$ is weakly dilating.

Lemma 3.1.1. *Let $\rho : \Gamma \to \mathsf{GL}(d,\mathbb{R})$ be a representation. Suppose there exists a pair of continuous, ρ-equivariant transvserse maps $\xi^+ : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_\infty\Gamma \to \mathsf{Gr}_{d-1}(\mathbb{R}^d)$. Then for any $x = [\hat{m}]_\Gamma \in \hat{\Gamma}$ and $u \in \{0\} \times \mathbb{R}^{d-1}$ we have:*

$$\varliminf_{t \to \infty} ||\varphi_t(X_u^+)||_{\varphi_t(x)} \cdot ||\varphi_t(X_u^-)||_{\varphi_t(x)} > 0$$

Proof. For two sequences of positive real numbers $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ we write $a_n \asymp b_n$ if there exists $R > 0$ such that $\frac{1}{R} a_n \leqslant b_n \leqslant R a_n$ for every $n \in \mathbb{N}$. We may assume that $\rho(\Gamma)$ is contained in $\mathsf{SL}^\pm(d,\mathbb{R})$, otherwise we replace ρ with $\hat{\rho}(\gamma) = |\det(\rho(\gamma))|^{-1/d}\rho(\gamma)$, $\gamma \in \Gamma$. Let $(t_n)_{n\in\mathbb{R}}$ be an increasing unbounded sequence. For each $n \in \mathbb{N}$, we may choose $\gamma_n \in \Gamma$ such that $\gamma_n \varphi_{t_n}(\hat{m})$ lies in the compact fundamental domain F. There exist $k_{1n}, k_{2n} \in K$ so that $\rho(\gamma_n) h = k_{1n} \begin{bmatrix} \lambda_n & * \\ 0 & A_n \end{bmatrix} = k_{2n} \begin{bmatrix} s_n & 0 \\ * & B_n \end{bmatrix}$. Notice that for $g \in P_1^\pm$ we have $dL_g \circ d\pi^\pm = d\pi^\pm \circ \mathrm{Ad}(g)$. Then, an elementary calculation shows that

$$\begin{aligned} dL_{\rho(\gamma_n)h} d\pi^+ \left(\begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix}\right) &= dL_{k_{1n}} \left(d\pi^+ \left(\mathrm{Ad}\left(\begin{bmatrix} \lambda_n & * \\ 0 & A_n \end{bmatrix}\right) \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix}\right)\right) \\ &= dL_{k_{1n}} \left(d\pi^+ \left(\begin{bmatrix} 0 & 0 \\ \frac{1}{\lambda_n} A_n u & 0 \end{bmatrix}\right)\right) \end{aligned}$$

and similarly

$$dL_{\rho(\gamma_n)h} d\pi^- \left(\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}\right) = dL_{k_{2n}} \left(d\pi^- \left(\begin{bmatrix} 0 & s_n B_n^{-t} u \\ 0 & 0 \end{bmatrix}\right)\right)$$

The continuity of the family of norms $\left(||\cdot||_x\right)_{x\in\Gamma\backslash\hat{\Gamma}}$ and the fact that k_{1n}, k_{2n} lie in the compact group K imply

$$\|\varphi_{t_n}(X_u^+)\|_{\varphi_{t_n}(x)} \asymp \frac{\|A_n u\|}{|\lambda_n|}$$
$$\|\varphi_{t_n}(X_u^-)\|_{\varphi_{t_n}(x)} \asymp |s_n|\,\|B_n^{-t}u\|$$

where $||\cdot||$ denotes the usual Euclidean norm on $\mathbb{R}^{d-1}$. Up to passing to a subsequence, we may assume that $\lim_n \gamma_n\varphi_{t_n}(\hat{m}) = \hat{m}'$. Since the maps $\tau^{\pm}$ are continuous we conclude up to subsequence that $(\gamma_n\tau^+(\hat{m}))_{n\in\mathbb{N}}$ and $(\gamma_n\tau^-(\hat{m}))_{n\in\mathbb{N}}$ converge to $\tau^+(\hat{m}')$ and $\tau^-(\hat{m}')$ respectively. We have $\xi^+(\tau^+(\gamma_n\hat{m})) = k_{1n}P_1^+$ and $\xi^-(\tau^+(\gamma_n\hat{m})) = k_{2n}P_1^-$ and by transversality, there exist $p_n \in P_1^+, q_n \in P_1^-$ and $g \in G$ such that $\lim_n k_{1n}p_n = \lim_n k_{2n}q_n = g$. Then there exist $z_n, z_n' \in \mathbb{R}$ so that $\lim_n z_n k_{1n}e_1 = ge_1$ and $\lim_n z_n' k_{2n}e_1 = g^{-t}e_1$ and we observe that $|z_n|, |z_n'|$ converge respectively to $||ge_1||$ and $||g^{-t}e_1||$. Notice that $\lim_n z_n z_n'\langle k_{1n}e_1, k_{2n}e_1\rangle = |\langle g^{-t}e_1, ge_1\rangle| = 1$ and so $\lim_n\langle k_{1n}e_1, k_{2n}e_1\rangle = \frac{1}{||ge_1||\cdot||g^{-t}e_1||}$. Recall that $k_{2n}^{-1}k_{1n}\begin{bmatrix}\lambda_n & * \\ 0 & A_n\end{bmatrix} = \begin{bmatrix}s_n & 0 \\ * & B_n\end{bmatrix}$, hence by looking at the $(1,1)$ entry of both sides we obtain $\left|\frac{s_n}{\lambda_n}\right| = |\langle k_{1n}e_1, k_{2n}e_1\rangle|$ and so $L := \inf_{n\in\mathbb{N}}\left|\frac{s_n}{\lambda_n}\right| > 0$. Furthermore, we observe that $\begin{bmatrix}\lambda_n & 0 \\ * & A_n^t\end{bmatrix} k_{1n}^t = \begin{bmatrix}s_n & * \\ 0 & B_n^t\end{bmatrix} k_{2n}^t$ and hence $\begin{bmatrix}* & * \\ * & B_n^{-t}A_n^t\end{bmatrix} = k_{2n}^{-1}k_{1n}$ since $k_{1n}k_{1n}^t = k_{2n}k_{2n}^t = I_d$. Up to extracting, we may assume that $\lim_n B_n^{-t}A_n^t = Q$ exists. Since $|\lambda_n||\det(A_n)| = |s_n||\det(B_n)|$ we have $|\det(B_n^{-t}A_n^t)| = \left|\frac{s_n}{\lambda_n}\right| \geqslant L > 0$. In particular, Q is invertible and there exists $M > 0$ with $\frac{1}{M} \leqslant \max(||B_n^{-t}A_n^t||, ||A_n^{-t}B_n^t||) \leqslant M$ for every $n \in \mathbb{N}$. Therefore, for every $n \in \mathbb{N}$

$$\frac{\|A_n u\|}{|\lambda_n|} \geqslant \frac{\|u\|^2}{|\lambda_n|\,\|A_n^{-t}u\|} = \frac{\|u\|^2}{|\lambda_n|\,\|A_n^{-t}B_n^t(B_n^{-t}u)\|} \geqslant \frac{\|u\|^2}{|\lambda_n|\,||A_n^{-t}B_n^t||\cdot\|B_n^{-t}u\|} \geqslant \frac{L\,\|u\|^2}{M|s_n|\,\|B_n^{-t}u\|}$$

since $||A_n u||\cdot||A_n^{-t}u|| \geqslant ||u||^2$. Finally,

$$\varliminf_{n\to\infty}\|\varphi_{t_n}(X_u^+)\|_{\varphi_{t_n}(x)}\cdot\|\varphi_{t_n}(X_u^-)\|_{\varphi_{t_n}(x)} > 0$$

and since the sequence $(t_n)_{n\in\mathbb{N}}$ was arbitrary the conclusion follows. $\square$

Proposition 3.1.2. *Let $\rho : \Gamma \to \mathsf{GL}(d,\mathbb{R})$ be a representation which admits a pair of continuous ρ-equivariant transverse maps $\xi^+ : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_\infty\Gamma \to \mathsf{Gr}_{d-1}(\mathbb{R}^d)$. We fix $x = [\hat{m}]_\Gamma$, $u \in \{0\}\times\mathbb{R}^{d-1}$ and suppose $\xi^+(\tau^+(\hat{m})) = hP_1^+$ and $\xi^-(\tau^-(\hat{m})) = hP_1^-$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of elements of Γ such that $\left(\gamma_n\varphi_{t_n}(\hat{m})\right)_{n\in\mathbb{N}}$ lies in a compact subset of $\hat{\Gamma}$. Then*

(i) $\lim_{n\to\infty} ||\varphi_{t_n}(X_u^+)||_{\varphi_{t_n}(x)} = +\infty$ *if and only if*

$$\lim_{n\to\infty} \frac{\|\rho(\gamma_n)hu\|}{\|\rho(\gamma_n)he_1\|} = +\infty$$

(ii) $\lim_{n\to\infty} ||\varphi_{t_n}(X_u^-)||_{\varphi_{t_n}(x)} = 0$ *if and only if*

$$\lim_{n\to\infty} \frac{\|\rho^*(\gamma_n)h^{-t}u\|}{\|\rho^*(\gamma_n)h^{-t}e_1\|} = 0$$

Proof. Suppose that $\rho(\gamma_n)h = k_{1n}\begin{bmatrix}\lambda_n & * \\ 0 & A_n\end{bmatrix} = k_{2n}\begin{bmatrix}s_n & 0 \\ * & B_n\end{bmatrix}$. Let $(\gamma_{r_n})_{n\in\mathbb{N}}$ be a subsequence of $(\gamma_n)_{n\in\mathbb{N}}$. A calculation shows that

$$\frac{\|A_{r_n}u\|}{|\lambda_{r_n}|} = \frac{\|\rho(\gamma_{r_n})hu\|}{\|\rho(\gamma_{r_n})he_1\|} \cdot \sin\measuredangle(\rho(\gamma_{r_n})he_1, \rho(\gamma_{r_n})hu)$$

where $\xi^+(x) = hP_1^+$ and $hu \in \xi^-(y)$. Up to subsequence, we may assume that $\lim_n \gamma_{r_n}\varphi_{t_{r_n}}(\hat{m})$ exists and so $\lim_n \gamma_{r_n}\tau^+(\hat{m}) \neq \lim_n \gamma_{r_n}\tau^-(\hat{m})$. The maps ξ^+ and ξ^- are transverse, hence there exists $g \in G$ and $p_n \in P_1^+$, $q_n \in P_1^-$ such that $\lim_n \rho(\gamma_{r_n})hp_n = \lim_n \rho(\gamma_{r_n})hq_n = g$. Let $v_\infty \in e_1^\perp$ be a limit point of the sequence $\left(\frac{q_n^{-1}u}{||q_n^{-1}u||}\right)_{n\in\mathbb{N}}$. Then we have $\lim_n \frac{1}{||q_n^{-1}u||}\rho(\gamma_{r_n})hu = gv_\infty$ and hence $\lim_n \sin\measuredangle(\rho(\gamma_{r_n})he_1, \rho(\gamma_{r_n})hu) = \sin\measuredangle(gv_\infty, ge_1) > 0$. Since we started with an arbitrary subsequence, there exists $\varepsilon > 0$ with $|\sin\measuredangle(\rho(\gamma_{r_n})he_1, \rho(\gamma_{r_n})hu)| \geqslant \varepsilon$ for every $n \in \mathbb{N}$. Therefore, $\frac{\|A_n u\|}{|\lambda_n|} \asymp \frac{\|\rho(\gamma_n)hu\|}{\|\rho(\gamma_n)he_1\|}$. By Proposition 3.1.1 we have that

$$||\varphi_{t_n}(X_u^+)||_{\varphi_{t_n}(x)} \asymp \frac{\|A_n u\|}{|\lambda_n|}$$

and so part (i) follows. The argument for part (ii) is similar. □

3.2 The Cartan property and the uniform gap summation property

Let G be a linear, non-compact, semisimple Lie group with finitely many components, K a maximal compact subgroup of G, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ and consider the Cartan decomposition $G = K\exp(\overline{\mathfrak{a}}^+)K$ with Cartan projection $\mu : G \to \overline{\mathfrak{a}}^+$. Let Γ be an infinite, finitely generated group and $\rho : \Gamma \to G$ be a representation. We say that ρ is *P_θ-divergent* if

$$\lim_{|\gamma|_\Gamma \to \infty} \langle \alpha, \mu(\rho(\gamma)) \rangle = +\infty$$

for every $\alpha \in \theta$. Notice that the representation ρ is P_θ-divergent if and only if ρ is $P_{\theta^\star}$-divergent. Recall that for an element $g = k_g \exp(\mu(g)) k'_g$ written in the Cartan decomposition of G, we define

$$\Xi_\theta^+(g) = k_g P_\theta^+ \quad \text{and} \quad \Xi_\theta^-(g) = k_g w P_\theta^-$$

For an equivariant map $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$, the map $\xi^* : \partial_\infty \Gamma \to G/P_{\theta^*}^+$ is defined as follows $\xi^*(x) = k_x w P_{\theta^*}^+$, where $\xi^-(x) = k_x P_\theta^-$ and $k_x \in K$.

Definition 3.2.1. *Let G be a real semisimple Lie group, Γ be a word hyperbolic group and suppose that $\rho : \Gamma \to G$ is a P_θ-divergent representation.*

(1) *Suppose that ρ admits a continuous ρ-equivariant map $\xi^+ : \partial_\infty \Gamma \to G/P_\theta^+$. The map ξ^+ satisfies the Cartan property if for any $x \in \partial_\infty \Gamma$ and every infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of Γ with $\lim_n \gamma_n = x$ we have*

$$\xi^+(x) = \lim_{n \to \infty} \Xi_\theta^+(\rho(\gamma_n))$$

(2) *Suppose that ρ admits a continuous ρ-equivariant map $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$. The map ξ^- satisfies the Cartan property if the map $\xi^* : \partial_\infty \Gamma \to G/P_{\theta^\star}^+$ satisfies the Cartan property. In other words, for every $x \in \partial_\infty \Gamma$ and every infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of Γ with $\lim_n \gamma_n = x$, we have*

$$\xi^-(x) = \lim_{n \to \infty} \Xi_\theta^-(\rho(\gamma_n))$$

Remark 3.2.2. Let $\rho : \Gamma \to G$ be a P_θ-divergent representation. The Cartan property for a continuous ρ-equivariant map $\xi^+ : \partial_\infty \Gamma \to G/P_\theta^+$ (resp. ξ^-) is independent of the choice of the Cartan decomposition of G. This follows by the fact that all Cartan subspaces of G are conjugate under the adjoint action of G and the second part of [Gué+17, Corollary 5.9].

The following fact is immediate from the definition of the Cartan property:

Fact 3.2.3. *Suppose that ρ, Γ, G and θ are defined as in Definition 3.2.1 and let $\xi : \partial_\infty \Gamma \to G/P_\theta^+$ be a continuous ρ-equivariant map. Suppose that $\tau : G \to \mathsf{GL}(d, \mathbb{R})$ is an irreducible θ-proximal representation as in Proposition 2.5.4 so that $\tau(P_\theta^+)$ stabilizes a line in $\mathbb{R}^d$ and induces a τ-equivariant embedding $\iota^+ : G/P_\theta^+ \hookrightarrow \mathbb{P}(\mathbb{R}^d)$. The map ξ satisfies the Cartan property if and only if $\iota^+ \circ \xi$ satisfies the Cartan property.*

Let M be a compact metrizable space and Γ a group acting on M by homeomorphisms. The action is called a *convergence group action* if for any infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ there exists a subsequence $(\gamma_{k_n})_{n\in\mathbb{N}}$ and $a,b\in M$ such that for every compact subset $C\subset M-\{a\}$, $\gamma_{k_n}|_C$ converges uniformly to the constant map b. For an infinite order element $\gamma\in\Gamma$, we denote by $\gamma^{\pm}$ the local uniform limit of the sequence $(\gamma^{\pm n})_{n\in\mathbb{N}}$. Examples of convergence group actions include:
(i) the action of a non-elementary word hyperbolic group on its Gromov boundary (see [Gro87]).
(ii) the action of a finitely generated group Γ on its Floyd boundary $\partial_f\Gamma$ (see [Gro96, §8] and [Kar03, Theorem 2]).

We prove a version of [CT20, Lemma 9.2] which shows, in many cases, that a representation ρ is P_θ-divergent when it admits a continuous ρ-equivariant limit map.

Lemma 3.2.4. *Let M be a compact metrizable perfect space and let Γ be a torsion free group acting on M by homeomorphisms and $\rho:\Gamma\to\mathsf{GL}(d,\mathbb{R})$ be a representation. Suppose that Γ acts on M as a convergence group and there exists a continuous ρ-equivariant non-constant map $\xi:M\to\mathbb{P}(\mathbb{R}^d)$. Then for every infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ we have*

$$\overline{\lim_{n\to\infty}}\,\mu_1(\rho(\gamma_n))-\mu_{d-p+2}(\rho(\gamma_n))=+\infty$$

where $p=\dim_{\mathbb{R}}\langle\xi\,(M)\rangle$.

Proof. We first prove the statement when $p=d$. If the result does not hold, then there exists $\varepsilon>0$ and a subsequence which we continue to denote by $(\gamma_n)_{n\in\mathbb{N}}$ such that $\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}\geqslant\varepsilon$. We may write $\rho(\gamma_n)=k_n\exp(\mu(\rho(\gamma_n))k_n'$, where $k_n,k_n'\in\mathsf{O}(d)$. Up to subsequence, there exist $\eta,\eta'\in M$ such that if $x\neq\eta'$ then $\lim_n\gamma_n x=\eta$ and hence $\lim_n\rho(\gamma_n)\xi(x)=\xi(\eta)$. We may also assume that the sequences $(k_n)_{n\in\mathbb{N}},(k_n')_{n\in\mathbb{N}}$ converge to $k,k'\in\mathsf{O}(d)$ respectively and $\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}$ converges to some $C>0$. Let $x\neq\eta$ and write $\xi(x)=k_xP_1^+$ for some $k_x\in\mathsf{O}(d)$. Since $\lim_n\rho(\gamma_n)\xi(x)=\xi(\eta)$, up to passing to a subsequence, we may assume that

$$\lim_{n\to\infty}\frac{\exp\left(\mu(\rho(\gamma_n))\right)k_n'k_xe_1}{||\exp\left(\mu(\rho(\gamma_n))\right)k_n'k_xe_1||}=\epsilon\cdot k^{-1}k_\eta e_1$$

where $\xi(\eta)=k_\eta P_1^+$ and $\epsilon\in\{-1,1\}$. We conclude that for every $x\in X$, there exits $\lambda_x\in\mathbb{R}$ such that $\langle k'k_xe_1,e_1\rangle=\lambda_x\langle k^{-1}k_\eta e_1,e_1\rangle$ and $\langle k'k_xe_1,e_2\rangle=\frac{\lambda_x}{C}\langle k^{-1}k_\eta e_1,e_2\rangle$. Since $\langle\xi\,(M\setminus\{\eta'\})\rangle=\mathbb{R}^d$ and M is perfect, there exists $x_0\neq\eta'$ such that $\lambda_{x_0}\neq 0$. Then for every $x\neq\eta'$ we observe that

$$\langle k'k_xe_1,e_1\rangle=\frac{\lambda_x}{\lambda_{x_0}}\langle kk_{x_0}e_1,e_1\rangle\quad\text{and}\quad\langle k'k_xe_1,e_2\rangle=\frac{\lambda_x}{\lambda_{x_0}}\langle kk_{x_0}e_1,e_2\rangle$$

Therefore, for every $x \neq \eta'$, $k\xi(x)$ lies in the subspace $V = \langle kk_{x_0}e_1\rangle + e_1^{\perp} \cap e_2^{\perp}$, a contradiction since $\dim(V) \leqslant d-1$.

In the case where $p < d$, we consider the subspace $V = \langle \xi(M)\rangle$ and the restriction $\hat{\rho} : \Gamma \to \mathsf{GL}(V)$ of ρ. The map ξ is $\hat{\rho}$-equivariant and a spanning map for $\hat{\rho}$. The conclusion follows by observing that for any $\gamma \in \Gamma$ we have $\frac{\sigma_1(\hat{\rho}(\gamma))}{\sigma_2(\hat{\rho}(\gamma))} \leqslant \frac{\sigma_1(\rho(\gamma))}{\sigma_{d-p+2}(\rho(\gamma))}$. $\square$

Corollary 3.2.5. *Let Γ be a word hyperbolic group, G a real semisimple Lie group and $\theta \subset \Delta$ a subset of simple restricted roots of G.*

(i) *Suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is an irreducible representation which admits a continuous ρ-equivariant map $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$. Then ρ is P_1-divergent and ξ satisfies the Cartan property.*

(ii) *Suppose that $\rho' : \Gamma \to G$ is a Zariski dense representation which admits a continuous ρ'-equivariant map $\xi' : \partial_\infty \Gamma \to G/P_\theta^+$. Then ρ' is P_θ-divergent and ξ' satisfies the Cartan property.*

Proof. (i) We first claim that if $\rho(\gamma)$ is P_1-proximal, then $\xi(\gamma^+)$ is the attracting fixed point in $\mathbb{P}(\mathbb{R}^d)$. Indeed, since ρ is irreducible we have $\langle \xi(\partial_\infty \Gamma)\rangle = \mathbb{R}^d$. If $\rho(\gamma)$ is P_1-proximal, we can find $x \in \partial_\infty \Gamma - \{\gamma^-\}$ such that $\xi(x)$ is not in the repelling hyperplane V_γ^-. Since $\lim_n \gamma^n x = \gamma^+$, we have $\xi(\gamma^+) = x^+_{\rho(\gamma)}$.

Since ρ is irreducible it follows by Lemma 3.2.4 that ρ is P_1-divergent. Let $(\gamma_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements of Γ such that $\lim_n \gamma_n = x$. By the sub-additivity of the Cartan projection μ (see [Gué+17, Fact 2.18]) and Theorem 2.6.1, there exists a finite subset F and $C > 0$ such that for every $\gamma \in \Gamma$, there exists $f \in F$ with $||\lambda(\rho(\gamma f)) - \mu(\rho(\gamma f))|| \leqslant C$. Then, for large $n \in \mathbb{N}$ there exist $f_n \in F$ such that $\rho(\gamma_n f_n)$ is P_1-proximal and $\lambda_1(\rho(\gamma_n f_n)) - \mu_1(\rho(\gamma_n f_n)) \geqslant -C$. Notice also $\lim_n \gamma_n = \lim_n \gamma_n f_n = \lim_n (\gamma_n f_n)^+ = x$ in the compactification $\Gamma \cup \partial_\infty \Gamma$ and so $\lim_n x^+_{\rho(\gamma_n f_n)} = \lim_n \xi((\gamma_n f_n)^+) = \xi(x)$. Then, by using Proposition 2.5.5, for every $n \in \mathbb{N}$ we obtain the estimate:

$$\begin{aligned} d_{\mathbb{P}}\Big(x^+_{\rho(\gamma_n f_n)}, \Xi_1^+(\rho(\gamma_n))\Big) &\leqslant d_{\mathbb{P}}\Big(x^+_{\rho(\gamma_n f_n)}, \Xi_1^+(\rho(\gamma_n f_n))\Big) + d_{\mathbb{P}}\Big(\Xi_1^+(\rho(\gamma_n f_n)), \Xi_1^+(\rho(\gamma_n))\Big) \\ &\leqslant \Big(e^C + \sup_{f\in F} C_{d,f}\Big)\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} \end{aligned}$$

where $C_{d,f} > 0$ is defined as in Proposition 2.5.5 (ii). This shows $\xi(x) = \lim_n \Xi_1^+(\rho(\gamma_n))$ and finally that ξ satisfies the Cartan property.

(ii) Let τ be as in Proposition 2.5.4. Since ρ' is Zariski dense the representation $\tau \circ \rho'$ is irreducible. By Lemma 3.2.4 the representation $\tau \circ \rho'$ is P_1-divergent and hence ρ' is P_θ-divergent. By part (i), the $\tau \circ \rho'$-equivariant map $\iota^+ \circ \xi'$ satisfies the Cartan property. It follows by Fact 3.2.3 that ξ' satisfies the Cartan property. $\square$

3.2.1 The uniform gap summation property

We are now aiming to generalize the uniform gap summation property (see [Gué+17, Definition 5.2]) for representations of arbitrary finitely generated groups.

Definition 3.2.6. *Let Γ be a finitely generated group, $\rho : \Gamma \to G$ be a representation and $\theta \subset \Delta$ a finite subset of restricted roots of G. We say that ρ satisfies the uniform gap summation property with respect to θ and the Floyd function $f : \mathbb{N} \to \mathbb{R}^+$ if there exists $C > 0$ such that*

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \geqslant -\log\left(f(|\gamma|_\Gamma)\right) - C$$

for every $\gamma \in \Gamma$ and $\alpha \in \theta$.

We say that the representation ρ satisfies the uniform gap summation property if there exists a Floyd function f, a subset of simple roots $\theta \subset \Delta$ and $C > 0$ with the previous properties.

Let $\rho : \Gamma \to G$ be a representation. In [Gué+17, Theorem 5.3 (3)] it is proved that if Γ is word hyperbolic group and ρ satisfies the uniform gap summation property, then it admits a pair of ρ-equivariant, continuous limit maps which satisfy the Cartan property. If Γ is not word hyperbolic, we may obtain a pair of ρ-equivariant maps from a Floyd boundary $\partial_f\Gamma$ of Γ into the flag spaces G/P_θ^+ and G/P_θ^-. Note that when $\partial_f\Gamma$ is non-trivial, the action of Γ on $\partial_f\Gamma$ is a convergence group action (see [Kar03, Theorem 2]) so we obtain additional information for the action of $\rho(\Gamma)$ on its limit set in $G/P_\theta^\pm$.

Lemma 3.2.7. *Let Γ be a finitely generated group, G a real semsimiple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G and $\rho : \Gamma \to G$ a representation. Suppose that ρ satisfies the uniform gap summation property with respect to θ and the Floyd function $f : \mathbb{N} \to \mathbb{R}^+$. Then, there exists a constant $C > 0$ with*

$$d_{G/P_\theta^\pm}\Big(\Xi_\theta^\pm(\rho(g)), \Xi_\theta^\pm(\rho(h))\Big) \leqslant C \cdot d_f(g,h)$$

for all but finitely many $g, h \in \Gamma$. In particular, there exists a pair of continuous ρ-equivariant maps

$$\xi_f^+ : \partial_f\Gamma \to G/P_\theta^+ \quad \text{and} \quad \xi_f^- : \partial_f\Gamma \to G/P_\theta^-.$$

Moreover, if $\rho(\Gamma)$ contains a P_θ-proximal element, then $\xi_f^+(\partial_f\Gamma)$ maps onto the P_θ-proximal limit set of $\rho(\Gamma)$ in G/P_θ^+.

Proof. As in the proof of Proposition 2.5.4, we may assume that $\theta=\{\varepsilon_1-\varepsilon_2\}$ and $G=\mathsf{SL}(d,\mathbb{R})$ and $G/P_\theta^+=\mathbb{P}(\mathbb{R}^d)$. Let S be a fixed generating set of Γ defining a left invariant metric $|\cdot|_\Gamma$. There exists a constant $K>0$ such that $\frac{\sigma_2(\rho(\gamma)}{\sigma_1(\rho(\gamma))}\leqslant Kf(|\gamma|_\Gamma)$ for all $\gamma\in\Gamma$. Let $\mathbf{p}$ be a path in the Cayley graph of Γ defined by the sequence of adjacent vertices $g_0=g,...,h=g_n$ with $L_f(\mathbf{p})=d_f(g,h)$. Then, by using Proposition 2.5.5, we may find $C'>0$ depending only on S and ρ such that:

$$\begin{aligned}d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g)),\Xi_1^+(\rho(h))\Big)&\leqslant\sum_{i=0}^{n-1}d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g_i)),\Xi_1^+(\rho(g_{i+1}))\Big)\\&\leqslant C'\sum_{i=0}^{n-1}\frac{\sigma_2(\rho(g_i))}{\sigma_1(\rho(g_i))}\leqslant C'K\sum_{i=0}^{n-1}f(|g_i|_\Gamma)=C'Kd_f(g,h)\end{aligned}$$

Now define the map $\xi_f^+:\partial_f\Gamma\to\mathbb{P}(\mathbb{R}^d)$ as follows: for a point $x\in\partial_f\Gamma$ represented by a Cauchy sequence $(\gamma_n)_{n\in\mathbb{N}}$ (with respect to d_f), then

$$\xi_f^+(x)=\lim_{n\to\infty}\Xi_1^+\big(\rho(\gamma_n)\big)$$

The previous inequality shows that the limit $\lim_n\Xi_1^+\big(\rho(\gamma_n)\big)$ is independent of the choice of the sequence $(\gamma_n)_{n\in\mathbb{N}}$ representing x, since for any other sequence $(\gamma_n')_{n\in\mathbb{N}}$ with $x=\lim_n\gamma_n'$, we have $\lim_n d_f(\gamma_n,\gamma_n')=0$. Finally, ξ_f^+ is well defined and Lipshitz and hence continuous. By identifying G/P_θ^- with $G/P_{\theta^\star}^+$, we similarly obtain the limit map ξ_f^-.

Suppose that $\rho(\Gamma)$ is P_1-proximal. By the definition of the map ξ_f^+, if $\rho(\gamma_0)$ is P_1-proximal, then $\xi_f^+(\gamma_0^+)$ is the unique attracting fixed point of $\rho(\gamma_0)$ in $\mathbb{P}(\mathbb{R}^d)$. Since Γ acts minimally on $\partial_f\Gamma$ we have $\xi_f^+(\partial_f\Gamma)=\Lambda_\Gamma$. □

We remark that in the case where Γ is a geometrically finite Kleinian subgroup of $\mathsf{SO}(3,1)$, the natural inclusion $\Gamma\hookrightarrow\mathsf{SL}(4,\mathbb{R})$ satisfies the uniform gap summation property (see [Flo80]).

3.3 Proof of Theorem 1.4.1 and 1.4.4

This section is devoted to the proof of Theorems 1.4.1 and 1.4.4. Note that in the statement of Theorem 1.4.1 we do not assume that the group $\rho(\Gamma)$ contains a P_θ-proximal element, the pair of limit maps (ξ^+,ξ^-) is compatible or the map ξ^- satisfies the Cartan property.

Theorem 1.4.1. *Let Γ be a word hyperbolic group, G a real semisimple Lie group, $\theta\subset\Delta$ a subset* *of simple restricted roots of G and $\rho : \Gamma \to G$ a representation. Then ρ is P_θ-Anosov if and only if the following conditions hold:*

(i) *ρ is P_θ-divergent.*

(ii) *There exists a pair of continuous, ρ-equivariant transverse maps*

$$\xi^+ : \partial_\infty \Gamma \to G/P_\theta^+ \quad \text{and} \quad \xi^- : \partial_\infty \Gamma \to G/P_\theta^-$$

and the map ξ^+ satisfies the Cartan property.

Proof. If ρ is P_θ-Anosov, the Anosov limit maps of ρ are transverse and dynamics preserving and ρ is P_θ-divergent by Theorem 2.5.3 (ii). Also, the fact that the Anosov limit maps satisfy the Cartan property is contained in [Gué+17, Theorem 1.3 (4) & 5.3 (4)].

Now we assume that ρ satisfies (i) and (ii). We first reduce to the case where Γ is torsion free. Since ρ is P_θ-divergent, every element of the kernel $\ker(\rho)$ has finite order, hence $\ker(\rho)$ is finite. The quotient group $H = \Gamma/\ker(\rho)$ is quasi-isometric to Γ and by Selberg's lemma [Sel62] H contains a torsion free and finite-index subgroup H_1. It is enough to prove that the induced representation $\hat{\rho} : H_1 \to G$ is P_θ-Anosov. Notice that $\hat{\rho}$ satisfies the same assumptions as ρ and the source group is torsion free.

By Proposition 2.5.4, we may assume that $G = \mathsf{SL}(d, \mathbb{R})$, $\theta = \{\varepsilon_1 - \varepsilon_2\}$, $P_\theta^+ = \mathrm{Stab}(\mathbb{R}e_1)$ and $P_\theta^- = \mathrm{Stab}(e_1^\perp)$. We consider the section $\sigma : \Gamma\backslash\hat{\Gamma} \to \mathcal{X}_\rho$, $\sigma([\hat{m}]_\Gamma) = \big[\hat{m}, \xi^+(\tau^+(\hat{m})), \xi^-(\tau^-(\hat{m}))\big]_\Gamma$ inducing the splitting $\sigma_*\mathcal{E} = \sigma_*\mathcal{E}^+ \oplus \sigma_*\mathcal{E}^-$. Then we fix $x = [\hat{m}]_\Gamma$ and choose an element $h \in G$ so that $\xi^+(\tau^+(\hat{m})) = hP_1^+$ and $\xi^-(\tau^-(\hat{m})) = hP_1^-$. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing unbounded sequence. We consider a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ such that $(\gamma_n\varphi_{t_n}(\hat{m}))_{n\in\mathbb{N}}$ lies in a compact subset of $\hat{\Gamma}$. We observe that $\lim_n \gamma_n^{-1} = \tau^+(\hat{m})$ in the bordification $\Gamma \cup \partial_\infty\Gamma$. We write $\rho(\gamma_n^{-1}) = (k_n')^{-1} w \exp(\mu(\rho(\gamma_n^{-1})) w k_n^{-1}$ in the Cartan decomposition of G, where $w = \sum_{i=1}^d E_{i(d+1-i)} \in \mathsf{O}(d)$. Since ξ^+ is assumed to satisfy the Cartan property and $(\gamma_n)_{n\in\mathbb{N}}$ is P_θ-divergent, up to subsequence, we may assume that $\lim_n \Xi_1^+(\rho(\gamma_n^{-1})) = \lim_n (k_n')^{-1} w P_\theta^+ = hP_\theta^+$. Equivalently, if $k' = \lim_n k_n'$ then $k'h = w \begin{bmatrix} s & * \\ 0 & B \end{bmatrix}$ for some $B \in \mathsf{GL}(d-1, \mathbb{R})$. Fix $u \in \{0\} \times \mathbb{R}^{d-1}$. Then, since $k'(k')^t = I_d$, we observe that $k'h^{-t}u = w_{d-1}B^{-t}u + 0e_d$ and $k'h^{-t}e_1 = \frac{1}{s}e_d + \sum_{i=1}^{d-1} \zeta_i e_i$ for some $s \neq 0$, $\zeta_1, ..., \zeta_{d-1} \in \mathbb{R}$ and $w_{d-1} \in \mathsf{O}(d-1)$ is a permutation matrix with $w_{d-1}e_1 = e_{d-1}$ and $w_{d-1}e_{d-1} = e_1$. Equivalently, we write:

$$k_n'h^{-t}u = \sum_{i=1}^d \chi_{i,n} e_i \quad \text{and} \quad k_n'h^{-t}e_1 = \sum_{i=1}^d \zeta_{i,n} e_i$$

we have that $\lim_n \chi_{d,n} = 0$ and $\lim_n \zeta_{d,n} = \frac{1}{s}$. A computation shows that

$$\begin{aligned} \frac{\|\rho^*(\gamma_n)h^{-t}u\|^2}{\|\rho^*(\gamma_n)h^{-t}e_1\|^2} &= \frac{\sum_{i=1}^d \chi_{i,n}^2 e^{-2\mu_i(\rho(\gamma_n))}}{\sum_{i=1}^d \zeta_{i,n}^2 e^{-2\mu_i(\rho(\gamma_n))}} \\ &= \frac{\chi_{1,n}^2 e^{-2\mu_1(\rho(\gamma_n))+2\mu_d(\rho(\gamma_n))} + \sum_{i=2}^{d-1} \chi_{i,n}^2 e^{-2\mu_1(\rho(\gamma_n))+2\mu_d(\rho(\gamma_n))} + \chi_{d,n}^2}{\sum_{i=1}^{d-1} \zeta_{i,n}^2 e^{-2\mu_1(\rho(\gamma_n))+2\mu_d(\rho(\gamma_n))} + \zeta_{d,n}^2} \end{aligned}$$

and we deduce $\lim_n \frac{||\rho(\gamma_n)^* h^{-t}u||}{||\rho(\gamma_n)^* h^{-t}e_1||} = 0$. By Proposition 3.1.2 (ii) we obtain

$$\lim_{n\to\infty} ||\varphi_{t_n}(X_u^-)||_{\varphi_{t_n}(x)} = 0$$

The sequence we started with was arbitrary, therefore the geodesic flow on $\sigma_*\mathcal{E}^-$ is weakly contracting. By Lemma 3.1.1 we conclude that the flow on $\sigma_*\mathcal{E}^+$ is weakly dilating. The compactness of $\Gamma\backslash\hat{\Gamma}$ implies that the geodesic flow on $\sigma_*\mathcal{E}^+$ (resp. $\sigma_*\mathcal{E}^-$) is uniformly dilating (resp. contracting). Finally, ρ is P_θ-Anosov with Anosov limit maps ξ^+ and ξ^-. □

Proof of Corollary 1.4.2. Assume that conditions (i) and (ii) hold. Let $\tau : G \to \mathsf{GL}(d,\mathbb{R})$ be an irreducible and θ-proximal representation as in Proposition 2.5.4. It is enough to prove that $\rho' = \tau\circ\rho$ is P_1-Anosov. By using [Gué+17, Theorem 5.3 (1)] (see also Lemma 3.2.7), there exists a pair of continuous, ρ'-equivariant maps $\xi^+ : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_\infty\Gamma \to \mathsf{Gr}_{d-1}(\mathbb{R}^d)$ satisfying the Cartan property. Let $x, y \in \partial_\infty\Gamma$ be two distinct points and $(\gamma_n)_{n\in\mathbb{N}}$ a sequence of elements of Γ with $x = \lim_n \gamma_n$ and $y = \lim_n \gamma_n^{-1}$. The second condition, shows that $\sup_n\langle\varepsilon_1, 2\mu(\rho'(\gamma_n)) - \mu(\rho'(\gamma_n^2))\rangle < +\infty$. By Proposition 3.6.2 we have that $\mathrm{dist}(\xi^+(x), \xi^-(y)) \cdot \mathrm{dist}(\xi^+(y), \xi^-(y)) > 0$ so the pair $(\xi^+(x), \xi^-(y))$ is transverse. The maps ξ^+ and ξ^- are transverse, ρ' is P_1-divergent by (i), so by Theorem 1.4.1 ρ' is P_1-Anosov.

Conversely, part (i) follows by Theorem 2.5.3 (ii). By Proposition 1.4.9 (i) we can find $A, b > 0$ such that for every $\alpha \in \theta$ and $\gamma \in \Gamma$, $\langle\omega_\alpha, 2\mu(\rho(\gamma)) + 2\mu(\rho(\gamma^{-1})) - \mu(\rho(\gamma^2)) - \mu(\rho(\gamma^{-2}))\rangle \leqslant A(\gamma \cdot \gamma^{-1})_e + b$. There exists $N \geq 1$ such that $N\omega_\alpha$ is the highest weight χ_{τ_α} of an irreducible proximal representation τ_α of G (see for example [Gué+17, Lemma 3.2]). In particular, $N\langle\omega_\alpha, \mu(h)\rangle = \langle\varepsilon_1, \mu(\tau_\alpha(h))\rangle$ for every $h \in G$. Therefore, $\langle\omega_\alpha, 2\mu(g) - \mu(g^2)\rangle \geqslant 0$ for every $g \in G$. Now part (ii) follows. □

Let Γ be a word hyperbolic group and H be a subgroup of Γ. The group H is *quasiconvex* in Γ if and only if H is finitely generated and quasi-isometrically embedded in Γ. In this case, there exists a continuous injective H-equivariant map $\iota_H : \partial_\infty H \hookrightarrow \partial_\infty\Gamma$ called the *Cannon-Thurston map* extending the inclusion $H \hookrightarrow \Gamma$.

Theorem 1.4.4. *Let Γ be a word hyperbolic group, H a quasiconvex subgroup of Γ, G a semisimple Lie group, $\theta \subset \Delta$ a subset of simple restricted roots of G and $\rho : \Gamma \to G$ a Zariski dense representation. Suppose that ρ admits continuous ρ-equivariant maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta^+$ and $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$. Then $\rho|_H$ is P_θ-Anosov if and only if the maps $\xi^+ \circ \iota_H$ and $\xi^- \circ \iota_H$ are transverse.*

Proof of Theorem 1.4.4. Corollary 3.2.5 shows that the representation ρ is P_θ-divergent and ξ^+ satisfies the Cartan property. Since ι_H is an H-equivariant embedding, the map $\xi^+ \circ \iota_H$ also satisfies the Cartan property. Theorem 1.4.1 then implies that the representation $\rho|_H$ is P_θ-Anosov. $\square$

We observe (see Example 3.10.4) that there exists a closed hyperbolic surface S and a Zariski dense representation $\rho_1 : \pi_1(S) \to \mathsf{PSL}(4, \mathbb{R})$ which is not P_1-Anosov and admits a pair of continuous ρ_1-equivariant maps (ξ^+, ξ^-). The representation ρ_1 is P_1-divergent and $\rho_1(\gamma)$ is P_1-proximal for every $\gamma \in \pi_1(S)$ non-trivial. However, for every finitely generated free subgroup F of $\pi_1(S)$, the maps $\xi^+ \circ \iota_F$ and $\xi^- \circ \iota_F$ are transverse and $\rho_1|_F$ is P_1-Anosov.

3.4 Proof of Theorem 1.4.7

In this section, we prove Theorem 1.4.7 which was established by Richard Canary and the author in [CT20]. Let us first recall that a representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$, $d \geq 3$ of a torsion-free group Γ is a *Benoist representation* if and only if $\rho(\Gamma)$ is a discrete subgroup of $\mathsf{SL}(d, \mathbb{R})$ which acts properly discontinuosuly and cocompactly on a strictly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$.

Theorem 1.4.7 ([CT20, Theorem 1.5 & 1.7]). *Let Γ be a torsion free word hyperbolic group of cohomological dimension at least $d-1 \geq 3$ and suppose that $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a representation. The following conditions are equivalent:*

(i) *ρ is a Benoist representation.*

(ii) *ρ is P_1-Anosov.*

(iii) *There exists a non-constant continuous ρ-equivariant map $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$.*

Since Benoist representations into $\mathsf{SL}(d, \mathbb{R})$ are P_1-Anosov, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow. Therefore, it is enough to establish the implication (iii) $\Rightarrow$ (i).

First, we prove:

Proposition 3.4.1. ([CT20, Proposition 9.3]) *Suppose that Γ is a torsion-free hyperbolic group of cohomological dimension $m \geq d-1$ which does not admit a cyclic splitting. If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is irreducible and there exists a ρ-equivariant continuous map $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$, then $m = d-1$ and ρ is a Benoist representation.*

Proof. Since ρ is irreducible we have $\langle\xi(\partial_\infty\Gamma)\rangle = \mathbb{R}^d$. By Lemma 3.2.4, ρ is P_1-divergent. Since ρ is irreducible, by Theorem 2.6.1, we may choose $\gamma_0 \in \Gamma$ so that $\rho(\gamma_0)$ is biproximal. We may assume that the attracting eigenlines of $\rho(\gamma_0)$ and $\rho(\gamma_0^{-1})$ are $\langle e_1\rangle$ and $\langle e_d\rangle$ respectively and the corresponding attracting hyperplanes are $e_d^\perp$ and $e_1^\perp$. In particular, since Γ acts as a convergence group on $\partial_\infty\Gamma$ and $\langle\xi(\partial_\infty\Gamma)\rangle = \mathbb{R}^d$, we deduce that $\xi(\gamma_0^+) = [e_1]$ and $\xi(\gamma_0^-) = [e_d]$. Suppose that $x \in \partial_\infty\Gamma - \{\gamma_0^+, \gamma_0^-\}$. Since $\lim_n \gamma_0^n x = \gamma_0^+$ and $\lim_n \gamma_0^{-n} x = \gamma_0^-$, $\xi(x)$ cannot lie in either $\mathbb{P}(e_1^\perp)$ or $\mathbb{P}(e_d^\perp)$. Since the group Γ does not split over a cyclic subgroup, the set $\partial\Gamma - \{\gamma_0^\pm\}$ is connected (see Bowditch [Bow98, Theorem 6.2]), so we may assume that $\xi(\partial_\infty\Gamma - \{\gamma_0^\pm\})$ is contained in the connected component $\{[1 : x_2 : ... : x_d :] : x_d > 0\}$ of $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(e_1^\perp) \cup \mathbb{P}(e_d^\perp)$. It follows that $\xi(\partial_\infty\Gamma)$ lies in the affine chart $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(V)$ where $V = \{(x_1, ..., x_d) \in \mathbb{R}^{n+1} : x_1 = -x_d\}$. By [CT20, Proposition 2.8] $\rho(\Gamma)$ preserves a properly convex domain Ω in $\mathbb{P}(\mathbb{R}^d)$. Since $\rho(\Gamma)$ is P_1-divergent, it is discrete and faithful, so it must acts properly discontinuously on Ω (see [Ben05, Fact 2.10]). Since $\rho(\Gamma)$ has cohomological dimension $m \geq d-1$, it must have compact quotient. Hence, by Benoist [Ben04, Theorem 1.1], Ω is strictly convex, so ρ is a Benoist representation and $m = d-1$. $\square$

In order to complete the proof of Theorem 1.4.7 we need to reject the cases where ρ is reducible or Γ admits a non-trivial cyclic splitting.

Proposition 3.4.2. *Let Γ be a word hyperbolic group and $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ be a representation. If there exists a continuous ρ-equivariant non-constant map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$, then ρ is discrete and* $\ker(\rho)$ *is finite.*

Proof. Assume that there exists an infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ of distinct elements of Γ with $\lim_n \rho(\gamma_n) = I_d$. The group Γ acts on $\partial_\infty\Gamma$ as a convergence group, hence up to subsequence, there exists $\eta, \eta' \in \partial_\infty\Gamma$ with $\lim_n \gamma_n x = \eta$ for $x \neq \eta'$ and $\xi(x) = \xi(\eta)$ for $x \neq \eta'$. Since $\partial_\infty\Gamma$ is perfect, ξ has to be constant, a contradiction. $\square$

We also need the following lemma which asserts that for a non-elementary hyperbolic group Γ and a linear representation $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$, a continuous, spanning ρ-equivariant limit map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ is non-constant when restricted on the Gromov boundary $\partial_\infty\Gamma_0$ of a quasiconvex non-cyclic subgroup Γ_0 of Γ.

Lemma 3.4.3. ([CT20, Lemma 9.5]) *Suppose that Γ is a torsion-free hyperbolic group and Γ_0 is a non-abelian quasiconvex subgroup of Γ. If $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ admits a continuous ρ-equivariant map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ so that $\xi(\partial\Gamma)$ spans $\mathbb{R}^d$, then the restriction of ξ to $\partial_\infty\Gamma_0$ is non-constant.*

Proof. Proposition 3.4.2 implies that ρ is discrete and faithful. Suppose that ξ is constant on $\partial_\infty\Gamma_0$. By conjugating, we may assume $\xi(\partial_\infty\Gamma_0) = \{[e_1]\}$. Then $\rho|_{\Gamma_0}$ has the form

$$\rho(\gamma) = \begin{bmatrix} \varepsilon(\gamma) & u(\gamma) \\ 0 & |\varepsilon(\gamma)|^{-\frac{1}{d-1}}\rho_0(\gamma) \end{bmatrix}$$

for some homomorphism $\varepsilon : \Gamma \to \mathbb{R}^*$ and some representation $\rho_0 : \Gamma_0 \to \mathsf{SL}(d-1,\mathbb{R})$. Notice that the representation of $\hat{\rho} : \Gamma_0 \to \mathsf{SL}(d,\mathbb{R})$ given by

$$\hat{\rho}(\gamma) = \begin{bmatrix} \varepsilon(\gamma) & 0 \\ 0 & |\varepsilon(\gamma)|^{-\frac{1}{d-1}}\rho_0(\gamma) \end{bmatrix}$$

is the limit of the discrete faithful representations $\{Q_n^{-1} \circ \rho|_{\Gamma_0} \circ Q_n\}$, where Q_n is a diagonal matrix with $a_{11} = n$ and all other diagonal entries equal to 1, so $\hat{\rho}$ is discrete and faithful (see Kapovich [KK01, Theorem 8.4]). We next show that if $\gamma \in \Gamma_0$ and $\varepsilon(\gamma) = 1$, then $\lambda_i(\hat{\rho}(\gamma)) = 1$ for all i. Suppose that there exists $\gamma \in \Gamma_0$ which has an eigenvalue of modulus strictly greater than 1. We may consider the Jordan normal formal for $\rho_0(\gamma)$, regarded as a matrix in $\mathsf{SL}(d-1,\mathbb{C})$, i.e.

$$\rho_0(\gamma) = P \begin{bmatrix} J_{q_1,k_1} & & \\ & \ddots & \\ & & J_{q_r,k_r} \end{bmatrix} P^{-1}$$

where $P \in \mathsf{SL}(d-1,\mathbb{C})$ and $J_{q,k}$ is the k-dimensional Jordan block with the value $q \in \mathbb{C}$ along the diagonal. We may assume that $|q_1| \geqslant \cdots \geqslant |q_r|$ and that if $|q_i| = |q_{i+1}|$, then $k_i \geq k_{i+1}$. Notice that, if n is sufficiently large, the co-efficient of $J_{q,k}^n$ with largest modulus has modulus exactly $\binom{n}{k-1}|q|^{n-k+1}$ It follows that there exists $C > 1$ so that

$$\frac{1}{C}n^{k_1-1}|q_1|^{n-k_1+1} \leq \|\rho_0(\gamma^n)\| \leq Cn^{k_1-1}|q_1|^{n-k_1+1}$$

for all $n \in \mathbb{N}$. Therefore, $\left\{\left(n^{k_1-1}|q_1|^{n-k_1+1}\right)^{-1} \rho_0(\gamma^n)\right\}_{n\in\mathbb{N}}$ has a subsequence which converges to a non-zero matrix $A_\infty \in \mathsf{GL}(d-1,\mathbb{R})$. Now let us choose $w = x_1e_1 + v$, where $v \in \mathbb{R}^{d-1} - \ker(A_\infty)$. Note that we can write:

$$\rho(\gamma^n) = \begin{bmatrix} 1 & u_n \\ 0 & \rho_0(\gamma^n) \end{bmatrix}, u_n = \sum_{i=0}^{n} \rho_0(\gamma^i)^t u(\gamma)$$

$$\rho(\gamma^n)w = x_n e_1 + \rho_0(\gamma^n)v,\ x_n := x_1 + \sum_{i=0}^{n} \langle \rho_0(\gamma^i)v, u(\gamma_0)\rangle$$

and observe that

$$|x_n| \leq |x_1| + \sum_{i=0}^{n} ||\rho_0(\gamma^i)|| \cdot ||u(\gamma_0)|| \cdot ||v|| \leq C_1 \sum_{i=0}^{n} i^{k_1-1}|q_1|^i \leq C_2 n^{k_1-1}|q_1|^n$$

for some constants $C_1, C_2 > 0$ independent of n. Up to passing to a sub-sequence, we deduce that

$$\lim_{n\to\infty} \frac{1}{n^{k_1-1}|q_1|^{n-k_1+1}} \rho(\gamma^n)w = \Big(\lim_{n\to\infty} \frac{x_n}{n^{k_1-1}|q_1|^{k_1-1}}\Big)e_1 + A_\infty v$$

We conclude that $\{[\rho(\gamma)^n(w)]\}$ does not converge to $[e_1]$. Since $\xi(\partial_\infty\Gamma)$ spans $\mathbb{R}^d$ and $\ker(A_\infty)$ is a proper subspace of $\mathbb{R}^d$, there exists $x \in \partial_\infty\Gamma - \partial_\infty\Gamma_0$ so that $\xi(x) \notin \mathbb{P}(\mathbb{R} \times \ker(A_\infty))$. Since ξ is ρ-equivariant, $\{\rho(\gamma)^n(\xi(x))\}_{n\in\mathbb{N}}$ must converge to $\xi(\gamma^+) = [e_1]$, which contradicts the previous calculation. Therefore, it follows that for every $\gamma \in \Gamma_0$ with $\varepsilon(\gamma_0) = 1$, all of the eigenvalues of the matrix $\hat{\rho}(\gamma)$ have modulus 1.

Notice that if N is the commutator subgroup of Γ_0, then $\varepsilon(N) = \{1\}$. Since Γ_0 is a non-abelian torsion-free hyperbolic group, N contains a free subgroup Δ of rank 2. Let $\psi = \hat{\rho}|_\Delta^{ss}$ be a semisimplification of $\hat{\rho}|_\Delta$. Since ψ is a limit of conjugates of $\hat{\rho}|_\Delta$ and $\hat{\rho}|_\Delta$ is discrete and faithful, ψ is also discrete and faithful [KK01, Theorem 8.4]. Since $\lambda_i(\psi(\gamma)) = \lambda_i(\hat{\rho}(\gamma)) = 0$ for all $\gamma \in \Delta$ and all i, Theorem 2.6.1 guarantees that there exists M so that $\big|\mu_i(\psi(\gamma))\big| \leq M$ for all $\gamma \in \Delta$ and all $1 \leq i \leq d$. Therefore, $\psi(\Delta)$ is bounded which contradicts the fact that ψ is discrete and faithful and that Δ is infinite. $\square$

Moreover, we need the following proposition showing that when Γ does not split over a cyclic subgroup and has large cohomological dimension, a representation of Γ which admits a spanning limit map has to be irreducible:

Proposition 3.4.4. ([CT20, Proposition 9.4]) *Suppose that Γ is a torsion-free hyperbolic group of cohomological dimension $d-1 \geq 3$ which does not admit a cyclic splitting. If $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a representation and there exists a ρ-equivariant continuous non-constant map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ so that $\xi(\partial_\infty\Gamma)$ spans $\mathbb{R}^d$, then ρ is irredicuble.*

Proof. If ρ is reducible, one may conjugate it to have the form

$$\begin{bmatrix} \rho_1 & * & * & * \\ 0 & \rho_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \rho_k \end{bmatrix}$$

where $k \geq 2$, each $\rho_i : \Gamma \to \mathsf{GL}(V_i)$ is an d_i-dimensional irreducible representation and $\mathbb{R}^d = \oplus_{i=1}^k V_i$. Notice that if $x \in \partial_\infty\Gamma$ and $\xi(x)$ lies in $\hat{V} = \mathbb{R}^{d-d_k} \times \{0\}^{d_k}$ then, since Γ acts minimally on $\partial_\infty\Gamma$ and $\rho(\Gamma)$ preserves $\hat{V}$, $\xi(\partial_\infty\Gamma)$ would be contained in the proper subspace $\hat{V}$, which would contradict our assumption that $\xi(\partial_\infty\Gamma)$ spans $\mathbb{R}^d$. It follows that there exists a ρ_k-equivariant map $\xi_k : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^{d_k})$, obtained by letting $\xi_k(x)$ denote the orthogonal projection of $\xi(x)$ onto V_k. Notice that $\xi_k(\partial_\infty\Gamma)$ spans V_k, since $\xi(\partial_\infty\Gamma)$ spans $\mathbb{R}^d$. Proposition 3.4.1, applied to the representation ρ_k, implies that Γ has cohomological dimension $d_k - 1$, which is a contradiction. $\square$

We also need the following result, following by the work of Louder-Touikan [LT17] and allows us to find quasiconvex subgroups in a torsion-free word hyperbolic group Γ which do not split.

Proposition 3.4.5. ([CT20, Proposition 9.6]) *Let Γ be a torsion-free hyperbolic group of cohomological dimension ≥ 3 which splits over a cyclic subgroup. Then Γ contains an infinite index, quasiconvex subgroup of cohomological dimension* $\mathrm{cd}(\Gamma)$ *which does not split over a cyclic subgroup.*

Proof. One first considers a maximal splitting of Γ along cyclic subgroups. One of the factors, say Δ has cohomological dimension m (see Bieri [Bie75, Corollary 4.1] and Swan [Swa69, Theorem 2.3]). A result of Bowditch [Bow98, Proposition 1.2], implies that Δ is a quasiconvex subgroup of Γ. If Δ itself splits along a cyclic subgroup, we consider a maximal splitting of Δ along cyclic subgroups. We then again find a factor Δ_1 of this decomposition which has cohomological dimension m and is quasiconvex in Δ, hence in Γ. Louder and Touikan [LT17, Corollary 2.7] implies that this process terminates after finitely many steps, so one obtains the desired quasiconvex subgroup of cohomological dimension $\mathrm{cd}(\Gamma)$. $\square$

Proof of Theorem 1.4.7 (iii) $\Rightarrow$ (i). Suppose that Γ has cohomological dimension $d-1$ and there is a non-constant ρ-equivariant map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$. There are two cases to consider.

Case 1. Suppose that Γ does not admit a non-trivial splitting over a cyclic subgroup. If $W = \langle \xi(\partial_\infty\Gamma) \rangle = \mathbb{R}^d$, by Proposition 3.4.4, ρ is irreducible. Therefore, by Proposition 3.4.1 ρ is a Benoist representation.

Now suppose that $V := \langle \xi(\partial_\infty \Gamma)\rangle$ is a proper subspace of $\mathbb{R}^d$. Let $\pi_W : \mathsf{GL}(V) \to \mathsf{SL}^{\pm}(V)$ be the obvious projection map. Consider $\hat\rho = \pi_V \circ \rho|_V : \Gamma \to \mathsf{SL}^{\pm}(V)$ and the non-constant $\hat\rho$-equivariant map $\hat\xi : \partial_\infty \Gamma \to \mathbb{P}(V)$ (which is simply ξ with the range regarded as $\mathbb{P}(V)$). Since ξ is non-constant, V has dimension at least 2. If V has dimension 2, then, by Proposition 3.4.2, ρ is discrete and faithful, which implies that Γ is a free group or surface group, contradicting our assumptions on Γ. If W has dimension at least 3, then Proposition 3.4.4 implies that $\hat\rho$ is irreducible, while Proposition 3.4.1 provides a contradiction in this case.

Case 2. Suppose that Γ admits a non-trivial splitting over a cyclic group. We will derive a contradiction. By Proposition 3.4.5, Γ contains an infinite index, quasiconvex subgroup Γ_1 of cohomological dimension $d-1$ which does not split over a cyclic subgroup.

We next observe that $\xi(\partial_\infty \Gamma_1)$ must span $\mathbb{R}^d$. If it doesn't, let W be the subspace spanned by $\xi(\partial_\infty \Gamma_1)$. We obtain a representation $\hat\rho : \Gamma \to \mathsf{SL}^{\pm}(W)$, given by $\pi_W \circ \rho|_W$ and a continuous $\hat\rho$-equivariant map $\hat\xi : \partial_\infty \Gamma \to \mathbb{P}(W)$, which is simply ξ with the range regarded as $\mathbb{P}(W)$, so that $\hat\xi(\partial_\infty \Gamma)$ spans W. There exists a subgroup Γ_2 of index at most two, so that $\hat\rho(\Gamma_2)$ lies in $\mathsf{SL}(W)$. Notice that Γ_2 also has cohomological dimension $d-1$. By Lemma 3.4.3, $\hat\xi|_{\partial_\infty \Gamma_2}$ is non-constant, so Propositions 3.4.1 and 3.4.4 imply that $\hat\rho|_{\Gamma_2}$ is a Benoist representation and that W has dimension d, which is a contradiction. It follows that $\langle \xi(\partial_\infty \Gamma_1)\rangle = \langle \xi(\partial_\infty \Gamma)\rangle = \mathbb{R}^d$.

Lemma 3.4.3 implies that $\xi|_{\partial_\infty \Gamma_2}$ is non-constant, so Proposition 3.4.1 implies that $\rho_1 = \rho|_{\Gamma_1}$ is a Benoist representation. Therefore, $\rho(\Gamma_1)$ acts properly discontinuously and cocompactly on

$$\Omega = \mathbb{P}(\mathbb{R}^d) - \bigcup_{x \in \partial_\infty \Gamma_1} \mathbb{P}(\xi^{d-1}_{\rho_1}(x))$$

where $\xi^{d-1}_{\rho_1}$ is the limit map for ρ_1. Moreover, $\xi(\partial_\infty \Gamma_1) = \partial\Omega$.

Let $\rho(\alpha)$ be a bi-proximal element of $\rho(\Gamma) - \rho(\Gamma_1)$ and let $V^-_{\rho(\alpha)}$ be the repelling hyperplane of $\rho(\alpha)$. Since ξ is equivariant, if $[v] \in \partial\Omega$, then $\{[\rho(\alpha^n)v]\}_{n\in\mathbb{N}}$ converges to $\xi(\alpha^+)$. Therefore, $V^-_{\rho(\alpha)}$ is disjoint from $\partial\Omega$. It follows that $\mathbb{P}(\mathbb{R}^d) - \Omega$ is the closure of the set of repelling hyperplanes of biproximal elements o $\rho(\Gamma)$. Therefore, the complement of Ω, and hence Ω itself, is invariant under the full group $\rho(\Gamma)$. By Proposition 3.4.2, $\rho(\Gamma)$ is discrete and since $\rho(\Gamma_1)$ acts cocompactly on Ω, $\rho(\Gamma_1)$ must have finite index in $\rho(\Gamma)$ which contradicts the fact that ρ is faithful. $\square$

3.5 Property U, weak eigenvalue gaps and the uniform gap summation property

In this section, we discuss Property U, its relation with the uniform gap summation property and prove Theorem 1.4.12.

Property U and weak Property U were introduced by Delzant-Guichard-Labourie-Mozes [Del+11] and Kassel-Potrie [KP20] respectively and are related with the growth of the translation length and stable translation length of group elements in terms of their word length.

Definition 3.5.1. *Let Γ be a finitely generated group and fix $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ a left invariant word metric on Γ. The group Γ satisfies Property U (resp. weak Property U) if there exists a finite subset F of Γ and $C, c > 0$ with the following property: for every $\gamma \in \Gamma$ there exists $f \in F$ such that*

$$\ell_\Gamma(f\gamma) \geqslant c|\gamma|_\Gamma - C \quad (\text{resp. } |f\gamma|_\infty \geqslant c|\gamma|_\Gamma - C)$$

Fact 3.5.2. (a) Note that (weak) Property U is independent of the choice of the left invariant word metric on Γ since any two such metrics on Γ are quasi-isometric. Moreover, for every $\gamma \in \Gamma$, $\ell_\Gamma(\gamma) \geqslant |\gamma|_{\Gamma,\infty}$, so if Γ satisfies weak Property U then Γ also satisfies Property U.

(b) Let Γ_1 and Γ_2 be two finitely generated groups satisfying (weak) Property U. The product $\Gamma_1 \times \Gamma_2$ also satisfies (weak) Property U. In particular, finitely generated abelian groups satisfy weak Property U.

(c) Suppose that $\phi : \Gamma_1 \to \Gamma_2$ is a surjective group homomorphism which is also a quasi-isometry. If Γ_1 satisfies (weak) Property U then so does the group Γ_2. In particular, (weak) Property U is preserved under taking finite extensions.

Delzant-Guichard-Labourie-Mozes [Del+11] proved that the following classes of groups satisfy Property U:

(i) the class of word hyperbolic groups.

(ii) every finitely generated group Γ admitting a semisimple quasi-isometric embedding into a reductive real algebraic Lie group.

We prove that a virtually torsion free finitely generated group with non-trivial Floyd boundary satisfies weak Property U. Recall that the Floyd boundary $\partial_f\Gamma$ of Γ with respect to f is called non-trivial if $|\partial_f\Gamma| \geq 3$. Let H be subgroup of Γ. We define $\Lambda(H) \subset \partial_f\Gamma$ to be the set of accumulation points of infinite sequences of elements of H in $\partial_f\Gamma$.

Theorem 3.5.3. *Let Γ be a finitely generated group and fix $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ a left invariant metric on Γ. Suppose that the Floyd boundary $\partial_f \Gamma$ of Γ is non-trivial for some Floyd function $f : \mathbb{N} \to \mathbb{R}^+$. Let H be a torsion free subgroup of Γ with $|\Lambda(H)| \geq 3$. Then there exists a finite subset F of H and $L > 0$ with the property: for every $\gamma \in H$ there exists $g \in F$ such that $|g\gamma|_\Gamma - |g\gamma|_{\Gamma,\infty} \leqslant L$. In particular, if Γ is virtually torsion free then it satisfies weak Property U.*

Proof. Let $G : [1, \infty) \to \mathbb{R}^+$ be the function $G(x) := 10 \sum_{k=\lfloor x/2 \rfloor}^{\infty} f(k)$. Note that G is decreasing and since f is a Floyd function we have $\lim_{x\to\infty} G(x) = 0$. By Karlsson's estimate [Kar03, Lemma 1] we have

$$d_f(g, h) \leqslant G((g \cdot h)_e) \quad \text{and} \quad d_f(g, g^+) \leqslant G\Big(\frac{1}{2}|g|_\Gamma\Big)$$

for every $g, h \in \Gamma$, where g has infinite order. The group H is torsion free, so the second inequality shows that $\Lambda(H)$ is the closure of the attracting fixed points of elements of H. Since $\Lambda(H)$ contains at least 3 points, by [Kar03, Proposition 5] we may find $f_1, f_2 \in H$ non-trivial such that the sets $\{f_1^+, f_1^-\}$ and $\{f_2^+, f_2^-\}$ are disjoint. Let $\varepsilon = \frac{1}{100} \min\{d_f(f_1^+, f_2^\pm), d_f(f_1^-, f_2^\pm)\}$. We make the following three choices of constants $M, R, N > 0$. First, we choose $M > 0$ such that $G(x) \geqslant \frac{\varepsilon}{100}$ if and only if $x \leq M$. We also choose $R > 0$ such that $G(x) \leq \frac{\varepsilon}{100}$ for every $x \geqslant R$. We also consider $N > 0$ large enough such that $\min\{|f_1^N|_\Gamma, |f_2^N|_\Gamma\} \geqslant 10(M + R)$.

Let $F := \{f_1^N, f_2^N, e\}$. Now we claim that for every non-trivial $\gamma \in H$, there exists $g \in F$ such that $d_f(g\gamma^+, \gamma^-) \geqslant \varepsilon$. If $d_f(\gamma^+, \gamma^-) \geqslant \varepsilon$ we choose $g = e$. So we may assume that $d_f(\gamma^+, \gamma^-) \leqslant \varepsilon$. We can choose $n_0 \in \mathbb{N}$ such that $G(\frac{1}{2}|\gamma^n|_\Gamma) < \varepsilon$ for $n \geqslant n_0$. Notice that we can find $i \in \{1, 2\}$ such that $d_f(\gamma^+, f_i^+) \geqslant 50\varepsilon$ and $d_f(\gamma^+, f_i^-) \geqslant 50\varepsilon$. Indeed, if we assume that $\mathrm{dist}(\gamma^+, \{f_1^+, f_1^-\}) < 50\varepsilon$ then $d_f(\gamma^+, f_2^\pm) \geqslant \mathrm{dist}(f_2^\pm, \{f_1^+, f_1^-\}) - 50\varepsilon \geqslant 50\varepsilon$. Without loss of generality we may assume that $d_f(\gamma^+, f_1^+) \geqslant 50\varepsilon$ and $d_f(\gamma^+, f_1^-) \geqslant 50\varepsilon$. By our choices of N and n_0 we have that

$$d_f(\gamma^n, f_1^{-N}) \geqslant d_f(\gamma^+, f_1^-) - d_f(f_1^-, f_1^{-N}) - d_f(\gamma^+, \gamma^n)$$

$$\geqslant 50\varepsilon - G\Big(\frac{1}{2}|f_1^N|_\Gamma\Big) - G\Big(\frac{1}{2}|\gamma^n|_\Gamma\Big) \geqslant 48\varepsilon$$

hence $G\big((\gamma^n \cdot f_1^{-N})_e\big) \geq \varepsilon$ for $n \geqslant n_0$. By the choice of $M > 0$ we have that $\big(\gamma^n \cdot f_1^{-N}\big)_e \leqslant M$ for $n \geqslant n_0$. Then, we choose a sequence $(k_n)_{n\in\mathbb{N}}$ such that $|f_1^{k_n - N}|_\Gamma < |f_1^{k_n}|_\Gamma$ for every $n \in \mathbb{N}$. For $n \geqslant n_0$ we have

$$\begin{aligned} 2(f_1^N\gamma^n \cdot f_1^{k_n})_e &= |f_1^N\gamma^n|_\Gamma + |f_1^{k_n}|_\Gamma - |f_1^{N-k_n}\gamma^n|_\Gamma \\ &= |\gamma^n|_\Gamma + |f_1^N|_\Gamma - 2(\gamma^n \cdot f_1^{-N})_e + |f_1^{k_n}|_\Gamma - |f_1^{N-k_n}\gamma^n|_\Gamma \end{aligned}$$

$$
\begin{aligned}
&\geqslant -2M + |f_1^N|_\Gamma + |f_1^{k_n}|_\Gamma - |f_1^{N-k_n}|_\Gamma \\
&\geqslant |f_1^N|_\Gamma - 2M \geqslant \frac{|f_1^N|_\Gamma}{2} \geqslant 2R
\end{aligned}
$$

Therefore, by the choice of $R > 0$ we have $G\big((f_1^N\gamma^n \cdot f_1^{k_n})_e\big) \leq \varepsilon$ for $n \geqslant n_0$. It follows that $d_f(f_1^N\gamma^+, f_1^+) \leqslant \varepsilon$ so

$$
d_f(f_1^N\gamma^+, \gamma^-) \geqslant d_f(\gamma^+, f_1^+) - d_f(f_1^N\gamma^+, f_1^+) - d_f(\gamma^+, \gamma^-) \geqslant 48\varepsilon
$$

The claim follows.

Now, let $L := 10\max_{g\in F} |g|_\Gamma + 2R$ and $A := \{\gamma \in H : |\gamma|_\Gamma \geqslant L\}$. If $\gamma \notin A$ then we choose $g = e$ and obviously $|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty} \leq L$. Suppose that $\gamma \in A$. We may choose $g \in F$ such that $d_f\big((g\gamma g^{-1})^+, \gamma^-\big) \geqslant \varepsilon$, where $(g\gamma g^{-1})^+ = g\gamma^+$ in $\partial_f\Gamma$. We observe that

$$
\begin{aligned}
d_f\big((g\gamma g^{-1})^+, (g\gamma)^+\big) &\leqslant d_f\big((g\gamma g^{-1})^+, g\gamma g^{-1}\big) + d_f(g\gamma g^{-1}, g\gamma) + d_f\big((g\gamma)^+, g\gamma\big) \\
&\leqslant G\Big(\frac{1}{2}|g\gamma g^{-1}|_\Gamma\Big) + G((g\gamma g^{-1} \cdot g\gamma)_e) + G\Big(\frac{1}{2}|g\gamma|_\Gamma\Big) \\
&\leqslant 3G\Big(\frac{1}{2}|\gamma|_\Gamma - 2|g|_\Gamma\Big) \leqslant \frac{3\varepsilon}{100} \\
\text{and} \quad d_f(\gamma^-, \gamma^{-1}g^{-1}) &\leqslant d_f(\gamma^-, \gamma^{-1}) + d_f(\gamma^{-1}, \gamma^{-1}g^{-1}) \\
&\leqslant G\Big(\frac{1}{2}|\gamma|_\Gamma\Big) + G((\gamma^{-1} \cdot \gamma^{-1}g^{-1})_e) \\
&\leqslant 2G\Big(\frac{1}{2}|\gamma|_\Gamma - 2|g|_\Gamma\Big) \leqslant \frac{\varepsilon}{50}
\end{aligned}
$$

since $|\gamma|_\Gamma - 2|g|_\Gamma > R$. Therefore, we have

$$
d_f\big((g\gamma)^+, \gamma^{-1}g^{-1}\big) \geqslant d_f(g\gamma^+, \gamma^-) - d_f(g\gamma^+, (g\gamma)^+) - d_f(\gamma^{-1}, \gamma^{-1}g^{-1}) \geqslant \frac{\varepsilon}{2}
$$

This shows that there exists $n_1 > 0$ such that $G\big(((g\gamma)^{k_n} \cdot (g\gamma)^{-1})_e\big) \geqslant \frac{\varepsilon}{3}$ and $\big((g\gamma)^n \cdot (g\gamma)^{-1}\big)_e \leqslant M$ for $n \geqslant n_1$. We can find a sequence $(m_n)_{n\in\mathbb{N}}$ such that

$$
\lim_{n\to\infty} \big(|(g\gamma)^{m_n+1}|_\Gamma - |(g\gamma)^{m_n}|_\Gamma\big) \leqslant |g\gamma|_{\Gamma,\infty}
$$

so $\lim_n 2\big((g\gamma)^{m_n} \cdot (g\gamma)^{-1}\big)_e \geqslant |g\gamma|_\Gamma - |g\gamma|_{\Gamma,\infty}$. Finally, since $R > M$, we conclude that

$$
|g\gamma|_\Gamma - |g\gamma|_{\Gamma,\infty} \leqslant 2M \leqslant L
$$

This completes the proof of the theorem. □

We need the following proposition which allows us to control the quasi-isometry constants of the induced quasi-isometry between two Gromov hyperbolic spaces (X, d_X) and (Y, d_Y) acted on geometrically by the discrete group Γ.

Proposition 3.5.4. *Let Γ be a non-elementary torsion free word hyperbolic group acting properly discontinously and cocompactly by isometries on the Gromov hyperbolic spaces (X, d_X) and (Y, d_Y) respectively. We set $a^-_{X,Y} := \inf_{\gamma\in\Gamma_\infty} \frac{|\gamma|_{X,\infty}}{|\gamma|_{Y,\infty}}$ and $a^+_{X,Y} := \sup_{\gamma\in\Gamma_\infty} \frac{|\gamma|_{X,\infty}}{|\gamma|_{Y,\infty}}$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are two given points. Then there exists $M > 0$ such that*

$$a^-_{X,Y}\big(\gamma y_0 \cdot \delta y_0\big)_{y_0} - M \leqslant \big(\gamma x_0 \cdot \delta x_0\big)_{x_0} \leqslant a^+_{X,Y}\big(\gamma y_0 \cdot \delta y_0\big)_{y_0} + M$$

for every $\gamma, \delta \in \Gamma$.

Proof. For $\gamma \in \Gamma$, let $|\gamma|_X = d_X(\gamma x_0, x_0)$ and $|\gamma|_Y = d_Y(\gamma y_0, y_0)$. By the Svarc-Milnor lemma (see Proposition 2.2.2) and the fellow traveller property for Gromov hyperbolic spaces (see Theorem 3.5.4) we can find constants $C, c > 0$ such that

$$\max\Big\{\big(\gamma^+ \cdot \gamma^{-1}x_0\big)_{x_0}, \big(\gamma^+ \cdot \gamma^{-1}y_0\big)_{y_0}\Big\} \leqslant C\big(\gamma^+ \cdot \gamma^{-1}\big)_e + c$$

for every $\gamma \in \Gamma$. By Lemma 2.2.7 we obtain a constant $m > 0$ such that

$$\max\Big\{|\gamma|_X - |\gamma|_{X,\infty}, |\gamma|_Y - |\gamma|_{Y,\infty}\Big\} \leqslant C(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}) + m$$

for every $\gamma \in \Gamma$. By using Theorem 3.5.3 and the previous inequalities, we can find a finite subset F of Γ and $L > 0$ with the property: for every $\gamma \in \Gamma$ there exists $f \in F$ such that $\big||f\gamma|_{X,\infty} - |f\gamma|_X\big| \leqslant L$ and $\big||f\gamma|_{Y,\infty} - |f\gamma|_Y\big| \leqslant L$. By definition we have $a^-_{X,Y}|f\gamma|_{Y,\infty} \leqslant |f\gamma|_{X,\infty} \leqslant a^+_{X,Y}|f\gamma|_{Y,\infty}$, hence

$$|f\gamma|_X \geqslant |f\gamma|_{X,\infty} \geqslant a^-_{X,Y}|f\gamma|_{Y,\infty} \geqslant a^-_{X,Y}|f\gamma|_Y - a^-_{X,Y}L$$
$$a^+_{X,Y}|f\gamma|_Y \geqslant a^+_{X,Y}|f\gamma|_{Y,\infty} \geqslant |f\gamma|_{X,\infty} \geqslant |f\gamma|_X - a^+_{X,Y}L$$

Therefore, since $\big||f\gamma|_X - |\gamma|_X\big| \leq |f|_X$ and $\big||f\gamma|_Y - |\gamma|_Y\big| \leq |f|_Y$ for every $\gamma \in \Gamma$, there exists $K > 0$ with

$$a^-_{X,Y}|\gamma|_Y - K \leqslant |\gamma|_X \leqslant a^+_{X,Y}|\gamma|_Y + K$$

for every $\gamma \in \Gamma$. The fellow traveller property combined with previous double inequality shows that

$$-K' + a^{-}_{X,Y}\mathrm{dist}_Y(y_0, [\gamma y_0, \delta y_0]) \leqslant \mathrm{dist}_X(x_0, [\gamma x_0, \delta x_0]) \leqslant a^{+}_{X,Y}\mathrm{dist}_Y(y_0, [\gamma y_0, \delta y_0]) + K'$$

for some $K' > 0$. The conclusion follows since X and Y are Gromov hyperbolic. $\square$

Remark 3.5.5. Kassel-Potrie established an analogue of the Abels-Margulis-Soifer lemma [AMS95, Theorem 5.17] simultaneously for a linear representation $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ of a word hyperbolic group and the abstract group Γ equipped with a left invariant word metric (see [KP20, page 16]). Note that in the case Γ is word hyperbolic, Theorem 3.5.3 follows by their result.

3.5.1 Weak uniform gaps in eigenvalues.

Recall that a linear representation $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ has a weak uniform i-gap in eigenvalues if there exists $c > 0$ such that $\lambda_i(\rho(\gamma)) - \lambda_{i+1}(\rho(\gamma)) \geq c|\gamma|_{\Gamma,\infty}$ for every $\gamma \in \Gamma$.
For a group Γ the *lower central series*

$$... \trianglelefteq \mathbf{g}_3(\Gamma) \trianglelefteq \mathbf{g}_2(\Gamma) \trianglelefteq \mathbf{g}_1(\Gamma) \trianglelefteq \mathbf{g}_0(\Gamma) := \Gamma$$

is inductively defined as $\mathbf{g}_{k+1}(\Gamma) = \big[\Gamma, \mathbf{g}_k(\Gamma)\big]$ for $k \geq 1$. For every k, $\mathbf{g}_k(\Gamma)$ is a characteristic subgroup of Γ and the quotient $\mathbf{g}_k(\Gamma)/\mathbf{g}_{k+1}(\Gamma)$ is a central subgroup of $\Gamma/\mathbf{g}_{k+1}(\Gamma)$. The group Γ is *nilpotent* if there exists $m \geqslant 0$ with $\mathbf{g}_m(\Gamma) = 1$.

First, we prove that a nilpotent group Γ admitting a representation with a uniform weak eigenvalue i-gap has to be virtually cyclic. We remark that the following proposition fails to be true when Γ is assumed to be solvable. For example, the solvable Baumslag-Solitar group $\mathsf{BS}(1, 2) = \langle a, t | ta^{-1}t^{-1}a^2\rangle$ admits a faithful representation into $\mathsf{GL}(2, \mathbb{R})$ with a weak uniform 1-gap (see [KP20, Example 4.8]).

Proposition 3.5.6. *Let Γ be a finitely generated nilpotent group. Suppose that $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ has a weak uniform i-gap in eigenvalues for some $1 \leq i \leq d-1$. Then Γ is virtually cyclic.*

Proof. Let G_1 be a group and $G_2 \subset Z(G_1)$ be a central subgroup of G_1. Observe that if the quotient G_1/G_2 is virtually cyclic, then G_1 is virtually abelian.

Let G be the Zariski closure of $\rho(\Gamma)$ in $\mathsf{GL}(d, \mathbb{R})$. We consider the Levi decomposition $G = L \ltimes U$, where U is a connected normal unipotent subgroup of G and L is a reductive Lie group.

The projection $\pi \circ \rho : \Gamma \to L$ is Zariski dense and $\lambda(\pi(\rho(\gamma)) = \lambda(\rho(\gamma))$ for every $\gamma \in \Gamma$. The Lie group L is reductive and $\pi(\rho(\Gamma))$ is solvable, so L has to be virtually abelian since it has finitely many connected components. We may find a finite-index subgroup H of Γ such that $\mathbf{g}_1(H)$ is a subgroup of $\ker(\pi \circ \rho)$. Therefore, for every $k \geqslant 1$ we obtain a well defined representation $\rho_k : H/\mathbf{g}_k(H) \to \mathsf{GL}(d, \mathbb{R})$ such that $\rho_k \circ \pi_k = \pi \circ \rho$, where $\pi_k : H \to H/\mathbf{g}_k(H)$ is the quotient map. Note that for every $k \geq 1$ there exists $c_k \geq 1$ such that $|\pi_k(h)|_{H/\mathbf{g}_k(H),\infty} \leqslant c_k |h|_{H,\infty}$ for every $h \in H$. Since $\lambda(\rho_k(h)) = \lambda(\rho(h))$ for every $h \in H$, ρ_k has a weak uniform i-gap in eigenvalues for every $k \geq 1$. We may use induction on k to see that $H/\mathbf{g}_k(H)$ is virtually cyclic. The group $H/\mathbf{g}_1(H)$ is abelian and satisfies weak Property U, so ρ_1 is P_i-Anosov by [KP20, Proposition 4.12] and $H/\mathbf{g}_1(H)$ has to be virtually cyclic. Now suppose that $H/\mathbf{g}_k(H)$ is virtually cyclic. Note that $\mathbf{g}_k(H)/\mathbf{g}_{k+1}(H)$ is a central subgroup of $H/\mathbf{g}_{k+1}(H)$ with virtually cyclic quotient $H/\mathbf{g}_k(H)$. It follows that $H/\mathbf{g}_{k+1}(H)$ is virtually abelian. In particular, $H/\mathbf{g}_{k+1}(H)$ satisfies weak Property U, so ρ_{k+1} is P_i-Anosov and $H/\mathbf{g}_{k+1}(H)$ is virtually cyclic. Therefore, $H/\mathbf{g}_k(H)$ has to be virtually cylic for every $k \geq 1$ and H is virtually cyclic since $\mathbf{g}_m(H) = 1$ for some $m \geq 1$. □

By using Theorem 3.5.3 we obtain the following relation between the uniform gap summation property and weak Property U.

Corollary 3.5.7. *Let Γ be a finitely generated group which is not virtually nilpotent, G be a semisimple Lie group and $\theta \subset \Delta$ a subset of simple restricted roots of G. Suppose that there exists a representation $\rho : \Gamma \to G$ satisfying the uniform gap summation property with respect to θ. Then Γ satisfies weak Property U.*

Proof. By Proposition 2.5.4 we may assume that $G = \mathsf{SL}(d, \mathbb{R})$ and $\theta = \{\varepsilon_1 - \varepsilon_2\}$. Since ρ satisfies the uniform gap summation property $\ker(\rho)$ is finite. It suffices to prove that a finite-index subgroup of $\Gamma' = \Gamma/\ker(\rho)$ satisfies weak Property U. By Selberg's lemma [Sel62] Γ' is virtually torsion free, so we may assume that Γ is torsion free and ρ is faithful. By Lemma 3.2.7 there exists a continuous ρ-equivariant map $\xi_f : \partial_f \Gamma \to \mathbb{P}(\mathbb{R}^d)$ for some Floyd function f. We first prove that $\partial_f \Gamma$ is not a singleton.

Suppose that $|\partial_f \Gamma| = 1$. By the definition of the map ξ_f, the image $\xi_f(\partial_f \Gamma)$ identifies with the τ_{mod}-limit set of Γ in $\mathbb{P}(\mathbb{R}^d)$. Since Γ is not virtually nilpotent, we may use [KL18, Corollary 5.10] to reach a contradiction. We provide here the following different argument. Since $\partial_f \Gamma$ is a singleton, up to conjugation, we may assume that $\xi_f(\partial_f \Gamma) = [e_1]$ and find a group homomorphism $a : \Gamma \to \mathbb{R}^*$ such that $\rho(\gamma) e_1 = a(\gamma) e_1$ for every $\gamma \in \Gamma$. We consider the representation $\hat{\rho}(\gamma) = \frac{1}{a(\gamma)} \rho(\gamma)$. Note that $\hat{\rho}$ satisfies the uniform gap summation property, ξ_f is $\hat{\rho}$-equivariant and we can write $\hat{\rho}(\gamma) = \begin{bmatrix} 1 & u(\gamma) \\ 0 & \rho_0(\gamma) \end{bmatrix}$ for some group homomorphism $\rho_0 : \Gamma \to \mathsf{GL}(d-1, \mathbb{R})$. Let $g \in \Gamma$ be a non-trivial element. Since ξ_f is constant we have $\lim_n \Xi_1^+(\hat{\rho}(g^n)) = \lim_n \Xi_1^+(\hat{\rho}(g^{-n})) = [e_1]$. Let us write $\hat{\rho}(g^n) = k_n \exp\big(\mu(\hat{\rho}(g^n))\big)k_n'$ in the Cartan decomposition of G. Up to passing to a subsequence we may assume that $\lim_n k_n = k_\infty$ and $\lim_n k_n' = k_\infty'$. Then $k_\infty' P_1^+ = wP_1^+$, $\langle k_\infty' e_1, e_1\rangle = 0$ and $|\langle k_\infty e_1, e_1\rangle| = 1$, so $\lim_n \frac{1}{\sigma_1(\hat{\rho}(g^n))}\hat{\rho}(g^n) = k_\infty E_{11} k_\infty' \in \oplus_{i=2}^d \mathbb{R}E_{1i}$. If $\ell_1(\hat{\rho}(g)) > 1$, then $\ell_1(\rho_0(g)) = \ell_1(\hat{\rho}(g))$. Let p_1 and p_2 be the largest possible dimension of a Jordan block for an eigenvalue of maximum modulus of $\hat{\rho}(g)$ and $\rho_0(g)$ respectively. We have $\sigma_1(\hat{\rho}(g^n)) \asymp n^{p_1-1}\ell_1(\hat{\rho}(g^n))$ and $\sigma_1(\rho_0(g^n)) \asymp n^{p_2-1}\ell_1(\hat{\rho}(g^n))$ and $p_1 > p_2$ since $\lim_n \frac{1}{\sigma_1(\hat{\rho}(g^n))}\rho_0(g^n) = 0$. We can find $C > 0$ such that

$$||u(g^n)|| = \Big|\Big|\sum_{i=0}^{n} \rho_0(g^i)^t u(g)\Big|\Big| \leqslant ||u(g)|| \sum_{i=0}^{n} i^{p_2-1}\ell_1(\hat{\rho}(g))^i \leq Cn^{p_2-1}\ell_1(\hat{\rho}(g))^n$$

for every $n \in \mathbb{N}$. Since $p_1 > p_2$ and $\ell_1(\rho_0(g)) > 1$ we have $\lim_n \frac{1}{n^{p_1-1}\ell_1(\hat{\rho}(g^n))} \sum_{i=0}^n i^{p_2-1}\ell_1(\hat{\rho}(g))^i = 0$. Therefore, $\lim_n \frac{1}{\sigma_1(\hat{\rho}(g^n))}||u(g^n)|| = 0$ which is impossible since $\lim_n \frac{\hat{\rho}(g^n)}{\sigma_1(\hat{\rho}(g^n))}$ has at least one of its $(1,2),...,(1,d)$ entries non-zero. It follows that $\ell_1(\hat{\rho}(g)) \leq 1$ and $\ell_1(\rho(g)) \leq |a(g)|$. Similarly, we obtain $\frac{1}{\ell_d(\rho(g))} = \ell_1(\rho(g^{-1})) \leq \frac{1}{|a(g)|}$. It follows that all the eigenvalues of $\rho(g)$ have modulus equal to 1. Therefore, by Theorem 2.6.1, any semisimplification of ρ has compact Zariski closure. Then, by using [Aus61, Theorem 3] and [KL18, Theorem 10.1], we conclude that $\rho(\Gamma)$ (and hence Γ) is virtually nilpotent. We have reached a contradiction. Therefore, ξ_f is non-constant and $|\partial_f\Gamma| \geqslant 2$.

Now we conclude that Γ has weak Property U. If $|\partial_f\Gamma| = 2$, we consider the restriction $\rho_V : \Gamma \to \mathsf{GL}(V)$ where $V = \langle \xi_f(\partial_f\Gamma)\rangle$ and $\dim(V) = 2$. Since $\xi_f(\partial_f\Gamma)$ contains two points, up to passing to a finite-index subgroup of Γ and conjugating ρ_V, we may assume that $\rho_V(\Gamma)$ lies in the diagonal subgroup $\mathsf{GL}(V)$. Let $g \in \ker(\rho_V)$. We may write $\rho(g^n) = k_n \exp(\mu(g^n))k_n'$ and assume that $k_\infty P_1^+ = P_1^+$. We see that $\lim_n \frac{\rho(g)^n}{||\rho(g)^n||} = k_\infty E_{11} k_\infty' \in \sum_{i=1}^d \mathbb{R}E_{1i}$. We may write

$$\rho(g^n) = \begin{bmatrix} I_2 & \big(\sum_{i=0}^n A^i\big)^t B \\ 0 & A^n \end{bmatrix},\ \rho(g) = \begin{bmatrix} I_2 & B \\ 0 & A \end{bmatrix}$$

and so $\lim_n \frac{1}{||\rho(g^n)||}A^n$ is the zero matrix. If A has an eigenvalue of modulus greater than 1, then $\ell_1(A) = \ell_1(\rho(g))$. By working similarly as before, we have $\lim_n \frac{1}{||\rho(g^n)||}\sum_{i=0}^n ||A^i|| = 0$. This would show that $\lim_n \frac{1}{||\rho(g^n)||}\rho(g^n)$ has all of its $(1,i)$ entries equal to zero, a contradiction. It follows that all elements of $\rho(\ker(\rho_V))$ have all of their eigenvalues of modulus 1. We deduce that $\rho(\ker(\rho_V))$ (and hence $\ker(\rho_V)$) is virtually nilpotent and finitely generated. The quotient $\Gamma/\ker(\rho_V)$ is abelian, so Γ has to be virtually polycyclic. Since $|\partial_f\Gamma| > 1$, a theorem of Floyd [Flo80, page 211] implies that Γ has two ends, so Γ is virtually cyclic. Since Γ is assumed not to be virtually nilpotent, this is again a contradiction.

Finally, it follows that $|\partial_f\Gamma| \geq 3$. Therefore, Theorem 3.5.3 shows that Γ satisfies weak Property U. $\square$

We obtain the following theorem providing consitions under which a linear representation $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ of a finitely generated group Γ with a weak uniform i-gap in eiganvalues is P_i-Anosov.

Theorem 1.4.12. *Let Γ be a non-virtually cyclic finitely generated group and $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ be a left invariant word metric on Γ. Suppose that $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ is a representation which has a weak uniform i-gap in eigenvalues for some $1 \leqslant i \leqslant d-1$. Then the following are equivalent:*

(i) Γ *is word hyperbolic and ρ is P_i-Anosov.*

(ii) *There exists a Floyd function $f : \mathbb{N} \to \mathbb{R}^+$ such that the Floyd boundary $\partial_f \Gamma$ of Γ contains at least three points.*

(iii) Γ *admits a representation $\rho_1 : \Gamma \to \mathsf{GL}(m, \mathbb{R})$ satisfying the uniform gap summation property.*

(iv) Γ *admits a semisimple representation $\rho_2 : \Gamma \to \mathsf{GL}(n, \mathbb{R})$ with the property*

$$\lim_{|\gamma|_\Gamma \to \infty} \frac{||\mu(\rho_2(\gamma))||}{\log |\gamma|_\Gamma} = +\infty$$

Proof. Suppose that (i) holds. Then (ii) holds since the Floyd boundary identifies with the Gromov boundary of Γ. Moreover, by Theorem 2.5.3 and Proposition 2.6.2, (iii) and (iv) hold true for any semisimplification ρ^{ss} of the P_i-Anosov representation ρ. Now let us prove the other implications. There exists $c > 0$ such that $\lambda_i(\rho(\gamma)) - \lambda_{i+1}(\rho(\gamma)) \geqslant c|\gamma|_{\Gamma,\infty}$ for every $\gamma \in \Gamma$. By [KP20, Proposition 4.12] it is enough to prove that Γ satisfies weak Property U.
(ii) $\Rightarrow$ (i). We first observe that for every element $g \in \ker(\rho)$ we have $|g|_{\Gamma,\infty} = 0$. We next show that $N := \ker\rho$ is finite. If not, N is an infinite normal subgroup of Γ and $\Lambda(N) = \partial_f \Gamma$ since Γ acts minimally on $\partial_f \Gamma$. By [Kar03, Theorem 1] there exists a free subgroup H of N of rank at least 2 and $|\Lambda(H)| \geq 3$. By Theorem 3.5.3 we can find $\gamma \in H$ such that $|\gamma|_{\Gamma,\infty} > 0$. This is a contradiction since $\gamma \in N$. It follows that N is finite.

The Floyd boundary of $\Gamma' = \Gamma/N$ is non-trivial since Γ' is quasi-isometric to Γ. Note that the representation ρ induces a faithful representation $\rho' : \Gamma' \to \mathsf{GL}(d, \mathbb{R})$ which also has a weak uniform i-gap in eigenvalues. Selberg's lemma [Sel62] implies that Γ' is virtually torsion free so, by Theorem 3.5.3, Γ' satisfies weak Property U. We conclude that Γ' and Γ are word hyperbolic and ρ is P_i-Anosov.

(iii) $\Rightarrow$ (i). If Γ is virtually nilpotent, Proposition 3.5.6 implies that Γ is virtually cyclic, a contradiction. Therefore, Γ is not virtually nilpotent. Since ρ_1 satisfies the uniform gap summation property, by Corollary 3.5.7, Γ has to satisfy weak Property U. Therefore, (i) holds.

(iv) $\Rightarrow$ (i). Let ρ^{ss} be a semisimplification of ρ. By Proposition 2.6.2, $\lambda(\rho(\gamma)) = \lambda(\rho^{ss}(\gamma))$ for every $\gamma \in \Gamma$ so there exists $\delta > 0$ such that

$$\lambda_i(\rho^{ss}(\gamma)) - \lambda_{i+1}(\rho^{ss}(\gamma)) \geqslant c|\gamma|_{\Gamma,\infty} \geqslant \delta||\lambda(\rho_2(\gamma))||$$

for every $\gamma \in \Gamma$. By Theorem 2.6.1 there exists a finite subset F of Γ and $C > 0$ such that for every $\gamma \in \Gamma$ there exists $f \in F$ with $||\mu(\rho^{ss}(\gamma)) - \lambda(\rho(\gamma f))|| \leqslant C$ and $||\mu(\rho_2(\gamma)) - \lambda(\rho_2(\gamma f))|| \leqslant C$. By the previous two inequalities we obtain $K > 0$ such that

$$\mu_i(\rho^{ss}(g)) - \mu_{i+1}(\rho^{ss}(g)) \geqslant \delta||\mu(\rho_2(g))|| - K$$

for all $g \in \Gamma$. By assumption, for all but finitely many $g \in \Gamma$ we have $||\mu(\rho_2(g))|| \geqslant \frac{2}{\delta}\log|g|_\Gamma$, so there exists $K' > 0$ such that

$$\mu_i(\rho^{ss}(g)) - \mu_{i+1}(\rho^{ss}(g)) \geqslant 2\log|g|_\Gamma - K'$$

for all $g \in \Gamma$. In particular, ρ^{ss} satisfies the uniform gap summation property. The conclusion follows by the implication (iii) $\Rightarrow$ (i). $\square$

Remark 3.5.8. The representation $\rho_2 : \Gamma \to \mathsf{GL}(p, \mathbb{R})$ in condition (iv) of Theorem 1.4.12 is far from being a quasi-isometric embedding. As mentioned above, it follows by [Del+11] that any group Γ admitting a semisimple quasi-isometric embedding into $\mathsf{GL}(d, \mathbb{R})$, $d \geq 2$, satisfies (weak) Property U.

3.6 Gromov products

In this section, we define the Gromov product associated to an Anosov representation of a word hyperbolic group Γ and prove Proposition 1.4.9, showing that it is comparable with the usual Gromov product on the group Γ. We also prove an analogue of Proposition 1.4.9 for representations satisfying the uniform gap summation property. First, let us recall the definition of the Gromov product:

Definition 3.6.1. *Let G be a real semisimple Lie group, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ and let $\mu : G \to \overline{\mathfrak{a}}^+$ be the Cartan projection. For every linear form $\varphi \in \mathfrak{a}^*$ the map $(\cdot)_\varphi : G \times G \to \mathbb{R}$ is called the Gromov product relative to φ and is defined as follows: for $g, h \in G$*

$$(g \cdot h)_\varphi := \frac{1}{4}\Big\langle \varphi, \mu(g) + \mu(g^{-1}) + \mu(h) + \mu(h^{-1}) - \mu(g^{-1}h) - \mu(h^{-1}g)\Big\rangle$$

For a line $\ell \in \mathbb{P}(\mathbb{R}^d)$ and a hyperplane $V \in \mathsf{Gr}_{d-1}(\mathbb{R}^d)$, the distance $\mathrm{dist}(\ell, V)$ is computed by the formula

$$\mathrm{dist}(\ell, V) = |\langle k_\ell e_1, k_V e_d\rangle|$$

where $\ell = [k_\ell e_1]$, $V = [k_V e_d^\perp]$ and $k_V, k_\ell \in \mathsf{O}(d)$. The following proposition relates the Gromov product with the limit maps of a representation ρ and will be used in the following chapters.

Proposition 3.6.2. *Let Γ be a word hyperbolic group and $\rho : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ be a representation. Suppose that ρ is P_1-divergent and there exists a pair of continuous ρ-equivariant maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_\infty\Gamma \to \mathsf{Gr}_{d-1}(\mathbb{R}^d)$ which satisfy the Cartan property. Then for $x, y \in \partial_\infty\Gamma$ and two sequences $(\gamma_n)_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$ of elements of Γ with $\lim_n \gamma_n = x$ and $\lim_n \delta_n = y$ we have*

$$\lim_{n\to\infty} \exp\Big(-4\big(\rho(\gamma_n)\cdot\rho(\delta_n)\big)_{\varepsilon_1}\Big) = \mathrm{dist}(\xi(x), \xi^-(y)) \cdot \mathrm{dist}(\xi(y), \xi^-(x))$$

Proof. We may write $\rho(\gamma_n) = w_{\gamma_n}\exp(\mu(\rho(\gamma_n)))w'_{\gamma_n}$ and $\rho(\delta_n) = w_{\delta_n}\exp(\mu(\rho(\delta_n)))w'_{\delta_n}$ where $w_{\gamma_n}, w'_{\gamma_n}, w_{\delta_n}, w'_{\delta_n} \in \mathsf{PO}(d)$. Since ρ is P_1-divergent we have $\lim_n \frac{\sigma_d(\rho(\gamma_n))}{\sigma_j(\rho(\gamma_n))} = \lim_n \frac{\sigma_d(\rho(\delta_n))}{\sigma_j(\rho(\delta_n))} = 0$ for $1 \leq j \leq d-1$. Then we notice that

$$\begin{aligned}
&\lim_{n\to\infty} \exp\Big(-4\Big(\rho(\gamma_n)\cdot\rho(\delta_n)\Big)_{\varepsilon_1}\Big) = \lim_{n\to\infty} \frac{\sigma_1(\rho(\gamma_n^{-1}\delta_n))\sigma_1(\rho(\delta_n^{-1}\gamma_n))}{\sigma_1(\rho(\gamma_n))\sigma_1(\rho(\gamma_n^{-1}))\sigma_1(\rho(\delta_n))\sigma_1(\rho(\delta_n^{-1}))} \\
&= \lim_{n\to\infty}\Bigg(\Big|\Big|\Big((w'_{\delta_n})^{-1}\mathrm{diag}\Big(\frac{\sigma_d(\rho(\delta_n))}{\sigma_1(\rho(\delta_n))}, ..., 1\Big)w_{\delta_n}^{-1}w_{\gamma_n}\mathrm{diag}\Big(1, ..., \frac{\sigma_d(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}\Big)w'_{\gamma_n}\Big|\Big|\cdot \\
&\Big|\Big|\Big((w'_{\gamma_n})^{-1}\mathrm{diag}\Big(\frac{\sigma_d(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}, ..., 1\Big)w_{\gamma_n}^{-1}w_{\delta_n}\mathrm{diag}\Big(1, ..., \frac{\sigma_d(\rho(\delta_n))}{\sigma_1(\rho(\delta_n))}\Big)w'_{\delta_n}\Big|\Big|\Bigg) \\
&\overset{P_1\text{div.}}{=} \lim_{n\to\infty}\Big|\Big|E_{1d}w_{\gamma_n}^{-1}w_{\delta_n}E_{11}\Big|\Big|\cdot\Big|\Big|E_{1d}w_{\delta_n}^{-1}w_{\gamma_n}E_{11}\Big|\Big| \\
&= \lim_{n\to\infty}|\langle w_{\gamma_n}^{-1}w_{\delta_n}e_1, e_d\rangle\cdot\langle w_{\delta_n}^{-1}w_{\gamma_n}e_1, e_d\rangle| \\
&= \lim_{n\to\infty}\mathrm{dist}\big(\Xi_1^+(\rho(\gamma_n)), \Xi_1^-(\rho(\delta_n))\big)\cdot\mathrm{dist}\big(\Xi_1^+(\rho(\delta_n)), \Xi_1^-(\rho(\gamma_n))\big) \\
&= \mathrm{dist}\big(\xi(x), \xi^-(y)\big)\cdot\mathrm{dist}\big(\xi(y), \xi^-(x)\big)
\end{aligned}$$

since ξ and ξ^- satisfy the Cartan property. The proposition follows. $\square$

Proof of Proposition 1.4.9. (i) Fix $\alpha \in \theta$. By [Gué+17, Lemma 3.2], there exists $N_\alpha > 0$ and an irreducible θ-proximal representation τ_α of highest weight $N_\alpha \omega_\alpha$. Since ρ is $P_{\{\alpha\}}$-Anosov, the representation $\tau_\alpha \circ \rho$ is P_1-Anosov. The difference between $\big(\rho(\gamma) \cdot \rho(\delta)\big)_{\omega_\alpha}$ and $N_\alpha\big(\tau_\alpha(\rho(\gamma)) \cdot \tau_\alpha(\rho(\delta))\big)_{\varepsilon_1}$ is bounded above and below by uniform constants depending only on τ_α and ρ. Therefore, for the proofs of (i) and (iii) it suffices to restrict to the case where $G = \mathsf{SL}(d, \mathbb{R})$, $\theta = \{\varepsilon_1 - \varepsilon_2\}$ and ρ is P_1-Anosov. By Lemmas 2.7.4 and 2.7.5, we may further assume that ρ is semisimple and $\rho(\Gamma)$ also preserves a properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$. By Lemma 2.7.2 we can find $M > 0$ such that $\big|(\rho(\gamma) \cdot \rho(\delta))_{\varepsilon_1} - \big(\rho(\gamma)x_0 \cdot \rho(\delta)x_0\big)_{x_0}\big| \leqslant M$. Then $\rho(\Gamma)$ acts cocompactly on a closed convex subset $\mathcal{C} \subset \Omega$. Fix $x_0 \in \mathcal{C}$. By applying Proposition 3.5.4 for $\big(X, d_X\big) = \big(\mathcal{C}, d_\Omega\big)$ and $\big(Y, d_Y\big) = (\Gamma, d_\Gamma)$ we can find $C > 0$ such that

$$a^-_{\mathcal{C},\Gamma}(\gamma \cdot \delta)_e - C \leqslant \big(\rho(\gamma)) \cdot \rho(\delta)\big)_{\varepsilon_1} \leqslant a^+_{\mathcal{C},\Gamma}(\gamma \cdot \delta)_e + C$$

This finishes the proof of (i).

By using Proposition 3.6.2 and our previous argument we deduce the following corollary:

Corollary 3.6.3. *Let (X, d) be a Gromov hyperbolic space and suppose that Γ is a word hyperbolic group acting properly discontinuously and cocompactly on X by isometries and fix $x_0 \in X$. Suppose that $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ is a P_1-Anosov representation and with Anosov limit maps $\xi^+ : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^- : \partial_\infty\Gamma \to \mathsf{Gr}_{d-1}(\mathbb{R}^d)$. We set $a^-_\rho := \inf_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho(\gamma)) - \lambda_d(\rho(\gamma))}{|\gamma|_{X,\infty}}$ and $a^+_\rho := \sup_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho(\gamma)) - \lambda_d(\rho(\gamma))}{|\gamma|_{X,\infty}}$. There exists $C > 0$ such that*

$$\frac{1}{C} e^{-4a^+_\rho (x \cdot y)_{x_0}} \leq \operatorname{dist}\big(\xi^+(x), \xi^-(y)\big) \cdot \operatorname{dist}\big(\xi^+(y), \xi^-(x)\big) \leq d_{\mathbb{P}}\big(\xi^+(x), \xi^+(y)\big)^2$$
$$\operatorname{dist}\big(\xi^+(x), \xi^-(y)\big) \cdot \operatorname{dist}\big(\xi^+(y), \xi^-(x)\big) \leq C e^{-4a^-_\rho (x \cdot y)_{x_0}}$$

for every $x, y \in \partial_\infty X$.

For the proof of Proposition 1.4.9 (ii) we need the following sharper bounds for the Gromov product. For a group Γ, we denote by Γ_∞ the set of all infinite order elements of Γ.

Lemma 3.6.4. *Let Γ be a word hyperbolic group, $\rho_1 : \Gamma \to \mathsf{GL}(d_1, \mathbb{R})$ and $\rho_2 : \Gamma \to \mathsf{GL}(d_2, \mathbb{R})$ be two P_1-Anosov representations and set*

$$A^-_{\rho_1,\rho_2} := \inf_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho_2(\gamma)) - \lambda_{d_2}(\rho_2(\gamma))}{\lambda_1(\rho_1(\gamma)) - \lambda_{d_1}(\rho_1(\gamma))} \quad \text{and} \quad A^+_{\rho_1,\rho_2} := \sup_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho_2(\gamma)) - \lambda_{d_2}(\rho_2(\gamma))}{\lambda_1(\rho_1(\gamma)) - \lambda_{d_1}(\rho_1(\gamma))}$$

There exists $C > 0$ such that

$$A^-_{\rho_1,\rho_2}\big(\rho_1(\gamma)\cdot\rho_1(\delta)\big)_{\varepsilon_1} - C \leqslant \big(\rho_2(\gamma)\cdot\rho_2(\delta)\big)_{\varepsilon_1} \leqslant A^+_{\rho_1,\rho_2}\big(\rho_1(\gamma)\cdot\rho_1(\delta)\big)_{\varepsilon_1} + C$$

for every $\gamma, \delta \in \Gamma$.

Proof. By Selberg's lemma [Sel62] we may find a finite-index subgroup Γ_0 such that $\rho_i(\Gamma_0)$ is torsion free for $i = 1, 2$. By Lemma 2.7.5 we may also assume that ρ_1 and ρ_2 are semisimple. Furthermore, by Lemma 2.7.4, for $i = 1, 2$, we may assume that $\rho_i(\Gamma)$ preserves a properly convex domain Ω_i of $\mathbb{P}(\mathbb{R}^{d_i})$ and acts cocompactly on a closed convex subset $\mathcal{C}_i$ of Ω_i. By applying Proposition 3.5.4 for $(X, d_X) = (\mathcal{C}_2, d_{\Omega_2})$ and $(Y, d_Y) = (\mathcal{C}_1, d_{\Omega_1})$ the inequality follows. $\square$

Proof of Proposition 1.4.9 (ii). Suppose that ρ is P_Δ-Anosov. We fix $\alpha \in \Delta$. By Proposition 2.5.4, there exists an irreducible $\{\alpha\}$-proximal representation $\tau : G \to \mathsf{GL}(V)$ such that $P^+_{\{a\}}$ stabilizes a line of V and $\tau \circ \rho$ is P_1-Anosov. Let χ_τ be the highest weight of τ. For each restricted weight $\chi_i \in \mathfrak{a}^*$ of τ we have $\chi_i = \chi_\tau - n_i\alpha - \sum_{\beta\in\Delta-\{\alpha\}} n_{\beta,i}\beta$ for some $n_i \in \mathbb{N}^+$, $n_{\beta,i} \geq 0$ and $\chi_1 = \chi_\tau - \alpha$. If $k = \dim(V^{\chi_1})$ and $g \in G$, then $\langle \varepsilon_i, \mu(\tau(g))\rangle = \langle \chi_\tau - \alpha, \mu(\tau(g))\rangle$ for $2 \leqslant i \leqslant k+1$ and $\langle \varepsilon_{k+1} - \varepsilon_{k+2}, \mu(\tau(g))\rangle = \min_{i\geq 2}\langle \chi_\tau - \alpha - \chi_i, \mu(g)\rangle$. Since ρ is P_Δ-Anosov there exist $C, c > 0$ with $\min_{\alpha\in\Delta}\langle \alpha, \mu(\rho(\gamma))\rangle \geqslant C|\gamma|_\Gamma - c$, hence we can find $L, \ell > 0$ with $\langle \chi_\tau - \alpha - \chi_i, \mu(g)\rangle \geqslant L|\gamma|_\Gamma - \ell$ for every $\gamma \in \Gamma$ and $i \geq 2$. By [BPS16], [KLP18] the representation $\phi := \tau \circ \rho$ is P_{k+1}-Anosov. For every $\gamma, \delta \in \Gamma$ we have

$$k\big(\rho(\gamma)\cdot\rho(\delta)\big)_\alpha = (k+1)\big(\phi(\gamma)\cdot\phi(\delta)\big)_{\varepsilon_1} - \big(\wedge^{k+1}\phi(\gamma)\cdot\wedge^{k+1}\phi(\rho(\delta))\big)_{\varepsilon_1}$$

and

$$\begin{aligned} k+1-A^+_{\phi,\wedge^{k+1}\phi} &= k\cdot\inf_{\gamma\in\Gamma_\infty}\frac{\langle \alpha, \lambda(\rho(\gamma)) + \lambda(\rho(\gamma^{-1}))\rangle}{\langle \chi_\tau, \lambda(\rho(\gamma)) + \lambda(\rho(\gamma^{-1}))\rangle} \\ k+1-A^-_{\phi,\wedge^{k+1}\phi} &= k\cdot\sup_{\gamma\in\Gamma_\infty}\frac{\langle \alpha, \lambda(\rho(\gamma)) + \lambda(\rho(\gamma^{-1}))\rangle}{\langle \chi_\tau, \lambda(\rho(\gamma)) + \lambda(\rho(\gamma^{-1}))\rangle} \end{aligned}$$

are well defined and positive since ρ is $P_{\{\alpha\}}$-Anosov. By applying Lemma 3.6.4 for $\rho_1 := \phi$ and $\rho_2 := \wedge^{k+1}\phi$ we obtain $M > 0$ such that

$$\frac{1}{k}\big(k+1-A^+_{\phi,\wedge^{k+1}\phi}\big)(\phi(\gamma)\cdot\phi(\delta))_{\varepsilon_1} - M \leqslant (\rho(\gamma)\cdot\rho(\delta))_\alpha \leqslant \frac{1}{k}\big(k+1-A^-_{\phi,\wedge^{k+1}\phi}\big)(\phi(\gamma)\cdot\phi(\delta))_{\varepsilon_1} + M$$

for every $\gamma, \delta \in \Gamma$. The conclusion follows by part (i) since ϕ is P_1-Anosov. $\square$

Remark 3.6.5. Note that when $G = \mathsf{SL}(d,\mathbb{R})$ and ρ is P_1 and P_2-Anosov the previous proof shows that the Gromov product with respect to the simple root $\{\varepsilon_1 - \varepsilon_2\}$ also grows linearly in terms of the Gromov product on Γ with respect to the metric $|\cdot|_\Gamma$.

We prove the following analogue of Proposition 1.4.9 for representations which satisfy the uniform gap summation property. For simplicity, we assume that $f(x) = x^{-\kappa-1}$ for some $\kappa > 0$, $G = \mathsf{SL}(d,\mathbb{R})$ and $\theta = \{\varepsilon_1 - \varepsilon_2\}$. In particular, part (i) of the following proposition shows that the Gromov product of $\rho(\Gamma)$ with respect to $\varepsilon_1 \in \mathfrak{a}^*$ grows at least logarithmically in terms of the Gromov product on Γ.

Proposition 3.6.6. *Let Γ be a finitely generated group and $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ be a left invariant word metric on Γ. Suppose that $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ is a representation satisfying the uniform gap summation property with respect to the set $\theta = \{\varepsilon_1 - \varepsilon_2\}$ and $f(x) = x^{-1-\kappa}$ where $\kappa > 0$.*

(i) *There exists $R > 0$ with the property*

$$(g \cdot h)_e \leqslant R\left(\frac{\sigma_1(\rho(g^{-1}))\sigma_1(\rho(h))}{\sigma_1(\rho(g^{-1}h))}\right)^{1/\kappa}$$

for every $g, h \in \Gamma$.

(ii) *There exists $L > 0$ with the property*

$$|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty} \leqslant L\left(\frac{\sigma_1(\rho(\gamma))}{\ell_1(\rho(\gamma))}\right)^{1/\kappa}$$

for every $\gamma \in \Gamma$ of infinite order.

We remark that the domain group of a representation with the uniform gap summation property is not necessarily hyperbolic and the representation might not be convex cocompact. We will need the following estimates. In the following lemma $w \in \mathsf{O}(d)$ is the permutation matrix with $w = \sum_{i=1}^d E_{i(d+1-i)}$.

Lemma 3.6.7. *Let $g, h \in \mathsf{SL}(d,\mathbb{R})$. Suppose that $g = k_g \exp(\mu(g))k_g'$ and $h = k_h \exp(\mu(h))k_h'$ are written in the standard Cartan decomposition of $\mathsf{SL}(d,\mathbb{R})$ and $k_g, k_g', k_h, k_h' \in \mathsf{SO}(d)$.*

(i) *The following inequality holds*

$$\frac{\sigma_1(gh)}{\sigma_1(g)\sigma_1(h)} \leqslant \frac{\sigma_2(g)}{\sigma_1(g)} + \frac{\sigma_2(h)}{\sigma_1(h)} + \frac{\sigma_2(g)}{\sigma_1(g)} \cdot \frac{\sigma_2(h)}{\sigma_1(h)} + d_{\mathbb{P}}(k_h P_1^+, (k_g')^{-1} w P_1^+)$$

(ii) *Suppose that* $\lim_n \frac{\sigma_1(g^n)}{\sigma_2(g^n)} = +\infty$. *Then*

$$\frac{\ell_1(g)}{\sigma_1(g)} \leqslant \frac{\sigma_2(g)}{\sigma_1(g)} + \overline{\lim_{n\to\infty}}\, d_{\mathbb{P}}\big(\Xi_1^+(g^n), (k_g')^{-1}wP_1^+\big)$$

Proof. (i) We may write $\exp(\mu(g)) = \sigma_1(g)E_{11} + S_g$ and $\exp(\mu(h)) = \sigma_1(h)E_{11} + S_h$ where S_g and S_h are diagonal matrices such that $||S_g|| \leqslant \sigma_2(g)$ and $||S_h|| \leqslant \sigma_2(h)$. Then we notice that

$$\begin{aligned}
\frac{\sigma_1(gh)}{\sigma_1(g)\sigma_1(h)} &= \frac{1}{\sigma_1(g)\sigma_1(h)}\Big|\Big|k_g(\sigma_1(g)E_{11} + S_g)k_g'k_h(\sigma_1(h)E_{11} + S_h)'k_h'\Big|\Big| \\
&= \frac{1}{\sigma_1(g)\sigma_1(h)}\Big|\Big|\sigma_1(g)\sigma_1(h)E_{11}k_g'k_hE_{11} + \sigma_1(g)E_{11}k_g'k_hS_h + \sigma_1(h)S_gk_g'k_hE_{11} + S_gk_g'k_hS_h\Big|\Big| \\
&\leqslant ||E_{11}k_g'k_hE_{11}|| + \frac{1}{\sigma_1(h)}||E_{11}k_g'k_hS_h|| + \frac{1}{\sigma_1(g)}||S_gk_g'k_hE_{11}|| + \frac{1}{\sigma_1(g)\sigma_1(h)}||S_gk_g'k_hS_h|| \\
&\leqslant |\langle k_g'k_he_1, e_1\rangle| + \frac{1}{\sigma_1(h)}||S_h|| + \frac{1}{\sigma_1(g)}||S_g|| + \frac{1}{\sigma_1(g)\sigma_1(h)}||S_g||\cdot||S_h|| \\
&\leqslant d_{\mathbb{P}}\big(k_hP_1^+, (k_g')^{-1}wP_1^+\big) + \frac{\sigma_2(h)}{\sigma_1(h)} + \frac{\sigma_2(g)}{\sigma_1(g)} + \frac{\sigma_2(h)}{\sigma_1(h)}\cdot\frac{\sigma_2(g)}{\sigma_1(g)}
\end{aligned}$$

The inequality follows.

(ii) We note that since $\ell_1(g) = \lim_n \sigma_1(g^n)^{1/n}$ we also have

$$\ell_1(g) \leqslant \overline{\lim_{n\to\infty}}\, \frac{\sigma_1(g^{n+1})}{\sigma_1(g^n)}$$

We may choose a sequence $(m_n)_{n\in\mathbb{N}}$ such that $\ell_1(g) \leqslant \lim_n \frac{\sigma_1(g^{m_n+1})}{\sigma_1(g^{m_n})}$. Let us write

$$g^{m_n} = k_n\mathrm{diag}(\sigma_1(g^{m_n}), ..., \sigma_d(g^{m_n}))k_n'$$

in the standard Cartan decomposition of $\mathsf{GL}(d,\mathbb{R})$. Note that $\Xi_1^+(g^{m_n}) = k_nP_1^+$ and up to passing to a subsequence we may assume that $\lim_n k_n = k_\infty \in \mathsf{O}(d)$. Therefore, we obtain

$$\lim_{n\to\infty} d_{\mathbb{P}}\big(\Xi_1^+(g^{m_n}), (k_g')^{-1}wP_1^+\big) = d_{\mathbb{P}}\big(k_\infty P_1^+, (k_g')^{-1}wP_1^+\big)$$

Since $\lim_n \frac{\sigma_1(g^n)}{\sigma_2(g^n)} = +\infty$, by part (i) we have that

$$\frac{\ell_1(g)}{\sigma_1(g)} \leqslant \lim_{n\to\infty} \frac{\sigma_1(g^{m_n+1})}{\sigma_1(g^{m_n})\sigma_1(g)} \leqslant \lim_{n\to\infty} d_{\mathbb{P}}(\Xi_1^+(g^{m_n}), (k'_g)^{-1}wP_1^+) + \frac{\sigma_2(g)}{\sigma_1(g)}$$

$\square$

Proof of Proposition 3.6.6. By assumption, there exists $C > 0$ such that $\frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} \geqslant C|\gamma|_\Gamma^{\kappa+1}$ for every $\gamma \in \Gamma$. Karlsson's estimate for the Floyd distance (see [Kar03, Lemma 1]) shows that there exists $M > 0$ such that $d_f(g,h) \leqslant \frac{M}{(g\cdot h)_e^\kappa}$ for every $g, h \in \Gamma$. Hence, by Lemma 3.2.7 (i) there exists $C_1 > 0$ and a finite subset A of Γ such that

$$d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g)), \Xi_1^+(\rho(h))\Big) \leqslant \frac{C_1}{(g\cdot h)_e^\kappa}$$

for all $g, h \in \Gamma - A$.

(i) We note that $k'_{g^{-1}} = wk_g^{-1}$. By using Lemma 3.6.7 (i) and since $\min\{|g|_\Gamma, |h|_\Gamma\} \geqslant \max\{1, (g\cdot h)_e\}$ we obtain:

$$\begin{aligned}\frac{\sigma_1\big(\rho(g^{-1}h)\big)}{\sigma_1\big(\rho(g^{-1})\big)\sigma_1\big(\rho(h)\big)} &\leqslant \frac{\sigma_2(\rho(h))}{\sigma_1(\rho(h))} + \frac{\sigma_{d-1}(\rho(g))}{\sigma_d(\rho(g))} + \frac{\sigma_2(\rho(h))}{\sigma_1(\rho(h))}\cdot\frac{\sigma_{d-1}(\rho(g))}{\sigma_d(\rho(g))} + d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g)), \Xi_1^+(\rho(h))\Big) \\ &\leqslant \frac{1}{C|g|_\Gamma^{\kappa+1}} + \frac{1}{C|h|_\Gamma^{\kappa+1}} + \frac{1}{C^2|g|_\Gamma^{\kappa+1}|h|_\Gamma^{\kappa+1}} + \frac{C_1}{(g\cdot h)_e^\kappa} \leqslant \frac{M}{(g\cdot h)_e^\kappa}\end{aligned}$$

where $M = \frac{2}{C} + \frac{1}{C^2} + \frac{1}{C_1}$. Since A is finite, part (i) follows.

(ii) Let $\gamma \in \Gamma - A$ be of infinite order. We recall that $\lim_n \frac{|\gamma^n|_\Gamma}{n} = |\gamma|_{\Gamma,\infty}$ and hence $\underline{\lim}_n \big(|\gamma^{n+1}|_\Gamma - |\gamma^n|_\Gamma\big) \leqslant |\gamma|_{\Gamma,\infty}$. We may find a sequence $(m_n)_{n\in\mathbb{N}}$ such that $\lim_n \big(|\gamma^{m_n+1}|_\Gamma - |\gamma^{m_n}|_\Gamma\big) \leqslant |\gamma|_{\Gamma,\infty}$. It follows that

$$\lim_{n\to\infty} (\gamma^{m_n}\cdot\gamma^{-1})_e \geqslant \frac{1}{2}(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty})$$

The uniform gap summation property implies that the limit $\lim_n \Xi_1^+(\rho(\gamma^n))$ exists. We obtain the bound

$$d_{\mathbb{P}}\Big(\lim_{n\to\infty}\Xi_1^+(\rho(\gamma^n)), \Xi_1^+\big(\rho(\gamma^{-1})\big)\Big) = \lim_{n\to\infty} d_{\mathbb{P}}\Big(\Xi_1^+\big(\rho(\gamma^{m_n})\big), \Xi_1^+\big(\rho(\gamma^{-1})\big)\Big) \leqslant \frac{2^\kappa C_1}{\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)^\kappa}.$$

By Lemma 3.6.7 (ii) we have that

$$\frac{\ell_1(\rho(\gamma))}{\sigma_1(\rho(\gamma))} \leqslant \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} + \frac{2^{\kappa}C_1}{\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)^{\kappa}} \leqslant \frac{1}{C|\gamma|_\Gamma^{1+\kappa}} + \frac{2^{\kappa}C_1}{\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)^{\kappa}} \leqslant \frac{2^{\kappa}C_1 + 1/C}{\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)^{\kappa}}.$$

Since A is a finite subset of Γ, the inequality follows. $\square$

Now we can finish the proof of Proposition 1.4.9.

Proof of Proposition 1.4.9 (iii). We may assume that $G = \mathsf{SL}^{\pm}(d,\mathbb{R})$, ρ is semisimple and P_1-Anosov, $\rho(\Gamma)$ preserves a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$ and acts cocompactly on a closed convex subset $\mathcal{C}$ of Ω. For $\gamma \in \Gamma$, let $\rho(\gamma)^+$ be the attracting fixed point of $\rho(\gamma)$ in the Gromov boundary of $\mathcal{C}$. By applying Proposition 3.5.4 for $(X, d_X) = (\Gamma, d_\Gamma)$ and $(Y, d_Y) = (\mathcal{C}, d_\Omega)$ and Lemma 2.2.7 we can find $L, L', a > 0$ such that

$$\big(\rho(\gamma)^+ \cdot \rho(\gamma)^{-1}x_0\big)_{x_0} \leqslant a\big(\gamma^+ \cdot \gamma^{-1}\big)_e + L \leqslant \frac{a}{2}\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big) + L'$$

for every $\gamma \in \Gamma$. By Lemma 2.2.7 we obtain uniform constants $M, m > 0$ such that

$$\begin{aligned}
\frac{1}{4}\log\frac{\sigma_1(\rho(\gamma))}{\ell_1(\rho(\gamma))} &\leqslant \frac{1}{4}\log\frac{\sigma_1(\rho(\gamma))\ell_d(\rho(\gamma))}{\sigma_d(\rho(\gamma))\ell_1(\rho(\gamma))} \\
&\leqslant \frac{1}{2}\big(d_\Omega(\rho(\gamma)x_0, x_0) - |\rho(\gamma)|_{\mathcal{C},\infty}\big) + m \\
&\leqslant \big(\rho(\gamma)^+ \cdot \rho(\gamma)^{-1}x_0\big)_{x_0} + M
\end{aligned}$$

for every $\gamma \in \Gamma$. The upper bound follows.

Let us fix d_a a visual metric on the compactification $\Gamma \cup \partial_\infty\Gamma$. Since ρ is P_1-Anosov, the map $\Xi^+ \cup \xi^+ : \big(\partial_\infty\Gamma \cup \Gamma, d_a\big) \to \big(\mathbb{P}(\mathbb{R}^d), d_{\mathbb{P}}\big)$ is s-Hölder for some $s > 0$ (see for example Section 3.9). By Lemma 3.6.7 (ii) and Lemma 2.2.7 there exist $C', C'', a_1 > 0$ such that:

$$\begin{aligned}
\frac{\ell_1(\rho(\gamma))}{\sigma_1(\rho(\gamma))} &\leqslant \frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))} + d_{\mathbb{P}}\big(x^+_{\rho(\gamma)}, \Xi_1^+(\rho(\gamma^{-1}))\big) \\
&\leqslant C'e^{-a_1|\gamma|_\Gamma} + d_{\mathbb{P}}\big(x^+_{\rho(\gamma)}, \Xi_1^+(\rho(\gamma^{-1}))\big) \\
&\leqslant C'e^{-a_1|\gamma|_\Gamma} + C''a^{-s(\gamma^+\cdot\gamma^{-1})_e} \\
&\leqslant C'e^{-a_1\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)} + C''e^{-\frac{s\log a}{2}\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)} \\
&\leqslant (C' + C'')e^{-\zeta\big(|\gamma|_\Gamma - |\gamma|_{\Gamma,\infty}\big)}
\end{aligned}$$

where $\zeta = \min\{a_1, \frac{s\log a}{2}\}$. This concludes the proof of the lower bound. $\square$

3.7 Strongly convex cocompact subgroups of $\mathsf{PGL}(d,\mathbb{R})$

In this section, we prove Theorem 1.4.6 which we recall here.

Theorem 1.4.6. *Let Γ be a finitely generated subgroup of $\mathsf{PGL}(d,\mathbb{R})$. Suppose that Γ preserves a strictly convex domain of $\mathbb{P}(\mathbb{R}^d)$ with C^1-boundary and the natural inclusion $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ is semisimple. Then the following conditions are equivalent:*

(i) Γ *is strongly convex cocompact in* $\mathbb{P}(\mathbb{R}^d)$.

(ii) $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ *is a quasi-isometric embedding,* Γ *preserves a properly convex domain* Ω *of* $\mathbb{P}(\mathbb{R}^d)$ *and there exists a* Γ*-invariant closed convex subset* $\mathcal{C}$ *of* Ω *such that* $(\mathcal{C}, d_\Omega)$ *is Gromov hyperbolic.*

For our proof we need the following proposition characterizing P_1-Anosov representations in terms of the Gromov product under the assumption that the group preserves a domain with strictly convex and C^1-boundary.

Proposition 3.7.1. *Let Γ be a word hyperbolic subgroup of $\mathsf{PGL}(d,\mathbb{R})$ which preserves a strictly convex domain Ω of $\mathbb{P}(\mathbb{R}^d)$ with C^1-boundary. Then the following are equivalent.*

(i) *The natural inclusion* $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ *is* P_1*-Anosov.*

(ii) *There exist constants* $J, k > 0$ *such that*

$$\frac{1}{J}(\gamma\cdot\delta)_e - k \leqslant (\gamma\cdot\delta)_{\varepsilon_1} \leqslant J(\gamma\cdot\delta)_e + k$$

for every $\gamma, \delta \in \Gamma$.

Proof. (ii) $\Rightarrow$ (i). We observe that Γ is a discrete subgroup of $\mathsf{PGL}(d,\mathbb{R})$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements of Γ and $x_0 \in \Omega$. We may pass to a subsequence such that $\lim_n \gamma_{k_n} x_0 \in \partial\Omega$ exists. Since $\partial\Omega$ is strictly convex we conclude that $\lim_n \gamma_{k_n} x_0$ is independent of the basepoint x_0. Therefore, as in [DGK17, Lemma 7.5] or Lemma 3.2.4, we conclude that $\lim_n \frac{\sigma_2}{\sigma_1}(\gamma_{k_n}) = 0$ and Γ has to be P_1-divergent.

Now let $(\gamma_n)_{n\in\mathbb{N}}, (\delta_n)_{n\in\mathbb{N}}$ be two sequences of elements of Γ which converge to a point $x \in \partial_\infty\Gamma$. We claim that the limits $\lim_n \gamma_n x_0$, $\lim_n \delta_n x_0$ exist and are equal. Note that the limits will be independent of the choice of x_0. We may write

$$\gamma_n = w_{\gamma_n} \exp(\mu(\gamma_n)) w'_{\gamma_n} \quad \text{and} \quad \delta_n = w_{\delta_n} \exp(\mu(\delta_n)) w'_{\delta_n}$$

where $w_{\gamma_n}, w'_{\gamma_n}, w_{\delta_n}, w'_{\delta_n} \in \mathsf{PO}(d)$. Since Γ is P_1-divergent, there exist subsequences $(\gamma_{k_n})_{n\in\mathbb{N}}$, $(\delta_{s_n})_{n\in\mathbb{N}}$ such that $a_1 = \lim_n \gamma_{k_n} x_0 = \lim_n \Xi_1^+(\gamma_{k_n})$, $a_2 = \lim_n \delta_{s_n} x_0 = \lim_n \Xi_1^+(\delta_{s_n})$, $\lim_n \Xi_1^-(\gamma_{k_n}) = a_1^-$ and $\lim_n \Xi_1^-(\delta_{s_n}) = a_2^-$, where $\Xi_1^+(\gamma_{k_n}) = [w_{\gamma_{k_n}} e_1]$ and $\Xi_1^-(\gamma_{k_n}) = [w_{\gamma_{k_n}} e_d^\perp]$. Proposition 3.6.2 and the fact that $(\gamma_{k_n} \cdot \delta_{s_n})_{\varepsilon_1} \to \infty$ show that

$$\lim_{n\to\infty} \operatorname{dist}\big(\Xi_1^+(\gamma_{k_n}), \Xi_1^-(\delta_{s_n})\big) \cdot \operatorname{dist}\big(\Xi_1^+(\delta_{s_n}), \Xi_1^-(\gamma_{k_n})\big) = 0$$

so either $a_1 \in a_2^-$ or $a_2 \in a_1^-$. Using the same argument, we see that

$$\lim_{n\to\infty} \operatorname{dist}\big(\Xi_1^+(\gamma_{k_n}), \Xi_1^-(\gamma_{k_n})\big) = \lim_{n\to\infty} \operatorname{dist}\big(\Xi_1^+(\delta_{s_n}), \Xi_1^-(\delta_{s_n})\big) = 0$$

so $a_i \in a_i^-$ for $i = 1, 2$. In each case, the previous calculation shows that $a_1, a_2 \in a_1^-$ or $a_1, a_2 \in a_2^-$. Without loss of generality, assume that $a_2 \in a_1^-$, so the projective line segment $[a_1, a_2]$ is contained in the projective hyperplane a_1^- and $\overline{\Omega}$. Since Γ is P_1-divergent, there exist $x_0^* \in \Omega^*$ such that $\lim_n \Xi_1^-(\gamma_{k_n}) = \lim_n \gamma_{k_n} x_0^*$ and $a_1^- \in \partial\Omega^*$. Therefore, a_1^- avoids Ω. We conclude that $[a_1, a_2]$ is contained in $\partial\Omega$ and $a_1 = a_2$.

Finally, for any two sequences of $(\gamma_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ converging to $x \in \partial_\infty\Gamma$ the limits $\lim_n \gamma_n x_0$ and $\lim_n \delta_n x_0$ exist and are equal. We obtain a Γ-equivariant map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ defined by the formula $\xi(\lim_n \gamma_n) = \lim_n \gamma_n x_0$. Let $x = \lim_n \delta_n$ and suppose $\lim_n x_n = x$ in $\partial_\infty\Gamma$. We may write $x_n = \lim_m \gamma_{n,m}$. For every n there exists $k_n, m_n \in \mathbb{N}$, such that $(\gamma_{n,k_n} \cdot \delta_{m_n})_e > n$ and $d_{\mathbb{P}}(\gamma_{n,k_n} x_0, \xi(x_n)) \leqslant \frac{1}{n}$. Then, $\lim_n \gamma_{n,k_n} x_0$ exists and is equal to $\xi(x) = \lim_n \delta_n x_0$. It follows, that $\lim_n \xi(x_n) = \xi(x)$. So the map ξ is continuous. By definition ξ has the Cartan property.

The dual convex set Ω^* has strictly convex boundary since the boundary of Ω is of class C^1. By considering the standard identification of $\mathbb{P}((\mathbb{R}^d)^*)$ with $\mathbb{P}(\mathbb{R}^d)$, we obtain a properly convex domain Ω' of $\mathbb{P}(\mathbb{R}^d)$ which is Γ^*-invariant and has strictly convex boundary. Since $(\gamma^{-t} \cdot \delta^{-t})_{\varepsilon_1} = (\gamma \cdot \delta)_{\varepsilon_1}$, we obtain a continuous Γ^*-equivariant limit map $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ satisfying the Cartan property. From ξ^* we obtain a Γ-equivariant continuous map $\xi^- : \partial_\infty\Gamma \to \mathsf{Gr}_{d-1}(\mathbb{R}^d)$ as follows: if $\xi^*(x) = [k_x e_1]$ where $k_x \in \mathsf{PO}(d)$ then $\xi^-(x) = [k_x e_1^\perp]$.

For two distinct points $x, y \in \partial_\infty\Gamma$, we choose sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ with $x = \lim_n \alpha_n$, $y = \lim_n \beta_n$ and $(x \cdot y)_e = \lim_n (\alpha_n \cdot \beta_n)_e$. By Proposition 3.6.2 and the assumption, we obtain the lower bound

$$\operatorname{dist}\big(\xi(x), \xi^-(y)\big) \cdot \operatorname{dist}\big(\xi(y), \xi^-(x)\big) \geqslant e^{-4J(x\cdot y)_e - 4k} > 0$$

Therefore, the pair of maps (ξ, ξ^-) is transverse. Finally, the inclusion $\Gamma \hookrightarrow \mathsf{PGL}(d, \mathbb{R})$ is P_1-divergent, admits a pair (ξ, ξ^-) of Γ-equivariant, continuous transverse maps with the Cartan property, so Theorem 1.4.1 shows that the inclusion $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ is P_1-Anosov.

The converse is a consequence of Corollary 1.4.9. $\square$

Proof of Theorem 1.4.6. The implication (i) $\Rightarrow$ (ii) follows immediately by the Svarc-Milnor lemma. Now assume that (ii) holds. By [DGK17, Theorem 1.4] it is enough to prove that $\Gamma \hookrightarrow \mathsf{PGL}(d,\mathbb{R})$ is P_1-Anosov. Let $x_0 \in \mathcal{C}$. Lemma 2.7.2 shows that the orbit map $x_0 \mapsto \gamma x_0$ is a quasi-isometric embedding of Γ into $(\mathcal{C}, d_\Omega)$, hence Γ is word hyperbolic. By Proposition 3.5.4 and Lemma 2.7.2 there exist constants $J, k > 0$ such that for every $\gamma, \delta \in \Gamma$

$$\frac{1}{J}(\gamma \cdot \delta)_e - k \leqslant \big(\rho(\gamma) \cdot \rho(\delta)\big)_{\varepsilon_1} \leqslant J(\gamma \cdot \delta)_e + k$$

Proposition 3.7.1 then finishes the proof. $\square$

3.8 Distribution of singular values

Let Γ be a word hyperbolic group and $\rho_L : \Gamma \to \mathsf{SL}(m,\mathbb{R})$ and $\rho_R : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ be two representations. By using Theorem 1.4.1 we exhibit conditions guaranteeing that the products $\rho_L \times \rho_R$ and $\rho_L \otimes \rho_R$ are P_1-Anosov and P_2-Anosov respectively. We deduce estimates on the distribution of the Cartan projections of the images of the representations ρ_L and ρ_R.

Recall that for a matrix $g \in \mathsf{GL}(d,\mathbb{R})$ and $1 \leqslant i \leqslant d$, $\mu_i(g)$ (resp. $\lambda_i(g)$) is the logarithm of the i-th singular value (resp. modulus of the i-th eigenvalue) of g. For $q \in \mathbb{N}$, let $\mathrm{Sym}^q(\mathbb{R}^d)$ be the q-symmetric power of $V = \mathbb{R}^d$ and $\mathrm{sym}^q : \mathsf{GL}(d,\mathbb{R}) \to \mathsf{GL}(\mathrm{Sym}^q(\mathbb{R}^d))$ the corresponding representation. Note that with respect to the standard Cartan decomposition we have $\sigma_1(\mathrm{sym}^q g) = (\sigma_1((g))^q$.

Theorem 3.8.1. *Let Γ be a word hyperbolic group and let $\rho_L : \Gamma \to \mathsf{SL}(m,\mathbb{R})$, $\rho_R : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ be two representations such that there exists an infinite order element $\gamma_0 \in \Gamma$ with $\lambda_1(\rho_L(\gamma_0)) > \lambda_1(\rho_R(\gamma_0))$. Furthermore, suppose that ρ_L is P_1-Anosov and ρ_R satisfies one of the following conditions:*

(i) *ρ_R is P_1-Anosov.*

(ii) *$\rho_R(\Gamma)$ is contained in a semisimple proximal Lie subgroup of $\mathsf{SL}(d,\mathbb{R})$ of real rank 1.*

Then, the following conditions are equivalent:

(1) *The representation $\rho_L \times \rho_R : \Gamma \to \mathsf{SL}(m+d,\mathbb{R})$ is P_1-Anosov.*

(2) $\displaystyle\lim_{|\gamma|_\Gamma \to \infty} \mu_1(\rho_L(\gamma)) - \mu_1(\rho_R(\gamma)) = +\infty.$

(3) *There exist $C, c > 0$ such that*

$$\left|\mu_1(\rho_L(\gamma)) - \mu_1(\rho_R(\gamma))\right| \geqslant c \log |\gamma|_\Gamma - C$$

for every $\gamma \in \Gamma$.

(4) $\lim_{|\gamma|_\infty \to \infty} \lambda_1(\rho_L(\gamma)) - \lambda_1(\rho_R(\gamma)) = +\infty$.

(5) *There exist $C, c > 0$ such that*

$$\left|\lambda_1(\rho_L(\gamma)) - \lambda_1\rho_R((\gamma))\right| \geqslant c \log |\gamma|_{\Gamma,\infty} - C$$

for every $\gamma \in \Gamma$ of infinite order.

Proof. Let G be a P_1-proximal Lie subgroup of $\mathsf{SL}(d,\mathbb{R})$ of real rank 1 with Cartan projection $\mu_G : G \to \mathbb{R}$. Up to conjugation, we may write $G = K_G \exp\big(\mu_G(G)X_0\big)K_G$, where $K_G \subset h\mathsf{SO}(d)h^{-1}$ for some $h \in \mathsf{SL}(d,\mathbb{R})$ and $\exp(tX_0) = \mathsf{diag}(e^{ta_1}, ..., e^{ta_k})$ with $a_1 > a_2 \geq ... \geq a_{d-1} > a_d$. The subadditivity of the Cartan projection shows that there exists $M > 0$ such that $\left|\mu_i(g) - a_i\mu_G(g)\right| \leqslant M$ for every $g \in G$ and $1 \leq i \leq d$. In particular, there exists $M' > 0$ such that

$$\mu_1(g) - \mu_2(g) \geq \frac{a_1 - a_2}{a_1}\mu_1(g) - M' \quad \forall g \in G.$$

Since we assume that ρ_R is P_1-Anosov or $\mu(\rho_R(\Gamma))$ is contained in a proximal, rank 1 Lie subgroup G of $\mathsf{SL}(d,\mathbb{R})$, by the previous remarks we can find $A, a > 0$ such that

$$\begin{aligned}\lambda_1(\rho_R(\gamma)) - \lambda_2(\rho_R(\gamma)) &\geqslant a\lambda_1(\rho_R(\gamma))\\ \mu_1(\rho_R(\gamma)) - \mu_2(\rho_R(\gamma)) &\geqslant a\mu_1(\rho_R(\gamma)) - A\end{aligned}$$

Let $\rho := \rho_L \times \rho_R$. We obtain continuous, ρ-equivariant and transverse maps $\xi^+_{LR} : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^{m+d})$ and $\xi^+_{LR} : \partial_\infty\Gamma \to \mathsf{Gr}_{m+d-1}(\mathbb{R}^{m+d})$ defined as follows:

$$\xi^+_{LR}(x) = \xi^+_L(x) \quad \text{and} \quad \xi^-_{LR}(x) = \xi^-_L(x) \oplus \mathbb{R}^d$$

where ξ^+_L and ξ^-_L are the Anosov limit maps of ρ_L. For every element $\gamma \in \Gamma$ we observe that the following estimates hold:

$$
\begin{aligned}
&\text{(a)}\quad \big|\mu_1(\rho_L(\gamma)) - \mu_1(\rho_R(\gamma))\big| \geqslant \mu_1(\rho(\gamma)) - \mu_2(\rho(\gamma))\\
&\qquad\ \big|\lambda_1(\rho_L(\gamma)) - \lambda_1(\rho_R(\gamma))\big| \geqslant \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma))\\
&\text{(b)}\quad \mu_1(\rho(\gamma)) - \mu_2(\rho(\gamma)) \geqslant \min\Big(\big|\mu_1(\rho_L(\gamma)) - \mu_1(\rho_R(\gamma))\big|, \mu_1(\rho_{L,R}(\gamma)) - \mu_2(\rho_{L,R}(\gamma))\Big)\\
&\qquad \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)) \geqslant \min\Big(\big|\lambda_1(\rho_L(\gamma)) - \lambda_1(\rho_R(\gamma))\big|, \lambda_1(\rho_{L,R}(\gamma)) - \lambda_2(\rho_{L,R}(\gamma))\Big)
\end{aligned}
$$

(2) $\Rightarrow$ (1). We observe that condition (2) and estimate (b) together show that ρ is P_1-divergent. Since ξ_L^+ satisfies the Cartan property and $\varepsilon_1, \mu_1(\rho_L(\gamma)) - \mu_1(\rho_R(\gamma)) > 0$ as $|\gamma|_\Gamma \to \infty$, the map ξ_{LR}^+ has the Cartan property. The maps ξ_{LR}^+ and ξ_{LR}^- are transverse, hence Theorem 1.4.1 shows that $\rho_L \times \rho_R$ is P_1-Anosov.

(3) $\Rightarrow$ (1). We first assume that $c > 1$. By estimate (b), there exists a constant $C_1 > 0$ such that

$$\mu_1(\rho(\gamma)) - \mu_2(\rho(\gamma)) \geqslant c \log |\gamma|_\Gamma - C_1$$

for every $\gamma \in \Gamma$. Therefore, by [Gué+17, Theorem 5.3], we obtain a ρ-equivariant map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^{m+d})$ which satisfies the Cartan property. Then, since $\rho(\gamma_0)$ is P_1-proximal, we have $\xi(\gamma_0^+) = \xi_{LR}^+(\gamma_0^+)$. The minimality of the action of Γ on $\partial_\infty\Gamma$ shows that $\xi = \xi_{LR}^+$. Then ξ_{LR}^+ satisfies the Cartan property, ξ_{LR}^- and ξ_{LR}^+ are transverse and ρ is P_1-divergent. Theorem 1.4.1 shows that ρ is P_1-Anosov. If $c \leqslant 1$, we choose $n \in \mathbb{N}$ large enough and consider the symmetric powers $\mathrm{sym}^n\rho_L, \mathrm{sym}^n\rho_R$ of ρ_L, ρ_R respectively. Then $\mathrm{sym}^n\rho_L$ is P_1-Anosov and $\mathrm{sym}^n\rho_R$ satisfies either (i) or (ii). Since $\mu_1(\mathrm{sym}^n\rho_R(\gamma)) = n\mu_1(\rho_R(\gamma))$ for $\gamma \in \Gamma$, the representation $\mathrm{sym}^n\rho_L \times \mathrm{sym}^n\rho_R$ satisfies condition (3) for $c > 1$. Therefore, the previous argument implies that the representation $\mathrm{sym}^n\rho_L \times \mathrm{sym}^n\rho_R$ is P_1-Anosov. Therefore, by estimate (a), we obtain uniform constants $k, K > 0$ such that $\big|\mu_1(\rho_L(\gamma)) - \mu_1(\rho_R(\gamma))\big| \geqslant k|\gamma|_\Gamma - K$ for every $\gamma \in \Gamma$. The first part again verifies that ρ is P_1-Anosov.

(4) $\Rightarrow$ (1). We are proving that (4) $\Rightarrow$ (2). Let ρ_L^{ss}, ρ_R^{ss} be semisimplifications of ρ_L, ρ_R respectively. By Proposition 2.6.2, it is enough to show that $\rho_L^{ss} \times \rho_R^{ss}$ is P_1-Anosov. By Theorem 2.6.1 there exists $C > 0$ and a finite subset F of Γ such that for every $\gamma \in \Gamma$, there exists $f \in F$ such that $\big|\lambda_1(\rho_L(\gamma f)) - \mu_1(\rho_L^{ss}(\gamma))\big| \leqslant C$ and $\big|\lambda_1(\rho_R(\gamma f)) - \mu_1(\rho_R^{ss}(\gamma))\big| \leqslant C$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements of Γ. For every $n \in \mathbb{N}$ we choose $f_n \in F$ satisfying the previous bounds. The triangle inequality shows $||\lambda(\rho_L(\gamma_n f_n)|| \geqslant ||\mu(\rho_L(\gamma_n))|| - C$, hence $\lim_n |\gamma_n f_n|_\infty = +\infty$. Therefore, $\lim_n \lambda_1(\rho_L^{ss}(\gamma_n f_n)) - \lambda_1(\rho_R^{ss}(\gamma_n f_n)) = +\infty$ so $\lim_n \mu_1(\rho_L^{ss}(\gamma_n)) - \mu_1(\rho_R^{ss}(\gamma_n)) = +\infty$. The

claim now follows by (2) $\Rightarrow$ (1).

(5) $\Rightarrow$ (1). It is enough to prove that the semisimplification $\rho_L^{ss} \times \rho_R^{ss}$ of ρ is P_1-Anosov. Note that the representation ρ_L^{ss} is P_1-Anosov and ρ_R^{ss} satisfies either (i) or (ii). By Theorem 2.6.1 there exists $L > 0$ and a finite subset F of Γ such that for every $\gamma \in \Gamma$ there exists $f \in F$ with $||\lambda(\rho_L(\gamma f)) - \mu(\rho_L^{ss}(\gamma))|| \leqslant L$ and $||\lambda(\rho_R(\gamma f)) - \mu(\rho_R^{ss}(\gamma))|| \leqslant L$. Since ρ_L is a quasi-isometric embedding, by using the previous inequality, we may find $M > 0$ such that $|\gamma f|_{\Gamma,\infty} \geqslant \frac{1}{M}|\gamma|_\Gamma - M$, where $\gamma \in \Gamma$ and $f \in F$ are as previously. Finally, we obtain $L', c > 0$ such that for every $\gamma \in \Gamma$ we have

$$\left|\mu_1(\rho_L^{ss}(\gamma)) - \mu_1(\rho_R^{ss}(\gamma))\right| \geqslant c \log|\gamma|_\Gamma - L'$$

Therefore, $\rho_L^{ss} \times \rho_R^{ss}$ is P_1-Anosov from (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2),(3),(4),(5). Since $\langle \varepsilon_1, \lambda\rho_L(\gamma_0))\rangle > \langle \varepsilon_1, \lambda(\rho_R(\gamma_0))\rangle$, $\xi_{LR}^+(\gamma_0^+)$ is the attracting fixed point of $\rho(\gamma_0)$ in $\mathbb{P}(\mathbb{R}^{m+d})$. The action of Γ on $\partial_\infty\Gamma$ is minimal, hence ξ_{LR}^+ is the Anosov limit map of ρ. In particular, ξ_{LR}^+ satisfies the Cartan property. This shows that for any sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ we have $\lim_n \langle \varepsilon_1, \mu(\rho_L(\gamma_n)) - \mu(\rho_R(\gamma_n))\rangle = +\infty$. The Anosov limit map of ρ has to be the map ξ_{LR}^+ since $\xi_{LR}^+(\gamma_0^+)$ is the attracting fixed point of $\rho(\gamma)$ in $\mathbb{P}(\mathbb{R}^{m+d})$ and hence there exists $\varepsilon > 0$ such that $(1-\varepsilon)\langle \varepsilon_1, \lambda(\rho_L(\gamma))\rangle \geqslant \langle \varepsilon_1, \lambda(\rho_R(\gamma))\rangle$ for every $\gamma \in \Gamma$. By estimates (a), (b) and Theorem 2.5.3 (ii) we deduce that (3), (4), (5) hold. $\square$

Let $\rho_i : \Gamma \to \mathsf{SL}(m_i, \mathbb{R})$, $i = 1, 2$ be two representations such that ρ_2 is P_1-Anosov. The stretch factors associated with the representations ρ_1 and ρ_2 are:

$$\upsilon_+(\rho_1, \rho_2) = \sup_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho_1(\gamma))}{\lambda_1(\rho_2(\gamma))} \quad \text{and} \quad \upsilon_-(\rho_1, \rho_2) = \inf_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho_1(\gamma))}{\lambda_1(\rho_2(\gamma))}$$

where Γ_∞ denotes the set of infinite order elements of Γ. Since ρ_2 is a quasi-isometric embedding both quantities are well defined.

Proof of Corollary 1.4.3. We consider the representation $\rho = \mathrm{sym}^q\rho_1 \times \mathrm{sym}^p\rho_2$. The representation $\mathrm{sym}^p\rho_2$ is P_1-Anosov and $\mathrm{sym}^q\rho_1$ satisfies either condition (i) or (ii) of Theorem 3.8.1. We consider the following cases:

Case 1. $\frac{\lambda_1(\rho_1(\gamma))}{\lambda_1(\rho_2(\gamma))} = \frac{p}{q}$ for every $\gamma \in \Gamma$ of infinite order. Note that ρ_1 is a quasi-isometric embedding. If $\rho_1(\Gamma)$ lies in a proximal semisimple rank 1 subgroup of $\mathsf{SL}(d, \mathbb{R})$, then ρ_1 is P_1-Anosov. In each case both ρ_1 and ρ_2 are P_1-Anosov and the conclusion follows immediately by Lemma 2.7.5 and Theorem 2.6.1.

Case 2. $\upsilon_-(\rho_1, \rho_2) < \frac{p}{q} \leq \upsilon_+(\rho_1, \rho_2)$. We may find $\gamma_0 \in \Gamma$ of infinite order such that $p\lambda_1(\rho_2(\gamma_0)) >$

$q\lambda_1(\rho_1(\gamma_0))$ and $\lambda_1(\mathrm{sym}^p\rho_2(\gamma_0)) > \lambda_1(\mathrm{sym}^q\rho_1(\gamma_0))$. Suppose that there exists $\delta > 0$ such that $|\mu_1(\mathrm{sym}^q\rho_1(\gamma)) - \mu_1(\mathrm{sym}^p\rho_2(\gamma))| \geq \delta \log|\gamma|_\Gamma$ for every $\gamma \in \Gamma$. Theorem 3.8.1 (3) implies that ρ is P_1-Anosov and $\mathrm{sym}^p\rho_2$ uniformly dominates $\mathrm{sym}^q\rho_1$. This contradicts the fact that $\frac{p}{q} \leq \upsilon_+(\rho_1, \rho_2)$.

Case 3. $\frac{p}{q} = \upsilon_-(\rho_1, \rho_2)$ and $\lambda_1(\rho_1(\gamma)) \geq \frac{p}{q}\lambda_1(\rho_2(\gamma))$ for every $\gamma \in \Gamma$. We deduce that ρ_1 is a quasi-isometric embedding. In each case, under the assumption of the corollary, ρ_1 is P_1-Anosov. By Case 1, we may find $\gamma_1 \in \Gamma$ of infinite order such that $q\lambda_1(\rho_1(\gamma_1)) > p\lambda_1(\rho_2(\gamma_1))$ and hence $\lambda_1(\mathrm{sym}^q\rho_1(\gamma_1)) > \lambda_1(\mathrm{sym}^p\rho_2(\gamma_1))$. If the conclusion fails to hold for some $\delta > 0$, by Theorem 3.8.1 (3) we deduce that ρ is P_1-Anosov and $\mathrm{sym}^q\rho_1$ uniformly dominates $\mathrm{sym}^p\rho_2$. This contradicts the fact that $\frac{p}{q} = \upsilon_-(\rho_1, \rho_2)$. $\square$

Remarks 3.8.2. (i) In Theorem 3.8.1, when both $\rho_L(\Gamma)$ and $\rho_R(\Gamma)$ are contained in a proximal real rank 1 subgroup of $\mathsf{SL}(m, \mathbb{R})$ and $\mathsf{SL}(d, \mathbb{R})$ respectively, the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow by [Gué+17, Theorem 1.14]. If ρ_L and ρ_R take values in $\mathrm{Aut}_{\mathbf{k}}(b)$ ($\mathbf{k} = \mathbb{R}, \mathbb{C}, \mathcal{H}$) for some bilinear form b (see [Gué+17, §7] for background), the implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (4) of Theorem 3.8.1 follow by [Gué+17, Proposition 7.13 & Lemma 7.11 & Theorem 1.3].

(ii) By Theorem 2.6.1 and Corollary 1.4.3 we deduce that the closure of the set

$$\left\{ \frac{\lambda_1(\rho_1(\gamma))}{\lambda_1(\rho_2(\gamma))} : \gamma \in \Gamma_\infty \right\}$$

is the closed interval $\big[\upsilon_-(\rho_1, \rho_2), \upsilon_+(\rho_1, \rho_2)\big]$. We may replace both ρ_1 and ρ_2 with their semisimplifications, so this fact also follows by the limit cone theorem of Benoist in [Ben97]. In the case where ρ_1 and ρ_2 are convex cocompact into a Lie group of real rank 1, the previous fact also follows by [Bur93, Theorem 2].

By using similar arguments as in Theorem 3.8.1 we obtain the following conditions for the tensor product $\rho_L \otimes \rho_R$ to be P_2-Anosov.

Proposition 3.8.3. *Let $m, d \geq 2$, Γ be a word hyperbolic group and $\rho_L : \Gamma \to \mathsf{GL}(m, \mathbb{R})$, $\rho_R : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ be two P_1 and P_2-Anosov representations. Then the following conditions are equivalent:*

(1) *The representation $\rho_L \otimes \rho_R : \Gamma \to \mathsf{GL}(md, \mathbb{R})$ is P_2-Anosov.*

(2) *There exist $C, c > 1$ such that*

$$\big|(\mu_1 - \mu_2)(\rho_L(\gamma)) - (\mu_1 - \mu_2)(\rho_R(\gamma))\big| \geqslant c \log|\gamma|_\Gamma - C$$

for every $\gamma \in \Gamma$.

(3) *There exist $C, c > 1$ such that*

$$\big|(\lambda_1-\lambda_2)(\rho_L(\gamma)) - (\lambda_1-\lambda_2)(\rho_R(\gamma))\big| \geqslant c\log|\gamma|_{\Gamma,\infty} - C$$

for every $\gamma \in \Gamma$ of infinite order.

Proof. We observe that for any $\gamma \in \Gamma$ the following estimates hold:

(a) $\big|(\mu_1-\mu_2)(\rho_L(\gamma)) - (\mu_1-\mu_2)(\rho_R(\gamma))\big| \geqslant (\mu_2-\mu_3)((\rho_L\otimes\rho_R)(\gamma))$

$\big|(\lambda_1-\lambda_2)(\rho_L(\gamma)) - (\lambda_1-\lambda_2)(\rho_R(\gamma))\big| \geqslant (\lambda_2-\lambda_3)((\rho_L\otimes\rho_R)(\gamma))$

(b) $(\mu_2-\mu_3)((\rho_L\otimes\rho_R)(\gamma)) \geqslant \min\Big(\big|(\mu_1-\mu_2)(\rho_L(\gamma)) - (\mu_1-\mu_2)(\rho_R(\gamma))\big|, (\mu_2-\mu_3)(\rho_{L,R}(\gamma))\Big)$

$(\lambda_2-\lambda_3)((\rho_L\otimes\rho_R)(\gamma)) \geqslant \min\Big(\big|(\lambda_1-\lambda_2)(\rho_L(\gamma)) - (\lambda_1-\lambda_2)(\rho_R(\gamma))\big|, (\lambda_2-\lambda_3)(\rho_{L,R}(\gamma))\Big)$

The two estimates in (a) immediately imply (1) $\Rightarrow$ (2), (3). Note that for this part, we did not use the fact that ρ_L, ρ_R are P_1-Anosov.

Suppose that (2) holds. Notice that since ρ_L and ρ_R are P_2-Anosov, by estimate (b) the representation $\rho_L\otimes\rho_R$ satisfies the uniform gap summation property with respect to $\theta = \{\varepsilon_2-\varepsilon_3\}$. Moreover, since ρ_L and ρ_R satisfy condition (ii) of Corollary 1.4.2 for $\alpha = \varepsilon_1-\varepsilon_2$ and $\varepsilon_2-\varepsilon_3$, by the previous estimates we deduce that $\rho_L\otimes\rho_R$ also satisfies condition (ii) of Corollary 1.4.2 for $\alpha = \varepsilon_2-\varepsilon_3$. Then $\rho_L\otimes\rho_R$ is P_2-Anosov and (2) $\Rightarrow$ (1) follows.

For the implication (3) $\Rightarrow$ (1), let ρ_L^{ss} and ρ_R^{ss} be two semsimplifications of ρ_L and ρ_R respectively. It is enough to prove that $\rho_L^{ss}\otimes\rho_R^{ss}$ is P_2-Anosov. We may work as in Theorem 3.8.1 (5) $\Rightarrow$ (1), to see that (2) holds for ρ_L^{ss} and ρ_R^{ss}. Then, $\rho_L^{ss}\otimes\rho_R^{ss}$ is P_2-Anosov by (2) $\Rightarrow$ (1). The proof follows. ☐

Let S be a closed orientable surface of genus at least 2. Recall that a *Fuchsian representation* is the composition of a discrete faithful representation $j : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ with the (unique up to conjugation) irreducible representation $\mathrm{i}_d : \mathsf{PSL}(2,\mathbb{R}) \hookrightarrow \mathsf{PSL}(d,\mathbb{R})$ and a continuous deformation of a Fuchsian representation is called a *Hitchin representation.* We use the calculation of the simple root entropy for Hitchin representations by Potrie-Sambarino in [PS17] to prove that the tensor product of two Hitchin representations is not P_2-Anosov and hence, by using Proposition 3.8.3, we deduce the following:

Corollary 3.8.4. *Let $d_1, d_2 \geqslant 2$, $\Gamma := \pi_1(S)$ and $\rho_1 : \Gamma \to \mathsf{PSL}(d_1,\mathbb{R})$, $\rho_2 : \Gamma \to \mathsf{PSL}(d_2,\mathbb{R})$ be two Hitchin representations. There exists a constant $C > 0$ depending only on ρ_1 and an infinite* *sequence of elements* $(\gamma_n)_{n\in\mathbb{N}}$ *of* Γ *such that*

$$\left|1-\frac{\mu_1(\rho_2(\gamma_n))-\mu_2(\rho_2(\gamma_n))}{\mu_1(\rho_1(\gamma_n))-\mu_2(\rho_1(\gamma_n))}\right| \leqslant \frac{C\log|\gamma_n|_\Gamma}{|\gamma_n|_\Gamma}$$

for every $n\in\mathbb{N}$.

Proof. We first conclude that the tensor product $\rho_1\otimes\rho_2$ is not P_2-Anosov. We fix a hyperbolic structure on S_g and let $\{g_t\}_{t\in\mathbb{R}}$ be the geodesic flow on T^1S. For a Hölder continuous function $f: T^1S\to\mathbb{R}$ we denote by $\{g_t^f\}_{t\in\mathbb{R}}$ the reparametrization of $\{g_t\}_{t\in\mathbb{R}}$ by f. For more details on the thermodynamical formalism we refer the reader to [Bri+15] and the references therein. Since ρ_1,ρ_2 are Borel Anosov [Lab06], there exist positive, Hölder continuous functions $f_{\rho_i}: T^1S\to\mathbb{R}$, $i=1,2$ with the following property: a periodic orbit represented by the conjugacy class of an element $\gamma\in\Gamma$ has period $\ell_{\rho_i}(\gamma) := \lambda_1(\rho_i(\gamma))-\lambda_2(\rho_i(\gamma))$ as a $g_t^{f_{\rho_i}}$-periodic orbit. Note that for a point x in this periodic orbit we have $\int_0^{\ell_{\rho_1}(\gamma)}\frac{f_{\rho_2}}{f_{\rho_1}}(g_s^{f_{\rho_2}}(x))ds=\ell_{\rho_2}(\gamma)$. By the theorem of Potrie-Sambarino [PS17], the topological entropy of $g^{f_{\rho_1}}$ and $g^{f_{\rho_2}}$ is equal to 1. Then, [Bri+15, Proposition 3.8] applied to f_{ρ_1} and f_{ρ_2}, provides sequences $(\delta_n)_{n\in\mathbb{N}}$ and $(\delta_n')_{n\in\mathbb{N}}$ of elements of Γ with $\underline{\lim}_n\frac{\ell_{\rho_2}(\delta_n')}{\ell_{\rho_1}(\delta_n')}\leqslant 1\leqslant\overline{\lim}_n\frac{\ell_{\rho_2}(\delta_n)}{\ell_{\rho_1}(\delta_n)}$. Up to passing to a subsequence, let m_1 and m_2 be the weak limit of the $g^{f_{\rho_1}}$-invariant measures supported on the periodic orbits represented by δ_n and δ_n' respectively. Then, $\int_{T^1S}\frac{f_{\rho_2}}{f_{\rho_1}}dm_1\leqslant 1\leqslant\int_{T^1S}\frac{f_{\rho_2}}{f_{\rho_1}}dm_2$ and we can find $0\leqslant t\leqslant 1$ such that $\int_{T^1S}\frac{f_{\rho_2}}{f_{\rho_1}}dm=1$, where $m=tm_1+(1-t)m_2$. By the Anosov closing lemma we obtain a sequence of periodic orbits represented by the elements $(\gamma_n')_{n\in\mathbb{N}}$ such that $\lim_n\frac{\ell_{\rho_2}(\gamma_n')}{\ell_{\rho_1}(\gamma_n')}=\int_{T^1S}\frac{f_{\rho_2}}{f_{\rho_1}}dm=1$. Therefore, estimate (a) of the proof of Proposition 3.8.3 shows that the representation $\rho_1\otimes\rho_2$ cannot be P_2-Anosov. For every $n\geqslant 2$, Proposition 3.8.3 (2) provides an element $\gamma_n\in\Gamma$ with $|\gamma_n|_\Gamma>n$ and $\big|(\mu_1-\mu_2)(\rho_1(\gamma_n))-(\mu_1-\mu_2)(\rho_2(\gamma_n))\big|\leqslant(1+\frac{1}{n})\log|\gamma_n|_\Gamma$. The conclusion follows. $\square$

3.9 The Hölder exponent of the Anosov limit maps

In this section, we express the Hölder exponent of the limit map of an Anosov representation $\rho:\Gamma\to G$ in terms of the Cartan and Lyapunov projection of $\rho(\Gamma)$. Let us recall the definition of the Hölder exponent of a continuous map between two metric spaces.

Definition 3.9.1. *Let* (X_1,d_1) *and* (X_2,d_2) *be two metric spaces and* $f:(X_1,d_1)\to(X_2,d_2)$ *be a Hölder continuous map. The Hölder exponent of* f *is defined to be*

$$\alpha_f(d_1,d_2) := \sup\Big\{\alpha>0 : \exists\ C>0,\ d_2(f(x),f(y))\leqslant C\cdot d_1(x,y)^\alpha\ \forall\ x,y\in X_1\Big\}$$

We have the following computation for the Hölder exponent of the limit map $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^q)$ of a P_1-Anosov representation $\rho : \Gamma \to \mathsf{GL}(q, \mathbb{R})$ when ξ is spanning. In the case where $V = \langle \xi(\partial_\infty \Gamma) \rangle$ is a proper subspace of $\mathbb{R}^q$, we can always consider the restirction of $\rho|_V$ which is also P_1-Anosov and its Anosov limit map ξ is spanning.

Theorem 1.4.13. *Let (X, d) be a Gromov hyperbolic space and let Γ be a word hyperbolic group acting properly discontinuously and cocompactly on X by isometries. We fix $x_0 \in X$ and $a > 0$ such that there exists a visual metric d_a on $\partial_\infty X$ with $d_a(x, y) \asymp a^{-(x \cdot y)_{x_0}}$ for $x, y \in \partial_\infty X$. Suppose that $q \geqslant 2$ and $\rho : \Gamma \to \mathsf{SL}(q, \mathbb{R})$ is a P_1-Anosov representation whose Anosov limit map $\xi : \big(\partial_\infty X, d_a\big) \to \big(\mathbb{P}(\mathbb{R}^q), d_{\mathbb{P}}\big)$ is spanning. Then*

$$\alpha_\xi(d_a, d_{\mathbb{P}}) = \frac{1}{\log a} \cdot \sup_{n \geqslant 1} \inf_{|\gamma|_X \geqslant n} \frac{\mu_1(\rho(\gamma)) - \mu_2(\rho(\gamma))}{|\gamma|_X}$$

where $|\gamma|_X = d(\gamma x_0, x_0)$.

Proof. We set $a_m := \inf_{|\gamma|_X \geqslant m} \frac{1}{|\gamma|_X} \langle \varepsilon_1 - \varepsilon_2, \mu(\rho(\gamma)) \rangle$ for $m \geqslant 1$. First, we prove that $\alpha(d_a, d_{\mathbb{P}}) \geqslant \frac{1}{\log a}\big(\sup_{m \geqslant 1} a_m\big)$. Fix $\varepsilon > 0$. Let $g, h \in \Gamma$ be two elements and $[gx_0, hx_0] \subset X$ be a geodesic joining gx_0 and hx_0. Let $p \in [gx_0, hx_0]$ such that $d(x_0, m) = \mathrm{dist}(x_0, [gx_0, hx_0])$. Then, we consider points $y_0 = p, x_1, ..., x_k = gx_0$ and $y_0 = p, y_1, ..., y_\ell = hx_0$ with the property $\frac{1}{2} \leqslant d(x_i, x_{i+1}) \leqslant 1$ and $\frac{1}{2} \leqslant d(y_j, y_{j+1}) \leqslant 1$ for every i, j. We can find $L > 0$ and $g_0, .., g_k, h_1, ..., h_\ell \in \Gamma$, with $g_k = g$, $h_\ell = h$, $d(g_i x_0, x_i) \leqslant L$ and $d(h_i x_0, y_i) \leqslant L$. We note that $g_i^{-1} g_{i+1}$ and $h_j^{-1} h_{j+1}$ always lie in a finite subset F of Γ. By Proposition 2.5.5 (i), there exists a constant $C_\rho > 0$ such that $d_{\mathbb{P}}(\Xi_1^+(\rho(gf)), \Xi_1^+(\rho(g))) \leqslant C_\rho \frac{\sigma_2(\rho(g))}{\sigma_1(\rho(g))}$ for every $f \in F$ and $g \in \Gamma$. By using the previous inequality and arguing as in [Kar03, Lemma 1] we obtain the bounds:

$$\begin{aligned}
&d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g))), \Xi_1^+(\rho(h))\Big) \leqslant \sum_{i=1}^{k} d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g_i)), \Xi_1^+(\rho(g_{i+1}))\Big) + \sum_{i=1}^{\ell} d_{\mathbb{P}}\Big(\Xi_1^+(\rho(h_i)), \Xi_1^+(\rho(h_{i+1}))\Big) \\
&= C_\rho \sum_{i:|g_i|_X \geqslant m} \frac{\sigma_2(\rho(g_i))}{\sigma_1(\rho(g_i))} + C_\rho \sum_{i:|h_i|_X \geqslant m} \frac{\sigma_2(\rho(h_i))}{\sigma_1(\rho(h_i))} + C_\rho \sum_{i:|g_i|_X < m} \frac{\sigma_2(\rho(g_i))}{\sigma_1(\rho(g_i))} + C_\rho \sum_{i:|h_i|_X < m} \frac{\sigma_2(\rho(h_i))}{\sigma_1(\rho(h_i))} \\
&\leqslant C_\rho \sum_{i:|g_i|_X \geqslant m} e^{-a_m |g_i|_X} + C_\rho \sum_{i:|h_i|_X \geqslant m} e^{-a_m |h_i|_X} + \\
&+ C_\rho \sum_{i:|g_i|_X < m} e^{(a_m - a_1)m} e^{-a_m |g_i|_\Gamma} + C_\rho \sum_{i:|h_i|_X < m} e^{(a_m - a_1)m} e^{-a_m |h_i|_X} \\
&\leqslant C_\rho e^{(a_m - a_1)m} \sum_{i=1}^{k} e^{-a_m |g_i|_X} + C_\rho e^{(a_m - a_1)m} \sum_{i=1}^{k} e^{-a_m |h_i|_X}
\end{aligned}$$

The choice of the midpoint $p \in X$ and the triangle inequality show that:

$$|g_i|_X \geqslant \max\{i - d(x_0,p) - L, d(x_0,p) - L\} \quad \text{and} \quad |h_j|_X \geqslant \max\{j - d(x_0,p) - L, d(x_0,p) - L\}$$

for every $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$. Therefore, for every $g,h \in \Gamma$

$$d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g)), \Xi_1^+(\rho(h))\Big) \leqslant 2C_\rho e^{(a_m-a_1)m}\Big(2d(x_0,p) + \frac{e^{a_m L}}{1-e^{-a_m}}\Big)e^{-a_m d(x_0,p)}$$

Since (X,d) is Gromov hyperbolic, there exists $\delta > 0$ such that $\big|d(x_0,[gx_0,hx_0]) - (gx_0 \cdot hx_0)_{x_0}\big| \leqslant \delta$ for every $g,h \in \Gamma$. If we set $R_m := \frac{e^{a_m L}}{1-e^{-a_m}}$ then:

$$\begin{aligned} d_{\mathbb{P}}\Big(\Xi_1^+(\rho(g)), \Xi_1^+(\rho(h))\Big) &\leqslant \frac{2C_\rho}{\varepsilon} \cdot e^{(a_m-a_1)m} \cdot e^{\varepsilon R_m} \cdot e^{-(a_m-2\varepsilon)d(x_0,p)} \\ &\leqslant \Big(\frac{2C_\rho}{\varepsilon} \cdot e^{(a_m-a_1)m+\varepsilon R_m+(a_m-2\varepsilon)\delta}\Big)e^{-(a_m-2\varepsilon)(gx_0\cdot hx_0)_{x_0}} \end{aligned}$$

for every $g,h \in \Gamma$. Since ξ satisfies the Cartan property we have

$$d_{\mathbb{P}}(\xi(x),\xi(y)) \leqslant \Big(\frac{2C_\rho}{\varepsilon} \cdot e^{(a_m-a_1)m+\varepsilon R_m+(a_m-2\varepsilon)\delta}\Big)e^{-(a_m-2\varepsilon)(x\cdot y)_{x_0}} \asymp d_a(x,y)^{\frac{a_m-2\varepsilon}{\log a}}$$

for every $x,y \in \partial_\infty X$. It follows that that ξ is $\frac{1}{\log a}(a_m - 2\varepsilon)$-Hölder. Note that since $\varepsilon > 0$ and $m \geqslant 1$ were arbitrary we have $\alpha_\xi(d_a, d_{\mathbb{P}}) \geqslant \frac{1}{\log a}\big(\sup_{m\geqslant 1} a_m\big)$.

Let $(\gamma_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements of Γ such that $\lim_n \frac{\langle \varepsilon_1-\varepsilon_2, \mu(\rho(\gamma_n))\rangle}{|\gamma_n|_X} = \sup_{m\geqslant 1} a_m$. We may write $\rho(\gamma_n) = k_{\rho(\gamma_n)} \exp(\mu(\rho(\gamma_n))k'_{\rho(\gamma_n)}$ in the standard Cartan decomposition of $\mathsf{SL}(q,\mathbb{R})$ and up to extracting we may assume that $\lim_n k'_{\rho(\gamma_n)} = k'$. Since ξ is spanning and Γ acts minimally on $\partial_\infty\Gamma$, for every open subset W of $\partial_\infty\Gamma$, the image $\xi(W)$ cannot be contained in a union of projective hyperplanes. Hence, we may choose W to satisfy:

-if $y \in W$ and $\xi(y) = k_y P_1^+$, then $\langle k'k_ye_1, e_1\rangle\langle k'k_ye_1,e_2\rangle \neq 0$.

-the function $a_y := \frac{\langle k'k_ye_1,e_2\rangle}{\langle k'k_ye_1,e_1\rangle}$, where $y \in W$ and $\xi(y) = k_yP_1^+$, is not constant.

Therefore, we may choose $z,z' \in W$ such that $a_z \neq a_{z'}$ and also $z,z' \neq \lim_n \gamma_n^{-1}$. Then we observe that if we write

$$a_{z,i,n} := \frac{\sigma_i(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} \cdot \frac{\langle k'_{\rho(\gamma_n)}k_ze_1, e_i\rangle}{\langle k'_{\rho(\gamma_n)}k_ze_1,e_1\rangle} \qquad a_{z',i,n} := \frac{\sigma_i(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))} \cdot \frac{\langle k'_{\rho(\gamma_n)}k_{z'}e_1, e_i\rangle}{\langle k'_{\rho(\gamma_n)}k_{z'}e_1,e_1\rangle}$$

$v_{z,n}=\sum_{i=3}^{d}a_{z,i,n}e_i$ and $v_{z',n}=\sum_{i=3}^{d}a_{z',i,n}e_i$, we have that

$$\begin{aligned}
&d_{\mathbb{P}}\big(\rho(\gamma_n)\xi(z),\rho(\gamma_n)\xi(z')\big)^2=d_{\mathbb{P}}\Big(\exp(\mu(\rho(\gamma_n))k'_{\rho(\gamma_n)}k_zP_1^+,\exp(\mu(\rho(\gamma_n))k'_{\rho(\gamma_n)}k_{z'}P_1^+\Big)^2\\
&=1-\frac{\big(1+a_{z,2,n}a_{z',2,n'}+\langle v_{z,n},v_{z',n}\rangle\big)^2}{\big(1+a_{z,2,n}^2+||v_{z,n}||^2\big)\big(1+a_{z',2,n}^2+||v_{z',n}||^2\big)}\\
&=\frac{(a_{z,2,n}-a_{z',2,n})^2+||a_{z',2,n}v_{z,n}-a_{z,2,n}v_{z',n}||^2+||v_{z,n}-v_{z',n}||^2+||v_{z,n}||^2||v_{z',n}||^2-\langle v_{z,n},v_{z',n}\rangle^2}{\big(1+a_{z,n}^2+||v_{z,n}||^2\big)\big(1+a_{z',2,n}^2+||v_{z',n}||^2\big)}\\
&\geqslant(a_{z,2,n}-a_{z',2,n})^2\cdot\frac{\langle k'_{\rho(\gamma_n)}k_ze_1,e_1\rangle^2}{d-1+\langle k'_{\rho(\gamma_n)}k_ze_1,e_1\rangle^2}\cdot\frac{\langle k'_{\rho(\gamma_n)}k_{z'}e_1,e_1\rangle^2}{d-1+\langle k'_{\rho(\gamma_n)}k_{z'}e_1,e_1\rangle^2}
\end{aligned}$$

Note that by the choice of $z,z'\in W$ we have $\lim_n\frac{\sigma_1(\rho(\gamma_n))}{\sigma_2(\rho(\gamma_n))}\big(a_{z,2,n}-a_{z',2,n}\big)=a_z-a_{z'}\neq 0$ as well as there exists $\delta>0$ such that $|\langle k_{\rho(\gamma_n)}k_ze_1,e_1\rangle|\geqslant\delta$ and $|\langle k_{\rho(\gamma_n)}k_{z'}e_1,e_1\rangle|\geqslant\delta$ for every $n\in\mathbb{N}$. Therefore, we can find $\nu=\nu_{z,z'}>0$ such that $d_{\mathbb{P}}\big(\xi(\gamma_nz),\xi(\gamma_nz')\big)\geqslant\nu\frac{\sigma_2(\rho(\gamma_n))}{\sigma_1(\rho(\gamma_n))}$ for every $n\in\mathbb{N}$. Since $z,z'\neq\lim_n\gamma_n^{-1}$, we can find $M>0$ with the following property: for every pair of sequences $(z_s)_{s\in\mathbb{N}}$ and $(z_s')_{s\in\mathbb{N}}$ in X converging to z and z' respectively and $n\in\mathbb{N}$ we have $\overline{\lim}_s\big(\gamma_n^{-1}x_0\cdot z_s\big)_{x_0}\leqslant M$, $\overline{\lim}_s\big(\gamma_n^{-1}x_0\cdot z_s'\big)_{x_0}\leqslant M$ and $\overline{\lim}_s\big(z_s\cdot z_s'\big)_{x_0}\leqslant M$. Notice that we can write

$$\big(\gamma_nz_s\cdot\gamma_nz_s'\big)_{x_0}=|\gamma_n|_X+(z_s\cdot z_s')_{x_0}-\big(\gamma_n^{-1}x_0\cdot z_s\big)_{x_0}-\big(\gamma_n^{-1}x_0\cdot z_s'\big)_{x_0}$$

Therefore, $\big|\big(\gamma_nz\cdot\gamma_nz'\big)_{x_0}-|\gamma_n|_X\big|\leqslant 3M$ for every $n\in\mathbb{N}$. Now suppose that there exists $C>0$ such that $d_{\mathbb{P}}(\xi(x),\xi(y))\leqslant Ca^{-\kappa(x\cdot y)_{x_0}}$ for every $x,y\in\partial_\infty X$. Then, $d_{\mathbb{P}}\big(\xi(\gamma_nz),\xi(\gamma_nz')\big)\leqslant Ca^{-\kappa(\gamma_nz\cdot\gamma_nz')_{x_0}}$ for every $n\in\mathbb{N}$ and by the previous step we conclude

$$\frac{\sigma_1(\rho(\gamma_n))}{\sigma_2(\rho(\gamma_n))}\geqslant\frac{\nu}{C}a^{\kappa(\gamma_nz\cdot\gamma_nz')_{x_0}}\geqslant\frac{\nu a^{3\kappa M}}{C}a^{\kappa|\gamma_n|_X}$$

for every $n\in\mathbb{N}$. We finally obtain $\kappa\leqslant\frac{1}{\log a}\lim_n\frac{1}{|\gamma_n|_X}(\mu_1(\rho(\gamma_n))-\mu_2(\rho(\gamma_n)))=\frac{1}{\log a}\sup_{m\geqslant 1}a_n$. This gives the upper bound $\alpha(d_a,d_{\mathbb{P}})\leqslant\frac{1}{\log a}\big(\sup_{m\geqslant 1}a_m\big)$ and the theorem follows. □

The previous formula works also for reducible P_1-Anosov representations which are not semisimple and whose Anosov limit map ξ is spanning. For Zariski dense Anosov representations we obtain the following general formula for the Hölder exponent of its Anosov limits maps in terms of the Lyapunov projection. The first equality between the exponents of the Anosov limit maps is in analogy to Guichard's result [Gui05, Corollaire 12] for the Hölder regularity of the boundary of a divisible properly convex domain in the projective space and its dual. Zhang-Zimmer in [ZZ19] established conditions under which the proximal limit set of a P_1-Anosov representation is a C^{α}-

submanifold of the corresponding projective space. In [ZZ19, Theorem 1.12] they provide a formula for the optimal value of α in terms of the Lyapunov projection of the image of the representation.

Recall that for a group Γ, Γ_∞ denotes the subset of infinite order elements of Γ. In the case where ρ is semisimple we are able to replace the Cartan projection in the formula of Theorem 1.4.13 with the Lyapunov projection as follows:

Corollary 3.9.2. *Let (X, d) be a Gromov hyperbolic space and let Γ be a word hyperbolic group acting properly discontinuously and cocompactly on X by isometries. We fix $x_0 \in X$ and $a > 0$ such that there exists a visual metric d_a on $\partial_\infty X$ with $d_a(x, y) \asymp a^{-(x\cdot y)_{x_0}}$ for $x, y \in \partial_\infty X$. Let G be a real semisimple Lie group and fix $\theta \subset \Delta$ a subset of restricted roots of G.*

(i) *Suppose that $d \geqslant 2$ and $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ is an irreducible P_1-Anosov representation and let $\eta^+ : (\partial_\infty X, d_a) \to (\mathbb{P}(\mathbb{R}^d), d_{\mathbb{P}})$ and $\eta^- : (\partial_\infty X, d_a) \to (\mathsf{Gr}_{d-1}(\mathbb{R}^d), d_{\mathsf{Gr}_{d-1}})$ be the Anosov limit maps of ρ. Then*

$$\alpha_{\eta^+}(d_a, d_{\mathbb{P}}) = \alpha_{\eta^-}(d_a, d_{\mathsf{Gr}_{d-1}}) = \frac{1}{\log a} \cdot \inf_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma))}{|\gamma|_{X,\infty}}$$

Moreover, if $\rho(\Gamma)$ is a subgroup of $\mathsf{SO}(d-1, 1)$ then $\alpha_{\eta^+}(d_a, d_{\mathbb{P}})$ is attained.

(ii) *Suppose that $\rho' : \Gamma \to G$ is a Zariski dense P_θ-Anosov representation and let $\xi^+ : (\partial_\infty X, d_a) \to (P_\theta^+, d_{G/P_\theta^+})$ and $\xi^- : (\partial_\infty X, d_a) \to (G/P_\theta^-, d_{G/P_\theta^-})$ be the Anosov limit maps of ρ'. Then*

$$\alpha_{\xi^+}(d_a, d_{G/P_\theta^+}) = \alpha_{\xi^-}(d_a, d_{G/P_\theta^+}) = \frac{1}{\log a} \cdot \inf_{\gamma \in \Gamma_\infty} \frac{\min_{\varphi \in \theta} \langle \varphi, \lambda(\rho'(\gamma)) \rangle}{|\gamma|_{X,\infty}}$$

Proof. (i) Let $\beta := \inf\limits_{\gamma \in \Gamma_\infty} \frac{\lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma))}{|\gamma|_{X,\infty}}$. The inequality $\alpha_{\eta^+}(d_a, d_{\mathbb{P}}) \leqslant \frac{\beta}{\log a}$ follows by Sambarino's lemma [Sam16, Lemma 6.8]. It is enough to prove the bound $\alpha_{\eta^+}(d_a, d_{\mathbb{P}}) \geqslant \frac{\beta}{\log a}$.

By Theorem 1.4.13 there exists an infinite sequence of elements $(\gamma_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{\mu_1(\rho(\gamma_n)) - \mu_2(\rho(\gamma_n))}{|\gamma_n|_X} = \log a \cdot \alpha_{\eta^+}(d_a, d_{\mathbb{P}})$$

Since ρ is semisimple, there exists a finite subset F of Γ and $C > 0$ satisfying the conclusion of Theorem 2.6.1. Up to passing to a subsequence, we may assume that $\lim_n \gamma_n x_0 = x$ and $\lim_n \gamma_n^{-1} x_0 = y$ for some $x, y \in \partial_\infty X$. Since $\partial_\infty X$ is perfect, we may choose $b \in \Gamma$ such that $b^{-1} f^{-1} y \neq x$ for every $f \in F$. Hence, $\lim_n (\gamma_n b f)^{-1} \neq \lim_n (\gamma_n b f)$ for every $f \in F$. By applying Theorem 2.6.1 for the sequence $(\gamma_n b)_{n\in\mathbb{N}}$ and the sub-additivity of the Cartan projection, we may pass to a subsequence still denoted by $(\gamma_n)_{n\in\mathbb{N}}$ and find $f_0 \in F$ such that $\left|(\mu_1-\mu_2)(\rho(\gamma_n)) - (\lambda_1-\lambda_2)(\rho(\gamma_n b f_0))\right| \leqslant C$ for every $n \in \mathbb{N}$. Let $\delta_n := \gamma_n b f_0$ and notice that $b \in \Gamma$ was chosen so that $\lim_n \delta_n \neq \lim_n \delta_n^{-1}$. Note that since $\lim_n \delta_n^+ = \lim_n \delta_n x_0$ we have that $\sup_{n\in\mathbb{N}} \left(\delta_n^+ \cdot \delta_n x_0\right)_{x_0} < +\infty$. By Lemma 2.2.7, we may find $L > 0$ such that $0 \leqslant |\delta_n|_X - |\delta_n|_{X,\infty} \leqslant L$ for every $n \in \mathbb{N}$. Note that $\lim_n \frac{|\gamma_n|_X}{|\delta_n|_X} = \lim_n \frac{|\delta_n|_{X,\infty}}{|\delta_n|_X} = 1$, hence $\lim_n \frac{\mu_1(\rho(\gamma_n))-\mu_2(\rho(\gamma_n))}{|\gamma_n|_X} = \lim_n \frac{\lambda_1(\rho(\gamma_n))-\lambda_2(\rho(\gamma_n))}{|\gamma_n|_X}$. This shows $\log a \cdot \alpha_{\eta^+}(d_a, d_{\mathbb{P}}) \geqslant \beta$ and proves the formula for the Hölder exponent of the map η^+.

Similarly, we obtain the formula for the exponent of the the Anosov limit map $\eta^* : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d)$ of the dual representation ρ^*. By definition of the metric $d_{\mathsf{Gr}_{d-1}}$ we have $\alpha_{\eta^-}(d_a, d_{\mathsf{Gr}_{d-1}}) = \alpha_{\eta^*}(d_a, d_{\mathbb{P}})$. Since for $\gamma \in \Gamma$ we have $\lambda_1(\rho(\gamma^{-1}))-\lambda_2(\rho(\gamma^{-1})) = \lambda_1(\rho^*(\gamma))-\lambda_2(\rho^*(\gamma))$ the conclusion follows.

Suppose that $\rho(\Gamma)$ is a subgroup of $\mathsf{SO}(d-1,1)$. Let $(\gamma_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ be two sequences of Γ with $x = \lim_n \gamma_n x_0$ and $y = \lim_n \delta_n x_0$. We may find $k_n, k_n' \in \mathsf{O}(d-1)\times\mathsf{O}(1)$ and write

$$\Xi_1^+(\rho(\gamma_n)) = [k_n g e_1] \qquad \Xi_1^+(\rho(\delta_n)) = [k_n' g e_1] \quad \text{and} \quad \Xi_1^-(\rho(\delta_n)) = k_n' g e_d^{\perp}$$

where $g = \frac{1}{\sqrt{2}}(E_{11}+E_{d1}+E_{1d}-E_{dd}) + \sum_{i=2}^{d-1} E_{ii} \in \mathsf{O}(d)$. A straightforward calculation shows that

$$d_{\mathbb{P}}\big(\Xi_1^+(\rho(\gamma_n)), \Xi_1^+(\rho(\delta_n))\big) \leqslant \sqrt{2}\sqrt{\operatorname{dist}\big(\Xi_1^+(\rho(\gamma_n)), \Xi_1^-(\rho(\delta_n))\big)}$$

for every $n \in \mathbb{N}$. Therefore, for $x, y \in \partial_\infty\Gamma$ we conclude $d_{\mathbb{P}}(\eta^+(x), \eta^+(y)) \leqslant \sqrt{2}\sqrt{\operatorname{dist}(\eta^+(x), \eta^-(y))}$.

By Lemma 3.6.4, since $\sigma_2(\rho(\gamma)) = 1$ for $\gamma \in \Gamma$, we can find $L > 0$ such that

$$(\rho(\gamma)\cdot\rho(\delta))_{\varepsilon_1} \geqslant \log a \cdot \alpha_{\eta^+}(d_a, d_{\mathbb{P}})(\gamma\cdot\delta)_e - L$$

for every $\gamma, \delta \in \Gamma$. By Corollary 3.6.3 we have that

$$\operatorname{dist}(\eta^+(y), \eta^-(x)) \cdot \operatorname{dist}(\eta^+(x), \eta^-(y)) \leqslant e^{4L} a^{-4\alpha_{\eta^+}(d_a,d_{\mathbb{P}})(x\cdot y)_e}$$

and therefore,

$$d_{\mathbb{P}}(\eta^+(x), \eta^+(y)) \leqslant \sqrt{2}\sqrt[4]{\operatorname{dist}(\eta^+(y), \eta^-(x)) \cdot \operatorname{dist}(\eta^+(x), \eta^-(y))} \leqslant \sqrt{2}e^L \cdot a^{-\alpha_{\eta^+}(d_a,d_{\mathbb{P}})}$$

for every $x, y \in \partial_\infty\Gamma$. In particular, $\alpha_{\eta^+}(d_a, d_{\mathbb{P}})$ is attained.

(ii) Let τ be as in Proposition 2.5.4 so that $\tau \circ \rho'$ is irreducible and P_1-Anosov. Let η and η^* be the Anosov limit maps of $\tau\circ\rho'$ and $(\tau\circ\rho')^*$ in $\mathbb{P}(\mathbb{R}^m)$ respectively. By the definition of the metrics d_{G/P_θ^+} and d_{G/P_θ^-} we have $\alpha_{\xi^+}(d_a, d_{G/P_\theta^+}) = \alpha_{\eta^+}(d_a, d_{\mathbb{P}})$ and $\alpha_{\xi^-}(d_a, d_{G/P_\theta^-}) = \alpha_{\eta^*}(d_a, d_{\mathbb{P}})$. The conclusion follows by the previous part and Proposition 2.5.4 (iv). □

For a metric space (X, d) denote by $\dim(X, d)$ its Hausdorff dimension.

Proof of Corollary 1.4.14. The first part of the corollary follows immediately by applying Theorem 1.4.13 where X is the convex hull of the limit set Λ_Γ in $\mathbb{H}^d$.

For $\delta > 0$ small, the Anosov limit map ξ is $(\alpha_\xi(d_{\mathrm{v}}, d_{\mathbb{P}}) - \delta)$-Hölder, therefore by the definition of Hausdorff dimension we obtain $\dim(\Lambda_{\rho(\Gamma)}, d_{\mathbb{P}}) \leqslant \frac{1}{\alpha_\xi(d_{a_{\mathcal{C}}}, d_{\mathbb{P}})}\dim(\Lambda_\Gamma, d_{\mathrm{v}})$. Now assume that Γ is a uniform lattice in $\mathsf{PO}(d,1)$. The Hausdorff dimension of the limit set Λ_Γ equipped with d_{v} is exactly $d-1$. On the other hand, since $\Lambda_{\rho(\Gamma)}$ is homeomorphic to $\partial_\infty\Gamma$, we have that $\dim(\Lambda_{\rho(\Gamma)}, d_{\mathbb{P}})$ is at least the topological dimension of $\partial_\infty\Gamma$ which is exactly $d-1$. Therefore, the previous inequality immediately shows that $\alpha_\xi(d_{a_{\mathcal{C}}}, d_{\mathbb{P}}) \leqslant 1$. The conclusion now follows by Corollary 3.9.2 (i). □

Remarks 3.9.3. (i) Let Γ, X, ρ, ξ and $a > 0$ be as in Theorem 1.4.13. The Hölder exponent $\alpha(d_a, d_{\mathbb{P}})$ can be arbitrarily large when Γ is virtually a free group. However, when Γ is not virtually free, $\alpha(d_a, d_{\mathbb{P}})$ satisfies the upper bound

$$\alpha(d_a, d_{\mathbb{P}}) \leqslant \frac{H_\Gamma}{\mathrm{vcd}(\Gamma) - 1}\cdot\frac{1}{\log a}$$

where $\mathrm{vcd}(\Gamma)$ denotes the cohomological dimension of a torsion free and finite-index subgroup of Γ and $H_\Gamma = \lim\limits_{n\to\infty}\frac{1}{n}\log\big|\{\gamma\in\Gamma : |\gamma|_X \leq n\}\big|$ is the critical exponent of Γ.

(ii) In Corollary 3.9.2 (i) the formula for the exponent $\alpha_{\eta^+}(d_a, d_{\mathbb{P}})$ in terms of the Lyapunov projection remains valid when ρ is P_1 and P_2-Anosov (e.g. $d = 3$) and η^+ is spanning.

3.10 Examples and counterexamples

In this section, we discuss examples of representations of surface groups with nice properties which are not P_1-Anosov. The examples show that the assumptions of the main results of this paper are necessary. Throughout this section S denotes a closed orientable surface of genus at least 2.

Example 3.10.1. *There exists a strongly irreducible representation $\rho: \pi_1(S) \to \mathsf{SL}(12, \mathbb{R})$ which satisfies the following properties:*

• *ρ is a quasi-isometric embedding, P_1-divergent and preserves a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^{12})$.*

• *ρ admits continuous, injective, ρ-equivariant maps $\xi_1 : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^{12})$ and $\xi_{11} : \partial_\infty \pi_1(S) \to \mathsf{Gr}_{11}(\mathbb{R}^{12})$ which satisfy the Cartan property. The proximal limit set of $\rho(\pi_1(S))$ in $\mathbb{P}(\mathbb{R}^{12})$ is $\xi_1(\partial_\infty \pi_1(S))$ and does not contain projective line segments.*

• *ρ admits continuous, ρ-equivariant and transverse maps $\xi_4 : \partial_\infty \pi_1(S) \to \mathsf{Gr}_4(\mathbb{R}^{12})$ and $\xi_8 : \partial_\infty \pi_1(S) \to \mathsf{Gr}_8(\mathbb{R}^{12})$.*

• *ρ is not P_k-Anosov for any $1 \leqslant k \leqslant 6$.*

The previous example shows that the assumption of transversality in Theorem 1.4.1 is necessary. Moreover, the maps ξ_4 and ξ_8 are transverse although ρ is not P_4-Anosov, therefore Zariski density is also necessary in Theorem 1.4.4.

Proof. Let S be a closed orientable surface of genus at least 2 and $\phi : S \to S$ be a pseudo Anosov homeomorphism of S. The mapping torus M of S with respect to ϕ is a closed 3-manifold whose fundamental group is isomorphic to the HNN extension

$$\pi_1(M) = \Big\langle \pi_1(S), t \ \Big| \ tat^{-1} = \phi_*(a), \ a \in \pi_1(S) \Big\rangle$$

where ϕ_* is the automorphism of $\pi_1(S)$ induced by ϕ. Thurston in [Thu98] (see also Otal [Ota96]) proved that there exists a convex cocompact reprsentation $\rho_0 : \pi_1(M) \to \mathsf{PO}(3,1)$. The representation ρ_0 lifts to a P_1-Anosov representation in $\mathsf{SL}(4,\mathbb{R})$ which we continue to denote by ρ_0 and let $\rho_{\mathrm{Fiber}} := \rho_0|_{\pi_1(S)}$. By a result of Cannon-Thurston [CT07], there exists a continuous equivariant surjection $\theta : \partial_\infty \pi_1(S) \twoheadrightarrow \partial_\infty \pi_1(M)$. By precomposing θ with the Anosov limit map of ρ_0 in $\mathbb{P}(\mathbb{R}^4)$, we obtain a ρ_{Fiber}-equivariant continuous map $\xi_{\mathrm{Fiber}} : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^4)$. Let $\gamma \in \pi_1(S)$ be an element representing a separating simple closed curve on S. We may choose a Zariski dense, Hitchin representation $\rho_{\mathrm{H}} : \pi_1(S) \to \mathsf{SL}(3,\mathbb{R})$ with $2\lambda_1(\rho_{\mathrm{Fiber}}(\gamma)) = \lambda_1(\rho_{\mathrm{H}}(\gamma))$. We claim that $\rho = \rho_{\mathrm{Fiber}} \otimes \rho_{\mathrm{H}} : \pi_1(S) \to \mathsf{SL}(12,\mathbb{R})$ satisfies the required properties.

Let $\otimes : \mathsf{SO}(3,1) \times \mathsf{SL}(3,\mathbb{R}) \to \mathsf{SL}(12,\mathbb{R})$ be the tensor product representation sending the pair (g_1, g_2) to the matrix $g_1 \otimes g_2$. The representation $\otimes$ is irreducible. Let G be the Zariski closure of $\rho_{\mathrm{Fiber}} \times \rho_{\mathrm{H}}$ into $\mathsf{SO}(3,1) \times \mathsf{SL}(3,\mathbb{R})$. Note that the projection of the identity component G^0 into $\mathsf{SO}(3,1)$ (resp. $\mathsf{SL}(3,\mathbb{R})$) is normalized by $\rho_{\mathrm{Fiber}}(\pi_1(S))$ (resp. $\rho_{\mathrm{H}}(\pi_1(S))$), so it has to be surjective. Since the Zariski closures of ρ_{Fiber} and ρ_{H} are simple and not locally isomorphic, it follows by Goursat's lemma that $G = \mathsf{SO}(3,1) \times \mathsf{SL}(3,\mathbb{R})$. We conclude that ρ is strongly irreducible.

We obtain a properly convex domain Ω of $\mathbb{P}(\mathbb{R}^{12})$ preserved by $\rho(\pi_1(S))$ as follows. Let Ω_1 and Ω_2 be properly convex domains of $\mathbb{P}(\mathbb{R}^4)$ and $\mathbb{P}(\mathbb{R}^3)$ preserved by $\rho_{\mathrm{Fiber}}(\pi_1(S))$ and $\rho_{\mathrm{H}}(\pi_1(S))$

respectively. Let Ω_i' be a properly convex cone lifting Ω_i for $i = 1, 2$. The compact set $\mathcal{C} = \{[u_1 \otimes u_2] : u_i \in \overline{\Omega_i'},\ i = 1, 2\}$ is connected, spans $\mathbb{R}^{12}$ and is contained in an affine chart A of $\mathbb{P}(\mathbb{R}^{12})$. We finally take Ω to be the interior of the convex hull of $\mathcal{C}$ in A.

The representations ρ_{Fiber} and ρ_{H} are P_1-divergent hence ρ is also P_1-divergent. Moreover, we notice that for $\delta \in \pi_1(S)$ we have $\mu_1(\rho(\delta)) = \mu_1(\rho_{\mathrm{Fiber}}(\delta)) + \mu_1(\rho_{\mathrm{H}}(\delta))$, hence ρ is a quasi-isometric embedding. Let $\xi_{\mathrm{H}} : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^3)$ and $\xi_{\mathrm{H}}^- : \partial_\infty \pi_1(S) \to \mathsf{Gr}_2(\mathbb{R}^3)$ be the Anosov limit maps of ρ_{H}. The map $\xi_1 : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^{12})$ defined as $\xi_1(x) = [k_x e_1 \otimes k_x' e_1]$, where $\xi_{\mathrm{Fiber}}(x) = [k_x e_1]$ and $\xi_{\mathrm{H}}(x) = [k_x' e_1]$, is continuous and ρ-equivariant. Since ρ is strongly irreducible, the proof of Corollary 3.2.5 shows that the map ξ_1 satisfies the Cartan property. The image of ξ_1 is the P_1-proximal limit set of $\rho(\pi_1(S))$ in $\mathbb{P}(\mathbb{R}^{12})$. Similarly, the dual reprsentation $\rho^* = \rho_{\mathrm{Fiber}}^* \otimes \rho_{\mathrm{H}}^*$ admits a ρ^*-equivariant map $\xi_1^* : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^{12})$, so we obtain the ρ-equivariant map ξ_{11}.

The maps $\xi_4 : \partial_\infty \pi_1(S) \to \mathsf{Gr}_4(\mathbb{R}^{12})$ and $\xi_8 : \partial_\infty \pi_1(S) \to \mathsf{Gr}_8(\mathbb{R}^{12})$ defined as

$$\xi_4(x) = \mathbb{R}^4 \otimes_{\mathbb{R}} \xi_{\mathrm{H}}(x) \quad \text{and} \quad \xi_8(x) = \mathbb{R}^4 \otimes_{\mathbb{R}} \xi_{\mathrm{H}}^-(x)$$

are ρ-equivariant, continuous and transverse. Also for every $x \in \partial_\infty \pi_1(S)$ we have $\xi_1(x) \in \xi_4(x)$, and hence ξ_1 is injective. It follows that $\xi_1(\partial_\infty \pi_1(S)) = \Lambda_{\rho(\pi_1(S))} \cong S^1$. For $x \neq y$ the projective line segment $[\xi_{\mathrm{H}}(x), \xi_{\mathrm{H}}(y)]$ intersects $\Lambda_{\rho_{\mathrm{H}}(\Gamma)}$ at exactly $\{\xi_{\mathrm{H}}(x), \xi_{\mathrm{H}}(y)\}$ hence $[\xi_1(x), \xi_1(y)] \cap \Lambda_{\rho(\Gamma)} = \{\xi_1(x), \xi_1(y)\}$.

The choice of the element $\gamma \in \pi_1(S)$ in the first paragraph shows that $\rho(\gamma)$ cannot be P_k-proximal for $k = 2, 4, 6$, so ρ is not P_k-Anosov for $k = 2, 4, 6$. Let $g \in \pi_1(S)$ be a non-trivial element. The infinite sequence of elements $g_n := \phi_*^{(n)}(g)$ has the property that $(|g_n|_{\pi_1(S),\infty})_{n \in \mathbb{N}}$ is unbounded and there exists $M > 0$ such that $\left|\lambda_1(\rho_{\mathrm{Fiber}}(g_n)) - \lambda_2(\rho_{\mathrm{Fiber}}(g_n))\right| \leqslant M$ for every $n \in \mathbb{N}$. Then, it is easy to check that the differences $\lambda_1(\rho(g_n)) - \lambda_2(\rho(g_n))$, $\lambda_3(\rho(g_n)) - \lambda_4(\rho(g_n))$ and $\lambda_5(\rho(g_n)) - \lambda_6(\rho(g_n))$ are uniformly bounded, so ρ is not P_k-Anosov for $k = 1, 3, 5$. $\square$

Example 3.10.2. *Necessity of the Cartan property.* The representation $\rho \times \rho_{\mathrm{H}} : \pi_1(S) \to \mathsf{SL}(15, \mathbb{R})$ (where ρ and ρ_{H} are from Example 3.10.1) is P_1-divergent and admits a pair of continuous, equivariant, compatible and transverse maps $\xi^+ : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^{15})$ and $\xi^- : \partial_\infty \pi_1(S) \to \mathsf{Gr}_{14}(\mathbb{R}^{15})$ induced from the Anosov limit maps of ρ_{H}. However, $\rho \times \rho_{\mathrm{H}}$ is not P_1-Anosov since ρ cannot uniformly dominate ρ_{H}. This shows that the assumption of the Cartan property for the map ξ^+ in Theorem 1.4.1 is necessary.

Example 3.10.3. *Necessity of regularity of $\partial\Omega$ in Proposition 3.7.1.* Let $n \geqslant 2$ and Γ be an convex cocompact subgroup of $\mathsf{SU}(n, 1) \subset \mathsf{SL}(n+1, \mathbb{C})$. Let $\tau_2 : \mathsf{SL}(n+1, \mathbb{C}) \hookrightarrow \mathsf{SL}(2n+2, \mathbb{R})$ be the standard inclusion defined as

$$\tau_2(g) = \begin{bmatrix} \mathrm{Re}(g) & -\mathrm{Im}(g) \\ \mathrm{Im}(g) & \mathrm{Re}(g) \end{bmatrix}, \quad g \in \mathsf{SL}(n+1, \mathbb{C})$$

The group $\mathrm{sym}^2\big(\tau_2(\Gamma)\big) \subset \mathsf{SL}(2n+2, \mathbb{R})$ is a P_2-Anosov subgroup for which there exist $J, k > 0$ such that

$$\frac{1}{J}(\gamma \cdot \delta)_e - k \leq \big(\mathrm{sym}^2(\tau_2(\gamma)) \cdot \mathrm{sym}^2(\tau_2(\delta))\big)_{\varepsilon_1} \leq J(\gamma \cdot \delta)_e + k$$

for every $\gamma, \delta \in \Gamma$. Moreover, $\mathrm{sym}^2\big(\tau_2(\Gamma)\big)$ preserves a properly convex domain in $\mathbb{P}\big(\mathrm{Sym}^2\mathbb{R}^{2n+2}\big)$ but it cannot preserve a strictly convex domain since it is not P_1-divergent. Similar counterexamples are given by convex cocompact subgroups of the rank 1 Lie group $\mathsf{Sp}(n,1) \subset \mathsf{GL}(n+1, \mathbb{H})$.

Example 3.10.4. *Necessity of transversality in Theorem 1.4.1 in the Zariski dense case.* There exists a Zariski dense representation $\rho_1 : \pi_1(S) \to \mathsf{PSL}(4, \mathbb{R})$ which admits a pair of continuous ρ_1-equivariant maps $\xi^+ : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^4)$ and $\xi^- : \partial_\infty \pi_1(S) \to \mathsf{Gr}_3(\mathbb{R}^4)$ but is not P_1-Anosov. Let M be a closed hyperbolic 3-manifold fibering over the circle (with fiber S) which also contains a totally geodesic surface. By Johnson-Millson [JM87] the natural inclusion $j : \pi_1(M) \hookrightarrow \mathsf{PO}(3,1)$ admits a non-trivial Zariski dense deformation $j' : \pi_1(M) \to \mathsf{PSL}(4, \mathbb{R})$ which by Theorem 2.5.3 can be chosen to be P_1-Anosov. Let ξ_1^+ and ξ_1^- be the Anosov limit maps of j' into $\mathbb{P}(\mathbb{R}^4)$ and $\mathsf{Gr}_3(\mathbb{R}^4)$ respectively. By the theorem of Cannon-Thurston [CT07] there exists a continuous, $\pi_1(S)$-equivariant map $\theta : \partial_\infty \pi_1(S) \to \partial_\infty \pi_1(M)$. The restriction $\rho_1 := j'|_{\pi_1(S)}$ is Zariski dense, not a quasi-isometric embedding and $\xi_1^+ \circ \theta$ and $\xi_1^- \circ \theta$ are continuous, non-transverse and ρ_1-equivariant maps. In addition, by [Can96], every finitely generated free subgroup F of $\pi_1(S)$ is a quasiconvex subgroup of $\pi_1(M)$. Hence, $\iota'|_F$ is P_1-Anosov and $\xi^+ \circ \iota_F$ and $\xi^- \circ \iota_F$ are transverse.

CHAPTER IV

Borel Anosov Representations in Even Dimensions

In this chapter, we prove Theorem 1.4.15 and Corollary 1.4.16, providing a characterization of the domain group of a Borel Anosov representation into $\mathsf{PSL}(4q+2,\mathbb{R})$, $q \geq 1$.

Let us recall that an element $g \in \mathsf{GL}(d,\mathbb{R})$ is called P_1-proximal if $\lambda_1(g) > \lambda_2(g)$. In this case, g admits a unique eigenvalue of maximum modulus which we denote by $\ell_1(g)$. The element $g \in \mathsf{GL}(d,\mathbb{R})$ is called *positively proximal* if g is P_1-proximal and $\ell_1(g) > 0$. From now, when we say that g is *proximal* we mean that g is P_1-proximal. A subgroup Γ of $\mathsf{GL}(d,\mathbb{R})$ is called *positively proximal* if it contains a proximal element and every proximal element of Γ is positively proximal.

4.1 The work of Benoist

First, let us summarize here some results that we use from [Ben00] and [Ben05]. An open cone $C \subset \mathbb{R}^d$ is called *properly convex* if it does not contain an affine line. A domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is called *properly convex* if it is contained in some affine chart of $\mathbb{P}(\mathbb{R}^d)$ in which Ω is bounded and convex.

Lemma 4.1.1. ([Ben05, Lemma 3.2]) *Let Γ be a subgroup of $\mathsf{GL}(d,\mathbb{R})$ which preserves a properly convex open cone C in $\mathbb{R}^d$. Then every $\gamma \in \Gamma$ is positively semi-proximal. In particular, every proximal element $\gamma \in \Gamma$ is positively proximal.*

Benoist characterized irreducible subgroups of $\mathsf{GL}(d,\mathbb{R})$ which preserve a properly convex cone in $\mathbb{R}^d$ as follows:

Theorem 4.1.2. ([Ben00, Proposition 1.1]) *Let Γ be an irreducible subgroup of $\mathsf{GL}(d,\mathbb{R})$. Then Γ preserves a properly convex open cone C in $\mathbb{R}^d$ if and only if Γ is positively proximal.*

We also have the following fact for subgroups of $\mathsf{GL}(d,\mathbb{R})$ which preserve properly convex domains in $\mathbb{P}(\mathbb{R}^d)$:

Fact 4.1.3. Let Γ be a subgroup of $\mathsf{GL}(d,\mathbb{R})$ which preserves a properly convex domain $\Omega\subset\mathbb{P}(\mathbb{R}^d)$. There exists a representation $\widetilde{\iota}:\Gamma\to\mathsf{GL}(d,\mathbb{R})$ and a group homomorphism $\varepsilon:\Gamma\to\mathbb{Z}/2$ such that: $\widetilde{\iota}(\gamma)=(-1)^{\varepsilon(\gamma)}\gamma$ for every $\gamma\in\Gamma$ and $\widetilde{\iota}(\Gamma)$ preserves a properly convex open cone C lifting Ω. Thus, if Γ is also finitely generated the group $\Gamma_2:=\bigcap\{H:[\Gamma:H]\leqslant 2\}$ has finite-index in Γ and preserves the properly convex cone C.

Let F_k be the free group on k generators. We close this section with the following proposition which follows by the work of Breuillard-Green-Guralnick-Tao (see [Bre+12, Theorem 4.1]):

Proposition 4.1.4. ([Bre+12]) *The set of Zariski dense representations from F_2 in $\mathsf{SL}(d,\mathbb{R})$ is dense in the representation variety* $\mathrm{Hom}(F_2,\mathsf{SL}(d,\mathbb{R}))$.

4.2 Proof of Theorem 1.4.15

We need the following lemma which is proved using a theorem of Kapovich-Leeb-Porti [KLP18] (see also [CLS17]).

Lemma 4.2.1. *Let Γ be a torsion free non-elementary word hyperbolic group and $\rho:\Gamma\to\mathsf{GL}(d,\mathbb{R})$ be a representation which admits a continuous ρ-equivariant map $\xi:\partial_\infty\Gamma\to\mathbb{P}(\mathbb{R}^d)$. Suppose there exists $\gamma\in\Gamma$ such that $\rho(\gamma)$ is biproximal, $\xi(\gamma^+)=x^+_{\rho(\gamma)}$ and $\xi(\gamma^-)=x^-_{\rho(\gamma)}$. Then, there exist $a,b\in\Gamma$ such that $\langle a,b\rangle$ is a free quasiconvex subgroup of Γ of rank 2 and the restricted representation $\rho:\langle a,b\rangle\to\mathsf{GL}(d,\mathbb{R})$ is P_1-Anosov with Anosov limit map ξ.*

Proof. By Proposition 3.4.2, the representation ρ is discrete and faithful. Let $t\in\Gamma$ be an infinite order element such that $\{\gamma^+,\gamma^-\}\cap\{t^+,t^-\}$ is empty. Note that $\lim_n t^n\gamma^\pm=t^+$ and $\lim_n t^{-n}\gamma^\pm=t^-$, so we may find $m>0$ such that $\{t^m\gamma^+,t^m\gamma^-\}\cap\{\gamma^+,\gamma^-\}$ and $\{t^{-m}\gamma^+,t^{-m}\gamma^-\}\cap\{\gamma^+,\gamma^-\}$ are empty. Up to conjugating ρ we may assume that $x^+_{\rho(\gamma)}=[e_1], x^+_{\rho(\gamma^{-1})}=[e_d]$ and $V^-_{\rho(\gamma)}=\langle e_2,...,e_d\rangle$, $V^-_{\rho(\gamma^{-1})}=\langle e_1,...,e_{d-1}\rangle$. Then we notice that

$$\rho(t^{\pm m})x^+_{\rho(\gamma)}\notin\mathbb{P}(V^-_{\rho(\gamma)})\cup\mathbb{P}(V^-_{\rho(\gamma^{-1})})\quad\text{and}\quad\rho(t^{\pm m})x^-_{\rho(\gamma)}\notin\mathbb{P}(V^-_{\rho(\gamma)})\cup\mathbb{P}(V^-_{\rho(\gamma^{-1})})$$

For example, suppose that $\rho(t^m)x^+_{\rho(\gamma)}\in\mathbb{P}(V^-_{\rho(\gamma)})$, then $\lim_n\rho(\gamma^n)\rho(t^m)x^+_{\rho(\gamma)}=\lim_n\xi(\gamma^n t^m\gamma^+)=\xi(\gamma^+)=[e_1]$ has to be in $\mathbb{P}(V^-_{\rho(\gamma)})$, a contradiction. Note that $\lim_n\gamma^n t^{-m}\gamma^+=\gamma^+$, hence $\lim_n\rho(\gamma^n t^{-m})\xi(\gamma^+)=x^+_{\rho(\gamma)}$ and $\rho(t^{-m})x^+_{\rho(\gamma^{-1})}\notin\mathbb{P}(V^-_{\rho(\gamma)})$. Then, by [KLP18, Theorem 7.40] (see also [CLS17, Theorem A2]), there exists $N>0$ such that the group $H=\langle\gamma^N,t^m\gamma^N t^{-m}\rangle$ is a free group of rank 2 and the restriction $\rho|_H$ is P_1-Anosov. The restriction $\rho|_H$ is also a quasi-isometric embedding, hence H is a quasiconvex subgroup of Γ and its Anosov limit map is the restriction of ξ on $\partial_\infty H$ considered as a subset of $\partial_\infty \Gamma$. $\square$

Recall that for a finitely generated group Γ, Γ_2 is defined to be the intersection of all finite-index subgroups of Γ of index at most 2.

Lemma 4.2.2. *Let Γ be a torsion free one-ended word hyperbolic group and $\rho : \Gamma * \mathbb{Z} \to \mathsf{GL}(d, \mathbb{R})$ be a representation which admits a ρ-equivariant continuous map $\xi : \partial_\infty(\Gamma * \mathbb{Z}) \to \mathbb{P}(\mathbb{R}^d)$. Suppose that $\delta \in \Gamma_2$ is a non-trivial element such that $\rho(\delta)$ is biproximal and $\xi(\delta^+) = x^+_{\rho(\delta)}$ and $\xi(\delta^-) = x^-_{\rho(\delta)}$. Then $\rho(\delta)$ is positively proximal.*

Proof. Let s be a generator of the free cyclic factor, $t = s\delta s^{-1} \in \Gamma$ and notice that $\rho(t)$ is proximal with $\rho(s)x^+_{\rho(\delta)} = x^+_{\rho(t)} = \xi(t^+)$ and $t^\pm \notin \partial_\infty \Gamma$. If $x \in \partial_\infty \Gamma$, $\lim_n \rho(t^n)\xi(x) = \lim_n \xi(t^n x) = \xi(t^+)$. Since $\rho(t)$ preserves $V^-_{\rho(t)}$ and $\lim_n t^n x = t^+$, $\xi(x)$ cannot lie in $\mathbb{P}(V^-_{\rho(t)})$. It follows that $\xi(\partial_\infty \Gamma)$ lies in the affine chart $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(V^-_{\rho(t)})$. Let $V = \langle \xi(\partial_\infty \Gamma) \rangle$ and we consider the representation $\rho' : \Gamma \to \mathsf{GL}(V)$ where $\rho'(\gamma) = \rho|_V(\gamma)$, $\gamma \in \Gamma$. The map ξ is not constant, hence ρ' is discrete and faithful. The map $\xi : \partial_\infty \Gamma \to \mathbb{P}(V)$ is ρ'-equivariant, $\rho'(\delta)$ is proximal with attracting fixed point $\xi(\delta^+)$ and $\ell_1(\rho(\delta)) = \ell_1(\rho'(\delta))$.

Then we notice that $\xi(\partial_\infty \Gamma)$ also lies in the affine chart $A = \mathbb{P}(V) - \mathbb{P}(V \cap V^-_{\rho(t)})$ of $\mathbb{P}(V)$. Since Γ is one-ended, $\partial_\infty \Gamma$ and $\xi(\partial_\infty \Gamma)$ are connected. The convex hull of $\xi(\partial_\infty \Gamma)$ in A, say $\mathcal{C}$, is bounded and convex in A and has non-empty interior since $\xi(\partial_\infty \Gamma)$ spans V. Then $\rho'(\Gamma)$ preserves $\xi(\partial_\infty \Gamma)$ and by [CT20, Proposition 2.8] it also preserves $\mathcal{C}$. It follows that $\rho'(\Gamma)$ preserves the non-empty properly convex set $\Omega = \mathrm{Int}(\mathcal{C}) \subset \mathbb{P}(V)$. Fact 4.1.3 shows that there exists a representation $\widetilde{\rho}' : \Gamma \to \mathsf{GL}(V)$ which preserves a properly convex cone $C \subset V$ and $\rho'(\gamma) = \widetilde{\rho}'(\gamma)$ for every $\gamma \in \Gamma_2$. By Lemma 4.1.1, $\rho(\delta)$ is positively proximal in $\mathbb{P}(V)$ and hence in $\mathbb{P}(\mathbb{R}^d)$. $\square$

A torsion free word hyperbolic group Γ is called *rigid* if it does not admit a non-trivial splitting over a cyclic subgroup. For example, the fundamental group of a closed negatively curved Riemannian manifold of dimension at least 3 is rigid. By a theorem of Bowditch [Bow98] the Gromov boundary $\partial_\infty \Gamma$ of a rigid hyperbolic group Γ does not contain local cut points.

Lemma 4.2.3. *Let Γ be a torsion free rigid one-ended word hyperbolic group. Let $\rho : \Gamma \to \mathsf{GL}(d, \mathbb{R})$ be a representation which admits a continuous ρ-equivariant map $\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$. Suppose that $\delta \in \Gamma_2$ is a non-trivial element such that $\rho(\delta)$ is biproximal and $\xi(\delta^+) = x^+_{\rho(\delta)}$ and $\xi(\delta^-) = x^-_{\rho(\delta)}$. Then $\rho(\delta)$ is positively proximal.*

Proof. Since $\partial_\infty \Gamma$ does not have any local cut points, the set $\partial_\infty \Gamma - \{\delta^+, \delta^-\}$ is connected. For $x \neq \delta^+, \delta^-$ we have that $\lim_n \delta^{\pm n} x = \delta^\pm$ and, as in Lemma 4.2.2, the conected set $\xi\big(\partial_\infty \Gamma -$

$\{\delta^+, \delta^-\})$ is contained in $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(V^-_{\rho(\delta)}) \cup \mathbb{P}(V^-_{\rho(\delta^{-1})})$. Note that the two $(d-1)$-planes $V^-_{\rho(\delta)}$ and $V^-_{\rho(\delta^{-1})}$ are distinct, hence by the connectedness of $\partial_\infty\Gamma - \{\delta^+, \delta^-\}$ we can find a hyperplane V_0 such that $\xi(\partial_\infty\Gamma)$ is contained in $\mathbb{P}(\mathbb{R}^d) - \mathbb{P}(V_0)$. Then we consider the restriction $\rho' : \Gamma \to \mathsf{GL}(V)$, $V = \langle \xi(\partial_\infty\Gamma)\rangle$, whose image preserves the compact connected subset $\xi(\partial_\infty\Gamma)$ of the affine chart $\mathbb{P}(V) - \mathbb{P}(V \cap V_0)$ of $\mathbb{P}(V)$. The element $\rho'(\gamma)$ is proximal in $\mathbb{P}(V)$ and $\ell_1(\rho(\gamma)) = \ell_1(\rho'(\gamma))$. We similarly conclude that $\rho'(\Gamma)$ preserves a properly convex domain Ω of $\mathbb{P}(V)$. Again, Fact 4.1.3 guarantees that $\rho'(\Gamma_2)$ preserves a properly convex cone of V and $\ell_1(\rho'(\delta)) > 0$. $\square$

Now we combine the previous results to prove Theorem 1.4.15.

Theorem 1.4.15: *Let Γ be a word hyperbolic group and $\rho : \Gamma \to \mathsf{GL}(4q+2, \mathbb{R})$ a representation. Suppose that there exists a continuous, ρ-equivariant dynamics preserving map $\xi : \partial_\infty\Gamma \to \mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$. Then Γ is virtually free or virtually a surface group.*

Proof. We first assume that Γ is a torsion free hyperbolic group. By Proposition 3.4.2, ρ is faithful and we may assume that $\rho(\Gamma)$ is a subgroup of $\mathsf{SL}(4q+2, \mathbb{R})$. If not, we replace ρ with the representation $\hat{\rho} : \Gamma \to \mathsf{SL}^{\pm}(n, \mathbb{R})$, $\hat{\rho}(\gamma) = |\det(\rho(\gamma))|^{-1/(4q+2)}\rho(\gamma)$ and Γ with a finite-index subgroup Γ_0 such that $\hat{\rho}(\Gamma_0)$ is a subgroup of $\mathsf{SL}(4q+2, \mathbb{R})$. Notice that $\hat{\rho}$ has to be faithful since ξ is $\hat{\rho}$-equivariant and dynamics preserving for $\hat{\rho}$.

Let $V_q = \wedge^{2q+1}\mathbb{R}^{4q+2}$, and notice by assumption that $\xi_q = \tau^+_{2q+1} \circ \xi$ is $\wedge^{2k+1}\rho$-equivariant and dynamics preserving. We consider the following two cases:

Case 1. Suppose that Γ has infinitely many ends. Then we show that Γ is free. If not, by Stallings' theorem [Sta68], there exists a splitting $\Gamma = \Gamma_1 * ... * \Gamma_k * F_s$, where $s \geqslant 0$ and for $1 \leqslant i \leqslant k$, Γ_i is an one-ended word hyperbolic group. In particular, there exists a quasiconvex subgroup of Γ of the form $\Delta * \mathbb{Z}$, with Δ one-ended. Lemma 4.2.1, shows that there exists a quasiconvex free subgroup H_0 of Δ_2 such that $\wedge^{2q+1}\rho(H_0)$ is P_1-Anosov in $\mathsf{SL}(V_q)$ and its limit map is the restriction $\xi_q : \partial_\infty H_0 \to \mathbb{P}(V_q)$.

Since $\wedge^{2q+1}\rho(\delta)$ is proximal for every $\delta \in H_0 \subset \Delta_2$, by Lemma 4.2.2, $\ell_1(\wedge^{2q+1}(\rho(\delta))) > 0$. The representation $\rho : H_0 \to \mathsf{SL}(4q+2, \mathbb{R})$ is P_{2q+1}-Anosov and $\wedge^{2q+1}\rho(\gamma)$ is positively proximal for every non-trivial $\gamma \in H_0$. By Theorem 2.5.3 (iii), we can find a path connected open neighbourhood U of $\rho_0 := \rho|_{H_0}$ in $\mathrm{Hom}(H_0, \mathsf{SL}(4q+2, \mathbb{R}))$ consisting of entirely of P_{2q+1}-Anosov representations. Proposition 4.1.4 guarantees that there exists $\rho_1 \in U$ such that $\rho_1(F_k)$ is Zariski dense in $\mathsf{SL}(4q+2, \mathbb{R})$. Let $\{\rho_t\}_{0 \leqslant t \leqslant 1}$ be a continuous path between ρ_0 and ρ_1 contained entirely in U. Observe that for every $\gamma \in H_0$, the map $t \mapsto \ell_1(\wedge^{2q+1}\rho_t(\gamma))$ is continuous with real values and nowhere vanishing. Hence $\ell_1(\wedge^{2q+1}\rho_1(\gamma)) > 0$ for every $\gamma \in H_0$. Therefore, since $\wedge^{2k+1}$ is an irreducible representation, the group $\wedge^{2q+1}\rho_1(H_0)$ is a strongly irreducible subgroup of $\mathsf{SL}(V_q)$ which is positively proximal. By Theorem 4.1.2, the group $\wedge^{2q+1}\rho_1(H_0)$ preserves a properly convex cone and hence a properly convex domain of $\mathbb{P}(V_q)$. On the other hand, the group $\wedge^{2q+1}\mathsf{SL}(4q+2,\mathbb{R})$ (and hence $\wedge^{2q+1}\rho_1(H_0)$) preserves the symplectic non-degenerate form $\omega_q : V_q \times V_q \to \mathbb{R}$ given by the formula $\omega_q(a,b) = a \wedge b \in \langle e_1 \wedge ... \wedge e_{4q+2}\rangle$. However, by [Ben00, Corollary 3.5], a strongly irreducible subgroup of $\mathsf{SL}(d,\mathbb{R})$ which preserves a symplectic form cannot preserve a properly convex domain of $\mathbb{P}(\mathbb{R}^d)$. We have reached a contradiction, so Γ cannot contain any non-trivial one-ended factors in its free product decomposition. Therefore, Γ is free.

Case 2. Suppose that Γ is one-ended and not virtually a surface group. Wilton's result [Wil18, Corollary B] ensures that Γ contains a quasiconvex subgroup Δ which is either isomorphic to a surface group or rigid. If Δ has infinite index in Γ, then there exists a quasiconvex subgroup of Γ isomorphic to $\Delta * \mathbb{Z}$. However, by the previous case we obtain a contradiction. Therefore, we may assume that Δ is rigid and has finite index in Γ. By Lemma 4.2.1, there exists H_1 a quasiconvex free subgroup of Δ_2 such that the restriction $\wedge^{2q+1}\rho|_{H_1}$ is P_1-Anosov. By Lemma 4.2.3, for every $h \in H_1$, $\wedge^{2q+1}\rho(h)$ is positively proximal in $\mathbb{P}(V_q)$. By continuing as previously, we obtain a P_{2q+1}-Anosov, Zariski dense deformation ρ_1 of $\rho|_{H_1}$ such that $\wedge^{2q+1}\rho_1(H_1)$ is positively proximal. Again, by Theorem 4.1.2, $\wedge^{2q+1}\rho_1(H_1)$ preserves a properly convex domain and the symplectic form ω_q, a contradiction.

We now consider the general case where Γ might have torsion or ρ is not faithful. If ρ is not faithful, Proposition 3.4.2 shows that $\ker(\rho)$ is finite. The group $\Gamma' = \Gamma/\ker\rho$ is word hyperbolic, $\partial_\infty\Gamma' = \partial_\infty\Gamma$, so ξ is a ρ'-equivariant dynamics preserving map, where $\rho' : \Gamma' \to \mathsf{GL}(4q+2,\mathbb{R})$ is the faithful representation induced by ρ. By Selberg's lemma, there exists a torsion free finite-index subgroup Γ_1 of Γ'. The previous arguments imply that Γ_1 is either a surface group or a free group. Therefore, Γ is either a finite extension of a virtually free group or a virtually surface group. In the second case, its boundary is the circle and by [Gab92], Γ is virtually a surface group. In the first case, by [Dun85], Γ has infinitely many ends and splits as the fundamental group of a finite graph of groups with finite edge groups and vertex groups of at most one end. The vertex groups of this splitting are also finite extensions of a virtually free group hence finite. It follows that Γ is virtually free. □

By following the argument of case 1 in the proof of Theorem 1.4.1 we obtain the following conclusion:

Theorem 4.2.4. *Let F_2 be the free group on two generators and $\rho : F_2 \to \mathsf{GL}(4q+2,\mathbb{R})$ a representation. Suppose that ρ is P_{2q+1}-Anosov. Then $\wedge^{2q+1}\rho(F_2)$ is not a positively proximal subgroup of $\mathsf{GL}(\wedge^{2q+1}\mathbb{R}^{4q+2})$.*

For the proof of Corollary 1.4.16 we need the following proposition for the existence of lifts of P_{2k+1}-Anosov representations into $\mathsf{PGL}(d,\mathbb{R})$. The proof is similar to Lemma 4.2.2 and 4.2.3. In the case ρ is irreducible and $k=0$, Zimmer has proved the existence of lifts in [Zim17, Theorem 3.1].

Proposition 4.2.5. *Let Γ be a torsion free word hyperbolic group and $\rho:\Gamma\to\mathsf{PGL}(d,\mathbb{R})$ is a P_{2k+1}-Anosov representation, where $0\leqslant k\leqslant \frac{d-1}{4}$.*

(i) *Suppose that Δ is an infinite index, one-ended quasiconvex subgroup of Γ and ρ_0 is the restriction of ρ on Δ. There exists a lift $\widetilde{\rho_0}:\Delta\to\mathsf{GL}(d,\mathbb{R})$ such that $\wedge^{2k+1}\widetilde{\rho_0}(\Delta)$ is positively proximal.*

(ii) *If Γ is a rigid word hyperbolic group then there exists a lift $\widetilde{\rho}:\Gamma\to\mathsf{GL}(d,\mathbb{R})$ of ρ such that $\wedge^{2k+1}\rho(\Gamma)$ is positively proximal.*

Proof. We begin with the following observation: suppose that $\varphi:\Gamma\to\mathsf{PGL}(V_1\oplus V_2)$ is a representation such that $\varphi(\gamma)$ preserves V_1 for every $\gamma\in\Gamma$. If $\rho(\gamma)=[g_\gamma]$ then the map $\varphi_0(\gamma)=[g_\gamma|_{V_1}]$ is a well defined representation $\varphi_0:\Gamma\to\mathsf{PGL}(V_1)$. If φ_0 admits a lift $\widetilde{\varphi}_0$, then there exists a lift $\widetilde{\varphi}$ of φ such that $\widetilde{\varphi}(\gamma)|_{V_1}=\widetilde{\varphi}_0(\gamma)$ for every $\gamma\in\Gamma$. The lift $\widetilde{\varphi}$ is defined as follows: for $\gamma\in\Gamma$, $\widetilde{\varphi}(\gamma)$ is the unique element $h_\gamma\in\mathsf{GL}(V_1\oplus V_2)$ such that the restriction of h_γ on V_1 is $\widetilde{\varphi}_0(\gamma)$ and $\varphi(\gamma)=[h_\gamma]$.

Notice that we may asssume that $k=0$, because the exterior power $\wedge^{2k+1}:\mathsf{GL}(d,\mathbb{R})\to\mathsf{GL}(\wedge^{2k+1}\mathbb{R}^d)$ is faithful. For part (i), we may consider $\delta\in\Gamma$ with $\delta^{\pm}\notin\partial_\infty\Delta$ and $\xi(\partial_\infty\Delta)$ is a connected compact subset of the affine chart $\mathbb{P}(\mathbb{R}^d)-\mathbb{P}(V^-_{\rho(\delta)})$. In particular, $\xi(\partial_\infty\Delta)$ lies in the affine chart $A=\mathbb{P}(V)-\mathbb{P}(V\cap V^-_{\rho(\delta)})$ of $\mathbb{P}(V)$, where $V=\langle\xi(\partial_\infty\Delta)\rangle$. Since $\rho_0(\Delta)$ preserves V there exists a well defined representation $\rho_1:\Delta\to\mathsf{PGL}(V)$. The image $\rho_1(\Delta)$ preserves the connected compact set $\xi(\partial_\infty\Delta)$ and hence the interior of the convex hull of $\xi(\partial_\infty\Delta)$ in A. There exists a lift $\widetilde{\rho_1}$ of ρ_1 into $\mathsf{GL}(V)$ such that $\widetilde{\rho_1}(\Delta)$ preserves a properly convex cone C of V. The representation $\widetilde{\rho_1}$ is P_1-Anosov, faithful and by Lemma 4.1.1, $\widetilde{\rho_1}(\gamma)$ is positively proximal for every $\gamma\in\Delta$ non-trivial. By our initial observation we obtain a lift $\widetilde{\rho_0}:\Delta\to\mathsf{GL}(d,\mathbb{R})$ of ρ_0 with $\widetilde{\rho_0}(\gamma)|_V=\widetilde{\rho_1}(\gamma)$. The representation $\widetilde{\rho_1}$ is P_1-Anosov with Anosov limit map ξ. For every non-trivial $\gamma\in\Delta$, the attracting fixed point of $\widetilde{\rho_0}(\gamma)$ is in V and $\ell_1(\widetilde{\rho_0}(\gamma))=\ell_1(\widetilde{\rho_1}(\gamma))>0$.

The proof of (ii) follows by observing, as in Lemma 4.2.3, that the image of $\partial_\infty\Gamma$ under the Anosov limit map ξ lies in an affine chart of $\mathbb{P}(\mathbb{R}^d)$. Then we continue as previously to obtain the lift $\widetilde{\rho}$. $\square$

Proof of Corollary 1.4.16. We first assume that Γ is torsion free. If Γ contains a quasiconvex infinite index one-ended subgroup Γ_0, there exists a lift $\widetilde{\rho}_0$ of $\rho|_{\Gamma_0}$ such that the group $\wedge^{2k+1}\widetilde{\rho}_0(\Gamma_0)$ is positively proximal, contradicting Theorem 4.2.4. Also Γ cannot be rigid again by part (ii) of the previous proposition. Therefore, Γ is either free or has one end and by [Wil18, Corollary B] there exists a quasiconvex surface subgroup which has to be of finite index in Γ.

Now suppose that Γ is not torsion free or $\ker\rho$ is non-trivial. We may find a torsion free finite-index subgroup Γ_1 of $\Gamma' = \Gamma/\ker(\rho)$ so that ρ induces the faithful P_{2q+1}-Anosov representation $\rho' : \Gamma_1 \to \mathsf{G}_{4q+2}$. The previous step shows that $\partial_\infty\Gamma_1 = \partial_\infty\Gamma$ is either a circle or totally disconnected. By working as in the last paragraph of Theorem 1.4.1 we conclude that Γ is virtually free or virtually a surface group. $\square$

Proof of Corollary 1.4.17. Let $\xi : \partial_\infty\Gamma \to \mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$ be a continuous ρ-equivariant map. We first show that ξ is nowhere dynamics preserving. Suppose not, i.e. there exists a P_{2q+1}-proximal element $\rho(\gamma) \in \rho(\Gamma)$ with $\xi(\gamma^+) = x^+_{\rho(\gamma)}$ and $\xi(\gamma^-) = x^-_{\rho(\gamma)}$. The map $\xi^+ := \tau^+_{2q+1} \circ \xi$ is $\wedge^{2q+1}\rho$-equivariant and by Lemma 4.2.1 there exist a free quasiconvex subgroup H of Γ_2 such that $\wedge^{2q+1}\rho|_H$ is P_1-Anosov. Lemma 4.2.3 shows that $\wedge^{2q+1}\rho(H)$ is positively proximal, a contradiction by Theorem 4.2.4.

Let $V_q = \wedge^{2q+1}\mathbb{R}^{4q+2}$ and $\xi^- = \tau^-_{2q+1} \circ \xi$. We show that the map ξ^+ cannot be spanning. Suppose that ξ^+ is spanning and $x_1, ..., x_r \in \partial_\infty\Gamma$ with $V_q = \oplus_{i=1}^r \xi^+(x_i)$, $r = \dim(V_q)$. Since Γ acts minimally on $\partial_\infty\Gamma$, for every open subset U of $\partial_\infty\Gamma$, $\xi^+(U)$ spans V_q and the union $\cup_{i=1}^r \xi^-(x_i)$ cannot contain $\xi^+(\partial_\infty\Gamma)$. There exists $y \in \partial_\infty\Gamma$ and $1 \leqslant j \leqslant r$ with $V_q = \xi^+(x_j) \oplus \xi^-(y) = \xi^+(y) \oplus \xi^-(x_j)$. By the density of pairs $\{(\delta^+, \delta^-) : \delta \in \Gamma\}$ in the set of 2-tuples of $\partial_\infty\Gamma$, we can find $\gamma \in \Gamma$ such that $V_q = \xi(\gamma^+) \oplus \xi^-(\gamma^-) = \xi^+(\gamma^-) \oplus \xi^-(\gamma^+)$.

Then we claim that $g = \wedge^{2q+1}\rho(\gamma)$ is a biproximal matrix. Up to conjugating g we may assume that $\xi^+(\gamma^+) = [e_1 \wedge \cdots \wedge e_{2q+1}]$ and $\xi^-(\gamma^-) = [W_{2q+1}]$, where W_{2q+1} is defined as in sub-section 2.1.2. We may write $g = \begin{bmatrix} a(g) & 0 \\ 0 & A \end{bmatrix}$ for some matrix $A \in \mathsf{GL}(W_{2q+1})$. Suppose that $|\ell_1(A)| \geqslant |a(g)|$. Let $p \geqslant 1$ be the largest possible dimension of a complex Jordan block corresponding to an eigenvalue of maximum modulus of A. Then there exists a subsequence $(k_n)_{n\in\mathbb{N}}$, A_∞ a non-zero matrix and $b \in \mathbb{R}$ with

$$\lim_{n\to\infty} \frac{1}{k_n^{p-1}\big|\ell_1(A)\big|^{k_n}} g^{k_n} = \begin{bmatrix} b & 0 \\ 0 & A_\infty \end{bmatrix}$$

Since $\partial_\infty\Gamma$ is perfect and $\xi^+(\partial_\infty\Gamma)$ spans V_q, we may choose $x \in \partial_\infty\Gamma - \{\gamma^-\}$ such that the projection of $\xi^+(x)$ into W_{2q+1} is not in $\ker(A_\infty)$. Thus, $\lim_n g^{k_n}\xi^+(x) = \lim_n \xi^+(\gamma^{k_n}x) = \xi^+(\gamma^+)$ cannot be the line $[e_1 \wedge ... \wedge e_{2q+1}]$, a contradiction. It follows that $|a(g)| > |\ell_1(A)|$ and $\wedge^{2q+1}\rho(\gamma)$ is proximal with attracting fixed point $\xi^+(\gamma^+)$. Since $V_q = \xi^+(\gamma^-) \oplus \xi^-(\gamma^+)$, the same argument shows that $\wedge^{2q+1}\rho(\gamma^{-1})$ is proximal with attracting fixed point $\xi^+(\gamma^-)$. The map ξ^+ (and hence ξ) preserves the dynamics of $\{\gamma^-, \gamma^+\}$. This contradicts the fact that ξ is nowhere dynamics preserving. Therefore, $\tau^+_{2q+1}(\xi(\partial_\infty\Gamma))$ lies in some proper vector subspace of V_q. $\square$

4.3 Examples

In this section, we provide an example showing that the analogue of Theorem 1.4.1 does not hold in dimensions which are multiples of 4. We also provide examples of Zariski dense Borel Anosov representations of free groups into $\mathsf{SL}(4q,\mathbb{R})$, $q \geq 1$, all of whose elements have positive eigenvalues. Moreover, we give an example of a surface group representation ρ into $\mathsf{SL}(4q+2,\mathbb{R})$ which is not P_{2q+1}-Anosov but admits a ρ-equivariant continuous dynamics preserving map ξ into $\mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$.

Let S be a closed orientable hyperbolic surface and $\tau_2 : \mathsf{SL}(2,\mathbb{C}) \to \mathsf{SL}(4,\mathbb{R})$ be the standard inclusion defined as

$$\tau_2(g) = \begin{bmatrix} \mathrm{Re}(g) & -\mathrm{Im}(g) \\ \mathrm{Im}(g) & \mathrm{Re}(g) \end{bmatrix}, \ g \in \mathsf{SL}(2,\mathbb{C}).$$

Example 4.3.1. *P_{2k}-Anosov representations into $\mathsf{SL}(4k,\mathbb{R})$ of non-surface groups.* Let F_2 be the free group on two generators. The group $\Gamma = \pi_1(S) * \mathbb{Z} * \mathbb{Z}$ admits an Anosov representation ρ into $\mathsf{SL}(2,\mathbb{C})$ and hence $\tau_2 \circ \rho$ is a P_2-Anosov representation into $\mathsf{SL}(4,\mathbb{R})$. For $k \in \mathbb{N}$, the representation $\rho_k = \times_{i=1}^{k}(\tau_2 \circ \rho)$ of Γ into $\mathsf{SL}(4k,\mathbb{R})$ is P_{2k}-Anosov. In fact, by Theorem 2.5.3 (iii) and Proposition 4.1.4 there exists a deformation ρ_k' of ρ_k which is Zariski dense and P_{2k}-Anosov.

Example 4.3.2. *For every $q \geq 1$, there exist Zariski dense Borel Anosov representations of $\mathbb{Z} * \mathbb{Z}$ in $\mathsf{SL}(4q,\mathbb{R})$ all of whose elements have all of their eigenvalues positive.*

Let us fix a presentation of the surface group

$$\pi_1(S) = \Big\langle a_1, b_1, ..., a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \Big\rangle$$

where $g \geq 2$ is the genus of S. We fix $\rho_0 : \pi_1(S) \to \mathsf{SL}(2,\mathbb{R})$ a discrete faithful representation and assume that $\rho_0(a_1) = \mathrm{diag}(s, s^{-1})$ for some $|s| > 1$. Note that since $\{a_1^+, a_1^-\} \cap \{b_1^+, b_1^-\}$ is empty and ρ_0 is P_1-Anosov, $\rho_0(b_1)x^+_{\rho_0(a_1)} \notin \{x^-_{\rho_0(a_1)}, x^+_{\rho_0(a_1)}\}$ and hence the $(1,1)$ entry of $\rho_0(b_1)$ is non-zero. For $t > 0$ define the map ρ_t on the generating set $\{a_1, b_1, ..., a_g, b_g\}$ of $\pi_1(S)$:

$$\rho_t(\gamma) := \begin{cases} \rho_0(\gamma), \ \gamma \in \{a_1, a_2, b_2, ..., a_g, b_g\} \\ \rho_0(b_1)\mathrm{diag}(e^t, e^{-t}) \end{cases}$$

Note that ρ_t extends to a well defined representation $\rho_t : \pi_1(S) \to \mathsf{SL}(2,\mathbb{R})$ which is P_1-Anosov for every $t > 0$. Since the $(1,1)$ entry of $\rho_0(b_1)$ is non-zero a direct computation shows that $\lim_t e^{-t}||\rho_t(b_1)|| > 0$. Since the representation $\rho_0|_{\langle a_2,b_2\rangle}$ is Zariski dense in $\mathsf{SL}(2,\mathbb{R})$, by [AMS95, Theorem 4.1], there exists $C > 0$ and a finite subset F_0 of $\langle a_2, b_2\rangle$ with the property: for every $t > 0$ there exists $f_t \in F_0$ with $\mu_1(\rho_t(b_1 f_t)) = \mu_1(\rho_t(b_1)\rho_0(f_t)) \leq \lambda_1(\rho_t(b_1)\rho_0(f_t)) + C = \lambda_1(\rho_t(b_1 f_t)) + C$. Therefore, we may find $t_0 > 0$ large enough such that $\lambda_1(\rho_{t_0}(g)) > (2q-1)\lambda_1(\rho_0(g))$ where $g := b_1 f_{t_0}$.

Now let $\mathrm{i}_{2q} : \mathsf{SL}(2,\mathbb{R}) \to \mathsf{SL}(2q,\mathbb{R})$ be the unique up to conjugation irreducible representation and consider the tensor product representation $\rho := (\mathrm{i}_{2q} \circ \rho_0) \otimes \rho_{t_0}$. Our example will be constructed as a deformation of the restriction $\rho|_F$ on a free subgroup F of $\pi_1(S)$.

The representation $\mathrm{i}_{2p} \circ \rho_0$ is Borel Anosov and let $\xi^i_{\mathrm{i}_{2p}\circ\rho_1} : \partial_\infty \pi_1(S) \to \mathsf{Gr}_i(\mathbb{R}^{2q})$, $1 \leq i \leq 2q$ be the Anosov limit maps of $\mathrm{i}_{2p} \circ \rho_0$. Let also $\xi_{\rho_{t_0}} : \partial_\infty \pi_1(S) \to \mathbb{P}(\mathbb{R}^2)$ be the Anosov limit map of ρ_{t_0}. Since $\lambda_1(\rho_{t_0}(g)) > (2q-1)\lambda_1(\rho_0(g))$, we may check that the matrix $\rho(g)$ (resp. $\rho(g^{-1})$) is P_i-proximal for every $1 \leq i \leq 2q$ and its attracting fixed point in $\mathsf{Gr}_i(\mathbb{R}^{2q})$ is the i-plane $\xi^i_{\mathrm{i}_{2p}\circ\rho_0}(g^+) \otimes \xi_{\rho_{t_0}}(g^+)$ (resp. $\xi^i_{\mathrm{i}_{2p}\circ\rho_0}(g^-) \otimes \xi_{\rho_{t_0}}(g^-)$). In particular, the ρ-equivariant map $\xi^i_{\mathrm{i}_{2q}\circ\rho_0} \otimes \xi_{\rho_{t_0}} : \partial_\infty \pi_1(S) \to \mathsf{Gr}_i(\mathbb{R}^{4q})$ is dynamics preserving restricted to $\{g^+, g^-\}$. By applying Lemma 4.2.1 for the representations $\wedge^i \rho$ and the maps $\tau_i \circ (\xi^i_{\mathrm{i}_{2q}\circ\rho_0} \otimes \xi_{\rho_{t_0}})$ (τ_i denotes the Plücker embedding), we may find $m \in \mathbb{N}$ and $h \in \pi_1(S)$ such that $F := \langle g^m, hg^m h^{-1}\rangle$ is a free group of rank 2 and $\wedge^i \rho|_F$ is P_1-Anosov for every $1 \leq i \leq 2q$. In particular, $\rho|_F$ is Borel Anosov into $\mathsf{SL}(4q,\mathbb{R})$.

Now observe that since ρ_0 and ρ_{t_0} are connected by a path of P_1-Anosov representations into $\mathsf{SL}(2,\mathbb{R})$, for every $\delta \in \pi_1(S)$ we have $\ell_1(\rho_0(\delta))\ell_1(\rho_{t_0}(\delta)) > 0$. We deduce that for every $\delta \in F$, $\rho(\delta) = (\mathrm{i}_{2q} \circ \rho_0)(\delta) \otimes \rho_{t_0}(\delta)$ is diagonizable and all of its eigenvalues are positive. By Proposition 4.1.4 and the stability of Anosov representations, we may find a Zariski dense deformation $\rho' : F \to \mathsf{GL}(4q,\mathbb{R})$ of $\rho|_F$ such that $\ell_1(\wedge^i \rho(\delta))\ell_1(\wedge^i \rho'(\delta)) > 0$ for every $\delta \in F$ and $1 \leq i \leq 4q$. In particular, for every $\delta \in F$ all the eigenvalues of $\rho'(\delta)$ are positive. $\square$

Example 4.3.3. Let M be the mapping torus of the closed hyperbolic surface S with respect to a fixed pseudo-Anosov homeomorphism $\phi : S \to S$. Recall (see Example 3.10.1) that the group $\pi_1(M)$ contains a normal and infinite index subgroup Γ isomorphic with $\pi_1(S)$. By Thurston's theorem [Thu98] (see also Otal [Ota96]), the group $\pi_1(M)$ admits a convex cocompact representation ι into $\mathsf{PSL}(2,\mathbb{C})$. In fact, by [Cul86], ι lifts to a quasi-isometric embedding $\widetilde{\iota} : \pi_1(M) \to \mathsf{SL}(2,\mathbb{C})$. By composing τ_2 with $\widetilde{\iota}$, we obtain a P_2-Anosov representation $\rho_1 : \pi_1(M) \to \mathsf{SL}(4,\mathbb{R})$. The Cannon-Thurston map (see [CT07]), $\theta : \partial_\infty \pi_1(S) \to \partial_\infty \pi_1(M)$ composed with the Anosov limit map $\xi^2_{\rho_1} : \partial_\infty \pi_1(M) \to \mathsf{Gr}_2(\mathbb{R}^4)$ provides a $\rho_1|_\Gamma$-equivariant dynamics preserving map $\xi_0 : \partial_\infty\Gamma \to \mathsf{Gr}_2(\mathbb{R}^4)$. Note that the representation $\rho_1|_\Gamma$ is not a quasi-isometric embedding, in particular not P_2-Anosov, since Γ is not a quasiconvex subgroup of $\pi_1(M)$. Let $\rho_F : \Gamma \to \mathsf{SL}(2,\mathbb{R})$ be a Fuchsian representation with limit map $\xi^1_{\rho_F}$. The representation $\rho = (\times_{i=1}^q \rho_1|_\Gamma) \times \rho_F$ into $\mathsf{SL}(4q+2,\mathbb{R})$ is not P_{2q+1}-Anosov, however the ρ-equivariant map $\xi = (\oplus_{i=1}^q \xi_0) \oplus \xi^1_{\rho_F} : \partial_\infty\Gamma \to \mathsf{Gr}_{2q+1}(\mathbb{R}^{4q+2})$ is dynamics preserving.

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