Continuous-time Optimal Dynamic Contracts

by

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ABSTRACT

Business operations often need long-term contracts to manage incentives over time. In this proposal, we discuss three projects in designing long-term contracts in different incentive management settings. In the **first** chapter, we study an optimal contract design problem, where a principal hires an agent to repair a machine when it is down and maintain it when it is up. If the agent exerts effort, the downtime is shortened, and uptime is prolonged. Effort, however, is costly to the agent and unobservable to the principal. The principal, therefore, devises a mechanism to always induce the agent to exert effort while maximizing the principal's profits. In the **second** chapter, we consider a service management setting, where the principal hires an agent to provide services to customers. Customers request service in one of two ways: either via an online or a traditional, walk-in, channel. The principal does not observe the walk-ins, nor does she observe whether the agent exerts (costly) effort that can increase the arrival rate of customers. This leads to a novel so far unexplored double moral hazard problem. We also present dynamic contracts that maximize the principal's profit. In the **third** chapter, we study the optimal incentive scheme for a long-term Poisson project with both moral hazard and adverse selection. The project has a flow cost that must be reimbursed by the principal, but the agent privately observes the cost that he incurs. The principal's optimal contract is a menu that contains several items, each of which is prepared for a specific group of agents. The agents reveal their costs immediately after they pick their preferred contract. We fully characterize the optimal contracts in the case of two types of agents. When the number of agent types is infinite and the cost distribution is continuous, we formulate an easy-to-compute upper bound optimization problem to the original problem. This optimization problem further provides a way for us to design a menu of contracts. Our numerical study illustrates that the proposed menu of contracts is indeed optimal with commonly used distributions.

CHAPTER 1

Introduction

Business operations often need long-term contracts to manage incentives over time. For example, a firm needs to continuously motivate its sales force or R&D team to work hard. Further, firms usually outsource maintenance activities to specialized companies in the airline, aerospace, defense, and mining industries, that often rely on complex, heavy, and critical equipment. However, it may be hard for firms to observe whether maintenance companies put sufficient resources into providing the best service, which gives rise to incentive issues. In service management settings, many service organizations have individual locations that are managed by agents on behalf of the owners. In such organizations, the owners do not have full visibility into agents' operations. Thus, the organizations need to provide incentives for the agent to behave in their best interest. Furthermore, in such incentive management settings, the firm not only cannot see the agents' actions but also does not know their capabilities. In this dissertation, we study three different inventive management problems. In each of the problems, we help firms design long-term incentive schemes to incentivize the agents to behave in their best interests. Specifically, we formulate constrained, continuous-time, stochastic optimal control problems, and derive optimal contracts with simple, intuitive structures.

In Chapter 2, we study an optimal contract design problem in a machine repair and maintenance setting. Specifically, a principal hires an agent to repair a machine when it is down and maintain it when it is up and earns a flow revenue when the machine is up. Both the up and down times follow exponential distributions. If the agent exerts effort, the downtime is shortened, and up-time is prolonged. The effort, however, is costly to the agent and unobservable to the principal. We study optimal dynamic contracts that always induce the agent to exert effort while maximizing the principal's profits. We formulate the contract design problem as a stochastic optimal control model with incentive constraints in continuous time over an infinite horizon. Although we consider the contract space that allows payments and potential contract termination time to take general forms, the optimal contracts demonstrate simple and intuitive structures, making them easy to describe and implement in practice. Most of the previous literature on maintenance and repair has largely ignored the agency issues. The few papers studying maintenance outsourcing contracts only consider static or repeated single-period settings. We contribute to the literature by considering a dynamic principal-agent framework where the agent's actions are time-dependent. Further, the agent is responsible for both repair and maintenance in our setting which makes the problem more complex, and the optimal solution more intricate. We find two intricate and interesting features in the optimal contract: First, it is possible that the principal rewards the agent with an amount more than the minimum necessary to incentivize the effort, i.e. the incentive compatibility constraints are not always binding. Second, to incentivize the agent, the principal may randomly terminate the agent.

In Chapter 3, we consider a stylized incentive management problem over an infinite time horizon, where the principal hires an agent to provide services to customers. Customers request service in one of two ways: either via an online or a traditional walk-in channel. The principal does not observe the walk-ins, nor does she observe whether the agent exerts (costly) effort that can increase the arrival rate of customers. This creates an opportunity for the agent (i) to divert cash (that is, to under-report the number of walk-in customers and pocket respective revenues) and also (ii) to shirk (that is, not to exert effort), thus, leading to a novel and so-far-unexplored double moral hazard problem. To address this problem, we formulate a constrained, continuous-time, stochastic optimal control problem and derive an optimal contract with a simple, intuitive structure that includes a payment scheme and a potential termination time of the agent. We extend the model to allow the principal to either (i) monitor the agent or (ii) manipulate the relative attractiveness of the online- against the walk-in- channel (by allowing the use of dynamic price discounting). Both such tools help the principal to alleviate the double moral hazard problem. We derive respective optimal strategies for using those tools that guarantee the highest profits. We show that the worse the agent's past performance is, the more the principal should monitor the agent and the more aggressive online channel price discounting should be. Finally, we demonstrate that sole monitoring of the walk-in arrivals or the usage of the channel manipulation tool can, sometimes, make the agent better off, as opposed to the situation when the principal monitors both the walk-in arrivals and effort, which always makes the agent worse off.

Most of the literature in dynamic moral hazard assumes that both the principal and the agent know the agent's capability, while the only asymmetric information is the hidden action that the agent is taking over time. In practice, however, the principal often does not know the agent's capabilities. For example, an employer (principal) may not know whether it is easy or hard for a sales representative (agent) to increase customers' arrival rate. In all these settings, the principal needs to motivate the agent's effort while not knowing the exact operating cost. In Chapter 4, we study the optimal incentive scheme for a long-term Poisson project with both moral hazard and adverse selection. Following standard results in mechanism design, the principal should provide a menu of contracts, such that an agent with a specific cost chooses a particular contract (including payments and termination) from this menu. In our setting, the contracts depend on the agent's past performance, and one type of agent should not have an incentive to choose a dynamic contract for another type. Consequently, the contract design problem is very complicated, and it can no longer be formulated as a classic dynamic program. We provide a novel solution approach based on deterministic continuous-time optimal control for this problem. If there are two possible types of agents with high or low cost, when the high-cost agent is too costly, he is asked to leave the system to avoid inefficiency. For him to tell the truth, an immediate payment that is equal to what he can get from mimicking the low-cost agent, namely the information rent, has to be rewarded to him. When the high-cost agent is less costly, he can possibly be hired. Yet it could be that even being hired; he would prefer mimicking the low-cost agent. Hence, an immediate payment that makes up the difference to induce him to tell the truth is still required. If there is a continuum of possible cost levels, we formulate an easy-to-compute upper bound optimization problem to the original problem thanks to the deterministic optimal control formulation. This optimization problem further provides a way for us to design a menu of contracts. Furthermore, we show that if the solution in the upper bound calculation satisfies a simple condition, then the upper and lower bounds match, which implies that our contract design is in fact optimal. Our numerical study illustrates that the condition is often satisfied with commonly used distributions. In this case, the principal designs a menu with a continuum of different items.

In summary, my research studies dynamic incentive design problems in different complex settings in operations management. Although we consider general contract spaces, the optimal contracts demonstrate simple and intuitive structures, making them easy to describe and implement in practice.

CHAPTER 2

Optimal Contract for Machine Repair and Maintenance

2.1 Introduction

In this paper, we study a dynamic contract design problem over an infinite horizon, in which a principal hires an agent to more efficiently operate a production process ("machine"), which changes between two states: up and down.[1] The state of the machine is public information. The "up" state yields a constant flow of revenue to the principal. The machine is subject to random shocks which causes it to go "down." When it is "down," the machine can be repaired to be "up" again. Without the agent, the machine stays in the up and down states for exponentially distributed random time periods with certain baseline rates. The agent has the expertise to improve maintenance and repair procedures by reducing the instantaneous rate for breaking down, and increasing the instantaneous rate to recover from the down state, if the agent exerts effort. Exerting effort is costly to the agent, and the effort cost may be different for repairing or maintaining the machine. Whether and when the agent puts in effort is the agent's private information. The principal would like to induce the agent's effort, and is able to commit to a long term contract, which involves payments and potential termination contingent on public information. We allow general forms of payments, including both instantaneous and flow payments. The principal is allowed to terminate the contract at any time, including terminating the contract with a probability less than one when the machine changes state. We also assume that the agent has limited liability. That is, the agent can decide to quit and never owes money to the principal.[2] Both players are risk neutral.

Although there is a wide literature on maintenance and repair, the majority of this literature is focused on optimal maintenance and repair conducted by a central decision maker, and has largely ignored the issues caused by agency. In many practical settings, however, maintenance and repair is conducted by an agent. Maintenance outsourcing is quite common in airline, aerospace, defense

[1] The material presented in this chapter is based on the paper [TSD21] co-authored with Peng Sun and Izak Duenyas.

[2] Limited liability is commonly assumed in contract theory, especially dynamic contract theory. Without it, the model and analysis becomes easy, or even trivial. For example, the principal could simply sell the entire enterprise to the agent up front, at a price that equals the efficient social surplus. This allows the principal to exact the entire surplus and leaves the agent with zero surplus. It is worth noting that in our optimal contract, the IC constraint may not be binding, which is also driven by limited liability.

and mining industries, that often rely on complex, heavy and critical equipment [TTMP06]. Instead of investing in the latest maintenance tools and facilities, and training in-house maintenance teams, firms outsource maintenance activities to specialized companies [MW12]. It may be hard for firms to observe whether maintenance companies put sufficient resources into providing best service, which gives rise to agency issues. Therefore, our paper makes a contribution to the maintenance/repair literature by tackling agency issues. In particular, we study a dynamic principal-agent framework, in which we obtain optimal contracts among history dependent ones. Despite the complexity of history dependent contracts, we demonstrate that the optimal contracts possess very simple structures that are easy to compute and implement. Further distinguished from the existing service/maintenance contract literature, we allow the agent to have limited liability and the ability to walk away at any point in time. Therefore, our contracts need to satisfy participation constraints, which guarantee that the agent stays before contract termination.

The paper also contributes to the dynamic contract design literature by considering an environment with *two* (machine) states. It is standard to formulate dynamic contract design problems as continuous time stochastic optimal control problems with incentive compatibility constraints, in which the agent's "promised utility" (see, for example, [SS87]) constitutes the state space. Our state space, however, also needs to include the machine state, which yields dynamics that do not appear to arise in traditional settings without such a multi-state environment. The paper studies all of the following three possible cases, although the main body of the paper is focused on the third one: (1) the principal only needs the agent when the machine is down; (2) the principal only needs the agent when the machine is up; and (3) the principal needs the agent for both types of work.

The classical maintenance literature is focused on optimal scheduling in a centralized context (see, for example, [PV76], [PL94], [McC65], [BP65], [GGS01]). Several papers consider maintenance outsourcing contracts involving a maintenance agent and a customer. In particular, [MA98] study a game-theoretic model in which an agent offers several options of contracts to a customer, including charging a fee for each repair during a given duration of time, or charging a lump sum fee for repairing the machine whenever it breaks down. The customer decides whether to hire the agent depending on the proposed contract. [MA99] extends the model to include multiple customers. [AM00] further allows the agent to choose the number of customers and the number of service channels besides a pricing strategy. Following this line of work, [TTMP06] develop incentive contracts to achieve channel coordination. [KCSS10] consider performance-based contracts for recovery services where the disruptions occur infrequently when the agent is risk-averse. They compare two types of widely used contracts, one based on sample-average downtime and the other based on cumulative downtime according to the supplier's ability to influence the frequency of disruptions. A clear distinction of our paper is that we consider time-dependent dynamic contracts while the aforementioned papers either consider static settings or repeated single-period settings. Other papers with static or repeated single-period settings include [TTT09], [Wan10], [PI15], [Bak06], [Coh87], [TPK14]. A common assumption in this literature is that the agent decides on the effort level (or, equivalently, capacity level) only once, then sticks to this level regardless of further outcomes. In many settings, an agent is often able to adjust effort choices over time. If a contract ignores such possibilities, the agent may lose the incentive to stick to the effort level as intended by the designer. Our model avoids this incentive issue because it is dynamic.

[PZ00] is the first paper to consider a dynamic principal-agent model of maintenance contract design in a discrete-time setting with a finite time horizon. In each period, if the machine is down, a risk-averse manager (agent) could choose between high and low effort levels, which further determine the probabilities that the machine comes back up in the following period. There is no moral hazard issue when the machine is up. More fundamentally, the agent in their model has access to borrowing and lending at the same rate, which means that they do not assume limited liability. Limited liability is a key assumption widely adopted in the dynamic contract literature (see, for example, [BMRV10], [GT16], [ST18]). Without it, the principal can essentially sell the entire enterprise to the agent, and therefore align incentives in a rather trivial fashion. The long-term optimal contract in [PZ00] is history-independent and renegotiation-proof. These nice properties rely critically on the borrowing and lending interest rates being exactly the same, and the agent's utility is additively separable and exponential. Following their optimal contract, in case the machine performance is bad for a period of time, the agent may have to borrow large amounts against future income, resulting in a negative total future utility. We, on the other hand, assume limited liability, which allows the agent to simply walk away (contract termination) instead of bearing a large debt.

The origin of the continuous time dynamic contract literature is often credited to the seminal paper [San08], which considers a principal hiring an agent to control the drift of a Brownian motion. Several papers have applied similar techniques in different settings with applications mostly in corporate finance (see, for example, to name a few [DS06b],[BMPR07],[Fu15]).

Instead of controlling the drift of a Brownian motion, in our model, the agent exerts effort to change the arrival rates of Poisson processes. Previous literature has studied one-sided problems, i.e., either decreasing or increasing the arrival rate of a Poisson process. [BMRV10], for example, considers a firm (principal) hiring a manager (agent) to exert private effort to decrease the arrival rate of large losses, modeled as a Poisson process, when the two players have different time discount rates. [Mye15] studies essentially the same model as in [BMRV10], except that the two players share the same time discount rate. In contrast, [ST18] consider the case of increasing the arrival rate of a Poisson process by the agent's private effort. [Var17], [Sha17a], and [GT16] study similar models with a finite number of arrivals and additional features, such as adverse selection issues or multiple players.

Because of limited liability, the optimal contract structures are different for decreasing versus increasing arrival rates. The common theme between the one-sided cases is that the optimal dynamic contracts often involve letting the promised utility to take a constant jump upon an arrival, which is upward for the case of increasing the arrival rate, and downward for decreasing the arrival rate. Our paper generalizes the previous literature by studying contracts that induce the agent to alternatively increase and decrease two different arrival rates over time (increase the rate of repair, and decrease the rate of failure). The combined control problem is more complex, and the optimal solution more intricate. In particular, the dynamics of the promised utility following our optimal control policy is not a mere combination of one-sided control policies.

Specifically, whenever the machine is repaired from a down state, the agent needs to be at least rewarded with an amount (denoted as $\beta_{\mathbf{d}}$). This amount $\beta_{\mathbf{d}}$ is set to exactly compensate the agent's effort to repair when the machine is down, so that the agent is (barely) willing to exert effort. The reward could take the form of either an increase in promised utility or a direct payment. Similarly, whenever the machine breaks down from an upstate, the agent needs to be penalized with an amount (denoted as $\beta_{\mathbf{u}}$), set to be exactly enough to induce the agent's effort to maintain when the machine is up. However, due to limited liability, the principal cannot charge the agent money. Therefore the penalty $\beta_{\mathbf{u}}$ takes the form of a reduction of promised utility. When the agent's promised utility is already lower than $\beta_{\mathbf{u}}$, we cannot reduce the promised utility by $\beta_{\mathbf{u}}$ anymore since the agent is protected by limited liability. Instead, in the optimal contract, the principal applies random termination to incentivize the agent to exert effort when the promised utility is low. The exact optimal contract structure differ between the cases of $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$. In both cases, the optimal contracts possess interesting structures only if the revenue rate the principal accrues when the machine is up is high enough.

If $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, the structure of the optimal contract is not quite surprising. The aforementioned reward and penalties are always set at the minimum levels $\beta_{\mathbf{d}}$ and $\beta_{\mathbf{u}}$ respectively, and random termination never happens. If $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, however, the optimal contract has a much more complex and delicate structure, and it has the following two intricate features.

First, it is possible that the principal rewards the agent with an amount more than the minimum necessary to incentivize the effort, i.e, the incentive compatibility constraints are not always binding. This is in contrast to previous papers (see, for example [San08], [BMRV10], [ST18]), where the incentive compatibility constraints are always binding in the optimal contract.

Second, because of limited liability and incentive compatibility, the agent's continuation utility cannot be attained below a threshold when the machine is up. To mitigate this possibility, we need to introduce random termination, where we randomly decide whether to terminate the agent or let the continuation utility increase back up to the threshold for free. This random termination also appears in [Mye15], in which the threshold is fixed at $\beta_{\mathbf{u}}$ (using our paper's notation). In our paper, however, the threshold is endogenously determined and may be higher than $\beta_{\mathbf{u}}$. We use a "smooth-pasting" technique to derive this threshold. This technique is classical in optimal stopping problem [DP94] and has been used in optimal contract literature (see, for example, [Zhu13], [CSX20]).

Overall, we consider the aforementioned two features as the most interesting and intricate results of this paper. In particular, the first one appears new in the literature, while the second one constitutes a major technical challenge in the analysis. Therefore, Section 2.4.2 contains the most interesting results, while earlier sections allow readers to gradually ease into them.

Specifically, we introduce the model and derive the incentive compatibility constraints in Section 2.2. In Section 2.3, we derive simple incentive compatible contracts without termination. In Section 2.4.1 and 2.4.2, we characterize the optimal incentive compatible contract under the condition $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, respectively. Section 2.4.3 summarizes all the results thus far. In certain settings, it may be better for the principal not to always induce effort from the agent. Therefore, in Section 2.5, we numerically compare the optimal incentive compatible contracts and two other alternative contracts that only induce effort for one of the machine states. Formal derivation and analysis of these alternative contracts are in the e-companion.

2.2 Model

Consider a principal operating a process (e.g. a "machine") in a continuous time setting. At any time t, the state of the machine, θ_t, is either up or down, denoted as $\mathbf{u}$ or $\mathbf{d}$, respectively. The principal receives a revenue at a positive rate R per unit of time when the machine is up. When the machine is down, the revenue is zero.[3] The machine remains in the up state for an exponentially distributed random time with rate $\bar{\mu}_{\mathbf{u}} > 0$ before breaking down. Once down, it takes an exponentially distributed random time with rate $\underline{\mu}_{\mathbf{d}} > 0$ to repair the machine back to state $\mathbf{u}$. There are many settings in which the above described situation arises. For example, factories produce products to be sold for revenue when their equipment are working. Similarly, airlines only generate revenue when their planes are functioning. (Obviously, most airlines have more than one plane, and factories more than one machine. Our model can be considered as a building block for multi-machine settings.)

The principal hires an agent to improve the process. Whenever the agent exerts effort (for example, assigning sufficient personnel to this job), the instantaneous rate of breaking down from state $\mathbf{u}$ is reduced to $\mu_{\mathbf{u}} \in (0, \bar{\mu}_{\mathbf{u}})$.[4] Similarly, at state $\mathbf{d}$, the agent's effort increases the instantaneous rate of recovering to $\mu_{\mathbf{d}} > \underline{\mu}_{\mathbf{d}}$. Effort is costly to the agent, and not observable to the principal. Specifically, denote $c_{\mathbf{u}}$ and $c_{\mathbf{d}}$ to be the effort costs in states $\mathbf{u}$ and $\mathbf{d}$, respectively. The

[3]It is without loss of generality to assume zero revenue rate when the machine is down. In fact, our results hold as long as the revenue rate is lower when the machine is down.

[4]We assume two levels of effort for simplicity. The results do not change if the effort level is from an interval, and the effort cost is linear in effort level.

corresponding effort cost rate at time t is

$$c(\theta_t) := c_{\mathbf{u}} \mathbb{1}_{\theta_t = \mathbf{u}} + c_{\mathbf{d}} \mathbb{1}_{\theta_t = \mathbf{d}}. \tag{2.1}$$

At any point in time t, the *public information* includes all the time epochs the machines changes state by time t. Formally, we denote a right-continuous counting process N_t to represent the total number of public events, i.e., change of machine states, up to time t. Let $\mathcal{F}^N$ be the filtration generated by the initial state θ_0 and the counting process $N = \{N_t\}$. Further denote an $\mathcal{F}^N$-predictable $\nu = \{\nu_t\}$ to represent the agent's effort process, such that $\nu_t \in \{0, 1\}$ for any time t. Specifically, $\nu_t = 1$ and $\nu_t = 0$ represent that the agent exerts effort and shirks at time t, respectively. Therefore, at any point in time t with the state of the machine θ_t and the effort level ν_t, the arrival rate of process N is

$$\mu(\theta_t, \nu_t) := \left[\mu_{\mathbf{u}} \nu_t + \bar{\mu}_{\mathbf{u}} (1 - \nu_t)\right] \mathbb{1}_{\theta_t = \mathbf{u}} + \left[\mu_{\mathbf{d}} \nu_t + \underline{\mu}_{\mathbf{d}} (1 - \nu_t)\right] \mathbb{1}_{\theta_t = \mathbf{d}}. \tag{2.2}$$

We assume that the agent has limited liability, and we mainly focus our attention on contracts that always induce effort from the agent. (In Section 2.5 and the e-companion, we also consider contracts that induce effort only in one of the machine states.) Therefore, the principal needs to reimburse the aforementioned effort costs in real time as flow payments whenever effort is expected. As a result, the effort cost $c(\theta_t)$ becomes shirking benefit if the agent shirks at time t. This is a standard treatment in the dynamic contracting literature (see, for example, [BMRV10]).

We further assume that the principal has the commitment power to a long-term contract based on public information. Overall, a dynamic contract $\Gamma = (L, \tau, q)$ includes a payment process L, a contract termination time τ, and a stochastic termination process q.

Specifically, denote an $\mathcal{F}^N$-adapted process $L = \{L_t\}_{t \geq 0}$ to represent the cumulative payment from the principal to the agent up to time t. The payment includes an instantaneous one, I_t, and a flow with rate ℓ_t *beyond the background payment* $c(\theta_t)$ *that reimburses effort*, such that $dL_t = I_t + \ell_t dt$. Limited liability implies $I_t \geq 0$ and $\ell_t \geq 0$.

The contract not only includes payments, but also the possibility of terminating the agent at a random time τ. We consider two ways of contract termination. First, at any point in time t when the machine changes state (i.e., $dN_t = 1$), we allow the principal to terminate the contract randomly, with probability $q_t \in [0, 1]$, where the probability q_t depends on all of the information on machine state changes until time t, i.e., $\mathcal{F}_t^N$-measurable. Therefore, the contract contains an $\mathcal{F}^N$-adapted process $q = \{q_t\}_{t \geq 0}$ for random contract termination. Second, we also allow the principal to terminate the agent depending (deterministically) on history $\mathcal{F}_t^N$ without randomization. As will be clear later in the paper, allowing random termination is crucial to construct optimal contracts for certain model parameter settings. The principal and the agent are both risk-neutral and discount future cash flows at rate r.[5]

The principal's expected total discounted profit under a contract Γ and effort process ν is defined as[6]

$$U(\Gamma,\nu,\theta_0) = \mathbb{E}\left[\int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t - c(\theta_t)dt) + e^{-r\tau}\underline{v}_\tau \,\middle|\, \theta_0\right], \tag{2.3}$$

where $\underline{v}_\tau$ is the principal's total discounted future profit after terminating the agent. This value clearly depends on the state of machine θ_τ at the termination time τ. It is easy to verify (see Lemma A.1 in the Appendix) that $\underline{v}_\tau$ takes the following values for $\theta_\tau = \mathbf{u}$ and $\theta_\tau = \mathbf{d}$, respectively,

$$\underline{v}_\mathbf{u} := \frac{R}{r}\cdot\frac{r+\underline{\mu}_\mathbf{d}}{r+\bar{\mu}_\mathbf{u}+\underline{\mu}_\mathbf{d}}, \text{ and } \underline{v}_\mathbf{d} := \frac{R}{r}\cdot\frac{\underline{\mu}_\mathbf{d}}{r+\bar{\mu}_\mathbf{u}+\underline{\mu}_\mathbf{d}}. \tag{2.4}$$

Given contract Γ that always reimburses effort cost before termination, and an effort process ν, the agent's expected total discounted utility is the cumulative payments minus the effort cost, expressed as the following

$$u(\Gamma,\nu,\theta_0) = \mathbb{E}\left\{\int_0^\tau e^{-rt}\left[dL_t + (1-\nu_t)c(\theta_t)dt\right] \,\middle|\, \theta_0\right\}. \tag{2.5}$$

Therefore, given either initial state $\theta_0 = \mathbf{u}$ or $\theta_0 = \mathbf{d}$, we can define a game between the two players, in which the principal designs an optimal contract Γ that maximizes utility $U(\Gamma,\nu,\theta_0)$, anticipating the agent's effort process ν that maximizes $u(\Gamma,\nu,\theta_0)$. Throughout the paper, we focus on studying contracts that induce the agent to always exert effort (so called "incentive compatible" contracts).[7] In the e-companion Section A.2.1 we provide a sufficient condition on model parameters such that it is indeed optimal for the principal to only focus on incentive compatible contracts.

Incentive Compatibility

A contract Γ is *incentive compatible* (IC) if in equilibrium, the agent has the incentive to always exert effort (to better maintain the machine so that the failure rate from state $\mathbf{u}$ drops to $\mu_\mathbf{u}$, and to faster repair the machine so that it comes back up at rate $\mu_\mathbf{d}$ from state $\mathbf{d}$), i.e. $\nu^* := \{\nu_t = 1\}_{\forall t\in[0,\tau]}$.

[5]We assume equal discount rate between the two players, similar to [Mye15]. This is mostly for simplicity, although one may also argue that having access to a complete financial market allows the two agents to hedge all risks and use the risk-free interest rate and risk-neutral probabilities. Interestingly, one of the main findings of [Mye15], the infinite back-loading issue when the two players share equal time discount, does not arise in our setting. We explain this phenomenon and the underlying reasons in more detail in Section 2.4. If the two players have different discount rates, the optimal contract structure appears to be much more intricate. We leave that for future research.

[6]Note that the expectation here, as well as in (2.5), is taken with respective to the stochastic process generated from the effort process ν. This explains that in (2.3) we need to specify ν as an argument of the function. For ease of exposition, we omit the explicit dependence between the expectation and ν in the main text of the paper.

[7]Allowing shirking complicates the analysis for dynamic contracts substantially. See, for example, [Zhu13] for a reference of optimal contract design allowing shirking in a Brownian motion framework.

That is, the contract is *incentive compatible* if[8]

$$u(\Gamma, \nu^*, \theta_0) \geq u(\Gamma, \nu, \theta_0)\ ,\ \ \forall \mathcal{F}^{\mathcal{N}}\text{-predictable effort process } \nu,\ \theta_{\prime} \in \{\sqcap, \lceil\}. \tag{IC}$$

In this paper we focus on the class of incentive compatible contracts that always induce effort.

The contract design problem may be formulated as a stochastic optimal control problem, in which the state is the agent's promised utility at time t, defined as,

$$W_t(\Gamma, \nu) = \mathbb{E}\left\{ \int_t^{\tau} e^{-r(s-t)} \left[dL_s + (1-\nu_s)c(\theta_s)ds\right] \middle| \mathcal{F}_t \right\} \mathbb{1}\{t \leq \tau\}. \tag{2.6}$$

It is clear that $W_0(\Gamma, \nu) = u(\Gamma, \nu, \theta_0)$ for θ_0 consistent with $\mathcal{F}_{\prime}$. It is worth noting that the $\mathcal{F}^N$-adapted process W_t is right-continuous, representing the agent's continuation utility *after* observing a potential arrival at time t and *after* a potential instantaneous payment I_t. In comparison, the principal's control processes, L_t and q_t, are $\mathcal{F}^N$-predictable and left-continuous. The principal schedules payments and stopping time through controlling the agent's promised utility. Therefore it is important to introduce process $W_{t-} = \lim_{s \uparrow t} W_s$, the left-hand limit of W_t, which is left-continuous and $\mathcal{F}^N$-predictable (assuming $W_{0-} = W_0$). At any time t, W_{t-} captures the agent's continuation utility *before* knowing about the potential arrival and instantaneous payment at time t. Similarly, for the right-continuous state process θ_t, we define left-continuous process $\theta_{t-} = \lim_{s \uparrow t} \theta_t$ to represent the state of the machine right before time t for any $t > 0$.

The contract also needs to ensure the agent's participation at any point in time. That is, the agent's promised utility needs to be non-negative (also called the *individual rationality* (IR) condition), i.e.,[9]

$$W_t \geq 0, \ \ \forall t \geq 0. \tag{IR}$$

The following lemma provides the evolution of the agent's promised utility W_t under any contract Γ, which is often called the *promise keeping* (PK) condition in the dynamic contract literature. It also provides an equivalent condition for (IC) in terms of the promised utility W_t.

Lemma 2.1 *For any contract Γ, there exists an $\mathcal{F}^N$-predictable process H_t such that for $t \in [0, \tau]$,*

$$\begin{aligned} dW_t = \Big\{ rW_{t-} - (1-\nu_t)c(\theta_t) - \left[(1-q_t)H_t - q_t W_{t-}\right]\mu(\theta_t, \nu_t) - \ell_t \Big\} dt \\ + \left[(1-X_t)H_t - X_t W_{t-}\right] dN_t - I_t, \end{aligned} \tag{PK}$$

in which Bernoulli random variable X_t takes value 1 with probability q_t.

---

[8]All the inequalities in this paper are to be understood as almost surely.

[9]If one only considers (IR) for time 0, the contract design is trivial. The principal can extract the entire surplus of the first best outcome by offering zero utility to the agent.

Furthermore, contract Γ satisfies (IC) if and only if

$$
\begin{aligned}
(1-q_t)H_t - q_t W_{t-} &\leq -\beta_{\mathbf{u}}, \quad \textit{for} \quad \theta_{t-} = \mathbf{u}, \textit{ and} \\
(1-q_t)H_t - q_t W_{t-} &\geq \beta_{\mathbf{d}}, \quad \textit{for} \quad \theta_{t-} = \mathbf{d},
\end{aligned} \tag{2.7}
$$

for all $t \in [0, \tau]$, where

$$
\beta_{\mathbf{u}} := \frac{c_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}}} \quad \textit{and} \quad \beta_{\mathbf{d}} := \frac{c_{\mathbf{d}}}{\mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}}}. \tag{2.8}
$$

Finally, we need $-H_t \leq W_{t-}$ for all $t \geq 0$ in order to satisfy (IR).

The (PK) condition is a standard result for the dynamics of the agent's promised utilities over time. To facilitate understanding, it is helpful to consider a heuristic derivation based on discrete time approximation. Consider a small time interval $[t, t+\delta)$. Assume that the agent's promised utility W_{t-} evolves continuously to W_{t+} over this interval, unless there is a change of machine state, with probability $\mu(\theta_t, \nu_t)\delta$. With a change of state, the agent's total future utility takes a jump to either $W_{t-} + H_t$ with probability $1 - q_t$, or to 0 with probability q_t (termination). Also taking into consideration the shirking benefit $(1-\nu_t)c(\theta_t)\delta$, flow payment $\ell_t\delta$, and time discount rate r (for simplicity, ignore the instantaneous payment I_t), the above description of the discrete time approximation of the promised utility implies the following,

$$
W_{t-} = (1-\nu_t)c(\theta_t)\delta + \ell_t\delta + \mu(\theta_t, \nu_t)\delta\left[q_t \cdot 0 + (1-q_t)(W_{t-} + H_t)\right] + \left[1 - (\mu(\theta_t, \nu_t) + r)\delta\right] W_{t+}.
$$

As δ approaches 0, replace it with dt, and rearrange terms, we observe that the smooth change $W_{t+} - W_{t-}$ equals

$$
\left\{rW_{t-} - (1-\nu_t)c(\theta_t) - \ell_t - \left[(1-q_t)H_t - q_t W_{t-}\right]\mu(\theta_t, \nu_t)\right\}dt,
$$

which recovers the terms involving dt in (PK). The change of machine state ($dN_t = 1$) results in the agent's total future utility changing by either H_t or $-W_{t-}$ (termination), depending on the outcome of the random variable X_t. Therefore, the change is $\left[(1-X_t)H_t - X_t W_{t-}\right] dN_t$. Finally, this total change can be delivered by a direct instantaneous payment I_t in addition to the change in the promised utility dW_t. That is, $dW_t + I_t = \left[(1-X_t)H_t - X_t W_{t-}\right] dN_t$ when $dN_t = 1$. Therefore, we can consider the process H_t as the total change of the agent's total future utility if the state changes at time t, which consists of the change in the agent's continuation utility dW_t and the instantaneous payment I_t that the agent receives. Rigorously, in the continuous time setting, the $\mathcal{F}^N$-predictable process H_t is left-continuous.

The values $\beta_{\mathbf{d}}$ and $\beta_{\mathbf{u}}$ defined in equation (2.8) reflect the ratios between effort cost and improvement in the repair or failure rates, which reveal the intuition behind the (IC) condition. For an intuitive interpretation of these two important quantities, consider, for example, the up state. If the principal could charge the agent an amount $\beta_{\mathbf{u}}$ upon the machine breaking down, the agent is then indifferent between exerting effort or not. This is because over a small time period δ, the shirking benefit, $c_{\mathbf{u}}\delta$, exactly compensates the additional expected charge, $\beta_{\mathbf{u}}(\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}})\delta$. Condition (2.7) states that instead of directly charging the agent, an incentive compatible contract needs to reduce the agent's promised utility by at least $\beta_{\mathbf{u}}$. The term $\beta_{\mathbf{d}}$ has a similar interpretation for the down state. Following standard IC conditions in [San08] and [BMRV10], one would only obtain the result that the magnitude of H_t is larger than $\beta_{\mathbf{d}}$ or smaller than $-\beta_{\mathbf{u}}$. Our (IC) condition in Lemma 2.1, however, generalizes the standard form due to the consideration of contract termination. In Section 2.4.2, we show that the probability of random termination, q_t, could indeed be positive in the the optimal contract.

Later in the paper we show that the structure of the optimal contract, including whether and when the incentive compatibility constraints (2.7) are binding, depends on whether $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ or $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$. The intuitive interpretation of these conditions follows the definition of $\beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}}$. For example, if the costs of effort are the same in the two machine states (i.e., $c_{\mathbf{d}} = c_{\mathbf{u}}$), then $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ means that the agent is able to decrease the break down rate more than increase the recovery rate ($\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}} \geq \mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}}$).

Finally, (IR) requires that the agent's promised utility must be non-negative at all times, including right after a downward jump of the promised utility. As explained above, (IC) requires that in the up state a downward jump has to be at least $\beta_{\mathbf{u}}$. Therefore, when the state $\theta_t = \mathbf{u}$, we can only satisfy constraint (2.7) when $W_{t-} \geq \beta_{\mathbf{u}}$. When the machine is up and W_{t-} becomes too low (say, lower than $\beta_{\mathbf{u}}$), however, the principal needs to randomize the promised utility to either 0 (termination), or back to a threshold. This is why we need the randomized termination process q_t for the optimal contract. Interestingly, as we will show in Section 2.4, random termination only occurs if $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$. In fact randomization may even occur when $W_{t-} > \beta_{\mathbf{u}}$. If $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, on the other hand, the optimal contract always guarantees $W_{t-} \geq \beta_{\mathbf{u}}$ when the machine is up.

In the next section, we first introduce two simple and stationary incentive compatible contracts, which help us lay the foundation of the optimal incentive compatible contracts.

2.3 Benchmark contracts

Before introducing the optimal contract, it is worth studying simple incentive compatible contracts in this section. These contracts are stationary in nature – the contract terms only depend on the state of the machine and its transitions, and not on time otherwise. This implies that they never terminate the agent. In fact, if we do not allow contract termination, they are indeed optimal incentive compatible contracts. In the next section, however, we show that optimal contracts that allow termination are based upon, but outperform these simple ones. In particular, it is important to distinguish between the two cases $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, which are studied separately in the two subsections, respectively.

2.3.1 $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$

The contract is indeed very simple: the principal pays an instantaneous bonus $\beta_{\mathbf{d}} - \beta_{\mathbf{u}}$ when the machine recovers from state $\mathbf{d}$, followed by a flow payment with rate

$$\ell_1^* = \mu_{\mathbf{d}}\beta_{\mathbf{d}} + (r + \mu_{\mathbf{u}})\beta_{\mathbf{u}} \tag{2.9}$$

when the machine remains in state $\mathbf{u}$. We denote $\bar{\Gamma}$ to represent this contract.

In order to prove that $\bar{\Gamma}$ is incentive compatible, it is important to derive the agent's promised utility following this contract. In fact, we claim that the promised utility remains a constant for each machine state. Define $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$ as these two promised utilities when the machine's state is $\mathbf{d}$ and $\mathbf{u}$, respectively,

$$\bar{w}_{\mathbf{d}} = \frac{\mu_{\mathbf{d}}\beta_{\mathbf{d}}}{r}, \text{ and } \bar{w}_{\mathbf{u}} = \bar{w}_{\mathbf{d}} + \beta_{\mathbf{u}}. \tag{2.10}$$

It is easy to verify that contract $\bar{\Gamma}$ is incentive compatible. In fact, whenever the machine breaks down, the promised utility changes from $\bar{w}_{\mathbf{u}}$ to $\bar{w}_{\mathbf{d}}$, with a downward jump of exactly $H_t = -\beta_{\mathbf{u}}$. Upon recovery from state $\mathbf{d}$, the promised utility first takes an upward jump of $\beta_{\mathbf{u}}$, and then the agent is given a direct payment of $\beta_{\mathbf{d}} - \beta_{\mathbf{u}}$ resulting in $H_t = \beta_{\mathbf{d}}$. Therefore, incentive compatibility constraints (2.7) are always binding, with aforementioned H_t and $q_t = 0$. This further ensures that the agent always exerts effort. Regarding the promise keeping constraint, for state $\theta_t = \mathbf{u}$, if we set $W_t = \bar{w}_{\mathbf{u}}$ and $dL_t = \ell_1^* dt$, then (PK) becomes $dW_t = -\beta_{\mathbf{u}} dN_t$. Similarly, for state $\theta_t = \mathbf{d}$, setting $W_t = \bar{w}_{\mathbf{d}}$, and $dL_t = (\beta_{\mathbf{d}} - \beta_{\mathbf{u}})dN_t$, (PK) becomes $dW_t = \beta_{\mathbf{u}} dN_t$. Therefore, contract $\bar{\Gamma}$ and our claimed promised utilities (2.10) indeed satisfy both (PK) and (IC) constraints.

Besides the mathematical arguments above, it is in fact intuitive that contract $\bar{\Gamma}$ provides the incentive for the agent to exert effort. When the machine is down, the prospect of an instantaneous bonus followed by a flow payment provides the incentive for the agent to repair the machine faster. When the machine is up, the flow payment incentivizes the agent to better maintain and prolong the period of payment. In particular, the flow payment ℓ_1^* has two components. The first component is the interest payment $r\bar{w}_{\mathbf{u}}$, so that the agent's promised utility is kept at $\bar{w}_{\mathbf{u}}$. The second component is the rent $\mu_{\mathbf{u}}\beta_{\mathbf{u}}$ whenever there is no arrival (machine breaking down).

Furthermore, because contract $\bar{\Gamma}$ never terminates the agent, we have the following expressions for the total discounted societal values (summation of the principal and the agent's utilities) at states

$\mathbf{u}$ and $\mathbf{d}$, respectively (see Lemma A.1 in the Appendix for the derivations).

$$\bar{v}_{\mathbf{d}} = \frac{\mu_{\mathbf{d}}(R - c_{\mathbf{u}}) - (r + \mu_{\mathbf{u}})c_{\mathbf{d}}}{r(r + \mu_{\mathbf{d}} + \mu_{\mathbf{u}})} \quad \text{and} \quad \bar{v}_{\mathbf{u}} = \frac{(r + \mu_{\mathbf{d}})(R - c_{\mathbf{u}}) - \mu_{\mathbf{u}} c_{\mathbf{d}}}{r(r + \mu_{\mathbf{d}} + \mu_{\mathbf{u}})}. \tag{2.11}$$

Consequently, the principal's utilities under contract $\bar{\Gamma}$ for state $\mathbf{u}$ and $\mathbf{d}$ are, respectively,

$$U(\bar{\Gamma}, \nu^*, \mathbf{u}) = \bar{U}_{\mathbf{u}} := \bar{v}_{\mathbf{u}} - \bar{w}_{\mathbf{u}} \text{ and } U(\bar{\Gamma}, \nu^*, \mathbf{d}) = \bar{U}_{\mathbf{d}} := \bar{v}_{\mathbf{d}} - \bar{w}_{\mathbf{d}}. \tag{2.12}$$

Although this simple contract $\bar{\Gamma}$ is incentive compatible, it is actually not optimal, because it only uses the "carrot" of payments without the "stick" of termination. At the end of this section, Proposition 2.7 shows that this simple contract is actually the optimal incentive compatible contract if the principal is not allowed to terminate the agent. Besides introducing this simple contract to build intuition, we would like to clarify the simple contract's connection with the optimal contract. According to the optimal contract, it is possible that the promised utilities eventually become $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$ for states $\mathbf{u}$ and $\mathbf{d}$, respectively. From that point on, the optimal contract becomes identical to the simple contract $\bar{\Gamma}$, and the agent is never terminated. However, following the optimal contract, it is also possible that the promised utilities never reach $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$ before the agent is terminated.

Finally, it is clear that the society is better off with contract $\bar{\Gamma}$ compared with not hiring the agent at all if $\bar{v}_{\mathbf{u}}$ and $\bar{v}_{\mathbf{d}}$ are at least as high as $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$ defined in (2.4). In fact, when $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, one can verify that $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$ readily implies $\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$. Furthermore,

$$\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}} \tag{2.13}$$

is equivalent to

$$R \geq h_{\mathbf{d}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \frac{\mu_{\mathbf{d}} c_{\mathbf{u}} + (r + \mu_{\mathbf{u}})\, c_{\mathbf{d}}}{\mu_{\mathbf{d}} \Delta\mu_{\mathbf{u}} + (r + \mu_{\mathbf{u}})\, \Delta\mu_{\mathbf{d}}}. \tag{2.14}$$

Intuitively, hiring the agent is beneficial only if the revenue rate R is high enough. In Section 2.4.1, we demonstrate that the structure of the the optimal contracts depends critically on whether condition (2.13) holds.

2.3.2 $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$

The simple contract in this case, denoted as $\hat{\Gamma}$, can be described in one sentence: it pays the agent a flow payment with rate

$$\ell_2^* = (r + \mu_{\mathbf{u}} + \mu_{\mathbf{d}})\beta_{\mathbf{u}} \tag{2.15}$$

at state $\mathbf{u}$.

The promised utilities are the following two constants for the two machine states, respectively,

$$\hat{w}_{\mathbf{d}} = \frac{\mu_{\mathbf{d}}\beta_{\mathbf{u}}}{r}, \quad \text{and} \quad \hat{w}_{\mathbf{u}} = \hat{w}_{\mathbf{d}} + \beta_{\mathbf{u}}. \tag{2.16}$$

similar to $\bar{w}_{\mathbf{d}}$ and $\bar{w}_{\mathbf{u}}$ defined in (2.10). Similar to the analysis for $\bar{\Gamma}$, we can verify that contract $\hat{\Gamma}$ together with $\hat{w}_{\mathbf{d}}$ and $\hat{w}_{\mathbf{u}}$ satisfy (PK) and (IC). The expressions for the societal utility still follow (2.11). The principal's utilities under contract $\hat{\Gamma}$ are, therefore,

$$U(\hat{\Gamma}, \nu^*, \mathbf{u}) = \hat{U}_{\mathbf{u}} := \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}} \text{ and } U(\hat{\Gamma}, \nu^*, \mathbf{d}) = \hat{U}_{\mathbf{d}} := \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}}, \tag{2.17}$$

for machine states $\mathbf{u}$ and $\mathbf{d}$, respectively.

The overall feature of $\hat{\Gamma}$ for the case of $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ is very similar to $\bar{\Gamma}$ for the case of $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$. Later in Section 2.4.2.1, we show that the agent's promised utility has a chance to eventually become $\hat{w}_{\mathbf{d}}$ and $\hat{w}_{\mathbf{u}}$ following the optimal contract. After reaching that point, the optimal contract becomes identical to $\hat{\Gamma}$, and the agent is never terminated. At the end of Section 2.4.2, we present Proposition 2.7, which shows that this simple contract is actually the optimal incentive compatible contract if terminating the agent is not allowed. Similar to ℓ_1^*, the flow payment ℓ_2^* can also be decomposed into two components, the interest payment $r\hat{w}_{\mathbf{u}}$, and the rent $\mu_{\mathbf{u}}\beta_{\mathbf{u}}$. Finally, parallel to the previous case, when $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, we have

$$\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}} \tag{2.18}$$

is equivalent to

$$R \geq h_{\mathbf{u}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \frac{\mu_{\mathbf{u}} c_{\mathbf{d}} + (r + \mu_{\mathbf{d}}) c_{\mathbf{u}}}{\mu_{\mathbf{u}} \Delta\mu_{\mathbf{d}} + (r + \mu_{\mathbf{d}})\, \Delta\mu_{\mathbf{u}}}, \tag{2.19}$$

and readily implies $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$.

Despite these similarities between $\bar{\Gamma}$ and $\hat{\Gamma}$, it is worth noting an important difference between them. Under contract $\hat{\Gamma}$ of the current case, the incentive compatibility constraint (2.7) is not binding. In fact, both the downward jump upon breaking down and the upward jump upon recovery are both $\beta_{\mathbf{u}}$ (i.e.,$H_t = \beta_{\mathbf{u}} \left(\mathbb{1}_{\theta_{t-}=\mathbf{d}} - \mathbb{1}_{\theta_{t-}=\mathbf{u}}\right)$). In particular, when the machine recovers, the upward jump is higher than what is required in constraint (2.7). Given our claim in the last paragraph, it means that the incentive compatibility constraint may not be binding in the optimal contract. This may appear surprising, given that we are not aware of other optimal dynamic contract with non-binding incentive compatibility constraint in the literature. We will explain why this phenomenon arises in our setting after introducing the optimal contract in Section 2.4.2.

2.4 Optimal Contract

In this section, we study and characterize in detail the optimal contracts that induce the agent to always exert effort before termination. Similar to the previous section, here we first study the case $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ before $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, in Sections 2.4.1 and 2.4.2, respectively. In the end we summarize main results for different cases in Section 2.4.3. It is worth noting that the more interesting and intricate results of this paper, including non-binding incentive compatible constraint, are presented in Section 2.4.2.

2.4.1 The Case $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$

The structure of the optimal contract in this case, although new, may not appear surprising to readers already familiar with the continuous time contracting literature ([BMRV10], [ST18]). However, this section provides a gentle preparation to the more complex and delicate structure in the optimal contract for the case $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ next.

In Section 2.4.1.1, we first introduce the optimal contract under condition (2.13), which is equivalent to (2.14). Section 2.4.1.2 further provides the principal's value functions under the optimal contract and the proof of optimality. Finally, Section 2.4.1.3 studies what happens when the condition (2.13) does not hold.

2.4.1.1 Optimal IC contract when $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$

In this subsection, we develop a contract Γ_1^*, and leave the proof of optimality to the next subsection. The contract keeps track of the agent's promised utility. Figure 2.1 depicts two sample trajectories of the agent's promised utility in the proposed contract where the machine starts at state $\theta_0 = \mathbf{d}$.

In Figure 2.1, we have $\bar{w}_{\mathbf{d}} = 0.74$, $\bar{w}_{\mathbf{u}} = 1.01$ and $\beta_{\mathbf{u}} = 0.27 < \beta_{\mathbf{d}} = 0.33$. The policy starts from $W_0 = w_{\mathbf{d}}^* = 0.4685$. The two dashed horizontal lines represent the level of $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$, respectively. The upward jump level when the machine is repaired is $\beta_{\mathbf{d}}$ and the downward drop level when the machine breaks down is $\beta_{\mathbf{u}}$.

The promised utility starts from an initial promised utility $W_0 = w_{\mathbf{d}}^* \in (0, \bar{w}_{\mathbf{d}})$. While repairing the machine, this utility keeps decreasing (the exact form to be specified later) until either the machine is repaired or the utility reaches 0. If the machine has not recovered before the utility W_t reaches 0, the principal terminates the agent. The dotted curve in Figure 2.1 represents this situation, where the promised utility decreases to zero at time τ.

On the other hand, if the machine recovers at time t with $W_{t-} > 0$, the utility W_t takes an upward jump of level $\min\{\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}} - W_{t-}\}$ and the agent is paid $(W_{t-} + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+$ instantaneously. See the solid curve in Figure 2.1, which represents another sample trajectory. In the time interval $[0, t_1)$,

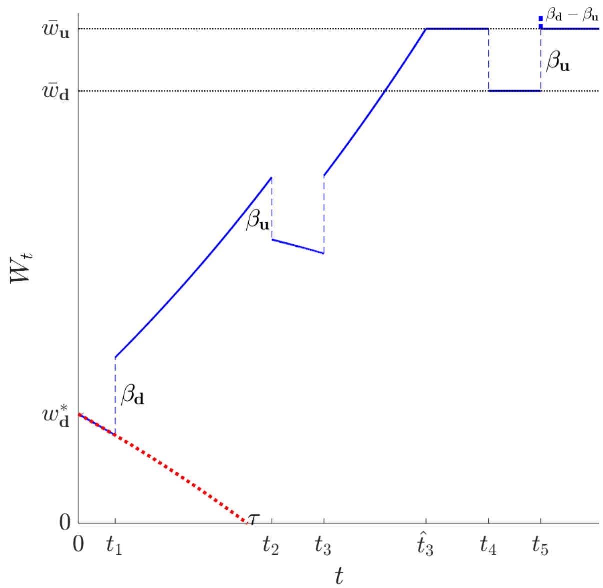


Figure 2.1: Two sample trajectories of promised utility with $\mu_{\mathbf{u}} = 6$, $\Delta\mu_{\mathbf{u}} = 3$, $\mu_{\mathbf{d}} = 5$, $\Delta\mu_{\mathbf{d}} = 2$, $c_{\mathbf{u}} = 0.8$, $c_{\mathbf{d}} = 1, r = 0.9, R = 7.5$.

the promised utility is decreasing over time. At t_1, it jumps up by $\beta_{\mathbf{d}}$ because $W_{t_1-} < \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}$. The corresponding instantaneous payment is 0. Then the contract continues with the agent maintaining the machine in the up state, while the promised utility keeps increasing until either it reaches $\bar{w}_{\mathbf{u}}$, or the machine breaks down. During (t_1, t_2), the promised utility is increasing over time. At time t_2, the machine breaks down and the promised utility drops by $\beta_{\mathbf{u}}$. Again, in (t_2, t_3), the agent is repairing the machine with the promised utility decreasing over time. After t_3, the machine does not break down before the promised utility reaches $\bar{w}_{\mathbf{u}}$ at time $\hat{t}_3$, at which point the flow payment ℓ_1^* (defined in (2.9)) starts. After time $\hat{t}_3$, the agent's promised utility jumps back and forth between $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$. The contract becomes exactly the same as the simple contract $\bar{\Gamma}$ introduced in the previous subsection. In the following, we provide a formal definition of the proposed optimal contract.

Definition 2.1 *For a machine starting from state $\theta \in \{\mathbf{u}, \mathbf{d}\}$, define contract $\Gamma_1^*(w) = (L^*, q^*, \tau^*)$ as the following, where $w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$ if the initial state is $\mathbf{u}$, and $w \in [0, \bar{w}_{\mathbf{d}}]$ if the initial state is $\mathbf{d}$.*

- *The dynamics of the agent's promised utility W_t follows*

$$dW_t = \big[r(W_{t-} - \bar{w}_{\mathbf{d}})dt + \min\{\bar{w}_{\mathbf{u}} - W_{t-}, \beta_{\mathbf{d}}\}dN_t\big]\mathbb{1}_{\theta_{t-}=\mathbf{d}} + \big[(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}}dt - \beta_{\mathbf{u}}dN_t\big]\mathbb{1}_{\theta_{t-}=\mathbf{u}}, \tag{DW1}$$

  from the initial promised utility $W_0 = w$.

- *The payment to the agent follows $dL_t^* = \ell_1^*\mathbb{1}_{W_{t-}=\bar{w}_{\mathbf{u}}}\mathbb{1}_{\theta_{t-}=\mathbf{u}}dt + (W_{t-} + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+\mathbb{1}_{\theta_{t-}=\mathbf{d}}dN_t$.*

- *The random termination probability is $q_t^* = 0$, (i.e. there is no random termination) and the termination time is $\tau^* = \min\{t : W_t = 0\}$.*

One can verify that the dynamics of W_t in the proposed optimal contract follows (PK), with $H_t = \beta_{\mathbf{d}} \mathbb{1}_{\theta_{t-}=\mathbf{d}} - \beta_{\mathbf{u}} \mathbb{1}_{\theta_{t-}=\mathbf{u}}$, $dL_t = dL_t^*$ and $q_t = q_t^*$. Also, in the proposed optimal contract, the incentive compatible constraints (2.7) are binding, and the principal never randomly terminates the agent. It is only possible to terminate the agent when the machine is down (note that we do not terminate the agent exactly at the point when the machine goes down but when the promised utility reaches zero, e.g., after a long enough down period). On the other hand, when the machine is up, the agent's promised utility is always greater than $\beta_{\mathbf{u}}$. This is because if the initial state of the machine is up, the initial promised utility would be at least $\beta_{\mathbf{u}}$ and keeps going up until the first break down; after the agent has finished repair once, the promised utility would always jump to a level above $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ to start the up state.

It is worth noting that payment in Definition 2.1 involves both instantaneous payment and flow payment. And payment only occur when the promised utility is high enough such that the optimal contract becomes the benchmark $\bar{\Gamma}$ defined in Section 2.3.1.

Remark 2.1 *We want to emphasize that the principal has three options to provide incentives: (1) a flow payment ℓ_t only in the up state motivates the agent to prolong the up state; (2) an instantaneous payment I_t when the state changes from down to up motivates the agent to speed up the machine recovery; and (3) when the machine has been down for too long, the agent is threatened with contract termination. In the next subsection, when we study the case of $\beta_{\mathbf{u}} > \beta_{\mathbf{d}}$, contract termination may also occur when the machine changes state. In the optimal contract, we usually do not initiate payments immediately. Instead, the agent's continuation utility keeps increasing in the up state, implying that the start of the flow payment is approaching. In the down state, the continuation utility keeps decreasing, as the threat of termination looms larger. The upward and downward jumps upon state changes also bring the continuation utility closer to either payment or termination. Therefore, the continuation utility W_t can be perceived as a proxy to the timing of the payment and contract termination.*

Remark 2.2 *The contract $\Gamma_1^*(w)$ is optimal but not unique. When the principal and the agent share the same discount rate, the principal can delay a payment to a later time with the corresponding interests. In particular, the principal can spread the instantaneous payment as additional flow payment when the machine is up. Correspondingly, the principal can increase the upper bound of the agent's continuation utility at state $\mathbf{u}$ to $\tilde{w}_{\mathbf{u}} := \bar{w}_{\mathbf{d}} + \beta_{\mathbf{d}} > \bar{w}_{\mathbf{u}}$, while increasing the flow payment from the original $\ell^* = r\bar{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{d}}$ to $r\tilde{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{d}}$ when the continuation utility reaches this upper bound. When the continuation utility is below $\bar{w}_{\mathbf{u}}$, the dynamics of the continuation utility is the same as it in contract $\Gamma_1^*(w)$. When the continuation utility $W_{t-} \in (\bar{w}_{\mathbf{u}}, \tilde{w}_{\mathbf{u}}]$ at state*

$\mathbf{u}$*, machine breaking down still brings the continuation utility to* $\bar{w}_{\mathbf{d}}$*. In this case, the increase rate of the continuation utility is* $rW_{t-} + \mu_{\mathbf{u}}(W_{t-} - \bar{w}_{\mathbf{d}})$*, which is higher than the original rate* $rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{d}}$*. This alternative contract does not involve instantaneous payment, but has the same performance as* $\Gamma_1^*(w)$ *starting from the same continuation utility* w*.*[10]

Remark 2.3 (Implementation) *In practice, a principal can implement contract* Γ_1^* *by stationing a meter that shows changing* W_t *(promised utility to the agent) over time. At time* t*, if the machine is up, the meter keeps increasing at an ever-increasing speed* $\mu_{\mathbf{u}}\beta_{\mathbf{u}} + rW_{t-}$ *per period of time (where* $\mu_{\mathbf{u}}\beta_{\mathbf{u}}$ *is the rent for keeping the machine running, and* rW_{t-} *is the interest to the agent), and stops at* $\bar{w}_{\mathbf{u}}$*. When the machine is down, the meter keeps decreasing with a speed* $-rW_{t-} + r\bar{w}_{\mathbf{d}}$ *(where* $r\bar{w}_{\mathbf{d}}$ *is a constant punishment for not having finished repairing, and* rW_{t-} *is again the interest to the agent). The agent is terminated when the meter reaches 0. When the machine breaks down, the meter jumps down* $\beta_{\mathbf{u}}$*. When the machine recovers, the meter jumps up by* $\beta_{\mathbf{d}}$*, unless the jump is clipped by* $\bar{w}_{\mathbf{u}}$*. The agent receives incentive payments of* $r\bar{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}$ *per unit of time only when the meter reaches* $\bar{w}_{\mathbf{u}}$*. In addition, the agent is continuously reimbursed at rate* c_θ *for his effort cost when the machine's state is* $\theta \in \{\mathbf{d}, \mathbf{u}\}$*. This form of payment can be interpreted as a "base rate" of pay in addition to the aforementioned incentive pay, which is easy to explain in practice.*

2.4.1.2 Value functions and proof of optimality when $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$

In this section, we first heuristically derive the dynamics of the principal's utility, as a function of the agent's promised utility under the proposed optimal contract Γ_1^* defined in Definition 2.1. Later, in Proposition 2.2, we prove that our derived value function is the actual optimal value function of the principal.

Specifically, let $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ represent the principal's utility at time t when the agent's promised utility is w if the machine's state is $\mathbf{d}$ and $\mathbf{u}$, respectively. Following a standard heuristic derivation (see Appendix A.1.4.1), we obtain the following system of differential equations. In particular, for state $\mathbf{d}$ and $w \in [0, \bar{w}_{\mathbf{d}}]$, the differential equation is

$$(\mu_{\mathbf{d}} + r)J_{\mathbf{d}}(w) = -c_{\mathbf{d}} + r(w - \bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w) + \mu_{\mathbf{d}}J_{\mathbf{u}}(\min\{w + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - \mu_{\mathbf{d}}(w + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+. \tag{2.21}$$

[10]Here we present the complete mathematical formulation of the alternative optimal contract. The dynamic of the continuation utility is

$$\begin{aligned} dW_t =& \big[r(W_{t-} - \bar{w}_{\mathbf{d}})dt + \beta_{\mathbf{d}}dN_t\big]\mathbb{1}_{\theta_{t-}=\mathbf{d}} + \big\{\big[(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})dt - \beta_{\mathbf{u}}dN_t\big]\mathbb{1}_{W_{t-}\leq \bar{w}_{\mathbf{u}}} \\ &+ \big[(rW_{t-} + \mu_{\mathbf{u}}(W_{t-} - \bar{w}_{\mathbf{d}}))\mathbb{1}_{W_{t-}<\tilde{w}_{\mathbf{u}}}dt - (W_{t-} - \bar{w}_{\mathbf{d}})dN_t\big]\mathbb{1}_{W_{t-}\in(\bar{w}_{\mathbf{u}},\tilde{w}_{\mathbf{u}}]}\big\}\mathbb{1}_{\theta_{t-}=\mathbf{u}}, \end{aligned} \tag{2.20}$$

and payment $dL_t = (r\tilde{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{d}})\mathbb{1}_{W_{t-}=\tilde{w}_{\mathbf{u}}}\mathbb{1}_{\theta_{t-}=\mathbf{u}}dt$.

For state $\mathbf{u}$, the differential equation for $w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$ is

$$(\mu_{\mathbf{u}} + r) J_{\mathbf{u}}(w) = R - c_{\mathbf{u}} + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}) J'_{\mathbf{u}}(w) + \mu_{\mathbf{u}} J_{\mathbf{d}}(w - \beta_{\mathbf{u}}) \,, \tag{2.22}$$

with at $w = \bar{w}_{\mathbf{u}}$,

$$(\mu_{\mathbf{u}} + r) J_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = R - c_{\mathbf{u}} + \mu_{\mathbf{u}} J_{\mathbf{d}}(\bar{w}_{\mathbf{u}} - \beta_{\mathbf{u}}) - \ell_1^*. \tag{2.23}$$

The boundary conditions are

$$J_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}} \quad \text{and} \quad J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}, \tag{2.24}$$

reflecting that the principal receives baseline revenues $\underline{v}_{\mathbf{d}}$ and $\underline{v}_{\mathbf{u}}$ (defined in (2.4)), upon terminating the agent in states $\mathbf{d}$ and $\mathbf{u}$, respectively.

For the interval $[0, \beta_{\mathbf{u}}]$, we simply extend the function $J_{\mathbf{u}}(w)$ to be linear, that is,

$$J_{\mathbf{u}}(w) = J_{\mathbf{u}}(0) + \frac{J_{\mathbf{u}}(\beta_{\mathbf{u}}) - J_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}} w, \ \text{for } w \in [0, \beta_{\mathbf{u}}]. \tag{2.25}$$

As we have demonstrated, the agent's promised utility never falls into the interior of this interval if we follow the optimal contract according to Definition 2.1. However, having an extended definition of $J_{\mathbf{u}}(w)$ for that interval is crucial for the the proof of optimality of the contract in Definition 2.1. This is because the optimality proof needs to argue that contract Γ_1^* outperforms any other contract, and a generic contract may bring the promised utility to this interval at state $\mathbf{u}$.

Proposition 2.1 *The system of differential equations (2.21)-(2.23) with boundary conditions* (2.24) *and* (2.25) *has a unique solution: the pair of functions $J_{\mathbf{u}}(w)$ on $[0, \bar{w}_{\mathbf{u}}]$ and $J_{\mathbf{d}}(w)$ on $[0, \bar{w}_{\mathbf{d}}]$, both of which are strictly concave with $J'_{\mathbf{u}}(w) \geq -1$ and $J'_{\mathbf{d}}(w) \geq -1$.*

Following proposition 2.1, we can define $w_{\mathbf{d}}^*$ and $w_{\mathbf{u}}^*$ as unique maximizers of $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ on $[0, \bar{w}_{\mathbf{d}}]$ and $[0, \bar{w}_{\mathbf{u}}]$, respectively. Next, we show that functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are indeed the value functions of the principal under contract $\Gamma_1^*(w)$, starting from a promised utility w at time 0 with the initial states $\theta_0 = \mathbf{d}$ and $\theta_0 = \mathbf{u}$, respectively.

Proposition 2.2 *For any state $\theta \in \{u, d\}$ and promised utility $w \in [0, \bar{w}_\theta]$, we have $U(\Gamma_1^*(w), \nu^*, \theta) = J_\theta(w)$. That is, functions $J_{\mathbf{u}}(w)$ and $J_{\mathbf{d}}(w)$ are equal to the principal's total discounted utilities of following contract Γ_1^* when the initial state of the machine is* $\mathbf{u}$ *and* $\mathbf{d}$*, respectively.*

Figure 2.2 provides a numerical example of the principal's value functions $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$. To implement the contract, the principal needs to designate the initial promised utility W_0. The initial promised utility should be $w^*_{\mathbf{d}}$ if the machine starts at state $\theta_0 = \mathbf{d}$ and should be $w^*_{\mathbf{u}}$ if the machine starts at state $\theta_0 = \mathbf{u}$. Note that due to concavity, if $J_{\mathbf{u}}(\beta_{\mathbf{u}}) \geq J_{\mathbf{u}}(0)$, then $w^*_{\mathbf{u}} \geq \beta_{\mathbf{u}}$. Otherwise, the optimal initial promised utility $w^*_{\mathbf{u}} = 0$, and, in this case, it is better not to hire the agent if the initial state of the machine is $\mathbf{u}$.

Furthermore, it is worth noting that $J_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}}$ and $J_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}}$, where $\bar{U}_{\mathbf{d}}$ and $\bar{U}_{\mathbf{u}}$, defined in (2.12), are the principal's utilities under the simple contract $\bar{\Gamma}$ introduced in Section 2.3.1. This implies that Γ^*_1 always (weakly) outperforms $\bar{\Gamma}$. The suboptimality of the benchmark contract $\bar{\Gamma}$ is the difference between the peak of the value function J_θ and $\bar{U}_\theta$ if the system starts from state θ.

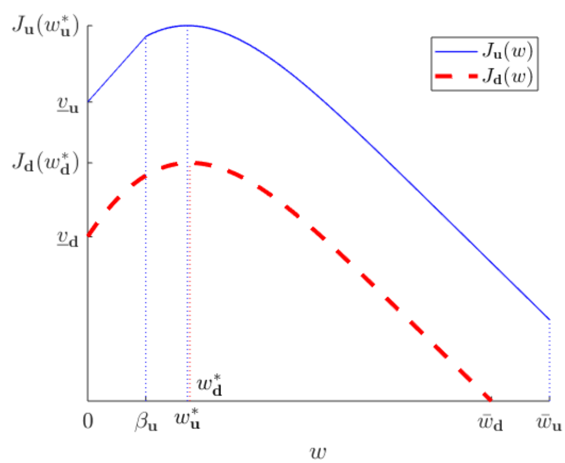


Figure 2.2: Principal's Value functions with $\mu_{\mathbf{u}} = 6$, $\Delta\mu_{\mathbf{u}} = 3$, $\mu_{\mathbf{d}} = 5$, $\Delta\mu_{\mathbf{d}} = 3$, $c_{\mathbf{u}} = 0.8$, $c_{\mathbf{d}} = 1$, $r = 0.9$, and $R = 7.5$.

In Figure 2.2, we have $\bar{w}_{\mathbf{d}} = 0.74$, $\bar{w}_{\mathbf{u}} = 1.01$ and $\beta_{\mathbf{u}} = 0.27 < \beta_{\mathbf{d}} = 0.33$. $J_{\mathbf{u}}(w^*_{\mathbf{u}}) = 2.388$, $\underline{v}_{\mathbf{u}} = 2.031$ and $J_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}} = 1.012$. $J_{\mathbf{d}}(w^*_{\mathbf{d}}) = 1.746$, $\underline{v}_{\mathbf{d}} = 1.4$ and $J_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}}$ = 0.632.

Finally, to show that the contract Γ^*_1 is indeed optimal, in the next proposition, we first demonstrate that functions $J_{\mathbf{u}}$ and $J_{\mathbf{d}}$ are upper bounds for the principal's utility under any incentive compatible contract Γ, if the machine starts at states $\mathbf{u}$ and $\mathbf{d}$, respectively.

Proposition 2.3 *For any incentive compatible contract Γ and any initial state $\theta \in \{\mathbf{u}, \mathbf{d}\}$, we have $J_\theta\big(u(\Gamma, \nu^*, \theta)\big) \geq U(\Gamma, \nu^*, \theta)$, in which we extend the function $J_\theta(w) = J_\theta(\bar{w}_\theta) - (w - \bar{w}_\theta)$ for $w > \bar{w}_\theta$.*

Therefore, we know that for any incentive compatible contract Γ and initial state θ,

$$U(\Gamma, \nu^*, \theta) \leq J_\theta\left(u(\Gamma, \nu^*, \theta)\right) \leq J_\theta(w^*_\theta) = U\left(\Gamma^*_1(w^*_\theta), \nu^*, \theta\right),$$

where the first inequality follows from Proposition 2.3, the second inequality follows from the fact that w^*_θ is the maximizer of J_θ, and the third equality follows from Proposition 2.2. This implies the following main result of this section.

Theorem 2.1 *The optimal incentive contract is* $\Gamma_1^*(w_\theta^*)$ *if* $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$*, condition* (2.14) *is satisfied and the machine starts from state* $\theta \in \{u, d\}$*. That is,* $U\left(\Gamma_1^*(w_\theta^*), \nu^*, \theta\right) \geq U(\Gamma, \nu^*, \theta)$ *for any incentive compatible contract* Γ *and state* θ*.*

2.4.1.3 $\bar{v}_d < \underline{v}_d$

In this section, we consider the case if (2.13), or equivalently, (2.14), is violated. That is, the revenue rate R when the machine is up is not very high. Consider the following contract structure. If the machine starts at state $\mathbf{d}$, the principal does not hire the agent. If the machine starts at state $\mathbf{u}$, on the other hand, the principal hires the agent only to maintain the machine until it breaks down for the first time. During the maintenance period, the principal pays a constant flow payment with rate $(r + \mu_{\mathbf{u}})\,\beta_{\mathbf{u}}$. Furthermore, the agent's corresponding promised utility is maintained at $\beta_{\mathbf{u}}$, because

$$\mathbb{E}\left[\int_0^{\tau_{\mathbf{u}}^*} e^{-rt}\,(r + \mu_{\mathbf{u}})\,\beta_{\mathbf{u}} dt\right] = \beta_{\mathbf{u}},$$

where $\tau_{\mathbf{u}}^*$, the time in state $\mathbf{u}$, follows an exponential distribution with rate $\mu_{\mathbf{u}}$.

Here is a formal definition of the proposed contract.

Definition 2.2 *Define contract* $\Gamma_{\mathbf{u}}^*$ *when the machine starts in state* $\mathbf{u}$ *as the following:*

i. In state $\mathbf{u}$*, the agent's promised utility* W_t *is maintained at* $\beta_{\mathbf{u}}$*, which drops to* 0 *as soon as the state switches to* $\mathbf{d}$*. In state* $\mathbf{d}$*,* W_t *remains to be* 0*.*

ii. The payment to the agent follows $dL_t^* = (r + \mu_{\mathbf{u}})\,\beta_{\mathbf{u}} dt$ *at state* $\mathbf{u}$*.*

iii. Termination occurs when the state switches to $\mathbf{d}$*, that is,* $q^* = 1$ *and* $\tau^* = \min\{t : \theta_t = \mathbf{d}\}$*.*

It can be verified that the corresponding expected societal value starting from state $\mathbf{u}$ is

$$v_{\mathbf{u}} := \mathbb{E}\left[\int_0^{\tau_{\mathbf{u}}^*} e^{-rt}(R - c_{\mathbf{u}})dt + e^{-r\tau_{\mathbf{u}}^*}\underline{v}_{\mathbf{d}}\right] = \frac{R - c_{\mathbf{u}} + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}}{r + \mu_{\mathbf{u}}}. \tag{2.26}$$

Intuitively, the aforementioned contract structure is desirable only if it out performs not hiring the agent at all starting from state $\mathbf{u}$. That is,

$$v_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}, \text{ or, equivalently, } R \geq g_{\mathbf{u}}, \tag{2.27}$$

in which we define

$$g_{\mathbf{u}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right)\beta_{\mathbf{u}}. \tag{2.28}$$

For $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, it is easy to verify that $h_{\mathbf{d}} \geq g_{\mathbf{u}}$. (In particular, $h_{\mathbf{d}} = g_{\mathbf{u}}$ if $\beta_{\mathbf{d}} = \beta_{\mathbf{u}}$.)

The next result formally states that such a contract is indeed optimal when condition (2.14) is violated while (2.27) holds, that is,

$$g_{\mathbf{u}} \leq R < h_{\mathbf{d}}. \tag{2.29}$$

Theorem 2.2 *1. Contract $\Gamma_{\mathbf{u}}^*$ is incentive compatible.*

2. The principal's utilities following Contract $\Gamma_{\mathbf{u}}^*$ are

$$U\left(\Gamma_{\mathbf{u}}^*, \nu^*, \mathbf{d}\right) = \underline{v}_{\mathbf{d}} \;\; \textit{and} \;\; U\left(\Gamma_{\mathbf{u}}^*, \nu^*, \mathbf{u}\right) = v_{\mathbf{u}} - \beta_{\mathbf{u}}.$$

3. Assume that condition (2.29) *holds.*

(i) For any incentive compatible contract Γ, we have

$$\underline{v}_{\mathbf{d}} \geq U(\Gamma, \nu^*, \mathbf{d}),$$

or, it is better not to hire the agent starting from state $\mathbf{d}$.

(ii) Furthermore, if $v_{\mathbf{u}} - \beta_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$, we have

$$U\left(\Gamma_{\mathbf{u}}^*, \nu^*, \mathbf{u}\right) \geq U(\Gamma, \nu^*, \mathbf{u}),$$

That is, $\Gamma_{\mathbf{u}}^*$ is the optimal incentive compatible contract.

(iii) Finally, if $v_{\mathbf{u}} - \beta_{\mathbf{u}} < \underline{v}_{\mathbf{u}}$, for any incentive compatible contract Γ, we have

$$\underline{v}_{\mathbf{u}} \geq U(\Gamma, \nu^*, \mathbf{u}),$$

or, it is better not to hire the agent starting from state $\mathbf{u}$ as well.

Contract $\Gamma_{\mathbf{u}}^*$ suggests that the principal hire the agent only if the machine starts in the up state, and terminate the agent as soon as it breaks down. This is driven by the fact that we do not allow the agent to shirk so far in the paper.[11] If we allow shirking instead, the principal may benefit from hiring the agent to exert effort only when the machine is up, while allowing the agent to shirk when the machine is down. In Section A.2.2 of the e-companion, we provide the optimal contracts that motivate the agent to exert effort only when the machine is up (resp. down), and call it "maintenance contract" (resp, "repair contract"). It is clear that contract $\Gamma_{\mathbf{u}}^*$ is a particular "maintenance contract." Therefore, under condition (2.29), the optimal "maintenance contract" always outperforms the contract $\Gamma_{\mathbf{u}}^*$. Generally speaking, the principal may prefer the maintenance contract over a contract that always induces effort when, for example, when the agent's cost of effort to repair ($c_{\mathbf{d}}$) is so expensive that the principal is better off just hiring the agent to conduct maintenance and not repair.

[11] "Shirk" usually refers to the situation that the agent does not make the required effort. Here, we follow the convention of [Zhu13] to refer to the situation where the contract instructs the agent not to exert effort.

The next result further states that if condition (2.27) is violated, it is also better for the principal to not hire the agent than motivating effort.

Theorem 2.3 *If*

$$R < g_{\mathbf{u}}, \tag{2.30}$$

we have $\underline{v}_\theta \geq U(\Gamma, \nu^*, \theta)$ *for any incentive compatible contract* Γ *and state* $\theta \in \{\mathbf{u}, \mathbf{d}\}$*, where* $g_{\mathbf{u}}$ *is defined in* (2.28).

Theorem 2.3 is intuitive in the sense that when revenue rate R is not large enough compared to the cost, it is not worthwhile for the principal to pay the cost and payment to induce the agent to work.

2.4.2 The case $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$

If $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, the contract Γ_1^* in Definition 2.1 is no longer incentive compatible. To see this, consider the situation where the promised utility $W_{t-} < \beta_{\mathbf{u}} - \beta_{\mathbf{d}}$ before the machine recovers. If the promised utility still jumps up by $\beta_{\mathbf{d}}$ upon the machine recovery at time t, then $W_t < \beta_{\mathbf{u}}$. At that point constraint (2.7) cannot be satisfied. That is, the principal cannot incentivize the agent to exert effort in maintaining the machine. As we will show in the following, the optimal contract needs to involve random termination when the agent's promised utility is low. Furthermore, when the promised utility is high, the optimal contract involves a region where one of the incentive compatible constraints in (2.7) is not binding. As we have alluded to in Section 2.3.2, this is quite peculiar, because, as far as we know, IC constraints are always binding in optimal contracts studied in the continuous time moral hazard literature (see, for example, [San08], [BMRV10], [Sha17a], [ST18]).

The structure of this section mirrors Section 2.4.1. In Sections 2.4.2.1 and 2.4.2.2, we first study incentive compatible optimal contracts under condition (2.18). Finally, Section 2.4.2.3 studies what happens when condition (2.18) is violated.

2.4.2.1 Optimal IC contract when $\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$

We first illustrate the structure of the optimal contract using Figure 2.3 before formally defining the optimal contract. Once again, the contract keeps track of the agent's promised utility W_t over time. The dynamics of W_t, however, are more complicated than the optimal contract in Section 2.4.1.1. In particular, if $W_{t-} \in (0, \bar{w}_{\mathbf{d}})$ in state $\mathbf{d}$, the promised utility keeps decreasing until either the machine is repaired, or the promised utility reaches 0 and the agent is terminated. If $W_{t-} \in [\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}]$ in state $\mathbf{d}$, on the other hand, the promised utility remains a constant until the machine is repaired. If, upon recovery to state $\mathbf{u}$, the promised utility is below $\beta_{\mathbf{u}}$, however, the incentive compatibility constraint (2.7) implies that the machine cannot stay in state $\mathbf{u}$ at the current promised utility level. Instead, the principal randomly terminates the contract or resets the promised utility to be at or above $\beta_{\mathbf{u}}$.

Figure 2.3 depicts two sample trajectories following the proposed contract starting at state $\theta_0 = \mathbf{d}$ from an initial promised utility $W_0 = w^*_{\mathbf{d}} \in (0, \hat{w}_{\mathbf{d}})$. First, focus on the solid curve. The promised utility decreases over time while the agent exerts effort to repair the machine, until time t_1, when the machine recovers. At this point, the promised utility jumps up by $\beta_{\mathbf{d}}$ and the agent starts maintaining the machine at state $\mathbf{u}$. The promised utility keeps increasing until time t_2, when the machine breaks down. In the time interval (t_2, t_4), the promised utility behaves the same way as it does in $(0, t_2)$, with the machine recovering at t_3. When the machine breaks down again at time t_4, however, the promised utility is already so high that it will still be above $\bar{w}_{\mathbf{d}}$ after a downward jump of $\beta_{\mathbf{u}}$. Because $W_{t-} \geq \bar{w}_{\mathbf{d}}$ at state $\mathbf{d}$, the promised utility is kept at this level as a constant, until the machine recovers at time t_5. At this point in time the promised utility takes an upward jump $rW_{t_5-}/\mu_{\mathbf{d}} > \beta_{\mathbf{d}}$, or, the IC constraint (2.7) at state $\mathbf{d}$ is not binding. After time t_5, the machine stays in state $\mathbf{u}$ while the promised utility increases to reach $\hat{w}_{\mathbf{u}}$ at time $\tilde{t}_5$, at which point the contract follows $\hat{\Gamma}$ as defined in Section 2.3.2. Note that following this sample trajectory, the structure of the optimal contract after time t_4 behaves differently from the optimal contract Γ^*_1 defined in Section 2.4.1 (because the promised utility remains constant even though the machine is down).

Now we focus on the other sample trajectory in Figure 2.3[12], the dotted curve. The machine is in state $\mathbf{d}$ during time intervals $[0, \hat{t}_1)$ and $[\hat{t}_2, \hat{t}_3)$, and in state $\mathbf{u}$ during $[\hat{t}_1, \hat{t}_2)$. The promised utility decreases in state $\mathbf{d}$ and increases in state $\mathbf{u}$. Right before the machine recovers for the second time, at $\hat{t}_3$, the promised utility is below $\beta_{\mathbf{u}} - \beta_{\mathbf{d}}$. Therefore, even an upward jump of $\beta_{\mathbf{d}}$ cannot raise the promised utility above $\beta_{\mathbf{u}}$. In light of the discussion in the beginning of this section, the agent is terminated with probability $q^*_{\hat{t}_3} = (\beta_{\mathbf{u}} - W_{\hat{t}_3})/\beta_{\mathbf{u}}$. On the other hand, with probability $1 - q^*_{\hat{t}_3}$, the agent's promised utility is reset to $\beta_{\mathbf{u}}$.

The policy starts from $w^*_{\mathbf{d}} = 1.194$. The solid curve represents a sample trajectories which brings the agent to the point of never terminated. The dotted curve represents another sample trajectory in which the agent is terminated due to a random draw at a point when the machine recovers.

It is clear that randomization needs to occur at state $\mathbf{u}$ if the promised utility is below the threshold $\beta_{\mathbf{u}}$. In fact, the threshold below which the random termination occurs does not have to be exactly $\beta_{\mathbf{u}}$. It can be at a more general level of $\hat{\beta} \geq \beta_{\mathbf{u}}$. In the contract depicted in Figure 2.3, we have $\hat{\beta} = \beta_{\mathbf{u}}$, but this equality does not necessarily always hold, and we may have $\hat{\beta} > \beta_{\mathbf{u}}$. That is, as long as the promised utility W_t is below $\hat{\beta}$ in state $\mathbf{u}$, the agent is randomly terminated with probability $q^*_t = (\hat{\beta} - W_t)/\hat{\beta}$. If termination does not happen at the random draw, the promised utility is reset to $\hat{\beta}$.

[12]we have $\bar{w}_{\mathbf{d}} = 3$, $\hat{w}_{\mathbf{d}} = 6$, $\hat{w}_{\mathbf{u}} = 7$ and $\beta_{\mathbf{u}} = 1 > \beta_{\mathbf{d}} = 0.6$.

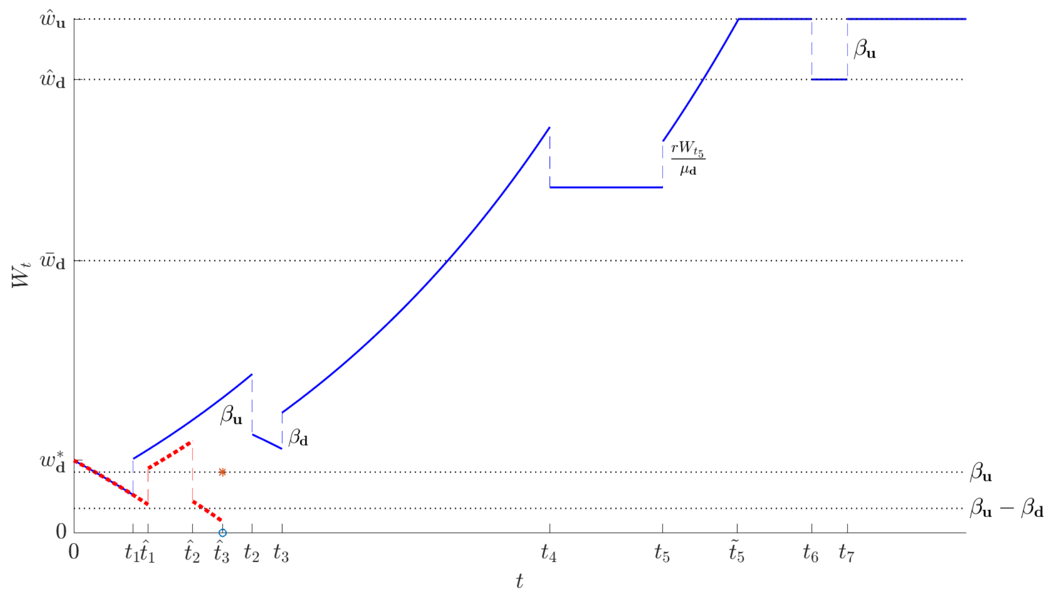


Figure 2.3: Two sample trajectories of promised utility with model parameters $\mu_{\mathbf{u}} = 2, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 6, \Delta\mu_{\mathbf{d}} = 2, c_{\mathbf{u}} = 1, c_{\mathbf{d}} = 1.2, r = 0.8, R = 20$.

Formally, we define the following contract, $\Gamma^*_{\hat{\beta}}$, and later show that the optimal contract follows this definition with an appropriately chosen value of $\hat{\beta} \geq \beta_{\mathbf{u}}$.

Definition 2.3 *For any $\hat{\beta} \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$, define contract $\Gamma^*_{\hat{\beta}}(w) = (L^*, q^*, \tau^*)$ for $w \in [0, \hat{w}_\theta]$ if the initial state of the machine is $\theta \in \{\mathbf{u}, \mathbf{d}\}$.*

i. The dynamics of the agent's promised utility W_t, follows

$$\begin{aligned} dW_t = &\Bigg\{ r(W_{t-} - \bar{w}_{\mathbf{d}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{d}}} dt + \bigg\{ \mathbb{1}_{W_{t-}\in(\bar{w}_{\mathbf{d}},\hat{w}_{\mathbf{d}}]} \frac{rW_{t-}}{\mu_{\mathbf{d}}} + \mathbb{1}_{W_{t-}\in(\hat{\beta}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}]}\beta_{\mathbf{d}} \\ &+ \mathbb{1}_{W_{t-}<\hat{\beta}-\beta_{\mathbf{d}}} \Big[(1-X_t)(\hat{\beta} - W_{t-}) - X_t W_{t-} \Big] \bigg\} dN_t \Bigg\} \mathbb{1}_{\theta_{t-}=\mathbf{d}} \\ &+ [(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})dt \mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}} - \beta_{\mathbf{u}} dN_t] \mathbb{1}_{\theta_{t-}=\mathbf{u}}, \end{aligned} \tag{DW2}$$

from an initial promised utility $W_0 = w$.

ii. The payment to the agent follows $dL^*_t = \ell^*_2 \mathbb{1}_{\theta_{t-}=\mathbf{u}} \mathbb{1}_{W_{t-}=\hat{w}_{\mathbf{u}}} dt$.

iii. The random termination probability is $q^*_t = \hat{q}(W_{t-})\mathbb{1}_{W_{t-}+\beta_{\mathbf{d}}<\hat{\beta}}\mathbb{1}_{\theta_{t-}=\mathbf{d}} dN_t$, in which

$$\hat{q}(w) = \frac{\hat{\beta} - (w + \beta_{\mathbf{d}})}{\hat{\beta}}, \tag{2.31}$$

and the termination time is $\tau^* = \min\{t : W_t = 0\}$.

It is worth noting that in contract $\Gamma^*_{\hat{\beta}}(w)$, constraint (2.7) is not always binding. Specifically, if $W_{t-} > \bar{w}_{\mathbf{d}}$, following the definition we have $q^*_t = 0$ and $H_t = rW_{t-}/\mu_{\mathbf{d}} > \beta_{\mathbf{d}}$. Before we rigorously prove the optimality of the contract, let us explain the intuition why constraint (2.7) is not always binding in the optimal contract, in two steps. First, we explain that social efficiency can be achieved in the optimal contract. Then we explain why achieving efficiency introduces slacks in the incentive compatible constraint (2.7) when $\beta_{\mathbf{d}} < \beta_{\mathbf{d}}$.

The principal and agent having the same time discount rate implies that they have the same total discounted valuation for any payments. Therefore, the societal value function is simply the principal's value function plus the agent's promised utility. Consequently, a contract that maximizes the principal's value function must also maximize the societal value function. Under condition (2.18), contract $\hat{\Gamma}$ introduced in Section 2.3.2 achieves social efficiency (maximizes the societal value functions at promised utility levels $\bar{w}_{\mathbf{u}}$ or $\bar{w}_{\mathbf{u}}$). Therefore, social efficiency must also be achievable at the same promised utility levels under the optimal contract.

If we had to force incentive compatible constraints to be always binding, the upward jump in the promised utility would be $\beta_{\mathbf{d}}$, smaller than the downward jump $\beta_{\mathbf{u}}$. Therefore, no matter where the promised utility starts from, a downward jump of $\beta_{\mathbf{u}}$ cannot be fully compensated by an upward jump of $\beta_{\mathbf{d}}$. As a result, starting from any finite promised utility value, a sample trajectory (however unlikely) with a sequence of very frequent state switches eventually drives the promised utility down to 0. The existence of such sample trajectories implies that the agent would be terminated with positive probability, and, hence, social efficiency would not be achievable. This contradicts the arguments in the last paragraph that the optimal contract should be able to achieve social efficiency. Therefore, in the optimal contract we cannot enforce IC constraints to be binding all the time.

2.4.2.2 Value functions and proof of optimality when $\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$

There are some important distinctions in the approach to determine the principal's value functions, in the case of $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, compared with the one in Section 2.4.1.2. This is because here we need to specify the threshold $\hat{\beta}$ that defines when/if the agent will be randomly terminated.

First, let $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ represent the principal's value functions for states $\mathbf{u}$ and $\mathbf{d}$, respectively. Following Definition 2.3 and similar heuristic derivation steps as in Appendix A.1.4.1, we obtain the following system of differential equations. In particular, for state $\mathbf{d}$, equation (2.21) in Section 2.4.1.2 becomes the following three equations

$$(\mu_{\mathbf{d}} + r)J_{\mathbf{d}}(w) = \mu_{\mathbf{d}} J_{\mathbf{u}}\left(\frac{r + \mu_{\mathbf{d}}}{r} w\right) - c_{\mathbf{d}},\ w \in [\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}], \tag{2.32}$$

$$-c_{\mathbf{d}} + r(w - \bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w) = (\mu_{\mathbf{d}} + r)J_{\mathbf{d}}(w) - \mu_{\mathbf{d}} J_{\mathbf{u}}(w + \beta_{\mathbf{d}}), \;\; w \in [\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}], \text{ and} \tag{2.33}$$

$$-c_{\mathbf{d}} + r(w - \bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w) = (\mu_{\mathbf{d}} + r)J_{\mathbf{d}}(w) - \mu_{\mathbf{d}} \left[\hat{q}(w)J_{\mathbf{u}}(0) + \big(1 - \hat{q}(w)\big)J_{\mathbf{u}}(\hat{\beta})\right], \; w \in [0, \hat{\beta} - \beta_{\mathbf{d}}]. \tag{2.34}$$

For state $\mathbf{u}$, the differential equation is similar to (2.22) for $w \in [\hat{\beta}, \hat{w}_{\mathbf{u}}]$. That is,

$$-c_{\mathbf{u}} + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\hat{w}_{\mathbf{u}}} J'_{\mathbf{u}}(w) = (\mu_{\mathbf{u}} + r)J_{\mathbf{u}}(w) - R - \mu_{\mathbf{u}} J_{\mathbf{d}}(w - \beta_{\mathbf{u}}) + \ell^* \mathbb{1}_{w=\hat{w}_{\mathbf{u}}}, \; w \in [\hat{\beta}, \hat{w}_{\mathbf{u}}] \tag{2.35}$$

Due to randomization, we may further extend function $J_{\mathbf{u}}(w)$ to the interval $[0, \hat{\beta}]$ as a linear function with a slope a, that is,

$$J_{\mathbf{u}}(w) = J_{\mathbf{u}}(0) + aw, \;\; w \in [0, \hat{\beta}]. \tag{2.36}$$

The principal receives baseline revenues $\underline{v}_{\mathbf{d}}$ and $\underline{v}_{\mathbf{u}}$, as defined in (2.4), upon termination in states $\mathbf{d}$ and $\mathbf{u}$, respectively, which implies the following boundary conditions

$$J_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}} \quad \text{and} \quad J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}. \tag{2.37}$$

As we described in Section 2.4.2.1, we have $\hat{\beta} \geq \beta_{\mathbf{u}}$. Before specifying how to obtain the threshold $\hat{\beta}$ next, here we provide some intuition on why the optimal threshold may be higher than $\beta_{\mathbf{u}}$. Intuitively, in optimal control problems with a finite number of actions, randomization between actions allows us to obtain a concave upper envelope of the value function. In our setting, randomization between contract termination (setting the promised utility w to 0) and resetting the promised utility to $\beta_{\mathbf{u}}$ allows us to achieve a value function that is linear between 0 and $\beta_{\mathbf{u}}$. If the resulting value function is concave, then we can show that the control policy is indeed optimal. However, if the aforementioned randomization yields a value function such that the left derivative at $\beta_{\mathbf{u}}$ is smaller than the right derivative at this point, then the resulting value function is not concave. Whenever a value function is non-concave, it must be sub-optimal. This is because using randomization we should at least achieve its concave upper envelope. In our setting, this implies that we can increase the point where we reset the promised utility, from $\beta_{\mathbf{u}}$ to somewhere above it, until the value function becomes concave. Smooth pasting captures the intuition that the value function becomes "barely concave." In this later case, randomization between 0 and $\beta_{\bar{a}}$ (instead of $\beta_{\mathbf{u}}$) yields a value function that is smooth at $\beta_{\bar{a}}$ for state $\mathbf{u}$. In the following, we present the technical details to find $\hat{\beta}$.

We first present the following result regarding general solutions to the aforementioned differential equations.

Lemma 2.2 *For any $a > -1$, there exists a unique pair of functions $J_{\mathbf{d}}^{a\hat{\beta}}$ and $J_{\mathbf{u}}^{a\hat{\beta}}$, in place of $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$, respectively, that satisfy* (2.32)-(2.37)*, in which slope "a" appears in* (2.36)*.*

Furthermore, functions $J_{\mathbf{d}}^{a\hat{\beta}}(w)$ and $J_{\mathbf{u}}^{a\hat{\beta}}(w)$ are twice continuously differentiable, except for $J_{\mathbf{u}}^{a\hat{\beta}}(w)$ at $w = \hat{\beta}$.

It is straightforward to show that it is sufficient to focus only on the case $a > -1$. Intuitively, this is because the slope a represents how much the the principal's utility changes as the agent's promised utility increases. It can never be less than -1 because otherwise, decreasing the agent's promised utility by a direct monetary payment would generate a profit to the principal, which is impossible.

Next, we determine the threshold $\hat{\beta}$ for a given slope a. The idea is to set $\hat{\beta}$ such that function $J_{\mathbf{u}}^{a\hat{\beta}}(w)$ is differentiable at $\hat{\beta}$ if possible, so that we achieve "smooth pasting"[13] between (2.35) and (2.36). For this purpose we define the following function for $\hat{\beta} \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$,

$$f_a(\hat{\beta}) := \left(J_{\mathbf{u}}^{a\hat{\beta}\prime}(\hat{\beta}_-) - J_{\mathbf{u}}^{a\hat{\beta}\prime}(\hat{\beta}_+) \right) (r\hat{\beta} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}). \tag{2.38}$$

Function f_a is a technical construction, and has good properties for us to study when the function $J_{\mathbf{u}}^{a\hat{\beta}}$'s left and right derivatives are the same at $\hat{\beta}$. Clearly, we can achieve "smooth pasting" if there exists a $\hat{\beta}$ such that $f_a(\hat{\beta}) = 0$. The following result guarantees that there exists at most one such $\hat{\beta}$.

Lemma 2.3 *For any $a > -1$, function $f_a(\hat{\beta})$ is increasing in $\hat{\beta}$ on $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$, and $\lim_{\beta \uparrow \bar{w}_{u-}} f(\hat{\beta}) > 0$. Therefore, the following quantity β_a is well defined,*

$$\beta_a := \begin{cases} \beta_{\mathbf{u}}, & f_a(\beta_{\mathbf{u}}) \geq 0, \\ f_a^{-1}(0), & f_a(\beta_{\mathbf{u}}) < 0, \end{cases} \tag{2.39}$$

in which f_a^{-1} is the invervse function of f_a.

Furthermore, as soon as the promised utility reaches $\hat{w}_{\mathbf{u}}$ in state $\mathbf{u}$, the contract $\Gamma_{\hat{\beta}}^*$ becomes identical to $\hat{\Gamma}$, and the agent will no longer be terminated. This implies the following boundary conditions

$$J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{d}}) = \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}} \quad \text{and} \quad J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}}\ , \tag{2.40}$$

in which $\bar{v}_{\mathbf{d}}$ and $\bar{v}_{\mathbf{u}}$ are the societal value function when the agent is never terminated, as defined in (2.11).

[13]The "smooth pasting" condition requires that the value function is differentiable at $\hat{\beta}$. This condition often arises in optimal stopping problems [DP94] and optimal contract design [Zhu13, CSX20].

Now we are ready to uniquely determine the value a in equation (2.36) for the value function.

Proposition 2.4 *There exists a unique $\bar{a} > 0$ such that*

$$\lim_{w\uparrow\hat{w}_{\mathbf{u}}} J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w) = J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}} \quad \textit{and} \quad \lim_{w\uparrow\hat{w}_{\mathbf{d}}} J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w) = J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{d}}) = \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}}, \quad (2.41)$$

where threshold $\beta_{\bar{a}}$ is defined according to (2.39). *Furthermore, functions $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ are both strictly concave, and,*

$$\lim_{w\uparrow\hat{w}_{\mathbf{u}}} \frac{d}{dw} J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w) = \lim_{w\uparrow\hat{w}_{\mathbf{d}}} \frac{d}{dw} J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w) = -1.$$

Similar to Proposition 2.2, the following result shows that $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ specified in Proposition 2.4 are indeed the principal's total discounted utility under contract $\Gamma_{\beta_{\bar{a}}}(w)$, as stated in the next result.

Proposition 2.5 *For any state $\theta \in \{\mathbf{u}, \mathbf{d}\}$ and promised utility $w \in [0, \bar{w}_\theta]$, we have $U(\Gamma^*_{\beta_{\bar{a}}}(w), \nu^*, \theta) = J_\theta^{\bar{a}\beta_{\bar{a}}}(w)$. That is, values $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ are equal to the principal's total discounted utilities of following contract $\Gamma^*_{\beta_{\bar{a}}}$ from the initial promised utility w when the initial state of the machine is $\mathbf{u}$ and $\mathbf{d}$, respectively.*

Figures 2.4 and 2.5 depict the principal's value functions $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$, similar to Figure 2.2 of Section 2.4.1. In particular, Figure 2.4[14] depicts a case where the threshold $\beta_{\bar{a}} = \beta_{\mathbf{u}}$, while Figure 2.5[15] depicts a case with $\beta_{\bar{a}} > \beta_{\mathbf{u}}$ with smooth pasting in play.

Now we are ready to show that the contract $\Gamma^*_{\beta_{\bar{a}}}$ is indeed optimal. The following main result is parallel to a combination of Proposition 2.3 and Theorem 2.1 of the previous subsection.

Theorem 2.4 *For any incentive compatible contract Γ and initial state $\theta \in \{\mathbf{u}, \mathbf{d}\}$, we have $J_\theta^{\bar{a}\beta_{\bar{a}}}\left(u(\Gamma, \nu^*, \theta)\right) \geq U(\Gamma, \nu^*, \theta)$, in which we extend the function $J_\theta^{\bar{a}\beta_{\bar{a}}}(w) = J_\theta^{\bar{a}\beta_{\bar{a}}}(\bar{w}_\theta) - (w - \bar{w}_\theta)$ for $w > \bar{w}_\theta$.*

Therefore, denoting w^*_θ to represent a maximizer of function $J_\theta^{\bar{a}\beta_{\bar{a}}}$, we have $U\left(\Gamma^*_{\beta_{\bar{a}}}(w^*_\theta), \nu^*, \theta\right) \geq U(\Gamma, \nu^*, \theta)$ for any incentive compatible contract Γ and state θ. That is, the optimal incentive compatible contract is $\Gamma^*_{\beta_{\bar{a}}}(w^*_\theta)$, if $\beta_{\mathbf{u}} > \beta_{\mathbf{d}}$, condition (2.19) *holds, and the machine starts from state $\theta \in \{\mathbf{u}, \mathbf{d}\}$.*

[14]In this case, $\bar{w}_{\mathbf{d}} = 1.5$, $\hat{w}_{\mathbf{d}} = 1.75$ and $\hat{w}_{\mathbf{u}} = 2.45$ and $\beta_{\mathbf{u}} = 0.7 > \beta_{\mathbf{d}} = 0.6$. $J_{\mathbf{u}}(w^*_{\mathbf{u}}) = 8.195$, $\underline{v}_{\mathbf{u}} = 5.602$ and $J_{\mathbf{u}}(\hat{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}} = 7.147$. $J_{\mathbf{d}}(w^*_{\mathbf{d}}) = 5.414$, $\underline{v}_{\mathbf{d}} = 2.546$ and $J_{\mathbf{d}}(\hat{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}} = 4.819$.

[15]In this case, $\hat{w}_{\mathbf{d}} = 3.33$, $\hat{w}_{\mathbf{u}} = 4.33$ and $\beta_{\mathbf{u}} = 1.2 > \beta_{\mathbf{d}} = 0.6$. $\bar{a} = -0.499$ and $\beta_{\bar{a}} = 1.259$, $w^*_{\mathbf{d}} = 0.222$. $J_{\mathbf{u}}(w^*_{\mathbf{u}}) = 2.066$, $\underline{v}_{\mathbf{u}} = 2.066$ and $J_{\mathbf{u}}(\hat{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}} = -4.095$. $J_{\mathbf{d}}(w^*_{\mathbf{d}}) = 0.964$, $\underline{v}_{\mathbf{d}} = 0.939$ and $J_{\mathbf{d}}(\hat{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}} = -3.829$.

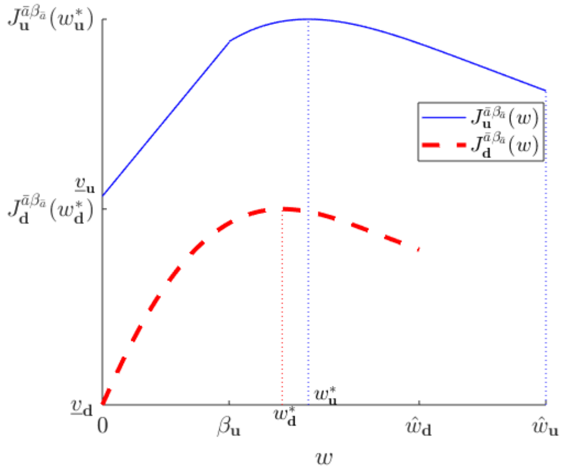


Figure 2.4: Principal's Value functions with $\mu_{\mathbf{u}} = 1.5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 1.5, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 0.7, c_{\mathbf{d}} = 0.6, r = 0.6, R = 11$.

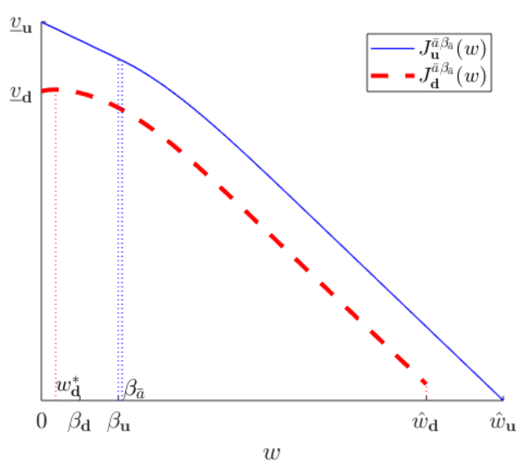


Figure 2.5: Principal's Value functions with smooth-pasting, where $\mu_{\mathbf{u}} = 8, \Delta\mu_{\mathbf{u}} = 4, \mu_{\mathbf{d}} = 6, \Delta\mu_{\mathbf{d}} = 5, c_{\mathbf{u}} = 4.8, c_{\mathbf{d}} = 3, r = 1.2, R = 16$.

It is worth noting that $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{d}}) = \hat{U}_{\mathbf{d}}$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{u}}) = \hat{U}_{\mathbf{u}}$ where $\hat{U}_{\mathbf{d}}$ and $\hat{U}_{\mathbf{u}}$, defined in (2.17), are the principal's utility under the simple contract $\hat{\Gamma}$ of Section 2.3.2. This also implies that contract $\Gamma^*_{\hat{\beta}}$ always (weakly) outperforms $\hat{\Gamma}$. The difference between the peak of the value function $J_{\theta}^{\bar{a}\beta_{\bar{a}}}$ and $\hat{U}_{\theta}$ demonstrates the suboptimality of the benchmark contract $\hat{\Gamma}$ if the machine starts from state θ. For example, in Figure 2.4, the difference between the optimal contract and the benchmark contract is captured in the difference between $J_{\mathbf{u}}(w^*_{\mathbf{u}}) = 8.195$ and $J_{\mathbf{u}}(\hat{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}} = 7.147$, or $J_{\mathbf{d}}(w^*_{\mathbf{d}}) = 5.414$ and $J_{\mathbf{d}}(\hat{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}} = 4.819$, when the machine starts from state $\mathbf{u}$ and $\mathbf{d}$, respectively. In Figure 2.5, we have $J_{\mathbf{u}}(w^*_{\mathbf{u}}) = 2.066$ and $J_{\mathbf{u}}(\hat{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}} = -4.095$; $J_{\mathbf{d}}(w^*_{\mathbf{d}}) = 0.964$ and $J_{\mathbf{d}}(\hat{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}} = -3.829$. Therefore, in the case of Figure 2.5, the optimal contract is profitable, while the benchmark contract is not.

Furthermore, as we can see from Figure 2.5, where the threshold $\beta_{\bar{a}} > \beta_{\mathbf{u}}$, the function $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ is monotonically decreasing, or, the maximizer $w^*_{\mathbf{u}} = 0$. That is, if the initial state of the machine is $\mathbf{u}$, it is better for the principal not to hire the agent than to motivate the agent's full effort. This is generally true, as confirmed in the following result.

Proposition 2.6 *If $\beta_{\bar{a}} > \beta_{\mathbf{u}}$, then we have the slope $\bar{a} < 0$.*

In other words, if it is optimal to hire the agent at the initial state $\mathbf{u}$, then the threshold $\beta_{\bar{a}}$ in contract $\Gamma^*_{\beta_{\bar{a}}}$ must be equal to $\beta_{\mathbf{u}}$. On the other hand, in Figure 2.5, we have $w^*_{\mathbf{d}} > 0$. Therefore, when smooth pasting is at work ($\beta_{\bar{a}} > \beta_{\mathbf{u}}$), it is better not to hire the agent if the initial state is $\mathbf{u}$. However, it may still be beneficial to hire the agent if the initial state is $\mathbf{d}$, although this benefit tends to be small.

2.4.2.3 $\bar{v}_{\mathbf{u}} < \underline{v}_{\mathbf{u}}$

Now we consider the case that (2.18), or, equivalently, (2.19), is violated. First, similar to (2.26) in Section 2.4.1.3, we define the following societal value for the case where the agent starts in state $\mathbf{d}$, exerts effort to repair the machine and is terminated once the machine is repaired,

$$v_{\mathbf{d}} := \mathbb{E}\left[-\int_0^{\tau^*_{\mathbf{d}}} e^{-rt} c_{\mathbf{d}} dt + e^{-r\tau^*} \underline{v}_{\mathbf{u}}\right] = \frac{\mu_{\mathbf{d}} \underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{r + \mu_{\mathbf{d}}}. \tag{2.42}$$

Here $\tau^*_{\mathbf{d}}$ represents the time that the machine is in state $\mathbf{d}$, which follows an exponential distribution with rate $\mu_{\mathbf{d}}$ when the agent exerts effort.

Similar to condition (2.27) in Section 2.4.1.3, we first consider the optimal contract under the following condition,

$$v_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}, \text{ or, equivalently, } R \geq g_{\mathbf{d}}, \tag{2.43}$$

in which we define

$$g_{\mathbf{d}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \beta_{\mathbf{d}}. \tag{2.44}$$

And we have $g_{\mathbf{d}} < (=) h_{\mathbf{u}}$ for $\beta_{\mathbf{d}} < (=) \beta_{\mathbf{u}}$.

If the machine starts at state $\mathbf{u}$, the principal does not hire the agent. On the other hand, if the machine starts at state $\mathbf{d}$, then the promised utility starts from an initial value $W_0 \leq \bar{w}_{\mathbf{d}}$ and keeps decreasing according to $dW_t = r(W_{t-} - \bar{w}_{\mathbf{d}})dt$ until termination, when either W_t reaches 0 or the machine recovers. If the machine recovers at a positive W_{t-}, then the agent is paid this promised utility W_{t-} plus $\beta_{\mathbf{d}}$, which provides the incentive for the agent to exert effort to repair the machine. Formally, we have the following definition of a contract.

Definition 2.4 *Define contract $\Gamma^*_{\mathbf{d}}(w)$ for $w \in [0, \bar{w}_{\mathbf{d}}]$ if the machine starts in state $\mathbf{d}$ as the following.*

i. In state $\mathbf{d}$, the agent's promised utility W_t follows

$$dW_t = r(W_{t-} - \bar{w}_{\mathbf{d}})dt - W_{t-}dN_t\ , \tag{DWd}$$

starting from $W_0 = w$. In state $\mathbf{u}$, W_t remains 0.

ii. The payment to the agent follows $dL^*_t = (W_{t-} + \beta_{\mathbf{d}})dN_t$.

iii. The random termination probability is $q^*_t = \mathbb{1}_{\{\theta_t = \mathbf{u}\}}$, and the termination time is $\tau^* = \{t : W_t = 0\}$.

According to Definition 2.4, termination may occur when the machine is down for a long enough period, or at the time it recovers.

The next result formally establishes the optimality of the contract.

Theorem 2.5 *1. Contract $\Gamma^*_{\mathbf{d}}(w)$ is incentive compatible.*

2. The principal's value functions under contract $\Gamma^*_{\mathbf{d}}(w)$ are

$$U\left(\Gamma^*_{\mathbf{d}}(w), \nu^*, \mathbf{u}\right) = \underline{v}_{\mathbf{u}} - w,$$
$$U\left(\Gamma^*_{\mathbf{d}}(w), \nu^*, \mathbf{d}\right) = (\underline{v}_{\mathbf{d}} - v_{\mathbf{d}}) \left(1 - \frac{w}{\bar{w}_{\mathbf{d}}}\right)^{1 + \frac{\mu_{\mathbf{d}}}{r}} - w + v_{\mathbf{d}},$$

3. Assume that condition (2.19) is violated while (2.43) holds, that is

$$g_{\mathbf{d}} \leq R < h_{\mathbf{u}}. \tag{2.45}$$

For any incentive compatible contract Γ*, we have*

$$U\left(\Gamma^*_{\mathbf{d}}(w^*), \nu^*, \mathbf{d}\right) \geq U(\Gamma, \nu^*, \mathbf{d}) \text{ and } \underline{v}_{\mathbf{u}} \geq U(\Gamma, \nu^*, \mathbf{u}),$$

where w^* *is a maximizer of* $U\left(\Gamma^*_{\mathbf{d}}(w), \nu^*, \mathbf{d}\right)$ *as a function of* w*.*

Contract $\Gamma^*_{\mathbf{d}}$ suggests that the principal hires the agent only if the machine starts in the down state, and terminates the agent as soon as the machine recovers. This is intuitive because $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ implies that it is cheaper to motivate effort to repair than to maintain. The fact that the agent is terminated as soon as the machine is up is, again, due to the fact Theorem 2.5 is focused on incentive compatible contracts. If we allow shirking instead, the principal may benefit from hiring the agent to exert effort only when the machine is down, while allowing the agent to shirk when the machine is up. As mentioned in Section 2.4.1.3, we call this class of contract "repair contract," which also includes $\Gamma^*_{\mathbf{d}}$. Therefore, under condition (2.45), the optimal repair contract outperforms the contract $\Gamma^*_{\mathbf{d}}$ (the repair contract is analyzed in the e-companion).

Despite similarities, Theorem 2.5 is not quite the same as Theorem 2.2 for the previous case. Most prominently, the value function in (A.85) is non-linear, while in (A.50) it is piece-wise linear.

If the maximizer $w^* = 0$, Theorem 2.5 indicates that the principal should not hire the agent at all. Similar to Theorem 2.3 in Section 2.4.1.3, the following result indicates that the principal is also better off not hiring the agent if condition (2.43) is violated.

Theorem 2.6 *If*

$$R \leq g_{\mathbf{d}}, \tag{2.46}$$

we have $\underline{v}_\theta \geq U(\Gamma, \nu^*, \theta)$ *for any incentive compatible contract* Γ *and state* $\theta \in \{\mathbf{u}, \mathbf{d}\}$*, where* $g_{\mathbf{d}}$ *is defined in* (2.44)*.*

2.4.3 A summary

It is helpful to summarize the main results that we obtained throughout this section. For the case of $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, we have characterized model parameters into three regions that can be easily characterized by focusing on the revenue rate R, fixing other model parameters.

- $R > h_{\mathbf{d}}$: The incentive compatible constraints in equation (2.7) are always binding, and the dynamic contract Γ^*_1 demonstrates rich structures.
- $R \in [g_{\mathbf{u}}, h_{\mathbf{d}}]$: The principal may hire the agent and motivate effort only to maintain the machine.

- $R < g_{\mathbf{u}}$: No incentive compatible contract (including hiring an agent only to maintain or repair–analyzed in the e-companion) performs better for the principal than not hiring the agent at all. Furthermore, as we will demonstrate in the e-companion Section A.2.1, for these model parameters, not hiring the agent is the best strategy for the principal, even among contracts that allow shirking.

Similarly, when $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, we also characterize model parameters into three regions of revenue R.

- $R > h_{\mathbf{u}}$: The optimal contract follows $\Gamma^*_{\hat{\beta}}(w)$, where the incentive compatible constraints in equation (2.7) may not be always binding and the agent may need to be terminated randomly.
- $R \in [g_{\mathbf{d}}, h_{\mathbf{u}}]$: The principal may hire the agent and motivate effort only to repair the machine.
- $R < g_{\mathbf{d}}$: Not hiring the agent is the best strategy for the principal.

Finally, if we do not allow contract termination, the following Proposition shows that the simple contracts $\bar{\Gamma}$ and $\hat{\Gamma}$ introduced in Section 2.3 are optimal.

Proposition 2.7 *For any state $\theta \in \{\mathbf{u}, \mathbf{d}\}$ and incentive compatible contracts Γ such that $\tau = \infty$, we have*

- $U(\bar{\Gamma}, \nu^*, \theta) \geq U(\Gamma, \nu^*, \theta)$ *if* $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$,
- $U(\hat{\Gamma}, \nu^*, \theta) \geq U(\Gamma, \nu^*, \theta)$ *if* $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$.

2.5 Numerical Comparison

So far, we have focused on analyzing optimal contracts that induce full effort from the agent before termination. However, these contracts are not necessarily optimal if the principal does not need to always induce full effort from the agent. In the e-companion, we provide sufficient conditions based on principal's value functions. One can use these conditions to verify if the optimal incentive compatible contracts that induce full effort are, in fact, optimal, even if we allow shirking. When the sufficient conditions are not satisfied, it may be preferable for the principal to hire the agent just to maintain or just to repair, and to allow the agent to shirk ("maintenance contract" and "repair contract" formally studied in the e-companion Section A.2.2).

In the following, we numerically compare the performance of the full effort incentive compatible contracts versus the repair only contract and maintenance only contract. In Figures 2.6, 2.7 and 2.8, 2.9, we vary revenue rate R, while keeping all other parameters the same. For each choice of model parameters, we calculate the principal's value for the three contracts and the value without any agents when the machine starts from state $\mathbf{d}$ and $\mathbf{u}$, respectively. As R increases, the principal's value under all the three contracts increase.

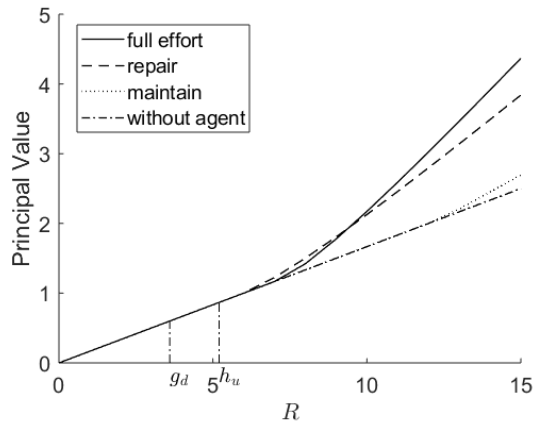


Figure 2.6: Principal's value start at **d**, under three contracts

Figures 2.6 and 2.7[16] depict the case of $\beta_{\mathbf{u}} > \beta_{\mathbf{d}}$. When $R \leq g_{\mathbf{d}}$, according to Theorem 2.6, it is not worthwhile to hire the agent when we only consider the full effort incentive compatible contracts. In the e-companion, Propositions A.4 and A.7 show that even under the maintenance only contract and repair only contract, the principal should not hire the agent when $R \leq g_{\mathbf{d}}$ (note that $g_{\mathbf{d}} < g_{\mathbf{u}}$ in this case). This is consistent with what we see in Figures 2.6 and 2.7, where the four curves coincide when $R < g_{\mathbf{d}}$. In fact, the region that they are all the same extends to $R > g_{\mathbf{d}}$, indicating that the optimal initial promised utility $w^*_{\mathbf{d}}$ or $w^*_{\mathbf{u}}$ is 0 in the optimal contracts, which is equivalent to not hiring the agent at all. Increasing R further, hiring the agent starts making sense. First, the repair only contract outperforms the other two. In the e-companion, Theorem A.2 implies that when $R \in [g_{\mathbf{d}}, h_{\mathbf{u}}]$, the repair only contract always outperforms the full effort contract $\Gamma^*_{\mathbf{d}}$. Intuitively, the repair only contract outperforms the maintenance only contract because $\beta_{\mathbf{u}} > \beta_{\mathbf{d}}$ implies that motivating effort to maintain is more costly than motivating effort to repair. When R becomes large enough, on the other hand, full effort contract outperforms the other two one-sided contracts.

Similarly, Figures 2.8 and 2.9[17] depicts the case $\beta_{\mathbf{u}} < \beta_{\mathbf{d}}$. The observations and underlying reasons are parallel to Figures 2.6 and 2.7 and we do not repeat.

It is clear that a very interesting extension of our paper would be one that studies optimal dynamic contracts that allow the agent to shirk. Unfortunately, this case seems to be very difficult to analyze, and the optimal contracts may involve complex structures that renders them impractical even for simple settings. [Zhu13], for example, considers the optimal contract with shirking under Brownian motion. The evolution of the promised utility involves a sticky Brownian motion that is a

[16] $\mu_{\mathbf{u}} = 2, \bar{\mu}_{\mathbf{u}} = 4, \mu_{\mathbf{d}} = 3, \underline{\mu}_{\mathbf{d}} = 1, c_{\mathbf{u}} = 2, c_{\mathbf{d}} = 1.2, r = 1, R \in [0, 20]$ and $\beta_{\mathbf{u}} = 1 > \beta_{\mathbf{d}} = 0.6$. Here $g_{\mathbf{d}}$ and $h_{\mathbf{u}}$ are defined in (2.44) and (2.19), respectively.

[17] $\mu_{\mathbf{u}} = 1.5, \bar{\mu}_{\mathbf{u}} = 3.5, \mu_{\mathbf{d}} = 3.5, \underline{\mu}_{\mathbf{d}} = 1.5, c_{\mathbf{u}} = 0.6, c_{\mathbf{d}} = 2, r = 1, R \in [0, 20]$ and $\beta_{\mathbf{u}} = 0.3 < \beta_{\mathbf{d}} = 1$. Here $g_{\mathbf{u}}$ and $h_{\mathbf{d}}$ are defined in (2.28) and (2.14), respectively.

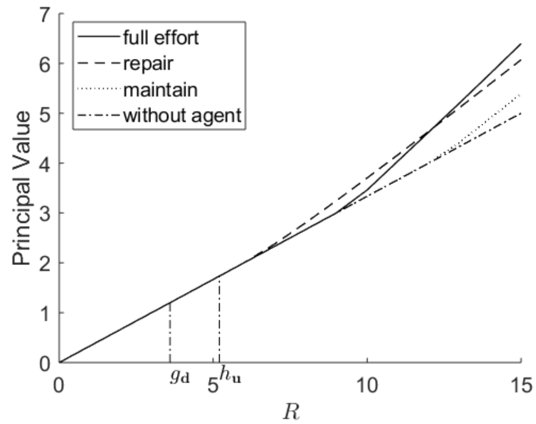


Figure 2.7: Principal's value start at **u**, under three contracts

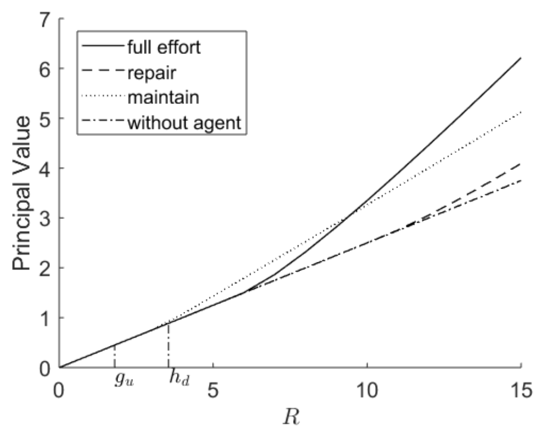


Figure 2.8: Principal's value start at **d**, under three contracts

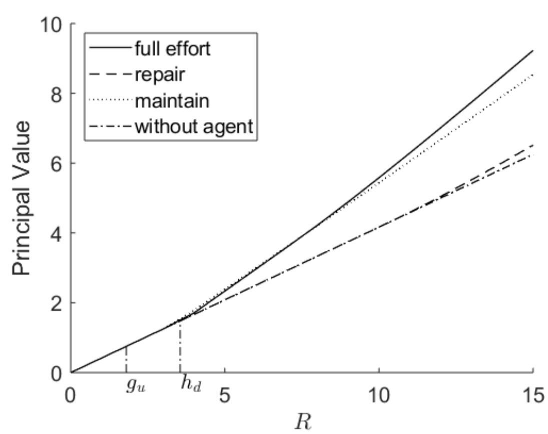


Figure 2.9: Principal's value start at **u**, under three contracts

mathematical construct with very little practical relevance. Therefore, we consider the pursuit of optimal contracts that allow shirking outside the scope of this paper, and leave it for future research.

2.6 Conclusion

We study an incentive design problem in continuous time over an infinite horizon. Specifically, a principal hires an agent to exert effort in order to repair a machine when the machine is down, and maintain the machine when it is up. The agent can adjust the effort level at any time, which is not observable to the principal. Our paper contributes to the service/maintenance literature by studying the optimal dynamic contract. Although we allow a general form of payment and random termination in the contract design, the structure of the optimal contract is overall simple and intuitive. In particular, payment over time and potential termination decisions are all based on the evolution of the agent's promised utility. Payment only occurs when the promised utility is high enough. Intuitively, the principal pays the agent a flow when the machine is up, which can be decomposed into an interest payment to maintain the promised utility, and a rent. In the case that $\beta_{\mathbf{d}} > \beta_{\mathbf{u}}$, the principal also needs to use an instantaneous payment upon machine recovery to provide an appropriate incentive for the agent to repair the machine fast enough.

Our paper also contributes to the dynamic contract literature where an agent exerts effort to either increase or decrease the arrival rates of Poisson processes. We instead combine both directions (increase and decrease), which turns out to be a non-trivial generalization. In particular, we find two new features in the optimal contract, which are new in the dynamic contract literature: (1) The incentive compatibility constraints are not always binding. (2) When the agent's promised utility is low, the optimal contract needs to involve a random termination, if the agent is not terminated from the random draw, the promised utility is brought back up to a certain threshold. Different from [Mye15], which also involves random termination, our threshold is not fixed at one level, but endogenously determined depending on model parameters.

Our general approach applies to other operational settings beyond maintenance/repair. For example, consider a queuing control system where an agent needs to exert effort in order to increase either the service rate or the arrival rate (e.g., by marketing efforts). In this case and the number of customers in the queue could be considered as the state of the system, which is more than the two states studied in our model. We believe that the techniques and results derived in our paper serve as a necessary step for solving these more general problems.

CHAPTER 3

Dynamic Contract Design in the Presence of Double Moral Hazard

3.1 Introduction

We consider a stylized incentive management problem over an infinite time horizon, in which a principal hires an agent to provide services to customers.[1] Customers arrive according to a Poisson process and request service in one of two ways: either via an *online* channel, or via a traditional, *walk-in*, channel. An *online* customer makes a reservation for the service through the firm's online reservation system and prepays for the service, while a *walk-in* customer simply shows up to receive the service and pays the agent upon the service completion. The agent is tasked with providing the requested service to both types of customers as well as collecting the payments from the walk-in customers. In addition, the agent can exert costly effort to increase the combined (across-two-channels) arrival rate of customers. The principal is not able to observe the arrivals of the walk-in customers, nor does she observe whether the agent exerts effort.[2] This creates an opportunity for the agent (i) to *divert cash* (that is, to under-report the number of walk-in customers and pocket respective revenues) and also (ii) to *shirk* (that is, not to exert costly effort), thus, leading to a novel and so-far-unexplored *double moral hazard* problem. We are interested in how, in the presence of such a double moral hazard problem, a principal can design a dynamic contract that maximizes her profits.

The situation we have described is ubiquitous in a variety of service and franchise settings. For instance, many service organizations have individual locations that are managed by agents on behalf of the owners. In such organizations, the owners do not have full visibility into agents' operations. Thus, the agents have the potential ability to deliver services but to not report that the services were delivered (practically, diverting some of the revenues from the owners to themselves). [KLS18], [NRST02], and [PSM15] provide an empirical confirmation of the existence of such revenue

[1] The material presented in this chapter is based on the paper [TAD20] co-authored with Ekaterina Astashkina and Izak Duenyas.

[2] The inability of the principal to observe the arrivals of the walk-in customers, for instance, can be motivated by (i) the principal being located far away from the agent, or (ii) the principal being busy with managing multiple business units.

under-reporting/diversion problem across various service settings. In addition, approximately 15%-20% of franchise locations are estimated to under-report "sales" by 15% or more (`http://www.audigence.com/franchise.html`). A number of forensic accounting firms (Audience, MDD Forensic Accountants, Tacit, etc.) advertise services that claim to discover and document how agents have diverted cash. They specifically advertise services that *deter under-reporting, identify fraud schemes, gain visibility into franchise owner practices, develop monitoring tools*, thus confirming the existence of moral hazard problems associated with under-reporting. However, the cost of hiring an accounting firm and then litigating can be very high.

Make-to-order manufacturing environments are also known to have been exposed to similar moral hazard issues. As such, according to a classical law case "General Automotive Manufacturing Co. v. Singer", a general manager of the shop diverted substantial amount of arriving-at-the-shop orders to subcontractors of his choice and pocketed respective profits. (General Automotive sued the manager for breach of contract, – the case was finally decided by the Wisconsin Supreme Court.) Most organizations, however, do not have the resources to litigate and may prefer to tackle the moral hazard problems differently, for instance, by setting up appropriate contracts with the agents and/or adopting mechanisms of monitoring the agents. Despite the fact that this problem is common place, there is a lack of literature focusing on how, in such situations, a principal can (i) provide incentives to the agent and (ii) use monitoring mechanisms to address the moral hazard issues and capture the highest possible profits. This is what we are addressing in this paper.

To induce the desired behavior from the agent, we let the principal employ a carrot-and-stick approach. The "carrot" is a general form compensation transferred from the principal to the agent: it is allowed to be anything from an instantaneous one-time payment to a flow payment with a time-varying rate. The "stick" is the threat of the contract termination that the principal holds over the agent. Both the compensation and the termination are, at any point in time, contingent on the information available to the principal: (observable to the principal) past arrival times of the online customers and (unobservable to the principal and only reported by the agent) past arrival times of the walk-in customers. To induce the desired agent's behavior, the principal, thus, designs and commits to a contract that combines a payment scheme and a potential termination time.[3] In the extended versions of the model, in addition to using payments and termination, we allow the principal to also (i) *monitor* the agent or (ii) *manipulate* the relative attractiveness of the online channel against the walk-in channel. Such channel manipulation can be done via the dynamic adjustment of the prices charged for the services reserved online. In the remainder of this section, we describe our findings, discuss the contributions relative to the existing academic literature, and, finally, conclude by outlining the structure of the remaining sections of the paper.

[3]Our model can be easily extended to allow the principal to find a replacement at a fixed cost when terminating the current agent. The nature of our results remains unchanged.

Select Findings

We focus on the contracts that induce the agent (i) to *truthfully report* the arrivals of and the revenues from the walk-in customers (i.e., to not divert cash) and (ii) to *exert effort* to increase the customer arrival rate (i.e., to not shirk). We show that focusing on these types of contracts is without loss of optimality for the principal. We formulate the dynamic contract design problem as a continuous-time, stochastic optimal control problem with two incentive compatibility constraints. The incentive compatibility constraints of the moral hazard problems are not separable, which makes the combined control problem complex and leads to a rich optimal contract structure. In particular, depending on the relative severity of the moral hazard problems (that is, depending on which of the incentive compatibility constraints ends up being binding — the one related to *effort* or the one related to *truthful reporting*), the optimal contract has one of three possible structures.

All three contract structures have the following elements in common: (i) a *probationary period* during which the agent works for the principal while being offered a so-called (ii) *promised future utility* (see [SS87]), that the principal keeps track of during this period; (iii) a *tenure period* during which the agent keeps all the revenues collected from the walk-in customers to himself; and (iv) a *termination threshold* on the promised future utility. The agent's promised future utility serves as an indicator of the agent's past performance; during the probationary period, it takes instantaneous upward jumps on certain customer arrivals but otherwise decreases. In each of the three contracts, the probationary period is over when the promised future utility reaches either the *termination threshold* (which marks an immediate contract termination) or the *tenure threshold* (which marks the start of the tenure period, that lasts indefinitely).

The variation in the three contract structures, in turn, is captured by the differences in (i) tenure thresholds, and (ii) the rules for the evolution of the promised future utility (the rate of its decrease, and the when-and-how for the upward jumps). As such: (1) under the first of the three optimal contracts, the principal "rewards" the agent with an upward jump of the promised future utility only for arrivals that are originally unobservable by the principal (but are reported by the agent under this contract); (2) under the second contract, the principal "rewards" the agent for any arrival type but the jumps associated with the arrivals from the observable channel are smaller in magnitude; (3) finally, under the third contract, the principal "rewards" the agent for both arrival types equally.

In the extended version of the model, in addition to controlling payments and termination, the principal can also *monitor* the agent (see Section 3.3.1). We, first, consider the *full monitoring* setting where the principal could pay a flow cost to monitor two types of (agent's) private information: walk-in customer arrivals and whether the agent exerts effort. We find that the principal will conduct full monitoring only when the agent's promised future utility reaches a *monitoring threshold*. Once full monitoring is initiated, it continues indefinitely and the agent is never terminated.

We also consider the *partial monitoring* setting, where the principal could pay a flow cost to monitor only one type of private information: the walk-in customer arrivals (see Section 3.3.2). In contrast to *full* monitoring, the *partial* one is rather natural and can be realistically implemented in service management environments. When devising an optimal contract with *partial monitoring schedule*, the principal is trading off the partial monitoring costs against the rents that she has to pay to the agent to address the incentive compatibility constraint for truthful reporting. The principal also needs to incentivize the agent to exert effort. We find that the principal will conduct partial monitoring whenever the agent's promised future utility drops below a *partial monitoring* threshold, which does not necessarily last indefinitely. Under partial monitoring, with poor enough performance, the agent will get terminated, while, with good enough performance, monitoring stops, and the agent can possibly achieve tenure. Intuitively, partial monitoring ensures that the agent will not divert any cash, hence, reducing the likelihood of agent's termination. Since the agent's promised future utility serves as an indicator of the agent's past performance, our results indicate that it is optimal for the principal to monitor agents with worse past performance more, while there is less of a need to monitor the agents with a high enough performance record.

In another extension, the principal is allowed to manipulate the relative attractiveness of the online channel against the walk-in channel, instigating customers to place orders through the channels observable to the principle. In our setting, lowering the price for the services reserved online would lure more customers from the walk-in to the online channel. In Section 3.4, we, therefore, consider the case where the principal can adjust pricing (by providing a discount) to induce customers to place orders online. We endogenize the customer's channel choice, by adding the customer's utility into the baseline model. When deciding on the optimal dynamic discounting strategy, the principal is trading off the revenue loss from the provided discount versus the incentive payments to the agent. We show that the principal will adjust prices less aggressively when the agent's promised future utility is high (that is, when the agent's past performance has been good). It is worth noting that applying the dynamic discounting tool to address a moral hazard problem is unique to our model and novel in the literature.

Finally, in Section 3.5, we study how the agent's utility changes when the principal uses one of the additional tools: full monitoring, partial monitoring, and dynamic discounting. Unsurprisingly, under full monitoring (which, when activated, reveals all the agent's private information), the agent is always worse off. However, under partial monitoring, (which, when activated, reveals only one type of private information) or under dynamic discounting, – this is no longer the case. In fact, under certain conditions (discussed in the respective section of the paper), imposing partial monitoring or dynamic discounting can sometimes increase the expected duration of the agent's employment or, equivalently, reduces the likelihood of agent's termination, thus, making the agent better off.

Literature Review

Our paper contributes to the contract theory literature with moral hazard, where the principal hires an agent who could take a hidden action that affects the principal's outcome. In the moral hazard literature, a common focus area has been situations where the agent may shirk (see, e.g., [Mir76], [Höl79], [CL13]) and situations where the agent may divert cash (see, e.g., [Tow79], [Dia84], [BS90]). One clear difference between our paper and most of the previous literature on contract theory is that our contract design problem is subject to both moral hazard issues (shirking and diverting cash).

Among the few papers that do allow for multiple dimensions of moral hazard, [HM91] develop a multi-task model of moral hazard in which the agent has multiple effort choices that affect multiple outputs. Following [HM91], [GP00] analyze optimal contracts with joint moral hazard in effort and risk, and [GK14] analyze optimal contracts where the agent chooses management effort and effective labor effort. However, these papers only consider static contracts, while we are solving for optimal long-term contracts where the principal can *dynamically* change the agent's incentives based on the agent's performance to-date. In our setting, it is easy to show that being able to adjust incentives dynamically makes a significant difference on the principal's overall profit level.

Our paper is also related to the theoretical literature on franchising, where a franchising chain hires store managers to manage stores and potentially faces a moral hazard problem. Most of these papers focus on the level of private effort exerted by the franchisee, that is, on shirking (see [Sti74] and [BL95]). The problem of cash diversion is much less explored in this literature stream, except for a few empirical papers such as [A^+00] and [MY12], and a theoretical paper by [PRS11]. [PRS11] build a static model in which a franchising chain has franchised and company-owned units. The franchised unit manager may misreport the sales and steal the rest of the profit (i.e., divert cash) but not shirk, while the company-owned unit manager may shirk but cannot divert cash. In contrast to [PRS11], the "manager" in our model can do both of these simultaneously: shirk and divert cash. In addition, our paper focuses on dynamic rather than static contracts.

Our paper is also related to the dynamic moral hazard literature which adopts the "promised utility " framework. In this literature, there are two streams closely related to our paper. The first stream considers the contract design problem in which the principal hires an agent to exert private effort to change the arrival rate of a Poisson process: the agent either increases the arrival rate of good arrivals (see [GT16], [Sha17b], [ST18]), decreases the arrival rate of bad arrivals (see [BMRV10], [Mye15], [LSTZ20]), or does both (see [TSD21]). All these papers deal with a single moral hazard problem, while we address double moral hazard. A distinct feature of our problem setting is that, whereas, in problems with a single moral hazard, the incentive compatibility constraint is always binding, in our setting, one of the incentive compatibility constraints might not be binding, hence, leading to a more complex problem and a richer contract structure.

The second stream considers the dynamic moral hazard problem caused by "diverting cash" originating from [DS06b] and [CH06]. [DS06b] consider a firm that hires a manager to run a project, in which the project generates cash flow with Brownian motion uncertainty, and the manager could privately divert funds from the project for his own benefit. [CH06] consider a similar problem in discrete time. Following [DS06b], several papers had used similar solution techniques but applied their models mostly to the area of corporate finance (see, e.g., [BMPR07], [Fu17], [Mal19]). Another difference between this literature and our work is that, in our setting, the agent can divert some of the cash coming from customers who arrive according to a Poisson process, whereas the finance literature typically assumes that the agent is managing the principal's cash which is subject to Brownian motion uncertainty. This results in a contract, that is very different from the contracts considered in those papers. Finally, our paper considers the agent's private effort to increase sales, which is missing from this literature stream.

Our extended model with monitoring is related to the literature that addresses dynamic moral hazard problems using monitoring tools. [PW16] embed a costly monitoring technology into the setting introduced by [DS06b]. This costly monitoring technology allows the principal to check whether the agent diverts cash. [CSX20] design a monitoring schedule when the principal hires an agent who privately exerts effort to decrease the rate of adverse events; whenever the principal is monitoring, the agent's effort is guaranteed. In our model, the agent may exert effort to increase the arrival rate of customers (generating benefits to the principal) but may also divert cash. Hence, our paper clearly differs from these papers as we focus on double moral hazard problem (that includes both effort and cash diversion). In addition, we also consider two types of monitoring tools: the *full monitoring*, which deals with both moral hazard problems, and the *partial monitoring*, which *only* deals with the cash diversion problem.

Further, some recent empirical studies have demonstrated that monitoring tools are commonly used in service management environments and have empirically shown that monitoring tools can mitigate moral hazard and enable firms to design more efficient contracts. For example, [NRST02], [PSM15], and [KLS18] consider settings (agents soliciting donations, agents serving customers at a restaurant, and agents driving minibuses), where the principal monitors agents' effort and/or agents' sales. Specifically, [KLS18] consider the setting, where a firm in Kenya hires mini-bus drivers who can shirk and also divert cash, which is similar to ours. However, customers in their setting arrive from only one channel, which is unobservable by the principal. In contrast, we allow for two customer channels, one observable and one unobservable by the principal. Also, our focus is on the analytical derivation of optimal dynamic contracts, while these papers are empirical.

The approach of "promised utility" is not the only framework to address the dynamic contract problem. In operations management field, [PZ00], [ZZ08], and [ZTH19] develop a dynamic principal-agent framework to delegate operational control of a system that can be modeled as a

Markov decision process. They assume that the agent is risk-averse and can access efficient banking with the same rate of borrowing and lending. Therefore, they do not assume limited liability. Limited liability is a common assumption in the contract literature (see, e.g., [BMRV10], [CL13], [GT16], [GAKR20]). Without it, the principal can simply sell the business to the agent to resolve the incentive issues if both the principal and the agent are risk-neutral. Furthermore, none of these papers consider the double moral hazard problem. The assumption of limited liability for the agent and the existence of two dimensions of moral hazard result in significant complexity and very different control structures in the problem that we address.

Finally, our paper is related to recent operations management literature which studies the off-platform transactions in online marketplaces (see [GZ20] and [HSGT20]). Specifically, they explore the phenomenon where, after initially connecting on the platform, the customer can transact with the service provider directly "off-platform". If we see the platform as a principal and the service provider as an agent, our model can be applied to deal with the off-platform transaction problem. Neither of these papers considers a dynamic principal-agent framework. In particular, [GZ20] conduct an empirical study while [HSGT20] adopt a queueing game-theoretic framework but have no contract structure. Our paper's framework, therefore, can also be applied to design contracts in online marketplaces.

The rest of this paper is organized as follows. We present the baseline model and the corresponding optimal contracts in Section 3.2. In Section 3.3, we extend the baseline model to allow monitoring. Section 3.4 generalizes the baseline model to allow the principal to manipulate the relative attractiveness of the two channels. Finally, in Section 3.5, we look at how agent's utility changes when the principal adds either the monitoring tool or the dynamic discounting tool.

3.2 Baseline Model

This section is organized as follows. In Section 3.2.1, we introduce the baseline model. In Section 3.2.2, we present the structure of the optimal contracts and the corresponding value functions. In Section 3.2.3, we show the optimality of the contracts in the space of contracts with full-effort and truth-telling (i.e., the contracts that induce the agent to always exert effort and truthfully report arrivals). In Section 3.2.4, we show that these contracts are also optimal in the expanded contract space. Finally, we conclude with the numerical examples and comparative statics in Section 3.2.5.

3.2.1 Baseline Model Setup

A principal hires an agent to run her store. Customers arrive at the store from two different channels. Customers arriving from one of the channels are unobservable by the principal (we call it an "unobservable channel"). We can think of the unobservable channel as the walk-in channel and the observable channel as the online reservation channel. With probability p, a customer arrives to the unobservable channel and, with probability $1-p$, a customer arrives to the observable channel. We assume that customers from the unobservable channel are charged a fixed fee, R_u, for the service, while customers from the observable channel are charged R_o (in Section 3.4, we extend the model to the case, where the fees are decision variables and p is a function of the fees). The agent pays an effort flow cost c (i.e., cost of exerting effort) to achieve a customer arrival rate of μ (i.e., by marketing/advertising the services).[4]

Formally, we denote the agent's effort process by $\nu_t \in \{0, \mu\}$.[5] Let N_t^u and N_t^o denote, respectively, the counting processes of customers from unobservable and observable channels, arriving to the store prior to time t. With the agent always exerting effort (i.e, $\nu_t = \mu$), N_t^u is a Poisson random variable with rate $p\mu t$, and N_t^o is a Poisson random variable with rate $(1-p)\mu t$. Arrival process N_t^o is observable to both principal and agent, while only the agent observes the process N_t^u. The agent reports arrivals $\{\hat{N}_t^u; t \geq 0\}$ to the principal. The principal and the agent have the same discount rate denoted by r.[6] We assume that the principal has the power to commit to a long-term contract, which is a function of the reported arrivals and the public information. The principal designs the contract $\gamma = (L, \tau)$ which includes payment L and termination time τ. Formally, the payment could be an instantaneous payment I_t, or a flow payment with rate ℓ_t, such that $dL_t = I_t + \ell_t dt$.[7] We assume the agent has limited liability. That is, the agent can decide to quit and never owes money to the principal.[8] To guarantee the limited liability constraint, we need to have $I_t \geq 0$ and $\ell_t \geq 0$. We define Γ as the contract space that includes all the contracts ($\gamma \in \Gamma$) that satisfy the limited liability constraint.

By the revelation principle, it is without loss of generality that we can focus on the contract that induces the agent to truthfully report the arrivals, i.e., $\hat{N}_t^u = N_t^u$.[9] Further, in the current section, we also just focus our analysis on finding the optimal contract when the agent is necessarily forced to always exert effort. We denote the space of the contracts that induce the agent to truthfully report and exert effort as Γ^{IC}. Therefore, with the limited liability constraint, the principal needs to reimburse the agent's effort cost cdt. Section 3.2.4 confirms that the optimal contract in the space

[4] Without loss of generality, we assume that the cost of serving a customer that has arrived to the store is zero. Generalizing it to the case where the agent incurs a positive cost for every customer served does not change our results.

[5] We can easily generalize it to the case where the agent's effort process is given by $\nu_t \in \{\underline{\mu}, \mu\}$, where $\underline{\mu} \in [0, \mu)$, which does not change the nature of our results.

[6] It is common to assume that the principal and the agent share the same discount rate (e.g., see [Mye15], [TSD21]).

[7] Formally, ν is $\mathcal{F}$-predictable, where $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by (N^u, N^o); L is adapted to $\hat{\mathcal{F}} = \{\hat{\mathcal{F}}_t, t \geq 0\}$, which is the filtration generated by $(\hat{N}^u, N^o)$; and τ is an $\hat{\mathcal{F}}-$measurable stopping time.

[8] Limited liability is commonly assumed in contract theory, especially dynamic contract theory. Without it, the model and analysis become easy, or even trivial. For example, the principal could simply sell the entire enterprise to the agent upfront, at a price, that equals the efficient social surplus. This allows the principal to exact the entire surplus and leaves the agent with zero surplus.

[9] By the revelation principle in [Mye86], any outcome that can be achieved by a general mechanism can also be achieved by a truth-telling direct mechanism.

Γ^{IC} is also optimal in a larger contract space Γ.

Agent's Utility

Given a contract $\gamma \in \Gamma^{IC}$, an effort process ν, and a reported arrival process $\hat{N}_t^u$, the agent's expected total discounted utility is[10]

$$u(\gamma, \nu, \hat{N}^u) = E\left[\int_0^\tau e^{-rt}\left[R_u(dN_t^u - d\hat{N}_t^u) + dL_t + c(1 - \mathbb{I}[\nu_t = \mu])dt\right]\right], \qquad (3.1)$$

where the first term represents the income from diverting cash (if the agent fails to report the customer arrivals from the unobservable channel), second term is the payment that the agent receives from the principal, and the third term is the monetary benefit from shirking (if the agent exerts zero effort).[11]

Principal's Utility

We assume that the principal is very busy owning multiple stores and needs an agent to run the store. Therefore, the principal only earns revenues while the agent is employed. Hence, the principal's expected total discounted profit under a contract $\gamma \in \Gamma^{IC}$, effort process ν, and reported arrival process $\hat{N}^u$ is defined as[12]

$$U(\gamma, \nu, \hat{N}^u) = \mathbb{E}\left[\int_0^\tau e^{-rt}\left[R_u d\hat{N}_t^u + R_o dN_t^o - dL_t - cdt\right]\right]. \qquad (3.2)$$

If the agent is never terminated and always exerts effort, then the sum of principal's and agent's utilities corresponds to the total utility (as if the principal could operate the store by herself):

$$\bar{V} := \frac{\mu(pR_u + (1-p)R_o) - c}{r}. \qquad (3.3)$$

We define the adjusted effort cost as

$$\beta := \frac{c}{\mu}. \qquad (3.4)$$

[10]For the current setting, we do not allow the agent to save. The agent needs to immediately consume cash if he misreports the arrival and steals the money. Furthermore, following [DS06b], as long as the saving rate is smaller or equal to the discount rate, it is without loss of generality to assume that saving is not allowed.

[11]The expectation is taken given the agent's reporting strategy and effort choices $\mathcal{S} = \{\hat{N}_t^u, \nu_t : t \in [0, \tau]\}$.

[12]For ease of presentation, in our setting, we assume that once the agent is fired, the principal has no one to run the business. We can easily generalize the model to the case where the principal can fire the agent at any time and incur a fixed cost to hire a new agent; such generalization does not change our main results.

The contract design problem is only meaningful when, at least from the societal perspective (when the objective is the sum of the agent's and the principal's utility), inducing effort from the agent is worthwhile, which formally translates into the below condition:

$$\bar{V} \geq 0 \Longleftrightarrow pR_u + (1-p)R_o \geq \beta.$$

Let us now formally define the contract space Γ^{IC}. A contract γ is incentive compatible ($\gamma \in \Gamma^{IC}$) if, in equilibrium, the agent has the incentive to always exert effort and truthfully report the arrivals, i.e., $\nu^* := \{\nu_t = \mu\}$ and $\hat{N}^u_t = N^u_t$, for $t \in [0, \tau]$. That is, the contract is incentive compatible if

$$u(\gamma, \nu^*, N^u) \geq u(\gamma, \nu, \hat{N}^u), \text{for any effort process } \nu \text{ and any reported arrival process } \hat{N}. \quad \text{(IC)}$$

The contract design problem of the principal is to find a contract $\gamma = (L, \tau) \in \Gamma^{IC}$ that maximizes her utility $U(\gamma, \nu, \hat{N}^u)$ under the incentive compatibility constraint (IC).

In what follows, we characterize the incentive compatible contracts in a recursive manner, so that the contract design problem can be formulated as a stochastic optimal control problem. Hence, we introduce the agent's *promised future utility* (for brevity, henceforth referred to as a *promised utility*) $W_t(\gamma, \nu, \hat{N}^u)$ at time t, following contract $\gamma \in \Gamma^{IC}$, effort process ν, and after a history of reports $\hat{N}^u_s$ $(0 \leq s \leq t)$, to be the total expected payoff the agent receives if he tells the truth after time t:

$$W_t(\gamma, \nu, \hat{N}^u) = \mathbb{E}\left[\int_t^\tau e^{-r(s-t)}\left[dL_s + c(1 - \mathbb{I}[\nu_s = \mu])ds\right]\right].$$

It is clear that $W_0(\gamma, \nu, \hat{N}^u) = u(\gamma, \nu, N^u)$. It is convenient to introduce the notation $W_{t-}(\gamma, \nu, \hat{N}^u) = \lim_{s\uparrow t} W_s(\gamma, \nu, \hat{N}^u)$ to denote the left-hand limit of the process $W(\gamma, \nu, \hat{N}^u)$ at $t \geq 0$. In the sequel, we omit W_t and W_{t-}'s dependence on γ, ν, and $\hat{N}^u$ when there is no confusion. We assume that the agent can decide to quit if future payments do not compensate for effort costs. That is, the promised utility can never be negative (we refer to it as the *individual rationality* (IR) condition):

$$W_t \geq 0, \forall t \geq 0. \quad \text{(IR)}$$

In what follows, we characterize the evolution of the agent's promised utility W_t under a contract $\gamma \in \Gamma^{IC}$, which is also known as the *promise keeping* (PK) condition in the dynamic contract literature (see [DS06b], [ST18]).

Lemma 3.1 *For any contract $\gamma \in \Gamma^{IC}$, there exists a pair of predictable processes, H^u_t and H^o_t, such that if the agent follows a truth-telling strategy $\hat{N}^u_t = N^u_t$ and effort choice ν_t for $t \in [0, \tau]$,* *his promised utility follows:*

$$dW_t = rW_{t-}dt - c(1 - \mathbb{I}[\nu_t = \mu])dt - dL_t + H_t^u(d\hat{N}_t^u - \nu_t p dt) + H_t^o(dN_t^o - \nu_t(1-p)dt). \tag{PK}$$

H_t^u and H_t^o capture the instantaneous change of the promised utility W_t upon arrival before a potential instantaneous payment I_t at time t. Later, we will show that the processes H_t^u and H_t^o are the key tools to incentivize the agent to exert effort and report the truth. Therefore, the key of the contract design problem is to choose H_t^u and H_t^o optimally.

The next lemma shows that, to guarantee truth-telling, we need H_t^u to be greater or equal to the revenue R_u generated by the unobservable arrival.

Lemma 3.2 *Truth-telling is incentive compatible if and only if*

$$H_t^u \geq R_u, \tag{IC-truthful}$$

for all $t \leq \tau$, where H_t^u is defined in Lemma 3.1.

Similarly, Lemma 3.3 shows that, to guarantee that the agent exerts effort, we also need the expected jump associated with the customer arrivals, $pH_t^u + (1-p)H_t^o$, to be greater than or equal to the adjusted effort cost β.

Lemma 3.3 *For the agent, full-effort is incentive compatible if and only if*

$$pH_t^u + (1-p)H_t^o \geq \beta, \tag{IC-effort}$$

for all $t \leq \tau$, where H_t^u and H_t^o are defined in Lemma 3.1.

The above two lemmas imply that to provide incentives to the agent, we only need to reward him at the times when new customers arrive. Further, it follows that the (IC-truthful) and (IC-effort) conditions are equivalent to the (IC) condition.

3.2.2 Structure of the Optimal Contract $\&$ the Corresponding Value Function

Let $F(w)$ denote the principal's value function. Then the Hamiltonian-Jacobi-Bellman (HJB) equation is

$$\begin{aligned} rF(W_{t-}) = rW_{t-}F'(W_{t-}) - c + \mu[pR_u + (1-p)R_o] + \max_{H_t^u, H_t^o} \mu \{p[F(W_{t-} + H_t^u) - F(W_{t-})] \\ +(1-p)[F(W_{t-} + H_t^o) - F(W_{t-})] - (pH_t^u + (1-p)H_t^o)F'(W_{t-})\Big\}, \end{aligned} \tag{3.5}$$

subject to

$$H_t^u \geq R_u, pH_t^u + (1-p)H_t^o \geq \beta.$$

We need to appropriately pick H_t^u and H_t^o, so that they satisfy the optimality condition of the above optimization problem, where F is a concave function in w.[13] Hence, the optimization problem can be easily solved by invoking the Karush-Kuhn-Tucker (KKT) conditions.

As we move further with the analysis, we will show that the optimal contract structure depends on the relative values of μR_u and c. In fact, the parameters of the model give rise to three possibilities: (1) $c \leq \mu p R_u$, (2) $\mu p R_u < c < \mu R_u$, and (3) $c \geq \mu R_u$. Each of these scenarios leads to a distinct optimal contract structure which we describe next.

Scenario 1.

If $c \leq \mu p R_u$, then $H_t^u = R_u$ and $H_t^o = 0$.

In this case, if (IC-truthful) holds, then $c \leq \mu p R_u$ implies $pH_t^u + (1-p)H_t^o \geq pR_u \geq \beta$, which means that the (IC-effort) also holds. The condition $c \leq \mu p R_u$ means that the walk-in customers are very valuable to the principal (that is, either there is a high proportion of walk-in customers, captured by large p, or the walk-in-customer revenue R_u is large).

In such a setting, the principal will not waste money rewarding the agent for the customers from the observable channel (i.e., customers who reserved and paid online), and the agent's promised utility will only jump up when a walk-in customer arrival is reported. Note, that there exists an upper bound on the agent's promised utility, and, once the agent achieves this upper bound, he is never terminated. We denote this upper bound as

$$\bar{w}_1 := \mu p R_u / r, \tag{3.6}$$

and refer to it as a "tenure threshold". The evolution of the promised utility follows the below:

$$dW_t = \{r(W_{t-} - \bar{w}_1)dt + \min\{R_u, \bar{w}_1 - W_{t-}\}dN_t^u\} \, \mathbb{I}[W_{t-} \geq 0]. \tag{DW1}$$

Denote the outlined above contract as γ_1^*. We present the formal definition of γ_1^* next.

Definition 3.1 *Contract $\gamma_1^*(w) = (L_1^*, \tau_1^*)$ is generated from a process $\{W_t\}_{t\geq 0}$ following (DW1) with a given $W_{0-} = w \in [0, \bar{w}_1]$, such that $dL_{1t}^* = (W_{t-} + R_u - \bar{w}_1)^+ dN_t^u$ and $\tau_1^* = \min\{t : W_t = 0\}$, in which the reported arrival process $\hat{N}^u = N^u$ and dL_{1t}^* are generated from the agent's effort process ν^*.*

[13] We will verify the concavity in Lemma 3.4.

Following (3.5), the corresponding principal's value function should satisfy

$$rF(w) = r\bar{V} + \mu p[F(w + R_u) - F(w)] + r(w - \bar{w}_1)F'(w), \tag{3.7}$$

with the boundary condition $F(0) = 0$, where $\bar{V}$ is defined in (3.3). In Section 3.2.3, we will prove that the solution of the differential equation is the principal's expected total utility under contract γ_1^*, i.e, $F(w) = U(\gamma_1^*(w), \nu, N^u)$.

Scenario 2.

If $\mu p R_u < c < \mu R_u$, then $H_t^u = R_u$ and $H_t^o = \beta_1$, where $\beta_1 := \frac{\beta - pR_u}{1-p} = \frac{c - \mu p R_u}{\mu(1-p)} \in [0, R_u)$.

In this case, $H_t^u = R_u$ is not enough to satisfy the constraint (IC-effort), which guarantees that the agent is motivated to exert effort. To induce effort, the principal needs to reward the agent for both types of arrivals. Therefore, the promised utility jumps up when arrivals from the unobservable channel are reported as well as when customers from the observable channel reserve online (with the jump for arrivals from the unobservable channel being larger than for arrivals from the observable channel).[14] Again, it is possible for the agent to achieve tenure when his promised utility reaches the tenure threshold denoted by

$$\bar{w}_2 := c/r. \tag{3.8}$$

The evolution of the promised utility follows

$$dW_t = \{r(W_{t-} - \bar{w}_2)dt + \min\{R_u, \bar{w}_2 - W_{t-}\}dN_t^u + \min\{\beta_1, \bar{w}_2 - W_{t-}\}\, dN_t^o\}\, \mathbb{I}[W_{t-} \geq 0]. \tag{DW2}$$

Denote the outlined above contract as γ_2^*. The formal definition of γ_2^* is given next.

Definition 3.2 *Contract $\gamma_2^*(w) = (L_2^*, \tau_2^*)$ is generated from a process $\{W_t\}_{t\geq 0}$ following (DW2) with a given $W_{0-} = w \in [0, \bar{w}_2]$, such that $dL_{2t}^* = (W_{t-} + R - \bar{w}_2)^+ dN_t^u + (W_{t-} + \beta_1 - \bar{w}_2)^+\, dN_t^o$ and $\tau_2^* = \min\{t : W_t = 0\}$, in which the counting processes $\hat{N}^u = N^u$ and N^o as well as dL_{2t}^* are generated from the agent's effort process ν^*.*

Following (3.5), the corresponding principal's value function satisfies the following differential

[14] This follows from the fact that both incentive constraints are binding in this setting, and $\beta < R_u$ immediately implies $H_t^o < R_u$.

equation

$$
\begin{aligned}
rF(w) = r\bar{V} + \mu p[F(w+R_u) - F(w)] + \mu(1-p)\left[F\left(w+\beta_1\right) - F(w)\right] \\
- rF(w) + r(w - \bar{w}_2)F'(w),
\end{aligned}
\tag{3.9}
$$

with the boundary condition $F(0) = 0$, where $\bar{V}$ is defined in (3.3). Similar to contract γ_1^*, we will prove in Section 3.2.3 that the solution of the differential equation is the principal's expected total utility under contract γ_2^*, i.e, $F(w) = U(\gamma_2^*(w), \nu, N^u)$.

Scenario 3.

If $c \geq \mu R_u$, then $H_t^u = \beta$ and $H_t^o = \beta$.

In this case, condition (IC-truthful) always holds whenever condition (IC-effort) is satisfied, therefore, the constraint (IC-truthful) can be ignored. The principal lets the promised utility jump up by the same amount every time an arrival occurs (regardless of the channel of the arrival). The tenure threshold in this case is given by $\bar{w}_3 := \bar{w}_2 = c/r$, and the promised utility follows

$$
dW_t = \{r(W_{t-} - \bar{w}_3)dt + \min\{\beta, \bar{w}_3 - W_{t-}\}dN_t^u + \min\left\{\beta, \bar{w}_3 - W_{t-}\right\}dN_t^o\} \mathbb{I}[W_{t-} \geq 0].
\tag{DW3}
$$

Denote the outlined above contract as γ_3^*. The formal definition of γ_3^* is given next.

Definition 3.3 *Contract $\gamma_3^*(w) = (L_3^*, \tau_3^*)$ is generated from a process $\{W_t\}_{t\geq 0}$ following (DW3) with a given $W_{0-} = w \in [0, \bar{w}_3]$, such that $dL_{3t}^* = (W_{t-} + \beta - \bar{w}_3)^+ dN_t^u + (W_{t-} + \beta - \bar{w}_3)^+ dN_t^o$ and $\tau_3^* = \min\{t : W_t = 0\}$, in which the counting processes $\hat{N}^u = N^u$ and N^o as well as dL_{3t}^* are generated from the agent's effort process ν^*.*

The corresponding principal's value function F is a solution to the following differential equation

$$
0 = r\bar{V} + \mu[F(w+\beta) - F(w)] - rF(w) + r(w - \bar{w}_3)F'(w),
\tag{3.10}
$$

with the boundary condition $F(0) = 0$, where $\bar{V}$ is defined in (3.3). Similar to γ_1^* and γ_2^*, we will prove in Section 3.2.3 that the solution of the differential equation is indeed the principal's expected total utility under contract γ_3^*, i.e, $F(w) = U(\gamma_3^*(w), \nu, N^u)$.[15]

All three contract structures from above have the following elements in common: (i) a *probationary period* during which the agent works for the principal while being offered a (ii) *promised*

[15] It is worth noting here that the optimal contract γ_3^* has the same structure as Γ^* in [ST18]. The same discount rate case of [ST18] can be thought of a special case of one of our baseline model sub-cases, where the contract design problem only induces the agent to exert effort to increase the arrival rate of the customers with no truth-telling constraint and where the arriving customers are homogeneous.

utility; (iii) a *tenure period* during which the agent keeps all the revenues collected from the walk-in customers to himself; and (iv) a *termination threshold* on the promised utility. During the probationary period, the agent's promised future utility takes instantaneous upward jumps on certain customer arrivals but otherwise decreases. In each of the three contracts, the probationary period is over when the promised utility reaches either the *termination threshold* or the *tenure threshold*. The variation in the three contract structures, in turn, is captured by the differences in (i) tenure thresholds, and (ii) the rules for the evolution of the promised utility (the rate of its decrease, and the when-and-how for the upward jumps).

The contract structures that we have outlined above indicate the important role that the revenue from the unobservable (walk-in) channel and flow effort cost play in the optimal contract structure, and which of the incentive constraints end up being binding. When the effort cost is smaller than the expected revenue from the unobservable channel ($c \leq \mu p R_u$), the principal only rewards the agent for arrivals from the unobservable channel in the optimal contract γ_1^*, and the truth-telling constraint is the only binding constraint (scenario 1). When the effort cost is moderate ($\mu p R_u < c < \mu R_u$), the principal rewards both types of arrivals, and both incentive constraints end up being binding in the optimal contract γ_2^* (scenario 2). Finally, when the effort cost c is large enough ($c \geq \mu R_u$), the principal just rewards every arrival with the amount of adjusted cost β, and only the effort constraint is binding (scenario 3).

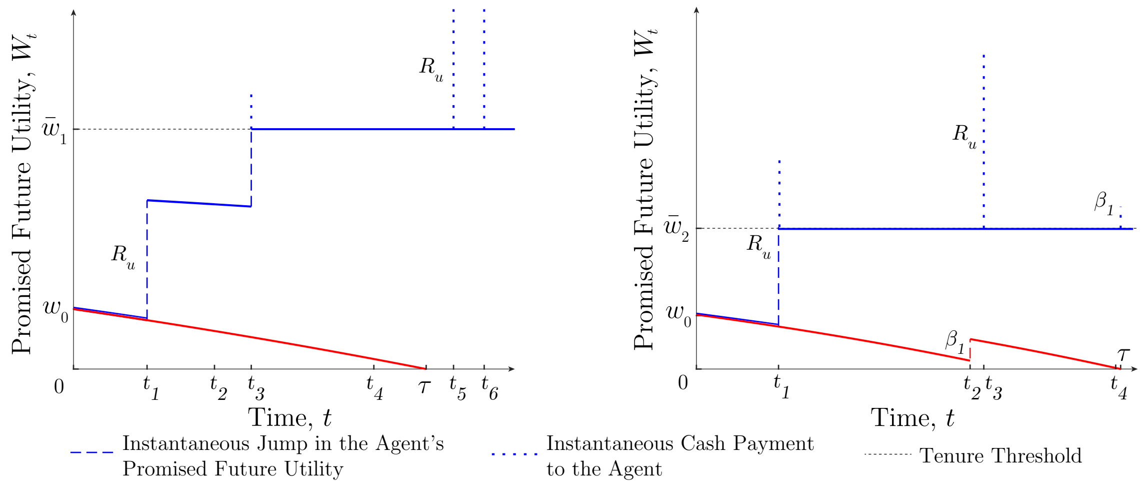
Furthermore, to implement the contracts, we need to let the starting point of the promised utility be $W_{0-} = w^*$, which is the maximizer of the corresponding principal's value function. Later, we will verify that the value functions are concave, which implies that the maximizer is unique.

3.2.2.1 Illustrative Examples of Sample Trajectories under Optimal Contract

Figure 3.1 provides an illustration of the sample trajectories of the agent's promised utility under the optimal contract for two distinct scenarios.

Panel (a) is an example under the contract γ_1^*. The promised utility starts from an initial point w_0 at time $t = 0$. As time passes, customers arrive at the store. On the blue trajectory, unobservable channel arrivals occur at t_1, t_3, t_5, and t_6, while observable channel arrivals occur at t_2 and t_4. During the probationary period (i.e., for $t \in [0, t_3]$), the promised utility keeps decreasing between the arrivals and jumps up by R_u every time a customer arrival from the unobservable channel is reported by the agent (the promised utility jumps are depicted by vertical dashed lines at t_1 and t_3).[16] The promised utility keeps this dynamic until it reaches the tenure threshold $\bar{w}_1$. Once the agent's promised utility reaches $\bar{w}_1$, it is kept at this level, and the agent is paid $(W_{t-} + R_u - \bar{w}_1)^+$

[16] In this example, t_3 marks the end of the probationary period and the start of the tenure period for the blue trajectory scenario. At this moment, the agent's utility jumps up by the dashed vertical line segment (which is smaller than R_u) so as to hit exactly the tenure threshold $\bar{w}_1$, while the remaining part of R_u is received by the agent as the first actual payment (the dotted vertical line segment). The dashed and dotted line segments at t_3 together comprise exactly R_u.



(a) Under Contract $\gamma_1^*(w_0)$.
$p = 0.5, w_0 = 1$

(b) Under Contract $\gamma_2^*(w_0)$.
$p = 0.1, w_0 = 0.6, \beta_1 = 0.22$

Figure 3.1: Blue and red colored lines illustrate different sample trajectories of the agent's promised future utility. For (a) and (b): $R_u = 2, R_o = 2, c = 0.8, \mu = 2, \Delta\mu = 2, r = 0.5$.

for every arrival from the unobservable channel (the payments are depicted by vertical dotted lines at t_3, t_5, and t_6). Note that, under γ_1^*, the agent is never rewarded for any customer arrival from the observable channel. Contrary to the blue trajectory scenario, on the red trajectory, the agent fails to get any arrivals from the unobservable channel before the promised utility reaches zero (captured by time τ), and, thus, the agent is terminated.

Panel (b) is an example under the contract γ_2^*. The promised utility starts from an initial point w_0. On the blue trajectory, unobservable channel arrivals occur at t_1 and t_3, while an observable channel arrival occurs at t_4. On the red trajectory, there are no other arrivals except for a customer arrival from the observable channel at t_2. Similarly to contract γ_1^*, during the probationary period in contract γ_2^*, the agent's promised utility (i) keeps decreasing between the arrivals, and (ii) jumps up by R_u for every customer arrival from the unobservable channel which is reported by the agent (such promised utility jump is depicted by the vertical dashed line at t_1). In contrast to γ_1^*, under the contract γ_2^*, the arrivals from the observable channel are also rewarded. In particular, during the probationary period, the promised utility jumps up by β_1 for every arrival from the observable channel (see the vertical dashed line at t_2 on the red trajectory). Once the promised utility reaches the tenure threshold $\bar{w}_2$ (as it is the case for the blue trajectory at t_1), the agent's promised utility is kept at this level, and the agent is paid $(W_{t-} + R_u - \bar{w}_2)^+$ for every arrival from the unobservable channel and $(W_{t-} + \beta_1 - \bar{w}_2)^+$ for every arrival from the observable channel (the payments are depicted by vertical dotted lines at t_1, t_3, and t_4). Note that, on the red trajectory, despite the fact that the agent receives an arrival, his promised utility still eventually reaches zero at time τ, thus, the agent is terminated.

3.2.3 Proof of Contract Optimality

To prove the optimality of the contract presented in Section 3.2.2, we, first, prove the concavity of the value functions in Lemma 3.4.

Lemma 3.4 *If $c \leq \mu p R_u$, then equation* (3.7) *with boundary condition $F(0) = 0$ has a unique solution $F_1(w)$, which is concave in w, and $F_1'(w) \geq -1$. If $\mu p R_u < c < \mu R_u$, then equation* (3.9) *with boundary condition $F(0) = 0$ has a unique solution $F_2(w)$, which is concave in w, and $F_2'(w) \geq -1$. If $c \geq \mu R_u$, then equation* (3.10) *with boundary condition $F(0) = 0$ has a unique solution $F_3(w)$, which is concave in w, and $F_3'(w) \geq -1$.*

In what follows, Proposition 3.1 shows that the function $F_i(w)$ is indeed the principal's utility under the contract γ_i^*, for $i = 1, 2, 3$.

Proposition 3.1 *For any promised utility $w \in [0, \bar{w}_i]$, we have $U(\gamma_1^*(w), \nu^*, N^u) = F_1(w)$, $U(\gamma_2^*(w), \nu^*, N^u) = F_2(w)$, $U(\gamma_3^*(w), \nu^*, N^u) = F_3(w)$. That is, when agent's total discounted utility is equal to w, a function $F_i(w)$, $i \in \{1, 2, 3\}$ is equal to the principal's total discounted utility under the contract γ_i^*.*

Next, we show that the value functions F_i, $i \in \{1, 2, 3\}$ are upper bounds on the principal's utility under any other incentive compatible contract.

Proposition 3.2 *For any contract $\gamma \in \Gamma^{IC}$ and promised utility $w \in [0, \bar{w}_i]$, if $u(\gamma, \nu^*, N^u) = w$, then*

$$\begin{aligned} &F_1(u(\gamma, \nu^*, N^u)) \geq U(\gamma, \nu^*, N^u) \text{ if } c \leq \mu p R_u, \\ &F_2(u(\gamma, \nu^*, N^u)) \geq U(\gamma, \nu^*, N^u) \text{ if } \mu p R_u < c < \mu R_u, \\ &F_3(u(\gamma, \nu^*, N^u)) \geq U(\gamma, \nu^*, N^u) \text{ if } c \geq \mu R_u, \end{aligned}$$

where $F_1(w) = F_1(\bar{w}_1) - (w - \bar{w}_1)$ for $w > \bar{w}_1$, $F_2(w) = F_2(\bar{w}_2) - (w - \bar{w}_2)$ for $w > \bar{w}_2$ and $F_3(w) = F_3(\bar{w}_3) - (w - \bar{w}_3)$ for $w > \bar{w}_3$.

Since functions $F_i(w)$, $i = 1, 2, 3$ are concave in w and $F_i'(w) \geq -1$, the maximizer of the function $F_i(w)$ is unique and can be denoted as

$$w_i^* = \arg\max_{w \geq 0} F_i(w). \tag{3.11}$$

Therefore, we know that for any contract $\gamma \in \Gamma^{IC}$,

$$U(\gamma, \nu^*, N^u) \leq F_i(u(\gamma, \nu^*, N^u)) = F_i(w) = U(\gamma_i^*(w), \nu^*, N^u) \leq U(\gamma_i^*(w_i^*), \nu^*, N^u) = F_i(w_i^*), \tag{3.12}$$

which implies that γ_i^* is the optimal incentive contract with full-effort and truth-telling. We summarize this result in the following Theorem.

Theorem 3.1 $\gamma_1^*(w_1^*)$ *in definition 3.1 is the optimal contract in the contract space* Γ^{IC} *if* $c \leq \mu p R_u$. $\gamma_2^*(w_2^*)$ *in definition 3.2 is the optimal contract in the contract space* Γ^{IC} *if* $\mu p R_u < c < \mu R_u$. $\gamma_3^*(w_3^*)$ *in definition 3.3 is the optimal contract in the contract space* Γ^{IC} *if* $c \geq \mu R_u$.

So far, we have assumed that the agent has to be incentivized to provide effort to generate arrivals at all times. In the next section, we consider a broader contract space where the agent could be allowed to shirk (i.e., not to exert effort), and show that Theorem 3.1 still holds in such expanded contract space.

3.2.4 Optimality of the Contract in the Expanded Contract Space

In the previous sections, we derived the optimal contract under the assumption that the agent has to exert effort and truthfully report arrivals at all times. In this section, we show that the contracts that we derived are still optimal even if we drop the requirement for the contract space to necessarily include only those contracts that induce full-effort and truth-telling.

Theorem 3.2 $\gamma_1^*(w_1^*)$ *in definition 3.1 is the optimal contract in the contract space* Γ *if* $c \leq \mu p R_u$. $\gamma_2^*(w_2^*)$ *in definition 3.2 is the optimal contract in the contract space* Γ *if* $\mu p R_u < c < \mu R_u$. $\gamma_3^*(w_3^*)$ *in definition 3.3 is the optimal contract in the contract space* Γ *if* $c \geq \mu R_u$.

The intuition behind the result of Theorem 3.2 is as follows. First, as we mentioned in the beginning of Section 3.2.1, due to the revelation principle, without loss of generality, we can focus on contracts that induce the agent to truthfully report the arrivals. Second, it turns out that the principal also does not lose anything if we only focus on contracts that induce full-effort from the agent. The reason that always inducing effort is optimal for the principal is that instead of allowing the agent to shirk, the principal can offer the agent another contract that uses direct payments for every arrival to induce full-effort, which is Pareto improving. Hence, shirking is inefficient to the system, and direct payments to the agent (that ensure he does not shirk) can make both principal and agent better off. For more details, please refer to the proof of Theorem 3.2 in Appendix B.1.2.7.

3.2.5 Numerical Examples & Comparative Statics

In this section, we provide some additional insights into how the principal's and the agent's utilities change with system parameters, such as R_u, R_o, and p, and how the portion of the total system profit, that is captured by the principal, changes as a function of these parameters.

When p is large, the agent captures most of the revenues generated by the business after he gets tenure, so hiring an agent may not seem too appealing to the principal at first sight. However, the principal chooses $W_{0-} = w^*$ in such a way that it takes the agent a while to get to the tenure threshold, and, in the meantime, the principal generates significant revenues. This is because the actual payments to the agent start only when the agent's promised utility reaches the tenure threshold $\bar{w}$, so the principal still gets to capture all the revenues from the unobservable (walk-in) arrivals during the probationary period plus all the revenues from the observable arrivals collected over the entire time horizon. The next lemma shows that both principal and agent's utilities are increasing in R_o.

Lemma 3.5 *The agent's utility w^* and principal's utility $F(w^*)$ are strictly increasing in R_o.*

Let $f^* := \dfrac{F(w^*)}{w^* + F(w^*)}$ be the fraction of the principal's utility out of the total system's utility. Figure 3.2 shows that f^* increases as R_o increases. This is an intuitive observation since the increase of R_o alleviates the moral hazard issue. Similarly, figure 3.3 shows that, when p increases, then f^* decreases, while figure 3.4 shows that, when R_u increases, then f^* decreases. This is because the increase of p or R_u only exacerbates the moral hazard issue.

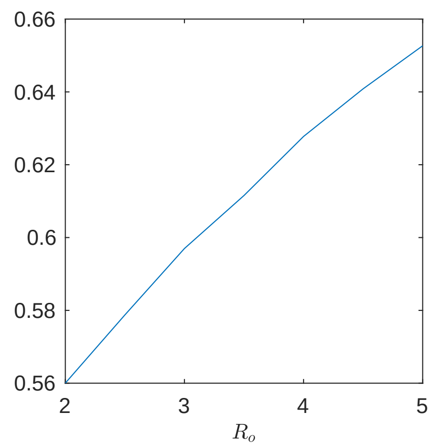


Figure 3.2: $F(w^*)/\,(w^* + F(w^*))$ as a function of R_o. $p = 0.7$, $R_u = 3$, $\mu = \Delta\mu = 2$, $r = 0.5$.

3.3 Monitoring

In the previous section, we characterized the contracts that the principal will have to offer to the agent to ensure that the agent does not divert cash and also exerts effort to attract more customers to the business. These contracts require the principal to pay rent to the agent, and we characterized situations where this rent could be fairly significant. One way that the principal could avoid paying this much rent is by monitoring the agent whenever necessary. Although monitoring in moral hazard problems has been addressed in other papers (see, e.g., [PW16], [CSX20]), our setting is unique in

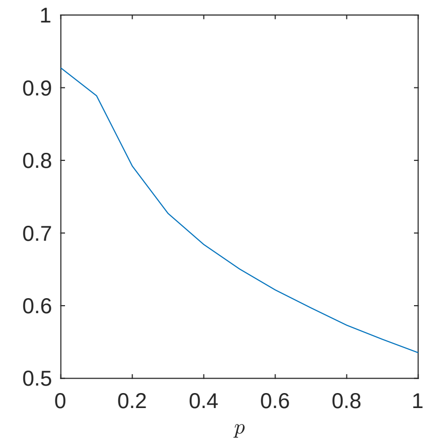


Figure 3.3: $F(w^*)/\left(w^* + F(w^*)\right)$ as a function of p. $R_u = 3$, $R_o = 3$, $\mu = \Delta\mu = 2$, $r = 0.5$.

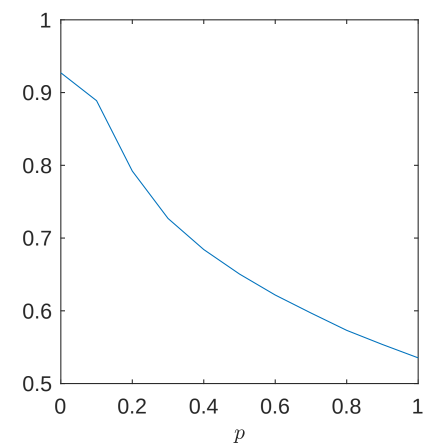


Figure 3.4: $F(w^*)/\left(w^* + F(w^*)\right)$ as a function of R_u. $p = 0.7$, $R_o = 3$, $\mu = \Delta\mu = 2$, $r = 0.5$.

that the principal could potentially decide either to monitor the agent as a protection against both moral hazard issues, or potentially just monitor partially to see if the agent is diverting cash.

In particular, in this section, we consider the setting where the principal could pay a flow cost m to monitor the agent. In our baseline model, the principal is facing two moral hazard problems: "diverting cash" and "shirking". Section 3.3.1 considers the case of *full monitoring*, that is, when monitoring can reveal both types of private information, which means the principal can monitor both unobservable arrivals and effort. This type of monitoring could be fairly expensive, requiring the principal (or someone the principal hires) to be at the place of business and observe the agent, to guarantee that the agent always exerts effort and reports arrivals truthfully. However, we will show that the principal does not need to conduct this expensive monitoring all the time and only needs to conduct it when the agent's promised utility becomes very low. In fact, when deciding whether to monitor or not, the principal is trading off the monitoring cost against the payment to incentivize the agent.

Additionally, Section 3.3.2 considers the case of *partial monitoring*, where the principal can only monitor (otherwise unobservable) arrivals. For example, if the agent delivers the services at the principal's business unit, the principal could install video monitoring equipment to continuously observe all arrivals and services delivered to customers, or the principal could install a GPS tracker on all company-owned equipment required to deliver the service. In fact, recent empirical papers ([KLS18], [NRST02], and [PSM15]) considered settings (agents driving minibuses, agents soliciting donations, and agents serving customers at a restaurant), where the principal monitors agents' effort and/or agents' sales, and showed that such monitoring can significantly improve productivity and reduce theft.

3.3.1 Full Monitoring

Throughout this section we focus on the derivation of an optimal contract, when a new costly option of full monitoring is, now, made available to the principal. In particular, during the time interval $[t, t+\delta]$, the principal could pay a monitoring cost $m\delta$ to observe all the arrivals and agent's effort choices during this time interval. Let a process $\{M_t\}_{t\geq 0}$ represent the full monitoring schedule under the contract, where $M_t \in \{0, m\}$ captures the monitoring cost at time t. We expand the definition of a contract to include the monitoring schedule as follows: $\gamma = (L, \tau, M) \in \Gamma_m$.[17]

The expression for the agent's utility is not affected by the presence of monitoring, and so is still captured by the equation (3.1). In contrast, the monitoring option does alter the expression of

[17]Formally, L and M are adapted to $\hat{\mathcal{F}} = \{\hat{\mathcal{F}}_t, t \geq 0\}$, the filtration generated by $\hat{N}^u, N^o$.

the principal's utility (which now has an additional term capturing the cost paid for monitoring):

$$U_m(\Gamma, \nu, \hat{N}) = \mathbb{E}\left[\int_0^\tau e^{-rt}\left(R_u d\hat{N}_t^u + R_o dN_t^o - dL_t - cdt - M_t dt\right)\right]. \tag{3.13}$$

Note that the promise keeping constraint does not undergo any change when full monitoring is introduced, so the (PK) constraint stays the same. In contrast, the incentive compatibility constraint that induces truthful reporting and the agent's effort does require updating to adjust for the occurrence of full monitoring, and, thus, is described next.

Lemma 3.6 *Truth-telling and full-effort are incentive compatible if and only if*

$$H_t^u \geq R_u, pH_t^u + (1-p)H_t^o \geq \beta \text{ if } M_t = 0, \tag{3.14}$$

for $t \leq \tau$*, where* H_t^u *and* H_t^o *are defined in Lemma 3.1.*

Lemma 3.6 is directly adapted from Lemma 3.2. We define the contract space that only includes the contracts where the agent always exerts effort and reports truthfully as Γ_m^{IC}. We, first, focus on the contracts in Γ_m^{IC}. We, then, prove that the optimal contract in Γ_m^{IC} is still optimal in a larger contract space Γ_m.

We find that, in the optimal contract, the principal only monitors the agent when the agent's promised utility reaches the *full monitoring threshold* equal to 0. While the principal conducts full monitoring, the agent's promised utility is kept at zero. In fact, under the optimal contract, once full monitoring is initiated, it continues indefinitely. Based on definitions 3.1-3.3, we now define such optimal contract by adding the monitoring schedule m_t^*.

Definition 3.4 *For* $i = 1$ *(*$2, 3$*), contract* $\gamma_{mi}^*(w) = (L_i^*, \tau_i^*, m_i^*)$ *is generated from a process* $\{W_t\}_{t\geq 0}$ *following* (DW1) *((*DW2*),* (DW3)*) with a given* $w \in [0, \bar{w}_i]$*, such that* L_i^* *has the same expression in* $\gamma_i^*(w)$*,* $\tau_i^* = \infty$*, and* $M_{it}^* = m\mathbb{I}[W_{t-} = 0]$*, where the counting process* N^u *and* dL_{it}^* *are generated from the agent's effort process* ν^**.*

According to the above definitions, the principal monitors the agent only when the agent's promised utility is 0. If monitoring were not available, the principal would have to terminate the agent and would make zero profits thereafter. Hence, monitoring enables the principal's utility to become $(\mu[pR_u + (1-p)R_o] - c - m)/r$ instead of zero as soon as the agent's promised utility hits 0. So, monitoring is profitable to the principal only if $(\mu[pR_u + (1-p)R_o] - c - m)/r \geq 0$, or equivalently:

$$m \leq \mu[pR_u + (1-p)R_o] - c. \tag{3.15}$$

The next proposition shows that the contracts γ_{m1}^*, γ_{m2}^*, and γ_{m3}^* are optimal under condition (3.15). Denote the principal's value function under contract $\Gamma_{mi}^*(w)$ by $F_{mi}(w)$. Then, F_{m1} (F_{m2}, F_{m3}) is the solution of equation (3.7) (eq. (3.9), eq. (3.10)) with the boundary condition $F(0) = (\mu[pR_u + (1-p)R_o] - c - m)/r$. We further define w_{mi}^* as the maximizer of $F_{mi}(w)$.

Proposition 3.3 *If condition* (3.15) *holds, then:*

- $\gamma_{m1}^*(w_{m1}^*)$ *in definition 3.4 is the optimal contract in* Γ_m *if* $c \leq \mu p R_u$*;*
- $\gamma_{m2}^*(w_{m2}^*)$ *in definition 3.4 is the optimal contract in* Γ_m *if* $\mu p R_u < c < \mu R_u$*;*
- $\gamma_{m3}^*(w_{m3}^*)$ *in definition 3.4 is the optimal contract in* Γ_m *if* $c \geq pR_u$*.*

Conversely, if condition (3.15) *does not hold, then:*

- $\gamma_1^*(w_1^*)$ *in definition 3.1 is the optimal contract in* Γ_m *if* $c \leq \mu p R_u$*;*
- $\gamma_2^*(w_2^*)$ *in definition 3.2 is the optimal contract in* Γ_m *if* $\mu p R_u < c < \mu R_u$*;*
- $\gamma_3^*(w_3^*)$ *in definition 3.3 is the optimal contract in* Γ_m *if* $c \geq pR_u$*.*

In the optimal contract with full monitoring, the principal conducts monitoring to avoid inefficient termination. When monitoring, the principal can guarantee that the agent will exert effort and report arrivals truthfully. Hence, the agent's promised utility does not need to decrease, and termination never happens in these contracts. Further, the principal monitors the agent if and only if the agent's promised utility hits zero. Once the principal initiates full monitoring, she will continue to monitor the agent and will never terminate the contract.

To show the robustness of our results, we consider two generalizations. First, in Appendix B.1.3.2, we are able to extend the model to the case where the agent has an outside option, which guarantees the agent a future utility $\underline{w} \geq 0$. Second, in Appendix B.1.3.3, we generalize the current setting to the case where the principal could pay a fixed cost $k > 0$ to fire the current agent and find a new one.

3.3.2 Partial Monitoring

In this section, we consider the situation where the principal can only monitor the (otherwise unobservable) customer arrivals but is unable to monitor the agent's effort. We refer to this setup as *partial monitoring*. In particular, during the time interval $[t, t+\delta]$, the principal could pay a monitoring cost $m\delta$ to observe all the arrivals during this time interval. Similar to the setting in Section 3.3.1, the contract is denoted as $\gamma = (L, \tau, M) \in \Gamma_{ma}$ with the partial monitoring schedule $M = \{M_t\}_{t \geq 0}$, where $M_t \in \{0, m\}$.

The agent's utility still follows equation (3.1) and the principal's utility follows equation (3.13). The promise keeping and full-effort incentive constraints follow (PK) and (IC-effort), respectively. In contrast, the incentive compatibility constraint that induces truthful reporting does require updating to adjust for the occurrence of partial monitoring, and is described next.

Lemma 3.7 *Truth-telling is incentive compatible if and only if*

$$H_t^u \geq R_u \text{ if } M_t = 0, \tag{IC-truthfulm}$$

for all $t \leq \tau$, where H_t^u is defined in Lemma 3.1.

Lemma 3.7 is directly adapted from Lemma 3.2. We define the contract space that only includes contracts where the agent always exerts effort and reports truthfully as Γ_{ma}^{IC}. That is, for any $\gamma \in \Gamma_{ma}^{IC}$, the constraints (IC-effort) and (IC-truthfulm) have to be satisfied. In the current section, we again focus on the optimal contract in Γ_{ma}^{IC}; and then show that the optimal contract in Γ_{ma}^{IC} is still optimal in a larger contract space Γ_{ma}. In what follows, we derive the optimal contracts which allow partial monitoring. As before, the parameter space of the problem naturally gives rise to three distinct scenarios ($c \leq \mu p R_u$, $\mu p R_u < c < \mu R_u$, and $\mu R_u \leq c$) with their respective distinct contract structures, that we characterize next.

3.3.2.1 $c \leq \mu p R_u$

In this Section, we first present the Hamilton-Jacobi-Bellman (HJB) equation and the optimal contract that allows partial monitoring. Next, we prove the optimality of the contract and provide an illustrative example. Finally, we provide the intuition behind this optimal contract.

Let $F(w)$ denote the principal's value function. When the principal does not monitor the agent, then, based on Section 3.2.2, the agent's promised utility should follow (DW1) and the principal's value function follows (3.7). When the principal conducts monitoring, it is optimal to set $H_t^u = H_t^o = \beta$. Hence, the dynamics of the agent's promised utility becomes (DW3). Then the principal's value function satisfies another differential equation:

$$rF(w) = r\bar{V} - m + \mu[F(w+\beta) - F(w)] + r(w - \bar{w}_2)F'(w). \tag{3.16}$$

Given that the principal is freely deciding whether to monitor the agent, the principal's value function follows the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rF(w) = r\bar{V} + \max\big\{&\mu p[F(w+R_u) - F(w)] + r(w - \bar{w}_1)F'(w), \\ &-m + \mu[F(w+\beta) - F(w)] + r(w - \bar{w}_2)F'(w)\big\}, \end{aligned} \tag{3.17}$$

with boundary condition $F(0) = 0$. The HJB equation implies that the principal would monitor the agent if and only if the first term of (3.17) is less than or equal to the second term, which is equivalent to

$$\mathcal{M}(w) := \mu[F(w+\beta) - F(w)] - \mu p[F(w+R_u) - F(w)] + (\bar{w}_1 - \bar{w}_2)F'(w) \geq m. \quad (3.18)$$

We can show that if $w \geq \bar{w}_1$, not monitoring is better than monitoring, i.e, $m > \mathcal{M}(w)$. As w decreases, monitoring potentially becomes more profitable, there are optimal switching points w_L^*, such that $m = \mathcal{M}(w_L^*)$. For simplicity, we assume a unique switching point.[18] We formally define the optimal contract in definition 3.5.

Definition 3.5 *For any $w \in [0, \bar{w}_1]$, define the contract $\gamma_m^*(w) = (L_m^*, \tau_m^*, m^*)$ as follows:*

1. *Set $W_{0-} = w$ and $L_0^* = (W_{0-} - \bar{w}_1)^+$.*
2. *For $t \geq 0$, let the payment be*

$$dL_t^* = (W_{t-} + R_u - \bar{w}_1)^+ dN_t^u \mathbb{I}[W_{t-} \geq w_L^*] \\ + \left[(W_{t-} + \beta - \bar{w}_1)^+ dN_t^u + (W_{t-} + \beta - \bar{w}_1)^+ dN_t^o\right] \mathbb{I}[W_{t-} < w_L^*],$$

and the dynamics of the promised utility follow

$$dW_t = \{r(W_{t-} - \bar{w}_1)dt + \min\{R_u, \bar{w}_1 - W_{t-}\}dN_t^u\} \mathbb{I}[W_{t-} \geq w_L^*] \quad \text{(DWm)} \\ + \{r(W_{t-} - \bar{w}_2)dt + \min\{\beta, \bar{w}_1 - W_{t-}\}dN_t^u + \min\{\beta, \bar{w}_1 - W_{t-}\}dN_t^o\} \mathbb{I}[W_{t-} < w_L^*],$$

3. *The partial monitoring schedule is $M_t^* = m\mathbb{I}[W_{t-} < w_L^*]$.*
4. *The termination time is $\tau_m^* = \min\{t : W_t = 0\}$.*

We further define w_m^* as the maximizer of $F(w)$, and the principal starts the agent's promised utility at w_m^*. Next, we show that $\gamma_m^*(w_m^*)$ is the optimal contract among the class of contracts that allow the agent to shirk or not truthfully report (in the space Γ_{ma}).

Proposition 3.4 *If the switching point w_L^* is unique and $\mu p R_u \geq c$, then the value function F defined in* (3.17) *is concave in w and differentiable and $\gamma_m^*(w_m^*)$ in definition 3.5 is the optimal contract in the contract space Γ_{ma}.*

Illustrative Example

Figure 3.5 depicts the sample trajectories of the agent's promised utility in the optimal contract γ_m^*

[18]We ran extensive numerical simulations, and in all instances we observed the single switching of the value function. Further, there is a recently published paper by [Won19], which makes a similar assumption of the unique switching point of the value function. In this paper, the authors also compare two terms in a HJB equation, however, their setting is very different from ours.

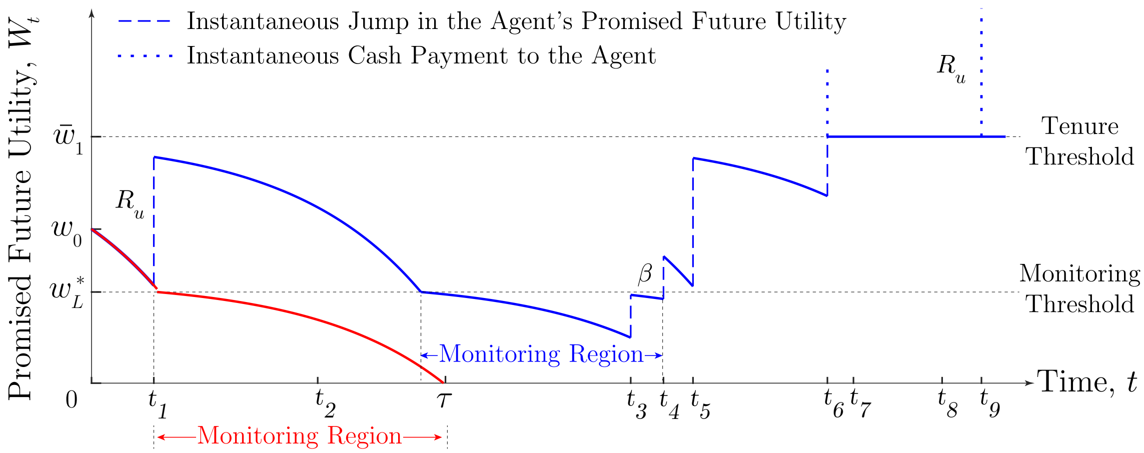


Figure 3.5: Blue and red colored lines illustrate two distinct sample trajectories of the agent's promised future utility under partial monitoring. Parameters: $p = 0.8, R_u = R_o = 2, c = 0.8, \mu = \Delta\mu = 1.2, r = 0.5,$ $\beta = 2/3 < pR_u$, $\bar{w}_1 = 3.2$, $w_0 = 2.4$, $w_L = 1.42$.

with partial monitoring (see Definition 3.5). This is the result of a numerical simulation. The initial value of the agent's promised utility (at $t = 0$) is w_0. As time passes, customers arrive at the store. The times t_1, t_4, t_5, t_6, and t_9 mark arrivals unobservable to the principal, and the times t_2, t_3, t_7, and t_8 mark arrivals observable to the principal. In contrast to the baseline contracts considered in Section 3.2, the contract with partial monitoring, γ_m^*, has two regimes which result in different dynamics of the evolution of the promised utility. In particular, when the agent's promised utility is equal to or above the monitoring threshold, $w \geq w_L^*$ (regime 1), the principal will not monitor the agent and the dynamics is exactly the same as in γ_1^*.[19] However, when the agent's promised utility is below the monitoring threshold, $w < w_L^*$ (regime 2), the principal will monitor the agent and let the promised utility decrease at a lower rate, compared to the regime 1. In regime 2, the principal will also reward both types of arrivals no matter the type, increasing the promised utility by β with every arrival. Note that each such jump of the promised utility in regime 2 is strictly smaller than the jump made under the regime 1 (because $c < \mu p R_u < \mu R_u$). On the blue trajectory, the agent manages to get out of the monitoring region and eventually gets tenure. On the red trajectory, once the agent reaches the monitoring region, he fails to receive any arrivals and is terminated at time τ.

Note that in the case of such partial monitoring, (unlike, in the case of full monitoring, where the principal can monitor both the arrivals and the effort), the agent is still possibly terminated if he does not receive sufficient number of customer arrivals once trapped in the monitoring region. This is because partial monitoring addresses only one of the moral hazard problems – the one caused by unobservable arrivals. Hence, when the principal conducts partial monitoring, she does not have to pay the agent to induce truth-telling. As a result, to decide when to monitor, the principal is

[19]We revert back to the structure of the contract γ_1^* because the parameters satisfy $c \leq \mu p R_u$, thus, placing us into the scenario 1 in Section 3.2.2 .

trading off the cost of partial monitoring versus the payment to incentivize the agent to tell the truth about the number of arrivals. However, this payment does not incentivize the agent to exert effort, and, therefore, the principal still needs to use the threat of termination to incentivize the agent to exert effort, unlike in the case of full monitoring. Another key difference between partial and full monitoring is that, under partial monitoring, the agent can potentially get out of the monitoring region if the promised utility bounces back to a high enough level, whereas under full monitoring, when both arrivals and effort are monitored, once monitoring starts, it continues indefinitely.

3.3.2.2 $\mu p R_u < c < \mu R_u$

The analysis and main results in this case are similar to the case in Section 3.3.2.1, thus, we only discuss the structure of the optimal contract and we leave the formal definition of the optimal contract and the statement of the optimality result to the Appendix. When the principal does not conduct partial monitoring, the dynamics of the agent's promised utility is given by (DW2), and when the principal monitors the agent, the dynamics of the agent's promised utility follows (DW3). Hence, similar to the previous section, the optimal contract has two regimes. There exists a monitoring threshold w_L^{2*}, such that: when $w \leq w_L^{2*}$, the principal monitors the agent and rewards him with β for every arrival; while, when $w > w_L^{2*}$, the principal does not monitor the agent and provides different rewards for different types of arrivals.

Similar to the case discussed in Section 3.3.2.1, the principal still needs to induce agent's effort when she conducts partial monitoring. The difference is: in this section, the incentive constraint to induce effort is always binding regardless of whether the agent is being monitored. As a result, unlike in the case, where $c \leq \mu p R_u$, in this case the agent's promised utility does not decrease slower when partial monitoring is utilized, so monitoring does not directly lead to a decrease in probability of firing the agent. However, the principal can still benefit from partial monitoring in the following way. In the case without monitoring, the principal has to reward the agent with a bigger jump in his promised utility for every arrival from the unobservable channel to ensure that the agent is incentivized to tell the truth about arrivals; however, partial monitoring automatically reveals arrivals from the unobservable channel, therefore, the principal no longer has to provide an extra reward for arrivals from the unobservable channel, which can lead to higher profits to the principal.

We have finished deriving the optimal contracts with partial monitoring when $\mu R_u > c$. If $\mu R_u \leq c$, as we have described in Section 3.2.2, if the principal incentivizes the agent to exert effort, then this automatically guarantees truth-telling. Thus, the principal can provide the reward of β for both types of arrivals, and this way the agent will be incentivized to report the truth. Hence, the optimal contract is still described by γ_3^* (Definition 3.3) even when we allow the principal to monitor the unobservable arrivals.

3.4 Dynamic Discounting as an Incentive Mechanism

In the previous section, we considered how the principal could use dynamic monitoring along with payments to the agent and the threat of termination to incentivize the agent to tell the truth about the number of arrivals and to exert effort. In this section, we consider how the principal could achieve the same aims by dynamically adjusting the prices charged to customers arriving from the observable channel (i.e., the customers who reserve online and, therefore, are directly observable by the principal).

We consider a situation where the price for the service is fixed at $R_u = R$ but the principal could decide to provide a discount to customers reserving and paying for the service online by dynamically adjusting the online channel price to $R_o = (1 - d_t)R$, $d_t \in [0, 1]$. The "dynamic discount" is a unique tool in our setting because customers arrive at the store through two different channels, and customers arriving through one of the channels are unobservable by the principal. If the principal provides a discount to customers who choose to arrive via the observable channel, customers will be more willing to order their service through this channel, which will partially address the moral hazard problem caused by the cash diversion. Hence, when deciding on the dynamic discounting, the principal is trading off the revenue loss from the discount against the incentive payments to the agent to induce truthful reporting.

Recall that, up until this point, we assumed that a customer chooses the unobservable channel (walk-in) with probability p and chooses the observable channel (online) with probability $1 - p$. Since the principal is dynamically adjusting the price of service through the observable channel, the probability that a customer chooses the unobservable channel or the observable channel will be influenced by the prices. Hence, we endogenize the customer channel choice, by adding a separate, micro-founded, stylized, customer utility consideration to the baseline model. The rest of this section is organized as follows. We first introduce the customer utility model in Section 3.4.1. Then, we present the main properties of the optimal contract in Section 3.4.2. Finally, we present the formal definition of the optimal contract in Section 3.4.3.

3.4.1 Customer Utility

Let U_{iu}^t and U_{io}^t denote a customer i's utility, that she obtains when choosing to receive the service via the unobservable channel or the observable channel, respectively:

$$U_{iu}^t = U - R, U_{io}^t = U - (1 - d_t)R + \epsilon_i^t.$$

Here, U denotes the reservation utility of the service and ϵ_i^t is the relative utility difference that the customer i experiences when receiving service via the observable channel compared to the unobservable channel. The cumulative distribution function and probability distribution function of ϵ_i^t are assumed to be F_ϵ and f_ϵ. Further, we assume that the utility that the customer gets if she leaves without ordering is zero. In this Section, we focus on the case $U \geq R$, otherwise no customer will choose the unobservable channel. Therefore, the probability p_u that a customer chooses the *walk-in*, unobservable channel is a function of d_t:

$$p_u(d_t) = \mathbb{P}(U_{iu}^t \geq U_{io}^t, U_{iu} \geq 0) = \mathbb{P}(\epsilon_i^t \leq (1-d_t)R - R, U - R \geq 0) = F_\epsilon(-d_t R).$$

The probability p_o that an arriving customer chooses the *online*, observable channel is also a function of d_t:

$$p_o(d_t) = \mathbb{P}(U_{io}^t > U_{iu}^t, U_{io} \geq 0) = \mathbb{P}(\epsilon_i^t \geq (1-d_t)R - U, \epsilon_i^t > (1-d_t)R - R) = 1 - F_\epsilon(-d_t R).$$

We need to expand the definition of contracts to include the dynamic discounts, i.e., $\gamma = (L, \tau, D) \in \Gamma_D$.[20] The overall expression for the agent's utility remains the same as equation (3.1), while the payment L could be affected by the choice of discounts. Similarly, the expression of the principal's utility still follows equation (3.2), while $R_u = R$**,** $R_o = (1-d_t)R$ and the payment L could be affected by the choice of discounts.

With the new definition of the probability functions, we update the promise keeping constraint:

$$dW_t = rW_{t-}dt - c(1 - \mathbb{I}[\nu_t = \mu])dt - dL_t + H_t^u(d\hat{N}_t^u - \mu p_u(d_t)dt) + H_t^o(dN_t^o - \mu p_o(d_t)dt). \tag{PKd}$$

Note that the incentive compatibility constraint that induces truthful reporting of arrivals from the unobservable channel does not change when the discount d_t is applied. This is because it requires that $H_t^u \geq R$ and the discount only influences the price of purchasing the service through the observable channel. In contrast, the incentive compatibility constraint that induces full-effort does require updating to adjust for the new probability functions. Therefore, we provide an updated version of the incentive compatibility constraint for full-effort which is adapted from Lemma 3.3.

Lemma 3.8 *Full-effort is incentive compatible if and only if, for all* $t \leq \tau$*,*

$$p_u(d_t)H_t^u + p_o(d_t)H_t^o \geq \beta. \tag{ICeffortd}$$

We define the contract space, Γ_D^{IC}, to include the contracts where the agent reports truthfully and always exert effort. (Constraints (ICeffortd) and (IC-truthful) need to be always satisfied.) In the current section, we focus on the contracts in Γ_D^{IC}. However, in Section (3.4.3), we also show that

[20]Formally, L and $D = \{d_t\}_{t\geq 0}$ are adapted to $\hat{\mathcal{F}} = \{\hat{\mathcal{F}}_t, t \geq 0\}$, the filtration generated by $\hat{N}^u, N^o$.

the optimal contract in contract space Γ_D^{IC} is also optimal in the larger contract space Γ_D.

For analytical simplicity, we make the following standard assumption: $\epsilon_i^t \sim U([-a, b])$ where $a > 0, b > 0$.[21] We, now, can derive the following claim on the choice of the discount.

Claim 3.1 $d_t^* \in [0, \bar{d}]$, *where* $\bar{d} = \min\left(\frac{a}{R}, 1\right)$.

The above claim shows that the optimal discount to the observable channel lies within the following interval $[0, \bar{d}]$. This is because, the upper bound of the interval is the minimal value of the discount ($d_t = \frac{a}{R}$) that could place all customers to the observable (online) channel, and, thus, an even bigger discount would only lead to revenue loss and no appreciable benefit to the principal. As a result of Claim 3.1, we have

$$p_u(d_t) = \frac{-d_t R + a}{a+b}, p_o(d_t) = \frac{b + d_t R}{a+b}. \tag{3.19}$$

3.4.2 Properties of the Optimal Contract

In this section, we first present the main properties (Properties 1-5) of the optimal contracts based on the Hamiltonian-Jacobi-Bellman (HJB) equation. Then we prove that the contracts following these properties are indeed optimal.

We define the system's value function as the sum of the principal's utility and the agent's promised utility: $V(w) := F(w) + w$. Then, $V(w)$ should satisfy the following HJB equation:

$$\begin{aligned} rV(w) = -c + rwV'(w) + \mu \max_{H_u, H_o, d} \Big\{ & p_u(d)\left[R + V(w + H_u) - V(w) - H_u V'(w)\right] \\ & + p_o(d)\left[(1-d)R + V(w + H_o) - V(w) - H_o V'(w)\right] \Big\}, \end{aligned} \tag{3.20}$$

subject to

$$H_u \geq R; p_u(d)H_u + p_o(d)H_o \geq \beta,$$

where $p_u(d)$ and $p_o(d)$ are defined in (3.19). Hence, given the choice of the discount d and following the results of Section 3.2, we obtain the optimal values of H_u and H_o in the HJB equation next.

Property 1.

$$H_u = R, H_o(d) = \begin{cases} 0 & \text{if} \quad \mu p_u(d)R \geq c, \\ \dfrac{c - \mu p_u(d)R}{\mu p_o(d)} = \dfrac{\beta - p_u(d)R}{p_o(d)} & \text{if} \quad \mu p_u(d)R < c \leq \mu R. \end{cases} \tag{3.21}$$

[21] We numerically test that the main results in this section do not change when ϵ_i^t follows other distributions (for example, Normal, Logistic, U-quadratic, and Triangular).

Next, we derive the upper bound of the agent's promised utility in the optimal contract. We first define the upper bound of the system's value by choosing the value of a discount d:

$$\bar{V}_d = \max_{d \in [0, \bar{d}]} \frac{\mu[p_u(d)R + p_o(d)(1-d)R] - c}{r}, \tag{3.22}$$

where $p_u(d)$ and $p_o(d)$ are defined in (3.19). Since the objective function is quadratic in d and, hence, concave in d, the optimization problem is easy to solve, and the solution is $d^* = 0$. Therefore, $\bar{V}_d = \dfrac{\mu R - c}{r}$. We further define

$$\bar{w} := \max\left(\frac{\mu R}{r} \cdot \frac{a}{a+b}, \frac{\mu \beta}{r}\right), \tag{3.23}$$

which will be shown to be the upper bound of the agent's promised utility in the optimal contract. The above confirms that, once the agent reaches the upper bound of the promised utility, discounting is no longer needed ($d^* = 0$). Later, with Property 5, we will further emphasize that the optimal discount is decreasing in the agent's promised utility w.

Property 2. $V(w) = \bar{V}_d$ for $w \geq \bar{w}$, where $\bar{w}$ follows (3.23).

Property 2 shows that when the agent's promised utility reaches a certain threshold $\bar{w}$, the system's value function achieves the upper bound $\bar{V}_d$. This implies that the agent gets tenure once his promised future utility is equal to or above the "tenure threshold" ($w \geq \bar{w}$), and discount is zero.

We have determined the values for H_u, H_o, and the upper bound of the agent's utility in the optimal contract. The only decision variable left is the discount d. Next, we define

$$\begin{aligned} G(d,w) :=& p_u(d)\left[R + V(w+R) - V(w) - RV'(w)\right], \\ &+ p_o(d)\left[(1-d)R + V(w + H_o(d)) - V(w) - H_o(d)V'(w)\right]. \end{aligned} \tag{3.24}$$

Lemma 3.9 $G(d,w)$ *is concave in* d *for any* $w \geq 0$.

Lemma 3.9 implies that we can choose d using Karush-Kuhn-Tucker (KKT) conditions.

Property 3. The optimal discount $d^*(w)$ follows the optimality condition:

$$\begin{aligned} &\frac{\partial G(d,w)}{\partial d}\Big|_{d=d^*} - z + y = 0; \\ &zd^* = 0, z \geq 0; \quad y\left(d^* - \bar{d}\right) = 0, y \geq 0. \end{aligned}$$

When d increases, the net price charged to customer ordering through the observable channel decreases, and more customers choose the observable channel, which further makes the cash diversion moral hazard problem less significant. Therefore, the greater the discount d is, the less the principal has to pay to the agent to incentivize him to report customer arrivals truthfully. Hence, by adjusting the dynamic discounts in an optimal manner (d^*), and, hence, changing the distribution of customers across the two channels, the principal optimally trades-off the revenues collected for the services against the payments made to the agent.

Furthermore, following Property 2, we learn that $d^* = 0$, for $w \geq \bar{w}$. Before we look into the characterization of $d^*(w)$, we first prove a technical result that the system's value function is strictly concave in the agent's promised utility w, which implies that, under the optimal contract, the principal would not randomize over the agent's promised utility.[22]

Property 4. V defined in (3.20) is strictly concave in w on $[0, \bar{w})$.

Finally, we prove that the optimal discount $d^*(w)$ is decreasing in w (Property 5), that is, as the agent's promised utility increases, the principal sets a higher price for ordering through the observable channel. This means that when the agent's promised utility increases, principal sees less of a need to divert customers from the unobservable channel to the observable channel. This echoes our findings of Section 3.3.2, where the principal can monitor the agent: she will only do so when the agent's promised utility is sufficiently low.

Property 5. $d^*(w)$ is decreasing in w.

Based on the Properties 1-5 established above, the system's value function can be obtained by solving the following differential equation:

$$\begin{aligned} rV(w) = -c + rwV'(w) + \mu \{ p_u(d^*(w)) \left[R + V(w+R) - V(w) - RV'(w) \right] \\ + p_o(d) \left[(1-d^*)R + V(w + H_o(d^*(w))) - V(w) - H_o(d^*)V'(w) \right] \Big\}, \end{aligned} \tag{3.25}$$

with the boundary condition $V(0) = 0$ since the termination threshold equals 0 when the contract is terminated. The principal's value function can be expressed as $F(w) := V(w) - w$. Further, define w_d^* as the maximizer of $F(w)$. We, now, can formally define the optimal contract that the principal adopts when the channel manipulation (via the observed channel's price discounting) is allowed.

3.4.3 Optimal Contract under Channel Manipulation

Definition 3.6 *For any $w \in [0, \bar{w}]$, define a contract $\gamma_d^*(w) = (L^*, \tau^*, D^*)$ as follows:*

1. *Set $W_{0-} = w$ and $L_0^* = (W_{0-} - \bar{w})^+$.*

2. *For $t \geq 0$, set $d_t^* = d^*(W_{t-})$, where function $d^*(w)$ is defined by Property 3. $dL_t^* =$*

[22] If the optimal value function is linear in the promised utility w, the principal implements the optimal contract by randomizing between two levels of the promised utility. Hence, when the value function is strictly concave, randomization is not needed.

$(W_{t-} + R - \bar{w})^+ dN_t^u + (W_{t-} + H_o(d_t^*) - \bar{w})^+ dN_t^o$, *such that*

$$dW_t = rW_{t-}dt - dL_t^* + R(dN_t^u - \mu p_u(d_t^*)dt) + H_o(d_t^*)(dN_t^u - \mu p_o(d_t^*)dt),$$

where H_o is defined in (3.21)*, and functions p_u and p_o are defined in* (3.19)*.*

3. *The termination time is $\tau^* = \min\{t : W_t = 0\}$.*

Similar to the previous sections, to implement the contract, the principal keeps track of the agent's promised utility while the payment and the termination at time t are contingent on the agent's promised utility at time t_-. What is new here is that the dynamic discount at time t is also a function of the agent's promised utility at time t_- (i.e., $d^*(W_{t_-})$), and it further affects the dynamics of the promised utility through dW_t.

Finally, in the next theorem, we prove that $\gamma_d^*(w_d^*)$ is not only the optimal contract in the class of contracts which induce full-effort and truth-telling Γ_D^{IC}, but also the optimal contract in the class of contracts that allow the agent to shirk or not truthfully report Γ_D.

Theorem 3.3 *$\gamma_d^*(w_d^*)$ in definition 3.6 is the optimal contract in the contract space Γ_D.*

3.5 Welfare Analysis

It is clear that the principal is better off when she uses the monitoring or dynamic discounting tools along with the dynamic contracts levers (payments and termination) to incentivize the agent, however, how the agent's utility changes as these tools are used is not as obvious. In this section, we are interested in how the agent's utility (w^*) changes when the principal adds the monitoring tool or the dynamic discounting tool. Recall that the agent's utility represents the agent's total utility under the optimal contract which is chosen by the principal at the beginning of the contract, while the principal's utility is a function of the agent's utility. Further, the system's utility is the sum of the agent's utility and the principal's utility. First, we look at the case in which the principal uses the monitoring tool to reveal both types of information (unobservable arrivals and agent's effort).

Lemma 3.10 *$w_{mi}^* \leq w_i^*$ for $i = 1, 2, 3$.*

Lemma 3.10 shows that, unsurprisingly, the agent is always worse off when the principal conducts full monitoring (that is, when the monitoring can reveal both types of private information). In the baseline model (i.e., absent of monitoring), the positive utility, w_i^*, that the agent receives can be understood as the information rent since the agent has two types of private information. The monitoring tool has two effects on the system and the agent: first, it increases the system's total utility (i.e. the system's value function with monitoring is larger than the system's value function without monitoring) by reducing the probability of termination. Furthermore, in the case when monitoring reveals both types of private information, the contract is never terminated. Recall that termination hurts the agent as well as the principal, therefore, in the baseline model, the principal is motivated to start the agent with a baseline utility that is high enough that the agent will not be terminated with a high enough probability. However, once monitoring is used, the principal can monitor both truth-telling and effort and can avoid termination, therefore, the second effect of monitoring is that the principal starts the agent at a lower promised utility. Thus, while monitoring eliminates the chance that the agent will be terminated, it also causes the agent's expected utility to decrease. In fact, in the extreme, trivial case where monitoring cost is zero, the principal would monitor at all times and the agent's expected utility would, therefore, go down to zero.

The situation is different, however, if the principal can only conduct partial monitoring (by, monitoring arrivals). Once again, the monitoring tool increases the system's total utility (i.e. the system's value function with monitoring is larger than the system's value function without monitoring) by reducing the probability of termination. Since monitoring makes the contract more profitable, the principal may have the incentive to increase the agent's expected duration of employment in the contract compared with the baseline contract. However, it is still possible that the agent will be terminated in the contract with monitoring. Hence, the principal may have motivation to further increase the agent's starting promised utility to increase the agent's expected duration of employment (i.e., to reduce the probability of termination). In the following numerical example, we show a situation where the agent's utility increases when the principal conducts partial monitoring which only reveals unobservable arrivals.

Example 3.1 $c = 1.6, \mu = 1, p = 0.9, R_u = R_o = R = 2, r = 0.1, m = 0.05$, $w_m^* = 1.2$, *and* $w_1^* = 0.96$.

In Example 1, when partial monitoring is added, the agent's utility increases from 0.96 to 1.2. In this example, the probability of termination in the baseline contract is 0.82 which is quite large. Hence, the principal has the motivation to decrease this probability by increasing the agent's utility. As a result, the agent's utility increases and the probability of termination in the contract with monitoring goes down to 0.75.

However, if we fix other parameters and only change the service fee as follows: $R_u = R_o = R = 5$, the agent becomes worse off ($w_m^* = 5.4 < w_1^* = 9.12$) with monitoring compared to the baseline case. In this case, R is large, and the probability of termination in the baseline contract decreases to 0.26 which is already small. Hence, the principal does not have the motivation to further decrease this probability by increasing the agent's utility. In addition, in the contract with monitoring, although we reduce the agent's utility, the termination probability is still only 0.1. The next lemma confirms our numerical findings: if the service fee R is large enough, the agent will be worse off when the principal conducts partial monitoring.

Lemma 3.11 *If $m = 0$, there exists $\hat{R}$ such that if $R_u = R_o > \hat{R}$, then $w_m^* \leq w_1^*$.*

Our previous analysis indicates that the partial monitoring tool is more likely to hurt the agent as R increases. We tested this conjecture by conducting a numerical study. In the study, we took the model parameters as follows: $r \in \{0.5, 1, 1.5, 2\}$, $R_u = R_o = R \in \{1, 2.5, 4, 5.5, 7\}$, $m \in \{0.2, 0.4, 0.6, 0.8\}$, $\mu \in \{0.5, 1, 1.5, 2\}$, $c \in \{0.8, 1.2, 1.6, 2, 2.4\}$. In the numerical study, when we fixed other parameters and let R increase, we found that there exists a threshold where the monitoring tool hurts the agent if and only if R is above that threshold.

Finally, we analyzed the case when the principal uses the dynamic discounting tool. We first discuss how the dynamic discounting tool may affect the agent's utility. Similar to the case with monitoring, the dynamic discounting tool increases the system's utility by decreasing the probability of termination. The dynamic discounting tool alleviates the moral hazard problem by guiding customers from the unobservable channel to the observable channel which makes the promised utility of the agent decrease slower. Since the dynamic discounting makes the contract more profitable, the principal may have the incentive to decrease the probability of termination in such contract compared with the baseline contract. However, it is still possible for the agent to be terminated in the contract with dynamic discounting. Hence, the principal may have motivation to further increase the agent's utility to increase the agent's expected duration of employment (i.e., to reduce the probability of termination). In the following numerical example, we show a situation where the agent's utility increases when the principal uses the dynamic discounting tool.

Example 3.2 $c = 0.9, \mu = 1, a = 0.8, b = 0.2, p = 0.8, R_u = R_o = R = 1.6, r = 0.7, U = 2$, $w_d^* = 0.35$*, and* $w_1^* = 0.25$.

In Example 2, when the dynamic discounting tool is used, the agent's utility increases to 0.22 from 0, the latter is the utility when no dynamic discounting is undertaken. In this example, the principal has the motivation to reduce the probability of termination by increasing the agent's utility while also employing dynamic discounting. However, if we fix other parameters in Example 2 and only change the revenue such that $R_u = R_o = R = 3$, then dynamic discounting makes the agent worse off, i.e. $w_d^* = 1$, $w_1^* = 1.46$. In this case, R is large, the probability of termination in the baseline contract decreases to 0.54 which is already relatively small. Hence, the principal does not have the motivation to further decrease this probability by increasing the agent's utility while using dynamic discounting.

We conduct a numerical study with the same set of the parameters as we did in the case of partial monitoring. We find that, as R increases, the dynamic discounting tool is more likely to hurt the agent. The explanation is similar to the case of partial monitoring: as R increases, the probability of termination in the baseline contract becomes small, and the principal's motivation to further reduce the probability of termination is weakened, hence, she chooses to reduce the agent's utility by discounting more.

CHAPTER 4

Dynamic Moral Hazard with Adverse Selection

4.1 Introduction

Many business environments involve situations where an agent is supposed to exert effort and obtain results over time for a principle. For example, a firm's R&D department (principal) funds researchers (agent) for extended durations of time, hoping to generate "breakthrough results."[1] Similarly, many companies expect their employees in charge of business development activities to acquire new target customers. In politics, many firms hire lobbying agencies in the hope of influencing politicians to pass legislation benefiting the firm. In all these situations, the agent's activities are hard to observe, leading to a dynamic moral hazard problem. Furthermore, the agent's capabilities to achieve results, reflected in its operating cost, may be only known by the agent, and not observable to the principal. For example, firms may have a hard time estimating how much expenditure it will take the researcher team to achieve breakthroughs, or how much lobbying expenditure it will take a lobbyist to achieve legislative results. An agent may either claim a higher expenditure than necessary, or, if under-funded, may choose not to exert effort. This therefore gives rise to a principal-agent problem with both dynamic hazard and adverse selection.

In particular, we consider a risk-neutral setting in which a principal hires an agent to create positive results (new businesses, research breakthroughs, favorable legislation, etc.). When the agent exerts effort, positive results arrive according to a Poisson process with a given rate. This instantaneous arrival rate is a positive constant if and only if the agent exerts costly effort. If the agent shirks, the arrival rate drops to zero. A distinct feature of our model, compared to previous literature, is that the agent's cost rate (of exerting effort) is private information, which represents heterogeneity in the cost/outcome ratio of agents. Thus, an agent with a lower cost structure is able to obtain more positive results for the same effort than an agent with a higher cost structure. However, the principal does not have a priori knowledge of an agent's cost structure and could be stuck with either a high cost/incapable agent or an agent that is capable but does not exert the right effort if the principal does not provide the right incentives.

[1] The material presented in this chapter is based on the paper [TZSD21] co-authored with Feifan Zhang, Peng Sun and Izak Duenyas.

We formulate the problem as a continuous-time dynamic moral hazard problem with adverse selection. The continuous-time dynamic moral hazard literature originates from [San08], which considers a principal hiring an agent to affect the cash flow of a project by controlling the drift of a Brownian motion. [San08] develops a continuation utility formulation (similar to the "promised utility" framework in [SS87]) to transform the contract design problem into a dynamic programming problem in which the agent's continuation utility is the state variable. This formulation has been further applied to the Poisson model by [BMRV10]. However, their Poisson arrival is "bad news" that causes losses to the principal, rather than "good news" that is beneficial to the principal in our model. Some recent papers also consider "good" Poisson arrivals ([ST18],[GT16],[Sha17a]). However, the operating cost of the agent in these papers is known by the principal.

Following standard results in mechanism design in [LM09], the principal should provide a menu of contracts, such that an agent with a specific cost chooses a particular contract from this menu. Following the revelation principle [Mye81], it is without loss of generality for us to consider direct mechanisms. In traditional adverse selection models, such mechanisms only involve allocation and payment decisions that depend on the agent's type. In our setting, however, after the agent reports the operating cost, the two players still face a dynamic moral hazard game. That is, payments and allocation (and contract termination) should also depend on the agent's dynamic performance, which is stochastic in nature. Therefore, the principal needs to optimize over menus of dynamic incentive compatible contracts that motivate agents with different operating costs to continuously exert effort before contract termination. Furthermore, one type of agent should not have an incentive to choose a dynamic contract for another type. Consequently, the optimal design problem can no longer be formulated as a classic dynamic program. In this paper, we contribute by providing a solution approach based on deterministic continuous time optimal control.

There have been previous attempts on the problem with both dynamic moral hazard and adverse selection. [Ma91] focuses on renegotiation and actions with long-term effects, whereas we give the principal full commitment power on contracting, and hence the issue of renegotiation does not exist. [May20] and [RCW21] both consider dynamic moral hazard problems with adverse selection where an agent is hired to exert effort to reach a single breakthrough. Hence, their model only considers a single arrival while we consider an infinite horizon Poisson process. Furthermore, the adverse selection in their model comes from the information about the arrival (timing of the arrival or the status of the arrival) but not a characteristic of the agent (capability of the agent in our model). Similarly, [CCW$^+$18] considers an infinite horizon Poisson model where the adverse selection also comes from the feature of the arrivals. [CWY13] and [SPZ20] study continuous-time moral hazard problems in infinite horizon with adverse selection under Brownian and Poisson stochasticity, respectively. To solve the adverse selection problem, they adopt the methodology of a credible set regarding the agents' continuation and temptation values. Rather than resorting to their method, which involves stochastic differential equations with variational inequalities, we formulate our optimization problem with a deterministic optimal control approach. Our formulation enables us to provide closed-form solutions of optimal menu of contracts with intuitive implementations, such as *probation contract* and *sign-on-bonus* contract. Another distinct feature of our paper from previous papers is that we can tackle the continuous type problem by taking advantage of our deterministic optimal control approach, while they only consider the two type problem.

There are recent papers in OR/OM considering both moral hazard and adverse selection. [CGDJ20] consider a principal who periodically provide a non-monetary reward to agents who have private information to incentivize the agents to invest effort over the long-run. Our paper differs from theirs in two aspects. First, the principal's objective in their model is a long-run average payoff while the principal maximizes the expected discounted revenue in our model. Second, they focus on finding near-optimal policies while we focus on characterizing the structure of the optimal contracts. [ZTH19] considers a delegated search model where the agent's search effort and the result of the search process are private information. Hence, the adverse selection in their model comes from the search result but not the agent's capability/cost in our paper, and they consider a finite-horizon setting. Furthermore, none of these papers adopt the continuation utility formulation.

Another related strand of literature combines dynamic moral hazard with learning. Unlike our private information setting where the principal can elicit truthful information, under their setting with learning, the uncertainty is unobservable to either party, and hence the contract has to update the belief of the state and adjust contracts accordingly, such as [Bha12], and [Kwo11]. [HKL16] considers long-term contracting that involves adverse selection, moral hazard, and learning. Our model differs from theirs in three aspects. First, our agents are protected by limited liability, which greatly impacts the contract structure; second, there is no experimentation in our model; third, our paper is built in the continuous-time setting instead of their discrete-time setting, which facilitates our deterministic optimal control approach.

In our model, the principal offers the agent a menu of contracts, with each item in the menu designed specifically for an agent with a particular operating cost. In each contract, the principal keeps track of the agent's performance score, which is the continuation utility as in [San08]. The performance score takes an upward jump once a Poisson arrival occurs and keeps decreasing between arrivals. There are two absorbing states, a lower threshold of zero at which the contract terminates, and an upper threshold at which the agent will never be terminated, is rewarded with a monetary payment for each future arrival. If the agent is of two possible types, with the cost being high or low, when the high-cost agent is too costly, he is asked to leave the system to avoid inefficiency. In order to induce truthful revelation of high cost, we should adopt a *pay-to-leave* contract: an immediate payment that is equal to what the agent can get from mimicking the low-cost agent. This immediate payment is the information rent rewarded to the agent. If the high-cost agent is less costly, he can possibly be hired. Yet it could be that even though he is hired, the agent would still prefer to mimic the low-cost agent. Hence, the principal should use a *sign-on-bonus* contract which provides the agent an immediate payment to induce truth telling. The contract to the low-cost agent only reimburses the low operating cost. This implies that the high-cost agent cannot afford to exert effort, and therefore is unable to generate any arrival by mimicking the low-cost one. Consequently, the first arrival is the particular evidence that differentiates the low-cost agent from the high-cost one who attempts to mimic without working. Therefore, the principal gives the low-cost agent a *probation contract*, which starts with a probation period for the low-cost agent to prove his type. We believe our model is especially appropriate in settings like new business generation, R&D or legislative lobbying where 'arrivals' are likely to be rare but significant.

If an arrival occurs during the probation period, the identity of the low-cost agent has been confirmed. Hence, to design the menu of contracts, we can focus on designing the length of the probation period and the magnitude of the sign-on bonus. This enables us to transform the original contract design problem into a deterministic optimal control problem. If there is a continuum of possible cost levels, we formulate an easy-to-compute upper bound optimization problem to the original problem thanks to the deterministic optimal control formulation. This optimization problem further provides a way for us to design a menu of contracts. Furthermore, we show that if the solution in the upper bound calculation satisfies a simple condition, then the upper and lower bounds match, which implies that our contract design is in fact optimal. Our numerical study illustrates that the condition is often satisfied with commonly used distributions. In this case, the principal designs a menu with a continuum of different items, each of which has the form of *sign-on-bonus contract* or *probation contract*.

Our paper is one of the first papers that combine the continuous-time moral hazard problem and the adverse selection problem. Luckily, we can solve the two-type problem and show that the optimal contracts take a simple and intuitive structure. Furthermore, ours is the first paper that can tackle the continuous-type problem. Although the space of dynamic contracts is enormous, the optimal contracts only take two intuitive and easy-to-implement forms both in two-type and continuous-type problems.

The rest of the paper is organized as follows. We introduce the model at the beginning of Section 2.2. In Section 4.3, we present three contracts that are candidates for optimal contracts when the agent type is unknown by the principal. In Section 4.4, we solve the optimal contracts for the two-type case. In Section 4.5, we consider the contract design problem for the continuous-type case.

4.2 Model

A principal contracts an agent to increase the arrival rate of a Poisson process over an infinite time horizon. At any point of time t, the agent can privately choose to either work or shirk. Whenever working, the agent incurs a constant flow of cost, and generates Poisson arrivals with an instantaneous rate μ. Shirking costs the agent nothing, and also generates no arrivals. Each arrival yields a revenue R to the principal, and is observable to both the principal and the agent. Therefore, we denote $\nu_t \in \{0, \mu\}$ to represent the agent's effort level at time t, such that $\nu_t = 0$ represents shirking and $\nu_t = \mu$ working. Further denote a right-continuous counting process $N = \{N_t\}_{t\geq 0}$ to represent the total number of arrivals up to time t, which generates a filtration $\mathcal{F}^N = \{\mathcal{F}_t^N\}_{t\geq 0}$. Therefore, the instantaneous arrival rate of the counting process at time t is ν_t, and the left-continuous effort process $\nu = \{\nu_t\}_{t\geq 0}$ is $\mathcal{F}^{\mathcal{N}}$-predictable.

The agent's capability, reflected by his operating cost per unit of time, is uncertain to the principal a priori. That is, a more capable agent can generate arrivals with a lower operating cost. In this paper, we use "capability" and "cost" interchangably. We assume that the operating cost is the agent's private information, and stays the same throughout the time horizon. The common prior distribution of the operating cost has a support $\mathcal{C}$. In this paper we consider $\mathcal{C}$ to be either a binary set $\{g, b\}$, or a continuous interval. We also refer to the operating cost $c \in \mathcal{C}$ as the agent's *type*. We assume that the principal needs to cover the operating cost because the agent has limited liability and is cash constrained, a standard assumption in the dynamic contracting literature. In particular, at any point in time, in order for the agent to exert effort, the principal needs to reimburse for the agent's reported operating cost c. (This situation is pretty common in the contexts such as R&D and lobbying where the principals have to provide a continuous flow of payments to the agents to let them operate. It may take the form of retainers in the case of lobbyists or a fixed amount of repetitive payments in the case of contract R&D.)[2]

Therefore, if the agent's type is c but pretends to be of a better (lower-cost) type $c' < c$, and the principal only pays operating cost c', then this agent is not able to generate any arrivals. If the agent shirks, a fraction $\rho \in (0, 1]$ of the operating cost payment by the principal can be diverted as a shirking benefit to the agent. For ease of exposition, we assume that $\rho = 1$, such that all the operating cost can be converted to the agent's shirking benefit. Following [GT16], we assume

[2] The charging of retainers by lobbyists is common, see for example, https://lobbyit.com/pricing/, https://arnoldpublicaffairs.com/faq/ and https://lobbying101.wordpress.com/about-lobbyists/how-much-do-they-charge/. Furthermore, it is common that R&D projects are funded for long durations of time and may not bring any results at the end. For example, 50% of registered clinical trials are never published in full and at least 50% of published reports are not sufficiently clear, complete, or accurate for others to interpret, use, or replicate the research correctly, see https://blogs.bmj.com/bmj/2016/01/14/paul-glasziou-and-iain-chalmers-is-85-of-health-research-really-wasted/ and https://www.fiercebiotech.com/special-report/2019-s-top-15-clinical-trial-flops-and-a-dishonorable-mention. Other famous failure examples include VCF developed by FBI ([Wik21]) and South Carolina's nuclear power plant construction project. ([Wik21]).

whenever the agent's operating cost is c, the principal has to pay the agent at least a flow of c at any point in time in order for the enterprise to continue operating. That is, whenever the payment flow from the principal drops below c, the operations stop completely and there will be no more future arrivals. This further implies that a high-cost agent who mimicks a low cost agent can enjoy the low operating cost as a shirking benefit until contract termination.[3] Because the agent knows the operating cost in the beginning of the time horizon, following the *Revelation Principle*, it is without loss of generality to consider direct mechanisms (see, for example, [Mye86],[PST14]). In our context, the principal designs a menu of contracts $\Gamma_{\mathcal{C}} = \{\gamma^c\}_{c\in\mathcal{C}}$, such that type c agent chooses contract γ^c. Any contract $\gamma^c = (L^c, \tau^c)$ includes an $\mathcal{F}$-predictable payment process L^c, and a $\mathcal{F}$-random time τ^c representing contract termination. When stressing the operating cost c is not necessary, we also use notation $\gamma = (L, \tau)$ without superscripts to represent a generic contract. As for the contract termination time τ, if $\tau = \infty$, the contract continues throughout the infinite time horizon.

Specifically, for the payment process $L = \{L_t\}_{t\geq 0}$, at each time epoch $t \geq 0$, L_t represents the cumulative payment from the principal to the agent up to time t. For simplicity of expressions, in the rest of the paper we consider $\mathrm{d}L_t = \ell_t \mathrm{d}t + I_t$, in which ℓ_t represents the flow, and I_t the instantaneous payment at time t. Limited liability of the agent and the assumption that the agent is cash constrained imply that payment is from the principal to the agent but not the other way around, or,

$$L_{t_1} \geq L_{t_2},\ \forall t_1 \leq t_2.$$

Furthermore, before contract termination, the payment needs to cover the operating cost, or,

$$L^c_{t_2} - L^c_{t_1+} \geq c(t_2 - t_1),\ \forall t \in (t_1, t_2],\ t_2 \leq \tau,$$

where we use notation $X_{t+} := \lim_{s\downarrow t} X_t$ to represent the right limit of any left-continuous process $\{X_t\}_{t\geq 0}$ at time t. Similarly, we define notation $X_{t-} := \lim_{s\uparrow t} X_t$. For simplicity of expressions, in the rest of the paper we consider $\mathrm{d}L_t = \ell_t \mathrm{d}t + I_t$, in which ℓ_t represents the flow, and I_t the instantaneous payment at time t. Therefore, the aforementioned constraints on payments can be summarized in the following *limited liability* (LL) constraint for all contract $\gamma^c = (L^c, \tau^c) \in \Gamma_{\mathcal{C}}$,

$$I_t \geq 0,\ \ \ell^c_t \geq c,\ \ \forall t \in [0, \tau] \text{ and } c \in \mathcal{C}. \tag{LL}$$

[3]Shirking and misuse of research funds are surprisingly common in R&D settings, see for example, https://www.chron.com/news/houston-texas/article/Prof-accused-of-spending-NASA-grants-on-cars-1722521.php, https://www.nbcnews.com/news/us-news/philadelphia-professor-accused-spending-185-000-grant-funds-strip-clubs-n1118571, https://www.newsweek.com/fund-meant-vaccine-research-misused-least-145m-unrelated-expenses-almost-decade-1564954, and https://www.theguardian.com/higher-education-network/2015/mar/27/research-grant-money-spent.

Both the principal and the agent discount future costs, and payments with a discount rate r. Without loss of generality, and for simplicity of expressions, we normalize time unit such that

$$\mu + r = 1. \tag{4.1}$$

In order to formally define direct mechanisms, we need to start with expressing the agent's utility.

Agent utility

Given a dynamic contract $\gamma = (L, \tau)$ and an effort process ν, the expected discounted utility of the agent with an operating cost c is

$$u(\gamma, \nu; c) = \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(\mathrm{d}L_t - c\mathbb{1}_{\nu_t=\mu}\mathrm{d}t)\right], \tag{4.2}$$

in which the expectation $\mathbb{E}^{\nu}$ is taken with respect to probabilities generated from the effort process ν.

Next, we describe the agent's cash constraint. The agent's resource to conduct the project is provided solely by the principal, and insufficient resources would render the agent unable to exert effort. Formally, any effort process of a type c agent facing a contract $\gamma = (L, \tau)$ needs to satisfy,

$$\nu_t = \mu, \text{ only if } \ell_s \geq c, \ \forall s \leq t. \tag{4.3}$$

Use $\mathcal{N}(\gamma, c)$ to denote the set of all $\mathcal{F}^{\mathcal{N}}$-predictable effort processes ν that satisfy condition (4.3) for a type c agent facing a contract γ. Further use $\mathfrak{N}(\gamma, c) \subseteq \mathcal{N}$ to denote the set of *best-response effort processes*, that is,

$$u\big(\gamma, \nu; c\big) \geq u(\gamma, \nu'; c), \ \forall \nu \in \mathfrak{N}(\gamma, c) \text{ and } \nu' \in \mathcal{N}. \tag{4.4}$$

We denote $\mathcal{F}_t^N$-predictable effort process $\nu^0 = \{\nu_t^0\}_{t\geq 0}$ to be the *always shirking process* such that $\nu_t^0 = 0$ almost surely for all t before contract termination. Similarly, we denote $\mathcal{F}_t^N$-predictable effort process $\bar{\nu} = \{\bar{\nu}_t\}_{t\geq 0}$ to be the *always exerting effort process* such that $\bar{\nu}_t = \mu$ almost surely for all t before contract termination. Define quantities

$$\beta_c := \frac{c}{\mu}, \text{ and } \bar{w}_c = \frac{\mu\beta_c}{r}. \tag{4.5}$$

A simple contract that induces the agent to always exert effort is to pay the agent a constant β_c for each arrival besides reimbursing the operating cost rate c. That is, we can expression such a simple contract as $\bar{\gamma}^c = (L^c, \tau^c)$ with $\mathrm{d}L_t^c = \beta^c \mathrm{d}N_t + c\mathrm{d}t$ and $\tau^c = \infty$. One can verify that the corresponding agent's utility is

$$u(\bar{\gamma}^c, \bar{\nu}, c) = \bar{w}_c.$$

Although this simple contract is not optimal, the quantities β_c and $\bar{w}_c$ are useful for describing the optimal contracts.

Furthermore, the revelation principal implies that we can focus on direct mechanisms. Therefore, we need the following *Truth-Telling* (TT) contraint on the menu $\Gamma_{\mathcal{C}}$, which ensures that an agent with operating cost c indeed chooses contract γ^c from the menu.

$$u(\gamma^c, \nu^c; c) \geq u\left(\gamma^{c'}, \nu; c\right),\ \forall c, c' \in \mathcal{C},\ \nu^c \in \mathcal{N}(\gamma^c, c), \nu \in \mathcal{N}(\gamma^{c'}, c). \tag{TT}$$

It is standard to consider the agent's continuation utility (also called *promised utility*) at time t, defined as (see, for example,[BMRV10]),

$$W_t(\gamma, \nu; c) = \mathbb{E}^\nu \left[\int_t^\tau e^{-r(s-t)} (\mathrm{d}L_s - c\mathbb{1}_{\nu_s=\mu}\mathrm{d}s) \middle| \mathcal{F}_t^N \right] \mathbb{1}_{t<\tau}. \tag{4.6}$$

In this literature it is standard to assume that the principal has the commitment power to a long term contract, while the agent does not need to commit to staying in the contract. That is, we need the following *Individual Rationality* (IR) constraint to guarantee participation before contract termination,

$$W_t(\gamma, \nu; c) \geq 0,\ \forall t \in [0, \tau],\ c \in \mathcal{C}. \tag{IR}$$

The following result depicts the dynamics of the process W_t, and provides an equivalent condition to the best response effort process.

Lemma 4.1 *For any contract γ, effort process ν, and operating cost c, there exists an $\mathcal{F}^{\mathcal{N}}$-adaptive process H_t such that*

$$\mathrm{d}W_t(\gamma, \nu; c) = \{[rW_{t-}(\gamma, \nu; c) - \nu_t H_t + c\mathbb{1}_{\nu_t=\mu}]\mathrm{d}t + H_t \mathrm{d}N_t - \mathrm{d}L_t\} \mathbb{1}_{0\leq t<\tau}. \tag{PK}$$

Furthermore, the following defined effort process is a best response to contract γ, or, $\{\nu_t\}_{t\in[0,\tau]} \in \mathfrak{N}(\gamma, c)$, in which

$$\nu_t = \begin{cases} \mu, & \textit{if } H_t \geq \beta_c, \\ 0, & \textit{o.w.} \end{cases} \tag{IC}$$

Lemma 4.1 implies that the principal can motivate a type c agent to exert effort if and only if each arrival yields an upward jump of at least β_c in the agent's promised utility. Later in the paper we show that in the optimal contract, the *incentive compatibility* (IC) constraint may not always be binding. That is, for certain operating cost c and time t, we need $H_t > \beta_c$, greater than the minimum amount necessary to induce effort.

Principal utility.

Denote $U(\gamma, \nu)$ to represent the principal's total expected discounted utility from a contract γ while the agent's type is c and uses an effort process $\nu \in \mathcal{N}(\gamma, c)$. That is,

$$U(\gamma, \nu) := \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}\left(R\mathrm{d}N_t - \mathrm{d}L_t\right)\right]. \tag{4.7}$$

Now we define $\mathcal{U}(\Gamma_{\mathcal{C}}) := \mathbb{E}\left[U(\gamma^c, \nu^c)\right]$ to represents the principal's total expected discounted utility from the menu of contracts $\Gamma_{\mathcal{C}}$ when the agent's effort process $\nu^c \in \mathfrak{N}(\gamma, c)$ satisfies (IC). The principal's contract design problem is

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) := \sup_{\Gamma_{\mathcal{C}}} \quad & \mathcal{U}(\Gamma_{\mathcal{C}}) \\ & \text{s.t. (LL), (PK), (IC), (IR), and (TT).} \end{aligned} \tag{4.8}$$

Note that the expectation in the objective function is taken with respect to the operating cost c, while constraints (LL), (PK), (IC) and (IR) are for all $c \in \mathcal{C}$. In contrast, the constraint (TT) is for all pairs of operating costs c and c', which implies that the maximization problem (4.8) cannot decouple in c. Finally, the objective function value $\mathcal{Z}(\mathcal{C})$ is the principal's optimal expected utility.

4.3 Implementable Contracts

In this section, we present all possible contract forms that will appear in an optimal menu of contracts before rigorously deriving them in the next section. Note that the space of the dynamic contracts could be enormous. Here we greatly narrow down the possibilities to two structures: *sign-on-bonus* contract (including *pay-to-leave* contract as a special case) and *probation* contract, all of which are mathematically tractable and possess managerial interpretations. In later sections, we will formally show how to derive the optimal contracts and verify that these three contract structures suffice.

4.3.1 Sign-on-bonus contract

First, we introduce the so-called *sign-on-bonus* contract. In particular, we allow the principal to pay a sign-on-bonus in the beginning.

Definition 4.1 *For any initial promised utility $w \geq 0$ and sign-on-bonus $B \geq 0$, define a sign-on-bonus contract $\gamma^c_{\mathsf{B}}(w, B) = (L^c, \tau^c_B)$, which pays the agent $dL_0 = B + \max\{w - \bar{w}_c, 0\}$ at time 0, and then generates a promised utility process W^c_t according to*

$$\mathrm{d}W^c_t = \left[r(W^c_{t-} - \bar{w}_c)\mathrm{d}t + \min\left\{\bar{w}_c - W^c_{t-}, \beta_c\right\}\mathrm{d}N_t\right]\mathbb{1}_{W_{t-}\geq 0} \tag{4.9}$$

following $W^c_0 = \min\{w, \bar{w}_c\}$. Furthermore, the payment process L^c_t follows

$$\mathrm{d}L^c_t = \left[c\mathrm{d}t + (W^c_{t-} + \beta_c - \bar{w}_c)^+\mathrm{d}N_t\right]\mathbb{1}_{W_{t-}\geq 0} \tag{4.10}$$

and the termination time τ^c_B is according to

$$\tau^c_B = \min\{t : W^c_{t-} = 0\}. \tag{4.11}$$

According to contract $\gamma^c_{\mathsf{B}}(w, B)$, the principal pays the agent a sign-on-bonus B at the beginning if the initial promosed utility w is below the upper bound $\bar{w}_c$. A special case $\gamma^c_{\mathsf{B}}(0, B)$ gives the agent a bonus B without asking the agent to work. Such a *pay-to-leave* contract may be useful if the agent's operating cost c is so high that it is not worth hiring the agent. The initial payment induces the agent to truthly reveal his type, as will be evident later in the paper.

As long as $w \leq \bar{w}_c$, the promised utility W^c_t starts from $W^c_0 = w$, and its dynamics (4.9) is consistent with (PK) with $H_t = \beta_c$. That is, the promised utility takes an upward jump of β_c upon each arrival, and gradually decreases at rate $r(\bar{w}_c - W^c_{t-})$ as long as $W^c_{t-} < \bar{w}_c$. An instantaneous payment occurs when the promised utility W^c_t jumps above $\bar{w}_c$. After that, the promised utility stays at $\bar{w}_c$ and the principal pays the agent β_c for each future arrival, in addition to the flow payment $c\mathrm{d}t$, which reimburses the operating cost. In this case the termination time τ^c is infinity. If W^c_t does not reach $\bar{w}_c$ but decreases to 0 instead, the contract is terminated. Therefore, contract $\gamma^c_{\mathsf{B}}(w, B)$ motivates the agent to always exert effort before contract termination by setting $H^c_t = \beta_c$ at all times in the (IC) constraint.

The sign-on-bonus generalizes the optimal contract structure for the case without adverse selection ([ST18]), by adding a sign-on-bonus B and allowing the initial promised utility w to be higher than $\bar{w}_c$. That is, the optimal contract for the known cost case is another special case of the sign-on-bonus contract. We call this special case, $\gamma^c_{\mathsf{B}}(w, 0)$ for some initial value w, a *standard* contract. In the next section we show exactly when a sign-on bonus contract structure is optimal, and the corresponding optimal w and B values.

4.3.2 Probation contract

The next contract structure is more intricate, and it is important to understand why it may arise. First, recall that it is necessary for the principal to pay a flow rate of at least c in order to induce effort from a type c agent. Consider, for simplicity of illustration, an agent whose true operating cost is c' can only cheat by mis-reporting type $c < c'$. Assume that the principal then pays the agent a flow cost of c with the intention of reimbursing the operating cost. In reality, the agent cannot afford to exert effort, but rather pretends to work while collecting the payment c over time. When there is no effort, there is no arrival. In order to mitigate cheating, the principal should not allow such a no-arrival state to last forever, but rather allow a finite *probation* period. If there is no arrival during this probation period, the agent will be terminated at the end of probation. On the other hand, an arrival during this period would reveal that the agent's true type is indeed c. In this case, uncertainty of the agent's type is resolved, and the principal can follow the contract structure $\gamma^c_{\mathsf{B}}(w, 0)$ of Definition 4.1 after the first arrival. In order to ensure effort during probation for an agent who has truthfully reported his type, the promised utility needs to take an upward jump of at least β_c, and possibly higher, at the first arrival.

We now specify and derive the dynamics of the probation contract. The agent is only paid $\mathrm{d}L_t = c\mathrm{d}t$ to cover the operating cost during the probation period. A key element of this probation contract is a threshold $z \geq 0$. If the first arrival occurs at time t with $W^c_t \geq z$, the promised utility jumps up by exactly β_c. If $W^c_t < z$, on the other hand, the promised utility jumps to $z + \beta_c$ upon an arrival at time t, which means the magnitude of the jump is higher than β_c. Therefore, when the promised utility W^c_t is below the threshold z, it evolves according to (PK) with $H_t = z + \beta_c - W^c_t$, that is,

$$\frac{\mathrm{d}W^c_t}{\mathrm{d}t} = rW^c_t - \mu(z + \beta_c - W^c_t) = W^c_t - \mu(z + \beta_c). \tag{4.12}$$

Consider a probation period with length τ, that is, $W^c_\tau = 0$. With this boundary condition, we have a closed-form solution to (4.12), which is

$$W^c_t = \mu(z + \beta_c)\left(1 - e^{t-\tau}\right). \tag{4.13}$$

Consequently, if $z < \bar{w}_c$ the time it takes for W^c_t to decrease from the threshold z to 0 is

$$\tau_z := \ln \frac{\mu(z + \beta_c)}{r(\bar{w}_c - z)}. \tag{4.14}$$

If $\tau \leq \tau_z$, then (4.13) fully specifies the dynamics of the promised utility before the first arrival or end of probation, and

$$W_0^c(\tau, z) = \mu(z + \beta_c)\left(1 - e^{-\tau}\right), \quad \text{for } \tau < \tau_z. \tag{4.15}$$

If $\tau > \tau_z$, on the other hand, then (4.13) only captures the dynamics from $t = \tau_z$ to $t = \tau$, when $W_t^c \leq z$. For the initial period of time from $t = 0$ to $t = \tau - \tau_z$, the promised utility $W_t^c > z$, and evolves according to (PK) with $H_t = \beta_c$, that is,

$$\frac{\mathrm{d}W_t^c}{\mathrm{d}t} = rW_t^c - \mu\beta_c = r(W_t^c - \bar{w}_c).$$

With boundary condition $W_{\tau-\tau_z}^c = z$, we once again have a closed form solution for $t \in [0, \tau - \tau_z]$,

$$W_t^c = \bar{w}_c - (\bar{w}_c - z)e^{r(t+\tau_z-\tau)}. \tag{4.16}$$

Overall, if $\tau > \tau_z$, at time 0, we have

$$W_0^c(\tau, z) = \bar{w}_c - (\bar{w}_c - z)e^{-r(\tau-\tau_z)}, \quad \text{for } \tau < \tau_z. \tag{4.17}$$

and, if $\tau \leq \tau_z$,

If $z \geq \bar{w}_c$, on the other hand, W_t^c evolves according to (4.13) from the very beginning. In this case the initial promised utility is still (4.15).

The probation period ends if either the first arrival occurs or W_t^c becomes zero, whichever happens first. Therefore, we further define the first arrival time as

$$\tau_1^N := \min\{t \mid \mathrm{d}N_t = 1\}.$$

With the above set up we present the following definition.

Definition 4.2 *For any probation time period $\tau \geq 0$ and threshold $z \geq 0$, we define a probation contract $\gamma_{\mathtt{P}}^c(\tau, z) = (L^c, \tau^c)$, which pays $\mathrm{d}L_t^c = c\mathrm{d}t$ and generates a promised utility process W_t^c that evolves according to the following rules for $t \in [0, \tau)$ if $\tau_1^N > \tau$.*

- *If $z < \bar{w}_c$, then W_t^c follows (4.16) for $t \in [0, \max\{\tau - \tau_z, 0\})$, and (4.13) for $t \in [\max\{\tau - \tau_z, 0\}, \tau]$, starting from (4.17) if $\tau > \tau_z$ and (4.15) if $\tau \leq \tau_z$.*
- *If $z \geq \bar{w}_c$, then W_t^c follows (4.13) for $t \in [0, \tau]$, starting from (4.15).*

If $\tau_1^N \leq \tau$, then the aforementioned dynamics lasts until $\tau^c = \tau_1^N$. After that point, and the contract continues with $\gamma_{\mathtt{B}}^c\left(\max\left\{W_{\tau_1^N}^c, z\right\} + \beta_c, 0\right)$ by resetting time τ_1^N to 0.

In the next section we demonstrate when this contract structure may be optimal, as well as how to specify the corresponding τ and z values. Here, we use a figure to better illustrate this contract structure and the associated dynamic.

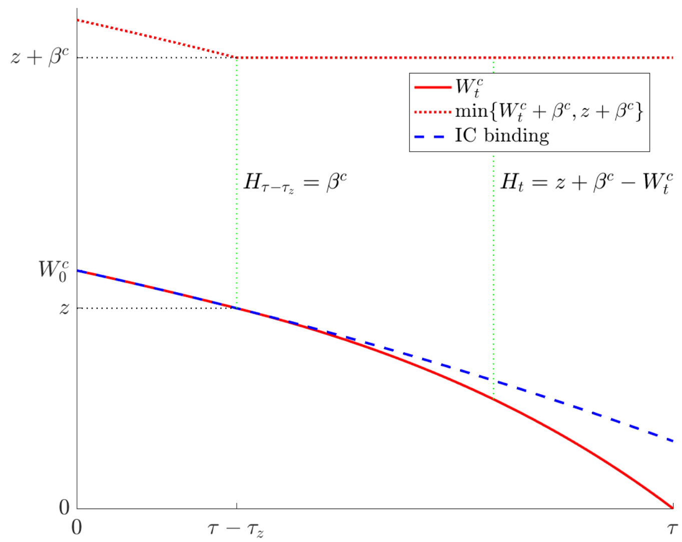


Figure 4.1: Sample trajectories of agent's promised utility before the first arrival.

Figure 4.1 gives an illustrative example of contract $\gamma_{\text{P}}^c(\tau, z)$ for the case that $\tau > \tau_z$. The agent's promised utility trajectory W_t^c starts from W_0^c. Over time, if no arrival has occurred, the agent's promised utility drifts down, following the solid curve. If the first arrival occurs before τ, the promised utility jumps up to the dotted curve $\min\{W_t^c + \beta_c, z + \beta_c\}$. Conceptually, the difference $H_t^c = \min\{\beta_c, z + \beta_c - W_t^c\}$ represents the scale of upward jump in the agent's promised utility upon the first arrival. It is fixed and equal to β_c before time $\tau - \tau_z$. After $\tau - \tau_z$, however, the jump $H_t^c = z + \beta_c - W_t^c > \beta_c$, and the (IC) constraint is not binding. After this first arrival, the contract $\gamma_{\text{P}}^c(\tau, z)$ sets H_t^c to β_c. Finally, the dashed curve, which overlaps with the solid curve when $t < \tau - \tau_z$, characterizes the movement of the promised utility following the dynamic (4.9) of the regular contract when H_t^c is kept at β_c. The figure implies that allowing the upward jump H_t^c to be higher than β_c effectively shrinks the probation period.

4.3.3 Principal and agent utilities

To conclude this section, we present results on the agent's and principal's utilities under the aforementioned contract structures. First, we formally establish the agent's utility under the sign-on-bonus contract and the probation contract.

Proposition 4.1 *(i) For any $w \geq 0$ and $B \geq 0$, we have $\bar{\nu} \in \mathfrak{N}(\gamma^c_{\mathtt{B}}(w, B), c)$, and*

$$u\Big(\gamma^c_{\mathtt{B}}(w, B), \bar{\nu}; c\Big) = w + B.$$

(ii) For any $\tau \geq 0$ and $z \geq 0$, we have $\bar{\nu} \in \mathfrak{N}(\gamma^c_{\mathtt{P}}(\tau, z), c)$, and

$$u\Big(\gamma^c_{\mathtt{P}}(\tau, z), \bar{\nu}; c\Big) = W^c_0(\tau, z),$$

in which $W^c_0(\tau, z)$ follows (4.17) *and* (4.15)*.*

We will formally show in the next section that after the type c becomes known, it is optimal for the principal to follow the standard contract $\gamma^c_{\mathtt{B}}(w, 0)$. In this case, Proposition 4.1 verifies that the agent's utility under the standard contract $\gamma^c_{\mathtt{B}}(w, 0)$ is indeed w; the principal's value function, $F_c(w)$, as a function of the promised utility w, is defined by the following differential equation,[4]

$$r(w - \bar{w}_c)F_c'(w) = (c - \mu R) + F_c(w) - \mu F_c\left(w + \beta_c\right), \; \forall w \in [0, \bar{w}_c], \tag{4.18}$$

$$\text{with boundary conditions } \; F_c(0) = 0 \; \text{ and } \; F_c(w) = \frac{\mu R - c}{r} - w \text{ for } w \geq \bar{w}_c. \tag{4.19}$$

The following proposition adopts some results from [ST18] into our setting.

Proposition 4.2 *If $R \geq \beta_c$, the differential equation* (4.18) *with boundary condition* (4.19) *has a unique solution, $F_c(w)$, which is strictly concave on $[0, \bar{w}_c)$ and $F_c'(w) \geq -1$; furthermore, we have $F_c(w) = U(\gamma^c_{\mathtt{B}}(w, 0), \bar{\nu})$. If $R < \beta_c$, on the other hand, define $F_c(w) = -w$. Overall, we have $\mathcal{Z}(\{c\}) = \max_{w \geq 0} F_c(w)$.*

Proposition 4.2 implies that the function $F_c(w)$ is indeed the principal's value under contract $\gamma^c_{\mathtt{B}}(w, 0)$, which is the optimal contract if the cost is known to be c. Furthermore, the maximizer of function $F_c(w)$ is the agent's utility under the optimal contract when the cost c is known to the principal.

In the next section, we conisder the simple case when there are only two types of agents. We will prove that it is sufficient to only consider the sign-on-bonus and probation contract structures to construct an optimal menu of contracts, and we will show how to determine the parameters in these optimal contracts.

[4] This differential equation is similar to Equation (12) in [ST18]. The main difference is due to differences in model set-ups. In [ST18] the principal can pay reimburse the effort cost at a later time, while in our setting the principal needs to reimburse the effort cost immediately.

4.4 Two-Type Case

It is natural to first consider the simple case when there are only two types $\mathcal{C} = \{g, b\}$ with $g < b$. We can completely solve this two-type case, and the insights and results we derived in this section will also be useful in Section 4.5 when we consider the continuous-type case. The prior probabilities of types g and b are p and $1 - p$, respectively. We refer to the agent with the lower cost g as the *good* agent, and b as the *bad* agent. In this section we require the following assumption.

Assumption 4.1

$$\beta_g \leq R, \textit{ or, equivalently, } \mu R \geq g.$$

This assumption guarantees that the good agent is efficient, which means that this agent generates a positive societal value whenever exerting effort. With this assumption, we exclude the trivial case where both agents are inefficient, because it is obviously dominant for the principal to offer a null contract with no payment and immediate termination in that trivial case. Note that we have not made any assumption on the bad agent's cost b yet. Indeed the bad agent can either be efficient ($\beta_b \leq R$) or inefficient ($\beta_b > R$), which leads to somewhat different optimal contract structures, as will be discussed later in this section.

Following the problem set up in the last section, the menu Γ offered by the principal contains two items, (γ^g, γ^b). The objective function of the principal's contract design problem (4.8) becomes

$$\mathcal{U}\left(\Gamma_{\{g,b\}}\right) = pU(\gamma^g, \nu^g) + (1-p)U(\gamma^b, \nu^b). \tag{4.20}$$

In the remainder of this section, we demonstrate the construction of the optimal menu of contracts in steps. First, we construct an optimization problem, which provides an upper bound for the optimization (4.8), in Subsection 4.4.1. Then, in Subsection 4.4.2, we construct a menu of contracts based on the optimal solution to the upper bound optimization problem, and show that this menu of contracts achieves the upper bound, and therefore is indeed the optimal menu. The first two subsections are focused on technical results constructing the optimal menu of contracts. Finally, in Subsection 4.4.3 we discuss additional economic insights of our optimal solution.

4.4.1 Upper bound optimization

In this subsection we present a new optimization problem, which provides an upper bound to the original contract design problem (4.8). In the following result, we use functions F_g and F_b as defined in (4.18)-(4.19) for $c \in \{g, b\}$.

Proposition 4.3 *The following optimization problem yields an upper bound to the optimal value of the contract design problem* (4.8)*. That is,* $\mathcal{Y} \geq \mathcal{Z}(\{g, b\})$*, where*

$$\mathcal{Y} := \max_{w_g, w_b, \tau, \xi} p \cdot G(w_g, \tau) + (1-p)\xi, \tag{4.21}$$

$$\textit{s.t. } w_g \geq w_b \geq g(1 - e^{-r\tau})/r, \tag{4.22}$$

$$\tau \geq 0, \tag{4.23}$$

$$\xi \leq F_b(w_b), \tag{4.24}$$

$$\xi \leq \frac{w_g - w_b}{b - g}(\mu R - b)^+ - w_b, \tag{4.25}$$

in which we define operator $(x)^+ := \max\{x, 0\}$*, and function* $G(w, \tau)$ *through the following optimal control problem, if* $\tau < \infty$*,*

$$\begin{aligned} G(w, \tau) := \max_{W_t, H_t} & \int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]\mathrm{d}t - g(1 - e^{-\tau}), \\ \textit{s.t. } & \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-} - \mu H_t, \textit{ for } t \in [0, \tau];\ W_0 = w,\ W_\tau = 0, \\ & H_t \geq \beta_g,\ \forall t \in [0, \tau]; \end{aligned} \tag{4.26}$$

if $\tau = \infty$*, with a slight abuse of notation, we define*

$$\begin{aligned} G(w, \tau) := \max_{W_t, H_t} & \int_0^\infty \mu e^{-t}[R + F_g(W_t + H_t)]\mathrm{d}t - g, \\ \textit{s.t. } & \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-} - \mu H_t, \textit{ for } t \geq 0;\ W_0 = w, \\ & W_t \geq 0, H_t \geq \beta_g,\ \forall t \geq 0. \end{aligned} \tag{4.27}$$

It is instructive to explain the terms in the optimization problem (4.21)-(4.25). First, the decision variables w_g and w_b represent the utilities of type g and b agent under their respective contracts. The decision variable τ is the duration of the probation period for the type g agent. And the variable ξ represents the principal's expected utility facing a type b agent. The constraint (4.22) states that the good agent's utility, w_g, needs to be as good as or better than the bad agent's w_b. Furthermore, the last inequality in (4.22) states that w_b needs to be no less than the total discounted expected operating cost that the agent would receive by pretending to be a good agent. This is because receiving the operating cost g without working yields a utility $\int_0^\tau ge^{-rt}\mathrm{d}t = g(1 - e^{-r\tau})/r$.

Next, the term $G(w_g, \tau)$ represents the principal's expected utility if the agent is the good type (with a low operating cost g). We explain what $G(w_g, \tau)$ means in the following remark.

Remark 4.1 *First, $G(w_g, \tau)$ calculates the principal's expected utility where the agent always exerts effort. The reason we can focus on full-effort contract is that any contract in which the agent shirks can be improved by a direct payment and no shirking. According to the optimal control problems* (4.26) *and* (4.27)*, the principal designs a contract with an initial promised utility w and a (probation) time period τ. The decision variables include the promised utility process W_t, and the upward jump H_t associated with a potential arrival if $t \leq \tau$. During this period of time, if an arrival occurs, with rate μ, then the principal receives a revenue R, and the promised utility jumps to $W_t + H_t$, at which point the principal follows the contract that keeps the IC constraint binding, i.e., $\gamma^g_{\mathtt{B}}(W_t + H_t, 0)$ and earns a future utility $F_g(W_t + H_t)$, following Lemma 4.2. Recall $\mu + r = 1$, which explains the term $e^{-t} = e^{-(\mu+r)t}$. The constraints further captures the* (PK), (IC) *and* (IR) *constraints. Finally, the second term of the objective function in* (4.26)*, $g(1 - e^{-\tau})$, captures the total discounted operating cost that the principal needs to pay before the first arrival or the end of the period, whichever comes first. Here, again, we use $\mu + r = 1$ as the effective discount rate.*

Finally, we focus on constraints (4.24) and (4.25). First, constraint (4.24) states that the principal's utility ξ is upper bounded by $F_b(w_b)$ when offering the type b agent a promised utility w_b, consistent Proposition 4.2. Finally, constraint (4.25) ensures a type g agent does not pretend to be of type b, which is elaborated in the following remark.

Remark 4.2 *Should the type g agent receive the type b contract, the agent is able to exert effort, and receive the same trajectory of payments as a type b agent. In addition to receiving the w_b reward, the type g agent also collects the extra operating cost $b - g$ for the duration of the contract. This (discounted) duration can be calculated as the (discounted) societal utility, $\xi + w_b$, divided by the societal utility rate, $\mu R - b$, if $\mu R > b$. This implies the following inequality,*

$$w_g \geq w_b + (b - g)\frac{\xi + w_b}{\mu R - b}, \text{ or, equivalently, } \xi + w_b \leq \frac{w_g - w_b}{b - g}(\mu R - b). \tag{4.28}$$

If $\mu R \leq b$, on the other hand, the societal value of hiring the agent is negative, and, therefore, $\xi + w_b \leq 0$. Constraint (4.25) *captures both cases of $\mu R > b$ and $\mu R \leq b$.*

So far we have provided intuitive interpretations of various components of the optimization problem (4.21)-(4.25). This optimization plays a central role in our contract design problem. In the next subsection we convert the non-convex optimization problem (4.21)-(4.25) into an equivalent convex optimization problem, and obtain a menu of contracts based on its optimal solution. We further show that the performance of such a menu of contracts indeed achieves the upper bound $\mathcal{Y}$ of $\mathcal{Z}(\{g, b\})$. Therefore, this menu of contracts is optimal. In our construction, each contract in the menu is either a probation contract or a sign-on-bonus contract defined in the previous section.

In order to solve the optimization (4.21)-(4.25), we first solve the deterministic optimal control problem (4.26). The solution approach is based on the Pontryagin minimum principle, as illustrated in the proof of the following Lemmas, presented in the Appendix.

Lemma 4.2 *For any $\tau \in [0, \infty)$, define thresholds*

$$\breve{\omega}(\tau) := \frac{1 - e^{-r\tau}}{r} g, \text{ and } \hat{\omega}(\tau) := \frac{1 - e^{-\tau}}{r + \mu e^{-\tau}} g.$$

(i) If $w \in [\breve{\omega}(\tau), \hat{\omega}(\tau))$, then there exists a unique value $\mathrm{z}(w, \tau) \in [0, \hat{\omega}(\tau))$ (also call it z for simplicity) such that

$$w = \bar{w}_g - (\bar{w}_g - \mathrm{z}) e^{r(\tau_{\mathrm{z}} - \tau)}, \tag{4.29}$$

where $\bar{w}_g$ and τ_{z} are defined in (4.5) and (4.14), respectively, with $c = g$ and $z = \mathrm{z}$. Furthermore, the following W_t and H_t solves the optimization $G(w, \tau)$ in (4.26),

$$W_t = \begin{cases} \bar{w}_g - (\bar{w}_g - \mathrm{z}) e^{r(t + \tau_{\mathrm{z}} - \tau)}, & \text{for } t \in [0, \tau - \tau_{\mathrm{z}}], \\ \mu(\mathrm{z} + \beta_g)(1 - e^{t - \tau}), & \text{for } t \in [\tau - \tau_{\mathrm{z}}, \tau], \end{cases} \tag{4.30}$$

and

$$H_t = \begin{cases} \beta_g, & \text{for } t \in [0, \tau - \tau_{\mathrm{z}}], \\ \mathrm{z} + \beta_g - W_t, & \text{for } t \in [\tau - \tau_{\mathrm{z}}, \tau]. \end{cases} \tag{4.31}$$

(ii) If $w \geq \hat{\omega}(\tau)$, then define

$$\mathrm{z}(w, \tau) := \frac{w}{\mu(1 - e^{-\tau})} - \beta_g. \tag{4.32}$$

For any $t \in [0, \tau]$, the following H_t and W_t solves the optimization $G(w, \tau)$ in (4.26),

$$W_t = \mu(\mathrm{z} + \beta_g)(1 - e^{t - \tau}), \text{ and } H_t = \mathrm{z} + \beta_g - W_t. \tag{4.33}$$

(iii) If $w < \breve{\omega}(\tau)$, the optimization problem is infeasible, or, by convention, $G(w, \tau) = -\infty$.

Similarly, we solve optimization problem (4.27) in the following Lemma.

Lemma 4.3 *If $\tau = \infty$, then define*

$$\mathrm{z}(w, \tau) := \frac{w}{\mu} - \beta_g, \tag{4.34}$$

(i) If $w \geq \frac{g}{r}$, then the following W_t and H_t solves the optimization (4.27),

$$W_t = w, \text{ and } H_t = \frac{w}{\mu} - w. \tag{4.35}$$

(ii) If $w < \frac{g}{r}$, the optimization problem (4.27) *is infeasible, or, by convention, $G(w, \tau) = -\infty$.*

We define the *discounted length* of the probation period to be

$$\bar{\tau} := \frac{1 - e^{-r\tau}}{r}. \tag{4.36}$$

Further define function

$$J(w, \bar{\tau}) := G\left(w, -\frac{\log(1 - r\bar{\tau})}{r}\right), \text{ for } \bar{\tau} \in \left[0, \frac{1}{r}\right], \ w \geq g\bar{\tau}. \tag{4.37}$$

Based on Lemma 4.2, we have the following result.

Proposition 4.4 *Function $J(w, \bar{\tau})$ is jointly concave in w and $\bar{\tau}$, and increasing in $\bar{\tau}$. Furthermore, we have*

$$\mathcal{Y} = \max_{w_g, w_b, \bar{\tau}} p \cdot J(w_g, \bar{\tau}) + (1-p) \min\left\{F_b(w_b), \ \frac{w_g - w_b}{b - g}(\mu R - b)^+ - w_b\right\} \tag{4.38}$$

$$\text{s.t. } w_g \geq w_b \geq g \cdot \bar{\tau} \tag{4.39}$$

$$0 \leq \bar{\tau} \leq \frac{1}{r}. \tag{4.40}$$

Because the minimum of two concave functions is concave, the objective function in (4.38) is concave. Therefore, Proposition 4.4 implies that we can convert the non-convex optimization problem (4.21)-(4.25) into a convex optimization problem with linear constraints, which can be solved efficiently.

4.4.2 Optimal menu of contracts

Now we define a menu of contracts based on the upper bound optimization problem (4.38)-(4.40). Let $(w_g^*, w_b^*, \bar{\tau}^*)$ represent an optimal solution of the convex optimization (4.38)-(4.40). Define

$$\tau^* := -\frac{1}{r}\log(1 - r\bar{\tau}^*), \text{ and } z^* := \mathrm{z}(w_g^*, \tau^*), \tag{4.41}$$

in which the function $\mathrm{z}(w, \tau)$ is defined according to Lemma 4.2 and 4.3. We can then construct a probation contract $\gamma_{\mathrm{P}}^g(\tau^*, z^*)$ of Definition 4.2 for the good agent.

To construct the contract for the bad agent, we first present the following lemma.

Lemma 4.4 *We have $w_b^* \le \bar{w}_b$. Furthermore, if $\mu R > b$, there exists a quantity $w \in [0, w_b^*]$ such that*

$$F_b(w) \le \frac{\left(w_g^* - w_b^*\right)(\mu R - b)}{b - g} - w. \tag{4.42}$$

Lemma 4.4 implies that the following threshold is well-defined if $\mu R > b$,

$$w_B := \max\left\{ w \in [0, w_b^*] \;\middle|\; F_b(w) \le \frac{\left(w_g^* - w_b^*\right)(\mu R - b)}{b - g} - w \right\}, \tag{4.43}$$

and the principal should give the bad agent a sign-on-bonus contract $\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B)$ of Definition 4.1.

To summarize, we define the following menu of contracts and show that it is optimal.

Definition 4.3 *Given the optimal solution $(w_g^*, w_b^*, \bar{\tau}^*)$ to the convex optimization* (4.38)-(4.40)*, define a menu of contracts $\Gamma^*_{\{g,b\}} := \left\{\gamma_{\mathtt{P}}^g(\tau^*, z^*),\ \gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B)\right\}$, in which τ^* and z^* are defined in* (4.41)*, and w_B in* (4.43) *if $\mu R > b$, and $w_B = 0$ otherwise.*

Lemma 4.5 *There exists $\bar{b} \in [g, \mu R]$, such that $w_B = 0$ for $b \ge \bar{b}$ and $w_B > 0$ for $b < \bar{b}$.*

Lemma 4.5 shows that it is still possible that $w_B = 0$ even for $b < \mu R$.

Therefore, the good agent is always given a probation contract. The bad agent's contract depends on how high his operating cost is. If $b \ge \bar{b}$, or, the operating cost of the bad agent is too high to be worth hiring, then $w_B = 0$, and $\gamma_{\mathtt{B}}^b(0, w_b^*)$ is a pay-to-leave contract. That is, the bad agent is paid w_b^* upfront payment and is asked to leave. If $b < \bar{b}$, on the other hand, the bad agent is still socially efficient, and would be hired by contract $\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B)$ which allows the bad agent to work from an initial promised utility w_B. Before proving optimality, we first discuss the following properties of the menu of contracts $\Gamma^*_{\{g,b\}}$.

Property 1:

$$\ell_t^g = g, \text{ and } \ell_t^b = b.$$

The optimal contracts never over compensate operating costs. That is, before contract termination, the good agent receives a flow payment g, and the bad agent receives a flow payment b.

Property 2:

$$\bar{\nu} \in \mathfrak{N}\Big(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B, 0),\ g\Big), \text{ and } \mathcal{N}\Big(\gamma_{\mathtt{P}}^g(\tau^*, z^*),\ b\Big) = \{\nu^0\},$$

where $\nu^0 := \{\nu_t^0\}_{t \ge 0}$ is the *always shirking effort process*.

The first part of the property states that a good agent who pretends to be bad would exert full effort in response to the contact for the bad agent. It is rigorously proved in Lemma C.4 in the

Table 4.1: Optimal menu of contracts

	$b \geq \bar{b}$	$b < \bar{b}$
Good agent	Probation contract $\gamma^g_{\mathtt{P}}(\tau^*, z^*)$	Probation contract $\gamma^g_{\mathtt{P}}(\tau^*, z^*)$
Bad agent	Pay-to-leave contract $\gamma^b_{\mathtt{B}}(0, w^*_b)$	Sign-on-bonus contract $\gamma^b_{\mathtt{B}}(w_B, w^*_b - w_B)$

appendix. The second part of the property states that a bad agent who pretends to be good has to shirk until the end. This is because according to Property 1, the contract for good agent only compensates the operating cost at rate g, which is too low to cover the bad agent's effort. As a result, under the probation contract $\gamma^g_{\mathtt{P}}(\tau^*, z^*)$ for the good agent, the first arrival would confirm that the agent's type is indeed good, resolving the adverse selection issue. This property plays an important role in showing that $\Gamma^*_{\{g,b\}}$ is feasible to the (TT) constraint.

Property 3:

$$H^g_t = \beta_g, \text{ if } N_t \geq 1 \text{ and } H^b_t = \beta_b, \ \forall t.$$

This property indicates that under the menu $\Gamma^*_{\{g,b\}}$, the (IC) constraint is binding in the good agent's contract after the first arrival, and in the bad agent's contract the entire time. As mentioned earlier, the first arrival under the good agent's contract resolves adverse selection. Therefore the principal follows the most efficient contract, by setting the (IC) constraint binding. For the bad agent, on the other hand, arrivals do not resolve adverse selection because the good agent is able to mimick bad agent and generate arrivals. Therefore the principal always uses a dynamically efficient contract (binding (IC)) for the bad agent's contract, and adjust other paramters in the menu to achieve optimality.

We present in table 4.1 a summary of the menu of optimal contracts in the two-type case. Although the good agent's contract is always probation contract regardless of b, τ^* and z^* are still functions of b.

Now we are ready to present the main result of this section.

Theorem 4.1 *The menu of contracts* $\Gamma^*_{\{g,b\}}$ *satisfies* (LL), (PK), (IC), (IR), *and* (TT) *with* $\mathcal{C} = \{g, b\}$. *Furthermore, we have* $\mathcal{U}(\Gamma^*_{\{g,b\}}) = \mathcal{Y}$, *in which* $\mathcal{Y}$ *is defined in* (4.21)-(4.25). *Therefore, we have* $\mathcal{U}(\Gamma^*_{\{g,b\}}) = \mathcal{Z}(\{g, b\})$, *or, the menu of contract* $\Gamma^*_{\{g,b\}}$ *solves the optimal contract design problem* (4.8) *with two types.*

It is worth reflecting incentives around the optimal menu of contracts $\Gamma^*_{\{g,b\}}$. In the case of $b \geq \bar{b}$, the initial bonus w^*_b to the bad agent equals the discounted total operating cost g that the agent can collect by pretending to be the good agent and shirk until the end of the probation period so that the bad agent has no incentive to lie about the bad type. It is also worth noting that the (TT) constraint for the good agent is not binding. That is, the good agent's promised utility w_g^* under the probation contract is strictly higher than the bonus w_b^*.

If $b < \bar{b}$, however, the principal may allow the bad agent to work following contract $\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B)$, and provides sufficient information rent, w_b^*, for the bad agent to tell the truth. In order to discourage the good agent from pretending to be bad and exerting effort while collecting a higher operating cost reimbursement, b, the principal needs to lower the bad agent's contract's initial promised utility w_B. If this initial promised utility w_B is lower than the information rent w_b^*, however, the principal needs to pay the difference as an initial sign-on-bonus to the bad agent.

4.4.3 Welfare implications of unknown capability

In this section, we present how unknown capability affects the welfare of the principal and the agent, compared to the situations with known capability. We show that unknown capability always hurts the principal, but may hurt or benefit the agent, depending on whether or not the bad agent is efficient.

Denote $\bar{\mathcal{Y}}$ to represent the principal's expected payoff when cost is observable, and either takes value g with probability p, or b with probability $1 - p$. That is,

$$\bar{\mathcal{Y}} := p\,\mathcal{Z}(\{g\}) + (1-p)\,\mathcal{Z}(\{b\}), \tag{4.44}$$

in which $\mathcal{Z}(\{g\})$ and $\mathcal{Z}(\{b\})$ are the principal's optimal utility earned from the good agent and the bad agent, respectively, following (4.8).[5] It is clear that the principal is always better off knowing the cost of the agent before issuing the contract, or, $\mathcal{Y} \leq \bar{\mathcal{Y}}$. Intuitively, this conclusion follows from the basic idea of value of information. In particular, with known cost, the principal does not need to pay the information rent associated with unknown cost.

Now we consider the agent's utility in the following two different cases. Define w_*^g and w_*^b to be the maximizers of functions $F_g(w)$ and $F_b(w)$, respectively. Following Proposition 4.2, we know that they are the good and bad agents' utilities when the cost is observable.. First, consider the situation that the bad agent is not efficient, or, $\beta_b \geq R$. In this case, the good agent is worse off and the bad agent is better off in the unknown cost situation, compared with the known cost one, as stated in the following result.

Proposition 4.5 *If $\beta_b \geq R$, we have*

$$w_g^* \leq w_*^g, \text{ and } w_b^* \geq w_*^b = 0, \tag{4.45}$$

where w_g^* and w_b^* are from the optimal solution of (4.38)-(4.40).

[5] In appendix C.3.2, we formally present the result when the agent's cost is known by the principal.

Apparently, the bad agent can earn the information rent if capability is unknown. Such an information rent does not exist if capability is known. Therefore the bad agent is better off with unknown capability. The good agent is worse off because with unknown capability, the bad agent could mimic the good agent, triggering the principal to curtail the good agent's payoff to save the bad agent's information rent.

If the bad agent is efficient, or, $\beta_b < R$, then either agents can be better or worse off with unknown capability. We illustrate this with the following two examples.

Example 1. $p = 0.4$, $\mu = 0.8$, $r = 1 - \mu = 0.2$, $R = 20$, $g = 0.3$ and $b = 0.5$. In this case, we have

$$w_g^* = 2.76 > w_*^g = 1.16 \text{ and } w_b^* = 1.76 < w_*^b = 1.86.$$

Furthermore, in this example, $\bar{\tau}^* = 1/r$, which means that the probation period of the good agent's contract is infinite. That is, the principal is willing to wait arbitrarily long for the first arrival of the good agent, i.e. the good agent is never terminated.

Example 2. $p = 0.9$, $\mu = 0.8$, $r = 1 - \mu = 0.2$, $R = 3$, $g = 0.3$ and $b = 3.7$. In this case, we have

$$w_g^* = 0.93 < w_*^g = 0.941 \text{ and } w_b^* = 0.93 > w_*^b = 0 \tag{4.46}$$

Furthermore, in this example, the bad agent is paid an amount $w_b^* = 0.93$ at time 0 to leave.

The reason that either agent can be better or worse off is because two competing forces influence agents' welfare when capability is unknown. First, as explained earlier, the bad agent benefits from mimicking the good agent, while this behavior hurts the good agent. This force is present no matter whether the bad agent is efficient. The second force is unique to the efficient bad agent case, where the good agent can potentially mimic the bad agent. This possibility can benefit the good agent while hurting the bad agent. Therefore, whether or not an agent is better off depends on which of the two forces dominates. In fact, in Example 1, the second force dominates the first, while in Example 2, the first dominates the second.

4.5 Continuous-type case

In this section we generalize the two-type case to a situation where the agent's operating cost may take value from an interval $\mathcal{C} := [\underline{c}, \bar{c}]$ with $\underline{c} < \bar{c}$, following a commonly known cumulative distribution function $P(c)$ with probability density function $\rho(c)$. In this section we require the following assumption.

Assumption 4.2

$$\int_{\underline{c}}^{\mu R} \rho(c)dc > 0.$$

This assumption is similar to the condition $\mu R \geq g$ in Assumption 4.1 for the two-type case, which guarantees that the agent is efficient with a positive probability, which excludes the trivial case where the agent is inefficient with probability 1. (If the agent is known to be inefficient, it is obviously a dominant strategy for the principal to immediately terminate the agent with no payment.) Note that we make no assumption on the worst cost $\bar{c}$. If $\bar{c} > \mu R$, an agent with cost $c > \mu R$ is not efficient, that is, not worth hiring.

In the contract design optimization (4.8) defined in Section 4.2, the objective function becomes

$$\mathcal{U}(\Gamma_{\mathcal{C}}) = \int_{\underline{c}}^{\bar{c}} \rho(c)U(\gamma^c, \nu^c)dc \tag{4.47}$$

in which the menu $\Gamma_{\mathcal{C}}$ offered by the principal contains a continuum of contracts γ^c for $c \in [\underline{c}, \bar{c}]$. Unlike for the two-type case, it appears hard to characterize the optimal solution for this infinite-dimensional optimization problem. Therefore, in this section, we focus on good approximations.

In Section 4.5.1, we first construct an optimization formulation similar to Section 4.4.1. However, this upper bound is hard to solve. Therefore, in Section 4.5.2, we provide a further relaxation that is easy to compute, using a dynamic programming approach. This upper bound calculation not only yields an upper bound for the optimal contract design, but also a way for us to design a menu of contracts. Therefore, in Section 4.5.3, we specify this menu of contracts, and compare its performance (a lower bound) with the upper bound. Furthermore, we show that if the solution in the upper bound calculation satisfies a simple condition, then the upper and lower bounds match, which implies that our contract design is in fact optimal. Numerical test illustrates that the condition is often satisfied with commonly used distributions.

4.5.1 Upper bound optimization

Similar to Section 4.4.1, we present a new optimization problem, which provides an upper bound to the contract design problem (4.47). First, we expand the definition of function J in (4.37) to include the cost variable as the following

$$\mathcal{J}(w, \bar{\tau}, c) := \mathcal{G}\left(w, -\frac{\log(1 - r\bar{\tau})}{r}, c\right), \text{ for } \bar{\tau} \in \left[0, \frac{1}{r}\right], \ c \in [\underline{c}, \min\{\bar{c}, \mu R\}], \text{ and } w \geq c\bar{\tau}, \tag{4.48}$$

in which function function $\mathcal{G}$ is defined similar to function G in (4.26) and (4.27) as

$$\mathcal{G}(w,\tau,c) := \max_{W_t,H_t} \int_0^\tau \mu e^{-t}[R + F_c(W_t + H_t)]\mathrm{d}t - c(1-e^{-\tau}), \tag{4.49}$$
$$\text{s.t. } \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-} - \mu H_t, \text{ for } t\in[0,\tau];\ W_0 = w,\ W_\tau = 0,$$
$$H_t \geq \beta_c,\ \forall t\in[0,\tau].$$

Therefore, function $\mathcal{J}(w,\bar{\tau},c)$ represents the principal's optimal utility when offering a type c agent a probation period with discounted length $\bar{\tau}$ and an initial promised utility level w. The relationship between the discounted length $\bar{\tau}$ and the real length τ of the probation period is defined in (4.36). Note that function $\mathcal{J}$ is well-defined only for $\bar{\tau}\in[0,1/r]$, $c\in[\underline{c},\ \min\{\bar{c},\mu R\}]$, and $w\geq c\bar{\tau}$, when the corresponding optimal control problem (4.49) is admissible.

Lemma 4.6 *For $\bar{\tau}\in\left[0,\frac{1}{r}\right]$, $c\in[\underline{c},\min\{\bar{c},\mu R\}]$, and $w\geq c\bar{\tau}$, we have the following properties for function $\mathcal{J}(w,\bar{\tau},c)$,*

(i) $\mathcal{J}(w,\bar{\tau},c)$ *is jointly concave in w and $\bar{\tau}$;*

(ii) $\mathcal{J}(w,\bar{\tau},c)$ *is increasing in $\bar{\tau}$ and $\mathcal{J}(w,0,c) = -w$;*

(iii) $\mathcal{J}(w,\bar{\tau},c)+w$ *is non-decreasing in w;*

(iv) $0\leq\mathcal{J}(w,\bar{\tau},c)+w\leq\dfrac{\mu R - c}{r}$.

Based on the definition of the value function $\mathcal{J}$ we are ready to present the following upper bound optimization problem,

$$\mathcal{Y}^{\mathcal{C}} := \sup_{\mathsf{w}(\cdot)} \int_{\underline{c}}^{\min\{\bar{c},\mu R\}} \xi\big(c;\mathsf{w}(\cdot)\big)\rho(c)dc - \int_{\min\{\bar{c},\mu R\}}^{\bar{c}} \mathsf{w}(c)\rho(c)dc \tag{4.50}$$
$$s.t.\ \ \mathsf{w}(c) \text{ is non-increasing in } c\in[\underline{c},\bar{c}], \tag{4.51}$$

in which for any $c\in[\underline{c},\bar{c}]$ we define,

$$\xi\big(c;\mathsf{w}(\cdot)\big) := \min\left\{\mathcal{J}\left(\mathsf{w}(c),\ \min\left\{\frac{\mathsf{w}(\bar{c})}{c},\ \frac{1}{r}\right\},\ c\right),\ \inf_{\tilde{c}\in[\underline{c},c)}\left[\frac{\mathsf{w}(\tilde{c})-\mathsf{w}(c)}{c-\tilde{c}}\right]\cdot(\mu R - c) - \mathsf{w}(c)\right\}. \tag{4.52}$$

Theorem 4.2 *For any feasible menu of contracts $\Gamma_{\mathcal{C}}$ that satisfies* (LL), (PK), (IC), (IR), *and* (TT)*, we have $\mathcal{Y}^C\geq\mathcal{U}(\Gamma_{\mathcal{C}})$, where $\mathcal{U}(\Gamma_{\mathcal{C}})$ is defined in* (4.47).

The optimization problem (4.50)-(4.51) is a generalization of one defined in Proposition 4.4 for the two type case. First, the decision variables $\mathsf{w}(c)$ in the maximization problem (4.50)-(4.51) represent the initial promised utility assigned to the type c agent. If the agent is efficient ($c \leq \mu R$), the function $\xi(c, \mathsf{w}(\cdot))$ represents the principal's utility facing a type c agent when initial promised utility function is w, similar to the variable ξ in Proposition 4.3. If the agent is inefficient ($c > \mu R$), on the other hand, the objective function (4.50) implies that the principal's utility is $-\mathsf{w}(c)$, that is, the agent should be paid off and terminated immediately.

The constraint (4.51) states that the principal needs to offer a higher promised utility to the agent with a better type (lower cost) than to a worse type (higher cost). This monotonicity constraint partly mitigates the agent's incentive to mimic a *worse* type. The first term of the ξ function in (4.52) represents that the principal's utility from type c agent is upper bounded by the function $\mathcal{J}$, the reason of which follows Remark 4.1. The second term in ξ also helps mitigating such a incentive. The intuition follows Remark 4.2, with type c and a better type $\tilde{c}$ replacing b and g, respectively.

To understand how to mitigate the agent's incentive to mimic a *better* type, we need to look at the term $\mathcal{J}$ inside (4.52). If an agent with a higher cost $\tilde{c}$ mimics the lower cost c and shirks through the probation period, the total discounted utility would be $c\bar{\tau}$, in which $\bar{\tau}$ represents the discounted probation period offered to type c. A constraint $c\bar{\tau} \leq \mathsf{w}(\tilde{c})$, or, equivalently, $\bar{\tau} \leq \mathsf{w}(\tilde{c})/c$, would mitigate such an incentive. Monotonicity of w following (4.51) implies that we only need to require $\bar{\tau} \leq \mathsf{w}(\bar{c})/c$. Following Lemma 4.6(ii), the function $\mathcal{J}(w, \bar{\tau}, c)$ is increasing in $\bar{\tau}$. Therefore, in order to maximize $\mathcal{J}$, it helps to to set $\bar{\tau}$ to the upper bound $\mathsf{w}(\bar{c})/c$. However, by definition, the discounted probation period $\bar{\tau}$ cannot be longer than $1/r$. This explains the second argument in function $\mathcal{J}$.

Finally, given constraint (4.51), if $\mu R < \bar{c}$, it is clear that the optimal $\mathsf{w}(c)$ value for any $c > \mu R$ should be a constant, $\mathsf{w}(\bar{c})$.

4.5.2 Computing an upper bound of $\mathcal{Y}^{\mathcal{C}}$

The optimization problem (4.50)-(4.51) is infinite-dimensional, which is hard to solve. Therefore, we provide an efficient algorithm to compute an upper bound of $\mathcal{Y}^{\mathcal{C}}$ based on a finite-dimensional approximation of. In the next subsection, we provide conditions to verify if the bound is tight.

Towards this goal, we divide the interval $[\underline{c},\ \min\{\bar{c}, \mu R\}]$ into a finite number of pieces. In particular, for a positive integer N, define $\delta := (\min\{\bar{c}, \mu R\} - \underline{c})/N$ and $c_i := \underline{c} + i\delta$ for $i \in \{0, \ldots, N\}$, such that $c_0 = \underline{c}$ and $c_N = \min\{\bar{c}, \mu R\}$.

Proposition 4.6 *Define*

$$\hat{y}(N) := \max_{w_0,\ldots,w_N} \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] \min \left\{ \mathcal{J}\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\},\ c_i\right), \right.$$
$$\left. \frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i \right\} - w_N \int_{\min\{\mu R, \bar{c}\}}^{\bar{c}} \rho(c) dc \tag{4.53}$$
$$\text{s.t. } w_i \geq w_{i+1},\ \forall i \in \{0, \ldots, N-1\}.$$

We have

$$\mathcal{Y}^C \leq \liminf_{N \to \infty} \hat{y}(N). \tag{4.54}$$

Therefore, we use w_i to approximate $\mathsf{w}(c_i)$ to obtain a finite-dimensional optimization (4.53). It is worth noting that the key difference between this upper bound optimization and the original problem. For a type $c = c_i$ and the corresponding $\mathsf{w}(c) = w_i$, the term $\inf_{\tilde{c} \in [\underline{c}, c)} \left[\frac{\mathsf{w}(\tilde{c}) - \mathsf{w}(c)}{c - \tilde{c}}\right]$ in the original formulation is replaced with a higher value $(w_{i-1} - w_i)/\delta$, which yields the upper bound. (In fact, if the optimal w function is concave, then this upper bound is tight. In the next subsection, we explore this further in the context of contract design.) The benefit of this change is computational efficiency. In fact, the optimization problem (4.53) can be solved using a dynamic programming approach.

Given a value $w_N \geq 0$, define the following deterministic dynamic programming recursion for any $i = \{1, \ldots, N\}$, starting from the boundary condition $\mathfrak{J}_0(w|w_N) = 0$ for all $w \geq w_N$,

$$\mathfrak{J}_i(w_i|w_N) = \max_{w_{i-1} \in [w_i, \infty)} [P(c_i) - P(c_{i-1})] \min \left\{ \frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i, \right.$$
$$\left. \mathcal{J}\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right) \right\} + \mathfrak{J}_{i-1}(w_{i-1}|w_N), \quad \forall w_i \geq w_N, \tag{4.55}$$

It is clear that

$$\hat{y}(N) = \max_{w_N \in [0, \infty)} \mathfrak{J}_N(w_N|w_N) - w_N(P(\bar{c}) - P(c_N)), \tag{4.56}$$

which implies that we can obtain the upper bound approximation $\hat{y}(N)$ by solving a sequence of the dynamic programming formulation (4.55) together with a one-dimensional search for the w_N value. Furthermore, we have the following result, which provides the closed-form solution to the maximization problem in (4.55).

Proposition 4.7 *For any given $i = 1, \ldots, N$ and $w_N \geq 0$, function $\mathfrak{J}_i(w|w_N)$ is concave in w. Use*

$\mathfrak{J}'_{i-1}(w|w_N)$ *to represent the its left-derivative at* w. *Further fix a value* $w_i \geq w_N$, *and define*

$$\breve{w} := \sup \left\{ w \;\middle|\; w \geq w_i \text{ and } \mathfrak{J}'_{i-1}(w|w_N) \geq 0 \right\},$$
$$\hat{w} := \inf \left\{ w \;\middle|\; w \geq w_i \text{ and } \mathfrak{J}'_{i-1}(w|w_N) \leq -(\mu R - c_i)\frac{P(c_i) - P(c_{i-1})}{\delta} \right\}, \text{ and}$$
$$\bar{u} := \begin{cases} \dfrac{\mathcal{J}\left(w_i, \min\left\{w_N/c_i, 1/r\right\}, c_i\right) + w_i}{\mu R - c_i}, & \text{if } \mu R - c_i > 0 \\ 0, & \text{if } \mu R - c_i = 0. \end{cases}$$

We have $\breve{w} \leq \hat{w}$, *and the following defined* w^*_{i-1} *solves the right-hand-side optimization problem in* (4.55)*,*

$$w^*_{i-1} := \begin{cases} \breve{w}, & \text{if } w_i \leq \breve{w} - \bar{u}\delta, \\ w_i + \bar{u}\delta, & \text{if } w_i \in (\breve{w} - \bar{u}\delta,\ \hat{w} - \bar{u}\delta], \\ \hat{w}, & \text{if } w_i \in (\hat{w} - \bar{u}\delta,\ \hat{w}], \\ w_i, & \text{if } w_i > \hat{w}. \end{cases}$$

Concavity of $\mathfrak{J}_i(w|w_N)$ follows from an induction proof showing that the objective of the maximization in (4.55) is jointly concave in w_i and w_{i-1}. This concavity property is crucial for us to obtain the closed-form optimal solution u^*.

Finally, we have the following result, which provides an upper bound for the optimal w_N.

Proposition 4.8 *Define* $\bar{\mathrm{w}} := \min\{\mu R - \underline{c}, \bar{c}\}/r$. *For any* $w_N \geq \mathrm{w}$, *we have* $\mathfrak{J}_N(w_N|w_N) \leq \mathfrak{J}_N(\bar{\mathrm{w}}|\bar{\mathrm{w}})$.

Proposition 4.8 implies that we can focus the search for the optimal w_N that solves (4.56) in the interval $[0, \bar{\mathrm{w}}]$.

In the following subsection, we construct a menu of contracts based on the optimal sequence of promised utilities obtained in Proposition 4.7 using the optimal w_N value that solves (4.56).

4.5.3 Contract design

Note that the upper bound computation from the last subsection generates a non-increasing sequence w_i of initial promised utilities. Now we construct a menu of contracts based on this sequence.

From a generic non-negative and non-increasing sequence $\mathsf{w} := \{w_i\}_{i=0,\ldots,N}$, we propose a menu of contracts. In particular, if the cost c is higher than μR, then the agent is paid w_N and terminated immediately, which corresponds to contract $\gamma^c_{\mathtt{B}}(0, w_N, 0)$ following Definition 4.1. If $c \in (c_{i-1}, c_i]$, on the other hand, the agent is given a probation contract $\gamma^{c_i}_{\mathtt{P}}(\tau, z)$, where we need to specify a probation time τ and a threshold z according to Definition 4.2. The following *discounted* probation period length is well-defined for $i = 1, \ldots, N$, according to Lemma 4.6,

$$\bar{\tau}^i_{\mathsf{w}} := \max \left\{ \bar{\tau} \in \left[0, \min\left\{ \frac{w_N}{\bar{c}}, \frac{1}{r} \right\} \right] \;\middle|\; \mathcal{J}\left(w_i, \bar{\tau}, c_i\right) \le \min_{j \in \{0,\ldots,i-1\}} \frac{w_j - w_i}{(i-j)\delta}(\mu R - c_i) - w_i \right\}. \tag{4.57}$$

That is, the discounted probation period $\bar{\tau}^i_{\mathsf{w}}$ allows the corresponding principal's utility to mimic $\xi(c_i; w_i)$ defined in (4.52).

Based on the definition of the discounted probation period, we can define the *actual* probation period length and the threshold as, respectively,

$$\tau^i_{\mathsf{w}} := -\frac{1}{r} \log(1 - r\bar{\tau}^i_{\mathsf{w}}), \text{ and } z^i_{\mathsf{w}} := \frac{w_i}{\mu(1 - e^{-\tau^i_{\mathsf{w}}})} - \beta_{c_i}, \tag{4.58}$$

in which the calculation of the threshold z^i_{w} is consistent with the function z defined in (4.32), with c_i replacing g and w_i replacing w.

Lemma 4.7 *For any non-negative and non-increasing sequence* $\mathsf{w} := \{w_i\}_{i=0,\ldots,N}$*, define a menu of contracts* $\hat{\Gamma}^{\mathsf{w}}_{\mathcal{C}} := \{\gamma^c_{\mathsf{w}}\}_{c \in \mathcal{C}}$*, in which*

$$\gamma^c_{\mathsf{w}} = \begin{cases} \gamma^{c_i}_{\mathsf{P}}(\tau^i_{\mathsf{w}}, z^i_{\mathsf{w}}), & \textit{if } c \in (c_{i-1}, c_i],\ \forall i = 1, \ldots, N, \\ \gamma^{c}_{\mathsf{B}}(0, w_N, 0), & \textit{if } c \in (\mu R, \bar{c}], \end{cases}$$

and $\gamma^{\underline{c}}_{\mathsf{w}} = \gamma^{c_0}_{\mathsf{P}}(\tau^0_{\mathsf{w}}, z^0_{\mathsf{w}})$*. The menu of contracts* $\hat{\Gamma}^{\mathsf{w}}_{\mathcal{C}}$ *satisfies* (LL)*,* (PK)*,* (IC)*,* (IR)*, and* (TT).

In particular, following (4.56) and Proposition 4.7, we have a sequence of initial promised utilities $\boldsymbol{w}^*_N := \{w^*_i\}_{i=1,\ldots,N}$ that are optimal solutions to the upper bound problem $\hat{y}(N)$. Lemma 4.7 and Proposition 4.2 imply that

$$\mathcal{Y}^C \ge \mathcal{U}\left(\hat{\Gamma}^{\boldsymbol{w}^*_N}_{\mathcal{C}}\right). \tag{4.59}$$

The following result provides a condition that one can use to verify when the inequality (4.59) holds as an equality.

Theorem 4.3 *If the sequence* $\boldsymbol{w}^*_N$ *is "convex," in the sense that* $2w_i \le w_{i-1} + w_{i+1}$ *for all* $i = 1, \ldots, N-1$*, we have*

$$\hat{y}(N) = \mathcal{U}\left(\hat{\Gamma}^{\boldsymbol{w}^*_N}_{\mathcal{C}}\right).$$

If $\boldsymbol{w}^*_N$ is always convex when N is large enough, then Theorem 1, together with (4.54), implies that as N approaches infinity, (4.59) holds as an equality, or, the menu of contracts $\hat{\Gamma}^{\boldsymbol{w}^*_N}_{\mathcal{C}}$ is optimal in the limit.

We conduct a numerical study to see the performance of $\hat{\Gamma}_{\mathcal{C}}^{\boldsymbol{w}_N^*}$. In the numerical study, we keep the support of the cost as $[\underline{c}, \bar{c}]$ where $\underline{c} = 1$ and $\bar{c} = 5$. Further, we let the density function of the cost be uniform distribution, symmetric triangular distribution, truncated exponential distribution ($\lambda = 1$), truncated normal distribution ($\mu = 3, \sigma = 1$) and U-quadratic distribution, respectively. We take $\mu = \{0.1, 0.2, ..., 0.9\}$, $R = \{2, 4, ..., 20\}$, $r = 1 - \mu$ and $N = 200$. For each distribution, there are 80 cases that $\mu R \geq \underline{c}$. In each of the cases, we calculate the optimal solution to the upper bound problem $\hat{y}(N)$, i.e., $\boldsymbol{w}_N^*$. Numerical result shows that in all of these cases, $\boldsymbol{w}_N^*$ is "convex". Therefore, the menu of contracts $\hat{\Gamma}_{\mathcal{C}}^{\boldsymbol{w}_N^*}$ is a good approximation of the optimal solution to the original contract design problem when N is large.

4.6 Conclusion

We study an optimal incentive design problem in continuous time over an infinite horizon with both moral hazard and adverse selection. Specifically, the principal hires an agent to exert effort to increase the arrival rate of a Poisson process where the agent's efforts are unobservable by the principal and the agent's capabilities are unknown by the principal. This type of problem is common in many businesses, R&D, and political environments.

Although combining dynamic moral hazard and adverse selection is generally hard, we can completely solve the problem when there are two types of agents. We show that the optimal contracts take simple and intuitive forms: the low-cost agent always takes the form of a probation contract, and the high-cost agent takes the form of a sign-on-bonus contract or the form of a pay-to-leave contract which depend on how high his cost is. All of these forms of contracts are easy to compute and implement. Furthermore, based on the solution of the two-type problem, we can tackle the continuous-type problem. In the continuous-type problem, the principal designs a menu of contracts, each of which has a form of pay-to-leave, sign-on-bonus, or probation contract.

Our model serves as a foundation for dynamic contract design problems with both moral hazard and adverse selection. First, in our model, the agent's effort cost is unknown by the principal. It is also worthwhile to think about the model when the arrival rate is the agent's private information. Second, in our model, the agent exerts effort to increase the arrival rate of 'good' events. We have learned from the literature that the optimal contract shares a very different structure when the arrivals are bad to the principal. (See [BMRV10]) Hence, it is worth considering when the agent is hired to decrease the arrival rate of adverse events, and the agent's capabilities/costs are unknown by the principal. Another possible extension is to consider the replacement of agents. Each time the principal terminates an agent, she can find another agent as the replacement from the agent pool with uncertain capabilities. We leave these extensions to future studies.

APPENDIX A

Appendix to Chapter 2

A.1 Proofs of Statements

A.1.1 Summary of Notations

Model parameters

- R: flow revenue rate to the principal when the machine is up.
- $\bar{\mu}_{\mathbf{u}}$ and $\mu_{\mathbf{u}}$: base case and low break down rates of the machine, respectively.
- $\mu_{\mathbf{d}}$ and $\underline{\mu}_{\mathbf{d}}$: base case and high recovery rates of the machine, respectively.
- $c_{\mathbf{u}}$ and $c_{\mathbf{d}}$: cost of effort in maintaining and in repairing the machine, respectively, per unit of time.
- r: principal and agent's discount rates.

Contracts and utilities

- ν and ν^*: generic and full effort process under the contracts.
- I and ℓ: instantaneous and flow payments, respectively.
- L: payment process $dL_t = I_t + \ell_t dt$.
- q: a stochastic firing probability at time t.
- τ: termination time.
- Γ: generic contract, $\Gamma = (L, \tau, q)$.
- $\bar{\Gamma}$ and $\hat{\Gamma}$: simple contract introduced in Section 2.3.1 and 2.3.2, respectively.
- Γ_1^* and $\Gamma_{\beta_{\bar{a}}}^*$: optimal contracts for the case in Section 2.4.1.1 and 2.4.2.1, respectively.
- u and U: agent's and principal's utilities, respectively.
- W_t: agent's promised utility.

Derived quantities

- $\beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}}$: defined in Lemma 2.1.
- $\underline{v}_{\mathbf{d}}, \underline{v}_{\mathbf{u}}$: defined in (2.4).
- $\bar{v}_{\mathbf{d}}, \bar{v}_{\mathbf{u}}$: defined in (2.11).
- $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$: defined in (2.10).
- $\hat{w}_{\mathbf{u}}$ and $\hat{w}_{\mathbf{d}}$: defined in (2.16).
- $w_\theta^*, \theta \in \{\mathbf{u}, \mathbf{d}\}$: maximizers of function $J_\theta(w)$.

Value functions

- $J_{\mathbf{d}}$, $J_{\mathbf{u}}$: the principal's value function of the optimal contract under state $\mathbf{d}$ and $\mathbf{u}$, respectively.
- $V_{\mathbf{d}}$, $V_{\mathbf{u}}$: the societal value function of the optimal contract under state $\mathbf{d}$ and $\mathbf{u}$, respectively.

A.1.2 Proofs in Section 2.2

A.1.2.1 Proof of Lemma 2.1

Since the proof would not depend on θ_0, we omit the θ_0 of the equations (2.3) and (2.5) throughout the proof. We first define a 2-variate counting process $\{N_t^n, N_t^f\}_{t\in[0,\tau]}$, in which $dN_t^f = X_t dN_t$, and $N_t^n = N_t - N_t^f$. If $\tau < \infty$, the principal terminates the collaboration with the agent, while the collaboration continues throughout the infinite time horizon if $\tau = \infty$. Also, $dN_t = dN_t^f + dN_t^n = X_t dN_t + (1 - X_t) dN_t$.

For a generic contract Γ and effort process ν, we introduce the agent's total expected utility conditional on the information available at time t as the following $\mathcal{F}_t^N$-adapted random variable,

$$u_t(\Gamma, \nu) = \mathbb{E}\left[\int_0^\tau e^{-rs}(dL_s + (1-\nu_s)c(\theta_s)ds)\middle|\, \mathcal{F}_t^N\right] \tag{A.1}$$
$$= \int_0^{t\wedge\tau^-} e^{-rs}(dL_s + (1-\nu_s)c(\theta_s)ds) + e^{-rt}W_t(\Gamma, \nu).$$

Therefore, $u_0(\Gamma, \nu) = u(\Gamma, \nu)$.

Process $\{u_t\}_{t\geq 0}$ is an $\mathcal{F}^N$-martingale. Define processes

$$M_t^{n,\nu} = N_t^n - \int_0^t \mu(\theta_s, \nu_s)(1 - q_s)ds, \text{and} \tag{A.2}$$

$$M_t^{f,\nu} = N_t^f - \int_0^t \mu(\theta_s, \nu_s) q_s ds, \tag{A.3}$$

which are $\mathcal{F}^N$-martingales. Following the Martingale Representation Theorem, see [Bré81], there exists a $\mathcal{F}^N$-predictable processes $H(\Gamma, \nu) = \{H_t(\Gamma, \nu)\}_{t\geq 0}$ such that

$$u_t(\Gamma, \nu) = u_0(\Gamma, \nu) + \int_0^{t\wedge\tau} e^{-rs}[H_s(\Gamma, \nu)dM_s^{n,\nu} - W_{s-}dM_s^{f,\nu}]\,, \quad \forall t\geq 0\,. \tag{A.4}$$

Differentiating (A.1) and (A.4) with respect to t yields

$$\begin{aligned} du_t &= e^{-rt}[H_t(\Gamma,\nu)dM_t^{n,\nu} - W_{t-}dM_t^{f,\nu}] \\ &= e^{-rt}(dL_t + (1-\nu_t)c(\theta_t)dt) - re^{-rt}W_t(\Gamma,\nu)dt + e^{-rt}dW_t(\Gamma,\nu), \end{aligned}$$

which implies (PK).

Denote $\tilde{u}_t(\Gamma,\nu',\nu)$ to be a $\mathcal{F}_t^N$-measurable random variable, representing the agent's total payoff following an effort process ν' before time t and ν after t, that is,

$$\tilde{u}_t(\Gamma,\nu',\nu) = \int_0^{t\wedge\tau} e^{-rs}(dL_s + (1-\nu_s)c(\theta_s))ds) + e^{-rt}W_t(\Gamma,\nu).$$

Therefore,

$$\tilde{u}_0(\Gamma,\nu',\nu) = u_0(\Gamma,\nu) = u(\Gamma,\nu)\ , \tag{A.5}$$

$$\mathbb{E}\left[\tilde{u}_\tau(\Gamma,\nu',\nu)\middle|\,\mathcal{F}_0^N\right] = u(\Gamma,\nu')\ , \text{ and} \tag{A.6}$$

$$\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu)\middle|\,\mathcal{F}_0^N\right] = u(\Gamma,\nu)\ , \ \forall t\geq 0\ . \tag{A.7}$$

For any given sample trajectory $\{N_s\}_{0\leq s\leq t}$ and effort processes ν and ν^*.

$$\begin{aligned} \tilde{u}_t(\Gamma,\nu,\nu^*) =& u_t(\Gamma,\nu^*) + \int_0^{t\wedge\tau} e^{-rs}(1-\nu_s)c(\theta_s)ds \\ =& u_0(\Gamma,\nu^*) + \int_0^{t\wedge\tau} e^{-rs}[H_s(\Gamma,\nu^*)dM_s^{n,\nu^*} - W_{s-}dM_s^{f,\nu^*}] + \int_0^{t\wedge\tau} e^{-rs}(1-\nu_s)c(\theta_s)ds \\ =& u_0(\Gamma,\nu^*) + \int_0^{t\wedge\tau} e^{-rs}[H_s(\Gamma,\nu^*)dM_s^{n,\nu} - W_{s-}dM_s^{f,\nu}] + \int_0^{t\wedge\tau} e^{-rs}(1-\nu_s)c(\theta_s)ds \\ &+ \int_0^{t\wedge\tau} e^{-rs}[(1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}](\mu(\theta_s,\nu_s)-\mu(\theta_s,1))ds, \end{aligned}$$

where the first equality follows from (A.1), the second equality follows (A.4) and the third equality follows from (A.2) and (A.3). Consider any two times $t' < t$,

$$\begin{aligned} &\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)\middle|\,\mathcal{F}_{t'}^N\right] \\ =& u_0(\Gamma,\nu^*) + \int_0^{t'\wedge\tau} e^{-rs}[H_s(\Gamma,\nu^*)dM_s^{n,\nu} - W_{s-}dM_s^{f,\nu}] \\ &+ \int_0^{t'\wedge\tau} e^{-rs}\{(1-\nu_s)c(\theta_s) + [(1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}](\mu(\theta_s,\nu_s)-\mu(\theta_s,1))\}ds \\ &+ \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}\{(1-\nu_s)c(\theta_s) + [(1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}](\mu(\theta_s,\nu_s)-\mu(\theta_s,1))\}\middle|\,\mathcal{F}_{t'}^N\right] \\ =& \tilde{u}_{t'}(\Gamma,\nu,\nu^*) + \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}(\bar{\mu}_{\mathbf{u}}-\mu_{\mathbf{u}})(\nu_s-1)[-\beta_{\mathbf{u}} - (1-q_s)H_s(\Gamma,\nu^*) + q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{u}}ds\middle|\,\mathcal{F}_{t'}^N\right] \\ &+ \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}(\mu_{\mathbf{d}}-\underline{\mu}_{\mathbf{d}})(\nu_s-1)[-\beta_{\mathbf{d}} + (1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{d}}ds\middle|\,\mathcal{F}_{t'}^N\right], \end{aligned} \tag{A.8}$$

where the second equality follows from equation (2.8).

If condition (2.7) holds for all $s\geq 0$, then (A.8) implies that $\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)\middle|\,\mathcal{F}_{t'}^N\right] \leq \tilde{u}_{t'}(\Gamma,\nu,\nu^*)$.

Therefore, $\{\tilde{u}_t\}_{t\geq 0}$ is a super-martingale. Take $t'=0$, we have

$$u(\Gamma,\nu^*)=\tilde{u}_0(\Gamma,\nu,\nu^*)\geq \mathbb{E}\left[\tilde{u}_\tau(\Gamma,\nu,\nu^*)|\,\mathcal{F}_0^N\right]=u(\Gamma,\nu),$$

in which the first equality follows from (A.5) and the last equality from (A.6), while the inequality follows from Doob's Optional Stopping Theorem. Therefore, the agent prefers the effort process ν^* to any other effort process ν, which implies that Γ satisfies (IC) if condition (2.7) holds for all $s\geq 0$.

If, on the other hand, $(1-q_s)H_s(\Gamma,\nu^*)-q_sW_{s-}>-\beta_{\mathbf{u}}$ for $s\in\Omega_{\mathbf{u}}\subset[0,t]$ with $\theta_{s-}=\mathbf{u}$, where $\Omega_{\mathbf{u}}$ is a positive measure set, define effort process ν to be such that

$$\nu_s=\begin{cases}1\,, & (1-q_s)H_s(\Gamma,\nu^*)-q_sW_{s-}\leq-\beta_{\mathbf{u}}\\ 0\,, & (1-q_s)H_s(\Gamma,\nu^*)-q_sW_{s-}>-\beta_{\mathbf{u}}\end{cases}\quad \text{for } s\in[0,t] \text{ where } \theta_{s-}=\mathbf{u},$$

and $\nu_s=1$ for $s>t$ where $\theta_{s-}=\mathbf{u}$ and $\nu_s=1$ for $\theta_{s-}=\mathbf{d}\ \forall s$. Therefore, $\tilde{u}_t(\Gamma,\nu,\nu^*)=\tilde{u}_t(\Gamma,\nu,\nu)$, and

$$\mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau}e^{-rs}(\bar{\mu}_{\mathbf{u}}-\mu_{\mathbf{u}})(\nu_s-1)[-\beta_{\mathbf{u}}-(1-q_s)H_s(\Gamma,\nu^*)+q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{u}}ds\right|\mathcal{F}_{t'}^N\Bigg]>0,$$

while

$$\mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau}e^{-rs}(\mu_{\mathbf{d}}-\underline{\mu}_{\mathbf{d}})(\nu_s-1)[-\beta_{\mathbf{d}}+(1-q_s)H_s(\Gamma,\nu^*)-q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{u}}ds\right|\mathcal{F}_{t'}^N\Bigg]=0.$$

Equation (A.8) then implies that $\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)|\,\mathcal{F}_0^N\right]>\tilde{u}_0(\Gamma,\nu,\nu^*)$, and, therefore,

$$u(\Gamma,\nu^*)=\tilde{u}_0(\Gamma,\nu,\nu^*)<\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)|\,\mathcal{F}_0^N\right]=\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu)|\,\mathcal{F}_0^N\right]=u(\Gamma,\nu),$$

in which the last equality follows from (A.7). The same logic applies if we can consider the situation when $(1-q_s)H_s(\Gamma,\nu^*)-q_sW_{s-}<\beta_{\mathbf{d}}$ for $s\in\Omega_{\mathbf{d}}\subset[0,t]$ with $\theta_{s-}=\mathbf{d}$ and a positive measure set $\Omega_{\mathbf{d}}$. Therefore, the agent prefers effort process ν' over ν^*, which implies that Γ does not satisfy (IC) if condition (2.7) does not hold.

A.1.2.2 Lemma A.1 and its proof

Lemma A.1 *Define* $\underline{\nu}:=\{\nu_t=0\}_{\forall t}$. *For* $\theta_0\in\{u,d\}$, *we have*

$$\mathbb{E}\left[\int_0^\infty e^{-rt}R\mathbb{1}_{\theta_t=\mathbf{u}}dt\right|\theta_0,\underline{\nu}\Bigg]=\underline{v}_{\theta_0}\,. \tag{A.9}$$

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}-c(\theta_t))dt\right|\theta_0,\nu^*\Bigg]=\bar{v}_{\theta_0}\,. \tag{A.10}$$

where $\underline{v}_{\theta_0}$ *and* $\bar{v}_{\theta_0}$ *are defined in equation (2.4) and (2.11), respectively.*

Proof. We first calculate (A.10) with $\theta_0=\mathbf{d}$ which is the societal value when the machine starts with state $\mathbf{d}$ and the agent always exerts effort. We define t_k as the time of occurrence of the kth transition of the states, and $t_0=0$. Further define $\tau_k:=t_k-t_{k-1}$. Therefore τ_{2k+1} follows an exponential distribution with rate

$\mu_{\mathbf{u}}$, and τ_{2k+2} follows an exponential distribution with rate $\mu_{\mathbf{d}}$ where $k \in \mathbb{N}$. Then

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}} - c(\theta_t, \nu_t^*))dt \middle| d, \nu^*\right] = \sum_{k=0}^{\infty}\left\{\mathbb{E}\left[\int_{t_{2k}}^{t_{2k+1}} e^{-rt}(R - c_{\mathbf{u}})dt\right]\right.$$
$$\left. +\mathbb{E}\left[\int_{t_{2k+1}}^{t_{2k+2}} e^{-rt} - c_{\mathbf{d}}dt\right]\right\}$$
$$=\sum_{k=0}^{\infty}\left\{\mathbb{E}\left[\int_{t_{2k}}^{t_{2k+1}} e^{-rt}dt\right](R - c_{\mathbf{u}}) + \mathbb{E}\left[\int_{t_{2k+1}}^{t_{2k+2}} e^{-rt}dt\right]\cdot(-c_{\mathbf{d}})\right\}, \tag{A.11}$$

where

$$\mathbb{E}\left[\int_{t_{2k}}^{t_{2k+1}} e^{-rt}dt\right] = \frac{\mathbb{E}\left[e^{-rt_{2k}}\right](1 - \mathbb{E}\left[e^{-r\tau_{2k+1}}\right])}{r}$$
$$= \frac{\mathbb{E}\left[e^{-r\sum_{i=1}^{2k}\tau_{2k}}\right](1 - \mathbb{E}\left[e^{-r\tau_{2k+1}}\right])}{r}$$
$$= \frac{\alpha^k\beta^k(1-\alpha)}{r},$$

where $\alpha = \mathbb{E}\left[e^{\tau_1}\right] = \dfrac{\mu_{\mathbf{d}}}{r + \mu_{\mathbf{d}}}$ and $\beta = \mathbb{E}\left[e^{\tau_2}\right] = \dfrac{\mu_{\mathbf{u}}}{r + \mu_{\mathbf{u}}}$. In the same way, $\mathbb{E}\left[\int_{t_{2k+1}}^{t_{2k+2}} e^{-rt}dt\right] = \dfrac{\alpha^{k+1}\beta^k(1-\beta)}{r}$. Furthermore, the expression of α and β yields $\dfrac{1-\alpha}{r} = \dfrac{1}{r + \mu_{\mathbf{d}}}$ and $\dfrac{1-\beta}{r} = \dfrac{1}{r + \mu_{\mathbf{u}}}$. Following equation (A.11),

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}} - c(\theta_t))dt \middle| d, \nu^*\right]$$
$$=\sum_{k=0}^{\infty}\left\{\frac{\alpha^{k+1}\beta^k}{r+\nu_{\mathbf{u}}}(R - c_{\mathbf{u}}) + \frac{\alpha^k\beta^k}{r+\nu_{\mathbf{d}}}(-c_{\mathbf{d}})\right\}$$
$$=\frac{\alpha}{1-\alpha\beta}\frac{1}{r+\nu_{\mathbf{u}}}(R - c_{\mathbf{u}}) + \frac{1}{1-\alpha\beta}\frac{1}{r+\nu_{\mathbf{d}}}(-c_{\mathbf{d}})$$
$$=\frac{\mu_{\mathbf{d}}(R - c_{\mathbf{u}}) - (r+\mu_{\mathbf{u}})c_{\mathbf{d}}}{r(r+\mu_{\mathbf{u}}+\mu_{\mathbf{d}})}.$$

The same logical steps yields (A.10) for the case of $\theta_0 = \mathbf{d}$, and also (A.9) and (A.10) for the case of $\theta_0 = \mathbf{u}$.

A.1.3 Optimality Condition

The following lemma states conditions for functions $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$ such that they are upper bounds of the principal's utility $U(\Gamma)$ under any contract Γ. This verification result serves as an optimality condition for later sections.

Lemma A.2 *Suppose $J_{\mathbf{d}}(w) : [0, \infty) \to \mathbb{R}$ and $J_{\mathbf{u}}(w) : [\beta_{\mathbf{u}}, \infty) \to \mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J_{\mathbf{d}}'(w) \geq -1$, $J_{\mathbf{u}}'(w) \geq -1$, and $J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$. Consider any incentive compatible contract Γ, which yields the agent's expected utility $u(\Gamma, \nu^*) = W_0$, followed by the promised utility process*

$\{W_t\}_{t\geq 0}$ *according to (PK) and satisfy* (IC). *Define a stochastic process* $\{\Phi_t\}_{t\geq 0}$ *as*

$$\begin{aligned}\Phi_t :=& R\mathbb{1}_{\theta_t=\mathbf{u}} + J'_{\theta_t}(W_{t-})(rW_{t-} - [-q_tW_{t-} + (1-q_t)H_t]\mu(\theta_t,\nu_t)) - rJ_{\theta_t}(W_{t-}) \\ &+ \mu(\theta_t,\nu_t)q_t[J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})] + \mu(\theta_t,\nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-}+H_t) - J_{\theta_t}(W_{t-})] - c(\theta_t)\ .\end{aligned} \tag{A.12}$$

where $\theta_t \in \{u, d\}$ *and* $\hat{\theta}_t = \mathbb{1}_{\theta_t=\mathbf{d}} \cdot u + \mathbb{1}_{\theta_t=\mathbf{u}} \cdot d$, *and* $J_{\mathbf{u}}$ *is extended such that* $J_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}}$. *If the process* $\{\Phi_t\}_{t\geq 0}$ *is non-positive almost surely, then we have* $J_\theta(u(\Gamma,\nu^*,\theta)) \geq U(\Gamma,\nu^*,\theta)$.

Proof. We define the following function to represent the value function as a function of time t,

$$J(t) = \begin{cases} J_{\mathbf{d}}(W_{t-}) & \text{if} \quad \theta_{t-} = \mathbf{d}\ , \\ J_{\mathbf{u}}(W_{t-}) & \text{if} \quad \theta_{t-} = \mathbf{u}\ . \end{cases} \tag{A.13}$$

Following the Itô's Formula for jump processes (see, for example, Theorem 17.5 in [Bas11]) and (PK), we obtain

$$\begin{aligned} e^{-r\tau}J(\tau) =& e^{-r0}J(0) + \int_0^\tau [e^{-rt}dJ(t) - re^{-rt}J(t)dt] = J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt + c(\theta_t)dt + dL_t) \\ &+ \int_0^\tau e^{-rt}\mathcal{A}_t, \end{aligned} \tag{A.14}$$

where

$$\begin{aligned}\mathcal{A}_t =& dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t \\ =& J'(t)[rW_{t-} - \mu(\theta_t,\nu_t)(-q_tW_{t-} + (1-q_t)H_t) - \ell_t]dt - rJ(t)dt \\ &+ J(t+) - J(t) + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t \\ =& J'(t)[rW_{t-} - \mu(\theta_t,\nu_t)(-q_tW_{t-} + (1-q_t)H_t) - \ell_t]dt - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t \\ &+ [J_{\theta_t}(W_{t-} - I_t) - J_{\theta_t}(W_{t-})](1 - dN_t) + [J_{\hat{\theta}_t}(W_{t-} + [(1-X_t)H_t - X_tW_{t-}] - I_t) - J_{\theta_t}(W_{t-})]dN_t \\ =& J'(t)[rW_{t-} - \mu(\theta_t,\nu_t)(-q_tW_{t-} + (1-q_t)H_t) - \ell_t]dt - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t \\ &+ [J_{\theta_t}(W_{t-} - I_t) - J_{\theta_t}(W_{t-})](1 - dN_t) + \big[J_{\hat{\theta}_t}(W_{t-} + [(1-X_t)H_t - X_tW_{t-}] - I_t) \\ &- J_{\hat{\theta}_t}(W_{t-} + [(1-X_t)H_t - X_tW_{t-}])\big]dN_t + \big[J_{\hat{\theta}_t}(W_{t-} + [(1-X_t)H_t - X_tW_{t-}]) - J_{\theta_t}(W_{t-})\big]dN_t.\end{aligned}$$

Further define

$$\begin{aligned}\mathcal{B}_t :=& [J_{\hat{\theta}_t}(W_{t-} + H_t) - J_{\theta_t}(W_{t-})](dN_t^n - \mu(\theta_t,\nu_t)(1-q_t)dt) \\ &+ [J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})](dN_t^f - \mu(\theta_t,\nu_t)q_tdt).\end{aligned}$$

Because function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are concave, $J'_{\mathbf{d}}(w) \geq -1$ and $J'_{\mathbf{u}}(w) \geq -1$, we have

$$
\begin{aligned}
\mathcal{A}_t &= dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t \\
&\leq J'(t)[rW_{t-} - \mu(\theta_t,\nu_t)(-q_tW_{t-} + (1-q_t)H_t)]dt - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t - J'(t)\ell_t dt \\
&\quad -J'_{\theta_t}(W_{t-})I_t(1-dN_t) - J'_{\hat{\theta}_t}(W_{t-} + [(1-X_t)H_t - X_tW_{t-}])I_t dN_t \\
&\quad +\big[J_{\hat{\theta}_t}(W_{t-} + [(1-X_t)H_t - X_tW_{t-}]) - J_{\theta_t}(W_{t-})\big]dN_t - c(\theta_t)dt \\
&\leq J'(t)[rW_{t-} - \mu(\theta_t,\nu_t)(-q_tW_{t-} + (1-q_t)H_t)]dt - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt \\
&\quad + \big[J_{\hat{\theta}_t}(W_{t-} + [(1-X_t)H_t - X_tW_{t-}]) - J_{\theta_t}(W_{t-})\big]dN_t - c(\theta_t)dt \\
&= R\mathbb{1}_{\theta_t=\mathbf{u}}dt + J'_{\theta_t}(t)[rW_{t-} - \mu(\theta_t,\nu_t)(-q_tW_{t-} + (1-q_t)H_t)]dt - rJ_{\theta_t}(t)dt \\
&\quad + \big[J_{\hat{\theta}_t}(W_{t-} + H_t) - J_{\theta_t}(W_{t-})\big]dN_t^n + \big[J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})\big]dN_t^f - c(\theta_t)dt \\
&= \mathcal{B}_t + \Phi_t dt\ .
\end{aligned}
$$

Therefore, if $\Phi_t \leq 0$, we must have $\mathcal{A}_t \leq \mathcal{B}_t$ almost surely. Taking the expectation on both sides of (A.14), we immediately have

$$
J_{\theta_0}(u(\Gamma,\nu,\theta_0)) = J(0) \geq \mathbb{E}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t)\bigg|\,\theta_0\right] = u(\Gamma,\nu,\theta_0),
$$

where we use the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t dt$ is a martingale and $J(\tau) = J_{\theta_\tau}(0) = v_\tau$.

To prove that a contract is optimal among all incentive compatible contracts, we only need to verify if Φ_t defined in (A.12) is non-positive.

A.1.4 Proofs and derivations in Section 2.4.1

A.1.4.1 Heuristic derivation of equations (2.21)-(2.23)

If the machine's current state is $\mathbf{d}$, consider a small time interval $[t, t+\delta]$, during which the principal reimburses the agent's effort cost $c_{\mathbf{d}}\delta$. With probability $\mu_{\mathbf{d}}\delta$, the machine recovers after this interval and changes to state $\mathbf{u}$, the principal pays the agent $(w + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+$, and, correspondingly, the promised utility jumps up to $\min\{w + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}$. With probability $1 - \mu_{\mathbf{d}}\delta$, on the other hand, the machine stays in $\mathbf{d}$, and the promised utility evolves to $w + r(w - \bar{w}_{\mathbf{d}})\delta$. Therefore, we have

$$
\begin{aligned}
J_{\mathbf{d}}(w) = &- c_{\mathbf{d}}\delta + e^{-r\delta}\Big\{\mu_{\mathbf{d}}\delta\big[-(w + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+ + J_{\mathbf{u}}(\min\{w + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\})\big] \\
&+ (1 - \mu_{\mathbf{d}}\delta)J_{\mathbf{d}}(w + r(w - \bar{w}_{\mathbf{d}})\delta)\Big\} + o(\delta).
\end{aligned}
$$

Subtracting $J_{\mathbf{d}}(w)$ and dividing δ on both sides, then letting δ approach 0, we obtain equation (2.21).

Similarly, consider the machine's current state at $\mathbf{u}$. and a small time interval $[t, t+\delta]$, when the principal collects revenue $R\delta$ and the agent's promised utility $w \geq \beta_{\mathbf{u}}$. With probability $\mu_{\mathbf{u}}\delta$, the machine breaks down and changes to state $\mathbf{d}$, and the promised utility drops to $w - \beta_{\mathbf{u}}$. With probability $1 - \mu_{\mathbf{u}}\delta$, on the other hand, the machine stays in $\mathbf{u}$, the promised utility evolves to $w + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\delta$ if $w < \bar{w}_{\mathbf{u}}$, and the principal pays the agent $\ell^*\delta$ if $w = \bar{w}_{\mathbf{u}}$ while the promised utility stays at $\bar{w}_{\mathbf{u}}$. Therefore,

$$
\begin{aligned}
J_{\mathbf{u}}(w) =&(R - c_{\mathbf{u}})\delta + e^{-r\delta}\Big\{\mu_{\mathbf{u}}\delta J_{\mathbf{d}}(w - \beta_{\mathbf{u}}) + (1 - \mu_{\mathbf{u}}\delta)\big[J_{\mathbf{u}}(w + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\delta\mathbb{1}_{w<\bar{w}_{\mathbf{u}}}) - \ell^*\mathbb{1}_{w=\bar{w}_{\mathbf{u}}}\big]\Big\} \\
&+o(\delta).
\end{aligned}
$$

Following similar steps as before, we obtain equations (2.22) and (2.23).

A.1.5 Proof of Proposition 2.1

It is helpful to consider the societal value functions, defined below as the summation of the principal and the agent's utilities,

$$V_{\mathbf{d}}(w) = J_{\mathbf{d}}(w) + w \quad \text{and} \quad V_{\mathbf{u}}(w) = J_{\mathbf{u}}(w) + w. \tag{A.15}$$

Following (2.21)-(2.25), we obtain the following system of differential equations for $V_{\mathbf{d}}$ and $V_{\mathbf{u}}$,

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = \mu_{\mathbf{d}} V_{\mathbf{u}}(\min\{w + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w)\ ,\ w \in [0, \bar{w}_{\mathbf{d}}], \tag{A.16}$$

$$(\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(w) = -c_{\mathbf{u}} + R + \mu_{\mathbf{u}} V_{\mathbf{d}}(w - \beta_{\mathbf{u}}) + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_{\mathbf{u}}} V'_{\mathbf{u}}(w)\ ,\ w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}], \tag{A.17}$$

$$V_{\mathbf{u}}(w) = V_{\mathbf{u}}(0) + \frac{V_{\mathbf{u}}(\beta_{\mathbf{u}}) - V_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}} w, \tag{A.18}$$

$$V_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}} \quad \text{and} \quad V_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}. \tag{A.19}$$

Furthermore, as soon as the promised utility reaches $\bar{w}_{\mathbf{u}}$ at state $\mathbf{u}$, contract Γ_1^* becomes identical to the simple contract studied in Section 2.3.1. This implies the following boundary conditions

$$V_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) = \bar{v}_{\mathbf{d}} \quad \text{and} \quad V_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}}, \tag{A.20}$$

in which $\bar{v}_{\mathbf{d}}$ and $\bar{v}_{\mathbf{u}}$ are defined in (2.11). Equivalently, we prove that the system of differential equations (A.16) and (A.17) with boundary conditions (A.18), (A.19) and (A.20) has a unique solution: the pair of functions $V_{\mathbf{u}}(w)$ on $[0, \bar{w}_{\mathbf{u}}]$ and $V_{\mathbf{d}}(w)$ on $[0, \bar{w}_{\mathbf{d}}]$, both of which are increasing and strictly concave.

First, we prove that (A.16) and (A.17) with boundary conditions (A.19) and (A.20) has a unique solution: the pair of functions $V_{\mathbf{u}}(w)$ on $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$ and $V_{\mathbf{d}}(w)$ on $[0, \bar{w}_{\mathbf{d}}]$. Next we write the proof for the two cases $\beta_{\mathbf{d}} > \beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}} = \beta_{\mathbf{u}}$ separately.

$\beta_{\mathbf{d}} > \beta_{\mathbf{u}}$

Recall that function $V_{\mathbf{d}}$ and $V_{\mathbf{u}}$ satisfies the system of differential equations (A.16) and (A.17).

Case 1. $\bar{w}_{\mathbf{u}} \leq \beta_{\mathbf{d}}$. Since for $w \in [0, \bar{w}_{\mathbf{d}}]$, $V_{\mathbf{u}}(\min\{w + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) = V_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}}$, we could rearrange equation (A.16) as

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = \mu_{\mathbf{d}}\bar{v}_{\mathbf{u}} - c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w).$$

The above equation in $[0, \bar{w}_{\mathbf{d}})$ is a linear differential equation with boundary condition. The solution is

$$V_{\mathbf{d}}(w) = \bar{v}_{\mathbf{d}} + b_1(\bar{w}_{\mathbf{d}} - w)^{\frac{r+\mu_{\mathbf{d}}}{r}} \quad \text{for } w \in [0, \bar{w}_{\mathbf{d}}], \tag{A.21}$$

with $b_1 = (\underline{v}_{\mathbf{d}} - \bar{v}_{\mathbf{d}})\bar{w}_{\mathbf{d}}^{(r+\mu_{\mathbf{d}})/r} < 0$. (Followed by the condition (2.13).)

Then, (A.21) implies that $V'_{\mathbf{d}}(w) = -b_1(r + \mu_{\mathbf{d}})(\bar{w}_{\mathbf{d}} - w)^{\mu_{\mathbf{d}}/r}/r > 0$, $V''_{\mathbf{d}}(w) = b_1(r + \mu_{\mathbf{d}})\mu_{\mathbf{d}}(\bar{w}_{\mathbf{d}} - w)^{\mu_{\mathbf{d}}-r/r}/r^2 < 0$ for $w \in [0, \bar{w}_{\mathbf{d}}]$. Hence, $V_{\mathbf{d}}$ is increasing and strictly concave in $[0, \bar{w}_{\mathbf{d}}]$. Furthermore, it can be verified that $V'_{\mathbf{d}}(\bar{w}_{\mathbf{d}}-) = 0$. Next, we show that $V_{\mathbf{u}}$ is also increasing and strictly concave in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$. Rearranging equation (A.17) in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$ as

$$(\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(w) = -c_{\mathbf{u}} + R + \mu_{\mathbf{u}}\left(\bar{V}_{\mathbf{d}} + b_1(\bar{w}_{\mathbf{d}} - w + \beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right) + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_{\mathbf{u}}} V'_{\mathbf{u}}(w). \tag{A.22}$$

The above equation in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$ is a linear differential equation with boundary condition. It is easy to verify that $\lim_{w\to\bar{w}_{\mathbf{u}}-} V'_{\mathbf{u}}(w) = 0$ with $V_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}}$. Equation (A.22) implies that

$$V''_{\mathbf{u}}(w) = \frac{\mu_{\mathbf{u}}(V'_{\mathbf{u}}(w) - V'_{\mathbf{d}}(w-\beta_{\mathbf{u}}))}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \quad \text{for } w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}), \tag{A.23}$$

and

$$V'''_{\mathbf{u}}(w) = \frac{\mu_{\mathbf{u}}(V''_{\mathbf{u}}(w) - V''_{\mathbf{d}}(w-\beta_{\mathbf{u}})) - rV''_{\mathbf{u}}(w)}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \quad \text{for } w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}). \tag{A.24}$$

Since $V'_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) = 0$, then equation (A.23) implies that $\lim_{w\to w_{\mathbf{u}}-} V''_{\mathbf{u}}(w) = 0$. Furthermore, with $V'_{\mathbf{d}}(w - \beta_{\mathbf{u}}) < 0$ for $w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$, we can show that there exists $\epsilon > 0$ such that $V''_{\mathbf{u}}(w) < 0$ and $V'_{\mathbf{u}}(w) > 0$ for $w \in [\bar{w}_{\mathbf{u}} - \epsilon, \bar{w}_{\mathbf{u}})$. Hence, $V_{\mathbf{u}}(w)$ is increasing and strictly concave in $[\bar{w}_{\mathbf{u}} - \epsilon, \bar{w}_{\mathbf{u}})$. Assume there exists $\tilde{w} \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}} - \epsilon)$ such that $V''_{\mathbf{u}}(\tilde{w}) \geq 0$. There must be $\hat{w} = \max\{w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}} - \epsilon) | V''_{\mathbf{u}}(w) = 0\}$, and $V''_{\mathbf{u}}(w) < 0, \forall w > \hat{w}$. However, this contradicts $V'''_{\mathbf{u}}(\hat{w}) = \dfrac{-\mu_{\mathbf{u}} V''_{\mathbf{d}}(\hat{w} - \beta_{\mathbf{u}})}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} > 0$ which is implied by equation (A.24). Therefore, we must have $V_{\mathbf{u}}$ to be increasing and strictly concave in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$. Furthermore, it can be verified that $V_{\mathbf{u}}(w) = \bar{v}_{\mathbf{u}}$ for $w \in [\bar{w}_{\mathbf{u}}, \infty)$ and $V_{\mathbf{d}}(w) = \bar{v}_{\mathbf{d}}$ for $w \in [\bar{w}_{\mathbf{d}}, \infty)$ solves (A.16) and (A.17).

Case 2. $\bar{w}_{\mathbf{u}} > \beta_{\mathbf{d}}$. Rearranging (A.16) as

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = \mu_{\mathbf{d}}\bar{v}_{\mathbf{u}} - c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w), \text{ for } w \in [\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \infty) \text{ , and} \tag{A.25}$$

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = \mu_{\mathbf{d}}V_{\mathbf{u}}(w + \beta_{\mathbf{d}}) - c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w), \text{ for } w \in [0, \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}). \tag{A.26}$$

We then show the result according to the following steps.

1. Demonstrate the solution of (A.25) as a parametric function $V^b_{\mathbf{d}}$, with parameter b.
2. Show that the solution of (A.26) and (A.17) are a pair of unique and twice continuously differentiable equations for any b, called as $V^b_{\mathbf{d}}$ and $V^b_{\mathbf{u}}$.
3. Show that for $b < 0$, $V^b_{\mathbf{d}}$ and $V^b_{\mathbf{u}}$ are concave and increasing.
4. Show that $V^b_{\mathbf{d}}(0)$ is increasing in b, which implies that the boundary condition $V_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$ uniquely determines b, and therefore the solution of the original system of differential equations.

Step 1. The solution to the linear ordinary differential equation (A.25) on $[\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$ must have the following form, for any scalar b.

$$V^b_{\mathbf{d}}(w) = \bar{v}_{\mathbf{d}} + b(w_{\mathbf{d}} - w)^{\frac{r+\mu_{\mathbf{d}}}{r}} \quad \text{for } w \in [\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}], \tag{A.27}$$

Also define $V^b_{\mathbf{d}}(w) = \bar{v}_{\mathbf{d}}$ for $w \in [\bar{w}_{\mathbf{d}}, \infty]$, which satisfies (A.25), so that $V^b_{\mathbf{d}}$ is continuously differentiable on $[\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \infty)$.

Step 2. Using (A.27) as the boundary condition, we show that the system of differential equations (A.26) and (A.17) has a unique pair of solutions (called $V^b_{\mathbf{d}}$ and $V^b_{\mathbf{u}}$, on $(0, \bar{w}_{\mathbf{d}})$, $(\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$), which are continuously differentiable. In fact, the system of differential equations (A.26) and (A.17) are equivalent to a sequence of initial value problems over the intervals $[\bar{w}_{\mathbf{d}} - (k+1)(\beta_{\mathbf{d}} - \beta_{\mathbf{u}}), \bar{w}_{\mathbf{d}} - k(\beta_{\mathbf{d}} - \beta_{\mathbf{u}})]$ for $V_{\mathbf{d}}$ and $[\bar{w}_{\mathbf{u}} - k(\beta_{\mathbf{d}} - \beta_{\mathbf{u}}), \bar{w}_{\mathbf{u}} - (k-1)(\beta_{\mathbf{d}} - \beta_{\mathbf{u}}))$ for $V_{\mathbf{u}}$, $k = 1, 2, ...$ This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem and, therefore, bear unique solutions. Also define $V^b_{\mathbf{u}}(w) = \bar{v}_{\mathbf{u}}$ for $w \in [\bar{w}_{\mathbf{u}}, \infty)$, which satisfies (A.17), so that $V^b_{\mathbf{u}}$ is continuously differentiable on $[\bar{w}_{\mathbf{u}}, \infty)$. Also, computing $V'_b(\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}})$ from (A.27), and comparing it with (A.26), we see that $V^b_{\mathbf{d}}$ is continuously differentiable at $\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}$, and therefore $V^b_{\mathbf{d}}$ and $V^b_{\mathbf{u}}$ are continuously differentiable $[0, \infty)$ and $[\beta_{\mathbf{u}}, \infty)$, respectively. Furthermore, we could derive the expressions for $V^{b''}_{\mathbf{d}}$ and $V^{b''}_{\mathbf{u}}$ following (A.26) and (A.17), respectively,

$$V^{b''}_{\mathbf{u}}(w) = \frac{\mu_{\mathbf{u}}(V^{b'}_{\mathbf{u}}(w) - V^{b'}_{\mathbf{d}}(w - \beta_{\mathbf{u}}))}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \text{ , and} \tag{A.28}$$

$$V^{b''}_{\mathbf{d}}(w) = \frac{\mu_{\mathbf{d}}(V^{b'}_{\mathbf{u}}(w + \beta_{\mathbf{d}}) - V^{b'}_{\mathbf{d}}(w))}{r(w_{\mathbf{d}} - w)} . \tag{A.29}$$

Step 3. Next, we argue that for $b < 0$, $V^b_{\mathbf{d}}$ and $V^b_{\mathbf{u}}$ are concave and increasing. Equation (A.27) implies that $V^b_{\mathbf{d}}$ is increasing and strictly concave on $[\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$, and therefore $V^{d''}_{b}(w) < 0$ in this interval. We could firstly prove that $V^b_{\mathbf{u}}$ is strictly concave and increasing in $[\bar{w}_{\mathbf{u}} + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$ in the same way in Case 1. Next, we want to show that $V^b_{\mathbf{d}}$ is strictly concave in $[\bar{w}_{\mathbf{u}} + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}})$.

In the following, we prove two lemmas to establish the result.

Lemma A.3 *For any $w \leq \bar{w}_{\mathbf{u}}$, if $V^b_{\mathbf{u}}$ is strictly concave in $[w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$ and $V^b_{\mathbf{d}}$ is strictly concave in $[w - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}})$, then $V^b_{\mathbf{d}}$ is strictly concave in $[w + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}})$.*

Proof. $V^b_{\mathbf{d}}$ is strictly concave in $[w - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}})$ implies that $V^{b''}_{\mathbf{d}}(w) < 0$ in this interval. Assume that there exists $\tilde{w}_b \in [w + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, w - \beta_{\mathbf{d}})$ such that $V^{b''}_{\mathbf{d}}(\tilde{w}_b) \geq 0$, then following step 2, $V^b_{\mathbf{d}}$ twice continuously differentiable implies that there must exist $\hat{w}_b = \max\{w \in [w + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, w - \beta_{\mathbf{d}}) | V^{b''}_{\mathbf{d}}(w) = 0\}$, and $V^{b''}_{\mathbf{d}}(w) < 0, \forall w > \hat{w}_b$. Equation (A.29) implies that

$$V^{b'}_{\mathbf{u}}(\hat{w}_b + \beta_{\mathbf{d}}) = V^{b'}_{\mathbf{d}}(\hat{w}_b) . \tag{A.30}$$

Furthermore, since $V^b_{\mathbf{u}}$ is strictly concave in $[w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$ and $\hat{w}_b + \beta_{\mathbf{d}} \geq w + \beta_{\mathbf{u}}$, we have $V^{b''}_{u}(\hat{w}_b + \beta_{\mathbf{d}}) < 0$. Then equation (A.28) implies that $V^{b'}_{\mathbf{u}}(\hat{w}_b + \beta_{\mathbf{d}}) - V^{b'}_{\mathbf{d}}(\hat{w}_b + \beta_{\mathbf{d}} - \beta_{\mathbf{u}}) < 0$. With equation (A.30), we have $V^{b'}_{\mathbf{d}}(\hat{w}_b) < V^{b'}_{\mathbf{d}}(\hat{w}_b + \beta_{\mathbf{d}} - \beta_{\mathbf{u}})$ which contradicts with $V^{b''}_{\mathbf{d}}(w) < 0, \forall w > \hat{w}_b$.

Lemma A.4 *For $w \leq \bar{w}_{\mathbf{u}} + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}$, if $V^b_{\mathbf{d}}$ is strictly concave in $[w - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$ and $V^b_{\mathbf{u}}$ is strictly concave in $[w, \bar{w}_{\mathbf{u}}]$, then $V^b_{\mathbf{u}}$ is strictly concave in $[w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$.*

The proof of Lemma A.4 follows the same steps as the proof of Lemma A.3.

With Lemmas A.3 and A.4, we prove that if $V^b_{\mathbf{u}}$ is strictly concave in $[w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$ and $V^b_{\mathbf{d}}$ is strictly concave in $[w - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}})$, then $V^b_{\mathbf{u}}$ is strictly concave in $[w + 2\beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$ and $V^b_{\mathbf{d}}$ is strictly concave in $[w + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}})$. Hence, by induction, we can prove that $V^b_{\mathbf{d}}$ is strictly concave and increasing in $[0, \bar{w}_{\mathbf{d}})$ and $V^b_{\mathbf{u}}$ is strictly concave and increasing in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$.

Step 4. Finally, we show that $V^b_{\mathbf{d}}(0)$ is strictly increasing in b for $b < 0$, which allows us to uniquely determine b that satisfies $V^b_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$. For given $b_1 < b_2 < 0$, define $X_{\mathbf{d}}(w) := V^{b1}_{\mathbf{d}}(w) - V^{b2}_{\mathbf{d}}(w)$ and $X_{\mathbf{u}}(w) := V^{b1}_{\mathbf{u}}(w) - V^{b2}_{\mathbf{u}}(w)$. Equations (A.16) and (A.17) imply that

$$(\mu_{\mathbf{d}} + r)X_{\mathbf{d}}(w) = \mu_{\mathbf{d}} X_{\mathbf{u}}(w + \beta_{\mathbf{d}}) - r(\bar{w}_{\mathbf{d}} - w)X'_{\mathbf{d}}(w) \text{ , and}$$

$$(rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w < \bar{w}_{\mathbf{u}}} X'_{\mathbf{u}}(w) = -\mu_{\mathbf{u}} X_{\mathbf{d}}(w - \beta_{\mathbf{u}}) + (\mu_{\mathbf{u}} + r)X_{\mathbf{u}}(w).$$

Equation (A.27) implies that $X_{\mathbf{d}}(w) = (b_1 - b_2)(\bar{w}_{\mathbf{d}} - w)^{\frac{r + \mu_{\mathbf{d}}}{r}}$ for $[\bar{w}_{\mathbf{u}} + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$, which is strictly concave and increasing. Following the same logic as in step 3, we can prove that $X_{\mathbf{d}}$ is strictly concave and increasing on $[0, \bar{w}_{\mathbf{d}}]$ and $X_{\mathbf{u}}$ is strictly concave and increasing on $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$. Hence,

$$V^{b1}_{\mathbf{d}}(0) - V^{b2}_{\mathbf{d}}(0) = X_{\mathbf{d}}(0) < X_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) = 0.$$

Because $V^0_{\mathbf{d}}(0) = \bar{v}_{\mathbf{d}} > \underline{v}_{\mathbf{d}}$, and $\lim_{b\to-\infty} V^b_{\mathbf{d}}(0) < V^b_{\mathbf{d}}(\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}) = -\infty$, there must exist a unique $b^* < 0$ such that $V^{b*}_{\mathbf{d}}(0) = \bar{v}_{\mathbf{d}}$, and $V^{b*}_{\mathbf{d}}(w)$ and $V^{b*}_{\mathbf{u}}(w)$ are strictly concave and increasing on $[0, \bar{w}_{\mathbf{d}}]$ and $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$, respectively.

$\beta_{\mathbf{d}} = \beta_{\mathbf{u}}$

Let $\beta_{\mathbf{d}} = \beta_{\mathbf{u}} = \beta$, then equations (A.16) and (A.17) become

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = \mu_{\mathbf{d}} V_{\mathbf{u}}(w + \beta) - c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w), \text{ for } w \in [0, \bar{w}_{\mathbf{d}}), \text{ and} \tag{A.31}$$

$$(\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(w) = -c_{\mathbf{u}} + R + \mu_{\mathbf{u}} V_{\mathbf{d}}(w - \beta) + (rw + \mu_{\mathbf{u}}\beta)\mathbb{1}_{w<\bar{w}_{\mathbf{u}}} V'_{\mathbf{u}}(w), \text{ for } w \in [\beta, \bar{w}_{\mathbf{u}}), \tag{A.32}$$

since $w + \beta \le w_{\mathbf{u}}$ for $w \in [0, \bar{w}_{\mathbf{d}})$. Let $\breve{w} = w - \beta$ in equation (A.32), we have

$$(\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(\breve{w} + \beta) = -c_{\mathbf{u}} + R + \mu_{\mathbf{u}} V_{\mathbf{d}}(\breve{w}) + (rw + (r + \mu_{\mathbf{u}})\beta)V'_{\mathbf{u}}(\breve{w} + \beta), \text{ for } \breve{w} \in [0, \bar{w}_{\mathbf{d}}). \tag{A.33}$$

Differentiate (A.33) with respect to $\breve{w}$ on both sides, we obtain

$$\mu_{\mathbf{u}} V'_{\mathbf{u}}(\breve{w} + \beta) = \mu_{\mathbf{u}} V'_{\mathbf{d}}(\breve{w}) + (rw + (r + \mu_{\mathbf{u}})\beta)V''_{\mathbf{u}}(\breve{w} + \beta), \text{ for } \breve{w} \in [0, \bar{w}_{\mathbf{d}}). \tag{A.34}$$

Equations (A.31), (A.33) and (A.34) together imply that

$$\begin{aligned} &(\mu_{\mathbf{u}} + r)[r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w) + c_{\mathbf{d}} + (\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w)] \\ =&\mu_{\mathbf{d}}(-c_{\mathbf{u}} + R) + \mu_{\mathbf{d}}\mu_{\mathbf{u}} V_{\mathbf{d}}(w) + (rw + (r + \mu_{\mathbf{u}})\beta)[r(\bar{w}_{\mathbf{d}} - w)V''_{\mathbf{d}}(w) + \mu_{\mathbf{d}} V'_{\mathbf{d}}(w)], \text{ for } w \in [0, \bar{w}_{\mathbf{d}})\ , \end{aligned} \tag{A.35}$$

Differentiate (A.35) with respect to w on both sides, we obtain

$$\begin{aligned} &[\mu_{\mathbf{u}} r(\bar{w}_{\mathbf{d}} - w) - (rw + (r + \mu_{\mathbf{u}})\beta)(\mu_{\mathbf{d}} - r)]V''_{\mathbf{d}}(w) \\ =&(rw + (r + \mu_{\mathbf{u}})\beta_{\mathbf{u}})r(\bar{w}_{\mathbf{d}} - w)V'''_{\mathbf{d}}(w), \text{ for } w \in [0, \bar{w}_{\mathbf{d}}). \end{aligned} \tag{A.36}$$

Further, we define:

$$z(w) := \frac{[\mu_{\mathbf{u}} r(\bar{w}_{\mathbf{d}} - w) - (rw + (r + \mu_{\mathbf{u}})\beta)(\mu_{\mathbf{d}} - r)]}{(rw + (r + \mu_{\mathbf{u}})\beta_{\mathbf{u}} + c_{\mathbf{u}})r(\bar{w}_{\mathbf{d}} - w)}, \text{ for } w \in [0, \bar{w}_{\mathbf{d}})\ .$$

Then equation (A.36) is equivalent to

$$\frac{V'''_{\mathbf{d}}(w)}{V''_{\mathbf{d}}(w)} = z(w)\ ,$$

Solving the differential equation, we obtain $V''_{\mathbf{d}}(w) = C_0 e^{\int z(w)}$. With the boundary condition $V_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}} < \bar{v}_{\mathbf{d}}$, we could calculate C_0 with $C_0 < 0$. Hence, $V_{\mathbf{d}}$ is strictly concave and increasing in $[0, \bar{w}_{\mathbf{d}})$. In the same way we used in the step 4 of the case $\beta_{\mathbf{d}} > \beta_{\mathbf{u}}$, we could establish that $V_{\mathbf{u}}$ is also strictly concave and increasing in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$.

Second, combining with boundary condition (A.18), we further prove that $V_{\mathbf{u}}$ is increasing and concave in $[0, \bar{w}_{\mathbf{u}}]$. Following condition (2.13), (2.18) and $\beta_{\mathbf{d}} \ge \beta_{\mathbf{u}}$, we have

$$R \ge (r + \bar{\mu}_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})\beta_{\mathbf{u}}. \tag{A.37}$$

Following (A.17), we have

$$V'_{\mathbf{u}}(\beta_{\mathbf{u}+}) = \frac{(\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(\beta_{\mathbf{u}}) + c_{\mathbf{u}} - R - \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}}{r\beta_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \ge 0,$$

which implies that

$$V_{\mathbf{u}}(\beta_{\mathbf{u}}) \geq \frac{R - c_{\mathbf{u}} + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}}{\mu_{\mathbf{u}} + r} = \left[(r + \mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \frac{\Delta\mu_{\mathbf{u}} R}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} - c_{\mathbf{u}}\right] / (r + \mu_{\mathbf{u}}) \geq \underline{v}_{\mathbf{u}}, \tag{A.38}$$

where the second inequality follows from (A.37). Also, this implies that $V'_{\mathbf{u}}(\beta_{\mathbf{u}-}) = \frac{V_{\mathbf{u}}(\beta_{\mathbf{u}}) - \underline{v}_{\mathbf{u}}}{\beta_{\mathbf{u}}} \geq 0$ and,

$$\begin{aligned}
(r + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{u}}(V'_u(\beta_{\mathbf{u}-}) - V'_{\mathbf{u}}(\beta_{\mathbf{u}+})) &= (r + \bar{\mu}_{\mathbf{u}})(V_{\mathbf{u}}(\beta_{\mathbf{u}}) - \underline{v}_{\mathbf{u}}) - (\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(\beta_{\mathbf{u}}) - c_{\mathbf{u}} + R + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} \\
&\geq \Delta\mu_{\mathbf{u}}\underline{v}_{\mathbf{u}} - (r + \bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}} - c_{\mathbf{u}} + R + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} \\
&\geq R + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} - (r + \mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} - c_{\mathbf{u}} \\
&= R + \frac{[\mu_{\mathbf{u}}\underline{\mu}_{\mathbf{d}} - (r + \mu_{\mathbf{u}})(r + \underline{\mu}_{\mathbf{d}})]R}{r(r + \bar{\mu}_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})} - c_{\mathbf{u}} \\
&= \frac{\Delta\mu_{\mathbf{u}} R}{(r + \bar{\mu}_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})} - c_{\mathbf{u}} \geq 0,
\end{aligned} \tag{A.39}$$

where the first inequality follows from (A.38) and the last inequality follows from (A.37). Finally, (A.39) implies that $V'_{\mathbf{u}}(\beta_{\mathbf{u}-}) \geq V'_u(\beta_{\mathbf{u}+})$.

Furthermore, equations (A.16) and (A.17) imply that

$$\begin{aligned}
V''_{\mathbf{u}}(w) &= \frac{\mu_{\mathbf{u}}(V'_{\mathbf{u}}(w) - V'_{\mathbf{d}}(w - \beta_{\mathbf{u}}))}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}}, \text{ for } w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}), \\
V''_{\mathbf{d}}(w) &= \frac{\mu_{\mathbf{d}}(V'_{\mathbf{u}}(w + \beta_{\mathbf{d}}) - V'_{\mathbf{d}}(w))}{r(w_{\mathbf{d}} - w)}, \text{ for } w \in [0, \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}), \text{ and} \\
V''_{\mathbf{d}}(w) &= \frac{-V'_{\mathbf{d}}(w)}{r(w_{\mathbf{d}} - w)}, \text{ for } w \in [\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}).
\end{aligned}$$

Then the concavity of $V_{\mathbf{d}}$ and $V_{\mathbf{u}}$ implies that

$$V'_{\mathbf{u}}(w) < V'_{\mathbf{d}}(w - \beta_{\mathbf{u}}), \text{ for } w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}), \text{ and} \tag{A.40}$$

$$V'_{\mathbf{u}}(w + \beta_{\mathbf{d}}) < V'_{\mathbf{d}}(w), \text{ for } w \in [0, \bar{w}_{\mathbf{d}}). \tag{A.41}$$

A.1.5.1 Proof of Proposition 2.2

Following (A.13) and (A.14), we obtain that under contract Γ_1^* in Definition 2.1,

$$e^{-r\tau}J(\tau) = J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt + c(\theta_t)dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t^*, \tag{A.42}$$

where

$$
\begin{aligned}
\mathcal{A}_t^* &= dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t - c(\theta_t)dt \\
&= J'(t)[rW_{t-} - \mu(\theta_t,1)H_t^* - \ell_t^*]dt - rJ(t)dt + J(t+) - J(t) + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t^* \\
&= \left\{J_{\mathbf{u}}'(t)(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}} - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}}\right\} dt\mathbb{1}_{\theta_t=\mathbf{u}} \\
&\quad + \left\{J_{\mathbf{d}}'(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}})dt - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}\right\} dt\mathbb{1}_{\theta_t=\mathbf{d}} \\
&\quad + [J_{\mathbf{u}}(\min\{W_{t-} + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - J_{\mathbf{d}}(W_{t-})]dN_t\mathbb{1}_{\theta_t=\mathbf{d}} \\
&\quad + [J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})]dN_t\mathbb{1}_{\theta_t=\mathbf{u}} + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^* \\
&= \{R - c_{\mathbf{u}} + J_{\mathbf{u}}'(W_{t-})(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}} - rJ_{\mathbf{u}}(W_{t-})dt + \mu_{\mathbf{u}}(J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})) \\
&\quad + (r\bar{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}=\bar{w}_{\mathbf{u}}}\}\mathbb{1}_{\theta_t=\mathbf{u}}dt \\
&\quad + \{J_{\mathbf{d}}'(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}}) - rJ_{\mathbf{d}}(W_{t-})dt + \mu_{\mathbf{d}}(J_{\mathbf{u}}(\min\{W_{t-} + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - J_{\mathbf{d}}(W_{t-})) \\
&\quad - \mu_{\mathbf{d}}(W_{t-} + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+ - c_{\mathbf{d}}\}\mathbb{1}_{\theta_t=\mathbf{d}}dt + \mathcal{B}_t^* = \mathcal{B}_t^*,
\end{aligned}
$$

in which the last equality follows from (2.21) and (2.22), and

$$
\begin{aligned}
\mathcal{B}_t^* &= [J_{\mathbf{u}}(\min\{W_{t-} + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - J_{\mathbf{d}}(W_{t-}) - (W_{t-} + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+](dN_t - \mu_{\mathbf{d}}dt)\mathbb{1}_{\theta_t=\mathbf{d}} \\
&\quad + [J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})](dN_t - \mu_{\mathbf{u}}dt)\mathbb{1}_{\theta_t=\mathbf{u}}.
\end{aligned}
$$

Taking the expectation on both sides of (A.42), we immediately have

$$
J_{\theta_0}(w) = J(0) = \mathbb{E}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t^*)\middle|\theta_0\right] = u(\Gamma_1^*(w), \nu^*, \theta_0),
$$

where $u(\Gamma_1^*, \nu^*, \theta_0) = w$ and we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^* dt$ is a martingale and $J(\tau) = J_{\theta_\tau}(0) = v_\tau$.

A.1.5.2 Proof of Proposition 2.3

From Proposition 2.1, we know that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are concave, $J_{\mathbf{d}}'(w) \geq -1$, and $J_{\mathbf{u}}'(w) \geq -1$. Recall Lemma A.2, to show that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are upper bounds of principal's utility under any incentive compatible contract, we only need to show that $\Phi_t \leq 0$ holds almost surely if $\nu_t = 1$. From (A.12), we have

$$
\Phi_t = \Phi_t^u \mathbb{1}_{\theta_t=\mathbf{u}} + \Phi_t^d \mathbb{1}_{\theta_t=\mathbf{d}},
$$

where

$$
\begin{aligned}
\Phi_t^u :=& R + J_{\mathbf{u}}'(W_{t-})rW_{t-} + \mu_{\mathbf{u}}q_t[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] \\
&+ \mu_{\mathbf{u}}(1 - q_t)[-H_tJ_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}},
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_t^d :=& J_{\mathbf{d}}'(W_{t-})rW_{t-} + \mu_{\mathbf{d}}q_t[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] \\
&+ \mu_{\mathbf{d}}(1 - q_t)[-H_tJ_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})] - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}.
\end{aligned}
$$

We have $\Phi_t \leq 0$ if $\Phi_t^d \leq 0$ and $\Phi_t^u \leq 0$. First, we prove that $\Phi_t^u \leq 0$ by considering the following optimization problem,

$$\max_{q_t,H_t} \quad q_t[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})],$$
$$s.t. \quad 0 \le q_t \le 1, \;\; -q_tW_{t-} + (1-q_t)H_t \le -\beta_{\mathbf{u}}.$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\beta_{\mathbf{u}}. \tag{A.43}$$

by the KKT conditions. Define the following dual variables for the binding constraints

$$x_{\mathbf{u}} = -(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}})) \ge 0,$$

in which the inequality follows from (A.40), and

$$y_{\mathbf{u}} = (W_{t-} - \beta_{\mathbf{u}})\left(\frac{J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{d}}(0)}{W_{t-} - \beta_{\mathbf{u}}} - J'_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}})\right) \ge 0,$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$ and the fact that $W_{t-} \ge \beta_{\mathbf{u}}$ for any incentive compatible contract. One can verify that

$$[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H_t^*J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t^*) - J_{\mathbf{u}}(W_{t-})] \tag{A.44}$$
$$= -y_{\mathbf{u}} - (H_t^* + W_{t-})x_{\mathbf{u}},$$
$$(1-q_t^*)(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-} + H_t^*)) = (q_t^* - 1)x_{\mathbf{u}}. \tag{A.45}$$

Therefore, (A.43) implies that

$$\Phi_t^u \le R + J'_{\mathbf{u}}(W_{t-})rW_{t-} + \mu_{\mathbf{u}}[\beta_{\mathbf{u}}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}} = 0,$$

where the equality follows from (2.22).

Following similar logic, we prove that $\Phi_t^u \le 0$ by considering the following optimization problem,

$$\max_{q_t,H_t} \quad q_t[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_tJ'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t) - J_{\mathbf{d}}(W_{t-})],$$
$$s.t. \quad 0 \le q_t \le 1, \;\; -q_tW_{t-} + (1-q_t)H_t \ge \beta_{\mathbf{d}}, W_{t-} + H_t \ge \beta_{\mathbf{u}},$$

and verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = \beta_{\mathbf{d}}. \tag{A.46}$$

by the KKT conditions. Again define the following dual variables for the binding constraints

$$x_{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) \ge 0, \;\; \text{and}$$
$$y_{\mathbf{d}} = (W_{t-} + \beta_{\mathbf{d}})(\frac{J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{u}}(0)}{W_{t-} - \beta_{\mathbf{d}}} - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}})) \ge 0.$$

We can verify that

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_tJ'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H^*_t) - J_{\mathbf{d}}(W_{t-})] \tag{A.47}$$
$$= -y_{\mathbf{d}} + (H^*_t + W_{t-})x_{\mathbf{d}},$$
$$(1 - q^*_t)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H^*_t)) = (1 - q^*_t)x_{\mathbf{d}}. \tag{A.48}$$

Therefore, (A.46) implies that

$$\Phi^u_t \le J'_{\mathbf{d}}(W_{t-})rW_{t-} + \mu_{\mathbf{d}}[-\beta_{\mathbf{d}}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})] - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} = 0,$$

where the equality follows from (2.21).

A.1.5.3 Proof of Theorem 2.2

First, it is easy to verify that contract $\Gamma^*_{\mathbf{u}}$ is incentive compatible. Next, we define two functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ as

$$J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w. \tag{A.49}$$

and

$$J_{\mathbf{u}}(w) = \begin{cases} v_{\mathbf{u}} - w, & \text{for } w \in [\beta_{\mathbf{u}}, \infty), \\ \underline{v}_{\mathbf{u}} + (v_{\mathbf{u}} - \underline{v}_{\mathbf{u}} - \beta_{\mathbf{u}})w/\beta_{\mathbf{u}}, & \text{for } w \in [0, \beta_{\mathbf{u}}). \end{cases} \tag{A.50}$$

Under condition (2.29), $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$ are concave, $J'_{\mathbf{d}}(w) \ge -1$, and $J'_{\mathbf{u}} \ge -1$. Hence, following Lemma A.2, we have $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are upper bounds of the principal's utility under state $\mathbf{d}$ and $\mathbf{u}$, respectively if $\Phi_t \le 0$, where Φ_t is defined by (A.12). Furthermore,

$$\Phi_t = \Phi^{\mathbf{u}}_t \mathbb{1}_{\theta_t = \mathbf{u}} + \Phi^{\mathbf{d}}_t \mathbb{1}_{\theta_t = \mathbf{d}},$$

where

$$\Phi^{\mathbf{u}}_t = R - rW_{t-} + \mu_{\mathbf{u}}[-q_tW_{t-} + (1 - q_t)H_t] - r(v_{\mathbf{u}} - W_{t-}) + \mu_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}} + \mu_{\mathbf{u}}(1 - q_t)(\underline{v}_{\mathbf{d}} - W_{t-} - H_t)$$
$$- \mu_{\mathbf{u}}(v_{\mathbf{u}} - W_{t-}) - c_{\mathbf{u}} = R - c_{\mathbf{u}} - rv_{\mathbf{u}} + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} - \mu_{\mathbf{u}}v_{\mathbf{u}} = 0,$$

where the first equality follows from $J'_{\mathbf{u}}(W_{t-}) = -1$ for $W_{t-} \ge \beta_{\mathbf{u}}$,and the third equality follows from (2.26). Therefore,

$$\Phi^{\mathbf{d}}_t = -rW_{t-} + \mu_{\mathbf{d}}[-q_tW_{t-} + (1 - q_t)H_t] - r(\underline{v}_{\mathbf{d}} - W_{t-}) + \mu_{\mathbf{d}}q_t\underline{v}_{\mathbf{u}} + \mu_{\mathbf{d}}(1 - q_t)J_{\mathbf{u}}(W_{t-} + H_t)$$
$$- \mu_{\mathbf{d}}(\underline{v}_{\mathbf{d}} - W_{t-}) - c_{\mathbf{d}} = -c_{\mathbf{d}} - (r + \mu_{\mathbf{d}})\underline{v}_{\mathbf{d}} + \mu_{\mathbf{d}}q_t\underline{v}_{\mathbf{u}} + \mu_{\mathbf{d}}(1 - q_t)V_{\mathbf{u}}(W_{t-} + H_t)$$
$$\le -c_{\mathbf{d}} - (r + \mu_{\mathbf{d}})\underline{v}_{\mathbf{d}} + \mu_{\mathbf{d}}v_{\mathbf{u}} \le 0,$$

where the first inequality follows by taking $q_t = 0$ and $H_t = \beta_{\mathbf{d}}$, and the second inequality follows from (2.29).

Next, we can easily verify that the performance of $\Gamma^*_{\mathbf{u}}$ is

$$U(\Gamma^*_{\mathbf{u}}, \nu^*, \mathbf{d}) = J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$$

and

$$U(\Gamma^*_{\mathbf{u}}, \nu^*, \mathbf{u}) = J_{\mathbf{u}}(\beta_{\mathbf{u}}) = v_{\mathbf{u}} - \beta_{\mathbf{u}}$$

Starting from state $\mathbf{d}$, it is optimal to let $W_0 = 0$, hence $\underline{v}_d \geq U(\Gamma, \nu^*, \mathbf{d})$. Starting from state $\mathbf{u}$, if $v_{\mathbf{u}} - \beta_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$, it is optimal to let $W_0 = \beta_{\mathbf{u}}$ and if $v_{\mathbf{u}} - \beta_{\mathbf{u}} < \underline{v}_{\mathbf{u}}$, it is optimal to let $W_0 = 0$. Hence, $U(\Gamma^*(\beta_{\mathbf{u}}).\nu^*, \mathbf{u}) \geq U(\Gamma, \nu^*, \mathbf{u})$ if $v_{\mathbf{u}} - \beta_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{u}} \geq U(\Gamma, \nu^*, \mathbf{u})$ if $v_{\mathbf{u}} - \beta_{\mathbf{u}} < \underline{v}_{\mathbf{u}}$.

A.1.5.4 Proof of Theorem 2.3

It suffices to show that if (2.30) is satisfied, then the principal's value functions $J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w$ and $J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w$ satisfy the optimality condition $\Phi_t \leq 0$ where Φ_t is defined by (A.12). In fact,

$$\Phi_t = \Phi_t^{\mathbf{u}} \mathbb{1}_{\theta_t = \mathbf{u}} + \Phi_t^{\mathbf{d}} \mathbb{1}_{\theta_t = \mathbf{d}},$$

where

$$\begin{aligned}
\Phi_t^{\mathbf{u}} &= R - rW_{t-} + \mu_{\mathbf{u}}[-q_t W_{t-} + (1-q_t)H_t] - r(\underline{v}_{\mathbf{u}} - W_{t-}) + \mu_{\mathbf{u}} q_t \underline{v}_{\mathbf{d}} + \mu_{\mathbf{u}}(1-q_t)(\underline{v}_{\mathbf{d}} - W_{t-} - H_t) \\
&\quad - \mu_{\mathbf{u}}(\underline{v}_{\mathbf{u}} - W_{t-}) - c_{\mathbf{u}} \\
&= R - c_{\mathbf{u}} - r\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} - \mu_{\mathbf{u}}\underline{v}_{\mathbf{u}} = R - c_{\mathbf{u}} - (r + \mu_{\mathbf{u}})\frac{(r + \underline{\mu}_{\mathbf{d}})R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} + \mu_{\mathbf{u}}\frac{\underline{\mu}_{\mathbf{d}} R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} \\
&= \frac{\Delta\mu_{\mathbf{u}} R}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} - c_{\mathbf{u}} = \frac{\Delta\mu_{\mathbf{u}}}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}}(R - (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{u}}) < 0,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_t^{\mathbf{d}} &= -rW_{t-} + \mu_{\mathbf{d}}[-q_t W_{t-} + (1-q_t)H_t] - r(\underline{v}_{\mathbf{d}} - W_{t-}) + \mu_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} + \mu_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) \\
&\quad - \mu_{\mathbf{d}}(\underline{v}_{\mathbf{d}} - W_{t-}) - c_{\mathbf{d}} \\
&= -c_{\mathbf{d}} - r\underline{v}_{\mathbf{d}} + \mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - \mu_{\mathbf{d}}\underline{v}_{\mathbf{d}} = -c_{\mathbf{d}} - (r + \mu_{\mathbf{d}})\frac{\underline{\mu}_{\mathbf{d}} R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} + \mu_{\mathbf{d}}\frac{(r + \underline{\mu}_{\mathbf{d}})R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} \\
&= \frac{\Delta\mu_{\mathbf{d}} R}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} - c_{\mathbf{d}} = \frac{\Delta\mu_{\mathbf{d}}}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}}(R - (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{d}}) < 0,
\end{aligned}$$

where the inequalities follow from (2.30).

A.1.6 Results and Proofs in Section 2.4.2

A.1.6.1 Proof of Lemma 2.2

Using (2.36) and (2.37) as boundary conditions, (2.34) is a linear differential equation with boundary condition. The solution is

$$J_{\mathbf{d}}^{a\hat{\beta}}(w) = aw + \frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} + C_1(\bar{w}_{\mathbf{d}} - w)^{\frac{r + \mu_{\mathbf{d}}}{r}}, \text{ for } w \in [0, \min\{\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\}], \tag{A.51}$$

with

$$C_1 = -\frac{\left[\frac{\Delta\mu_{\mathbf{d}} R}{r+\mu_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}} - c_{\mathbf{d}}\right] \bar{w}_{\mathbf{d}}^{-\frac{r+\mu_{\mathbf{d}}}{r}}}{r+\mu_{\mathbf{d}}} < 0, \tag{A.52}$$

in which the inequality follows from (2.18). Therefore, we can solve $J_{\mathbf{u}}^{a\hat{\beta}}$ for $\left[\hat{\beta}, \min\{\hat{\beta}-\beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\}+\beta_{\mathbf{u}}\right]$ using (2.35), (2.36) and (A.51). By induction, we can solve $J_{\mathbf{d}}^{a\hat{\beta}}$ in $[0, \bar{w}_{\mathbf{d}}]$ and $J_{\mathbf{u}}^{a\hat{\beta}}$ in $[0, \bar{w}_{\mathbf{u}}]$. These are a sequence of initial value problems satisfying the Cauchy-Lipschitz Theorem, and, therefore, bear unique solutions. Furthermore, $J_{\mathbf{u}}^{a\hat{\beta}}$ is $C^2\left([0,\bar{w}_{\mathbf{u}}]\setminus\{\hat{\beta}\}\right)$ and $J_{\mathbf{d}}^{a\hat{\beta}}$ is $C^3\left([0,\bar{w}_{\mathbf{d}}]\setminus\{\hat{\beta}-\beta_{\mathbf{d}}\}\right)$. For $w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}}]$, (2.32) and (2.35) together imply that

$$(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\hat{w}_{\mathbf{u}}} J_{\mathbf{u}}'(w) = (\mu_{\mathbf{u}}+r)J_{\mathbf{u}}(w) - R - \frac{\mu_{\mathbf{u}}\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} J_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(w-\beta_{\mathbf{u}})\right) + \frac{\mu_{\mathbf{u}} c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} + \ell^* \mathbb{1}_{w=\hat{w}_{\mathbf{u}}}. \tag{A.53}$$

If we define $w_0 := \bar{w}_{\mathbf{u}}$ and $w_n := (\mu_{\mathbf{d}} w_{n-1})/(r+\mu_{\mathbf{d}}) + \beta_{\mathbf{u}}$ for $n = 1, 2, 3...$, then $\hat{w}_{\mathbf{u}} = \lim_{n\to\infty} w_n$. Furthermore, (A.53) is equivalent to a sequence of initial value problems over the intervals $[w_n, w_{n+1}]$, $n = 1, 2, ...$ This sequence of initial value problem again satisfy the Cauchy-Lipschitz Theorem and bear unique solutions. Furthermore, if $\hat{\beta} < \bar{w}_{\mathbf{d}} + \beta_{\mathbf{d}}$, then $J_{\mathbf{u}}^{a\hat{\beta}}$ is $C^2\left([0, \hat{w}_{\mathbf{u}}) \setminus \{\hat{\beta}\}\right)$, $J_{\mathbf{d}}^{a\hat{\beta}}$ is $C^3\left([0, \hat{w}_{\mathbf{d}}) \setminus \{\hat{\beta}-\beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\}\right)$ and if $\hat{\beta} \geq \bar{w}_{\mathbf{d}} + \beta_{\mathbf{d}}$, then $J_{\mathbf{u}}^{a\hat{\beta}}$ is $C^2([0, \hat{w}_{\mathbf{u}}) \setminus \{\hat{\beta}, (\mu_{\mathbf{d}}\hat{\beta})/(r+\mu_{\mathbf{d}}) + \beta_{\mathbf{u}}\})$, $J_{\mathbf{d}}^{a\hat{\beta}}$ is $C^3([0, \hat{w}_{\mathbf{d}}) \setminus \{\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}, \hat{\beta} + (\mu_{\mathbf{d}}\beta_{\mathbf{u}})/(r+\mu_{\mathbf{d}})\})$. Then, we could derive the expressions for $J_{\mathbf{u}}^{a\hat{\beta}''}$, $J_{\mathbf{d}}^{a\hat{\beta}''}$ and $J_{\mathbf{d}}^{a\hat{\beta}'''}$ following (2.32), (2.34) and (2.35), respectively,

$$J_{\mathbf{u}}^{a\hat{\beta}''}(w) = \frac{\mu_{\mathbf{u}}\left(J_{\mathbf{u}}^{a\hat{\beta}'}(w) - J_{\mathbf{d}}^{a\hat{\beta}'}(w-\beta_{\mathbf{u}})\right)}{rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}}}, \text{ for } w \in (\hat{\beta}, \hat{w}_{\mathbf{u}}), \tag{A.54}$$

$$J_{\mathbf{u}}^{a\hat{\beta}''}(w) = \frac{\mu_{\mathbf{u}}\left(J_{\mathbf{u}}^{a\hat{\beta}'}(w) - J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(w-\beta_{\mathbf{u}})\right)\right)}{rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}}}, \text{ for } w \in [\bar{w}_u, \hat{w}_{\mathbf{u}}), \tag{A.55}$$

$$J_{\mathbf{d}}^{a\hat{\beta}''}(w) = \frac{\mu_{\mathbf{d}}\left(J_{\mathbf{u}}^{a\hat{\beta}'}(w+\beta_{\mathbf{d}}) - J_{\mathbf{d}}^{a\hat{\beta}'}(w)\right)}{r(\bar{w}_{\mathbf{d}}-w)}, \text{ for } w \in [0, \bar{w}_{\mathbf{d}}) \setminus \{\hat{\beta}-\beta_{\mathbf{d}}\}, \tag{A.56}$$

$$J_{\mathbf{d}}^{a\hat{\beta}''}(w) = J_{\mathbf{u}}^{a\hat{\beta}''}\left(w + \frac{rw}{\mu_{\mathbf{d}}}\right), \text{ for } w \in (\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}), \text{ and} \tag{A.57}$$

$$J_{\mathbf{d}}^{a\hat{\beta}'''}(w) = \frac{\mu_{\mathbf{d}}\left(J_{\mathbf{u}}^{a\hat{\beta}''}(w+\beta_{\mathbf{d}}) - J_{\mathbf{d}}^{a\hat{\beta}''}(w)\right) + rJ_{\mathbf{d}}^{a\hat{\beta}''}(w)}{r(\bar{w}_{\mathbf{d}}-w)}, \text{ for } w \in [0, \bar{w}_{\mathbf{d}}) \setminus \{\hat{\beta}-\beta_{\mathbf{d}}\}. \tag{A.58}$$

A.1.6.2 Proof of Lemma 2.3

Following (2.35), we can calculate for $\hat{\beta} \in [\beta_u, \bar{w}_u)$:

$$\begin{aligned} J_{\mathbf{u}}^{a\hat{\beta}'}(\hat{\beta}_+) &= \frac{(r+\mu_{\mathbf{u}})J_{\mathbf{u}}(\hat{\beta}) - \mu_{\mathbf{u}} J_{\mathbf{d}}(\hat{\beta}-\beta_{\mathbf{u}}) - R + c_{\mathbf{u}}}{r\hat{\beta} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \\ &= \frac{(r+\mu_{\mathbf{u}})(\underline{v}_{\mathbf{u}} + a\hat{\beta}) - \mu_{\mathbf{u}}\left[a(\hat{\beta}-\beta_{\mathbf{u}}) + \frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} + C_1(\bar{w}_{\mathbf{d}} - \hat{\beta} + \beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right] - R + c_{\mathbf{u}}}{r\hat{\beta} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \\ &= a + \frac{(r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} - \mu_{\mathbf{u}}\left[\frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} + C_1(\bar{w}_{\mathbf{d}} - \hat{\beta} + \beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right] - R + c_{\mathbf{u}}}{r\hat{\beta} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}}, \end{aligned} \tag{A.59}$$

where C_1 follows (A.52). Furthermore, following equation (2.38) and (A.59), we have for $\hat{\beta} \in [\beta_u, \bar{w}_u)$,

$$f_a(\hat{\beta}) = -(r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}}\left[\frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} + C_1(\bar{w}_{\mathbf{d}} - \beta + \beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right] + R - c_{\mathbf{u}}, \tag{A.60}$$

and $f_a(\hat{\beta})$ is increasing in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$ because $C_1 < 0$. Therefore,

$$\begin{aligned} \lim_{\hat{\beta}\uparrow\bar{w}_{u-}} f_a(\hat{\beta}) &= -(r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}}\left[\frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r}\right] + R - c_{\mathbf{u}} \\ &= \frac{r\Delta\mu_{\mathbf{u}} + \mu_{\mathbf{u}}\Delta\mu_{\mathbf{d}} + \mu_{\mathbf{d}}\Delta\mu_{\mathbf{u}}}{(\mu_{\mathbf{d}}+r)(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})}R - \frac{\mu_{\mathbf{u}}c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} - c_{\mathbf{u}} \geq 0, \end{aligned}$$

where the last inequality follows from the condition (2.19).

A.1.6.3 Proof of Proposition 2.4

We show the result following three steps.

1. Show that $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, \hat{w}_{\mathbf{d}})$, and $J_{\mathbf{u}}^{a\beta_a}$ is concave in $[0, \hat{w}_{\mathbf{u}})$ and strictly concave in $[\beta_a, \hat{w}_{\mathbf{u}})$.
2. Show that for any $w \geq 0$, derivatives $\frac{d}{dw}J_{\mathbf{u}}^{a\beta_a}(w)$ and $\frac{d}{dw}J_{\mathbf{d}}^{a\beta_a}(w)$ are increasing in a.
3. There exists unique $\bar{a} > -1$ such that (2.41) is satisfied, and the corresponding functions $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ are both concave with derivatives greater than or equal to -1.

Step 1. For any $a > -1$, if $\beta_a = \beta_{\mathbf{u}}$, then $J_{\mathbf{u}}^{a\beta_a}$ is $C^2([0, \bar{w}_{\mathbf{u}}) \setminus \{\beta_{\mathbf{u}}\})$ and $J_{\mathbf{d}}^{a\beta_a}$ is $C^3([0, \bar{w}_{\mathbf{d}}) \setminus \{\beta_{\mathbf{u}} - \beta_{\mathbf{d}}\})$. Otherwise, if $\beta_a > \beta_{\mathbf{u}}$, then $J_{\mathbf{u}}^{a\beta_a}$ is $C^2([0, \hat{w}_{\mathbf{u}}))$ and $J_{\mathbf{d}}^{a\beta_a}$ is $C^3([0, \bar{w}_{\mathbf{d}}) \setminus \{\bar{w}_{\mathbf{d}}\})$. Following (A.51) and (A.52), we have $J_{\mathbf{d}}^{a\beta_a}(w)$ is strictly concave with $J_{\mathbf{d}}^{a\beta'_a}(w) > a$ in the interval $[0, \beta_a - \beta_{\mathbf{d}})$. We claim that $J_{\mathbf{d}}^{a\beta''_a}((\beta_a - \beta_{\mathbf{d}})_+) < 0$. If $\beta_a > \beta_{\mathbf{u}}$, then this result directly follows by smooth pasting. Otherwise, if $\beta_a = \beta_{\mathbf{u}}$, equation (A.56) implies that

$$J_{\mathbf{d}}^{a\beta''_a}((\beta_{\mathbf{u}} - \beta_{\mathbf{d}})_+) = \frac{\mu_{\mathbf{d}}\left(J_{\mathbf{u}}^{a\beta'_a}(\beta_{\mathbf{u}+}) - J_{\mathbf{d}}^{a\beta'_a}(\beta_{\mathbf{u}} - \beta_{\mathbf{d}})\right)}{r(\bar{w}_{\mathbf{d}} - w)} < 0,$$

where the inequality follows from $a - J_{\mathbf{u}}^{a\beta'_a}(\beta_{\mathbf{u}+}) \geq 0$ which is implied by the definition of β_a and $J_{\mathbf{d}}^{a\beta'_a}(\beta_{\mathbf{u}} - \beta_{\mathbf{d}}) > a$. Next, we prove that $J_{\mathbf{u}}^{a\beta_a}(w)$ is strictly concave in $[\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}]$. First, following (A.54), we have

$$J_{\mathbf{u}}^{a\beta''_a}(\beta_{a+}) = \frac{\mu_{\mathbf{u}}\left(J_{\mathbf{u}}^{a\beta'_a}(\beta_{a+}) - J_{\mathbf{d}}^{a\beta'_a}(\beta_a - \beta_{\mathbf{u}})\right)}{r\beta_a + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} < 0,$$

where the inequality follows from $J_{\mathbf{u}}^{a\beta'_a}(\beta_{a+}) \leq a$ and $J_{\mathbf{d}}^{a\beta'_a}(\beta_a - \beta_{\mathbf{u}}) > a$. Assume that there exists $w \in (\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}]$ such that $J_{\mathbf{u}}^{a\beta''_a}(w) \geq 0$, then $J_{\mathbf{u}}^{a\beta_a}$ being twice continuously differentiable implies that there must exist $\hat{w} = \min\{w \in (\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}] | J_{\mathbf{u}}^{a\beta''_a}(w) = 0\}$, such that $J_{\mathbf{u}}^{a\beta''_a}(w) < 0$ for $w < \hat{w}$. Equation (A.54) implies that

$$J_{\mathbf{u}}^{a\beta'_a}(\hat{w}) = J_{\mathbf{d}}^{a\beta'_a}(\hat{w} - \beta_{\mathbf{u}}).$$

Since $J_{\mathbf{d}}^{a\beta_a}$ is concave in the interval $[0, \min\{\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\}]$, equation (A.56) implies that $J_{\mathbf{u}}^{a\beta'_a}(\hat{w} + \beta_{\mathbf{d}} - \beta_{\mathbf{u}}) < J_{\mathbf{d}}^{a\beta'_a}(\hat{w} - \beta_{\mathbf{u}})$, which further implies that

$$J_{\mathbf{u}}^{a\beta'_a}(\hat{w} + \beta_{\mathbf{d}} - \beta_{\mathbf{u}}) < J_{\mathbf{u}}^{a\beta'_a}(\hat{w}),$$

which contradicts with $J_{\mathbf{u}}^{a\beta''_a}(w) < 0$ for $w < \hat{w}$. Hence, $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}]$.

Next we prove two lemmas.

Lemma A.5 *For any $w \geq 0$, if $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, w + \beta_a - \beta_{\mathbf{d}}]$ and $J_{\mathbf{u}}^{a\beta_a}$ is concave in $[\beta_a, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $J_{\mathbf{d}}^{a\beta_a}$ is also strictly concave in $[0, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$.*

Proof. Assume that there exists $w \in [w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$ such that $J_{\mathbf{d}}^{a\beta''_a}(w) \geq 0$, then the fact that $J_{\mathbf{d}}^{a\beta_a}$ is twice continuously differentiable implies that there must exist $\tilde{w} = \min\{w \in (w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}] | J_{\mathbf{d}}^{a\beta''_a}(\tilde{w}) = 0\}$, such that $J_{\mathbf{d}}^{a\beta''_a}(w) < 0$ for $w < \tilde{w}$. Equation (A.58) implies that

$$J_{\mathbf{d}}^{a\beta'''_a}(\hat{w}) = \frac{\mu_{\mathbf{d}} J_{\mathbf{u}}^{a\beta''_a}(w + \beta_{\mathbf{d}})}{r(\bar{w}_{\mathbf{d}} - \hat{w})} < 0,$$

where the inequality follows from $J_{\mathbf{u}}^{a\beta_a}$ being concave in $[0, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$. This contradicts with $J_{\mathbf{d}}^{a\beta''_a}(\tilde{w}) = 0$ and $J_{\mathbf{d}}^{a\beta''_a}(w) < 0$ for $w < \tilde{w}$.

Lemma A.6 *For any $w \geq 0$, if $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[0, w]$ and $J_{\mathbf{d}}^{a\beta_a}$ is concave in $[0, w - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $J_{\mathbf{u}}^{a\beta_a}$ is also strictly concave in $[w, w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$.*

The proof for Lemma A.6 follows the same logic as Lemma A.5, and is omitted here. Equipped with Lemmas A.5 and A.6, we prove that if $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ and $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, w + \beta_a - \beta_{\mathbf{d}}]$, then $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, w + \beta_a + 2\beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$ and $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$. Hence, by induction, $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, \bar{w}_{\mathbf{u}})$ and $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, \bar{w}_{\mathbf{d}})$.

We have $J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}-) > J_{\mathbf{u}}^{a\beta'_a}(\bar{w}_{\mathbf{u}})$ from (A.54) and $J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}+) = J_{\mathbf{u}}^{a\beta'_a}(\bar{w}_{\mathbf{u}})$ from (2.32). Hence, $J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}-) > J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}+)$. Finally, we prove that $J_{\mathbf{u}}^{a\beta''_a}(w+) < 0$ for $w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}})$. If there exists

$w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}})$ such that $J_{\mathbf{u}}^{a\beta_a''}(w+) \geq 0$, then there must exist $\breve{w} = \min\{w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}}) | J_{\mathbf{u}}^{a\beta_a''}(w+) = 0\}$, such that $J_{\mathbf{u}}^{a\beta_a''}(w+) < 0$ for $w < \breve{w}$. Finally, (A.55) implies that

$$J_{\mathbf{u}}^{a\hat{\beta}'}(\breve{w}) - J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})_+\right) = 0,$$

which contradicts with

$$J_{\mathbf{u}}^{a\hat{\beta}'}(\breve{w}) = J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})_+\right) + \int_{\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})}^{\breve{w}} J_{\mathbf{u}}^{a\hat{\beta}''}(x)dx < J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})\right),$$

where the inequality follows from $\breve{w} > \frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})$ and $J_{\mathbf{u}}^{a\beta_a''}(w_+) < 0$ for $w < \breve{w}$. Following (A.57), $J_{\mathbf{d}}^{a\beta_a''}$ is also strictly concave in $[\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}})$.

Step 2. We show that for any $w \geq 0$, $dJ_{\mathbf{u}}^{a\beta_a}/dw$ and $dJ_{\mathbf{d}}^{a\beta_a}/dw$ are increasing in a. To do so, we define

$$g_{\mathbf{d}}(w) := \frac{dJ_{\mathbf{d}}^{a\beta_a}}{da}(w_+) \text{ and } g_{\mathbf{u}}(w) := \frac{dJ_{\mathbf{u}}^{a\beta_a}}{da}(w_+).$$

It suffices to prove that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ are well-defined and strictly increasing in w.

- For $w \in [0, \beta_a)$, we have $g_{\mathbf{u}}(w) = w$, which is strictly increasing in w. For $w \in [0, \beta_a - \beta_{\mathbf{d}}]$, $g_{\mathbf{d}}(w) = w$ which is also strictly increasing in w.
- For $w = \beta_a$, we have

$$\begin{aligned} g_{\mathbf{u}}(\beta_{a+}) &= \lim_{\epsilon \downarrow 0} \frac{J_{\mathbf{u}}^{a+\epsilon\beta_a}(\beta_a) - J_{\mathbf{u}}^{a\beta_a}(\beta_a)}{\epsilon} + \frac{J_{\mathbf{u}}^{a\beta_{a+\epsilon}}(\beta_a) - J_{\mathbf{u}}^{a\beta_a}(\beta_a)}{\epsilon} \cdot \frac{d\beta_a}{da} \\ &= \lim_{\epsilon \downarrow 0} \frac{J_{\mathbf{u}}^{a+\epsilon\beta_a}(\beta_a) - J_{\mathbf{u}}^{a\beta_a}(\beta_a)}{\epsilon} = \beta_a = g_{\mathbf{u}}(\beta_{a-}), \end{aligned}$$

  where the second equality follows from $J_{\mathbf{u}}^{a\beta_{a+\epsilon}}(\beta_a) = J_{\mathbf{u}}^{a\beta_a}(\beta_a)$ because $\beta_{a+\epsilon} \geq \beta_a$ for any $\epsilon \geq 0$.
- For $J_{\mathbf{u}}^{a\beta_a}(w)$ on $[\beta_a, \bar{w}_{\mathbf{u}}]$ and $J_{\mathbf{d}}^{a\beta_a}(w)$ on $[\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$, taking derivatives with respect to a on both sides of (2.33) and (2.35), we know that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ satisfies the following system of equations:

$$(\mu_{\mathbf{d}} + r)g_{\mathbf{d}}(w) = \mu_{\mathbf{d}} g_{\mathbf{u}}(w + \beta_{\mathbf{d}}) - r(\bar{w}_{\mathbf{d}} - w)g_{\mathbf{d}}'(w), \quad w \in [0, \bar{w}_{\mathbf{d}}], \text{ and} \tag{A.61}$$
$$(\mu_{\mathbf{u}} + r)g_{\mathbf{u}}(w) = \mu_{\mathbf{u}} g_{\mathbf{d}}(w - \beta_{\mathbf{u}}) + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})g_{\mathbf{u}}'(w). \quad w \in [\beta_a, \bar{w}_{\mathbf{u}}] \tag{A.62}$$

  In the following, we prove that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ are also strictly increasing on $[\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$ and $[\beta_a, \bar{w}_{\mathbf{u}}]$, respectively. Following equation (A.62), we have

$$\begin{aligned} g_{\mathbf{u}}'(\beta_{a+}) &= \frac{(\mu_{\mathbf{u}} + r)g_{\mathbf{u}}(\beta_a) - \mu_{\mathbf{u}} g_{\mathbf{d}}(\beta_a - \beta_{\mathbf{u}})}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \\ &= \frac{(\mu_{\mathbf{u}} + r)\beta_a - \mu_{\mathbf{u}}[\beta_a - \beta_{\mathbf{u}}]}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \\ &\geq \frac{(\mu_{\mathbf{u}} + r)\beta_{\mathbf{u}}}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} > 0, \end{aligned}$$

  where the second inequality follows from $\beta_a \geq \beta_{\mathbf{u}}$. Then we claim that $g_{\mathbf{u}}(w)$ is strictly increasing in $[\beta_a, \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$. If not, then there exists $w \in (\beta_a, \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ such that $g_{\mathbf{u}}'(w) \geq 0$. Therefore, we must have $\hat{w} = \min\{w \in (\beta_a, \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}] | g'_{\mathbf{u}}(w) = 0\}$ and $g'_{\mathbf{u}}(w) > 0$ for $w < \hat{w}$. Equation (A.62) implies that

$$(r + \mu_{\mathbf{u}}) g_{\mathbf{u}}(\hat{w}) = \mu_{\mathbf{u}} g_{\mathbf{d}}(\hat{w} - \beta_{\mathbf{u}}).$$

The fact that $g_{\mathbf{d}}(w)$ is increasing in $[0, \beta_a - \beta_{\mathbf{d}}]$ implies that $(\mu_{\mathbf{d}} + r) g_{\mathbf{d}}(w - \beta_{\mathbf{u}}) < \mu_{\mathbf{d}} g_{\mathbf{u}}(\hat{w} - \beta_{\mathbf{u}} + \beta_{\mathbf{d}})$, which further implies that

$$(r + \mu_{\mathbf{u}}) g_{\mathbf{u}}(\hat{w}) = \mu_{\mathbf{u}} g_{\mathbf{d}}(\hat{w} - \beta_{\mathbf{u}}) < \mu_{\mathbf{u}} \frac{\mu_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} g_{\mathbf{u}}(\hat{w} - \beta_{\mathbf{u}} + \beta_{\mathbf{d}}),$$

which contradicts $g'_{\mathbf{u}}(w) > 0$ for $w < \hat{w}$. We establish the final results by proving the next two claims.

Lemma A.7 *If $g_{\mathbf{d}}$ is strictly increasing in $[0, w + \beta_a - \beta_{\mathbf{d}}]$ and $g_{\mathbf{u}}$ is strictly increasing in $[0, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $g_{\mathbf{d}}$ is also increasing in $[w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$.*

Proof. If there exists $w \in (w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$ such that $g'_{\mathbf{d}}(w) \leq 0$, then we must have $\tilde{w} = \min\{w \in (w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}] | g'_{\mathbf{d}}(w) = 0\}$ such that $g'_{\mathbf{d}}(w) > 0$ for $w < \tilde{w}$. Differentiating (A.61), we obtain that

$$g''_{\mathbf{d}}(\hat{w}) = \frac{\mu_{\mathbf{d}}(g'_{\mathbf{u}}(\hat{w} + \beta_{\mathbf{d}}) - g'_{\mathbf{d}}(\hat{w}))}{r(\bar{w}_{\mathbf{d}} - w)} > 0,$$

where the inequality holds because $g_{\mathbf{u}}$ is increasing on $[0, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$. However, this contradicts $g'_{\mathbf{d}}(\hat{w}) = 0$ and $g'_{\mathbf{d}}(w) > 0$ for $w < \tilde{w}$.

Lemma A.8 *If $g_{\mathbf{u}}$ is strictly concave in $[0, w]$ and $g_{\mathbf{d}}$ is concave in $[0, w - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $g_{\mathbf{u}}$ is also strictly concave in $[w, w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$.*

The logic of the proof of Lemma A.8 is similar to that of Lemma A.7, and is therefore omitted here. Following Lemmas A.7 and A.8, we can prove by induction that $g_{\mathbf{u}}$ is strictly concave in $[\beta_a, \bar{w}_{\mathbf{u}})$ and $g_{\mathbf{d}}$ is strictly concave in $[0, \bar{w}_{\mathbf{d}})$.

- For $J_{\mathbf{u}}^{a\beta_a}(w)$ on $[\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}})$ and $J_{\mathbf{d}}^{a\beta_a}(w)$ on $[\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}]$, taking derivatives with respect to a on both sides of (2.32) and (A.53), we know that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ satisfies the following system of equations,

$$(\mu_{\mathbf{d}} + r) g_{\mathbf{d}}(w) = \mu_{\mathbf{d}} g_{\mathbf{u}} \left(w + \frac{rw}{\mu_{\mathbf{d}}} \right) \text{ for } w \in [\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}], \text{ and} \tag{A.63}$$

$$(\mu_{\mathbf{u}} + r) g_{\mathbf{u}}(w) = \frac{\mu_{\mathbf{u}}\mu_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} g_{\mathbf{u}} \left(\frac{r + \mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(w - \beta_{\mathbf{u}}) \right) + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}) \mathbb{1}_{w < \hat{w}_{\mathbf{u}}} g'_{\mathbf{u}}(w) \text{ for } w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}}]. \tag{A.64}$$

Since (A.63) implies that $g'_{\mathbf{d}}(w) = g'_{\mathbf{u}}\left(w + \frac{rw}{\mu_{\mathbf{d}}}\right)$ for $w \in [\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}]$, we just need to show that $g'_{\mathbf{u}}(w) > 0$ for $w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}})$. We have proved that $g'_u(w) > 0$ for $w \in [0, \bar{w}_{\mathbf{u}}]$. If there exists $w \in (\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}})$ such that $g'_{\mathbf{u}}(w) \leq 0$, then there must be $\breve{w} = \min\{w \in (\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}}) | g'_{\mathbf{u}}(w) = 0\}$, such that $g'_u(w) > 0$ for $w > \breve{w}$. Then, (A.64) implies that

$$(\mu_{\mathbf{u}} + r) g_{\mathbf{u}}(\breve{w}) = \frac{\mu_{\mathbf{u}}\mu_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} g_{\mathbf{u}} \left(\frac{r + \mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w} - \beta_{\mathbf{u}}) \right).$$

However, this contradicts with

$$g_{\mathbf{u}}(\breve{w}) = g_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})\right) + \int_{\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})}^{w} g'_{\mathbf{u}}(x)dx > g_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})\right).$$

Step 3. Since for any $w \geq 0$, derivatives $\frac{d}{dw}J_{\mathbf{u}}^{a\beta_a}(w)$ and $\frac{d}{dw}J_{\mathbf{d}}^{a\beta_a}(w)$ are increasing in a, with boundary condition (2.37), $J_{\mathbf{u}}^{a\beta_a}(w)$ and $J_{\mathbf{d}}^{a\beta_a}(w)$ are also increasing in a. For a approaching -1, we have $\lim_{w\uparrow\bar{w}_{\mathbf{u}}} J_{\mathbf{u}}^{a\beta_a}(w) < \underline{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}} \leq \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}}$. For a approaching ∞, we have $\lim_{w\uparrow\bar{w}_{\mathbf{u}}} J_{\mathbf{u}}^{a\beta_a}(w) \to \infty$. Hence, there exists a unique $a > 0$, denoted as $\bar{a}$, such that $\lim_{w\uparrow\bar{w}_{\mathbf{u}}} J_{\mathbf{u}}^{a\beta_a}(w) = \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}}$. Following (2.32), we have $\lim_{w\uparrow\hat{w}_{\mathbf{d}}} J_{\mathbf{d}}^{a\beta_a}(w) = \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}}$.

Then, (2.32) and (A.53) imply that $J_{\mathbf{d}}^{a\beta_a}(\hat{w}_{\mathbf{d}}) = \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}}$, $J_{\mathbf{u}}^{a\beta_a}(\hat{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}}$, $\lim_{w\uparrow\bar{w}_{\mathbf{u}}} J_{\mathbf{u}}^{a\beta'_a}(w) = -1$, and $\lim_{w\uparrow\hat{w}_{\mathbf{d}}} J_{\mathbf{d}}^{a\beta'_a}(w) = -1$. Hence, (2.41) is satisfied and the corresponding functions $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}$ on $[0, \hat{w}_{\mathbf{u}}]$ and $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}$ on $[0, \hat{w}_{\mathbf{d}}]$ are strictly concave. Further, the derivatives of $J_{\mathbf{u}}^{a\beta_a}$ and $J_{\mathbf{d}}^{a\beta_a}$ are greater than or equal to -1.

Finally, following (A.54), (A.56), (A.57) and the concavity of $J_{\mathbf{d}}^{a\beta_a}$ and $J_{\mathbf{u}}^{a\beta_a}$, we have

$$J_{\mathbf{u}}^{a\beta'_a}(w) < J_{\mathbf{d}}^{a\beta'_a}(w-\beta_{\mathbf{u}}), \text{ for } w \in (\beta, \hat{w}_{\mathbf{u}}), \tag{A.65}$$

$$J_{\mathbf{u}}^{a\beta'_a}(w+\beta_{\mathbf{d}}) < J_{\mathbf{d}}^{a\beta'_a}(w), \text{ for } w \in [0, \bar{w}_{\mathbf{d}}) \setminus \{\beta_{\bar{a}} - \beta_{\mathbf{d}}\},$$

$$J_{\mathbf{u}}^{a\beta'_a}(\beta_{\bar{a}+}), J_{\mathbf{u}}^{a\beta'_a}(\beta_{\bar{a}-}) < J_{\mathbf{d}}^{a\beta'_a}(\beta_{\bar{a}} - \beta_{\mathbf{d}}), \text{ and} \tag{A.66}$$

$$J_{\mathbf{d}}^{a\beta'_a}(w) = J_{\mathbf{u}}^{a\beta'_a}\left(w + \frac{rw}{\mu_{\mathbf{d}}}\right) \text{ for } w \in (\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}]. \tag{A.67}$$

A.1.6.4 Proof of Proposition 2.5

Following definition (A.13) and equation (A.14), we obtain that under contract $\Gamma^*_{\beta_a}$ in Definition 2.3,

$$e^{-r\tau}J(\tau) = J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt + c(\theta_t)dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t^*, \tag{A.68}$$

where

$$
\begin{aligned}
\mathcal{A}_t^* &= dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t \\
&= J'(t)[rW_{t-} - \mu(\theta_t,1)H_t^* - \ell_t^*]dt - rJ(t)dt + J(t+) - J(t) + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^* - c(\theta_t)dt \\
&= \left\{J_{\mathbf{u}}'(t)(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\hat{w}_{\mathbf{u}}} - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}}\right\} dt\mathbb{1}_{\theta_t=\mathbf{u}} \\
&+ \left\{J_{\mathbf{d}}'(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{d}}}dt - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}\right\} dt\mathbb{1}_{\theta_t=\mathbf{d}} \\
&+ \left\{\left[J_{\mathbf{u}}\left(W_{t-} + \frac{rW_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\geq\bar{w}_{\mathbf{d}}} + [J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})}\right. \\
&\left.+[(J_{\mathbf{u}}(\beta_a) - J_{\mathbf{d}}(W_{t-}))(1-X_t) + (J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-}))X_t]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}}\right\}dN_t\mathbb{1}_{\theta_t=\mathbf{d}} \\
&+ [J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})]dN_t\mathbb{1}_{\theta_t=\mathbf{u}} + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^* \\
&=\{R + J_{\mathbf{u}}'(W_{t-})(rW_t + c_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}} - rJ_{\mathbf{u}}(W_{t-})dt + \mu_{\mathbf{u}}(J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})) \\
&+(r\bar{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}} + c_{\mathbf{u}})\mathbb{1}_{W_{t-}=\bar{w}_{\mathbf{u}}}\}\mathbb{1}_{\theta_t=\mathbf{u}} - c_{\mathbf{u}}dt \\
&+ \left\{J_{\mathbf{d}}'(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{d}}} - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} + [\mu_{\mathbf{d}}q_t^*(J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-}))\right. \\
&+\mu_{\mathbf{d}}(1-q_t^*)(J_{\mathbf{u}}(\beta_a) - J_{\mathbf{d}}(W_{t-}))]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}} + \mu_{\mathbf{d}}\left[J_{\mathbf{u}}\left(W_{t-} + \beta_{\mathbf{d}}\right) - J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})} \\
&\left.+\mu_{\mathbf{d}}\left[J_{\mathbf{u}}\left(W_{t-} + \frac{rW_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{d}}(W_{t-})\right]\right\}\mathbb{1}_{\theta_t=\mathbf{d}}dt + \mathcal{B}_t^* \\
&= \mathcal{B}_t^*,
\end{aligned}
$$

in which the last equality follows from (2.33), (2.34), (2.35) and

$$
\begin{aligned}
\mathcal{B}_t^* =\{&\left[J_{\mathbf{u}}\left(W_{t-} + \frac{rW_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\geq\bar{w}_{\mathbf{d}}}(dN_t - \mu_{\mathbf{d}}dt) \\
&+ [J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})}(dN_t - \mu_{\mathbf{d}}dt) \\
&+ [(J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-}))(X_t dN_t - \mu_{\mathbf{d}}q_t^* dt) \\
&+ (J_{\mathbf{u}}(\beta_a) - J_{\mathbf{d}}(W_{t-}))((1-X_t)dN_t - \mu_{\mathbf{d}}(1-q_t^*)dt)]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}}\}\mathbb{1}_{\theta_t=\mathbf{d}} \\
&+ [J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})](dN_t - \mu_{\mathbf{u}}dt)\mathbb{1}_{\theta_t=\mathbf{u}}.
\end{aligned}
$$

Taking the expectation on both sides of (A.68), we obtain

$$
J_{\theta_0}(w) = J(0) = \mathbb{E}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t^*)\right] = u(\Gamma_{\beta_{\bar{a}}}^*(w), \nu^*, \theta_0),
$$

where $u(\Gamma_{\beta_{\bar{a}}}^*, \nu^*, \theta_0) = w$, and we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^* dt$ is a martingale and $J(\tau) = J_{\theta_\tau}(0) = v_\tau$.

A.1.6.5 Proof of Theorem 2.4

From Proposition 2.4, we know that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are concave, $J_{\mathbf{d}}'(w) \geq -1$ and $J_{\mathbf{u}}'(w) \geq -1$. Given Lemma A.2, we only need to show $\Phi_t \leq 0$ holds almost surely if $\nu_t = 1$. From (A.12), we have

$$
\Phi_t = \Phi_t^u\mathbb{1}_{\theta_t=\mathbf{u}} + \Phi_t^d\mathbb{1}_{\theta_t=\mathbf{d}},
$$

where

$$\Phi_t^u := R + J_{\mathbf{u}}'(W_{t-})rW_{t-} + \mu_{\mathbf{u}} q_t[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})]$$
$$+ \mu_{\mathbf{u}}(1-q_t)[-H_t J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}},$$

and

$$\Phi_t^d := J_{\mathbf{d}}'(W_t)rW_{t-} + \mu_{\mathbf{d}} q_t[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})]$$
$$+ \mu_{\mathbf{d}}(1-q_t)[-H_t J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t) - J_{\mathbf{d}}(W_{t-})] - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}.$$

We have $\Phi_t \le 0$ if $\Phi_t^d \le 0$ and $\Phi_t^u \le 0$. First, we prove that $\Phi_t^u \le 0$ by considering the following optimization problem,

$$\begin{aligned} \max_{q_t, H_t} \quad & q_t[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_t J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})] \\ s.t. \quad & 0 \le q_t \le 1,\ -q_t W_{t-} + (1-q_t)H_t \le -\beta_{\mathbf{u}}. \end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\beta_{\mathbf{u}}. \tag{A.69}$$

using the KKT conditions. Define the following dual variables for the binding constraints

$$x_{\mathbf{u}} = -(J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}'(W_{t-} - \beta_{\mathbf{u}})) \ge 0,$$

in which the inequality follows from (A.71), and

$$\begin{aligned} y_u &= (W_{t-} - \beta_{\mathbf{u}})(J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}'(W_{t-} - \beta_{\mathbf{u}})) - W_{t-}J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}(0) + \beta_{\mathbf{u}} J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) \\ &= (W_{t-} - \beta_{\mathbf{u}}) \left(\frac{J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{d}}(0)}{W_{t-} - \beta_{\mathbf{u}}} - J_{\mathbf{d}}'(W_{t-} - \beta_{\mathbf{u}}) \right) \ge 0, \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H_t^* J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t^*) - J_{\mathbf{u}}(W_{t-})] \tag{A.70}$$
$$= -y_{\mathbf{u}} - (H_t^* + W_{t-})x_{\mathbf{u}},$$
$$(1-q_t^*)(J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}'(W_{t-} - H_t^*)) = (q_t^* - 1)x_{\mathbf{u}}. \tag{A.71}$$

Therefore, (A.69) implies that

$$\Phi_t^u \le R + J_{\mathbf{u}}'(W_{t-})rW_{t-} + \mu_{\mathbf{u}}[\beta_{\mathbf{u}} J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}} = 0,$$

where the equality follows from (2.35).

Following similar logic, we prove that $\Phi_t^d \le 0$ by considering the following optimization problem,

$$\begin{aligned} \max_{q_t, H_t} \quad & q_t[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_t J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t) - J_{\mathbf{d}}(W_{t-})], \\ s.t. \quad & 0 \le q_t \le 1,\ -q_t W_{t-} + (1-q_t)H_t \ge \beta_{\mathbf{d}},\ W_t + H_t \ge \beta_{\mathbf{u}}. \end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \text{ and } H_t^* = \frac{rW_{t-}}{\mu_{\mathbf{d}}} \quad \text{if} \quad W_{t-} \geq \bar{w}_{\mathbf{d}}, \tag{A.72}$$

$$q_t^* = 0 \text{ and } H_t^* = \beta_{\mathbf{d}} \quad \text{if} \quad W_{t-} \in [\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}], \text{ and} \tag{A.73}$$

$$q_t^* = \frac{\beta_a - \beta_{\mathbf{d}} - W_{t-}}{\beta_a} \text{ and } H_t^* = -W_{t-} + \beta_a \quad \text{if} \quad W_{t-} < \beta_a - \beta_{\mathbf{d}}, \tag{A.74}$$

using the KKT conditions.

- If $W_{t-} \geq \bar{w}_{\mathbf{d}}$, define the following dual variable for the binding constraint

$$\begin{aligned} y_{\mathbf{d}} &= J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{u}}(0) - (W_{t-} + H_t^*)J'_{\mathbf{d}}(W_{t-}) \\ &= \frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}} \left[\frac{J_{\mathbf{u}}\left(\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{u}}(0)}{\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}} - J'_{\mathbf{u}}\left(\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}\right) \right] \geq 0, \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] = -y_{\mathbf{d}}, \tag{A.75}$$

$$(1 - q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H_t^*)) = J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}\left(\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}\right) = 0, \tag{A.76}$$

where (A.76) follows from (A.67).

- If $W_{t-} \in [\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$, define the following dual variables for the binding constraints,

$$x_{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) \geq 0,$$

in which the inequality follows from (A.66), and

$$\begin{aligned} y_{\mathbf{d}} &= (W_{t-} + \beta_{\mathbf{d}})(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}})) - W_{t-}J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{u}}(0) - \beta_{\mathbf{d}} J'_{\mathbf{u}}(W_{t-}) \\ &+ J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) = (W_{t-} + \beta_{\mathbf{d}}) \left(\frac{J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{u}}(0)}{W_{t-} + \beta_{\mathbf{d}}} - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) \right) \geq 0, \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$\begin{aligned} &[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] \quad \text{(A.77)} \\ =& - y_{\mathbf{d}} + (W_{t-} + H_t^*)x_{\mathbf{d}}, \\ &(1 - q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H_t^*)) = (1 - q_t^*)x_{\mathbf{d}}. \quad \text{(A.78)} \end{aligned}$$

- If $W_{t-} < \beta_a - \beta_{\mathbf{d}}$ and $\beta_a = \beta_{\mathbf{u}}$, define the following dual variables for the binding constraints

$$x_{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-}) - \frac{J_{\mathbf{u}}(\beta_{\mathbf{u}}) - J_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}} = J'_{\mathbf{d}}(W_{t-}) - a > 0$$

in which the inequality follows from (A.51), and

$$\alpha = (1 - q_t^*)(a - J'_{\mathbf{u}}(\beta_{\mathbf{u}})) \geq 0,$$

in which the inequality follows from the definition of β_a. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H^*_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H^*_t) - J_{\mathbf{d}}(W_{t-})] \tag{A.79}$$
$$= (W_{t-} + H^*_t)x_{\mathbf{d}},$$
$$(1 - q^*_t)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H^*_t)) = (1 - q^*_t)x_{\mathbf{d}} + \alpha. \tag{A.80}$$

- If $W_{t-} < \beta_a - \beta_{\mathbf{d}}$ and $\beta_a > \beta_{\mathbf{u}}$, define the following dual variable for the binding constraint

$$x_{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-}) - a > 0$$

in which the inequality follows from (A.51). One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H^*_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H^*_t) - J_{\mathbf{d}}(W_{t-})] \tag{A.81}$$
$$= (W_{t-} + H^*_t)x_{\mathbf{d}},$$
$$(1 - q^*_t)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H^*_t)) = (1 - q^*_t)x_{\mathbf{d}}. \tag{A.82}$$

where (A.82) follows from $J'_{\mathbf{u}}(\beta_a) = a$.

Therefore, (A.72), (A.73) and (A.74) together imply that

$$\begin{aligned}\Phi^d_t \le & -rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} + \mu_{\mathbf{d}}\left[J_{\mathbf{u}}\left(W_{t-} + \frac{rW_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{u}}(W_{t-})\right]\mathbb{1}_{W_{t-}\ge\bar{w}_{\mathbf{d}}} \\ & + [J'_{\mathbf{d}}(W_{t-})rW_{t-} + \mu_{\mathbf{d}}[-\beta_{\mathbf{d}}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})]]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}]} \\ & + [J'_{\mathbf{d}}(W_{t-})rW_{t-} + \mu_{\mathbf{d}}q^*_t[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] \\ & + \mu_{\mathbf{d}}(1 - q^*_t)[(\beta_a - W_{t-})J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(\beta_a) - J_{\mathbf{d}}(W_{t-})]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}} = 0,\end{aligned}$$

where the equality follows from (2.32), (2.33) and (2.34).

A.1.6.6 Proof of Proposition 2.6

For any $\bar{a} \ge 0$, (A.60) implies that $f_a(\beta_{\mathbf{u}}) \ge 0$. Therefore, the definition of β_a implies that $\beta_{\bar{a}} = \beta_{\mathbf{u}}$. Hence, if $\beta_{\bar{a}} > \beta_{\mathbf{u}}$, then $\bar{a} < 0$.

A.1.6.7 Proof of Theorem 2.5

First, it is easy to verify that $\Gamma^*_{\mathbf{d}}(w)$ is incentive compatible. Following definition 2.4, we obtain the following equation for the principal's value function at state $\mathbf{d}$,

$$(\mu_{\mathbf{d}} + r)J_{\mathbf{d}}(w) = r(w - \bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w) + \mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - \mu_{\mathbf{d}}(w + \beta_{\mathbf{d}}) - c_{\mathbf{d}}, w \in [0, \bar{w}_{\mathbf{d}}]\ , \tag{A.83}$$

with boundary condition $J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$. By solving this differential equation, we obtain that under state $\mathbf{d}$,

$$J_{\mathbf{d}}(w) = (\underline{v}_{\mathbf{d}} - v_{\mathbf{d}})\left(1 - \frac{w}{\bar{w}_{\mathbf{d}}}\right)^{1+\frac{\mu_{\mathbf{d}}}{r}} - w + v_{\mathbf{d}}\ . \tag{A.84}$$

For state $\mathbf{u}$, the societal value function is a constant,

$$J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w, \tag{A.85}$$

Following similar logic to the one we use in the proof of proposition 2.2, we can show that the principal's utilities following contract $\Gamma^*_{\mathbf{d}}(w)$ are $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ in states $\mathbf{d}$ and $\mathbf{u}$, respectively. Under condition (2.45), $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$ are concave, $J'_{\mathbf{d}}(w) \geq -1$, and $J'_{\mathbf{u}} \geq -1$. Hence, it suffices to prove that $\Phi_t \leq 0$ where Φ_t is defined in (A.12). To this end, we let

$$\Phi_t = \Phi^{\mathbf{u}}_t \mathbb{1}_{\theta_t=\mathbf{u}} + \Phi^{\mathbf{d}}_t \mathbb{1}_{\theta_t=\mathbf{d}} \ ,$$

where

$$\begin{aligned}\Phi^{\mathbf{u}}_t &= R - rW_{t-} + \mu_{\mathbf{u}}[-q_t W_{t-} + (1-q_t)H_t] - r(\underline{v}_{\mathbf{u}} - W_{t-}) + \mu_{\mathbf{u}} q_t \underline{v}_{\mathbf{d}} + \mu_{\mathbf{u}}(1-q_t) J_{\mathbf{d}}(W_{t-} + H_t) \\ &\quad - \mu_{\mathbf{u}}(\underline{v}_{\mathbf{u}} - W_{t-}) - c_{\mathbf{u}} = R - c_{\mathbf{u}} - (r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}} q_t \underline{v}_{\mathbf{d}} + \mu_{\mathbf{u}}(1-q_t) V_{\mathbf{d}}(W_{t-} + H_t) \\ &\quad \leq R - c_{\mathbf{u}} - (r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}} v_d \leq 0,\end{aligned}$$

where the first equality follows from taking $q_t = 0$ and $\underline{v}_{\mathbf{d}} \leq V_{\mathbf{d}}(W_{t-} + H_t) \leq v_{\mathbf{d}}$, and the second inequality from the opposition of (2.18). Therefore,

$$\begin{aligned}\Phi^{\mathbf{d}}_t &= J'_{\mathbf{d}}(W_{t-})(rW_{t-} - \mu_{\mathbf{d}}[-q_t W_{t-} + (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-}) + \mu_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} + \mu_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) \\ &\quad - \mu_{\mathbf{d}} J_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}.\end{aligned}$$

We prove that $\Phi^{\mathbf{d}}_t \leq 0$ by considering the following optimization problem,

$$\begin{aligned}\max_{q_t, H_t} \quad & J'_{\mathbf{d}}(W_{t-})[q_t W_{t-} - (1-q_t)H_t] + q_t \underline{v}_{\mathbf{u}} + (1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t), \\ s.t. \quad & 0 \leq q_t \leq 1, \ q_t W_{t-} + (1-q_t)H_t \geq \beta_d,\end{aligned}$$

and verify that its optimal solution is

$$q^*_t = 0 \quad \text{and} \quad H^*_t = \beta_{\mathbf{d}} \ , \tag{A.86}$$

following the KKT conditions. Define the following dual variable for the binding constraint

$$\alpha = J'_{\mathbf{d}}(W_{t-}) + 1 \geq 0 \ ,$$

in which the inequality follows from $J'_{\mathbf{d}}(W_{t-}) \geq -1$. One can verify

$$J'_{\mathbf{d}}(W_{t-})(W_{t-} + H^*_t) + W_{t-} + H^*_t = (W_{t-} + H^*_t)\alpha, \text{ and} \tag{A.87}$$

$$(1-q^*_t)(J'_{\mathbf{d}}(W_{t-}) + 1) = (1-q^*_t)\alpha. \tag{A.88}$$

Therefore, (A.86) implies that

$$\begin{aligned}\Phi^{\mathbf{d}}_t &\leq J'_{\mathbf{d}}(W_{t-})(rW_{t-} - \mu_{\mathbf{d}}\beta_{\mathbf{d}}) - rJ_{\mathbf{d}}(W_{t-}) + \mu_{\mathbf{d}}(\underline{v}_{\mathbf{u}} - W_{t-} - \beta_{\mathbf{d}}) - \mu_{\mathbf{d}} J_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} \\ &= J'_{\mathbf{d}}(W_{t-}) r(W_{t-} - \bar{w}_{\mathbf{d}}) - (r+\mu_{\mathbf{d}}) J_{\mathbf{d}}(W_{t-}) + \mu_{\mathbf{d}}(\underline{v}_{\mathbf{u}} - W_{t-} - \beta_{\mathbf{d}}) - c_{\mathbf{d}} = 0,\end{aligned}$$

where the second equality follows from (A.83). In summary, we have $U(\Gamma^*_{\mathbf{d}}(w), \nu^*, \mathbf{d}) \geq U(\Gamma, \nu^*, \mathbf{d})$ and $\underline{v}_{\mathbf{u}} \geq U(\Gamma, \nu^*, \mathbf{u})$.

A.1.6.8 Proof of Theorem 2.6

The proof of this theorem follows the same logic as the proof of Theorem 2.3, and is omitted here.

A.1.7 Proofs in Section 2.4.3

A.1.7.1 Proof of Proposition 2.7

Case $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$: According to Lemma 2.1, under any incentive compatible contract without termination, the agent's promised utility satisfies equation (PK) with $q_t = 0$, $H_t \geq \beta_{\mathbf{d}}$ if $\theta_t = \mathbf{d}$ and $H_t \leq -\beta_{\mathbf{u}}$ if $\theta_t = \mathbf{u}$. Rearranging equation (PK) and replacing ν with ν^*, $q_t = 0$ and $X_t = 0$, we obtain that

$$dW_t = \{(rW_{t-} - \mu_{\mathbf{d}} H_t)dt + H_t dN_t\}\mathbb{1}_{\theta_t = \mathbf{d}} + \{(rW_{t-} - \mu_{\mathbf{u}} H_t)dt + H_t dN_t\}\mathbb{1}_{\theta_t = \mathbf{u}} - dL_t.$$

For any contract that starts at state $\mathbf{d}$ and agent's utility $W_{t-} < \bar{w}_{\mathbf{d}}$, we have $rW_{t-} - \mu_{\mathbf{d}} H_t \leq rW_{t-} - \mu_{\mathbf{d}}\beta_{\mathbf{d}} = r(W_{t-} - \bar{w}_{\mathbf{d}}) < 0$. This implies that before the machine recovers, the utility W_t keeps decreasing. Therefore, starting from any promised utility below $\bar{w}_{\mathbf{d}}$ when the machine's state is $\mathbf{d}$, there is a positive probability that the promised utility decreases to 0 before the machine is repaired, which contradicts the requirement of $\tau = \infty$.

Similarly, for any contract that starts at state $\mathbf{u}$ and agent's utility $W_{t-} < \bar{w}_{\mathbf{u}}$, there is a positive probability that the agent is terminated. This is because at state $\mathbf{u}$, in order to incentivize the agent, the utility needs to drop by at least $\beta_{\mathbf{u}}$ when the machine breaks down, which implies that it is possible that the utility at state $\mathbf{d}$ is smaller than $\bar{w}_{\mathbf{u}} - \beta_{\mathbf{u}} = \bar{w}_{\mathbf{d}}$.

Furthermore, Propositions 2.2 and 2.3 imply that $J_{\mathbf{d}}(w)$ is decreasing for $w > \bar{w}_{\mathbf{d}}$ and $J_{\mathbf{u}}(w)$ is decreasing for $w > \bar{w}_{\mathbf{u}}$, and are optimal value functions starting from the agent's initial utility w and with initial state $\mathbf{d}$ and $\mathbf{u}$, respectively. Therefore, the initial w for the required optimal contract should be $\bar{w}_{\mathbf{d}}$ and $\bar{w}_{\mathbf{u}}$ with the initial state $\mathbf{d}$ and $\mathbf{u}$, respectively. The corresponding optimal contract is the simple contract $\bar{\Gamma}$.

Case $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$: At state $\mathbf{d}$, the machine should start the promised utility with $W_{t-} \geq \bar{w}_{\mathbf{d}}$, and, at state $\mathbf{u}$, the machine should start the promised utility with $W_{t-} \geq \bar{w}_{\mathbf{u}}$.

Furthermore, at state $\mathbf{d}$, the promised utility starts with $W_{t-} \in [\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}})$. If the upward jump $-H_t > rW_{t-}/\mu_{\mathbf{d}}$, then (PK) implies that $rW_{t-} - \mu_{\mathbf{d}} H_t < 0$, and the agent is terminated with positive probability. On the other hand, if $H_t \geq -rW_{t-}/\mu_{\mathbf{d}}$, since $W_t < \hat{w}_{\mathbf{d}}$, we have $rW_{t-}/\mu_{\mathbf{d}} < \beta_{\mathbf{u}}$. If the machine recovers and then breaks down soon afterwards, then the upward jump of the promised utility is $rW_{t-}/\mu_{\mathbf{d}}$, while the downward jump is at least $\beta_{\mathbf{u}}$. Hence, in a cycle of up and down, the continuation utility can decrease by at least $\beta_{\mathbf{u}} - rW_{t-}/\mu_{\mathbf{d}}$. Therefore, after a finite number of such cycles, the promised utility at state $\mathbf{d}$ will drop below $\bar{w}_{\mathbf{d}}$. Again, the agent is then terminated, with a positive probability.

Hence, in order to ensure $\tau = \infty$, the starting promised utility at state $\mathbf{d}$ needs to be greater than $\hat{w}_{\mathbf{d}}$, and at state $\mathbf{u}$ greater than $\hat{w}_{\mathbf{u}}$. Furthermore, Propositions 2.4 and 2.5 imply that $J_{\mathbf{d}}(w)$ is decreasing for $w > \hat{w}_{\mathbf{d}}$ and $J_{\mathbf{u}}(w)$ is decreasing for $w > \hat{w}_{\mathbf{u}}$. Therefore, the initial promised utility w for the required optimal contract should be $\hat{w}_{\mathbf{d}}$ and $\hat{w}_{\mathbf{u}}$ for initial states $\mathbf{d}$ and $\mathbf{u}$, respectively. The corresponding optimal contract is the simple contract $\hat{\Gamma}$.

A.2 E-companion: Optimal One Sided Contracts

The main body of the paper studies the optimal contract when the agent is responsible for both maintaining and repairing the machine (call it "combined contract") and these contracts induce full effort from the agent before termination. Results in Section 2.4 indicate that for a set of given model parameters, it is fairly easy to obtain optimal incentive compatible contracts and the corresponding value functions. In this e-companion, we first provide sufficient conditions based on computed corresponding value functions, which can be used to verify if the optimal incentive compatible contracts that obtain full effort from the agent are, in fact, optimal, even if we allow shirking.

When the sufficient conditions are not satisfied, it may be preferable for the principal to hire the agent just to maintain or just to repair, and to allow the agent to shirk. In Section A.2.2 and A.2.3 of this e-companion, we consider two one sided contracts where the agent is only responsible for one of the two duties. A "maintenance contract" only induces the agent to exert effort when the machine is up in order to decrease the arrival rate of failures. Similarly, a "repair contract" only induces the agent to exert effort when the machine is down to increase the rate of recovery. Studying these two types of contracts is relevant because as we showed in Section 2.5, one of these two contracts may outperform the optimal combined contract.

As it turns out, these two contract design problems are not special cases of the model studied in the main body of the paper. To see this, consider the example of maintenance contracts. In this setting, the machine recovers with a rate of $\underline{\mu}_{\mathbf{d}}$ without the agent's effort. In the optimal combined contract, the agent's promised utility is increased by at least $\beta_{\mathbf{d}}$ when the state changes from down to up, in which $\beta_{\mathbf{d}} = c_{\mathbf{d}} / \left(\mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}} \right)$. In the maintenance contract setting, we cannot simply set $c_{\mathbf{d}} = 0$ and $\mu_{\mathbf{d}} = \underline{\mu}_{\mathbf{d}}$, because the corresponding $\beta_{\mathbf{d}}$ would not be well defined. In fact, the principal does not need to reward the agent when the state changes from down to up. Consequently, how the promised utility should change in this case is not immediately clear.

A.2.1 Incentive Compatibility where agents are responsible for both maintenance and repair

Following the optimality condition presented in Lemma A.12, we first obtain the following sufficient condition for optimality of maintaining incentive compatibility in the problem where agents are responsible for both maintenance and repair. Since the sufficient condition is based on the principal's value functions, it is convenient to summarize the definition of value functions under different parameter regions:

- $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, $R \geq h_d$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (2.21)-(2.25) in Section 2.4.1.2. ($h_{\mathbf{d}}$ is defined in (2.14))
- $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, $R \in [g_{\mathbf{u}}, h_d)$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (A.49)-(A.50) in the proof of theorem 2.2. ($g_{\mathbf{u}}$ is defined in (2.28))
- $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, $R < g_{\mathbf{u}}$: Principal's value function $J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w$ and $J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w$.
- $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, $R \geq h_u$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (2.32)-(2.37) in Section 2.4.2.2 with $\bar{a}$ defined in proposition 2.4. ($h_{\mathbf{u}}$ is defined in (2.19))
- $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, $R \in [g_{\mathbf{d}}, h_u)$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (A.84)-(A.85) in the proof of theorem 2.5. ($g_{\mathbf{d}}$ is defined in (2.44))
- $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, $R < g_{\mathbf{d}}$: Principal's value function $J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w$ and $J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w$.

Proposition A.1 *It is optimal to always induce full effort from the agent before contract termination if function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ summarized above satisfy the following two conditions,*

$$\varphi_{\mathbf{d}}(w) := rJ_{\mathbf{d}}(w) + \underline{\mu}_{\mathbf{d}} J_{\mathbf{d}}(w) - rwJ'_{\mathbf{d}}(w) - \underline{\mu}_{\mathbf{d}} \max_{-h \leq w} \left\{ -hJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(w+h) \right\} \geq 0, \text{ for } w \geq 0, \tag{A.89}$$

and

$$\varphi_{\mathbf{u}}(w) := rJ_{\mathbf{u}}(w) + \bar{\mu}_{\mathbf{u}} J_{\mathbf{u}}(w) - R - rwJ'_{\mathbf{u}}(w) - \bar{\mu}_{\mathbf{u}} \max_{-h \leq w} \left\{ -hJ'_{\mathbf{u}}(w) + J_{\mathbf{d}}(w+h) \right\} \geq 0, \text{ for } w \geq 0. \tag{A.90}$$

It is worth noting that Proposition A.1 is a parallel result to condition (54) in [BMRV10], Proposition 8 in [DS06b] and Proposition 6 in [Var17]. However, our conditions are more complex than the corresponding conditions in the literature, involving solving a single dimensional maximization problem in both (A.89) and (A.90). This complexity is due to the key difference between our paper and the aforementioned continuous time dynamic contracting papers: in all the other papers, the agent is only responsible for one task whereas in ours, the agent is responsible for two tasks. This induces complexity because the principal's value function will further depend on the machine's states $\mathbf{u}$ and $\mathbf{d}$.

Specifically, imagine, for the moment, that we replace the term $J_{\mathbf{u}}(w-h)$ in (A.89) by $J_{\mathbf{d}}(w-h)$, so that there would be only one state. (the down state) It is easy to verify that in this case, concavity of the value function $J_{\mathbf{d}}(w-h)$ implies that the optimal h in this maximization problem should be 0. (The intuitive interpretation is that there is no change in the agent's promised utility associated with arrivals during the period when the agent is allowed to shirk.) Consequently, the expression $\varphi_{\mathbf{d}}(w)$ would be greatly simplified to be a monotone function, which yields a sufficient condition only involving evaluating the value function at its boundaries. In our case, however, concavity of functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ do not guarantee that the optimal h takes value 0. (That is, in general contracts allowing shirking, the agent's promised utility still needs to include jumps as the machine changes states when the agent shirks.) This exactly explains the reason why our verification conditions are more complex than those in the aforementioned literature, and highlights the distinct feature of our set-up with two machine states.

Fortunately, the principal's value functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ defined in the previous sections are, in fact, quite easy to compute. Therefore, conditions (A.89) and (A.90) can be easily verified numerically for any model parameter settings. From Sections 2.4.1 and 2.4.2, we learn that the optimal incentive compatible contracts take three forms depending on model parameters. Specifically, the three regions can be characterized by dividing the value of revenue rate R into three intervals, fixing all other model parameters. The following result indicates that if the value of R belongs to the lowest interval, sufficient conditions (A.89) and (A.90) are guaranteed to hold. If R is moderate, on the other hand, sufficient conditions (A.89) and (A.90) do not hold. Therefore, we only need to check conditions (A.89) and (A.90) if revenue R is high enough.

Corollary A.1 *(i) If $\beta_d \geq \beta_u$ and condition* (2.30) *holds, or, if $\beta_d < \beta_u$ and condition* (2.46) *holds, then conditions* (A.89) *and* (A.90) *hold.*

(ii) If $\beta_d \geq \beta_u$ and condition (2.29) *holds, or, if $\beta_d < \beta_u$ and condition* (2.45) *holds, then conditions* (A.89) *and* (A.90) *do not hold.*

Corollary A.1(i) implies that if R is in the lowest interval, then not hiring the agent is not only the optimal incentive compatible contract, but also the best strategy among all contracts. In this case the principal's value function is a linear function with slope -1, which allows us to easily verify conditions (A.89) and (A.90). In comparison, Corollary A.1(ii) implies that if R takes moderate values (in the middle interval defined in (2.29) or (2.45)), the principal may be better off allowing shirking at some point in time before terminating the contract.

Note that the intervals defined in (2.29) and (2.45) are empty when $\beta_{\mathbf{u}} = \beta_{\mathbf{d}}$. That is, the middle interval only occurs if the ratios between effort cost and repair rate and maintenance rate improvement are not balanced, or, between the two types of efforts (repairing and maintaining) one of them is more favored than the other. In this case, the optimal incentive compatible contract dictates the principal to hire the agent only if the machine starts in the favored state, and to terminate the agent as soon as the state changes. If we allow shirking instead, the principal may benefit from hiring the agent to exert effort when the machine is in the favored state, while allowing the agent to shirk when the machine is in the other state and wait for the favored state to come back. This, again, provides us the motivation to study the optimal one-sided contracts.

A.2.2 Optimal Maintenance Contract

In this section, we consider the contract design problem where the agent only has the expertise of maintenance work which means when the machine is up, he could decrease the rate that machine breaks down from $\bar{\mu}_{\mathbf{u}}$ to $\mu_{\mathbf{u}}$ and when the machine is down, agent does not work and the machine recovers with rate $\underline{\mu}_{\mathbf{d}}$. Correspondingly, we need to change the arrival rate of process N in (2.2) as

$$\mu_m(\theta_t, \nu_t) = [\mu_{\mathbf{u}}\nu_t + \bar{\mu}_{\mathbf{u}}(1-\nu_t)]\mathbb{1}_{\theta_t=\mathbf{u}} + \underline{\mu}_{\mathbf{d}}\mathbb{1}_{\theta_t=\mathbf{d}},$$

and the effort cost rate (2.1) at t as

$$c_m(\theta_t) = c_{\mathbf{u}}\mathbb{1}_{\theta_t=\mathbf{u}}.$$

With these new definitions, we need to change the agent's expected total utility (2.5) by substituting $c(\theta_t)$ with $c_m(\theta_t)$. Without the agent, the principal's total discounted future profit for states $\mathbf{u}$ and $\mathbf{d}$ are $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$, respectively where $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$ are defined in equation (2.4). The principal's expected total discounted profit under a contract Γ_m and effort process $\nu = \{\nu_t\}_{\forall t\in[0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{d}$ is still defined as (2.3). Denote the full effort process as $\nu_m := \{(\nu_m)_t = \mathbb{1}_{\theta_t=\mathbf{u}}\}_{\forall t\in[0,\tau]}$. A maintenance contract Γ_m is incentive compatible if $u(\Gamma_m, \nu_m, \theta_0) \geq u(\Gamma_m, \nu, \theta_0)$ for any effort process $\nu = \{\nu_t\}_{\forall t\in[0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{d}$. Furthermore, the following result is parallel to Lemma 2.1.

Lemma A.9 *In the maintenance setting, for any contract Γ_m, there exists $\mathcal{F}_t$-predictable processes H_t such that for $t \in [0, \tau)$,*

$$dW_t = \{rW_{t-} - (1-\nu_t)c_m(\theta_t) - [(1-q_t)H_t - q_tW_{t-}]\mu_m(\nu_t, \theta_t)\}dt - dL_t + [(1-X_t)H_t - X_tW_{t-}]dN_t, \tag{PKm}$$

in which Bernoulli random variable X_t takes value 1 with probability q_t. Furthermore, contract Γ_m is incentive compatible if and only if

$$-q_tW_{t-} + (1-q_t)H_t \leq -\beta_{\mathbf{u}} \quad \textit{for } \theta_{t-} = \mathbf{u},\ \forall t \in [0, \tau]. \tag{A.91}$$

Finally, we need $-H_t \leq W_{t-}$ for all $t \geq 0$ in order to satisfy (IR).

Similar to the combined contract, constraint (A.91) implies that any incentive compatible maintenance contract must satisfy the condition $W_{t-} \geq \beta_{\mathbf{u}}$ when $\theta_{t-} = \mathbf{u}$.

Next, we propose a maintenance contract and prove its optimality following similar approaches in Sections 2.4.1 and 2.4.2. The general idea is that the promised utility increases at rate $rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}$ in state $\mathbf{u}$, and drops $\beta_{\mathbf{u}}$ whenever the machine breaks down. In state $\mathbf{d}$, the promised utility stays at a constant, and takes an upward jump of $rW_{t-}/\underline{\mu}_{\mathbf{d}}$ when the machine recovers, which collects the expected interest accrued during state $\mathbf{d}$. At the end of an up state, if a downward jump brings the promised utility to below the following threshold,

$$\underline{w}_m := \frac{\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}} + r}\beta_{\mathbf{d}},$$

the upward jump at the end of the down state cannot bring it back to $\beta_{\mathbf{u}}$ anymore. Because the promised utility has to be higher than $\beta_{\mathbf{u}}$ in state $\mathbf{u}$ in order to induce full effort, if the promised utility jumps down to below $\underline{w}_m$, then the principal should randomly terminate the agent, or reset it back to $\underline{w}_m$. Similar to before, payment starts when the promised utility reaches the upper threshold

$$\bar{w}_m := \frac{\underline{\mu}_{\mathbf{d}} + r}{r}\beta_{\mathbf{d}}.$$

The exact dynamics is represented in the following definition.

Definition A.1 *The contract $\Gamma^*_m(w) = (L^*, q^*, \tau^*)$ is defined as the following.*

i. The dynamics of the agent's promised utility W_t follows

$$dW_t = \begin{cases} (rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})dt - \beta_{\mathbf{u}}dN_t, & \theta_t = \mathbf{u},\ \beta_{\mathbf{u}} \le W_{t-} \le \bar{w}_m \\ -X_t W_{t-} + (1 - X_t)\left(\underline{w}_m - W_{t-}\right), & \theta_t = \mathbf{d},\ W_{t-} < \underline{w}_m \\ \left(rW_{t-}/\underline{\mu}_{\mathbf{d}}\right) dN_t, & \theta_t = \mathbf{d},\ W_{t-} \ge \underline{w}_m \end{cases}, \tag{DWm}$$

from an initial promised utility $W_0 = w$.

ii. The payment process follow $dL^*_t = \left(2\underline{\mu}_{\mathbf{d}} + r\right) \beta_{\mathbf{u}} \mathbb{1}_{W_{t-}=\bar{w}_m} \mathbb{1}_{\theta_t=\mathbf{u}} dt$.

iii. The random termination probability process for $W_{t-} < \underline{w}_m$ is $q^*_t = \hat{q}(W_{t-})$, in which

$$\hat{q}(w) := 1 - w/\underline{w}_m,$$

and the termination time is $\tau^* = \min\{t : W_t = 0\}$.

Furthermore, the following set of differential equations define the principal's value functions $J^m_{\mathbf{d}}$ and $J^m_{\mathbf{u}}$.

$$(\underline{\mu}_{\mathbf{d}} + r)J^m_{\mathbf{d}}(w) = \underline{\mu}_{\mathbf{d}} J^m_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} w\right), \quad w \ge \underline{w}_m, \tag{A.92}$$

$$J^m_{\mathbf{d}}(w) = \hat{q}(w)J^m_{\mathbf{d}}(0) + (1 - \hat{q}(w))J^m_{\mathbf{d}}\left(\underline{w}_m\right), \quad w < \underline{w}_m, \text{ and} \tag{A.93}$$

$$\begin{aligned} &- c_{\mathbf{u}} + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_m} J^{m\prime}_{\mathbf{u}}(w) = (\mu_{\mathbf{u}} + r)J^m_{\mathbf{u}}(w) + \left(2\underline{\mu}_{\mathbf{d}} + r\right) \beta_{\mathbf{u}} \mathbb{1}_{w=\bar{w}_m} \\ &\qquad - R - \mu_{\mathbf{u}} J^m_{\mathbf{d}}(w - \beta_{\mathbf{u}}),\ w \in [\beta_{\mathbf{u}}, \bar{w}_m]\,. \end{aligned} \tag{A.94}$$

with boundary conditions

$$J^m_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}, \qquad J^m_{\mathbf{d}}\left(\bar{w}_m - \beta_{\mathbf{u}}\right) = \frac{\underline{\mu}_{\mathbf{d}}(R - c_{\mathbf{u}})}{r\left(r + \mu_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}}\right)} - (\bar{w}_m - \beta_{\mathbf{u}}), \tag{A.95}$$

$$J^m_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}}, \qquad J^m_{\mathbf{u}}\left(\bar{w}_m\right) = \frac{(r + \underline{\mu}_{\mathbf{d}})(R - c_{\mathbf{u}})}{r\left(r + \mu_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}}\right)} - \bar{w}_m. \tag{A.96}$$

Similar to Proposition 2.1, the next proposition establishes the concavity of the principal's value functions.

Proposition A.2 *The system of differential equations* (A.92)-(A.94) *with boundary conditions* (A.95) *and* (A.96) *has a unique solution: the pair of functions, $J^m_{\mathbf{u}}(w)$ on $[0, \bar{w}_m]$, and $J^m_{\mathbf{d}}(w)$ on $[0, \bar{w}_m - \beta_{\mathbf{u}}]$, both of which are strictly concave and $J^{m\prime}_{\mathbf{u}}(w) \ge -1$, $J^{m\prime}_{\mathbf{d}}(w) \ge -1$.*

The next result shows that $J^m_{\mathbf{d}}$ and $J^m_{\mathbf{u}}$ are indeed the principal's value function if the initial promised utility is w starting from states $\mathbf{d}$ and $\mathbf{u}$, respectively.

Proposition A.3 *For promised utility $w \in [0, \bar{w}_m - \beta_{\mathbf{u}}]$, we have $U(\Gamma^*_m(w), \nu_m, \mathbf{d}) = J^m_{\mathbf{d}}(w)$. For promised utility $w \in [0, \bar{w}_m]$, we have $U(\Gamma^*_m(w), \nu_m, \mathbf{u}) = J^m_{\mathbf{u}}(w)$.*

Furthermore, we can find $w^{m*}_{\mathbf{d}}$ and $w^{m*}_{\mathbf{u}}$ as the maximizers of $J^m_{\mathbf{u}}$ and $J^m_{\mathbf{d}}$ respectively, and start the promised utility from them.

Similar to Section 2.4.1 and 2.4.2, we define the societal value functions as the summation of the principal and the agent's utilities, $V^m_{\mathbf{d}}(w) = J^m_{\mathbf{d}}(w) + w$ and $V^m_{\mathbf{u}}(w) = J^m_{\mathbf{u}}(w) + w$. Figures A.1 and A.2[1] provide a numerical example of societal value functions $V^m_{\mathbf{d}}$ and $V^m_{\mathbf{u}}$ and the principal's value functions $J^m_{\mathbf{d}}$ and $J^m_{\mathbf{u}}$.

[1] $\mu_{\mathbf{u}} = 5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 2, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 0.1, c_{\mathbf{d}} = 1.3, r = 0.5, R = 10.$

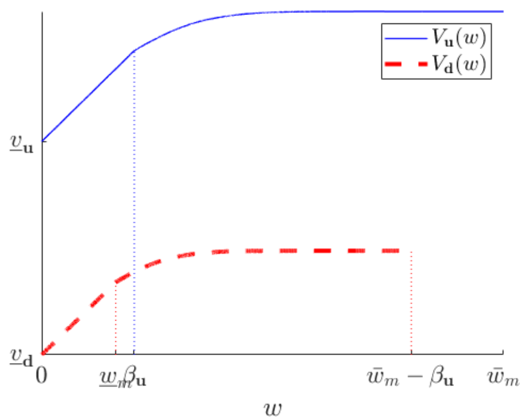


Figure A.1: Societal's Value functions

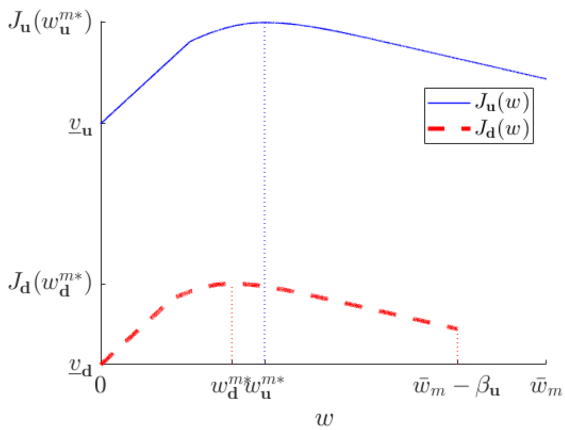


Figure A.2: Principal's Value functions

From Section A.2.1, we know that for the combined contract, the sufficient conditions that guarantee the optimality of the full effort contract are relatively complicated. For the maintenance contract setting, we can show that, the following simple condition is necessary and sufficient for the principal to want to hire and induce full effort from the agent,

$$R \geq \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \beta_{\mathbf{u}} = g_{\mathbf{u}}. \tag{A.97}$$

The next theorem shows that under condition (A.97), functions $J_{\mathbf{d}}^m$ and $J_{\mathbf{u}}^m$ are upper bounds for the principal's utility under any maintenance contract Γ_m.

Theorem A.1 *Under condition* (A.97)*, for any contract Γ_m and any initial state $\theta \in \{\mathbf{u}, \mathbf{d}\}$ that satisfies (A.91), we have $J_\theta\big(u(\Gamma_m^*, \nu, \theta)\big) \geq U(\Gamma_m, \nu, \theta)$, in which we extend the function $J_{\mathbf{d}}^m(w) = J_{\mathbf{d}}^m(\bar{w}_m - \beta_{\mathbf{u}}) - (w - \bar{w}_m + \beta_{\mathbf{u}})$ for $w > \bar{w}_m - \beta_{\mathbf{u}}$ and $J_{\mathbf{u}}^m(w) = J_{\mathbf{u}}^m(\bar{w}_m) - (w - \bar{w}_m)$ for $w > \bar{w}_m$.*

We have $U\big(\Gamma_m^*(w_\theta^{m*}), \nu_m, \theta\big) \geq U(\Gamma_m, \nu, \theta)$ for any contract Γ_m and state θ. That is, the optimal contract is $\Gamma_m^*(w_\theta^{m*})$ and the machine starts from state $\theta \in \{\mathbf{u}, \mathbf{d}\}$.

It is worth noting that Theorem A.1 shows that contract Γ_m^* defined in Definition A.1 is optimal among any maintenance contract Γ_m. This result is stronger than Theorem 2.1 and 2.4, which only show that contracts Γ_1^* and $\Gamma_{\hat{\beta}}^*$ in Sections 2.4.1.1 and 2.4.2.1 are optimal among incentive compatible contracts.

The next proposition shows that if condition (A.97) is violated, then the principal is better off not hiring the agent, even if we take contracts that allow shirking into consideration.

Proposition A.4 *Assuming condition* (A.97) *does not hold, that is,*

$$R < \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \beta_{\mathbf{u}} = g_{\mathbf{u}}. \tag{A.98}$$

We have $\underline{v}_\theta \geq U(\Gamma_m, \nu_m, \theta)$ for any maintenance contract Γ_m and state $\theta \in \{\mathbf{d}, \mathbf{u}\}$.

Figure A.3 depicts two sample trajectories of the agent's promised utility according to $\Gamma_m^*(w_{\mathbf{u}}^{m*})$ where the machine starts at state $\theta_0 = \mathbf{u}$. In state $\mathbf{u}$, the promised utility increases over time until the machine breaks down or the promised utility reaches $\bar{w}_m$. According to the solid curve in Figure A.3, the machine changes states at times t_1, t_2, t_3, t_4, and t_5. Between $[0, t_1]$, the promised utility increases over time while the agent is maintaining the machine. At time t_1, the machine breaks down and the promised utility drops by $\beta_{\mathbf{u}}$. Once the machine is in state $\mathbf{d}$, the agent does not need to work, and the promised utility remains a constant, until the machine recovers at time t_2. Whenever the machine recovers at time t, the utility W_{t-} takes an upward jump of $\dfrac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}}$. This upward jump happens at time t_2 following the solid curve. After t_2, the promised utility increases again while the agent maintains the machine, until time $\hat{t}_3$ when the promised utility reaches $\bar{w}_m$. At this point, the flow payment starts. After time t_3, the agent's promised utility is jumping back and forth between $\bar{w}_m$ when the machine is up and $\bar{w}_m - \beta_{\mathbf{u}}$ when the machine is down.

Now we focus on the other sample trajectory in Figure A.3, the dotted curve. The machine is in state $\mathbf{u}$ during time intervals $[0, \tilde{t}_1]$, $[\tilde{t}_2, \tau]$ and in state $\mathbf{d}$ during $[\tilde{t}_1, \tilde{t}_2]$. The promised utility increases in state $\mathbf{u}$ and stays at a constant in state $\mathbf{d}$. Right after the machine breaks down at time τ, the promised utility jumps to below $\underline{w}_m$. Consequently, even an upward jump of $\dfrac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}}$ cannot raise the promised utility to above $\beta_{\mathbf{u}}$. Therefore, at time $\tilde{t}_3$ the agent is terminated with probability $\hat{q}(W_{\tilde{t}_3-})$. On the other hand, with probability $1 - \hat{q}(W_{\tilde{t}_3-})$, the agent's promised utility is reset to $\beta_{\mathbf{u}}$ (the "*" in the figure) and continues increasing.

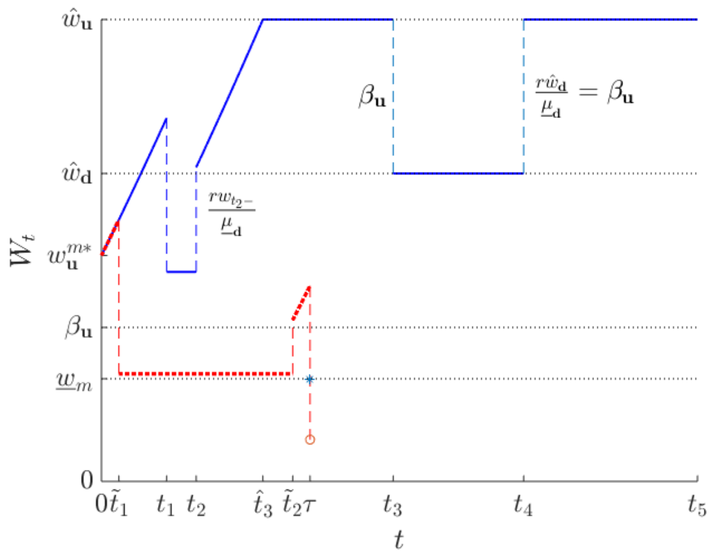


Figure A.3: Two sample trajectories of promised utility with model parameters $\mu_{\mathbf{u}} = 5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 2, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 0.1, c_{\mathbf{d}} = 1.3, r = 0.5, R = 10$. The policy starts from $w_{\mathbf{u}}^{m*} = 0.146$. The solid curve represents a sample trajectory which brings the agent to the point of never to be terminated. The dotted curve represents another sample trajectory in which the agent is terminated due to a random draw at a point when the machine breaks down.

A.2.3 Optimal Repair Contract

In this section, we consider the contract design problem where the agent only has the expertise to repair. That is, when the machine is down, the agent is able to decrease the recovery rate from $\underline{\mu}_{\mathbf{d}}$ to $\mu_{\mathbf{d}}$ with effort. When the machine is up, on the other hand, the agent does not work and the machine breaks down with rate $\bar{\mu}_{\mathbf{u}}$. Correspondingly, we need to change the the arrival rate of process N in (2.2) as

$$\mu_r(\theta_t, \nu_t) = \bar{\mu}_{\mathbf{u}} \mathbb{1}_{\theta_t=\mathbf{u}} + [\mu_{\mathbf{d}}\nu_t + \underline{\mu}_{\mathbf{d}}(1-\nu_t)]\mathbb{1}_{\theta_t=\mathbf{d}},$$

and the effort cost rate (2.1) at t as

$$c_r(\theta_t) = c_{\mathbf{d}} \mathbb{1}_{\theta_t=\mathbf{d}}.$$

With these new definitions, we need to change the agent's expected total utility (2.5) by substituting $c(\theta_t)$ with $c_r(\theta_t)$. Without the agent, the principal's total discounted future profit for states $\mathbf{u}$ and $\mathbf{d}$ are $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$, respectively, where $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$ are defined in (2.4). The principal's expected total discounted profit under a contract Γ_r and effort process $\nu = \{\nu_t\}_{\forall t\in[0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{u}$ is still defined as (2.3). Denote the full effort process as $\nu_r := \{(\nu_r)_t = \mathbb{1}_{\theta_t=\mathbf{d}}\}_{\forall t\in[0,\tau]}$. A contract Γ_r is incentive compatible if $u(\Gamma_r, \nu_r, \theta_0) \geq u(\Gamma_r, \nu, \theta)$ for any effort process $\nu = \{\nu_t\}_{\forall t\in[0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{u}$. Again, the following result is parallel to Lemma 2.1.

Lemma A.10 *In a repair setting, for any contract Γ_r, there exists $\mathcal{F}_t$-predictable processes H_t such that*

$$\begin{aligned} dW_t =& \{rW_{t-} - (1-\nu_t)c_r(\theta_t) - [(1-q_t)H_t - q_tW_{t-}]\mu_r(\nu_t, \theta_t)\}dt - dL_t \\ &+ [(1-X_t)H_t - X_tW_{t-}]dN_t, t \in [0, \tau) \end{aligned} \quad \text{(PKr)}$$

in which Bernoulli random variable X_t takes value 1 with probability q_t. Furthermore, contract Γ_r is incentive compatible if and only if

$$-q_t W_{t-} + (1-q_t)H_t \geq \beta_{\mathbf{d}} \quad \text{for } \theta_t = \mathbf{d},\ \forall t \in [0,\tau]. \tag{A.99}$$

Finally, we need $-H_t \leq W_{t-}$ for all $t \geq 0$ in order to satisfy (IR).

In the following, we directly propose a repair contract and prove the optimality following the similar approach in Section 2.4.1 and 2.4.2.

Definition A.2 *The contract $\Gamma_r^*(w) = (L^*, q^*, \tau^*)$ is defined as*

i. The dynamics of the agent's promised utility W_t, follows

$$dW_t = \left[r(W_{t-} - \bar{w}_{\mathbf{d}})dt + \min\left\{ \frac{\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}} + r}\bar{w}_{\mathbf{d}} - W_{t-}, \beta_{\mathbf{d}} \right\} dN_t \right] \mathbb{1}_{\theta_t = \mathbf{d}} + \frac{rW_{t-}}{\bar{\mu}_{\mathbf{u}}} dN_t \mathbb{1}_{\theta_t = \mathbf{u}} \tag{DWr}$$

from an initial promised utility $W_0 = w$.

ii. The payment to the agent follows $dL_t^* = (W_{t-} + \beta_{\mathbf{d}} - \bar{\mu}_{\mathbf{u}}/(\bar{\mu}_{\mathbf{u}} + r)\bar{w}_{\mathbf{d}})^+ dN_t \mathbb{1}_{\theta_t = \mathbf{d}}$.

iii. The random termination probability $q_t^* = 0$ and the termination time $\tau^* = \min\{t : W_t = 0\}$.

Furthermore, the principal's value functions are determined by the following set of differential equations

$$(\mu_{\mathbf{d}} + r)J_{\mathbf{d}}^r(w) = -c_{\mathbf{d}} + r(w - \bar{w}_{\mathbf{d}})J_{\mathbf{d}}^{r'}(w) + \mu_{\mathbf{d}} J_{\mathbf{u}}^r\left(\min\left\{ w + \beta_{\mathbf{d}}, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r} \right\} \right) - \mu_{\mathbf{d}} \left(w + \beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r} \right)^+, \tag{A.100}$$

$$(\bar{\mu}_{\mathbf{u}} + r)J_{\mathbf{u}}^r(w) = R + \bar{\mu}_{\mathbf{u}} J_{\mathbf{d}}^r\left(\frac{r + \bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}} w \right), \tag{A.101}$$

with boundary conditions

$$J_{\mathbf{d}}^r(0) = \underline{v}_d, \qquad J_{\mathbf{d}}^r(\bar{w}_{\mathbf{d}}) = \frac{\mu_{\mathbf{d}} R - (r + \bar{\mu}_{\mathbf{u}})c_{\mathbf{d}}}{r(r + \bar{\mu}_{\mathbf{u}} + \mu_{\mathbf{d}})} - \bar{w}_{\mathbf{d}}, \tag{A.102}$$

$$J_{\mathbf{u}}^r(0) = \underline{v}_u, \qquad J_{\mathbf{u}}^r\left(\frac{\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}} + r}\bar{w}_{\mathbf{d}} \right) = \frac{(r + \mu_{\mathbf{d}})R - \bar{\mu}_{\mathbf{u}} c_{\mathbf{d}}}{r(r + \bar{\mu}_{\mathbf{u}} + \mu_{\mathbf{d}})} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r}. \tag{A.103}$$

Similar to Proposition 2.1, the next Proposition establishes the concavity of the principal's value functions.

Proposition A.5 *The system of differential equations* (A.100) *and* (A.101) *with boundary conditions* (A.102) *and* (A.103) *has a unique solution: the pair of functions, $J_{\mathbf{u}}^r(w)$ on $\left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r}\right]$, and $J_{\mathbf{d}}^r(w)$ on $[0, \bar{w}_{\mathbf{d}}]$, both of which are strictly concave and $J_{\mathbf{u}}^{r'}(w) \geq -1$, $J_{\mathbf{d}}^{r'}(w) \geq -1$.*

The next result shows that $J_{\mathbf{d}}^r(w)$ and $J_{\mathbf{u}}^r(w)$ are indeed the principal's value function if the initial promised utility is w starting from states $\mathbf{d}$ and $\mathbf{u}$, respectively.

Proposition A.6 *For promised utility $w \in [0, \bar{w}_{\mathbf{d}}]$, we have $U(\Gamma_r^*, \nu_r, \theta) = J_{\mathbf{d}}^r(w)$. For promised utility $w \in \left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r}\right]$, we have $U(\Gamma_r^*(w), \nu_r, \theta) = J_{\mathbf{u}}^r(w)$.*

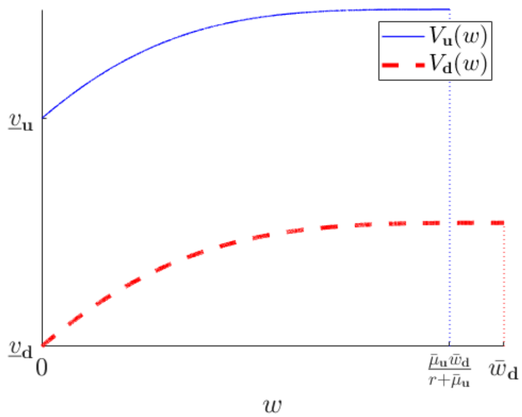


Figure A.4: Societal's Value functions

Furthermore, we can find $w_{\mathbf{d}}^{r*}$ and $w_{\mathbf{u}}^{r*}$ as the maximizers of $J_{\mathbf{d}}^{r}(w)$ and $J_{\mathbf{u}}^{r}(w)$ respectively, and start the promised utility from them.

Similar to Section 2.4.1 and 2.4.2, we define the societal value functions as the summation of the principal and the agent's utilities, $V_{\mathbf{d}}^{r}(w) = J_{\mathbf{d}}^{r}(w) + w$ and $V_{\mathbf{u}}^{r}(w) = J_{\mathbf{u}}^{r}(w) + w$. Figures A.5 and A.4[2] provide a numerical example of societal value functions $V_{\mathbf{d}}^{r}$ and $V_{\mathbf{u}}^{r}$ and the principal's value functions $J_{\mathbf{d}}^{r}$ and $J_{\mathbf{u}}^{r}$.

From Section A.2.1, we know that for the combined contract, the sufficient conditions that guarantee the optimality of the full effort contract are relatively complicated. For the repair contract setting, we can show that, the following simple condition is necessary and sufficient for the principal to want to hire and induce full effort from the agent,

$$R \geq (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{d}} = g_{\mathbf{d}} \tag{A.104}$$

The next theorem shows that under condition (A.104), functions $J_{\mathbf{d}}^{r}$ and $J_{\mathbf{u}}^{r}$ are upper bounds for the principal's utility under any (not necessarily incentive compatible) contract Γ_r.

Theorem A.2 *Under condition* (A.104)*, for any repair contract Γ_r and any initial state $\theta \in \{\mathbf{u}, \mathbf{d}\}$ that satisfies (A.99), we have $J_\theta\big(u(\Gamma_r^*, \nu, \theta)\big) \geq U(\Gamma_r, \nu, \theta)$, in which we extend the function $J_{\mathbf{d}}(w) = J_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) - (w - \bar{w}_{\mathbf{d}})$ for $w > \bar{w}_{\mathbf{d}}$ and $J_{\mathbf{u}}(w) = J_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r}\right) - \left(w - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r}\right)$ for $w > \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}} + r}$.*

We have $U\big(\Gamma_r^*(w_\theta^{r*}), \nu_r, \theta\big) \geq U(\Gamma_r, \nu, \theta)$ for any repair contract Γ_r and state θ. That is, the optimal contract is $\Gamma_r^*(w_\theta^{r*})$ when the machine starts from state $\theta \in \{\mathbf{u}, \mathbf{d}\}$.

Therefore, Γ_r^* is, in fact, the optimal contract among any repair contract Γ_r. Similar to Proposition A.4, the next proposition shows that if condition (A.104) is violated, then the principal is better off not hiring the agent.

Proposition A.7 *Assuming condition* (A.104) *does not hold, that is,*

$$R < (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{d}} = g_{\mathbf{d}}. \tag{A.105}$$

We have $\underline{v}_\theta \geq U(\Gamma_r, \nu_r, \theta)$ for any repair contract Γ_r and state $\theta \in \{\mathbf{d}, \mathbf{u}\}$.

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[2] $\mu_{\mathbf{u}} = 5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 2, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 1.3, c_{\mathbf{d}} = 0.9, r = 0.8, R = 16.$

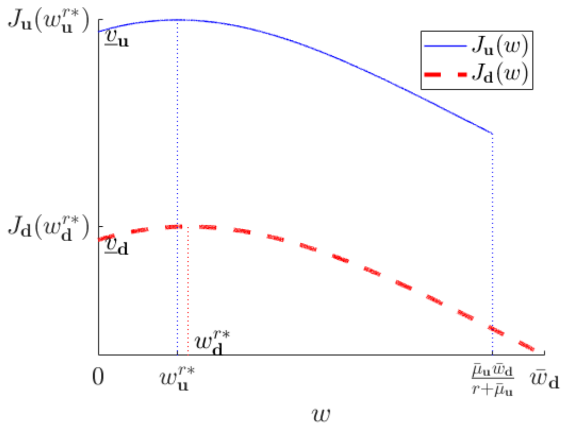

Figure A.5: Principal's Value functions

Figure A.6 depicts two sample trajectories of the agent's promised utility according to contract $\Gamma_r^*(w_{\mathbf{u}}^{r*})$, where the machine starts at state $\theta_0 = \mathbf{u}$. In state $\mathbf{u}$, the agent does not need to work, and the promised utility always remains a constant, until the machine breaks down, at time t_1 on the solid curve and at $\hat{t}_1$ on the dotted curve. Whenever the machine breaks down, the utility W_{t-} takes an upward jump of level $\dfrac{r}{\bar{\mu}_{\mathbf{u}}}W_{t-}$, after which the machine is in state $\mathbf{d}$ and the agent starts to exert effort. In this state, the promised utility keeps decreasing until either the machine recovers, as depicted by the solid curve between time t_1 and t_2, or the promised utility decreases to zero and the contract terminates, as depicted by time τ on the dotted curve. If the machine recovers at time t with $W_{t-} > 0$, the utility takes an upward jump of level $\min\left\{\dfrac{\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}+r}\bar{w}_{\mathbf{d}} - W_{t-}, \beta_{\mathbf{d}}\right\}$, and the agent is paid $\left(W_{t-} + \beta_{\mathbf{d}} - \dfrac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{(\bar{\mu}_{\mathbf{u}}+r)}\right)^+$ instantaneously, as what happens at time t_4 or t_6 following the solid curve. After the first payment, the promised utility remains constant $\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}/(\bar{\mu}_{\mathbf{u}}+r)$ at state $\mathbf{u}$ and $\bar{w}_{\mathbf{d}}$ at state $\mathbf{d}$.

A.2.4 Proofs

This section collects all the proofs in this e-companion.

A.2.4.1 Proofs in Section A.2.1

To discuss the optimality of the full effort incentive compatible contract under any contracts that even allow shirking, we need to consider a larger contract space in which the principal does not need to induce full effort from the agent. First, the principal's utility is revised to be

$$U(\Gamma, \nu, \theta_0) = \mathbb{E}\left[\int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t) + e^{-r\tau}\underline{v}_\tau \middle| \theta_0\right], \tag{A.106}$$

and the agent's utility is changed to be

$$u(\Gamma, \nu, \theta_0) = \mathbb{E}\left[\int_0^\tau e^{-rt}\left[dL_t - \nu_t c(\theta_t)dt\right] \middle| \theta_0\right]. \tag{A.107}$$

A more general version of Lemma 2.1 is presented in the following:

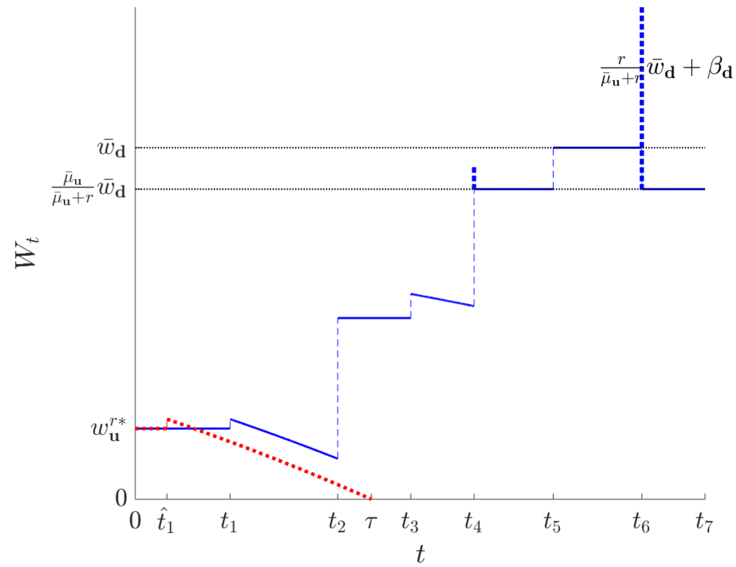

Figure A.6: Two sample trajectories of promised utility with model parameters $\mu_{\mathbf{u}} = 5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 2, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 1.3, c_{\mathbf{d}} = 0.9, r = 0.8, R = 16$. The policy starts from $w_{\mathbf{u}}^{r*} = 0.4525$. The solid curve represents a sample trajectory which brings the agent to the point of never terminated. The dotted curve represents another sample trajectory in which the agent is terminated.

Lemma A.11 *For any contract Γ, there exists an $\mathcal{F}^N$-predictable process H_t such that for $t \in [0, \tau]$,*

$$dW_t = \{rW_{t-} + \nu_t c(\theta_t) - [(1-q_t)H_t - q_t W_{t-}]\mu(\theta_t, \nu_t) - \ell_t\}dt + [(1-X_t)H_t - X_t W_{t-}]dN_t - I_t, \tag{A.108}$$

in which Bernoulli random variable X_t takes value 1 with probability q_t. Furthermore, the necessary and sufficient condition for the effort process ν to maximize agent's utility (A.107) *given Γ is that*

$$\begin{aligned} \nu_t = 1 \text{ if and only if } & -q_t W_{t-} + (1-q_t)H_t \le -\beta_{\mathbf{u}}, \ell_t \ge c_{\mathbf{u}}, \text{ for } \theta_t = \mathbf{u}, \text{and} \\ & -q_t W_{t-} + (1-q_t)H_t \ge \beta_{\mathbf{d}}, \ell_t \ge c_{\mathbf{d}}, \text{ for } \theta_t = \mathbf{d} \end{aligned} \tag{A.109}$$

for all $t \in [0, \tau]$.

Correspondingly, a more general optimality condition (compared to Lemma A.2) is presented in the following,

Lemma A.12 *Suppose $J_{\mathbf{d}}(w) : [0,\infty) \to \mathbb{R}$ and $J_{\mathbf{u}}(w) : [0,\infty) \to \mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J'_{\mathbf{d}}(w) \ge -1$, $J'_{\mathbf{u}}(w) \ge -1$, and $J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$. Consider any contract Γ, which yields the agent's expected utility $u(\Gamma, \nu) = W_0$, followed by the continuation utility process $\{W_t\}_{t\ge 0}$ according to (PK). Define a stochastic process $\{\Phi_t\}_{t\ge 0}$ as*

$$\begin{aligned} \Phi_t :=& R\mathbb{1}_{\theta_t=\mathbf{u}} + J'_{\theta_t}(W_{t-})(rW_{t-} - [-q_t W_{t-} + (1-q_t)H_t]\mu(\theta_t, \nu_t)) - rJ_{\theta_t}(W_{t-}) \\ & + \mu(\theta_t, \nu_t)q_t[J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})] + \mu(\theta_t, \nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-} + H_t) - J_{\theta_t}(W_{t-})] - \nu_t c(\theta_t)\,. \end{aligned} \tag{A.110}$$

where $\theta_t \in \{u, d\}$ and $\hat{\theta}_t = \mathbb{1}_{\theta_t=\mathbf{d}} \cdot u + \mathbb{1}_{\theta_t=\mathbf{u}} \cdot d$. Also, $\nu_t = 0$ if constraints (A.109) are not satisfied at time t and $\nu_t = 1$ if constraints (A.109) are satisfied at time t. If the process $\{\Phi_t\}_{t\ge 0}$ is non-positive almost *surely, then we have* $J_\theta(u(\Gamma,\nu,\theta)) \geq U(\Gamma,\nu,\theta)$.

Proof of Proposition A.1 We have shown that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ summarized at the beginning of Section A.2.1 are upper bounds of the societal utility of any incentive compatible contracts starting from states $\mathbf{d}$ and $\mathbf{u}$, respectively, under different conditions. Or, equivalently, they satisfy that $J'_{\mathbf{d}}(w) \geq -1$, $J'_{\mathbf{u}}(w) \geq -1$, and boundary conditions $J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$ and $J_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}}$, and that Φ_t defined in (A.12) (or equivalently (A.110) with $\nu_t = 1$) is non-positive almost surely. Hence, to prove that they are upper bounds of any contracts, we need to further verify that if Φ_t defined in (A.110) is non-positive almost surely when $\nu_t = 0$. Hence, following (A.110), the following conditions

$$\begin{aligned} rJ_{\mathbf{d}}(W_{t-}) \geq &-\underline{\mu}_{\mathbf{d}} J_{\mathbf{d}}(W_{t-}) + rW_{t-}J'_{\mathbf{d}}(W_{t-}) + \underline{\mu}_{\mathbf{d}}[q_t W_{t-} - (1-q_t)H_t]J'_{\mathbf{d}}(W_{t-}) \\ &+ \underline{\mu}_{\mathbf{d}}(q_t J_{\mathbf{u}}(0) + (1-q_t)J_{\mathbf{u}}(W_{t-} + H_t)), W_{t-} \geq 0, \end{aligned}$$

and

$$\begin{aligned} rJ_{\mathbf{u}}(W_{t-}) \geq &-\bar{\mu}_{\mathbf{u}} J_{\mathbf{u}}(W_{t-}) + rW_{t-}J'_{\mathbf{u}}(W_{t-}) + \bar{\mu}_{\mathbf{u}}[q_t W_{t-} + (1-q_t)H_t]J'_{\mathbf{u}}(W_{t-}) \\ &+ \bar{\mu}_{\mathbf{u}}(q_t J_{\mathbf{d}}(0) + (1-q_t)J_{\mathbf{d}}(W_{t-} - H_t)), W_t \geq 0, \end{aligned}$$

for any $-H_t \leq W_{t-}$ and $q_t \in [0,1]$ imply that it is optimal to induce effort from the agent before contract termination. They are further equivalent to

$$\begin{aligned} rJ_{\mathbf{d}}(w) \geq &-\underline{\mu}_{\mathbf{d}} J_{\mathbf{d}}(w) + rwJ'_{\mathbf{d}}(w) \\ &+ \underline{\mu}_{\mathbf{d}} \max_{q\in[0,1],-h\leq w} \left\{[qw-(1-q)h]J'_{\mathbf{d}}(w) + (qJ_{\mathbf{u}}(0) + (1-q)J_{\mathbf{u}}(w+h))\right\}, w \geq 0, \end{aligned} \tag{A.111}$$

and

$$\begin{aligned} rJ_{\mathbf{u}}(w) \geq &-\bar{\mu}_{\mathbf{u}} J_{\mathbf{u}}(w) + rwJ'_{\mathbf{u}}(w) \\ &+ \bar{\mu}_{\mathbf{u}} \max_{q\in[0,1],-h\leq w} \left\{[qw-(1-q)h]J'_{\mathbf{u}}(w) + (qJ_{\mathbf{d}}(0) + (1-q)J_{\mathbf{d}}(w+h))\right\}, w \geq 0, \end{aligned} \tag{A.112}$$

respectively. In the following, we first consider the optimization problem in (A.111),

$$\begin{aligned} &\max_{q\in[0,1],-h\leq w} \left\{[qw-(1-q)h]J'_{\mathbf{d}}(w) + (qJ_{\mathbf{u}}(0) + (1-q)J_{\mathbf{u}}(w+h))\right\} \\ =&\max_{q\in[0,1],-h\leq w} \left\{q[wJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(0)] + (1-q)[-hJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(w+h)]\right\} \\ =&\max_{q\in[0,1]} \left\{q[wJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(0)] + (1-q)\max_{-h\leq w}[-hJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(w+h)]\right\}. \end{aligned}$$

Because $\max_{-h\leq w}[-hJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(w+h)] \geq [wJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(0)]$, we know that the optimal solution to the above optimization problem should be $q^* = 0$. Similarly, the optimal solution in the optimization problem in (A.112) should also be $q^* = 0$. Plugging $q^* = 0$ into (A.111) and (A.112) yields (A.89) and (A.90), respectively.

Proof of Corollary A.1 If $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and condition (2.30) holds, or, if $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ and condition (2.46) holds, then $J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w$ and $J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w$. In these cases, we have

$$\begin{aligned}\varphi_{\mathbf{d}}(w) &= (r + \underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - w) + rw - \underline{\mu}_{\mathbf{d}}[h + \underline{v}_{\mathbf{u}} - w - h] \\ &= (r + \underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - w) + rw - \underline{\mu}_{\mathbf{d}}(-w + \underline{v}_{\mathbf{u}}) \\ &= (r + \underline{\mu}_{\mathbf{d}})\underline{v}_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}}\underline{v}_{\mathbf{u}} = 0,\end{aligned}$$

and

$$\begin{aligned}\varphi_{\mathbf{u}}(w) &= (r + \bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}} - w) - R + rw - \bar{\mu}_{\mathbf{u}}[h + \underline{v}_{\mathbf{d}} - w - h] \\ &= (r + \bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}} - w) - R + rw - \bar{\mu}_{\mathbf{u}}(\underline{v}_{\mathbf{d}} - w) \\ &= (r + \bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}} - R - \bar{\mu}_{\mathbf{u}}\underline{v}_{\mathbf{d}} = 0.\end{aligned}$$

Hence, conditions (A.89) and (A.90) hold in these situations.

If $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and condition (2.29) holds, then $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ follow (A.49) and (A.50). Then we have

$$\begin{aligned}\varphi_{\mathbf{d}}(w) &= (r + \underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - w) + rw - \underline{\mu}_{\mathbf{d}} \max_{-h \leq w} \left\{-hJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(w + h)\right\} \\ &= (r + \underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - w) + rw - \underline{\mu}_{\mathbf{d}}(\beta_{\mathbf{u}} - w + v_{\mathbf{u}} - \beta_{\mathbf{u}}) \\ &= (r + \underline{\mu}_{\mathbf{d}})\underline{v}_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}} v_{\mathbf{u}} \\ &= \underline{\mu}_{\mathbf{d}}(\underline{v}_{\mathbf{u}} - v_{\mathbf{u}}) < 0 \,,\end{aligned}$$

where the second equality follows from $h^* = \beta_{\mathbf{u}} - w$ and the inequality follows from (2.29).

If $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ and condition (2.45) holds, then $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ follow (A.84) and (A.85). Then we have

$$\begin{aligned}\varphi_{\mathbf{u}}(w) &= (r + \bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}} - w) - R + rw - \bar{\mu}_{\mathbf{u}} \max_{h \leq w} \left\{-hJ'_{\mathbf{u}}(w) + J_{\mathbf{d}}(w + h)\right\} \\ &= (r + \bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}} - w) - R + rw - \bar{\mu}_{\mathbf{u}}(\bar{w}_{\mathbf{d}} - w + v_{\mathbf{d}} - \bar{w}_{\mathbf{d}}) \\ &= (r + \bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}} - R - \bar{\mu}_{\mathbf{u}} v_{\mathbf{d}} \\ &= \bar{\mu}_{\mathbf{u}}(\underline{v}_{\mathbf{d}} - v_{\mathbf{d}}) < 0 \,,\end{aligned}$$

where the second equality follows from $h^* = \bar{w}_{\mathbf{d}} - w$ and the inequality follows from (2.45). Hence, in the above two scenarios, the sufficient conditions (A.89) and (A.90) do not hold.

A.2.4.2 Proofs in Section A.2.2

Proof of Proposition A.2 First, we could rearrange (A.92)-(A.94) as the following

$$(\underline{\mu}_{\mathbf{d}} + r)J^m_{\mathbf{d}}(w) = \underline{\mu}_{\mathbf{d}} J^m_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} w\right) , \qquad w \in [0, \bar{w}_m - \beta_{\mathbf{u}}] \,. \tag{A.113}$$

and

$$-c_{\mathbf{u}} + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_m} J_{\mathbf{u}}^{m'}(w) = (\mu_{\mathbf{u}} + r)J_{\mathbf{u}}^{m}(w) - R - \mu_{\mathbf{u}} J_{\mathbf{d}}^{m}(w - \beta_{\mathbf{u}}) + \left[\left(2\underline{\mu}_{\mathbf{d}} + r\right)\beta_{\mathbf{u}}\right]\mathbb{1}_{w=\bar{w}_m}, \tag{A.114}$$

$$w \in [\beta_{\mathbf{u}}, \bar{w}_m] \ ,$$

$$J_{\mathbf{u}}^{m}(w) = J_{\mathbf{u}}^{m}(0) + \frac{J_{\mathbf{u}}^{m}(\beta_{\mathbf{u}}) - J_{\mathbf{u}}^{m}(0)}{\beta_{\mathbf{u}}} w \ , \quad w \in [0, \beta_{\mathbf{u}}] \ .$$

Then, the corresponding differential equations for $V_{\mathbf{d}}^{m}(w)$ and $V_{\mathbf{u}}^{m}(w)$ are

$$(\underline{\mu}_{\mathbf{d}} + r)V_{\mathbf{d}}^{m}(w) = \underline{\mu}_{\mathbf{d}} V_{\mathbf{u}}^{m}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} w\right), \qquad w \in [0, \bar{w}_m - \beta_{\mathbf{u}}], \tag{A.115}$$

$$(rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_m} V_{\mathbf{u}}^{m'}(w) = (\mu_{\mathbf{u}} + r)V_{\mathbf{u}}^{m}(w) + c_{\mathbf{u}} - R - \mu_{\mathbf{u}} V_{\mathbf{d}}^{m}(w - \beta_{\mathbf{u}}), \quad w \in [\beta_{\mathbf{u}}, \bar{w}_m] \ , \tag{A.116}$$

$$V_{\mathbf{u}}^{m}(w) = aw + \underline{v}_{\mathbf{u}}, \quad w \in [0, \beta_{\mathbf{u}}]. \tag{A.117}$$

From equation (A.115), we observe that $V_{\mathbf{d}}^{m'}(w) = V_{\mathbf{u}}^{m'}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} w\right)$ and $V_{\mathbf{d}}^{m''}(w) = \frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} V_{\mathbf{u}}^{m''}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} w\right)$. Hence, $V_{\mathbf{d}}^{m}$ is increasing and strictly concave if and only if so is $V_{\mathbf{u}}^{m}$. Combining (A.115) and (A.116), we obtain

$$(rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}} + c_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_m} V_{\mathbf{u}}^{m'}(w) = (\mu_{\mathbf{u}} + r)V_{\mathbf{u}}^{m}(w) - \frac{\mu_{\mathbf{u}}\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}} + r} V_{\mathbf{u}}^{m}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}}(w - \beta_{\mathbf{u}})\right) - (R - c_{\mathbf{u}}) \quad , \tag{A.118}$$

$$w \in [\beta_{\mathbf{u}}, \bar{w}_m] \, .$$

We then show the result according to the following steps.

1. Show that the solution to (A.118) is unique and twice continuously differentiable except at $w = \beta$ for any $a > 0$. Call it V_a.
2. Argue that $V_{\mathbf{u}}^{m}$ is left-continuous at $\bar{w}_m$, which is $\lim_{w \to \bar{w}_m -} V_{\mathbf{u}}(w) = V_{\mathbf{u}}\left(\bar{w}_m\right)$.
3. For any $a > 0$, show that V_a is concave.
4. Show that $\lim_{w \to \bar{w}_m -} V_a(w)$ is increasing in a for $a > 0$, which implies that the boundary condition $V_a(\bar{w}_m) = \frac{(r + \underline{\mu}_{\mathbf{d}})(R - c_{\mathbf{u}})}{r(r + \mu_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})}$ uniquely determines a, and therefore the solution to the original differential equation. Furthermore, $\lim_{w \to \bar{w}_m -} V_{\mathbf{u}}(w) = V_{\mathbf{u}}\left(\bar{w}_m\right)$ implies that $\lim_{w \to \bar{w}_m -} V_{\mathbf{u}}'(w) = 0$. Hence, the solution $V_{\mathbf{u}}$ is increasing and concave.

Step 1. Define $w_0 := 0$ and $w_n := \frac{\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}} + r} w_{n-1} + \beta_{\mathbf{u}}$ for $n = 1, 2, 3....$ Then, we can verify that $\lim_{n \to \infty} w_n = \bar{w}_m$. Applying (A.117) as the boundary condition, we show that differential equation (A.118) has a unique solution (call it $V_a(w)$, on the interval $(\beta_{\mathbf{u}}, \bar{w}_m)$), which is continuously differentiable. In fact, differential equation (A.118) is equivalent to a sequence of initial value problems over the intervals

$[w_n, w_{n+1}), n = 1, 2, ...$ This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem and, therefore, bear unique solutions.

Furthermore, we could derive the expression of $V_a''(w)$ following (A.118), as

$$V_a''(w) = \frac{\mu_{\mathbf{u}} \left[V_a'(w) - V_a' \left(\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(w - \beta_{\mathbf{u}}) \right) \right]}{rw + \mu_{\mathbf{u}} \beta_{\mathbf{u}}}, \text{ for } w \in (\beta_{\mathbf{u}}, \bar{w}_m). \tag{A.119}$$

Step 2. The sequence of initial value problems in step 1 do not attain $\bar{w}_m$, so we first argue that $V_{\mathbf{u}}$ is left-continuous at $\bar{w}_m$. According to the contract Γ_r^*, if the contract starts with $W_0 = \bar{w}_m - \epsilon$ with sufficiently small $\epsilon > 0$, the probability that W_t eventually reaches $\bar{w}_m$ approaches 1 as ϵ approaches 0. Therefore, we have $\lim_{\epsilon \to 0+} V_a(\bar{w}_m - \epsilon) = V_a(\bar{w}_m)$.

Step 3. Next, we show that if $a > 0$, V_a is increasing and concave on $[0, \bar{w}_m)$. Equation (A.118) implies that

$$V_{a+}'(\beta_{\mathbf{u}}) = a + \frac{c_{\mathbf{u}} - \frac{\Delta \mu_{\mathbf{u}} R}{r + \underline{\mu}_{\mathbf{d}} + \mu_{\mathbf{u}}}}{(r + \bar{\mu}_{\mathbf{u}}) \beta_{\mathbf{u}}} < a,$$

where the inequality follows from (A.97). Also, equation (A.119) implies that $V_{a+}''(\beta_{\mathbf{u}}) < 0$. Then, we claim that $V_a''(w) < 0$ for $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$. We proceed the proof by contradiction. Assuming that there exists $\breve{w} \in (\beta_{\mathbf{u}}, \bar{w}_m)$ such that $V_a''(\breve{w}) \geq 0$, because V_a is twice continuously differentiable on $(\beta_{\mathbf{u}}, \bar{w}_m)$, there must exist $\tilde{w} = \max\{ w \in (\beta_{\mathbf{u}}, \bar{w}_m) | V_a''(w) = 0\}$, and $V_a''(w) < 0, \forall w < \tilde{w}$. Equation (A.119) implies that $V_a'(\tilde{w}) = V_a' \left(\dfrac{r + \underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}} (\tilde{w} - \beta_{\mathbf{u}}) \right)$. However, this contradicts

$$V_a'(\tilde{w}) = V_a' \left(\frac{r + \underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}} (\tilde{w} - \beta_{\mathbf{u}}) \right) + \int_{\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(\tilde{w} - \beta_{\mathbf{u}})}^{\tilde{w}} V_a''(x) dx < V_a' \left(\frac{r + \underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}} (\tilde{w} - \beta_{\mathbf{u}}) \right),$$

in which the inequality follows from the fact that for any $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$, we must have $w > \dfrac{r + \underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}} (w - \beta_{\mathbf{u}})$. Hence, V_a should be concave on the interval $[0, \bar{w}_m)$.

Step 4. Finally, we show that $\lim\limits_{w \uparrow \bar{w}_m} V_a(w)$ is strictly increasing in a for $a > 0$, which allows us to uniquely determine a that satisfies $V_a(\bar{w}_m) = \dfrac{(r + \underline{\mu}_{\mathbf{d}})(R - c_{\mathbf{u}})}{r(r + \mu_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})}$. For any $0 < a_1 < a_2$, it can be seen that $V_{a_1}(w) < V_{a_2}(w)$, $V_{a_1}'(w) < V_{a_2}'(w)$, for $w \in [0, \beta_{\mathbf{u}})$ from (A.117). We claim that $V_{a_1}' < V_{a_2}'$ for $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$. Otherwise, because $V_{a_1} - V_{a_2}$ is continuously differentiable, there must exist $w' = \max\left\{ w \left| V_{a_1}'(w) = V_{a_2}'(w), w \in (\beta_{\mathbf{u}}, \bar{w}_m) \right. \right\}$ and $V_{a_1}'(w) < V_{a_2}'(w)$ for $w < w'$. Equation (A.118) implies that

$$(r + \mu_{\mathbf{u}})(V_{a_1}(w') - V_{a_2}(w')) = \frac{\underline{\mu}_{\mathbf{d}} \mu_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r} \left[V_{a_1} \left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} (w' - \beta_{\mathbf{u}}) \right) - V_{a_2} \left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} (w' - \beta_{\mathbf{u}}) \right) \right].$$

However, this contradicts

$$
\begin{aligned}
0 > & V_{a_1}(w') - V_{a_2}(w') - \left[V_{a_1}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}}(w' - \beta_{\mathbf{u}}) \right) - V_{a_2}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}}(w' - \beta_{\mathbf{u}}) \right) \right] \\
= & \int_{\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}(w'-\beta_{\mathbf{u}})}^{w'} V'_{a_1}(x) - V'_{a_2}(x) dx \ .
\end{aligned}
$$

Therefore, we must have $V'_{a_1}(w) - V'_{a_2}(w) < 0$ for $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$, which implies that $V_{a_1}(w) - V_{a_2}(w) < 0$ for $w \in (0, \bar{w}_m)$. This implies that $\lim_{w \uparrow \bar{w}_m} V_a(w)$ is strictly increasing in a for $a > 0$. Because $\lim_{a \downarrow 0} \lim_{w \uparrow \bar{w}_m} V_a(w) \leq \underline{v}_{\mathbf{u}}$ and $\lim_{a \uparrow \infty} \lim_{w \uparrow \bar{w}_m} V_a(w) > \lim_{a \uparrow \infty} V_a(\beta_{\mathbf{u}}) = \infty$, there must exist a unique $\bar{a} > 0$ such that $\lim_{w \uparrow \bar{w}_m} V_{\bar{a}}(w) = \bar{V}_{\mathbf{u}}$. Further, with equation (A.118), we are able to verify that $\lim_{w \uparrow \bar{w}_m} V'_{\mathbf{u}}(w) = 0$. Hence, the solution V_1 is concave and increasing on $[0, \bar{w}_m]$ and strictly concave on $(\beta_{\mathbf{u}}, \bar{w}_m)$.

A.2.4.3 Proof of Proposition A.3

Following Itô's Formula for jump processes (see, for example, Theorem 17.5 in [Bas11]) and (DWm), we obtain

$$
e^{-r\tau} J(\tau) = e^{-r0} J(0) + \int_0^\tau [e^{-rt} dJ(t) - r e^{-rt} J(t) dt] = J(0) + \int_0^\tau e^{-rt}(-R \mathbb{1}_{\theta_t = \mathbf{u}} dt + c_m(\theta_t) dt + dL_t) \tag{A.120}
$$

$$
+ \int_0^\tau e^{-rt} \mathcal{A}_t.
$$

Following definition (A.3) and equation (A.120), we obtain, under contract Γ^*_r,

$$
e^{-r\tau} J(\tau) = J(0) + \int_0^\tau e^{-rt}(-R \mathbb{1}_{\theta_t = \mathbf{u}} dt + c_m(\theta_t) dt + dL_t) + \int_0^\tau e^{-rt} \mathcal{A}^*_t, \tag{A.121}
$$

where

$$
\begin{aligned}
\mathcal{A}^*_t =& dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t = \mathbf{u}} dt - dL_t - c_m(\theta_t) dt \\
=& \left\{ J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}) \mathbb{1}_{W_{t-} < \bar{w}_m} - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}} \right\} dt \mathbb{1}_{\theta_t = \mathbf{u}} - rJ_{\mathbf{d}}(W_{t-}) dt \mathbb{1}_{\theta_t = \mathbf{d}} \\
&+ \left[J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-} \right) - J_{\mathbf{d}}(W_{t-}) \right] \mathbb{1}_{W_{t-} \geq \underline{w}_m} dN_t \mathbb{1}_{\theta_t = \mathbf{d}} \\
&+ \left[J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-}) \right] \mathbb{1}_{W_{t-} - \beta_{\mathbf{u}} \geq \underline{w}_m} dN_t \mathbb{1}_{\theta_t = \mathbf{u}} + R\mathbb{1}_{\theta_t = \mathbf{u}} dt - dL^*_t \\
&+ \left[(J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-}))(1 - X_t) + (J_{\mathbf{d}}(\underline{w}_m) - J_{\mathbf{u}}(W_{t-}))X_t \right] \mathbb{1}_{W_{t-} - \beta_{\mathbf{u}} < \underline{w}_m} \\
=& \left\{ R - c_{\mathbf{u}} + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}) \mathbb{1}_{W_{t-} < \bar{w}_m} - rJ_{\mathbf{u}}(W_{t-}) + \mu_{\mathbf{u}}(J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})) \right. \\
&\left. - \left[\left(2\underline{\mu}_{\mathbf{d}} + r \right) \beta_{\mathbf{u}} \right] \mathbb{1}_{W_{t-} = \bar{w}_m} \right\} \mathbb{1}_{\theta_t = \mathbf{u}} dt \\
&+ \left\{ -rJ_{\mathbf{d}}(W_{t-}) dt + \underline{\mu}_{\mathbf{d}} \left[J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-} \right) - J_{\mathbf{d}}(W_{t-}) \right] \right\} \mathbb{1}_{W_{t-} \geq \underline{w}_m} \mathbb{1}_{\theta_t = \mathbf{d}} dt + \mathcal{B}^*_t \\
=& \mathcal{B}^*_t,
\end{aligned}
$$

in which the last equality follows from (A.113) and (A.114), and

$$\mathcal{B}_t^* = \left[J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}W_{t-}\right) - J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\geq \underline{w}_m}(dN_t - \underline{\mu}_{\mathbf{d}}dt)\mathbb{1}_{\theta_t=\mathbf{d}}$$
$$+\Big\{\Big[(J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-}))(X_t dN_t - \mu_{\mathbf{u}}\hat{q}_t(W_{t-}-\beta_{\mathbf{u}})dt)$$
$$+ (J_{\mathbf{d}}(\underline{w}_m) - J_{\mathbf{u}}(W_{t-}))((1-X_t)dN_t - \mu_{\mathbf{u}}(1-\hat{q}(W_{t-}-\beta_{\mathbf{u}}))dt)\Big]\mathbb{1}_{W_{t-}-\beta_{\mathbf{u}}<\underline{w}_m}$$
$$+ \left[J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})\right](dN_t - \mu_{\mathbf{u}}dt)\mathbb{1}_{W_{t-}-\beta_{\mathbf{u}}\geq \underline{w}_m}\Big\}\mathbb{1}_{\theta_t=\mathbf{u}}.$$

Taking the expectation on both sides of (A.121), we obtain

$$J_{\mathbf{d}}(w) = J(0) = \mathbb{E}^{\Gamma(w),\nu^*}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c_m(\theta_t)dt - dL_t^*)\right],$$

where we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^* dt$ is a martingale.

A.2.4.4 Proof of Theorem A.1

First, we provide a parallel result of Lemma A.11,

Lemma A.13 *For any contract Γ_m, there exists an $\mathcal{F}^N$-predictable process H_t such that for $t\in[0,\tau]$,*

$$dW_t = \{rW_{t-} + \nu_t c_m(\theta_t) - [(1-q_t)H_t - q_tW_{t-}]\mu(\theta_t,\nu_t) - \ell_t\}dt + [(1-X_t)H_t - X_tW_{t-}]dN_t - I_t, \tag{A.122}$$

in which Bernoulli random variable X_t takes value 1 with probability q_t. Furthermore, the necessary and sufficient condition for the effort process ν to maximize agent's utility given Γ_m is that

$$\nu_t = 1 \text{ if and only if } -q_tW_{t-} + (1-q_t)H_t \leq -\beta_{\mathbf{u}}, \ell_t \geq c_{\mathbf{u}},\ \text{for } \theta_t = \mathbf{u} \tag{A.123}$$

for all $t\in[0,\tau]$.

Correspondingly, a general optimality condition (parallel to Lemma A.12) is presented in the following,

Lemma A.14 *Suppose $J_{\mathbf{d}}(w):[0,\infty)\to\mathbb{R}$ and $J_{\mathbf{u}}(w):[0,\infty)\to\mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J'_{\mathbf{d}}(w)\geq -1$, $J'_{\mathbf{u}}(w)\geq -1$, and $J_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}$. Consider any contract Γ, which yields the agent's expected utility $u(\Gamma,\nu)=W_0$, followed by the continuation utility process $\{W_t\}_{t\geq 0}$ according to* (A.122). *Define a stochastic process $\{\Phi_t\}_{t\geq 0}$ as*

$$\begin{aligned}\Phi_t :=& R\mathbb{1}_{\theta_t=\mathbf{u}} + J'_{\theta_t}(W_{t-})(rW_{t-} - [-q_tW_{t-} + (1-q_t)H_t]\mu(\theta_t,\nu_t)) - rJ_{\theta_t}(W_{t-})\\ &+ \mu(\theta_t,\nu_t)q_t[J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})] + \mu(\theta_t,\nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-}+H_t) - J_{\theta_t}(W_{t-})] - \nu_t c_m(\theta_t)\,.\end{aligned} \tag{A.124}$$

where $\theta_t\in\{u,d\}$ and $\hat{\theta}_t = \mathbb{1}_{\theta_t=\mathbf{d}}\cdot u + \mathbb{1}_{\theta_t=\mathbf{u}}\cdot d$. Also, $\nu_t=0$ if constraints (A.123) are not satisfied at time t and $\nu_t=1$ if constraints (A.123) are satisfied at time t. If the process $\{\Phi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $J_\theta(u(\Gamma,\nu,\theta))\geq U(\Gamma,\nu,\theta)$.

From Proposition A.2, we know that $J^m_{\mathbf{d}}(w)$ and $J^m_{\mathbf{u}}(w)$ are concave and $J^{m'}_{\mathbf{d}}(w)\geq -1$, $J^{m'}_{\mathbf{u}}(w)\geq -1$. Then to prove Theorem A.1, we only need to show that $\Phi_t\leq 0$ holds almost surely. First, if $\theta_t=\mathbf{d}$, then

$\nu_t = 0$. Following (A.124), we have

$$\Phi_t := J'_{\mathbf{d}}(W_{t-})(rW_{t-} + [q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-})$$
$$+ \underline{\mu}_{\mathbf{d}} q_t [J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + \underline{\mu}_{\mathbf{d}}(1-q_t)[J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})] \,. \tag{A.125}$$

We need to consider the following optimization problem,

$$\max_{q_t, H_t} \quad q_t[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})],$$
$$s.t. \quad 0 \le q_t \le 1 (y_{\mathbf{d}}, z_{\mathbf{d}}), W_{t-} + H_t \ge \beta_{\mathbf{u}}, -H_t \le W_{t-}.$$

In the following, we verify that its optimal solution is

$$q_t^* = 0,\ H_t^* = \frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}} \quad \text{if} \quad W_{t-} \ge \underline{w}_m, \text{ and} \tag{A.126}$$

$$q_t^* = 1 - \frac{W_{t-}(\underline{\mu}_{\mathbf{d}} + r)}{\beta_{\mathbf{u}} \underline{\mu}_{\mathbf{d}}},\ H_t^* = \beta_{\mathbf{u}} - W_{t-} \quad \text{if} \quad W_{t-} < \underline{w}_m. \tag{A.127}$$

by the KKT conditions.

- If $W_{t-} \ge \dfrac{\underline{\mu}_{\mathbf{d}} \beta_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r}$, define the following dual variable for the binding constraint

$$y_{\mathbf{d}} = W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) + H_t^* J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{u}}(W_{t-} + H_t^*)$$
$$= \frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-} \left[J'_{\mathbf{u}} \left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-} \right) - \frac{J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}} W_{t-}\right) - J_{\mathbf{u}}(0)}{\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}} W_{t-}} \right] \ge 0,$$

  where the inequality follows from concavity. One can verify that

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] = -y_{\mathbf{d}}, \tag{A.128}$$

$$(1 - q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H_t^*)) = 0, \tag{A.129}$$

  where (A.129) follows from $J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}\left(\dfrac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-}\right) = 0$.

- If $W_{t-} < \dfrac{\underline{\mu}_{\mathbf{d}} \beta_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r}$, one can verify that

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} - H_t^*) - J_{\mathbf{d}}(W_{t-})] = 0, \tag{A.130}$$

$$(1 - q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} - H_t^*)) = 0, \tag{A.131}$$

  where (A.130) follows from

$$J_{\mathbf{u}}(0) - J_{\mathbf{u}}(\beta_{\mathbf{u}}) + \beta_{\mathbf{u}} J'_{\mathbf{d}}(W_{t-}) = \beta_{\mathbf{u}} \left[J'_{\mathbf{d}}(W_{t-}) - \frac{J_{\mathbf{u}}(\beta_{\mathbf{u}}) - J_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}} \right] = 0,$$

and (A.131) follows from $J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(\beta_{\mathbf{u}}) = 0$.

Therefore, (A.126) and (A.127) implies that in (A.125),

$$\begin{aligned}\Phi_t \leq & - rJ_{\mathbf{d}}(W_{t-}) + \left\{ J'_{\mathbf{d}}(W_{t-})rW_{t-} + \underline{\mu}_{\mathbf{d}} \left[-\frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}} J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}} \left(\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}} W_{t-} \right) - J_{\mathbf{d}}(W_{t-}) \right] \right\} \\ & \cdot \mathbb{1}_{W_{t-} \geq \underline{w}_m} + \Big\{ J'_{\mathbf{d}}(W_{t-})rW_{t-} + \underline{\mu}_{\mathbf{d}} q_t^* [W_{t-} J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] \\ & \quad + \underline{\mu}_{\mathbf{d}} (1-q_t^*)[(W_{t-} - \beta_{\mathbf{u}}) J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(\beta_{\mathbf{u}}) - J_{\mathbf{d}}(W_{t-})] \Big\} \mathbb{1}_{W_{t-} < \underline{w}_m} = 0.\end{aligned}$$

where the last equality follows from equation (A.113),(A.114) and $J'_{\mathbf{u}}(W_{t-}) = -1$ for $W_{t-} \geq \bar{w}_m$ and $J'_{\mathbf{d}}(W_{t-}) = -1$ for $W_{t-} \geq \bar{w}_m - \beta_{\mathbf{u}}$.

If $\theta_t = \mathbf{u}$ and $\nu_t = 1$, then following (A.124), we have

$$\begin{aligned}\Phi_t :=& R + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + [q_t W_{t-} - (1-q_t)H_t]\mu_{\mathbf{u}}) - rJ_{\mathbf{u}}(W_{t-}) \\ & + \mu_{\mathbf{u}} q_t [J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + \mu_{\mathbf{u}}(1-q_t)[J_{\mathbf{d}}(W_{t-} + H_t) - J_{\mathbf{u}}(W_{t-})] \,. \end{aligned} \tag{A.132}$$

We need to consider the following optimization problem,

$$\begin{aligned}\max_{q_t, H_t} \quad & q_t[W_{t-} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t) - J_{\mathbf{u}}(W_{t-})], \\ s.t. \quad & 0 \leq q_t \leq 1, \; -q_t W_{t-} + (1-q_t)H_t \leq -\beta_{\mathbf{u}}.\end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\beta_{\mathbf{u}} \tag{A.133}$$

by the KKT conditions. Define the following dual variables for the binding constraints

$$\begin{aligned}x_{\mathbf{u}} &= -(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}})) \\ &= -\left(J'_{\mathbf{d}} \left(\frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r} \right) - J'_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) \right) \geq 0,\end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$, and

$$\begin{aligned}y_u &= (W_{t-} - \beta_{\mathbf{u}})(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}})) - W_{t-} J'_{\mathbf{u}}(W_{t-}) - J_{\mathbf{d}}(0) + \beta_{\mathbf{u}} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) \\ &= (W_{t-} - \beta_{\mathbf{u}})(\frac{J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{d}}(0)}{W_{t-} - \beta_{\mathbf{u}}} - J'_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}})) \geq 0.\end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H_t^* J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t^*) - J_{\mathbf{u}}(W_{t-})] \tag{A.134}$$
$$= -y_{\mathbf{u}} - (H_t^* + W_{t-}) x_{\mathbf{u}},$$
$$(1-q_t^*)(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-} + H_t^*)) = (q_t^* - 1) x_{\mathbf{u}}. \tag{A.135}$$

Therefore, (A.133) implies that in (A.132),

$$\Phi_t \leq R + J'_{\mathbf{u}}(W_{t-})rW_{t-} + \mu_{\mathbf{u}}[\beta_{\mathbf{u}} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}} = 0.$$

where the equality follows from (A.94).

If $\theta_t = \mathbf{u}$ and $\nu_t = 0$, then following (A.124), we have

$$\begin{aligned}\Phi_t :=& R + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \bar{\mu}_{\mathbf{u}}[q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{u}}(W_{t-}) \\ &+ \bar{\mu}_{\mathbf{u}} q_t [J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + \bar{\mu}_{\mathbf{u}}(1-q_t)[J_{\mathbf{d}}(W_{t-} + H_t) - J_{\mathbf{u}}(W_{t-})] \,. \end{aligned} \tag{A.136}$$

We need to consider the following optimization problem,

$$\begin{aligned} \max_{q_t, H_t} \quad & q_t[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t) - J_{\mathbf{u}}(W_{t-})], \\ s.t. \quad & 0 \le q_t \le 1 (y, z), \ -H_t \le W_{t-} (x). \end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}} + r} \tag{A.137}$$

following the KKT conditions. Define the following dual variable for the binding constraint

$$\begin{aligned} y &= -H_t^* J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t^*) - W_{t-}J'_{\mathbf{u}}(W_{t-}) - J_{\mathbf{d}}(0) \\ &= J_{\mathbf{d}}(\frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r}) - J_{\mathbf{d}}(0) - \frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r} J'_{\mathbf{d}}(\frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r}) \ge 0, \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$. One can verify

$$[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H_t^* J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t^*) - J_{\mathbf{u}}(W_{t-})] = -y, \tag{A.138}$$

$$(1-q_t^*)(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-} + H_t^*)) = 0, \tag{A.139}$$

Therefore, (A.137) implies that in (A.136),

$$\begin{aligned} \Phi_t \le& R + J'_{\mathbf{u}}(W_{t-}) \left(rW_{t-} + \frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}} + r} \right) - rJ_{\mathbf{u}}(W_{t-}) + \bar{\mu}_{\mathbf{u}} \left[J_{\mathbf{d}}(\frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r}) - J_{\mathbf{u}}(W_{t-}) \right] \\ =& R + \left(r + \frac{r\bar{\mu}_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r} \right) W_{t-}J'_{\mathbf{u}}(W_{t-}) - (r + \bar{\mu}_{\mathbf{u}} - \frac{\bar{\mu}_{\mathbf{u}}\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}} + r}) J_{\mathbf{u}}(W_{t-}) \\ \le& R + \left(r + \frac{r\bar{\mu}_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r} \right) \left(W_{t-} \frac{J_{\mathbf{u}}(W_{t-}) - J_{\mathbf{u}}(0)}{W_{t-}} - J_{\mathbf{u}}(W_{t-}) \right) \\ =& R - \left(r + \frac{r\bar{\mu}_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r} \right) \underline{v}_{\mathbf{u}} = 0, \end{aligned}$$

where the second inequality follows from the concavity of $J_{\mathbf{u}}$ and the last equality follows from (2.4).

A.2.4.5 Proof of Proposition A.4

It suffices to show that if (A.97) is not satisfied, then the following principal's value functions

$$J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w, \text{ and } J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w.$$

satisfies the optimality condition $\Phi_t \le 0$, where Φ_t is defined in (A.124). If $\theta_t = \mathbf{d}$, then $\nu_t = 0$ and

$$\Phi_t = -rW_{t-} - \underline{\mu}_{\mathbf{d}}[q_tW_{t-} - (1-q_t)H_t] - (r+\underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - W_{t-}) + \underline{\mu}_{\mathbf{d}}q_t\underline{v}_{\mathbf{u}}$$
$$+ \underline{\mu}_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) = -(r+\underline{\mu}_{\mathbf{d}})\underline{v}_{\mathbf{d}} + \underline{\mu}_{\mathbf{d}}\underline{v}_{\mathbf{u}} = 0,$$

and when $\theta_t = \mathbf{u}$ and $\nu_t = 1$, then

$$\begin{aligned}\Phi_t &= R - rW_{t-} - c_{\mathbf{u}} - \mu_{\mathbf{u}}[q_tW_{t-} - (1-q_t)H_t] - r(\underline{v}_{\mathbf{u}} - W_{t-}) + \mu_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}} \\ &\quad + \mu_{\mathbf{u}}(1-q_t)(\underline{v}_{\mathbf{d}} - W_{t-} - H_t) - \mu_{\mathbf{u}}(\underline{v}_{\mathbf{u}} - W_{t-}) \\ &= R - c_{\mathbf{u}} - r\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} - \mu_{\mathbf{u}}\underline{v}_{\mathbf{u}} = R - c_{\mathbf{u}} - (r+\mu_{\mathbf{u}})\frac{(r+\underline{\mu}_{\mathbf{d}})R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})} + \mu_{\mathbf{u}}\frac{\underline{\mu}_{\mathbf{d}}R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})} \\ &= \frac{\Delta\mu_{\mathbf{u}}R}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}} - c_{\mathbf{u}} = \frac{\Delta\mu_{\mathbf{u}}}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}}(R - (r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})\beta_{\mathbf{u}}) < 0,\end{aligned}$$

where the inequalities follow from the fact that (A.97) is not satisfied. If $\theta_t = \mathbf{u}$ and $\nu_t = 0$, then

$$\Phi_t = R - rW_{t-} - \bar{\mu}_{\mathbf{u}}[q_tW_{t-} - (1-q_t)H_t] - (r+\bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}} - W_{t-}) + \bar{\mu}_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}}$$
$$+ \bar{\mu}_{\mathbf{u}}(1-q_t)(\underline{v}_{\mathbf{d}} - W_{t-} - H_t) = R - (r+\bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \bar{\mu}_{\mathbf{u}}\underline{v}_{\mathbf{d}} = 0.$$

A.2.5 Proofs in Section A.2.3

A.2.5.1 Proof of Proposition A.5

First, based on (A.100) and (A.101), we write the differential equations for $V^r_{\mathbf{d}}$ and $V^r_{\mathbf{u}}$ as the following,

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = -c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w) + \mu_{\mathbf{d}}V_{\mathbf{u}}\left(\min\left\{w+\beta_{\mathbf{d}}, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}\right\}\right), \text{ for } w \in [0, \bar{w}_{\mathbf{d}}], \tag{A.140}$$

$$(\bar{\mu}_{\mathbf{u}} + r)V_{\mathbf{u}}(w) = R + \bar{\mu}_{\mathbf{u}}V_{\mathbf{d}}\left(\frac{r+\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}}w\right), \text{ for } w \in \left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}\right]. \tag{A.141}$$

Combining equations (A.140) and (A.141) yields

$$r(\bar{w}_{\mathbf{d}} - w)V'_{\mathbf{d}}(w) + (r+\mu_{\mathbf{d}})V_{\mathbf{d}}(w) = -c_{\mathbf{d}} + \mu_{\mathbf{d}}\left[\frac{R + \bar{\mu}_{\mathbf{u}}V_{\mathbf{d}}\left(\min\left\{\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w+\beta_{\mathbf{d}}), \bar{w}_{\mathbf{d}}\right\}\right)}{r+\bar{\mu}_{\mathbf{u}}}\right]. \tag{A.142}$$

Rearrange equation (A.142) as

$$(r+\mu_{\mathbf{d}})V_{\mathbf{d}}(w) - rV'_{\mathbf{d}}(w)(w-\bar{w}_{\mathbf{d}}) + c_{\mathbf{d}} - \frac{\mu_{\mathbf{d}}R}{r+\bar{\mu}_{\mathbf{u}}} = \frac{\mu_{\mathbf{d}}\bar{\mu}_{\mathbf{u}}}{r+\bar{\mu}_{\mathbf{u}}}V_{\mathbf{d}}(\bar{w}_{\mathbf{d}}), \text{ for } w \in \left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}, \infty\right), \tag{A.143}$$

$$(r+\mu_{\mathbf{d}})V_{\mathbf{d}}(w) - rV'_{\mathbf{d}}(w)(w-\bar{w}_{\mathbf{d}}) + c_{\mathbf{d}} - \frac{\mu_{\mathbf{d}}R}{r+\bar{\mu}_{\mathbf{u}}} = \frac{\mu_{\mathbf{d}}\bar{\mu}_{\mathbf{u}}}{r+\bar{\mu}_{\mathbf{u}}}V_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w+\beta_{\mathbf{d}})\right)$$
$$, \text{ for } w \in \left(0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}\right). \tag{A.144}$$

We then show the result according to the following steps.

1. Demonstrate the solution of (A.143) as a parametric function V_b, with parameter b.

2. Show that the solution of (A.144) (which we call V_b) is unique and twice continuously differentiable for any b.

3. Show that V_b is convex and decreasing for $b > 0$ and concave and increasing for $b < 0$.

4. Show that $V_b(0)$ is increasing in b for $b < 0$, which implies that the boundary condition $V_b(0) = \underline{v}_{\mathbf{d}}$ uniquely determines b, and therefore the solution of the original differential equation.

Step 1. The solution to the linear ordinary differential equation (A.143) on $\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}, \bar{w}_{\mathbf{d}}\right)$ must have the following form, for any scalar b,

$$V_b(w) = \frac{\mu_{\mathbf{d}}R - (r+\bar{\mu}_{\mathbf{u}})c_{\mathbf{d}}}{r\left(r+\bar{\mu}_{\mathbf{u}}+\mu_{\mathbf{d}}\right)} + b(\bar{w}_{\mathbf{d}} - w)^{\frac{r+\mu_{\mathbf{d}}}{r}}, \quad \text{for } w \in \left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\right). \tag{A.145}$$

Also define $V_b(w) = \frac{\mu_{\mathbf{d}}R - (r+\bar{\mu}_{\mathbf{u}})c_{\mathbf{d}}}{r\left(r+\bar{\mu}_{\mathbf{u}}+\mu_{\mathbf{d}}\right)}$ for $w \in [\bar{w}_{\mathbf{d}}, \infty)$, which satisfies (A.143), so that V_b is continuously differentiable on $\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}, \infty\right)$.

Step 2. Using (A.145) as the boundary condition, we show that differential equation (A.144) has a unique solution, (which we call $V_b(w)$) on $\left(0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}\right)$, which is continuously differentiable. In fact, differential equation (A.144) is equivalent to a sequence of initial value problems. This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem and, therefore, bear unique solutions. Also, computing $V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}\right)$ from (A.145), and comparing it with (A.144), we see that V_b is continuously differentiable at $\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}$, and therefore on $[0, \infty)$.

Furthermore, deriving expressions for $V_b''(w)$ following (A.144) and (A.145), respectively, confirms that V_b is twice continuously differentiable on $[0, \infty)$. In particular, (A.144) implies that

$$V_b''(w) = \frac{\mu_{\mathbf{d}}\left[V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w+\beta_{\mathbf{d}})\right) - V_b'(w)\right]}{r(\bar{w}_{\mathbf{d}} - w)}. \tag{A.146}$$

Step 3. Next, we argue that in order to satisfy the boundary condition $V_b(0) = \underline{v}_{\mathbf{d}}$, we must have $b < 0$. Equivalently, we show that if $b > 0$, V_b must be convex and decreasing, which violates $V_b(0) = \underline{v}_{\mathbf{d}} < V_b(\bar{w}_{\mathbf{d}})$. In fact, if $b > 0$, (A.145) implies that V_b is decreasing and convex on $\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\right)$, and therefore $V_b''(w) > 0$ in this interval. Then, we show that $V_b''(w) > 0$ for $w \in [0, \bar{w}_{\mathbf{d}})$. We prove this by contradiction. If there exists $\breve{w} \in [0, \bar{w} - \beta_{\mathbf{d}})$, such that $V_b''(\breve{w}) \leq 0$, then V_b being twice continuously differentiable implies that there must exist $\tilde{w} = \max\left\{ w \in \left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}\right) \middle| V_b''(w) = 0\right\}$ such that $V_b''(w) > 0$ for all $w > \tilde{w}$. Equation (A.146) implies that $V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(\tilde{w}+\beta_{\mathbf{d}})\right) = V_b'(\tilde{w})$. However, we this contradicts with

$$V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(\tilde{w}+\beta_{\mathbf{d}})\right) = V_b'(w) + \int_{\tilde{w}}^{\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(\tilde{w}+\beta_{\mathbf{d}})} V_b''(x)dx > V_b'(\tilde{w}).$$

Therefore, we must have $V_b''(w) > 0$ and V_b is decreasing on $[0, \bar{w}_{\mathbf{d}})$ if $b > 0$. If $b = 0$, $V_b(w)$ is a constant following (A.144) and (A.145), which also contradicts the boundary condition. Therefore we must have $b < 0$.

The same logic implies that for $b < 0$, V_b must best be increasing and strictly concave on $[0, \bar{w}_{\mathbf{d}})$.

Step 4. Finally, we show that $V_b(0)$ is strictly increasing in b for $b < 0$, which allows us to uniquely determine b that satisfies $V_b(0) = \underline{v}_{\mathbf{d}}$. For any $b_1 < b_2 < 0$, it can be verified that $V_{b_1}(w) < V_{b_2}(w)$, $V_{b_1}'(w) > V_{b_2}'(w)$, for $w \in \left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\right)$ from (A.145). We claim that $V_{b_1}' > V_{b_2}'$ for $w \in [0, \bar{w}]$. Otherwise, because $V_{b_1} - V_{b_2}$ is continuously differentiable, there must exist

$$w' = \max\left\{w \,\middle|\, V_{b_1}'(w) = V_{b_2}'(w), w \in \left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}\right)\right\},$$

such that $V_{b_1}'(w) > V_{b_2}'(w)$ for $w > w'$. Equation (A.144) implies that

$$\frac{\mu_{\mathbf{d}}\bar{\mu}_{\mathbf{u}}}{r+\bar{\mu}_{\mathbf{u}}}\left[V_{b_1}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right) - V_{b_2}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right)\right] = (r+\mu_{\mathbf{d}})(V_{b_1}(w') - V_{b_2}(w')).$$

However, this contradicts

$$\begin{aligned} 0 &> V_{b_1}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right) - V_{b_2}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right) \\ &= V_{b_1}(w') - V_{b_2}(w') + \int_0^{\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})-w'} V_{b_1}'(w'+x) - V_{b_2}'(w'+x)dx\,. \end{aligned}$$

Therefore, we must have $V_{b_1}'(w) - V_{b_2}'(w) > 0$ for $w \in [0, \bar{w}_{\mathbf{d}})$, which further implies that $V_{b_1}(w) - V_{b_2}(w) < 0$ for $w \in [0, \bar{w}_{\mathbf{d}})$. As a result, $V_b(0)$ is strictly increasing in b for $b < 0$. Because $\lim_{b\uparrow 0} V_b(0) \le \underline{v}_{\mathbf{d}}$ and $\lim_{b\downarrow -\infty} V_b(0) > V_b(\bar{w}_1 - \beta_{\mathbf{d}}) = -\infty$, there must exist a unique $b^* < 0$ such that $V_{b^*}(0) = \underline{v}_{\mathbf{d}}$. Hence, the solution V_{b^*} is strictly concave and increasing in $\left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}\right]$.

A.2.5.2 Proof of Proposition A.6

Following definition (A.2) and equation (A.120), we obtain, under contract Γ_r^*,

$$e^{-r\tau}J(\tau) = J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt + c_r(\theta_t)dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t^*, \tag{A.147}$$

where

$$
\begin{aligned}
\mathcal{A}_t^* =& dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c_r(\theta_t)dt - dL_t \\
=& - rJ_{\mathbf{u}}(W_{t-})dt\mathbb{1}_{\theta_t=\mathbf{u}} + \left\{J'_{\mathbf{d}}(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}})dt - rJ_0(W_{t-})dt\right\}\mathbb{1}_{\theta_t=\mathbf{d}} + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^* \\
&+ \left[J_{\mathbf{u}}\left(\min\left\{\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}, W_{t-} + \beta_{\mathbf{d}}\right\}\right) - J_{\mathbf{d}}(W_{t-})\right]dN_t\mathbb{1}_{\theta_t=\mathbf{u}} \\
&+ \left[J_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}\right) - J_{\mathbf{u}}(W_{t-})\right]dN_t\mathbb{1}_{\theta_t=\mathbf{u}} \\
=& \left\{R - rJ_{\mathbf{u}}(W_t) + \bar{\mu}_{\mathbf{u}}\left(J_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}W_{t-}\right) - J_{\mathbf{u}}(W_{t-})\right)\right\}\mathbb{1}_{\theta_t=\mathbf{u}}dt \\
&+ \left\{J'_{\mathbf{d}}(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}}) - rJ_{\mathbf{d}}(W_{t-})dt + \mu_{\mathbf{d}}\left(J_{\mathbf{u}}\left(\min\left\{W_{t-} + \beta_{\mathbf{d}}, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right\}\right) - J_{\mathbf{d}}(W_{t-})\right)\right. \\
&\left. -\mu_{\mathbf{d}}\left(W_{t-} + \beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)^+ - c_{\mathbf{d}}\right\}\mathbb{1}_{\theta_t=\mathbf{d}}dt + \mathcal{B}_t^* \\
=& \mathcal{B}_t^*,
\end{aligned}
$$

in which the last equality follows from (A.100), (A.101) and

$$
\begin{aligned}
\mathcal{B}_t^* =& \left[J_{\mathbf{u}}\left(W_{t-} + \beta_{\mathbf{d}} - \left(W_{t-} + \beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)^+\right) - J_{\mathbf{d}}(W_{t-}) - \left(W_{t-} + \beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)^+\right] \\
&\cdot(dN_t - \mu_{\mathbf{d}}dt)\mathbb{1}_{\theta_t=\mathbf{d}} + \left[J_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}W_{t-}\right) - J_{\mathbf{u}}(W_{t-})\right](dN_t - \bar{\mu}_{\mathbf{u}}dt)\mathbb{1}_{\theta_t=\mathbf{u}}.
\end{aligned}
$$

Taking the expectation on both sides of (A.147), we immediately have

$$
J_{\mathbf{d}}(w) = J(0) = \mathbb{E}^{\Gamma(w),\nu^*}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c_r(\theta_t)dt - dL_t^*)\right],
$$

where we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^* dt$ is a martingale.

A.2.5.3 Proof of Theorem A.2

Again, we provide a parallel result of Lemma A.11,

Lemma A.15 *For any contract Γ_r, there exists an $\mathcal{F}^N$-predictable process H_t such that for $t \in [0, \tau]$,*

$$
dW_t = \{rW_{t-} + \nu_t c_r(\theta_t) - [(1-q_t)H_t - q_t W_{t-}]\mu(\theta_t, \nu_t) - \ell_t\}dt + [(1-X_t)H_t - X_t W_{t-}]dN_t - I_t, \tag{A.148}
$$

in which Bernoulli random variable X_t takes value 1 with probability q_t. Furthermore, the necessary and sufficient condition for the effort process ν to maximize agent's utility given Γ_m is that

$$
\nu_t = 1 \text{ if and only if } -q_t W_{t-} + (1-q_t)H_t \geq \beta_{\mathbf{d}}, \ell_t \geq c_{\mathbf{d}}, \text{ for } \theta_t = \mathbf{d} \tag{A.149}
$$

for all $t \in [0, \tau]$.

Correspondingly, a more general optimality condition (parallel to Lemma A.12) is presented in the following,

Lemma A.16 *Suppose $J_{\mathbf{d}}(w):[0,\infty)\to\mathbb{R}$ and $J_{\mathbf{u}}(w):[0,\infty)\to\mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J'_{\mathbf{d}}(w)\geq -1$, $J'_{\mathbf{u}}(w)\geq -1$, and $J_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}$. Consider any contract Γ, which yields the agent's expected utility $u(\Gamma,\nu)=W_0$, followed by the continuation utility process $\{W_t\}_{t\geq 0}$ according to* (A.148). *Define a stochastic process $\{\Phi_t\}_{t\geq 0}$ as*

$$\begin{aligned}\Phi_t :=& R\mathbb{1}_{\theta_t=\mathbf{u}} + J'_{\theta_t}(W_{t-})(rW_{t-} - [-q_tW_{t-} + (1-q_t)H_t]\mu(\theta_t,\nu_t)) - rJ_{\theta_t}(W_{t-}) \\ &+ \mu(\theta_t,\nu_t)q_t[J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})] + \mu(\theta_t,\nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-}+H_t) - J_{\theta_t}(W_{t-})] - \nu_t c_r(\theta_t)\ .\end{aligned} \tag{A.150}$$

where $\theta_t\in\{u,d\}$ and $\hat{\theta}_t=\mathbb{1}_{\theta_t=\mathbf{d}}\cdot u+\mathbb{1}_{\theta_t=\mathbf{u}}\cdot d$. Also, $\nu_t=0$ if constraints (A.149) are not satisfied at time t and $\nu_t=1$ if constraints (A.149) are satisfied at time t. If the process $\{\Phi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $J_\theta(u(\Gamma,\nu,\theta))\geq U(\Gamma,\nu,\theta)$.

From Proposition A.5, we know that $J^r_{\mathbf{d}}(w)$ and $J^r_{\mathbf{u}}(w)$ are concave and $J^{r'}_{\mathbf{d}}(w)\geq -1$, $J^{r'}_{\mathbf{u}}(w)\geq -1$. In order to show Theorem A.2, we only need to show that $\Phi_t\leq 0$ holds almost surely. First, if $\theta_t=\mathbf{u}$, then $\nu_t=0$ and following (A.150), we have

$$\begin{aligned}\Phi_t :=& R + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \bar{\mu}_{\mathbf{u}}[q_tW_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{u}}(W_{t-}) \\ &+ \bar{\mu}_{\mathbf{u}}q_t[J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + \bar{\mu}_{\mathbf{u}}(1-q_t)[J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_t)]\ .\end{aligned} \tag{A.151}$$

We need to consider the following optimization problem,

$$\begin{aligned}\max_{q_t,H_t}\quad & q_t[W_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})], \\ s.t.\quad & 0\leq q_t\leq 1, -H_t\leq W_{t-}.\end{aligned}$$

In the following, we verify that the optimal solution is

$$q^*_t = 0 \quad \text{and} \quad H^*_t = \frac{rW_{t-}}{\bar{\mu}_{\mathbf{u}}}, \tag{A.152}$$

using the KKT conditions. Define the following dual variable for the binding constraint

$$\begin{aligned}y &= -H^*_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H^*_t) - W_{t-}J'_{\mathbf{u}}(W_{t-}) - J_{\mathbf{d}}(0) \\ &= J_{\mathbf{d}}\left(\frac{(r+\bar{\mu}_{\mathbf{u}})W_{t-}}{\bar{\mu}_{\mathbf{u}}}\right) - J_{\mathbf{d}}(0) - \frac{(r+\bar{\mu}_{\mathbf{u}})W_{t-}}{\bar{\mu}_{\mathbf{u}}}J'_{\mathbf{d}}\left(\frac{(r+\bar{\mu}_{\mathbf{u}})W_{t-}}{\bar{\mu}_{\mathbf{u}}}\right)\geq 0,\end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$. One can verify

$$[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H^*_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H^*_t) - J_{\mathbf{u}}(W_{t-})] = -y, \tag{A.153}$$

$$(1-q^*_t)(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-}+H^*_t)) = 0. \tag{A.154}$$

Therefore, (A.152) implies that in (A.151),

$$\Phi_t := R - rJ_{\mathbf{u}}(W_t) + \bar{\mu}_{\mathbf{u}}\left[J_{\mathbf{d}}\left(W_t + \frac{rW_t}{\bar{\mu}_{\mathbf{u}}}\right) - J_{\mathbf{u}}(W_t)\right] = 0\ .$$

where the equality follows from (A.101).

If $\theta_t = \mathbf{d}$ and $\nu_t = 1$, then following (A.150), we have

$$\begin{aligned}\Phi_t :=& J'_{\mathbf{d}}(W_{t-})(rW_{t-} + \mu_{\mathbf{d}}[q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-}) \\ &+ \mu_{\mathbf{d}} q_t [J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + \mu_{\mathbf{d}}(1-q_t)[J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})] \,. \end{aligned} \tag{A.155}$$

We need to consider the following optimization problem,

$$\begin{aligned} \max_{q_t, H_t} \quad & q_t[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})], \\ s.t. \quad & 0 \le q_t \le 1, -q_t W_{t-} + (1-q_t)H_t \ge \beta_{\mathbf{d}}. \end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = \beta_{\mathbf{d}}. \tag{A.156}$$

using the KKT conditions. Define the following dual variables for the binding constraints,

$$x_{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) = J'_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{r + \bar{\mu}_{\mathbf{u}}}\right) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) \ge 0.$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$, and

$$\begin{aligned} y_{\mathbf{d}} &= (W_{t-} + \beta_{\mathbf{d}})(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}})) - W_{t-}J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{u}}(0) - \beta_{\mathbf{d}} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) \\ &= (W_{t-} + \beta_{\mathbf{d}})\left(\frac{J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{u}}(0)}{W_{t-} - \beta_{\mathbf{d}}} - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}})\right) \ge 0, \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$\begin{aligned} & [W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] \qquad \text{(A.157)} \\ & = -y_{\mathbf{d}} - (H_t^* + W_{t-})x_{\mathbf{d}}, \\ & (1-q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H_t^*)) = (1-q_t^*)x_{\mathbf{d}}. \qquad \text{(A.158)} \end{aligned}$$

Therefore, (A.156) implies that in (A.151),

$$\Phi_t \le J'_{\mathbf{d}}(W_{t-})rW_{t-} + \mu_{\mathbf{d}}[-\beta_{\mathbf{d}} J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})] - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} = 0,$$

where the equality follows from (A.100).

If $\theta_t = \mathbf{d}$ and $\nu_t = 0$, then following (A.151), we have

$$\begin{aligned}\Phi_t :=& J'_{\mathbf{d}}(W_{t-})(rW_{t-} + \underline{\mu}_{\mathbf{d}}[q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-}) \\ &+ \underline{\mu}_{\mathbf{d}} q_t [J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + \underline{\mu}_{\mathbf{d}}(1-q_t)[J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})] \,. \end{aligned} \tag{A.159}$$

We need to consider the following optimization problem,

$$\begin{aligned} \max_{q_t, H_t} \quad & q_t[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})], \\ s.t. \quad & 0 \le q_t \le 1, -H_t \le W_{t-}. \end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\frac{rW_{t-}}{\bar{\mu}_{\mathbf{u}} + r}. \tag{A.160}$$

using the KKT conditions. Define the following dual variable for the binding constraint

$$\begin{aligned} y_{\mathbf{d}} &= W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) + H_t^* J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{u}}(W_{t-} + H_t^*) \\ &= \frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r}\left[J'_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r}\right) - \frac{J_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}}+r}\right) - J_{\mathbf{u}}(0)}{\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}}+r}}\right] \geq 0\,. \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] = -y_{\mathbf{d}}, \tag{A.161}$$

$$(1 - q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H_t^*)) = 0, \tag{A.162}$$

where (A.162) follows from $J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r}\right) = 0$. Therefore, (A.160) implies that in (A.159),

$$\begin{aligned} \Phi_t :=& J'_{\mathbf{d}}(W_{t-})\left(rW_{t-} + \frac{r\underline{\mu}_{\mathbf{d}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r}\right) - rJ_{\mathbf{d}}(W_{t-}) + \underline{\mu}_{\mathbf{d}}\left[J_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r}\right) - J_{\mathbf{d}}(W_{t-})\right] \\ =& J'_{\mathbf{d}}(W_{t-})\left(rW_{t-} + \frac{r\underline{\mu}_{\mathbf{d}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r}\right) - rJ_{\mathbf{d}}(W_{t-}) + \underline{\mu}_{\mathbf{d}}\left[\frac{R}{\bar{\mu}_{\mathbf{u}} + r} - \frac{r}{r + \bar{\mu}_{\mathbf{u}}} J_{\mathbf{d}}(W_{t-})\right] \\ =& \frac{\underline{\mu}_{\mathbf{d}} R}{\bar{\mu}_{\mathbf{u}} + r} + \frac{r(\underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}} + r)}{\bar{\mu}_{\mathbf{u}} + r}(W_{t-}J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{d}}(W_{t-})) \\ \leq& \frac{\underline{\mu}_{\mathbf{d}} R}{\bar{\mu}_{\mathbf{u}} + r} + \frac{r(\underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}} + r)}{\bar{\mu}_{\mathbf{u}} + r}\left(W_{t-}\frac{J_{\mathbf{d}}(W_{t-}) - \underline{v}_{\mathbf{d}}}{W_{t-}} - J_{\mathbf{d}}(W_{t-})\right) \\ =& 0\,. \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{d}}(w)$ and the last equality follows from equation (2.4).

A.2.5.4 Proof of Proposition A.7

It suffices to show that if (A.97) is not satisfied, then the following principal's value functions,

$$J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w, \text{ and } J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w,$$

satisfy the optimality condition $\Phi_t \leq 0$, where Φ_t is defined by (A.150). If $\theta_t = \mathbf{u}$, then $\nu_t = 0$ and

$$\begin{aligned} \Phi_t &= R - rW_{t-} - \bar{\mu}_{\mathbf{u}}[q_t W_{t-} - (1 - q_t)H_t] - (r + \bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}} - W_{t-}) + \bar{\mu}_{\mathbf{u}} q_t \underline{v}_{\mathbf{d}} \\ &\quad + \bar{\mu}_{\mathbf{u}}(1 - q_t)(\underline{v}_{\mathbf{d}} - W_{t-} - H_t) = R - (r + \bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \bar{\mu}_{\mathbf{u}}\underline{v}_{\mathbf{d}} = 0. \end{aligned}$$

When $\theta_t = \mathbf{d}$ and $\nu_t = 0$, then

$$\begin{aligned} \Phi_t &= -rW_{t-} - \underline{\mu}_{\mathbf{d}}[q_t W_{t-} + (1 - q_t)H_t] - (r + \underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - W_{t-}) + \underline{\mu}_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} \\ &\quad + \underline{\mu}_{\mathbf{d}}(1 - q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) = -\left(r + \underline{\mu}_{\mathbf{d}}\right)\underline{v}_{\mathbf{d}} + \underline{\mu}_{\mathbf{d}}\underline{v}_{\mathbf{u}} = 0. \end{aligned}$$

When $\theta_t = \mathbf{d}$ and $\nu_t = 1$, then

$$\begin{aligned}
\Phi_t &= -rW_{t-} - c_{\mathbf{d}} - \mu_{\mathbf{d}}[q_t W_{t-} - (1-q_t)H_t] - (r+\mu_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - W_{t-}) + \mu_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} \\
&\quad + \mu_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) \\
&= -c_{\mathbf{d}} - (r+\mu_{\mathbf{d}})\underline{v}_{\mathbf{d}} + \mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} = -c_{\mathbf{d}} - (r+\mu_{\mathbf{d}})\frac{\underline{\mu}_{\mathbf{d}} R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})} + \mu_{\mathbf{d}}\frac{(r+\underline{\mu}_{\mathbf{d}})R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})} \\
&= \frac{\Delta\mu_{\mathbf{d}} R}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}} - c_{\mathbf{d}} = \frac{\Delta\mu_{\mathbf{d}}}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}}(R - (r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})\beta_{\mathbf{d}}) < 0,
\end{aligned}$$

where the inequalities follow from the fact that (A.104) is not satisfied.

APPENDIX B

Appendix to Chapter 3

B.1 Proofs of Statements

B.1.1 Summary Of Notations

Summary of Model Primitives

- R_u and R_o: revenue per each arrival from the unobservable and observable channels, respectively.
- μ: combined arrival rate of customers if the agent works.
- c: cost of effort per unit of time.
- ν and ν^*: generic and full-effort process.
- r: principal and agent's discount rates.
- m:cost of monitoring per unit of time.
- U:customer's reservation utility.
- a, b: random shock ϵ_i^t on customer's utility in the unobservable channel is distributed uniformly on $[-a, b]$.
- p:fraction of customers from the unobservable channel before Section 3.4.
- $p_u(.)$ and $p_o(.)$: probability functions that customer goes to the unobservable and observable channels, respectively.

Contracts and Utilities

- I and ℓ: nstantaneous and flow payments, respectively.
- L: payment process $dL_t = I_t + \ell_t dt$.
- τ: stopping time.

- M: monitoring schedule.
- D: discount to the observable channel.
- γ: generic contract, $\gamma = (L, \tau)$ in Section 3.2, $\gamma = (L, \tau, M)$ in Section 3.3, $\gamma = (L, \tau, D)$ in Section 3.4.
- γ_1^*, γ_2^*, γ_3^*: optimal contracts in Section 3.2.
- γ_m^* andγ_m^{2*}: optimal contracts in Section 3.3.2.
- Γ_d^*: optimal contracts in Section 3.4.
- u and U: agent's and principal's utilities, respectively.
- U_m: principal's utility when monitoring is allowed.
- W_t: agent's promised utility.

Summary of Derived Parameters of the Model

- β: adjusted cost, defined in (3.4).
- $\bar{V}$: defined in and (3.3).
- $\bar{w}_1$ and $\bar{w}_2$: defined in (3.6) and (3.8), respectively.
- $\bar{w}_d$ and $\bar{V}_d$: defined in (3.23) and (3.22), respectively.
- w_L^*: the switching point defined in Section (3.3.2.1).

Value functions

- F: generic principal's value function.
- V: generic system's value function.
- F_1, F_2, F_3: optimal principal's value function in Section 3.2.
- F_{m1}, F_{m2}, F_{m3}: optimal principal's value function in Section 3.3.1.

B.1.2 Proofs in Section 3.2

B.1.2.1 Proof of Lemma 3.1

For $\hat{N}^u = N^u$, we define the agent's expected lifetime utility evaluated at time t conditional on past information by:

$$\begin{aligned} u_t(\gamma, \nu, \hat{N}^u) &:= \mathbb{E}\left[\int_0^\tau e^{-rs} dL_s + \int_0^\tau e^{-rs} \cdot c(1 - \mathbb{I}[\nu_s = \mu]) ds \middle| F_t\right] \\ &= \int_0^t e^{-rs}(dL_s + c(1 - \mathbb{I}[\nu_s = \mu]) ds) + e^{-rt} W_t \end{aligned} \tag{B.1}$$

which by construction is a martingale and, by the martingale representation theorem, there is a pair of processes H^u and H^o, such that $du_t = e^{-rt}(H_t^u(dN_t^u - \mu p dt) + H_t^o(dN_t^o - \mu(1-p)dt))$, where $dN_t^u - \mu p dt$ and $dN_t^o - \mu(1-p)dt$ are also martingales. Differentiating (B.1) with respect to t, we have $du_t = e^{-rt}(H_t^u(dN_t^u - \mu p dt) + H_t^o(dN_t^o - \mu(1-p)dt)) = e^{-rt}(dL_t + c(1 - \mathbb{I}[\nu_t = \mu])dt) - re^{-rt}W_t dt + e^{-rt} dW_t$, and, thus, we have (PK).

B.1.2.2 Proof of Lemma 3.2

The proof of this lemma is adapted from the proof of Lemma 3 in [DS06a]. If the agent diverts $R_u(dN_t^u - d\hat{N}_t^u)$ at time t when arrival from the unobservable channel comes, he gains immediate income R_u but loses H_t^u in his future expected payoff. Therefore, the payoff from reporting strategy $\hat{N}^u$ gives the agent the payoff of

$$W_{0-} + \mathbb{E}\left[e^{-rt} R_u(dN_t^u - d\hat{N}_t^u) - \int_0^\tau e^{-rt} H_t^u (dN_t^u - d\hat{N}_t^u)\right], \tag{B.2}$$

where W_{0-} denotes the agent's payoff under truth-telling. We see that if $H_t^u \geq R_u$ for all t, then (B.2) is maximized when the agent chooses $dN_t^u = d\hat{N}_t^u$ because the agent cannot over-report cash flows. If $H_t^u < R_u$ on a set of positive measure, then the agent is better off under-reporting the sales (and diverting cash) on this set instead of always telling the truth. Therefore, truth-telling is incentive compatible if and only if the constraint (IC-truthful) holds.

B.1.2.3 Proof of Lemma 3.3

Similar to Lemma 3.2, if the agent does not exert effort at time t, he saves the effort cost, c, but loses $\mu(pH_t^u + (1-p)H_t^o)$ on average because the agent loses the opportunity to get an arrival. Hence, the agent would exert effort if $\mu(pH_t^u + (1-p)H_t^o) \geq c$, which is equivalent to $pH_t^u + (1-p)H_t^o \geq \beta = c/\mu$. If $pH_t^u + (1-p)H_t^o < \beta$ on a set of positive measure, then the agent is better off shirking on this set instead of always exerting effort. Therefore, exerting effort is incentive compatible if and only if the constraint (IC-effort) holds.

B.1.2.4 Proof of Lemma 3.4

First, if $\mu p R_u \geq c$ ($pR_u \geq \beta$), the principal's value function follows the differential equation (3.7), which implies that the system's value function, $V(w) = F(w) + w$, follows

$$rV(w) = r\bar{V} + \mu p[V(w + R_u) - V(w)] + r(w - \bar{w}_1)V'(w). \tag{B.3}$$

We show the result according to the following steps. 1.Demonstrate the solution of (B.3) over $[\max\{0, \bar{w}_1 - R_u\}, \bar{w}_1]$ as a parametric function V_{b_1}, with parameter b_1 ,and show that the solution of (B.3) over $[0, \bar{w}_1]$ is unique and twice continuously differentiable for any b_1, also called V_{b_1}. 2. Show that V_{b_1} is convex and decreasing for $b_1 > 0$ and concave (in w) and increasing for $b_1 < 0$. 3. Show that $V_{b_1}(0)$ is increasing in b_1 for $b_1 < 0$, which implies that the boundary condition $V_{b_1}(0) = 0$ uniquely determines b_1, and, therefore the solution of the original differential equation.

Step 1. For $w \in [\max\{0, \bar{w}_1 - R_u\}, \bar{w}_1]$, the differential equation becomes an ordinary differential equation (ODE) and the solution is

$$V_{b_1}(w) = \bar{V} + b_1(\bar{w}_1 - w)^{\frac{r+\mu p}{r}}. \tag{B.4}$$

Using (B.4) as the boundary condition, we show that the differential equation (B.3) has a unique solution (also called $V_{b_1}(w)$, on $[0, \max\{0, \bar{w}_1 - R_u\})$, which is continuously differentiable. In fact, the differential equation (B.3) on $[0, \max\{0, \bar{w}_1 - R_u\})$ is equivalent to a sequence of initial value problems over the intervals $[\bar{w}_1 - (k+1)R_u, \bar{w}_1 - kR_u), k = 1, 2, ...$ This sequence of initial value problems satisfies the Cauchy-Lipschitz Theorem and, therefore, bears unique solutions. Further, computing $V'_{b_1}(\bar{w}_1 - R_u)$ from (B.4), and comparing it with (B.3), we see that V_{b_1} is continuously differentiable at $\bar{w} - R_u$, and therefore on $[0, \bar{w}_1)$.

Next, deriving the expressions for $V''_{b_1}(w)$ by following (B.4) and (B.3), confirms that V_b is twice continuously differentiable on $[0, \bar{w}_1]$.

Step 2. It is obvious that equation (B.4) is strictly concave in w if and only if $b_1 < 0$. Next, we prove by contradiction that if $b_1 < 0$, then $V_{b_1}(w)$ is strictly concave in w on $[0, \max\{0, \bar{w}_1 - R_u\}]$. If there exists, $\hat{w}$, such that $V''(\hat{w}) \geq 0$, then, by continuity, we can define $\tilde{w} = \sup\{w \in [0, \max\{0, \bar{w}_1 - R_u)\} : V''(\tilde{w}) = 0\}$. Hence, following (B.3), we have $V''_{b_1}(\tilde{w}) = \frac{\mu p(V'_{b_1}(\tilde{w}+R_u) - V'_{b_1}(\tilde{w}))}{r(\bar{w}_1 - \tilde{w})} = 0$, which contradicts $V'_{b_1}(\tilde{w} + R_u) = V'_{b_1}(\tilde{w}) + \int_{x=0}^{R_u} V''_{b_1}(\tilde{w} + x)dx < V'(\tilde{w})$, where the inequality follows from $V''_{b_1}(w) < 0$ for $w \in [\tilde{w}, \bar{w}_1)$. Hence, V_{b_1} is strictly concave in w on $[0, \bar{w}_1)$ if $b_1 < 0$. In case $b_1 = 0$, V_{b_1} is a constant in $[0, \bar{w}_1)$ following (B.4) and (B.3). Applying the same logic, we can prove that V_{b_1} is strictly convex in w on $[0, \bar{w}_1)$ if $b_1 > 0$. Hence, to satisfy the boundary condition $V_{b_1}(0) = 0$, we need to let $b_1 < 0$.

Step 3. Finally, we can show that V_{b_1} is monotonically increasing in b_1. First, for any $b_1 < b'_1$, it can be verified that $V_{b_1}(w) < V_{b'_1}(w)$, $V'_{b_1}(w) > V'_{b'_1}(w)$ for $w \in [\max\{\bar{w}_1 - R_u, 0\}, \bar{w}_1)$ following (B.3). Next, we can claim that $V'_{b_1} > V'_{b'_1}, \forall w \in [0, \bar{w}_1]$. Otherwise, there must exist $w' = \max\{w | V'_{b_1}(w) - V'_{b'_1}(w) = 0, w \in [0, \max\{0, \bar{w}_1 - R_u\})\}$ and $V'_{b_1}(w) > V'_{b'_1}(w) \forall w > w'$ because $V_{b_1} - V_{b'_1}$ is continuously differentiable. Equation (3.7) implies that $\mu p(V_{b_1}(w' + R_u) - V_{b'_1}(w' + R_u)) = (r + \mu p)(V_{b_1}(w) - V_{b'_1}(w))$. However, it contradicts with $0 > V_{b_1}(w'+R_u) - V_{b_1'}(w'+R_u) = V_{b_1}(w) - V_{b_1'}(w) + \int_0^{R_u} V_{b_1}(w'+x) - V_{b_1'}(w'+x)dx$. Hence, we must have $V_{b_1}' > V_{b_1'}'$, $\forall w \in [0, \bar{w}_1)$, and it implies that $V_{b_1} > V_{b_1'}$, $\forall w \in [0, \bar{w}_1)$. This implies that $V_{b_1}(0)$ is strictly increasing in b for $b < 0$. Because $V_0(0) = [\mu(pR_u + (1-p)R_o) - c]/r > 0$ and $\lim_{b_1 \to \infty} V_{b_1}(0) \to -\infty$, there must exist a unique b_1^* such that $V_{b_1^*}(0) = \underline{v}$. Further, $V_{b_1^*}$ is strictly concave and increasing in w on $[0, \bar{w}_1]$. Therefore, there exists b_1^*, such that $V_{b_1^*}(0) = 0$. This completes the proof.

The proof of the cases $\mu pR_u < c < \mu R_u$ ($pR_u < \beta < R_u$) and $c > \mu R_u$ ($R_u < \beta$) are similar to the case when $pR_u \geq \beta$, and, hence are omitted here.

B.1.2.5 Proof of Proposition 3.1

If $\mu pR_u \geq c$ ($pR_u \geq \beta$), we need to prove that $U(\gamma_1^*(w), \nu^*, N^u) = F_1(w)$. Following (DW1), we have

$$\begin{aligned} e^{-r\tau}F_1(W_\tau) &= e^{-r0}F_1(W_{0-}) + \int_0^\tau [e^{-rt}dF_1(W_{t-}) - re^{-rt}F(W_{t-})dt] \\ &= F_1(W_{0-}) + \int_0^\tau e^{-rt} - (R_u dN_t^u + R_o dN_t^o - cdt - dL_{1t}^*) + \int_0^\tau e^{-rt}\mathcal{A}_t, \end{aligned} \quad \text{(B.5)}$$

where

$$\begin{aligned} \mathcal{A}_t &= dF_1(W_{t-}) - rF_1(W_{t-})dt + (R_u dN_t^u + R_o dN_t^o - cdt - dL_{1t}^*) \\ &= F_1'(W_{t-})(rW_{t-} - \mu pR_u)dt - rF_1(W_{t-})dt + F_1(W_t) - F_1(W_{t-}) \\ &+ (R_u dN_t^u + R_o dN_t^o - cdt - dL_{1t}^*) \\ &= F'(W_{t-})(rW_{t-} - \mu pR_u)dt - rF(W_{t-})dt + F(W_{t-} + R_u - (W_{t-} + R_u - \bar{w}_1)^+)dN_t^u \quad \text{(B.6)} \\ & F(W_{t-})dN_t^u + (R_u dN_t^u + R_o dN_t^o - cdt - (W_{t-} + R_u - \bar{w}_1)^+ dN_t^u) \\ &= F'(W_{t-})(rW_{t-} - \mu pR_u)dt - rF(W_{t-})dt + (F(W_{t-} + R_u) - F(W_{t-}))dN_t^u \\ &+ (R_u dN_t^u + R_o dN_t^o - cdt) = \mathcal{B}_t, \end{aligned}$$

where the fourth equality follows from $F(W_{t-} + R_u - (W_{t-} + R_u - \bar{w}_1)^+) - (W_{t-} + R_u - \bar{w}_1)^+ = F(W_{t-} + R_u)$, and the last equality follows from (3.7) and $\mathcal{B}_t := (R_u + F(W_{t-} + H_t^u) - F(W_{t-}))(dN_t^u - \mu pdt) + (R_o + F(W_{t-} + H_t^o) - F(W_{t-}))(dN_t^o - \mu(1-p)dt)$. Taking the expectation on both sides of (B.5), and following (B.6), we immediately obtain

$$\begin{aligned} F(u(\gamma_1^*(w), \nu^*, N^u)) = F(w) &= \mathbb{E}\left[e^{-r\tau}F(W_\tau) + \int_0^\tau e^{-rt}(R_u dN_t^u + R_o dN_t^o - cdt - dL_t)\right] \\ &= U(\gamma_1^*(w), \nu^*, N^u) \end{aligned}$$

where we use the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t$ is a martingale and that $F(W_\tau) = F(0) = \underline{v}$. For the cases when $\mu pR_u < c < \mu R_u$ and $\mu R_u \leq c$, the proof can be directly adapted from the case when $\mu pR_u \geq c$ and hence is omitted here. To conclude, we have $U(\gamma_1^*(w), \nu^*, N^u) = F_1(w)$ if $\mu pR_u \geq c$, $U(\gamma_2^*(w), \nu^*, N^u) = F_2(w)$ if $\mu pR_u < c < \mu R_u$, and $U(\gamma_3^*(w), \nu^*, N^u) = F_3(w)$ if $\mu R_u \leq c$.

B.1.2.6 Proof of Proposition 3.2

To prove Proposition 3.2, we first present a verification result in Lemma B.1.

Lemma B.1 *Let $F(w) : [0, \infty) \to \mathbf{R}$ be differentiable, concave, upper-bounded function, with $F'(w) \geq -1$, and $F(0) \geq 0$. Consider any contract $\gamma \in \Gamma^{IC}$, such that it (i) yields the agent's expected utility $u(\gamma, \nu^*, N^u) = W_{0-} = w$, followed by the promised utility process $\{W_t\}_{t \geq 0}$ (according to* (PK)*), and (ii) satisfies* (IC-truthful) *and* (IC-effort)*. Define a stochastic process $\{\Phi_t\}_{t \geq 0}$ as*

$$\begin{aligned}\Phi_t = \mu p R_u + \mu(1-p)R_o + F'(W_{t-})[rW_{t-} - \mu p H_t^u - \mu(1-p)H_t^o] - rF(W_{t-}) \\ + (F(W_{t-} + H_t^u) - F(W_{t-}))\mu p + (F(W_{t-} + H_t^o) - F(W_{t-}))\mu(1-p) - c. \qquad \text{(B.7)}\end{aligned}$$

If the process $\{\Phi_t\}_{t \geq 0}$ is non-positive almost surely, then we have $F(w) \geq U(\gamma, \nu^*, N^u)$.

Proof of Lemma B.1: Following Ito's Formula for a jump process (see, for example, Theorem 17.5 in [Bas11]), and considering (PK), we obtain

$$\begin{aligned}e^{-r\tau}F(W_\tau) &= e^{-r0}F(W_{0-}) + \int_0^\tau [e^{-rt}dF(W_{t-}) - re^{-rt}F(W_{t-})dt] \\ &= F(W_{0-}) + \int_0^\tau e^{-rt} - (R_u dN_t^u + R_o dN_t^o - cdt - dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t, \qquad \text{(B.8)}\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_t &= dF(W_{t-}) - rF(W_{t-})dt + (R_u dN_t^u + R_o dN_t^o - cdt - dL_t) \\ &= F'(W_{t-})(rW_{t-} - \mu p H_t^u - \mu(1-p)H_t^o - \ell_t)dt - rF(W_{t-})dt + F(W_t) - F(W_{t-}) \\ &+ (R_u dN_t^u + R_o dN_t^o - cdt - dL_t) \\ &= F'(W_{t-})(rW_{t-} - \mu p H_t^u - \mu(1-p)H_t^o - \ell_t)dt - rF(W_{t-})dt + F(W_{t-} - I_t(1 - dN_t^u - dN_t^o)) \\ &- F(W_{t-}) + (F(W_{t-} + H_t^u - I_t dN_t^u) - F(W_{t-}))dN_t^u + (F(W_{t-} + H_t^o - I_t dN_t^o) - F(W_{t-}))dN_t^o \\ &+ (R_u dN_t^u + R_o dN_t^o - cdt - dL_t). \qquad \text{(B.9)}\end{aligned}$$

Then, the concavity of F and $F' \geq -1$ implies that $-\ell_t F'(W_{t-}) \leq \ell_t, F(W_{t-} - I_t(1 - dN_t^u - dN_t^o)) - F(W_{t-}) \leq I_t(1 - dN_t^u - dN_t^o)$,

$$\begin{aligned}F(W_{t-} + H_t^u - I_t dN_t^u) - F(W_{t-}) = F(W_{t-} + H_t^u - I_t dN_t^u) - F(W_{t-} + H_t^u) \\ + F(W_{t-} + H_t^u) - F(W_{t-}) \leq I_t dN_t^u + F(W_{t-} + H_t^u) - F(W_{t-}),\end{aligned}$$

and $F(W_{t-} + H_t^o - I_t dN_t^o) - F(W_{t-}) = I_t dN_t^o + F(W_{t-} + H_t^o) - F(W_{t-})$. Hence, we have

$$\begin{aligned}\mathcal{A}_t \le\ & F'(W_{t-})(rW_{t-} - \mu p H_t^u - \mu(1-p)H_t^o)dt + \ell_t dt - rF(W_{t-})dt + I_t(1 - dN_t^u - dN_t^o) + I_t dN_t^u \\ & + I_t dN_t^o + (F(W_{t-} + H_t^u) - F(W_{t-}))dN_t^u + (F(W_{t-} + H_t^o) - F(W_{t-}))dN_t^o \\ & + (R_u dN_t^u + R_o dN_t^o - cdt - dL_t) = \mathcal{B}_t + \Phi_t dt,\end{aligned} \tag{B.10}$$

where the equality follows from $dL_t = I_t + \ell_t dt$ and $\mathcal{B}_t := (R_u + F(W_{t-} + H_t^u) - F(W_{t-}))(dN_t^u - \mu p dt) + (R_o + F(W_{t-} + H_t^o) - F(W_{t-}))(dN_t^o - \mu(1-p)dt)$. Therefore, if $\Phi_t \le 0$, we must have

$$\mathcal{A}_t \le \mathcal{B}_t \tag{B.11}$$

almost surely. Taking the expectation on both sides of (B.8), we immediately obtain

$$\begin{aligned}F(u(\gamma, \nu^*, N^u)) = F(W_{0-}) &\ge \mathbb{E}\left[e^{-r\tau}F(W_\tau) + \int_0^\tau e^{-rt}(R_u dN_t^u + R_o dN_t^o - cdt - dL_t)\right] \\ &\ge U(\gamma, \nu^*, N^u),\end{aligned}$$

where we use the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t$ is a martingale and $F(W_\tau) = F(0) \ge 0$. Therefore, if the process $\{\Phi_t\}_{t\ge 0}$ is non-positive almost surely, then we have $F(u(\gamma, \nu^*, N^u)) \ge U(\gamma, \nu^*, N^u)$.

With the extended definition of F_i for $w \ge \bar{w}_i$, it is clear that $F_i'(w) = -1$ for $w \ge \bar{w}_i$, $i = 1, 2, 3$. Following Lemma B.1, to prove that the principal's value function is the upper bound of the principal's utility under any other contract, we only need to check if Φ_t (defined in equation (B.7)) is non-positive at $F = F_1$ (F_2, F_3) under the condition $\mu p R_u \ge c$ ($\mu p R_u < c < \mu R_u$, $\mu R_u \le c$). Here, F_1 (F_2, F_3) is defined in equation (3.7) ((3.9), (3.10)). To prove this, first, we consider the following optimization problem with a generic concave function $F(w)$ (with dual variables, $x$and y denoted after the constraints),

$$\begin{aligned}\max_{H_t^u, H_t^o}\ & \mu p(F(W_{t-} + H_t^u) - F(W_{t-})) + \mu(1-p)(F(W_{t-} + H_t^o) - F(W_{t-})) \\ & - (\mu p H_t^u + \mu(1-p)H_t^o)F'(W_{t-}), \\ s.t.\ & H_t^u \ge R_u; x, pH_t^u + (1-p)H_t^o \ge \beta; y.\end{aligned} \tag{B.12}$$

If $\mu p R_u \ge c$, then in the following, we verify by Karush-Kuhn-Tucker (KKT) conditions that the optimal solution is

$$H_t^{u*} = R_u, H_t^{o*} = 0. \tag{B.13}$$

We plug in the following values for the dual variables x and y: $x = p(F'(W_{t-}) - F'(W_{t-} + R_u)), y = 0$. Then, we can easily verify that $-pF'(W_{t-}) + pF'(W_{t-} + H_t^{u*}) + x = 0$, $-(1-p)F'(W_{t-}) + (1-p)F'(W_{t-} + H_t^{o*}) = 0$, and $x \ge 0$, where the inequality follows from the concavity of F. Hence, if $\mu p R_u \ge c$, then the optimal solution of the problem (B.12) is (B.13). Similarly, we can prove that, if

$\mu pR_u < c < \mu R_u$, then the optimal solution of (B.12) is

$$H_t^{u*} = R_u, H_t^{o*} = (\beta - pR_u)/(1-p) = \beta_1, \tag{B.14}$$

and, if $\mu R_u \leq c$, then the optimal solution of (B.12) is

$$H_t^{u*} = \beta, H_t^{o*} = \beta. \tag{B.15}$$

With these optimal solutions and equations, (3.7), (3.9) and (3.10), we could verify that $\Phi_t \leq 0$, where Φ_t is defined in (B.7). If $\mu pR_u \geq c$, then, by plugging $F = F_1$ into (B.7), we get

$$\begin{aligned} \Phi_t \leq \mu pR_u + \mu(1-p)R_o + F_1'(W_{t-})[rW_{t-} - \mu pR_u] - rF_1(W_{t-}) \\ + (F_1(W_{t-} + R_u) - F_1(W_{t-}))\mu p - c = 0, \end{aligned} \tag{B.16}$$

where the inequality follows from (B.13), and the last equality follows from equation (3.7). If $\mu pR_u < c < \mu R_u$, then, by plugging $F = F_2$ into (B.7), we obtain

$$\begin{aligned} \Phi_t \leq \mu pR_u + \mu(1-p)R_o + F_2'(W_{t-})[rW_{t-} - \mu pR_u] - rF(W_{t-}) + (F(W_{t-} + R_u) - F(W_{t-}))\mu p \\ + (F(W_{t-} + \beta_1) - F(W_{t-}))\mu(1-p) - c = 0, \end{aligned}$$

where the inequality follows from (B.14), and the last equality follows from equation (3.9). If $\mu R_u \leq c$, then plugging $F = F_3$ into (B.7), we obtain

$$\Phi_t \leq \mu pR_u + \mu(1-p)R_o + F'(W_{t-})[rW_{t-} - \mu\beta] - rF(W_{t-}) + (F(W_{t-} + \beta) - F(W_{t-}))\mu - c = 0,$$

where the inequality follows from (B.15) and the last equality follows from equation (3.10). Hence, by applying Lemma B.1, we have $F_1(u(\gamma, \nu^*, N^u)) \geq U(\gamma, \nu^*, N^u)$ if $\mu pR_u \geq c$, $F_2(u(\gamma, \nu^*, N^u)) \geq U(\gamma, \nu^*, N^u)$ if $\mu pR_u < c < \mu R_u$ and $F_2(u(\gamma, \nu^*, N^u)) \geq U(\gamma, \nu^*, N^u)$ if $\mu R_u \leq c$.

B.1.2.7 Proof of Theorem 3.2

As shown by the revelation principle, it is without loss of generality that we can focus on the contract, where the agent truthfully reports the arrivals. Hence, in the following, we prove that the contract, that we derived in Section 3.2.2, is still optimal, even where we include contracts that allow the agent to shirk. First, we shall update the promise keeping, and incentive compatibility constraints, which allow the principal to let the agent not exert effort. At any time t, if the principal induces the agent to work and truthfully report the arrivals, then W_t follows (PK) with $\nu_t = \mu$ and incentive constraints (IC-truthful) and (IC-effort) should be satisfied. If the principal only induces the agent to truthfully report and lets the agent not to work, then the principal does not need to pay cdt, so W_t follows

$$dW_t = rW_{t-}dt - dL_t, \tag{B.17}$$

and the incentive constraint only includes constraint (IC-truthful), since there are no arrivals when the agent does not work, constraint (IC-truthful) is directly satisfied. Next, we present the following lemma which is helpful for proving the optimality of the contract in the contract space Γ.

Lemma B.2 *Let $F(w) : [0, \infty) \to \mathbf{R}$ be differentiable, concave, upper-bounded function, with $F'(w) \geq -1$, and $F(0) \geq 0$. Consider any contract $\gamma \in \Gamma$, such that it yields the agent's expected utility $u(\gamma, \nu, N^u) = W_{0-} = w$. Define a stochastic process $\{\Phi_t\}_{t\geq 0}$, such that: (1) if the principal induces the agent to exert effort and report arrivals truthfully (i.e., constraints* (IC-truthful) *and* (IC-effort) *are satisfied) where W_t follows* (PK)*, then Φ_t follows equation* (B.7)*; (2) if the principal only induces agent's truth-telling (i.e., constraint* (IC-truthful) *is satisfied) where W_t follows* (B.17)*, then*

$$\Phi_t = F'(W_{t-})rW_{t-} - rF(W_{t-}). \tag{B.18}$$

Furthermore, if the process $\{\Phi_t\}_{t\geq 0}$ is non-positive almost surely, then $F(w) \geq U(\gamma, \nu, N^u)$.

Proof of Lemma B.2: By applying Ito's Formula for a jump process (see, for example, Theorem 17.5 of [Bas11]) and considering (PK) and equation (B.17), we have

$$\begin{aligned}
e^{-r\tau}F(W_\tau) &= e^{-r0}F(W_{0-}) + \int_0^\tau [e^{-rt}dF(W_{t-}) - re^{-rt}F(W_{t-})dt] \\
&= F(W_{0-}) + \int_0^\tau e^{-rt} - (R_u dN_t^u + R_o dN_t^o - cdt)\mathbb{I}[\nu_t = \mu] + dL_t \\
&+ \int_0^\tau e^{-rt}\mathcal{A}_t\mathbb{I}[\nu_t = \mu] + \mathcal{A}_{tn}\mathbb{I}[\nu_t = 0],
\end{aligned} \tag{B.19}$$

where $\mathcal{A}_t$ follows equation (B.9) and

$$\begin{aligned}
\mathcal{A}_{tn} &= dF(W_{t-}) - rF(W_{t-})dt - dL_t \\
&= F'(W_{t-})(rW_{t-} - \ell_t)dt - rF(W_{t-})dt + F(W_t) - F(W_{t-}) - dL_t \\
&= F'(W_{t-})(rW_{t-} - \ell_t)dt - rF(W_{t-})dt + F(W_{t-} - I_t) - F(W_{t-}) - dL_t \\
&\leq [F'(W_{t-})rW_{t-} - rF(W_{t-})]dt = \Phi_t dt,
\end{aligned} \tag{B.20}$$

where the inequality follows from the concavity of F and $F' \geq -1$, and Φ_t follows equation (B.18). Therefore, if $\Phi_t \leq 0$, then $\mathcal{A}_{tn} \leq 0$. Taking the expectation on both sides of (B.19), we immediately obtain

$$\begin{aligned}
F(w) = F(u(\gamma, \nu, N^u)) &\geq \mathbb{E}\left[e^{-r\tau}F(W_\tau) + \int_0^\tau e^{-rt}([R_u dN_t^u + R_o dN_t^o - cdt]\mathbb{I}[\nu_t = \mu] - dL_t)\right] \\
&\geq U(\gamma, \nu, N^u),
\end{aligned}$$

where the inequality follows from equation (B.11), $\mathcal{A}_{tn} \leq 0$, $\int_0^\tau e^{-rt}\mathcal{B}_t$ is a martingale, and $F(W_\tau) = F(0) \geq 0$.

Next, we will apply Lemma B.2 to prove the optimality of the contracts in the contract space Γ. We, first, prove that the principal's value function F_i is an upper bound of the principal's utility under any other contract in Γ.

Case 1. Under the condition $\mu p R_u \geq c$, the agent is always willing to exert effort since (IC-truthful) directly implies (IC-effort). Furthermore, we have verified in the proof of Proposition 3.2 that equation (B.7) is non-positive. Hence, we have for any $\gamma \in \Gamma$, $U(\gamma, \nu, N^u) \leq F_1(w)$ if $u(\gamma, \nu, N^u) = w$ and $\mu p R_u \geq c$.

Case 2. Under the condition $\mu p R_u < c < \mu R_u$, first, if the principal induces the agent to exert effort and report arrivals truthfully (constraints (IC-truthful) and (IC-effort) should be satisfied), then Φ_t follows equation (B.7) (at $F = F_2$), and we have verified in the proof of Proposition 3.2 that Φ_t is non-positive. Second, if the principal does not induce the agent to exert effort, then Φ_t follows equation (B.18) (at $F = F_2$), and we get

$$\begin{aligned} \Phi_t &\leq F_2'(W_{t-}) r W_{t-} - r F_2(W_{t-}) = V_2'(W_{t-}) r W_{t-} - r V_2(W_{t-}) \\ &= r W_{t-} \left[V_2'(W_{t-}) - (V_2(W_{t-}) - V_2(0))/W_{t-} \right] \leq 0, \end{aligned} \tag{B.21}$$

where the first equality follows from $V_2(w) = F_2(w) + w$ and the last inequality follows from the concavity of V_2 and $V_2(0) = 0$. Hence, we have verified that, for any $\gamma \in \Gamma$, $U(\gamma, \nu, N^u) \leq F_2(w)$ if $u(\gamma, \nu, N^u) = w$, and $\mu p R_u < c < \mu R_u$.

Case 3. Under the condition $\mu R_u \leq c$, first, if the principal induces the agent to exert effort and report arrivals truthfully (constraints (IC-truthful) and (IC-effort) should be satisfied), then Φ_t follows equation (B.7) (with $F = F_3$) and we have verified in the proof of Proposition 3.2 that Φ_t is non-positive. Second, if the principal does not induce the agent to exert effort, then Φ_t follows equation (B.18) (with $F = F_3$), we can verify that Φ_t is also non-positive as we have shown in the inequality (B.21). Hence, we have verified that, for any $\gamma \in \Gamma$, $U(\gamma, \nu, N^u) \leq F_3(w)$ if $u(\gamma, \nu, N^u) = w$ and $\mu R_u \leq c$.

Finally, similar to inequality (3.12), we have for any contract $\gamma \in \Gamma$, $i = 1$ (2, 3),

$$U(\gamma, \nu, N^u) \leq F_i(u(\gamma, \nu, N^u)) = F_i(w) = U(\gamma_i^*(w), \nu^*, N^u) = F_i(w_i^*) \leq U(\gamma_i^*(w_i^*), \nu^*, N^u),$$

under the condition $\mu p R_u \geq c$ ($\mu p R_u < c < \mu R_u$, $\mu R_u \leq c$), where the last inequality follows from the fact that w_1^* (w_2^*, w_3^*) is the maximizer of F_1 (F_2, F_3). Therefore, γ_1^* in definition 3.1 is the optimal contract in the contract space Γ under the condition $\mu p R_u \geq c$; γ_2^* in definition 3.2 is the optimal contract in the contract space Γ under the condition $\mu p R_u < c < \mu R_u$; γ_3^* in definition 3.3 is the optimal contract in the contract space Γ under the condition $\mu R_u \leq c$.

B.1.2.8 Proof of Lemma 3.5

First, we consider the case when $\mu p R_u \geq c$. We define $f_2(w) := \dfrac{\partial F_1(w)}{\partial R_o}$. Following equation (3.7), we have $f_2(0) = 0$, $f_2(w) = \mu(1-p)/r$ for $w \geq \bar{w}_1$, and $r f_2(w) = \mu(1-p) + \mu[f_2(w + R_u) - f_2(w)] + r(w - \bar{w}_1) f_2'(w)$. Hence, for $w \in [\bar{w}_1 - R_u, \bar{w}_1]$, we have $f_2(w) = \mu(1-p)/r + b(\bar{w}_1 - w)^{(r+\mu)/r}$.

Following Step 2 of the proof of the Case 1 in equation (3.4), we have if $b \geq 0$**,** then $f_2(w)$ is decreasing and convex, which leads to a contradiction with $f_2(0) = 0$. Hence, we have $b < 0$ and $f_2(w)$ is increasing and concave in w. Therefore, $f_2(w) > 0, \forall w$ and $f_2'(w) > 0$ for $w \in [0, \bar{w}_1]$. Next, $f_2(w) > 0$ implies that $F(w^*)$ is increasing in R_o. Furthermore, by the Implicit Function Theorem, $F'(w^*) = 0$ implies that $\frac{\partial w^*}{\partial R_o} = -\frac{f_2'(w^*)}{F_1''(w^*)} > 0$, where the inequality follows from the concavity of F_1, and $f_2'(w) > 0$. The proof for the case when $\mu p R_u < c < \mu R_u$ and the case when $\mu R_u \leq c$ is similar to the proof for the case when $\mu p R_u \geq c$, and hence, is omitted here. Therefore, we have shown that the agent's total utility w^* and the principal's total utility are strictly increasing in R_o.

B.1.3 Proofs For Section 3.3

B.1.3.1 Proof of Proposition 3.3

In this setting, when the principal conducts full monitoring, the agent will exert effort and report arrivals truthfully. By applying Ito's Formula, (PK) and equation (B.17) (adapted from the proof of Lemma B.2), we have the following lemma. It is worth noting that if the principal does not induce the agent's effort, there are no arrivals, hence, the agent cannot misreport anything and the principal will not monitor.

Lemma B.3 *Suppose $F(w) : [0, \infty) \to \mathbf{R}$ is differentiable, concave, upper-bounded function, with $F'(w) \geq -1$, and $F(0) \geq 0$. Consider any contract $\gamma \in \Gamma_m$, which yields the agent's expected utility $u(\gamma, \nu, N^u) = W_{0-} = w$. Define a stochastic process $\{\Phi_t\}_{t \geq 0}$, such that: (1) if the principal induces the agent to report arrivals truthfully and exert effort (i.e., incentive constraint* (3.14) *is satisfied) where W_t follows* (PK)*, then*

$$\begin{aligned} \Phi_t = \mu p R_u + \mu(1-p) R_o + F'(W_{t-})[rW_{t-} - \mu p H_t^u - \mu(1-p) H_t^o] - rF(W_{t-}) \\ + (F(W_{t-} + H_t^u) - F(W_{t-}))\mu p + (F(W_{t-} + H_t^o) - F(W_{t-}))\mu(1-p) - c - m_t; \end{aligned} \tag{B.22}$$

(2) if the principal does not induce agent to exert effort where W_t follows (B.17)*, then Φ_t follows equation* (B.18)*. Furthermore, if the process $\{\Phi_t\}_{t \geq 0}$ is non-positive almost surely, then $F(w) \geq U(\gamma, \nu, N^u)$.*

By applying Lemma B.3, we first show that the principal's value function ($F_{mi}, i = 1, 2, 3$) is an upper bound of the principal's utility under any other contract in Γ_m. Next, we conduct the proof for the case when $\mu p R_u \geq c$ by considering two cases.

Case 1. If condition (3.15) holds, then, following definition 3.4, the principal's value function is F_{m1} which is the solution of (3.7) with the boundary condition $F_{m1}(0) = \frac{\mu[pR_u + (1-p)R_o] - c - m}{r} \geq 0$. Following case 1 of the proof of Lemma 3.4, we can verify that F_{m1}, is concave in w and $F_{m1}'(w) \geq -1$. If the principal induces the agent to exert effort and report arrivals truthfully with $m_t = 0$ (does not conduct monitoring), then, by plugging $F = F_{m1}$ into equation (B.22), we get

$$\begin{aligned} \Phi_t \leq \mu p R_u + \mu(1-p) R_o + F_{m1}'(W_{t-})[rW_{t-} - \mu p R_u] - rF_{m1}(W_{t-}) \\ + [F_{m1}(W_{t-} + R_u) - F_{m1}(W_{t-})]\mu p - c = 0, \end{aligned} \tag{B.23}$$

where the inequality follows from equation (B.13), and the equality follows from equation (3.7). Before we proceed, we present an inequality which will be useful in proving $\Phi_t \leq 0$: if F is concave in w and $W_t + H_t^u \geq 0$, we have

$$F(W_t + H_t^u) - H_t^u F'(W_t) \leq F(W_t). \tag{B.24}$$

Next, if the principal induces the agent to exert effort with $m_t = m$ (the principal conducts monitoring), then, by plugging $F = F_{m1}$ into (B.22), we get

$$\begin{aligned}
\Phi_t &= \mu p R_u + \mu(1-p)R_o - c - m + F_{m1}'(W_{t-})rW_{t-} - rF_{m1}(W_{t-}) - \mu F_{m1}(W_{t-}) \\
&\quad + \mu p[F_{m1}(W_{t-} + H_t^u) - H_t^u F_{m1}'(W_{t-})] + \mu(1-p)[F_{m1}(W_{t-} + H_t^o) - H_t^o F_{m1}'(W_{t-})] \\
&\leq \mu p R_u + \mu(1-p)R_o - c - m + F_{m1}'(W_{t-})rW_{t-} - rF_{m1}(W_{t-}) \\
&= V_{m1}'(W_{t-})rW_{t-} - rV_{m1}(W_{t-}) \leq rW_t\left[V_{m1}'(W_{t-}) - (V_{m1}(W_{t-}) - V_{m1}(0))/W_{t-}\right] \leq 0,
\end{aligned}$$

where the first inequality follows from the inequality (B.24), the second equality follows from $V_{m1} := F_{m1}(w) + w$, and the last inequality follows from the concavity of V_{m1}. Finally, if the principal does not induce the agent to exert effort, then, by plugging $F = F_{m1}$ into (B.18), (similar to (B.21)), we can obtain $\Phi_t \leq 0$. Hence, following Lemma B.3, we have $F_{m1}(w) \geq U(\gamma, \nu, N^u)$. Next, similar to Proposition 3.1, we can verify that $U(\gamma_{m1}^*(w), \nu^*, N^u) = F_{m1}(w)$. Hence, we have for any $\gamma \in \Gamma_m$,

$$U(\gamma_{m1}^*(w_{m1}^*), \nu^*, N^u) = F_{m1}(w_{m1}^*) \geq U(\gamma_{m1}^*(w), \nu^*, N^u) = F_{m1}(w) \geq U(\gamma, \nu, N^u),$$

where the first inequality follows from the fact that w_{m1}^* is the maximizer of F_{m1}. To conclude, we have shown that if condition (3.15) and $c \leq \mu p R_u$ hold, then $\gamma_{m1}^*(w_{m1}^*)$ in definition 3.4 is the optimal contract in Γ_m.

Case 2. If condition (3.15) does not hold, then $(\mu[pR_u + (1-p)R_o] - m - c)/r \leq 0 \leq F(0) = V(0)$ and we will show that the optimal contract is still γ_1^* (following definition 3.1). Hence, the principal's value function is F_1 defined in Lemma 3.4. If the principal induces the agent to report arrivals truthfully and exert effort with $m_t = 0$ (the principal does not conduct monitoring), then $\Phi_t \leq 0$ since (B.16) is still valid. Next, if the principal induces the agent to report arrivals truthfully and exert effort with $m_t = m$ (the principal conducts monitoring), then, by plugging $F = F_1$ into (B.22), we get

$$\begin{aligned}
\Phi_t &\leq \mu p R_u + \mu(1-p)R_o + F_1'(W_{t-})rW_{t-} - rF_1(W_{t-}) - c - m \\
&\leq rV_1(0) + V_1'(W_{t-})rW_{t-} - rV_1(W_{t-}) = rW_{t-}\left[V_1'(W_{t-}) - (V_1(W_{t-}) - V_1(0))/W_{t-}\right] \leq 0,
\end{aligned}$$

where the first inequality follows from the inequality (B.24), the second inequality follows from $F_1(0) = V_1(0) \geq 0$, and the last inequality follows from the concavity of V_1. Finally, if the principal does not induce the agent to exert effort, then, by plugging $F = F_1$ into (B.18), (similar to (B.21)), we can obtain $\Phi_t \leq 0$. Next, following Proposition 3.1, we have $U(\gamma_1^*(w), \nu^*, N^u) = F_1(w)$. Therefore, we have $U(\gamma_1^*(w_1^*), \nu^*, N^u) = F_1(w_1^*) \geq U(\gamma_1^*(w), \nu^*, N^u) = F_1(w) \geq U(\gamma, \nu, N^u)$, where the first inequality follows from the fact that w_1^* is the maximizer of F_1. To conclude, we have shown that if condition (3.15) does not hold but $c \leq \mu p R_u$ holds, then $\gamma_1^*(w_1^*)$ (in definition3.1) is the optimal contract in the contract space Γ_m. For the cases $\mu p R_u < c < \mu R_u$ and $\mu R_u \leq c$, the proofs are very similar to the proof for the case $\mu p R_u \geq c$, and, hence, are omitted here.

B.1.3.2 Model with Non-zero Outside Option

Baseline Model with Outside Option Assume that the agent has an outside option which delivers him a future utility $\underline{w} > 0$. Then, the contract design problem will include an additional constraint imposed on the agent's promised utility $W_t \geq \underline{w}$, for any $t < \tau$. Further, if the agent's promised utility reaches the level $\underline{w}$, the principal can freely terminate the agent without paying anything since the agent is willing to leave the contract and earns his future utility outside. Therefore, the principal has two options: 1. Not to hire the agent, and both principal and agent have the total utility equal to zero. 2. Hire the agent and set his starting promised utility as $W_{0-} \geq \underline{w}$. For the second case, we can prove that the structure of the optimal contract before termination will not change (as defined in γ_1^*, γ_2^* and γ_3^*). The proof relies on two changes made to the optimal contract and the corresponding principal's value function: (1) The contract is terminated when the agent's promised utility reaches $\underline{w}$. (2) The boundary condition of the system's value function has to be changed from $V(0) = 0$ to $V(\underline{w}) = \underline{w}$. (The societal value at the time of termination equals to $\underline{w}$ where the principal's future utility is 0 and the agent earns $\underline{w}$ afterwards.) With the new definition of the societal value function V, the parallel results of Lemma 3.4, Proposition 3.1, and Proposition 3.2 are easy to establish and, hence, are, omitted here.

Monitoring with Outside Option Again, we assume that the agent has an outside option which delivers him a future utility $\underline{w}$. Then, the contract design problem will include an additional constraint imposed on the agent's promised utility $W_t \geq \underline{w}$ for any $t < \tau$. We have established the optimal contract for the model where we add the non-zero outside option to the baseline model. Now, in the model with monitoring, the principal has another action which is to conduct monitoring.

We can show that the principal only monitors the agent when his promised utility reaches $\underline{w}$, and the structure of the optimal policy does not change when $w > \underline{w}$. We have shown in the main body of the paper that, if $\underline{w} = 0$, then the principal conducts monitoring only when the agent's promised utility reaches 0. Furthermore, the principal will never stop monitoring once she starts and the agent's promised utility is kept at 0. However, when $\underline{w} > 0$ and when the principal conducts monitoring, the agent's promised utility follows $dW_{t-} = rW_{t-}dt > 0$. Hence, the agent's promised utility will increase, which is different from the case when $\underline{w} = 0$. When the principal conducts monitoring, the agent's promised utility will be greater than $\underline{w}$ in the next second, then the principal stops monitoring and the agent's promised utility starts to decrease again. In fact, the trajectory of the agent's promised utility is such that the agent's promised utility is trembling near the threshold $\underline{w}$ until an arrival occurs, which jumps up his promised utility. Furthermore, the agent is never terminated in this contract.

To prove the optimality of this contract, we need to, first, redefine the boundary condition of the system's value function. At $W_{t-} = \underline{w}$, the agent's promised utility follows $dW_t = rW_{t-}dt$. Hence,

$V(W_{t-}) = [\mu(pR_u + (1-p)R_o) - c - m]\delta + e^{-r\delta}V(W_{t-} + rW_{t-}\delta)$ where the first term of the right-hand side of the equation represents the societal value in the current period, and the second term represents the discounted future value. By letting $\delta \to 0$, we have $[\mu(pR_u + (1-p)R_o) - c - m] + r\underline{w}V'(\underline{w}) - rV(\underline{w}) = 0$, which is equivalent to

$$V'(\underline{w}) = \{V(\underline{w}) - [\mu(pR_u + (1-p)R_o) - c - m]/r\}/\underline{w}. \tag{B.25}$$

For $w \geq \underline{w}$, the system's value equation follows the original differential equations (equation (3.7) under the condition $\mu pR_u \geq c$, equation (3.9) under the condition $\mu pR_u < c < \mu R_u$, and equation (3.10) under the condition $\mu R_u \leq c$). As long as $m > 0$, similar to Lemma 3.4, we can prove that there exists (1) a unique concave function that satisfies equation (3.7) and the boundary condition (B.25) under the condition $\mu pR_u \geq c$; (2) a unique concave function that satisfies equation (3.9) and the boundary condition (B.25) under the condition $\mu pR_u < c < \mu R_u$; and (3) a unique concave function that satisfies equation (3.10) and the boundary condition (B.25) under the condition $\mu R_u \leq c$.

Proof: Under the condition $\mu pR_u \geq c$, for any $V_0 \in [[\mu(pR_u + (1-p)R_o) - c - m]/r, \bar{V}]$, following the proof of Lemma 3.4, we can show that there exists a unique concave function that satisfies equation (3.7) and the boundary condition $V(\underline{w}) = V_0$. First, if $V_0 = \bar{V}$, then we have $V'(\underline{w}_+) = 0 < \{V_0 - [\mu(pR_u + (1-p)R_o) - c - m]/r\}/\underline{w}$. Second, if $V_0 = [\mu(pR_u + (1-p)R_o) - c - m]/r$, then $V'(\underline{w}_+) > \{V_0 - [\mu(pR_u + (1-p)R_o) - c - m]/r\}/\underline{w} = 0$. Hence, there exists $V^* \in ([\mu(pR_u + (1-p)R_o) - c - m]/r, \bar{V})$, such that the boundary condition (B.25) is satisfied. For $\mu pR_u < c < \mu R_u$ and $\mu R_u \leq c$, the proofs are very similar to this case, and, hence, are omitted here.

We have defined the principal's value function with monitoring (denoted as $V_m(w)$). The principal would conduct monitoring if it brings her a future value greater than or equal to the value without any agent, i.e. $V_m(\underline{w}) \geq \underline{w}$. With the definition of V_m, a parallel result to Proposition 3.3 is easy to establish, and, hence, is omitted here. To conclude, the principal has the following three options: 1. Not to hire the agent, and both the principal and the agent have total utility of zero. 2. Hire the agent and set his starting promised utility as $W_{0-} \geq \underline{w}$. The principal will terminate the agent when the agent's promised utility reaches $\underline{w}$. 3. Hire the agent and set his starting promised utility as $W_{0-} \geq \underline{w}$. The principal will monitor the agent whenever the agent's promised utility reaches $\underline{w}$ (For $w > \underline{w}$, the optimal contract has the same structure as γ_1^* if $\mu pR_u \geq c$, γ_2^* if $\mu pR_u < c < \mu R_u$, and γ_3^* if $\mu R_u \leq c$). The principal will choose the one which brings her the highest profit.

B.1.3.3 Monitoring with Replacement of the Agent

In this subsection, we generalize the current setting to the case where the principal could pay a fixed cost $k > 0$ to fire the current agent and find a new agent. Formally, the principal's utility changes from (3.13) to

$$U_m(\Gamma, \nu, \hat{N}) = \mathbb{E}\left[\int_0^\tau e^{-rt}\left(R_u d\hat{N}_t^u + R_o dN_t^o - dL_t - cdt - M_t dt\right) \right. \\ \left. + e^{-r\tau}\max(0, U_m(\Gamma, \nu, \hat{N}) - k)\right], \tag{B.26}$$

where the first term in the expectation remains unchanged and the second term represents the principal's utility after terminating the contract with the current agent. To solve this problem, we need to slightly change the principal's value function. For the value functions $F_i(w), i = 1, 2, 3$ defined in Lemma 3.4 and w_i^* defined in (3.11), we let $\hat{F}_i(w) = F_i(w)$ if $F_i(w_i^*) - k < 0$. Alternatively, if $F_i(w_i^*) - k \geq 0$, then we need to change the boundary condition of the principal's value function from $F_i(0) = 0$ to $\hat{F}_i(0) = \max(\max_w \hat{F}_i(w) - k, 0)$. The uniqueness of the value function $\hat{F}_i(w)$ is proved in the following lemma.

Lemma B.4 *If $c \leq \mu p R_u$, then differential equation* (3.7) *with boundary condition $F(0) = \max(\max_w F(w) - k, 0)$ has a unique solution $\hat{F}_1(w)$, which is strictly concave in w, and $F_1'(w) \geq -1$. If $\mu p R_u < c < \mu R_u$, then differential equation* (3.9) *with boundary condition $F(0) = \max(\max_w F(w) - k, 0)$ has a unique solution $\hat{F}_2(w)$, which is strictly concave in w, and $F_2'(w) \geq -1$. If $c \geq \mu R_u$, then equation* (3.10) *with boundary condition $F(0) = \max(\max_w F(w) - k, 0)$ has a unique solution $\hat{F}_3(w)$, which is strictly concave in w, and $F_3'(w) \geq -1$.*

Proof of Lemma B.4: The proof of this lemma is similar to the proof of Lemma 3.4. Under the condition $\mu p R_u \geq c$, if the solution to (3.7) satisfies $0 \geq F_1(w_1^*) - k$, then the result is established and $\hat{F}_1(w) = F_1(w)$. The rest of the proof focuses on the case $0 < F_1(w_1^*) - k$, in which the boundary condition is $\hat{F}_1(0) = \max_w \hat{F}_1(w) - k$. In the proof of Lemma 3.4, we have established that, for $b_1 < b_1' < 0$, $V_{b_1}' > V_{b_1'}'$ for $w \in [0, \bar{w}_1]$. Then $F_{b_1}' > F_{b_1'}'$, for $w \in [0, \bar{w}_1]$. Hence,

$$F_{b_1'}(w_{b_1'}^*) - F_{b_1'}(0) = \int_0^{w_{b_1'}^*} F_{b_1'}'(w)dw < \int_0^{w_{b_1'}^*} F_{b_1}'(w)dw = F_{b_1}(w_{b_1'}^*) - F_{b_1}(0),$$

where $w_b^* = \arg\max_w F_b(w)$ for any b. Further, we have $w_{b_1'}^* < w_{b_1}^*$ and $F_{b_1}'(w) > 0$, for $w \in [w_{b_1'}^*, w_{b_1}^*]$. Therefore, $F_{b_1'}(w_{b_1'}^*) - F_{b_1'}(0) < F_{b_1}(w_{b_1}^*) - F_{b_1}(0)$. Hence, $F_{b_1}(w_{b_1}^*) - F_{b_1}(0)$ decreases in b. Finally, if we let $b_1 \to -\infty$, $F_{b_1}(w_{b_1}^*) - F_{b_1}(0) \to \infty$; further, as $b_1 \to 0$, $F_{b_1}(w_{b_1}^*) - F_{b_1}(0) \to 0$. Therefore, there must be a unique b_k, such that $F_{b_k}(w_{b_k}^*) - F_{b_k}(0) = k$, and we can let $\hat{F}_1(w) = F_{b_k}(w)$. To conclude, we have shown that the differential equation (3.7) with boundary condition $F(0) = \max(\max_w F(w) - k, 0)$ has a unique solution $\hat{F}_1(w)$, which is strictly concave in w, and $F_1'(w) \geq -1$. For the cases $\mu p R_u < c < \mu R_u$ and $\mu R_u \leq c$, the proofs are very similar to the case $\mu p R_u \geq c$, and, hence, are omitted here.

Suppose that full monitoring, termination, or replacement only happens when the agent's promised utility reaches 0, then we only need to focus on the principal's value when the agent's promised utility is 0.

Therefore, full monitoring is profitable to the principal only if

$$\{\mu[pR_u + (1-p)R_o] - c - m\}/r \geq \hat{F}_i(0). \tag{B.27}$$

Hence, when the agent's promised utility decreases to 0, the principal would start to monitor the agent if condition (B.27) holds; the principal would terminate the contract and find a new agent if condition (B.27) does not hold but $F_i(w_i^*) - k \geq 0$ holds; the principal would terminate the contract if (B.27) does not hold but $F_i(w_i^*) - k < 0$ holds. Therefore, even with the opportunity to replace the agent, the structure of the optimal contracts with full monitoring remains the same before the agent's promised utility reaches zero. The proof of this statement is easily adapted from Proposition 3.3 and is omitted here.

B.1.3.4 Proof of Proposition 3.4

In fact, it is more informative to consider the societal value function $V(w) := F(w) + w$, which, together with (3.17), leads to:

$$\begin{aligned} rV(w) = \max \big\{ & r\bar{V} + \mu p[V(w+R_u) - V(w)] + r(w - \bar{w}_1)V'(w), \\ & r\bar{V} - m + \mu[V(w+\beta) - V(w)] + r(w - \bar{w}_2)V'(w) \big\}. \end{aligned} \tag{B.28}$$

If $w = \bar{w}_1$ and the agent is paid R_u for every reported arrival from the unobservable channel, then the incentive compatibility constraints are satisfied. Therefore, the agent reports the truth, always exerts effort and is never terminated, and in this case the societal value is $\bar{V}$ which equals the upper bound. Hence, $V(w) \geq \bar{V}$ if $w \geq \bar{w}_1$. Obviously, the principal achieves this by not monitoring. Therefore, we have $w_L^* < \bar{w}_1$. In the following, we show the desired result according in three steps: 1. We, first, show that the system's value function defined in (B.28) is concave and differentiable in w (which implies that F is also concave and differentiable). 2. We, then, present a lemma which will be useful to prove that the principal's value function is an upper bound on the principal's utility under any other contract. 3. Finally, we complete the proof of the optimality of γ_m^* by combining steps 1 and 2.

Step 1. We assumed that the solution of (B.28) has a unique switching point at w_L^*. Hence, we have $\mathcal{M}(w) \geq m$ if and only if $w \leq w_L^*$. Further, the system's value function V follows the differential equation (B.3) for $w \geq w_L^*$ and for $w \in [0, w_L^*]$, V follows the differential equation

$$rV(w) = r\bar{V} - m + \mu[V(w+\beta) - V(w)] + r(w - \bar{w}_2)V'(w). \tag{B.29}$$

On the one hand, if $w_L^* = 0$, then V follows the differential equation (B.3) for the entire region of w. Hence, the proof of concavity directly follows from Case 1 in the proof of Lemma 3.4. On the other hand, if $w_L^* > 0$, we prove that V is also concave and differentiable in w. The structure of (B.28) requires that the solution V is smoothly pasted at the optimal switching point w_L^*. Formally, at $w = w_L^*$, we have $\mathcal{M}(w_L^*) = m$, and $V'(w_L^*-) = V'(w_L^*+)$.

Since the value function follows the differential equation (B.3) for $w \in [w_L^*, \bar{w}_1]$, following Lemma 3.4,

V is concave and differentiable in w on $[w_L^*, \bar{w}_1]$ and $V'(\bar{w}_1) = 0$. Further, since $V'(w_L^*-) = V'(w_L^*+)$, we only need to verify that V is also concave and differentiable in w on $[0, w_L^*)$. First, at $w = w_L^*$, following (B.29), we have $V''(w_L^*-) = \frac{\mu(V'(w_L^*+\beta)-V'(w_L^*))}{r(\bar{w}_2-w)} < 0$, where the inequality follows from the concavity of V in $(w_L^*, \bar{w}_1]$. Next, we can prove the concavity of V in $[0, w_L^*)$ by contradiction. Suppose that there exists $\hat{w} \in [0, w_L^*)$, such that $V''(\hat{w}) \leq 0$. By continuity, we can define $\tilde{w} = \sup\{w \in [0, w_L^*) : V''(w) = 0\}$. Furthermore, we have $V''(w) < 0$ for $w > \tilde{w}$. At $w = \tilde{w}$, following (B.29), we have $V''(\tilde{w}) = \frac{\mu(V'(w+\beta)-V'(w))}{r(\bar{w}_2-w)} = 0$, which contradicts $V'(\tilde{w}+\beta) = \int_0^\beta V''(\tilde{w}+x)dx + V'(\tilde{w}) < V'(\tilde{w})$,where the inequality follows from the fact that $V'' < 0$ on $(\tilde{w}, \bar{w}_1]$. Hence, we have $V'' < 0$ for $w \in [0, w_L^*)$. The differentiability of V follows from the fact that $V'(w_L^*-) = V'(w_L^*+)$ and that V' is well-defined by the differential equation (B.29) on $[0, w_L^*)$. Furthermore, $V'(\bar{w}_1) = 0$ and the concavity of V implies that $V' \geq 0$. Hence, the principal's value function F is concave, differentiable in w, and $F'(w) \geq -1$.

Step 2. By applying the Ito's formula and (PK) (adapted from the proof of Lemma B.1), we have the following Lemma B.5. It is worth noting here that if the principal does not induce agent's effort, then there is no arrivals, hence, the agent cannot misreport anything, and, the principal will never monitor.

Lemma B.5 *Suppose $F(w) : [0, \infty) \to \mathbf{R}$ is differentiable, concave, upper-bounded function, with $F'(w) \geq -1$, and $F(0) \geq 0$. Consider any $\gamma \in \Gamma_{ma}$, which yields the agent's expected utility $u(\gamma, \nu, N^u) = W_{0-} = w$. Define a stochastic process $\{\Phi_t\}_{t \geq 0}$, such that: (1) if the principal induces the agent to exert effort and report arrivals truthfully (i.e., constraints* (IC-truthfulm) *and* (IC-effort) *are satisfied) where $\{W_t\}_{t\geq 0}$ follows* (PK)*, then*

$$\begin{aligned}\Phi_t = \mu p R_u + \mu(1-p)R_o + F'(W_{t-})[rW_{t-} - \mu p H_t^u - \mu(1-p)H_t^o] - rF(W_{t-}) \\ + (F(W_{t-} + H_t^u) - F(W_{t-}))\mu p + (F(W_{t-} + H_t^o) - F(W_{t-}))\mu(1-p) - c - m_t;\end{aligned} \tag{B.30}$$

(2) if the principal does not induce the agent to exert effort where W_t follows (B.17)*, then Φ_t follows equation* (B.18)*. Furthermore, if the process $\{\Phi_t\}_{t\geq 0}$ is non-positive almost surely, then $F(w) \geq U(\Gamma, \nu, N^u)$.*

Next, we show that the principal's value function F is an upper bound on the principal's utility under any other contract in Γ_{ma}. Following Lemma B.5, we only need to verify that Φ_t, defined in (B.30) is non-positive (with F as the principal's value function defined in (3.16)). Before we proceed, we present an inequality which will be useful in proving $\Phi_t \leq 0$: if F is concave in w and $pH_t^u + (1-p)H_t^o \geq \beta$, we have

$$F'(W_t)[-pH_t^u - (1-p)H_t^o] + pF(W_t + H_t^u) + (1-p)F(W_t + H_t^o) \leq -F'(W_t)\beta + F(W_t + \beta). \tag{B.31}$$

Hence, first, if the principal induces the agent to report arrivals truthfully and exert effort with $m_t = m$ (the principal conducts monitoring and, hence constraint (IC-effort) has to be satisfied), therefore, Φ_t follows from equation (B.30), and we obtain

$$\begin{aligned}\Phi_t \leq \mu p R_u + \mu(1-p)R_o + F'(W_{t-})[rW_{t-} - \mu\beta] - rF(W_{t-}) \\ + (F(W_{t-} + \beta) - F(W_{t-}))\mu - c - m \leq 0,\end{aligned}$$

where the first inequality follows from the inequality (B.31) and the last inequality follows from the HJB equation (3.17). Second, if the principal induces the agent to report arrivals truthfully and exert effort with $m_t = 0$ (the principal does not conduct monitoring, and, hence, constraints (IC-truthfulm) and (IC-effort) have to be satisfied), then Φ_t follows from equation (B.30), and we get

$$\begin{aligned}\Phi_t &\leq \mu p R_u + \mu(1-p)R_o + F'(W_{t-})[rW_{t-} - \mu p R_u] - rF(W_{t-}) \\ &\quad + (F(W_{t-} + R_u) - F(W_{t-}))\mu p - c \leq 0,\end{aligned}$$

where the first inequality follows from the optimization problem (B.12) (with the optimal solution (B.13) under the condition $\mu p R_u \geq c$), and the last inequality follows from the HJB equation (3.17). Third, if the principal does not induce the agent to exert effort, then Φ_t follows (B.18), and (similar to (B.21)) we obtain $\Phi_t \leq 0$. Hence, we have shown that $F(w) \geq U(\gamma, \nu, N^u)$ for any $\gamma \in \Gamma_{ma}$ if $\mu p R_u \geq c$.

Step 3. Finally, similar to Proposition 3.1, we can easily verify that $U(\gamma_m^*(w), \nu^*, N^u) = F(w)$. Therefore, we obtain, under the condition $\mu p R_u \geq c$, for any $\gamma \in \Gamma_{ma}$, $U(\gamma_m^*(w_m^*), \nu^*, N^u) = F(w_m^*) \geq U(\gamma_m^*(w), \nu^*, N^u) = F(w) \geq U(\gamma, \nu, N^u)$,where the first inequality follows from the fact that w_m^* is the maximizer of F. To conclude, we have shown that $\gamma_m^*(w_m^*)$ is the optimal contract in the contract space Γ_{ma} under the condition $c \leq \mu R_u$.

B.1.3.5 Partial Monitoring under the Condition $\mu p R_u < c < \mu R_u$

When the principal conducts partial monitoring, the dynamics of the agent's promised utility is given by (DW2) and when the principal monitors the agent, the dynamics of the agent's promised utility follows (DW3). Hence, the societal value function should satisfy the HJB equation:

$$\begin{aligned}rV(w) = \max\Big\{&r\bar{V} + \mu p[V(w+R_u) - V(w)] + \mu(1-p)\left[V\left(w+\beta_1\right) - V(w)\right] + r(w - \bar{w}_2)V'(w), \\ &r\bar{V} - m + \mu[V(w+\beta) - V(w)] + r(w - \bar{w}_2)V'(w)\Big\}. \end{aligned} \tag{B.32}$$

with boundary condition $V(0) = 0$. Hence, the principal would monitor the agent if and only if the second term of (B.32) greater or equal to the first term which is also equivalent to $\mathcal{M}_2(w) := \mu[V(w+\beta) - pV(w+R_u) - pV(w)] \geq m$.

If $w = \bar{w}_2$ and the agent is paid R_u for every arrival from the unobservable channel that he reports, and is paid β_1 for every arrival from the observable channel, then the incentive constraints are binding. Hence, the agent always exerts effort, reports the truth and is never terminated, which results in the societal value achieving the upper bound $\bar{V}$. Therefore, for $w \geq \bar{w}_2$, we have $V(w) \geq \bar{V}$. In contrast, if the principal monitors the agent, the highest societal value that can be achieved is $\bar{V} - \dfrac{m}{r}$. Hence, for $w \geq \bar{w}_2$, not monitoring is better than monitoring, i.e., $m > \mathcal{M}_2(w)$. As w decreases, monitoring becomes more profitable, and there may be optimal switching points w_L^{2*}, such that $m = \mathcal{M}_2(w_L^{2*})$. For simplicity, we assume a unique switching point. Similar to the case in Section (3.3.2.1), in all the numerical examples, we see the single switching of the value function. Next, we formally define the contract in the following.

Definition B.1 *For any $w \in [0, \bar{w}_2]$, define contract $\gamma_m^{2*}(w) = (L_m^{2*}, \tau_m^{2*}, m_2^*)$ as follows:*

1. *Set $W_{0-} = w$ and $L_0^* = (W_{0-} - \bar{w}_2)^+$.*
2. *For $t \geq 0$, let payment be*

$$
\begin{aligned}
dL_{mt}^{2*} = &\left\{(W_{t-} + R_u - \bar{w}_2)^+ dN_t^u + (W_{t-} + \beta_1 - \bar{w}_2)^+ \, dN_t^o\right\} \mathbb{I}[W_{t-} \geq w_L^{2*}] \\
&+ (W_{t-} + \beta - \bar{w}_2)^+ \, dN_t \mathbb{I}[W_{t-} < w_L^{2*}],
\end{aligned}
$$

   and the dynamics of promised utility follows

$$
\begin{aligned}
dW_t = &\{r(W_{t-} - \bar{w}_2)dt + \min\{R_u, \bar{w}_2 - W_{t-}\}dN_t^u + \min\{\beta_1, \bar{w}_2 - W_{t-}\} \, dN_t^o\} \, \mathbb{I}[W_{t-} \geq w_L^{2*}] \\
&+ \{r(W_{t-} - \bar{w}_2)dt + \min\{\beta, \bar{w}_1 - W_{t-}\}dN_t\} \, \mathbb{I}[W_{t-} < w_L^{2*}].
\end{aligned}
\tag{B.33}
$$

3. *The partial monitoring schedule is $M_t^* = m\mathbb{I}[W_{t-} < w_L^{2*}]$.*
4. *The termination time is $\tau_m^{2*} = \min\{t : W_t = 0\}$.*

The following Proposition shows that γ_m^{2*} is the optimal contract in the class of contracts that allow the agent to shirk or not truthfully report (in the space Γ_{ma}). We further define w_m^{2*} as the maximizer of $F(w)$ and the principal starts the agent's promised utility at $W_{0-} = w_m^{2*}$.

Proposition B.1 *Suppose that the switching point w_L^{2*} is unique and $\mu p R_u < c < \mu R_u$, the value function V defined in* (B.32) *is concave in w and differentiable and $\gamma_m^{2*}(w_m^{2*})$ in definition B.1 is the optimal contract in the contract space Γ_{ma}.*

The proof of Proposition B.1 is very similar to the proof of Proposition 3.4, and hence is omitted here.

B.1.4 Proofs in Section 3.4

Throughout this section, we assume that the solution to the HJB equation (3.20) exists. Given the existence, we characterize the optimal incentive compatible contract. A recent contract theory paper, [Mal19], has the same assumption, they study the optimal contract design of a dynamic capital allocation process in a firm. Following the main body of the paper, we denoted by F, the principal's value function, and by V, the system's value function, under the optimal contract. Hence, both F and V must be weakly concave, otherwise, public randomization over two levels of promised utility can improve both the system's and the principal's value. Later, in Section B.1.4.3, we will prove that F and V are strictly concave. Hence, the principal does not need to use public randomization in the optimal contract.

B.1.4.1 Proof of Property 2

When $w = \bar{w}$, we are able to find a contract such that the system's value function achieves the upper bound $\bar{V}_d$, which implies that $V(w) \geq \bar{V}_d$ for $w \geq \bar{w}$. In that contract, the principal makes direct payments to the agent: the payment R for every customer from the unobservable channel and $\max\left((a+b)\beta - aR/b, 0\right)$ for every customer from the observable channel. Further, since $\bar{V}_d$ is the highest value that the system can achieve, we have $V(w) \leq \bar{V}_d$ for $w \geq \bar{w}$. Hence, $V(w) = \bar{V}_d$ for $w \geq \bar{w}$.

B.1.4.2 Proof of Lemma 3.9

For a given $w \geq 0$, we denote $g(d) := G(d, w)$. If $p_u(d)R \geq \beta$, then $H_o(d) = 0$ and

$$\begin{aligned} g(d) =& p_u(d)\left[R + V\left(w+R\right) - V(w) - RV'(w)\right] + p_o(d)(1-d)R \\ =& p_u(d)\left[V\left(w+R\right) - V(w) - RV'(w)\right] + R - p_o(d)dR, \end{aligned} \tag{B.34}$$

where the first term is linear in d and the last term is quadratic in d. Hence, $g(d)$ is concave in d if $p_u(d)R \geq \beta$. If $p_u(d)R < \beta$, before we calculate $g''(d)$, we, first, calculate $\frac{\partial H_o(d)}{\partial d}$ and $\frac{\partial^2 H_o(d)}{\partial^2 d}$ as follows:

$$\begin{aligned} \frac{\partial H_o(d)}{\partial d} &= \frac{-p_u'(d)Rp_o(d) - p_o'(d)(\beta - p_u(d)R)}{p_o(d)^2} = \frac{[-p_u'(d)p_o(d) + p_o'(d)p_u(d)]R - p_o'(d)\beta}{p_o(d)^2} \\ &= \frac{p_o'(d)(R-\beta)}{p_o(d)^2}, \end{aligned} \tag{B.35}$$

where the last equality follows from $p_u'(d) + p_o'(d) = 0$ and $p_u(d) + p_o(d) = 1$, $\frac{\partial^2 H_o(d)}{\partial^2 d} = \frac{-2(p_o'(d))^2(R-\beta)}{p_o(d)^3}$. Hence, we immediately obtain

$$2p_o'(d)\frac{\partial H_o(d)}{\partial d} + p_o(d)\frac{\partial^2 H_o(d)}{\partial^2 d} = 0 \tag{B.36}$$

Then, the expressions of $g'(d)$ and $g''(d)$ are

$$\begin{aligned} g'(d) = p_u'(d)\left[R + V\left(w+R\right) - V(w) - RV'(w)\right] + p_o'(d)\left[(1-d)R + V\left(w + H_o(d)\right) - V(w)\right] \\ - H_o(d)V'(w) + p_o(d)\left[-R + V'\left(w + H_o(d)\right)\frac{\partial H_o(d)}{\partial d} - \frac{\partial H_o(d)}{\partial d}V'(w)\right], \end{aligned} \tag{B.37}$$

and

$$\begin{aligned} g''(d) = \left[2p_o'(d)\frac{\partial H_o(d)}{\partial d} + p_o(d)\frac{\partial^2 H_o(d)}{\partial^2 d}\right]\left[V'\left(w + H_o(d)\right) - V'(w)\right] - 2p_o'(d)R \\ + p_o(d)V''\left(w + H_o(d)\right)\left(\frac{\partial H_o(d)}{\partial d}\right)^2 = -2p_o'(d)R + p_o(d)V''\left(w + H_o(x)\right)\left(\frac{\partial H_o(d)}{\partial d}\right)^2 \leq 0, \end{aligned}$$

where the last equality follows from equation (B.36), and the last inequality follows from $p_o'(d) > 0$ and the concavity of V. Finally, (B.34) and (B.37) imply that $g'(d)$ is continuous at d if $p_u(d)R = \beta$. To conclude, we have shown that $g(d)$ defined in (3.24) is concave in d. Further, Property 3 is a direct result of the concavity of $g(d)$. Hence, Lemma 3.9 directly implies Property 3 and we have

$$\frac{\partial G(d,w)}{\partial d} = p_u'(d)\left[R + V(w+R) - V(w) - RV'(w)\right] + p_o'(d)\left[dR + V(w + H_o(d)) - V(w) - H_o(d)V'(w)\right] + p_o(d)\left[R + V'(w + H_o(d))\frac{\partial H_o(d)}{\partial d} - \frac{\partial H_o(d)}{\partial d}V'(w)\right]. \tag{B.38}$$

B.1.4.3 Proof of Property 4

First, we show that there exists $\delta > 0$, such that, for $w \in [\bar{w} - \delta, \bar{w}]$, $d^*(w) = 0$. Property 2 implies that $d^*(w) = 0$ if $w \geq \bar{w}$. Hence, at $w = \bar{w}$, we have $V'(\bar{w}) = 0$ and $V(w + R) = V(w + H_o(d^*(\bar{w}))) = V(w) = \bar{V}$. Furthermore, at $w = \bar{w}$, $g'(0) = p_u'(0)R + p_o'(0)R + p_o(0)R > 0$. Due to the continuity of V, there exists $\delta > 0$, such that for $w \in [\bar{w} - \delta, \bar{w}]$, we have $\frac{\partial G(d,w)}{\partial d}|_{d=0} \geq 0$. Hence, for $w \in [\bar{w} - \delta, \bar{w}]$, we let $y = 0$ and $z = \frac{\partial G(d,w)}{\partial d}|_{d=0} \geq 0$ which implies that $d^*(w) = 0$ for $w \in [\bar{w} - \delta, \bar{w}]$. Next, we prove that $V(w)$ is strictly concave in w on $[0, \bar{w})$.

First, if $p_u(0)R \geq \beta$, then $H_o(0) = 0$. For $w \in [\max\{\bar{w} - \delta, \bar{w} - R_u\}, \bar{w}]$, since $d^*(w) = 0$, the equation (3.20) becomes an ordinary differential equation. We can solve it in closed-form, and it immediately implies that $V(w)$ is strictly concave in w on $[\max\{\bar{w} - \delta, \bar{w} - R_u\}, \bar{w}]$. We rewrite the equation (3.20) as follows:

$$rV(w) = -c + rwV'(w) + \mu\left\{p_u(d^*(w))\left[R + V(w+R) - V(w) - RV'(w)\right] + p_o(d^*(w))\left[(1 - d^*(w))R + V(w + H_o(d^*(w))) - V(w) - H_o(d^*(w))V'(w)\right]\right\}.$$

By taking derivative with respect to w on both sides and putting the term with $V''(w)$ on the left-hand side of the equation, we get

$$\begin{aligned}
&\left[\mu p_u(d^*(w))R_u + \mu p_o(d^*(w))H_o(d^*(w)) - rw\right]V''(w) = \mu\Big\{p_u'(d^*(w))d^{*'}(w) \\
&+\left[R + V(w+R) - V(w) - RV'(w)\right] + p_u(d^*(w))\left[V'(w+R) - V'(w)\right] \\
&+p_o'(d^*(w))d^{*'}(w)\cdot\left[(1 - d^*(w))R + V(w + H_o(d^*(w))) - V(w) - H_o(d^*(w))V'(w)\right] \\
&+p_o(d^*(w))\cdot\Big[-d^{*'}(w)R + V'(w + H_o(d^*(w)))(1 + H_o'(d^*(w))d^{*'}(w)) - V'(w) \\
&-H_o'(d^*(w))d^{*'}(w)V'(w)\Big]\Big\},
\end{aligned} \tag{B.39}$$

where the right-hand side of the equation can be simplified as

$$\begin{aligned}
&\mu\left\{p_u(d^*(w))\left[V'(w+R) - V'(w)\right] + p_o(d^*(w))\left[V'(w + H_o(d^*(w)) - V'(w)\right]\right\} + \mu g'(d^*(w))d^{*'}(w) \\
&= \mu\left\{p_u(d^*(w))\left[V'(w+R) - V'(w)\right] + p_o(d^*(w))\left[V'(w + H_o(d^*(w)) - V'(w)\right]\right\},
\end{aligned} \tag{B.40}$$

where the last equality follows from the optimality condition in Property 3. Next, we prove that $V(w)$ is strictly concave in w on $[0, \bar{w} - \delta)$ by contradiction. If there exists $\hat{w}$, such that $V''(\hat{w}) \geq 0$, then we can define $\tilde{w} = \max\{w : V''(w) \geq 0\}$. Therefore, $V''(\tilde{w}) = 0$ and $V''(w) < 0$ for $w \geq \tilde{w}$. $V''(\tilde{w}) = 0$ implies that the right-hand side of (B.39) equals to zero. However, $V''(w) < 0$ for $w \geq \tilde{w}$ implies that at $w = \tilde{w}$, we have $V'(\tilde{w} + R_u) < V'(\tilde{w})$, $V'(\tilde{w} + H_o(d^*(\tilde{w}))) < V'(\tilde{w})$, and, hence, the right-hand side of (B.39) is smaller than zero. This leads to a contradiction. Hence, $V(w)$ is strictly concave in w on $[0, \bar{w})$.

Then, if $p_u(0)R < \beta$, the proof is very similar to the case when $p_u(0)R \geq \beta$, and, hence, is omitted here. To conclude, we have shown that $V(w)$ is strictly concave in w on $[0, \bar{w})$. Further, since $F(w) = V(w) - w$, $F(w)$ is also strictly concave in w on $[0, \bar{w})$. In the following, we present an inequality which is useful for the proof of Property 5. Since $V(w)$ is strictly concave in w on $[0, \bar{w})$, we have $V''(w) < 0$ and the right-hand side of equation (B.39) is smaller than 0 which implies that for $w \in [0, \bar{w})$,

$$\mu p_u(d^*(w))R_u + \mu p_o(d^*(w))H_o(d^*(w)) - rw \geq 0. \tag{B.41}$$

B.1.5 Proof of Property 5

Property 3 implies that, for the region of w, such that $d^*(w) \neq 0$ and $d^*(w) \neq \bar{d}$, we have $\frac{\partial G(d,w)}{\partial d}|_{d=d^*(w)} = 0$. By the Implicit Function Theorem, we have $\frac{\partial d^*(w)}{\partial w} = -\frac{\partial^2 G(d,w)}{\partial^2 w} / \frac{\partial^2 G(d,w)}{\partial d \partial w}$. Hence, our goal is to show that $\frac{\partial d^*(w)}{\partial w} \leq 0$. Further, following Lemma 3.9, we have $\frac{\partial^2 G(d,w)}{\partial^2 d} = g''(d) \leq 0$, and

$$\begin{aligned} \frac{\partial^2 G(d,w)}{\partial w \partial d} &= p_u'(d)[V'(w+R) - V'(w) - RV''(w)] + p_o'(d)[V'(w + H_o(d)) - V'(w) - H_o(d)V''(w)] \\ &\quad + p_o(d)[V''(w + H_o(d)) - V''(w)]\frac{\partial H_o(d)}{\partial d}. \end{aligned} \tag{B.42}$$

Hence, we can prove that $\frac{\partial d^*(w)}{\partial w} \leq 0$ if and only if $\frac{\partial^2 G(d,w)}{\partial w \partial d} \leq 0$. If $p_u(d)R_u \geq \beta$, then $H_o(d) = 0$. Hence, following (B.42), we have

$$\frac{\partial^2 G(d,w)}{\partial w \partial d} = p_u'(d)[V'(w+R) - V'(w) - RV''(w)] = p_u'(d)R\left[\frac{V'(w+R) - V'(w)}{R} - V''(w)\right]. \tag{B.43}$$

Following (B.39) and (B.40), we have

$$V''(w) = \frac{\mu p_u(d^*(w))\left[V'(w+R) - V'(w)\right]}{\mu p_u(d^*(w))R - rw}. \tag{B.44}$$

Following (B.43), (B.44), we have $\frac{\partial^2 G(d,w)}{\partial w \partial d} = p_u'(d)\left[V'(w+R) - V'(w)\right]\frac{-rw}{\mu p_u(d^*(w))R - rw} \leq 0$, where the inequality follows from the concavity of V, $p_u'(d) \leq 0$, and equation (B.41). If $p_u(d)R_u < \beta$, then $H_o(d) = (\beta - p_u(d)R)/p_o(d)$. Following (B.39) and (B.40), we have

$$V''(w) = \frac{\mu p_u(d^*(w))\left[V'(w+R) - V'(w)\right] + \mu p_o(d^*(w))\left[V'(w + H_o(d)) - V'(w)\right]}{\mu\beta - rw}. \tag{B.45}$$

Next, following (B.42), we have

$$
\begin{aligned}
\frac{\partial^2 G(d,w)}{\partial w \partial d} &= p_u'(d)[V'(w+R) - V'(w) - RV''(w)] + p_o'(d)[V'(w+H_o(d)) - V'(w) - H_o(d)V''(w)] \\
&+ p_o(d)[V''(w+H_o(d)) - V''(w)]p_o'(d)(R-\beta)(p_o(d))^{-2} \\
&= p_u'(d)\left\{[V'(w+R) - V'(w) - RV''(w)] - [V'(w+H_o(d)) - V'(w) - H_o(d)V''(w)]\right. \\
&\left. - [V''(w+H_o(d)) - V''(w)](R-\beta)(p_o(d))^{-1}\right\} \\
&= p_u'(d)\left\{[V'(w+R) - V'(w)]\left[1 - \frac{\mu p_u(d) R_u}{\mu\beta - rw} + \frac{\mu p_u(d) H_o(d)}{\mu\beta - rw} + \frac{(R-\beta)}{p_o(x)}\frac{\mu p_u(d)}{\mu\beta - rw}\right]\right. \\
&+ [V'(w+H_o(d)) - V'(w)]\left[-1 - \frac{\mu p_o(d) R_u}{\mu\beta - rw} + \frac{\mu p_o(d) H_o(d)}{\mu\beta - rw} + \frac{(R-\beta)}{p_o(x)}\frac{\mu p_o(d)}{\mu\beta - rw}\right] \\
&\left. - V''(w+H_o(d))(R_u-\beta)(p_o(d))^{-1}\right\} \\
&= p_u'(d)\left\{-[V'(w+R_u) - V'(w)] - [V'(w+H_o(d)) - V'(w)]\right\} \\
&- p_u'(d)V''(w+H_o(d))(R-\beta)(p_o(d))^{-1} \le 0,
\end{aligned}
$$

where the first equality follows from equation (B.35), the second equality follows from $p_u'(d) = -p_o'(d)$, the third equality follows from equation (B.45), and the last inequality follows from the concavity of V and $p_u'(d) \le 0$. For the region of w, where $d^*(w) = 0$ or $d^*(w) = \bar{d}$, $\frac{\partial d^*(w)}{\partial w} = 0$. To conclude, we have shown that $d^*(w)$ is decreasing in w.

B.1.5.1 Proof of Theorem 3.3

To prove the optimality of the contract, adapted from Lemma B.2, we first present Lemma B.6 below. It is worth noting here that if the principal does not induce the agent's effort, there are no arrivals, hence, the agent cannot misreport anything and the principal will never monitor.

Lemma B.6 *Suppose $F(w) : [0,\infty) \to \mathbf{R}$ is differentiable, concave, upper-bounded function, with $F'(w) \ge -1$, and $F(0) \ge 0$. Consider any contract $\gamma \in \Gamma_D$, which yields the agent's expected utility $u(\Gamma, \nu, N^u) = W_{0-} = w$. Define a stochastic process $\{\Phi_t\}_{t\ge 0}$, such that: (1) if the principal incentivizes the agent to report arrivals truthfully and to exert effort (i.e., incentive constraints* (ICeffortd) *and* (IC-truthful) *are satisfied) where W_t follows* (PKd)*, then*

$$
\begin{aligned}
\Phi_t &= \mu p_u(d_t)R + \mu p_o(d_t)(1-d_t)R + F'(W_{t-})[rW_{t-} - \mu p_u(d_t)H_t^u - \mu p_o(d_t)H_t^o] - rF(W_{t-}) \\
&+ [F(W_{t-} + H_t^u) - F(W_{t-})]\mu p_u(d_t) + [F(W_{t-} + H_t^o) - F(W_{t-})]\mu p_o(d_t) - c;
\end{aligned}
\tag{B.46}
$$

(2) if the principal does not induce effort where W_t follows (B.17)*, then Φ_t follows equation* (B.18)*. Furthermore, if the process $\{\Phi_t\}_{t\ge 0}$ is non-positive almost surely, then $F(w) \ge U(\gamma, \nu, N^u)$.*

Following the system's value function $V(w)$ defined in (3.25), define $F(w) := V(w) - w$. Next, Property 2 implies that $V'(w) = 0$ for $w \ge \bar{w}$. Furthermore, following Property 4, we have that $V(w)$ is increasing and concave in w, hence, we have $F'(w) \geq -1$, and $F(w)$ is concave in w. Hence, applying Lemma B.6, to show that $F(w) \geq U(\Gamma, \nu, N^u)$ for any $\gamma \in \Gamma_D$, we only need to verify that Φ_t is non-positive. Before we proceed, we present an inequality which will be helpful in proving $\Phi_t \leq 0$: under the constraints (ICeffortd) and (IC-truthful), if F is concave in w and $p_u(d_t) + p_o(d_t) = 1$, then the optimization problem (B.12) (with the optimal solution (B.13) and (B.14)) implies that

$$\begin{aligned} &F'(W_{t-})[-p_u(d_t)H_t^u - p_o(d_t)H_t^o] + F(W_{t-} + H_t^u)p_u(d_t) + F(W_{t-} + H_t^o)p_o(d_t) \\ \leq &F'(W_{t-})[-p_u(d_t)R_u - p_o(d_t)H_o(d_t)] + F(W_{t-} + R_u)p_u(d_t) + F(W_{t-} + H_o(d_t))p_o(d_t) \end{aligned}, \tag{B.47}$$

where $H_o(d_t)$ is defined in Property 1 of Section 3.4.2. Hence, first, if the principal induces the agent to report arrivals truthfully and exert effort (i.e., incentive constraints (ICeffortd) and (IC-truthful) should be satisfied), then Φ_t follows (B.46) and we get

$$\begin{aligned} \Phi_t \leq\ & \mu p_u(d_t)R + \mu p_o(d_t)(1-d_t)R + F'(W_{t-})[rW_{t-} - \mu p_u(d_t)R_u - \mu p_o(d_t)H_o(d_t)] - rF(W_{t-}) \\ &+ [F(W_{t-} + R_u) - F(W_{t-})]\mu p_u(d_t) + [F(W_{t-} + H_o(d_t)) - F(W_{t-})]\mu p_o(d_t) - c \\ &\leq \mu p_u(d_t^*)R + \mu p_o(d_t^*)(1-d_t)R + F'(W_{t-})[rW_{t-} - \mu p_u(d_t^*)R_u - \mu p_o(d_t^*)H_o(d_t^*)] - rF(W_{t-}) \\ &+ [F(W_{t-} + R_u) - F(W_{t-})]\mu p_u(d_t^*) + [F(W_{t-} + H_o(d_t^*)) - F(W_{t-})]\mu p_o(d_t^*) - c \\ &= \mu p_u(d_t^*)R + \mu p_o(d_t^*)(1-d_t^*)R + V'(W_{t-})[rW_{t-} - \mu p_u(d_t^*)R_u - \mu p_o(d_t^*)H_o(d_t^*)] - rV(W_{t-}) \\ &+ [V(W_{t-} + R_u) - V(W_{t-})]\mu p_u(d_t^*) + [V(W_{t-} + H_o(d_t^*)) - V(W_{t-})]\mu p_o(d_t^*) - c = 0, \end{aligned}$$

where the first inequality follows from the inequality (B.47), the second inequality follows from $d_t^* = d^*(W_{t-})$ and Property 3 of Section 3.4.2 ($d^*(W_{t-})$ maximizes $G(d, W_{t-})$), the first equality follows from $V(w) = F(w) + w$, and the last equality follows from the HJB equation (3.25). Second, if the principal does not induce the agent to exert effort, then Φ_t follows (B.18), and (similar to (B.21)) we obtain $\Phi_t \leq 0$. Hence, we have shown that $F(w) \geq U(\gamma, \nu, N^u)$ for any $\gamma \in \Gamma_D$. Similar to Proposition 3.1, we can easily verify that $F(w) = U(\gamma_d^*, \nu^*, N^u)$. Finally, we have for any $\gamma \in \Gamma_D$, $U(\gamma_d^*(w_d^*), \nu^*, N^u) = F(w_d^*) \geq U(\gamma_d^*(w), \nu^*, N^u) = F(w) \geq U(\gamma, \nu, N^u)$, where the first inequality follows from the fact that w_d^* the maximizer of F. To conclude, we have shown that $\gamma_d^*(w_d^*)$ is the optimal contract in the contract space Γ_D.

B.1.6 Proofs in Section 3.5

B.1.6.1 Proof of Lemma 3.10

We take the case $\mu p R_u \geq c$ as an example. The optimal contract in the baseline model is γ_1^* while (1) the optimal contract with monitoring is γ_{m1}^* if condition (3.15) holds, and (2) the optimal contract with monitoring is still γ_1^* if condition (3.15) does not hold. Hence, we only need to consider the case when condition (3.15) holds. Note that the principal's value function under γ_1^* is F_1, following (3.7), with boundary condition $F_1(0) = 0$, and the principal's value function under γ_{m1}^* is F_{m1}, following (3.7), with boundary condition $F_{m1}(0) = \dfrac{\mu[pR_u + (1-p)R_o] - c - m}{r} > 0$. Therefore, the only difference is in the boundary condition.

Following the proof of Lemma 3.4 (Step 1 and Step 2 of Case 1): if we let $V_{m1}(w) = F_{m1}(w) + w$ and $V_1(w) = F_1(w) + w$, we have $V_{m1}(w) > V_1(w)$ and $V'_{m1}(w) < V'_1(w)$ for any $w \in [0, \bar{w}_1)$. Since $w^*_{m1} = \arg\max_w F_{m1}(w)$, we have $w^*_{m1} = w$, such that $V'_{m1}(w) = 1$ if $V'_{m1}(0) > 1$, and $w^*_{m1} = 0$ if $V'_{m1}(0) \leq 1$. Similarly, $w^*_1 = w$, such that $V'_1(w) = 1$ if $V'_1(0) > 1$, and $w^*_1 = 0$ if $V'_1(0) \leq 1$. Hence, $V'_{m1}(w) < V'_1(w)$ for any $w \in [0, \bar{w}_1)$, and $V'_{m1}(w) = V'_1(w) = 0$ for $w \geq \bar{w}_1$ implies that $w^*_m \leq w^*_1$. The proofs for cases $\mu p R_u < c < \mu R_u$ and $\mu R_u \leq c$ are similar to the case when $\mu p R_u \geq c$, and, hence, are omitted here. To conclude, we have shown that if the principal conducts full monitoring (i.e., monitoring both effort and arrivals), then the agent is always worse off, i.e., $w^*_{mi} \leq w^*_i$ for $i = 1, 2, 3$.

B.1.6.2 Proof of Lemma 3.11

In this lemma, we consider the case when the principal conducts partial monitoring (i.e., by only monitoring arrivals). First, we consider a special case where $p = 0$, when all arrivals are observable to the principal. Hence, the principal never monitors. Hence, $w^*_m = w^*_1$. In what follows, we prove the result for the case when $p > 0$. If $R_u = R_o = R \geq \beta/p$ ($\mu p R_u \geq c$), then the optimal contract in the baseline model is γ^*_1 and the optimal contract with monitoring is γ^*_m. If $m = 0$, then the principal always monitors the agent and $w^*_L = \bar{w}_1$ in the contract γ^*_m. Hence, the principal's value function follows the differential equation (3.16), and $V(w) = \bar{V}$ for $w \geq \bar{w}_2 = \mu\beta/r < \bar{w}_1$. Therefore, $V'(\bar{w}_2) = 0$. Since $w^*_m = \arg\max_w F(w)$, we have $w^*_m = w$ such that $V'(w) = 1$ if $V'(0) > 1$, and $w^*_m = 0$ if $V'(0) \leq 1$. Following the concavity of V, we have $w^*_m < \mu\beta/r = c/r$.

Next, we show that when R is sufficiently large, then $w^*_1 \geq c/r$. The principal's value function F_1 follows (3.7) with boundary condition $F(0) = 0$. Hence, the societal value function $V_1(w) = F_1(w) + w$ solves the following:

$$rV_1(w) = r\bar{V} + \mu p[V_1(w + R) - V_1(w)] + r(w - \bar{w}_1)V'_1(w). \tag{B.48}$$

For any $\hat{R} > \check{R} > \beta/p$, let $\check{V}_1(w)$ and $\hat{V}_1(w)$ denote the solutions of the system's value function for the cases when $R = \check{R}$ and $R = \hat{R}$, respectively. Following (B.48), we have $\hat{V}_1(w) = \dfrac{\mu\hat{R} - c}{\mu\check{R} - c}\check{V}_1\left(\dfrac{w\check{R}}{\hat{R}}\right)$. Hence, $\hat{V}'_1(w) = \dfrac{\mu - c/\hat{R}}{\mu - c/\check{R}}\check{V}'_1\left(\dfrac{w\check{R}}{\hat{R}}\right)$, and $\hat{V}'_1(w)$ increases in $\hat{R}$, given w and $\check{R}$ fixed. Since V_1 is strictly concave in w on $[0, \bar{w}_1)$, we have $\check{V}'_1(0) > \dfrac{\check{V}_1(\bar{w}_1) - \check{V}_1(0)}{\bar{w}_1 - 0} = \dfrac{\bar{V} - 0}{\bar{w}_1} = \dfrac{\mu\check{R} - c}{\mu p \check{R}}$, where the inequality follows from the concavity of $\hat{V}$. Hence, we obtain $\lim_{\hat{R}\to\infty} \hat{V}'_1(c/r) = \lim_{\hat{R}\to\infty} \dfrac{\mu - c/\hat{R}}{\mu - c/\check{R}}\check{V}'_1\left(\dfrac{w\check{R}}{\hat{R}}\right) = \dfrac{\mu}{\mu - c/\check{R}}\check{V}'_1(0) > \dfrac{\mu}{\mu - c/\check{R}}\dfrac{\mu\check{R} - c}{\mu p \check{R}} = 1/p \geq 1$. Therefore, there exist $\hat{R}$, such that, for any $R \geq \hat{R}$, we have $\hat{V}'_1(c/r) \geq 1$. Since $w^*_1 = w$, where $\hat{V}'(w) = 1$ if $\hat{V}'_1(0) > 1$, and $w^*_1 = 0$ if $\hat{V}'_1(0) \leq 1$, we have $w^*_1 \geq c/r$. Finally, we obtain $w^*_1 \geq c/r > w^*_m$. To conclude, we have shown that if $m = 0$, there exists $\hat{R}$, such that, if $R_u = R_o > \hat{R}$, then $w^*_m \leq w^*_1$.

APPENDIX C

Appendix to Chapter 4

C.1 Proofs of Statements

C.2 Proof in Section 2

C.2.1 Proof of Lemma 4.1

To characterize how the agent's continuation utility evolves over time, it is useful to consider her lifetime expected utility, evaluated conditionally upon the information available at time t

$$\begin{aligned} u_t(\gamma, \nu; c) &= \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rs}\left(\mathrm{d}L_s - c\mathbb{1}_{\nu_s=\mu}\mathrm{d}s\right)\Big| \mathcal{F}_t^N\right] \\ &= \int_0^{t\wedge\tau-} e^{-rs}(dL_s - c\mathbb{1}_{\nu_s=\mu}\mathrm{d}s) + e^{-rt}W_t(\gamma,\nu;c) \end{aligned} \tag{C.1}$$

Since $u_t(\gamma, \nu; c)$ is the expectation of a given random variable conditional on $\mathcal{F}_t^N$, the process $\mathbf{u}(\gamma, \nu; c) = \{u_t(\gamma, \nu; c)\}_{t\geq 0}$ is an martingale under the probability measure $\mathbf{P}^{\nu}$. Relying on this martingale property, we now offer an alternative representation of $\mathbf{u}(\gamma, \nu; c)$. Consider the process $M^{\nu} = \{M_t^{\nu}\}_{t=\geq 0}$ defined by

$$M_t^{\nu} = N_t - \int_0^t \nu_s \mathrm{d}s \tag{C.2}$$

for all $t \geq 0$. The martingale representation theorem for point processes implies that the martingale $\mathbf{u}(\gamma, \nu; c)$ satisfies

$$u_t(\gamma, \nu; c) = u_0(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs} H_s(\gamma, \nu; c)\mathrm{d}M_s^{\nu} \tag{C.3}$$

for all $t \geq 0$, $\mathbf{P}^{\nu}$-almost surely, for some $\mathcal{F}^N$-predictable process $H(\gamma, \nu; c) = \{H_t(\gamma, \nu; c)\}_{t\geq 0}$. Then, (C.1) and (C.3) imply (PK). Next, we show that $\{\nu_t\}_{t\in[0,\tau]}$ defined in (IC) is a best response to contract γ.

Let u_t' denote the agent's lifetime expected payoff, given the information available at date t, when he acts according to $\nu' = \{\nu_t'\}_{t\geq 0}$ until date t and then reverts to $\nu = \{\nu_t\}_{t\geq 0}$:

$$u_t' = \int_0^{t\wedge\tau^-} e^{-rs}\left(\mathrm{d}L_s - 1_{\nu_s'=\mu}\cdot c\mathrm{d}s\right) + e^{-rt}W_t(\gamma, \nu; c) \tag{C.4}$$

Following [San08] (Proposition 2), the proof now proceeds as follows. First, we show that if $u' = \{u_t'\}_{t\geq 0}$

is an $\mathcal{F}^N$-submartingale under $\mathbf{P}^{\nu}$ that is not a martingale, then ν is suboptimal for the agent. Indeed, in that case there exists some $t > 0$ such that

$$u_{0-}(\gamma, \nu; c) = u'_{0-} < \mathbb{E}^{\nu'}[u'_t] \tag{C.5}$$

where $u_{0-}(\gamma, \nu; c)$ and u'_{0-} correspond to unconditional expected payoffs at date 0. By (C.4), the agent is then strictly better off acting according to ν' until date t and then reverting to ν. The claim follows. Next, we show that if u' is a $\mathcal{F}^N$-supermartingale under $\mathbf{P}^{\nu'}$, then ν is at least as good as ν' for the agent. From (C.1) and (C.4),

$$u'_t = u_t(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs}(1_{\nu'_s=0} - 1_{\nu_s=0})cds \tag{C.6}$$

for all $t \geq 0$. Hence, since $u_t(\gamma, \nu; c)$ is right-continuous with left-hand limits, so is u'. Moreover, since u' is non-negative, it has a last element. Hence, by the optional sampling theorem ([DM11], Chapter VI, Theorem 10)),

$$u'_0 \geq \mathbb{E}^{\nu'}[u'_\tau] = u_0(\gamma, \nu'; c) \tag{C.7}$$

where again $u_{0-}(\gamma, \nu')$ is an unconditional expected payoff at date 0. Since $u'_0 = u_0(\gamma, \nu)$ by (C.4), the claim follows. Now, for each $t \geq 0$,

$$\begin{aligned}
u'_t &= u_t(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs}(1_{\nu'_s=0} - 1_{\nu_s=0})cds \\
&= u_0(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs}H_s(\gamma, \nu; c)dM_s^{\nu} + \int_0^{t\wedge\tau} e^{-rs}(1_{\nu'=0} - 1_{\nu=0})cds \\
&= u_0(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs}H_s(\gamma, \nu; c)dM_s^{\nu'} + \int_0^{t\wedge\tau} e^{-rs}H_s(\gamma, \nu; c)(\nu'_s - \nu_s)ds \\
&+ \int_0^{t\wedge\tau} e^{-rs}(1_{\nu'_s=0} - 1_{\nu_s=0})cds \\
&= u_0(\gamma, \nu; c) + \int_0^{t\wedge\tau} e^{-rs}H_s(\gamma, \nu; c)dM_s^{\nu'} + \int_0^{t\wedge\tau} e^{-rs}\mu(1_{\nu'_s=0} - 1_{\nu_s=0})\left[\frac{c}{\mu} - H_s(\gamma, \nu; c)\right]ds
\end{aligned} \tag{C.8}$$

Since $H(\gamma, \nu; c)$ is $\mathcal{F}^N$-predictable and $M^{\nu'}$ is an $\mathcal{F}^N$-martingale under $P^{\nu'}$, the drift of u' has the same sign as

$$(1_{\nu'_s=0} - 1_{\nu_s=0})\left[\frac{c}{\mu} - H_s(\gamma, \nu; c)\right]$$

for all $t \in [0, \tau)$. If (IC) holds, then this drift remains non-positive for all $t \in [0, \tau)$ and all choices of ν'. This implies that for any effort process ν', u' is an $\mathcal{F}^N$-supermartingale under $P^{\nu'}$ and, thus, that ν is at least as good as ν' for the agent. If (IC) does not hold for the effort process ν, then choose ν' such that for each $t \in [0, \tau)$, $\nu'_t = \mu$ if $H_t \geq \beta_c$ and $\nu'_t = 0$ if $H_t < \beta_c$. The drift of u' is then everywhere non-negative and strictly positive over a set of $P^{\nu'}$-strictly positive measure. As a result of this, u' is an $\mathcal{F}^N$-submartingale under $P^{\nu'}$ that is not a martingale and, thus, ν is suboptimal for the agent. This concludes the proof.

C.3 Proofs in Section 4.3

C.3.1 Proof of Proposition 4.1

(i) Following contract $\gamma^c_{\mathtt{B}}(w, B, 0)$, since the promised utility process W^c_t follows (4.9) and the payment process follows (4.10), then $H^c_t = \beta_c$. Hence, $\bar{\nu} \in \mathfrak{N}(\gamma^c_{\mathtt{B}}(w, B), c)$. Meanwhile,

$$
\begin{aligned}
u\Big(\gamma^c_{\mathtt{B}}(w, B), \bar{\nu}; c\Big) &= \mathbb{E}^{\bar{\nu}}\left[\int_{0-}^{\tau} e^{-rs}(\mathrm{d}L_s - c\mathbb{1}_{\bar{\nu}_s=\mu}\mathrm{d}s)\bigg|\, \mathcal{F}^N_t\right] \\
&= B + \mathbb{E}^{\nu}\left[\int_{0+}^{\tau} e^{-rs}(\mathrm{d}L_s - c\mathbb{1}_{\bar{\nu}_s=\mu}\mathrm{d}s)\bigg|\, \mathcal{F}^N_t\right] = B + w
\end{aligned}
\tag{C.9}
$$

where the third equality follows from the definition of $\gamma^c_{\mathtt{B}}(w, B, 0)$.

(ii) By definition 4.2, we have that the promised utility of $\gamma^c_{\mathtt{P}}(\tau, z)$ follows (PK), where $H^c_t = \max\{\beta_c, z + \beta_c - W^c_t\}\mathbb{1}_{t \le \tau^N_1} + \beta_c \mathbb{1}_{t > \tau^N_1}$, $dL^c_t = cdt + (W^c_t + H^c_t - \bar{w}_c)^+ dN_t$, and $\tau^c = \min\{t : W^c_t = 0\}$. Hence, following (IC), we have $\bar{\nu} \in \mathfrak{N}(\gamma^c_{\mathtt{P}}(\tau, z), c)$. Further, by definition of W_t, we have

$$
u\Big(\gamma^c_{\mathtt{P}}(\tau, z), \bar{\nu}; c\Big) = W^c_0.
$$

in which W^c_0 follows (4.17) if $z < \bar{w}_c$ and $\tau > \tau_z$, and (4.15) otherwise.

C.3.2 Proof of Proposition 4.2

The goal of this proposition is to solve the case $\mathcal{Z}(\{c\})$. This benchmark case differs from (4.8) with the omission of the (TT) constraint, and contributes to an important building block to our general adverse selection problem. This benchmark setting is very similar, although not identical, to the model in [ST18]. In their setting the principal does not have to reimburse the operating cost rate c in real time. That is, the constrain (LL) reduces to L^c_t monotonically non-decreasing in time t. In the following, we show the claims.

If $R \ge \beta_c$, the differential equation (4.18) with boundary condition (4.19) has a unique solution, $F_c(w)$, which is strictly concave on $[0, \bar{w}_c)$ and $F'_c(w) \ge -1$. The proof of this technical result can be directly adapted from the proof of Lemma 3 in [ST18], hence, is omitted here. Furthermore, the proof of $F_c(w) = U(\gamma^c_{\mathtt{B}}(w, 0), \bar{\nu})$ can be directly adapted from the proof of Proposition 1 in [ST18] and, hence, is omitted here.

Next, we show that it is optimal for the principal to always induce effort from the agent before contract termination.

Lemma C.1 *For any contract γ^c, define a probation period*

$$
\tau^0(\gamma^c) := \inf\{t : W_t(\gamma^c, \nu^0; c) = 0\}.
\tag{C.10}
$$

Then, for any effort process $\nu^c \in \mathfrak{N}(\gamma^c, c)$ that satisfies (IC)*, there exists a contract $\hat{\gamma}^c$ such that $\bar{\nu} \in \mathfrak{N}(\hat{\gamma}^c, c)$, $u(\gamma^c, \nu^c; c) = u(\hat{\gamma}^c, \bar{\nu}; c)$, $\tau^0(\gamma^c) = \tau^0(\hat{\gamma}^c)$ and*

$$
U(\hat{\gamma}^c, \bar{\nu}) \ge U(\gamma^c, \nu^c),
$$

as long as $R \ge \beta_c$.

Proof. Consider the contract $\gamma^c = \{L^c, \tau^c\}$ and the best response effort process $\nu^c \in \mathfrak{N}(\gamma^c, c)$ such that $H_t < \beta_c$ for $t \in \mathcal{T} \subset [\prime, \tau^{\rfloor}]$. Define $\hat{\gamma}^c = \{\hat{L}^c, \tau^c\}$ such that $\hat{H}^c_t = H^c_t$, $\hat{\ell}^c_t = \ell^c_t$ and $\hat{I}^c_t = I^c_t$ except

$\hat{H}_t^c = \beta_c$, $\hat{I}_t^c = \beta_c \mathrm{d}N_t$ for $t \in \mathcal{T}$. Following (PK), we know that $W_t(\gamma^c, \nu; c) = W_t(\hat{\gamma}^c, \bar{\nu}; c)$ for $t \in [0, \tau^c]$. Hence, $\bar{\tau}(\hat{\gamma}^c) = \bar{\tau}(\gamma^c)$. Furthermore,

$$U(\hat{\gamma}^c, \bar{\nu}) - U(\gamma^c, \nu^c) = \mathbb{E}^{\bar{\nu}}\left[\int_{t\in\mathcal{T}} e^{-rt}(R - \beta_c)\mathrm{d}N_t\right] \geq 0,$$

in which the inequality follows from $R \geq \beta_c$.

The quantity $\tau^0(\gamma^c)$ represents the time when contract γ^c terminates if there is no arrival. Proposition C.1 implies that it is without loss of generality to focus on the contracts that induce full effort from the agent. It is worth noting here that [ST18] claimed optimality of full effort contracts in their equal time discount case without providing a proof. Next, we show that $\mathcal{Z}(\{c\}) = \max_{w\geq 0} F_c(w)$.

Following Proposition C.1, we know that it is without loss of generality to focus on contracts that induce full effort from the agent. If $R \geq \beta_c$, then the rest of proof can be easily adapted from the proof of Proposition 2 in [ST18].

If $R < \beta_c$, then for any γ^c that satisfies (LL), (PK), and (IR), and $\nu^c \in \mathfrak{N}(\gamma^c, c)$,

$$\begin{aligned} U(\gamma^c, \nu^c) + w = U(\gamma^c, \nu^c) + u(\gamma^c, \nu^c; c) &= \mathbb{E}^{\nu^c}\left[\int_0^{\tau} e^{-rt}(R\mathrm{d}N_t - c\mathbb{1}_{\nu_t^c=\mu}\mathrm{d}t)\right] \\ = \mathbb{E}^{\nu^c}\left[\int_0^{\tau} e^{-rt}(R\mu\mathbb{1}_{\nu_t^c=\mu}\mathrm{d}t - c\mathbb{1}_{\nu_t^c=\mu}\mathrm{d}t)\right] &= \mu(R-\beta_c)\mathbb{E}^{\nu^c}\left[\int_0^{\tau} e^{-rt}\mathbb{1}_{\nu_t^c=\mu}\mathrm{d}t\right] \leq 0, \end{aligned}$$

which verifies $\mathcal{Z}(\{c\}) = 0$. This completes the proof.

Finally, Proposition 4.2 implies that for any contract γ^c that satisfies (LL), (PK), and (IR), $\nu^c \in \mathfrak{N}(\gamma^c, c)$, and $u(\gamma^c, \nu^c; c) = w$, we have

$$U(\gamma^c, \nu^c) \leq F_c(w) \tag{C.11}$$

C.4 Proofs in Section 4.2

C.4.1 Proof of Proposition 4.1

(i) Following contract $\gamma_{\mathtt{B}}^c(w, B)$, since the promised utility process W_t^c follows (4.9) and the payment process follows (4.10), then $H_t^c = \beta_c$. Hence, $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathtt{B}}^c(w, B), c)$. Meanwhile,

$$\begin{aligned} u\Big(\gamma_{\mathtt{B}}^c(w,B), \bar{\nu}; c\Big) &= \mathbb{E}^{\bar{\nu}}\left[\int_{0-}^{\tau} e^{-rs}(\mathrm{d}L_s - c\mathbb{1}_{\bar{\nu}_s=\mu}\mathrm{d}s)\bigg|\mathcal{F}_t^N\right] \\ &= B + \mathbb{E}^{\nu}\left[\int_{0+}^{\tau} e^{-rs}(\mathrm{d}L_s - c\mathbb{1}_{\bar{\nu}_s=\mu}\mathrm{d}s)\bigg|\mathcal{F}_t^N\right] = B + w \end{aligned} \tag{C.12}$$

where the third equality follows from the definition of $\gamma_{\mathtt{B}}^c(w, B, 0)$.

(ii) By definition 4.2, we have that the promised utility of $\gamma_{\mathtt{P}}^c(\tau, z)$ follows (PK), where $H_t^c = \max\{\beta_c, z + \beta_c - W_t^c\}\mathbb{1}_{t\leq\tau_1^N} + \beta_c\mathbb{1}_{t>\tau_1^N}$, $dL_t^c = cdt + (W_t^c + H_t^c - \bar{w}_c)^+ dN_t$, and $\tau^c = \min\{t : W_t^c = 0\}$. Hence, following (IC), we have $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathtt{P}}^c(\tau, z), c)$. Further, by definition of W_t, we have

$$u\Big(\gamma_{\mathtt{P}}^c(\tau, z), \bar{\nu}; c\Big) = W_0^c.$$

in which W_0^c follows (4.17) if $z < \bar{w}_c$ and $\tau > \tau_z$, and (4.15) otherwise.

C.5 Proof in Section 4

C.5.1 Useful Definitions and Results

Working Duration:

$$\bar{T}(\gamma,\nu) := \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}\mathbb{1}_{\nu_t=\mu}\mathrm{d}t\right], \tag{C.13}$$

which measures the agent's expected working time under contract γ when the agent chooses the effort process ν.

Societal Value:

$$S(\gamma,\nu;c) = \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(R\mathrm{d}N_t - c\mathbb{1}_{\nu_t=\mu}\mathrm{d}t)\right], \tag{C.14}$$

which measures the expected total value net of cost produced with effort ν when the agent's cost is c.

Lemma C.2 *The societal value produced is fractional to the working duration, i.e., $S(\gamma,\nu;c) = (\mu R - c)\bar{T}(\gamma,\nu)$.*

Proof:

$$\begin{aligned} S(\gamma,\nu;c) &= \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(R\mathrm{d}N_t - c\mathbb{1}_{\nu_t=\mu}\mathrm{d}t)\right] = \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-rt}(R\mu\mathbb{1}_{\nu_t=\mu}\mathrm{d}t - c\mathbb{1}_{\nu_t=\mu}\mathrm{d}t)\right] \\ &= (\mu R - c)\bar{T}(\gamma,\nu). \end{aligned}$$

Hence, for each moment the agent exerts effort, he produces an expected revenue of μR with a cost c.

C.5.2 Proof of Proposition 4.3

For any contract pair (γ^g,γ^b), denote $w_g := u(\gamma^g,\nu^g;g)$, $w_b := u(\gamma^b,\nu^b;b)$ and $\tau := \tau^0(\gamma^g)$ where $\bar{\tau}(.)$ is defined in (C.10). In the following, we establish the proof in two steps.

Step 1: Constraint is more relaxed: We prove that the constraint of $\mathcal{Z}(\{g,b\})$ implies the constraint of $\mathcal{Y}$. First, (TT) implies that

$$w_g \geq \max_{\nu} u(\gamma^b,\nu;g) \geq u(\gamma^b,\nu^b;g) = w_b + (b-g)\bar{T}(\gamma^b,\nu^b) \geq w_b, \tag{C.15}$$

where $\nu^b \in \mathfrak{N}(\gamma^b,b)$ and $\bar{T}$ is defined in (C.13).

$$w_b \geq \max_{\nu} u(\gamma^g,\nu;b) \geq u(\gamma^g,\nu^0;b) = g\int_0^{\tau} e^{-rt}\mathrm{d}t = g/r\cdot(1-e^{-r\tau}). \tag{C.16}$$

Hence, constraint (TT) implies constraint (4.22).

Step 2: Objective is higher We prove that the objective of $\mathcal{Y}$ is greater or equal to the objective of $\mathcal{Z}(\{g,b\})$.

Step 2.1: If $R > \beta_b$, then following (C.15), we have

$$w_g \geq w_b + (b-g)\bar{T}(\gamma^b,\nu^b) = w_b + \frac{(b-g)S(\gamma^b,\nu^b;b)}{\mu R - b} = w_b + \frac{(b-g)(U(\gamma^b,\nu^b)+w_b)}{\mu R - b} \tag{C.17}$$

where the first equality follows from Lemma C.2 and the last equality follows from $S(\gamma^b,\nu^b;b)=U(\gamma^b,\nu^b)+u(\gamma^b,\nu^b;b)$. Rearrange (C.17), we have

$$U(\gamma^b,\nu^b)\le \frac{(w_g-w_b)(\mu R-b)}{b-g}-w_b,\ \text{if}\ R>\beta_b. \tag{C.18}$$

Further, following C.11, we have

$$U(\gamma^b,\nu^b)\le F_b(w_b). \tag{C.19}$$

On the other hand, if $R\le\beta_b$, then

$$U(\gamma^b,\nu^b)\le F_b(w_b)=-w_b,\ \text{if}\ R\le\beta_b \tag{C.20}$$

Hence, following (C.18) - (C.20), we have

$$U(\gamma^b,\nu^b)\le \min\left\{\frac{(w_g-w_b)}{b-g}\max\{\mu R-b,0\}-w_b, F_b(w_b)\right\}, \tag{C.21}$$

which corresponds to constraints (4.24) and (4.25).

Step 2.2: Next, following Proposition C.1, there exists $\hat\gamma^g=(\hat L,\hat\tau)$ such that $\bar\nu\in\mathfrak{N}(\hat\gamma^g,g)$, $u(\hat\gamma^g,\bar\nu;g)=w_g$ and $\bar\tau(\hat\gamma^g)=\tau$. Denote $\hat W_t$ as the agent's continuation utility under contract $\hat\gamma^g$ and τ_1^g as the time of the first arrival. Then, following Lemma 4.1, we have

$$d\hat W_t=[r\hat W_t-\mu\hat H_t+g]dt-d\hat L_t, \hat H_t\ge\beta_g, \quad \text{for} \quad t<\min\{\tau_1^g,\tau\}. \tag{C.22}$$

Furthermore, denote $\hat I_{\tau_1^g}$ as the payment upon the first arrival. Thus,

$$\begin{aligned}
U(\hat\gamma^g,\bar\nu)&=\mathbb{E}^{\bar\nu}\left[\int_0^\tau e^{-rt}(RdN_t-d\hat L_t)\right]\\
&\le \mathbb{E}_{\tau_1^g}\left[e^{-r\tau_1^g}\left(R-I_{\tau_1^g}+U_{\tau_1^g}(\hat\gamma^g,\bar\nu)\right)\mathbb{1}_{\tau_1^g<\tau}-\int_0^{\min\{\tau_1^g,\tau\}}e^{-rt}g\mathrm{d}t\right]\\
&=\int_0^\tau\left[e^{-r\tau_1^g}\left(R-I_{\tau_1^g}+U_{\tau_1^g}(\hat\gamma^g,\bar\nu)\right)-\int_0^{\tau_1^g}e^{-rt}g\mathrm{d}t\right]\mu e^{-\mu\tau_1^g}d\tau_1^g\\
&-\int_\tau^\infty\int_0^\tau e^{-rt}g\mathrm{d}t\cdot\mu e^{-\mu\tau_1^g}d\tau_1^g\\
&=\left[\int_0^\tau \mu e^{-t}(R-I_t+U_t(\hat\gamma^g,\bar\nu))\mathrm{d}t-\int_0^\tau ge^{-t}\mathrm{d}t\right]
\end{aligned} \tag{C.23}$$

where the first inequality follows from that $d\hat L_t\ge 0$ and $\hat\ell_t\ge g$ for $t<\tau$ and the inequality is binding if and only if $d\hat L_t=gdt$. Finally, following Proposition 4.2 (since W_t is the state variable of the optimal control problem, we can easily generalize (C.11) to it at time t), we have

$$-I_t+U_t(\hat\gamma^g,\bar\nu)\le -I_t+F_g(\hat W_{t-}+\hat H_t-I_t)\le F_g(\hat W_{t-}+\hat H_t), \tag{C.24}$$

where the first inequality follows from that the agent's continuation utility $\hat W_t=\hat W_{t-}+H_t-I_t$ and the second inequality follows from $F_g' \geq -1$. Therefore, (C.22) - (C.24) imply that

$$U(\hat{\gamma}^g, \bar{\nu}) \leq G(w_g, \tau) \tag{C.25}$$

which further implies that

$$U(\gamma^g, \nu^g) \leq U(\hat{\gamma}^g, \bar{\nu}) \leq G(w_g, \tau). \tag{C.26}$$

To conclude, compared with the optimization problem $\mathcal{Z}(\{g, b\})$, $\mathcal{Y}$ has more relaxed constraint and higher objective. Hence, $\mathcal{Z}(\{g, b\}) \leq \mathcal{Y}$.

C.5.3 Proof of Lemma 4.2

First, we verify (iii). If $H_t = \beta_g, \forall t \in [0, \tau]$, then $W_0 = \breve{w}(\tau)$. Further since $H_t \geq \beta_g, \forall t \in [0, \tau]$, we have $W_0 \geq \breve{w}(\tau)$. Hence, if $w < \breve{w}(\tau)$, then the optimization problem (4.26) is infeasible, or, by convention, $G(w, \tau) = -\infty$.

Next we verify (i) and (ii) by solving the optimiation problem (4.26). Since $g(1 - e^{-\tau})$ is fixed when τ is given, we only need to maximize the integral $\int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]dt$. To solve the optimization problem, we can write down the Hamiltonian:

$$\mathcal{H} = e^{-t}\{\mu[R + F_g(W_t + H_t)]\} + \lambda(t)(rW_t - \mu H_t) + \eta(t)(H_t - \beta_g). \tag{C.27}$$

The optimality conditions are

$$\frac{\partial \mathcal{H}}{\partial H} = \mu e^{-t} F_g'(W_t + H_t) - \lambda(t)\mu + \eta(t) = 0, \tag{C.28}$$

$$\eta(t)(H_t - \beta_g) = 0; \quad \eta(t) \geq 0, \tag{C.29}$$

$$\frac{\partial \mathcal{H}}{\partial W} = \mu e^{-t} F_g'(W_t + H_t) + \lambda(t) r = -\lambda'(t). \tag{C.30}$$

Since the objective of the optimal control problem is jointly concave in (W_t, H_t), it is sufficient to verify the above optimality conditions.

Next, we verify (ii). If $W_0 = w \geq \hat{w}(\tau)$, then $W_t + H_t = \mathrm{z} + \beta_g, \forall t \in [0, \tau]$, where $\mathrm{z} + \beta_g = w/(\mu(1 - e^{-\tau}))$. We can easily verify the optimality conditions (C.28) - (C.30) by letting

$$\lambda(t) = F_g'(\mathrm{z} + \beta_g)e^{-t} \quad \text{and} \quad \eta(t) = 0.$$

Furthermore, we can verify that

$$W_t = \mu(\mathrm{z} + \beta_g) - \mu(\mathrm{z} + \beta_g)e^{t-\tau},$$

$$H_t = \mathrm{z} + \beta_g - W_t \geq \mathrm{z} + \beta_g - W_0 \geq w\left(\frac{1}{\mu(1 - e^{-\tau})} - 1\right) \geq \beta_g.$$

where the last inequality follows from $w = W_0 \geq \hat{w}(\tau)$.

Finally, we verify (i). If $w = W_0 \in [\breve{w}(\tau), \hat{w}(\tau))$, we firstly prove that there exists a unique $\mathrm{z} \in [0, g(1 - e^{-\tau})/(r + \mu e^{-\tau}))$ such that

$$\bar{w}_g - (\bar{w}_g - z)e^{r(\tau_z - \tau)} = w.$$

Define

$$h(z) := \bar{w}_g - (\bar{w}_g - z)e^{r(\tau_z - \tau)},$$

then we can easily obtain that

$$h(0) = \bar{w}_g - \bar{w}_g \cdot e^{-r\tau} = g/r \cdot (1 - e^{-r\tau}),$$

where the first equality follows from $\tau_0 = 0$ and

$$\lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})} h(z) = \bar{w}_g - (\bar{w}_g - z) = g(1 - e^{-\tau})/(r + \mu e^{-\tau}),$$

where the first equality follows from that $\lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})} \tau_z = \tau$. Furthermore, we have

$$\begin{aligned} h'(z) &= e^{-r\tau} e^{r\tau_z} \left(1 + (z - \bar{w}_g) r \cdot \frac{\partial \tau_z}{\partial z_1}\right) \\ &= e^{r(\tau_z - \tau)} \left(1 + (z - \bar{w}_g) \frac{r(\bar{w}_g + \beta_g)}{(z_1 + \beta_g)(\bar{w}_g - z_1)}\right) \\ &= e^{r(\tau_z - \tau)} \left(1 - \frac{r(\bar{w}_g + \beta_g)}{z + \beta_g}\right) = e^{r(\tau_z - \tau)} \frac{z}{z + \beta_g} > 0. \end{aligned}$$

Since $w \in [h(0), \lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})} h(z))$ and h is continuous, we have that there exists a unique $z \in [0, g(1 - e^{-\tau})/(r + \mu e^{-\tau}))$ such that $w = h(z)$, denoted by z. It is easy to verify that $\tau_{\mathrm{z}} < \tau$ since $\mathrm{z} < g(1 - e^{-\tau})/(r + \mu e^{-\tau})$. Hence, if we let H_t follows (4.31), then W_t follows (4.30).

We can easily verify the optimality conditions (C.28) - (C.30) by letting

$$\lambda(t) = \begin{cases} \left[\int_t^{\tau - \tau_{\mathrm{z}}} \mu e^{-\mu\xi} F_g'(W_\xi + \beta) d\xi + F_g'(\mathrm{z} + \beta_g) e^{-\mu(\tau - \tau_{\mathrm{z}})}\right] e^{-rt}, & t \in [0, \tau - \tau_{\mathrm{z}}], \\ F_g'(\mathrm{z} + \beta_g) e^{-t}, & t \in [\tau - \tau_{\mathrm{z}}, \tau], \end{cases} \tag{C.31}$$

and

$$\eta(t) = \begin{cases} \mu e^{-rt} \gamma(t), & t \in [0, \tau - \tau_{\mathrm{z}}], \\ 0, & t \in [\tau - \tau_{\mathrm{z}}, \tau], \end{cases} \tag{C.32}$$

where

$$\gamma(t) := \left[\int_t^{\tau - \tau_{\mathrm{z}}} \mu e^{-\mu\xi} F_g'(W_\xi + \beta_g) d\xi + F_g'(\mathrm{z} + \beta_g) e^{-\mu s} - e^{-\mu t} F_g'(W_t + \beta_g)\right] \geq 0,$$

and $\gamma(t)$ follows from that $\gamma(t)$ is decreasing in t and $\gamma(\tau - \tau_{\mathrm{z}}) = 0$. Furthermore, we have

$$\begin{aligned} &H_t = \beta, t \in [0, \tau - \tau_{\mathrm{z}}], \\ &H_t = \mathrm{z} + \beta_g - W_t \geq \mathrm{z} + \beta_g - W_{\tau - \tau_{\mathrm{z}}} = \beta, t \in [\tau - \tau_{\mathrm{z}}, \tau]. \end{aligned}$$

C.5.4 Proof of Lemma 4.3

Similar to the proof of Lemma 4.2, since the objective of the optimal control problem is jointly concave in (W_t, H_t), it is sufficient to verify the optimality conditions (C.28) - (C.30). We can verify (C.28) - (C.30) by simply letting

$$\lambda(t) = F_g'\left(\frac{w}{\mu}\right)e^{-t}, \eta(t) = 0,$$

and

$$W_t = w > 0, H_t = \frac{w}{\mu} - w \geq \frac{r}{\mu} \cdot \frac{g}{r} = \beta_g.$$

C.5.5 Proof of Proposition 4.4

First, we look at the function when $\bar{\tau} \in [0, 1/r)$. If $w \geq \hat{w}(\tau)$, according to the optimal solution in (4.33), we have

$$G(w,\tau) = \int_0^\tau \mu e^{-t} F_g(y^*)\mathrm{d}t + \int_0^\tau (\mu R - c)e^{-t}\mathrm{d}t, \tag{C.33}$$

where $y^* = \mathrm{z} + \beta_g = \dfrac{w}{\mu(1-e^{-\tau})}$. Hence,

$$\begin{aligned}\frac{\partial G(w,\tau)}{\partial \tau} &= \mu e^{-\tau} F_g(y^*) + \int_0^T \mu e^{-t} F_g'(y^*)\frac{\partial y^*}{\partial \tau}\mathrm{d}t + (\mu R - g)e^{-\tau} \\ &= \mu e^{-\tau} F_g(y^*) + (\mu R - g)e^{-\tau} + F_g'(y^*)\frac{-we^{-\tau}}{\mu(1-e^{-\tau})^2}\int_0^T \mu e^{-t}\mathrm{d}t \\ &= \mu e^{-\tau}(F_g(y^*) - y^* F_g'(y^*)) + (\mu R - c)e^{-\tau} \\ &= \mu e^{-\tau} y^* \left(\frac{F_g(y^*) - F_g(0)}{y^*} - F_g'(y^*)\right) + (\mu R - c)e^{-\tau} > 0.\end{aligned} \tag{C.34}$$

where the inequality follows from the concavity of F. As a result, we have

$$\frac{\partial J(w,\bar{\tau})}{\partial \bar{\tau}} = \frac{\partial \tau}{\partial \bar{\tau}}\frac{\partial G(w,\tau)}{\partial \tau} > 0, \tag{C.35}$$

where the inequality follows from (C.34) and $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and

$$\frac{\partial J(w,\bar{\tau})}{\partial w} = F_g'(y^*). \tag{C.36}$$

Hence, J is increasing in $\bar{\tau}$ when $w \geq \hat{w}(\tau)$. Next, we verify the concavity of J. Following (C.35) and (C.36), we have that the Hessian matrix of J is

$$\begin{bmatrix} \dfrac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} & \dfrac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} \\ \dfrac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} & \dfrac{\partial^2 J(w,\bar{T})}{\partial^2 \bar{T}} \end{bmatrix}, \tag{C.37}$$

where

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} = \frac{F_g''(y^*)}{\mu(1-e^{-\tau})} < 0, \tag{C.38}$$

where the inequality follows from the concavity of F_g,

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} = \frac{\partial y^*}{\partial \bar{\tau}} F_g''(y^*) = \frac{\partial y^*}{\partial \tau}\frac{\partial \tau}{\partial \bar{\tau}} F_g''(y^*) > 0, \tag{C.39}$$

where the inequality follows from $\frac{\partial y^*}{\partial \tau} < 0$, $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and the concavity of F_g, and

$$\begin{aligned}\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 \bar{\tau}} &= -\mu(1-r\bar{\tau})^{\frac{\mu}{r}} \cdot y^* \cdot F_g''(y^*) \cdot \frac{\partial y^*}{\partial \bar{\tau}} - \mu^2(1-r\bar{\tau})^{\frac{\mu-r}{r}}[F(y^*) - y^* F_g'(y^*)] \\ &\quad - (\mu R - g)\mu(1-r\bar{\tau})^{\frac{\mu-r}{r}} < 0.\end{aligned} \tag{C.40}$$

where the inequality follows from $\frac{\partial y^*}{\partial \bar{\tau}} < 0$ and $F_g(y^*) - y^* F_g'(y^*) = y^*[(F_g(y^*) - F_g(0))/y^* - F_g'(y^*)] \geq 0$ (implied by the concavity of F_g).

Further, with (C.38) - (C.40), we can show that the Hessian matrix of J is negative definite which implies that J is jointly concave when $w \geq \hat{w}(\tau)$.

Second, we look at the function if $w \in [\check{w}(\tau), \hat{w}(\tau))$. According to (4.30) and (4.31), we have

$$W_{\tau_1} = \mathrm{z} = \mu(\mathrm{z} + \beta_g)(1 - e^{\tau_1 - \tau}) = \bar{w}_g + (w - \bar{w}_g)e^{r\tau_1},$$

by denoting $\tau_1(w,\tau) := \tau - \tau_{\mathrm{z}}$ and simplifying $\tau_1(w,\tau)$ with τ_1. Further, we denote $y_1^* = \mathrm{z} + \beta_g$. Hence,

$$y_1^* = \frac{g}{\mu(r + \mu e^{\tau_1 - \tau})},$$

and $\tau_1(w,\tau)$ is the solution of

$$\frac{g}{r + \mu e^{\tau_1 - \tau}}(1 - e^{\tau_1 - \tau}) = \bar{w}_g + (w - \bar{w}_g)e^{r\tau_1}, \tag{C.41}$$

where we again simplify $\tau_1(w,\tau)$ with τ_1. Therefore,

$$G(w,\tau) = \int_0^{\tau_1(w,\tau)} \mu e^{-t} F_g\left(\bar{w}_g + (w - \bar{w}_g)e^{rt} + \beta_g\right) \mathrm{d}t + \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} F_g(y_1^*) \mathrm{d}t + \int_0^{\tau} (\mu R - g)e^{-t} \mathrm{d}t. \tag{C.42}$$

Then,

$$\begin{aligned}\frac{\partial G(w,\tau)}{\partial \tau} &= \left[\mu e^{-\tau_1} F_g\left(\bar{w}_g + (w - \bar{w}_g)e^{r\tau_1} + \beta_g\right) - \mu e^{-\tau_1} F_g(y_1^*)\right] \frac{\partial \tau_1(w,\tau)}{\partial \tau} \\ &\quad + \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} F'(y_1^*) \frac{\partial y_1^*}{\partial \tau} \mathrm{d}t + \mu e^{-\tau} F_g(y_1^*) \\ &= \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} \mathrm{d}t \cdot F_g'(y_1^*) \frac{-g\mu e^{\tau_1(w,\tau) - \tau} \cdot \left(\frac{\partial \tau_1(w,\tau)}{\partial \tau} - 1\right)}{\mu(r + \mu e^{\tau_1(w,\tau) - \tau})^2} + \mu e^{-\tau} F_g(y_1^*) + (\mu R - g)e^{-\tau},\end{aligned} \tag{C.43}$$

Since (C.41) implies that

$$\frac{\partial \tau_1(w,\tau)}{\partial \tau} - 1 = \frac{r + \mu e^{\tau_1 - \tau}}{\mu(1 - e^{\tau_1 - \tau})}, \tag{C.44}$$

we have

$$\frac{\partial G(w,\tau)}{\partial \tau} = \mu e^{-\tau}(F_g(y_1^*) - y_1^* F_g'(y_1^*)) + (\mu R - g)e^{-\tau} > 0, \tag{C.45}$$

where the inequality follows from $F_g(y_1^*) - y_1^* F_g'(y_1^*) = y_1^*[(F_g(y_1^*) - F_g(0))/y_1^* - F_g'(y_1^*)] \geq 0$. As a result, we have

$$\frac{\partial J(w,\bar{\tau})}{\partial \bar{\tau}} = \frac{\partial \tau}{\partial \bar{\tau}} \frac{\partial G(w,\tau)}{\partial \tau} > 0, \tag{C.46}$$

where the inequality follows from (C.45) and $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and

$$\frac{\partial J(w,\bar{\tau})}{\partial w} = \mu \int_0^{\tau_1} e^{-\mu t} F_g'(\bar{w}_g + (w - \bar{w}_g)e^{rt} + \beta_g)\mathrm{d}t + F_g'(y_1^*) \cdot e^{-\mu \tau_1}. \tag{C.47}$$

Hence, J is increasing in $\bar{\tau}$ when $w \in [\check{w}(\tau), \hat{w}(\tau))$. Next, we verify the concavity of J. Following (C.46) and (C.47), we have that the Hessian matrix of J is

$$\begin{bmatrix} \frac{\partial^2 J(w,\bar{T})}{\partial^2 w} & \frac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} \\ \frac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} & \frac{\partial^2 J(w,\bar{T})}{\partial^2 \bar{T}} \end{bmatrix}, \tag{C.48}$$

where

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} = \mu \int_0^{\tau_1} e^{-\mu t} F''(\bar{w}_g + (w - \bar{w}_g)e^{rt} + \beta_g)e^{rt}\mathrm{d}t + F''(y_1^*) \cdot e^{-\mu \tau_1} \cdot \frac{\partial y_1^*}{\partial w}, \tag{C.49}$$

and

$$\begin{aligned} \frac{\partial^2 J(w,\bar{\tau})}{\partial^2 \bar{\tau}} &= -\mu(1 - r\bar{\tau})^{\frac{\mu}{r}} \cdot y_1^* \cdot F_g''(y_1^*) \cdot \frac{\partial y_1^*}{\partial \bar{\tau}} - \mu^2(1 - r\bar{\tau})^{\frac{\mu - r}{r}} \cdot [F_g(y_1^*) - y_1^* F_g'(y_1^*)] \\ &\quad - (\mu R - g) \cdot \mu \cdot (1 - r\bar{\tau})^{\frac{\mu - r}{r}} < 0, \end{aligned} \tag{C.50}$$

where the inequality follows from $\frac{\partial y^*}{\partial \bar{\tau}} < 0$ and $F_g(y^*) - y^* F_g'(y^*) = y^*[(F_g(y^*) - F_g(0))/y^* - F_g'(y^*)] \geq 0$ (implied by the concavity of F_g), and

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} = -\mu \cdot \frac{\partial \tau}{\partial \bar{\tau}} \cdot e^{-\tau} \cdot y_1^* \cdot F''(y_1^*) \cdot \frac{\partial y_1^*}{\partial w}. \tag{C.51}$$

Further, the concavity of F_g and

$$\frac{\partial y_1^*}{\partial w} = \frac{\partial y_1^*}{\partial \tau_1} \cdot \frac{\partial \tau_1}{\partial w} = -e^{r\tau_1} \frac{(r + \mu e^{-(\tau - \tau_1)})^2}{g e^{-(\tau - \tau_1)} \mu (1 - e^{-(\tau - \tau_1)})} \cdot \frac{\partial y_1^*}{\partial \tau_1} > 0.$$

implies that

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} < 0, \frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} > 0. \tag{C.52}$$

Furthermore, following (C.49)-(C.51) we can show that the Hessian matrix of J is negative definite which implies that J is jointly concave when $w \in [\check{w}(\tau), \hat{w}(\tau))$. And following (C.33) and (C.42), we can show that $G(w, \tau)$ is continuously differentiable when $w = \hat{w}(\tau)$.

Finally, we show that $\lim_{\bar{\tau} \to 1/r} J(w, \bar{\tau}) = J(w, 1/r)$.

$$\lim_{\bar{\tau} \to 1/r} J(w, \bar{\tau}) = \lim_{\tau \to \infty} G\left(w, -\frac{\log(1 - r\bar{\tau})}{r}\right) \tag{C.53}$$

Since $\lim_{\tau \to \infty} \hat{w}(\tau) = g/r$, if $w < g/r$, we have $\lim_{\bar{\tau} \to 1/r} J(w, \bar{\tau}) = -\infty = J(w, \bar{\tau})$. If $w \geq g/r$, then since

$$\lim_{\tau \to \infty} z(w, \tau) = \lim_{\tau \to \infty} \frac{w}{\mu(1 - e^{-\tau})} - \beta_g = \frac{w}{\mu} - \beta_g$$

and

$$\lim_{\tau \to \infty} W_t = \lim_{\tau \to \infty} \mu(z + \beta_g)(1 - e^{t-\tau}) = w; \lim_{\tau \to \infty} H_t = \lim_{\tau \to \infty} z + \beta_g - W_t = w/\mu - w$$

Hence, following (4.35), we have $\lim_{\bar{\tau} \to \infty} J(w, \bar{\tau}) = J(w, 1/r)$ if $w \geq g/r$. This concludes the proof.

C.5.6 Proof of Lemma 4.4

First, it is clear that at optimality, either (4.24) or (4.25) holds as equality. Otherwise we can increase ξ^* to improve the objective value without violating any constraint, which contradicts optimality.

If (4.24) is binding at optimality, (4.42) holds with $w = w_b^*$, following (4.25). If (4.25) is binding, on the other hand, (4.24) implies $F_b\left(w_b^*\right) + w_b^* \geq \xi^* \geq 0$. Furthermore, (4.19) implies $F_b(0) + 0 = 0$. Finally, $F_b(w) + w$ is increasing following Lemma 4.2. Therefore, there exists a $w \in [0, w_b^*]$ such that (4.42) holds as an equality.

C.5.7 Proof of Lemma 4.5

Define

$$V_b(w) := F_b(w) + w. \tag{C.54}$$

First, we present a technical lemma.

Lemma C.3 *For any $k \geq 0$, we have*

$$\frac{V_{b_1}(k \cdot b_1)}{\mu R - b_1} = \frac{V_{b_2}(k \cdot b_2)}{\mu R - b_2}, \forall b_1, b_2 < \mu R, \tag{C.55}$$

where V_b is defined in (C.54).

Proof. Following Proposition 4.2, we have

$$\begin{aligned} V_b(w) &= F_b(w) + w = U(\gamma_{\mathtt{B}}^b(w, 0), \bar{\nu}) + w \\ &= S(\gamma_{\mathtt{B}}^b(w, 0), \bar{\nu}; b) = (\mu R - b)\bar{T}(\gamma_{\mathtt{B}}^b(w, 0), \bar{\nu}) \\ &= (\mu R - b)\mathbb{E}\left[\int_0^{\tau_B^b} e^{-rt} dt\right], \end{aligned} \tag{C.56}$$

where the fourth equality follows from Lemma 9, and the fifth equality follows from (C.13). Following (4.9) and (4.11), we have for any b,

$$dW_t^b = [r(W_{t-}^b - b) + \min\{b/r - W_{t-}^b, b/\mu\}dN_t]\mathbb{1}_{W_{t-}^b \geq 0}, \tau_B^b = \min\{t : W_{t-}^b = 0\}.$$

Define process $w_t := W_t^b/b$, then we have

$$dw_t = [r(w_t - 1) + \min\{1/r - w_{t-}, 1/\mu\}dN_t]\mathbb{1}_{w_{t-} \geq 0}, \tau_B^b = \min\{t : w_{t-} = 0\},$$

Hence, for any b_1, b_2, $\tau_B^{b_1}$ and $\tau_B^{b_2}$ follows the same distribution if $W_0^{b_1}/b_1 = W_0^{b_2}/b_2$. Therefore,

$$\frac{V_{b_1}(k \cdot b_1)}{\mu R - b_1} = \mathbb{E}\left[\int_0^{\tau_B^{b_1}} e^{-rt}dt\right] = \mathbb{E}\left[\int_0^{\tau_B^{b_2}} e^{-rt}dt\right] = \frac{V_{b_2}(k \cdot b_2)}{\mu R - b_2},$$

which verifies (C.55).

Lemma C.3 implies that for any $w > 0$ and $b_1 < b_2$, we have

$$V_{b_1}(w) = V_{b_2}\left(w\frac{b_2}{b_1}\right)\frac{\mu R - b_1}{\mu R - b_2} > V_{b_2}(w)\frac{\mu R - b_1}{\mu R - b_2} > V_{b_2}(w) \tag{C.57}$$

Next, we prove the desired results. For any given b, we denote the solution of optimization problem (4.38)-(4.40) as $(w_g^*(b), w_b^*(b), \bar{\tau}^*(b))$ and w_B defined in (4.43) as $w_B(b)$. If there exists $\check{b} \in [b, \mu R]$ such that $w_B(\check{b}) = 0$, then we prove by contradcition that for any $\hat{b} > \check{b}$, we have $w_B(\hat{b}) = 0$. It implies that there exists $\bar{b} \in [g, \mu R]$ such that $w_B = 0$ if and only if $w_B > 0$.

If $w_B(\hat{b}) > 0$, following (4.43), we have $\mu R > b$ and $w_g^*(\hat{b}) > w_b^*(\hat{b})$. Following (4.39) and (4.40), $(w_g^*(\hat{b}), w_b^*(\hat{b}), \bar{\tau}^*(\hat{b}))$ and $(w_g^*(\check{b}), w_b^*(\check{b}), \bar{\tau}^*(\check{b}))$ are feasible for both $b = \hat{b}$ and $b = \check{b}$. Therefore,

$$p \cdot J(w_g^*(\hat{b}), \bar{\tau}^*(\hat{b})) + (1-p)\min\left\{F_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} - w_b^*(\hat{b})\right\} > p \cdot J(w_g^*(\check{b}), \bar{\tau}^*(\check{b}))$$

$$+(1-p)w_b^*(\check{b}) \geq p \cdot J(w_g^*(\hat{b}), \bar{\tau}^*(\hat{b})) + (1-p)\min\left\{F_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g} - w_b^*(\hat{b})\right\}$$

which implies that

$$\min\left\{F_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} - w_b^*(\hat{b})\right\} > \min\left\{F_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g} - w_b^*(\hat{b})\right\}$$

which is equivalent to

$$\min\left\{V_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g}\right\} \geq \min\left\{V_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g}\right\}$$

which contradicts with

$$V_{\hat{b}}(w_b^*(\hat{b})) < V_{\check{b}}(w_b^*(\hat{b})), \text{and}, \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} < \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g}.$$

where the first inequality follows from (C.57).

C.5.8 Proof of Theorem 4.1

Following proposition 4.1, $(\gamma_{\mathtt{P}}^{g}(\tau^*, z^*), \gamma_{\mathtt{B}}^{b}(w_B, w_b^* - w_B))$ satisfy constraints (LL), (PK), (IC), and (IR). In the following, we verify (TT).

$$u(\gamma_{\mathtt{B}}^{b}(w_B, w_b^* - w_B, 0), \bar{\nu}; b) = w_b^* \geq g/r \cdot (1 - e^{-r\tau^*}) = u(\gamma_{\mathtt{P}}^{g}(\tau^*, z^*), \nu^0; b) = \max_{\nu \in \mathcal{N}} u(\gamma_{\mathtt{P}}^{g}(\tau^*, z^*), \nu; b),$$

where the first equality follows from Proposition 4.1, the first inequality follows from constraint (4.22), and the last inequality follows from $\{\nu^0\} = \mathfrak{N}(\gamma_{\mathtt{P}}^{g}(\tau^*, z^*), b)$. If $\mu R \leq b$,

$$u(\gamma_{\mathtt{P}}^{g}(\tau^*, z^*), \bar{\nu}; g) = w_g^* \geq w_b^* = \max_{\nu \in \mathcal{N}} u(\gamma_{\mathtt{B}}^{b}(0, w_b^*, 0), \nu; g),$$

where the first equality follows from proposition 4.1 and the first inequality follows from constraint (4.22). On the other hand, if $\mu R > b$,

$$\begin{aligned} u(\gamma_{\mathtt{P}}^{g}(\tau^*, z^*), \bar{\nu}; g) &= w_g^* \geq w_b^* + (b - g)\frac{\xi^*}{(\mu R - b)} = w_b^* + (b - g)\frac{V_b(w_B)}{(\mu R - b)} \\ &= w_b^* + (b - g)\bar{T}(\gamma_{\mathtt{B}}^{b}(w_B, w_b^* - w_B), \bar{\nu}) \\ &= u(\gamma_{\mathtt{B}}^{b}(w_B, w_b^* - w_B), \bar{\nu}; g) = \max_{\nu \in \mathcal{N}} u(\gamma_{\mathtt{B}}^{b}(w_B, w_b^* - w_B), \nu; g), \end{aligned}$$

where the first equality follows from proposition 4.1, the first inequality follows from constraint (4.25), the last equality follows from the following lemma.

Lemma C.4 *For any γ^b that $\bar{\nu} \in \mathfrak{N}(\gamma^b, b)$, we have $\bar{\nu} \in \mathfrak{N}(\gamma^b, g)$.*

Define bad agent's lifetime expected utility, evaluated conditionally upon the information available at time t under contract γ^b and effort process $\bar{\nu}$ as u_t^b, then

$$u_t^b = \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(\mathrm{d}L_s^b - b\mathrm{d}s)\middle|\mathcal{F}_t^N\right] = u_0^b + \int_0^t H_s^b dM_s^{\bar{\nu}},$$

where $M_t^{\bar{\nu}} = N_t - \mu t$ and $H_s^b \geq \beta_b$ for any s. Define good agent's lifetime expected utility, evaluated conditionally upon the information available at time t under contract γ^b and effort process $\bar{\nu}$ as u_t^g, then

$$\begin{aligned} u_t^g &= \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(\mathrm{d}L_s^b + (b - g)\mathrm{d}s)\middle|\mathcal{F}_t^N\right] = u_t^b + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b - g)\mathrm{d}s\middle|\mathcal{F}_t^N\right] \\ &= u_0^b + \int_0^t H_s^b dM_s^{\bar{\nu}} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b - g)\mathrm{d}s\middle|\mathcal{F}_t^N\right]. \end{aligned}$$

Next, we denote $u_t^{g'}$ as good agent's lifetime expected payoff, given the information available at time t, when he acts according to $\nu' = \{\nu_t'\}_{t\geq 0}$ until time t and then reverts to $\bar{\nu}$, then

$$\begin{aligned}
u_t^{g'} &= u_t^g + \int_0^{t\wedge\tau-} e^{-rs}(1-\mathbb{1}_{\nu_s'=\mu})g ds \\
&= u_0^b + \int_0^{t\wedge\tau-} H_s^b dM_s^{\bar{\nu}} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] + \int_0^{t\wedge\tau-} e^{-rs}(1-\mathbb{1}_{\nu_s'=\mu})g ds \\
&= u_0^b + \int_0^{t\wedge\tau^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] + \int_0^{t\wedge\tau^b-} e^{-rs}\mu(1-\mathbb{1}_{\nu_s'=\mu})(\beta_g - H_s^b)ds,
\end{aligned}$$

Then, for any $t' > t$,

$$\begin{aligned}
\mathbb{E}^{\nu'}[u_{t'}^{g'}|\mathcal{F}_t^N] &= \mathbb{E}^{\nu'}\left[u_0^b + \int_0^{t'\wedge\tau^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_{t'}^N\right]\right. \\
&\quad \left. + \int_0^{t'\wedge\tau^b-} e^{-rs}\mu(1-\mathbb{1}_{\nu_s'=\mu})(\beta_g - H_s^b)ds \middle| \mathcal{F}_t^N\right] \\
&= u_0^b + \int_0^{t\wedge\tau^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] \\
&\quad + \mathbb{E}^{\nu'}\left[\int_0^{t'\wedge\tau^b-} e^{-rs}\mu(1-\mathbb{1}_{\nu_s'=\mu})(\beta_g - H_s^b)ds \middle| \mathcal{F}_t^N\right] \\
&\leq u_0^b + \int_0^{t\wedge\tau^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] \\
&\quad + \int_0^{t\wedge\tau^b-} e^{-rs}\mu(1-\mathbb{1}_{\nu_s'=\mu})(\beta_g - H_s^b)ds = u_t^{g'},
\end{aligned}$$

where the second equality follows from law of iterated expectation and the first inequality follows from that $\mu(1-\mathbb{1}_{\nu_s'=\mu})(\beta_g - H_s^b) \leq 0, \forall t$. Hence, $u_t^{g'}$ is $\mathcal{F}^N$-supermartingale under $P^{\nu'}$. Therefore, by the optional sampling theorem ([DM11], Chapter VI, Theorem 10),

$$u(\gamma^b, \bar{\nu}; g) = u_0^{g'} \geq \mathbb{E}^{\nu'}[u_\tau^{g'}] = u(\gamma^b, \nu'; g).$$

which implies that $\bar{\nu}$ is at least as good as ν' for the agent.

C.5.9 Proof of Proposition 4.5

Following Proposition 4.2, we have if $\beta_b \geq R$, then $U(\gamma^b, \nu^b) \leq -w$ which implies that $w_*^b = 0$. As a result, $w_b^* \geq w_*^b = 0$.

If $\beta_b \geq R$, then in the optimization problem (4.4), $\xi \leq 0$. Hence, the optimization problem (4.4) becomes

$$\begin{aligned}
\max_{w_g, w_b, \bar{\tau}} \quad & p \cdot J(w_g, \bar{\tau}) - (1-p)w_b \\
& w_g \geq w_b \geq g \cdot \bar{\tau}, \\
& \bar{\tau} \geq 0, \bar{\tau} \leq 1/r.
\end{aligned}$$

Since the objective function is decreasing in w_b, we can let $w_b = g \cdot \bar{\tau}$. Hence, optimization problem (4.4) becomes

$$\begin{aligned} \max_{w_g, \bar{\tau}} \quad & p \cdot J(w_g, \bar{\tau}) - (1-p)g \cdot \bar{\tau} \\ & w_g \geq g \cdot \bar{\tau}, \\ & \bar{\tau} \geq 0, \bar{\tau} \leq 1/r. \end{aligned}$$

We firstly consider the following optimization problem (corresponding dual variable defined after each constraint):

$$\begin{aligned} \max_{w_g, \bar{\tau}} \quad & p \cdot J(w_g, \bar{\tau}) - (1-p)g \cdot \bar{\tau} \\ & \bar{\tau} \geq 0; \eta_1, \\ & w_g \geq g \cdot \bar{\tau}; \eta_2, \\ & w_g \leq \bar{w}_g = g/r; \eta_3. \end{aligned} \tag{C.58}$$

The Lagrangian of the above optimization problem is given as

$$\mathcal{L} = pJ(w_g, \bar{\tau}) - (1-p)g\bar{\tau} + \eta_1\bar{\tau} + \eta_2(w_g - g\bar{\tau}) + \eta_3(\bar{w}_g - w_g)$$

Optimality condition requires

$$\begin{aligned} & \eta_1 \geq 0, \eta_1\bar{\tau} = 0, \\ & \eta_2 \geq 0, \eta_2(w_g - g\bar{\tau}) = 0, \\ & \eta_3 \geq 0, \eta_3(\bar{w}_g - w_g) = 0, \\ & \frac{\partial \mathcal{L}}{w_g} = pJ_1 + \eta_2 - \eta_3 = 0, \\ & \frac{\partial \mathcal{L}}{\partial \bar{\tau}} = pJ_2 - (1-p)g + \eta_1 - \eta_2 g = 0. \end{aligned}$$

If $\eta_3 > 0$, then $w_g = \bar{w}_g$. Hence, $J_1 = F_g'(y^*) = -1$ (following (C.36)). Hence, $\eta_3 = -\eta_2 - pJ_1 < 0$ which leads to a contradiction. Hence, $\eta_3 = 0$. We further consider the following two cases:

1. If $\eta_2 > 0$, then $w = g\bar{\tau}$. Hence, $\eta_2 = -pJ_1(w, w/g) > 0$ and $\eta_1 = -pJ_1(w, w/g)g - pJ_2(w, w/g) + (1-p)g = -pgF_g'(w) + (1-p)g \geq 0$. Hence, in this case, the optimal solution will be either $w_g^* = \bar{\tau}^* = 0$ or $w_g^* = w^{**}$ and $\bar{\tau}^* = w^{**}/g$ where $F_g'(w^{**}) = (1-p)/p$. In both cases, $w_g^* < w_*^g$

2. If $\eta_2 = 0$, then we need to find $w_g^*, \bar{\tau}^*$ such that

$$\begin{aligned} & \eta_2 = -pJ_1(w_g^*, \bar{\tau}^*) = 0, \\ & \eta_1 = (1-p)g - pJ_2, \\ & \eta_1\bar{\tau} = 0. \end{aligned}$$

   Since $J_1(w_g^*, \bar{\tau}^*) = 0$, we have $\bar{\tau}^* > 0$ and $\eta_1 = 0$. (If $\eta_1 > 0$ and $\bar{\tau}^* = 0$, then $J_1(w_g^*, 0) = -1$) Hence, we require $J_1(w_g^*, \bar{\tau}^*) = 0$ and $J_2(w_g^*, \bar{\tau}^*) = (1-p)g/p$.

   Finally, we show that $w_g^* < w_*^g$. If $w_*^g > 0$, then $w_*^g = \{w : F_g'(w) = 0\}$. For any $w \geq w_*^g$, $F_g'(w) = J_1(w, w/g) + 1/g \cdot J_2(w, w/g) \leq 0$. Further, since $J_2(w, w/g) > 0$ (following (C.46)), we have $J_1(w, w/g) < 0$. Furthermore, since $J_{12} > 0$ (following (C.52)), we have $J_1(w, \bar{\tau}) < J_1(w, w/g) < 0$ for any $w \geq w_g^*$ and $\bar{\tau} \leq w/g$. In this case, $w_g^* < w_*^g$.

On the other hand, if $w_*^g = 0$, then $F_g'(0) \leq 0$ which further implies that $F_g'(w) = J_1(w, w/g) + 1/g \cdot J_2(w, w/g) \leq 0$ for any $w \geq 0$. Again, since $J_2(w, w/g) > 0$(following (C.46)), we have $J_1(w, w/g) < 0$. Furthermore, since $J_{12} > 0$ (following (C.52)), we have $J_1(w, \bar{\tau}) < J_1(w, w/g) < 0$ for any $w \geq 0$ and $\bar{\tau} \leq w/g$. In this case, $(w_g^*, \bar{\tau}^*)$ does not exist.

Hence, $w_g^* \leq w_*^g$.

Since $w \leq w_*^g < \bar{w}_g$ in the optimization problem (C.58), the constraint $w \leq \bar{w}_g$ is redundant. Hence, in the optimization problem (4.4), we have if $\beta_b \geq R$, $w_g^* \leq w_*^g$. This concludes the proof.

C.6 Proofs in Section 5

C.6.1 Proof of Lemma 4.6

(i) Following Proposition 4.4, $\mathcal{J}(w, \bar{\tau}, c)$ is jointly concave in $(w, \bar{\tau})$.

(ii) Following Proposition 4.4, $\mathcal{J}(w, \bar{\tau}, c)$ is is increasing in $\bar{\tau}$. If $\bar{\tau} = 0$, then $\tau = -\log(1 - r\bar{\tau})/r = 0$. Hence, $\hat{w}(0) = 0$. Hence, for any $w \geq 0$, following (C.33), we have

$$\begin{aligned}\mathcal{J}(w, 0, c) &= \lim_{\tau \to 0} \int_0^\tau \mu e^{-t} F_g\left(\frac{w}{\mu(1-e^{-\tau})}\right) dt = \lim_{\tau \to 0} \int_0^\tau \mu e^{-t}\left[\frac{\mu R - c}{r} - \frac{w}{\mu(1-e^{-\tau})}\right] dt \\ &= \lim_{\tau \to 0} -\mu(1-e^{-\tau}) \frac{w}{\mu(1-e^{-\tau})} = -w.\end{aligned}$$

(iii) It is equivalent to show that $\mathcal{J}_1'(w, \bar{\tau}, c) = J_1'(w, \bar{\tau}) \geq -1$. First, J is concave in w. Hence, we only need to show $J_1'(w, \bar{\tau}) \geq -1$ when w is large enough. Denote $\tau := \dfrac{\log(1 - r\bar{\tau})}{r}$. Following (C.36), we have, for $w \geq \hat{w}(\tau)$, $J_1'(w, \bar{\tau}) = F_c'\left(\dfrac{w}{\mu(1-e^{-\tau})}\right) \geq -1$.

(iv) Following (ii) and (iii), we have

$$\mathcal{J}(w, \bar{\tau}, c) + w \geq \mathcal{J}(w, 0, c) + w = 0.$$

Again, following (ii) and definition of $\mathcal{J}$, we have

$$\mathcal{J}(w, \bar{\tau}, c) + w \leq \mathcal{J}\left(w, \frac{w}{c}, c\right) + w = U(\gamma_{\mathrm{P}}^c(\tau, 0), \bar{\nu}) = U(\hat{\gamma}^c, \bar{\nu}) + w = F_c(w) + w = V_c(w) \leq \frac{\mu R - c}{r},$$

where $\tau := \dfrac{\log(1 - r\bar{\tau})}{r}$ and $\bar{\tau} = \dfrac{w}{c}$, the last inequality follows from 4.2.

C.6.2 Proof of Theorem 4.2

For any contract menu $\Gamma_{\mathcal{C}}, \mathcal{C} = [\underline{c}, \bar{c}]$, denote $\mathsf{w}(c) := u(\gamma^c, \nu^c; c)$, and $\tau(c) := \tau^0(\gamma^c)$ for any $c \in [\underline{c}, \bar{c}]$ where $\tau^0(.)$ is defined in (C.10).

Step 1: Constraint is more relaxed: We prove that the constraint of $\mathcal{Z}(\mathcal{C})$ implies the constraint of $\mathcal{Y}^C$. First, (TT) implies that for any $c_1 < c_2 \in \mathcal{C}$,

$$\mathsf{w}(c_1) \geq \max_\nu u(\gamma^{c_2}, \nu; c_1) \geq u(\gamma^{c_2}, \nu^{c_2}; c_1) = \mathsf{w}(c_2) + (c_2 - c_1)\bar{T}(\gamma^{c_2}, \nu^{c_2}) \geq \mathsf{w}(c_2), \tag{C.59}$$

where $\nu^{c_2} \in \mathfrak{N}(\gamma^{c_2}, c_2)$ and $\bar{T}$ is defined in (C.13). Hence, (TT) implies that $\mathsf{w}(c)$ should be non-increasing.

Furthermore, for any $c < \bar{c}$, we have

$$\mathsf{w}(\bar{c}) \geq \max_{\nu} u(\gamma^c, \nu; \bar{c}) \geq u(\gamma^c, \nu^0; \bar{c}) = c \int_0^{\tau} e^{-rt} \mathrm{d}t = c/r \cdot (1 - e^{-r\tau}). \tag{C.60}$$

Step 2: Objective is higher We prove that the objective of $\mathcal{Y}^{\mathcal{C}}$ is greater or equal to the objective of $\mathcal{Z}(\mathcal{C})$.

If $R > \beta_c$, then for any $\tilde{c} < c$, following (C.59), we have

$$\mathsf{w}(\tilde{c}) \geq \mathsf{w}(c) + (c - \tilde{c})\bar{T}(\gamma^c, \nu^c) = \mathsf{w}(c) + \frac{(c - \tilde{c})S(\gamma^c, \nu^c; c)}{\mu R - c} = \mathsf{w}(\tilde{c}) + \frac{(c - \tilde{c})(U(\gamma^c, \nu^c) + w_c)}{\mu R - c}, \tag{C.61}$$

where the first equality follows from Lemma C.2 and the last equality follows from $S(\gamma^c, \nu^c; c) = U(\gamma^c, \nu^c) + u(\gamma^c, \nu^c; c)$. Rearrange (C.61), we have, for any $\tilde{c} < c$,

$$U(\gamma^c, \nu^c) \leq \frac{(\mathsf{w}(\tilde{c}) - \mathsf{w}(c))(\mu R - c)}{c - \tilde{c}} - \mathsf{w}(c), \text{ if } R > \beta_c. \tag{C.62}$$

Hence, for any $c < \mu R$, we have

$$U(\gamma^c, \nu^c) \leq \inf_{\tilde{c} < c} \left[\frac{\mathsf{w}(\tilde{c}) - \mathsf{w}(c)}{c - \tilde{c}} \right] (\mu R - c) - \mathsf{w}(c). \tag{C.63}$$

On the other hand, if $R \leq \beta_c$ ($c \geq \mu R$), then

$$U(\gamma^c, \nu^c) \leq F_c(\mathsf{w}(c)) = -\mathsf{w}(c), \text{ if } R \leq \beta_c. \tag{C.64}$$

Finally, following Step 2.2 of the proof of Proposition 4.3 and definition of $\mathcal{J}$, we have for any $c < \mu R$,

$$U(\gamma^c, \nu^c) \leq \mathcal{J}(\mathsf{w}(\mathsf{c}), \tau(\mathsf{c}), \mathsf{c}) \leq \mathcal{J}\left(\mathsf{w}(\mathsf{c}), \min\left\{\frac{\mathsf{w}(\bar{\mathsf{c}})}{\mathsf{c}}, \frac{1}{\mathsf{r}}\right\}, \mathsf{c}\right), \tag{C.65}$$

where the second inequality follows from that the function $\mathcal{J}(w, \bar{\tau}, c)$ is increasing in $\bar{\tau}$ (Lemma 4.6(ii)) and (C.61). Therefore, (C.63) and (C.65) imply that for any $c < \mu R$,

$$U(\gamma^c, \nu^c) \leq \xi\big(c; \mathsf{w}(\cdot)\big). \tag{C.66}$$

With (C.64), we established that the objective of $\mathcal{Y}^{\mathcal{C}}$ is higher than the objective of $\mathcal{Z}(\mathcal{C})$. To conclude, compared with the optimization problem $\mathcal{Z}(\mathcal{C})$, $\mathcal{Y}^{\mathcal{C}}$ has more relaxed constraint and higher objective. Hence, $\mathcal{Z}(\mathcal{C}) \leq \mathcal{Y}^{\mathcal{C}}$.

C.6.3 Proof of Proposition 4.6

Define

$$y(N, \mathsf{w}(.)) := \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] \min \left\{ \mathcal{J}\left(\mathsf{w}(c_i), \min\left\{ \frac{\mathsf{w}(\bar{c})}{c_i}, \frac{1}{r} \right\},\ c_i \right) \right. \tag{C.67}$$

$$\left. ,\ \inf_{\tilde{c}<c_i} \left[\frac{\mathsf{w}(\tilde{c}) - \mathsf{w}(c_i)}{\delta} \right] (\mu R - c_i) - \mathsf{w}(c_i) \right\} - \mathsf{w}(c_N) \int_{\min\{\mu R, \bar{c}\}}^{\bar{c}} \rho(c) dc, \tag{C.68}$$

Following (4.50), we have

$$\mathcal{Y}^c = \sup_{\mathsf{w}(.)} \quad \lim_{N\to\infty} y(N, \mathsf{w}(.)),$$
$$s.t\ \mathsf{w}(c) \text{ is non-increasing in } c. \tag{C.69}$$

As a result, we have for any $\epsilon > 0$, there exists non-increasing $\tilde{\mathsf{w}}(c)$ such that

$$\mathcal{Y}^c \le \lim_{N\to\infty} y(N, \tilde{\mathsf{w}}(.)) + \epsilon,$$

Since for any N,

$$y(N, \tilde{\mathsf{w}}(.)) \le \hat{y}(N),$$

we have

$$\lim_{N\to\infty} y(N, \tilde{\mathsf{w}}(.)) = \liminf_{N\to\infty} y(N, \tilde{\mathsf{w}}(.)) \le \liminf_{N\to\infty} \hat{y}(N).$$

Hence, for any $\epsilon > 0$, we have

$$\mathcal{Y}^c \le \liminf_{N\to\infty} \hat{y}(N) + \epsilon,$$

which finally implies (4.54).

C.6.4 Proof of Proposition 4.7

First, we show that $\mathfrak{J}_i(w|w_N)$ is concave in w by induction. $\mathfrak{J}_0(w|w_N) = 0$ is clearly concave in w. Next, if $\mathfrak{J}_{i-1}(w|w_N)$ is concave in w, we verify that $\mathfrak{J}_i(w|w_N)$ is also concave in w.

Denote

$$f(w_{i-1}, w_i) := [P(c_i) - P(c_{i-1})] \min\left\{ \frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i,\ \mathcal{J}\left(w_i, \min\left\{ \frac{w_N}{c_i}, \frac{1}{r} \right\}, c_i \right) \right\}$$
$$+ \mathfrak{J}_{i-1}(w_{i-1}|w_N).$$

Since $\frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i$ is linear in (w_{i-1}, w_i) and $\mathcal{J}\left(w_i, \min\left\{ \frac{w_N}{c_i}, \frac{1}{r} \right\}, c_i \right)$ is concave in w_i (follows Lemma 4.6), then $G(w_{i-1}, w_i)$ is jointly concave in (w_{i-1}, w_i). Hence, $\mathfrak{J}_i(w_i|w_N)$ is concave in w_i.

Since $\mathfrak{J}_{i-1}(w_{i-1}|w_N)$ is concave in w_{i-1}, $\breve{w}$ and $\hat{w}$ are well-defined and $\breve{w} \le \hat{w}$. Next, we verify the optiaml solution in the following 3 cases. Further, following Lemma 4.6 (iii) and $c_i < \mu R$, we have

Case 1. If $w_i \le \breve{w} - \bar{u}\delta$, then we verify that $w^*_{i-1} = \breve{w}$. If $w_{i-1} \ge \breve{w}$, then $w_{i-1} \ge \breve{w} \ge w_i + \bar{u}\delta$.

Hence

$$f(w_{i-1}, w_i) = [P(c_i) - P(c_{i-1})]\mathcal{J}\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right) + \mathfrak{J}_{i-1}(w_{i-1}|w_N),$$

and

$$f_1'(w_{i-1}, w_i) = \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \leq 0,$$

for $w_{i-1} \geq \check{w}$, where the last inequality follows from the definition of $\check{w}$. If $w_{i-1} < \check{w}$, then

$$f_1'(w_{i-1}, w_i) = \begin{cases} \frac{[P(c_i)-P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}_{i-1}'(w_{i-1}|w_N) > 0, & \text{if } w_{i-1} \leq w_i + \bar{u}\delta, \\ \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \geq 0, & \text{if } w_i \in (w_i + \bar{u}\delta,\ \check{w}], \end{cases}$$

where the second inequality follows from the definition of $\check{w}$. Hence, $f(w_{i-1}, w_i)$ is increasing in w_{i-1} if $w_{i-1} < \check{w}$ and decreasing in w_{i-1} if $w_{i-1} \geq \check{w}$which imply that $w_{i-1}^* = \check{w}$.

Case 2. If $w_i \in (\check{w} - \bar{u}\delta, \hat{w} - \bar{u}\delta]$, then we verify that $w_{i-1}^* = w_i + \bar{u}\delta$. For $w_{i-1} < w_i + \bar{u}\delta$,

$$f_1'(w_{i-1}, w_i) = \frac{[P(c_i) - P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \geq 0,$$

where the inequality follows from $w_{i-1} < w_i + \bar{u}\delta \leq \hat{w}$. Further, for $w_{i-1} > \check{w} + \bar{u}\delta$,

$$f_1'(w_{i-1}, w_i) = \mathfrak{J}_{i-1}'(w_{i-1}|w_N) < 0,$$

where the inequality follows from $w_{i-1} > w_i + \bar{u}\delta > \check{w}$.

Case 3. If $w_i \in (\hat{w} - \bar{u}\delta, \hat{w}]$, then we verify that $w_{i-1}^* = \hat{w}$. For $w_{i-1} < \hat{w} < w_i + \bar{u}\delta$, we have

$$f_1'(w_{i-1}, w_i) = \frac{[P(c_i) - P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \geq 0,$$

where the inequality follows from $w_{i-1} < \hat{w}$. And for $w_{i-1} > \hat{w}$, then

$$f_1'(w_{i-1}, w_i) = \begin{cases} \frac{[P(c_i)-P(c_{i-1})](\mu R - c_i)}{\delta} + \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \leq 0, & \text{if } w_{i-1} \in (\hat{w}, w_i + \bar{u}\delta], \\ \mathfrak{J}_{i-1}'(w_{i-1}|w_N) \leq 0, & \text{if } w_{i-1} > w_i + \bar{u}\delta, \end{cases} \quad \text{(C.70)}$$

where the first inequality follows from $w_{i-1} > \hat{w}$. Hence, $f(w_{i-1}, w_i)$ is increasing in w_{i-1} if $w_{i-1} < \hat{w}$ and decreasing in w_{i-1} if $w_{i-1} \geq \hat{w}$ which imply that $w_{i-1}^* = \hat{w}$.

Case 4. If $w_i > \hat{w}$, we verify that $w_{i-1}^* = w_i$. Folloing (C.70), we have $f(w_{i-1}, w_i)$ is decreasing in w_{i-1} for $w_{i-1} \geq w_i$. Hence, $w_{i-1}^* = w_i$.

C.6.5 Proof of Proposition 4.8

First, if $w_N \geq \dfrac{\mu R - \underline{c}}{r}$, then

$$\begin{aligned}\mathfrak{J}_N(w_N|w_N) &= \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})] \min\left\{\frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i,\ \mathcal{J}\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ &\leq \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})]\mathcal{J}\left(w_i, \min\left\{\frac{w_N}{c_i}, \frac{1}{r}\right\}, c_i\right) \\ &\leq \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})]\left[\frac{\mu R - c_i}{r} - w_i\right] \leq 0 \leq \mathfrak{J}_N(0|0),\end{aligned}$$

where the last inequality follows from $w_i \geq w_N \geq \dfrac{\mu R - \underline{c}}{r}$.

On the other hand, for any $w_N > \bar{c}/r$, then denote the corresponding optimal solution as $\{w_i^*\}_{i=0,\dots,N}$. Since $w_N > \bar{c}/r$, we have $w_i^* > \bar{c}/r \geq c_i/r$ for $i = 0, \dots, N$. Hence, $\min\{w_N/c_i, 1/r\} = 1/r$ for $i = 1, \dots, N$.

Following (C.36), we have

$$\frac{\partial \mathcal{J}(w, 1/r, c)}{\partial w} = -1, \tag{C.71}$$

if $w \geq \hat{w}(\infty) = c/r$ and $w \geq \mu c/r$. Hence,

$$\mathcal{J}(w, 1/r, c_i) \leq \mathcal{J}(\bar{c}/r, 1/r, c), \tag{C.72}$$

for any $i \in \{0, \dots, N\}$ and $w > \bar{c}/r$. Define $\{\tilde{w}_i\}_{i=0,\dots,N}$ as

$$\tilde{w}_i = w_i^* - (w_N^* - \bar{c}/r).$$

Hence, if $w_N \geq \bar{c}/r$,

$$\begin{aligned}\mathfrak{J}_N(w_N|w_N) &= \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})] \min\left\{\frac{w_{i-1}^* - w_i^*}{\delta}(\mu R - c_i) - w_i^*,\ \mathcal{J}\left(w_i^*, \min\left\{\frac{w_N^*}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ &\leq \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})] \min\left\{\frac{\tilde{w}_{i-1} - \tilde{w}_i}{\delta}(\mu R - c_i) - \tilde{w}_i,\ \mathcal{J}\left(\tilde{w}_i, \min\left\{\frac{\tilde{w}_N}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ &\leq \sum_{i=1}^{N}[P(c_i) - P(c_{i-1})]\left[\frac{\mu R - c_i}{r} - w_i\right] \leq 0 \leq \mathcal{J}_N(0|0) \leq \mathfrak{J}_N(\bar{c}/r|\bar{c}/r),\end{aligned}$$

where the first inequality follows from (C.72). Therefore,

$$\mathfrak{J}_N(w_N|w_N) \leq \mathfrak{J}_N(\bar{\mathrm{w}}|\bar{\mathrm{w}}).$$

C.6.6 Proof of Lemma 4.7

Since $c \leq c_i$, $\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i)$ satisfies (LL) for $c \in (c_{i-1}, c_i]$. Then, following the defition of $\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i)$ and γ_{B}^{c}, the menu of contracts $\hat{\Gamma}_{\mathcal{C}}^{\mathsf{w}}$ satisfies (LL), (PK), (IC) and (IR). Hence, what is left is to verify (TT).

First, we present a technical lemma to show that for any $c \leq c_i$, $\bar{\nu}$ is type c agent's best response if he takes contract $\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i)$.

Lemma C.5 *For any $c \leq c_i$, we have $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), c)$.*

The proof of this lemma is the same as the proof of Lemma C.4, which is omitted here. Next, we verify (TT) for type c_i, $i = 0, ..., N$. Given any c_j such that $\underline{c} \leq c_i < c_j \leq c_N$. We have

$$\begin{aligned}
&\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c_i\right) = u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_i\right) = u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_j\right) + (c_j - c_i)\bar{\tau}(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}) \\
&= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_j\right) + (c_j - c_i)\bar{\tau}_{\mathsf{w}}^i \leq \mathsf{w}_j + \mathsf{w}_i - \mathsf{w}_j = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c_i\right), \qquad \text{(C.73)}
\end{aligned}$$

where the first equality follows from Lemma C.5, the second equality follows from (C.13), the third equality follows from Lemma C.2, and the first inequality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (4.57). On the other hand, given any type c_j such that $\underline{c} \leq c_j < c_i \leq c_N$,

$$\begin{aligned}
\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c_i\right) &= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu^0; c_i\right) = c_j \bar{\tau}_{\mathsf{w}}^j \leq c_j \min\left\{\frac{\mathsf{w}_N}{c_j}, \frac{1}{r}\right\} \leq \mathsf{w}_N \leq \mathsf{w}_i \\
&= u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c_i\right), \qquad \text{(C.74)}
\end{aligned}$$

where the first inequality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (4.57), and the third inequality follows from that w is non-increasing. Furthermore,

$$\max_{\nu} u\left(\gamma_{\mathsf{B}}^{c}(0, w_N, 0), \nu; c_i\right) = \mathsf{w}_N \leq \mathsf{w}_i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c_i\right), \qquad \text{(C.75)}$$

where the inequality follows from that w is non-increasing. Hence, (C.73)-(C.75) imply that type c_i would not mimic any other type.

Before we consider a general type $c \in (c_{i-1}, c_i]$, we present a technical lemma.

Lemma C.6 *For any $k_0 \geq \bar{\tau} \geq 0$,*

$$\frac{\mathcal{V}(k_0 c_1, \bar{\tau}; c_1)}{\mu R - c_1} = \frac{\mathcal{V}(k_0 c_2, \bar{\tau}; c_2)}{\mu R - c_2}; \quad \forall c_1, c_2 < \mu R. \qquad \text{(C.76)}$$

Proof: Let $\tau := -\log(1 - r\bar{\tau})/r$. First, $k_0 c_1 \geq \hat{w}_{c_1}(\tau)$ is equivalent to $k_0 c_2 \geq \hat{w}_{c_2}(\tau)$. Further,

$$\frac{z_{c_1}(k_0 c_1, \tau)}{z_{c_2}(k_0 c_2, \tau)} = \frac{c_1}{c_2}.$$

Similarly, $k_0 c_1 \in [\check{w}_{c_1}(\tau), \hat{w}_{c_1}(\tau))$ is equivalent to $k_0 c_2 \in [\check{w}_{c_2}(\tau), \hat{w}_{c_2}(\tau))$. Further, $\tau_z(c_1) = \tau_z(c_2)$ and

$$\frac{z_{c_1}(k_0 c_1, \tau)}{z_{c_2}(k_0 c_2, \tau)} = \frac{c_1}{c_2}.$$

Hence, for any agent c with $w = k_0 c$, $\bar{\tau}$, and let $\tau^P(w, \bar{\tau}; c)$ be the stochastic stopping time that an agent with cost c and initial promised utility $k_0 c$ and probation length $\bar{\tau}$ is terminated, when exerting full effort in a probation contract. This implies that $\tau^P(w, \bar{\tau}; c)$ are identically distributed for any c. Therefore,

$$\frac{\mathcal{V}(k_0 c, \bar{\tau}; c)}{\mu R - c}; \quad \forall c < \mu R,$$

is a constant.

Now consider an agent with type $c \in (c_{i-1}.c_i]$, given $c_j > c_i$, then

$$\begin{aligned}
&\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c\right) = u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c\right) = u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \bar{\nu}; c_j\right) + (c_j - c)\bar{\tau}_{\mathsf{w}}^j \\
&= \mathsf{w}_j + (c_j - c)\bar{\tau}_{\mathsf{w}}^j \leq \mathsf{w}_j + \mathsf{w}_i - \mathsf{w}_j + (c_i - c)\bar{\tau}_{\mathsf{w}}^j \\
&= \mathsf{w}_i + (c_i - c)\min\left\{\frac{\mathcal{V}(\mathsf{w}_j, \min\{\mathsf{w}_N/c_j, 1/r\}, c_j)}{\mu R - c_j}, \inf_{c_k: k<j, k\in 1:N}\left[\frac{\mathsf{w}_k - \mathsf{w}_j}{c_j - c_k}\right]\right\} \\
&= \mathsf{w}_i + (c_i - c)\min\left\{\frac{\mathcal{V}(\mathsf{w}_j \cdot c_i/c_j, \min\{\mathsf{w}_N/c_j, 1/r\}, c_i)}{\mu R - c_i}, \inf_{c_k: k<i, k\in 1:N}\left[\frac{\mathsf{w}_k - \mathsf{w}_i}{c_i - c_k}\right]\right\} \\
&\leq \mathsf{w}_i + (c_i - c)\min\left\{\frac{\mathcal{V}(\mathsf{w}_i, \min\{\mathsf{w}_N/c_i, 1/r\}, c_i)}{\mu R - c_i}, \inf_{c_k: k<i, k\in 1:N}\left[\frac{\mathsf{w}_k - \mathsf{w}_i}{c_i - c_k}\right]\right\} \\
&= \mathsf{w}_i + (c_i - c)\bar{\tau}_{\mathsf{w}}^i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c\right) \leq \max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \nu; c\right),
\end{aligned} \tag{C.77}$$

where the first equality follows from Lemma C.5, the second equality follows from (C.13), the third equality follows from Lemma C.2, the first inequality and the forth equality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (4.57), the fifth equality follows from Lemma C.6, the second inequality follows from that $\mathcal{V}$ is increasing in w and $\bar{\tau}$, and the sixth equality follows from (C.13). Meanwhile, given $c_j < c_i$,

$$\begin{aligned}
\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c\right) &= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu^0; c\right) = c_j\bar{\tau}_{\mathsf{w}}^j \leq c_j \min\left\{\frac{\mathsf{w}_N}{c_j}, \frac{1}{r}\right\} \\
&\leq \mathsf{w}_N \leq \mathsf{w}_i \leq \mathsf{w}_i + (c_i - c)\bar{\tau}_{\mathsf{w}}^i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c\right),
\end{aligned} \tag{C.78}$$

where the first inequality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (4.57), and the third inequality follows from that w is non-increasing. Furthermore,

$$\max_{\nu} u\left(\gamma_{\mathsf{B}}^{c}(0, w_N, 0), \nu; c\right) = \mathsf{w}_N \leq \mathsf{w}_i = u\left(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu}; c\right), \tag{C.79}$$

where inequality follows from that w is non-increasing. Hence, (C.77)-(C.79) imply that $c \in [\underline{c}, c_N]$ would not mimic any other type. Finally, for any type $c \in [\min\{\mu R, \bar{c}\}, \bar{c}]$ and $j \in \{1, ..., N\}$,

$$\begin{aligned}
\max_{\nu} u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu; c\right) &= u\left(\gamma_{\mathsf{P}}^{c_j}(\tau_{\mathsf{w}}^j, z_{\mathsf{w}}^j), \nu^0; c\right) = c_j\bar{\tau}_{\mathsf{w}}^j \\
&\leq c_j \min\left\{\frac{1}{r}, \frac{\mathsf{w}_N}{c_j}\right\} \leq w_N = u\left(\gamma_{\mathsf{B}}^{c}(0, w_N, 0), \nu; c\right),
\end{aligned} \tag{C.80}$$

which implies that type $c \in [\min\{\mu R, \bar{c}\}, \bar{c}]$ does not mimic any other type. This concludes the proof.

C.6.7 Proof of Theorem 4.3

If $2w_i^* \leq w_{i-1}^* + w_{i+1}^*$, then $w_i^* - w_{i+1}^* \leq w_{i-1}^* - w_i^*$. Hence, for any $j < i$,

$$\frac{w_j^* - w_i^*}{i - j} = \frac{\sum_{k=j+1}^{i}(w_{k-1}^* - w_k^*)}{i - j} \geq \frac{(i - j)(w_{i-1}^* - w_i^*)}{i - j} = w_{i-1}^* - w_i^*. \tag{C.81}$$

Therefore,

$$
\begin{aligned}
\mathcal{U}\left(\hat{\Gamma}_{\mathcal{C}}^{\boldsymbol{w}_N^*}\right) &= \int_{\underline{c}}^{\bar{c}} U(\gamma_{\boldsymbol{w}_N^*}^{c}, \nu^c)\rho(c)dc \\
&= \sum_{i=1}^{N} \int_{c_{i-1}}^{c_i} U(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \nu^c)\rho(c)dc + \int_{\mu R}^{\bar{c}} U(\gamma_{\mathsf{B}}^{c}(0, w_N, 0), \nu^c)\rho(c)dc \\
&= \sum_{i=1}^{N} \int_{c_{i-1}}^{c_i} U(\gamma_{\mathsf{P}}^{c_i}(\tau_{\mathsf{w}}^i, z_{\mathsf{w}}^i), \bar{\nu})\rho(c)dc - w_N \int_{\mu R}^{\bar{c}} \rho(c)dc \\
&= \sum_{i=1}^{N} (P(c_i) - P(c_{i-1}))\mathcal{J}(\sqsupseteq_{\rangle}^*, \bar{\tau}_{\boldsymbol{w}_{\mathcal{N}}^*}^{\rangle}, \rfloor_{\rangle}) - \sqsupseteq_{\mathcal{N}}(\mathcal{P}(\bar{\rceil}) - \mathcal{P}(\rfloor_{\mathcal{N}})) \\
&= \sum_{i=1}^{N} (P(c_i) - P(c_{i-1})) \min\left\{\mathcal{J}\left(w_i^*, \min\left\{\frac{w_N^*}{c_i}, \frac{1}{r}\right\}, c_i\right),\right. \\
&\left. \min_{j \in \{0,\ldots,i-1\}} \frac{w_j - w_i}{(i-j)\delta}(\mu R - c_i) - w_i\right\} - w_N^*(P(\bar{c}) - P(c_N)) \\
&= \sum_{i=1}^{N} (P(c_i) - P(c_{i-1})) \min\left\{\mathcal{J}\left(w_i^*, \min\left\{\frac{w_N^*}{c_i}, \frac{1}{r}\right\}, c_i\right)\right. \\
&\left., \frac{w_{i-1} - w_i}{\delta}(\mu R - c_i) - w_i\right\} - w_N^*(P(\bar{c}) - P(c_N)) = \hat{y}(N),
\end{aligned}
$$

where the third equality follows from Lemma C.5, the fifth equality follows from the definition of $\bar{\tau}_{\mathsf{w}}^j$ in (4.57) and sixth equality follows from (C.81).

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