Rank One Phenomena in Convex Projective Geometry

by

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Dedicated to my parents, my brother, and my younger self

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ABSTRACT

The Hilbert geometry of properly convex domains is a generalization of real hyperbolic geometry using the real projective space. In this dissertation, we study the Hilbert geometry of properly convex domains by developing analogies with various notions of non-positive curvature in geometry and geometric group theory. We use the resulting geometric tools to study convex co-compact groups, which are a generalization of convex co-compact Kleinian groups.

In the first part, we introduce a notion of rank one properly convex domains and prove that rank one groups are either acylindrically hyperbolic or contain a finite index cyclic subgroup. This is analogous to rank one non-positively curved Riemannian manifolds. In the second part, we develop the notion of "properly convex domains with strongly isolated simplices" which is a 'finer' notion than rank one. We prove that this notion completely characterizes convex co-compact groups that are relatively hyperbolic with respect to Abelian subgroups of rank at least two. This answers a question of Danciger-Guéritaud-Kassel and provides a plausible direction for generalizing Anosov representations beyond Gromov hyperbolic groups. We also establish an analogue of the Flat Torus Theorem from CAT(0) geometry for studying Abelian subgroups of convex co-compact groups.

CHAPTER I

Introduction

The Beltrami-Klein model of the real hyperbolic space $\mathbb{H}^2$ is an open disk in an affine chart in $\mathbb{P}(\mathbb{R}^3)$ where the distance between points is determined by projective cross-ratios. This is a motivating example in convex projective geometry. Convex projective geometry is a generalization of real hyperbolic geometry where we replace the open disk with properly convex domains in the real projective space $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$.

A properly convex domain Ω is an open subset of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ that can be realized as a Euclidean bounded convex domain in some affine chart. The projective cross-ratio distance function on the Beltrami-Klein model also generalizes to such domains - it is called the Hilbert metric on Ω and denoted by d_Ω. The symmetries of a properly convex domain Ω consist of all projective linear transformations that preserve Ω. This is the automorphism group $\mathrm{Aut}(\Omega)$ of the domain and it acts properly and isometrically on $(\Omega, \mathrm{d}_\Omega)$. Then, convex projective geometry can be defined as the study of manifolds (more generally, orbifolds) diffeomorphic to Ω/Λ where Ω is a properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup of $\mathrm{Aut}(\Omega)$. In this sense, it generalizes real hyperbolic geometry.

A different much-studied generalization of real hyperbolic geometry is the geometry of Riemannian manifolds of variable negative curvature, and more generally, non-positive curvature. This dissertation begins with the following question:

Question 1. *Is there a similarity between the geometry of properly convex domains (induced by the Hilbert metric) and the Riemannian geometry of non-positively curved manifolds?*

An old theorem of Kelly-Strauss [KS58] says that a naive answer to this question is "no", since the geometry induced by the Hilbert metric lacks global non-positive curvature. In particular they prove that: $(\Omega, \mathrm{d}_\Omega)$ is a $\mathrm{CAT}(0)$ space (a much-studied generalization of non-positively curved Riemannian manifolds in metric geometry) if and only if $(\Omega, \mathrm{d}_\Omega)$ is isometric to the real hyperbolic space $\mathbb{H}^{d-1}$ (in which case Ω is an open ball in an affine chart in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$).

The goal of this thesis is to overcome this obstacle and develop analogies between the geometry of properly convex domains and geometry of $\mathrm{CAT}(0)$ spaces. Our approach will be to develop tools motivated by geometric group theory to study properly convex domains. Informed by the analogy with $\mathrm{CAT}(0)$ geometry, we will then use these tools to develop a good understanding of properly convex domains and groups that act on such domains. This work draws inspiration from earlier results of Y. Benoist on strictly convex domains (i.e. properly convex domains Ω whose topological boundary $\partial\Omega$ does not contain non-trivial projective line segments). Benoist showed that if Ω is a strictly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a torsion-free discrete subgroup that acts co-compactly on Ω, then the compact manifold Ω/Λ has some properties reminiscent of compact manifolds of negative curvature.

Theorem I.1 ([Ben04])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain that is strictly convex and $\Lambda \leq \mathrm{Aut}(\Omega)$ acts co-compactly on Ω. Then:*

(1) Λ is a Gromov hyperbolic group,

(2) the projective geodesic flow is Anosov,

(3) $\partial\Omega$ is a C^1-submanifold.

This analogy between strictly convex domains and Riemannian negative curvature has been generalized further to finite volume and geometrically finite actions by several authors, see for instance [CLT15, CM14]. But until recently, not much was known about the geometry of more general properly convex domains (i.e. properly convex domains Ω that contain non-trivial projective line segments in $\partial\Omega$). The results of this dissertation aim to fill this gap by studying general properly convex domains from the perspective of CAT(0) geometry using geometric group theory.

1.0.1 Rank One Hilbert Geometries

Hyperbolicity of the geodesic flow has played a key role in the study of the geometry and dynamics of non-positively curved Riemannian manifolds. Although properly convex domains have a projective geodesic flow, they are not even C^1 for a generic properly convex domain. Thus the dynamical notion of hyperbolicity is not as useful in convex projective geometry. Hence, our starting point is to identify a notion of weak hyperbolicity that is well-suited in this setting.

In Riemannian non-positive curvature, the rank rigidity theorem establishes a remarkable dichotomy between the presence of weak hyperbolicity and a complete lack thereof. We need some terminology to state the rank rank rigidity theorem. A compact manifold M is said to have rank one if its unit tangent bundle has a vector v such that the space of parallel Jacobi fields along the geodesic γ_v determined by v is one dimensional. The Riemannian rank rigidity theorem then says:

Theorem I.2 ([Bal85, BS87]). *Suppose M is a compact non-positively curved Riemannian manifold. Then,* *(1) either M has rank one,*

(2) or M splits as a Riemannian product,

(3) or the universal cover $\widetilde{M}$ of M is a higher rank symmetric space.

In the case of irreducible Riemanninan manifolds, the non-hyperbolicity then corresponds to the higher rank Riemannian symmetric spaces. Their analogues in convex projective geometry are higher rank symmetric convex domains [Ben08, Zim20]. Taking a cue from Riemannian non-positive curvature, one can then ask the following natural questions.

Question 2. *Is there a notion of rank one properly convex domains? Does a rank rigidity theorem hold for properly convex domains?*

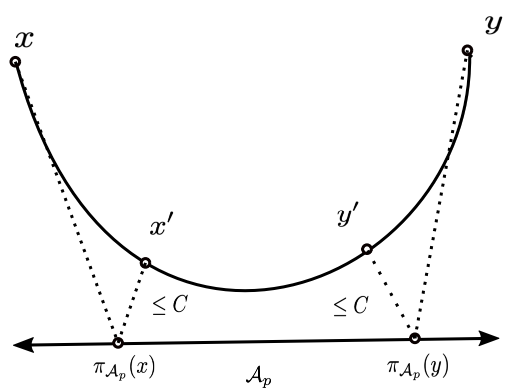

Figure 1.1: Illustration of a contracting element (see Definition III.20)

We answer the first question by characterizing a notion of weak hyperbolicity (the so called, rank one phenomena) in properly convex domains using ideas from geometric theory. The required notion from geometric group theory is that of contracting elements that we now informally state (see III.20 for a more precise definition). Suppose a group G acts properly and isometrically on a proper metric space (X, d) and $\mathcal{PS}^X$ is a path system on X that G preserves. Informally, an element $g \in G$ is called $(X, \mathcal{PS}^X)$-contracting if for some (hence any) $\langle g \rangle$-orbit $\mathcal{A}_p := \langle g \rangle \cdot p$ in X, the following holds: for any two points $x, y \in X$, either the path in $\mathcal{PS}^X$ joining them travels close to the $\langle g \rangle$-orbit for a long time, or the closest-point projection $\pi_{\mathcal{A}_p}$ of x and y on the $\langle g \rangle$-orbit is within a uniformly bounded distance (see Figure 1.1). Prominent examples of contracting elements are pseudo-Anosov homeomorphisms in most mapping class groups, rank one isometries in rank one CAT(0) groups, etc.

We introduce the notion of rank one automorphisms of a properly convex domain and prove that they are precisely the contracting elements. Before stating this theorem, we will briefly discuss the notion of rank one automorphisms. An automorphism $g \in \mathrm{Aut}(\Omega)$ of a properly convex domain is called a rank one automorphism if g has positive translation length in Ω, has an axis in Ω, and none of its axes are contained in a half triangle in Ω (see Definition IV.12). This definition is motivated by a property in CAT(0) geometry - the axis of CAT(0) rank one isometry is not contained in a half flat, i.e. an isometrically embedded copy of $\mathbb{R} \times [0, \infty)$. Although the definition of rank one automorphisms do not require any compactness assumption, the rank one automorphisms admit a simpler characterization when the action is co-compact. If a discrete group $\Lambda \leq \mathrm{Aut}(\Omega)$ acts co-compactly on Ω, then $g \in \Lambda$ is a rank one automorphism if and only if g is biproximal (as a matrix in $\mathrm{PGL}_d(\mathbb{R})$) and g has an axis in Ω (see Lemma IV.16).

We now state the theorem connecting rank one automorphisms and contracting elements.

Theorem I.3 ([Isl19, Theorem 1.4], Chapter V)**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $\mathcal{PS}^\Omega := \{[x, y] : x, y \in \Omega\}$. An element $g \in \mathrm{Aut}(\Omega)$ is contracting for $(\Omega, \mathcal{PS}^\Omega)$ if and only if g is a rank one automorphism.*

Remark I.4. *An element $g \in \mathrm{Aut}(\Omega)$ is contracting for $(\Omega, \mathcal{PS}^\Omega)$ if and only if it is contracting in the sense of BF, as $\mathcal{PS}^\Omega$ is a geodesic path system (cf. III.24 and*

III.26).

We say that a discrete group Λ is a rank one group provided Λ preserves a properly convex domain Ω and some element of Λ is a rank one automorphism in $\mathrm{Aut}(\Omega)$. Using the above correspondence between contracting elements and rank one automorphisms, we prove in Chapter V that rank one groups are examples of acylindrically hyperbolic groups. Acylindrically hyperbolic groups embody a notion of generalized "non-positive curvature" in geometric group theory and includes many important classes of groups like mapping class groups of non-exceptional surfaces, the outer automorphism groups of free group on at least two generators, rank-one CAT(0) groups, etc. Before stating our theorem, we recall that a group is called "virtually cyclic" if it contains a finite index cyclic subgroup.

Theorem I.5 ([Isl19, Theorem 1.5], Chapter V)**.** *Suppose Λ is a rank one group. Then either Λ is virtually cyclic or Λ is an acylindrically hyperbolic group.*

This is analogous to a theorem characterizing rank one CAT(0) groups [Sis18]. This theorem allows us to use the rich theory of acylindrically hyperbolic groups to derive several results about rank one groups. But before discussing those applications, we mention some recent results which complement our above discussion. While our results on rank one properly convex domains work without any compactness assumption on the action, the complementary results that we are going to discuss now will require the action to be co-compact. To state the precise result, we will need to make some definitions. A properly convex domain is irreducible if it does not split as a non-trivial direct sum of two properly convex domains (cf. II.9). A prototypical k-dimensional projective simplex in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is

$$S_k := \mathbb{P}\left(\{[x_1 : x_2 : \ldots : x_{k+1} : 0 : \ldots : 0] : x_1 > 0, \ldots, x_{k+1} > 0\}\right).$$

Definition I.6. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain. We will say that $S \subset \Omega$ is a properly embedded k-dimesional simplex if $S = gS_k$ for some $g \in \mathrm{PGL}_d(\mathbb{R})$ and $S \hookrightarrow \Omega$ is a proper map.

Now we state the complementary results proven by A. Zimmer in [Zim20]. Zimmer introduces a notion of "higher rank" properly convex domains: a properly convex domain Ω has higher rank if for any two points $x, y \in \Omega$, there exists a properly embedded simplex S such that $[x, y] \subset S$. Zimmer proves that, under some assumptions, his notion of higher rank is 'special' and exactly complementary to our definition of rank one.

Theorem I.7 ([Zim20])**.** *Suppose Ω is an irreducible properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a discrete group that acts co-compactly on Ω. Then:*

(1) either Λ is a rank one group

(2) or Ω is a higher rank symmetric domain.

The higher rank symmetric domains are classified, see [Ben08, Zim20]. They are projective analogues of the higher rank Riemannian symmetric spaces. Thus Theorem I.7 is the convex projective analogue of the rank rigidity theorem (in the co-compact case) for Riemannian non-positive curvature. This also shows that our proposed definition of rank one does indeed capture a good notion of hyperbolicity for properly convex domains. We also remark that recently Blayac has studied a notion of rank one that is closely related to ours [Bla20]. He studies the projective geodesic flow on rank one domains and proves hyperbolicity results reminiscent of rank one manifolds of non-positive curvature.

We now end our detour and discuss applications of Theorem I.5 in studying rank one properly convex domains.

The first application of Theorem I.5 deals with 'non-trivial' quasi-morphisms. If G is a group, a quasi-morphism of G is a function $f : G \to \mathbb{R}$ such that

$$\sup_{g,h \in G} |f(gh) - f(g) - f(h)|$$

is finite. The 'trivial' quasi-morphisms are bounded functions and group homomorphisms (of G into $\mathbb{R}$). We say that two quasi-morphisms of G are equivalent if they differ by a 'trivial' quasi-morphism. The equivalence classes of quasi-morphisms constitute the space of 'non-trivial' quasi-morphisms $\widetilde{QH}(G)$ of G. This classical object (or its dimension as a $\mathbb{R}$-vector space to be precise) has connections with weak hyperbolicity and has played an important role in various rigidity theorems.

Theorem I.8 ([Isl19, Theorem 1.6], Section 5.5.1)**.** *Suppose Λ is a torsion-free rank one group that is not virtually cyclic. Then $\dim(\widetilde{QH}(\Lambda)) = \infty$.*

The vector space $\widetilde{QH}(\Lambda)$ can also be interpreted as the kernel of the comparison map between the second bounded cohomology and the second (ordinary) cohomology groups of Λ, modulo the subspace generated by bounded functions and homomorphisms. By virtue of this interpretation, there are more general analogues of $\widetilde{QH}(\Lambda)$ arising from cohomology with more general coefficients. See Section 5.5.1 for more general versions of this theorem in that setting.

In the co-compact case, the higher rank rigidity theorem I.7 and Theorem I.8 provide a rigidity result. This is analogous to a rigidity theorem of Bestvina-Fujiwara for Riemannian non-positive curvature [BF09].

Corollary I.9 ([Isl19, Corollary 1.7], Section 5.5.1)**.** *Suppose Ω is an irreducible properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a discrete torsion-free group that acts co-compactly on Ω. Then Λ is a rank one group if and only if $\dim \widetilde{QH}(\Lambda) = \infty$.*

Otherwise, $\dim \widetilde{QH}(\Lambda) = 0$ *(and* Λ *is isomorphic to a uniform lattice in a Lie group* G *where* G *is locally isomorphic to a simple Lie group of real rank at least two).*

The second application of Theorem I.5 is in counting of conjugacy classes. Let $\tau_\Omega(g)$ be the translation length of g in Ω (cf. II.32).

Theorem I.10 ([Isl19, Theorem 1.8], Section 5.5.2)**.** *Suppose* Ω *is a properly convex domain,* $\Lambda \leq \mathrm{Aut}(\Omega)$ *is a rank one group that is not virtually cyclic, and* Λ *acts co-compactly on* Ω*. Let* $\mathcal{C}(t) := \{[[g]] \in \Lambda : \tau_\Omega([[g]]) \leq t\}$ *where* $[[g]]$ *denotes the conjugacy class of* g *in* Λ*. Then there exist* $D' \geq 1$ *such that for all* $t \geq 1$*,*

$$\frac{1}{D'} \frac{\exp\left(t\omega_\Lambda\right)}{t} \leq \mathcal{C}(t) \leq D' \frac{\exp\left(t\omega_\Lambda\right)}{t}$$

where

$$\omega_\Lambda := \limsup_{n \to \infty} \frac{1}{n} \log\left(\#\{g \in \Lambda : \mathrm{d}_\Omega(x, gx) \leq n\}\right).$$

The third application of Theorem I.5 is to random walks. Let Λ be a finitely generated rank one group that is not virtually cyclic. Consider a simple random walk on Λ, i.e. a random walk generated by a measure supported on a finite symmetric generating set of Λ (see Definition V.17). Then the probability that such a random walk does not encounter a rank one automorphism after n steps decays exponentially fast as n goes to infinity.

Theorem I.11 ([Isl19, Proposition 1.9], Section 5.5.3)**.** *Suppose* Λ *is a finitely generated rank one group that is not virtually cyclic. If* $\{X_n\}$ *is a simple random walk on* Λ*, then there exists a constant* $C \geq 1$ *such that for all* $n \geq 1$*,*

$$\mathbb{P}\left[X_n \textit{ is not a rank one automorphism }\right] \leq Ce^{-Cn}.$$

1.0.2 Convex Co-compact Groups and Relative Hyperbolicity

In the first part of the thesis, our goal was a broad classification of properly convex domains into rank one and higher rank domains in the spirit of Riemannian non-positive curvature. Here we did not have any other assumptions on the action like compactness. The second part of the thesis aims for a finer classification and a more detailed analysis of the structure of properly convex domains. The trade-off is that we will work with a more restricted class of properly convex domains, i.e. domains that are associated to the so-called convex co-compact groups. The notion of convex co-compact subgroups of $\mathrm{PGL}_d(\mathbb{R})$ generalize convex co-compact Kleinian groups from the rank one Lie group $\mathrm{SO}(d,1)$ $(d \geq 2)$ to higher rank Lie groups like $\mathrm{PGL}_d(\mathbb{R})$ for $d \geq 3$. We now define convex co-compact groups.

Definition I.12 ([DGK17])**.** A discrete subgroup $\Lambda \leq \mathrm{PGL}_d(\mathbb{R})$ is called convex co-compact if:

(1) there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ such that $\Lambda \leq \mathrm{Aut}(\Omega)$, and

(2) the set $\mathcal{C}_\Omega(\Lambda) \subset \Omega$ is non-empty and Λ acts co-compactly on $\mathcal{C}_\Omega(\Lambda)$, where $\mathcal{C}_\Omega(\Lambda)$ is the convex hull in Ω of the full orbital limit set $\mathcal{L}^{\mathrm{orb}}_\Omega(\Lambda) := \bigcup_{x \in \Omega} \left(\overline{\Lambda x} \setminus \Lambda x\right)$.

Since the Ω in the definition of convex co-compact groups is not canonical, we will remove this ambiguity by explicitly mentioning the properly convex domain wherever necessary, i.e. we will say that "$\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group" instead of "Λ is a convex co-compact group".

A recent result of Danciger-Guéritaud-Kassel, independently Zimmer, establishes a connection between the Hilbert geometry of the properly convex domain $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ and Anosov representations. More precisely, they prove the following.

Theorem I.13 ([DGK17, Zim17]). *Suppose Ω is a properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then the following are equivalent:*

(1) $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ *is a Gromov hyperbolic space,*

(2) Λ *is a Gromov hyperbolic group, and*

(3) the inclusion $\Lambda \hookrightarrow \mathrm{PGL}_d(\mathbb{R})$ *is a projective Anosov representation.*

Anosov representations are a class of representations of Gromov hyperbolic groups into real semi-simple Lie groups that generalizes classical Teichmüller theory, i.e. the study of discrete faithful representations of hyperbolic surface groups into $\mathrm{PSL}_2(\mathbb{R})$. They are discrete faithful representations that have good dynamical and geometric properties. Introduced by Labourie [Lab06] and studied subsequently by many authors, this area has received much attention lately, see for instance [GGKW17, KLP17, BIW14, Poz19].

The above Theorem I.13 can be interpreted as a way associating convex projective structures to projective Anosov representations. This opens up a more geometric way of thinking about Anosov representations. In their paper, Danciger-Guéritaud-Kassel asked the following natural question [DGK17, Appendix A, Question A.2].

Question 3. *What geometric conditions on* $\mathcal{C}_\Omega(\Lambda)$ *will correspond to* Λ *relatively hyperbolic with respect to virtually Abelian subgroups of rank at least two?*

By virtue of Theorem I.13, this question can be interpreted as seeking a generalization of projective Anosov representations to relatively hyperbolic groups. Note that the current definition of Anosov representations work only for Gromov hyperbolic groups and generalizing it beyond Gromov hyperbolicity is an area of active research, see for instance [Kas18, Gui19]. The notion of relative Anosov representations due to [KL18] and [Zhu19] provide an approach. But we note that those approaches do not provide an answer to the above question. Indeed, the peripheral subgroups in the work of [KL18, Zhu19] consist of unipotent elements while it is easy to verify that convex co-compact subgroups cannot contain unipotent elements other than the identity.

There are many interesting examples of convex co-compact groups coming from Coxeter groups and 3-manifold theory that satisfy the conditions in Question 3, see for instance Figure 1.2 or the papers [Ben06, BDL18, DGK17].

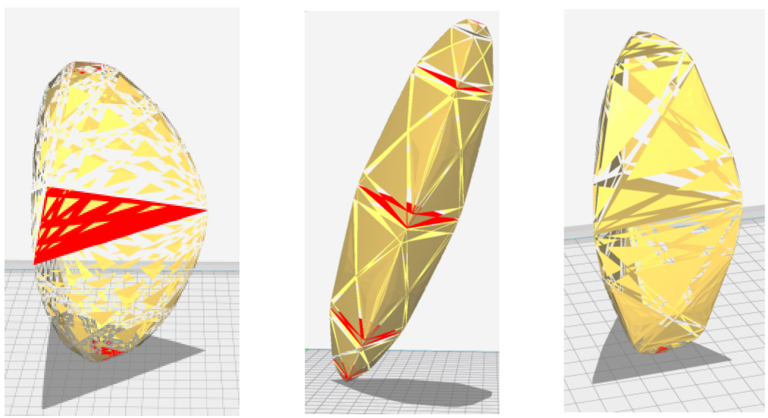

Figure 1.2: Examples of three-dimensional properly convex domains Ω that admit co-compact action by Λ where the groups Λ are relatively hyperbolic with respect to subgroups virtually isomorphic to $\mathbb{Z}^2$ [RSS+19]

We answer Question 3 in joint work with A. Zimmer in [IZ19] by introducing the notion of properly convex domains with strongly isolated simplices. We will now introduce the definition here. We will say that a properly embedded simplex is maximal if it is not properly contained in any other properly embedded simplex. Note that this notion of maximality does not mean that we are looking only at simplices of the maximal possible dimension.

Definition I.14 ([IZ19, Definition 1.15], Chapter VI)**.** Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $\mathcal{S}_\Lambda$ is the collection of all maximal properly embedded simplices in $\mathcal{C}$ of dimension at least two. We will say that $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has *strongly* *isolated simplices* provided: for any $r \geq 0$, there exists $D(r) \geq 0$ such that if $S_1, S_2 \in \mathcal{S}$ are distinct, then

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r) \right) \leq D(r).$$

We answer Question 3 by proving a theorem connecting relative hyperbolicity (with respect to virtually Abelian subgroups of rank at least two) of a convex co-compact group and properly convex domains with strongly isolated simplices. The precise statement is as follows.

Theorem I.15 ([IZ19, Theorem 1.7], Chapter VII)**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \operatorname{Aut}(\Omega)$ is a convex co-compact group, and $\mathcal{S}_\Lambda$ is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two. Then the following are equivalent:*

(1) $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices,

(2) $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ is a relatively hyperbolic space with respect to $\mathcal{S}_\Lambda$,

(3) Λ is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank at least two.

Theorem I.15 can be viewed as a real projective analogue of a CAT(0) result. In [HK05], Hruska-Kleiner study CAT(0) spaces with isolated flats and proves an analogoues result in that setting. In this analogy, maximal properly embedded simplices correspond to maximal totally geodesic flats in CAT(0) spaces (see [IZ21, Ben04]).

We also establish some finer geometric properties of $\mathcal{C}_\Omega(\Lambda)$ when $\Lambda \leq \operatorname{Aut}(\Omega)$ is a convex co-compact group and $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices. We will need some terminology to state the theorem precisely. The ideal boundary of $\mathcal{C}_\Omega(\Lambda)$ is defined as $\partial_{\mathrm{i}}\, \mathcal{C}_\Omega(\Lambda) := \overline{\mathcal{C}_\Omega(\Lambda)} \cap \partial\Omega$, i.e. it is the part of the boundary of $\mathcal{C}_\Omega(\Lambda)$ that is at "infinity" in $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$. If $C \subset \Omega$ is a convex subset and $x \in \overline{C}$, then

$$F_C(x) = \{x\} \cup \{y \in \overline{C} : \exists \text{ an open line segment in } \overline{C} \text{ containing both } x \text{ and } y\}.$$

Theorem I.16 ([IZ19, Theorem 1.8], Chapter VI)**.** *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices. Then*

(1) Λ has finitely many orbits in $\mathcal{S}_\Lambda$.

(2) If $S \in \mathcal{S}_\Lambda$, then $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S and contains a finite index subgroup isomorphic to $\mathbb{Z}^k$ where $k = \dim S$.

(3) If $A \leq \Lambda$ is an infinite Abelian subgroup of rank at least two, then there exists a unique $S \in \mathcal{S}_\Lambda$ with $A \leq \mathrm{Stab}_\Lambda(S)$.

(4) If $S \in \mathcal{S}_\Lambda$ and $x \in \partial S$, then $F_\Omega(x) = F_{\mathcal{C}_\Omega(\Lambda)}(x) = F_S(x)$.

(5) If $S_1, S_2 \in \mathcal{S}_\Lambda$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \emptyset$.

(6) If $\ell \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ is a non-trivial line segment, then there exists $S \in \mathcal{S}_\Lambda$ with $\ell \subset \partial S$.

(7) If $x, y, z \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ form a half triangle in $\mathcal{C}_\Omega(\Lambda)$ (i.e. $[x, y] \cup [y, z] \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ and $(x, z) \subset \mathcal{C}_\Omega(\Lambda)$)*, then there exists $S \in \mathcal{S}_\Lambda$ such that $x, y, z \in \partial S$.*

(8) If $x \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ is not a C^1-smooth point of $\partial\Omega$ (i.e. Ω does not have a unique supporting hyperplane at x), then there exists $S \in \mathcal{S}_\Lambda$ with $x \in \partial S$.

At this point, it is natural to ask how the notion of strongly isolated simplices connects with the notion of rank one introduced in the first part of the dissertation. To answer this, we need to adapt the notion of rank one to the convex co-compact setting (because the dynamics of a convex co-compact group can only "see" the limit set $\mathcal{L}^{\mathrm{orb}}_\Omega(\Lambda)$ instead of the entire boundary $\partial\Omega$). We can show that if $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices, then Λ is a convex co-compact rank one group (see

Section 6.3). Thus “strongly isolated simplices” identifies a very special class of rank one properly convex domains, in the case of convex co-compact actions.

It is worthwhile to note at this point that, to the best of our knowledge, all known examples of “indecomposable” convex co-compact groups correspond to properly convex domains $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ with strongly isolated simplices. We are using the word “indecomposable” here as an informal term that excludes many of the obvious counterexamples. For example, one could consider the convex co-compact group $\pi_1(\Sigma_g) \times \mathbb{Z}$ where Σ_g is a closed hyperbolic surface. This is clearly not a relatively hyperbolic group with respect to the $\mathbb{Z}^2$-subgroups. But one can rule out these examples by requiring that Ω is irreducible. For a more sophisticated example, one could take a free product of two uniform lattices in $\mathrm{PGL}_d(R)$ where $d \geq 3$. By a result of [DGK17], it is a convex co-compact group but it is not relatively hyperbolic with respect to virtually Abelian subgroups (see [Wei20, Section 2.6.3]). But such examples can also be ruled out by requiring that the convex co-compact group Λ is freely indecomposable. It is not clear to us whether these two conditions are enough to provide a good definition of “indecomposable convex co-compact groups”.

1.0.3 Convex Projective Flat Torus Theorem

In joint work with A. Zimmer, we prove a key technical result concerning the Abelian subgroups of a convex co-compact group [IZ21]. It is a convex projective analog of the well-known Flat Torus theorem in CAT(0) geometry [BH99].

Theorem I.17 ([IZ21, Theorem 1.6], Chapter VIII)**.** *Suppose that $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. If $A \leq \Lambda$ is a maximal Abelian subgroup of Λ, then there exists a properly embedded simplex $S \subset \mathcal{C}_\Omega(\Lambda)$ such that*

(1) S is A-invariant,

(2) A acts co-compactly on S, and

(3) A fixes each vertex of S.

Moreover, A has a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

A key step in the proof of Theorem I.17 is studying the centralizer $C_\Lambda(A)$ of an Abelian subgroup A of a convex co-compact group Λ. Recall that

$$C_\Lambda(A) := \bigcap_{a \in A} \{g \in \Lambda : ga = ag\}.$$

We will denote the minimal translation set of $g \in \Lambda$ by

$$\mathrm{Min}_\Omega(g) := \{x \in \Omega : \mathrm{d}_\Omega(x, gx) = \tau_\Omega(g) = \inf_{y \in \Omega} \mathrm{d}_\Omega(y, gy)\}.$$

Theorem I.18 ([IZ21, Theorem 1.10], Chapter VIII)**.** *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $A \leq \Lambda$ is an Abelian subgroup. Then*

$$\mathrm{Min}_{\mathcal{C}_\Omega(\Lambda)}(A) := \mathcal{C}_\Omega(\Lambda) \cap \bigcap_{a \in A} \mathrm{Min}(a)$$

is non-empty and $C_\Lambda(A)$ acts co-compactly on $\mathrm{ConvHull}_\Omega\left(\mathrm{Min}_{\mathcal{C}_\Omega(\Lambda)}(A)\right)$.

CHAPTER II

Preliminaries

2.1 Properly Convex Domains and Hilbert Geometry

2.1.1 Basics of Projective Geometry

Suppose V is a vector space over $\mathbb{R}$. Consider the equivalence relation $\sim$ on V defined by: $v \sim w$ if and only if there exists $r \in \mathbb{R} \setminus \{0\}$ such that $v = r \cdot w$. Then the real projective space of V is defined as

$$\mathbb{P}(V) := (V \setminus \{0\}) / \sim .$$

Since we can always identify V with $\mathbb{R}^{\dim(V)}$ by choosing a basis of V, we will mostly work with $\mathbb{P}\left(\mathbb{R}^{\dim(V)}\right)$. If $v \in V \setminus \{0\}$, we will denote by $[v]$ or $\pi(v)$ its image in $\mathbb{P}(V)$. Conversely, if $u \in \mathbb{P}(V)$, $\widetilde{u}$ will denote a lift of u in V.

If $W \subset V$ is a non-zero linear subspace, we will call $\mathbb{P}(W)$ the projective subspace W. Let $\mathrm{Span}(X)$ denote the linear span of a non-empty subset X of V. Taking linear span is well-defined operation in $\mathbb{P}(V)$: if $Y \subset \mathbb{P}(V)$ is non-empty, let

$$\mathrm{Span}(Y) := \mathrm{Span}\left(\{\widetilde{y} \in V : [\widetilde{y}] \in Y\}\right).$$

An affine chart in $\mathbb{P}(\mathrm{\mathbb{R}^d})$ is an open subset of $\mathbb{P}(\mathrm{\mathbb{R}^d})$ obtained by removing a projective linear subspace of co-dimension one. For instance, if we remove $\mathbb{P}(H_d) := \mathbb{P}(\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d = 0\})$ from $\mathbb{P}(\mathrm{\mathbb{R}^d})$, then we obtain an affine chart $\mathbb{A} :=$

$\mathbb{P}(\mathbb{R}^{\mathrm{d}}) \setminus \mathbb{P}(H_d)$. The affine chart $\mathbb{A}$ is diffeomorphic to $\mathbb{R}^{d-1}$ and the diffeomorphism is given by $(x_1, \ldots, x_d) \mapsto (\frac{x_1}{x_d}, \ldots, \frac{x_{d-1}}{x_d})$. If $s_1, \ldots, s_{d-1} \in \mathbb{R}$, then $[s_1 : \ldots : s_{d-1} : 1]$ are the homogeneous coordinates on $\mathbb{A}$.

The group of projective linear transformations is defined as

$$\mathrm{PGL}_d(\mathbb{R}) := \mathrm{GL}_d(\mathbb{R}) / \sim_G$$

where $\sim_G$ is an equivalence relation on $\mathrm{GL}_d(\mathbb{R})$ given by: $A \sim_G B$ if and only if $A = cB$ for some $c \in \mathbb{R} \setminus \{0\}$. If $A \in \mathrm{GL}_d(\mathbb{R})$, $[A]$ denotes its image in $\mathrm{PGL}_d(\mathbb{R})$. If $B \in \mathrm{PGL}_d(\mathbb{R})$, $\widetilde{B}$ will denote a lift of B in $\mathrm{GL}_d(\mathbb{R})$. The group $\mathrm{PGL}_d(\mathbb{R})$ acts transitively on $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. Let $\mathrm{SL}^{\pm}_d(\mathbb{R})$ be the subgroup of $\mathrm{GL}_d(\mathbb{R})$ consisting of matrices of determinant ± 1. Then the homomorphism $\mathrm{SL}^{\pm}_d(\mathbb{R}) \to \mathrm{PGL}_d(\mathbb{R})$ taking A to $[A]$ descends to an isomorphism between $\mathrm{SL}^{\pm}_d(\mathbb{R})/\{\pm \mathrm{Id}\}$ and $\mathrm{PGL}_d(\mathbb{R})$.

The set of all linear transformations on $\mathbb{R}^d$, denoted by $\mathrm{End}(\mathbb{R}^d)$, forms a $\mathbb{R}$-vector space. Hence we can define $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ which is a compact set. We will often think of the non-compact group $\mathrm{PGL}_d(\mathbb{R})$ as a subset of this compact set.

Suppose $A, B, C, D \in \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ lie in a projective subspace $\mathbb{P}(L)$ where L is two dimensional subspace $L := \mathrm{Span}\{A, B, C, D\} \subset \mathbb{R}^d$. We can find homogeneous coordinates such that

$$\mathbb{P}(L) := \{[x : 0 : \ldots : 0 : 1] : x \in \mathbb{R}\} \sqcup \{[1 : 0 : \ldots : 0 : 0]\}$$

and $\{A, B, C, D\} \cap \{[1 : 0 : \ldots : 0 : 0]\} = \emptyset$. Then there exist $a, b, c, d \in \mathbb{R}$ such that $A := [a : 0 : \ldots : 0 : 1]$, $B := [b : 0 : \ldots : 0 : 1]$, $C := [c : 0 : \ldots : 0 : 1]$ and $D := [d : 0 : \ldots : 0 : 1]$. Assume that A, B, C, D are ordered in such a way that $a \leq b \leq c \leq d$. Then the projective cross-ratio determined by the four points A, B, C, D is given by:

$$[A, B, C, D] := \frac{(c-a)(d-b)}{(b-a)(d-c)}.$$

The definition of cross-ratio is independent of the choice of homogeneous coordinates [Fau06, Chapter II, 10]. We now recall two classical facts about cross-ratios.

Proposition II.1 ([Fau06, Chapter II, 11])**.** *The projective cross-ratio is invariant under the action of* $\mathrm{PGL}_d(\mathbb{R})$ *on* $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$.

Proposition II.2 ([Fau06, Chapter II, 11])**.** *Suppose* L_1, L_2, L_3, L_4 *are four lines in* $\mathbb{R}^2$ *concurrent at* p *(the lines* L_i *are ordered clockwise around* p *by their indices). Let* ℓ *(resp.* ℓ'*) be a line in* $\mathbb{R}^2$ *that does not pass through* p *and intersects* L_1, L_2, L_3, L_4 *at* A, B, C, D *(resp.* A', B', C', D'*), in this order. Then*

$$[A, B, C, D] = [A', B', C', D'].$$

2.1.2 Convex Subsets

A set $C \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is convex if it is a convex set in some affine chart $\mathbb{A} \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$. Note that this notion of convexity is independent of the affine chart since the corresponding affine charts are related by an element of $\mathrm{PGL}_d(\mathbb{R})$. A convex set set $C \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is properly convex if there exists an affine chart $\mathbb{A}$ such that $\overline{C}$ is a bounded convex subset of $\mathbb{A}$.

Definition II.3. A properly convex set $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ which is also open in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is called a properly convex domain.

A properly convex domain Ω inherits the subspace topology from $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$. The closure (resp. the boundary) of Ω is its closure (resp. boundary) in $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and is denoted by $\overline{\Omega}$ (resp. $\partial\Omega$).

Definition II.4. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $X \subset \overline{\Omega}$ is non-empty. Then the convex hull of X in $\overline{\Omega}$, denoted by $\mathrm{ConvHull}_{\overline{\Omega}}(X)$, is the smallest convex subset in $\overline{\Omega}$ that contains X. We define the convex hull of X in Ω

as:

$$\mathrm{ConvHull}_{\Omega}(X) := \mathrm{ConvHull}_{\overline{\Omega}}(X) \cap \Omega.$$

For any two points $x, y \in \mathbb{P}(\mathbb{R}^{\mathrm{d}})$, $\overleftrightarrow{xy} := \mathbb{P}(\mathrm{Span}\{x, y\})$ is the projective line through x and y. Any connected proper subset of $\overleftrightarrow{xy}$ is a projective line segment. We observe that given two points $x, y \in \mathbb{P}(\mathbb{R}^{\mathrm{d}})$, there is no canonical way to associate a projective line segment to them. However, when a properly convex domain is present, we can make the following canonical choice.

Convention: If Ω is a properly convex domain and $x, y \in \overline{\Omega}$, then we define $[x, y]$ to be the unique closed projective line segment in $\overleftrightarrow{xy} \cap \overline{\Omega}$ that joins x and y. We will refer to $[x, y]$ as *the* projective line segment between x and y.

We also set $(x, y) := [x, y] \setminus \{x, y\}$, $[x, y) := [x, y] \setminus \{y\}$ and $(x, y] := [x, y] \setminus \{x\}$.

2.1.3 Hilbert Metric and Hilbert Geometry

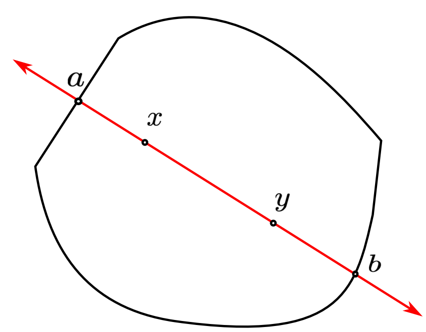

Figure 2.1: Definition of the Hilbert metric

Suppose Ω is a properly convex domain. We now introduce the Hilbert metric on Ω (see Figure 2.1). Let us fix some system of homogeneous coordinates on Ω, i.e. an affine chart $\mathbb{A}$. If $x, y \in \Omega$, then there exist $a, b \in \partial\Omega$ such that $\overleftrightarrow{xy} \cap \overline{\Omega} = [a, b]$ where the points are in the order a, x, y, b. The distance between x and y in the Hilbert metric is defined by

$$\mathrm{d}_{\Omega}(x, y) := \frac{1}{2} \log[a, x, y, b].$$

We now collect some basic facts about the metric space $(\Omega, \mathrm{d}_\Omega)$.

Proposition II.5. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain.*

(1) Projective line segments in Ω are geodesics in $(\Omega, \mathrm{d}_\Omega)$.

(2) The metric balls $\mathcal{B}_\Omega(p, r) := \{x \in \Omega : \mathrm{d}_\Omega(p, x) < r\}$ are convex and relatively compact for all $p \in \Omega$ and $r > 0$.

(3) The metric space $(\Omega, \mathrm{d}_\Omega)$ is proper, complete, and geodesic.

Although $(\Omega, \mathrm{d}_\Omega)$ is a geodesic metric space, the geodesics in Ω are not necessarily unique. Consider the example of the two dimensional simplex $S_2 := \mathbb{P}(\mathbb{R}^+ e_1 \oplus \mathbb{R}^+ e_2 \oplus \mathbb{R}^+ e_3)$ in Figure 2.2. It shows that there are uncountably many geodesics between x and y. The non-uniqueness of geodesics can be traced back to line segments in $\partial\Omega$ (see [dlH93] for details).

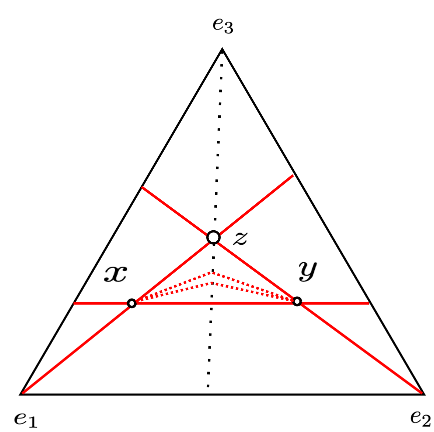

Figure 2.2: Non-uniqueness of geodesics in (S_2, d_{S_2}): $[x, z] \cup [z, y]$ and $[x, y]$ are both geodesics between x and y

Convexity is preserved under taking r-neighbourhoods in the Hilbert metric of closed convex sets.

Proposition II.6 ([CLT15, Corollary 1.10])**.** *If $C \subset \Omega$ is a closed convex set and $r > 0$ then*

$$\mathcal{N}_\Omega(C; r) := \{x \in \Omega : \mathrm{d}_\Omega(x, C) < r\}$$

is a convex set. The corresponding closed neighbourhood of C, i.e. $\overline{\mathcal{N}_\Omega(C;r)}$, is also convex.

The group of automorphisms of a properly convex domain Ω is

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}_d(\mathbb{R}) : g\Omega = \Omega\}.$$

Proposition II.7. *The group* $\mathrm{Aut}(\Omega)$ *acts on* $(\Omega, \mathrm{d}_\Omega)$ *properly and by isometries.*

Definition II.8. If Ω is a properly convex domain, then the triple $(\Omega, \mathrm{d}_\Omega, \mathrm{Aut}(\Omega))$ is called a Hilbert geometry.

We will often shorten the notation and say that Ω is a Hilbert geometry. A primary example of a Hilbert geometry is the projective model of the real hyperbolic space $\mathbb{H}^d$. Another example is the projective triangle (see Figure 2.2).

A cone in $\mathbb{R}^d$ is a set $D \subset \mathbb{R}^d$ such that: if $d_1, d_2 \in \mathbb{R}^d$ and $r_1, r_2 > 0$, then $r_1 d_1 + r_2 d_2 \in D$.

Definition II.9. A properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is reducible if there exist convex cones $C_1 \subset \mathbb{R}^{d_1}$ and $C_2 \subset \mathbb{R}^{d_2}$ with $d_1, d_2 \geq 1$ and $d = d_1 + d_2$ such that $\Omega = \mathbb{P}(C_1 \oplus C_2)$. A properly convex domain that is not reducible is called irreducible.

Projective triangle is a reducible domain while the projective model of $\mathbb{H}^d$ is irreducible.

2.1.4 Topological Preliminaries

Suppose $A \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a non-empty convex set. Then the relative interior of A is, denoted by rel-int(A), is defined as

$$\text{rel-int}(A) := \mathrm{int}(A) \cap \mathbb{P}(\mathrm{Span}(A)).$$

The A is relatively open (or open in its span) if $\text{rel-int}(A) = A$. Since $\text{rel-int}(A)$ is an open subset of the projective space $\mathbb{P}(\text{Span}(A))$, its dimension is well defined. The dimension of A can then be defined as

$$\dim(A) := \dim(\text{rel-int}(A)).$$

Note that $\text{rel-int}(A)$ is homeomorphic to $\mathbb{R}^{\dim(A)}$.

Recall that the boundary of a properly convex set $A \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is $\partial A := \overline{A} \setminus \text{int}(A)$. The ideal boundary of A is $\partial_{\mathrm{i}} A := \partial A \setminus A$ while the non-ideal boundary is $\partial_{\mathrm{n}} A := \partial A \cap A$. The ideal and non-ideal boundaries decompose ∂A into two disjoint sets.

If $A \subset B \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$, then A is said to be properly embedded in B provided the inclusion map $A \hookrightarrow B$ is proper.

Proposition II.10. *Suppose $A \subset B \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$. Then A is properly embedded in B if and only if $\partial_{\mathrm{i}} A \subset \partial_{\mathrm{i}} B$.*

2.1.5 Structure of the Boundary

For this subsection, fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$. Consider the equivalence relation $\sim_\Omega$ on $\overline{\Omega}$ is given by: $x \sim_\Omega y$ if and only if there exists an open projective line segment in $\overline{\Omega}$ containing x and y. The equivalence class of $x \in \overline{\Omega}$ is denoted by $F_\Omega(x)$ [CM14, Section 3.3]. The following results are simple consequences of convexity, see for instance [IZ21, Isl19].

Proposition II.11.

(1) $F_\Omega(x)$ is open in its span.

(2) $F_\Omega(x) = \Omega$ whenever $x \in \Omega$ and $F_\Omega(x) \subset \partial\Omega$ whenever $x \in \partial\Omega$.

(3) $y \in F_\Omega(x)$ if and only if $x \in F_\Omega(y)$ if and only if $F_\Omega(x) = F_\Omega(y)$.

(4) Suppose $x, y \in \overline{\Omega}$, $p \in F_\Omega(x)$, $q \in F_\Omega(y)$, and $z \in (x, y)$. Then

$$(p, q) \subset F_\Omega(z).$$

In particular, $(p, q) \subset \Omega$ if and only if $(x, y) \subset \Omega$.

(5) If $y \in \partial F_\Omega(x)$, then $F_\Omega(y) \subset \partial F_\Omega(x)$,

Proposition II.11 shows that $F_\Omega(x)$ is a relatively open convex subset of $\partial\Omega$ for all $x \in \partial\Omega$. Thus $F_\Omega(x)$ can be equipped with a Hilbert metric $\mathrm{d}_{F_\Omega(x)}$ for any $x \in \partial\Omega$. We will now state some estimates that relate the Hilbert metric in the interior of Ω with the Hilbert metric on the faces $F_\Omega(x)$. These results are elementary and can be found in many places, for instance [IZ21].

Proposition II.12. *Suppose $\{x_n\}$ is a sequence in Ω and $x_n \to x \in \overline{\Omega}$. If $\{y_n\}$ is another sequence in Ω, $y_n \to y \in \overline{\Omega}$, and*

$$\liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n) < +\infty,$$

then $y \in F_\Omega(x)$ and

$$\mathrm{d}_{F_\Omega(x)}(x, y) \leq \liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n).$$

Corollary II.13. *Suppose $A, B \subset \Omega$ be non-empty subsets such that $A \subset \mathcal{N}_\Omega(B; r)$ for some $r > 0$. If $a \in \overline{A}$, then there exists $b \in \overline{B}$ such that $a \in F_\Omega(b)$ and $\mathrm{d}_{F_\Omega(b)}(a, b) \leq r$.*

Proof of Corollary. If $a \in \Omega$, then any $b \in B$ works since $F_\Omega(b) = \Omega$. So suppose $a \in \partial\Omega$. Choose a sequence $\{a_n\}$ in Ω such that $a_n \to a$. Then there exists a sequence $\{b_n\}$ in B such that $\mathrm{d}_\Omega(a_n, b_n) < r$. Up to passing to a subsequence, we can assume that $b_n \to b \in \overline{\Omega}$. Then, by Proposition II.12, $a \in F_\Omega(b)$ and $\mathrm{d}_{F_\Omega(b)}(a, b) \leq r$. $\square$

Corollary II.14 ([DGK17, Corollary 3.5])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain, $y \in \partial\Omega$, and $\{y_m\}$ and $\{z_m\}$ are two sequences in Ω. If $y_m \to y$ and $\mathrm{d}_\Omega(y_m, z_m) \to 0$, then $z_m \to y$.*

Proof. Up to passing to a subsequence, we can assume that $z := \lim_{m\to\infty} z_m$ exists. Then Proposition II.12 implies that $\mathrm{d}_{F_\Omega(y)}(y, z) = 0$. Thus $z = y$. $\square$

Lemma II.15 ([Cra09, Lemma 8.3])**.** *Suppose that $\sigma_1, \sigma_2 : [0, T] \to \Omega$ are two unit speed projective line geodesics, then for $0 \le t \le T$,*

$$\mathrm{d}_\Omega(\sigma_1(t), \sigma_2(t)) \le \mathrm{d}_\Omega(\sigma_1(0), \sigma_2(0)) + \mathrm{d}_\Omega(\sigma_1(T), \sigma_2(T)).$$

Let $\mathrm{d}_\Omega{}^{\mathrm{Hauss}}$ denote the *Hausdorff distance* on subsets of Ω induced by d_Ω, that is: for subsets $A, B \subset \Omega$ define

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} \mathrm{d}_\Omega(a, b),\ \sup_{b\in B}\inf_{a\in A} \mathrm{d}_\Omega(a, b)\right\}.$$

Proposition II.16. *Assume $p_1, p_2, q_1, q_2 \in \overline{\Omega}$, $F_\Omega(p_1) = F_\Omega(p_2)$, and $F_\Omega(q_1) = F_\Omega(q_2)$. If $(p_1, q_1) \cap \Omega \neq \emptyset$, then $(p_2, q_2) \subset \Omega$ and*

$$\mathrm{d}_\Omega^{\mathrm{Hauss}}\Big((p_1, q_1), (p_2, q_2)\Big) \le \max\{\mathrm{d}_{F_\Omega(p_1)}(p_1, p_2), \mathrm{d}_{F_\Omega(q_1)}(q_1, q_2)\}.$$

The local Hausdorff topology is a natural topology on the set of all closed subsets of Ω induced by the Hausdorff distance $\mathrm{d}_\Omega{}^{\mathrm{Hauss}}$. For a closed subset $C_0 \subset \Omega$, $r_0, \varepsilon_0 > 0$, and $x_0 \in \Omega$, define $U(C_0, r_0, \varepsilon_0, x_0)$ to be the set of all closed subsets C of Ω such that

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(\mathcal{B}_\Omega(x_0, r_0) \cap C, \mathcal{B}_\Omega(x_0, r_0) \cap C_0) < \varepsilon_0.$$

The local Hausdorff topology is the topology generated by $U(\cdot, \cdot, \cdot, \cdot)$ on the set of closed subsets of Ω.

2.2 Projective Simplices

For $0 \leq k \leq d$, consider the following subsets of $\mathbb{P}(\mathbb{R}^{\mathrm{d}})$:

$$S_k := \left\{[x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \ldots, x_k + 1 > 0\right\}.$$

Definition II.17. A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is a $k-$dimensional *simplex* if there exists $g \in \mathrm{PGL}_d(\mathbb{R})$ such that $S = gS_k$. In this case, the k points

$$g[1 : 0 : \cdots : 0], g[0 : 1 : 0 : \cdots : 0], \ldots, g[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \partial S$$

are the vertices of S.

We now discuss some basic properties of projective simplices. All these properties are fairly elementary, see for instance [IZ19, Section 5]. Choosing suitable projective coordinates, we write a $(d-1)$ dimensional projective simplex as

$$S = \left\{[x_1 : \cdots : x_d] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \ldots, x_d > 0\right\}.$$

The Hilbert metric on S can be explicitly computed as:

$$\mathrm{d}_S\Big([x_1 : \cdots : x_d], [y_1 : \cdots : y_d]\Big) = \max_{1 \leq i,j \leq d} \frac{1}{2}\left|\log \frac{x_i y_j}{y_i x_j}\right|.$$

For details, see [Nus88, Proposition 1.7], [dlH93], or [Ver14]. Let $G \leq \mathrm{GL}_d(\mathbb{R})$ denote the group generated by the group of diagonal matrices with positive entries and the group of permutation matrices. Then

$$\mathrm{Aut}(S) = \{[g] \in \mathrm{PGL}_d(\mathbb{R}) : g \in G\}.$$

Proposition II.18. *If $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex, then (S, H_S) is quasi-isometric to real Euclidean space of dimension* $\dim S$.

We will frequently use the following observation about the faces of properly embedded simplices.

Observation II.19. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. If $x \in \partial S$, then*

(1) $F_S(x)$ is properly embedded in $F_\Omega(x)$.

(2) $F_S(x) = \overline{S} \cap F_\Omega(x)$.

The following lemma allows us to "wiggle" the vertices of a properly embedded simplex S and obtain a new properly embedded simplex "parallel" to S.

Proposition II.20 ([IZ19, Lemma 3.18])**.** *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex with vertices $v_1, \dots, v_p$. If $w_j \in F_\Omega(v_j)$ for $1 \leq j \leq p$, then*

$$S' := \Omega \cap \mathbb{P}(\mathrm{Span}\{w_1, \dots, w_p\})$$

is a properly embedded simplex with vertices $w_1, \dots, w_p$. Moreover,

$$\mathrm{d}_\Omega^{\mathrm{Hauss}}(S, S') \leq \max_{1 \leq j \leq p} \mathrm{d}_{F_\Omega(v_j)}(v_j, w_j).$$

Proof. The first part follows from Proposition II.11 part (4) by an induction argument. The moreover part follows from a similar induction argument and Proposition II.16. See [IZ19, Section 3.6] for details. □

Proposition II.21. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. The set of all properly embedded simplices in Ω of dimension at least two is a closed set in the local Hausdorff topology.*

2.3 Linear Projection on Simplices

In this section we construct certain linear projection maps associated to a properly embedded simplex in a properly convex domain. This notion was introduced in [IZ19] and all results in this section appear in [IZ19].

Definition II.22. Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex with $\dim S = (q-1) \geq 1$. A set of co-dimension one linear subspaces $\mathcal{H} := \{H_1, \dots, H_q\}$ is *S-supporting* when:

(1) Each $\mathbb{P}(H_j)$ is a supporting hyperplane of Ω,

(2) If $F_1, \dots, F_q \subset \partial S$ are the boundary faces of maximal dimension, then (up to relabelling) $F_j \subset \mathbb{P}(H_j)$ for all $1 \leq j \leq q$.

Proposition II.23. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex, and $\mathcal{H}$ is a set of S-supporting hyperplanes. Then*

$$\operatorname{Span} S \oplus (\cap_{H \in \mathcal{H}} H) = \mathbb{R}^d \quad \textit{and} \quad \Omega \cap \mathbb{P}\left(\cap_{H \in \mathcal{H}} H\right) = \emptyset.$$

Proof. Suppose $\mathcal{H} := \{H_1, \dots, H_q\}$, $F_1, \dots, F_q \subset \partial S$ are the boundary faces of maximal dimension, and $v_1, \dots, v_q$ are the vertices of S labelled so that $F_j \subset \mathbb{P}(H_j)$ and $v_j \notin \overline{F_j}$. Let $\overline{v}_1, \dots, \overline{v}_q \in \mathbb{R}^d \setminus \{0\}$ be lifts of $v_1, \dots, v_q$ respectively.

First notice that

$$\Omega \cap \mathbb{P}\left(\cap_{H \in \mathcal{H}} H\right) = \emptyset$$

since $\mathbb{P}(H_j) \cap \Omega = \emptyset$ for every j.

Since $S \subset \mathbb{P}(v_j + H_j)$ and $S \cap \mathbb{P}(H_j) = \emptyset$, we must have $v_j \notin \mathbb{P}(H_j)$ and hence

$$v_j \oplus H_j = \mathbb{R}^d \tag{2.1}$$

for every j. Further,

$$v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_q \in \overline{F_j} \subset \mathbb{P}(H_j) \tag{2.2}$$

for each j.

Define $W := \cap_{H \in \mathcal{H}} H$. We claim that

$$\operatorname{Span} S \oplus W = \mathbb{R}^d.$$

Since

$$\dim W + \dim \operatorname{Span} S \geq (d-q) + q = d,$$

it suffices to show that

$$\operatorname{Span} S \cap W = \{0\}.$$

If not, we can find $\alpha_1, \dots, \alpha_q \in \mathbb{R}$ such that

$$0 \neq \sum_{j=1}^{q} \alpha_j \overline{v}_j \in W.$$

By relabelling we can assume that $\alpha_1 \neq 0$. Then by Equation (2.2)

$$v_1 \subset \operatorname{Span}\{v_2, \dots, v_q\} + W \subset H_1$$

which contradicts Equation (2.1). So

$$\operatorname{Span} S \oplus W = \mathbb{R}^d. \qquad \square$$

Using Proposition II.23, we define the following linear projection.

Definition II.24. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex, and $\mathcal{H}$ is a set of S-supporting hyperplanes. Define $L_{S,\mathcal{H}} \in \operatorname{End}(\mathbb{R}^d)$ to be the linear projection

$$\operatorname{Span} S \oplus (\cap_{H \in \mathcal{H}} H) \longrightarrow \operatorname{Span} S$$

We call $L_{S,\mathcal{H}}$ the *linear projection of* Ω *onto* S *relative to* $\mathcal{H}$.

We now derive some basic properties of these projection maps. We use the notation

$$F_\Omega(X) = \cup_{x \in X} F_\Omega(x)$$

where $\Omega \subset \mathbb{P}(\mathrm{R^d})$ and $X \subset \overline{\Omega}$.

Proposition II.25. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $S \subset \Omega$ is a properly embedded simplex, and $\mathcal{H}$ is a set of S-supporting hyperplanes. Then*

(1) $L_{S,\mathcal{H}}(\Omega) = S$.

(2) If $x \in \partial\Omega \cap \mathbb{P}\left(\cap_{H\in\mathcal{H}} H\right)$ and $y \in \partial S$, then $[x, y] \subset \partial\Omega$.

(3) $\mathbb{P}\left(\cap_{H\in\mathcal{H}} H\right) \cap F_\Omega(\partial S) = \emptyset$.

Proof. (1) By Proposition II.23, $\mathbb{P}(\ker L_{S,\mathcal{H}}) \cap \Omega = \emptyset$, so $L_{S,\mathcal{H}}$ is well-defined on Ω. The set $L_{S,\mathcal{H}}(\Omega) \subset \mathbb{P}(\operatorname{Span} S)$ is connected and contains $S = L_{S,\mathcal{H}}(S)$. Further

$$L_{S,\mathcal{H}}^{-1}(\partial S) = \cup_{j=1}^{q} L_{S,\mathcal{H}}^{-1}\left(\overline{F_j}\right) \subset \cup_{j=1}^{q} \mathbb{P}(H_j)$$

and so $\Omega \cap L_{S,\mathcal{H}}^{-1}(\partial S) = \emptyset$. Thus $L_{S,\mathcal{H}}(\Omega) = S$.

(2) Suppose $x \in \partial\Omega \cap \mathbb{P}(\cap_{H\in\mathcal{H}} H)$ and $y \in \partial S$. Then there exists a boundary face $F \subset \partial S$ of maximal dimension such that $y \in \overline{F}$. Then there exists some $H \in \mathcal{H}$ such that $F \subset \mathbb{P}(H)$. Then $[x, y] \subset \mathbb{P}(H)$ and so $[x, y] \cap \Omega = \emptyset$. Thus $[x, y] \subset \partial\Omega$.

(3) Next, suppose for a contradiction that

$$x \in \mathbb{P}(\cap_{H\in\mathcal{H}} H) \cap F_\Omega(\partial S).$$

Then there exists $y \in \partial S$ with $x \in F_\Omega(y)$. Pick $y' \in \partial S$ such that $(y, y') \subset S$. Then by Proposition II.11 part (4) we also have $(x, y') \subset \Omega$. But this contradicts part (1). $\square$

For a general properly embedded simplex, there could be many different sets of supporting hyperplanes, but the next result shows that the corresponding linear projections form a compact set.

Definition II.26. Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. Define

$$\mathcal{L}_S := \{L_{S,\mathcal{H}} : \ \mathcal{H} \text{ is a set of } S\text{-supporting hyperplanes}\} \subset \operatorname{End}(\mathbb{R}^d).$$

Proposition II.27. *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. Then $\mathcal{L}_S$ is a compact subset of* $\mathrm{End}(\mathbb{R}^d)$.

Proof. Suppose that $F_1, \ldots, F_q \subset \partial S$ are the boundary faces of S of maximal dimension. Fix a sequence $L_{S,\mathcal{H}_n}$ of projections. Then

$$\mathcal{H}_n = \{H_{n,1}, \ldots, H_{n,q}\}$$

where $F_j \subset \mathbb{P}(H_{n,j})$. Since $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ is compact we can find $n_k \to \infty$ such that

$$H_j := \lim_{k \to \infty} H_{n_k,j}$$

exists in $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ for every $1 \leq j \leq q$. Then $F_j \subset \mathbb{P}(H_j)$ and $\mathbb{P}(H_j) \cap \Omega = \emptyset$ for every $1 \leq j \leq q$. So $\mathcal{H} = \{H_1, \ldots, H_q\}$ is a set of S-supporting hyperplanes. Further, by definition,

$$L_{S,\mathcal{H}} = \lim_{k \to \infty} L_{S,\mathcal{H}_{n_k}}$$

in $\mathrm{End}(\mathbb{R}^d)$. Since $L_{S,\mathcal{H}_n}$ was an arbitrary sequence, $\mathcal{L}_S$ is compact. $\square$

2.4 Closest-point Projection

Definition II.28. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $A \subset \Omega$ is a non-empty closed convex set. If $p \in \Omega$, the *closest-point projection of p on A* is

$$\pi_A(p) := A \cap \overline{\mathcal{B}_\Omega(p, \mathrm{d}_\Omega(p, A))}.$$

Lemma II.29. *If $p \in \Omega$, $\pi_A(p)$ is a compact convex set.*

Proof. By Proposition II.5, $\overline{\mathcal{B}_\Omega(p, \mathrm{d}_\Omega(p, A))}$ is a compact convex set. Moreover, the intersection of two closed convex sets is a closed convex set. Hence the result. $\square$

Lemma II.30. *If $g \in \mathrm{Aut}(\Omega)$, then $g \circ \pi_A = \pi_{gA} \circ g$.*

2.5 Center of Mass

There is a notion of "center of mass" for compact subsets of a properly convex domain. Let $\mathcal{K}_d$ denote the set of all pairs (Ω, K) where $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $K \subset \Omega$ is a compact subset.

Proposition II.31 ([IZ21])**.** *There exists a function*

$$(\Omega, K) \in \mathcal{K}_d \longmapsto \mathrm{CoM}_\Omega(K) \in \mathbb{P}(\mathbb{R}^d)$$

such that:

(1) $\mathrm{CoM}_\Omega(K) \in \mathrm{ConvHull}_\Omega(K)$*,*

(2) $\mathrm{CoM}_\Omega(K) = \mathrm{CoM}_\Omega(\mathrm{ConvHull}_\Omega(K))$*, and*

(3) if $g \in \mathrm{PGL}_d(\mathbb{R})$*, then* $g\mathrm{CoM}_\Omega(K) = \mathrm{CoM}_{g\Omega}(gK)$*,*

for every $(\Omega, K) \in \mathcal{K}_d$*.*

There are several ways of proving the existence of such a "center of mass". Proposition II.31 appears in [IZ21] and their argument is inspired by Frankel [Fra89]. An alternative approach to this construction appears in [Mar14, Lemma 4.2].

2.6 Dynamics of Automorphisms in Hilbert Geometry

If $g \in \mathrm{GL}_d(\mathbb{R})$, let $\lambda_1(g),\ \lambda_2(g),\ \dots,\ \lambda_d(g)$ denote the absolute values of eigenvalues of g (over $\mathbb{C}$), indexed such that

$$\lambda_1(g) \geq \lambda_2(g) \geq \ldots \geq \lambda_d(g).$$

In particular, we will use the notation $\lambda_{\max}(g) := \lambda_1(g)$ and $\lambda_{\min}(g) := \lambda_d(g)$.

If $h \in \mathrm{PGL}_d(\mathbb{R})$, we define

$$\frac{\lambda_i}{\lambda_j}(h) := \frac{\lambda_i(\widetilde{h})}{\lambda_j(\widetilde{h})}$$

where $\widetilde{h} \in \mathrm{GL}_d(\mathbb{R})$ is some (hence any) lift of h.

Proposition II.32 ([CLT15]). *Suppose Ω is a properly convex domain and $g \in \mathrm{Aut}(\Omega)$. Then the translation length of g, defined as*

$$\tau_\Omega(g) := \inf_{x\in\Omega} \mathrm{d}_\Omega(x, gx),$$

is given by

$$\tau_\Omega(g) = \log\left(\frac{\lambda_1}{\lambda_d}(g)\right).$$

2.6.1 Geometry of ω-limit Sets of Automorphisms

For the rest of this section, fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$. Let $\gamma \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(\gamma) > 0$. We will describe the set of all accumulation points of $\{\gamma^n x : x \in \Omega, n \in \mathbb{N}\}$ in $\overline{\Omega}$. It is called the ω-limit set of γ and denoted by $\omega(\gamma, \Omega)$. Let $\widetilde{\gamma}$ be any lift of γ in $\mathrm{GL}_d(\mathbb{R})$ and let $\mathcal{E}_{\widetilde{\gamma}}$ be its set of eigenvalues (over $\mathbb{C}$). If $\lambda \in \mathcal{E}_{\widetilde{\gamma}}$, let

- W_λ be the subspace of $\mathbb{R}^d$ such that $(W_\lambda)_\mathbb{C}$ is the generalized eigenspace of $\widetilde{\gamma}$ for λ, and
- E_λ be the subspace of $\mathbb{R}^d$ such that $(E_\lambda)_\mathbb{C}$ is the eigenspace of $\widetilde{\gamma}$ for λ.

Define the subspace $E_{\widetilde{\gamma}}$, $L_{\widetilde{\gamma}}$ and $K_{\widetilde{\gamma}}$ such that

$$(E_{\widetilde{\gamma}})_\mathbb{C} := \bigoplus_{\substack{\lambda\in\mathcal{E}_{\widetilde{\gamma}} \\ |\lambda|=\lambda_{\max}(\widetilde{\gamma})}} E_\lambda \quad , \quad (L_{\widetilde{\gamma}})_\mathbb{C} := \bigoplus_{\substack{\lambda\in\mathcal{E}_{\widetilde{\gamma}} \\ |\lambda|=\lambda_{\max}(\widetilde{\gamma})}} W_\lambda \quad \text{and} \quad (K_{\widetilde{\gamma}})_\mathbb{C} := \bigoplus_{\substack{\lambda\in\mathcal{E}_{\widetilde{\gamma}} \\ |\lambda|<\lambda_{\max}(\widetilde{\gamma})}} W_\lambda.$$

An elementary computation using Jordan blocks shows that if $w \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(K_{\widetilde{\gamma}})$, then the accumulation points of $\{\gamma^n w : n > 0\}$ lie in $\mathbb{P}(E_{\widetilde{\gamma}})$ (see for instance [Mar91, II.1] or [CLT15, Lemma 2.5]).

Further observe that, after scaling by $\lambda_{\max}(\widetilde{\gamma})$, the action of $\widetilde{\gamma}$ on $E_{\widetilde{\gamma}}$ can be conjugated into $\mathrm{O}(E_{\widetilde{\gamma}})$, the group of orthogonal linear transformations on $E_{\widetilde{\gamma}}$. This implies that $\Omega \cap \mathbb{P}(E_{\widetilde{\gamma}}) = \emptyset$. Otherwise, $\Omega \cap \mathbb{P}(E_{\widetilde{\gamma}})$ is a properly convex open set in

$\mathbb{P}(E_{\widetilde{\gamma}})$ and $< \gamma|_{\mathbb{P}(E_{\widetilde{\gamma}})} > \subset \mathrm{O}(E_{\widetilde{\gamma}})$ is a compact subgroup of $\mathrm{Aut}(\Omega \cap \mathbb{P}(E_{\widetilde{\gamma}}))$. Then by [Mar14, Lemma 2.1], γ has a fixed point in Ω implying $\tau_\Omega(\gamma) = 0$, a contradiction. We also note that $\Omega \cap \mathbb{P}(K_{\widetilde{\gamma}}) = \emptyset$. Otherwise,

$$\tau_{\Omega \cap \mathbb{P}(K_{\widetilde{\gamma}})}(\gamma) = \log\left(\frac{\lambda_{\max}(\widetilde{\gamma}|_{K_{\widetilde{\gamma}}})}{\lambda_{\min}(\widetilde{\gamma}|_{K_{\widetilde{\gamma}}})}\right) < \log\left(\frac{\lambda_{\max}(\widetilde{\gamma})}{\lambda_{\min}(\widetilde{\gamma})}\right) = \tau_\Omega(\gamma)$$

which is impossible.

Thus $\Omega \subset \mathbb{P}(\mathbb{R}^d) \backslash \mathbb{P}(K_{\widetilde{\gamma}})$, which implies that $\omega(\gamma, \Omega) \subset \mathbb{P}(E_{\widetilde{\gamma}})$. Moreover $\omega(\gamma, \Omega) \subset \overline{\Omega}$. Thus $\omega(\gamma, \Omega) \subset \big(\overline{\Omega} \cap \mathbb{P}(E_{\widetilde{\gamma}})\big) = \big(\partial\Omega \cap \mathbb{P}(E_{\widetilde{\gamma}})\big)$ where last equality holds because $\Omega \cap \mathbb{P}(E_{\widetilde{\gamma}}) = \emptyset$.

Finally also note that the subspaces $E_{\widetilde{\gamma}}$, $L_{\widetilde{\gamma}}$ and $K_{\widetilde{\gamma}}$ defined above are independent of the lift $\widetilde{\gamma}$ chosen. Thus we introduce:

$$E_\gamma^+ := \mathbb{P}\big(E_{\widetilde{\gamma}}\big) \cap \overline{\Omega} \quad \text{and} \quad K_\gamma^+ := \mathbb{P}\big(K_{\widetilde{\gamma}}\big) \cap \overline{\Omega}$$

$$E_\gamma^- := \mathbb{P}\big(E_{\widetilde{\gamma}^{-1}}\big) \cap \overline{\Omega} \quad \text{and} \quad K_\gamma^- := \mathbb{P}\big(K_{\widetilde{\gamma}^{-1}}\big) \cap \overline{\Omega}$$

We can sum up the above discussion in the following proposition. Note that the same proposition is true if we replace γ by γ^{-1} and E_γ^+ by E_γ^-.

Proposition II.33 ([Isl19]). *If Ω is a properly convex domain, $\gamma \in \mathrm{Aut}(\Omega)$ and $\tau_\Omega(\gamma) > 0$, then:*

(1) $\omega(\gamma, \Omega) \subset E_\gamma^+$.

(2) the action of γ on E_γ^+ is conjugated into the projective orthogonal group $\mathrm{PO}(E_{\widetilde{\gamma}})$.

(3) there exists an unbounded sequence of positive integers $\{m_k\}$ such that

$$\lim_{k \to \infty} \left(\gamma\Big|_{E_\gamma^+}\right)^{m_k} = \mathrm{Id}\Big|_{E_\gamma^+}.$$

We prove the following proposition about faces $F_\Omega(x)$ for $x \in E_\gamma^-$.

Proposition II.34 ([Isl19]). *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $\gamma \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(\gamma) > 0$.*

(1) If $y \in E_\gamma^-$, then $F_\Omega(y) \cap E_\gamma^+ = \emptyset$.

(2) If $y \in E_\gamma^-$, $z \in F_\Omega(y)$ and $\{i_k\}$ is a sequnece in $\mathbb{Z}$ such that $z_\infty := \lim_{k\to\infty} \gamma^{i_k} z$ exists, then

$$\textit{either} \ \ z_\infty \in E_\gamma^-, \quad \textit{or}, \ \ z_\infty \in \partial\Omega \setminus \left(E_\gamma^+ \sqcup E_\gamma^-\right).$$

Proof. (1) Suppose there exists $x \in F_\Omega(y) \cap E_\gamma^+$. Since $E_\gamma^+ \cap E_\gamma^- = \emptyset$, $x \neq y$. Then there exists a maximal line segment $\mathcal{I} = [a, b] \subset \partial\Omega$ containing x and y as its interior points (order: a, x, y, b). Since $y \in E_\gamma^-$, by Proposition II.33 part (3), there exists an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} \gamma^{d_k} y = y$. Up to passing to a subsequence, the following limits exist in $\overline{\Omega}$:

$$x_\infty := \lim_{k\to\infty} \gamma^{d_k} x, \ \ a_\infty := \lim_{k\to\infty} \gamma^{d_k} a \ \ \text{and} \ \ b_\infty := \lim_{k\to\infty} \gamma^{d_k} b.$$

Since $x \in \mathcal{I} \cap E_\gamma^+$ and E_γ^+ is a closed γ-invariant set, $x_\infty \in E_\gamma^+$. Hence, $x_\infty \neq y$.

The sequence $\gamma^{d_k}\mathcal{I}$ converges to the projective line segment $\mathcal{I}_\infty = [a_\infty, b_\infty] \subset \partial\Omega$ in the same affine chart. Since $x_\infty \neq y$, $\mathcal{I}_\infty$ is a non-degenerate projective line segment in $\partial\Omega$ containing both of them. We claim that x_∞ and y are interior points of the line segment $\mathcal{I}_\infty$. Observe that since γ is a projective transformation and preserves cross-ratios, $\lim_{k\to\infty}[\gamma^{d_k} a, \gamma^{d_k} x, \gamma^{d_k} y, \gamma^{d_k} b] = [a, x, y, b]$. Thus

$$[a_\infty, x_\infty, y, b_\infty] = \lim_{k\to\infty} [\gamma^{d_k} a, \gamma^{d_k} x, \gamma^{d_k} y, \gamma^{d_k} b]$$

exists and is finite. However

$$[\gamma^{d_k} a, \gamma^{d_k} x, \gamma^{d_k} y, \gamma^{d_k} b] = \left(1 + \frac{|\gamma^{d_k} x - \gamma^{d_k} y|}{|\gamma^{d_k} a - \gamma^{d_k} x|}\right)\left(1 + \frac{|\gamma^{d_k} x - \gamma^{d_k} y|}{|\gamma^{d_k} y - \gamma^{d_k} b|}\right).$$

Since $x_\infty \neq y$, $\lim_{k\to\infty} |\gamma^{d_k} x - \gamma^{d_k} y| \neq 0$. Then $[a_\infty, x_\infty, y, b_\infty] < \infty$ implies

$$\lim_{k\to\infty} |\gamma^{d_k} a - \gamma^{d_k} x| \neq 0 \quad \text{and} \quad \lim_{k\to\infty} |\gamma^{d_k} y - \gamma^{d_k} b| \neq 0.$$

Thus $a_\infty \neq x_\infty$ and $y \neq b_\infty$ which shows that both x_∞ and y are interior points of $\mathcal{I}_\infty$.

Recall that $x \in E_\gamma^+$ and $y \in E_\gamma^-$. Since $d_k > 0$, up to passing to a subsequence, $\lim_{k\to\infty} \gamma^{d_k} w = x_\infty$ for any $w \in (x, y)$. Thus

$$\lim_{k\to\infty} [\gamma^{d_k} a, \gamma^{d_k} w, \gamma^{d_k} y, \gamma^{d_k} b] = [a_\infty, x_\infty, y, b_\infty].$$

By projective invariance of cross-ratios,

$$\lim_{k\to\infty} [\gamma^{d_k} a, \gamma^{d_k} w, \gamma^{d_k} y, \gamma^{d_k} b] = [a, w, y, b].$$

Then $[a, w, y, b]$ is the constant number $[a_\infty, x_\infty, y, b_\infty]$ for all $w \in (x, y)$. This is impossible since x and y are distinct interior points of the line segment $\mathcal{I} = [a, b]$. Hence we have a contradiction and this finishes the proof of (1).

(2) Since $\partial\Omega$ is a closed set, $z_\infty \in \partial\Omega$. If $z_\infty \in E_\gamma^-$, there is nothing to prove. So, let $z_\infty \notin E_\gamma^-$.

Up to passing to a subsequence, we can assume that $y_\infty := \lim_{k\to\infty} \gamma^{i_k} y$ exists. As E_γ^- is a closed γ-invariant set, $y_\infty \in E_\gamma^-$. Thus, $z_\infty \neq y_\infty$.

We claim that $z_\infty \in F_\Omega(y_\infty)$. Since $z \in F_\Omega(y)$, there exists a maximal projective line segment $\mathcal{J} := [a', b'] \subset \partial\Omega$ that contains both z and y as its interior points (order: a', z, y, b'). Up to passing to a subsequence, we can assume that $a'_\infty := \lim_{k\to\infty} \gamma^{i_k} a'$ and $b'_\infty := \lim_{k\to\infty} \gamma^{i_k} b'$ exist in $\overline{\Omega}$. Then $J_\infty := \lim_{k\to\infty} \gamma^{i_k} \mathcal{J} = [a'_\infty, b'_\infty]$. By projective invariance of cross-ratios, $\lim_{k\to\infty} [\gamma^{i_k} a', \gamma^{i_k} z, \gamma^{i_k} y, \gamma^{i_k} b'] = [a', z, y, b']$. Thus

$$[a'_\infty, z_\infty, y_\infty, b'_\infty] = \lim_{k\to\infty} [\gamma^{i_k} a', \gamma^{i_k} z, \gamma^{i_k} y, \gamma^{i_k} b']$$

exists and is finite. Since $z_\infty \neq y_\infty$, then arguing as in part (1) of this proposition (using cross-ratios) we can show that $a'_\infty \neq z_\infty$ and $y_\infty \neq b'_\infty$. Thus z_∞ and y_∞ are interior points of $\mathcal{J}_\infty = [a_\infty, b_\infty] \subset \partial\Omega$. Thus $z_\infty \in F_\Omega(y_\infty)$.

Since $y_\infty \in E_\gamma^-$ and $z_\infty \in F_\Omega(y_\infty)$, part (1) of this proposition implies $z_\infty \notin E_\gamma^+$. Moreover $z_\infty \notin E_\gamma^-$ by assumption. Hence we have completed the proof. $\square$

2.6.2 Limits of Automorphisms and Boundary Faces

The results of this section are in [IZ21]. The next two results relate the faces of a convex domain with the behavior of automorphisms.

Proposition II.35 ([IZ21]). *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$, and $g_n \in \mathrm{Aut}(\Omega)$ is a sequence such that*

(1) $g_n(p_0) \to x \in \partial\Omega$,

(2) $g_n^{-1}(p_0) \to y \in \partial\Omega$, and

(3) g_n converges to T in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$.

Then $\mathrm{image}(T) \subset \mathrm{Span}\, F_\Omega(x)$, $\mathbb{P}(\ker T) \cap \Omega = \emptyset$, and $y \in \mathbb{P}(\ker T)$.

Proof. For $v \in \mathbb{R}^d$ let $\|v\|$ be the standard Euclidean norm of v and for $S \in \mathrm{End}(\mathbb{R}^d)$ let $\|S\|$ denote the associated operator norm. Also let $e_1, \dots, e_d$ denote the standard basis of $\mathbb{R}^d$.

Notice that

$$T(p) = \lim_{n\to\infty} g_n(p)$$

for all $p \notin \mathbb{P}(\ker T)$.

We can pick a lift $\overline{g}_n \in \mathrm{GL}_d(\mathbb{R})$ of each g_n with $\|\overline{g}_n\| = 1$ such that $\overline{g}_n \to \overline{T}$ in $\mathrm{End}(\mathbb{R}^d)$ and $\overline{T}$ is a lift of T.

Claim 1: $\mathbb{P}(\ker T) \cap \Omega = \emptyset$.

Proof of Claim 1: Using the singular value decomposition, we can find $k_{n,1}, k_{n,2} \in \mathrm{O}(d)$ and $1 = \sigma_{1,n} \geq \cdots \geq \sigma_{d,n} > 0$ such that

$$\overline{g}_n = k_{n,1} \begin{pmatrix} \sigma_{1,n} & & \\ & \ddots & \\ & & \sigma_{d,n} \end{pmatrix} k_{n,2}.$$

By passing to a subsequence we can suppose that $k_{n,1} \to k_1$, $k_{n,2} \to k_2$, and

$$\chi_j := \lim_{n \to \infty} \sigma_{j,n} \in [0,1]$$

exists for every $1 \leq j \leq d$. Then

$$\overline{T} = k_1 \begin{pmatrix} 1 & & & \\ & \chi_2 & & \\ & & \ddots & \\ & & & \chi_d \end{pmatrix} k_2.$$

Let

$$m := \max \{ j : \chi_j > 0 \}. \tag{2.3}$$

Then $\ker T = k_2^{-1} \operatorname{Span}\{e_{m+1}, \dots, e_d\}$.

Suppose for a contradiction that there exists $[v] \in \mathbb{P}(\ker T) \cap \Omega$. Let

$$v_n := k_{n,2}^{-1} k_2 v \in k_{n,2}^{-1} \operatorname{Span}\{e_{m+1}, \dots, e_d\}.$$

Since Ω is open and $v_n \to v$, by passing to a tail we can assume that there exists some $\epsilon > 0$ such that

$$\left\{ \left[v_n + s k_{n,2}^{-1} e_1 \right] : |s| < \epsilon \right\} \subset \Omega$$

for all $n \geq 0$. By passing to a subsequence we can suppose that

$$w := \lim_{n \to \infty} \frac{1}{\|\overline{g}_n v_n\|} \overline{g}_n v_n \in \mathbb{R}^d$$

exists. Now fix $t \in \mathbb{R}$ and let $t_n := \|\overline{g}_n v_n\| \, t$. Since $\|\overline{g}_n v_n\| \leq \sigma_{m+1,n} \|v_n\|$ and

$$\lim_{n\to\infty} \sigma_{m+1,n} = 0,$$

for n sufficiently large we have $|t_n| < \epsilon$. Then

$$\begin{aligned}
[w + tk_1 e_1] &= \lim_{n\to\infty} \left[\frac{1}{\|\overline{g}_n v_n\|} \left(\overline{g}_n v_n + t_n k_{n,1} e_1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{1}{\|\overline{g}_n v_n\|} \left(\overline{g}_n v_n + t_n \overline{g}_n k_{n,2}^{-1} e_1 \right) \right] \\
&= \lim_{n\to\infty} g_n \left[v_n + t_n k_{n,2}^{-1} e_1 \right] \in \overline{\Omega}.
\end{aligned}$$

Since t is arbitrary, we see that

$$\{[w + tk_1 e_1] : t \in \mathbb{R}\} \subset \overline{\Omega}$$

which contradicts the fact that Ω is properly convex. So $\mathbb{P}(\ker T) \cap \Omega = \emptyset$.

Claim 2: $T(\Omega) \subset F_\Omega(x)$. In particular,

$$\mathrm{image}(T) \subset \mathrm{Span}\, F_\Omega(x).$$

Proof of Claim 2: Since $\mathbb{P}(\ker T) \cap \Omega = \emptyset$,

$$T(p) = \lim_{n\to\infty} g_n(p)$$

for all $p \in \Omega$. Since $g_n(p_0) \to x$ and

$$\mathrm{d}_\Omega(g_n(p), g_n(p_0)) = \mathrm{d}_\Omega(p, p_0),$$

Proposition II.12 implies that $T(\Omega) \subset F_\Omega(x)$.

Claim 3: $y \in \mathbb{P}(\ker T)$.

Proof of Claim 3: Notice that

$$\overline{h}_n := k_{n,2}^{-1} \begin{pmatrix} \sigma_{1,n}^{-1} & & \\ & \ddots & \\ & & \sigma_{d,n}^{-1} \end{pmatrix} k_{n,1}^{-1}$$

is a lift of g_n^{-1}. Since $1 = \sigma_{1,n} \geq \cdots \geq \sigma_{d,n} > 0$, we can pass to a subsequence and assume that $\sigma_{d,n}\overline{h}_n$ converges in $\mathrm{End}(\mathbb{R}^d)$ to some non-zero $S \in \mathrm{End}(\mathbb{R}^d)$. Then g_n^{-1} converges in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ to $[S] \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Claim 1 applied to g_n^{-1} implies that $\mathbb{P}(\ker S) \cap \Omega = \emptyset$. So

$$S(p_0) = \lim_{n\to\infty} g_n^{-1}(p_0) = y.$$

Further, Equation (2.3) implies that

$$\mathrm{image}(S) \subset k_2^{-1}\,\mathrm{Span}\{e_{m+1}, \dots, e_d\} = \ker T.$$

So $y \in \mathbb{P}(\ker T)$. □

Given a group $G \leq \mathrm{PGL}_d(\mathbb{R})$ define $\overline{G}^{\mathrm{End}}$ to be the closure of the set

$$\{g \in \mathrm{GL}_d(\mathbb{R}) : [g] \in G\}$$

in $\mathrm{End}(\mathbb{R}^d)$.

Proposition II.36 ([IZ21])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\mathcal{C} \subset \Omega$ is a non-empty closed convex subset, and $G \leq \mathrm{Stab}_\Omega(\mathcal{C})$ acts co-compactly on $\mathcal{C}$. If $x \in \partial_{\mathrm{i}}\,\mathcal{C}$, then there exists $T \in \overline{G}^{\mathrm{End}}$ such that*

(1) $\mathbb{P}(\ker T) \cap \Omega = \emptyset$,

(2) $T(\Omega) = F_\Omega(x)$, and

(3) $T(\mathcal{C}) = F_\Omega(x) \cap \partial_{\mathrm{i}}\,\mathcal{C}$.

Proof. Fix some $p_0 \in \mathcal{C}$ and a sequence $p_n \in [p_0, x)$ with $p_n \to x$. Since G acts co-compactly on $\mathcal{C}$, there exists $R > 0$ and a sequence $g_n \in G$ such that

$$\mathrm{d}_\Omega(g_n p_0, p_n) \leq R$$

for all $n \geq 0$.

As before, for $S \in \mathrm{End}(\mathbb{R}^d)$ let $\|S\|$ be the operator norm associated to the standard Euclidean norm. Let $\overline{g}_n \in \mathrm{GL}_d(\mathbb{R})$ be a lift of g_n with $\|\overline{g}_n\| = 1$. By passing to a subsequence we can suppose that $\overline{g}_n \to T$ in $\mathrm{End}(\mathbb{R}^d)$. Proposition II.35 implies that $\mathbb{P}(\ker T) \cap \Omega = \emptyset$ and $T(\Omega) \subset F_\Omega(x)$. Then

$$T(p) = \lim_{n \to \infty} g_n(p)$$

for all $p \in \Omega$.

Claim 1: $T(\Omega) = F_\Omega(x)$.

Proof of Claim 1: We only need to show that $F_\Omega(x) \subset T(\Omega)$. So fix $y \in F_\Omega(x)$. Then we can pick $y_n \in [p_0, y)$ such that

$$\sup_{n \geq 0} \mathrm{d}_\Omega(y_n, p_n) < \infty.$$

Thus

$$\sup_{n \geq 0} \mathrm{d}_\Omega(g_n^{-1} y_n, p_0) < \infty.$$

So there exists $n_j \to \infty$ so that the limit

$$q := \lim_{j \to \infty} g_{n_j}^{-1} y_{n_j}$$

exists in Ω. Then

$$T(q) = \lim_{n \to \infty} g_n(q) = \lim_{j \to \infty} g_{n_j} g_{n_j}^{-1} y_{n_j} = \lim_{j \to \infty} y_{n_j} = y.$$

Hence $F_\Omega(x) \subset T(\Omega)$.

Claim 2: $T(\mathcal{C}) = F_\Omega(x) \cap \partial_{\mathrm{i}} \mathcal{C}$.

Proof of Claim 2: This is almost identical to the proof of Claim 1. $\square$

2.7 Convex Co-compact Groups

Definition II.37. A discrete subgroup $\Lambda \leq \mathrm{PGL}_d(\mathbb{R})$ is called convex co-compact if:

(1) there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ such that $\Lambda \leq \mathrm{Aut}(\Omega)$, and

(2) the set $\mathcal{C}_\Omega(\Lambda) \subset \Omega$ is non-empty and Λ acts co-compactly on $\mathcal{C}_\Omega(\Lambda)$, where $\mathcal{C}_\Omega(\Lambda)$ is the convex hull in Ω of the full orbital limit set $\mathcal{L}_\Omega^{\mathrm{orb}}(\Lambda) := \bigcup_{x\in\Omega}\left(\overline{\Lambda x} \setminus \Lambda x\right)$.

Remark II.38. *If Λ acts co-compactly on a properly convex domain Ω, then $\mathcal{C}_\Omega(\Lambda) = \Omega$ and Λ is a convex co-compact group.*

If Λ is convex co-compact, the boundary of $\mathcal{C}_\Omega(\Lambda)$ splits into the *ideal boundary* $\partial_{\mathrm{i}}\,\mathcal{C}_\Omega(\Lambda) := \partial\Omega \cap \overline{\mathcal{C}_\Omega(\Lambda)}$ and the *non-ideal boundary* $\partial_{\mathrm{n}}\,\mathcal{C}_\Omega(\Lambda) := \Omega \cap \overline{\mathcal{C}_\Omega(\Lambda)}$. We recall some results from [DGK17] regarding properties of convex co-compact groups.

Theorem II.39 ([DGK17])**.** *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Then:*

(1) $\mathcal{C}_\Omega(\Lambda)$ is the minimal non-empty Λ-invariant closed convex subset of Ω,

(2) $\mathcal{L}_\Omega^{\mathrm{orb}}(\Lambda) = \partial_{\mathrm{i}}\,\mathcal{C}_\Omega(\Lambda)$,

(3) if $x \in \partial_{\mathrm{i}}\,\mathcal{C}_\Omega(\Lambda)$, then $F_{\mathcal{C}_\Omega(\Lambda)}(x) = F_\Omega(x)$.

CHAPTER III

Weak Hyperbolicity in Geometric Group Theory

3.1 Introduction

The key idea of geometric group theory is to study infinite groups using geometry. We can view a group as a geometric object via its Cayley graph that we now explain. If G is a group with a finite generating set S, we will use the notation: (G, S) is a finitely generated group. If (G, S) is a finitely generated group, its Cayley graph $\mathrm{Cay}(G, S)$ is an unordered graph whose vertex set is G with an edge between g and gs for any $g \in G$ and $s \in S$. Assigning length 1 to each edge, this graph can be given the structure of a proper metric space that we denote by (G, d_S). Note that this metric structure depends on the generating set S. To show that this notion is "coarsely" well-defined (up to quasi-isometries), we need some definitions.

Definition III.1. Consider two metric spaces (X, d_X) and (Y, d_Y). If $K \geq 1$ and $C \geq 0$, a map $F : X \to Y$ is called:

(1) a (K, C)-quasi-isometric embedding if for any $x, x' \in X$

$$\frac{1}{K}\,\mathrm{d}_X(x, x') - C \leq \mathrm{d}_Y(F(x), F(x')) \leq K\,\mathrm{d}_X(x, x') + C$$

(2) a (K, C)-quasi-isometry if F is a (K, C)-quasi-isometric embedding and there

exists $D \geq 0$ such that:

$$\sup_{y \in Y} \mathrm{d}_Y(y, F(X)) \leq D.$$

The two metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there exists a (K, C)-quasi-isometry between them for some $K \geq 1$ and $C \geq 0$.

Proposition III.2. *If S and S' are two finite generating sets of G, then (G, d_S) and $(G, \mathrm{d}_{S'})$ are quasi-isometric.*

The fundamental lemma of geometric group theory establishes a coarse equivalence between the geometry of the $\mathrm{Cay}(G, S)$ and the geometry of a space X on which G acts.

Theorem III.3 ([BH99, Part I, Proposition 8.19])**.** *Suppose (X, d_X) is a proper geodesic metric space and G is a group that acts on X properly and co-compactly by isometries. Then G is finitely generated and if S is a finite generating set of G, then $F : (G, \mathrm{d}_S) \to (X, \mathrm{d}_X)$ defined by $F(g) = g \cdot x_0$ is a quasi-isometry for any choice of base-point $x_0 \in X$.*

We now introduce some standard notations that we will be used frequently. If (X, d) is a metric space and $r > 0$, we denote the metric r-tubular neighborhood of $A \subset X$ as

$$\mathcal{N}_X(A; r) := \{x \in X : \mathrm{d}(x, A) < r\}$$

and the metric r-ball around $x \in X$ as

$$\mathcal{B}_X(x, r) := \{y \in X : \mathrm{d}(x, y) < r\}.$$

We also introduce the notion of paths and path systems.

Definition III.4 ([Sis18])**.** Suppose (X, d) is a proper geodesic metric space and a group G acts on X properly and by isometries.

(1) A path in X is the image of $f : \mathbb{R} \to X$ where f is a (K, C)-quasi-geodesic embedding for some $K \geq 1$ and $C \geq 0$. If $\mathcal{I} \subset \mathbb{R}$ is an interval, then $f(\mathcal{I})$ is a subpath.

(2) A path system $\mathcal{PS}$ on X is a collection of paths in X for some fixed constants $K \geq 1$ and $C \geq 0$ such that:

   (a) any subpath of a path in $\mathcal{PS}$ is also in $\mathcal{PS}$, and

   (b) any pair of distinct points in X can be connected by a path in $\mathcal{PS}$.

(3) A path system $\mathcal{PS}$ is called a geodesic path system if all paths in $\mathcal{PS}$ are geodesics in (X, d).

(4) If G preserves $\mathcal{PS}$, then $(X, \mathcal{PS})$ is called a path system for the group G.

3.2 Gromov Hyperbolic Groups

Gromov hyperbolicity is a formulation of hyperbolicity or 'negative curvature-like' behavior in coarse geometry. For introducing this class of groups, we will require the notion of slim triangles. If x and x' are two points in a metric space X, let $\sigma_{x,x'}$ denote a geodesic joining x and x'.

Definition III.5 ([BH99, Part III, Definition 1.1])**.** Suppose (X, d) is a metric space and $x_1, x_2, x_3 \in X$. A geodesic triangle $\sigma_{x_1,x_2} \cup \sigma_{x_2,x_3} \cup \sigma_{x_3,x_1}$ is called δ-slim for some $\delta \geq 0$ if

$$\sigma_{x_{i-1},x_i} \subset \mathcal{N}_X\left(\sigma_{x_i,x_{i+1}} \cup \sigma_{x_{i+1},x_{i-1}};\ \delta\right) \tag{3.1}$$

for $i \in \{1, 2, 3\}$ with the indices of x_i counted modulo 3 in (3.1).

Definition III.6 ([BH99, Part III, Definition 2.1])**.**

(1) A geodesic metric space (X, d) is a Gromov hyperbolic space if there exists $\delta \geq 0$ such that all geodesic triangles in X are δ-slim.

(2) A finitely generated group (G, S) is a Gromov hyperbolic (or word hyperbolic) group if (G, d_S) is a Gromov hyperbolic space.

This notion is well-behaved under quasi-isometries.

Proposition III.7. *If (X, d_X) and (Y, d_Y) are quasi-isometric, then (X, d_X) is a Gromov hyperbolic space if and only if (Y, d_Y) is a Gromov hyperbolic space.*

Some examples of Gromov hyperbolic spaces are a regular tree of finite valence and the real hyperbolic space $\mathbb{H}^d$ (equipped with the Riemannian metric of constant curvature -1). Some examples of Gromov hyperbolic groups are $\mathbb{Z}$, free groups on finitely many generators, and fundamental groups of compact manifolds of negative curvature. A non-example of Gromov hyperbolic groups is any group that contains a subgroup isomorphic to $\mathbb{Z}^r$ for $r \geq 2$.

While the motivating examples of Gromov hyperbolic groups are discrete subgroup of $\mathrm{Isom}(\mathbb{H}^d)$, not all such groups are Gromov hyperbolic. For instance, the fundamental group of a finite volume non-compact hyperbolic manifold of dimension $d \geq 3$ is not Gromov hyperbolic. It contains subgroups isomorphic to $\mathbb{Z}^{d-1}$ that correspond to the cusp stabilizers. It is thus natural to seek generalizations of Gromov hyperbolic groups. We will discuss two such generalizations: relatively hyperbolic groups in Section 3.3 and acylindrically hyperbolic groups in Section 3.4.

3.3 Relatively Hyperbolic Groups

Relatively hyperbolic groups generalize fundamental groups of finite volume non-compact hyperbolic manifolds. There are several equivalent definitions due to Bowditch, Farb, and Druţu-Sapir to name a few. In this section, we will follow the approach taken by Druţu-Sapir in [DS05]. For this, we recall the notion of asymptotic cones and asymptotically tree-graded metric spaces.

Definition III.8. Suppose ω is a non-principal ultrafilter, (X, d) is a metric space, (x_n) is a sequence of points in X, and (λ_n) is a sequence of positive numbers with $\lim_\omega \lambda_n = \infty$. The *asymptotic cone of X with respect to (x_n) and (λ_n)*, denoted by $C_\omega(X, x_n, \lambda_n)$, is the ultralimit $\lim_\omega(X, \lambda_n^{-1}\,\mathrm{d}, x_n)$.

For more background on asymptotic cones, see [Dru02].

Definition III.9 ([DS05, Definition 2.1])**.** Let (X, d) be a complete geodesic metric space and let $\mathcal{S}$ be a collection of closed geodesic subsets (called *pieces*).

(1) We say that (X, d) is *tree-graded with respect to $\mathcal{S}$* if:

  (a) every two different pieces have at most one common point.

  (b) every simple geodesic triangle (a simple loop composed of three geodesics) in X is contained in one piece.

(2) We say that (X, d) is *asymptotically tree-graded with respect to $\mathcal{S}$* if all its asymptotic cones, with respect to a fixed non-principal ultrafilter, are tree-graded with respect to the collection of ultralimits of the elements of $\mathcal{S}$.

Using asymptotically tree-graded metric spaces, we now introduce the definition of relatively hyperbolic spaces and groups respectively.

Definition III.10 ([DS05])**.**

(1) A complete geodesic metric space (X, d) is *relatively hyperbolic with respect to a collection of subsets $\mathcal{S}$* if (X, d) is asymptotically tree-graded with respect to $\mathcal{S}$.

(2) A finitely generated group (G, S) is *relatively hyperbolic with respect to a family of subgroups $\{H_1, \ldots, H_k\}$* if (G, d_S) is relatively hyperbolic with respect to the collection of left cosets $\{gH_i : g \in G, i = 1, \ldots, k\}$.

Relative hyperbolicity is well-behaved under quasi-isometries.

Proposition III.11. *Suppose (X, d_X) is a relatively hyperbolic space with respect to $\mathcal{S}_X$ and $f : (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is a quasi-isometry. Then, (Y, d_Y) is a relatively hyperbolic space with respect to $\mathcal{S}_Y := f(\mathcal{S}_X)$.*

We will require the following results about relatively hyperbolic spaces.

Theorem III.12. *Suppose (X, d) is a relatively hyperbolic space with respect to $\mathcal{S}$.*

(1) [DS05, Theorem 4.1] *For any $r > 0$ there exists $Q(r) > 0$ such that: if $S_1, S_2 \in \mathcal{S}$ are distinct, then*

$$\mathrm{diam}_X \big(\mathcal{N}_X(S_1; r) \cap \mathcal{N}_X(S_2; r)\big) \leq Q(r).$$

(2) [DS05, Corollary 5.8] *If $A \geq 1$, $B \geq 0$, and $f : \mathbb{R}^k \to X$ is an (A, B)-quasi-isometric embedding, then there exists $M = M(A, B)$ such that: if $k \geq 2$, then there exists some $S \in \mathcal{S}$ such that*

$$f(\mathbb{R}^k) \subset \mathcal{N}_X(S; M).$$

(3) [DS05, Theorem 5.1] *If (Y, d_Y) are complete geodesic metric spaces and $f : X \to Y$ is a quasi-isometry, then (X, d_X) is relatively hyperbolic with respect to $\mathcal{S}$ if and only if (Y, d_Y) is relatively hyperbolic with respect to $f(\mathcal{S})$.*

We end this section by stating a characterization of relative hyperbolicity due to Sisto [Sis13]. In order to state his characterization, we introduce two notions: "almost-projection system" and "asymptotically transverse-free with respect to a geodesic path system".

Definition III.13 ([Sis13])**.** Let (X, d) be a complete geodesic metric space and $\mathcal{S}$ a collection of subsets of X. A family of maps $\Pi_{\mathcal{S}} = \{\pi_S : X \to S\}_{S \in \mathcal{S}}$ is an *almost-projection system for $\mathcal{S}$* if there exists $C > 0$ such that for all $S \in \mathcal{S}$:

(1) if $x \in X$ and $p \in S$, then $\mathrm{d}(x,p) \geq \mathrm{d}(x,\pi_S(x)) + \mathrm{d}(\pi_S(x),p) - C$,

(2) $\mathrm{diam}_X\, \pi_S(S') \leq C$ for all $S, S' \in \mathcal{S}$ distinct, and

(3) if $x \in X$ and $\mathrm{d}(x,S) = R$, then $\mathrm{diam}_X\, \pi_S(\mathcal{B}_X(x,R)) \leq C$.

Recall the notion of geodesic path systems from Definition III.4.

Definition III.14 ([Sis13])**.** Let (X,d) be a complete geodesic metric space and $\mathcal{S}$ a collection of subsets of X.

(1) A geodesic triangle $\mathcal{T}$ in X is $\mathcal{S}$*-almost-transverse with constants* κ *and* Δ if

$$\mathrm{diam}_X(\mathcal{N}_X(S;\kappa) \cap \gamma) \leq \Delta$$

for every $S \in \mathcal{S}$ and edge γ of $\mathcal{T}$.

(2) The collection $\mathcal{S}$ is *asymptotically transverse-free relative to a geodesic path system* $\mathcal{G}$ if there exists λ, σ such that for each $\Delta \geq 1, \kappa \geq \sigma$ the following holds: if $\mathcal{T}$ is a geodesic triangle in X whose sides are in $\mathcal{G}$ and is $\mathcal{S}$-almost-transverse with constants κ and Δ, then $\mathcal{T}$ is $(\lambda\Delta)$-thin.

We finally state Sisto's characterization of relative hyperbolicity.

Theorem III.15 ([Sis13, Theorem 2.14])**.** *Let* (X,d) *be a complete geodesic metric space and* $\mathcal{S}$ *a collection of subsets of* X*. Then the following are equivalent:*

(1) X *is relatively hyperbolic with respect to* $\mathcal{S}$*,*

(2) $\mathcal{S}$ *is asymptotically transverse-free relative to a geodesic path system and there exists an almost-projection system for* $\mathcal{S}$*,*

In [Sis13], the theorem is stated for path systems instead of geodesic path systems. But his methods also imply this result, see [IZ19, Appendix] for details.

3.4 Acylindrically Hyperbolic Groups

3.4.1 Definition and examples

Osin introduced the notion of acylindrically hyperbolic groups in [Osi16] as a generalization of Gromov hyperbolic groups.

Definition III.16. Suppose that a group G acts isometrically on a metric space (X, d_X). The action is called *acylindrical* provided: for every $\varepsilon > 0$, there exist $R_\varepsilon, N_\varepsilon > 0$ such that if $x, y \in X$ with $\mathrm{d}_X(x, y) \geq R_\varepsilon$, then

$$\#\left\{g \in G : \mathrm{d}_Y(x, gx) \leq \varepsilon \text{ and } \mathrm{d}_Y(y, gy) \leq \varepsilon\right\} \leq N_\varepsilon.$$

Suppose G acts properly and isometrically on a proper Gromov hyperbolic space (X, d). There is a notion of bordification (or compactification) of (X, d) by adding the Gromov boundary $\partial_\infty X$ to X (see [BH99] for details). We will denote this compactification by $\overline{X} := X \cup \partial_\infty X$. The limit set of the action $\Lambda_X(G)$ is the set of accumulation points of some (hence any) orbit of G in $\overline{X}$. It is a well-known fact in the classification of group actions on Gromov hyperbolic spaces that either $0 \leq \#\Lambda_X(G) \leq 2$ or $\Lambda_X(G)$ is an infinite set [Osi16, Section 3]. The group action is called elementary in the first case and non-elementary in the second case. The case of elementary actions is particularly simple (G is finite or virtually cyclic, see [BH99]). We will be interested in non-elementary actions.

Definition III.17 ([Osi16])**.** A group G is called *acylindrically hyperbolic* if it admits an isometric non-elementary acylindrical action on a Gromov hyperbolic metric space (X, d_X).

Examples: Mapping class group of closed surfaces of genus at least 2 are acylindrically hyperbolic because they act non-elementarily and acylindrically on the curve complex [PS15]. This is a particularly interesting example as these groups are neither Gromov hyperbolic nor relatively hyperbolic [AAS05]. Some other prominent examples are outer automorphism groups of finitely generated free group on at least two generators and rank one CAT(0) groups that are not virtually cyclic.

We now introduce the notion of hyperbolically embedded subgroups which was used in [Osi16] to characterize acylindrically hyperbolic groups. Let H be a subgroup of G and a (possibly infinite) subset S of G such that $S \sqcup (H \setminus \{1\})$ generates G. Then S is called the relative generating set and let $\mathrm{d}^S_{\mathrm{rel}} : H \times H \to [0, +\infty]$ be the relative metric, where $\mathrm{d}^S_{\mathrm{rel}}(g, h)$ is the length of the shortest path in $\mathrm{Cay}(G, S \sqcup (H \setminus \{1\}))$ connecting g and h that has no edges in $\mathrm{Cay}(H, H \setminus \{1\})$.

Definition III.18 ([Sis18, Definition 4.6])**.** A subgroup H of G is hyperbolically embedded if there exists a relative generating set S such that $\mathrm{Cay}(G, S \sqcup (H \setminus \{1\}))$ is Gromov hyperbolic and $(H, \mathrm{d}^S_{\mathrm{rel}})$ is a locally finite metric space.

Example: The subgroup $\mathbb{Z} * \{e\}$ is a hyperbolically embedded subgroup of $\mathbb{Z} * \mathbb{Z}$. On the other hand, $\mathbb{Z} \times \{e\}$ is not hyperbolically embedded in $\mathbb{Z} \times \mathbb{Z}$. See [Sis18].

Osin characterizes acylindrically hyperbolic groups based on the existence of hyperbolically embedded subgroups.

Proposition III.19 ([Osi16, Definition 1.3])**.** *A group G is acylindrically hyperbolic if and only if G contains a proper infinite hyperbolically embedded subgroup.*

In the next section, we will see another characterization of acylindrically hyperbolic groups (using contracting elements) that will be particularly useful in Chapter V.

3.4.2 Contracting Elements I: Characterizing Acylindrical Hyperbolicity

Recall the definition of path and path system in Definition III.4.

Definition III.20 ([Sis18])**.** Suppose (X, d) is a geodesic metric space, $\mathcal{PS}$ is a path system on X, and G acts on X properly and by isometries.

(1) A set $\mathcal{A} \subset X$ is called $\mathcal{PS}$-*contracting* (with constant C) if there exists a map $\pi_{\mathcal{A}} : X \to \mathcal{A}$ such that:

(a) if $x \in \mathcal{A}$, then $\mathrm{d}\left(x, \pi_{\mathcal{A}}(x)\right) \leq C$

(b) if $x, y \in X$ with $\mathrm{d}\left(\pi_{\mathcal{A}}(x), \pi_{\mathcal{A}}(y)\right) \geq C$ and $\sigma \in \mathcal{PS}$ is a path joining x and y, then

$$\mathrm{d}\left(\sigma, \pi_{\mathcal{A}}(x)\right) \leq C \quad \text{and} \quad \mathrm{d}\left(\sigma, \pi_{\mathcal{A}}(y)\right) \leq C.$$

(2) If $(X, \mathcal{PS})$ is a path system for G, then $g \in G$ is a *contracting element* for $(X, \mathcal{PS})$ provided for some (hence any) $x_0 \in X$:

(a) g is an infinite order element,

(b) $< g > x_0$ is a quasi-geodesic embedding of $\mathbb{Z}$ in X,

(c) there exists $\mathcal{A} \subset X$ containing x_0 that is $< g >$-invariant, $\mathcal{PS}$-contracting and has co-bounded $< g >$ action.

The following proposition will illustrate the geometric intuition behind the definition of contracting elements.

Proposition III.21. *Suppose $(X, \mathcal{PS})$ is a path system for G and $g \in G$ is a contracting element for $(X, \mathcal{PS})$. Then:*

(1) $\tau_X(g) := \inf_{x \in X} \mathrm{d}(x, gx) > 0$.

(2) for any $x_0 \in X$, $\mathcal{A}_{\min}(x_0) :=< g > x_0$ is the minimal $\mathcal{PS}$-contracting, $< g >$-invariant subset of X containing x_0 with a co-bounded $< g >$ action.

Proof. (1) Recall the definition of stable translation length:

$$\tau_X^{\text{stable}}(g) := \lim_{n \to \infty} \frac{\mathrm{d}(x, g^n x)}{n}.$$

Then triangle inequlaity shows that

$$\tau_X(g) \geq \tau_X^{\text{stable}}(g)$$

and it suffices to show $\tau_X^{\text{stable}}(g) > 0$. Fix any $x_0 \in X$. Since g is contracting, $< g > x_0$ is a quasi-geodesic, that is, there exists $K' \geq 1$ and $C' \geq 0$ such that for every $n \in \mathbb{Z}$,

$$\mathrm{d}(x_0, g^n x_0) \geq \frac{1}{K'}|n| - C'.$$

Then, $\tau_X^{\text{stable}}(g) \geq 1/K' > 0$.

(2) Let $\mathcal{A}$ be $\mathcal{PS}$-contracting with constant $C_{\mathcal{A}}$ and the map $\pi_{\mathcal{A}} : X \to \mathcal{A}$. Fix any $x_0 \in X$ and set $R_{\mathcal{A}} := \mathrm{diam}\big(\mathcal{A}/ < g >\big)$, $C_0 := C_{\mathcal{A}} + 2R_{\mathcal{A}}$ and $\mathcal{A}_{\min}(x_0) :=< g > x_0$.

Since $\mathcal{A}_{\min}(x_0) \subset \mathcal{A}$, if $x \in X$, then there exists $m \in \mathbb{Z}$ such that

$$\mathrm{d}(\pi_{\mathcal{A}}(x), g^m x_0) \leq R_{\mathcal{A}}.$$

Define $\pi_{\min} : X \to \mathcal{A}_{\min}(x_0)$ by setting $\pi_{\min}(x) = g^m x_0$. Then, if $x \in \mathcal{A}_{\min}(x_0)$, $\pi_{\min}(x) = x$. If $x, y \in X$ and $\mathrm{d}(\pi_{\min}(x), \pi_{\min}(y)) \geq C_0$, then

$$\mathrm{d}(\pi_{\mathcal{A}}(x), \pi_{\mathcal{A}}(y)) \geq C_{\mathcal{A}}.$$

Thus, if $\sigma \in \mathcal{PS}$ is any path from x to y,

$$\mathrm{d}(\pi_{\mathcal{A}}(x), \sigma) \leq C_{\mathcal{A}} \text{ and } \mathrm{d}(\pi_{\mathcal{A}}(y), \sigma) \leq C_{\mathcal{A}}.$$

Hence,

$$\mathrm{d}(\pi_{\min}(x), \sigma) \leq C_0 \text{ and } \mathrm{d}(\pi_{\min}(y), \sigma) \leq C_0.$$

□

We now prove a characterization of acylindrically hyperbolic groups using contracting elements. We will say that a group is virtually cyclic if it contains a finite index cyclic subgroup.

Theorem III.22 ([Osi16, Sis18]). *Suppose G has a proper isometric action on a geodesic metric space (X, d_X), $(X, \mathcal{PS})$ is a path system for G and $g \in G$ is a contracting element for $(X, \mathcal{PS})$. Then, either G is virtually $\mathbb{Z}$ or G is acylindrically hyperbolic.*

We will spend the rest of this section discussing a proof of this theorem. The key result comes from a result connecting the subgroup generated by a contracting element with the notion of hyperbolically embedded subgroups [REF]. Fix a path system $(X, \mathcal{PS})$ for G where G acts properly and isometrically on the metric space (X, d_X). Suppose $g \in G$ is contracting. Then, there exist a $< g >$-invariant set $\mathcal{A}$ with a co-bounded $< g >$-action equipped with a projection $\pi : G \to \mathcal{A}$. Further, assume that G is not virtually cyclic. Then we will prove that G is acylindrically hyperbolic.

Consider

$$\begin{aligned} E(g) &:= \left\{h \in G : \mathrm{d}_X^{\mathrm{Hauss}}(\pi_A(h\mathcal{A}), \mathcal{A}) < \infty\right\} \\ &= \left\{h \in G : \max\left\{\sup_{b \in h\mathcal{A}} \mathrm{d}_X(b, \mathcal{A}), \sup_{a \in \mathcal{A}} \mathrm{d}_X(a, h\mathcal{A})\right\} < \infty\right\}. \end{aligned}$$

Sisto proves:

Proposition III.23 ([Sis18, Theorem 4.7]). *If g is a contracting element in G for the path system $(X, \mathcal{PS})$, then $E(g)$ is a hyperbolically embedded subgroup which is virtually cyclic.*

Since $E(g)$ is virtually cyclic while G is not, $E(g)$ is a proper subgroup. Moreover, $E(g)$ is a hyperbolically embedded subgroup. Then Proposition III.19 implies that G is an acylindrically hyperbolic group.

3.4.3 Contracting Elements II: Other Notions of Contraction

In this section, we will discuss another notion of contraction that is due to Bestvina-Fujiwara [BF09]; we will use the symbol BF to denote this. Fix a geodesic metric space (X, d) and a group G that acts properly and isometrically on X. If $\mathcal{A} \subset X$ and $x \in X$, let the closest-point projection on $\mathcal{A}$ be defined by:

$$\rho_{\mathcal{A}}(x) := \{y \in X : \mathrm{d}(x, y) = \mathrm{d}(x, \mathcal{A})\}.$$

The following definition comes from [BF09] and [GY18].

Definition III.24. A set $\mathcal{A} \subset X$ is a *contracting subset in the sense of* BF if there exists a constant C such that: if $x \in X$, $R > 0$ and $B(x, R) \cap \mathcal{A} = \emptyset$, then

$$\operatorname{diam}\left(\rho_{\mathcal{A}}(B(x, R))\right) \leq C.$$

An element $g \in G$ is a *contracting element in the sense of* BF if for any $x_0 \in X$:

(1) g has infinite order,

(2) $< g > x_0$ is a quasi-geodesic embedding of $\mathbb{Z}$ in X, and

(3) $< g > x_0$ is a contracting subset in the sense of BF.

We will now connect the two notions of contraction. The proof is fairly elementary and this argument is given in [Isl19, Appendix]. For that, we first prove the following lemma.

Lemma III.25. *If $\mathcal{A} \subset X$ is $\mathcal{PS}$-contracting (with constant C) for the projection map $\pi_{\mathcal{A}} : X \to \mathcal{A}$, then $\mathrm{d}(\pi_{\mathcal{A}}(x), \rho_{\mathcal{A}}(x)) \leq 2C$ for all $x \in X$.*

Proof. If $x, y \in X$, let $\sigma_{x,y} \in \mathcal{PS}$ denote a path joining x and y.

Suppose there exists $x \in X$ such that

$$\mathrm{d}(\pi_{\mathcal{A}}(x), \rho_{\mathcal{A}}(x)) > 2C. \tag{3.2}$$

Since $\mathcal{A}$ is $\mathcal{PS}$-contracting and $\rho_{\mathcal{A}}(x) \in \mathcal{A}$,

$$\mathrm{d}(\pi_{\mathcal{A}}(\rho_{\mathcal{A}}(x)), \rho_{\mathcal{A}}(x)) \leq C.$$

Then

$$\mathrm{d}(\pi_{\mathcal{A}}(x), \pi_{\mathcal{A}}(\rho_{\mathcal{A}}(x))) \geq \mathrm{d}(\pi_{\mathcal{A}}(x), \rho_{\mathcal{A}}(x)) - \mathrm{d}(\pi_{\mathcal{A}}(\rho_{\mathcal{A}}(x)), \rho_{\mathcal{A}}(x)) > C.$$

Since $\mathcal{A}$ is $\mathcal{PS}$-contracting, there exists $z \in \sigma_{x,\rho_{\mathcal{A}}(x)}$ such that $\mathrm{d}(z, \pi_{\mathcal{A}}(x)) \leq C$. Then

$$\begin{aligned}\mathrm{d}(\rho_{\mathcal{A}}(x), z) &= \mathrm{d}(\rho_{\mathcal{A}}(x), x) - \mathrm{d}(z, x) \leq \mathrm{d}(\pi_{\mathcal{A}}(x), x) - \mathrm{d}(z, x) \\ &\leq \mathrm{d}(\pi_{\mathcal{A}}(x), z) + \mathrm{d}(z, x) - \mathrm{d}(z, x) \leq C.\end{aligned}$$

Hence

$$\mathrm{d}(\rho_{\mathcal{A}}(x), \pi_{\mathcal{A}}(x)) \leq \mathrm{d}(\rho_{\mathcal{A}}(x), z) + \mathrm{d}(z, \pi_{\mathcal{A}}(x)) \leq 2C.$$

This contradicts (3.2). $\square$

We now prove the equivalence of the two notions of contraction for geodesic path systems.

Proposition III.26. *If $(X, \mathcal{PS})$ is a geodesic path system for G, then:*

(1) $\mathcal{A} \subset X$ is $\mathcal{PS}$-contracting if and only if $\mathcal{A}$ is contracting in the sense of BF*.*

(2) $g \in G$ is a contracting element for $(X, \mathcal{PS})$ if and only if $g \in G$ is a contracting element in the sense of BF*.*

Proof. (1) Suppose $\mathcal{A}$ is contracting in the sense of BF. If $x \in X$, $\mathrm{d}(x, \rho_{\mathcal{A}}(x)) = \mathrm{d}(x, \mathcal{A})$. Then, by [Sis18, Lemma 2.3], $\mathcal{A}$ is $(X, \mathcal{PS})$ contracting.

Conversely, suppose $\mathcal{A}$ in $\mathcal{PS}$-contracting (with constant C and the map $\pi_{\mathcal{A}} : X \to \mathcal{A}$). Suppose $x \in X$ and $0 < R < d(x, \rho_{\mathcal{A}}(x))$. If $y \in B(x, R)$, let $\delta := \sigma_{x,y} \in \mathcal{PS}$

be a geodesic joining x and y. Let $x_1 \in \delta$ be a point closest to $\pi_{\mathcal{A}}(x)$, that is, $x_1 \in \rho_\delta(\pi_{\mathcal{A}}(x))$. There are two cases to consider.

Case 1: If $\mathrm{d}(x_1, \pi_{\mathcal{A}}(x)) > C$, then $\mathrm{d}(\delta, \pi_{\mathcal{A}}(x)) > C$. Since $\mathcal{A}$ is $\mathcal{PS}$-contracting (in the sense of Sisto), this implies that $\mathrm{d}(\pi_{\mathcal{A}}(x), \pi_{\mathcal{A}}(y)) < C$.

Case 2: If $\mathrm{d}(x_1, \pi_{\mathcal{A}}(x)) \leq C$, then by Lemma III.25, $\mathrm{d}(x_1, \rho_{\mathcal{A}}(x)) \leq 3C$. Thus

$$\mathrm{d}(x, \rho_{\mathcal{A}}(x)) \leq \mathrm{d}(x, x_1) + \mathrm{d}(x_1, \rho_{\mathcal{A}}(x)) \leq \mathrm{d}(x, x_1) + 3C.$$

Since $y \in B(x, R)$ and $R < \mathrm{d}(x, \rho_{\mathcal{A}}(x))$,

$$\mathrm{d}(y, x_1) = \mathrm{d}(y, x) - \mathrm{d}(x, x_1) \leq \mathrm{d}(x, \rho_{\mathcal{A}}(x)) - \mathrm{d}(x, x_1) \leq 3C.$$

Then,

$$\mathrm{d}(y, \rho_{\mathcal{A}}(y)) \leq \mathrm{d}(y, \pi_{\mathcal{A}}(x)) \leq \mathrm{d}(y, x_1) + \mathrm{d}(x_1, \pi_{\mathcal{A}}(x)) \leq 4C.$$

Then

$$\mathrm{d}(\rho_{\mathcal{A}}(y), \rho_{\mathcal{A}}(x)) \leq \mathrm{d}(\rho_{\mathcal{A}}(y), y) + \mathrm{d}(y, x_1) + \mathrm{d}(x_1, \rho_{\mathcal{A}}(x)) \leq 10C$$

Thus, if $x \in X$ and $0 < R < \mathrm{d}(x, \rho_{\mathcal{A}}(x))$, $\mathrm{diam}(\rho_{\mathcal{A}}(B(x, R)) \leq 20C$.

(2) Follows from definitions and part (1). $\square$

CHAPTER IV

Rank One Hilbert Geometries

This chapter is based on results that appear in [Isl19].

4.1 Axis of Automorphisms

Definition IV.1. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $g \in \mathrm{Aut}(\Omega)$. An *axis* of g is a non-trivial projective line segment $\ell_g := \mathbb{P}(V_g) \cap \Omega$ where $V_g \leq \mathbb{R}^d$ is a two-dimensional g-invariant linear subspace.

If ℓ_g is an axis of g, then $\overline{\ell_g} \cap \partial\Omega$ consists of two points both of which are fixed points of g. Let $\overline{\ell_g} \cap \partial\Omega = \{g_+, g_-\}$. Assume that $\tau_\Omega(g) > 0$ and recall the notation $E_g^+, E_g^- \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ from Section 2.6.1.

Lemma IV.2. *Suppose Ω is a properly convex domain and $g \in \mathrm{Aut}(\Omega)$ such that $\tau_\Omega(g) > 0$ and g has an axis $\ell_g = (g_+, g_-)$. Assume that $\widetilde{g}$, $\widetilde{g_+}$, and $\widetilde{g_-}$ are lifts of g, g_+, and g_- respectively such that*

$$\widetilde{g} \cdot \widetilde{g_+} = \lambda_+ \widetilde{g_+} \quad \textit{and} \quad \widetilde{g} \cdot \widetilde{g_-} = \lambda_- \widetilde{g_-}$$

where $\lambda_+, \lambda_- \in \mathbb{R}$ and $|\lambda_-| \leq |\lambda_+|$. Then

(1) $|\lambda_+| = \lambda_{\max}(\widetilde{g})$, $|\lambda_-| = \lambda_{\min}(\widetilde{g})$, $\tau_\Omega(g) = \log\left(\left|\dfrac{\lambda_+}{\lambda_-}\right|\right)$,

(2) $g_+ \in E_g^+$, and $g_- \in E_g^-$.

Proof. Let $W = \mathrm{Span}\{g_+, g_-\}$. Then $\widetilde{g}\big|_W$ can be conjugated to $\begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$ in $\mathrm{PGL}_2(\mathbb{R})$. Then $\tau_\Omega(g) \leq \tau_{\Omega \cap \mathbb{P}(W)}(g\big|_{\Omega \cap \mathbb{P}(W)})$ which implies that

$$\log \frac{\lambda_{\max}}{\lambda_{\min}}(\widetilde{g}) \leq \log \left| \frac{\lambda_+}{\lambda_-} \right|.$$

But $|\lambda_+| \leq \lambda_{\max}(\widetilde{g})$ while $|\lambda_-| \geq \lambda_{\min}(\widetilde{g})$. Thus

$$\left| \frac{\lambda_+}{\lambda_-} \right| = \frac{\lambda_{\max}}{\lambda_{\min}}(\widetilde{g}).$$

Then $|\lambda_+| \leq \lambda_{\max}(\widetilde{g})$ and $|\lambda_-| \geq \lambda_{\min}(\widetilde{g})$ implies that $|\lambda_+| = \lambda_{\max}(\widetilde{g})$ and $|\lambda_-| = \lambda_{\min}(\widetilde{g})$. This finishes the proof. ☐

An isometry $g \in \mathrm{Aut}(\Omega)$ may or may not have any axis. Hence we introduce the notion of a pseudo-axis.

Definition IV.3. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $g \in \mathrm{Aut}(\Omega)$. A *pseudo-axis* of g is a non-trivial projective line segment $\sigma_g := \mathbb{P}(W_g) \cap \overline{\Omega}$ where $W_g \leq \mathbb{R}^d$ is a two-dimensional linear subspace and $\mathbb{P}(W_g) \cap \Omega = \emptyset$.

Observation IV.4. *If $\tau_\Omega(g) > 0$, then either g has an axis or a pseudo-axis.*

This observation essentially comes from the following result of Benoist (also see [Mar14, Proposition 2.2]). For stating the result, we will require some notation. If Ω is a properly convex domain, then a cone above Ω is a connected component of $\pi^{-1}(\Omega) := \{v \in \mathbb{R}^d : \pi(v) \in \Omega\}$. We will denote a cone above Ω by $\widetilde{\Omega}$. In other words, $\pi^{-1}(\Omega) = \widetilde{\Omega} \sqcup -\widetilde{\Omega}$ where $\widetilde{\Omega}$ and $-\widetilde{\Omega}$ are the connected components of $\pi^{-1}(\Omega)$. Note that if $g \in \mathrm{Aut}(\Omega)$, then we can choose a lift $\widetilde{g}$ of g such that $\widetilde{g} \cdot \widetilde{\Omega} = \widetilde{\Omega}$. Indeed, if a lift $\widetilde{g}$ does not preserve $\widetilde{\Omega}$, then $\widetilde{g} \cdot \widetilde{\Omega} = -\widetilde{\Omega}$ which implies that $-\widetilde{g}$ preserves $\widetilde{\Omega}$. Now we state Benoist's result.

Proposition IV.5 ([Ben05, Lemma 3.2]). *Suppose Ω is a properly convex domain, $g \in \mathrm{Aut}(\Omega)$, and $\tau_\Omega(g) > 0$. Assume that $\widetilde{g}$ is a lift of g such that $\widetilde{g} \cdot \widetilde{\Omega} = \widetilde{\Omega}$ where $\widetilde{\Omega}$ is a cone above Ω. Then $\lambda_{\max}(\widetilde{g})$ is an eigenvalue of $\widetilde{g}$ and*

$$\mathbb{P}\Big(\ker\left(\widetilde{g} - \lambda_{\max}(\widetilde{g}) \cdot \mathrm{Id}\right)\Big) \cap \overline{\Omega} \neq \emptyset.$$

The same remains true if we replace $\lambda_{\max}(\widetilde{g})$ by $\lambda_{\min}(\widetilde{g})$.

We will now discuss a few examples to illustrate the notions introduced. An isometry may have a unique axis, infinitely many axes, or no axes at all. An isometry can have pseudo-axes without having an axis and vice versa.

Example A. (Unique axis, no pseudo-axes) Consider the properly convex domain Ω in $\mathbb{P}(\mathbb{R}^3)$ which is an open ball in an affine chart. It is the projective model of $\mathbb{H}^2$ and $\mathrm{Aut}(\Omega) = \mathrm{PO}(2,1)$. If $g \in \mathrm{SO}(2,1)$ has $\tau_\Omega([g]) > 0$ (i.e. is a hyperbolic isometry in $\mathrm{Isom}(\mathbb{H}^2)$), then $[g]$ has a unique axis. This is essentially because Ω is a strictly convex domain.

Example B. Consider the two-dimensional simplex S_2.

Uncountably many axes, several pseudo-axes: Let $g_2 = [\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)]$ where $\lambda_1 > \lambda_2 > 0$ and $\lambda_1\lambda_2^2 = 1$. For $0 \leq t \leq 1$, let $Q_t := (e_1, te_2 + (1-t)e_3)$. Then, $\{Q_t\}_{t\in(0,1)}$ is an uncountable family of axes of g_2. There are three pseudo-axes: $[e_1, e_2]$, $[e_2, e_3]$, and $[e_1, e_3]$.

Several pseudo-axes, no axis: Let $g_1 := [\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)]$ where $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_1\lambda_2\lambda_3 = 1$. The pseudo-axes of g_1 are $[e_1, e_2]$, $[e_2, e_3]$ and $[e_1, e_3]$. But g_1 does not have an axis.

We conclude this section by establishing three lemmas that will be used in the next section. Recall the notation E_g^+, E_g^- from Section 2.6.1.

The first lemma is a simple consequence of Lemma IV.2.

Lemma IV.6 ([Isl19, Lemma 5.8]). *Suppose Ω is a properly convex domain, $g \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(g) > 0$, and a, b, c are three fixed points of g. If $a \in E_g^+$, $b \in E_g^-$ and $c \notin E_g^+ \cup E_g^-$, then $[a, c] \cup [b, c] \subset \partial\Omega$.*

Proof. If $(a, c) \subset \Omega$, then (a, c) is an axis of g by definition. Then Lemma IV.2 implies that $c \in E_g^-$ which is a contradiction. Thus $[a, c] \subset \partial\Omega$. Similarly, $[b, c] \subset \partial\Omega$. $\square$

The next lemma shows that if $g \in \mathrm{Aut}(\Omega)$ (with $\tau_\Omega(g) > 0$) has an axis (a, b) and $\#(E_g^+) > 1$, then $F_\Omega(a)$ conatins a non-trivial projective line segment in $\partial\Omega$.

Lemma IV.7 ([Isl19, Lemma 5.6]). *Suppose Ω is a properly convex domain, $g \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(g) > 0$, and g has an axis $\ell_g = (g_+, g_-)$ with $g_+ \in E_g^+$ and $g_- \in E_g^-$. If $u \in E_g^+ \setminus \{g_+\}$, then*

$$\mathrm{diam}_{F_\Omega(g_+)} \left(F_\Omega(g_+) \cap \mathbb{P}(\mathrm{Span}\{g_+, u\})\right) > 0.$$

Remark IV.8.

(1) The conclusion of this lemma is that $F_\Omega(g_+) \cap \mathbb{P}(\mathrm{Span}\{g_+, u\})$ is a non-trivial projective line segment in $\partial\Omega$ containing g_+.

(2) The same result is true if we replace g by g^{-1} and g_+ by g_-.

Proof. Let $\widetilde{\Omega}$ be a cone over Ω. Fix the lifts $\widetilde{g}$, $\widetilde{u}$, $\widetilde{g_+}$, $\widetilde{g_-}$ of g, u, g_+, g_- such that $\widetilde{g} \cdot \widetilde{\Omega} = \widetilde{\Omega}$ and $\widetilde{u}, \widetilde{g_+}, \widetilde{g_-} \in \widetilde{\Omega}$. Since $u \in E_g^+$, there exist a sequence of positive integers $\{m_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \to \infty} \left(\frac{\widetilde{g}}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \widetilde{u} = \widetilde{u}.$$

Let $\widetilde{p_t} := \dfrac{\widetilde{g_+} + \widetilde{g_-}}{2} + t\widetilde{u}$. Then $\pi(\widetilde{p_0}) \in (g_+, g_-) \subset \Omega$. Since Ω is open, there exists

$\varepsilon > 0$ such that $p_t := \pi(\widetilde{p_t}) \in \Omega$ for all $t \in (-\varepsilon, \varepsilon)$. Then, if $t \in (-\varepsilon, \varepsilon)$

$$\begin{aligned} p_{t,\infty} &:= \lim_{k\to\infty} g^{m_k} p_t \\ &= \lim_{k\to\infty} \pi\left(\left(\frac{\widetilde{g}}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \widetilde{p_t}\right) = \lim_{k\to\infty} \pi\left(\frac{\widetilde{g_+}}{2} + \left(\frac{\lambda_{\min}(\widetilde{g})}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \frac{\widetilde{g_-}}{2} + t\left(\frac{\widetilde{g}}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \widetilde{u}\right) \\ &= \pi\left(\widetilde{g_+} + 2t\widetilde{u}\right). \end{aligned}$$

Note that $p_{0,\infty} = g_+$. By Proposition II.12,

$$\mathrm{d}_{F_\Omega(g_+)}(g_+, p_{t,\infty}) \leq \liminf_{k\to\infty} \mathrm{d}_\Omega(g^{m_k} p_0, g^{m_k} p_t) = \mathrm{d}_\Omega(p_0, p_t) < \infty.$$

Thus for all $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$, $p_{t,\infty} \in F_\Omega(g_+)$ and $p_{t,\infty} \neq g_+$. This finishes the proof. $\square$

The next lemma shows that if $\gamma \in \mathrm{Aut}(\Omega)$ has an axis and $\#(E_\gamma^-) = 1$, then γ^{-1} is a proximal element in $\mathrm{PGL}_d(\mathbb{R})$ (i.e. $\frac{\lambda_1}{\lambda_2}(\gamma^{-1}) > 1$).

Lemma IV.9. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a Hilbert geometry and $\gamma \in \mathrm{Aut}(\Omega)$ where $\tau_\Omega(\gamma) > 0$, and γ has an axis. If $\#(E_\gamma^-) = 1$, then $\frac{\lambda_{d-1}}{\lambda_d}(\gamma) > 1$.*

Remark IV.10. *Replacing γ by γ^{-1}, we see that $\#(E_\gamma^+) = 1$ implies $\frac{\lambda_1}{\lambda_2}(\gamma) > 1$.*

Proof. Suppose the axis of γ is (a, b) with $a \in E_\gamma^+$ and $b \in E_\gamma^-$. Let us fix $\widetilde{\Omega}$, a cone over Ω. Fix lifts $\widetilde{\gamma}$, $\widetilde{a}$ and $\widetilde{b}$ where $\widetilde{a}, \widetilde{b} \in \widetilde{\Omega}$ and $\widetilde{\gamma} \cdot \widetilde{\Omega} = \widetilde{\Omega}$. Set $\lambda_{\max} := \lambda_{\max}(\widetilde{\gamma})$ and $\lambda_{\min} := \lambda_{\min}(\widetilde{\gamma})$. Since $\widetilde{\gamma} \cdot \widetilde{\Omega} = \widetilde{\Omega}$, $b \in E_\gamma^-$ and $\widetilde{b} \in \widetilde{\Omega}$, we have

$$\widetilde{\gamma} \cdot \widetilde{b} = \lambda_{\min} \cdot \widetilde{b}.$$

Similarly, $\widetilde{\gamma} \cdot \widetilde{a} = \lambda_{\max} \cdot \widetilde{a}$.

Since $\#(E_\gamma^-) = 1$, there is a one-dimensional eigenspace of $\widetilde{\gamma}$ (namely $\mathbb{R}\widetilde{b}$) and a single Jordan block $J_{\min}$ corresponding to eigenvalues of modulus $\lambda_{\min}$. Thus in order to prove $\frac{\lambda_{d-1}}{\lambda_d}(\gamma) > 1$, it is enough to show that the Jordan block $J_{\min}$ has size 1. Suppose this is false. Then there exists $\widetilde{w} \in \mathbb{R}^{d+1}$ such that if $k \in \mathbb{Z}$, then

$$\widetilde{\gamma}^k \widetilde{w} = k\lambda_{\min}^{k-1}\widetilde{b} + \lambda_{\min}^k \widetilde{w}. \tag{4.1}$$

Setting $w := \pi(\widetilde{w})$, $\lim_{k\to\infty} \gamma^k w = b$. Since $\gamma^k a = a$ for all k, $\lim_{k\to\infty} \gamma^k[a, w] = [a, b]$. Fix $p \in (a, b) \subset \Omega$. Then there exists $y_k \in (a, w)$ such that

$$\lim_{k\to\infty} \gamma^k y_k = p. \tag{4.2}$$

Since $p \in \Omega$ and Ω is open, $\gamma^k y_k \in \Omega$ for k large enough. Thus, up to truncating finitely many terms of the sequence $\{y_k\}$, we can assume that for $k \geq 1$,

$$y_k \in (a, w) \cap \Omega.$$

Let us fix lifts $\widetilde{y}_k'$ of y_k in $\widetilde{\Omega}$. Then $\widetilde{y_k'} = c_k'\widetilde{a} + d_k'\widetilde{w}$ where $c_k', d_k' \geq 0$ and $c_k' + d_k' > 0$. Setting $\widetilde{y}_k := \widetilde{y}_k'/(c_k' + d_k')$, we get $c_k, d_k \in [0, 1]$ such that

$$\widetilde{y}_k := c_k\widetilde{a} + d_k\widetilde{w}. \tag{4.3}$$

Thus, upto passing to a subsequence, we can assume that $c_\infty := \lim_{k\to\infty} c_k$ and $d_\infty := \lim_{k\to\infty} d_k$ exist. Then $\widetilde{y_\infty} := \lim_{k\to\infty} \widetilde{y}_k$ exists and set

$$y_\infty := \pi(\widetilde{y_\infty}) = \pi(c_\infty\widetilde{a} + d_\infty\widetilde{w}).$$

We now claim that $y_\infty = w$. If this is not true, then $c_\infty \neq 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $c_k > 0$ for all $k > k_0$ and $\lim_{k\to\infty}(d_k/c_k) = d_\infty/c_\infty$ exists in $\mathbb{R}$. Then using equations (4.2) and (4.3),

$$\begin{aligned}
p = \lim_{k\to\infty} \gamma^k y_k &= \lim_{k\to\infty} \pi\left(\frac{\widetilde{\gamma}^k \widetilde{y}_k}{c_k \lambda_{\max}^k}\right) \\
&= \lim_{k\to\infty} \pi\left(\widetilde{a} + \frac{d_k}{c_k}\left(\frac{k}{\lambda_{\max}}\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{k-1}\widetilde{b} + \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^k \widetilde{w}\right)\right) \\
&= \pi(\widetilde{a}) = a.
\end{aligned}$$

This is a contradiction since $p \in \Omega$ while $a \in \partial\Omega$. Thus $y_\infty = w$.

Since $y_k \in \Omega$ for $k \geq 1$, $w = y_\infty$ implies that $w \in \overline{\Omega}$. Then for all $k \in \mathbb{Z}$,

$$[w, \gamma^k w] \subset \overline{\Omega}. \tag{4.4}$$

For $t > 0$, let

$$\mathcal{H}_t := \left\{ \pi\left(\widetilde{w} + r\widetilde{b}\right) : -t \leq r \leq t \right\}.$$

Let $\mathcal{H} := \cup_{t>0} \mathcal{H}_t$ and note that

$$\overline{\mathcal{H}} = \mathbb{P}(\mathrm{Span}\{b, w\}).$$

Observe that if $k \in \mathbb{Z}$, then equation (4.1) implies that

$$\gamma^k w = \pi\left(\frac{\widetilde{\gamma}^k \widetilde{w}}{\lambda_{\min}^k}\right) = \pi\left(\widetilde{w} + \frac{k}{\lambda_{\min}}\widetilde{b}\right).$$

For every $t > 0$, set $k_t := \lceil \lambda_{\min} t \rceil$. Then

$$\mathcal{H}_t \subset [\gamma^{-(k_t - 1)} w, w] \cup [w, \gamma^{k_t} w].$$

Then

$$\mathbb{P}(\mathrm{Span}\{w, b\}) = \overline{\mathcal{H}} \subset \bigcup_{t>0} \overline{\mathcal{H}_t} \subset \overline{\Omega},$$

where the last containment comes from equation (4.4). Thus $\overline{\Omega}$ contains the projective line $\overline{\mathcal{H}}$, which is impossible as Ω is a properly convex domain. Hence we have a contradiction. $\square$

4.2 Definition of Rank One

In this section, we introduce the notion of rank one automorphisms for Hilbert geometries following Ballmann-Brin's definition for CAT(0) spaces [Bal82, BB95].

Some of the dynamical properties of rank one automorphisms for Hilbert geometries are reminiscent of Ballmann's early results in rank-one Riemannian non-positive curvature [Bal82, Bal95].

We first introduce the notion of half triangles which are analogues of half flats used in the CAT(0) setting.

Definition IV.11. If Ω is a properly convex domain, then three points $x, y, z \in \partial\Omega$ form a *half triangle* if

$$[x, y] \cup [y, z] \subset \partial\Omega \text{ and } (x, z) \subset \Omega.$$

A projective line segment $(x, y) \subset \Omega$ is said to be contained in a half triangle in Ω if there exists a point $z \in \partial\Omega \setminus \{x, y\}$ such that x, y, z form a half triangle.

We now introduce the notion of rank one automorphisms for Hilbert geometries.

Definition IV.12. Suppose Ω is a properly convex domain. Then $\gamma \in \mathrm{Aut}(\Omega)$ is called *rank one automorphism* if:

(1) $\tau_\Omega(\gamma) = \log \dfrac{\lambda_1}{\lambda_d}(\gamma) > 0$ and γ has an axis

(2) none of the axes ℓ_γ of γ are contained in a half triangle in Ω.

A projective line segment $\ell \subset \Omega$ is a *rank one axis* if ℓ is the axis of a rank one automorphism $\gamma \in \mathrm{Aut}(\Omega)$.

Using the definition of rank one automorphisms, we now introduce rank one Hilbert geometries and rank one groups.

Definition IV.13.

(1) A pair (Ω, Λ) is called a *rank one Hilbert geometry* if Ω is a properly convex domain, $\Lambda \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup, and Λ contains a rank one automorphism.

(2) A discrete group $\Lambda \leq \mathrm{PGL}_d(\mathbb{R})$ is called a *rank one group* if there exists a Hilbert geometry Ω_Λ such that $(\Omega_\Lambda, \Lambda)$ is a rank one Hilbert geometry.

A prime example of a rank one Hilbert geometry is the projective model $\Omega_0 \subset \mathbb{P}(\mathbb{R}^3)$ of the real hyperbolic space $\mathbb{H}^2$. Consider a discrete subgroup $\Lambda_1 \leq \mathrm{Aut}(\Omega_0) = \mathrm{PO}(2,1)$ that contains a hyperbolic isometry (i.e. it has a positive translation distance in $\mathbb{H}^2$). Then (Ω_0, Λ_1) is a rank one Hilbert geometry. A prime non-example of rank one is the two-dimensional projective simplex S_2. Suppose $\Lambda_2 := \{[\mathrm{diag}(2^m, 2^n, 2^{-(m+n)})] : m, n \in \mathbb{Z}\} \leq \mathrm{PGL}_3(\mathbb{R})$. Then (S_2, Λ_2) is non-rank one Hilbert geometry.

4.3 Properties of Rank One Automorphisms

In this section, we will establish some key geometric and dynamical properties of rank one automorphisms. We will use the following terminology: $g \in \mathrm{PGL}_d(\mathbb{R})$ is called biproximal if $\dfrac{\lambda_1}{\lambda_2}(g) > 1$ and $\dfrac{\lambda_{d-1}}{\lambda_d}(g) > 1$.

Proposition IV.14 ([Isl19, Proposition 6.3])**.** *Suppose γ is a rank one automorphism with a rank-one axis $\ell_\gamma = (a, b)$ where $a \in E_\gamma^+$ and $b \in E_\gamma^-$. Then:*

(1) γ is biproximal,

(2) ℓ_γ is the unique axis of γ in Ω,

(3) the only fixed points of γ in $\overline{\Omega}$ are a and b,

(4) if $z' \in \partial\Omega \setminus \{a, b\}$, then $(a, z') \cup (b, z') \subset \Omega$,

(5) if $z \in \partial\Omega \setminus \{a, b\}$, then neither (a, z) nor (b, z) is contained in a half triangle in Ω.

Proof. Let $\widetilde{\Omega}$ be cone above Ω. For the rest of this proof, fix lifts $\widetilde{\gamma}$, $\widetilde{a}$ and $\widetilde{b}$ where $\widetilde{a}, \widetilde{b} \in \widetilde{\Omega}$ and $\widetilde{\gamma} \cdot \widetilde{\Omega} = \widetilde{\Omega}$. Set $\lambda_{\max} := \lambda_{\max}(\widetilde{\gamma})$ and $\lambda_{\min} := \lambda_{\min}(\widetilde{\gamma})$. Note that $\widetilde{a}$ is an eigenvector of $\widetilde{\gamma}$ corresponding to the eigenvalue $\lambda_{\max}$ or $-\lambda_{\max}$. But since $\widetilde{\gamma}\cdot\widetilde{\Omega}=\widetilde{\Omega}$,

$$\widetilde{\gamma}\ \cdot\ \widetilde{a}=\lambda_{\max}\ \cdot\widetilde{a}.$$

Thus $a\in E_{\gamma}^{+}$ and $\#(E_{\gamma}^{+})\geq 1$. Similarly, $\widetilde{\gamma}\ \cdot\ \widetilde{b}=\lambda_{\min}\ \cdot\widetilde{b}$, $b\in E_{\gamma}^{-}$ and $\#(E_{\gamma}^{-})\geq 1$.

(1) In order to prove that γ is biproximal, we first prove the following.

Claim IV.15. $\#(E_{\gamma}^{+})=\#(E_{\gamma}^{-})=1$.

Proof of Claim. It suffices to prove the claim for E_{γ}^{+} since the same arguments with γ replaced by γ^{-1} implies the result for E_{γ}^{-}. Now suppose the claim is false and there exists $u\in E_{\gamma}^{+}\setminus\{a\}$. Then Lemma IV.7 implies that there exist $z^{-},z^{+}\in\partial\Omega$ such that $a\in(z^{-},z^{+})$ and

$$F_{\Omega}(a)\cap\mathbb{P}(\mathrm{Span}\{a,u\}))=(z^{-},z^{+}).$$

Then, $\mathcal{I}_{z}:=\left[z_{-},z_{+}\right]$ is the maximal projective line segment in $\partial\Omega$ containing both z_{-} and z_{+}.

Since γ is a rank-one isometry, its axis (a,b) cannot be contained in a half triangle in Ω. But $[a,z_{+}]\subset\partial\Omega$ which implies that $(z_{+},b)\subset\Omega$. Similarly, $(z_{-},b)\subset\Omega$. Choose $x_{+}\in(z_{+},b)\cap\Omega$ and $x_{-}\in(z_{-},b)\cap\Omega$. By Proposition II.33 part (3), there exists an unbounded sequence $\{m_{k}\}$ of positive integers such that

$$\lim_{k\to\infty}\left(\gamma\big|_{E_{\gamma}^{+}}\right)^{m_{k}}=\mathrm{Id}_{E_{\gamma}^{+}}.$$

Since $z_{+}\in\mathbb{P}(\mathrm{Span}\{a,u\})$, $z_{+}\in E_{\gamma}^{+}$. Fix a lift $\widetilde{z_{+}}\in\widetilde{\Omega}$ of z_{+}. Then

$$\lim_{k\to\infty}\left(\frac{\widetilde{\gamma}}{\lambda_{\max}}\right)^{m_{k}}\widetilde{z_{+}}=\widetilde{z_{+}}.$$

On the other hand, $\left(\frac{\widetilde{\gamma}}{\lambda_{\min}}\right)^{m_{k}}\widetilde{b}=\widetilde{b}$. Then, since $x_{+}\in(z_{+},b)$,

$$\lim_{k\to\infty}\gamma^{m_{k}}x_{+}=z_{+}.$$

Similarly

$$\lim_{k\to\infty} \gamma^{m_k} x_- = z_-.$$

Since $\lim_{k\to\infty} \mathrm{d}_\Omega(\gamma^{m_k}x_+, \gamma^{m_k}x_-) = \mathrm{d}_\Omega(x_+, x_-) < \infty$, Proposition II.12 implies that $z_+ \in F_\Omega(z_-)$. Thus there is an open projective line segment in $\partial\Omega$ containing both z_+ and z_-. This contradicts the maximality of $\mathcal{I}_z$ and finishes the proof of Claim IV.15.

By the above claim $\#(E_\gamma^+) = \#(E_\gamma^-) = 1$ where $\tau_\Omega(\gamma) > 0$ and γ has an axis (a, b). Then Lemma IV.9 implies that γ is biproximal.

(2) This follows from biproximality of γ.

(3) Suppose c is a fixed point of γ in $\partial\Omega$ that is distinct from both a and b. By part (1) of this Proposition, γ is biproximal. Thus $c \notin E_\gamma^+ \cup E_\gamma^-$. Then, by Lemma IV.6, $[a, c] \subset \partial\Omega$ and $[b, c] \subset \partial\Omega$. Thus, the axis $\ell_\gamma = (a, b)$ of γ is contained in a half triangle, contradicting that γ is a rank one automorphism.

(4) Let $v \in \partial\Omega \setminus \{a, b\}$. Suppose $[a, v] \subset \partial\Omega$. Since γ is biproximal, there exists a γ invariant decomposition of $\mathbb{R}^d$ given by:

$$\mathbb{R}^d = \mathbb{R}\widetilde{a} \oplus \mathbb{R}\widetilde{b} \oplus \widetilde{E}.$$

Choose any lift $\widetilde{v}$ of v. Then $\widetilde{v}$ decomposes as

$$\widetilde{v} = c_1\widetilde{a} + c_2\widetilde{b} + \widetilde{v_0}$$

where $c_1, c_2 \geq 0$ and $\widetilde{v_0} \neq 0$ (since $v \in \partial\Omega \setminus \{a, b\}$). If $c_2 \neq 0$, then $\lim_{n\to\infty} \gamma^{-n} v = b$, that is, $\lim_{n\to\infty} \gamma^{-n}[a, v] = [a, b]$. Since $[a, v] \subset \partial\Omega$ (by assumption) and $\partial\Omega$ is $\mathrm{Aut}(\Omega)$ invariant, $[a, b] \subset \partial\Omega$. This is a contradiction since $(a, b) \subset \Omega$. Thus, $c_2 = 0$.

Set $\lambda_{\widetilde{E}} := \lambda_{\max}\left(\widetilde{\gamma}\big|_{\widetilde{E}}\right)$. Since γ is biproximal, $\lambda_{\widetilde{E}} < \lambda_{\max}$. Then, for every $n > 0$,

$$\left(\frac{\widetilde{\gamma}}{\lambda_{\widetilde{E}}}\right)^{-n} \widetilde{v} = c_1\left(\frac{\lambda_{\max}}{\lambda_{\widetilde{E}}}\right)^{-n} \widetilde{a} + \left(\frac{\widetilde{\gamma}\big|_{\widetilde{E}}}{\lambda_{\widetilde{E}}}\right)^{-n} \widetilde{v_0}.$$

Then, up to passing to a subsequence, we can assume that $v_\infty := \lim_{n\to\infty} \gamma^{-n} v$ exists and $v_\infty \in \overline{E_\Omega}$, where $E_\Omega = \Omega \cap \mathbb{P}(\widetilde{E})$. Then $\overline{E_\Omega}$ is a non-empty convex compact subset of $\mathbb{R}^d$ and Brouwer's fixed point theorem implies that γ has a fixed point in $\overline{E_\Omega}$. But $\overline{E_\Omega} \subset \overline{\Omega} \setminus \{a, b\}$. This contradicts part (3). Hence, $(a, v) \subset \Omega$. Similarly we can show that $(b, v) \subset \Omega$.

(5) This is a consequence of part (4). $\square$

We can prove a simpler characterization of rank one automorphisms for co-compact actions.

Lemma IV.16 ([Isl19, Proposition 6.4])**.** *Suppose Ω is a properly convex domain, $\Lambda \leq \mathrm{Aut}(\Omega)$ is a discrete group that acts co-compactly on Ω, and $\gamma \in \Lambda$ with $\tau_\Omega(\gamma) > 0$. If γ has an axis, then the following are equivalent:*

(1) γ is biproximal.

(2) none of the axes of γ are contained in half triangles in Ω.

(3) γ is a rank one automorphism.

Proof. Note that (2) $\iff$ (3) is by definition (cf. IV.12) and (3) $\implies$ (1) is Proposition IV.14 part (1). We will prove (1) $\implies$ (2), under the assumption that Ω/Λ is compact.

Let (a, b) be the axis of γ with $a \in E_\gamma^+$ and $b \in E_\gamma^-$. We first show that γ has no other fixed points in $\partial\Omega$ except a and b. If this is not true, let v be such a fixed point of γ. Since γ is biproximal, $v \notin E_\gamma^+ \cup E_\gamma^-$. Then Lemma IV.6 implies that

$$[a, v] \cup [v, b] \subset \partial\Omega. \tag{4.5}$$

Let $A_\gamma := \langle \gamma \rangle$. Recall the notation $\mathrm{Min}_\Omega(A_\gamma) = \cap_{h \in A_\gamma} \{x \in \Omega : \mathrm{d}_\Omega(x, h \cdot x) = \tau_\Omega(h)\}$.

Claim: $\mathrm{ConvHull}_\Omega\{a, v, b\} \subset \mathrm{Min}_\Omega(A_\gamma)$.

Proof of Claim. If $x \in \mathrm{ConvHull}_\Omega\{a, v, b\}$, there exists $y \in (a, b)$ such that $x \in [y, v)$. Since $\gamma v = v$, $\gamma x \in [\gamma y, v)$. Let $L_{x',x''}$ denote the Euclidean line in the affine chart through x' and x''. Then $L_{v,a}$, $L_{v,y}$, $L_{v,\gamma y}$ and $L_{v,b}$ are four distinct lines concurrent at v. Both $L_{x,\gamma x}$ and $L_{y,\gamma y}$ intersect these four lines and do not pass through v. Then, by Proposition II.2, $\mathrm{d}_\Omega(x, \gamma x) = \mathrm{d}_\Omega(y, \gamma y)$. But since $y \in (a, b)$, $\mathrm{d}_\Omega(y, \gamma y) = \tau_\Omega(\gamma)$ which implies that $x \in \mathrm{Min}_\Omega(A_\gamma)$. This finishes the proof of this claim.

The group Λ acts co-compactly on Ω. Then, Theorem I.18 implies that $C_\Lambda(A_\gamma)$ acts co-compactly on $\mathrm{ConvHull}_\Omega(\mathrm{Min}_\Omega(A_\gamma))$. Fix $p \in (a, b)$ and choose $v_n \in [p, v)$ such that $\lim_{n\to\infty} v_n = v$. By the above claim, $v_n \in \mathrm{Min}_\Omega(A_\gamma)$. Then there exists $h_n \in C_\Lambda(A_\gamma)$ such that $q := \lim_{n\to\infty} h_n v_n$ exists in Ω. Thus $\lim_{n\to\infty} \mathrm{d}_\Omega(h_n^{-1}q, v_n) = 0$. Then Proposition II.12 implies that, up to passing to a subsequence,

$$\lim_{n\to\infty} h_n^{-1}q = \lim_{n\to\infty} v_n = v.$$

Pick a point $q' \in (a, b)$. Up to passing to a subsequence, $v' := \lim_{n\to\infty} h_n^{-1}q'$ exists in $\overline{\Omega}$. Since $\lim_{n\to\infty} \mathrm{d}_\Omega(h_n^{-1}q, h_n^{-1}q') = \mathrm{d}_\Omega(q, q')$, Proposition II.12 implies that $v \in F_\Omega(v')$. Now we show that $v' \in \{a, b\}$. Since $h_n \in C_\Lambda(A_\gamma)$, $h_n(a, b)$ is an axis of γ. As γ is biproximal, Lemma IV.2 implies that $h_n(a, b) = (a, b)$. Thus $v' \in \{a, b\}$. Hence

$$v \in F_\Omega(a) \cup F_\Omega(b).$$

By equation 4.5, $[a, v] \cup [v, b] \subset \partial\Omega$. Now, by Proposition II.11 part (4), $v \in F_\Omega(a)$ and $[v, b] \subset \partial\Omega$ implies that $[a, b] \subset \partial\Omega$. This is a contradiction as $(a, b) \subset \Omega$. Thus, $v \notin F_\Omega(a)$. By a similar reasoning, $v \notin F_\Omega(b)$. Thus we have a contradiction.

So we have shown that if γ has an axis (a, b) and is biproximal, then γ has no fixed points in $\partial\Omega$ other than a and b. Then the proof of Proposition IV.14 part (4) goes through verbatim. Thus $(a, z) \cup (z, b) \subset \Omega$ for all $z \in \partial\Omega \setminus \{a, b\}$, that is, the axis (a, b) cannot be contained in a half triangle in Ω. This finishes the proof. $\square$

4.4 Rank One Axis and Slim Triangles

A projective geodesic triangle in a properly convex domain Ω is defined as follows: if $v_1, v_2, v_3 \in \Omega$, let $\Delta(v_1, v_2, v_3) := [v_1, v_2] \cup [v_2, v_3] \cup [v_3, v_1]$. Recall the notion of slim triangles in a geodesic metric space from III.5. In this section, we will prove that any projective geodesic triangle in Ω with one of its edges on a rank one axis ℓ is $\mathcal{D}_\ell$-slim for some constant $\mathcal{D}_\ell$.

Theorem IV.17 ([Isl19, Theorem 8.1])**.** *If ℓ is a rank one axis in a properly convex domain Ω, then there exists a constant $\mathcal{D}_\ell \geq 0$ such that: if $\Delta(x, y, z)$ is a projective geodesic triangle in Ω with $[y, z] \subset \ell$, then $\Delta(x, y, z)$ is $\mathcal{D}_\ell$-slim.*

Remark IV.18. *The constant $\mathcal{D}_\ell$ depends only on the axis ℓ (and not on a rank one automorphism whose axis is ℓ).*

In order to simplify the proof, we first establish a simple criteria for determining when a geodesic triangle in Ω is D-slim.

Lemma IV.19 ([IZ19, Lemma 13.8])**.** *If $R \geq 0$ and $\Delta(x, y, z)$ is projective geodesic triangle in Ω with $[y, z] \subset \mathcal{N}_R\big([x, y] \cup [x, z]\big)$, then $\Delta(x, y, z)$ is $(2R)$-slim.*

Proof of Lemma. Since $[y, z] \subset \mathcal{N}_R\big([x, y] \cup [x, z]\big)$, there exists $m_{yz} \in [y, z]$, $m_{x,y} \in [x, y]$ and $m_{xz} \in [x, z]$ such that $\mathrm{d}_\Omega(m_{yz}, m_{xy}) \leq R$, and $\mathrm{d}_\Omega(m_{yz}, m_{xz}) \leq R$. By Proposition II.16, ${\mathrm{d}_\Omega}^{\mathrm{Hauss}}([y, m_{yz}], [y, m_{xy}) \leq R$, ${\mathrm{d}_\Omega}^{\mathrm{Hauss}}([z, m_{yz}], [z, m_{xz}]) \leq R$, and ${\mathrm{d}_\Omega}^{\mathrm{Hauss}}([x, m_{xy}], [x, m_{xz}]) \leq 2R$. Hence, $\Delta(x, y, z)$ is $(2R)$-slim. This finishes the proof of Lemma IV.19.

Now we begin the proof of the theorem. Fix a properly convex domain Ω and a rank one axis ℓ in Ω. The remark following the theorem will follow as a consequence of the arguments we give during the course of the proof - we only use the fact that there is some rank one automorphism γ that translates along ℓ; we do not require any other property of γ.

Note that Lemma IV.19 implies that is enough to prove the following proposition.

Proposition IV.20. *If ℓ is a rank one axis, then there exists a constant $\mathcal{B}_\ell$ with the following property: if $\Delta(x, y, z)$ is any projective geodesic triangle in Ω with $[y, z] \subset \ell$, then $[y, z] \subset \mathcal{N}_{\mathcal{B}_\ell}\big([x, y] \cup [x, z]\big)$. Moreover, this constant $\mathcal{B}_\ell$ depends only on the rank one axis ℓ.*

We spend the rest of this section proving this result. The moreover statement will follow from the proof since the proof works independent of the choice of the rank one automorphism which has ℓ as its axis.

Suppose the proposition is false. Then for every $n \geq 0$, there exists a sequence of geodesic triangles $\Delta(a_n, b_n, c_n) \subset \Omega$ with $[a_n, b_n] \subset \ell$, $c_n \in \Omega \setminus \ell$ and $e_n \in (a_n, b_n)$ such that

$$\mathrm{d}_\Omega(e_n, [c_n, a_n]) \geq n \quad \text{and} \quad \mathrm{d}_\Omega(e_n, [c_n, b_n]) \geq n.$$

Since ℓ is a rank one axis, there exists a rank one automorphism γ' whose axis is ℓ. Thus, translating $\Delta(a_n, b_n, c_n)$ by $< \gamma' >$, we can assume that $e := \lim_{n\to\infty} e_n$ exists and $e \in \ell$. Up to passing to a subsequence, we can assume that $a := \lim_{n\to\infty} a_n$, $b := \lim_{n\to\infty} b_n$ and $c := \lim_{n\to\infty} c_n$ exist. Observe that:

$$\mathrm{d}_\Omega(e, [a, c] \cup [c, b]) = \lim_{n\to\infty} \mathrm{d}_\Omega(e_n, [a_n, c_n] \cup [c_n, b_n]) \geq \lim_{n\to\infty} n = \infty.$$

Thus $[a, c] \cup [c, b] \subset \partial\Omega$. But $(a, b) \subset \Omega$ since $e \in (a, b) \cap \Omega$. Thus a, b, c form a half triangle in Ω.

But since $a_n, b_n \in \ell$, $\ell = (a, b)$. Thus the rank one axis ℓ is contained in a half triangle, which is a contradiction. This concludes the proof the proposition.

CHAPTER V

Acylindrical Hyperbolicity of Rank One Groups

This chapter is based on results that appear in [Isl19].

5.1 Outline

In this chapter, we will prove Theorem I.5 which we now restate.

Theorem I.5. *If Λ is a rank one group, then either Λ is virtually cyclic or Λ is an acylindrically hyperbolic group.*

The proof of this theorem involves two steps. The first (and the key) step is proving the following result.

Theorem I.3. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ is a properly convex domain and $\mathcal{PS}^{\Omega} := \{[x,y] : x, y \in \Omega\}$. An element $g \in \mathrm{Aut}(\Omega)$ is contracting for $(\Omega, \mathcal{PS}^{\Omega})$ if and only g is a rank one automorphism.*

Remark V.1. *An element $g \in \mathrm{Aut}(\Omega)$ is contracting for $(\Omega, \mathcal{PS}^{\Omega})$ if and only if it is contracting in the sense of BF, as $\mathcal{PS}^{\Omega}$ is a geodesic path system (cf. III.26).*

In the second step, we observe that Theorem I.3 implies that a rank one group Λ necessarily contains at least one contracting element. Then Theorem III.22 implies Theorem I.5.

The chapter will be structured as follows. In Section 5.2, we prove that rank one automorphisms are contracting. We prove the converse in Section 5.3. These two sections taken together proves Theorem I.3 - this is done in Section 5.4. Then, we discuss several applications of Theorem I.5 in Section 5.5:

(1) Cohomological characterization of rank one groups and infinite dimensionality of group of quasi-morphisms (Section 5.5.1)

(2) Asymptotic counting results for conjugacy classes (Section 5.5.2)

(3) Genericity of rank one automorphisms from the viewpoint of random walks (Section 5.5.3)

5.2 Rank one automorphisms are contracting

Theorem V.2 ([Isl19, Theorem 10.1])**.** *If $\gamma \in \mathrm{Aut}(\Omega)$ is a rank one automorphism, then γ is a contracting element for $(\Omega, \mathcal{PS}^{\Omega})$.*

We devote the rest of this section for the proof of this theorem. The key step will be part (2) of Lemma V.3 which shows that a rank one axis is $\mathcal{PS}^{\Omega}$-contracting.

First, we construct a suitable projection map onto the rank one axis. Suppose ℓ is a rank one axis parametrized by $\sigma : \mathbb{R} \to \Omega$ to have unit speed. Let π_ℓ be the closest-point projection onto the closed convex set ℓ (see Definition II.28). If $p \in \Omega$, there exist $T_p^-, T_p^+ \in \mathbb{R}$ with $T_p^- \leq T_p^+$ such that $\pi_\ell(p) = [\sigma(T_p^-), \sigma(T_p^+)]$. Define the map $\pi : \Omega \to \ell$ via

$$\pi(p) := \sigma\left(\frac{T_p^- + T_p^+}{2}\right).$$

Lemma V.3. *If $\ell \subset \Omega$ is a rank one axis, then there exists $\mathcal{C}_\ell \geq 0$ such that*

(1) if $x \in \Omega$ and $z \in \ell$, then there exists $p_{xz} \in [x, z]$ such that

$$\mathrm{d}_\Omega\left(\pi(x), p_{xz}\right) \leq 3\,\mathcal{C}_\ell\,.$$

(2) ℓ is $\mathcal{PS}^{\Omega}$-contracting with constant $\mathcal{C}_\ell$ (and the map π).

Proof. (1) Let $x \in \Omega$ and $z \in \ell$. Set $C_\ell \geq D_\ell$, where D_ℓ is the constant from Theorem IV.17. By Theorem IV.17, $\Delta(x, \pi(x), z)$ is $\mathcal{D}_\ell$-thin. Then there exists $p \in [x, \pi(x)]$, $q \in [\pi(x), z]$ and $r \in [z, x]$ such that

$$\mathrm{d}_\Omega(q,p) \leq \mathcal{D}_\ell \quad \text{and} \quad \mathrm{d}_\Omega(q,r) \leq \mathcal{D}_\ell\,.$$

Then

$$\begin{aligned}
\mathrm{d}_\Omega\big(\pi(x), p\big) &= \mathrm{d}_\Omega\big(\pi(x), x\big) - \mathrm{d}_\Omega\big(p, x\big) \\
&\leq \mathrm{d}_\Omega\big(q, x\big) - \mathrm{d}_\Omega\big(p, x\big), \\
&\leq \mathrm{d}_\Omega\big(p, q\big) \leq \mathcal{D}_\ell\,.
\end{aligned}$$

Thus

$$\mathrm{d}_\Omega\big(\pi(x), q\big) \leq \mathrm{d}_\Omega\big(\pi(x), p\big) + \mathrm{d}_\Omega\big(q, p\big) \leq 2\,\mathcal{D}_\ell\,.$$

Set $p_{xz} := r$. Then

$$\mathrm{d}_\Omega\big(\pi(x), p_{xz}\big) \leq \mathrm{d}_\Omega\big(\pi(x), q\big) + \mathrm{d}_\Omega\big(q, r\big) \leq 3\,\mathcal{D}_\ell \leq 3C_\ell.$$

(2) Let us label the endpoints of ℓ such that $\ell := (a, b)$. Observe that it suffices to verify part (1b) in Definition III.20. Suppose that it is not satisfied. Then, for $n \geq 1$, there exists $x_n, y_n \in \Omega$ such that

$$\mathrm{d}_\Omega\big(\pi(x_n), \pi(y_n)\big) \geq n$$

and

$$\mathrm{d}_\Omega\big([x_n, y_n], \pi(x_n)\big) \geq n.$$

Since ℓ is a rank one axis, fix a rank one automorphism γ whose axis is ℓ. Then $\gamma \circ \pi = \pi \circ \gamma$. Hence, up to translating x_n and y_n using elements in $< \gamma >$, we can assume that $\alpha := \lim_{n\to\infty} \pi(x_n)$ exists in $\ell \subset \Omega$. Up to passing to a subsequence, we can further assume that the following limits exist in $\overline{\Omega}$: $x := \lim_{n\to\infty} x_n$, $y := \lim_{n\to\infty} y_n$, $\beta := \lim_{n\to\infty} \pi(y_n)$. Then $\lim_{n\to\infty}[x_n, y_n] = [x, y]$. We will now show that

$$[x,y] \subset \partial\Omega. \tag{5.1}$$

This follows from the following estimate:

$$\begin{aligned} \mathrm{d}_\Omega(\alpha,[x,y]) = \lim_{n\to\infty} \mathrm{d}_\Omega(\alpha,[x_n,y_n]) &\geq \lim_{n\to\infty} \left(\mathrm{d}_\Omega\left(\pi(x_n),[x_n,y_n]\right) - \mathrm{d}_\Omega\left(\pi(x_n),\alpha\right) \right) \\ &\geq \lim_{n\to\infty} \left(n - \mathrm{d}_\Omega\left(\pi(x_n),\alpha\right) \right) = \infty. \end{aligned}$$

We also observe that:

$$\begin{aligned} \mathrm{d}_\Omega(\alpha,\beta) = \lim_{n\to\infty} \mathrm{d}_\Omega\left(\alpha,\pi(y_n)\right) &\geq \lim_{n\to\infty} \left(\mathrm{d}_\Omega\left(\pi(x_n),\pi(y_n)\right) - \mathrm{d}_\Omega\left(\pi(x_n),\alpha\right) \right) \\ &\geq \lim_{n\to\infty} \left(n - \mathrm{d}_\Omega\left(\pi(x_n),\alpha\right) \right) = \infty. \end{aligned}$$

Thus $\beta \in \partial\Omega$. However, since $\beta \in \overline{\ell} = [a,b]$, $\beta \in \{a,b\}$. Thus, up to switching labels of endpoints of ℓ, we can assume that

$$\beta = b. \tag{5.2}$$

Claim V.4. $x = y = b$.

Proof of Claim We first show that $y = b$. Since $y_n \in \Omega$ and $\alpha \in \ell$, part (1) of Lemma V.3 implies that there exists $p_n \in [y_n, \alpha]$ such that

$$\mathrm{d}_\Omega\left(p_n, \pi(y_n)\right) \leq 3\,\mathcal{C}_\ell\,.$$

Up to passing to a subsequence, we can assume $p := \lim_{n\to\infty} p_n$ exists in $\overline{\Omega}$. Then, by Proposition II.12, $p \in F_\Omega(\beta)$. By equation (5.2), $\beta = b$ which implies $p \in F_\Omega(b)$. Since b is an endpoint of the rank one axis ℓ, part (4) of Proposition IV.14 implies that $F_\Omega(b) = b$. Thus $p = b$. On the other hand, since $p_n \in [y_n, \alpha]$, we have $p \in [y, \alpha]$. Since $p = b$, $p \in \partial\Omega$. Thus,

$$p \in [\alpha, y] \cap \partial\Omega = \{y\}.$$

Hence,

$$y = p = b.$$

We now show that $x = b$. By (5.1), $[x, y] \subset \partial\Omega$. But since $y = b$, this contradicts part (4) of Proposition IV.14 unless $x = y$. Hence $x = y = b$. This concludes the proof of Claim V.4.

Consider points $x_n \in \Omega$ and $\pi(y_n) \in \ell$. By part (2) of Lemma V.3, there exists $q_n \in \big[x_n, \pi(y_n)\big]$ such that $\mathrm{d}_\Omega\big(\pi(x_n), q_n\big) \leq 3\,\mathcal{C}_\ell$. Up to passing to a subsequence, we can assume that $q := \lim_{n\to\infty} q_n$ exists in $\overline{\Omega}$. Then by Proposition II.12, $q \in F_\Omega(\alpha) = \Omega$. Thus $\lim_{n\to\infty}[x_n, \pi(y_n)]$ is a projective line segment containing q and hence intersects Ω . However, $\lim_{n\to\infty}[x_n, \pi(y_n)] = [x, \beta] = \{b\} \subset \partial\Omega$ which is a contradiction. This shows that the rank one axis ℓ is $\mathcal{PS}^\Omega$-contracting. $\square$

We will now use Lemma V.3 to prove Theorem V.2. Suppose $\gamma \in \mathrm{Aut}(\Omega)$ is a rank one automorphism. Then $\tau_\Omega(\gamma) > 0$ which implies that γ has infinite order. Since γ is a rank one automorphism, part (2) of Proposition IV.14 implies that γ has a unique axis ℓ_γ. Fix $x_0 \in \ell_\gamma$. Then $<\gamma> x_0$ is a quasi-geodesic embedding of $\mathbb{Z}$ in Ω. Part (3) of Lemma V.3 implies that ℓ_γ is a $\mathcal{PS}^\Omega$-contracting set. Then ℓ_γ is $<\gamma>$-invariant, contains x_0, and has a co-bounded γ action. Thus γ is a contracting element for $(\Omega, \mathcal{PS}^\Omega)$ (see Definition III.20).

5.3 Contracting automorphisms are rank one

Theorem V.5 ([Isl19, Theorem 11.1])**.** *If $\gamma \in \mathrm{Aut}(\Omega)$ is a contracting element for $(\Omega, \mathcal{PS}^{\Omega})$, then γ is a rank one automorphism.*

The rest of this section is devoted to the proof of this theorem. We begin by recalling a result of Sisto which says that contracting elements are 'Morse' in the following sense.

Proposition V.6 ([Sis18, Lemma 2.8])**.** *If $\mathcal{PS}$ is a path system on (X, d) and $\mathcal{A} \subset X$ is $\mathcal{PS}$-contracting with constant C, then there exists a constant $M(C)$ such that: if θ is a (C,C)-quasi-geodesic with endpoints in $\mathcal{A}$, then $\theta \subset \mathcal{N}_{M(C)}(\mathcal{A}) := \big\{x \in X : \mathrm{d}(x, \mathcal{A}) < M(C)\big\}$.*

We use this Morse property to show that a contracting element has at least one axis and none of the axes are contained in half triangles in $\partial\Omega$. The first step is the next lemma.

Lemma V.7. *Suppose $\gamma \in \mathrm{Aut}(\Omega)$ is a contracting element. If there exist $x_0 \in \Omega$ and two sequences of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ such that*

$$p := \lim_{k\to\infty} \gamma^{n_k} x_0 \quad \textit{belongs to } E_\gamma^+$$

and

$$q := \lim_{k\to\infty} \gamma^{-m_k} x_0 \quad \textit{belongs to } E_\gamma^-,$$

then

(1) $(p,q) \subset \Omega$,

(2) (p,q) is not contained in any half triangle in $\partial\Omega$.

Proof. Since γ is contracting, by Proposition III.21, $\tau_\Omega(\gamma) > 0$. Thus results of Section 2.6.1 apply.

(1) Suppose this is false. Then $[p,q] \subset \partial\Omega$. Choose any $r \in (p,q)$. Set $L_k := \left[\gamma^{-m_k} x_0, \gamma^{n_k} x_0\right]$. Then $L_\infty := \lim_{k\to\infty} L_k = [q,p]$. Thus we can choose $r_k \in L_k$ such that $\lim_{k\to\infty} r_k = r$.

Since γ is contracting, (2) of Proposition III.21 implies that $\mathcal{A}_{\min}(x_0) :=< \gamma > x_0$ is $\mathcal{PS}^\Omega$-contracting. Since the L_k are geodesics with endpoints in $\mathcal{A}_{\min}(x_0)$, Proposition V.6 implies that there exists a constant M such that for all $k \geq 1$, $L_k \subset \mathcal{N}_M(\mathcal{A}_{\min}(x_0))$. Thus for every $k \geq 1$, there exists $\gamma^{t_k} x_0 \in \mathcal{A}_{\min}(x_0)$ such that

$$\mathrm{d}_\Omega(r_k, \gamma^{t_k} x_0) \leq M. \tag{5.3}$$

Up to passing to a subsequence, we can assume that

$$t := \lim_{k\to\infty} \gamma^{t_k} x_0$$

exists in $\overline{\Omega}$. Since r_k leaves every compact subset of Ω, $\{t_k\}$ is an unbounded sequence. Then by Proposition II.33 part (1), $t \in \left(E_\gamma^+ \sqcup E_\gamma^-\right)$. On the other hand, by Proposition II.12 and (5.3), $t \in F_\Omega(r) \subset \partial\Omega$. We now analyze the two possibilities:

Possibility 1: Suppose $t \in \left(E_\gamma^- \cap \partial\Omega\right)$.

Consider the sequence $\{\gamma^{n_k} r\}_{k=1}^\infty$. Up to passing to a subsequence, we can assume that $r_\infty := \lim_{k\to\infty} \gamma^{n_k} r$ exists in $\partial\Omega$. Since $r \in (p,q)$ with $n_k > 0$, $p \in E_\gamma^+$ and $q \in E_\gamma^-$, we observe that

$$r_\infty = \lim_{k\to\infty} \gamma^{n_k} r = \lim_{k\to\infty} \gamma^{n_k} p \in E_\gamma^+. \tag{5.4}$$

Recall however that $r \in F_\Omega(t)$. Since $t \in E_\gamma^-$ and $r_\infty = \lim_{k\to\infty} \gamma^{n_k} r$, part (2) of Proposition II.34 implies that either $r_\infty \in E_\gamma^-$ or $r_\infty \in \partial\Omega \setminus (E_\gamma^+ \sqcup E_\gamma^-)$. Both of these contradict equation (5.4).

Possibility 2: Suppose $t \in \left(E_\gamma^+ \cap \partial\Omega\right)$.

We can repeat the same argument as in Possibility 1 by considering the sequence

$\{\gamma^{-m_k} r\}_{k=1}^{\infty}$ and arrive at a contradiction (we need a version of Proposition II.34 with γ replaced by γ^{-1}; see the comments preceding the proposition).

The contradiction to both of these possibilities finishes the proof of (1).

(2) By part (1), $(p, q) \subset \Omega$. Suppose there exists $z \in \partial\Omega$ such that the points p, q and z form a half triangle. Choose any sequence of points $z_k \in [\gamma x_0, z] \cap \Omega$ such that $\lim_{k\to\infty} z_k = z$. Since γ is contracting, part (2) of Proposition III.21 implies that $\mathcal{A}_{\min}(x_0) =< \gamma > x_0$ is $\mathcal{PS}^{\Omega}$-contracting (with constant, say C). Thus there exists a projection $\pi : \Omega \to \mathcal{A}_{\min}(x_0)$ that satisfies Definition III.20. We will analyze the sequence $\pi(z_k)$. Since $\pi(z_k) \in \mathcal{A}_{\min}(x_0) =< \gamma > x_0$, there exists a sequence of integers $\{i_k\}$ such that $\pi(z_k) = \gamma^{i_k} x_0$. Up to passing to a subsequence, we can assume that the following limit exists in $\overline{\Omega}$,

$$w := \lim_{k\to\infty} \pi(z_k) = \lim_{k\to\infty} \gamma^{i_k} x_0. \tag{5.5}$$

Claim V.8. $w \in (E_\gamma^+ \sqcup E_\gamma^-) \cap \partial\Omega$.

Proof of Claim. Suppose $w \in \Omega$. Then, by (5.5), $\lim_{k\to\infty} \mathrm{d}_\Omega(w, \pi(z_k)) = 0$. Since $\gamma^{n_k} x_0 \in \mathcal{A}_{\min}(x_0)$, (1a) of Definition III.20 implies that $\mathrm{d}_\Omega(\gamma^{n_k} x_0, \pi(\gamma^{n_k} x_0)) \leq C$. This implies that

$$\begin{aligned}\lim_{k\to\infty} \mathrm{d}_\Omega(\pi(z_k), \pi(\gamma^{n_k} x_0) &\geq \lim_{k\to\infty} \Big(\mathrm{d}_\Omega(w, \gamma^{n_k} x_0) - \mathrm{d}_\Omega(w, \pi(z_k)) - \mathrm{d}_\Omega(\gamma^{n_k} x_0, \pi(\gamma^{n_k} x_0)) \Big) \\ &\geq \lim_{k\to\infty} \mathrm{d}_\Omega(w, \gamma^{n_k} x_0) - C = \infty\end{aligned}$$

Thus, for k large enough, $\mathrm{d}_\Omega\left(\pi(z_k), \pi(\gamma^{n_k} x_0)\right) \geq C$. Since π is a projection into a $\mathcal{PS}^{\Omega}$-contracting set, (1b) of Definition III.20 implies that

$$\mathrm{d}_\Omega\left(\pi(z_k), [z_k, \gamma^{n_k} x_0]\right) \leq C.$$

Thus

$$\mathrm{d}_\Omega(w,[z,p]) = \lim_{k\to\infty} \mathrm{d}_\Omega(\pi(z_k),[z_k,\gamma^{n_k}x_0]) \leq C.$$

Then $[z,p]\cap\Omega \neq \emptyset$. But since p, q and z form a half triangle, $[z,p]\subset\partial\Omega$. This is a contradiction, hence $w\in\partial\Omega$.

Since $w\in\partial\Omega$ and $w=\lim_{k\to\infty}\gamma^{i_k}x_0$ with $x_0\in\Omega$, Proposition II.33 part (1) implies that $w\in E_\gamma^+\sqcup E_\gamma^-$. This concludes the proof of this claim.

Claim V.9. $w\in F_\Omega(z)$.

Proof of Claim: Since $w=\lim_{k\to\infty}\pi(z_k)\in\partial\Omega$ and $\pi(\gamma x_0)\in\Omega$, $\lim_{k\to\infty}\mathrm{d}_\Omega(\pi(z_k),\pi(\gamma x_0))=\infty$. Thus, for k large enough, $\mathrm{d}_\Omega\big(\pi(z_k),\pi(\gamma x_0)\big)\geq C$. Again, as π is a projection into a $\mathcal{PS}^\Omega$-contracting set, we have

$$\mathrm{d}_\Omega(\pi(z_k),[\gamma x_0,z_k])\leq C.$$

Choose $\eta_k\in[\gamma x_0,z_k]$ such that $\mathrm{d}_\Omega(\pi(z_k),\eta_k)\leq C$. Up to passing to a subsequence, we can assume that $\eta:=\lim_{k\to\infty}\eta_k$ exists. By Proposition II.12, $\eta\in F_\Omega(w)$. By Proposition II.11(1), $\eta\in\partial\Omega$. But $\eta\in[\gamma x_0,z]$, which intersects $\partial\Omega$ at exactly one point, namely z. Thus, $\eta=z$ implying $z\in F_\Omega(w)$, or equivalently, $w\in F_\Omega(z)$. This concludes the proof of Claim V.9.

Since $p,\ q,\ z$ form a half triangle, $[q,z]\cup[p,z]\subset\partial\Omega$. By Claim V.9, $w\in F_\Omega(z)$. Then part (4) of Proposition II.11 implies that

$$[p,w]\cup[q,w]\subset\partial\Omega. \tag{5.6}$$

By Claim V.8, $w\in E_\gamma^+\sqcup\ E_\gamma^-$. We will now show that (5.6) contradicts this.

Suppose $w\in E_\gamma^+$. Since $\lim_{k\to\infty}\gamma^{i_k}x_0=w\in E_\gamma^+$ and $\lim_{k\to\infty}\gamma^{-m_k}x_0=q\in E_\gamma^-$, then part (1) of Lemma V.7 implies that $(w,q)\subset\Omega$. This contradicts (5.6). On the other hand, if we suppose $w\in E_\gamma^-$, then similar arguments show that $(p,w)\subset\Omega$

which again contradicts (5.6). These contradictions show that p, q and z cannot form a half triangle. $\square$

We now prove Theorem V.5 using the above lemma. Let $\gamma \in \mathrm{Aut}(\Omega)$ be a contracting element for $(\Omega, \mathcal{PS}^{\Omega})$. Then the following will imply that γ is a rank one automorphism:

Translation distance $\tau_{\Omega}(\gamma) > 0$: This is part (1) of Proposition III.21.

γ has an axis: By Proposition IV.5, there exists $(a, b) \subset \overline{\Omega}$ where a, b are fixed points of γ, $a \in E_{\gamma}^{+}$, and $b \in E_{\gamma}^{-}$. We will show that $(a, b) \subset \Omega$, hence it is an axis of γ.

Fix $x_0 \in \Omega$. Proposition II.33 part (1) implies $\{\gamma^n x_0 : n \in \mathbb{N}\}$ has an accumulation point p in E_{γ}^{+} and $\{\gamma^{-n} x_0 : n \in \mathbb{N}\}$ has accumulation point q in E_{γ}^{-}. By part (1) of Lemma V.7, $(p, q) \subset \Omega$. Moreover, since $a, p \in E_{\gamma}^{+}$ and $E_{\gamma}^{+} \subset \partial\Omega$ (cf. II.33), $[a, p] \subset \partial\Omega$. Similarly, $[b, q] \subset \partial\Omega$.

By Part (2) of Lemma V.7, $(p, q) \subset \Omega$ is not contained in any half triangle. Since $[b, q] \subset \partial\Omega$, this implies that $(p, b) \subset \Omega$. Let $y_0 \in (p, b)$. By Proposition II.33 part (3), there exists a sequence of positive integers $\{n_k\}$ such that $\lim_{k\to\infty} \left(\gamma|_{E_{\gamma}^{+}}\right)^{n_k} = \mathrm{Id}_{E_{\gamma}^{+}}$. Then $\lim_{k\to\infty} \gamma^{n_k} p = p$ which implies $\lim_{k\to\infty} \gamma^{n_k} y_0 = p \in E_{\gamma}^{+}$. On the other hand, $\lim_{k\to\infty} \gamma^{-k} y_0 = b \in E_{\gamma}^{-}$. Then, by Part (2) of Lemma V.7, $(p, b) \subset \Omega$ cannot be contained in a half triangle. But we know that $[a, p] \subset \partial\Omega$. Thus, $(a, b) \subset \Omega$.

None of the axes of γ are contained in a half triangles in $\partial\Omega$: Let $(a', b') \subset \Omega$ be any axis of γ with $a' \in E_{\gamma}^{+}$ and $b' \in E_{\gamma}^{-}$. If $z_0 \in (a', b')$, then $\lim_{k\to\infty} \gamma^k z_0 = a'$ and $\lim_{k\to\infty} \gamma^{-k} z_0 = b'$. Then, by Part (2) of Lemma V.7, (a', b') cannot be contained in a half triangle in Ω.

5.4 Proof of Theorem I.5

Since Λ is a rank one group, Λ contains a rank one automorphism. Then Theorem I.3 implies that Λ contains a contracting element for $(\Omega, \mathcal{PS}^{\Omega})$. The result follows from Theorem III.22.

5.5 Applications of Theorem I.5

5.5.1 Cohomological Characterization of Rank One and Quasi-morphisms

The goal of this section is to prove Theorem I.8 and its generalizations. In order to state our theorem, we first introduce some definitions from group cohomology. See [BBF16, Section 1] or [Fri17] for details.

Suppose G is a group, $(E, ||\cdot||)$ is a complete normed $\mathbb{R}$-vector space, and $\rho : G \to \mathcal{U}(E)$ is a unitary representation. If $F : G \to E$, let

$$\Delta(F) := \sup_{g,g' \in G} ||F(gg') - F(g) - \rho(g)F(g')||.$$

F is called a cocycle if $\Delta(F) = 0$ and a quasi-cocycle if $\Delta(F)$ is finite. We say that two quasi-cocyles are equivalent if they differ by a bounded function (on G, taking values in E) or a cocycle. The set of equivalence classes of quasi-cocycles is denoted by $\widetilde{QC}(G;\rho)$. Group cohomology of G affords a different interpretation of $\widetilde{QC}(G;\rho)$: it is the kernel of the comparison map $H^2(G,\rho) \to H^2(G;\rho)$ modulo the subspace generated by bounded functions and cocycles.

We now mention two important special cases of $\widetilde{QC}(G;\rho)$.

- Suppose $\rho_{\mathrm{triv}} : G \to \mathbb{R}$ is the trivial representation. Then cocycles are homomorphisms of G into $\mathbb{R}$ and quasi-cocycles are quasi-morphisms of G. Then

$\widetilde{QC}(G;\rho)$ recovers a classical object $\widetilde{QH}(G)$, the space of 'non-trivial' quasi-morphisms of G (see the definitions preceding Theorem I.8).

- Suppose G is a discrete group and $\rho^p_{\text{reg}} : G \to \mathcal{U}(\ell^p(G))$ is the left regular representation of G on $\ell^p(G)$. When $1 < p < \infty$, $\ell^p(G)$ is a uniformly convex Banach space and $\widetilde{QC}(G;\rho^p_{\text{reg}})$ will be of interest to us.

This group cohomology data often carries important geometric information. For non-positively curved Riemannian manifolds, Bestvina-Fujiwara proves:

Theorem V.10 ([BF09, Theorem 1.1])**.** *Suppose M is a complete finite volume Riemannian manifold of non-positive curvature such that $\Gamma := \pi_1(M)$ is neither virtually cyclic nor is a product of two infinite groups. Then, either* $\dim(\widetilde{QH}(\Gamma)) = \infty$ *or $\widetilde{M}$ is a higher rank symmetric space.*

A more general result is proven in [BBF16].

Theorem V.11. *([BBF16, Corollary 1.2]) If G is an acylindrically hyperbolic group, $E \neq 0$ is a uniformly convex Banach space, $\rho : G \to \mathcal{U}(E)$ is a unitary representation and any maximal finite normal subgroup of G has a non-zero fixed vector, then* $\dim\left(\widetilde{QC}(G;\rho)\right) = \infty$.

On the other hand, higher rank lattices have no 'non-trivial' quasi-morphisms.

Theorem V.12 ([BM02, Theorem 21])**.** *Suppose $\Gamma \leq G$ is an irreducible lattice in a semi-simple Lie group G with finite center. Then $\widetilde{QH}(\Gamma) = 0$.*

Proof of Theorem I.8 and Generalizations

In the same spirit as Riemannian non-positive curvature, we now prove a cohomological characterization of rank one for properly convex domains.

Theorem V.13 ([Isl19, Theorem 13.1]). *Suppose Λ is torsion-free rank one group and ρ is any unitary representation of Λ on a uniformly convex Banach space $E \neq 0$. Then either Λ is virtually cyclic or $\dim\left(\widetilde{QC}(\Lambda;\rho)\right) = \infty$.*

Proof. If Λ is not virtually cyclic, then Theorem I.5 implies that Λ is an acylindrically hyperbolic group. Since Λ is torsion-free, there are no finite normal subgroups. The claim then follows from Theorem V.11. □

This theorem implies some straightforward corollaries.

Corollary V.14 ([Isl19, Theorem 1.6]). *Suppose Λ is a torsion-free rank one group and Λ is not virtually cyclic. Then*

(1) $\dim\left(\widetilde{QH}(\Lambda)\right) = \infty$*, and*

(2) $\dim\left(\widetilde{QC}(\Lambda;\rho^p_{\mathrm{reg}})\right) = \infty$ *if* $1 < p < \infty$.

Note that part (1) of this Corollary is Theorem I.8. Theorem I.8, along with Theorem V.12 and the higher rank rigidity theorem I.7, proves Corollary I.9 which we restate here.

Corollary I.9. *Suppose Ω is an irreducible properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a discrete torsion-free group that acts co-compactly on Ω. Then Λ is a rank one group if and only if $\dim \widetilde{QH}(\Lambda) = \infty$. Otherwise, $\dim \widetilde{QH}(\Lambda) = 0$.*

5.5.2 Counting of Conjugacy Classes

We will prove Theorem I.10 in this section. If Ω is a Hilbert geometry and $g \in \mathrm{Aut}(\Omega)$, define the translation length (cf. 2.6)

$$\tau_\Omega(g) := \inf_{x\in\Omega} \mathrm{d}_\Omega(x, gx)$$

and the stable translation length

$$\tau_\Omega^{\mathrm{stable}}(g) := \lim_{n\to\infty} \frac{\mathrm{d}_\Omega(x, g^n x)}{n}.$$

Note that $\tau_\Omega^{\mathrm{stable}}(g)$ is independent of the base point $x \in \Omega$. Now suppose (Ω, Λ) is a rank one Hilbert geometry. Let $[[g]]$ denote the conjugacy class of $g \in \Lambda$. Both τ_Ω and $\tau_\Omega^{\mathrm{stable}}$ are well-defined on the set of conjugacy classes in Λ. Then for $t > 0$, define

$$\mathcal{C}(t) := \#\{[[g]] : g \in \Lambda,\ \tau_\Omega([g]) \leq t\} \quad \text{ and }$$
$$\mathcal{C}^{\mathrm{stable}}(t) := \#\{[[g]] : g \in \Lambda, \tau_\Omega^{\mathrm{stable}}([g]) \leq t\}.$$

Here $\mathcal{C}(t)$ (resp. $\mathcal{C}^{\mathrm{stable}}(t)$) counts the number of conjugacy classes in Λ whose translation length (resp. stable translation length) is at most t. We also introduce the notion of pointed length for a conjugacy class $[[g]]$ of $g \in \Lambda$. Fix a base point $p \in \Omega$. The pointed length of $[[g]]$ is

$$\mathcal{L}_p([[g]]) := \inf_{g' \in [[g]]} \mathrm{d}_\Omega(p, g'p).$$

Let $\mathcal{C}^{\mathcal{L}_p}(t) := \#\{[[g]] : g \in \Lambda, \mathcal{L}_p([[g]]) \leq t\}$.

For co-compact rank one Hilbert geometries, we prove an asymptotic growth formula for $\mathcal{C}(t)$ and $\mathcal{C}^{\mathrm{stable}}(t)$. To state our result, we will require the critical exponent of Λ which is defined as

$$\omega_\Lambda := \limsup_{n \to \infty} \frac{\log \#\{g \in \Lambda : \mathrm{d}_\Omega(x, gx) \leq n\}}{n}$$

for some (and hence any) base point $x \in \Omega$.

Theorem V.15 ([Isl19, Theorem 1.8])**.** *Suppose Ω is a properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a rank one group that acts co-compactly on Ω. Assume that Λ is not virtually cyclic. Then there exists a constant D' such that if $t \geq 1$,*

$$\frac{1}{D'} \frac{\exp(t\omega_\Lambda)}{t} \ \leq\ \mathcal{C}(t) \ \leq\ D' \frac{\exp(t\omega_\Lambda)}{t}. \tag{5.7}$$

The function $\mathcal{C}^{\mathrm{stable}}(t)$ and $\mathcal{C}^{\mathcal{L}_p}(t)$ also satisfy a similar growth formula as above.

Remark V.16. *Counting of conjugacy classes in Λ is usually connected to counting of closed geodesics in Ω/Λ. However this connection is subtle for Hilbert geometries since there could be isometries in Λ that do not act by a translation along any projective line in Ω (i.e. do not have an axis, see Section 4.1).*

We will devote the rest of this section to the proof of this theorem. Fix a rank one Hilbert geometry (Ω, Λ). We first show that

$$\tau_\Omega([[g]]) = \tau_\Omega^{\text{stable}}([[g]]).$$

Indeed, triangle inequality implies $\tau_\Omega^{\text{stable}}(g) \leq \tau_\Omega(g)$. On the other hand, using Proposition II.32,

$$\tau_\Omega^{\text{stable}}(g) \geq \lim_{n\to\infty} \frac{\tau_\Omega(g^n)}{n} = \frac{1}{n}\log\frac{\lambda_{\max}(\widetilde{g}^n)}{\lambda_{\min}(\widetilde{g}^n)} = \log\frac{\lambda_{\max}(\widetilde{g})}{\lambda_{\min}(\widetilde{g})} = \tau_\Omega(g).$$

Next, we show that if Ω/Λ is compact and $R := \operatorname{diam}(\Omega/\Lambda)$, then

$$\tau_\Omega([[g]]) \leq \mathcal{L}_p([[g]]) \leq \tau_\Omega([[g]]) + 2R.$$

Clearly $\tau_\Omega([[g]]) \leq \mathcal{L}_p([[g]])$. On the other hand, if $x \in \Omega$ then there exists $h_x \in \Lambda$ such that $\mathrm{d}_\Omega(x, h_x p) \leq R$. Then

$$\mathcal{L}_p([[g]]) \leq \mathrm{d}_\Omega(p, h_x^{-1} g h_x p) \leq 2\,\mathrm{d}_\Omega(h_x p, x) + \mathrm{d}_\Omega(x, gx) \leq 2R + \mathrm{d}_\Omega(x, gx).$$

Thus $\mathcal{L}_p([[g]]) \leq \tau_\Omega([[g]]) + 2R$.

Based on the above discussion,

$$\mathcal{C}(t) = \mathcal{C}^{\text{stable}}(t).$$

If Ω/Λ is compact and $R = \operatorname{diam}(\Omega/\Lambda)$, then

$$\mathcal{C}^{\mathcal{L}_p}(t) \leq \mathcal{C}(t) \leq \mathcal{C}^{\mathcal{L}_p}(t + 2R).$$

Thus, it is enough to prove the asymptotic growth formula for $\mathcal{C}^{\mathcal{L}_p}(t)$. This is a direct consequence of the Main Theorem in [GY18]. The Main Theorem part (1) in [GY18] implies that if Λ is a non-elementary group with a co-compact action (more generally, statistically convex co-compact action) on a geodesic metric space and Λ contains a contracting element (in the sense of BF; cf. 3.4.3), then $\mathcal{C}^{\mathcal{L}_p}(t)$ satisfies the growth formula in (5.7). If (Ω, Λ) is as above, then it satisfies all of these conditions (cf. I.3 and III.26). Then $\mathcal{C}^{\mathcal{L}_p}(t)$ satisfies equation (5.7) and it finishes our proof.

5.5.3 Genericity from the Viewpoint of Random Walks

Suppose Λ is a finitely generated rank one group that is not virtually cyclic. If S is a finite symmetric generating set of Λ, let $W_n(S)$ be the set of words of length n in the elements of S.

Definition V.17. A simple random walk on Λ (with support S) is a sequence of Λ-valued random variables $\{X_n\}_{n\in\mathbb{N}}$ with laws μ_n defined by: if $n \geq 1$ and $g \in \Lambda$

$$\mu_n(\{g\}) = \frac{\#\{w \in W_n(S) : w \text{ represents } g\}}{\#W_n(S)}.$$

We now prove Theorem I.11. Note that under the hypotheses of Theorem I.11, Λ is an acylindrically hyperbolic group. The result then follows from [Sis18, Theorem 1.6].

CHAPTER VI

Properly Convex Domains with Strongly Isolated Simplices

This chapter is based on results that appear in [IZ19] which is a joint work with A. Zimmer. In [IZ19], we prove all our results for naive convex co-compact groups, a class that strictly contains all convex co-compact groups. In this chapter (and the ones that follow), we work only with convex co-compact groups (see Section 2.7). This makes for a cleaner exposition, simplifies many of the proofs, and we hope that it will make the proof ideas clearer.

6.1 Definitions

In this chapter, we will introduce a special class of properly convex domains called "properly convex domains with strongly isolated simplices". This definition is motivated by Hruska-Kleiner's work on CAT(0) spaces with isolated flats [HK05]. We will work with convex co-compact groups; recall the definition from Section 2.7.

If Ω' is a properly convex domain and $S \subset \Omega'$ is a properly embedded simplex of dimension at least two, then S is called maximal provided S is not properly contained in any other properly embedded simplex in Ω'. If $X \subset \Omega$, let $\operatorname{diam}_\Omega(X) := \sup_{x_1, x_2 \in X} \mathrm{d}_\Omega(x_1, x_2)$.

Definition VI.1 ([IZ19, Definition 1.15])**.** Suppose $\Lambda \leq \operatorname{Aut}(\Omega)$ is a convex co-compact group and $\mathcal{S}_\Lambda$ is the collection of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two.

(1) We will say that $\mathcal{S} \subset \mathcal{S}_\Lambda$ is *strongly isolated* provided: for any $r \geq 0$, there exists $D(r) \geq 0$ such that if $S_1, S_2 \in \mathcal{S}$ are distinct, then

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)\right) \leq D(r).$$

(2) We will say that $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has *strongly isolated simplices* if $\mathcal{S}_\Lambda$ is strongly isolated.

Observation VI.2. *If $\mathcal{S} \subset \mathcal{S}_\Lambda$ is strongly isolated, then $\mathcal{S}$ is closed and discrete in the local Hausdorff topology induced by d_Ω.*

Proof. See the proof of Proposition VI.3 part (1). □

In the next chapter, we will discuss our key result, Theorem I.15, on properly convex domains with strognly isolated simplices. In this result proven jointly with A. Zimmer [IZ19], we show that for a convex co-compact group Λ, the properly convex domain $\mathcal{C}_\Omega(\Lambda)$ having strongly isolated simplices is equivalent to the property that Λ is a relatively hyperbolic group with respect to virtually Abelian subgroups of rank at least two. In order to prove Theorem I.15, we need to understand the geometric consequences of the property - "strongly isolated simplices". This will be our focus in this chapter. In particular, we will prove Theorem I.16 in Section 6.2. We will prove in Section 6.3 that if $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices, then Λ is a convex co-compact rank one group.

6.2 Geometric Properties: Proof of Theorem I.16

In this section, we will prove the Theorem I.16. It establishes some key geometric properties of $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ with strongly isolated simplices where $\Lambda \leq \mathrm{Aut}(\Omega)$ is convex co-compact. We will use this theorem in the next chapter for proving Theorem I.15. We restate Theorem I.16 before beginning the proof.

Theorem I.16 ([IZ19, Theorem 1.8]) *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices. Then*

(1) Λ has finitely many orbits in $\mathcal{S}_\Lambda$.

(2) If $S \in \mathcal{S}_\Lambda$, then $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S and contains a finite index subgroup isomorphic to $\mathbb{Z}^k$ where $k = \dim S$.

(3) If $A \leq \Lambda$ is an infinite Abelian subgroup of rank at least two, then there exists a unique $S \in \mathcal{S}_\Lambda$ with $A \leq \mathrm{Stab}_\Lambda(S)$.

(4) If $S \in \mathcal{S}_\Lambda$ and $x \in \partial S$, then $F_\Omega(x) = F_{\mathcal{C}_\Omega(\Lambda)}(x) = F_S(x)$.

(5) If $S_1, S_2 \in \mathcal{S}_\Lambda$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \emptyset$.

(6) If $\ell \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ is a non-trivial line segment, then there exists $S \in \mathcal{S}_\Lambda$ with $\ell \subset \partial S$.

(7) If $x, y, z \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ form a half triangle in $\mathcal{C}_\Omega(\Lambda)$ (i.e. $[x,y] \cup [y,z] \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ and $(x,z) \subset \mathcal{C}_\Omega(\Lambda)$)*, then there exists $S \in \mathcal{S}_\Lambda$ such that $x, y, z \in \partial S$.*

(8) If $x \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ is not a C^1-smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_\Lambda$ with $x \in \partial S$.

For the rest of this chapter, fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{\mathrm{d}})$ and a convex co-compact subgroup $\Lambda \leq \mathrm{Aut}(\Omega)$. Let $\mathcal{S}_\Lambda$ denote the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two. For ease of notation, we set

$$\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$$

The proof of Theorem I.16 is split into the next few sections in the following order:

□ parts (1) – (3) of is proven in Section 6.2.1,

- ☐ part (5) is proven in Section 6.2.2,
- ☐ part (4) is proven in Section 6.2.3
- ☐ parts (6) and (7) are proven in Section 6.2.4, and
- ☐ part (8) is proven in Section 6.2.5.

6.2.1 Maximal Simplices are Periodic

In this section we show that if $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices, then each simplex $S \in \mathcal{S}_\Lambda$ is periodic, i.e. $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S.

Proposition VI.3 ([IZ19, Proposition 8.1])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. Then the collection $\mathcal{S}_\Lambda$ satisfies the following properties:*

(1) $\mathcal{S}_\Lambda$ is closed and discrete in the local Hausdorff topology.

(2) $\mathcal{S}_\Lambda$ is a locally finite collection, that is, for any compact set $K \subset \Omega$ the set $\{S \in \mathcal{S}_\Lambda : S \cap K \neq \emptyset\}$ is finite.

(3) Λ has finitely many orbits in $\mathcal{S}$.

(4) If $S \in \mathcal{S}_\Lambda$, then $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S and contains a finite index subgroup isomorphic to $\mathbb{Z}^k$ where $k = \dim S$.

(5) If $A \leq \Lambda$ is an infinite Abelian subgroup of rank at least two, then there exists a unique $S \in \mathcal{S}_\Lambda$ with $A \leq \mathrm{Stab}_\Lambda(S)$.

We spend the rest of this section proving this proposition. The proofs are almost analogous to results in the CAT(0) setting, see Wise [Wis96, Proposition 4.0.4], Hruksa [Hru05, Theorem 3.7], or Hruska-Kleiner [HK05, Section 3.1].

(1) Suppose S_n is a sequence in $\mathcal{S}_\Lambda$ that converges to S in the local Hausdorff topology. By Proposition II.21, S is a properly embedded simplex in Ω of dimension at least two. It is enough to show that $S_n = S$ for n large enough.

Fix $\varepsilon > 0$. Since $\mathcal{S}_\Lambda$ is strongly isolated, there exists $D(\varepsilon) \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Lambda$ are distinct, then

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_1; \varepsilon) \cap \mathcal{N}_\Omega(S_2; \varepsilon) \right) \leq D(\varepsilon).$$

Let $r_\varepsilon := D(\varepsilon) + 1$ and fix $x \in S$. Since $S_n \to S$, there exists $N_0 \in \mathbb{N}$ for all $n \geq N_0$,

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(S_n \cap \mathcal{B}_\Omega(x, r_\varepsilon), S \cap \mathcal{B}_\Omega(x, r_\varepsilon)) < \varepsilon.$$

Observe that there exists $x_1, x_2 \in S$ such that $(x_1, x_2) \subset S \cap \mathcal{B}_\Omega(x, r_\varepsilon)$ and $\mathrm{d}_\Omega(x_1, x_2) = r_\varepsilon$. Thus, for any $m \neq n \geq N_0$,

$$(x_1, x_2) \subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_m; \varepsilon).$$

Thus

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_m; \varepsilon) \right) \geq r_\varepsilon > D(\varepsilon)$$

implying that $S_m = S_n$ for all $m, n \geq N_0$. Thus $S_n = S$ for all n large enough.

(2) Follows from part (1).

(3) Follows from part (2).

(4) Fix $S \in \mathcal{S}_\Lambda$ and a compact set $K \subset \Omega$. Let

$$X := \{ g \in \Lambda : S \cap gK \neq \emptyset \}.$$

Then $S = \cup_{g \in X} S \cap gK$. Since $(g^{-1}S) \cap K \neq \emptyset$ when $g \in X$, Part (2) implies that the set

$$\{ g^{-1}S : g \in X \}$$

is finite. Since $g^{-1}S = h^{-1}S$ if and only if $gh^{-1} \in \operatorname{Stab}_\Lambda(S)$ if and only if $\operatorname{Stab}_\Lambda(S)g = \operatorname{Stab}_\Lambda(S)h$, there exists $g_1, \ldots, g_m \in X$ such that

$$\bigcup_{g \in X} \operatorname{Stab}_\Lambda(S)g = \bigcup_{j=1}^{m} \operatorname{Stab}_\Lambda(S)g_j.$$

Then the set $\widehat{K} := \cup_{j=1}^m S \cap g_j K$ is compact and

$$\mathrm{Stab}_\Lambda(S) \cdot \widehat{K} = \cup_{g \in X} S \cap gK = S.$$

So $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S.

It is now easy to show that $\mathrm{Stab}_\Lambda(S)$ contains a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

(5) This is a straightforward application of the convex projective Flat Torus Theorem. Suppose $\widehat{A} \geq A$ is a maximal abelian subgroup containing A. By Theorem I.17, there exists $\widehat{S} \in \mathcal{S}_\Lambda$ such that $\widehat{A}$ acts co-compactly on $\widehat{S}$. Thus $A \leq \mathrm{Stab}_\Lambda(\widehat{S})$. If A preserves another simplex, then it violates the strong isolation property because A is infinite. Thus $\widehat{S}$ is the unique properly embedded simplex preserved by A.

6.2.2 Intersections of Simplices

This result follows easily from the strong isolation property.

Proposition VI.4 ([IZ19, Section 12]). *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $S_1, S_2 \in \mathcal{S}_\Lambda$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \emptyset$.*

Suppose $x \neq y \in S_1 \cap S_2$. Let $(x_1, y_1) \subset S_1$ be the maximal projective line segment in S_1 containing $[x, y]$. By convexity, $(x_1, y_1) \subset S_1 \cap S_2$. However $\mathrm{diam}_\Omega((x_1, y_1)) = \infty$ which implies $S_1 = S_2$ since $\mathcal{S}_\Lambda$ is a strongly isolated. This is a contradiction.

For the second part, suppose $y \in \partial S_1 \cap \partial S_2$. Let $p_1 \in S_1$ and $p_2 \in S_2$. By Proposition II.16

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}([p_1, y), [p_2, y)) \leq R := \mathrm{d}_\Omega(p_1, p_2).$$

Then,

$$[p_1, y) \subset S_1 \cap \mathcal{N}_\Omega(S_2; R).$$

As $\mathcal{S}_\Lambda$ is strongly isolated, if S_1 and S_2 are distinct, then there exists $D(R) \geq 0$ such that

$$\operatorname{diam}_\Omega(\mathcal{N}_\Omega(S_1; R) \cap \mathcal{N}_\Omega(S_2; R)) \leq D(R) < \infty.$$

But $\operatorname{diam}_\Omega(p_1, y) = \infty$. Thus $S_1 = S_2$, a contradiction.

6.2.3 Boundary Faces of Simplices

In this subsection, we will prove the following result about boundary faces of simplices.

Proposition VI.5 ([IZ19]). *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $S \in \mathcal{S}_\Lambda$ and $x \in \partial S$, then $F_\Omega(x) = F_\mathcal{C}(x) = F_S(x)$.*

Fix $S \in \mathcal{S}_\Lambda$ and $x \in \partial S$. By Theorem II.39 part (3), $F_\Omega(x) = F_\mathcal{C}(x)$. So it is enough to show that $F_S(x) = F_\mathcal{C}(x)$. Also observe that if $\dim(F_\mathcal{C}(x)) = 0$, then $F_\mathcal{C}(x) = F_S(x) = \{x\}$ and the result is immediate. So, without loss of generality, we can assume that $\dim(F_\mathcal{C}(x)) \geq 1$.

In order to prove this theorem, it is enough to show that $\partial F_\mathcal{C}(x) \subset \partial F_S(x)$. Indeed,

$$F_\mathcal{C}(x) = \text{rel-int}(\text{ConvHull}_\Omega(\partial F_\mathcal{C}(x)))$$

and

$$F_S(x) = \text{rel-int}(\text{ConvHull}_\Omega(\partial F_S(x))).$$

So, $\partial F_\mathcal{C}(x) \subset \partial F_S(x)$ implies that $F_\mathcal{C}(x) \subset F_S(x)$. On the other hand, $F_S(x) \subset F_\mathcal{C}(x)$ since $S \subset \mathcal{C}$. This shows that proving $\partial F_\mathcal{C}(x) \subset \partial F_S(x)$ is enough to prove the theorem.

In order to prove $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$, we will require the following general result about convex co-compact groups. Note that the following lemma does not require the assumption that $\mathcal{S}_\Lambda$ is strongly isolated.

Lemma VI.6. *Suppose $w \in \partial_{\mathrm{i}} \mathcal{C}$ with $\dim(F_{\mathcal{C}}(w)) \geq 1$ and $w' \in \partial_{\mathrm{i}} F_{\mathcal{C}}(w)$. For any $r, \varepsilon > 0$ and $p \in \mathcal{C}$, there exists $N \geq 0$ such that: if $y \in (w, w')$ with $\mathrm{d}_{F_\Omega(w)}(w, y) > N$, then there exists $p_y \in [p, y)$ such that whenever $q \in [p_y, y)$,*

$$\mathbb{P}(\mathrm{Span}\{w, w', p\}) \cap \mathcal{B}_\Omega(q, r) \subset \mathcal{N}_\Omega(S_q; \varepsilon).$$

for some $S_q \in \mathcal{S}_\Lambda$.

Proof of Lemma VI.6. Suppose this fails. Then there exist $r, \varepsilon > 0$ and $p \in \mathcal{C}$ such that: if $n \geq 1$, there exist $y_n \in (w, w')$ with $\mathrm{d}_{F_\Omega(w)}(w, y_n) \geq n$ and $q_{n,m} \in [p, y_n)$ with $\lim_{m \to \infty} q_{n,m} = y_n$ such that

$$\mathbb{P}\left(\mathrm{Span}\{w, w', p\}\right) \cap \mathcal{B}_\Omega(q_{n,m}, r) \not\subset \mathcal{N}_\Omega(S; \varepsilon) \tag{6.1}$$

for any properly embedded simplex $S \in \mathcal{S}_\Lambda$. By Proposition II.12,

$$\liminf_{m \to \infty} \mathrm{d}_\Omega(q_{n,m}, [p, w] \cup [p, w']) \geq \mathrm{d}_{F_\Omega(w)}(y_n, w) \geq n.$$

Then for each n, we choose m_n large enough such that

$$\mathrm{d}_\Omega(q_{n,m_n}, [p, w] \cup [p, w']) \geq n. \tag{6.2}$$

Set $q'_n := q_{n,m_n}$.

Since Λ acts co-compactly on $\mathcal{C}$, we can pass to a subsequence and choose $\gamma_n \in \Lambda$ such that $\gamma_n q'_n \to q'_\infty \in \mathcal{C}$. Up to passing to another subsequence, we can assume that

$$\gamma_n w', \gamma_n w, \gamma_n p, \gamma_n y_n \to w'_0, w_0, p_0, y_\infty \in \overline{\mathcal{C}}.$$

By construction and by Equation (6.2),

$$[p_0, w_0'] \cup [w_0', w_0] \cup [w_0, p_0] \subset \partial_{\mathrm{i}}\, \mathcal{C}.$$

But $(p_0, y_\infty) \subset \mathcal{C}$ since $q_\infty' \in (p_0, y_\infty) \cap \mathcal{C}$. Thus,

$$S := \text{rel-int}\left(\text{ConvHull}\{w_0, w_0', p_0\}\right)$$

is a properly embedded two dimensional simplex in $\mathcal{C}$. Note that

$$S_n := \text{rel-int}\left(\text{ConvHull}\{\gamma_n w, \gamma_n w', \gamma_n p\}\right)$$

converges to S in the local Hausdorff topology. Thus, for n large enough,

$$\mathrm{d}_\Omega{}^{\text{Hauss}}\left(\mathcal{B}_\Omega(q_\infty', r) \cap S, \mathcal{B}_\Omega(q_\infty', r) \cap S_n\right) < \varepsilon/2.$$

Since $\gamma_n q_n' \to q_\infty'$,

$$\mathrm{d}_\Omega{}^{\text{Hauss}}(\mathcal{B}_\Omega(q_\infty', r), \mathcal{B}_\Omega(\gamma_n q_n', r)) < \varepsilon/2$$

when n is large enough. Thus, for large enough n,

$$\mathrm{d}_\Omega{}^{\text{Hauss}}\left(\mathcal{B}_\Omega(q_\infty', r) \cap S, \mathcal{B}_\Omega(\gamma_n q_n', r) \cap S_n\right) < \varepsilon.$$

Since $q_\infty' \in \mathcal{S}$, this implies that

$$\mathcal{B}_\Omega(q_n', r) \cap \gamma_n^{-1} S_n \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \varepsilon).$$

Now observe that

$$\mathcal{B}_\Omega(q_n', r) \cap \gamma_n^{-1} S_n = \mathcal{B}_\Omega(q_n', r) \cap \mathbb{P}\left(\text{Span}\{w, w', p\}\right).$$

Thus, for n large enough,

$$\mathcal{B}_\Omega(q_n', r) \cap \mathbb{P}\left(\text{Span}\{w, w', p\}\right) \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \varepsilon).$$

Let $\widehat{S}_n \in \mathcal{S}_\Lambda$ be a maximal properly embedded simplex such that $\gamma_n^{-1} S \subset \widehat{S}_n$. Thus,

$$\mathcal{B}_\Omega(q_n', r) \cap \mathbb{P}\left(\mathrm{Span}\{w, w', p\}\right) \subset \mathcal{N}_\Omega(\widehat{S}_n; \varepsilon)$$

This contradicts Equation (6.1) and concludes the proof of this lemma. □

Now we finish the proof of Proposition VI.5. We want to prove $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$. Recall that $\dim(F_{\mathcal{C}}(x)) \geq 1$ which implies that $\partial F_{\mathcal{C}}(x) \neq \emptyset$. Let $x' \in \partial F_{\mathcal{C}}(x)$. We will show that $x' \in \partial F_S(x)$.

Fix $\varepsilon > 0$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, there exists $D(\varepsilon) \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Lambda$ are distinct, then

$$\mathrm{diam}_\Omega\left(\mathcal{N}_\Omega(S_1; \varepsilon) \cap \mathcal{N}_\Omega(S_2; \varepsilon)\right) \leq D(\varepsilon). \tag{6.3}$$

Fix $R_\varepsilon := D(\varepsilon) + 1$. Fix $p \in S$. Applying the above Lemma VI.6 with $R_\varepsilon, \varepsilon$ and p, we get $N \geq 0$ which satisfies the conclusions of the lemma. Choose $y \in (x, x')$ such that $\mathrm{d}_{F_\Omega(x)}(x, y) > N$. Then there exists $p_y \in [p, y)$ such that whenever $q \in [p_y, y)$, there exists $S_q \in \mathcal{S}_\Lambda$ such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q, r) \subset \mathcal{N}_\Omega(S_q; \varepsilon).$$

Pick a sequence $q_n \in [p_y, y)$ with $q_n \to y$ such that $\mathrm{d}_\Omega(q_n, q_{n+1}) = R_\varepsilon$ for all $n \geq 1$. There exist properly embedded simplices S_n such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q_n, R_\varepsilon) \subset \mathcal{N}_\Omega(S_n; \varepsilon)$$

for all $n \geq 1$. Then, for $n \geq 1$,

$$\begin{aligned}(q_n, q_{n+1}) \subset \mathcal{B}_\Omega(q_n, R_\varepsilon) \cap \mathcal{B}_\Omega(q_{n+1}, R_\varepsilon) \cap \mathbb{P}\left(\mathrm{Span}\{x, x', p\}\right) \\ \subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon).\end{aligned}$$

Thus

$$\operatorname{diam}_\Omega\left(\mathcal{N}_\Omega(S_n;\varepsilon)\cap\mathcal{N}_\Omega(S_{n+1};\varepsilon)\right)\geq \mathrm{d}_\Omega(q_n,q_{n+1})=R_\varepsilon>D_\varepsilon.$$

Then Equation (6.3) implies that $S_n=S_{n+1}=S'$ for all $n\geq 1$. Thus,

$$[p_y,y)\subset\mathcal{N}_\Omega(S';\varepsilon). \tag{6.4}$$

Let $p_x\in S$ be such that $\mathrm{d}_\Omega(p_x,p_y)=\mathrm{d}_\Omega(p_y,S)$. Then Proposition II.16 implies that

$$\mathrm{d}_\Omega{}^{\mathrm{Hauss}}([p_x,x),[p_y,y))\leq R_0:=\max\{\mathrm{d}_\Omega(p_x,p_y),\mathrm{d}_{F_\Omega(x)}(x,y)\}.$$

Then equation (6.4) implies that

$$[p_x,x)\subset S\cap\mathcal{N}_\Omega(S';R_0+\varepsilon),$$

that is, $\operatorname{diam}_\Omega(S\cap\,\mathcal{N}_\Omega(S';R_0+\varepsilon))=\infty$. This violates equation (6.3) unless $S=S'$. Thus, by equation (6.4),

$$[p_y,y)\subset\mathcal{N}_\Omega(S;\varepsilon).$$

Then, by Corollary II.13, there exists $a_y\in\partial S$ such that $y\in F_\Omega(a_y)$ and

$$\mathrm{d}_{F_\Omega(y)}(y,a_y)=\mathrm{d}_{F_\Omega(x)}(y,a_y)\leq\varepsilon.$$

Note that this is true for any $y\in(x,x')$ with $\mathrm{d}_{F_\Omega(x)}(x,y)>N$ (here N depends on ε, see Lemma VI.6). Thus, for $m\geq 1$, we can find a sequence $y_m\in(x,x')$ and $a_m\in\partial S$ with $y_m\to x'$ and $\mathrm{d}_{F_\Omega(x)}(y_m,a_m)<1/m$. Then, by Corollary II.14, $\lim_{m\to\infty}a_m=x'$. Thus, $x'\in\partial S\cap\partial F_\Omega(x)=\partial F_S(x)$. This finishes the proof.

6.2.4 Lines and Half Triangles in the Boundary

Proposition VI.7 ([IZ19]). *Suppose* $(\mathcal{C}, \mathrm{d}_\Omega)$ *has strongly isolated simplices. If* $\ell \subset \partial_{\mathrm{i}} \mathcal{C}$ *is a non-trivial line segment, then there exists* $S \in \mathcal{S}_\Lambda$ *with* $\ell \subset \partial S$.

Proof. We can assume that ℓ is an open line segment with x' as one of its endpoints. Fix some $x \in \ell$, that is $\ell \subset F_{\mathcal{C}}(x)$. Then $x' \in \partial_{\mathrm{i}} F_{\mathcal{C}}(x)$. Now fix $\varepsilon > 0$ and $p \in \mathcal{C}$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, there exists $D(\varepsilon) \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Lambda$ are distinct, then

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_1; \varepsilon) \cap \mathcal{N}_\Omega(S_2; \varepsilon)\right) \leq D(\varepsilon). \tag{6.5}$$

Fix $r_\varepsilon = D(\varepsilon) + 1$. Applying Lemma VI.6 with $r_\varepsilon, \varepsilon$, and p, let $N \geq 0$ be such that it satisfies the conclusions of the lemma. Choose $y \in \ell$ with $\mathrm{d}_{F_\Omega(x)}(x, y) > N$. Then there exists $p_y \in [p, y)$ such that: if $q \in [p_y, y)$, there exists $S_q \in \mathcal{S}_\Lambda$ such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q, r_\varepsilon) \subset \mathcal{N}_\Omega(S_q; \varepsilon).$$

Pick a sequence $q_n \in [p_y, y)$ with $q_n \to y$ such that $\mathrm{d}_\Omega(q_n, q_{n+1}) = r_\varepsilon$. Let $S_n \in \mathcal{S}_\Lambda$ be such that

$$\mathbb{P}(\mathrm{Span}\{x, x', p\}) \cap \mathcal{B}_\Omega(q_n, r_\varepsilon) \subset \mathcal{N}_\Omega(S_n; \varepsilon).$$

Then

$$\begin{aligned}(q_n, q_{n+1}) &\subset \mathcal{B}_\Omega(q_n, r_\varepsilon) \cap \mathcal{B}_\Omega(q_{n+1}, r_\varepsilon) \cap \mathbb{P}\left(\mathrm{Span}\{x, x', p\}\right) \\ &\subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon).\end{aligned}$$

Thus,

$$\operatorname{diam}_\Omega \left(\mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon)\right) \geq r_\varepsilon = D(\varepsilon) + 1 > D(\varepsilon)$$

Then equation (6.5) implies that $S_n = S_{n+1} = S$ for all $n \geq 1$. Then $\{q_n : n \geq 1\} \subset \mathcal{N}_\Omega(S;\varepsilon)$. Then Corollary II.13 implies that $y \in F_\Omega(c)$ for some $c \in \partial S$. As $c \in \partial S$, Proposition VI.5 implies that $F_\Omega(c) = F_S(c) \subset \partial S$. Since $y \in F_\Omega(x)$, this implies that

$$F_\Omega(x) = F_\Omega(y) = F_\Omega(c) \subset \partial S.$$

Finally, since $\ell \subset F_\Omega(x)$,

$$\ell \subset \partial S.$$

□

Proposition VI.8 ([IZ19])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $x, y, z \in \partial_{\mathrm{i}} \mathcal{C}$ form a half triangle in $\mathcal{C}$, then there exists $S \in \mathcal{S}_\Lambda$ such that $x, y, z \in \partial S$.*

Proof. By Proposition VI.7, there exist $S_1, S_2 \in \mathcal{S}_\Lambda$ such that $[x, y] \subset \partial S_1$ and $[y, z] \subset \partial S_2$. Thus $y \in \partial S_1 \cap \partial S_2$. Then Proposition VI.4 implies that $S_1 = S_2 = S$. Hence $x, y, z \in \partial S$. □

6.2.5 Corners in the Boundary

A supporting hyperplane of Ω at $z \in \partial\Omega$ is a co-dimension one projective subspace $\mathbb{P}(H)$ such that $\mathbb{P}(H) \cap \Omega = \emptyset$ and $z \in \mathbb{P}(H) \cap \overline{\Omega}$. We will say that a point $z \in \partial_{\mathrm{i}} \mathcal{C}$ is not C^1-smooth if Ω does not have a unique supporting hyperplane at z. We will show that such a point is necessarily contained in the boundary of a properly embedded simplex.

Proposition VI.9 ([IZ19])**.** *Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. If $z \in \partial_{\mathrm{i}} \mathcal{C}$ is not a C^1-smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_\Lambda$ with $z \in \partial S$.*

In order to prove this, we first establish the following lemma about general convex co-compact subgroups. Note that this lemma does not require that $\mathcal{S}_\Lambda$ is strongly isolated.

Lemma VI.10 ([IZ19]). *Suppose that $z \in \partial_{\mathrm{i}} \mathcal{C}$ is not a C^1-smooth point of $\partial\Omega$ and $q \in \mathcal{C}$. For any $r > 0$ and $\epsilon > 0$ there exists $N > 0$ such that: if $p \in [q, z)$ with $\mathrm{d}_\Omega(q, p) > N$, then there exists a properly embedded simplex $S_p \subset \mathcal{C}$ of dimension at least two such that*

$$\mathcal{B}_\Omega(p; r) \cap (z, q] \subset \mathcal{N}_\Omega(S_p; \epsilon). \tag{6.6}$$

Proof. Fix $r > 0$ and $\epsilon > 0$. Suppose for a contradiction that such a N does not exist. Then we can find $p_n \in (z, q]$ such that $\lim_{n\to\infty} p_n = z$ and

$$\mathcal{B}_\Omega(p, r) \cap (z; q] \not\subset \mathcal{N}_\Omega(S; \varepsilon)$$

for any properly embedded simplex S in $\mathcal{C}$ of dimension at least two.

We can find a 3-dimensional linear subspace V such that $(z, q] \subset \mathbb{P}(V)$ and $z \in \partial_{\mathrm{i}} \mathcal{C}$ is not a C^1-smooth boundary point of $\mathbb{P}(V) \cap \Omega$. By changing coordinates we can suppose that

$$\mathbb{P}(V) = \{[w : x : y : 0 : \cdots : 0] : w, x, y \in \mathbb{R}\},$$

$$\mathbb{P}(V) \cap \Omega \subset \{[1 : x : y : 0 : \cdots : 0] : x \in \mathbb{R},\ y > |x|\},$$

$$z = [1 : 0 : 0 : \cdots : 0], \text{ and}$$

$$q = [1 : 0 : 1 : 0 \cdots : 0].$$

We may also assume that $\mathbb{P}(V) \cap \Omega$ is bounded in the affine chart

$$\{[1 : x : y : 0 : \cdots : 0] : x, y \in \mathbb{R}\}.$$

Then

$$p_n = [1 : 0 : y_n : 0 : \cdots : 0]$$

where $0 < y_n < 1$ and y_n converges to 0. Let

$$L_n := \{[1 : x : y_n : 0 : \dots : 0] : x \in \mathbb{R}\} \cap \Omega.$$

By passing to a subsequence we can suppose that $(y_n)_{n\geq 1}$ is a decreasing sequence and

$$\lim_{n\to\infty} \mathrm{d}_\Omega(p_n, L_{n-1}) = \infty. \tag{6.7}$$

Then

$$\lim_{n\to\infty} \frac{y_{n-1}}{y_n} = \infty.$$

Let $a_n, b_n \in \partial\Omega$ be the endpoints of $L_n = (a_n, b_n)$. We claim that

$$\lim_{n\to\infty} \mathrm{d}_\Omega\Big(p_n, (z, a_{n-1})\Big) = \infty = \lim_{n\to\infty} \mathrm{d}_\Omega\Big(p_n, (z, b_{n-1})\Big). \tag{6.8}$$

Consider $g_n \in \mathrm{PGL}(V)$ defined by

$$g_n([w : x : y : 0 : \dots : 0]) = \left[w : \frac{1}{y_n}x : \frac{1}{y_n}y : \dots : 0\right].$$

Since $(y_n)_{n\geq 1}$ is a decreasing sequence converging to zero, $D_n := g_n(\mathbb{P}(V) \cap \Omega)$ is an increasing sequence of properly convex domains in $\mathbb{P}(V)$ and

$$D := \cup_{n\geq 1} D_n \subset \{[1 : x : y : 0 : \dots : 0] : x \in \mathbb{R},\ y > |x|\}$$

is also a properly convex domain. Notice that d_{D_n} converges to d_D uniformly on compact subsets of D. Also, by construction, there exist $t \leq -1$ and $1 \leq s$ such that

$$D = \{[1 : x : y : 0 : \dots : 0] : x \in \mathbb{R},\ y > \max\{sx, tx\}\}.$$

Then $a_n = [1 : t_n^{-1} y_n : y_n : 0 : \dots : 0]$ where $t_n \to t$.

Now pick $v_n \in (z, a_{n-1})$ such that

$$\mathrm{d}_\Omega\Big(p_n, (z, a_{n-1})\Big) = \mathrm{d}_\Omega(p_n, v_n).$$

Since

$$\lim_{n\to\infty} g_n a_{n-1} = \lim_{n\to\infty} \left[1 : t_{n-1}^{-1}\frac{y_{n-1}}{y_n} : \frac{y_{n-1}}{y_n} : 0 : \ldots : 0\right] = [0 : t^{-1} : 1 : 0 : \cdots : 0]$$

any limit point of $g_n v_n$ is in

$$\{[0 : t^{-1} : 1 : 0 : \cdots : 0]\} \cup \{[1 : rt^{-1} : r : 0 : \cdots : 0] : r \geq 0\} \subset \partial D.$$

Then

$$\lim_{n\to\infty} \mathrm{d}_\Omega \Big(p_n, (z, a_{n-1})\Big) = \lim_{n\to\infty} \mathrm{d}_\Omega(p_n, v_n) = \lim_{n\to\infty} \mathrm{d}_{D_n} \Big(g_n p_n, g_n v_n\Big) = \infty$$

since $g_n p_n \to [1 : 0 : 1 : 0 : \cdots : 0] \in D$.

For the same reasons,

$$\lim_{n\to\infty} \mathrm{d}_\Omega \Big(p_n, (z, b_{n-1})\Big) = \infty.$$

This establishes Equation (6.8).

Next we can pass to a subsequence and find $\gamma_n \in \Lambda$ such that $\gamma_n p_n \to p_\infty \in \mathcal{C}$. Passing to a further subsequence we can suppose that $\gamma_n a_{n-1} \to a_\infty$, $\gamma_n b_{n-1} \to b_\infty$, $\gamma_n z \to z_\infty$, and $\gamma_n q \to q_\infty$.

Equation (6.7) implies that $[a_\infty, b_\infty] \subset \partial\Omega$ and Equation (6.8) implies that

$$[z_\infty, a_\infty] \cup [z_\infty, b_\infty] \subset \partial\Omega.$$

Thus $a_\infty, b_\infty, z_\infty$ are the vertices of a properly embedded simplex $S \subset \Omega$ which contains p_∞. Further, for n sufficiently large we have

$$\mathcal{B}_\Omega(\gamma_n p_n, r) \cap \gamma_n(z, q] \subset \mathcal{N}_\Omega(S; \epsilon)$$

and so

$$\mathcal{B}_\Omega(p_n, r) \cap (z, q] \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \epsilon).$$

To obtain a contradiction we have to show that $\gamma_n^{-1} S \subset \mathcal{C}$ for every n or equivalently that $S \subset \mathcal{C}$. By construction, $q_\infty \in \partial_{\mathrm{i}} \mathcal{C} \cap (a_\infty, b_\infty)$. Then Theorem II.39 implies that

$$(a_\infty, b_\infty) \subset F_\Omega(q_\infty) = F_{\mathcal{C}}(q_\infty) \subset \partial_{\mathrm{i}} \mathcal{C}.$$

Thus $[a_\infty, b_\infty] \subset \partial_{\mathrm{i}} \mathcal{C}$. Since $z_\infty \in \partial_{\mathrm{i}} \mathcal{C}$ and S has vertices $a_\infty, b_\infty, z_\infty$ we then see that $S \subset \mathcal{C}$. $\square$

We now finish the proof of Proposition VI.9. The strategy is similar to the proof of Proposition VI.7. Fix $\varepsilon > 0$ and $q \in \mathcal{C}$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, there exists $D_\varepsilon \geq 0$ such that: if $S_1, S_2 \in \mathcal{S}_\Lambda$ are distinct, then

$$\mathrm{diam}_\Omega \left(\mathcal{N}_\Omega(S_1; \varepsilon) \cap \mathcal{N}_\Omega(S_2; \varepsilon) \right) \leq D(\varepsilon). \tag{6.9}$$

Fix $r_\varepsilon := D(\varepsilon) + 1$. Applying the above Lemma VI.6 with $r_\varepsilon, \varepsilon$ and q, we get $N \geq 0$ which satisfies the conclusions of the lemma. Pick a sequence $z_n \in [q, z)$ with $z_n \to z$, $\mathrm{d}_\Omega(z_n, z_{n+1}) = r_\varepsilon$, and $\mathrm{d}_\Omega(q, z_n) \geq N$ for $n \geq 1$. Then, for each $n \geq 1$, there exist $S_n \in \mathcal{S}_\Lambda$ such that:

$$[q, z) \cap \mathcal{B}_\Omega(z_n, r_\varepsilon) \subset \mathcal{N}_\Omega(S_n; \varepsilon).$$

Then, if $n \geq 1$,

$$\begin{aligned} (z_n, z_{n+1}) &\subset \mathcal{B}_\Omega(z_n, r_\varepsilon) \cap \mathcal{B}_\Omega(z_{n+1}, r_\varepsilon) \cap [q, z) \\ &\subset \mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon). \end{aligned}$$

Thus

$$\mathrm{diam}_\Omega \left(\mathcal{N}_\Omega(S_n; \varepsilon) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon) \right) \geq \mathrm{d}_\Omega(z_n, z_{n+1}) = r_\varepsilon > D_\varepsilon.$$

Then Equation (6.9) implies that $S_n = S_{n+1} = S$ for all $n \geq 1$. Thus $\{z_n : n \in \mathbb{N}\} \subset \mathcal{N}_\Omega(S; \varepsilon)$. Corollary II.13 then implies that there exists $c \in \partial S$ such that $z \in F_\Omega(c)$. As $c \in \partial S$, Proposition VI.5 implies that $F_\Omega(c) = F_S(c) \subset \partial S$. Thus, $z \in \partial S$.

6.3 Relationship with Convex Co-compact Rank One

This section is based on the Apppendix of [Isl19].

6.3.1 Definition of Convex Co-compact Rank One

If Λ is a convex co-compact group, then the ideal boundary $\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ is the only part of $\partial\Omega$ that is accessible by the dynamics of Λ on Ω. Thus it is necessary to modify the notion of rank one automrophisms for convex co-compact actions. For this, we consider half triangles in $\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ instead of $\partial\Omega$. We will say that x, y, z form a half triangle in $\mathcal{C}_\Omega(\Lambda)$ provided $[x, y] \cup [y, z] \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ and $(x, z) \subset \mathcal{C}_\Omega(\Lambda)$.

Definition VI.11 ([Isl19])**.** Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group.

(1) An element $g \in \Lambda$ is called a convex co-compact rank one automorphism if:

  (a) $\tau_{\mathcal{C}_\Omega(\Lambda)}(g) = \log \dfrac{\lambda_1}{\lambda_d}(g) > 0$ and g has an axis,

  (b) if ℓ_g is an axis of g, then ℓ_g is not contained in any half triangle in $\mathcal{C}_\Omega(\Lambda)$.

(2) Λ is called a convex co-compact rank one group if Λ contains a convex co-compact rank one automorphism.

Remark VI.12.

(1) The notion of a convex co-compact rank-one automorphism *that we just defined differs from the notion of a* rank-one automorphism *only in the half triangle condition: for the latter, we consider half triangles in* $\mathcal{C}_\Omega(\Lambda)$ *instead of* Ω.

(2) Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ *acts co-compactly on* Ω. *Then* $\mathcal{C}_\Omega(\Lambda) = \Omega$ *which implies that* Λ *is a rank one group if and only if* Λ *is a convex co-compact rank one group.*

We now have analogues of Proposition IV.14 and Lemma IV.16. The same proofs go through after replacing Ω by $\mathcal{C}_\Omega(\Lambda)$ and $\partial\Omega$ by $\partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$. This is essentially because

$E_\gamma^+ \cap \partial\Omega = E_\gamma^+ \cap \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda)$ (same for E_γ^-).

Proposition VI.13. *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact rank one group and $g \in \Lambda$ is a convex co-compact rank one automorphism with axis $\ell_g = (a, b)$ where $a \in E_g^+$ and $b \in E_g^-$. Then:*

(1) g is biproximal,

(2) ℓ_g is the unique axis of g in Ω,

(3) the only fixed points of g in $\overline{\mathcal{C}_\Omega(\Lambda)}$ are a and b,

(4) if $z' \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda) \setminus \{a, b\}$, then $(a, z') \cup (z', b) \subset \mathcal{C}_\Omega(\Lambda)$, and

(5) if $z \in \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Lambda) \setminus \{a, b\}$, then neither (a, z) nor (z, b) is contained in a half triangle in $\mathcal{C}_\Omega(\Lambda)$.

Lemma VI.14. *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact rank one group and $g \in \Lambda$ has an axis. Then the following are equivalent:*

(1) g is biproximal.

(2) none of the axes of g is contained in a half triangle in $\mathcal{C}_\Omega(\Lambda)$.

(3) g is a convex co-compact rank one automorphism.

6.3.2 Strongly Isolated Simplices imply Convex Co-compact Rank One

Proposition VI.15 ([Isl19])**.** *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices. Then either Λ is a virtually Abelian group or Λ is a convex co-compact rank one group.*

By virtue of Theorem I.15, this proposition is equivalent to the following:

Proposition VI.16 ([Isl19])**.** *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group that is relatively hyperbolic with respect to $\{A_1, A_2, \ldots, A_m\}$ where each A_i is a virtually*

Abelian group of rank at least two. Then either Λ is a virtually Abelian group or Λ is a convex co-compact rank one group.

We will spend the rest of this section proving Proposition VI.16. Note that we will rely heavily on Theorem I.15 and Theorem I.16 in this proof.

For the rest of this section, fix a convex co-compact group $\Lambda \leq \mathrm{Aut}(\Omega)$. Set $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$ and let $\mathcal{S}_\Lambda$ be the collection of all maximal properly embedded simplices in $\mathcal{C}$ of dimension at least two . We assume that $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices and that Λ is relatively hyperbolic with respect to $\{A_1, A_2, \dots, A_m\}$. By [IZ19, Theorem 1.18], we can assume that $A_i = \mathrm{Stab}_\Lambda(S_i)$ for $1 \leq i \leq m$ where $\mathcal{S}_\Lambda = \sqcup_{i=1}^m \Lambda S_i$.

Since Λ is relatively hyperbolic with respect to $\{A_1, A_2, \dots, A_m\}$, [DG18, Lemma 2.3] implies that either Λ is virtually contained in a conjugate of some A_i or Λ contains an infinite order element that is not contained in any conjugate of any A_i for $1 \leq i \leq m$. In the first case, Λ is a virtually Abelian group. So we can now assume that we are in the second case. Then there exists an infinite order element $\gamma \in \Lambda$ such that

$$\gamma \notin \bigcup_{g \in \Lambda} \bigcup_{i=1}^m g A_i g^{-1} = \bigcup_{S \in \mathcal{S}_\Lambda} \mathrm{Stab}_\Lambda(S).$$

We will show that γ is a convex co-compact rank one automorphism. As the action is convex co-compact $\tau_{\mathcal{C}}(\gamma) > 0$. We first show that γ has an axis in $\mathcal{C}_\Omega(\Lambda)$. Let $\mathcal{C}^+ := \overline{E_\gamma^+ \cap \mathcal{C}}$ and $\mathcal{C}^- := \overline{E_\gamma^- \cap \mathcal{C}}$. As $\mathcal{C}^+$ and $\mathcal{C}^-$ are non-empty, compact, convex, γ-invariant subsets of $\mathbb{R}^d$, the Brouwer fixed point theorem implies the existence of two fixed points γ^+ and γ^- of γ in $\mathcal{C}^+$ and $\mathcal{C}^-$ respectively. If $[\gamma^+, \gamma^-] \subset \partial_{\mathrm{i}} \mathcal{C}$, then Theorem I.16 part (6) implies that there exists $S \in \mathcal{S}_\Lambda$ such that $[\gamma^+, \gamma^-] \subset \partial S$. Then $\partial(\gamma S) \cap \partial S \supset [\gamma^+, \gamma^-]$. Theorem I.16 part (5) implies that $\gamma S = S$. Thus, $\gamma \in \mathrm{Stab}_\Lambda(S)$, a contradiction. Thus $(\gamma^+, \gamma^-) \subset \mathcal{C}$ and is an axis of γ.

Suppose $A_\gamma^+, z, A_\gamma^-$ is contained in a half triangle in $\mathcal{C}$ where $[A_\gamma^+, A_\gamma^-]$ is an axis of

γ. By Theorem I.16 part (7), there exists $S \in \mathcal{S}_\Lambda$ such that $A_\gamma^+, z, A_\gamma^- \in \partial S$. Then the axis of γ is contained in S. Arguing as above, $\gamma \in \mathrm{Stab}_\Lambda(S)$. This finishes the proof that γ is a rank one automorphism.

Hence Λ is a convex co-compact rank one group.

CHAPTER VII

Relative Hyperbolicity, Convex Co-compactness, and Strongly Isolated Simplices

This chapter is based on results that are contained in [IZ19] which is a joint work with A. Zimmer. In [IZ19], we prove all these results for naive convex co-compact groups, a class which is strictly larger than convex co-compact groups. Restricting to the case of convex co-compact groups makes many of the arguments much easier and hence affords a clearer exposition.

7.1 Outline

This chapter is devoted to the proof of Theorem I.15 which we now restate.

Theorem I.15.([IZ19, Theorem 1.7]) *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group, and $\mathcal{S}_\Lambda$ is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two. Then the following are equivalent:*

(1) $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ *has strongly isolated simplices,*

(2) $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ *is a relatively hyperbolic space with respect to $\mathcal{S}_\Lambda$,*

(3) Λ *is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank at least two.*

The most difficult part of the proof is (1) implies (2). This is done in Section 7.3. For this proof, we rely on Sisto's characterization of relative hyperbolicity (cf. Theorem III.15). A key ingredient of this proof is the notion of closest-point projection onto properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ (cf. Definition II.28) and its comparison with linear projections on simplices (cf. Definition II.24). Results proven in Section 7.2 play a key role in Section 7.3.

The proof of (3) implies (1) is also quite involved since we have to prove that $\mathcal{C}_\Omega(\Lambda)$ is relatively hyperbolic with respect to the collection of **all** simplices in $\mathcal{S}_\Lambda$. This is done in Section 7.4. The rest of the parts of the proof of Theorem I.15 is also in this section.

7.2 Closest-point Projections on Simplices

For the rest of this section fix a convex co-compact group $\Lambda \leq \mathrm{Aut}(\Omega)$. Set $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$ and $\mathcal{S} := \mathcal{S}_\Lambda$. We will assume that $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices for rest of this section.

Suppose S is a properly embedded simplex in $\mathcal{C}$. Since S is a closed convex subset, we can follow Definition II.28 and define the closest-point projection onto S. We will denote it by π_S. On the other hand, if $\mathcal{H}$ is a set of S-supporting hyperplanes, then we have a notion of linear projection onto S which we will denote by $L_{S,\mathcal{H}}$, see Definition II.24.

We will establish a coarse equivalence between the two projections. But first we need a continuity lemma for linear projections.

Lemma VII.1 ([IZ19, Lemma 13.4])**.** *If $S \in \mathcal{S}_\Lambda$, then the map*

$$(L, x) \in \mathcal{L}_S \times \overline{\mathcal{C}} \to L(x) \in \overline{S}$$

is continuous.

Proof. We first show that $\mathbb{P}(\ker L)\cap\overline{\mathcal{C}}=\emptyset$ for all $L\in\mathcal{L}_S$. Suppose for a contradiction that $L\in\mathcal{L}_S$ and

$$x\in\mathbb{P}(\ker L)\cap\overline{\mathcal{C}}.$$

Proposition II.23 implies that $x\in\partial_{\mathrm{i}}\mathcal{C}$. Then Proposition II.25 implies that $[y,x]\subset\partial_{\mathrm{i}}\mathcal{C}$ for every $y\in\partial S$. Next fix $y_1,y_2\in\partial S$ such that $(y_1,y_2)\subset S$. Then y_1,x,y_2 form a half triangle. By Theorem I.16, $y_1,x,y_2\in\partial S$ for some $S\in\mathcal{S}_\Lambda$. Since $x\in\partial S\subset\operatorname{Span}(S)$, $x\notin\ker L$, a contradiction.

Thus $\mathbb{P}(\ker L)\cap\overline{\mathcal{C}}=\emptyset$ for all $L\in\mathcal{L}_S$.

Now suppose that $\lim_{n\to\infty}(L_n,x_n)=(L,x)$ in $\mathcal{L}_S\times\overline{\mathcal{C}}$. Let $\widetilde{x}_n,\widetilde{x}$ denote lifts of x_n,x respectively such that $\lim_{n\to\infty}\widetilde{x}_n=\widetilde{x}$. Then

$$L(\widetilde{x})=\lim_{n\to\infty}L_n(\widetilde{x}_n)\in\mathbb{R}^d.$$

Since $\mathbb{P}(\ker L)\cap\overline{\mathcal{C}}=\emptyset$, we have $L(\widetilde{x})\neq 0$. So

$$L(x)=[L(\widetilde{x})]=\lim_{n\to\infty}[L_n(\widetilde{x}_n)]=\lim_{n\to\infty}L_n(x_n).\qquad\square$$

Now the proof of equivalence.

Proposition VII.2 ([IZ19, Proposition 13.7]). *There exists $\delta_1\geq 0$ such that: if $S\in\mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, and $x\in\mathcal{C}$, then*

$$\max_{p\in\pi_S(x)}\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x),p)\leq\delta_1.$$

Proof. Since $\mathcal{S}$ has finitely many Λ orbits (see Proposition VI.3), it is enough to prove the result for some fixed $S\in\mathcal{S}$.

Suppose the proposition is false. Then, for every $n\geq 0$, there exist $x_n\in\mathcal{C}$, a set of S-supporting hyperplanes $\mathcal{H}_n$, and $p_n\in\pi_S(x_n)$ such that

$$\mathrm{d}_\Omega(p_n,L_{S,\mathcal{H}_n}(x_n))\geq n.$$

Let m_n be the midpoint of the projective line segment $[p_n, L_{S,\mathcal{H}_n}(x_n)]$ in the Hilbert distance. Since $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S (see Proposition VI.3), translating by elements of $\mathrm{Stab}_\Lambda(S)$ and passing to a subsequence, we can assume that $m := \lim_{n\to\infty} m_n$ exists in S. Passing to a further subsequence and using Proposition II.27, we can assume that there exists $x, p, x' \in \partial_{\mathrm{i}}\,\mathcal{C}$ and $L_{S,\mathcal{H}} \in \mathcal{L}_S$ where $x := \lim_{n\to\infty} x_n$, $p := \lim_{n\to\infty} p_n$, $x' := \lim_{n\to\infty} L_{S,\mathcal{H}_n}(x_n)$, and $L_{S,\mathcal{H}} := \lim_{n\to\infty} L_{S,\mathcal{H}_n}$. By Lemma VII.1,

$$L_{S,\mathcal{H}}(x) = \lim_{n\to\infty} L_{S,\mathcal{H}_n}(x_n) = x'.$$

We first show that $[x', x] \subset \partial_{\mathrm{i}}\,\mathcal{C}$. Observe that $L_{S,\mathcal{H}}(v) = x'$ for all $v \in [x', x]$ since $L_{S,\mathcal{H}}$ is linear and $L_{S,\mathcal{H}}(x') = x' = L_{S,\mathcal{H}}(x)$. But $L_{S,\mathcal{H}}(\Omega) = S$, implying $[x', x] \cap \Omega = \emptyset$. Hence,

$$[x', x] \subset \partial_{\mathrm{i}}\,\mathcal{C}.$$

Next we show that $[p, x] \subset \partial_{\mathrm{i}}\,\mathcal{C}$. Suppose not, then $(p, x) \subset \mathcal{C}$. Choose any $v \in (p, x) \cap \mathcal{C}$ and a sequence $v_n \in [p_n, x_n]$ such that $v = \lim_{n\to\infty} v_n$. Since $p \in \partial_{\mathrm{i}}\,\mathcal{C}$ and $v \in \mathcal{C}$,

$$\lim_{n\to\infty} \mathrm{d}_\Omega(v_n, p_n) = \infty.$$

Fix any $v_S \in S$. Then, choosing n large enough so that $\mathrm{d}_\Omega(v_n, p_n) \geq 2 + \mathrm{d}_\Omega(v, v_S)$ and $\mathrm{d}_\Omega(v, v_n) \leq 1$,

$$\begin{aligned}
\mathrm{d}_\Omega(x_n, v_S) &\leq \mathrm{d}_\Omega(x_n, v_n) + \mathrm{d}_\Omega(v_n, v) + \mathrm{d}_\Omega(v, v_S) \\
&= \mathrm{d}_\Omega(x_n, p_n) - \mathrm{d}_\Omega(p_n, v_n) + \mathrm{d}_\Omega(v_n, v) + \mathrm{d}_\Omega(v, v_S) \\
&\leq \mathrm{d}_\Omega(x_n, p_n) - 1,
\end{aligned}$$

which is a contradiction since $p_n \in \pi_S(x_n)$. Hence, $[p, x] \subset \partial_{\mathrm{i}}\,\mathcal{C}$.

Thus, $[p,x] \cup [x,x'] \subset \partial_{\mathrm{i}}\mathcal{C}$ and by construction, $m \in (p,x) \subset \mathcal{C}$. Thus the three points p, x, x' form half triangle in $\mathcal{C}$. Then Theorem I.16 part (7) implies that $p, x, x' \in \partial S$ for some $S \in \mathcal{S}_\Lambda$. Then $x' = L_{S,\mathcal{H}}(x) = x$ which implies $[p,x] = [p,x'] \subset \partial_{\mathrm{i}}\mathcal{C}$. This is a contradiction since $(p,x) \subset \mathcal{C}$ by construction. $\square$

The next result proves δ-slimness of some special triangles built using linear projections.

Proposition VII.3 ([IZ19, Proposition 13.9])**.** *There exists $\delta_2 \geq 0$ such that: if $x \in \mathcal{C}$, $S \in \mathcal{S}$, $z \in S$, and $\mathcal{H}$ is a set of S-supporting hyperplanes, then the geodesic triangle*

$$\big[x,z\big] \cup \big[z, L_{S,\mathcal{H}}(x)\big] \cup \big[L_{S,\mathcal{H}}(x), x\big]$$

is δ_2-thin.

Proof. Since $\mathcal{S}$ has finitely many Λ orbits (see Proposition VI.3), it is enough to prove the result for some fixed $S \in \mathcal{S}$. By Lemma IV.19, it is enough to show that there exists $\delta_2 \geq 0$ such that

$$[L_{S,\mathcal{H}}(x), z] \subset \mathcal{N}_{\delta_2/2}([z,x] \cup [x, L_{S,\mathcal{H}}(x)])$$

for all $x \in \mathcal{C}$, $z \in S$, and $\mathcal{H}$ a set of S-supporting hyperplanes.

Suppose such a δ_2 does not exist. Then, for every $n \geq 0$, there exist $z_n \in S$, a set of S-supporting hyperplanes $\mathcal{H}_n$, $p_n := L_{S,\mathcal{H}_n}(x_n)$, and $u_n \in [z_n, p_n]$ such that

$$\mathrm{d}_\Omega(u_n, [z_n, x_n] \cup [x_n, p_n]) \geq n.$$

Since $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S, translating by elements of $\mathrm{Stab}_\Lambda(S)$ and passing to a subsequence, we can assume that $u := \lim_{n\to\infty} u_n$ exists and $u \in S$. Passing to a further subsequence and using Proposition II.27, we can assume there exist $x, z, p \in \overline{\mathcal{C}}$ and $L_{S,\mathcal{H}} \in \mathcal{L}_S$ where $x := \lim_{n\to\infty} x_n$, $z := \lim_{n\to\infty} z_n$, $p := \lim_{n\to\infty} p_n$, and $L_{S,\mathcal{H}} := \lim_{n\to\infty} L_{S,\mathcal{H}_n}$. Since

$$\begin{aligned}&\lim_{n\to\infty} \mathrm{d}_\Omega(u, [x_n, z_n] \cup [x_n, p_n])\\ &\qquad \geq \lim_{n\to\infty} \Big(\mathrm{d}_\Omega(u_n, [x_n, z_n] \cup [x_n, p_n]) - \mathrm{d}_\Omega(u, u_n)\Big) = \infty,\end{aligned}$$

we have

$$[x, z] \cup [x, p] \subset \partial_{\mathrm{i}} \mathcal{C}.$$

By construction, $u \in (p, z) \subset \mathcal{C}$. Thus, p, x, z form a half triangle in $\mathcal{C}$.

By Theorem I.16 part (7), $p, x, z \in \partial S$ for some $S \in \mathcal{S}_\Lambda$. Lemma VII.1 then implies that

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} L_{S,\mathcal{H}_n}(x_n) = L_{S,\mathcal{H}}(x) = x.$$

Thus $[p, z] = [p, x] \subset \partial_{\mathrm{i}} \mathcal{C}$ which is a contradiction since $(p, z) \subset \mathcal{C}$ by construction.

□

Let δ_1 and δ_2 be the constants as in Propositions VII.2 and VII.3.

Proposition VII.4 ([IZ19, Proposition 13.10])**.** *Set $\delta_3 := \delta_1 + 3\delta_2$. If $x \in \mathcal{C}$, $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, and $z \in S$, then $\mathrm{d}_\Omega\left(L_{S,\mathcal{H}}(x), [x, z]\right) \leq \delta_3$.*

Proof. By Proposition VII.3, the geodesic triangle

$$[x, z] \cup [z, L_{S,\mathcal{H}}(x)] \cup [L_{S,\mathcal{H}}(x), x]$$

is δ_2-thin. Thus, there exist $y \in [L_{S,\mathcal{H}}(x), z]$, $y_1 \in [x, L_{S,\mathcal{H}}(x)]$, and $y_2 \in [x, z]$ such that $\mathrm{d}_\Omega(y, y_1) \leq \delta_2$ and $\mathrm{d}_\Omega(y, y_2) \leq \delta_2$.

We claim that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), y_1) \leq \delta_1 + \delta_2$. Choose any $p \in \pi_S(x)$. Since $[L_{S,\mathcal{H}}(x), z] \subset S$,

$$\mathrm{d}_\Omega(x, p) = \mathrm{d}_\Omega(x, S) \leq \mathrm{d}_\Omega(x, y).$$

Then, using Proposition VII.2,

$$d_\Omega(x, L_{S,\mathcal{H}}(x)) \leq d_\Omega(x,p) + d_\Omega(p, L_{S,\mathcal{H}}(x)) \leq d_\Omega(x,y) + \delta_1.$$

Then,

$$\begin{aligned} d_\Omega(L_{S,\mathcal{H}}(x), y_1) &= d_\Omega(L_{S,\mathcal{H}}(x), x) - d_\Omega(y_1, x) \\ &\leq d_\Omega(x,y) + \delta_1 - d_\Omega(y_1, x) \\ &\leq d_\Omega(y, y_1) + \delta_1 \leq \delta_2 + \delta_1. \end{aligned}$$

Hence,

$$\begin{aligned} d_\Omega(L_{S,\mathcal{H}}(x), [x,z]) &\leq d_\Omega(L_{S,\mathcal{H}}(x), y_2) \\ &\leq d_\Omega(L_{S,\mathcal{H}}(x), y_1) + d_\Omega(y_1, y) + d_\Omega(y, y_2) \\ &\leq \delta_1 + 3\delta_2 = \delta_3. \end{aligned}$$

□

Our next goal is to prove if the distance between the linear projections of two points onto a simplex $S \in \mathcal{S}$ is large, then the geodesic between the two points spends a significant amount of time in a tubular neighborhood of S. This is accomplished in Corollary VII.6 using the next result.

Proposition VII.5 ([IZ19, Proposition 13.11])**.** *There exists a constant $\delta_4 \geq 0$ such that: if $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, $x, y \in \mathcal{C}$, and $d_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq \delta_4$, then*

$$d_\Omega(L_{S,\mathcal{H}}(x), [x,y]) \leq \delta_4 \quad \textit{and} \quad d_\Omega(L_{S,\mathcal{H}}(y), [x,y]) \leq \delta_4.$$

Proof. Observe that the linear projections are Λ-equivariant, that is,

$$L_{gS, g\mathcal{H}} \circ g = g \circ L_{S,\mathcal{H}}$$

for any $g \in \Lambda$, $S \in \mathcal{S}$, and $\mathcal{H}$ a set of S-supporting hyperplanes. Moreover, by Proposition VI.3 there are only finitely many Λ-orbits in $\mathcal{S}$. Thus, it is enough to prove this proposition for a fixed $S \in \mathcal{S}$.

Suppose the proposition is false. Then, for every $n \geq 0$, there exist $x_n, y_n \in \mathcal{C}$ and a set of S-supporting hyperplanes $\mathcal{H}_n$ with

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}_n}(x_n), L_{S,\mathcal{H}_n}(y_n)) \geq n$$

and

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}_n}(x_n), [x_n, y_n]) \geq n.$$

Let $a_n := L_{S,\mathcal{H}_n}(x_n)$ and $b_n := L_{S,\mathcal{H}_n}(y_n)$. Then pick $c_n \in [a_n, b_n]$ such that

$$\mathrm{d}_\Omega(c_n, a_n) = n/2. \tag{7.1}$$

Then,

$$\mathrm{d}_\Omega(c_n, b_n) \geq \mathrm{d}_\Omega(a_n, b_n) - \mathrm{d}_\Omega(c_n, a_n) \geq n/2 \tag{7.2}$$

and

$$\mathrm{d}_\Omega\Big(c_n, [x_n, y_n]\Big) \geq \mathrm{d}_\Omega\Big(a_n, [x_n, y_n]\Big) - \mathrm{d}_\Omega(c_n, a_n) \geq n/2. \tag{7.3}$$

Since $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S (see Proposition VI.3), translating by elements of $\mathrm{Stab}_\Lambda(S)$ and passing to a subsequence, we may assume that $c := \lim_{n\to\infty} c_n$ exists and $c \in S$. After taking a further subsequence, we can assume that the following limits exist in $\overline{\mathcal{C}}$: $a := \lim_{n\to\infty} a_n$, $b := \lim_{n\to\infty} b_n$, $x := \lim_{n\to\infty} x_n$ and $y := \lim_{n\to\infty} y_n$.

We now observe that $a, b, x, y \in \partial_{\mathrm{i}}\, \mathcal{C}$. Equation (7.1) and (7.2) imply that $a, b \in \partial_{\mathrm{i}}\, \mathcal{C}$. Equation (7.3) implies that $[x, y] \subset \partial_{\mathrm{i}}\, \mathcal{C}$.

We claim that $x \in F_\Omega(a)$ and $y \in F_\Omega(b)$. Since $c_n \in S$, by Proposition VII.4, there exists $a_n' \in [x_n, c_n]$ such that $\mathrm{d}_\Omega(a_n, a_n') \leq \delta_3$. Up to passing to a subsequence, we can assume that $a' := \lim_{n\to\infty} a_n'$ exists in $\overline{\mathcal{C}}$. Observe that $a' \in \partial_{\mathrm{i}} \mathcal{C}$ since

$$\lim_{n\to\infty} \mathrm{d}_\Omega(a_n', c) \geq \lim_{n\to\infty} \Big(\mathrm{d}_\Omega(a_n, c_n) - \mathrm{d}_\Omega(c_n, c) - \mathrm{d}_\Omega(a_n, a_n') \Big) = \infty.$$

Since $a_n' \in [x_n, c_n]$,

$$a' \in \partial_{\mathrm{i}} \mathcal{C} \cap [x, c] = \{x\}.$$

Thus, $\lim_{n\to\infty} a_n' = x$. Since $\lim_{n\to\infty} a_n = a$ and $\mathrm{d}_\Omega(a_n, a_n') \leq \delta_3$, Proposition II.12 implies that $x \in F_\Omega(a)$. Similar reasoning shows that $y \in F_\Omega(b)$.

Since $[x, y] \subset \partial_{\mathrm{i}} \mathcal{C}$, Proposition II.11 part (4) implies that $[a, b] \subset \partial_{\mathrm{i}} \mathcal{C}$. This is a contradiction since $c \in (a, b) \cap \mathcal{C} \neq \emptyset$. $\square$

Corollary VII.6 ([IZ19, Corollary 13.12]). *If $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, $R > 0$, $x, y \in \mathcal{C}$, and $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq R + 2\delta_4$, then:*

(1) there exists $[x_0, y_0] \subset [x, y]$ such that $[x_0, y_0] \subset \mathcal{N}_\Omega(S; \delta_4)$,

(2) $\big[L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)\big] \subset \mathcal{N}_\Omega\big([x, y]; \delta_4\big)$, and,

(3) $\mathrm{diam}_\Omega \Big(\mathcal{N}_\Omega(S; \delta_4) \cap [x, y]\Big) \geq R$.

Proof. Since $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) > \delta_4$, Proposition VII.5 implies that there exists $x_0, y_0 \in [x, y]$ such that

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x_0) \leq \delta_4 \quad \text{and} \quad \mathrm{d}_\Omega(L_{S,\mathcal{H}}(y), y_0) \leq \delta_4.$$

By Proposition II.16,

$$\mathrm{d}_\Omega^{\mathrm{Hauss}} \Big(\big[x_0, y_0\big], \big[L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)\big] \Big) \leq \delta_4$$

and, by convexity, $[L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)] \subset S$. This proves parts (1) and (2). To prove part (3), observe that

$$\begin{aligned} \mathrm{d}_\Omega(x_0, y_0) &\geq \mathrm{d}_\Omega\left(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)\right) - \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x_0) - \mathrm{d}_\Omega(L_{S,\mathcal{H}}(y), y_0) \\ &\geq R. \end{aligned}$$

Then, $\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S;\delta_4) \cap [x,y]\Big) \geq \mathrm{d}_\Omega(x_0, y_0) \geq R$. $\square$

7.3 Strongly Isolated Simplices implies Relative Hyperbolicity

For the rest of this section fix a convex co-compact group $\Lambda \leq \mathrm{Aut}(\Omega)$ for which $(\mathcal{C}_\Omega(\Lambda), \mathrm{d}_\Omega)$ has strongly isolated simplices. Set $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$ and $\mathcal{S} := \mathcal{S}_\Lambda$.

We will prove that (1) $\implies$ (2) in Theorem I.15, that is, $(\mathcal{C}, \mathrm{d}_\Omega)$ is a relatively hyperbolic space with respect to $\mathcal{S}_\Lambda$. Since $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices, the results of Section 7.2 hold. For each $S \in \mathcal{S}$, fix a set $\mathcal{H}_S$ of S-supporting hyperplanes. Consider the family of projection maps

$$\Pi_{\mathcal{S}} := \{L_{S,\mathcal{H}} : S \in \mathcal{S}, \mathcal{H} = \mathcal{H}_S\}$$

and the geodesic path system

$$\mathcal{G} := \{[x,y] : x, y \in \mathcal{C}\}$$

on $\mathcal{C}$. By Theorem III.15, it is enough to verify that $\Pi_{\mathcal{S}}$ is an almost projection system and that $\mathcal{S}$ is asymptotically transverse-free relative to $\mathcal{G}$. We complete this in the next two subsections (cf. 7.3.1 and 7.3.2).

7.3.1 $\Pi_{\mathcal{S}}$ is an Almost Projection System

Let δ_3 be the constant in Proposition VII.4.

Lemma VII.7 ([IZ19, Lemma 13.13]). *If $S \in \mathcal{S}$, $\mathcal{H}$ a set of S-supporting hyperplanes, $x \in \mathcal{C}$, and $z \in S$, then*

$$\mathrm{d}_\Omega(x,z) \geq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) + \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), z) - 2\delta_3.$$

Proof. By Proposition VII.4, there exists $q \in [x,z]$ such that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), q) \leq \delta_3$. Then,

$$\mathrm{d}_\Omega(x,z) = \mathrm{d}_\Omega(x,q) + \mathrm{d}_\Omega(q,z) \geq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) + \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), z) - 2\delta_3. \qquad \square$$

Lemma VII.8 ([IZ19, Lemma 13.14]). *There exists a constant $\delta_5 \geq 0$ such that: if $S \neq S' \in \mathcal{S}$ and $\mathcal{H}$ is a set of S-supporting hyperplanes, then*

$$\mathrm{diam}_\Omega(L_{S,\mathcal{H}}(S')) \leq \delta_5.$$

Proof. Since $\mathcal{S}$ is strongly isolated, for every $r > 0$ there exists $D(r) > 0$ such that

$$\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega\big(S_1; r\big) \cap \mathcal{N}_\Omega\big(S_2, r\big)\Big) \leq D(r) \tag{7.4}$$

for all $S_1, S_2 \in \mathcal{S}$ distinct.

Let δ_4 be the constant in Proposition VII.5. Set $\delta_5 := D\big(\delta_4\big) + 2\delta_4 + 1$. Fix $x, y \in S'$ and suppose for a contradiction that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq \delta_5$. Then, by Corollary VII.6,

$$\mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S;\delta_4) \cap S'\Big) \geq \mathrm{diam}_\Omega\Big(\mathcal{N}_\Omega(S;\delta_4) \cap [x,y]\Big) \geq D(\delta_4) + 1.$$

which contradicts Equation (7.4). $\square$

Let δ_1 and δ_4 be the constants in Proposition VII.2 and VII.5 respectively.

Lemma VII.9 ([IZ19, Lemma 13.15]). *If $x \in \mathcal{C}$, $S \in \mathcal{S}$, $\mathcal{H}$ is a set of S-supporting hyperplanes, and $R := \mathrm{d}_\Omega(x, S)$, then*

$$\mathrm{diam}_\Omega\bigg(L_{S,\mathcal{H}}\Big(\mathcal{B}_\Omega(x,R) \cap \mathcal{C}\Big)\bigg) \leq 8(\delta_4 + \delta_1).$$

Proof. Fix $y \in \mathcal{B}_\Omega(x,R) \cap \mathcal{C}$. We claim that

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \leq 4(\delta_4 + \delta_1).$$

It is enough to consider the case when $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) \geq \delta_4$. Then by Proposition VII.5, there exists $x' \in [x,y]$ such that $\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') \leq \delta_4$. By Proposition VII.2,

$$\mathrm{d}_\Omega(x,y) \leq R = \mathrm{d}_\Omega(x, \pi_S(x)) \leq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) + \delta_1.$$

Then,

$$\begin{aligned}
\mathrm{d}_\Omega(x',y) = \mathrm{d}_\Omega(x,y) - \mathrm{d}_\Omega(x,x') &\leq \mathrm{d}_\Omega(x, L_{S,\mathcal{H}}(x)) - \mathrm{d}_\Omega(x,x') + \delta_1 \\
&\leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') + \delta_1 \\
&\leq \delta_4 + \delta_1.
\end{aligned}$$

Thus,

$$\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), y) \leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') + \mathrm{d}_\Omega(x', y) \leq 2\delta_4 + \delta_1.$$

Since $L_{S,\mathcal{H}}(x) \in S$, using Proposition VII.2 again,

$$\begin{aligned}
\mathrm{d}_\Omega(y, L_{S,\mathcal{H}}(y)) \leq \mathrm{d}_\Omega(y, \pi_S(y)) + \delta_1 &\leq \mathrm{d}_\Omega(y, L_{S,\mathcal{H}}(x)) + \delta_1 \\
&\leq 2(\delta_4 + \delta_1).
\end{aligned}$$

Finally

$$\begin{aligned}
\mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), L_{S,\mathcal{H}}(y)) &\leq \mathrm{d}_\Omega(L_{S,\mathcal{H}}(x), x') + \mathrm{d}_\Omega(x', y) + \mathrm{d}_\Omega(y, L_{S,\mathcal{H}}(y)) \\
&\leq 4(\delta_4 + \delta_1).
\end{aligned}$$

□

7.3.2 $\mathcal{S}$ is Asymptotically Transverse-free relative to $\mathcal{G}$

This section is essentially the proof of [IZ19, Theorem 13.16]. Let δ_4 be the constant in Proposition VII.5. We will show that exists $\lambda > 0$ such that for each

$\Delta \geq 1$ and $\kappa \geq 2\delta_4$ the following holds: if $\mathcal{T} \subset \mathcal{C}$ is a geodesic triangle whose sides are in $\mathcal{G}$ and is $\mathcal{S}$-almost-transverse with constants κ and Δ, then $\mathcal{T}$ is $(\lambda\Delta)$-thin.

Suppose such a $\lambda > 0$ does not exist. Then, for every $n \geq 1$, there exist $\kappa_n \geq 2\delta_4$, $\Delta_n \geq 1$, and a $\mathcal{S}$-almost-transverse triangle $\mathcal{T}_n \subset \mathcal{C}$ with constants κ_n and Δ_n such that $\mathcal{T}_n$ is not $(n\Delta_n)$-thin. Let a_n, b_n, and c_n be the vertices of $\mathcal{T}_n$, labeled in a such a way that there exists $u_n \in [a_n, b_n] \subset \mathcal{T}_n$ with

$$\mathrm{d}_\Omega \Big(u_n, [a_n, c_n] \cup [c_n, b_n]\Big) > n\Delta_n \geq n. \tag{7.5}$$

Then the geodesic triangles $\mathcal{T}_n$ are also $\mathcal{S}$-almost-transverse with constants $2\delta_4$ and Δ_n since $\kappa_n \geq 2\delta_4$.

Since Λ acts co-compactly on $\mathcal{C}$, translating by elements of Λ and passing to a subsequence, we can assume that $u := \lim_{n\to\infty} u_n$ exists and $u \in \mathcal{C}$. By passing to a further subsequence, we can assume that $a := \lim_{n\to\infty} a_n$, $b := \lim_{n\to\infty} b_n$, and $c := \lim_{n\to\infty} c_n$ exist in $\overline{\mathcal{C}}$. By Equation (7.5),

$$[a, c] \cup [c, b] \subset \partial_{\mathrm{i}} \mathcal{C}$$

whereas, by construction, $u \in (a, b) \subset \mathcal{C}$. Thus, the points a, b, c form a half triangle. Then, by Theorem I.16 part (7), there exists $S \in \mathcal{S}_\Lambda$ such that $a, b, c \in \partial S$.

Fix a set of S-supporting hyperplanes $\mathcal{H}$. Let $a_n' := L_{S,\mathcal{H}}(a_n)$, $b_n' := L_{S,\mathcal{H}}(b_n)$, and $c_n' := L_{S,\mathcal{H}}(c_n)$. Up to passing to a subsequence, we can assume that the limits $a' := \lim_{n\to\infty} a_n'$, $b' := \lim_{n\to\infty} b_n'$ and $c' := \lim_{n\to\infty} c_n'$ exist. By Lemma VII.1,

$$a' = \lim_{n\to\infty} L_{S,\mathcal{H}}(a_n) = L_{S,\mathcal{H}}(a) = a.$$

Similarly, $b' = b$ and $c' = c$.

Since a, b, c form a half triangle, the faces $F_\Omega(a')$, $F_\Omega(b')$, and $F_\Omega(c')$, are pairwise disjoint. Then, by Proposition II.12,

$$\lim_{n\to\infty} \mathrm{d}_\Omega(a'_n, b'_n) = \infty.$$

Thus, for n large enough, Corollary VII.6 part (2) and part (3) implies

$$[a'_n, b'_n] \subset \mathcal{N}_\Omega([a_n, b_n]; \delta_4) \tag{7.6}$$

and

$$\mathrm{diam}_\Omega \Big(\mathcal{N}_\Omega(S; \delta_4) \cap [a_n, b_n] \Big) \geq \mathrm{d}_\Omega(a'_n, b'_n) - 2\delta_4. \tag{7.7}$$

Since $\mathcal{T}_n$ is $\mathcal{S}$-almost-transverse with constants $2\delta_4$ and Δ_n, by Equation (7.7),

$$\mathrm{d}_\Omega(a'_n, b'_n) \leq \Delta_n + 2\delta_4. \tag{7.8}$$

Similarly, for n large enough,

$$[b'_n, c'_n] \subset \mathcal{N}_\Omega([b_n, c_n]; \delta_4) \ \text{ and } \ \mathrm{d}_\Omega(b'_n, c'_n) \leq \Delta_n + 2\delta_4 \tag{7.9}$$

$$[c'_n, a'_n] \subset \mathcal{N}_\Omega([c_n, a_n]; \delta_4) \ \text{ and } \ \mathrm{d}_\Omega(c'_n, a'_n) \leq \Delta_n + 2\delta_4. \tag{7.10}$$

Let m_n^{ab}, m_n^{bc}, and m_n^{ca} be the Hilbert distance midpoints of $[a'_n, b'_n]$, $[b'_n, c'_n]$, and $[c'_n, a'_n]$ respectively. By Equations (7.6), (7.9), and (7.10), there exists w_n^{ab}, w_n^{bc}, and w_n^{ca} in $[a_n, b_n]$, $[b_n, c_n]$, and $[c_n, a_n]$ respectively such that:

$$\mathrm{d}_\Omega(w_n^{ab}, m_n^{ab}) \leq \delta_4,\ \mathrm{d}_\Omega(w_n^{bc}, m_n^{bc}) \leq \delta_4, \ \text{ and } \ \mathrm{d}_\Omega(w_n^{ca}, m_n^{ca}) \leq \delta_4.$$

Then,

$$\begin{aligned}
\mathrm{d}_\Omega(w_n^{ab}, w_n^{bc}) &\leq \mathrm{d}_\Omega(w_n^{ab}, m_n^{ab}) + \mathrm{d}_\Omega(m_n^{ab}, m_n^{bc}) + \mathrm{d}_\Omega(m_n^{bc}, w_n^{bc}) \\
&\leq \delta_4 + \mathrm{d}_\Omega(m_n^{ab}, b'_n) + \mathrm{d}_\Omega(b'_n, m_n^{bc}) + \delta_4 \\
&= 2\delta_4 + \frac{\mathrm{d}_\Omega(a'_n, b'_n) + \mathrm{d}_\Omega(b'_n, c'_n)}{2} \\
&\leq 4\delta_4 + \Delta_n \ \ \text{(by Equations (7.8) and (7.9))}
\end{aligned}$$

Similarly,

$$\mathrm{d}_\Omega(w_n^{bc}, w_n^{ca}) \leq \Delta_n + 4\delta_4 \quad \text{and} \quad \mathrm{d}_\Omega(w_n^{ca}, w_n^{ab}) \leq \Delta_n + 4\delta_4. \tag{7.11}$$

Then, for n large enough, the triangles $\mathcal{T}_n$ are $(\Delta_n + 4\delta_4)$-thin, since

$$\begin{aligned}
&{\mathrm{d}_\Omega}^{\text{Hauss}}\Big([a_n, w_n^{ab}], [a_n, w_n^{ca}]\Big) \leq \Delta_n + 4\delta_4, \\
&{\mathrm{d}_\Omega}^{\text{Hauss}}\Big([b_n, w_n^{bc}], [b_n, w_n^{ab}]\Big) \leq \Delta_n + 4\delta_4, \text{ and} \\
&{\mathrm{d}_\Omega}^{\text{Hauss}}\Big([c_n, w_n^{ca}], [c_n, w_n^{bc}]\Big) \leq \Delta_n + 4\delta_4.
\end{aligned}$$

Since $\Delta_n \geq 1$, we have $\Delta_n + 4\delta_4 \leq (1 + 4\delta_4)\Delta_n$. Thus, for n large enough, $\mathcal{T}_n$ is $(\lambda\Delta_n)$-thin for $\lambda := 1 + 4\delta_4$, which contradicts the assumption that $\mathcal{T}_n$ is not $(n\Delta_n)$-thin.

7.4 Proof of Theorem I.15

For the rest of the section suppose that $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. Set $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$ and let $\mathcal{S}_\Lambda$ be the family of all maximal properly embedded simplices in $\mathcal{C}$ of dimension at least two.

(1) implies (2). See Section 7.3.

(2) implies (1). This follows from Theorem III.12 part (1).

(1) and (2) implies (3). Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{S}_\Lambda$. Equivalence of (1) and (2) implies that $(\mathcal{C}, \mathrm{d}_\Omega)$ has strongly isolated simplices. Then, by Theorem I.16 part (1), there exists $m \in \mathbb{N}$ such that

$$\mathcal{S}_\Lambda = \sqcup_{i=1}^m \Lambda \cdot S_i. \tag{7.12}$$

Theorem I.16 part (2) implies that for each $i \in \{1, \dots, m\}$, there exists an Abelian subgroup $A_i \leq \Lambda$ of rank at least two such that A_i acts co-compactly on S_i. We will show that Λ is a relatively hyperbolic group with respect to the subgroups $\{A_1, \dots, A_m\}$.

Fix $p \in \mathcal{C}$. Since Λ acts co-compactly on $\mathcal{C}$, Theorem III.3 implies that the orbit map $F : (\Lambda, \mathrm{d}_S) \to (\mathcal{C}, \mathrm{d}_\Omega)$ defined by $F(g) = g \cdot p$ is a quasi-isometry . Here d_S is a word metric on Λ obtained by fixing a finite generating set S of Λ.

Since A_i acts co-compactly on S_i for $1 \leq i \leq m$, there exists $R > 0$ such that

$$\sup_{g \in \Lambda} \sup_{1 \leq i \leq m} {\mathrm{d}_\Omega}^{\mathrm{Hauss}}(gA_i \cdot p, gS_i) \leq R.$$

Then equation (7.12) implies that up to modifying the map F by a bounded quantity determined by R, we can assume that

$$F(\{gA_i : g \in \Lambda, 1 \leq i \leq m\}) = \mathcal{S}_\Lambda\,.$$

Then by Proposition III.11, (Λ, d_S) is a relatively hyperbolic space with respect to the collection of left cosets $\{gA_i : g \in \Lambda, 1 \leq i \leq m\}$. This completes the proof.

(3) implies (2). Suppose that Λ is a relatively hyperbolic group with respect to a collection of subgroups $\{H_1, \dots, H_k\}$ each of which is a virtually Abelian group of rank at least two. For each $1 \leq j \leq k$, let $A_j \leq H_j$ be a finite index Abelian subgroup with rank at least two. Then, by definition, Λ is a relatively hyperbolic group with respect to $\{A_1, \dots, A_k\}$.

Fix some $x_0 \in \Omega$ and consider the orbit map $F : (\Lambda, \mathrm{d}_\Lambda) \to (\mathcal{C}, \mathrm{d}_\Omega)$ defined by $F(g) = g \cdot x_0$. By Proposition III.3, F is a quasi-isometry. Let $G : \mathcal{C} \to \Lambda$ be a quasi-inverse. Fix a word metric d_Λ on Λ. We will use the following notation: if $U \subset \Lambda$ and $r > 0$, let

$$\mathcal{N}_\Lambda(U; r) := \{g \in \Lambda : \mathrm{d}_\Lambda(g, U) < r\}$$

and

$$\operatorname{diam}_\Lambda(U) = \sup\{\mathrm{d}_\Lambda(g_1, g_2) : g_1, g_2 \in U\}.$$

For each $1 \leq j \leq k$, let $\widehat{A}_j$ be a maximal Abelian subgroup of Λ that contains A_j. By Theorem I.17, there exists a properly embedded simplex $S_j \subset \mathcal{C}$ such that $\widehat{A}_j \leq \operatorname{Stab}_\Lambda(S_j)$, $\widehat{A}_j$ acts co-compactly on S_j, and $\widehat{A}_j$ has a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S_j)}$. Since A_j (and hence $\widehat{A}_j$) has rank at least two, this implies that $\dim S_j \geq 2$.

Claim VII.10. $(\mathcal{C}, \mathrm{d}_\Omega)$ is a relatively hyperbolic space with respect to

$$\mathcal{S}_0 := \{gS_j : g \in \Lambda, 1 \leq j \leq k\}.$$

Proof of Claim. We claim that $A_j \leq \widehat{A}_j$ has finite index and hence A_j also acts co-compactly on S_j. By Observation II.18, the metric space (S_j, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim S_j}$. So, by the fundamental lemma of geometric group theory [BH99, Chapter I, Proposition 8.19], $(\widehat{A}_j, \mathrm{d}_\Lambda)$ is also quasi-isometric to $\mathbb{R}^{\dim S_j}$. Since $\dim S_j \geq 2$, Theorem III.12 part (2) implies that there exists $r_1 > 0$, $g_j \in \Lambda$, and $1 \leq i_j \leq k$ such that

$$\widehat{A}_j \subset \mathcal{N}_\Lambda(g_j A_{i_j}; r_1).$$

Then

$$\operatorname{diam}_\Lambda \left(\mathcal{N}_\Lambda(g_j A_{i_j}; r_1) \cap \mathcal{N}_\Lambda(A_j; r_1)\right) \geq \operatorname{diam}_\Lambda (A_j) = \infty.$$

So Theorem III.12 part (1) implies that $g_j A_{i_j} = A_j$. Then,

$$\widehat{A}_j \subset \mathcal{N}_\Lambda(A_j; r_1)$$

and hence $A_j \leq \widehat{A}_j$ has finite index.

Then, using the fact that A_j acts co-compactly on S_j, there exists $r_2 > 0$ such that

$$F(gA_j) \subset \mathcal{N}_\Omega(gS_j; r_2)$$

and

$$G(gS_j) \subset \mathcal{N}_\Lambda(gA_j; r_2)$$

for all $g \in \Lambda$ and $1 \leq j \leq k$. Then, by Theorem III.12 part (3), $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{S}_0$. $\square$

In order to finish the proof, we will now show that $\mathcal{S}_0 = \mathcal{S}_\Lambda$.

We first show that $\mathcal{S}_0 \subset \mathcal{S}_\Lambda$. Suppose $S \in \mathcal{S}_0$ is properly contained in a maximal properly embedded simplex S'. Then, by Theorem III.12 part (2), there exists $S''' \in \mathcal{S}_0$ such that

$$S \subset S' \subset \mathcal{N}_\Omega(S'''; M).$$

This implies that $\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S'''; M) \cap \mathcal{N}_\Omega(S; M)) = \infty$. Then Theorem III.12 part (1) implies that $S''' = S$, i.e. $S' \subset \mathcal{N}_\Omega(S; M)$. Hence, $\dim(S') \leq \dim(S)$ which is a contradiction since S' properly contains S.

For proving $\mathcal{S}_\Lambda \subset \mathcal{S}_0$, we first need the following claim.

Claim VII.11. If $S \in \mathcal{S}_0$ and $x \in \partial S$, then $F_\Omega(x) = F_\mathcal{C}(x) = F_S(x)$.

Proof of Claim. Fix $S \in \mathcal{S}_0$. Recall that since Λ is a convex co-compact subgroup, $F_\Omega(x') = F_\mathcal{C}(x')$ for any $x' \in \partial_{\mathrm{i}} \mathcal{C}$. Then, as in the proof of Proposition VI.5, if $x \in \partial S$, then the claim fails only when

$$\partial F_S(x) \subsetneq \partial F_\mathcal{C}(x).$$

We will prove that $\partial F_S(x) = \partial F_\mathcal{C}(x)$ by induction on $\dim(F_S(x))$.

Base case: $\dim(F_S(x)) = 0$.

Then x is a vertex of S. Suppose the claim fails, i.e. $\partial F_S(x) \subsetneq \partial F_{\mathcal{C}}(x)$. Let $w_0 \in \partial F_{\mathcal{C}}(x) \setminus \overline{F_S(x)}$. Then $(w_0, x) \cap \overline{S} = \emptyset$. Otherwise, if there was $x'' \in (w_0, x) \cap \overline{S}$, then Observation II.19 would imply that

$$F_S(x'') = \overline{S} \cap F_\Omega(x'') = \overline{S} \cap F_\Omega(x) = F_S(x) = \{x\},$$

that is $x'' = x$, a contradiction.

Then, we fix $w \in (w_0, x) \subset F_\Omega(x)$ such that $M + 1 \leq \mathrm{d}_{F_\Omega(x)}(w, x)$. Set $R_w := \mathrm{d}_{F_\Omega(x)}(w, x)$. Let us label the vertices of S as $v_1, \dots, v_m$ where $m = \dim(S) + 1$ and $v_1 = x$. By Proposition II.20, $S' := \mathrm{ConvHull}_\Omega(w, v_2, \dots, v_m)$ is a properly embedded simplex in $\mathcal{C}$ such that ${\mathrm{d}_\Omega}^{\mathrm{Hauss}}(S', S) \leq R_w$. Observe that

$$S' \not\subset \mathcal{N}_\Omega(S; M).$$

Indeed, if $S' \subset \mathcal{N}_\Omega(S; M)$, then applying Corollary II.13 to $w \in \partial S'$, we get $s \in \partial S$ such that $w \in F_\Omega(s)$ and $\mathrm{d}_{F_\Omega(s)}(w, s) \leq M$. Since $w \in F_\Omega(x)$, this implies that $x \in F_\Omega(s)$. Since $x, s \in \partial S$ and x is a vertex of S, $s = x$. Thus $\mathrm{d}_{F_\Omega(x)}(w, x) \leq M$, a contradiction.

Then, by Theorem III.12 part (2), there exists $S_1 \neq S \in \mathcal{S}_0$ such that $S' \subset \mathcal{N}_\Omega(S_1; M)$. Then we have

$$S' \subset \mathcal{N}_\Omega(S; R_w) \cap \mathcal{N}(S_1; M)$$

which implies that $\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S; R_w) \cap \mathcal{N}(S_1; 2M)) = \infty$ for distinct $S, S_1 \in \mathcal{S}_0$. This contradicts Theorem III.12 part (1), as $\mathcal{C}$ is a relatively hyperbolic space with respect to $\mathcal{S}_0$. This completes the proof in the base case.

Induction step: Suppose the proposition is true when $\dim(F_S(x)) = k$ for some $k \geq 0$.

Now suppose $\dim(F_S(x)) = k+1$. Let $y \in \partial F_{\mathcal{C}}(x)$. We will show that $y \in \partial F_S(x)$. For $n \geq 1$, choose $y_n \in (y, x)$ such that $\mathrm{d}_{F_{\mathcal{C}}(x)}(y_n, x) = n$.

Let us label the vertices of S as $v_1, \dots, v_m$ where $m = \dim(S)+1$ and $v_1 = x$. By Proposition II.20,

$$S_n := \mathrm{ConvHull}_\Omega(y_n, v_2, \dots, v_m)$$

is a properly embedded simplex in $\mathcal{C}$ of dimension at least two and $\mathrm{d}_\Omega{}^{\mathrm{Hauss}}(S, S_n) \leq n$. Then, by Theorem III.12 part (2), there exist $T_n \in \mathcal{S}_0$ such that

$$S_n \subset \mathcal{N}_\Omega(T_n; M)$$

for each $n \geq 1$. Then

$$S \subset \mathcal{N}_\Omega(T_n; M+n).$$

Since $\mathrm{diam}_\Omega(\mathcal{N}_\Omega(T_n; M+n) \cap \mathcal{N}_\Omega(S; M+n)) = \infty$, Theorem III.12 part (1) implies that $S = T_n$ for all $n \geq 1$. Since $S_n \subset \mathcal{N}_\Omega(S; M)$ and $y_n \in \partial S_n$, Corollary II.13 implies that for each $n \geq 1$, there exists $z_n \in \partial S \cap F_\Omega(y_n)$ such that $\mathrm{d}_{F_\Omega(y_n)}(y_n, z_n) \leq M$. Since $y_n \in F_\Omega(x)$, $z_n \in \partial S \cap F_\Omega(x)$ and $\mathrm{d}_{F_\Omega(x)}(y_n, z_n) \leq M$. Up to passing to a subsequence, we can assume that $z_n \to z \in \overline{S}$. By Proposition II.12,

$$y \in F_{F_\Omega(x)}(z) = F_\Omega(z).$$

Observe that $z \in \partial F_S(x) = \partial S \cap \partial F_\Omega(x)$ since

$$\begin{aligned}
\mathrm{d}_{F_\Omega(x)}(x, z) = \lim_{n\to\infty} \mathrm{d}_{F_\Omega(x)}(x, z_n) &\geq \lim_{n\to\infty} \mathrm{d}_{F_\Omega(x)}(x, y_n) - \mathrm{d}_{F_\Omega(x)}(y_n, z_n) \\
&\geq \lim_{n\to\infty} (n - M) = \infty.
\end{aligned}$$

Since $z \in \partial F_S(x)$, $F_S(z) \subset \partial F_S(x)$. Then

$$\dim(F_S(z)) \leq \dim(\partial F_S(x)) = \dim(F_S(x)) - 1 = k.$$

The induction hypothesis then implies that $F_\Omega(z) = F_S(z)$. Thus,

$$y \in F_S(z) \subset \partial F_S(x).$$

Hence $\partial F_{\mathcal{C}}(x) \subset \partial F_S(x)$ which finishes the proof of this claim. $\square$

Now fix any $S \in \mathcal{S}_\Lambda$. By Theorem III.12 part (2), there exists M such that $S \subset \mathcal{N}_\Omega(S_0; M)$ for some $S_0 \in \mathcal{S}_0$. If $q \in \partial S$, then Corollary II.13 implies that there exists $q_0 \in \partial S_0$ such that $q \in F_\Omega(q_0)$. Since $S_0 \in \mathcal{S}_0$, the above claim implies that $F_\Omega(q_0) = F_{S_0}(q_0) \subset \partial S_0$. Thus $q \in \partial S_0$. This implies that $\partial S \subset \partial S_0$, that is, $S \subset S_0$. Since S is a maximal properly embedded simplex, $S = S_0$ and $S \in \mathcal{S}_0$. This finishes the proof that $\mathcal{S}_0 = \mathcal{S}_\Lambda$. Then, by Claim VII.10, $\mathcal{C}$ is a relatively hyperbolic space with respect to $\mathcal{S}_\Lambda$.

CHAPTER VIII

A Convex Projective Flat Torus Theorem

This chapter is based on results that appear in [IZ21] which is a joint work with A. Zimmer.

8.1 Outline

In this chapter, we will prove Theorem I.17 which we now restate.

Theorem I.17. ([IZ21, Theorem 1.6]) *Suppose that $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group. If $A \leq \Lambda$ is a maximal Abelian subgroup of Λ, then there exists a properly embedded simplex $S \subset \mathcal{C}_\Omega(\Lambda)$ such that*

(1) S is A-invariant,

(2) A acts co-compactly on S, and

(3) A fixes each vertex of S.

Moreover, A has a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

In Section 8.2, we show convex co-compact actions of Abelian groups. In Section 8.3, we prove Theorem I.18. It is a result about the centralizer of an Abelian subgroup in a convex co-compact group. Theorem I.17 is proven in 8.4 as a consequence of the aforementioned theorems.

8.2 Abelian Convex Co-compact Actions

In this section we show that every convex co-compact action of an Abelian group comes from "fattening" a properly embedded simplex. We will use the following notation to state our result: if $X \subset \Omega$, then $\mathrm{Stab}_\Omega(X) := \{g \in \mathrm{Aut}(\Omega) : gX = X\}$.

Theorem VIII.1 ([IZ21, Theorem 6.1])**.** *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\mathcal{C} \subset \Omega$ is a non-empty closed convex subset, and $G \leq \mathrm{Stab}_\Omega(\mathcal{C})$. If G is Abelian and acts co-compactly on $\mathcal{C}$, then there exists a properly embedded simplex $S \subset \mathcal{C}$ where*

(1) $G \leq \mathrm{Stab}_\Omega(S)$,

(2) G acts co-compactly on S, and

(3) G fixes each vertex of S.

Remark VIII.2. *Notice that we do not assume that G is a discrete subgroup of* $\mathrm{Aut}(\Omega)$.

The rest of the section is devoted to the proof of the theorem. We will induct on

$$\dim \Omega + \dim \mathcal{C}.$$

The base case, when $\dim \Omega = 1$ and $\dim \mathcal{C} = 0$, is trivial.

Suppose that $\Omega, \mathcal{C}, G$ satisfy the hypothesis of the theorem. From Proposition II.31 we immediately obtain the following.

Observation VIII.3. *If $\mathcal{C}$ is compact, then G fixes the point $\mathrm{CoM}_\Omega(\mathcal{C})$.*

Since a point is a 0-dimensional simplex, the above observation completes the proof in the case when $\mathcal{C}$ is compact. So for the rest of the argument we assume that $\mathcal{C}$ is non-compact and hence $\partial_{\mathrm{i}} \mathcal{C} \neq \emptyset$. Our first goal will be to find a finite number of fixed points $x_1, \dots, x_k$ of G in $\partial_{\mathrm{i}} \mathcal{C}$ such that

$$\mathrm{ConvHull}_\Omega\{x_1, \dots, x_k\} \cap \Omega$$

is non-empty.

Lemma VIII.4 ([IZ21, Lemma 6.4])**.** *If $x \in \partial_{\mathrm{i}} \mathcal{C}$ and $F := F_\Omega(x)$, then*

(1) $G \leq \mathrm{Stab}_\Omega(F)$,

(2) $G \leq \mathrm{Stab}_\Omega(F \cap \partial_{\mathrm{i}} \mathcal{C})$, and

(3) G acts co-compactly on $F \cap \partial_{\mathrm{i}} \mathcal{C}$.

Proof. By Proposition II.36 there exists some $T \in \overline{G}^{\mathrm{End}}$ such that $\mathbb{P}(\ker T) \cap \Omega = \emptyset$, $T(\Omega) = F$, and $T(\mathcal{C}) = F \cap \partial_{\mathrm{i}} \mathcal{C}$. Since G is Abelian, $T \circ g = g \circ T$ for every $g \in G$.

Then for $g \in G$ we have

$$gF = gT(\Omega) = T(g\Omega) = T(\Omega) = F.$$

Since $g \in G$ was arbitrary, $G \leq \mathrm{Stab}_\Omega(F)$. Then $G \leq \mathrm{Stab}_\Omega(F \cap \partial_{\mathrm{i}} \mathcal{C})$ since $G \leq \mathrm{Stab}_\Omega(\mathcal{C})$.

Since G acts co-compactly on $\mathcal{C}$, there exists a compact set $K \subset \mathcal{C}$ such that $G \cdot K = \mathcal{C}$. Since $\mathbb{P}(\ker T) \cap \Omega = \emptyset$, the map

$$p \in \Omega \ \mapsto \ T(p) \in F_\Omega(x)$$

is continuous. So $K_F := T(K)$ is a compact subset of $F \cap \partial_{\mathrm{i}} \mathcal{C}$. Then

$$G \cdot K_F = G \cdot T(K) = T(G \cdot K) = T(\mathcal{C}) = F \cap \partial_{\mathrm{i}} \mathcal{C}.$$

So G acts co-compactly on $F \cap \partial_{\mathrm{i}} \mathcal{C}$. $\square$

Lemma VIII.5. *There exists a properly embedded 1-dimensional simplex $\ell \subset \mathcal{C}$.*

Proof. Fix some $x_0 \in \mathcal{C}$. Since $\mathcal{C}$ is non-compact, there exists some $x \in \partial_{\mathrm{i}} \mathcal{C}$. Then pick $x_n \in [x_0, x)$ converging to x. Since $[x_0, x) \subset \mathcal{C}$ and G acts co-compactly on $\mathcal{C}$, there exist $r > 0$ and a sequence $g_n \in G$ such that

$$H_\Omega(g_n x_n, x_0) \leq r$$

for all $n \geq 0$. By passing to a subsequence we can suppose that $g_n x_n \to q \in \mathcal{C}$. By passing to another subsequence we can assume that $g_n \cdot (x_0, x)$ converges to a properly embedded 1-dimensionial simplex $\ell \subset \mathcal{C}$. $\square$

Lemma VIII.6. *There exists a finite number of fixed points $x_1, \dots, x_m$ of G in $\partial_{\mathrm{i}} \mathcal{C}$ such that*

$$\mathrm{ConvHull}_\Omega\{x_1, \dots, x_m\} \cap \Omega$$

is non-empty.

Proof. By the previous lemma there exists a properly embedded 1-dimensional simplex $\ell \subset \mathcal{C}$. Let y_1, y_2 be the endpoints of ℓ and let $F_j := F_\Omega(y_j)$.

First, we will find a finite number of fixed points $a_1, \dots, a_k$ of G in $\overline{F}_1 \cap \partial_{\mathrm{i}} \mathcal{C}$ such that

$$\mathrm{ConvHull}_\Omega \{a_1, \dots, a_k\} \cap F_1$$

is non-empty. By Lemma VIII.4 and induction there exists a properly embedded simplex $S_1 \subset F_1$ where G fixes each vertex of S_1. Let $a_1, \dots, a_k$ be the vertices of S_1. Then

$$S_1 = \mathrm{ConvHull}_\Omega \{a_1, \dots, a_k\} \cap F_1$$

is non-empty.

Applying the same argument to F_2 yields a finite number of fixed points $b_1, \ldots, b_n$ of G in $\overline{F_2} \cap \partial_{\mathrm{i}} \mathcal{C}$ such that

$$\mathrm{ConvHull}_\Omega \{b_1, \ldots, b_n\} \cap F_2$$

is non-empty.

Finally, we claim that

$$\mathrm{ConvHull}_\Omega \{a_1, \ldots, a_k, b_1, \ldots, b_n\} \cap \Omega \neq \emptyset.$$

is non-empty. By construction, this convex hull contains some $a' \in F_1$ and $b' \in F_2$. Since $y_1 \in F_1$, $y_2 \in F_2$, and $\ell = (y_1, y_2) \subset \Omega$, Proposition II.11 part (4) implies that $(a', b') \subset \Omega$. Then

$$(a', b') \subset \mathrm{ConvHull}_\Omega \{a_1, \ldots, a_k, b_1, \ldots, b_n\} \cap \Omega$$

and we are done. $\square$

By the previous lemma, there exist fixed points $x_1, \ldots, x_m$ of G in $\partial_{\mathrm{i}} \mathcal{C}$ such that

$$S := \mathrm{ConvHull}_\Omega\{x_1, \ldots, x_m\} \cap \Omega$$

is non-empty. We can also assume that m is minimal in the following sense: if $y_1, \ldots, y_k$ are fixed points of G in $\partial_{\mathrm{i}} \mathcal{C}$ with $k < m$, then

$$\mathrm{ConvHull}_\Omega\{y_1, \ldots, y_k\} \cap \Omega = \emptyset.$$

Also, notice that $m \geq 2$ since $x_1, \ldots, x_m \in \partial_{\mathrm{i}} \mathcal{C}$ and $S \neq \emptyset$. We complete the proof of Theorem VIII.1 by proving the following.

Lemma VIII.7. *S is a properly embedded simplex in Ω, $G \leq \mathrm{Stab}_\Omega(S)$, G acts co-compactly on S, and G fixes each vertex of S.*

Proof. Let $d_0 := \dim S$. We claim that $d_0 = m-1$. By definition,

$$d_0 = \dim \mathbb{P}(\mathrm{Span}\{x_1, \dots, x_m\}) \leq m-1.$$

For the reverse inequality, fix $p \in S$. Then by Carathéodory's convex hull theorem there exists $x_{i_1}, \dots, x_{i_k}$ with $k \leq d_0 + 1$ such that

$$p \in \mathrm{ConvHull}_\Omega\{x_{i_1}, \dots, x_{i_k}\}.$$

Hence

$$\emptyset \neq \mathrm{ConvHull}_\Omega\{x_{i_1}, \dots, x_{i_k}\} \cap \Omega.$$

So by our minimality assumption we must have $k = m$ and so $m \leq d_0 + 1$. So $m = d_0 + 1$. Thus $x_1, \dots, x_m$ are linearly independent and hence S is a simplex with vertices $\{x_1, \dots, x_m\}$.

By the minimality property, for any proper subset $\{x_{i_1}, \dots, x_{i_k}\} \subset \{x_1, \dots, x_m\}$ we have

$$\emptyset = \mathrm{ConvHull}_\Omega\{x_{i_1}, \dots, x_{i_k}\} \cap \Omega.$$

So S is a properly embedded simplex of Ω.

By construction $G \leq \mathrm{Stab}_\Omega(S)$ and G fixes each vertex of S. Finally, since $S \subset \mathcal{C}$ is a closed subset and G acts co-compactly on $\mathcal{C}$, we see that G acts co-compactly on S. $\square$

8.3 Centralizers and Minimal Translation Sets: Proof of Theorem I.18

In this section we prove Theorem I.18 which we restate here.

Theorem I.18. *Suppose $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $A \leq \Lambda$ is an Abelian subgroup. Then*

$$\mathrm{Min}_{\mathcal{C}_\Omega(\Lambda)}(A) := \mathcal{C}_\Omega(\Lambda) \cap \bigcap_{a \in A} \mathrm{Min}(a)$$

is non-empty and $C_\Lambda(A)$ *acts co-compactly on* $\mathrm{ConvHull}_\Omega\left(\mathrm{Min}_{\mathcal{C}_\Omega(\Lambda)}(A)\right)$.

For the rest of the section fix a co-compact group $\Lambda \leq \mathrm{Aut}(\Omega)$ and an Abelian subgroup $A \leq \Lambda$. Set $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$.

We will need the following elementary observations [IZ21, Section 7].

Observation VIII.8. *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain and* $g \in \mathrm{Aut}(\Omega)$. *If* $V \subset \mathbb{R}^d$ *is a linear subspace where* $\dim V > 1$, $\Omega \cap \mathbb{P}(V) \neq \emptyset$, *and* V *is* g*-invariant, then*

$$\tau_{\Omega \cap \mathbb{P}(V)}(g) = \tau_\Omega(g).$$

Observation VIII.9. *Suppose that* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain and* $S \subset \Omega$ *is a properly embedded simplex. If* $g \in \mathrm{Aut}(\Omega)$ *fixes every vertex of* S*, then* $S \subset \mathrm{Min}(g)$.

We will need the following fact about subgroups of solvable Lie groups.

Lemma VIII.10 ([Rag72, Proposition 3.8])**.** *Let* G *be a solvable Lie group with finitely many components and* $H \leq G$ *a closed subgroup. Let* H_0 *be the connected component of the identity in* H. *Then* H/H_0 *is finitely generated.*

Now we begin the proof of Theorem I.18. Let $\overline{A}^{\mathrm{Zar}}$ be the Zariski closure in $\mathrm{PGL}_d(\mathbb{R})$. Then $\overline{A}^{\mathrm{Zar}}$ is Abelian and has finitely many components. Since $A \leq \overline{A}^{\mathrm{Zar}}$ is discrete, Lemma VIII.10 implies that

$$A = \langle a_1, \dots, a_m \rangle$$

for some $a_1, \dots, a_m \in A$. In particular,

$$C_\Lambda(A) = \cap_{j=1}^m C_\Lambda(a_j).$$

Next for $r > 0$ define

$$M_r := \left\{ x \in \mathcal{C} : H_\Omega(x, a_j x) \leq r \text{ for all } 1 \leq j \leq m \right\}.$$

Lemma VIII.11. *$C_\Lambda(A) \leq \mathrm{Stab}_\Lambda(M_r)$.*

Proof. If $\gamma \in C_\Lambda(A)$ and $x \in M_r$, then

$$H_\Omega(\gamma x, a_j \gamma x) = H_\Omega(\gamma x, \gamma a_j x) = H_\Omega(x, a_j x) \leq r$$

Hence $\gamma x \in M_r$. So $\gamma M_r \subset M_r$. Applying the same argument to γ^{-1} shows that $M_r \subset \gamma M_r$. □

Lemma VIII.12. *For every $r > 0$, $C_\Lambda(A)$ acts co-compactly on M_r.*

The following argument comes from the proof of Theorem 3.2 in [Rua01].

Proof. If $M_r = \emptyset$, then there is nothing to prove. So we may assume that $M_r \neq \emptyset$.

Suppose for a contradiction that $C_\Lambda(A)$ does not act co-compactly on M_r. Fix some $x_0 \in M_r$. Then for each n there exists some $x_n \in M_r$ such that

$$H_\Omega\left(x_n, C_\Lambda(A) \cdot x_0\right) \geq n.$$

Since Λ acts co-compactly on $\mathcal{C}$, there exist $M > 0$ and a sequence $\beta_n \in \Lambda$ such that

$$H_\Omega(\beta_n x_0, x_n) \leq M.$$

for all $n \geq 0$. Then for $1 \leq j \leq m$

$$\begin{aligned} H_\Omega(\beta_n^{-1} a_j \beta_n x_0, x_0) &= H_\Omega(a_j \beta_n x_0, \beta_n x_0) \\ &\leq H_\Omega(a_j \beta_n x_0, a_j x_n) + H_\Omega(a_j x_n, x_n) + H_\Omega(x_n, \beta_n x_0) \\ &\leq M + r + M = r + 2M. \end{aligned}$$

Since Λ acts properly on Ω, for every $1 \leq j \leq m$ the set

$$\{\beta_n^{-1} a_j \beta_n : n \geq 0\}$$

must be finite. So by passing to a subsequence we can assume that

$$\beta_n^{-1} a_j \beta_n = \beta_1^{-1} a_j \beta_1$$

for all $1 \leq j \leq m$ and $n \geq 0$. Then $\beta_n \beta_1^{-1} \in \cap_{j=1}^m C_\Lambda(a_j) = C_\Lambda(A)$ for all $n \geq 0$. Then

$$\begin{aligned} n &\leq H_\Omega\left(x_n, C_\Lambda(A) \cdot x_0\right) \leq H_\Omega\left(x_n, \beta_n \beta_1^{-1} x_0\right) \\ &\leq H_\Omega\left(x_n, \beta_n x_0\right) + H_\Omega\left(\beta_n x_0, \beta_n \beta_1^{-1} x_0\right) \\ &\leq M + H_\Omega\left(x_0, \beta_1^{-1} x_0\right) \end{aligned}$$

for all $n \geq 0$, which is a contradiction. Hence $C_\Lambda(A)$ acts co-compactly on M_r. $\square$

Lemma VIII.13. *For any $r > 0$,*

$$\mathrm{ConvHull}_\Omega\left(M_r\right) \subset M_{2^{d-1} r}.$$

Remark VIII.14. *A similar estimate is established in [CLT15, Lemma 8.4].*

Proof. For $n \geq 0$, let $C_n \subset \mathrm{ConvHull}_\Omega\left(M_r\right)$ denote the elements which can be written as a convex combination of n elements in M_r. Then $C_1 = M_r$ and by Carathéodory's convex hull theorem, $C_d = \mathrm{ConvHull}_\Omega\left(M_r\right)$. We claim by induction that

$$C_n \subset M_{2^{(n-1)} r}$$

for every $1 \leq n \leq d$.

By definition $C_1 = M_r$ so the base case is true. Now suppose that

$$C_n \subset M_{2^{(n-1)} r}$$

and $p \in C_{n+1}$. Then there exists $p_1, p_2 \in C_n$ such that $p \in [p_1, p_2]$. Let $\sigma : [0, T] \to \mathcal{C}$ be the unit speed projective line geodesic with $\sigma(0) = p_1$ and $\sigma(T) = p_2$. Then

$p = \sigma(t_0)$ for some $t_0 \in [0, T]$. Next for $1 \leq j \leq m$ let $\sigma_j = a_j \circ \sigma$. Then Lemma II.15 implies that

$$\begin{aligned} H_\Omega(p, a_j p) &= H_\Omega(\sigma(t_0), \sigma_j(t_0)) \leq H_\Omega(\sigma(0), \sigma_j(0)) + H_\Omega(\sigma(T), \sigma_j(T)) \\ &= H_\Omega(p_1, a_j p_1) + H_\Omega(p_2, a_j p_2) \leq 2^{(n-1)} r + 2^{(n-1)} r = 2^n r \end{aligned}$$

Since $p \in C_{n+1}$ was arbitrary, we have

$$C_{n+1} \subset M_{2^n r}$$

and the proof is complete. $\square$

Combining Lemma VIII.12 and Lemma VIII.13 we have the following.

Lemma VIII.15. *For any $r > 0$, $C_\Lambda(A)$ acts co-compactly on* $\mathrm{ConvHull}_\Omega(M_r)$.

Now we can complete the final step of the proof.

Lemma VIII.16. $\mathrm{Min}_{\mathcal{C}}(A) \neq \emptyset$ *and $C_\Lambda(A)$ acts co-compactly on* $\mathrm{ConvHull}_\Omega(\mathrm{Min}_{\mathcal{C}}(A))$.

Proof. If $r > \max_{1 \leq j \leq d} \tau(a_j)$, then

$$\mathrm{Min}_{\mathcal{C}}(A) = \cap_{a \in A} \mathrm{Min}_{\mathcal{C}}(a) \subset \cap_{j=1}^m \mathrm{Min}_{\mathcal{C}}(a_j) \subset M_r.$$

So $\mathrm{ConvHull}_\Omega(\mathrm{Min}_{\mathcal{C}}(A))$ is a closed $C_\Lambda(A)$-invariant subset of $\mathrm{ConvHull}_\Omega(M_r)$. Further, Lemma VIII.15 implies that $C_\Lambda(A)$ acts co-compactly on $\mathrm{ConvHull}_\Omega(M_r)$. So $C_\Lambda(A)$ also acts co-compactly on $\mathrm{ConvHull}_\Omega(\mathrm{Min}_{\mathcal{C}}(A))$.

Next we show that $\mathrm{Min}_{\mathcal{C}}(A) \neq \emptyset$. Pick $A' \geq A$ a maximal Abelian subgroup in Λ. Then $A' = C_\Lambda(A')$. By Lemma VIII.10 and the discussion following the lemma

$$A' = \langle a_1', \dots, a_n' \rangle$$

for some $a_1', \dots, a_n' \in A'$. Notice that

$$\mathrm{Min}_{\mathcal{C}}(A') = \cap_{a \in A'} \mathrm{Min}_{\mathcal{C}}(a) \subset \cap_{a \in A} \mathrm{Min}_{\mathcal{C}}(a) = \mathrm{Min}_{\mathcal{C}}(A)$$

and so it is enough to show that $\mathrm{Min}_{\mathcal{C}}(A') \neq \emptyset$.

For $r > 0$ define

$$M'_r := \left\{ x \in \mathcal{C} : H_\Omega(x, a'_j x) \leq r \text{ for all } 1 \leq j \leq n \right\}.$$

Then for r sufficiently large, $M'_r \neq \emptyset$. Further, by applying Lemma VIII.15 to A', we see that A' acts co-compactly on the convex set

$$\mathcal{C}' := \mathrm{ConvHull}_\Omega(M'_r) \subset \mathcal{C}.$$

Then by Theorem VIII.1 there exists a properly embedded simplex $S \subset \mathcal{C}' \subset \mathcal{C}$ where $A' \leq \mathrm{Stab}_\Omega(S)$, A' acts co-compactly on S, and A' fixes each vertex of S. Then Proposition VIII.9 implies that

$$S \subset \mathrm{Min}_{\mathcal{C}}(A')$$

and hence $\mathrm{Min}_{\mathcal{C}}(A')$ is non-empty. $\square$

8.4 Proof of Theorem I.17

Theorem I.17 is a straightforward consequence of Theorems VIII.1 and I.18. Suppose that $\Lambda \leq \mathrm{Aut}(\Omega)$ is a convex co-compact group and $A \leq \Lambda$ is a maximal Abelian subgroup. Since A is a maximal Abelian subgroup, $A = C_\Lambda(A)$. Then Theorem I.18 implies that A acts co-compactly on the non-empty convex subset

$$\mathrm{ConvHull}_\Omega\left(\mathrm{Min}_{\mathcal{C}_\Omega(\Lambda)}(A)\right) \subset \mathcal{C}_\Omega(\Lambda).$$

Then by Theorem VIII.1 there exists a properly embedded simplex

$$S \subset \mathrm{ConvHull}_\Omega\left(\mathrm{Min}_{\mathcal{C}_\Omega(\Lambda)}(A)\right) \subset \mathcal{C}_\Omega(\Lambda)$$

where $A \leq \mathrm{Stab}_\Omega(S)$, A acts co-compactly on S, and A fixes each vertex of S. Now it is straightforward to argue that A contains a finite index subgroup isomorphic of $\mathbb{Z}^{\dim(S)}$, see for instance [IZ21].

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