Topics in Interacting Particle Systems and Random Schrödinger Operators

by

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To my parents, Shiguo Liao and Meirong Cheng

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ABSTRACT

There are two separate parts of this theis, discussing two separate problems of different strongly correlated random systems coming from mathematical physics.

The first part of this thesis is about multi-point space-time joint distributions of the totally symmetric simple exclusion process (TASEP) and some of its variants, both on the infinite lattice $\mathbb{Z}$ and on spatially-periodic domains. We obtained exact formulas involving contour integrals of Fredholm determinants for the joint distributions of arbitrarily many space-time points for the discrete time TASEP, both on the periodic domain and on $\mathbb{Z}$. The large time asymptotics for height fluctuations were considered, for both the relaxation time scale and the sub-relaxation time scale. These formulas are multi-time generalizations of the Tracy-Widom distributions and their periodic analogues. These results were generalized to inhomogeneous situations where there are two sets of parameters describing different waiting times for different particles or empty sites. In particular we obtained a description of the Baik-Ben Arous-Péché phase transition describing the effect of having finitely many slow particles for the joint height fluctuations at the multi-time level. A multi-time analogue of the Baik-Ben Arous-Péché distribution was obtained describing the height fluctuations with critically-tuned jumping rates.

The second part of this thesis is about spectrum of random Schrödinger operators. More specifically we studied the conservation properties for the point processes formed by eigenvalues of random Schrödinger operators under spatial conditioning.

We established number rigidity property for a large class of random Schrödinger operators, first for one-dimensional operators acting on continuous spaces, later for higher-dimensional (possibly non-selfadjoint) operators acting on discrete spaces. The number rigidity property for a point process roughly states the total number of points of a point process inside any compact set is a deterministic function of the point configurations outside of the compact set. The crucial techniques are exact integral formulas for the exponential linear spectral statistics obtained through a Feymann-Kac representation of the semigroup associated to the random Schrödinger operators.

CHAPTER 1

An Overview of this Thesis

This thesis focuses on the study of large random systems with strong spatial correlations. The thesis consists of two parts, each can be read independently.

The first part, consisting of Chapter 2 to Chapter 4, discusses a specific interacting particle system model, known as the totally asymmetric simple exclusion processes (TASEP). TASEP is the default model describing traffic transportation in one dimension and serves as a prototypical example among a large class of random growth models in $(1 + 1)$ (space + time) dimensions. The techniques used in Part I are heavily algebraic and combinatorial. Exact formulas for certain observables of the systems are obtained using the exact solvability (integrability) of the model and the long-time, large-scale behaviors of the sytems are then understood using these exact formulas. Studying probabilistic systems with rich algebraic structure that allows exact evaluations of certain observables has been an active research area over the last two decades, known as integrable probability.

The second part, consisting of Chapter 5 to Chapter 7, studies random Schr̈odinger operators and their spectrum. Random Schr̈odinger operators are Schrödinger operators with random potentials. They naturally arise from quantum physics and model solids and other materials with disorder. In Part II we will focus on the spectrum of certain classes of random Schrödinger operators and the main techniques used are probabilistic or analytic (spectral theoretic).

Below we provide a brief overview of the organization of each part and their connections, more detailed background material will be discussed in Chapter 2 for Part I and Chapter 5 for Part II.

1.1 An overview of the first part

The first part, including Chapter 2 to 4, focuses on understanding the space-time joint distributions in the $1+1$ (space + time) dimensional Kardar-Parisi-Zhang (KPZ) universality class, conjectured to be the class of models describing generic random interface growth with strong local spatial correlations, through the lens of one particular model (and some variants of it) in the universality class, the totally asymmetric simple exclusion processes (TASEP).

Chapter 2 introduces the necessary background on the KPZ universality class. We will mainly focus on the totally asymmetric simple exclusion processes (TASEP) and its variants, on both the infinite lattice $\mathbb{Z}$ and spatially-periodic domains (i.e., on the circle or ring). We give an overview of the main procedures of solving periodic TASEP exactly and discuss in particular how to obtain the transition probability using ideas known as the Coordinate Bethe Ansatz which comes from quantum integrable systems.

Chapter 3 is mainly drawn from my work [79]. It focuses on an inhomogeneous version of the TASEP depending on two sets of parameters. The finite-time multi-point distributions are obtained first for the model on periodic domains and then for the model on $\mathbb{Z}$. For the large time asymptotics we focus on the full-space model $\mathbb{Z}$ with finitely many particles and empty sites having non-uniform rates. Under proper scaling, a multi-time analogue of the famous Baik-Ben Arous-Péché phase transition first discovered in the study of spiked random matrix models [6] is obtained.

Chapter 4 is mainly drawn from my work [78]. We study a discrete time analogue of the TASEP, mainly on spatially periodic domains. The general strategy to get a finite-time joint distribution formula is similar to Chapter 3 but there are some extra difficulties for the algebraic part. For the large time asymptotics we focus on the relaxation time scale which is the scale when the periodicity affects the height fluctuations critically and thus has richer structure. We obtain the same limiting distribution as the one first discovered in [10] for the relaxation time limit of continuous time periodic TASEP. These results provide evidence of the universality of height fluctuations for models in the KPZ universality class with periodic boundary conditions.

1.2 An overview of the second part

The second part, including Chapter 5 to 7, studies random Schödinger operators (RSOs) and their spectrum, focuses on how the eigenvalue point processes behave under spatial conditioning, through a particular property known as the number rigidity. The main technical achievement is a novel method of establishing number rigidity using a Feynman-Kac type formula for the exponential linear spectral statistics.

Chapter 6 contains materials from my paper [52] jointly with Pierre Yves Gaudreau Lamarre and Promit Ghosal. We considered a large class of one-dimensional random Schrödinger operators acting on continuous spaces. Using the fact that the associated semigroups of the RSOs admit Feynman-Kac type integral formulas, we are able to obtain tractable formulas for the exponential linear spectral statistics of the form $\sum_{n=1}^{\infty} e^{-t\lambda_n}$ where $\{\lambda_n\}_{n=1}^{\infty}$ are eigenvalues of the RSO. Through analyzing the formula we obtain useful information on the eigenvalue point processes.

Chapter 7 comes from the work [53] jointly with Pierre Yves Gaudreau Lamarre and Promit Ghosal. It is a continuation of the previous work where we study the same type of problems for random Schrödinger operators acting on higher dimensional discrete spaces. The highlight here is due to the discreteness of the underlying spaces, we can treat much more general classes of noises (possibly correlated) and Markov generators (possibly non self-adjoint).

1.3 Connections between the two parts

There are nevertheless fruitful connections between the two parts of the thesis. Some common motivations for both parts come from random matrix theory and (determinantal) point processes. More specifically both parts are more or less related to the largest eigenvalues of certain classes of random matrices, or their liming fluctuations.

It is well-known that the one-point marginals of the height fluctuations of TASEP and the largest eigenvalues of a large class of random matrices are both described in the large scale limit by the Tracy-Widom law and its relatives. Even better at finite-time level it is known that the single-time (or multi-point equal time) distributions of the inhomogeneous TASEP model discussed in Chapter 3 agrees exactly with the laws of largest eigenvalues of a generalized Wishart random matrix model, where the entries of the random sample matrices are gaussian with covariance matrices depending on two sets of parameters (see [20]). The main achievement of Part I is the computation of multi-time generalizations of the single-time results on the interacting particle system side. Currently one can only see the spatial marginals of the KPZ fixed point (i.e., at the Airy processes level) from random matrix models while the time direction is missing. It remains an open and interesting question if one can find a natural random matrix model with the same multi-time distributions.

The central objects studied in Part II are random Schrödinger operators and their spectra. They can be viewed as continuum analoguess of random matrices and their eigenvalues. A very special random Schrödinger operator introduced in [95], usually called the stochastic Airy operator, naturally appears as the scaling limit of the Dumitriu-Edelman tri-diagonal random matrix models introduced in [42]. A natural motivation for the work in Part II is to understand the spectrum of the stochastic Airy operator, known as the Airy-β processes, from a stochastic analysis point of view. The framework extends nicely to more general random Schrödinger operators.

Part I

Multi-time Distribution in the KPZ Universality Class

CHAPTER 2

Introduction to the KPZ Universality Class

2.1 The KPZ universality class and KPZ fixed point

2.1.1 The KPZ equation and KPZ universality class

In 1986, three physicists Kardar, Parisi and Zhang [72] proposed the following stochastic partial differential equation as a natural continuum model describing random interface growth in $1+1$ (space+time) dimensions:

$$\partial_t h(x,t) = \partial_x^2 h(x,t) + (\partial_x h(x,t))^2 + \xi(x,t), \tag{2.1}$$

where $h(x,t) : I \times \mathbb{R}_+ \to \mathbb{R}$ for $I \subset \mathbb{R}$ and $\xi(x,t)$ is a Gaussian space-time white noise (i.e., a random Schwartz distribution with mean zero and δ covariance $\mathbb{E}[\xi(t,x)\xi(t',x')] = \delta(t-t')\delta(x-x')$). Through a dynamical renormalization group analysis, in [72] the authors predicted the dynamical scaling exponents for the random function $h(x,t)$ and argued that the same dynamic scaling exponents should be satisfied by a much larger class of random systems sharing similar strong spatial correlations regardless of the detailed microscopic mechanisms.

Following the seminal work of Kardar-Parisi-Zhang, a large class of random systems coming from interacting particle systems, random interface growths, directed polymers in random environments, random matrices and so on has been shown or conjectured to share the same dynamical scaling exponents and have the same long-time, large-scale behaviors. These models form the so-called Kardar-Parisi-Zhang (KPZ) universality class.

Though it remains an open and challenging question to describe the precise requirements for a random system to belong to the KPZ universality class, the following three vague mechanisms are believed to be shared among the class:

(i) (Locality) The interactions between height function at different locations are localized (long-range interactions are negligible).

(ii) (Nonlinear slope dependence) Vertical growth rate at each spatial point depends nonlinearly on the local slope of the height function.

(iii) (Space-time independent noise) The interface growth is driven by noise that decorrelates quickly in space and time and does not have heavy tails.

2.1.2 1:2:3 KPZ scaling and the KPZ fixed point

Models in the $(1+1)$-dimensional KPZ universality class are typically described by a random height function $h(x,t)$ (or some analogs of it) with $(x,t) \in I \times \mathbb{R}_+$ for $I \subset \mathbb{R}$. Here we take $I = \mathbb{R}$ be the full space. One can also consider half-space $\mathbb{R}_+$ or finite-volume spaces $I = [a,b]$ with certain boundary conditions at the end points. We will discuss the similarities and differences with different underlying spaces (mainly finite interval with periodic boundary conditions) in Section 2.3.

Despite rather different descriptions of the height functions for different models, it is believed that with the same type of dynamic scaling exponents, namely $T^{1/3}$: $T^{2/3}$: $T^{3/3}$ for height fluctuations, spatial correlation and temporal correlations, the large T limits for the height functions are universal. Mathematically we have

$$\lim_{T\to\infty} \frac{h(c_1 u T^{2/3}, c_2 \tau T) - c_3 \tau T}{c_4 T^{1/3}} = H(u,\tau),$$

for a unique (in distribution) random function $H(u, \tau)$ known as the KPZ fixed point. Here the constants $c_1, \cdots, c_4$ are model-dependent and the distribution of $H(u, \tau)$ depends on the initial data $H(u, 0)$.

The one-point marginal of $H(u, \tau)$ (i.e., the distribution of the random variable $H(u, \tau)$ for fixed u and τ) is known to be the Tracy-Widom distribution and its variants. Such limit laws for the one-time marginal has been obtained for a large class of models, mainly with very special algebraic or combinatorial structure so that one can obtain explicit formulas for certain observables related to the height functions, see [7, 65, 19, 105, 3, 15, 16, 17]. The spatial process $H(\cdot, \tau)$ for fixed τ is known as the Airy process (and some of its variants). Convergence at multi-point equal time level was mainly obtained for determinantal models (i.e., models related to the so-called determinantal point processes), see [92, 19, 83]. A complete description of $H(u, \tau)$ as a Markov process on the space of upper semicontinuous functions with a transition kernel described by certain Fredholm determinants was obtained recently in [83], by solving exactly the TASEP (will be introduced in Section 2.2) with general initial conditions and performing large time asymptotics. A different (and slightly more general) description of the full scaling limit was later obtained in [38]. By taking the scaling limit of a Brownian last passage percolation model (related to TASEP and is described briefly in Section 2.2), the authors in [38] obtained a four-parameter random field $\mathcal{L}(x, s; y, t)$, known as the direct landscape (or space-time Airy sheet conjecture in [36]) with $(x, s; y, t) \in \mathbb{R}^4$ and $s < t$. The KPZ fixed point $H(u, \tau)$ can be embedded in $\mathcal{L}(x, s; y, t)$ by defining

$$H(u, \tau) := \sup_{v \in \mathbb{R}} \left(H(v, 0) + \mathcal{L}(v, 0; u, \tau) \right).$$

More recently there is important progress of extending the previous results to non-determinantal (even without any solvability) models, see [93, 108].

The main goal of the first part of this thesis is an explicit description of the joint laws of $(H(u_1,\tau_1),\cdots,H(u_m,\tau_m))$ with arbitrary m points and possibly different τ_i's. Surprisingly the first rigorous result on the general multi-point space-time joint distributions was obtained for TASEP on the ring in [9] instead of on $\mathbb{Z}$. The results were extended to more general initial conditions in [10] for the periodic model. We will explain in Chapter 3 how the periodicity of the model makes the algebraic calculation simpler. For the full space models, in [67] the authors obtained a formula for the multi-time joint distributions of a geometric last passage percolation model (equivalent to a discrete time version of TASEP), following the idea of [66] where the special two-time distribution was obtained. Around the same time, in [81] a different formula for the multi-time distribution of TASEP on $\mathbb{Z}$ was obtained, by relating it to the periodic models. The contributions of part I are mostly extensions of [9, 10, 81, 67, 68]. We mainly follow the strategy in [81] by always solving the problem on periodic domains first which is easier due to nicer algebraic properties. Then the corresponding problem on $\mathbb{Z}$ will be obtained by taking the period large enough. See Chapter 3 and 4 for more details.

2.2 TASEP and some variants

2.2.1 TASEP and its height function

The totally asymmetric simple exclusion process (TASEP) is a prototypical stochastic model decribing transport. It was first introduced in [82] by biologists as a model for mRNA translation and independently in [102] by probabilists as a typical interacting particle system. Formally, the (continuous time, homogeneous) TASEP (on $\mathbb{Z}$) is a continuous time Markov process $\eta(t)=\{\eta_x(t)\}_{x\in\mathbb{Z}}$ with state space $S=\{0,1\}^{\mathbb{Z}}$ (meaning that $\eta_x(t)\in\{0,1\}$ for all $t\in\mathbb{R}_+$ and such $\eta_x(t)$ are usually called occupation variables, they represent whether the site $x\in\mathbb{Z}$ is occupied with a particle or not at time t) and infinitesimal generator $\mathcal{L}^{\text{TASEP}}$ acting on cylinder functions (those that are nonzero only for finitely many coordinates) $f : S \to \mathbb{R}$ given by

$$(\mathcal{L}^{\text{TASEP}} f)(\eta) = \sum_{x \in \mathbb{Z}} \eta_x (1 - \eta_{x+1})(f(\eta^{x,x+1}) - f(\eta)), \tag{2.2}$$

where the configuration $\eta^{x,x+1}$ is obtained from η by exchanging values at x and $x+1$, namely

$$\eta_y^{x,x+1} = \begin{cases} \eta_{x+1}, & \text{if } y = x, \\ \eta_x, & \text{if } y = x+1, \\ \eta_y, & \text{otherwise.} \end{cases}$$

In plain words the dynamics is simply several particles perform independent nearest neighbor Poisson random walk on $\mathbb{Z}$ where each particle tries to jump to their right neighboring site after an independent rate 1 exponential waiting time, however the jump is only allowed when the target site is empty. Note that the TASEP dynamics preserves the number and ordering of particles. Usually we are only interested in the locations of finitely many tagged particles and their dynamics will not be affected by the particles to their left. Hence it is sometimes more convenient to view TASEP as dynamics on $\vec{x}(t) = (x_1(t), \cdots, x_N(t))$, encoding the locations of the first N particles (from right to left), which lives on the state space

$$\Omega_N := \{\vec{x} = (x_1, \cdots, x_N) \in \mathbb{Z}^N : x_1 > x_2 > \cdots > x_N\}.$$

Note that here and throughout all the chapter we assume there is a right-most particle $x_1(t)$. From this dual point of view the infinitesimal generator $\hat{\mathcal{L}}^{\text{TASEP}}$ acting on bounded functions $g : \Omega_N \to \mathbb{R}$ takes the form

$$\left(\hat{\mathcal{L}}^{\text{TASEP}} g\right)(\vec{x}) = \sum_{i=1}^{N} \left(g(\vec{x}_i^-) - g(\vec{x})\right) \mathbf{1}_{\vec{x}_i^- \in \Omega_N}. \tag{2.3}$$

Here $\vec{x}_i^- = (x_1, \cdots, x_{i-1}, x_i - 1, x_{i+1}, \cdots, x_N)$.

Definition 2.2.1 (Height function associated to TASEP)**.** *We associate the following height function $h(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ to the TASEP with occupation variables $\{\eta_x(t)\}_{x\in\mathbb{Z}}$: For $x \in \mathbb{Z}$, we define*

$$h(x,t) := \begin{cases} 2J_0(t) + \sum_{y=1}^{x}(1 - 2\eta_x(t)), & \text{for } x \geq 1, \\ 2J_0(t), & \text{for } x = 0, \\ 2J_0(t) - \sum_{y=x+1}^{0}(1 - 2\eta_x(t)), & \text{for } x \leq -1. \end{cases} \tag{2.4}$$

The value of $h(x,t)$ for general $x \in \mathbb{R}$ is defined by linear interpolation. Here the function $J_0(t)$ counts the total number of particles that have passed the origin before time t.

Graphically we simply associate each particle with a line segment of slope -1 and each empty site with a line segment of slope 1. Any movement of the particles corresponds to switching a $\vee$ into a $\wedge$ for the height function. See Figure 2.1 for an illustration. By definition of $h(x,t)$ we then have the equality between the events $\{x_{\frac{H-X}{2}}(T) \geq X\} = \{h(X,T) \geq H\}$, where $x_k(t)$ is the location of the k-th particle at time t.

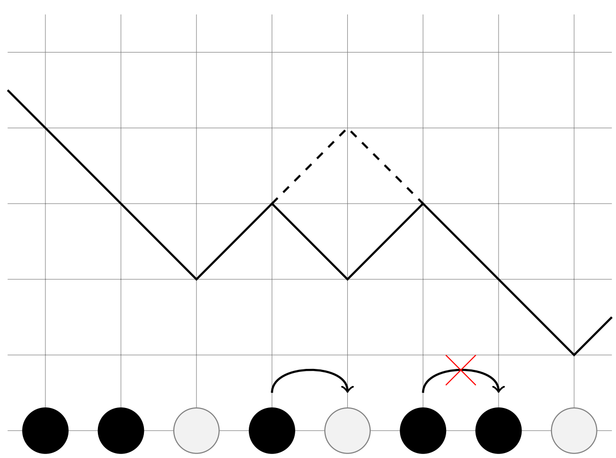

Figure 2.1: TASEP and the associated height function. The solid and dashed line represent the height function before and after a jump.

2.2.2 Exponential last passage percolation

The following statistical physics model, known as exponential last passage percolation (ExpLPP), is closely related to TASEP and will provide useful insights for our study on TASEP oftentimes. We associate each site $(i,j) \in \mathbb{Z}_+^2$ with an independent exponential clock w_{ij}. Then for any $(M,N) \in \mathbb{Z}_+^2$, we define a random variable, known as the (point-to-point) last passage time from $(1,1)$ to (M,N) by maximizing the total waiting times over all possible up-right paths:

$$G(M,N) := \max_{\pi\in\Pi} \sum_{(i,j)\in\pi} w_{ij}, \tag{2.5}$$

where the set Π consists of all up-right paths from $(1,1)$ to (M,N). The following Proposition summarizes the relationship between exponential last passage percolation and TASEP.

Proposition 2.2.2 (Coupling between TASEP and ExpLPP). *Let $x_k(t)$ be the location of the k-th particle under the TASEP dynamics with step initial condition $x_i(0) = -i$ for all $i \geq 1$. And let $(G(M,N))_{M,N\in\mathbb{Z}_+}$ be the last passage times at different sites for the same exponential last passage percolation model. Then for any integer $m \geq 1$,*

$$\begin{aligned} \mathbb{P}_{\text{TASEP}}\left(\bigcap_{\ell=1}^{m}\{x_{k_\ell}(t_\ell) \geq a_\ell\}\right) &= \mathbb{P}_{\text{ExpLPP}}\left(\bigcap_{\ell=1}^{m}\{G(k_\ell, a_\ell + k_\ell) \leq t_\ell\}\right) \\ &= \mathbb{P}_{\text{ExpLPP}}\left(\bigcap_{\ell=1}^{m}\{G(a_\ell + k_\ell, k_\ell) \leq t_\ell\}\right). \end{aligned} \tag{2.6}$$

Proof. Clearly the last passage times $G(M,N)$ satisfy a random recurrence relation

$$G(M,N) = \max\{G(M-1,N), G(M,N-1)\} + w_{M,N},$$

for all $(M,N) \in \mathbb{Z}_+^2$ where $G(0,N) = G(M,0) := 0$ and $w_{M,N}$ is an exponential random variable with rate 1 independent of $G(M-1,N)$ and $G(M,N-1)$. Now define another set of random variables $T(M,N)$ to be first time when the M-th particle x_M under a TASEP dynamics starting from the step initial condition reaches the site $N-M$. Then $T(M,N)$ satisfy the same recurrence relation as $G(M,N)$, namely

$$T(M,N) = \max\{T(M-1,N), T(M,N-1)\} + \hat{w}_{M,N},$$

for independent rate 1 exponential random variables $\hat{w}$. Thus $T(M,N)$ and $G(M,N)$ are equal in distribution and hence

$$\mathbb{P}_{\text{TASEP}}(x_M(t) \geq N-M) = \mathbb{P}_{\text{TASEP}}(T(M,N) \leq t) = \mathbb{P}_{\text{ExpLPP}}(G(M,N) \leq t).$$

The second equality in equation (2.6) follows from row/column symmetry. The coupling extends easily to multi-point joint distributions. □

This type of coupling between TASEP and LPP extends to general initial conditions for TASEP which correspond to point-to-curve last passage percolation (as opposed to point-to-point LPP). We will not explain this in detail here since it is not needed in this thesis.

2.2.3 Some variants

The models we will consider in Chapter 3 and Chapter 4 are both slightly different from (and more general than) the TASEP model introduced in Section 2.2.

In Chapter 3 we consider an inhomogeneous version of the exponential last passage percolation (or equivalently TASEP) introduced in [20]. Instead of taking the waiting time w_{ij} to be of rate 1 for all i,j, we take w_{ij} to be an exponential random variable with rate $\pi_i + \hat{\pi}_j$, for two sets of parameters $\{\pi_i\}$ and $\{\hat{\pi}_j\}$ with $\pi_i + \hat{\pi}_j > 0$ for all i,j. From the TASEP point of view this means different particles (and different empty sites) have different jumping rates (or speeds).

In Chapter 4 we consider a discrete time version of the TASEP. Instead of letting each particle run independent continuous time Poisson random walk, we let them run independent discrete time Bernoulli random walk where each particle tosses an independent coin with success probability p and tries to move to their right neighboring site after each discrete time step upon success, subject to the same exclusion rule. We will focus on the parallel update version which means the updates for all particles happen simultaneously at the end of each time step (so a particle will not move during a time step if its target site was occupied at the beginning of the time step). From the last passage percolation point of view this is equivalent to replacing the exponential waiting time at each site (i, j) with a geometric random variable with success probability p (meaning that $\mathbb{P}(w_{ij} = k) = (1-p)^{k-1}p$).

The continuous time TASEP can be obtained as a limit of the discrete time one by taking $p = \epsilon$, rescaling the time $T = t/\epsilon$ and sending $\epsilon \to 0$. There are other interesting degenerations of the discrete time TASEP (or geometric last passage percolation). For example it is also interesting to consider the Poissonian limit when we fix $\lambda > 0$ and take $p = 1 - \frac{\lambda^2}{N^2}$. The distribution of $\lim_{N\to\infty} G_p(N, N)$ is known as the Poissonized Plancherel measure and is related to the longest increasing subsequence problem, see [7, 92]. Another interesting limit is to rescale the columns in the geometric last passage percolation, send $M \to \infty$ and keep N fixed. The discrete random environment $(w_{1,k}, \cdots, w_{M,k})$ now becomes a continuous random environment (essentially a Brownian motion) and one is optimizing paths among N independent Brownian motions. The model is known as the Brownian last passage percolation and is closely related to Dyson's nonintersecting Brownian motions and Gaussian Unitary Ensembles from random matrix theory. See [11, 86].

2.3 Spatially periodic domain and other underlying spaces

So far we have been considering TASEP on $\mathbb{Z}$ or last passage percolation inside the first quadrant. The main spirit this thesis would like to convey is that it is also worth to consider other underlying spaces. Spatially periodic domains will serve as our main example to illustrate this philosophy. There are three reasons to consider TASEP on periodic domains:

(1) The periodic models are interesting on their own because they naturally interpolate equilibrium dynamics and KPZ dynamics on the infinite-volume spaces.

(2) The multi-time distribution of periodic TASEP is easier to compute comparing to TASEP on $\mathbb{Z}$ due to nicer algebraic properties.

(3) The multi-time distributions of TASEP on $\mathbb{Z}$ can be derived from the corresponding results for periodic TASEP by taking the period L large.

We will try to illustrate these philosophies briefly in this section. For more details see Chapter 3 and Chapter 4.

2.3.1 Periodic TASEP

For a fixed integer $L \in \mathbb{Z}_+$, we define the periodic TASEP with period L as the following minor modifications of the TASEP model on $\mathbb{Z}$ introduced in Section 2.2. Instead of considering the full state space $S = \{0,1\}^{\mathbb{Z}}$, we restrict ourselves to the subspace

$$S_L := \{(\eta_x)_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} : \eta_{i+jL} = \eta_i \quad \text{for all } i,j \in \mathbb{Z}\}.$$

The infinitesimal generator for the Markov process remains the same and the only difference is for the periodic model, we impose the extra assumption that particles with coordinates differ by a multiple of L are identical copies of each other and will all move simultaneously. If we identify all these particles that move together, we essentially get TASEP on a ring of size L. However we prefer to distinguish them and keep track of the winding number of each particle around the ring.

Now fixing L consecutive sites (say $\{-L, \cdots, -1\}$) and assume there are initially N particles in these sites (meaning that $\eta_i = 1$ for exactly N sites among $-L \leq i \leq -1$). Then the total number of particles in any L consecutive sites at any time will be N. Hence the cardinality of the state space $S_{L,N}$ for the occupation variable $(\eta_x)_{x\in\mathbb{Z}}$ will be $|S_{L,N}| = \binom{L}{N}$. Taking the dual point of view, the state space for the particle locations will then be

$$\Omega_{L,N} = \{\vec{x} = (x_1, \cdots, x_N) \in \mathbb{Z}^N : x_N + L > x_1 > x_2 > \cdots > x_N\}. \tag{2.7}$$

See Figure 2.2 for an illustration of periodic TASEP dynamics.

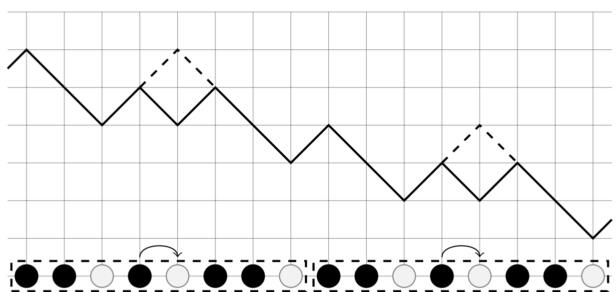

Figure 2.2: Periodic TASEP with $L = 8$ and $N = 5$. Particles within each dashed rectangle form a period and particles in different periods are identical copies of each other. The corresponding height functions inside each period are also identical copies of each other up to a global shift.

The variants introduced in Section 2.2.3 also have their analogues on the periodic domain and can be defined in a similar manner. We leave the formal introduction of these models to later chapters.

2.3.2 Periodic versus Infinite

For both the periodic and infinite models, we are mainly interested in the long time, large scale behaviors of the height functions. The periodic model contains an extra parameter L and by tuning the parameter L one can see phenomena that are not seen in the infinite-volume models. Depending on the relationship between the period L and the time parameter t (will be sent to infinity), there are three different regimes where the height fluctuations have completely different behaviors.

(i) The super-relaxation time scale $t \gg L^{3/2}$. This in particular includes the case when L remains bounded and $t \to \infty$. In this regime all the particles are strongly correlated and the height fluctuations at all the spatial locations are more or less the same (described by gaussian after a diffusive scaling).

(ii) The sub-relaxation time scale $t \ll L^{3/2}$. In this regime the period is extremely large and the time is not long enough so that the particles do not feel the effect of the boundary at time t. Thus the height fluctuations are expected to be the same as the infinite-volume models.

(iii) The relaxation time scale $t \sim L^{3/2}$. This is the scale when the height fluctuations are critically affected by the finite geometry. Thus the height fluctuations are expected to be a crossover between equilibrium dynamics and the KPZ dynamics on the full space.

The relaxation time scale $t \sim L^{3/2}$ was first indicated in [63], as the scaling exponent for the reciprocal of the spectral gap for the infinitesimal generator of periodic TASEP with period L (the latter is $O(L^{-3/2})$ according to their computation). Note that this is consistent with the definition of relaxation times for general Markov chains, see Chapter 12 of [77]. One can also understand the relaxation time scale from the $1:2:3$ KPZ scaling, from which we know the critical length for spatial correlations is $O(t^{2/3})$ for models in the KPZ universality class, namely particles of distance $\sim t^{2/3}$ are critically correlated. The relaxation time scale $t = O(L^{3/2})$ then can be identified with the scale when all particles in the same period are critically correlated, since $L \sim t^{2/3} \Leftrightarrow t \sim L^{3/2}$.

The discussion above indicates an alternative indirect way to study models in the KPZ universality class on the full space, namely first study the corresponding models in the periodic domain with period L (if it turns out to be easier) and then let $L \to \infty$. Even better, if we are only interested in the joint distribution of finitely many particles of TASEP at fixed finite-time, then we have exact equally in distribution between infinite model and periodic model with sufficiently large period L (but still finite).

Proposition 2.3.1 (Theorem 3.1 of [81]). *Consider periodic TASEP with period L and N particles in each period and TASEP on $\mathbb{Z}$ with N particles starting from the same initial condition $\vec{y} = (y_1, \cdots, y_N) \in \Omega_{L,N}$. Here $\Omega_{L,N}$ is defined in equation* (2.7)*. We denote the particle locations by $x_k^{(L)}(t)$ and $x_k^{(\infty)}(t)$ for the two models. Given any integer $m \geq 1$, for any m indices $\{k_1, \cdots, k_m\} \subset \{1, \cdots, N\}$ and m integers $a_1, \cdots, a_m$, if the period L satisfies*

$$L \geq \max\{y_1 + 1, a_1 + k_1, \cdots, a_m + k_m\} - y_N, \tag{2.8}$$

then we have

$$\mathbb{P}_{\vec{y}}^{(L)} \left(\bigcap_{\ell=1}^{m} \{x_{k_\ell}^{(L)}(t_\ell) \geq a_\ell\} \right) = \mathbb{P}_{\vec{y}}^{(\infty)} \left(\bigcap_{\ell=1}^{m} \{x_{k_\ell}^{(\infty)}(t_\ell) \geq a_\ell\} \right). \tag{2.9}$$

Here (L) and (∞) stand for periodic model and infinite model, respectively.

The proposition basically quantifies the intuition that when the period is large, particles will not feel the boundary effect if they have not gone far enough. We point out that a priori it is somehow surprising since the left hand side of equation (2.9) involves an extra parameter L and the Proposition states that it is independent of L when L is large. How to obtain a formula for the left hand side of (2.9) free of the parameter L is a separate important question.

A better way to understand the Proposition is through the related exponential last passage percolation model. The first observation is that introducing periodicity to TASEP corresponds to considering exponential last passage percolation on a cylinder under the coupling described in Section 2.2.2.

More precisely, we consider the following variant of the exponential last passage percolation model introduced in Section 2.2.2. Fixing positive integers $N < L$, we introduce the following equivalence relation among points in $\mathbb{Z}^2$:

$$(p_1, q_1) \sim (p_2, q_2) \quad \text{if } (p_1 - p_2, q_1 - q_2) = k(L - N, -N) \quad \text{for some } k \in \mathbb{Z}.$$

Then we associate the same rate 1 exponential random variable w_{ij} to all the sites in the same equivalence class as $(i, j) \in \mathbb{Z}^2$, while waiting times at sites in different equivalence classes are independent. The cylindrical last passage time $G^{(L,N)}(a, b)$ for $(a, b) \in \mathbb{Z}^2$ is then defined as the supremum over all usual last passage time $G(c, d)$ with $(c, d) \sim (a, b)$, namely

$$G^{(L,N)}(a, b) := \max_{(c,d)\sim(a,b)} \max_{\substack{\pi:(1,1)\to(c,d) \\ \pi \text{ up-right}}} \sum_{(i,j)\in\pi} w_{ij}.$$

Here up-right is in the usual sense in $\mathbb{Z}^2$, so if $c < 1$ or $d < 1$ then there is no up-right path from $(1, 1)$ to (c, d). See Figure 2.3 for an illustration. It is straightforward to check that with this definition we have similar coupling between Exponential LPP on cylinder and periodic TASEP (with step initial condition) as in equation (2.6).

Now for $L - N \geq \max\{a_1 + k_1, \cdots, a_m + k_m\} \geq 1$ and $N \geq \max\{k_1, \cdots, k_m\} \geq 1$, the m points $\{(a_\ell + k_\ell, k_\ell)\}_{\ell=1}^m$ all lie inside the rectangle $\{1, \cdots, L - N\} \times \{1, \cdots, N\}$. Then the only possible up-right paths from $(1, 1)$ to $(c, d) \sim (a_\ell + k_\ell, k_\ell)$ are those from $(1, 1)$ to $(a_\ell + k_\ell, k_\ell)$, for all $1 \leq \ell \leq m$. This is because for any $j \neq 0$, we either have $a_\ell + k_\ell + j(L-N) < 1$ or $k_\ell - jN < 1$. This then implies that

$$G^{(L,N)}(a_\ell + k_\ell, k_\ell) = G(a_\ell + k_\ell, k_\ell), \quad \text{for all } 1 \le \ell \le m.$$

This essentially proves equation (2.9). See Figure 2.3 for an illustration.

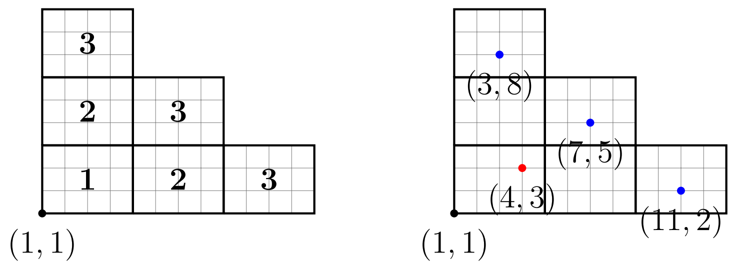

Figure 2.3: An illustration for cylindric ExpLPP with $L = 7$ and $N = 3$. On the left the solid rectangles with the same indexing (1,2 or 3) are identical copies of each other and waiting times for sites inside each solid rectangles are independent. On the right the cylindric last passage time $G^{(L,N)}(4,3)$ equals the usual last passage time $G(4,3)$. However the cylindric last passage time $G^{(L,N)}(3,8)$ by definition equals $\max(G(3,8), G(7,5), G(11,2))$.

2.3.3 Other underlying spaces

We mention briefly here some similar models with other interesting state spaces though we will not discuss them in details. There are at least two other interesting variants of TASEP with different underlying spaces:

(i) The half-space $\mathbb{Z}_+$. The state space is $\{0,1\}^{\mathbb{Z}_+}$ and usually there is a reservoir at the origin 0 where particles are created and they enter the system at a certain rate γ. This variant of TASEP is related to certain last passage percolation models in the half-quadrant, see [5, 106] for more precise descriptions and some generalizations.

(ii) The finite interval $\{1, \cdots, L\}$ with open boundaries. The state space is $\{0,1\}^{\{1,\cdots,L\}}$ while there are reservoirs at sites 1 and L. For TASEP particles are created with rate α at site 1 and are absorbed with rate β at site L. Height fluctuations are crucially affected by the boundary parameters α and β and hence are splitted into different phases. There is no rigorous descriptions yet of the height fluctuations in the so-called maximal current phase which should be the KPZ regime. See [60] and references therein for some physical computations.

One of the significant differences between the above two variants of TASEP with those discussed before (TASEP on $\mathbb{Z}$ and periodic TASEP) is that the total number of particles is no longer preserved. This is crucial in the computation using coordinate Bethe Ansatz as will be shown in the next section.

2.4 Solving periodic TASEP exactly

In this section we briefly summarize the whole procedure of solving the periodic TASEP exactly and obtaining formulas for the multi-point joint distribution. Such procedures will be described with much more details and greater generalities in the next two chapters. It roughly follows the following four steps:

Step 1(P): Derive an integral formula for the Markov transition probability $P_t(\vec{y} \to \vec{x})$. For TASEP this typically involves certain determinants and such approach was pioneered by Schütz [99].

Step 2(P): Perform a (multiple) summation over the transition probabilities in order to get finite-time joint distributions. This step typically involves nontrivial combinatorics.

Step 3(P): Rewrite the joint distributions obtained in Step 2 through certain orthogonalization procedures to get alternative formulas that are more suitable for taking large time asymptotics (such formulas are typically related to Fredholm determinants).

Step 4(P): Perform large time asymptotics under suitable scaling (typically a steepest-descent analysis).

The procedure above is described for periodic TASEP but one can go through the full procedure for TASEP on $\mathbb{Z}$ as well (of course with a different transition probability formula to start with). Historically for one-time (possibly multi spatial locations) marginals the $\mathbb{Z}$ formulas are obtained earlier (and they are simpler), due to the connection to determinantal point processes, see [65, 99, 18]. For the multi-time distributions there are significant extra difficulties in Step 2 and 3 and the calculation for TASEP on $\mathbb{Z}$ (or rather the discrete time analogue) was only carried through recently in [67], after the parallel and lighter computation was done in [9] for the periodic analogue.

Owing to the observation in Proposition 2.3.1, in [81] an alternative approach was proposed for studying multi-time distributions for TASEP on $\mathbb{Z}$. Instead of going through the parallel four-step procedure for the $\mathbb{Z}$ model (which we will denote by Step 1($\mathbb{Z}$)-4($\mathbb{Z}$) as opposed to Step 1(P)-4(P) for the periodic model), one starts with the result obtained in Step 3(P), then going through the following two steps:

Step 3.5 (P $\to$ $\mathbb{Z}$): Fix the parameters $\{k_\ell, a_\ell, t_\ell\}_{\ell=1}^m$ as in Proposition 2.3.1, taking the period L large so that the multi-time joint distributions for periodic TASEP agree with the one for TASEP on $\mathbb{Z}$ at fixed finite time by Proposition 2.3.1. Rewrite the periodic formula so that it is free of the extra parameter L (This is the hard part and relies heavily on the algebraic properties).

Step 4'($\mathbb{Z}$) Take the formula obtained from Step 3.5 and perform large time asymptotics under proper scaling.

See Figure 2.4 below for a summary of the procedures described above. Chapter 3 and Chapter 4 mainly follow the above strategy, proper modifications and generalizations are needed for the slightly more general models studied there.

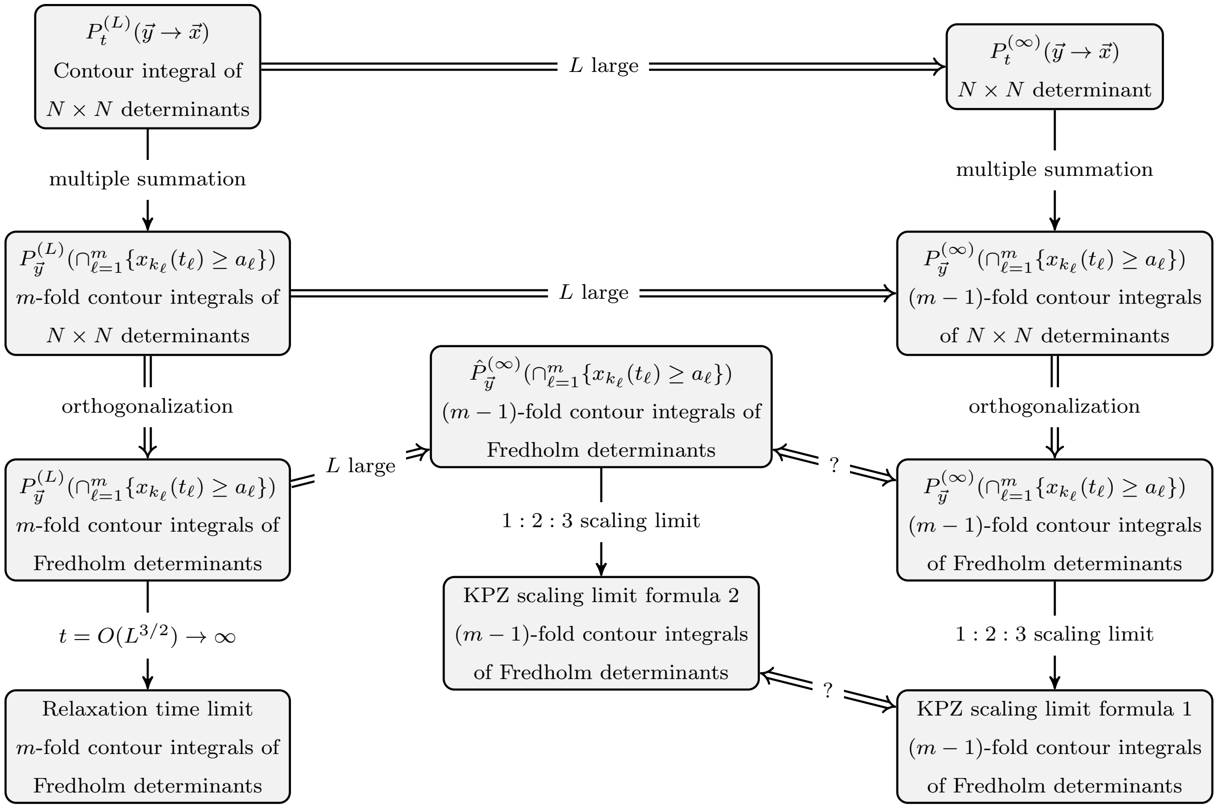

Figure 2.4: A diagram describing the procedure of computing the multi-time distributions of TASEP (periodic or on $\mathbb{Z}$). The approach used in this thesis, following [9, 9, 81], is going through the left-most column, solving the periodic models first, and derive multi-time formulas for the full-space model by taking L large (going through the mid column above). These lead to different formulas comparing to those obtained in [67, 68] which essentially go through the right-most column above.

Chapter 3 contains some discussions on the relationship between the two different approaches. What is shown there is that the two approaches give exactly the same formulas up to Step 2, for the finite-time joint distribution before the orthogonalization. However the two orthogonalization procedures are sufficiently different so that they lead to two different Fredholm determinants which should be equal since they are equal to the same $N \times N$ determinant. Nonetheless a direct verification of the equality is still missing at this moment (partly due to the fact that the kernels for the two Fredholm determinants do not seem to be simple conjugations of each other, for $m \geq 2$).

2.5 Transition probability and Coordinate Bethe Ansatz

In this concluding section we briefly explain how to obtain an exact integral formula for the transition probability of (homogeneous) periodic TASEP, such a formula was first obtained in [8] and is a special case of the more general inhomogeneous transition probability formula obtained in Proposition 3.3.1 in Chapter 3. The presentation there is self-contained but not so illuminating since the formula is directly given and we checked it satisfies the desired Kolmogorov forward equation with proper initial condition. Here we will try to illustrate how such formulas are constructed, using ideas motivated by Coordinate Behte Ansatz coming from quantum integrable systems.

Proposition 2.5.1 (Proposition 5.1 of [8]). *Given two particle configurations $\vec{x} = (x_1, \cdots, x_N),\ \vec{y} = (y_1, \cdots, y_N) \in \Omega_{L,N}$. Let $P_t(\vec{y} \to \vec{x})$ be the transition probability of observing configuration $\vec{x}$ at time t under the periodic TASEP dynamics with initial configuration $\vec{y}$. Then*

$$P_t(\vec{y} \to \vec{x}) = \oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det \left[\sum_{w \in \mathcal{S}_z} e^{tw} w^{i-j} (w+1)^{y_j - x_i + j - i - 1} J(w) \right]_{i,j=1}^{N}. \tag{2.10}$$

Here Γ is any simple closed contour with 0 inside and $\mathcal{S}_z$ consists of all the roots of the degree L polynomial $q(w) - z$, namely

$$\mathcal{S}_z := \{w \in \mathbb{C} : q(w) = z\}, \tag{2.11}$$

where $q(w) := w^N (w+1)^{L-N}$ and $J(w) := \frac{q(w)}{q'(w)} = \frac{w(w+1)}{Lw+N}$. Here

$$\Omega_{L,N} = \{\vec{x} = (x_1, \cdots, x_N) \in \mathbb{Z}^N : x_N + L > x_1 > \cdots > x_N\}.$$

2.5.1 Reduction of the Kolmogorov forward equation

By our choice of state space, the transition semigroup P_t is the semigroup with an infinitesimal genrator of a similar form as in equation (2.3), with a slightly different underlying space. It is the unique solution of the following Kolmogorov forward equation that for any $\vec{x}, \vec{y} \in \Omega_{L,N}$, one should have

$$\partial_t P_t(\vec{y} \to \vec{x}) = \hat{\mathcal{L}}^{\text{pTASEP}} P_t = \sum_{i=1}^{N} \big(P_t(\vec{y} \to \vec{x}_i^-) - P_t(\vec{y} \to \vec{x}) \big) \mathbf{1}_{\vec{x}_i^- \in \Omega_{L,N}}, \tag{2.12}$$

satisfying the initial condition $P_0 = \mathbf{1}_{\vec{x}=\vec{y}}$. Here $\vec{x}_i^- = (x_1, \cdots, x_{i-1}, x_i - 1, x_{i+1}, \cdots, x_N)$ and $\Omega_{L,N} = \{\vec{x} \in \mathbb{Z}^N : x_N + L > x_1 > \cdots > x_N\}$. The indicators on the right hand side of (2.12) encode the exclusion rule and make the whole differential difference equation not of constant coefficients. A standard starting point to find a solution of (2.12) is to get rid of the delta-interaction and replace them with extra two-body boundary conditions. Namely if we can find functions $u_t(\vec{x}; \vec{y}) : \mathbb{Z}^N \to \mathbb{R}$ satisfying

(i) (Free evolution equation) For functions $f : \mathbb{Z}^N \to \mathbb{R}$ we introduce the discrete difference operators $\nabla_i^- f(\vec{x}) := f(\vec{x}) - f(\vec{x}_i^-)$. Then the free evolution equation can be written as

$$\partial_t u_t(\vec{x}) = -\sum_{i=1}^{N} \nabla_i^- u_t(\vec{x}) = \sum_{i=1}^{N} (u_t(\vec{x}_i^-) - u_t(\vec{x})) \tag{2.13}$$

(ii) (Boundary conditions) In equation (2.12) even if we assume $\vec{x} \in \Omega_{L,N}$, the $\vec{x}_i^-$'s may not be in $\Omega_{L,N}$. And precisely when $\vec{x} \in \Omega_{L,N}$ but $\vec{x}_i^- \notin \Omega_{L,N}$ will the indicators make contribution to the equation and nullify the entry $P_t(\vec{y} \to \vec{x}_i^+) - P_t(\vec{y} \to \vec{x})$ in the sum. Thus if we get rid of the indicator, we need to impose the following extra boundary conditions on u_t to formally match the two evolution equations for P_t and u_t:

$$u_t(\cdots, x_i + 1, x_i, x_{i+1}, \cdots) = u_t(\cdots, x_i, x_i, x_{i+1}, \cdots), \quad i = 2, \cdots, N \quad (2.14)$$

$$u_t(x_N + L, x_2, \cdots, x_N) = u_t(x_N + L - 1, x_2, \cdots, x_N) \quad (2.15)$$

(iii) (Initial condition) We impose the same initial condition of u_t and P_t,

$$u_t(\vec{x}; \vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}. \quad (2.16)$$

A priori it is not obvious at all that there exist functions $\mathbb{Z}^N \to \mathbb{R}$ satisfying (i),(ii) and (iii), but if we can find such function u_t, then restricting on $\Omega_{L,N}$ we have $u_t(\vec{x}; \vec{y}) = P_t(\vec{y} \to \vec{x})$ since they have identical evolution equations and initial conditions. The remaining parts of this section will provide a construction of such function u_t.

2.5.2 Solving the free evolution equation

The free evolution equation (2.13) is solvable by Fourier methods with solutions typically of the form (sometimes called plane waves in physics literature)

$$A(\xi_1, \cdots, \xi_N) \prod_{j=1}^{N} \xi_j^{x_j} e^{(\xi_j^{-1} - 1)t},$$

for some undermined complex variables $\xi_1, \cdots, \xi_N \in \mathbb{C}$ and the coefficients $A(\xi_1, \cdots, x_N)$ are independent of $\vec{x}$ (they may depend on the initial condition $\vec{y}$). We will search for solutions that are linear combinations (superpositions) of the plane waves above and satisfy the initial and boundary conditions.

2.5.3 Satisfying the boundary conditions (2.14)

Owing to the fact that plane waves remain solutions of the free evolution equation under S_N-action and we are considering indistinguishable particles, we will take solutions of the following form as natural candidates for u_t:

$$\sum_{\sigma\in S_N} A_\sigma(\xi_1,\cdots,\xi_N)\prod_{j=1}^{N}\xi_{\sigma(j)}^{x_j}e^{(\xi_j^{-1}-1)t}$$

Boundary conditions (2.14) impose the following constraints on A_σ: for $2\le k\le N$,

$$\sum_{\sigma\in S_N} A_\sigma(\xi_1,\cdots,\xi_N)\prod_{j=1}^{N}e^{(\xi_j^{-1}-1)t}\prod_{j\neq k,k-1}\xi_{\sigma(j)}^{x_j}\cdot(\xi_{\sigma(k)}^{x_k}\xi_{\sigma(k-1)}^{x_k})(\xi_{\sigma_{(k-1)}}-1)=0.$$

Owing to the two-body nature, we impose the following stronger constraints on A_σ which directly implies the above cosntraints. For any $\sigma\in S_N$ and transposition $\tau_k:=(k-1,k)$ for $2\le k\le N$ we would like the coefficients to satisfy

$$A_\sigma(\xi,\cdots,\xi_N)(\xi_{\sigma_{(k-1)}}-1)+A_{\sigma\tau_k}(\xi,\cdots,\xi_N)(\xi_{\sigma_{(k)}}-1)=0.$$

A natural candidate for $A_\sigma(\xi_1,\cdots,\xi_N)$ takes the form

$$A_\sigma(\xi,\cdots,\xi_N)=\mathrm{sgn}(\sigma)\prod_{j=1}^{N}(1-\xi_{\sigma(j)})^j\cdot D(\xi_1,\cdots,\xi_N),$$

for some function $D(\xi_1,\cdots,\xi_N)$ independent of σ.

2.5.4 Satisfying the initial condition (2.16)

To find suitable candidates of u_t that satisfy the initial condition (2.16), we need further superpositions for the wave functions. This is achieved by taking multiple contour integrals, namely we consider wave functions $V_t(\vec{x};\vec{y})$ of the following form:

$$\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\oint\frac{\mathrm{d}\xi_1}{2\pi\mathrm{i}}\cdots\oint\frac{\mathrm{d}\xi_N}{2\pi\mathrm{i}}D(\xi_1,\cdots,\xi_N)\prod_{j=1}^{N}\xi_{\sigma(j)}^{x_j}(1-\xi_{\sigma(j)})^j e^{(\xi_j^{-1}-1)t}. \tag{2.17}$$

We would like $V_0(\vec{x};\vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}$. For $N = 1$ this is clearly achieved by taking $D(\xi_1) = \xi_1^{-y_1-1}(1-\xi_1)^{-j}$ and the integral contour be any circle $|\xi_1| = r < 1$. For general N we make the naive guess by taking $D(\xi_1,\cdots,\xi_N) = \prod_{j=1}^N \xi_j^{-y_j-1}(1-\xi_j)^{-j}$. By residue theorem the contribution comes from $\sigma = \mathrm{id}$ gives the desired indicator,

$$\oint\frac{\mathrm{d}\xi_1}{2\pi\mathrm{i}}\cdots\oint\frac{\mathrm{d}\xi_N}{2\pi\mathrm{i}}\prod_{j=1}^{N}\xi_j^{x_j-y_j-1} = \mathbf{1}_{\vec{x}=\vec{y}}.$$

One then hopes the contributions coming from $\sigma \neq \mathrm{id}$ sum to 0. Even better, it turns out that every single term with $\sigma \neq \mathrm{id}$ in the sum (2.17) vanishes. To see this, for each $\sigma \neq \mathrm{id}$ we set $\pi = \sigma^{-1}$ and write the term in (2.17) corresponding to σ as follows:

$$\prod_{j=1}^{N}\oint\frac{d\xi_j}{2\pi\mathrm{i}\xi_j}\xi_j^{x_{\pi(j)}-y_j}(1-\xi_j)^{\pi(j)-j}.$$

We will show at least one of the term in the product vanishes for $\pi \neq \mathrm{id}$. Recall an inversion of a permutation π is a pair of indices $a < b$ with $\pi(a) > \pi(b)$. We take the inversion of π with largest such b, namely $b \in \{1,\cdots,N\}$ be the largest index with $\pi(b) \neq b$. Then $\pi(b) < b$ and there exist index a with $a < b$ and $\pi(a) = b$. We claim that at least one of the integral above with $j = a$ or $j = b$ is zero. Since $\pi(b) < b$, a direct residue calculation shows

$$\oint\frac{d\xi_b}{2\pi\mathrm{i}\xi_b}\xi_b^{x_{\pi(b)}-y_b}(1-\xi_b)^{\pi(b)-b} = \sum_{j=0}^{\infty}c_j\mathbf{1}_{y_b-x_{\pi(b)}=j},$$

for some constants c_j. Similarly since $\pi(a) > a$, we have

$$\oint \frac{d\xi_b}{2\pi \mathrm{i}\xi_a} \xi_a^{x_{\pi(a)}-y_a} (1-\xi_b)^{\pi(a)-a} = \sum_{j=0}^{\pi(a)-a} d_j \mathbf{1}_{y_a - x_{\pi(a)}=j}.$$

Now if $y_b < x_{\pi(b)}$, then the integral with respect to ξ_b vanishes. On the other hand if $y_b \geq x_{\pi(b)}$, then since $y_a \geq y_b + b - a$ and $x_{\pi(a)} = x_b \leq x_a + a - b = x_{\pi(b)} + a - b$, we have

$$y_a - x_{\pi(a)} \geq y_b + b - a - x_{\pi(b)} - a + b \geq 2(b-a),$$

which implies the integral with respect to ξ_a vanishes. Thus the contributions from any $\sigma \neq \mathrm{id}$ vanish.

2.5.5 Cyclic invariance and boundary condition (2.15)

If we are working with TASEP on the full-space $\mathbb{Z}$, then the calculations in previous sections already give a solution

$$\begin{aligned} V_t(\vec{y} \to \vec{x}) &= \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \oint \frac{\mathrm{d}\xi_1}{2\pi \mathrm{i}} \cdots \oint \frac{\mathrm{d}\xi_N}{2\pi \mathrm{i}} \prod_{j=1}^{N} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)}-1} (1-\xi_{\sigma(j)})^{j-\sigma(j)} e^{(\xi_j^{-1}-1)t} \\ &= \det \left[\oint \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \xi^{x_j - y_i - 1} (1-\xi)^{j-i} e^{(\xi^{-1}-1)t} \right]_{i,j=1}^{N}, \end{aligned} \tag{2.18}$$

which is the famous TASEP transition probability formula of Schütz [99]. For periodic model there is yet another boundary condition (2.15) which is not satisfied by the above formula and one needs further superpositions. Owing to the periodic nature, the desired transition probability for periodic TASEP should have the following additional cyclic symmetry : If we replace $\vec{y} = (y_1, \cdots, y_N)$ by $\vec{y}' = (y_2, \cdots, y_N, y_1 - L)$ and $\vec{x} = (x_1, \cdots, x_N)$ by $\vec{x}' = (x_2, \cdots, x_N, x_1 - L)$, then the transition probability should remain the same. A function u_t with this cyclic symmetry satisfying the boundary conditions (2.14) will automatically satisfy the extra boundary condition

(2.15). Inserting $\vec{y}'$ and $\vec{x}'$ into the right hand side of (2.18) we obtain

$$V_t(\vec{x}';\vec{y}') = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} \oint \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \xi^{x'_{\sigma(j)}-y'_j-1}(1-\xi)^{\sigma(j)-j} e^{(\xi^{-1}-1)t} l,$$

where $\vec{x}' = (x'_1,\cdots,x'_N) = (x_2,\cdots,x_N,x_1-L)$ and similar for $\vec{y}'$. Take $\tau = (12\cdots N) \in S_N$ be a cyclic permutation, then clearly

$$x'_j = x_{\tau(j)} - L\mathbf{1}_{j=N}, \quad j = \tau(j) - 1 + N\mathbf{1}_{j=N}, \quad \text{for } 1 \le j \le N.$$

Hence

$$V_t(\vec{x}';\vec{y}') = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} \oint \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \xi^{(x_{\tau\sigma(j)}-L\mathbf{1}_{\sigma(j)=N})-(y_{\tau(j)}-L\mathbf{1}_{j=N})-1}$$
$$(1-\xi)^{(\tau\sigma(j)-1+N\mathbf{1}_{\sigma(j)=N})-(\tau(j)-1+N\mathbf{1}_{j=N})} e^{(\xi^{-1}-1)t}.$$

Replacing σ with $\tau\sigma\tau^{-1}$ we can rewrite $V_t(\vec{x}';\vec{y}')$ as

$$V_t(\vec{x}';\vec{y}') = \sum_{\pi\in S_N} \mathrm{sgn}(\pi) \prod_{j=1}^{N} \oint \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \xi^{(x_{\pi(j)}-L\mathbf{1}_{\pi(j)=1})-(y_j-L\mathbf{1}_{j=1})-1}$$
$$(1-\xi)^{\pi(j)+N\mathbf{1}_{\pi(j)=1})-(j+N\mathbf{1}_{j=1})} e^{(\xi^{-1}-1)t}.$$

This in general does not agree with $V_t(\vec{y} \to \vec{x})$. Note however that the contribution coming from $\pi = \mathrm{id}$ remains the same. To get the cyclic symmetry we further taking the sum over all possible wavefunctions $\{V_t(C^i(\vec{x},\vec{y}))\}_{i\in\mathbb{Z}}$ where $\{C^i(\vec{x},\vec{y})\}_{i\in\mathbb{Z}}$ is the orbit of $(\vec{x},\vec{y})$ under the cyclic shift $(\vec{x},\vec{y}) \to C(\vec{x},\vec{y}) = (\vec{x}',\vec{y}')$ for the same $\vec{x}'$ and $\vec{y}'$ above. Such averaging clearly leads to cyclic invariant solution and thus we arrive at solution $u_t(\vec{x};\vec{y})$ of the form

$$\sum_{\substack{\vec{m}\in\mathbb{Z}^N\\ m_1+\cdots+m_N=0}}\sum_{\pi\in S_N}\mathrm{sgn}(\pi)\prod_{j=1}^{N}\oint\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\xi^{x_{\pi(j)}-y_j+m_jL-1}(1-\xi)^{\pi(j)-j-m_jN}e^{(\xi^{-1}-1)t} \tag{2.19}$$

We check that the above infinite sum converges. First note that since $x_{\pi(j)}-y_j+\pi(j)-j\le x_1-y_N+N-1$, we have for large ξ

$$\xi^{x_{\pi(j)}-y_j+m_jL-1}(1-\xi)^{\pi(j)-j-m_jN}e^{(\xi^{-1}-1)t}=O(|\xi|^{m_j(L-N)+x_1-y_N+N-2}).$$

Hence if we take K large enough such that $-K(L-N)+x_1-y_N+N\le 0$. Then for any $\vec{m}\in\mathbb{Z}^N$ with $\min(m_i)\le -K$, at least one of the integrands is $O(\xi^{-2})$ and hence

$$\prod_{j=1}^{N}\oint\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\xi^{x_{\pi(j)}-y_j+m_jL-1}(1-\xi)^{\pi(j)-j-m_jN}e^{(\xi^{-1}-1)t}=0.$$

This reduces equation (2.19) to a smaller sum

$$\sum_{\substack{\vec{m}\in\mathbb{Z}^N\\ m_1+\cdots+m_N=0\\ \min(m_i)\ge -K}}\sum_{\pi\in S_N}\mathrm{sgn}(\pi)\prod_{j=1}^{N}\oint\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\xi^{x_{\pi(j)}-y_j+m_jL-1}(1-\xi)^{\pi(j)-j-m_jN}e^{(\xi^{-1}-1)t}$$

We recognize the above sum over $\vec{m}$ as taking the constant term in some Laurent series $\prod_{j=1}^{N}\left(\sum_{\ell=-\infty}^{\infty}c_{j\ell}z^{\ell}\right)$. Hence by residue theorem the above sum equals

$$\sum_{\pi\in S_N}\mathrm{sgn}(\pi)\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}z}\prod_{j=1}^{N}\left[\oint\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\xi^{x_{\pi(j)}-y_j-1}(1-\xi)^{\pi(j)-j}\frac{\left(\frac{(1-\xi)^N}{\xi^L z}\right)^K}{1-\frac{\xi^L z}{(1-\xi)^N}}e^{(\xi^{-1}-1)t}\right],$$

where the integral contour for z is any small circle such that $|\frac{\xi^L z}{(1-\xi)^N}| < 1$. Simplifying the sum over S_N as a determinant we have

$$u_t(\vec{x};\vec{y}) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\oint \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \xi^{x_i-y_j-1}(1-\xi)^{i-j} \frac{\left(\frac{(1-\xi)^N}{\xi^L z}\right)^K}{1-\frac{\xi^L z}{(1-\xi)^N}} e^{(\xi^{-1}-1)t}\right]_{i,j=1}^N$$

Finally introducing the change of variable $w = \xi^{-1} - 1$, the w contours will then be large circle $|w| = R$ and we have

$$\begin{aligned} u_t(\vec{x};\vec{y}) &= \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}} (w+1)^{y_j-x_i+j-i-1} w^{i-j} \frac{\left(w^N (w+1)^{L-N} z^{-1}\right)^K}{1-\frac{z}{w^N(w+1)^{L-N}}} e^{tw}\right]_{i,j=1}^N \\ &= \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\sum_{w:q(w)=z} (w+1)^{y_j-x_i+j-i-1} w^{i-j} \frac{q(w)}{q'(w)} e^{tw}\right]_{i,j=1}^N . \end{aligned}$$

Here $q(w) = w^N(1+w)^{L-N}$ and the last equality is a simple consequence of residue theorem. This is precisely the desired formula (2.18).

CHAPTER 3

Multi-time Distribution of Inhomogeneous TASEP

In this chapter we study an inhomogeneous generalization of the totally asymmetric simple exclusion processes, depending on two sets of parameters. The finite-time multi-point distributions are obtained, first for the model on a periodic domain and then for the model on the full-space $\mathbb{Z}$. For the full-space model we then obtain large time asymptotics for the multi-time distributions, these can be seen as a multi-time analogue of the Baik-Ben Arous-Péché phase transition.

3.1 The Models and main results

3.1.1 Inhomogeneous TASEP on $\mathbb{Z}$

Given two sets of real parameters $\{\pi_i\}_{i\in\mathbb{Z}}$ and $\{\hat{\pi}_j\}_{j\in\mathbb{Z}}$ satisfying $\pi_i + \hat{\pi}_j > 0$ for all i, j. We consider an inhomogeneous variant of the totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$ depending on the two sets of parameters $\{\pi_i\}$ and $\{\hat{\pi}_j\}$. There are two types of particles, black or white, located on the integer lattice $\mathbb{Z}$ such that each integer point is occupied by exactly one particle. For $j \in \mathbb{Z}$, we define the occupation function $\eta_j(t)$ as follows:

$$\eta_j(t) := \begin{cases} 1 & \text{if there is a black particle at site } j \text{ at time } t, \\ 0 & \text{if there is a white particle at site } j \text{ at time } t. \end{cases}$$

For $i, j \in \mathbb{Z}$, we denote the location of the i-th black particle at time t by $x_i(t)$ and the location of the j-th white particle at time t by $\hat{x}_j(t)$, where the index ordering for black particles is from right to left and the ordering for white particles is from left to right. In particular we have

$$\cdots > x_0(t) > x_1(t) > x_2(t) > \cdots,$$

and

$$\cdots < \hat{x}_0(t) < \hat{x}_1(t) < \hat{x}_2(t) < \cdots.$$

The particle configurations evolve according to the following dynamics: every pair of consecutive particles consisting of the i-th black particle on the left and j-th white particle on the right will exchange their locations after an independent exponential waiting time with rate $\pi_i + \hat{\pi}_j$. Particles with the same color will not exchange locations. Such model was first introduced in [20] in the equivalent form as a directed last passage percolation model with two sets of parameters.

3.1.2 Multi-point distribution

The first main theorem of this chapter is a formula for the finite-time multi-point joint distributions of arbitrary many tagged particles under inhomogeneous TASEP dynamics. It is a generalization of the corresponding result in [81] for the homogeneous degeneration when $\pi_i \equiv 0$ and $\hat{\pi}_j \equiv 1$.

Theorem 3.1.1. *Consider inhomogeneous TASEP on* $\mathbb{Z}$ *with parameters* $\{\pi_i\}$ *and*

$\{\hat{\pi}_i\}$. Let $\vec{y} = (y_1, \cdots)$ be the initial condition with $y_1 = -1$. Let m be a positive integer and $(k_1, t_1), \cdots, (k_m, t_m)$ be m distinct points in $\{1, \cdots, N\} \times [0, \infty)$. Assume that $0 \le t_1 \le \cdots \le t_m$. Then for any integers $a_1, \cdots, a_m$,

$$\mathbb{P}_{\vec{y}}\left(\bigcap_{\ell=1}^{m}\{x_{k_\ell}(t_\ell) \ge a_\ell\}\right) = \left[\prod_{\ell=1}^{m-1} \oint \frac{\mathrm{d}\theta_\ell}{2\pi \mathrm{i}\theta_\ell} \frac{1}{1-\theta_\ell}\right] \mathcal{D}_{\vec{y}}(\theta_1, \cdots, \theta_{m-1}). \tag{3.1}$$

where the integral is over circles centered at the origin with radius less than 1 and the function $\mathcal{D}_{\vec{y}}(\theta_1, \cdots, \theta_{m-1})$ is defined using a Fredholm determinant in Definition 3.1.4

Remark 3.1.2. For $m = 1$, there is no outer contour integral and our formula reduces to a single Fredholm determinant which can be shown to be equivalent to the one obtained in [20].

Remark 3.1.3. Dieker and Warren [40] showed the following remarkable equality in distribution between the equal time multi-point distribution of inhomogeneous exponential last passage percolation and the joint distribution of largest eigenvalues of corners of a generalized Wishart random matrix ensemble, namely

$$(G(N,1), \cdots, G(N,M)) \stackrel{d}{=} (\lambda_{\max}(N,1), \cdots, \lambda_{\max}(N,M)),$$

where the left hand side is the joint distribution of last passage time of locations at the same row (see Section 2.2.2 for details) and the right hand side above is the joint distribution of largest eigenvalues of M random matrices $W_1, \cdots, W_M$, where $W_k = A_k A_k^*$ for A_k the top $N \times k$ corner of a two-dimensional array of independent complex gaussian random variables with the (i,j)-th entry having variance $\frac{1}{\pi_i + \hat{\pi}_j}$. A Monte Carlo simulation indicates that this does not extend to multi-time joint laws, namely

$$(G(N_1, M_1), \cdots, G(N_k, M_k)) \stackrel{d}{\neq} (\lambda_{\max}(N_1, M_1), \cdots, \lambda_{\max}(N_k, M_k)),$$

for the same two-dimensional arrays $(G(n,m))_{m,n\in\mathbb{Z}_{>0}}$ and $(\lambda_{\max}(n,m))_{n,m\in\mathbb{Z}_{>0}}$ described above. It remains an interesting open question whether the two-dimensional random field $(G(n,m))_{m,n\in\mathbb{Z}_{>0}}$ will appear in certain random matrix model.

3.1.3 Fredholm determinant formula for $\mathcal{D}_{\vec{y}}(\theta_1,\cdots,\theta_{m-1})$

We will define the Fredholm determinant $\mathcal{D}_{\vec{y}}(\theta_1,\cdots,\theta_{m-1}) = \det(I-\mathcal{K}_1\mathcal{K}_{\vec{y}})$, with the two operators $\mathcal{K}_1$ and $\mathcal{K}_{\vec{y}}$ acting on two specific spaces of nested contours with complex measures depending on complex parameters $\theta=(\theta_1,\cdots,\theta_{m-1})$ for $1\le\ell\le m-1$.

3.1.3.1 Space of the operators

First we introduce the contours where the operators act on. Let Ω_{L} and Ω_{R} be two simply connected regions on the complex plane such that (1) Ω_{L} contains $\{-\hat{\pi}_i\}$, (2) Ω_{R} contains $\{\pi_i\}$, (3) Ω_{L} and Ω_{R} do not intersect.

Let $\Sigma_{m,\mathrm{L}}^+,\cdots,\Sigma_{2,\mathrm{L}}^+,\Sigma_{1,\mathrm{L}},\Sigma_{2,\mathrm{L}}^-,\cdots,\Sigma_{m,\mathrm{L}}^-$ be $2m-1$ nested simple closed contours, from outside to inside in Ω_{L} enclosing $\{-\hat{\pi}_i\}$. Let $\Sigma_{m,\mathrm{R}}^+,\cdots,\Sigma_{2,\mathrm{R}}^+,\Sigma_{1,\mathrm{R}},\Sigma_{2,\mathrm{R}}^-,\cdots,\Sigma_{m,\mathrm{R}}^-$ be $2m-1$ nested simple closed contours, from outside to inside in Ω_{R} enclosing $\{\pi_i\}$. For $2\le\ell\le m$, set

$$\Sigma_{\ell,\mathrm{L}} := \Sigma_{\ell,\mathrm{L}}^+\cup\Sigma_{\ell,\mathrm{L}}^-,\quad \Sigma_{\ell,\mathrm{R}} := \Sigma_{\ell,\mathrm{R}}^+\cup\Sigma_{\ell,\mathrm{R}}^-,\quad \ell=2,\cdots,m.$$

Finally we define the two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ where the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ act on:

$$\mathcal{S}_1 := \Sigma_{1,\mathrm{L}}\cup\Sigma_{2,\mathrm{R}}\cup\cdots\cup\begin{cases}\Sigma_{m,\mathrm{L}}, & \text{if } m \text{ is odd},\\ \Sigma_{m,\mathrm{R}}, & \text{if } m \text{ is even},\end{cases}$$

and

$$\mathcal{S}_2 := \Sigma_{1,\mathrm{R}} \cup \Sigma_{2,\mathrm{L}} \cup \cdots \cup \begin{cases} \Sigma_{m,\mathrm{R}}, & \text{if } m \text{ is odd,} \\ \Sigma_{m,\mathrm{L}}, & \text{if } m \text{ is even.} \end{cases}$$

See Figure 3.1 for an illustration. We associate complex measures to each of these contours as follows:

$$d\mu(w) = d\mu_\theta(w) := \begin{cases} \frac{-\theta_{\ell-1}}{1-\theta_{\ell-1}} \frac{dw}{2\pi \mathrm{i}}, & \text{if } w \in \Sigma_{\ell,\mathrm{L}}^{+} \cup \Sigma_{\ell,\mathrm{R}}^{+} \quad \text{for } \ell = 2, \cdots, m, \\ \frac{1}{1-\theta_{\ell-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}}, & \text{if } w \in \Sigma_{\ell,\mathrm{L}}^{-} \cup \Sigma_{\ell,\mathrm{R}}^{-} \quad \text{for } \ell = 2, \cdots, m, \\ \frac{\mathrm{d}w}{2\pi \mathrm{i}}, & \text{if } w \in \Sigma_{1,\mathrm{L}} \cup \Sigma_{1,\mathrm{R}}. \end{cases} \tag{3.2}$$

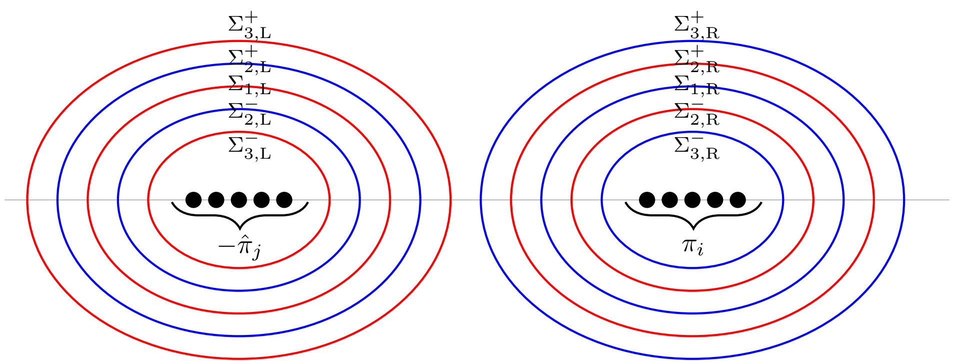

Figure 3.1: An illustration of the contours $\mathcal{S}_1$ and $\mathcal{S}_2$ for $m = 3$. $\mathcal{S}_1$ consists of union of the red contours and $\mathcal{S}_2$ consists of union of the blue contours.

3.1.3.2 The operators $\mathcal{K}_1$ and $\mathcal{K}_{\vec{y}}$

Now we are ready to introduce the operators $\mathcal{K}_1$ and $\mathcal{K}_{\vec{y}}$ and define $\mathcal{D}_{\vec{y}}(\theta_1, \cdots, \theta_{m-1})$. Given complex vector $\vec{\theta} = (\theta_1, \cdots, \theta_{m-1})$ with $\theta_\ell \neq 1$ for $1 \leq \ell \leq m-1$. Let

$$\begin{aligned} Q_1(j) &:= \begin{cases} 1-\theta_j, & \text{if } j < m \text{ is odd,} \\ 1-\frac{1}{\theta_{j-1}}, & \text{if } j \text{ is even,} \\ 1, & \text{if } j = m \text{ is odd,} \end{cases} \\ Q_2(j) &:= \begin{cases} 1-\theta_j, & \text{if } j < m \text{ is even,} \\ 1-\frac{1}{\theta_{j-1}}, & \text{if } j \text{ is odd and } j > 1, \\ 1, & \text{if } j = m \text{ is even, or } j = 1. \end{cases} \end{aligned} \tag{3.3}$$

Definition 3.1.4. *We define*

$$\mathcal{D}_{\vec{y}}(z_1, \cdots, z_{m-1}) = \det(I - \mathcal{K}_1 \mathcal{K}_{\vec{y}}), \tag{3.4}$$

where the two operators

$$\mathcal{K}_1 : L^2(\mathcal{S}_2, d\mu) \to L^2(\mathcal{S}_1, d\mu), \quad \mathcal{K}_{\vec{y}} : L^2(\mathcal{S}_1, d\mu) \to L^2(\mathcal{S}_2, d\mu)$$

are given by the kernels

$$\mathcal{K}_1(w, w') := (\delta_i(j) + \delta_i(j + (-1)^i)) \frac{f_i(w)}{w - w'} Q_1(j), \tag{3.5}$$

and

$$\mathcal{K}_{\vec{y}}(w', w) := \Lambda(i, w, w') \cdot (\delta_j(i) + \delta_j(i - (-1)^j)) \frac{f_j(w')}{w' - w} Q_2(i), \tag{3.6}$$

for any $w \in (\Sigma_{i,\mathrm{L}} \cup \Sigma_{i,\mathrm{R}}) \cap \mathcal{S}_1$ *and* $w' \in (\Sigma_{j,\mathrm{L}} \cup \Sigma_{j,\mathrm{R}}) \cap \mathcal{S}_2$ *for* $1 \le i, j \le m$. *Here*

$$\Lambda(i, w, w') := \begin{cases} \mathrm{ch}_{\vec{y}}(w', w), & \text{if } i = 1, \\ 1, & \text{if } i \ge 2. \end{cases} \tag{3.7}$$

Here the function $\mathrm{ch}_{\vec{y}}(w', w)$ *is an analytic function on* $(\Omega_{\mathrm{R}} \backslash \{\pi_j\}) \times (\Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_j\})$ *defined in Definition 3.1.6. Note that it is the only term in the kernel that depends on the initial condition and it only appears in the top-left corner of the matrix kernel* $\mathcal{K}_{\vec{y}}$. *The functions* $f_i(w)$ *are given by*

$$f_i(w) := \begin{cases} \frac{F_i(w)}{F_{i-1}(w)}, & w \in \Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_i\}, \\ \frac{F_{i-1}(w)}{F_i(w)}, & w \in \Omega_{\mathrm{R}} \backslash \{\pi_i\}, \end{cases} \tag{3.8}$$

for $1 \le i \le m$ *with*

$$F_i(w) := \begin{cases} e^{t_i w} \prod_{\ell=1}^{k_i} (w - \pi_\ell) \cdot \prod_{\ell=1}^{a_i + k_i} (w + \hat{\pi}_\ell)^{-1}, & i = 1, \cdots, m, \\ 1, & i = 0. \end{cases}$$

3.1.3.3 The function $\mathrm{ch}_{\vec{y}}(v, u)$

In our finite-time multi-point distribution formula (3.1), the quantities encoding information in the initial condition are related to the following symmetric function:

Definition 3.1.5. *Given* $\{\pi_i\}$ *and* $\{\hat{\pi}_i\}$. *For* $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N$ *with* $\lambda_1 \ge \cdots \ge \lambda_N$, *we define*

$$\mathcal{F}_\lambda(w_1, \cdots, w_N; \{\pi_j\}, \{\hat{\pi}_\ell\}) := \frac{\det \left[\prod_{\ell=i+1}^{N} (w_j - \pi_\ell) \cdot \prod_{\ell=1}^{\lambda_i} \frac{w_j + \hat{\pi}_\ell}{\pi_i + \hat{\pi}_\ell} \right]_{i,j=1}^{N}}{\det [w_j^{N-i}]_{i,j=1}^{N}}. \tag{3.9}$$

Since λ_i*'s may be negative, in general* $\mathcal{F}_\lambda$ *is a symmetric rational function. For later* *purposes we shift it to get a symmetric polynomial.*

$$\mathcal{F}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\}) := \prod_{j=1}^{N}\prod_{\ell=1}^{\lambda_N}\frac{w_j+\hat{\pi}_\ell}{\pi_j+\hat{\pi}_\ell}\cdot\hat{\mathcal{F}}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\}),$$

where $\hat{\mathcal{F}}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\})$ *is defined as*

$$\hat{\mathcal{F}}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\}) = \frac{\det\left[\prod_{\ell=i+1}^{N}(w_j-\pi_\ell)\cdot\prod_{\ell=\lambda_N+1}^{\lambda_i}\frac{w_j+\hat{\pi}_\ell}{\pi_i+\hat{\pi}_\ell}\right]_{i,j=1}^{N}}{\det[w_j^{N-i}]_{i,j=1}^{N}}. \tag{3.10}$$

Here to introduce $\mathrm{ch}_{\vec{y}}(v,u)$ it is convenient to introduce the shifted power sum symmetric functions as a basis for the ring of symmetric functions and expand $\mathcal{F}_{\lambda(\vec{y})}(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\})$ in terms of this basis. More precisely for $j\in\mathbb{Z}_{>0}$ we set

$$\hat{p}_j(w_1,\cdots,w_N) = \hat{p}_j(w_1,\cdots,w_N;\{\pi_j\}) := \sum_{i=1}^{N}(w_i^j-\pi_i^j).$$

And for partition $\mu=(\mu_1,\cdots,\mu_\ell)$ we set

$$\hat{p}_\mu(w_1,\cdots,w_N) := \prod_{j=1}^{\ell}\hat{p}_j(w_1,\cdots,w_N).$$

For $\mu=\emptyset$ we simply set $\hat{p}_\mu := 1$. Clearly $\{\hat{p}_\mu(\vec{w})\}_{\mu\in\mathbb{Y}_N}$ spans the ring of symmetric polynomials in N variables since the usual power sums are clearly spanned by the shifted ones. Here $\mathbb{Y}_N$ is the set of all partitions with at most N parts. Note that $\hat{p_\mu}(\pi_1,\cdots,\pi_N) = 0$ for any $\mu\neq\emptyset$. Now we expand the symmetric polynomial $\hat{F}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\})$ defined in Definition 3.1.5 in terms of the $\hat{p}_\mu$'s. For $\lambda=(\lambda_1,\cdots,\lambda_N)\in\mathbb{Z}^N$ we write

$$\hat{F}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\}) = 1+\sum_{\mu\neq\emptyset}c_{\lambda,\mu}\hat{p}_\mu(w_1,\cdots,w_N), \tag{3.11}$$

where the summation is over all nonempty partitions μ and the coefficient $c_{\lambda,\mu}$ may depend on the parameters $\{\pi_j\}$ and $\{\hat{\pi}_\ell\}$. Note that for $|\mu| > \sum_{i=1}^{N}(\lambda_i - \lambda_N)$ we have $c_{\lambda,\mu} = 0$ so the sum is finite. The constant 1 comes from evaluating $\hat{F}_\lambda$ at $w_j = \pi_j$ for $1 \le j \le N$.

Definition 3.1.6. *For $u \in \Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_j\}$ and $v \in \Omega_{\mathrm{R}} \backslash \{\pi_j\}$, we define*

$$\mathrm{ch}_{\vec{y}}(v,u) := \left(\prod_{\ell=1}^{y_N+N} \frac{u+\hat{\pi}_\ell}{v+\hat{\pi}_\ell}\right) \cdot \left(1 + \sum_{\mu \neq \emptyset} c_{\lambda,\mu} \left(\prod_{k=1}^{\ell(\mu)} (u^{\mu_k} - v^{\mu_k})\right)\right). \tag{3.12}$$

Where $\lambda = (y_1+1, \cdots, y_N+N)$ and the coefficients $c_{\lambda,\mu}$ is the same as in the expansion of $\hat{F}_\lambda$ in the $\hat{p}_\mu$'s.

Remark 3.1.7. For step initial condition $\vec{y} = (-1, -2, \cdots)$, it is straightforward to check that $\mathcal{F}_{\lambda(\vec{y})} = 1$ and hence $\mathrm{ch}_{\mathrm{step}} = 1$.

3.2 Large time asymptotics

We consider the large time asymtotics of the multi-time joint distribution of the inhomogeneous TASEP. For simplicity we will only study the step initial condition $y_i = -i$ for all $i \ge 1$. We are mainly interested in the case when $\pi_j = 0$ and $\hat{\pi}_\ell = 1$ for all but finitely many j's and ℓ's. Such asymptotics for the one-time distribution with one set of parameters ($\pi_j \equiv 0$ for all j) was first studied in [6]. In [20] the authors obtained the multi-point equal-time distribution with two sets of parameters in the critical regime.

3.2.1 A multi-time anologue of the Baik-Ben Arous-Péché transition

The following theorem is a multi-time analogue of Theorem 1.1 of [6]. For notational convenience we will only consider the case when $\frac{a+k}{k} \to 1$ as $T \to \infty$, which corresponds to the case when $\gamma = 1$ in Theorem 1.1 of [6]. The critical value for the strength of spike where the phase transition occurs will be $(1+\gamma^{-1})^{-1} = \frac{1}{2}$ under our assumptions.

Theorem 3.2.1. *Consider inhomogeneous TASEP on $\mathbb{Z}$ with step initial condition $\vec{x}(0) = -i$. Fix two integers $r, s \geq 0$. Assume that $\pi_\ell = 0$ for all $\ell \geq r+1$ and $\hat{\pi}_\ell = 1$ for all $\ell \geq s+1$. Depending on the relationship between $\{\pi_i\}_{1\leq i\leq r}$ with the critical value $-\frac{1}{2}$ and the relationship between $\{\hat{\pi}_j\}_{1\leq j\leq s}$ with $\frac{1}{2}$, one has the following three different behaviours of the joint height fluctuations:*

(i) (The critical regime) Assume that $\pi_\ell = -\frac{1}{2} + \frac{1}{2}\lambda_\ell \cdot T^{-1/3}$ for $1 \leq \ell \leq r$ and $\pi_\ell = \frac{1}{2} - \frac{1}{2}\mu_\ell \cdot T^{-1/3}$ for $1 \leq \ell \leq s$ where $\lambda_i > \mu_j$ for all i, j. Then under the scaling

$$a_\ell = 2x_\ell T^{2/3}, \quad k_\ell = \frac{1}{2}\tau_\ell T - \gamma_\ell T^{2/3} - \frac{1}{2}u_\ell T^{1/3}, \quad t_\ell = 2\tau_\ell T. \tag{3.13}$$

Where $\tau_1 < \cdots < \tau_m$ and $\gamma_1, \cdots, \gamma_m, u_1, \cdots, u_m \in \mathbb{R}$. We have

$$\begin{aligned} &\lim_{T\to\infty} \mathbb{P}_{\text{step}}\left(\bigcap_{\ell=1}^{m}\{x_{k_\ell}(t_\ell) \geq a_\ell\}\right) \\ &= \mathrm{F}_{\mathrm{BBP};\vec{\lambda},\vec{\mu}}(u_1, \cdots, u_m; (\gamma_1, \tau_1), \cdots, (\gamma_m, \tau_m)), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &\mathrm{F}_{\mathrm{BBP};\vec{\lambda},\vec{\mu}}(u_1, \cdots, u_m; (\gamma_1, \tau_1), \cdots, (\gamma_m, \tau_m)) \\ &= \oint \cdots \oint \left[\prod_{\ell=1}^{m-1} \frac{1}{1-\theta_\ell}\right] \mathrm{D}^{\mathrm{BBP};\vec{\lambda},\vec{\mu}}(\theta_1, \cdots, \theta_{m-1}) \frac{\mathrm{d}\theta_1}{2\pi\mathrm{i}\theta_1} \cdots \frac{\mathrm{d}\theta_{m-1}}{2\pi\mathrm{i}\theta_{m-1}}. \end{aligned} \tag{3.15}$$

Here $\mathrm{D}^{\mathrm{BBP};\vec{\lambda},\vec{\mu}}(\theta_1, \cdots, \theta_{m-1})$ is a Fredholm derminant defined in Definition 3.2.3.

(ii) (The sub-critical regime) Assume that π_ℓ stays in a compact subset of $(-\frac{1}{2}, 0)$ for $1 \leq \ell \leq r$ and π_ℓ stays in a compact subset of $(-1, -\frac{1}{2})$ for $1 \leq \ell \leq s$. Then

under the same scaling as in equation (3.13) *we have*

$$\lim_{T\to\infty} \mathbb{P}_{\text{step}} \left(\bigcap_{\ell=1}^{m} \{x_{k_\ell}(t_\ell) \geq a_\ell\} \right) = \mathrm{F}_{\text{step}}(u_1, \cdots, u_m; (\gamma_1, \tau_1), \cdots, (\gamma_m, \tau_m)), \tag{3.16}$$

where $\mathrm{F}_{\text{step}}(u_1, \cdots, u_m; (\gamma_1, \tau_1), \cdots, (\gamma_m, \tau_m))$ *is the same as the limiting distribution obtained in Thereom 2.20 in [81] as the limiting height fluctuation of homogeneous TASEP. It can be regarded as taking* $r = s = 0$ *in the critical limiting distribution defined above.*

(iii) (The super-critical regime) Assume $r = 1$ *and* $s = 0$ *for simplicity. If* $\pi_1 \in (-1, -\frac{1}{2})$*, then under the scaling*

$$k_\ell = \tau_\ell T, \quad a_\ell = 0, \quad , t_\ell = \frac{1}{-\pi_1(1+\pi_1)}\tau_\ell T - \frac{\sqrt{-(2\pi_1+1)}}{\pi_1(1+\pi_1)} x_\ell T^{1/2}, \tag{3.17}$$

where $\tau_1 < \cdots < \tau_m$ *and* $u_1, \cdots, u_m \in \mathbb{R}$*, we have*

$$\lim_{T\to\infty} \mathbb{P}_{\text{step}} \left(\bigcap_{\ell=1}^{m} \{x_{k_\ell}(t_\ell) \geq a_\ell\} \right) = \mathrm{G}_1(x_1, \cdots, x_m; \tau_1, \cdots, \tau_m). \tag{3.18}$$

Where

$$\mathrm{G}_1(x_1, \cdots, x_m; \tau_1, \cdots, \tau_m) = \mathbb{P} \left(\bigcap_{\ell=1}^{m} \{B(\tau_\ell) \leq u_\ell\} \right). \tag{3.19}$$

Where $B(t)$ *is a standard one-dimensional Brownian motion.*

Remark 3.2.2. Part (i) of Theorem 3.2.1 is our main contribution. Part (iii) was obtained in [35] using probabilistic arguments without knowing the finite-time multi-point distribution. We conjecture that for general rank k, the joint height fluctuations are described by the joint law of the largest eigenvlalues of a $k \times k$ Hermitian matrix Brownian motion at time $\tau_1, \cdots, \tau_m$.

3.2.2 Spaces of the operators

Given integer $m \geq 1$. We first fix two sets of real numbers $\{a_1, a_2^\pm, \cdots, a_m^\pm\}$ and $\{b_1, b_2^\pm, \cdots, b_m^\pm\}$ satisfying

$$\max_i \mu_i < b_m^+ < \cdots < b_1 < \cdots < b_m^- < a_m^- < \cdots < a_1 < \cdots < a_m^+ < \min_i \lambda_i.$$

Then we define the contours in the complex plane by

$$\Gamma_{1,\mathrm{R}} := \{w = a_1 + re^{\frac{\pi \mathrm{i}}{3}} : r \geq 0\} \cup \{w = a_1 + re^{\frac{-\pi \mathrm{i}}{3}} : r \geq 0\},$$

And for $2 \leq j \leq m$

$$\Gamma_{j,\mathrm{R}}^\pm := \{w = a_j^\pm + re^{\frac{\pi \mathrm{i}}{3}} : r \geq 0\} \cup \{w = a_j^\pm + re^{\frac{-\pi \mathrm{i}}{3}} : r \geq 0\}.$$

The contours are oriented from $e^{-\frac{\pi \mathrm{i}}{3}}\infty$ to $e^{\frac{\pi \mathrm{i}}{3}}\infty$. Similarly

$$\Gamma_{1,\mathrm{L}} := \{w = b_1 + re^{\frac{2\pi \mathrm{i}}{3}} : r \geq 0\} \cup \{w = b_1 + re^{\frac{-2\pi \mathrm{i}}{3}} : r \geq 0\},$$

And for $2 \leq j \leq m$

$$\Gamma_{j,\mathrm{L}}^\pm := \{w = b_j^\pm + re^{\frac{2\pi \mathrm{i}}{3}} : r \geq 0\} \cup \{w = b_j^\pm + re^{\frac{-2\pi \mathrm{i}}{3}} : r \geq 0\}.$$

The contours are oriented from $e^{-\frac{2\pi \mathrm{i}}{3}}\infty$ to $e^{\frac{2\pi \mathrm{i}}{3}}\infty$. Now set

$$\Gamma_{j,\mathrm{L}} := \Gamma_{j,\mathrm{L}}^+ \cup \Gamma_{\ell,\mathrm{L}}^-, \qquad \Gamma_{j,\mathrm{R}} := \Gamma_{j,\mathrm{R}}^+ \cup \Gamma_{j,\mathrm{R}}^-, \qquad \Gamma_j = \Gamma_{j,\mathrm{L}} \cup \Gamma_{j,\mathrm{R}}, \qquad j = 1, \cdots, m,$$

and

$$\mathbb{S}_1 := \Gamma_{1,\mathrm{L}} \cup \Gamma_{2,\mathrm{R}} \cup \cdots \cup \begin{cases} \Gamma_{m,\mathrm{L}}, & \text{if } m \text{ is odd,} \\ \Gamma_{m,\mathrm{R}}, & \text{if } m \text{ is even,} \end{cases}$$

and

$$\mathbb{S}_2 := \Gamma_{1,\mathrm{R}} \cup \Gamma_{2,\mathrm{L}} \cup \cdots \cup \begin{cases} \Gamma_{m,\mathrm{R}}, & \text{if } m \text{ is odd,} \\ \Gamma_{m,\mathrm{L}}, & \text{if } m \text{ is even.} \end{cases}$$

We associate complex measures to the contours depending on the parameters $\vec{\theta}$ in the same way as the finite-time distribution:

$$d\nu(w) = d\nu_\theta(w) := \begin{cases} \frac{-\theta_{\ell-1}}{1-\theta_{\ell-1}} \frac{dw}{2\pi \mathrm{i}}, & \text{if } w \in \Gamma^+_{\ell,\mathrm{L}} \cup \Gamma^+_{\ell,\mathrm{R}} \quad \text{for } \ell = 2, \cdots, m, \\ \frac{1}{1-\theta_{\ell-1}} \frac{dw}{2\pi \mathrm{i}}, & \text{if } w \in \Gamma^-_{\ell,\mathrm{L}} \cup \Gamma^-_{\ell,\mathrm{R}} \quad \text{for } \ell = 2, \cdots, m, \\ \frac{dw}{2\pi \mathrm{i}}, & \text{if } w \in \Gamma_{1,\mathrm{L}} \cup \Gamma_{1,\mathrm{R}}. \end{cases} \tag{3.20}$$

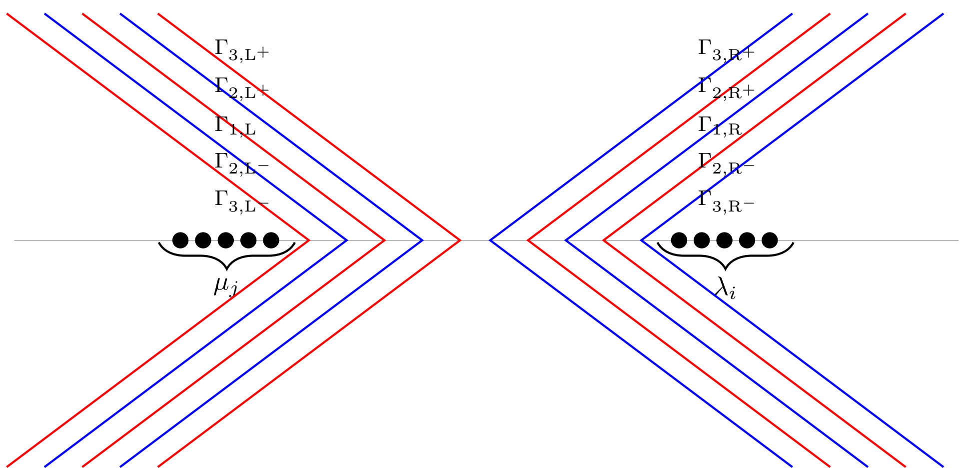

Figure 3.2: An illustration of the limiting contours $\mathbb{S}_1$ and $\mathbb{S}_2$ for $m = 3$. $\mathbb{S}_1$ consists of union of the red contours and $\mathbb{S}_2$ consists of union of the blue contours.

3.2.3 Operators $\mathrm{K}_1^{\vec{\lambda},\vec{\mu}}$ and $\mathrm{K}_{\mathrm{step}}^{\vec{\lambda},\vec{\mu}}$

Now we introduce the operators $\mathrm{K}_1^{\vec{\lambda},\vec{\mu}}$ and $\mathrm{K}_{\mathrm{step}}^{\vec{\lambda},\vec{\mu}}$ to define $\mathrm{D}^{\mathrm{BBP},\vec{\lambda},\vec{\mu}}$ in Theorem 3.2.1.

Definition 3.2.3. *We define*

$$\mathrm{D}^{\mathrm{BBP};\vec{\lambda},\vec{\mu}}(\theta_1, \cdots, \theta_{m-1}) = \det\left(I - \mathrm{K}_1^{\vec{\lambda},\vec{\mu}} \mathrm{K}_{\mathrm{step}}^{\vec{\lambda},\vec{\mu}}\right),$$

where the two operators

$$\mathrm{K}_1^{\vec{\lambda},\vec{\mu}} : L^2(\mathbb{S}_2, \mathrm{d}\nu) \to L^2(\mathbb{S}_1, \mathrm{d}\nu), \qquad \mathrm{K}_{\mathrm{step}}^{\vec{\lambda},\vec{\nu}} : L^2(\mathbb{S}_1, \mathrm{d}\nu) \to L^2(\mathbb{S}_2, \mathrm{d}\nu)$$

are defined by the kernels

$$\mathrm{K}_1^{\vec{\lambda},\vec{\mu}}(\zeta, \zeta') := \left(\delta_i(j) + \delta_i(j + (-1)^i)\right) \frac{\mathrm{f}_i(\zeta)}{\zeta - \zeta'} Q_1(j), \tag{3.21}$$

and

$$\mathrm{K}_{\mathrm{step}}^{\vec{\lambda},\vec{\mu}}(\zeta', \zeta) := \left(\delta_j(i) + \delta_j(i - (-1)^j)\right) \frac{\mathrm{f}_j(\zeta')}{\zeta' - \zeta} Q_2(i) \tag{3.22}$$

where $1 \le i, j \le m$ *and* $\zeta \in \Gamma_i \cap \mathbb{S}_1,\ \zeta' \in \Gamma_j \cap \mathbb{S}_2,$*. Here the functions* $\mathrm{f}_i(\zeta)$ *are given by*

$$\mathrm{f}_j(\zeta) = \mathrm{f}_j(\zeta; \vec{\lambda}, \vec{\mu}) = \begin{cases} \frac{\mathrm{F}_i(\zeta)}{\mathrm{F}_{i-1}(\zeta)}, & \text{for } \zeta \in \Gamma_{\mathrm{L}}, \\ \frac{\mathrm{F}_{i-1}(\zeta)}{\mathrm{F}_i(\zeta)}, & \text{for } \zeta \in \Gamma_{\mathrm{R}}, \end{cases} \tag{3.23}$$

where

$$\mathrm{F}_i(\zeta) := \frac{\prod_{\ell=1}^{r}(\lambda_\ell - \zeta)}{\prod_{\ell=1}^{s}(\zeta - \mu_\ell)} \cdot \exp\left(-\frac{1}{3}\tau_i\zeta^3 + x_i\zeta^2 + h_i\zeta\right), \tag{3.24}$$

for $1 \le i \le m$ *and* $\mathrm{F}_0(\zeta) := 1$*. The functions* Q_j*'s are the same as in equation* (3.3)*.*

3.2.4 Proof strategy and organizations

We derive the main theorems following the strategy described in Section 2.4 by first estalishing a joint distribution formula for the related inhomogeneous TASEP model on a periodic domain (will be described in Section 3.2.5 below) and then taking the period L large, see Figure 3.3 below for an illustration.

Section 3.3 establishes a novel transition probability formula for the inhomogeneous TASEP on periodic domains. In Section 3.4 we obtain a multi-point joint distribution by taking a multiple sum over the transition probabilities. The necessary combinatorial identity is discussed in Section 3.5. The proof of a key identity, Proposition 3.5.1, is shown by Zhipeng Liu to the author and we sincerely appreciate for the help. The proof is recorded as an appendix in Appendix A. Then in Section 3.6 we rewrite the multi-point distribution formula obtained in Section 3.4 for better asymptotic behaviors. Then in Section 3.7 we derive the multi-point distribution formula for inhomogeneous TASEP on $\mathbb{Z}$ by relating it to the periodic model. Finally in Section 3.8 we prove the limit theorem Theorem 3.2.1.

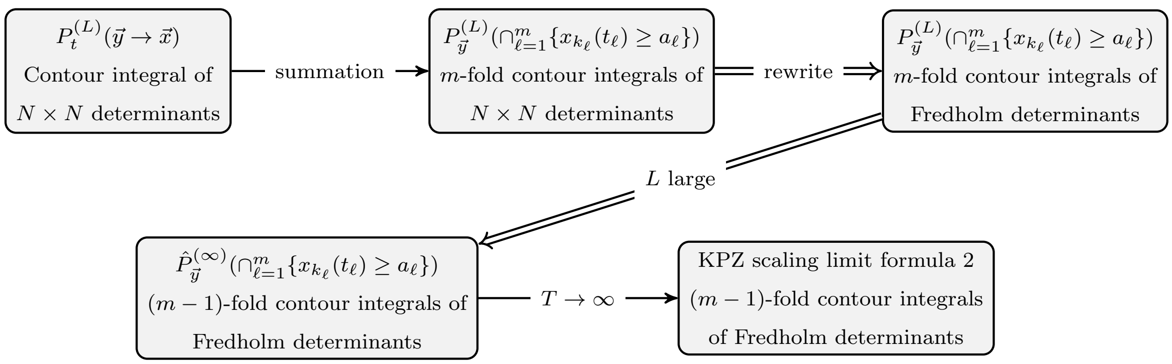

Figure 3.3: A diagram describing the procedure of computing the multi-time distributions of inhomogeneous TASEP on $\mathbb{Z}$

3.2.5 Inhomogeneous TASEP on periodic domain

Given positive integers $N < L$. We first consider the analogue of the inhomogeneous TASEP defined in Section 3.1 on a periodic domain of size L. Our periodicity assumption forces the occupation functions to satisfy $\eta_{j+kL}(t) = \eta_j(t)$ for all $j, k \in \mathbb{Z}$ and $t \geq 0$. Fix a single period consisting of L consecutive sites in $\mathbb{Z}$, we assume there are N black particles and $L - N$ white particles in this period (by periodicity this holds for any L consecutive sites in $\mathbb{Z}$). For $i, j \in \mathbb{Z}$, we denote the location of the i-th black particle at time t by $x_i(t)$ and the location of the j-th white particle at time t by $\hat{x}_j(t)$, where the indexing order for black particle is from right to left and the order for white particle is from left to right. In particular we have

$$\cdots > x_0(t) > x_1(t) > x_2(t) > \cdots,$$

and

$$\cdots < \hat{x}_0(t) < \hat{x}_1(t) < \hat{x}_2(t) < \cdots .$$

Due to periodicity we have $x_{j+kN}(t) = x_j(t) - kL$ and $\hat{x}_{j+k(L-N)}(t) = \hat{x}_j(t) + kL$ for all $j, k \in \mathbb{Z}$ and similarly the parameters need to satisfy $\pi_{j+kN} = \pi_j$ and $\hat{\pi}_{j+k(L-N)} = \hat{\pi}_j$ for all $j, k \in \mathbb{Z}$.

3.3 The periodic transition probability

Proposition 3.3.1. *Let $\vec{x} = (x_1, \cdots, x_N)$, $\vec{y} = (y_1, \cdots, y_N)$ be two given particle configurations in $\mathcal{X}_N^{(L)}$. Let $P_t(\vec{y} \to \vec{x})$ be the transition probability of observing configuration $\vec{x}$ at time t under the inhomogeneous TASEP dynamics with initial configuration $\vec{y}$. Let $j_0 \in \mathbb{Z}$ be the index such that*

$$\hat{x}_{j_0-1}(0) < x_1(0) < \hat{x}_{j_0}(0). \tag{3.25}$$

Then

$$P_t(\vec{y} \to \vec{x}; j_0) = \oint_\Gamma \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\sum_{w \in \mathcal{S}_z} F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) J(w)\right]_{i,j=1}^N . \tag{3.26}$$

Here Γ is any simple closed contour with 0 inside and $\mathcal{S}_z$ consists of all the roots of the degree L polynomial $q_z(w)$, namely

$$\mathcal{S}_z := \{w \in \mathbb{C} : q_z(w) = 0\}, \tag{3.27}$$

where

$$q_z(w) := \prod_{\ell=1}^N (w - \pi_\ell) \cdot \prod_{\ell=1}^{L-N} (w + \hat{\pi}_\ell) - z^L. \tag{3.28}$$

$F_{i,j}(w;t,\vec{x},\vec{y},k_0)$ and $J(w)$ are given by

$$F_{i,j}(w;\vec{x},\vec{y},t,k_0) = e^{t(w-\pi_i)} \frac{\prod_{\ell=j+1}^{N}(w-\pi_\ell)}{\prod_{\ell=i+1}^{N}(w-\pi_\ell)} \cdot \frac{\prod_{\ell=1}^{y_j-y_1+j+j_0-2}(w+\hat{\pi}_\ell)}{\prod_{\ell=1}^{x_i-y_1+i+j_0-1}(w+\hat{\pi}_\ell)} \cdot \frac{\prod_{\ell=1}^{x_i-y_1+i+j_0-2}(\pi_i+\hat{\pi}_\ell)}{\prod_{\ell=1}^{y_j-y_1+j+j_0-2}(\pi_j+\hat{\pi}_\ell)}, \tag{3.29}$$

for $1 \le i,j \le N$ and

$$J(w) := \frac{q_z(w)+z^L}{\frac{\mathrm{d}}{\mathrm{d}w}q_z(w)} = \frac{1}{\sum_{\ell=1}^{N}\frac{1}{w-\pi_\ell} + \sum_{\ell=1}^{L-N}\frac{1}{w+\hat{\pi}_\ell}}. \tag{3.30}$$

Note that for $p > q$, we define the product $\prod_{\ell=p}^{q} a_\ell$ as follows:

$$\prod_{\ell=p}^{q} a_\ell = \begin{cases} 1 & \text{if } q = p-1, \\ \prod_{\ell=q+1}^{p-1} a_\ell^{-1} & \text{otherwise.} \end{cases} \tag{3.31}$$

Remark 3.3.2. The index j_0 represents the index of the first white particle to the right of the first black particle. We will always assume $j_0 = 1$ at time $t = 0$. But in order to compute the multi-time joint distribution we need to keep track of the index of the first white particle to the right of the first black particle at several different times at which these indices change depending on how many jumps the first black particle has already made. For this purpose we add this extra parameter to the transition probability formula.

Proof of Proposition 3.3.1. The transition probability $P_t(\vec{y} \to \vec{x}; j_0)$ is the unique solution of the following Kolmogorov forward equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t(\vec{y}\to\vec{x};j_0) = \sum_{i=1}^{N} \mathbf{1}_{\vec{x}^{(i-)}\in\mathcal{X}_N^{(L)}} \cdot \left[(\pi_i + \hat{\pi}_{x_i-y_1+i+j_0-2})P_t(\vec{y}\to\vec{x}^{(i-)};j_0) - (\pi_{i+1}+\hat{\pi}_{x_{i+1}-y_1+i+j_0})P_t(\vec{y}\to\vec{x};j_0)\right], \tag{3.32}$$

satisfying the initial condition $P_0(\vec{y} \to \vec{x}; j_0) = \mathbf{1}_{\vec{x}=\vec{y}}$. Here $\vec{x}^{(i-)} = (x_1, \cdots, x_{i-1}, x_i - 1, x_{i+1}, \cdots, x_N)$ for $\vec{x} = (x_1, \cdots, x_N)$. Note that for $i = N$ we have

$$\pi_{N+1} + \hat{\pi}_{x_{N+1}-y_1+N+j_0} = \pi_1 + \hat{\pi}_{x_1-L-y_1+N+j_0} = \pi_1 + \hat{\pi}_{x_1-y_1+j_0}.$$

Following the usual coordinate Bethe ansatz method, we replace the Kolmogorov forward equation by a free evolution equation with extra boundary conditions. For $\vec{x}, \vec{y} \in \mathbb{Z}^N$, consider the free evolution equation for $G_t(\vec{x}) = G_t(\vec{x}; \vec{y})$

$$\frac{\mathrm{d}G_t}{\mathrm{d}t} = \sum_{i=1}^{N} \left[(\pi_i + \hat{\pi}_{x_i-y_1+i+j_0-2}) G_t(\vec{x}^{(i-)}; \vec{y}) - (\pi_{i+1} + \hat{\pi}_{x_{i+1}-y_1+i+j_0}) G_t(\vec{x}; \vec{y}) \right], \tag{3.33}$$

together with the boundary conditions

$$(\pi_i + \hat{\pi}_{x_i-y_1+i+j_0-2}) G_t(\vec{x}^{(i-)}) = (\pi_{i+1} + \hat{\pi}_{x_{i+1}-y_1+i+j_0}) G_t(\vec{x}), \quad \text{if } x_i = x_{i+1} + 1, \tag{3.34}$$

for $1 \le i \le N-1$ and

$$(\pi_N + \hat{\pi}_{x_N-y_1+i+j_0-2}) G_t(\vec{x}^{(N-)}) = (\pi_1 + \hat{\pi}_{x_1-y_1+j_0}) G_t(\vec{x}), \quad \text{if } x_1 = x_N + L - 1. \tag{3.35}$$

And the initial condition

$$G_0(\vec{x}; \vec{y}) = \mathbf{1}_{\vec{x}=\vec{y}}. \tag{3.36}$$

It is straightforward to check that $P_t(\vec{y} \to \vec{x}) = G_t(\vec{x}; \vec{y})$ for $\vec{y}, \vec{x} \in \mathcal{X}_N^{(L)}$. Hence it suffices to show that for all $\vec{x}, \vec{y} \in \mathbb{Z}^N$, we have

$$G_t(\vec{x}; \vec{y}) = \oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det \left[\sum_{w \in \mathcal{S}_z} F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) J(w) \right]_{i,j=1}^{N}. \tag{3.37}$$

To see this we check that the right-hand side of (3.37) satisfies the free evolution equation (3.33), the boundary conditions (3.34), (3.35) and the initial condition (3.36). For the free evolution equation note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_{i,j}(w;\vec{x}) &= (w-\pi_i)F_{i,j}(w;\vec{x})\\ &= (w+\hat{\pi}_{x_i-y_1+i+j_0-1})F_{i,j}(w;\vec{x}) - (\pi_i+\hat{\pi}_{x_i-y_1+i+j_0-1})F_{i,j}(w;\vec{x})\\ &= (\pi_i+\hat{\pi}_{x_i-y_1+i+j_0-2})F_{i,j}(w;\vec{x}^{(i-)}) - (\pi_i+\hat{\pi}_{x_i-y_1+i+j_0-1})F_{i,j}(w;\vec{x}).\end{aligned}$$

Hence (3.33) follows from linearity of the determinants and integrals. Here and throughout the proof we suppress the dependence of $F_{i,j}$ on $\vec{y},t$ and k_0 to make the notation light whenever there is no confusion.

Next we check the boundary conditions are satisfied. For $1\le i\le N-1$, suppose that $\vec{x}=(x_1,\cdots,x_N)\in\mathbb{Z}^N$ satisfies $x_i=x_{i+1}+1$. Then it is straightforward to check that

$$\begin{aligned}&(\pi_{i+1}+\hat{\pi}_{x_i-y_1+i+j_0-1})F_{i,j}(w;\vec{x}) + F_{i+1,j}(w;\vec{x})\cdot e^{t(\pi_{i+1}-\pi_i)}\cdot\prod_{\ell=1}^{x_i-y_1+i+j_0-2}\frac{\pi_i+\hat{\pi}_\ell}{\pi_{i+1}+\hat{\pi}_\ell}\\ &= (\pi_i+\hat{\pi}_{x_i-y_1+i+j_0-2})F_{i,j}(w;\vec{x}^{(i-)}).\end{aligned} \tag{3.38}$$

Hence by multiplying the $i+1$-th row of the determinant inside the contour integral on the right-hand side of (3.37) with $e^{t(\pi_{i+1}-\pi_i)}\cdot\prod_{\ell=1}^{x_i-y_1+i+j_0-2}\frac{\pi_i+\hat{\pi}_\ell}{\pi_{i+1}+\hat{\pi}_\ell}\cdot\frac{1}{\pi_{i+1}+\hat{\pi}_{x_i-y_1+i+j_0-1}}$ and adding to the i-th row we see that for $\vec{x}\in\mathbb{Z}^N$ with $x_i=x_{i+1}+1$ we have

$$\det\left[\sum_{w\in\mathcal{S}_z}F_{k,j}(w;\vec{x})J(w)\right]_{k,j} = \frac{\pi_i+\hat{\pi}_{x_i-y_1+i+j_0-2}}{\pi_{i+1}+\hat{\pi}_{x_i-y_1+i+j_0-1}}\det\left[\sum_{w\in\mathcal{S}_z}F_{k,j}(w;\vec{x}^{(i-)})J(w)\right]_{k,j},$$

which implies (3.34) by linearity of the contour integral. To see (3.35) recall that since $w\in\mathcal{S}_z$ we know

$$z^L=\prod_{\ell=\ell_1}^{N+\ell_1-1}(w-\pi_\ell)\cdot\prod_{\ell=\ell_2}^{L-N+\ell_2-1}(w+\hat{\pi}_\ell), \tag{3.39}$$

for any $\ell_1, \ell_2 \in \mathbb{Z}$. Hence similar as (3.38) we have

$$(\pi_1 + \hat{\pi}_{x_N - y_1 + N + j_0 - 1}) F_{N,j}(w; \vec{x}) + F_{1,j}(w; \vec{x}) \cdot z^L \cdot e^{t(\pi_1 - \pi_N)} \cdot \frac{\prod_{\ell=1}^{x_N - y_1 + N + j_0 - 2} (\pi_N + \hat{\pi}_\ell)}{\prod_{\ell=1}^{x_N - y_1 + L + j_0 - 2} (\pi_1 + \hat{\pi}_\ell)}$$
$$= (\pi_N + \hat{\pi}_{x_N - y_1 + N + j_0 - 2}) F_{N,j}(w; \vec{x}^{(N-)}).$$

A similar row operation between the first and last row as in the $1 \le i \le N-1$ cases then implies (3.35).

Finally we verify the initial condition (3.36). We write $q(w) = \prod_{j=1}^N (w - \pi_j) \cdot \prod_{j=1}^{L-N} (w + \hat{\pi}_j)$. Then since $J(w) = \frac{q(w)}{q'(w)}$, by residue theorem we have for any $|z| > 0$

$$\begin{aligned}
&\sum_{w \in \mathcal{S}_z} F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) J(w) \\
&= \oint_{|w|=R} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) \frac{q(w)}{q(w) - z^L} - \sum_{\ell=1}^{L-N} \oint_{|w + \hat{\pi}_\ell| = \epsilon} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) \frac{q(w)}{q(w) - z^L} \\
&:= \oint_{|w|=R} \frac{dw}{2\pi \mathrm{i}} F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) + \left(z^L E_1(i,j) + z^{-L} E_2(i,j) \right) \cdot C(i,j),
\end{aligned} \tag{3.40}$$

where $R = R(z) > 0$ is large enough and $\epsilon = \epsilon(z) > 0$ is small enough so that

$$\mathcal{S}_z \subset \{w \in \mathbb{C} : |w| < R\} \cap \bigcap_{\ell=1}^{L-N} \{w \in \mathbb{C} : |w + \hat{\pi}_\ell| > \epsilon\}.$$

The constant $C(i,j) := \frac{\prod_{\ell=1}^{x_i - y_1 + i + j_0 - 2} (\pi_i + \hat{\pi}_\ell)}{\prod_{\ell=1}^{y_j - y_1 + j + j_0 - 2} (\pi_j + \hat{\pi}_\ell)}$. Here we used the fact that for all $1 \le \ell \le N$, the functions $F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) \left(\frac{q_z(w) + z^L}{q_z(w)} \right)$ are analytic at π_ℓ. so the only possible poles of $F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) \left(\frac{q_z(w) + z^L}{q_z(w)} \right)$ are $\mathcal{S}_z \cup \{-\hat{\pi}_\ell\}_{\ell=1}^{L-N}$. The functions $E_1(i,j)$ and $E_2(i,j)$ are given by

$$E_1(i,j) = \oint_{|w|=R} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\prod_{\ell=1}^{j} (w - \pi_\ell)^{-1}}{\prod_{\ell=i+1}^{N} (w - \pi_\ell)} \cdot \frac{\prod_{\ell=1}^{y_j - y_1 + j + j_0 - 2 - L + N} (w + \hat{\pi}_\ell)}{\prod_{\ell=1}^{x_i - y_1 + i + j_0 - 1} (w + \hat{\pi}_\ell)} \cdot \frac{1}{1 - z^L q(w)^{-1}}, \tag{3.41}$$

$$E_2(i,j) = \sum_{\ell=1}^{L-N} \oint_{|w+\hat{\pi}_\ell|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\prod_{\ell=j+1}^{N}(w-\pi_\ell)}{\prod_{\ell=1}^{i}(w-\pi_\ell)^{-1}} \cdot \frac{\prod_{\ell=1}^{y_j-y_1+j+j_0-2+L-N}(w+\hat{\pi}_\ell)}{\prod_{\ell=1}^{x_i-y_1+i+j_0-1}(w+\hat{\pi}_\ell)} \cdot \frac{1}{1-z^{-L}q(w)}, \tag{3.42}$$

for $1 \le i,j \le N$. Now we split into two cases depending on the relationship between x_1 and y_1 and argue that in both cases at least one of E_1 and E_2 vanishes.

Case 1: $x_1 < y_1$. Note first that for $\vec{x}, \vec{y} \in \mathcal{X}_N^{(L)}$ we have

$$x_1 - L + N \le x_i + i - 1 \le x_1, \quad y_1 - L + N \le y_j + j - 1 \le y_1, \tag{3.43}$$

for all $1 \le i,j \le N$. Hence if $x_1 < y_1$ we further have

$$y_j + j - 1 - y_1 + L - N \ge 0, \quad x_i - y_1 + i \le x_1 + 1 - y_1 \le 0.$$

In this case the integrand in (3.42) is analytic at $\hat{\pi}_\ell$ for all $1 \le \ell \le L-N$ which implies that $E_2(i,j) = 0$ for all $1 \le i,j \le N$. Therefore by letting $z \to 0$ we have

$$\begin{aligned}
&\oint_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}z} \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},0,j_0)J(w)\right]_{i,j=1}^{N} \\
&= \oint_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}z} \det\left[\oint_{|w|=R} \frac{dw}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0,j_0) + z^L E_1(i,j)\right]_{i,j=1}^{N} \\
&= \det\left[\oint_{|w|=R} \frac{dw}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0,j_0)\right]_{i,j=1}^{N}.
\end{aligned}$$

Case 2: $x_1 \ge y_1$. Again by (3.43), for any $\vec{x}, \vec{y} \in \mathcal{X}_N^{(L)}$ with $x_1 \ge y_1$ we have

$$y_j - y_1 + j - 1 - L + N \le -L + N, \quad x_i - y_1 + i \ge x_1 + 1 - K + N - y_1 \ge 1 - L + N.$$

This implies that the integrand in (3.41) is $O(R^{-2})$ and by sending $R \to \infty$ and using the fact that $E_1(i,j)$ should be independent of R we see $E_1(i,j) = 0$ for all $1 \le i, j \le N$. Therefore by sending $z \to \infty$ we see

$$\begin{aligned}
&\oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\sum_{w \in \mathcal{S}_z} F_{i,j}(w; \vec{x}, \vec{y}, 0, j_0) J(w)\right]_{i,j=1}^{N} \\
&= \oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\oint_{|w|=R} \frac{dw}{2\pi \mathrm{i}} F_{i,j}(w; \vec{x}, \vec{y}, 0, j_0) + z^{-L} E_2(i,j)\right]_{i,j=1}^{N} \\
&= \det\left[\oint_{|w|=R} \frac{dw}{2\pi \mathrm{i}} F_{i,j}(w; \vec{x}, \vec{y}, 0, j_0)\right]_{i,j=1}^{N}.
\end{aligned}$$

Thus we have reduced checking the initial condition (3.36) to checking the following:

$$\det\left[\oint_{|w|=R} \frac{dw}{2\pi \mathrm{i}} F_{i,j}(w; \vec{x}, \vec{y}, 0, j_0)\right]_{i,j=1}^{N} = \mathbf{1}_{\vec{x}=\vec{y}}, \tag{3.44}$$

for any $\vec{x}, \vec{y} \in \mathcal{X}_N^{(L)}$. This can be done in exactly the same way as in [99] which corresponds to the special case when $\pi_\ell \equiv 0$ and $\hat{\pi}_\ell \equiv 1$ for all ℓ. See also [94] for the special case when one allows one set of parameters $\{\pi_\ell\}$ and set $\hat{\pi}_\ell \equiv 1$. □

Remark 3.3.3. Equation (3.40) is very important in this chapter since it connects two different ways of understanding the discrete nature of periodic TASEP. On the one side the entries of the determinant are sums over roots of certain polynomial equation depending on the parameter z. On the other side it can also be recognized as a analytic function in z on $\{|z| > 0\}$ defined through the contour integrals on the right hand side of (3.40) with a possible singularity at $z = 0$. However when L is large enough, using (3.43) one can check the integrand $F_{i,j}(w; \vec{x}, \vec{y}, t, j_0)\left(\frac{q_z(w)+z^L}{q_z(w)}\right)$ is at $w = -\hat{\pi}_\ell$ for all $1 \le \ell \le m$ so we can deform the contours $|w + \hat{\pi}_\ell| = \epsilon$'s all to 0 on the right hand side of (3.40). But then for fixed $R > 0$ large enough, the remaining integral over $|w| = R$ on the right hand side of (3.40) is well-defined and analytic at $z = 0$. So is the whole determinant. Deforming the z contour to 0 then gives

$$P_t(\vec{y} \to \vec{x}; j_0) = \det \left[\oint_{|w|=R} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w; \vec{x}, \vec{y}, t, j_0) \right]_{i,j=1}^{N}, \tag{3.45}$$

whenver $L > (x_1 + 1) - (y_N + N)$. This agrees with the transition probability of inhomogeneous TASEP on $\mathbb{Z}$ and generalizes the transition probability formulas of Schütz [99] and [94]. We are not able to find such a transition probability formula with two sets of parameters even for TASEP on $\mathbb{Z}$ in the literature. However see [68] Corollary 3.1 for a related transition probability formula for inhomogeneous exponential last passage percolations.

3.4 Multi-point distribution of periodic inhomogeneous TASEP

Theorem 3.4.1 (Multi-point joint distribution for inhomogeneous TASEP in $\mathcal{X}_N^{(L)}$). *Let $\vec{y} \in \mathcal{X}_N^{(L)}$ and $\vec{x}(t) = (x_1(t), \cdots, x_N(t)) \in \mathcal{X}_N^{(L)}$ be particle configurations evolving according to the inhomogeneous TASEP in $\mathcal{X}_N^{(L)}$ at time t with initial configuration $\vec{x}(0) = \vec{y}$ where we assume $y_1 = -1$. Fix a positive integer m. Let $(k_1, t_1), \cdots, (k_m, t_m) \in \{1, \cdots, N\} \times \mathbb{R}_{\geq 0}$ be distinct with $0 \leq t_1 \leq \cdots \leq t_m$. Let $a_i \in \mathbb{Z}$ for $1 \leq i \leq m$. Then*

$$\mathbb{P}_{\vec{y}}^{(L)} \left(\bigcap_{i=1}^{m} \{x_{k_i}(t_i) \geq a_i\} \right) = \oint \cdots \oint \frac{\mathrm{d}z_m}{2\pi \mathrm{i} z_m} \cdots \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1} \mathcal{C}^{(L)}(\vec{z}) \mathcal{D}_{\vec{y}}^{(L)}(\vec{z}), \tag{3.46}$$

where the contours for the integrals are nested circles $0 < |z_m| < \cdots < |z_1|$*. Here* $\vec{z} = (z_1, \cdots, z_m)$*. The functions* $\mathcal{C}^{(L)}(\vec{z})$ *and* $\mathcal{D}^{(L)}_{\vec{y}}(\vec{z})$ *are defined by*

$$\begin{aligned}\mathcal{C}^{(L)}(\vec{z}) = (-1)^{k_m(N-1)} z_1^{(N-k_1)L} \frac{\prod_{j=1}^{N} e^{-t_m \pi_j}}{\prod_{j=k_m+1}^{N} \prod_{\ell=1}^{L-N} (\pi_j + \hat{\pi}_\ell)} \\ \cdot \prod_{\ell=2}^{m} \left[z_\ell^{(k_{\ell-1}-k_\ell)L} \left(\left(\frac{z_\ell}{z_{\ell-1}} \right)^L - 1 \right)^{N-1} \right],\end{aligned} \tag{3.47}$$

and

$$\mathcal{D}^{(L)}_{\vec{y}}(\vec{z}) = \det \left[\sum_{\substack{w_\ell \in \mathcal{S}_{z_\ell} \\ \ell=1,\cdots,m}} \frac{p_i(w_1) q_j(w_m)}{\prod_{\ell=2}^{m} (w_\ell - w_{\ell-1})} \prod_{\ell=1}^{N} \frac{1}{w_1 - \pi_\ell} \cdot \prod_{\ell=1}^{m} G_\ell(w_\ell) \right]_{i,j=1}^{N}, \tag{3.48}$$

where for $1 \le i, j \le N$

$$p_i(w) = \prod_{\ell=i+1}^{N} (w - \pi_\ell) \cdot \prod_{\ell=1}^{y_i+i} \frac{w + \hat{\pi}_\ell}{\pi_i + \hat{\pi}_\ell}, \quad q_j(w) = \frac{\prod_{\ell=1}^{a_m+k_m} (\pi_j + \hat{\pi}_\ell)}{\prod_{\ell=j}^{N} (w - \pi_{\ell+k_m})}. \tag{3.49}$$

And for $1 \le \ell \le m$

$$G_\ell(w) := J(w) \cdot \frac{e^{t_\ell w} \cdot \prod_{j=1}^{k_\ell} (w - \pi_j) \cdot \prod_{j=1}^{a_\ell+k_\ell} (w + \hat{\pi}_j)^{-1}}{e^{t_{\ell-1} w} \cdot \prod_{j=1}^{k_{\ell-1}} (w - \pi_j) \cdot \prod_{j=1}^{a_{\ell-1}+k_{\ell-1}} (w + \hat{\pi}_j)^{-1}}. \tag{3.50}$$

Here $k_0 = t_0 = a_0 := 0$ *and we suppress the dependence on* a_i*,* k_i *and* t_i*'s in* $\mathcal{C}^{(L)}(\vec{z})$ *and* $\mathcal{D}^{(L)}_{\vec{y}}(\vec{z})$*. Recall that* $J(w) = \frac{q(w)}{q'(w)} = \frac{1}{\sum_{j=1}^{N} \frac{1}{w-\pi_j} + \sum_{j=1}^{L-N} \frac{1}{w+\hat{\pi}_j}}$.

Proof. We start with the case $m = 1$. By Cauchy-Binet formula the transition probability (3.26) equals

$$P_t(\vec{x} \to \vec{x}'; j_0 = 1) = \oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \sum_{\vec{w} \in (\mathcal{S}_z)^N} \psi^{\ell}_{\vec{x}}(\vec{w}) \psi^{r}_{\vec{x}'}(\vec{w}) \mathcal{Q}(\vec{w}; t),$$

where

$$\psi^r_{\vec{x}}(\vec{w}) = \det\left[\prod_{\ell=j+1}^{N}\frac{1}{w_i-\pi_\ell}\cdot\frac{\prod_{\ell=1}^{x_j+j}(\pi_j+\hat{\pi}_\ell)}{\prod_{\ell=1}^{x_j+j+1}(w_i+\hat{\pi}_\ell)}\right]_{i,j=1}^{N}, \tag{3.51}$$

$$\psi^\ell_{\vec{x}}(\vec{w}) = \det\left[\prod_{\ell=j+1}^{N}(w_i-\pi_\ell)\prod_{\ell=1}^{x_j+j}\frac{w_i+\hat{\pi}_\ell}{\pi_j+\hat{\pi}_\ell}\right]_{i,j=1}^{N}, \tag{3.52}$$

and

$$\mathcal{Q}(\vec{w};t) = \frac{1}{N!}\prod_{j=1}^{N}e^{t(w_j-\pi_j)}J(w_j) = \frac{1}{N!}\prod_{j=1}^{N}\frac{e^{t(w_j-\pi_j)}}{\sum_{\ell=1}^{N}\frac{1}{w_j-\pi_\ell}+\sum_{\ell=1}^{N}\frac{1}{w_j+\hat{\pi}_\ell}}.$$

Note that the transition probability formula simplifies due to our assumption $y_1 = -1$ and $j_0 = 1$. Now to get the one-point distribution $\mathbb{P}^{(L)}_{\vec{y}}(x_k(t) \geq a)$ we perform a summation over all configurations $\vec{x} \in \mathcal{X}^{(L)}_N$ with $x_k \geq a$ of the transition probability and interchange the order of integration and summation:

$$\begin{aligned}\mathbb{P}^{(L)}_{\vec{y}}(x_k(t) \geq a) &= \sum_{\vec{x}\in\mathcal{X}^{(L)}_N\cap\{x_k\geq a\}} P_t(\vec{y}\to\vec{x}) \\ &= \oint_\Gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\sum_{\vec{w}\in(\mathcal{S}_z)^N}\psi^\ell_{\vec{y}}(\vec{w})Q(\vec{w};t)\left(\sum_{\vec{x}\in\mathcal{X}^{(L)}_N\cap\{x_k\geq a\}}\psi^r_{\vec{x}}(\vec{w})\right).\end{aligned} \tag{3.53}$$

By Lemma 3.4.2 below we have $\sum_{\vec{x}\in\mathcal{X}^{(L)}_N\cap\{x_k\geq a\}}\psi^r_{\vec{x}}(\vec{w})$ equals

$$\begin{aligned}&(-1)^{k(N-1)}\cdot z^{(N-k)L}\cdot\prod_{j=k+1}^{N}\prod_{\ell=1}^{L-N}\frac{1}{\pi_j+\hat{\pi}_\ell}\cdot\prod_{j=1}^{N}\prod_{\ell=k+1}^{N}\frac{1}{w_j-\pi_\ell} \\ &\cdot\det\left[\prod_{\ell=j}^{N}\frac{1}{w_i-\pi_{k+\ell}}\cdot\prod_{\ell=1}^{a+k}\frac{\pi_j+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\right]_{i,j=1}^{N}.\end{aligned} \tag{3.54}$$

Inserting (3.54) back to (3.53) and using Cauchy-Binet formula backwards we conclude that

$$\mathbb{P}_{\vec{y}}^{(L)}(x_k(t) \geq a) = \frac{(-1)^{k(N-1)}}{2\pi \mathrm{i}} \cdot \prod_{j=k+1}^{N} \prod_{\ell=1}^{L-N} \frac{1}{\pi_j + \hat{\pi}_\ell} \cdot \oint_\Gamma \frac{\mathrm{d}z}{z^{1-(N-k)L}}$$
$$\det\left[\sum_{w\in\mathcal{S}_z} e^{t(w-\pi_j)} \frac{\prod_{\ell=i+1}^{N}(w-\pi_\ell)}{\prod_{\ell=j+k-N}^{N}(w-\pi_\ell)} \cdot \prod_{\ell=1}^{y_i+i} \frac{w+\hat{\pi}_\ell}{\pi_i+\hat{\pi}_\ell} \cdot \prod_{\ell=1}^{a+k} \frac{\pi_i+\hat{\pi}_\ell}{w+\hat{\pi}_\ell} J(w)\right]_{i,j=1}^{N}. \tag{3.55}$$

Here we need to ensure that the summation over $\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_k \geq a\}$ converges absolutely in order to interchange the order of summation and integration as in (3.53). This is allowed if we assume $\left|\frac{\pi_i+\hat{\pi}_\ell}{w+\hat{\pi}_\ell}\right| < 1$ for all $w \in \mathcal{S}_z$ and $1 \leq \ell \leq L-N$ (see Lemma 3.4.2). This can then be achieved by choosing the contour Γ to be circle with large radius R since for $|z| = R$ large we have $|w| = O(|z|) \gg 1$ for all w satisfying $\prod_{\ell=1}^{N}(w-\pi_\ell)\cdot\prod_{\ell=1}^{L-N}(w+\hat{\pi}_\ell) = z^L$. Finally it is not hard to check that the right-hand side of equation (3.55) does not depend on the choice of Γ (as long as it encloses the origin) so we can deform Γ to be any simple closed contour containing 0, not necessarily large circle.

Now assume $m \geq 2$. Then

$$\begin{aligned}&\mathbb{P}_{\vec{y}}^{(L)}\left(\cap_{i=1}^{m}\{x_{k_i}(t_i) \geq a_i\}\right)\\ &= \sum_{\substack{\vec{x}^{(\ell)}\in\mathcal{X}_N^{(L)}\cap\{x_{k_\ell}^{(\ell)}\geq a_\ell\}\\ \ell=1,\cdots,m}} P_{t_0\to t_1}(\vec{y}\to\vec{x}^{(1)};j_0)\cdots P_{t_{m-1}\to t_m}(\vec{x}^{(m-1)}\to\vec{x}^{(m)};j_{m-1}),\end{aligned}$$

where $t_0 := 0$ and j_k is the index of the first white particle (or hole) to the right of the first black particle at time $t = t_k$ for $0 \leq k \leq m-1$. Here we note that if we know at time $t = t_k$ the first black particle is at location $x_1^{(k)}$, then the index j_k is given by

$$j_k = j_0 + x_1^{(k)} - x_1^{(0)} = x_1^{(k)} + 2,$$

by our assumption that $j_0 = 1$ and $x_1^{(0)} = y_1 = -1$. Plugging into the formula (3.26) for the transition probability with parameter j_0 replaced by j_k we see $P_{t_k \to t_{k+1}}(\vec{x}^{(k)} \to \vec{x}^{(k+1)}; j_k)$ equals

$$\oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det \left[\sum_{w \in \mathcal{S}_z} e^{t(w-\pi_i)} \frac{\prod_{\ell=j+1}^{N}(w-\pi_\ell)}{\prod_{\ell=i+1}^{N}(w-\pi_\ell)} \cdot \frac{\prod_{\ell=1}^{x_j^{(k)}+j}(w+\hat{\pi}_\ell)}{\prod_{\ell=1}^{x_j^{(k)}+j}(\pi_j+\hat{\pi}_\ell)} \cdot \frac{\prod_{\ell=1}^{x_i^{(k+1)}+i}(\pi_i+\hat{\pi}_\ell)}{\prod_{\ell=1}^{x_i^{(k+1)}+i+1}(w+\hat{\pi}_\ell)} \right]_{i,j=1}^{N}.$$

Now we rewrite the transition probability using Cauchy-Binet formula as in the $m = 1$ case and interchange the order of summation and integration so that $\mathbb{P}_{\vec{y}}(\cap_{i=1}^{m}\{x_{k_i}(t_i) \geq a_i\})$ equals

$$\oint \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1} \cdots \oint \frac{\mathrm{d}z_m}{2\pi \mathrm{i} z_m} \sum_{\substack{\vec{w}^{(\ell)} \in (\mathcal{S}_{z_\ell})^N \\ \ell=1,\cdots,m}} \mathcal{P}(\vec{w}^{(1)}, \cdots, \vec{w}^{(m)}) \prod_{\ell=1}^{m} \mathcal{Q}(\vec{w}^{(\ell)}; t_\ell - t_{\ell-1}).$$

Here $\vec{w}^{(\ell)} = (w_1^{(\ell)}, \cdots, w_m^{(\ell)})$ and

$$\mathcal{P}(\vec{w}^{(1)}, \cdots, \vec{w}^{(m)}) = \psi_{\vec{y}}^{(\ell)}(\vec{w}^{(1)}) \cdot \left[\prod_{\ell=1}^{m-1} \mathcal{H}_{k_\ell, a_\ell}(\vec{w}^{(\ell)}; \vec{w}^{(\ell+1)}) \right] \cdot \left[\sum_{\substack{\vec{x} \in \mathcal{X}_N^{(L)} \\ x_{k_m} \geq a_m}} \psi_{\vec{x}}^{r}(\vec{w}^{(m)}) \right], \quad (3.56)$$

where

$$\mathcal{H}_{k,a}(\vec{w}; \vec{w}') := \sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_k \geq a\}} \psi_{\vec{x}}^{r}(\vec{w}) \psi_{\vec{x}}^{\ell}(\vec{w}'). \quad (3.57)$$

Evaluating the sums in (3.56) and (3.57) using Lemma 3.4.2 and Lemma 3.4.3 below and applying Cauchy-Binet formula we conclude that

$$\mathbb{P}_{\vec{y}}^{(L)}\left(\bigcap_{i=1}^{m} \{x_{k_i}(t_i) \geq a_i\} \right) = \oint \cdots \oint \frac{\mathrm{d}z_m}{2\pi \mathrm{i} z_m} \cdots \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1} \mathcal{C}^{(L)}(\vec{z}) \mathcal{D}_{\vec{y}}^{(L)}(\vec{z}),$$

for $\mathcal{C}^{(L)}(\vec{z})$ and $\mathcal{D}_{\vec{y}}^{(L)}(\vec{z})$ defined in (3.47) and (3.48). Similar as the discussion for $m = 1$ case, in order to interchange summation and integration we need the absolute convergence of all the infinite sums which holds if for all $1 \le \ell \le L-N$

$$\prod_{j=1}^{N} |w_j^{(1)} + \hat{\pi}_\ell| > \prod_{j=1}^{N} |w_j^{(2)} + \hat{\pi}_\ell| > \cdots > \prod_{j=1}^{N} |w_j^{(m)} + \hat{\pi}_\ell| > \prod_{j=1}^{N} |\pi_j + \hat{\pi}_\ell|. \tag{3.58}$$

By the same reasoning as in $m = 1$ case this can be achieved assuming the integral contours for z_i's are large nested contours $|z_\ell| = r_\ell$ with $r_\ell - r_{\ell+1}$ also large enough for all $1 \le \ell \le m$. Finally we can deform the integral contours in (3.46) into arbitrary nested contours with 0 inside, not necessarily with large radius thanks to the analyticity of $\mathcal{C}^{(L)}(\vec{z})$ and $\mathcal{D}_{\vec{y}}^{(L)}(\vec{z})$ in $\vec{z}$ for any z_i's nonzero and distinct. □

3.4.1 Summation identities over eigenfunctions

The following two summation identities are needed in our computation of the multi-point joint distribution formula. The first identity of summation over a single eigenfunction is relatively easy and we prove it in this section. The second identity is closely related to a Cauchy identity for some inhomogeneous variant of the Grothendieck polynomial and its dual which might be of independent interest so the proof is given in a separate section, together with some further discussions.

Lemma 3.4.2 (Summation over single eigenfunction)**.** *Let $z \in \mathbb{C}$ be nonzero. Let $\psi_{\vec{x}}^{r}(\vec{w})$ be as in* (3.51) *where $\vec{w} = (w_1, \cdots, w_N) \in (\mathcal{S}_z)^N$ such that $\prod_{j=1}^{N} |w_j + \hat{\pi}_\ell| > 1$ for all $1 \le \ell \le L-N$. Then*

$$\sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_k \ge a\}} \psi_{\vec{x}}^{r}(\vec{w}) = (-1)^{k(N-1)} \cdot z^{(N-k)L} \cdot \left(\prod_{j=k+1}^{N} \prod_{\ell=1}^{L-N} \frac{1}{\pi_j + \hat{\pi}_\ell} \right) \cdot \det \left[\prod_{\ell=j}^{N} \frac{1}{w_i - \pi_{k+\ell}} \right]_{i,j=1}^{N}$$
$$\cdot \prod_{j=1}^{N} \left[\prod_{\ell=k+1}^{N} \frac{1}{w_j - \pi_\ell} \cdot \prod_{\ell=1}^{a+k} \frac{\pi_j + \hat{\pi}_\ell}{w_j + \hat{\pi}_\ell} \right], \tag{3.59}$$

for all $1 \le k \le N$ and $a \in \mathbb{Z}$.

Lemma 3.4.3 (Summation over left and right eigenfunctions)**.** *Let $z, z' \in \mathbb{C}$ be nonzero with $z^L \neq (z')^L$. Let $\psi^r_{\vec{x}}(\vec{w})$ be as in* (3.51) *and $\psi^\ell_{\vec{x}}(\vec{w}')$ be as in* (3.52) *where $\vec{w} = (w_1, \cdots, w_N) \in (\mathcal{S}_z)^N$ and $w' = (w'_1, \cdots, w'_N) \in (\mathcal{S}_{z'})^N$. Assume further that*

$$\prod_{j=1}^{N} \left| \frac{w'_j + \hat{\pi}_\ell}{w_j + \hat{\pi}_\ell} \right| < 1, \quad \textit{for all } 1 \leq \ell \leq L - N.$$

Then

$$\sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_k \geq a\}} \psi^r_{\vec{x}}(\vec{w}) \psi^\ell_{\vec{x}}(\vec{w}') = \left(1 - \left(\frac{z'}{z}\right)^L\right)^{N-1} \cdot \left(\frac{z}{z'}\right)^{(N-k)L} \cdot \det\left[\frac{1}{w_i - w'_{i'}}\right]_{i,i'=1}^{N}$$
$$\cdot \prod_{j=1}^{N} \left[\prod_{\ell=k+1}^{N} \frac{w'_j - \pi_\ell}{w_j - \pi_\ell} \cdot \prod_{\ell=1}^{a+k} \frac{w'_j + \hat{\pi}_\ell}{w_j + \hat{\pi}_\ell} \right], \tag{3.60}$$

for all $1 \leq k \leq N$ and $a \in \mathbb{Z}$.

We start with the proof of Lemma 3.4.2. Using periodicity we first reduce the extra constraint $x_k \geq a$ to the last particle x'_N by choosing a different representative of the same configuration (due to periodicity any N consecutive particles can be a representative). The same trick will also be used in the proof of Lemma 3.4.3.

Proof of Lemma 3.4.2. First setting

$$\vec{x}' = (x'_1, \cdots, x'_N) = (x_{k+1} + L - N + k, \cdots, x_N + L - N + k, x_1 - N + k, \cdots, x_k - N + k).$$

Then a summation over $\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_k \geq a\}$ is the same as a summation over $\vec{x}' \in \mathcal{X}_N^{(L)} \cap \{x'_N \geq a - N + k\}$. Also it is straightforward to check that

$$x_j + j = \begin{cases} x'_{j-k+N} + j - k + N, & \text{for } 1 \leq j \leq k, \\ x'_{j-k} + j - k + N - L, & \text{for } k+1 \leq j \leq N. \end{cases}$$

We shift the indexing of the parameters $\{\pi_j\}$ accordingly, namely we define

$$(\pi'_1, \cdots, \pi'_N) := (\pi_{k+1}, \cdots, \pi_N, \pi_1, \cdots, \pi_k).$$

Then for $N - k + 1 \leq j \leq N$ we have

$$\prod_{\ell=j-N+k+1}^{N} \frac{1}{w - \pi_\ell} = \prod_{\ell=j-N+k+1}^{k} \frac{1}{w - \pi_\ell} \cdot \prod_{\ell=k+1}^{N} \frac{1}{w_i - \pi_\ell} = \prod_{\ell=j+1}^{N} \frac{1}{w - \pi'_\ell} \cdot \prod_{\ell=k+1}^{N} \frac{1}{w_i - \pi_\ell}.$$

Similarly for $1 \leq j \leq N - k$

$$\prod_{\ell=j+k+1}^{N} \frac{1}{w - \pi_\ell} = \prod_{\ell=j+1}^{N} \frac{1}{w - \pi'_\ell} \cdot \prod_{\ell=k+1}^{N} \frac{1}{w_i - \pi_\ell} \cdot \prod_{\ell=1}^{N} (w - \pi_\ell).$$

Note that we have used the fact that $\pi_{j+N} = \pi_j$ for all $j \in \mathbb{Z}$. Now we can re-express the function $\psi^r_{\vec{x}}(\vec{w})$ using the shifted variable $\vec{x}'$ as follows: First move the first k columns of the determinant to the last k columns. Then we factor out a common factor $\prod_{\ell=k+1}^{N} \frac{1}{w_i - \pi_\ell}$ from each row. Finally note that for the first $N - k$ columns (which was the last $N - k$ columns originally), there are extra common factors of the form

$$\prod_{\ell=1}^{L-N} \frac{1}{\pi'_j + \hat{\pi}_\ell} \cdot \prod_{\ell=1}^{N} (w_i - \pi_\ell) \prod_{\ell=1}^{L-N} (w_i + \hat{\pi}_\ell),$$

which equals $\prod_{\ell=1}^{L-N} \frac{1}{\pi'_j + \hat{\pi}_\ell} \cdot z^L$ by the assumption that $w_i \in \mathcal{S}_z$ for all $1 \leq i \leq N$. Factoring these common factors out from the first $N - k$ columns we conclude that

$$\psi^r_{\vec{x}}(\vec{w}; \{\pi_j\}) = (-1)^{k(N-1)} \cdot z^{(N-k)L} \cdot \prod_{j=k+1}^{N} \left(\prod_{\ell=1}^{N} \frac{1}{w_\ell - \pi_j} \cdot \prod_{\ell=1}^{L-N} \frac{1}{\pi_j + \hat{\pi}_\ell} \right) \cdot \psi^r_{\vec{x}'}(\vec{w}; \{\pi'_j\}). \tag{3.61}$$

This reduces the general $1 \leq k \leq N$ case to the special case $k = N$. Now it suffices to show that

$$\sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N\geq A\}} \psi^r_{\vec{x}}(w) = \det\left[\prod_{\ell=j}^{N}\frac{1}{w_i-\pi_\ell}\cdot\prod_{\ell=1}^{A+N}\frac{\pi_j+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\right]_{i,j=1}^{N}. \tag{3.62}$$

To see this we first fix $x_N = B$, perform the sum in the order $B < x_{N-1} < x_{N-2} < \cdots < x_1 < B+L$, then summing over B from A to ∞. Note that the first step is a finite sum so it converges for arbitrary w_i's and the extra assumption on w_i's guarantees the convergence of the sum in the second step. The following summation identity is easy to check and will be used several times in the whole chapter: for any distinct complex numbers w and z and a set of parameters $\{\alpha_j\}_{j\in\mathbb{Z}}$ such that $z\neq-\alpha_i$ for all i, we have

$$\sum_{x=B}^{A}\frac{\prod_{\ell=1}^{x}(w+\alpha_\ell)}{\prod_{\ell=1}^{x+1}(z+\alpha_\ell)} = \frac{1}{z-w}\cdot\left(\prod_{\ell=1}^{B}\frac{w+\alpha_\ell}{z+\alpha_\ell}-\prod_{\ell=1}^{A+1}\frac{w+\alpha_\ell}{z+\alpha_\ell}\right). \tag{3.63}$$

To see the identity one simply notes that

$$\begin{aligned}(z-w)\cdot\frac{\prod_{\ell=1}^{x}(w+\alpha_\ell)}{\prod_{\ell=1}^{x+1}(z+\alpha_\ell)} &= ((z+\alpha_{x+1})-(w+\alpha_{x+1}))\cdot\frac{\prod_{\ell=1}^{x}(w+\alpha_\ell)}{\prod_{\ell=1}^{x+1}(z+\alpha_\ell)}\\ &=\prod_{\ell=1}^{x}\frac{w+\alpha_\ell}{z+\alpha_\ell}-\prod_{\ell=1}^{x+1}\frac{w+\alpha_\ell}{z+\alpha_\ell},\end{aligned}$$

and the sum telescopes. Now by linearity we move the sum over x_{N-1} to the second last column of the determinant and applying the above summation identity, the $(i, N-1)$-th entry of the determinant becomes

$$\prod_{\ell=N-1}^{N}\frac{1}{w_i-\pi_\ell}\cdot\left(\prod_{\ell=1}^{B+N}\frac{\pi_{N-1}+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}-\prod_{\ell=1}^{x_{N-2}+N-1}\frac{\pi_{N-1}+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\right).$$

By multiplying the $N-2$-th column with $\prod_{\ell=1}^{x_{N-2}+N-1}(\pi_{N-1}+\hat{\pi}_\ell)/\prod_{\ell=1}^{x_{N-2}+N-2}(\pi_{N-2}+\hat{\pi}_\ell)$ and adding to the $N-1$-th column we get rid of the second term above. We repeat this procedure for $1 \leq j \leq N-1$. For the first column after the summation one should obtain

$$\prod_{\ell=2}^{N} \frac{1}{w_i - \pi_\ell} \cdot \sum_{x_1=B+N-1}^{B+L-1} \frac{\prod_{\ell=1}^{x_1+1}(\pi_1 + \hat{\pi}_\ell)}{\prod_{\ell=1}^{x_1+2}(w_i + \hat{\pi}_\ell)} = \prod_{\ell=1}^{N} \frac{1}{w_i - \pi_\ell} \cdot \left(\prod_{\ell=1}^{B+N} \frac{\pi_1 + \hat{\pi}_\ell}{w_i + \hat{\pi}_\ell} - \prod_{\ell=1}^{B+L+1} \frac{\pi_1 + \hat{\pi}_\ell}{w_i + \hat{\pi}_\ell} \right).$$

To get rid of the second term in the above equation we need to use the assumption that $w_i \in \mathcal{S}_z$, namely

$$\prod_{\ell=1}^{N}(w_i - \pi_\ell) \cdot \prod_{\ell=B+N+2}^{B+L+1}(w_i + \hat{\pi}_\ell) = z^L, \quad \text{for } 1 \leq i \leq N.$$

Here $\hat{\pi}_{j+L-N} = \hat{\pi}_j$ for all $j \in \mathbb{Z}$. Now multiplying the N-th column with $z^L \cdot \prod_{\ell=1}^{B+L+1}(\pi_1 + \hat{\pi}_\ell) / \prod_{\ell=1}^{B+N}(\pi_N + \hat{\pi}_\ell)$ and adding to the first column we conclude that

$$\sum_{B+L > x_1 > \cdots > x_N = B} \psi_{\vec{x}}^r(w) = \det \left[\prod_{\ell=j+\delta_N(j)}^{N} \frac{1}{w_i - \pi_\ell} \cdot \frac{\prod_{\ell=1}^{B+N}(\pi_j + \hat{\pi}_\ell)}{\prod_{\ell=1}^{B+N+\delta_N(j)}(w_i + \hat{\pi}_\ell)} \right]_{i,j=1}^{N}.$$

Here $\delta_N(j)$ equals 1 for $j = N$ and 0 otherwise. We rewrite (i, N)-th entry in the above determinant as

$$\frac{1}{w_i - \pi_N} \cdot \left(\prod_{\ell=1}^{B+N} \frac{\pi_N + \hat{\pi}_\ell}{w_i + \pi_\ell} - \prod_{\ell=1}^{B+N+1} \frac{\pi_N + \hat{\pi}_\ell}{w_i + \pi_\ell} \right).$$

Then using linearity and after some elementary column operations for the second determinant (multiplying the last column with some proper constant and adding to the second last row and repeating) we conclude that $\sum_{B+L > x_1 > \cdots > x_N = B} \psi_{\vec{x}}^\ell(w)$ equals

$$\det \left[\prod_{\ell=j}^{N} \frac{1}{w_i - \pi_\ell} \cdot \prod_{\ell=1}^{B+N} \frac{\pi_j + \hat{\pi}_\ell}{w_i + \hat{\pi}_\ell} \right] - \det \left[\prod_{\ell=j}^{N} \frac{1}{w_i - \pi_\ell} \cdot \prod_{\ell=1}^{B+N+1} \frac{\pi_j + \hat{\pi}_\ell}{w_i + \hat{\pi}_\ell} \right].$$

Finally summing over B which is a telescoping summation we conclude the proof of (3.62). In this step we need to use the assumption that $\prod_{j=1}^{N} \left| \frac{\pi_j + \hat{\pi}_\ell}{w_j + \hat{\pi}_\ell} \right| < 1$ for all

$1 \le \ell \le L-N$ to make sure the summation converges. Combining with (3.61) we see that $\sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_k\ge a\}} \psi^{\ell}_{\vec{x}}(w;\{\pi_j\})$ equals

$$(-1)^{k(N-1)} \cdot z^{(N-k)L} \cdot \prod_{j=k+1}^{N} \left(\prod_{\ell=1}^{N} \frac{1}{w_\ell - \pi_j} \cdot \prod_{\ell=1}^{L-N} \frac{1}{\pi_j + \hat{\pi}_\ell} \right) \cdot \det \left[\prod_{\ell=j}^{N} \frac{1}{w_i - \pi_{k+\ell}} \cdot \prod_{\ell=1}^{a+k} \frac{\pi_j + \hat{\pi}_\ell}{w_i + \hat{\pi}_\ell} \right]_{i,j=1}^{N}.$$

This completes the proof of Lemma 3.4.2. □

The proof of Lemma 3.4.3 works in a similar way where we first reduce the constraint $x_k \ge a$ on the k-th particle for some $1 \le k \le N$ to a constraint on the last particle x'_N. The corresponding summation identity is a Cauchy-like identity for some inhomogeneous variant of Grothendieck polynomial which is of independent interest, see Section 3.5 for more details.

Proof of Lemma 3.4.3. Similar as in equation (3.61) we have

$$\psi^r_{\vec{x}}(\vec{w};\{\pi_j\}) = (-1)^{k(N-1)} \cdot z^{(N-k)L} \cdot \prod_{j=k+1}^{N} \left(\prod_{\ell=1}^{N} \frac{1}{w_\ell - \pi_j} \cdot \prod_{\ell=1}^{L-N} \frac{1}{\pi_j + \hat{\pi}_\ell} \right) \cdot \psi^r_{\vec{x}'}(\vec{w};\{\pi'_j\}),$$

$$\psi^\ell_{\vec{x}}(\vec{w}';\{\pi_j\}) = (-1)^{k(N-1)} \cdot (z')^{(k-N)L} \cdot \prod_{j=k+1}^{N} \left(\prod_{\ell=1}^{N} (w'_\ell - \pi_j) \cdot \prod_{\ell=1}^{L-N} (\pi_j + \hat{\pi}_\ell) \right) \cdot \psi^\ell_{\vec{x}'}(\vec{w}';\{\pi'_j\}).$$

where $\vec{x}' = (x'_1, \cdots, x'_N) = (x_{k+1}+L-N+k, \cdots, x_N+L-N+k, x_1-N+k, \cdots, x_k - N + k)$ and $(\pi'_1, \cdots, \pi'_N) = (\pi_{k+1}, \cdots, \pi_N, \pi_1, \cdots, \pi_k)$. Hence

$$\sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_k\ge a\}} \psi^r_{\vec{x}}(\vec{w};\{\pi_j\}) \psi^\ell_{\vec{x}}(\vec{w}';\{\pi_j\}) = \left(\frac{z}{z'}\right)^{(N-k)L} \cdot \prod_{j=1}^{N} \prod_{\ell=k+1}^{N} \frac{w'_j - \pi_\ell}{w_j - \pi_\ell} \cdot \sum_{\vec{x}'\in\mathcal{X}_N^{(L)}\cap\{x_N\ge a-N+k\}} \psi^r_{\vec{x}'}(\vec{w};\{\pi'_j\}) \psi^\ell_{\vec{x}'}(\vec{w}';\{\pi'_j\}).$$

Evaluating the above sum over $\vec{x}'$ using Corollary 3.5.4 in Section 3.5 with $\lambda_j = x'_j + j$

we conclude the proof of Lemma 3.4.3. □

3.5 A generalized Cauchy identity for some Grothendieck-like polynomials

The goal of this section is to state the following (generalized) Cauchy-type identity for the Grothendieck-like polynomial (and its dual) which depend on two sets of parameters $\{\pi_i\}_{i\in\mathbb{Z}}$ and $\{\hat{\pi}_i\}_{i\in\mathbb{Z}}$. For notational convenience we set

$$\Psi^r_\lambda(\vec{w}) = \det\left(\prod_{\ell=j+1}^{N}\frac{1}{w_i-\pi_\ell}\prod_{\ell=1}^{\lambda_j+1}\frac{1}{w_i+\hat{\pi}_\ell}\right)_{i,j=1}^{N},$$

$$\Psi^\ell_\lambda(\vec{w}') = \det\left(\prod_{\ell=j+1}^{N}(w'_i-\pi_\ell)\prod_{\ell=1}^{\lambda_j}(w'_i+\hat{\pi}_\ell)\right)_{i,j=1}^{N}.$$

Proposition 3.5.1. *Given two sets of complex numbers $\{\pi_i\}_{i\in\mathbb{Z}}$ and $\{\hat{\pi}_i\}_{i\in\mathbb{Z}}$. Let n be a positive integer and $\{w_i\}_{i=1}^n$ and $\{w'_i\}_{i=1}^n$ be distinct complex numbers. Then for any integers A and B with $A\geq B$ we have*

$$\sum_{A\geq\lambda_1\geq\cdots\geq\lambda_n\geq B}\Psi^r_\lambda(\vec{w})\Psi^\ell_\lambda(\vec{w}') = \det\left(\frac{\prod_{\ell=1}^{B}\frac{w'_{i'}+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}-\prod_{\ell=2}^{n}\frac{w'_{i'}-\pi_\ell}{w_i-\pi_\ell}\prod_{\ell=1}^{A+1}\frac{w'_{i'}+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}}{w_i-w'_{i'}}\right)_{i,i'=1}^{n} \tag{3.64}$$

Remark 3.5.2. Proposition 3.5.1 and Corollary 3.5.4 are conjectured by the author and the proofs are provided by Zhipeng Liu (he observed Proposition 3.5.1 independently), the proofs will be recorded in Appendix A. It is worth to point out that if we set the parameters such that $\pi_i\equiv 0$ and $\hat{\pi}_i\equiv 1$ for all $i\in\mathbb{Z}$, then (3.64) reduces to the generalized Cauchy identity for the (homogeneous) Grothendieck polynomial obtained in Theorem 5.3 of [85] after a simple change of variable. It would be interesting to see whether the approach in [85] (which is different from ours) using algebraic Bethe ansatz and Izergin-Korepin analysis for some five vertex models can be generalized to the inhomogeneous case.

As a simple consequence, by setting $B = 0$ and letting $A \to \infty$ we obtain the usual Cauchy identity:

Corollary 3.5.3. *Given two sets of complex numbers $\{\pi_i\}_{i\in\mathbb{Z}}$ and $\{\hat{\pi}_i\}_{i\in\mathbb{Z}}$. Let n be a positive integer and $\{w_i\}_{i=1}^n$ and $\{w'_i\}_{i=1}^n$ be distinct complex numbers. Then*

$$\sum_{\lambda} \Psi^r_\lambda(\vec{w})\Psi^\ell_\lambda(\vec{w}') = \det\left[\frac{1}{w_i - w'_{i'}}\right]_{i,i'=1}^n, \tag{3.65}$$

where the summation is over all partitions $\lambda = (\lambda_1, \cdots, \lambda_n)$.

For our analysis on the inhomogeneous periodic TASEP, we will need the following less obvious corollary of Proposition 3.5.1 where the summation is over cylindrical partitions $\lambda = (\lambda_1, \cdots, \lambda_N)$ which satisfy $\lambda_N + L - N \geq \lambda_1 \cdots \geq \lambda_N \geq 0$ and the generalized Cauchy determinant on the right hand side of (3.64) further reduces to a genuine Cauchy determinant if we impose certain algebraic constraints (the Bethe equations) on the spectral parameters $\{w_i\}_{i=1}^N$ and $\{w'_i\}_{i=1}^N$.

Corollary 3.5.4. *Let $N < L$ be two positive integers. Let $\{\pi_i\}_{i\in\mathbb{Z}}$ and $\{\hat{\pi}_i\}_{i\in\mathbb{Z}}$ be two sets of complex numbers such that $\pi_{i+N} = \pi_i$ and $\hat{\pi}_{i+L-N} = \hat{\pi}_i$ for all $i \in \mathbb{Z}$. Suppose the spectral parameters $\{w_i\}_{i=1}^N$ and $\{w'_i\}_{i=1}^N$ are distinct complex numbers satisfying*

$$\prod_{\ell=1}^{N}(w_i - \pi_\ell)\prod_{\ell=1}^{L-N}(w_i + \hat{\pi}_\ell) = z^L, \quad \prod_{\ell=1}^{N}(w'_i - \pi_\ell)\prod_{\ell=1}^{L-N}(w'_i + \hat{\pi}_\ell) = (z')^L, \tag{3.66}$$

for all $1 \leq i \leq N$, where z, z' are complex numbers with $z^L \neq (z')^L$. Assume further that

$$\prod_{j=1}^{N}\left|\frac{w'_j + \hat{\pi}_\ell}{w_j + \hat{\pi}_\ell}\right| < 1, \quad \textit{for all } 1 \leq \ell \leq L - N.$$

Then

$$\sum_{\lambda_N+L-N\geq\lambda_1\geq\cdots\geq\lambda_N\geq A}\Psi_\lambda^r(\vec{w})\Psi_\lambda^\ell(\vec{w}')$$
$$=\left(1-\left(\frac{z'}{z}\right)^L\right)^{N-1}\cdot\prod_{j=1}^N\prod_{\ell=1}^A\frac{w_j'+\hat{\pi}_\ell}{w_j+\hat{\pi}_\ell}\cdot\det\left(\frac{1}{w_i-w_{i'}'}\right)_{i,j=1}^N. \tag{3.67}$$

3.6 Fredholm determinant representation

The multi-point distribution formula (3.46) for inhomogeneous periodic TASEP has the form of a multiple contour integral of a $N\times N$ determinant. It is not suitable for taking large N and large time limit so we would like to re-express the formula as a multiple contour integral of a Fredholm determinant with an underlying space independent of N. We will see that another advantage of working on the periodic domain instead of infinite lattice is that due to quantization of the eigenvalues, the kernels naturally act on ℓ^2 spaces with measures supported on certain finite sets with cardinality N (related to the Bethe roots or eigenvalues), thus one can freely rearranging the terms appearing in the series expansion of the determinant without worrying about the convergence issue since everything is finite. The particle-hole duality also plays an important role in such orthogonalization procedure. The precise statement is as follows:

Theorem 3.6.1 (Joint distribution of inhomogeneous TASEP in $\mathcal{X}_N(L)$ for general initial condition)**.** *Consider the inhomogeneous TASEP in $\mathcal{X}_N(L)$ with initial condition $x_i(0)=y_i$ for $1\leq i\leq N$. Fix a positive integer m. Let $(k_1,t_1),\cdots,(k_m,t_m)$ be m distinct pints in $\{1,\cdots,N\}\times[0,\infty)$. Assume that $0\leq t_1\leq\cdots\leq t_m$. Let $a_i\in\mathbb{Z}$*

for $1 \leq i \leq m$. *Then*

$$\mathbb{P}_{\vec{y}}\left(\bigcap_{i=1}^{m}\{x_{k_i}(t_i) \geq a_i\}\right) = \oint \cdots \oint \mathscr{C}_{\vec{y}}(\vec{z})\mathscr{D}_{\vec{y}}(\vec{z})\frac{dz_m}{2\pi i z_m} \cdots \frac{dz_1}{2\pi i z_1}, \tag{3.68}$$

where $\vec{z} = (z_1, \cdots, z_m)$ *and the contours are nested circles oriented counterclockwise satisfying*

$$0 < |z_m| < \cdots |z_1| < \mathbf{r}_0, \tag{3.69}$$

with $\mathbf{r}_0 > 0$ *sufficiently small so that the left and right Bethe roots associated to* z_i*'s are well-defined for all* $1 \leq i \leq m$*, see the discussion in Section 3.6.1 below. The functions* $\mathscr{C}_{\vec{y}}(\vec{z})$ *and* $\mathscr{D}_{\vec{y}}(\vec{z})$ *(the latter of which is a Fredholm determinant) are defined in* (3.78) *and* (3.84)*, respectively.*

3.6.1 Notations and Definitions

Recall for given parameter $z \in \mathbb{C}$ we have defined the polynomial $q_z(w) := \prod_{\ell=1}^{N}(w - \pi_\ell) \cdot \prod_{\ell=1}^{L-N}(w + \hat{\pi}_\ell) - z^L$ and the set of its roots

$$\mathcal{S}_z := \{w \in \mathbb{C} : q_z(w) = 0\} \tag{3.70}$$

We call the polynomial $q_z(w)$ the Bethe polynomial associated to z and its roots Bethe roots. For $\mathbf{r}_0 > 0$ small enough and $0 < |z| < \mathbf{r}_0$, the level set $|\prod_{\ell=1}^{N}(w - \pi_\ell) \cdot \prod_{\ell=1}^{L-N}(w + \hat{\pi}_\ell)| = |z|^L$ consists of L contours, with N of them enclosing $\{\pi_i\}_{i=1}^{N}$ and the other $L - N$ of them enclosing $\{-\hat{\pi}_i\}_{i=1}^{L-N}$. Here we are counting multiplicities so if some π_j's or $\hat{\pi}_\ell$'s coincide we will count the contours enclosing them multiple times. By our assumption that $\pi_j + \hat{\pi}_k > 0$ for all $1 \leq j \leq N$ and $1 \leq k \leq L - N$ we have $\max_{1 \leq i \leq L-N}(-\hat{\pi}_i) < \min_{1 \leq i \leq N} \pi_i$. Hence if we set

$$M := \frac{\max\{-\hat{\pi}_i\} + \min\{\pi_i\}}{2}. \tag{3.71}$$

Then for $|z| > 0$ sufficiently small we have $\mathcal{S}_z \cap \{\mathrm{Re}(w) > M\} = N$ and $\mathcal{S}_z \cap \{\mathrm{Re}(w) < M\} = L - N$. More concretely we fix any two simple closed contours $\Sigma_{\mathrm{R}} \subset \{\mathrm{Re}(w) > M\}$ and $\Sigma_{\mathrm{L}} \subset \{\mathrm{Re}(w) < M\}$ enclosing the sets $\{\pi_j\}_{j=1}^{N}$ and $\{-\hat{\pi}_j\}_{j=1}^{L-N}$, respectively. Let

$$m_\Sigma := \min\left\{\prod_{\ell=1}^{N} |w - \pi_\ell| \cdot \prod_{\ell=1}^{L-N} |w + \hat{\pi}_\ell| : w \in \Sigma_{\mathrm{R}} \cup \Sigma_{\mathrm{L}}\right\} > 0.$$

Then for any z with $|z| < m_\Sigma^{1/L} := \mathbf{r}_0$, by Rouche's theorem there are exactly the same amount of zeros inside Σ_{L} for $q_z(w)$ and $q_0(w)$ (same for Σ_{R}). Thus we can define the left and right parts of Bethe roots:

Definition 3.6.2 (Left and right Bethe roots)**.** *For $|z| > 0$ sufficiently small, we define*

$$\mathcal{L}_z := \mathcal{S}_z \cap \{\mathrm{Re}(w) < M\}, \quad \mathcal{R}_z := \mathcal{S}_z \cap \{\mathrm{Re}(w) > M\}. \tag{3.72}$$

We define the left and right Bethe polynomials as the monic polynomials with zeros at the left and right Bethe roots.

Definition 3.6.3. *Given $z \in \mathbb{C}$ with $|z| < \mathbf{r}_0$. Let $\mathcal{S}_z = \mathcal{L}_z \cap \mathcal{R}_z$ be the roots of the polynomial $q_z(w)$. We set*

$$q_{z,\mathrm{L}}(w) := \prod_{u \in \mathcal{L}_z} (w - u), \quad q_{z,\mathrm{R}}(w) := \prod_{u \in \mathcal{R}_z} (w - u). \tag{3.73}$$

In this section, we assume the contours of z_i to be nested circles satisfying $0 < |z_m| < \cdots |z_1| < \mathbf{r}_0$ with $\mathbf{r}_0 > 0$ sufficiently small, so that $\mathcal{L}_{z_i}$ and $\mathcal{R}_{z_i}$ are well-defined and the level sets $\{w \in \mathbb{C} : |\prod_{\ell=1}^{N}(w - \pi_\ell) \cdot \prod_{\ell=1}^{L-N}(w + \hat{\pi}_\ell)| = |z_i|^L\}$'s are (disjoint unions of) nested simple closed contours. See Figure 3.4 for a plot of the roots of a sufficiently generic Bethe equation and the corresponding level sets.

The following simple lemma whose proof is elementary summarizes the nesting behaviors of the level sets of the polynomial equation $q_z(w) = 0$ for different z as can been easily seen from Figure 3.4.

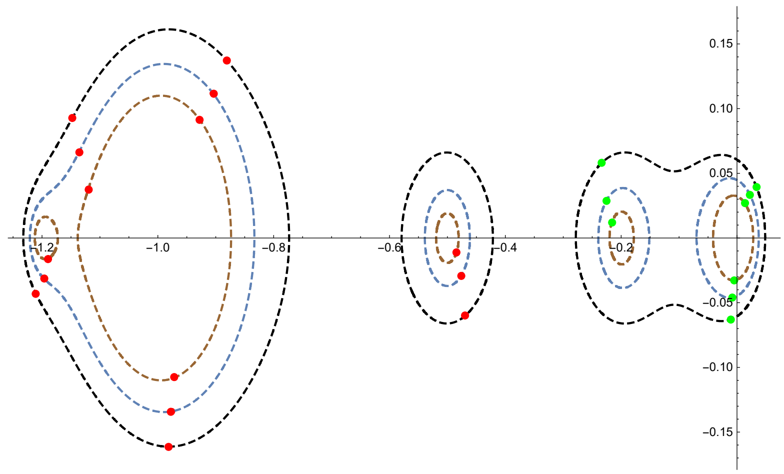

Figure 3.4: Roots of the polynomial equation $(w+1.2)(w+1)^3(w+0.5)(w+0.2)w^2 = z^8$ with $z = 0.31+0.1\mathrm{i}$, $0.34+0.1\mathrm{i}$, $0.37+0.1\mathrm{i}$ from inside to outside. The left roots are colored in red while the right roots are colored in green. Here $L = 8$ and $N = 3$ with $\{-\hat{\pi}_j\} = \{-1.2, -1, -1, -1, -0.5\}$ and $\{\pi_j\} = \{-0.2, 0, 0\}$. The dashed lines represent the corresponding level sets for different choices of z's and are displayed here merely for better visualization.

Lemma 3.6.4 (Winding numbers of the level sets of $q_z(w)$)**.** *Let $\{\pi_j\}_{j=1}^{N}$ and $\{\hat{\pi}_j\}_{j=1}^{L-N}$ be given real parameters satisfying $\pi_j + \hat{\pi}_\ell > 0$ for all j, ℓ. Let $q(w) = q_0(w) = \prod_{\ell=1}^{N}(w - \pi_j)\prod_{j=1}^{L-N}(w + \hat{\pi}_j)$. Then there exist $r_{\max} > 0$ such that for any $z, z' \in \mathbb{C}$ with $|z| \neq |z'|$, $|z| < r_{\max}$ and $|z'| < r_{\max}$, the level sets $\Gamma := \{w \in \mathbb{C} : |q(w)| = |z|\}$ and $\Gamma' := \{w \in \mathbb{C} : |q(w)| = |z'|\}$ are both disjoint unions of simple closed curves. Moreover for any $w' \in \Gamma'$ we have*

$$\mathrm{Ind}_\Gamma(w') := \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta - w'} = \begin{cases} 1 & \text{if } |z| > |z'|, \\ 0 & \text{if } |z| < |z'|. \end{cases}$$

In particular if we take $z' = 0$, we have $\mathrm{Ind}_\Gamma(\pi_j)$ and $\mathrm{Ind}_\Gamma(-\hat{\pi}_j) = 1$ for all $|z| > 0$.

Now we define the two functions $\mathscr{C}_{\vec{y}}(\vec{z})$ and $\mathscr{D}_{\vec{y}}(\vec{z})$ of $\vec{z} = (z_1, \cdots, z_m)$ appearing in the integrand of the contour integral formula for the multi-point distribution of inhomogeneous periodic TASEP under general initial condition. Here we suppress the dependence of $\mathscr{C}_{\vec{y}}(\vec{z})$ and $\mathscr{D}_{\vec{y}}(\vec{z})$ on the parameters $k_i, t_i, a_i, \{\pi_j\}, \{\hat{\pi}_j\}$.

The following two quantities related to the symmetric function $\mathcal{F}_\lambda$ defined in Definition 3.1.5 encode the initial condition:

Definition 3.6.5 (Global energy and characteristic function)**.** *For $\vec{y} \in \mathcal{X}_N^{(L)}$, let*

$$\lambda(\vec{y}) = (y_1 + 1, y_2 + 2, \cdots, y_N + N). \tag{3.74}$$

For $|z| < \mathbf{r}_0$, we define the global energy $\mathcal{E}_{\vec{y}}(z)$ by

$$\mathcal{E}_{\vec{y}}(z) := \mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z; \{\pi_j\}, \{\hat{\pi}_\ell\}). \tag{3.75}$$

When $\mathcal{E}_{\vec{y}}(z) \neq 0$, we define the characteristic function $\chi_{\vec{y}}(v, u; z)$ for a left Bethe root v and a right Bethe root v by

$$\chi_{\vec{y}}(v, u; z) := \frac{\mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z \cup \{u\} \backslash \{v\}; \{\pi_j\}, \{\hat{\pi}_\ell\})}{\mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z; \{\pi_j\}, \{\hat{\pi}_\ell\})}, \quad \text{for } u \in \mathcal{L}_z \text{ and } v \in \mathcal{R}_z. \tag{3.76}$$

Note that we can extend the definition of $\chi_{\vec{y}}(v, u; z)$ to any $u \in \Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_j\}$ by plugging in such u into right hand side of (3.76). For later purpose we need a further extension of $\chi_{\vec{y}}(v, u; z)$ as an analytic function on $(\Omega_{\mathrm{R}} \backslash \{\pi_j\}) \times (\Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_j\}) \times D(r_{\max})$.

Lemma 3.6.6 (Analytic continuation of $\chi_{\vec{y}}(v, u; z)$)**.** *There exists a function analytic in $(\Omega_{\mathrm{R}} \backslash \{\pi_j\}) \times (\Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_j\}) \times D(r_{\max})$, which we still denote by $\chi_{\vec{y}}(v, u; z)$, such that*

$$\chi_{\vec{y}}(v, u; z) = \frac{\mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z \cup \{u\} \backslash \{v\}; \{\pi_j\}, \{\hat{\pi}_\ell\})}{\mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z; \{\pi_j\}, \{\hat{\pi}_\ell\})}, \quad \text{for } u \in \mathcal{L}_z \text{ and } v \in \mathcal{R}_z.$$

Moreover there exists a function $h_{\vec{y}}(v, u; z)$ analytic in $(\Omega_{\mathrm{R}} \backslash \{\pi_j\}) \times (\Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_j\}) \times D(r_{\max})$ such that

$$\chi_{\vec{y}}(v, u; z) = \mathrm{ch}_{\vec{y}}(v, u) + h_{\vec{y}}(v, u; z), \tag{3.77}$$

with $\lim_{z \to 0} h_{\vec{y}}(v, u; z) = 0$ for all $(v, u) \in (\Omega_{\mathrm{R}} \backslash \{\pi_j\}) \times (\Omega_{\mathrm{L}} \backslash \{-\hat{\pi}_j\})$. Here the function $\mathrm{ch}_{\vec{y}}(v, u)$ is defined in Definition 3.1.6.

Proof. See [81] Lemma 5.5. □

3.6.2 Definition of $\mathscr{C}_{\vec{y}}(\vec{z})$ and $\mathscr{D}_{\vec{y}}(\vec{z})$

Definition 3.6.7 (Definition of $\mathscr{C}_{\vec{y}}(\vec{z})$)**.** *With the global energy function $\mathcal{E}_{\vec{y}}(z)$ defined in* (3.75)*, we define*

$$\mathscr{C}_{\vec{y}}(\vec{z}) := \mathcal{E}_{\vec{y}}(z_1)\mathscr{C}_{step}(\vec{z}), \tag{3.78}$$

where

$$\begin{aligned}\mathscr{C}_{step}(\vec{z}) := C(\{\pi_j\},\{\hat{\pi}_\ell\}) &\left[\prod_{\ell=1}^{m}\frac{E_{\ell-1}(z_\ell)}{E_\ell(z_\ell)}\right]\left[\prod_{\ell=2}^{m}\frac{z_{\ell-1}^L}{z_{\ell-1}^L - z_\ell^L}\right]\\ &\cdot\left[\prod_{\ell=1}^{m}\frac{\prod_{u\in\mathcal{L}_{z_\ell}}\prod_{\ell=1}^{N}(\pi_\ell - u)\cdot\prod_{v\in\mathcal{R}_{z_\ell}}\prod_{\ell=1}^{L-N}(v+\hat{\pi}_\ell)}{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_\ell})}\right]\\ &\cdot\left[\prod_{\ell=2}^{m}\frac{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_{\ell-1}})}{\prod_{u\in\mathcal{L}_{z_{\ell-1}}}\prod_{\ell=1}^{N}(\pi_\ell - u)\cdot\prod_{v\in\mathcal{R}_{z_\ell}}\prod_{\ell=1}^{L-N}(v+\hat{\pi}_\ell)}\right].\end{aligned} \tag{3.79}$$

Here

$$C(\{\pi_j\},\{\hat{\pi}_\ell\}) := \prod_{j=1}^{N} e^{-t_m\pi_j}\cdot\frac{\prod_{j=1}^{N}\prod_{\ell=1}^{a_m+k_m}(\pi_j+\hat{\pi}_\ell)}{\prod_{j=k_m+1}^{N}\prod_{\ell=1}^{L-N}(\pi_j+\hat{\pi}_\ell)}. \tag{3.80}$$

And

$$E_i(z) := \prod_{u\in\mathcal{L}_z}\prod_{\ell=1}^{k_i}(\pi_\ell - u)\cdot\prod_{v\in\mathcal{R}_z}\prod_{\ell=1}^{a_i+k_i}(v+\hat{\pi}_\ell)\cdot e^{-t_i v}, \tag{3.81}$$

for $1\le i\le m$ where $E_0(z) := 1$.

$\mathscr{D}_{\vec{y}}(\vec{z})$ is a Fredholm determinant with kernel acting on certain ℓ^2 space over discrete sets related to the Bethe roots. More precisely for m distinct complex numbers z_i satisfying $|z_i| < \mathbf{r}_0$, we define the discrete sets

$$\mathscr{S}_1 := \mathcal{L}_{z_1}\cup\mathcal{R}_{z_2}\cup\mathcal{L}_{z_3}\cup\cdots\cup\begin{cases}\mathcal{L}_{z_m}, & \text{if } m \text{ is odd},\\ \mathcal{R}_{z_m}, & \text{if } m \text{ is even},\end{cases} \tag{3.82}$$

and

$$\mathscr{S}_2 := \mathcal{R}_{z_1} \cup \mathcal{L}_{z_2} \cup \mathcal{R}_{z_3} \cup \cdots \cup \begin{cases} \mathcal{R}_{z_m}, & \text{if } m \text{ is odd}, \\ \mathcal{L}_{z_m}, & \text{if } m \text{ is even}. \end{cases} \tag{3.83}$$

Definition 3.6.8 (Definition of $\mathscr{D}_{\vec{y}}(\vec{z})$)**.** *Let $0 < |z_m| < \cdots < |z_1| < \mathbf{r}_0$. Assume $\mathcal{E}_{\vec{y}}(z_1) \neq 0$ so that $\chi_{\vec{y}}(v, u; z_1)$ is well defined. Define*

$$\mathscr{D}_{\vec{y}}(\vec{z}) = \det(I - \mathscr{K}^{\vec{y}}) \quad \textit{with} \quad \mathscr{K}^{\vec{y}} = \mathscr{K}_1^{\vec{y}} \mathscr{K}_2^{\vec{y}}, \tag{3.84}$$

where $\mathscr{K}_1^{\vec{y}} : \ell^2(\mathscr{S}_2) \to \ell^2(\mathscr{S}_1)$ and $\mathscr{K}_2^{\vec{y}} : \ell^2(\mathscr{S}_1) \to \ell^2(\mathscr{S}_2)$ have kernels given by $\mathscr{K}_1^{\vec{y}} = \mathscr{K}_1^{step}$ and

$$\mathscr{K}_2^{\vec{y}}(w, w') = \begin{cases} \chi_{\vec{y}}(w, w'; z_1) \mathscr{K}_2^{step}(w, w') & \textit{for } w \in \mathcal{R}_{z_1} \textit{ and } w' \in \mathcal{L}_{z_1}, \\ \mathscr{K}_2^{step}(w, w') & \textit{otherwise}. \end{cases} \tag{3.85}$$

Here the kernels $\mathscr{K}_1^{step}(w, w')$ and $\mathscr{K}_2^{step}(w', w)$ are given by

$$\mathscr{K}_1^{step}(w, w') := \frac{\delta_i(j) + \delta_i(j + (-1)^i)}{w - w'} \frac{J(w) f_i(w) (H_{z_i}(w))^2}{H_{z_{i-(-1)^i}}(w) H_{z_{j-(-1)^j}}(w')} Q_1(j) \tag{3.86}$$

and

$$\mathscr{K}_2^{step}(w', w) := \frac{\delta_j(i) + \delta_j(i - (-1)^j)}{w - w'} \frac{J(w') f_j(w') (H_{z_j}(w'))^2}{H_{z_{j+(-1)^j}}(w') H_{z_{i+(-1)^i}}(w)} Q_2(i) \tag{3.87}$$

for

$$w \in (\mathcal{L}_{z_i} \cup \mathcal{R}_{z_i}) \cap \mathscr{S}_1 \quad \textit{and} \quad w' \in (\mathcal{L}_{z_j} \cup \mathcal{R}_{z_j}) \cap \mathscr{S}_2$$

with $1 \leq i, j \leq m$. The functions $J(w)$, $f_i(w)$, $H_z(w)$ and $Q_i(j)$ are defined as follows:

$$J(w) := \frac{1}{\sum_{\ell=1}^{N} \frac{1}{w - \pi_\ell} + \sum_{\ell=1}^{L-N} \frac{1}{w + \hat{\pi}_\ell}}, \tag{3.88}$$

and

$$Q_1(j) := 1 - \left(\frac{z_{j-(-1)^j}}{z_j}\right)^L, \quad Q_2(j) := 1 - \left(\frac{z_{j+(-1)^j}}{z_j}\right)^L, \tag{3.89}$$

for $j = 1, \cdots, m$. *Here we set* $z_{m+1} = 0$ *for convenience.*

To define $H_z(w)$ *and* $f_i(w)$ *we recall the definition of left and right Bethe polynomials and Bethe roots discussed in Section 3.6.1. With the notation there we set*

$$H_z(w) := \begin{cases} \frac{q_{z,\mathrm{R}}(w)}{\prod_{\ell=1}^{N}(w-\pi_\ell)} & \text{for } Re(w) < M, \\ \frac{q_{z,\mathrm{L}}(w)}{\prod_{\ell=1}^{L-N}(w+\hat{\pi}_\ell)} & \text{for } Re(w) > M. \end{cases} \tag{3.90}$$

Finally the functions $f_i(w)$ *encodes the information of the parameters* a_i, k_i, t_i*'s:*

$$f_\ell(w) := \begin{cases} \frac{F_\ell(w)}{F_{\ell-1}(w)} & \text{for } Re(w) < M, \\ \frac{F_{\ell-1}(w)}{F_\ell(w)} & \text{for } Re(w) > M, \end{cases} \tag{3.91}$$

where

$$F_\ell(w) := \prod_{j=1}^{k_\ell}(w-\pi_j) \cdot \prod_{j=1}^{a_\ell+k_\ell}(w+\hat{\pi}_j)^{-1} \cdot e^{t_\ell w}, \tag{3.92}$$

for $1 \le \ell \le m$ *and* $F_0(w) := 1$.

To prove Theorem 3.6.1, one starts with re-writing the determinant in equation (3.48) using Cauchy-Binet formula, which leads to a multiple summation over Bethe roots for a product of several determinants (most of them are Cauchy determinants due to the Cauchy interaction $\frac{1}{w_\ell - w_{\ell-1}}$ on the denominator of the entries in (3.48)). Reorganizing the multiple sum over sets of Bethe roots according to the number of right roots used in each term will lead to a new summation which can be recognized as the series expansion for a Fredholm determinant. Using this idea, in [10] the authors obtained the following remarkable general identity between a Toeplitz-like determinant of the form of (3.48) and a Fredholm determinant acting on ℓ^2 spaces supported on finite sets.

Proposition 3.6.9 (Proposition 4.1 of [10]). *Define $N \times N$ matrices $T = (T_{ij})_{i,j=1}^N$ and $M = (M_{ij})_{i,j=1}^N$ with entries*

$$T_{ij} = \sum_{\substack{w_1 \in S_1 \\ \cdots \\ w_m \in S_m}} \frac{p_i(w_1) q_j(w_m)}{\prod_{\ell=2}^m (w_\ell - w_{\ell-1})} \prod_{\ell=1}^m h_\ell(w_\ell), \quad M_{ij} = \sum_{\substack{w_1 \in R_1 \\ \cdots \\ w_m \in R_m}} \frac{p_i(v_1) q_j(v_m)}{\prod_{\ell=2}^m (v_\ell - v_{\ell-1})} \prod_{\ell=1}^m h_\ell(v_\ell).$$

Assume that

$$\det \left[p_i(v_j^{(1)}) \right]_{i,j=1}^N \det \left[q_i(v_j^{(m)}) \right]_{i,j=1}^N \neq 0,$$

where $R_1 = \{v_1^{(1)}, \cdots, v_N^{(1)}\}$ and $R_m = \{v_1^{(m)}, \cdots, v_N^{(m)}\}$. Then

$$\det[T] = \det[M] \det(I - K_1 K_2). \tag{3.93}$$

Here $S_1, \cdots, S_m$ are finite subsets of $\mathbb{C}$ with at least N elements such that $S_i \cap S_{i+1} = \emptyset$ for $1 \leq i \leq m-1$ and R_i is an N element subset of S_i for $1 \leq i \leq m$. The h_i's are nonzero complex-valued functions on S_i. $p_1, \cdots, p_N$ are complex-valued functions on S_1 and $q_1, \cdots, q_N$ are complex-valued functions on S_m. The finite matrices K_1 and K_2 act on direct sums of some of the R_i's and $L_i := S_i \backslash R_i$'s for $1 \leq i \leq m$ (which are still finite sets) and are of similar block structure as the kernels $\mathscr{K}_1^{\vec{y}}$ and $\mathscr{K}_2^{\vec{y}}$ defined in Definition 3.6.8, for the precise form we refer to [10]. The determinant of M takes the form

$$\det[M] = (-1)^{(m-1)N(N-1)/2} \frac{\det[p_i(v_j^{(1)})]_{i,j=1}^N}{\Delta(v_1^{(1)}, \cdots, v_N^{(1)})} \frac{\det[q_i(v_j^{(m)})]_{i,j=1}^N}{\Delta(v_1^{(m)}, \cdots, v_N^{(m)})} \cdot \frac{\prod_{\ell=1}^m \Delta(R_\ell)^2}{\prod_{\ell=2}^m \Delta(R_\ell; R_{\ell-1})} \prod_{\ell=1}^m h_\ell(R_\ell). \tag{3.94}$$

Here $\Delta(v_1, \cdots, v_N)$ is the Vandermonde determinant $\prod_{1 \leq i < j \leq N}(v_j - v_i)$ and for finite *sets S and T*

$$\Delta(S)^2 := \prod_{\substack{\{w_1,w_2\}\subset S\\ w_1\neq w_2}} (w_1-w_2)^2, \quad \Delta(S;T) := \prod_{u\in S, v\in T} (u-v).$$

Finally for a complex-valued function h defined on a finite set S, $h(S)$ is defined as $\prod_{w\in S} h(w)$.

We have stated Theorem 3.4.1 in such a form that one can immediately recognize the applicability of the above Proposition in re-writing the determinant part $\mathcal{D}^{(L)}_{\vec{y}}(\vec{z})$ defined in equation (3.48).

Proof of Theorem 3.6.1. The goal is to prove $\mathcal{C}^{(L)}_{\vec{y}}(\vec{z})\mathcal{D}^{(L)}_{\vec{y}}(\vec{z}) = \mathscr{C}_{\vec{y}}(\vec{z})\mathscr{D}_{\vec{y}}(\vec{z})$ with the functions defined in Theorem 3.4.1 and Theorem 3.6.1. We view $\mathcal{D}^{(L)}_{\vec{y}}(\vec{z})$ in Theorem 3.4.1 and $\mathscr{D}_{\vec{y}}(\vec{z})$ in Theorem 3.6.1 as $\det[T]$ and $\det[I-K_1K_2]$ in Proposition 3.6.9 with $S_i := \mathcal{S}_{z_i}$ and $R_i := \mathcal{R}_{z_i}$ for $1\leq i\leq m$. The functions p_i's and q_j's take the same form in Theorem 3.4.1 and Proposition 3.6.9, namely

$$p_i(w) = \prod_{\ell=i+1}^{N}(w-\pi_\ell)\cdot\prod_{\ell=1}^{y_i+i}\frac{w+\hat{\pi}_\ell}{\pi_i+\pi_\ell}, \quad q_j(w) = \frac{\prod_{\ell=1}^{a_m+k_m}(\pi_j+\hat{\pi}_\ell)}{\prod_{\ell=j}^{N}(w-\pi_{\ell+k_m})}.$$

The functions h_ℓ in Proposition 3.6.9 are set to be

$$h_\ell(w) := \begin{cases} G_\ell(w)\prod_{j=1}^{N}(w-\pi_j)^{-1}, & \text{for } \ell=1,\\ G_\ell(w), & \text{for } 2\leq\ell\leq m.\end{cases}$$

Where the functions $G_\ell(w)$'s are defined in (3.50). Now by Proposition 3.6.9 we have $\mathcal{D}^{(L)}_{\vec{y}}(\vec{z}) = \det[M]\mathscr{D}_{\vec{y}}(\vec{z})$. Hence it suffices to prove that

$$\mathscr{C}_{\vec{y}}(\vec{z}) = \det[M]\cdot\mathcal{C}^{(L)}_{\vec{y}}(\vec{z}), \tag{3.95}$$

where $\mathcal{C}_{\vec{y}}^{(L)}(\vec{z})$ is defined in (3.47), $\mathscr{C}_{\vec{y}}(\vec{z})$ is defined in (3.78) and $\det[M]$ is given in (3.94). For this first note that for given $z, z' \in \mathbb{C}$ and $v' \in \mathcal{S}_{z'}$, we have

$$(z')^L - z^L = \prod_{j=1}^{N}(v'-\pi_j) \cdot \prod_{j=1}^{L-N}(v'+\hat{\pi}_j) - z^L = q_z(v') = \prod_{u\in\mathcal{L}_z}(v'-u) \prod_{v\in\mathcal{R}_z}(v'-v). \quad (3.96)$$

In particular if we take $z' = 0$ and $v' = \pi_j \in \mathcal{S}_0$ for some $1 \le j \le N$, this then implies

$$z^L = (-1)^{N-1} \prod_{u\in\mathcal{L}_z}(\pi_j - u) \prod_{v\in\mathcal{R}_z}(v - \pi_j).$$

Taking the product over j from 1 to N we then have

$$z^{NL} = \prod_{u\in\mathcal{L}_z}\prod_{j=1}^{N}(\pi_j - u) \cdot \prod_{v\in\mathcal{R}_z}\prod_{j=1}^{N}(v - \pi_j). \quad (3.97)$$

On the other hand for $w \in \mathcal{S}_z$ we have $z^L = \prod_{j=1}^{N}(w-\pi_j)\prod_{j=1}^{N}(w+\hat{\pi}_j)$. Taking the product over $v \in \mathcal{R}_z$ gives

$$z^{NL} = \prod_{v\in\mathcal{R}_z}\left(\prod_{j=1}^{N}(v-\pi_j)\prod_{j=1}^{L-N}(v+\hat{\pi}_j)\right). \quad (3.98)$$

Comparing (3.98) with (3.97) we see

$$\prod_{u\in\mathcal{L}_z}\prod_{j=1}^{N}(\pi_j - u) = \prod_{v\in\mathcal{R}_z}\prod_{j=1}^{L-N}(v+\hat{\pi}_j). \quad (3.99)$$

Now (3.96), (3.97) and (3.99) combined together implies that for any $2 \le \ell \le m$,

$$\left(\frac{z_\ell^L - z_{\ell-1}^L}{z_{\ell-1}^L}\right)^N = \frac{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_{\ell-1}})\Delta(\mathcal{R}_{z_\ell};\mathcal{R}_{z_{\ell-1}})}{\prod_{u\in\mathcal{R}_{z_{\ell-1}}}\prod_{j=1}^{N}(v-\pi_j)\cdot\prod_{v\in\mathcal{R}_{z_{\ell-1}}}\prod_{j=1}^{N}(v-\pi_j)}.$$

Taking the product over ℓ from 2 to m we have

$$
\begin{aligned}
&\prod_{\ell=2}^{m}\left(\left(\frac{z_\ell}{z_{\ell-1}}\right)^{L}-1\right)^{N} \\
&=\prod_{\ell=2}^{m}\frac{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_{\ell-1}})}{\prod_{u\in\mathcal{L}_{z_{\ell-1}}}\prod_{j=1}^{N}(\pi_j-u)\cdot\prod_{v\in\mathcal{R}_{z_\ell}}\prod_{j=1}^{L-N}(v+\hat{\pi}_j)} \\
&\cdot\prod_{\ell=2}^{m}\frac{\Delta(\mathcal{R}_{z_\ell};\mathcal{R}_{z_{\ell-1}})}{\prod_{v\in\mathcal{R}_{z_{\ell-1}}}\prod_{j=1}^{N}(v-\pi_j)\cdot\prod_{v\in\mathcal{R}_{z_\ell}}\prod_{j=1}^{L-N}(v+\hat{\pi}_j)^{-1}}.
\end{aligned}
\tag{3.100}
$$

Note that we have multiplied and divided by the same term $\prod_{\ell=2}^{m}\prod_{v\in\mathcal{R}_{z_\ell}}\prod_{j=1}^{L-N}(v+\hat{\pi}_j)$. Similarly

$$
\begin{aligned}
&(-1)^{k_m(N-1)}z_1^{(N-k_1)L}\prod_{\ell=2}^{m}z_\ell^{(k_{\ell-1}-k_\ell)L} \\
&=\prod_{u\in\mathcal{L}_{z_1}}\prod_{j=k_1+1}^{N}(\pi_j-u)\cdot\prod_{v\in\mathcal{R}_{z_1}}\prod_{j=k_1+1}^{N}(v-\pi_j) \\
&\cdot\prod_{\ell=2}^{m}\left[\prod_{u\in\mathcal{L}_{z_\ell}}\frac{\prod_{j=1}^{k_{\ell-1}}(\pi_j-u)}{\prod_{j=1}^{k_\ell}(\pi_j-u)}\cdot\prod_{v\in\mathcal{R}_{z_\ell}}\frac{\prod_{j=1}^{k_{\ell-1}}(v-\pi_j)}{\prod_{j=1}^{k_\ell}(v-\pi_j)}\right].
\end{aligned}
\tag{3.101}
$$

Finally taking the derivative with respect to w of $q_z(w)=q_{z,\mathrm{L}}(w)q_{z,\mathrm{R}}(w)$ and plugging in $v\in\mathcal{R}_z$ we have

$$
q'_{z,\mathrm{R}}(v)\prod_{u\in\mathcal{L}_z}(v-u)=\prod_{j=1}^{N}(v-\pi_j)\prod_{j=1}^{L-N}(v+\hat{\pi}_j)\cdot J(v)^{-1}.
$$

Taking the product over $v\in\mathcal{R}_z$ and using (3.99) then gives $q'_{z,\mathrm{R}}(\mathcal{R}_z)J(\mathcal{R}_z)$ equals

$$
\frac{\prod_{v\in\mathcal{R}_z}\prod_{j=1}^{L-N}(v+\hat{\pi}_j)\cdot\prod_{u\in\mathcal{L}_z}\prod_{j=1}^{N}(\pi_j-u)}{\Delta(\mathcal{R}_z;\mathcal{L}_z)}\cdot\prod_{v\in\mathcal{R}_z}\frac{\prod_{j=1}^{N}(v-\pi_j)}{\prod_{j=1}^{L-N}(v+\hat{\pi}_j)}.
\tag{3.102}
$$

But since $q_{z,\mathrm{R}}(w)=\prod_{v\in\mathcal{R}_z}(w-v)$, for any $v\in\mathcal{R}_z$ we have $q'_{z,\mathrm{R}}(v)=\prod_{\substack{v'\in\mathcal{R}_z\\ v'\neq v}}(v-v')$.

Hence

$$q'_{z,\mathrm{R}}(\mathcal{R}_z) = \prod_{\substack{v,v' \in \mathcal{R}_z \\ v' \neq v}} (v - v') = (-1)^{N(N-1)/2} \Delta(\mathcal{R}_z)^2.$$

Combining (3.100), (3.101) and (3.102), after some simplications we arrive at (3.95). This completes the proof of Theorem 3.6.1. □

3.7 Multi-time distribution for inhomogeneous TASEP on $\mathbb{Z}$

As already explained in the introduction through last passage percolation, for fixed parameters $\vec{a} = (a_1, \cdots, a_m)$ and $\vec{k} = (k_1, \cdots, k_m)$, the finite-time multipoint distribution of inhomogeneous TASEP on a periodic domain with sufficiently large period L agrees with the same multipoint distribution of inhomogeneous TASEP on the infinite lattice $\mathbb{Z}$ under the obvious coupling, because the particles will not feel the boundary effect if they have not gone far enough. The following proposition is nothing but a translation of the argument in the introduction to TASEP language where we allow general initial conditions

Proposition 3.7.1 (Coupling between periodic TASEP and TASEP on $\mathbb{Z}$)**.** *Consider inhomogeneous periodic TASEP with period L and N particles in each period and inhomogeneous TASEP on $\mathbb{Z}$ with N particles depending on the same set of parameters starting from the same initial condition $\vec{y} = (y_1, \cdots, y_N) \in \mathcal{X}_N^{(L)}$. We denote the particle locations by $x_k^{(L)}(t)$ and $x_k^{(\infty)}(t)$ for the two models. Given any integer $m \geq 1$, for any m indices $\{k_1, \cdots, k_m\} \subset \{1, \cdots, N\}$ and m integers $a_1, \cdots, a_m$, if the period L satisfies*

$$L \geq \max\{y_1 + 1, a_1 + k_1, \cdots, a_m + k_m\} - y_N, \tag{3.103}$$

then we have

$$\mathbb{P}^{(L)}_{\vec{y}}\left(\bigcap_{\ell=1}^{m}\{x^{(L)}_{k_\ell}(t_\ell)\geq a_\ell\}\right)=\mathbb{P}^{(\infty)}_{\vec{y}}\left(\bigcap_{\ell=1}^{m}\{x^{(\infty)}_{k_\ell}(t_\ell)\geq a_\ell\}\right). \tag{3.104}$$

Here (L) *and* (∞) *stand for periodic model and infinite model, respectively.*

Proof. Similar as the homogeneous case, see [9] Theorem 3.1. See also Chapter 2 for an illustration through last passage percolation. □

Theorem 3.1.1 is proved by re-writing the periodic multi-point formula (3.68) with sufficiently large period L so that we can get rid of the dependence on the discrete roots and the extra parameter L. The essential idea was partly illustrated in Remark 3.3.3. We rewrite the summation over Bethe roots as contour integrals using residue theorem. When L is large, the contour integrals as a function of the parameter z have analytic continuations to $z=0$ and by sending $z\to 0$ we get rid of dependence on L. For multi-point distribution the idea is similar. We would like to send $z_i\to 0$ for all $1\leq i\leq m$ in equation (3.68) but keep the nesting of the contours so that we do not introduce new singularities. Due to this consideration it is natural to introduce the following change of variables

$$\theta_j:=\frac{z_{j+1}^L}{z_j^L},\quad \text{for } 0\leq j\leq m-1, \tag{3.105}$$

where $z_0:=0$. The nesting relations $0<|z_m|<\cdots<|z_1|<\mathbf{r}_0$ then translates to

$$0<|\theta_0|<\mathbf{r}_0^L:=r_{\max},\quad 0<|\theta_j|<1,\quad \text{for } 1\leq j\leq m-1.$$

We will see that for large period L, the integrand in (3.68) has an analytic continuation to $\theta_0=0$ and we will arrive at the desired formula (3.1) by deforming the θ_0 contour to 0. For multipoint distribution formula, the main extra difficulties (compared with the simple argument in Remark 3.3.3) for this procedure come from the extra singularities resulting from the Cauchy-type interactions $\frac{1}{w_\ell - w_{\ell-1}}$ appearing in both equation (3.48) and the kernels $\mathscr{K}^{\vec{y}}$ for the Fredholm determinant defined in Definition 3.6.8. For $z, z' \in \mathbb{C}$ with $|z| \neq |z'|$, the term $\frac{1}{w-w'}$ is well-defined for any $w \in \mathcal{S}_z$ and $w' \in \mathcal{S}_{z'}$. However when $z, z' \to 0$, one needs to handle the singularities coming from $w = w'$.

3.7.1 An illustration through two-time distribution

Before providing the full proof of Theorem 3.1.1, in this section we will start with the simple but nontrivial case with $m = 2$ and the step initial condition $y_i = -i$. Moreover we will work with the Toeplitz-like determinant formula (3.48) to avoid overwhelming technicalities in the very beginning. We point out that this special case was obtained in [68] through a very different approach. Thus we proved that our results are really generalizations of the one in [68].

Proposition 3.7.2. *Let $\vec{x}(t) = (x_1(t), \cdots, x_N(t)) \in \mathcal{X}_N^{(L)}$ be particle configurations evolving according to the inhomogeneous TASEP in $\mathcal{X}_N^{(L)}$ at time t with initial configuration $\vec{x}_i(0) = -i$. Let $(k_1, t_1), (k_2, t_2) \in \{1, \cdots, N\} \times \mathbb{R}_{\geq 0}$ be distinct with $0 \leq t_1 \leq t_2$. Due to periodicity without loss of generality we assume $k_2 = N$. Let $a_i \in \mathbb{Z}$ for $1 \leq i \leq 2$. Assume that $L - N > \max\{a_1 + k_1, a_2 + k_2\}$, then*

$$\mathbb{P}^{(L)}_{\text{step}}\left(x_{k_1}(t_1) \geq a_1, x_{k_2}(t_2) \geq a_2\right) = \oint_{|\theta|=r} \frac{\mathrm{d}\theta}{2\pi\mathrm{i}\theta} \frac{\mathscr{D}_{\text{step}}(\theta)}{1-\theta}, \tag{3.106}$$

where $r < 1$. The function $\mathscr{D}_{\text{step}}(\theta)$ is an $N \times N$ determinant whose (i,j)-th entry is given by

$$\mathscr{D}^{(i,j)}_{\text{step}}(\theta) = \left[\theta^{\mathbf{1}_{i \leq k_1}} \oint_{|w|=R_2^+} \frac{\mathrm{d}w}{2\pi\mathrm{i}} - \theta^{-\mathbf{1}_{i > k_1}} \oint_{|w|=R_2^-} \frac{\mathrm{d}w}{2\pi\mathrm{i}}\right] \oint_{|z|=R_1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{t_1 z + (t_2 - t_1)w - t_2 \pi_j}}{w - z}$$
$$\cdot \frac{\prod_{\ell=1}^{k_1}(z - \pi_\ell)}{\prod_{\ell=1}^{i}(z - \pi_\ell)} \cdot \frac{\prod_{\ell=1}^{a_1+k_1}(\pi_j + \hat{\pi}_\ell)}{\prod_{\ell=1}^{a_1+k_1}(z + \hat{\pi}_\ell)} \cdot \frac{\prod_{\ell=1}^{j-1}(w - \pi_\ell)}{\prod_{\ell=1}^{k_1}(w - \pi_\ell)} \cdot \frac{\prod_{\ell=a_1+k_1+1}^{a_2+k_2}(\pi_j + \hat{\pi}_\ell)}{\prod_{\ell=a_1+k_1+1}^{a_2+k_2}(w + \hat{\pi}_\ell)}. \tag{3.107}$$

Here the contours are arranged so that $R_2^+ > R_1 > R_2^- > \max_{k,j}\{\pi_k, \hat{\pi}_j\}$.

Remark 3.7.3. Modulo the change of notations $(\{\hat{\pi}_j\}, \{\pi_j\}) \leftrightarrow (\{\alpha_j\}, \{\beta_j\})$, $(k_1, k_2) \leftrightarrow (n, N)$, $(a_1 + k_1, a_2 + k_2) \leftrightarrow (m, M)$, $(t_1, t_2) \leftrightarrow (h, H)$ and the change of variable $\theta \to \theta^{-1}$, it is easy to check that formula (3.106) agrees exactly with the two-time distribution of inhomogeneous exponential last passage percolation obtained in [68] Corollary 3.3. This is consistent with the well-known fact that under the standard coupling between TASEP and exponential last passage percolation, one has the following equality in distribution:

$$\mathbb{P}_{\text{TASEP}}(x_{k_1}(t_1) \geq a_1, x_{k_2}(t_2) \geq a_2) = \mathbb{P}_{\text{ExpLPP}}(G(a_1+k_1, k_1) \leq t_1, G(a_2+k_2, k_2) \geq t_2).$$

Proof of Proposition 3.7.2. We start with (3.46) in Theorem 3.4.1. First we introduce the following change of variables: For $0 < |z_2| < |z_1|$, we set

$$\theta_0 := z_1^L, \quad \theta := \frac{z_2^L}{z_1^L}. \tag{3.108}$$

Then it is straightforward to check that

$$\mathcal{C}^{(L)}(z_1, z_2) = \theta^{k_1 - N} \cdot \prod_{j=1}^{N} e^{-t_2 \pi_j} \cdot (\theta - 1)^{N-1}. \tag{3.109}$$

On the other hand the determinant $\mathcal{D}^{(L)}(z_1, z_2)$, as a function of (θ_0, θ), has an analytic continuation to $D(0, r_{\max}) \times D(0, 1)$ when $L - N > \max\{a_1 + k_1, a_2 + k_2\}$. Moreover, we can rewrite the entries of the determinant using residue theorem as follows: For $1 \leq i, j \leq N$ we set

$$D_{ij}(\theta_0, \theta) := \sum_{\substack{w_1 \in \mathcal{S}_{z_\ell} \\ \ell = 1,2}} \frac{p_i(w_1) q_j(w_2)}{w_2 - w_1} \prod_{\ell=1}^{2} J(w_\ell) \frac{F_\ell(w_\ell)}{F_{\ell-1}(w_\ell)}, \tag{3.110}$$

where z_1, z_2 and θ_0, θ are related by the equations $z_1^L = \theta_0$, $z_2^L = \theta_0 \cdot \theta$. Note that z_1

and z_2 are not uniquely determined by θ_0 and θ but the sets $\mathcal{S}_{z_\ell}$'s are. We claim that $D_{ij}(\theta_0, \theta)$ has an analytic continuation to $(\theta_0, \theta) \in D(r_{\max}) \times D(1)$ for some $r_{\max} > 0$ where $D(r)$ is a disk centered at the origin with radius r. Moreover

$$\lim_{\theta_0 \to 0} D_{ij}(\theta_0, \theta) = \left[\frac{1}{1-\theta} \oint_{|w|=R_2^-} \frac{\mathrm{d}w}{2\pi i} - \frac{\theta}{1-\theta} \oint_{|w|=R_2^+} \right] \oint_{|z|=R_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{p_i(z) q_j(w) F_2(w) F_1(z)}{(w-z) F_1(w)}. \tag{3.111}$$

Here

$$p_i(z) := \prod_{\ell=1}^{i} \frac{1}{z - \pi_\ell}, \quad q_j(w) := \frac{\prod_{\ell=1}^{a_2+k_2} (\pi_j + \hat{\pi}_\ell)}{\prod_{\ell=j}^{N} (w - \pi_\ell)}.$$

And

$$F_\ell(w) := e^{t_\ell w} \cdot \prod_{j=1}^{k_\ell} (w - \pi_j) \cdot \prod_{j=1}^{a_\ell + k_\ell} (w + \hat{\pi}_j)^{-1}, \quad \text{for } 1 \le \ell \le 2.$$

To see the claim we first note that $J(w) = \frac{q(w)}{q'(w)}$ where $q(w) := \prod_{j=1}^{N} (w - \pi_j) \prod_{j=1}^{L-N} (w + \hat{\pi}_j)$. Hence by the residue theorem we have

$$\sum_{w \in \mathcal{S}_z} H(w) J(w) = \oint_{|q(w)|=|z|^L+\epsilon} \frac{\mathrm{d}w}{2\pi \mathrm{i}} H(w) \frac{q(w)}{q(w) - z^L} - \oint_{|q(w)|=|z|^L-\epsilon} \frac{\mathrm{d}w}{2\pi \mathrm{i}} H(w) \frac{q(w)}{q(w) - z^L}, \tag{3.112}$$

for sufficiently small $\epsilon > 0$ and any $H(w)$ analytic in a neighborhood of the region $\{w \in \mathbb{C} : |z|^L - \epsilon < |q(w)| < |z|^L + \epsilon\}$. Here we used the nesting relations of the level sets discussed in Lemma 3.6.4. Hence

$$D_{ij}(\theta_0, \theta) = \left[\oint_{|q(z)|=|\theta_0|+\epsilon} \frac{\mathrm{d}z}{2\pi \mathrm{i}} - \oint_{|q(z)|=|\theta_0|-\epsilon} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \right] \left[\oint_{|q(w)|=|\theta_0\theta|+\epsilon} \frac{\mathrm{d}w}{2\pi \mathrm{i}} - \oint_{|q(w)|=|\theta_0\theta|-\epsilon} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \right] \frac{p_i(z) q_j(w)}{w - z} \frac{q(z)}{q(z) - \theta_0} \frac{q(w)}{q(w) - \theta_0 \theta} \frac{F_2(w) F_1(z)}{F_1(w)}, \tag{3.113}$$

for some $0 < |\theta_0| < r_{\max}$, $0 < \theta < 1$ and $\epsilon > 0$ sufficiently small. Here the products of integral signs $(\oint_{C_1} - \oint_{C_2})(\oint_{D_1} - \oint_{D_2}) H(z,w)\,\mathrm{d}z\mathrm{d}w$ should be understood as

$$\sum_{i,j=1}^{2} (-1)^{i+j} \oint_{C_i} \oint_{D_j} H(z,w)\,\mathrm{d}z\mathrm{d}w.$$

Starting from equation (3.113), we can first deform the inner w contour $\{|q(w)| = |\theta_0\theta| - \epsilon\}$ to a single point since the integrand is analytic inside this contour by our assumption that $L - N > \max\{a_1 + k_1, a_2 + k_2\}$ and $N \geq \max\{k_1, k_2\}$. We then deform the outer w contour $\{|q(w)| = |\theta_0\theta| + \epsilon\}$ to be the contour $|q(w)| = r_{\max} - \epsilon$, making it be outside of the two z-contours. Doing this we will pick up residues coming from $w = z$ on both the two z-contours. Hence

$$D_{ij}(\theta_0, \theta) = D_1 + D_2,$$

where

$$D_1 = \left[\oint_{|q(z)|=|\theta_0|+\epsilon} \frac{\mathrm{d}z}{2\pi\mathrm{i}} - \oint_{|q(z)|=|\theta_0|-\epsilon} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \right] \oint_{|q(w)|=r_{\max}-\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{p_i(z)q_j(w)}{w-z} \frac{q(z)}{q(z)-\theta_0} \frac{q(w)}{q(w)-\theta_0\theta} \frac{F_2(w)F_1(z)}{F_1(w)},$$

and

$$D_2 = \left[-\oint_{|q(z)|=|\theta_0|+\epsilon} \frac{\mathrm{d}z}{2\pi\mathrm{i}} + \oint_{|q(z)|=|\theta_0|-\epsilon} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \right] p_i(z)q_j(z) \frac{q(z)}{q(z)-\theta_0} \frac{q(z)}{q(z)-\theta_0\theta} F_2(z).$$

Now for D_1 we further deform the inner z contour to a single point without picking up any residue again by our assumption on L and N. Also we can deform the outer z contour to be $\{|q(z)| = r_{\max} - \epsilon'\}$ with $\epsilon' > \epsilon$ also sufficiently small. For D_2 we deform the outer z contour to $\{|q(z)| = r_{\max} - \epsilon'\}$ and deform the inner z contour to a single point but need to pick up extra residues at the roots of the equation $q(z) - \theta_0\theta = 0$.

Thus

$$D_1 = \oint_{|q(z)|=r_{\max}-\epsilon'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \oint_{|q(w)|=r_{\max}-\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{p_i(z)q_j(w)}{w-z} \frac{q(z)}{q(z)-\theta_0} \frac{q(w)}{q(w)-\theta_0\theta} \frac{F_2(w)F_1(z)}{F_1(w)}.$$

And

$$D_2 = \oint_{|q(z)|=r_{\max}-\epsilon'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} p_i(z)q_j(z) \frac{-q(z)}{q(z)-\theta_0} \frac{q(z)}{q(z)-\theta_0\theta} F_2(z) - \frac{\theta}{1-\theta} \sum_{\zeta\in\mathcal{S}_{z_2}} p_i(\zeta)q_j(\zeta)J(\zeta)F_2(\zeta).$$

Here $z_2^L = \theta_0\theta$ and the $\frac{\theta}{1-\theta}$ factor comes from evaluating $\frac{q(z)}{q(z)-\theta_0}$ at $z=\zeta$ for some $\zeta \in \mathcal{S}_{z_2}$. Finally apply (3.112) again for the sum over $\mathcal{S}_{z_2}$ above we conclude that

$$\begin{aligned} D_{ij}(\theta_0,\theta) = & \oint_{|q(z)|=r_{\max}-\epsilon'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \oint_{|q(w)|=r_{\max}-\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{p_i(z)q_j(w)}{w-z} \frac{q(z)}{q(z)-\theta_0} \frac{q(w)}{q(w)-\theta_0\theta} \frac{F_2(w)F_1(z)}{F_1(w)} \\ & - \oint_{|q(z)|=r_{\max}-\epsilon'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} p_i(z)q_j(z) \frac{q(z)}{q(z)-\theta_0} \frac{q(z)}{q(z)-\theta_0\theta} F_2(z) \\ & - \frac{\theta}{1-\theta} \oint_{|q(z)|=r_{\max}-\epsilon'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} p_i(z)q_j(z) \frac{q(z)}{q(z)-\theta_0\theta} F_2(z). \end{aligned} \tag{3.114}$$

Now note that the right hand side of (3.114) is analytic in θ_0 at $\theta_0 = 0$, moreover

$$\begin{aligned} \lim_{\theta_0\to 0} D_{ij}(\theta_0,\theta) = & \oint_{|q(z)|=r_{\max}-\epsilon'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \oint_{|q(w)|=r_{\max}-\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{p_i(z)q_j(w)}{w-z} \frac{F_2(w)F_1(z)}{F_1(w)} \\ & - \frac{1}{1-\theta} \oint_{|q(z)|=r_{\max}-\epsilon'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} p_i(z)q_j(z)F_2(z). \end{aligned} \tag{3.115}$$

It is straightforward to check the right hand side of (3.115) agrees with the right hand side of (3.111) after some contour deformations. This proves the claim. Combining (3.109) with (3.111) we conclude the proof of (3.106) by deforming the θ_0-contour to a single point. □

Starting from (3.106), it is possible to carry through a (rather different) orthogonalization procedure directly and obtain a (different from the one in Theorem 3.1.1) formula for the two-time distribution of inhomogeneous TASEP on $\mathbb{Z}$ as a contour integral of Fredholm determinant. This is the approach in [68] where they studied the related geometric last passage percolation model but with only one set of parameters (essentially $\pi_j \equiv 0$). We will present the slightly more general version with two sets of parameters of this orthogonalization procedure in Section 3.7.5 for comparison. With more effort this approach may also be extended to the general m-time joint distribution, we will not try to go further in this direction.

3.7.2 Proof of Theorem 3.1.1: Strategy

With the warm-up illustrated in Section 3.7.1, we are now ready to prove Theorem 3.1.1. As before we make the change of variable

$$\theta_j := \frac{z_{j+1}^L}{z_j^L}, \quad \text{for } 0 \leq j \leq m-1,$$

where $z_0 := 0$. Then we re-write equation (3.68) in Theorem 3.6.1 using the new variable $\vec{\theta} = (\theta_0, \cdots, \theta_{m-1})$, which we denote by

$$\mathbb{P}_{\vec{y}}^{(L)}\left(\bigcap_{i=1}^{m}\{x_{k_i}(t_i) \geq a_i\}\right) = \oint \cdots \oint \hat{\mathscr{C}}_{\vec{y}}(\vec{\theta})\hat{\mathscr{D}}_{\vec{y}}(\vec{\theta})\frac{d\theta_0}{2\pi \mathrm{i}\theta_0} \cdots \frac{\mathrm{d}\theta_{m-1}}{2\pi \mathrm{i}\theta_{m-1}}, \tag{3.116}$$

where the integral contours are now $0 < |\theta_0| = r_0 < r_{\max}$ and $0 < |\theta_i| = r_i < 1$ for $1 \leq i \leq m-1$. The functions in the integrand are defined by

$$\hat{\mathscr{C}}_{\vec{y}}(\vec{\theta}) := \mathscr{C}_{\vec{y}}(\vec{z}(\vec{\theta})), \quad \hat{\mathscr{D}}_{\vec{y}}(\vec{\theta}) := \mathscr{D}_{\vec{y}}(\vec{z}(\vec{\theta})), \tag{3.117}$$

where for given $\vec{\theta} \in D_0(r_{\max}) \times D_0(1)^{m-1}$, the corresponding $\vec{z} = \vec{z}(\vec{\theta})$ is defined by the change of variables

$$z_j^L = \prod_{\ell=0}^{j-1} \theta_\ell, \quad \text{for } 1 \leq j \leq m.$$

Here $D_0(r)$ is the disk centered at 0 with radius r and 0 removed. Note that z_j's are not uniquely determined by the θ_j's but the functions are. As before we will show that the integrand has an analytic continuation to $\theta_0 = 0$ when the period L is sufficiently large. Then Theorem 3.1.1 follows immediately once we establish the following two lemmas by deforming the θ_0 contour to the origin.

Lemma 3.7.4 (analytic continuation of $\hat{\mathscr{C}}_{\vec{y}}(\vec{\theta})$)**.** *Under the same assumption as in Theorem 3.1.1, the function* $\hat{\mathscr{C}}_{\vec{y}}(\vec{\theta})$ *defined in* (3.117) *has an analytic continuation to* $\theta_0 = 0$. *Moreover for fixed* $0 < |\theta_i| < 1$, $1 \le i \le m-1$,

$$\lim_{\theta_0 \to 0} \hat{\mathscr{C}}_{\vec{y}}(\theta_0, \cdots, \theta_{m-1}) = \prod_{\ell=1}^{m-1} \frac{1}{1-\theta_\ell}. \tag{3.118}$$

Lemma 3.7.5 (analytic continuation of $\hat{\mathscr{D}}_{\vec{y}}(\vec{\theta})$)**.** *Under the same assumption as in Theorem 3.1.1, the function* $\hat{\mathscr{D}}_{\vec{y}}(\vec{\theta})$ *defined in* (3.117) *has an analytic continuation to* $\theta_0 = 0$. *Moreover for fixed* $0 < |\theta_i| < 1$, $1 \le i \le m-1$,

$$\lim_{\theta_0 \to 0} \hat{\mathscr{D}}_{\vec{y}}(\theta_0, \cdots, \theta_{m-1}) = \mathcal{D}^{(\infty)}_{\vec{y}}(\theta_1, \cdots, \theta_{m-1}), \tag{3.119}$$

where $\mathcal{D}^{(\infty)}_{\vec{y}}(\theta_1, \cdots, \theta_{m-1})$ *is another Fredholm determinant defined in Definition 3.1.4.*

3.7.3 Proof of Theorem 3.1.1: $\hat{\mathscr{C}}_{\vec{y}}(\vec{\theta})$ part

We start with the simpler $\hat{\mathscr{C}}_{\vec{y}}(\vec{\theta})$ part. First we rewrite $\hat{\mathscr{C}}_{\vec{y}}(\vec{\theta}) = \mathscr{C}_{\vec{y}}(\vec{z})$ defined in Definition 3.6.7 as follows:

$$\hat{\mathscr{C}}_{\vec{y}}(\vec{\theta}) = \mathcal{E}_{\vec{y}}(z_1) \cdot \prod_{\ell=2}^{m} \frac{z_{\ell-1}^L}{z_{\ell-1}^L - z_\ell^L} \cdot \prod_{j=1}^{N} e^{-t_m \pi_j} \cdot \frac{\prod_{j=1}^{N} \prod_{\ell=1}^{a_m+k_m} (\pi_j + \hat{\pi}_\ell)}{\prod_{j=k_m+1}^{N} \prod_{\ell=1}^{L-N} (\pi_j + \hat{\pi}_\ell)}$$
$$\cdot \mathscr{A}_1(\theta_0, \cdots, \theta_{m-1}) \mathscr{A}_2(\theta_0, \cdots, \theta_{m-1}) \mathscr{A}_3(\theta_0, \cdots, \theta_{m-1}),$$

where θ_j's and z_j's are related by the change of variables (3.105). The global energy function $\mathcal{E}_{\vec{y}}(z_1)$ is defined in Definition 3.6.5. The functions $\mathscr{A}_j(\theta_0, \cdots, \theta_{m-1})$'s are as follows:

$$\mathscr{A}_1(\theta_0, \cdots, \theta_{m-1}) = \prod_{\ell=1}^{m} \frac{\prod_{u\in\mathcal{L}_{z_\ell}} \prod_{j=1}^{k_{\ell-1}} (\pi_j - u) \cdot \prod_{u\in\mathcal{R}_{z_\ell}} \prod_{j=1}^{a_{\ell-1}+k_{\ell-1}} (v + \hat{\pi}_j) e^{-t_{\ell-1}v}}{\prod_{u\in\mathcal{L}_{z_\ell}} \prod_{j=1}^{k_\ell} (\pi_j - u) \cdot \prod_{u\in\mathcal{R}_{z_\ell}} \prod_{j=1}^{a_\ell+k_\ell} (v + \hat{\pi}_j) e^{-t_\ell v}},$$

and

$$\mathscr{A}_2(\theta_0, \cdots, \theta_{m-1}) = \prod_{\ell=1}^{m} \frac{\prod_{u\in\mathcal{L}_{z_\ell}} \prod_{j=1}^{N} (\pi_j - u) \cdot \prod_{u\in\mathcal{R}_{z_\ell}} \prod_{j=1}^{L-N} (v + \hat{\pi}_j)}{\Delta(\mathcal{R}_{z_\ell}; \mathcal{L}_{z_\ell})},$$

and

$$\mathscr{A}_3(\theta_0, \cdots, \theta_{m-1}) = \prod_{\ell=2}^{m} \frac{\Delta(\mathcal{R}_{z_\ell}; \mathcal{L}_{z_{\ell-1}})}{\prod_{u\in\mathcal{L}_{z_{\ell-1}}} \prod_{j=1}^{N} (\pi_j - u) \cdot \prod_{u\in\mathcal{R}_{z_\ell}} \prod_{j=1}^{L-N} (v + \hat{\pi}_j)}.$$

Now when $\theta_0 \to 0$, we have $z_j \to 0$ for all $1 \le j \le m$. Thus the right roots $\mathcal{R}_{z_\ell}$'s all converge to $\{\pi_j\}_{j=1}^N$ and the left roots $\mathcal{L}_{z_\ell}$'s all converge to $\{-\hat{\pi}_j\}_{j=1}^{L-N}$ for $1 \le \ell \le m$. Hence

$$\lim_{z\to 0} \prod_{u\in\mathcal{L}_z} \prod_{j=1}^{k} (\pi_j - u) = \prod_{j=1}^{k} \prod_{\ell=1}^{L-N} (\pi_j + \hat{\pi}_\ell), \quad \lim_{z\to 0} \prod_{v\in\mathcal{R}_z} \prod_{j=1}^{a+k} (v + \hat{\pi}_j) e^{-tv} = \prod_{j=1}^{N} \prod_{\ell=1}^{a+k} (\pi_j + \hat{\pi}_\ell) e^{-t\pi_j}.$$

In particular

$$\lim_{z\to 0} \prod_{u\in\mathcal{L}_z} \prod_{j=1}^{N} (\pi_j - u) = \lim_{z\to 0} \prod_{v\in\mathcal{R}_z} \prod_{j=1}^{L-N} (v + \hat{\pi}_j) = \lim_{z\to 0} \Delta(\mathcal{R}_z; \mathcal{L}_z) = \prod_{j=1}^{N} \prod_{\ell=1}^{L-N} (\pi_j + \hat{\pi}_\ell).$$

For $\mathcal{E}_{\vec{y}}(z)$ we note that for $|z| = \epsilon$ sufficiently small we have $\mathcal{R}_z = \{\pi_j + O(\epsilon)\}_{j=1}^N$. But for $w_j = \pi_j$, one has

$$\prod_{\ell=i+1}^{N} (w_j - \pi_\ell) = \prod_{\ell=i+1}^{N} (\pi_j - \pi_\ell),$$

which equals 0 if $j > i$. Hence the matrix with (i,j)-th entry given by $\prod_{\ell=i+1}^{N}(w_j - \pi_j) \cdot \prod_{\ell=1}^{\lambda_i} \frac{w_j+\hat{\pi}_\ell}{\pi_i+\hat{\pi}_\ell}$ is low triangular with diagonal entries $\prod_{\ell=i+1}^{N}(\pi_i - \pi_\ell)$. Thus

$$\det\left[\prod_{\ell=i+1}^{N}(w_j - \pi_\ell)\cdot\prod_{\ell=1}^{\lambda_i}\frac{w_j+\hat{\pi}_\ell}{\pi_i+\hat{\pi}_\ell}\right]_{i,j=1}^{N} = \prod_{i=1}^{N}\prod_{j=i+1}^{N}(\pi_i - \pi_j).$$

This then implies that for π_j's all distinct we have

$$\mathcal{F}_\lambda(\pi_1,\cdots,\pi_N;\{\pi_j\},\{\hat{\pi}_\ell\}) = \frac{\det\left[\prod_{\ell=i+1}^{N}(w_j-\pi_\ell)\cdot\prod_{\ell=1}^{\lambda_i}\frac{w_j+\hat{\pi}_\ell}{\pi_i+\hat{\pi}_\ell}\right]_{i,j=1}^{N}}{\det[w_j^{N-i}]_{i,j=1}^{N}} = 1.$$

The general case follows from the fact that $\mathcal{F}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\})$ is a rational function in the parameters π_j's. Thus we have

$$\lim_{z\to 0}\mathcal{E}_{\vec{y}}(z) = \lim_{\substack{w_j\to\pi_j\\1\le j\le N}}\mathcal{F}_\lambda(w_1,\cdots,w_N;\{\pi_j\},\{\hat{\pi}_\ell\}) = 1.$$

Combine all the arguments above, after a lot of cancellations we see

$$\lim_{\theta_0\to 0}\hat{\mathscr{C}}_{\vec{y}}(\theta_0,\cdots,\theta_m) = \lim_{\theta_0\to 0}\prod_{\ell=1}^{m-1}\frac{\prod_{j=0}^{\ell-1}\theta_j}{\prod_{j=0}^{\ell-1}\theta_j - \prod_{j=0}^{\ell}\theta_j} = \prod_{\ell=1}^{m-1}\frac{1}{1-\theta_\ell}. \tag{3.120}$$

3.7.4 Proof of Theorem 3.1.1: $\hat{\mathscr{D}}_{\vec{y}}(\vec{\theta})$ part

We will show that the series expansions of the two Fredholm determinants match term by term. For this first using the block diagonal structure of the operators $\mathscr{K}_1^{\vec{y}}$ and $\mathscr{K}_2^{\vec{y}}$ defined in Definition 3.6.8 it is not hard to check (see also Proposition 2.10 of [81]) that

$$\hat{\mathscr{D}}_{\vec{y}}(\vec{\theta}) = \det(I - \mathscr{K}_1^{\vec{y}}\mathscr{K}_2^{\vec{y}}) = \sum_{\vec{n}\in(\mathbb{Z}_{\ge 0})^m}\frac{1}{(n_1!\cdots n_m!)^2}\mathscr{D}_{\vec{n},\vec{y}}(\vec{\theta}).$$

Where $\vec{n} = (n_1, \cdots, n_m)$ and $\mathscr{D}_{\vec{n},\vec{y}}(\vec{\theta})$ equals

$$\sum_{\substack{U^{(\ell)}=(u_1^{(\ell)},\cdots,u_{n_\ell}^{(\ell)})\in(\mathcal{L}_{z_\ell})^{n_\ell}\\ V^{(\ell)}=(v_1^{(\ell)},\cdots,v_{n_\ell}^{(\ell)})\in(\mathcal{R}_{z_\ell})^{n_\ell}\\ \ell=1,\cdots,m}} \left[(-1)^{n_1(n_1+1)/2}\frac{\Delta(U^{(1)};V^{(1)})}{\Delta(U^{(1)})\Delta(V^{(1)})}\det\left(\frac{\chi_{\vec{y}}(v_i^{(1)},u_j^{(1)};z_1)}{v_i^{(1)}-u_j^{(1)}}\right)_{i,j=1}^{n_1}\right]$$
$$\cdot\left[\prod_{\ell=1}^{m}\frac{\Delta(U^{(\ell)})^2\Delta(V^{(\ell)})^2}{\Delta(U^{(\ell)};V^{(\ell)})^2}f_\ell(U^{(\ell)})f_\ell(V^{(\ell)})H_{z_\ell}(U^{(\ell)})H_{z_\ell}(V^{(\ell)})J(U^{(\ell)})J(V^{(\ell)})\right]$$
$$\cdot\left[\prod_{\ell=1}^{m-1}\frac{\Delta(U^{(\ell)};V^{(\ell+1)})\Delta(V^{(\ell)};U^{(\ell+1)})}{\Delta(U^{(\ell)};U^{(\ell+1)})\Delta(V^{(\ell)};V^{(\ell+1)})}\cdot\frac{(1-z_{\ell+1}^L/z_\ell^L)^{n_\ell}(1-z_\ell^L/z_{\ell+1}^L)^{n_{\ell+1}}}{H_{z_{\ell+1}}(U^{(\ell)})H_{z_{\ell+1}}(V^{(\ell)})H_{z_\ell}(U^{(\ell+1)})H_{z_\ell}(V^{(\ell+1)})}\right].$$

Here we remind again that the variables z_i's and θ_i's are related by the equations (**??**). The functions $J(w)$, $H_z(w)$ and $f_\ell(w)$'s are the same as in Definition 3.6.8 and we adopt the convention that for a finite set S, $g(S) := \prod_{s\in S} g(s)$ for any single variable function g. Note that since $|\mathcal{L}_{z_\ell}| = L - N$ and $|\mathcal{R}_{z_\ell}| = N$ for all $1 \le \ell \le m$, the summand is nonzero only when $n_\ell \le \min\{L-N, N\}$ for all $1 \le \ell \le m$ since otherwise some of the Vandermonde determinants will vanish. Thus the series expansion is in fact a finite sum.

A similar series expansion for the other Fredholm determinant $\mathbf{D}_{\vec{y}}(\theta_1, \cdots, \theta_{m-1})$ with the summations over discrete sets replaced by contour integrals leads to the following

$$\mathbf{D}_{\vec{y}}(\theta_1,\cdots,\theta_{m-1}) = \det(I-\mathcal{K}_1\mathcal{K}_{\vec{y}}) = \sum_{\vec{n}\in(\mathbb{Z}_{\ge 0})^m}\frac{1}{(n_1!\cdots n_m!)^2}\mathbf{D}_{\vec{n},\vec{y}}(\theta_1,\cdots,\theta_{m-1}).$$

Where

$$
\begin{aligned}
&\mathbf{D}_{\vec{n},\vec{y}}(\theta_1,\cdots,\theta_{m-1})\\
=&\prod_{\ell=2}^{m}\prod_{i_\ell=1}^{n_\ell}\left[\frac{1}{1-\theta_{\ell-1}}\oint_{\Sigma_{\ell,\mathrm{L}}^-}\frac{\mathrm{d}u_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}}-\frac{\theta_{\ell-1}}{1-\theta_{\ell-1}}\oint_{\Sigma_{\ell,\mathrm{L}}^+}\frac{\mathrm{d}u_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}}\right]\cdot\prod_{i_1=1}^{n_1}\oint_{\Sigma_{1,\mathrm{L}}}\frac{\mathrm{d}u_{i_1}^{(1)}}{2\pi\mathrm{i}}\\
&\prod_{\ell=2}^{m}\prod_{i_\ell=1}^{n_\ell}\left[\frac{1}{1-\theta_{\ell-1}}\oint_{\Sigma_{\ell,\mathrm{R}}^-}\frac{\mathrm{d}v_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}}-\frac{\theta_{\ell-1}}{1-\theta_{\ell-1}}\oint_{\Sigma_{\ell,\mathrm{R}}^+}\frac{\mathrm{d}v_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}}\right]\cdot\prod_{i_1=1}^{n_1}\oint_{\Sigma_{1,\mathrm{R}}}\frac{\mathrm{d}v_{i_1}^{(1)}}{2\pi\mathrm{i}}\\
&\left[(-1)^{n_1(n_1+1)/2}\frac{\Delta(U^{(1)};V^{(1)})}{\Delta(U^{(1)})\Delta(V^{(1)})}\det\left(\frac{\mathrm{ch}_{\vec{y}}(v_i^{(1)},u_j^{(1)})}{v_i^{(1)}-u_j^{(1)}}\right)_{i,j=1}^{n_1}\right]\\
&\cdot\left[\prod_{\ell=1}^{m}\frac{\Delta(U^{(\ell)})^2\Delta(V^{(\ell)})^2}{\Delta(U^{(\ell)};V^{(\ell)})^2}f_\ell(U^{(\ell)})f_\ell(V^{(\ell)})\right]\\
&\cdot\left[\prod_{\ell=1}^{m-1}\frac{\Delta(U^{(\ell)};V^{(\ell+1)})\Delta(V^{(\ell)};U^{(\ell+1)})}{\Delta(U^{(\ell)};U^{(\ell+1)})\Delta(V^{(\ell)};V^{(\ell+1)})}(1-\theta_\ell)^{n_\ell}(1-\theta_\ell^{-1})^{n_{\ell+1}}\right].
\end{aligned}
\tag{3.121}
$$

Now Lemma 3.7.5 immediately follows from Lemma 3.7.6 below where we show the equality term by term between the two series expansions.

Lemma 3.7.6. *Under the same assumption as in Theorem 3.1.1, the functions* $\mathscr{D}_{\vec{n},\vec{y}}(\theta_0,\theta_1,\cdots,\theta_{m-1})$ *have analytic continuations to* $\theta_0=0$ *for any* $\vec{n}\in(\mathbb{Z}_{\geq 0})^m$. *Moreover for fixed* $0<|\theta_i|<1$, $1\leq i\leq m-1$,

$$
\lim_{\theta_0\to 0}\mathscr{D}_{\vec{n},\vec{y}}(\theta_0,\cdots,\theta_{m-1})=\mathbf{D}_{\vec{n},\vec{y}}(\theta_1,\cdots,\theta_{m-1}). \tag{3.122}
$$

Proof. This is a generalization of equation (3.111) in Proposition 3.7.2 and can be proved in a similar way. A related general statement which can be easily adapted to our purpose was given in Proposition 4.3 of [81] so we omit most of the details. One starts with replacing the summation over the Bethe roots as contour integrals using residue theorem. For example

$$\sum_{w\in\mathcal{L}_{z_\ell}} g(w)J(w) = \oint_{|q(w)|=|z_\ell|^L+\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} g(w)\frac{q(w)}{q(w)-z_\ell^L} - \oint_{|q(w)|=|z_\ell|^L-\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} g(w)\frac{q(w)}{q(w)-z_\ell^L},$$

for some function $g(w)$ analytic inside the region $\{w\in\mathbb{C} : |z_\ell|^L-\epsilon < |q(w)| < |z_\ell|^L+\epsilon\}$ with ϵ sufficiently small so that the contours for different ℓ do not intersect. Now equation (3.122) follows from carefully deforming all the inner contours $\{w : |q(w)| = |z_\ell|^L - \epsilon\}$ to a single point and the outer contours $\{w : |q(w)| = |z_\ell|^L + \epsilon\}$ to be sufficiently close to $\{w : |q(w)| = r_{\max}\}$.

The key fact here is due to our assumptions on L the integrand will always be analytic at π_j's and $-\hat{\pi}_j$'s because the possible poles at such points coming from the functions $f_\ell(w)$ will always be cancelled out by the zeros at these points coming from the $q(w)$ being multiplied in the integrand. Thus the only type of poles one needs to take care of comes from the Cauchy-type terms $\Delta(U^{(\ell)};U^{(\ell+1)})$ or $\Delta(V^{(\ell)};V^{(\ell+1)})$ when one deforms the contours. After the contour deformations we can send $\theta_0\to 0$ (or $z_j\to 0$ for all $1\le j\le m$ with $z_{j+1}^L/z_j^L=\theta_j$ fixed for $1\le j\le m-1$) and the Lemma follows by noting that

$$\lim_{z_j\to 0} H_{z_j}(w) = 1, \quad \lim_{z_j\to 0}\frac{q(w)}{q(w)-z_j^L} = 1, \quad \lim_{z\to 0}\chi_{\vec{y}}(v,u;z) = \mathrm{ch}_{\vec{y}}(v,u).$$

□

3.7.5 An equivalent formula for the two-time distribution

Starting from equation (3.106), we can also obtain a different Fredholm determinant representation (through a completely different orthogonalization procedure) of the $N\times N$ determinant, following the idea of [68] (they only worked out the case for geometric last passage percolation with one set of parameters). The precise statement is the following:

Theorem 3.7.7 (An equivalent Fredholm determinant representation of $\mathscr{D}_{\mathrm{step}}(\theta)$)**.** *Under the same assumption as in Theorem 3.7.2, we have*

$$\mathscr{D}_{\mathrm{step}}(\theta) = \det(I + F^{\mathrm{Exp}}(\theta))_{\ell^2(\{1,\cdots,N\})}. \tag{3.123}$$

Here

$$F_{i,j}^{\mathrm{Exp}}(\theta) = \theta^{-\mathbf{1}_{i>k_1}}(J_{\mathrm{LL}}^-(i,j) - J_{\mathrm{RL}}(i,j) + J_{\mathrm{LR}}(i,j)) - \theta^{\mathbf{1}_{i\le k_1}}(J_{\mathrm{LL}}^+(i,j) - J_{\mathrm{RL}}(i,j) + J_{\mathrm{LR}}(i,j)),$$

where

$$J_{\mathrm{LR}}(i,j) = \mathbf{1}_{j\le k_1}\cdot \oint_{\Sigma_{1,\mathrm{R}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \oint_{\Sigma_{1,\mathrm{L}}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{t_1(z-\zeta)}}{(z-\zeta)} \cdot \frac{\prod_{\ell=1}^{j-1}(z-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)} \cdot \prod_{\ell=1}^{a_1+k_1} \frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell},$$

$$J_{\mathrm{RL}}(i,j) = \mathbf{1}_{i>k_1}\cdot \oint_{\Sigma_{1,\mathrm{R}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \oint_{\Sigma_{1,\mathrm{L}}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{(t_2-t_1)(z-\zeta)}}{(z-\zeta)} \cdot \frac{\prod_{\ell=i+1}^{k_2}(z-\pi_\ell)}{\prod_{\ell=j}^{k_2}(\zeta-\pi_\ell)} \cdot \prod_{\ell=a_1+k_1+1}^{a_2+k_2} \frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell},$$

$$J_{\mathrm{LL}}^{\pm}(i,j) = \oint_{\Sigma_{1,\mathrm{R}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \oint_{\Sigma_{1,\mathrm{L}}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \oint_{\Sigma_{2,\mathrm{R}}^{\pm}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \oint_{\Sigma_{2,\mathrm{L}}^{\pm}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{e^{t_1(z-\zeta)+(t_2-t_1)(w-\omega)}}{(z-\zeta)(z-w)(w-\omega)}$$

$$\cdot \frac{\prod_{\ell=1}^{k_1}(z-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)} \cdot \frac{\prod_{\ell=k_1+1}^{k_2}(w-\pi_\ell)}{\prod_{\ell=j}^{k_2}(\omega-\pi_\ell)} \cdot \prod_{\ell=1}^{a_1+k_1} \frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell} \cdot \prod_{\ell=a_1+k_1+1}^{a_2+k_2} \frac{\omega+\hat{\pi}_\ell}{w+\hat{\pi}_\ell},$$

where the contours are chosen such that $\Sigma_{i,\mathrm{R}}$'s only enclose $\{\pi_j\}$ but not $\{-\hat{\pi}_j\}$, $\Sigma_{i,\mathrm{L}}$'s only enclose $\{-\hat{\pi}_j\}$ but not $\{\pi_j\}$ and they do not intersect. Moreover $\Sigma_{2,\mathrm{L}}^+$ lies outside of $\Sigma_{1,\mathrm{L}}$ while $\Sigma_{2,\mathrm{L}}^-$ lies inside $\Sigma_{1,\mathrm{L}}$, similar for the right contours. Here we recall that $k_2 = N$.

Proof. We start with the $N \times N$ determinant given in (3.107). First we write

$$D^-(i,j) := \oint_{|w|=R_2^-} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \oint_{|z|=R_1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{t_1 z+(t_2-t_1)w-t_2\pi_j}}{w-z}$$

$$\frac{\prod_{\ell=1}^{k_1}(z-\pi_\ell)}{\prod_{\ell=1}^{i}(z-\pi_\ell)}\cdot\frac{\prod_{\ell=1}^{a_1+k_1}(\pi_j+\hat{\pi}_\ell)}{\prod_{\ell=1}^{a_1+k_1}(z+\hat{\pi}_\ell)}\cdot\frac{\prod_{\ell=1}^{j-1}(w-\pi_\ell)}{\prod_{\ell=1}^{k_1}(w-\pi_\ell)}\cdot\frac{\prod_{\ell=a_1+k_1+1}^{a_2+k_2}(\pi_j+\hat{\pi}_\ell)}{\prod_{\ell=a_1+k_1+1}^{a_2+k_2}(w+\hat{\pi}_\ell)}.$$

$D^+(i,j)$ is defined similarly but with the ordering of the contours $|w|=R_2^+>R_1=|z|$. Then

$$\mathscr{D}_{\text{step}}^{(i,j)}(\theta)=\theta^{-\mathbf{1}_{i>k_1}}D^-(i,j)-\theta^{\mathbf{1}_{i\le k_1}}D^+(i,j).$$

Now set

$$A(i,j)=\oint_{\Sigma_{0,\mathrm{R}}}\frac{d\zeta}{2\pi\mathrm{i}}e^{-t_1(\zeta-\pi_j)}\cdot\frac{\prod_{\ell=1}^{j-1}(\zeta-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)}\cdot\prod_{\ell=1}^{a_1+k_1}\frac{\zeta+\hat{\pi}_\ell}{\pi_j+\hat{\pi}_\ell},$$
$$B(i,j)=\oint_{\Sigma_{0,\mathrm{R}}}\frac{d\omega}{2\pi\mathrm{i}}e^{-(t_2-t_1)(\omega-\pi_j)}\cdot\frac{\prod_{\ell=1}^{j-1}(\omega-\pi_\ell)}{\prod_{\ell=1}^{i}(\omega-\pi_\ell)}\cdot\prod_{\ell=a_1+k_1+1}^{a_2+k_2}\frac{\omega+\hat{\pi}_\ell}{\pi_j+\hat{\pi}_\ell},$$

where $\Sigma_{0,\mathrm{R}}$ is any simple closed contour enclosing merely the points $\{\pi_j\}$ (but not $\{-\hat{\pi}_j\}$) and inside $|w|=R_2^-$. Note that the matrices A and B are both lower triangular with 1's on their diagonals so the determinants are both 1. We will show that

$$\mathscr{D}_{\text{step}}(\theta)=\det(\theta^{-\mathbf{1}_{i>k_1}}AD^-B-\theta^{\mathbf{1}_{i\le k_1}}AD^+B)=\det(I+F^{\text{Exp}}(\theta)).$$

For this we decompose the integral contours for w and z in (3.106) into disjoint left and right parts, enclosing merely the left poles and right poles respectively, while keeping the ordering of the contours:

$$\{|z|=R_1\}\to\Sigma_{1,\mathrm{L}}\cup\Sigma_{1,\mathrm{R}},\quad\{|w|=R_2^\pm\}\to\Sigma_{2,\mathrm{L}}^\pm\cup\Sigma_{2,\mathrm{R}}^\pm.$$

Then we split the double contour integrals in the entries $D^\pm(i,j)$ according to this decomposition:

$$D^\pm(i,j)=I_{\mathrm{LL}}^\pm(i,j)+I_{\mathrm{LR}}^\pm(i,j)+I_{\mathrm{RL}}^\pm(i,j)+I_{\mathrm{RR}}^\pm(i,j).$$

Here I_{LR} means the double integral with z in left contour and w in right contour, similar for other parts. Now direct matrix multipication implies

$$AI_{\mathrm{RR}}^{-}B(i,j)=\oint_{\Sigma_{0,\mathrm{R}}}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\oint_{\Sigma_{1,\mathrm{R}}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\oint_{\Sigma_{0,\mathrm{R}}}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\oint_{\Sigma_{2,\mathrm{R}}^{-}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{e^{t_1(z-\zeta)+(t_2-t_1)(w-\omega)}}{z-w}\cdot\prod_{\ell=1}^{a_1+k_1}\frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell}$$
$$\cdot\prod_{\ell=a_1+k_1+1}^{a_2+k_2}\frac{\omega+\hat{\pi}_\ell}{w+\hat{\pi}_\ell}\cdot\frac{\prod_{\ell=1}^{j-1}(\omega-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)}\cdot\prod_{\ell=1}^{k_1}\frac{z-\pi_\ell}{w-\pi_\ell}\cdot\left[\sum_{p=1}^{k_2}\frac{\prod_{\ell=1}^{p-1}(\zeta-\pi_\ell)}{\prod_{\ell=1}^{p}(z-\pi_\ell)}\cdot\sum_{q=1}^{k_2}\frac{\prod_{\ell=1}^{q-1}(w-\pi_\ell)}{\prod_{\ell=1}^{q}(\omega-\pi_\ell)}\right].$$

We evaluating the sums over p and q using the identities (3.63). Note that among the four terms resulting from expanding the products $(1-\prod_{\ell=1}^{k_2}\frac{\zeta-\pi_\ell}{z-\pi_\ell})(\prod_{\ell=1}^{k_2}\frac{\omega-\pi_\ell}{w-\pi_\ell}-1)$, only the term $\prod_{\ell=1}^{k_2}\frac{\omega-\pi_\ell}{w-\pi_\ell}$ contributes to the quadruple contour integrals since the other terms are all 0 as they have no poles either in ζ or ω inside the contour $\Sigma_{0,\mathrm{R}}$. Hence

$$AI_{\mathrm{RR}}^{-}B(i,j)=\oint_{\Sigma_{0,\mathrm{R}}}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\oint_{\Sigma_{1,\mathrm{R}}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\oint_{\Sigma_{0,\mathrm{R}}}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\oint_{\Sigma_{2,\mathrm{R}}^{-}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{e^{t_1(z-\zeta)+(t_2-t_1)(w-\omega)}}{(z-\zeta)(z-w)(w-\omega)}$$
$$\cdot\prod_{\ell=1}^{a_1+k_1}\frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell}\cdot\prod_{\ell=a_1+k_1+1}^{a_2+k_2}\frac{\omega+\hat{\pi}_\ell}{w+\hat{\pi}_\ell}\cdot\frac{\prod_{\ell=1}^{j-1}(\omega-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)}\cdot\prod_{\ell=1}^{k_1}\frac{z-\pi_\ell}{w-\pi_\ell}\cdot\prod_{\ell=1}^{k_2}\frac{w-\pi_\ell}{\omega-\pi_\ell}.$$

Now first deforming the w contour to a single point with a residue at the pole $w=\omega$ we have $AI_{\mathrm{RR}}^{-}B(i,j)$ equals

$$\oint_{\Sigma_{0,\mathrm{R}}}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\oint_{\Sigma_{1,\mathrm{R}}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\oint_{\Sigma_{0,\mathrm{R}}}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{e^{t_1(z-\zeta)}}{(z-\zeta)(z-\omega)}\cdot\prod_{\ell=1}^{a_1+k_1}\frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell}\cdot\frac{\prod_{\ell=1}^{j-1}(\omega-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)}\cdot\prod_{\ell=1}^{k_1}\frac{z-\pi_\ell}{\omega-\pi_\ell}.$$

Next for $j>k_1$ we can deform the ω contour to a single point since there is no pole in ω at the π_j's and the integral will be 0. For $j\le k_1$ we instead deform the ω contour to ∞ with a residue at the pole $\omega=z$ but no residue at ∞ since the integrand is

$O(\omega^{-2})$. Hence

$$AI^{-}_{\mathrm{RR}}B(i,j) = \mathbf{1}_{j\leq k_1} \cdot \oint\limits_{\Sigma_{0,\mathrm{R}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \oint\limits_{\Sigma_{1,\mathrm{R}}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{t_1(z-\zeta)}}{z-\zeta} \cdot \prod_{\ell=1}^{a_1+k_1} \frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell} \cdot \frac{\prod_{\ell=1}^{j-1}(z-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)}.$$

Finally we deform the z contour to a single point with a residue at the pole $z=\zeta$ and conclude that

$$AI^{-}_{\mathrm{RR}}B(i,j) = \mathbf{1}_{j\leq k_1} \cdot \oint\limits_{\Sigma_{0,\mathrm{R}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{\prod_{\ell=1}^{j-1}(\zeta-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)} = \mathbf{1}_{i=j,j\leq k_1}.$$

For I^{-}_{LR} we repeat the same procedure except in the last step we can not deform the z contour since it encloses the left poles instead of the right ones :

$$AI^{-}_{\mathrm{LR}}B(i,j) = \mathbf{1}_{j\leq k_1} \cdot \oint\limits_{\Sigma_{0,\mathrm{R}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \oint\limits_{\Sigma_{1,\mathrm{L}}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{t_1(z-\zeta)}}{z-\zeta} \cdot \prod_{\ell=1}^{a_1+k_1} \frac{\zeta+\hat{\pi}_\ell}{z+\hat{\pi}_\ell} \cdot \frac{\prod_{\ell=1}^{j-1}(z-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)}.$$

For I^{-}_{RL} we start with deforming the z contour instead of the w contour (with a residue at $z=\zeta$ now):

$$AI^{-}_{\mathrm{RL}}B(i,j) = \oint\limits_{\Sigma_{0,\mathrm{R}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \oint\limits_{\Sigma^{-}_{2,\mathrm{R}}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \oint\limits_{\Sigma_{0,\mathrm{R}}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{e^{(t_2-t_1)(w-\omega)}}{(\zeta-\omega)(w-\omega)} \cdot \prod_{\ell=a_1+k_1+1}^{a_2+k_2} \frac{\omega+\hat{\pi}_\ell}{w+\hat{\pi}_\ell}$$
$$\cdot \frac{\prod_{\ell=1}^{j-1}(\omega-\pi_\ell)}{\prod_{\ell=1}^{i}(\zeta-\pi_\ell)} \cdot \prod_{\ell=1}^{k_1} \frac{\zeta-\pi_\ell}{w-\pi_\ell} \cdot \prod_{\ell=1}^{k_2} \frac{w-\pi_\ell}{\omega-\pi_\ell}.$$

Then for $i\leq k_1$ we can deform the ζ contour to a single point and the integral will vanish since the integrand is analytic in ζ inside the contour. For $i>k_1$ we instead deform the ζ contour to ∞ with a residue at $\zeta=w$ but no residue at ∞ since the integrand is $O(\zeta^{-2})$. Hence

$$AI^{-}_{\mathrm{RL}}B(i,j) = -\mathbf{1}_{i>k_1} \cdot \oint\limits_{\Sigma^{-}_{2,\mathrm{L}}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \oint\limits_{\Sigma_{0,\mathrm{R}}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{e^{(t_2-t_1)(w-\omega)}}{w-\omega} \cdot \prod_{\ell=a_1+k_1+1}^{a_2+k_2} \frac{\omega+\hat{\pi}_\ell}{w+\hat{\pi}_\ell} \cdot \frac{\prod_{\ell=i+1}^{k_2}(w-\pi_\ell)}{\prod_{\ell=j}^{k_2}(\omega-\pi_\ell)}.$$

For $AI^{-}_{\mathrm{LL}}B(i,j)$ we do not deform the contours. A similar manipulation with the $+$ contours now yields the desired results. $\square$

Remark 3.7.8. Here we have essentially showed that

$$\det(I + F^{\mathrm{Exp}}(\theta))_{\ell^2(\{1,2,\cdots,N\})} = \mathscr{D}_{\mathrm{step}}(\theta) = \det(I - \mathcal{K}_1\mathcal{K}_{\vec{y}})_{L^2(\mathcal{S}_2)}, \tag{3.124}$$

by relating the two Fredholm determinants to the same $N \times N$ determinants where $N = k_2$. A natural question is whether one can direct show the equality between the two Fredholm determinants and the subtlety here is the two kernels are not simply related by some direct conjugation. In fact one can check that $\mathrm{tr}(F^{\mathrm{Exp}}(\theta)) \neq \mathrm{tr}(-\mathcal{K}_1\mathcal{K}_{\vec{y}})$. We believe that the series expansions of the two Fredholm determinants should agree but we are not able to match them term by term.

3.8 Proof of Theorem 3.2.1

3.8.1 Proof of Theorem 3.2.1 (i)

The proof is a standard steepest descent analysis so we only provide essential calculations here. Note first that for any nonzero constants $c_1, \cdots, c_m$, the Fredholm determinant part $\mathcal{D}^{(\infty)}_{\mathrm{step}}$ is invariant if we replace the functions F_i appearing in the kernels by $\tilde{F}_i := c_i F_i$. This can be easily seen from equation (3.121) since the function $f_\ell(U^{(\ell)}) f_\ell(V^{(\ell)})$ is invariant when we replace F_i by $c_i F_i$. Now we set

$$\tilde{F}_i(w) = \frac{\prod_{\ell=1}^{k_i}(w - \pi_\ell) \cdot \prod_{\ell=1}^{a_i+k_i}(w + \hat{\pi}_\ell)^{-1} \cdot e^{t_i w}}{(-1/2)^{k_i} T^{-r/3} (1/2)^{-a_i-k_i} T^{s/3} e^{-t_i/2}}$$

Then for $w = \frac{1}{2} + \frac{1}{2}\zeta T^{-1/3}$ a straightforward Taylor expansion shows that for $|\zeta| \le T^{\epsilon/4}$

$$\tilde{F}_i(w) = \frac{\prod_{\ell=1}^{r}(\lambda_\ell - \zeta)}{\prod_{\ell=1}^{s}(\zeta - \mu_\ell)} \cdot \exp\left(-\frac{1}{3}\tau_i\zeta^3 + x_i\zeta^2 + h_i\zeta\right) \cdot \left(1 + O(T^{\epsilon-1/3})\right),$$

where $0 < \epsilon < 1/3$. To ensure the kernels have sufficiently fast decay on each variable we make the conjugations by setting

$$\tilde{\mathcal{K}}_1(w,w') := \big(\delta_i(j) + \delta_i(j+(-1)^i)\big)\, \frac{\sqrt{\tilde{f}_i(w)}\sqrt{\tilde{f}_j(w')}}{w-w'} Q_1(j),$$

$$\tilde{\mathcal{K}}_{\mathrm{step}}(w,w') := \big(\delta_j(i) + \delta_j(i+(-1)^j)\big)\, \frac{\sqrt{\tilde{f}_j(w')}\sqrt{\tilde{f}_i(w)}}{w-w'} Q_2(j),$$

for $w' \in \Sigma_j \cap \mathcal{S}_2,\ w \in \Sigma_i \cap \mathcal{S}_1$. Here $\tilde{f}_i$ is obtained by replacing F_i with $\tilde{F}_i$ in equation (3.23). We also conjugate the limiting kernels in a similar way:

$$\tilde{\mathrm{K}}_1^{\vec{\lambda},\vec{\mu}}(\zeta,\zeta') := \big(\delta_i(j) + \delta_i(j+(-1)^i)\big)\, \frac{\sqrt{\mathrm{f}_i(\zeta)}\sqrt{\mathrm{f}_j(\zeta')}}{\zeta-\zeta'} Q_1(j),$$

$$\tilde{\mathrm{K}}_{\mathrm{step}}^{\vec{\lambda},\vec{\mu}}(\zeta,\zeta') := \big(\delta_j(i) + \delta_j(i+(-1)^j)\big)\, \frac{\sqrt{\mathrm{f}_j(\zeta')}\sqrt{\mathrm{f}_i(\zeta)}}{\zeta-\zeta'} Q_2(j),$$

for $\zeta' \in \Gamma_j \cap \mathbb{S}_2,\ \zeta \in \Gamma_i \cap \mathbb{S}_1$. These conjugations do not change the Fredholm determinants because they do not change the $\mathscr{D}_{\vec{n},\vec{y}}$ and $\mathbf{D}_{\vec{n},\vec{y}}$ terms in the series expansions of the Fredholm determinants (3.121). For the same reason the choice of square root does not matter since such square roots appear an even number of times in each term of the series expansion. After the conjugation it is straightforward to show that for $w = -\frac{1}{2} + \frac{1}{2}\zeta T^{-1/3}$ and $0 < \epsilon < 1/3$ we have

$$\tilde{f}_j(w) = \begin{cases} \mathrm{f}_j(\zeta)(1 + O(T^{\epsilon-1/3})) & \text{if } |\zeta| \le T^{\epsilon/4}, \\ O(e^{-cT^{c'\cdot\epsilon}}) & \text{if } |\zeta| \ge T^{\epsilon/4}. \end{cases} \tag{3.125}$$

Now we deform the contours Σ_j to be sufficiently close to the critical points $w_c = -\frac{1}{2}$ such that locally they look like the limiting contours Γ_j. More precisely we deform $\Sigma_{j,\mathrm{R}}^{+}$ such that

$$\Sigma_{j,\mathrm{R}}^{+} \cap \{w \in \mathbb{C} : \left| w + \frac{1}{2} \right| \le \frac{1}{2} T^{\epsilon/4-1/3}\} = \{-\frac{1}{2} + T^{-1/3}(a_j^{+} + re^{\frac{\pi \mathrm{i}}{3}}) : 0 \le r \le T^{\epsilon/4}\}$$

$$\cup \{-\frac{1}{2} + T^{-1/3}(a_j^+ + re^{-\frac{\pi \mathrm{i}}{3}}) : 0 \le r \le T^{\epsilon/4}\},$$

and similar for other contours. Then by (3.125) it is straightforward to show that

1. For each $n \in \mathbb{N}$ and $0 < |\theta_i| < 1$ we have

$$\lim_{T\to\infty} \mathrm{Tr}\left(\tilde{\mathcal{K}}_1 \tilde{\mathcal{K}}_{\text{step}}\right)^n = \mathrm{Tr}\left(\tilde{\mathrm{K}}_1^{\vec{\lambda},\vec{\mu}} \tilde{\mathrm{K}}_{\text{step}}^{\vec{\lambda},\vec{\mu}}\right)^n.$$

2. There exists constant $C > 0$ independent of T and n such that for all $n \in \mathbb{N}$

$$\left| \oint_{\mathcal{S}_1} \cdots \oint_{\mathcal{S}_1} \det\left[\left(\tilde{\mathcal{K}}_1 \tilde{\mathcal{K}}_{\text{step}}\right)(w_i, w_j)\right]_{i,j=1}^n \frac{\mathrm{d}w_1}{2\pi \mathrm{i}} \cdots \frac{\mathrm{d}w_n}{2\pi \mathrm{i}} \right| \le C^n.$$

Now (1) and (2) immediately implies $\mathcal{D}^{(\infty)}_{\text{step}}(\theta_1, \cdots, \theta_{m-1})$ converges to $\mathrm{D}^{\mathrm{BBP};\vec{\lambda},\vec{\mu}}_{\text{step}}(\theta_1, \cdots, \theta_{m-1})$ locally uniformly for $0 < |\theta_i| < 1$ as $T \to \infty$, thus proving Theorem 3.2.1 part (i).

3.8.2 Proof of Theorem 3.2.1 (ii)

This part is almost identical to the previous part so we omit the details.

3.8.3 Proof of Theorem 3.2.1 (iii)

This part is proved in [35] Theorem 2.1 (b) using probabilistic arguments. Here we briefly explain how it can be obtained through steepest descent analysis of our finite-time formula. It follows immediately from the following two lemmas, each of which is a straightforward consequence of a steepest descent analysis of the corresponding contour integral representation. We omit most of the details.

Lemma 3.8.1. *For $\vec{n} = (n_1, n_2 \cdots, n_m) \in (\mathbb{Z}_{\ge 0})^m$ with $n_i = \mathbf{1}_{i \le k}$, we have*

$$\lim_{T\to\infty} \mathbf{D}_{\vec{n},\text{step}}(\theta_1, \cdots, \theta_{m-1}) = \int_{c_1 - \mathrm{i}\infty}^{c_1 + \mathrm{i}\infty} \cdots \int_{c_k - \mathrm{i}\infty}^{c_k + \mathrm{i}\infty} \prod_{i=1}^k \frac{\mathrm{d}\xi_i}{2\pi \mathrm{i}} \frac{e^{\frac{1}{2}(\tau_i - \tau_{i-1})\xi_i^2 + (x_i - x_{i-1})\xi_i}}{\xi_i - \xi_{i+1}}, \quad (3.126)$$

where $\xi_{k+1} = \tau_0 = x_0 := 0$ *and* $c_k < c_{k-1} < \cdots < c_1$ *with* $c_k < 0$.

Lemma 3.8.2. *For any* $\vec{n} \in (\mathbb{Z}_{\geq 0})^m$ *not of the form* $\vec{n} = (1, \cdots, 1, 0, \cdots, 0)$ *we have*

$$|\mathbf{D}_{\vec{n},\text{step}}(\theta_1, \cdots, \theta_{m-1})| \leq e^{-cT} \cdot C^{|\vec{n}|},$$

for some constants $c, C > 0$.

Proof of Theorem 3.2.1 (iii). By Lemma 3.8.1 and 3.8.2 above we have

$$\lim_{T\to\infty} \mathbb{P}_{\text{step}}\left(\bigcap_{\ell=1}^{m}\{x_{k_\ell}(t_\ell) \geq a_\ell\}\right) = \oint \cdots \oint \prod_{i=1}^{m-1} \frac{\mathrm{d}\theta_i}{2\pi\mathrm{i}\theta_i(1-\theta_i)} \cdot \left(1 + \sum_{k=1}^{m} U_k\right) = 1 + \sum_{k=1}^{m} U_k.$$

Where

$$U_k := \int_{c_1-\mathrm{i}\infty}^{c_1+\mathrm{i}\infty} \cdots \int_{c_k-\mathrm{i}\infty}^{c_k+\mathrm{i}\infty} \prod_{i=1}^{k} \frac{\mathrm{d}\xi_i}{2\pi\mathrm{i}} \frac{e^{\frac{1}{2}(\tau_i-\tau_{i-1})\xi_i^2+(x_i-x_{i-1})\xi_i}}{\xi_i - \xi_{i+1}}.$$

Here $\xi_{k+1} = \tau_0 = x_0 := 0$ and $c_k < c_{k-1} < \cdots < c_1$ with $c_k < 0$. Deforming all the contours to the right half-plane while preserving the orders of the vertical contours we have for $2 \leq k \leq m$

$$\begin{aligned} U_k = &\int_{\hat{c}_1-\mathrm{i}\infty}^{\hat{c}_1+\mathrm{i}\infty} \cdots \int_{\hat{c}_k-\mathrm{i}\infty}^{\hat{c}_k+\mathrm{i}\infty} \prod_{i=1}^{k} \frac{\mathrm{d}\xi_i}{2\pi\mathrm{i}} \frac{e^{\frac{1}{2}(\tau_i-\tau_{i-1})\xi_i^2+(x_i-x_{i-1})\xi_i}}{\xi_i - \xi_{i+1}} \\ &- \int_{\hat{c}_1-\mathrm{i}\infty}^{\hat{c}_1+\mathrm{i}\infty} \cdots \int_{\hat{c}_{k-1}-\mathrm{i}\infty}^{\hat{c}_{k-1}+\mathrm{i}\infty} \prod_{i=1}^{k-1} \frac{\mathrm{d}\xi_i}{2\pi\mathrm{i}} \frac{e^{\frac{1}{2}(\tau_i-\tau_{i-1})\xi_i^2+(x_i-x_{i-1})\xi_i}}{\xi_i - \xi_{i+1}}, \end{aligned}$$

where $0 < \hat{c}_k < \cdots < \hat{c}_1$ and we set $\xi_k = 0$ for the second term on the right hand side above. For $k = 1$ we simply have

$$U_1 = \int_{\hat{c}_1-\mathrm{i}\infty}^{\hat{c}_1+\mathrm{i}\infty} \frac{\mathrm{d}\xi_1}{2\pi\mathrm{i}} \frac{e^{\frac{1}{2}\tau_1\xi_1^2+x_1\xi_1}}{\xi_1} - 1.$$

Summing over $1 \le k \le m$ we see

$$1 + \sum_{k=1}^{m} U_k = \int_{\hat{c}_1 - \mathrm{i}\infty}^{\hat{c}_1 + \mathrm{i}\infty} \cdots \int_{\hat{c}_m - \mathrm{i}\infty}^{\hat{c}_m + \mathrm{i}\infty} \prod_{k=1}^{m} \frac{\mathrm{d}\xi_k}{2\pi \mathrm{i}} \frac{e^{\frac{1}{2}(\tau_k - \tau_{k-1})\xi_k^2 + (x_k - x_{k-1})\xi_k}}{\xi_k - \xi_{k+1}},$$

where $\xi_{m+1} = \tau_0 = x_0 := 0$. Introducing the change of variables $\xi_j := \mathrm{i}\eta_j$ for $1 \le j \le m$ we have

$$1 + \sum_{k=1}^{m} U_k = \int_{-\infty - \mathrm{i}\hat{c}_1}^{+\infty - \mathrm{i}\hat{c}_1} \cdots \int_{-\infty - \mathrm{i}\hat{c}_m}^{+\infty - \mathrm{i}\hat{c}_m} \prod_{k=1}^{m} \frac{\mathrm{d}\eta_k}{2\pi \mathrm{i}} \frac{e^{-\frac{1}{2}(\tau_k - \tau_{k-1})\eta_k^2 + \mathrm{i}(x_k - x_{k-1})\eta_k}}{\eta_k - \eta_{k+1}} := \hat{\mathrm{G}}_1(x_1, \cdots, x_m). \tag{3.127}$$

Taking derivatives with respect to $x_m, \cdots, x_1$ we see

$$\frac{\partial^m \hat{\mathrm{G}}_1(x_1, \cdots, x_m)}{\partial x_1 \cdots \partial x_m} = \int_{-\infty - \mathrm{i}\hat{c}_1}^{+\infty - \mathrm{i}\hat{c}_1} \cdots \int_{-\infty - \mathrm{i}\hat{c}_m}^{+\infty - \mathrm{i}\hat{c}_m} \prod_{k=1}^{m} \frac{\mathrm{d}\eta_k}{2\pi} \exp\left(-\frac{1}{2}(\tau_k - \tau_{k-1})\eta_k^2 + \mathrm{i}(x_k - x_{k-1})\eta_k \right).$$

Here $0 < \hat{c}_m < \cdots < \hat{c}_1$ and we have exchanged the order of integration and differentiation which follows directly from dominated convergence theorem.. A direct Gaussian integral gives

$$\int_{-\infty + \mathrm{i}\cdot c}^{+\infty + \mathrm{i}\cdot c} e^{-\frac{1}{2}\alpha u^2 + \mathrm{i}\beta u} \, \mathrm{d}u = \sqrt{\frac{2\pi}{\alpha}} e^{-\frac{\beta^2}{2\alpha}},$$

for all $\alpha > 0$ and $\beta, c \in \mathbb{R}$. Hence

$$\frac{\partial^m \hat{\mathrm{G}}_1(x_1, \cdots, x_m)}{\partial x_1 \cdots \partial x_m} = \frac{1}{\sqrt{(2\pi)^m \prod_{\ell=1}^{m} (\tau_\ell - \tau_{\ell-1})}} \exp\left(-\frac{1}{2} \sum_{\ell=1}^{m} \frac{(x_\ell - x_{\ell-1})^2}{\tau_\ell - \tau_{\ell-1}} \right).$$

Now note that for any $1 \le j \le m$, we have $\mathrm{Re}(i(\eta_j - \eta_{j+1})) = \hat{c}_j - \hat{c}_{j+1} > 0$ for any η_j and η_{j+1} with $\mathrm{Im}(\eta_j) = -\hat{c}_j$ and $\mathrm{Im}(\eta_{j+1}) = -\hat{c}_{j+1}$. Hence for all $1 \le k \le m$ and such η_k's we have

$$\lim_{x_k \to -\infty} e^{ix_k(\eta_k - \eta_{k+1})} = 0.$$

Thus by dominated convergence theorem we have for any $1 \leq i \leq m$

$$\lim_{x_i \to -\infty} \hat{\mathrm{G}}_1(x_1, \cdots, x_m) = 0.$$

Similar arguments holds for all derivatives, namely

$$\lim_{x_i \to -\infty} \frac{\partial^j \hat{\mathrm{G}}_1(x_1, \cdots, x_m)}{\partial x_j \cdots \partial x_m} = 0,$$

for all $1 \leq i, j \leq m$. Hence by integrating $\frac{\partial^m \hat{\mathrm{G}}_1(u_1, \cdots, u_m)}{\partial x_1 \cdots \partial x_m}$ from $-\infty$ to x_i for $1 \leq i \leq m$ we conclude that

$$\begin{aligned} &\hat{\mathrm{G}}_1(x_1, \cdots, x_m) \\ &= \frac{1}{\sqrt{(2\pi)^m \prod_{\ell=1}^m (\tau_\ell - \tau_{\ell-1})}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \exp\left(-\frac{1}{2} \sum_{\ell=1}^m \frac{(u_\ell - u_{\ell-1})^2}{\tau_\ell - \tau_{\ell-1}} \right) \mathrm{d}u_1 \cdots \mathrm{d}u_m. \end{aligned}$$

This completes the proof of Theorem 3.2.1 (iii). □

CHAPTER 4

Multi-point Distribution of Discrete Time Periodic TASEP

4.1 Introduction

The goal of this chapter is to study another classical model in the KPZ universality class, the discrete time totally asymmetric exclusion process with parallel updates, mainly on periodic domains. On the infinite lattice $\mathbb{Z}$, this model has been well studied. The one-point marginal distribution for height function was obtained in [65] for the equivalent geometric last passage percolation model and joint distributions of several locations at equal time was obtained in [19]. Recently the joint distributions along the time direction have also been studied, in [66] for two-time case and [67] for general multi-time joint distributions. However on the periodic domain there are fewer results concerning height fluctuations (see [91] for some results on transition probability and stationary distributions), which are the main focuses of this chapter. The main results of this chapter are summarized as follows:

1. For general initial conditions we obtain a finite-time multi-point (in both space and time) joint distribution formula for discrete time parallel periodic TASEP. The formula consists of an m-fold contour integral with integrand involving a Fredholm determinant, where m is the number of space-time points we are

considering and the Fredholm determinant has kernels acting on certain discrete sets related to roots of some polynomials.

2. Under the relaxation time scale $t = O(L^{3/2})$ and the $1:2:3$ KPZ scaling, we obtain large-time, large-period limits for the multi-point joint distributions under certain assumptions on the initial condition, which are verified for step and flat cases. These limiting formulas agree with those obtained in [9, 10], thus providing an evidence that the height fluctuations for periodic models in the KPZ class are in fact universal.

Comparing to the previous work [9, 10] and [81] on the multi-point distributions of continuous time TASEP, on either periodic domain or $\mathbb{Z}$, see also Chapter 3 for the inhomogeneous generalization, our work consists of formulas with similar structures but involves an extra parameter p describing the hopping probability, which makes the algebraic properties a bit more complicated. In particular the polynomial whose roots are related to the kernels in the periodic formulas now depends on the extra parameter p. We point out that all the formulas for continuous time TASEP can be obtained from our formulas by rescaling the time and taking $p \to 0$.

Outline of the chapter

In Section 4.2 we describe the discrete time parallel TASEP models and state the main results involving several multi-point joint distribution formulas, for both periodic domain and infinite lattice $\mathbb{Z}$, finite time and large time limit. From Sections 4.3 to Section 4.5 we derive the main finite-time algebraic formulas for multi-point distribution of discrete time parallel periodic TASEP and we regard them as the main technical novelties in this chapter. In particular in Section 4.3 we prove a novel transition probability formula for discrete time parallel TASEP on the periodic domain involving integral of determinants using coordinate Bethe ansatz. In Section 4.4

we derive the finite-time multi-point distribution formula by performing a multiple sum over the transition probabilities. The key ingredients are certain Cauchy-type summation identities over the eigenfunctions of the generator, which might be of independent interest so we discuss the proof in Section 4.5. In Section 4.6 and Section 4.7 we discuss the large time, large period asymptotics for the multi-point distribution under relaxation time scale $t = O(L^{3/2})$.

4.2 Models and main results

Let $N < L$ be positive integers. We consider discrete time TASEP with parallel updates with N particles on a spatially periodic domain of size L. It is convenient to view the dynamics as particles moving to the right on the integer lattice $\mathbb{Z}$ while periodicity forces particle configurations to be identical copies of each other every L sites. More precisely this means that the occupation functions $\eta_j(t)$ which equal 1 if there is a particle at site $j \in \mathbb{Z}$ at time t and equalts 0 otherwise should satisfy

$$\eta_j(t) = \eta_{j+kL}(t), \quad \text{for all } j, k \in \mathbb{Z} \text{ and } t \in \mathbb{N}. \tag{4.1}$$

We fix a single period of size L and denote the locations of N total particles in this period at time t as

$$x_1(t) > x_2(t) > \cdots > x_N(t).$$

Here $x_i(t) \in \mathbb{Z}$ for $1 \leq i \leq N$ and we index the particles from right to left. The locations of all the particles then satisfy $x_{i+kN}(t) = x_i(t) - kL$ for $1 \leq i \leq N$ and $k \in \mathbb{Z}$ so that we have

$$\cdots > x_N(t) + L = x_0(t) > x_1(t) > x_2(t) > \cdots > x_N(t) > x_{N+1}(t) = x_1(t) - L > \cdots .$$

Thus the natural configuration space for the particles should be

$$\mathcal{X}_N^{(L)} := \{\vec{x} = (x_1, \cdots, x_N) \in \mathbb{Z}^N : x_N + L > x_1 > x_2 > \cdots > x_N\}. \tag{4.2}$$

The discrete time parallel periodic TASEP with N particles, period L and hopping probability $0 < p < 1$, which we denote by dpTASEP(L, N, p), is the following Markovian dynamics on particle configurations $\vec{x}(t) \in \mathcal{X}_N^{(L)}$: at each time step, each particle in a single period hops to its right neighbor site independently with probability $p = 1 - q$ provided that the site is empty, otherwise it stays at its current position. As a special case, the discrete time parallel TASEP on $\mathbb{Z}$ which we will denote by dTASEP(p) corresponds to particles following the same evolution rules with the period $L \to \infty$ so that the configuration space for the first N particles (from right to left, we always assume the existence of right-most particle) becomes

$$\mathcal{X}_N^{(\infty)} := \{\vec{x} = (x_1, \cdots, x_N) \in \mathbb{Z}^N : x_1 > x_2 > \cdots > x_N\}. \tag{4.3}$$

Notation 4.2.1. Throughout the chapter there will be several very similar formulas and quantities corresponding to either discrete time parallel TASEP on a periodic domain with size L with the continuous time inhomogeneous TASEP discussed in Chapter 3. To emphasize the similarity we may use the same notation for two different but very similar objects in the two chapters when there is no confusion.

4.2.1 Multi-point distribution of dpTASEP(L, N, p)

The first theorem is a finite-time multi-point joint distribution formula for discrete time TASEP on a periodic domain, which is the starting point of all other results in this paper. It has an almost identical form as Theorem 3.6.1, except that the Bethe roots are deformed by the parameter p and the weight function F_i changes slightly.

Theorem 4.2.2 (Finite-time multi-point joint distribution for dpTASEP$(L, N, p, \vec{y})$)**.**

Let $N < L$ be integers. Consider discrete time periodic parallel TASEP with hopping probability $0 < p < 1$, N particles and L sites per period ($\mathrm{dpTASEP}(L, N, p, \vec{y})$) where $\vec{y} \in \mathcal{X}_N^{(L)}$ is the initial condition, i.e., $x_i(0) = y_i$ for $1 \le i \le N$. We set $\varrho = N/L$ and

$$\mathbf{r}_c := \left(\frac{-w_c}{1 + pw_c}\right)^{\varrho} (1 + w_c)^{1-\varrho}, \tag{4.4}$$

where

$$w_c = -\frac{2\varrho}{1 + \sqrt{1 - 4p \cdot \varrho(1 - \varrho)}} := -\frac{2\varrho}{1 + \nu}. \tag{4.5}$$

Fix a positive integer m and let (k_i, t_i), $1 \le i \le m$ be m distinct points in $\mathbb{Z} \times \mathbb{Z}_{\ge 0}$ satisfying $0 \le t_1 \le \cdots \le t_m$. Then for any integers $a_1, \cdots, a_m$,

$$\mathbb{P}_{\vec{y}}^{(L)}\left(\bigcap_{i=1}^{m}\{x_{k_i}(t_i) \ge a_i\}\right) = \oint \cdots \oint \mathscr{C}_{\vec{y}}^{(L)}(\vec{z}) \mathscr{D}_{\vec{y}}^{(L)}(\vec{z}) \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1} \cdots \frac{\mathrm{d}z_m}{2\pi \mathrm{i} z_m}. \tag{4.6}$$

Where we set $\vec{z} = (z_1, \cdots, z_m)$ and the contours are over nested circles centered at the origin: $0 < |z_m| < \cdots < |z_1| < \mathbf{r}_c$. Here and in all the remaining results we suppress the dependence of the integrand on the parameters a_i, k_i, t_i as well as the hopping probability $0 < p < 1$. The function $\mathscr{C}_{\vec{y}}^{(L)}(\vec{z})$ is defined in Section 4.2.4 and $\mathscr{D}_{\vec{y}}^{(L)}(\vec{z})$ is a Fredholm determinant

$$\mathscr{D}_{\vec{y}}^{(L)}(\vec{z}) = \det(1 - \mathscr{K}_1^{(L)} \mathscr{K}_{\vec{y}}^{(L)}), \tag{4.7}$$

where the operators $\mathscr{K}_1^{(L)}$ and $\mathscr{K}_{\vec{y}}^{(L)}$ are defined in Section 4.2.4.

Remark 4.2.3. Theorem 4.2.2 generalizes Theorem 3.1 of [10] (and also Theorem 3.6.1 in Chapter 3) on the finite-time multi-point distribution of continuous time periodic TASEP. In fact their formulas can be obtained from our formula (4.6) by taking $p = \epsilon$, $\hat{t} = \epsilon t$ and letting $\epsilon \to 0$.

Next we state the theorem on the large-time, large-period scaling limit of (4.6) under the relaxation time scale $t = O(L^{3/2})$. To emphasize the initial condition in the limit theorem, we add the subscript "ic" for the terms in the limit which depend on the initial conditions. We make the following choice of the labeling for convenience: we assume that $x_1(0) \leq 0 < x_0(0)$. This is equivalent to assuming that the initial condition satisfies $y_1 \leq 0 < y_N + L$.

Theorem 4.2.4 (Relaxation time limit)**.** *Consider a sequence* $\mathrm{dpTASEP}(L, N, \vec{y}(L))$ *where* $\varrho = \varrho_L = N/L$ *stays in a compact subset of* $(0,1)$ *and* $y_1 \leq 0 < y_N + L$. *Suppose that the sequence of initial conditions* $\vec{y}(L)$ *satisfies certain assumptions (see Assumption 4.6.1). Fix a positive integer* m *and let* $\mathrm{p}_j = (\gamma_j, \tau_j)$ *be* m *points in the region*

$$\mathrm{R} := [0,1] \times \mathbb{Z}_{>0}$$

satisfying

$$0 < \tau_1 < \tau_2 < \cdots < \tau_m.$$

Then for every fixed $\mathrm{x}_1, \cdots, \mathrm{x}_m \in \mathbb{R}$ *and parameters* a_i, k_i, t_i, $1 \leq i \leq m$ *given by*

$$t_i = c_1 \tau_i L^{3/2} + O(1), \quad a_i = c_2 t_i + \gamma_i L + O(1), \quad k_i = c_3 t_i + c_4 \gamma_i L + c_5 \mathrm{x}_i L^{1/2}, \tag{4.8}$$

we have

$$\lim_{L\to\infty} \mathbb{P}_{\vec{y}}^{(L)} \left(\bigcap_{j=1}^{m} \{x_{k_i}(t_i) \geq a_i\} \right) = \mathbb{F}_{\mathrm{ic}}^{\mathrm{per}}(\mathrm{x}_1, \cdots, \mathrm{x}_m; \mathrm{p}_1, \cdots, \mathrm{p}_m). \tag{4.9}$$

Here the constants c_i *depend explicitly on particle density* $0 < \varrho < 1$ *and hopping probability* $0 < p < 1$ *and are given by*

$$\begin{aligned} c_1 &= \frac{1}{p(1-p)} \frac{\nu^{5/2}}{\varrho^{1/2}(1-\varrho)^{1/2}}, \quad c_2 = \frac{p(1-2\varrho)}{\nu}, \\ c_3 &= \frac{2\varrho^2 \cdot p(1-p)}{\nu(1+\nu-2p\varrho)}, \quad c_4 = -\varrho, \quad c_5 = -\varrho^{1/2}(1-\varrho)^{1/2}\nu^{1/2}, \end{aligned} \tag{4.10}$$

where we set $\nu := \sqrt{1-4p\cdot\varrho(1-\varrho)}$ *for convenience. We recall that in equation* (4.9) $\mathbb{P}^{(L)}$ *denotes the probability associated to* $\mathrm{dpTASEP}(L, N, \vec{y}(L))$*. The function* $\mathbb{F}_{\mathrm{ic}}^{\mathrm{per}}$ *agrees with the one defined in Section 6.4 of [10] as the relaxation time limit of distribution of continuous time periodic TASEP. We recall the definition of* $\mathbb{F}_{\mathrm{ic}}^{\mathrm{per}}$ *in Section 4.6.3 for completeness. The convergence is locally uniform in* x_j, τ_j*, and* γ_j*.*

4.2.2 Bethe equations and Bethe roots

For our analysis on the discrete time periodic TASEP the following polynomial and its roots play essential role:

Definition 4.2.5 (Bethe roots)**.** *Given* $z \in \mathbb{C}$ *and* $0 < p < 1$*. Define the degree* L *polynomial* $q_z(w)$ *by*

$$q_z(w) := w^N(1+w)^{L-N} - z^L(1+pw)^N. \tag{4.11}$$

We call this polynomial the Bethe polynomial associated to z *and its roots Bethe roots. We denote the set of all roots of the Bethe polynomial* $q_z(w)$ *by* $\mathcal{S}_z$*:*

$$\mathcal{S}_z = \{w \in \mathbb{C} : q_z(w) = 0\}. \tag{4.12}$$

The Bethe root set $\mathcal{S}_z$ is contained in the level set $\{w \in \mathbb{C} : |w|^{\varrho}|1+w|^{1-\varrho} = |z|\cdot|1+pw|^{\varrho}\}$, which is sometimes called a deformed Cassini oval. It is not hard to check that for $|z| < \mathbf{r}_c$, the level set consists of two disjoint contours while for $|z| > \mathbf{r}_c$ the two contours merge to a single contour. For $z = \mathbf{r}_c$ there is a self-intersection point for the contour at $w = w_c$. Here $\mathbf{r}_c$ and w_c are defined in (4.4) and (4.5). See figure 4.1. We remark that the Bethe polynomials (and their roots) we are considering here are one-parameter generalizations of the one considered in [8, 9, 10] for continuous time periodic TASEP which corresponds to $p = 0$ degeneration of (4.11),

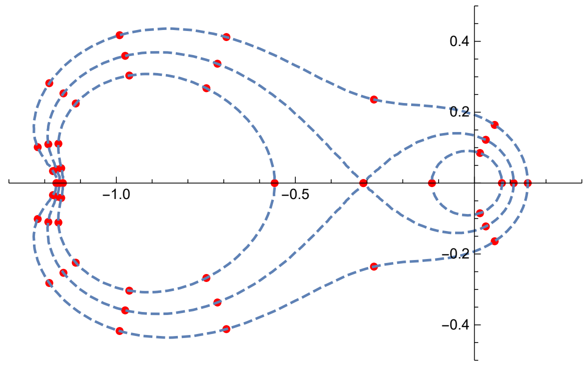

Figure 4.1: The solid dots are roots for $q_z(w)$ with $L = 16$, $N = 4$, $p = 5/6$ and $|z| = \frac{11}{10}\mathbf{r}_c, \mathbf{r}_c$ and $\frac{9}{10}\mathbf{r}_c$ from outside to inside. The dashed lines are the corresponding level sets.

Definition 4.2.6 (Left and right Bethe roots)**.** *For $|z| < \mathbf{r}_c$, we define the sets*

$$\mathcal{L}_z := \{w \in \mathcal{S}_z : Re(w) < w_c\}, \quad \mathcal{R}_z := \{w \in \mathcal{S}_z : Re(w) > w_c\}, \tag{4.13}$$

where w_c and $\mathbf{r}_c$ are defined in (4.5) and (4.4). Then it is straightforward to check that $|\mathcal{L}_z| = L - N$ and $|\mathcal{R}_z| = N$. Roots in $\mathcal{L}_z$ and $\mathcal{R}_z$ are called left and right Bethe roots, respectively. We also define the left and right Bethe polynomials $q_{z,\mathrm{L}}(w)$ and $q_{z,\mathrm{R}}(w)$ as the monic polynomials with roots in $\mathcal{L}_z$ and $\mathcal{R}_z$:

$$q_{z,\mathrm{L}}(w) := \prod_{u \in \mathcal{L}_z} (w - u), \quad q_{z,\mathrm{R}}(w) := \prod_{v \in \mathcal{R}_z} (w - v). \tag{4.14}$$

Then by definition we have

$$\mathcal{S}_z = \mathcal{L}_z \cup \mathcal{R}_z, \quad q_z(w) = q_{z,\mathrm{L}}(w) q_{z,\mathrm{R}}(w).$$

4.2.3 A symmetric function related to initial conditions

In our finite-time multi-point distribution formula (4.6), the quantities encoding information in the initial condition are all related to the following symmetric function:

Definition 4.2.7 (Symmetric function)**.** *Given $p \in \mathbb{C}$ and $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N$*

with $\lambda_1 \geq \cdots \geq \lambda_N$*. We define*

$$\mathcal{F}_\lambda(w_1, \cdots, w_N; p) := \frac{\det\left[w_j^{N-i}(pw_j+1)^{i-1}(w_j+1)^{\lambda_i}\right]_{i,j=1}^N}{\det[w_j^{N-i}]_{i,j=1}^N}. \tag{4.15}$$

Remark 4.2.8. The symmetric function $\mathcal{F}_\lambda$ defined here is a one-parameter generalization of the Grothendieck-like symmetric function defined in equation (3.6) of [10] which corresponds to the $p = 0$ degeneration in our situation.

The following two quantities related to $\mathcal{F}_\lambda$ encode the initial condition:

Definition 4.2.9 (Global energy and characteristic function)**.** *For* $\vec{y} \in \mathcal{X}_N^{(L)}$*, we set*

$$\lambda(\vec{y}) = (y_1 + 1, y_2 + 2, \cdots, y_N + N). \tag{4.16}$$

For $|z| < \mathbf{r}_c$*, we define the global energy* $\mathcal{E}_{\vec{y}}(z)$ *by*

$$\mathcal{E}_{\vec{y}}(z) := \mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z; p). \tag{4.17}$$

When $\mathcal{E}_{\vec{y}}(z) \neq 0$*, we define the characteristic function* $\chi_{\vec{y}}(v, u; z)$ *for a left Bethe root* u *and a right Bethe root* v *by*

$$\chi_{\vec{y}}(v, u; z) := \frac{\mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z \cup \{u\} \backslash \{v\}; p)}{\mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z; p)}, \quad \text{for } u \in \mathcal{L}_z \text{ and } v \in \mathcal{R}_z. \tag{4.18}$$

Remark 4.2.10. A straightforward calculation shows that for step initial condition $\vec{y} = (-1, \cdots, -N)$, we have $\mathcal{F}_{\lambda(\vec{y})} \equiv 1$. Hence the global energy function and characteristic function are both constant 1 for step initial condition. In general since the roots of the Bethe polynomial $q_z(w)$ depend analytically on z, the function $\mathcal{E}_{\vec{y}}(z)$ is analytic for $|z| < \mathbf{r}_c$. Furthermore $\mathcal{E}_{\vec{y}}(z)$ can not vanish identically. In fact $\mathcal{E}_{\vec{y}}(0) = 1$ since when $|z| \to 0$ all the right Bethe roots converge to 0. As a consequence $\mathcal{F}_{\lambda(\vec{y})}(\mathcal{R}_z; p)$ is nonzero for all but finitely many z in any compact subset of

$\{|z| < \mathbf{r}_c\}$, which means $\chi_{\vec{y}}(v, u; z)$ is a well-defined meromorphic function in z on $\{|z| < \mathbf{r}_c\}$ for fixed u, v.

4.2.4 Definition of $\mathscr{C}_{\vec{y}}^{(L)}(\vec{z})$ and $\mathscr{D}_{\vec{y}}^{(L)}(\vec{z})$

The functions $\mathscr{C}_{\vec{y}}^{(L)}(\vec{z})$ and $\mathscr{D}_{\vec{y}}^{(L)}(\vec{z})$ are defined in an almost identical way as the $\mathscr{C}_{\vec{y}}^{(L)}(\vec{z})$ and $\mathscr{D}_{\vec{y}}(\vec{z})$ functions defined in Section 3.6.2 except that the Bethe equation is now replaced by (4.11) and the weight functions are now

$$E_\ell(z) := \prod_{u \in \mathcal{L}_z} (-u)^{-k_\ell} \prod_{v \in \mathcal{R}_z} (v+1)^{-a_\ell - k_\ell} (pv+1)^{t_\ell - k_\ell},$$
$$F_\ell(w) := w^{k_\ell} (w+1)^{-a_\ell - k_\ell} (1+pw)^{t_\ell - k_\ell}.$$

4.3 Transition probability

In this section we give an explicit integral formula for the transition probability of discrete time parallel TASEP in the configuration space $\mathcal{X}_N^{(L)}$. This is the starting point for deriving the finite-time joint distribution formulas.

Proposition 4.3.1. *Given particle configurations $\vec{x} = (x_1, \cdots, x_N)$, $\vec{y} = (y_1, \cdots, y_N)$ in $\mathcal{X}_N^{(L)}$. Let $P_t(\vec{y} \to \vec{x})$ be the transition probability of observing configuration $\vec{x}$ at time t under the discrete time periodic TASEP dynamics with initial configuration $\vec{y}$. With the convention $x_0 := x_N + L$ we have*

$$P_t(\vec{y} \to \vec{x}) = \prod_{i=1}^{N} (1 - p\mathbf{1}_{x_{i-1} - x_i = 1}) \oint_\Gamma \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det \left[\sum_{w \in \mathcal{S}_z} F_{i,j}(w; \vec{x}, \vec{y}, t) J(w) \right]_{i,j=1}^{N}. \quad (4.19)$$

Here Γ is any simple closed contour with 0 inside and $\mathcal{S}_z$ consists of all the roots of *the degree L polynomial* $q_z(w) := w^N(w+1)^{L-N} - z^L(1+pw)^N$*, i.e.,*

$$\mathcal{S}_z := \{w \in \mathbb{C} : w^N(w+1)^{L-N} - z^L(1+pw)^N = 0\}. \tag{4.20}$$

The functions $F_{i,j}(w;t)$ *and* $J(w)$ *are given by*

$$F_{i,j}(w;\vec{x},\vec{y},t) = w^{j-i}(w+1)^{-x_{N-i+1}+y_{N-j+1}+i-j-1}(1+pw)^{t+i-j}, \quad 1 \le i,j \le N, \tag{4.21}$$

and

$$J(w) := \frac{w(w+1)(1+pw)}{N+Lw+p(L-N)w^2}. \tag{4.22}$$

Remark 4.3.2. We remark that a different formula for the transition probability of discrete time parallel TASEP on a ring was obtained in [91]. The key difference is the formula in [91] is expressed as an infinite sum of determinants while our formula is a single contour integral of determinants. The main reason for this is that in [91] the authors do not distinguish particle configurations differing by a translation of an integer multiple of the period, so in our language their transition probability really is

$$P_t([\vec{y}] \to [\vec{x}]) = \sum_{k\in\mathbb{Z}} P_t(\vec{y} \to \vec{x} + (kL, kL, \cdots, kL)).$$

We believe our formula is simpler and more suitable for deriving finite-time joint distributions.

Remark 4.3.3. If we take the continuous time limit by setting $p = \epsilon$, $t = T/\epsilon$ and send $\epsilon \to 0$, the dynamics then becomes continuous time periodic TASEP considered in [8] and our formula (4.19) reduces to equation (5.4) in [8] for the transition probability of continuous time periodic TASEP up to an index reversing.

We first list a few elementary properties of the transition probability formula (4.19) before discussing the proof of Proposition 4.3.1.

Proposition 4.3.4 (Properties of the transition probability formula)**.** *The right hand side of* (4.19) *satisfies the following properties:*

(i) The right hand side of (4.19) *can also be written as:*

$$\prod_{i=1}^{N}(1-p\mathbf{1}_{x_{i-1}-x_i=1})\oint_{\Gamma}\frac{\mathrm{d}z}{2\pi\mathrm{i}z}\det\left[\oint_{\Gamma_{\mathcal{S}_z}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}F_{i,j}(w;\vec{x},\vec{y},t)\frac{\hat{q}_z(w)+z^L}{\hat{q}_z(w)}\right]_{i,j=1}^{N}, \quad (4.23)$$

where $\hat{q}_z(w) = q_z(w)(1+pw)^{-N} = w^N(1+pw)^{-N}(1+w)^{L-N} - z^L$ *has the same roots as* $q_z(w)$ *for any* $|z| > 0$ *and* $\Gamma_{\mathcal{S}_z}$ *is any simple closed contour with all the roots in* $\mathcal{S}_z$ *inside and* -1 *and* $-1/p$ *outside.*

(ii) The outer integral with respect to z *in* (4.19) *(and also* (4.23)*) does not depend on the contour* Γ*.*

(iii) Assume further that $L \geq x_1 - y_N + 2$*. Then the right-hand side of* (4.19) *can be further written as*

$$\prod_{i=1}^{N}(1-p\mathbf{1}_{x_{i-1}-x_i=1})\det\left[\oint_{\Gamma_{0,-1}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}F_{i,j}(w;\vec{x},\vec{y},t)\right]_{i,j=1}^{N}, \quad (4.24)$$

where $\Gamma_{0,-1}$ *is any simple closed contour enclosing* 0 *and* -1 *as the only possible poles for the integrand. Note that* (4.24) *agrees with the transition probability for discrete time parallel TASEP on* $\mathbb{Z}$*, see for example equation (3.21) of [19].*

(iv) The right-hand side of (4.19) *is invariant under cyclic translation. Namely, for any fixed* $1 \leq k \leq N$*, set* $\vec{x}' := (x_k, x_{k+1}, \cdots, x_N, x_1 - L, x_2 - L, \cdots, x_{k-1} - L)$ *and* $\vec{y}' := (y_k, y_{k+1}, \cdots, y_N, y_1 - L, y_2 - L, \cdots, y_{k-1} - L)$*. Then the right-hand side of* (4.19) *is invariant if we replace* $\vec{x}$ *and* $\vec{y}$ *by* $\vec{x}'$ *and* $\vec{y}'$*.*

Proof. (i) It is easy to check that $J(w) = \frac{\hat{q}_z(w)+z^L}{\frac{\mathrm{d}}{\mathrm{d}w}\hat{q}_z(w)}$. Hence by the residue theorem we have

$$\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w) = \oint_{\Gamma_{\mathcal{R}_z}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},t)\frac{\hat{q}_z(w)+z^L}{\hat{q}_z(w)}.$$

Note that $F_{i,j}(w,\vec{x},\vec{y},t)\frac{\hat{q}_z(w)+z^L}{\hat{q}_z(w)}$ is analytic at $w=0$ for any $1\le i,j\le N$ and the only possible poles besides $\mathcal{S}_z$ are $w=-1$ and $w=-1/p$.

(ii) Choose $R>0$ large enough and $\epsilon>0$ small enough so that all the roots in $\mathcal{S}_z$ are inside the region $\{\epsilon<|w+1|<R\}\backslash\{|1+pw|\le\epsilon\}$. Then

$$\begin{aligned}&\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\\ &= \oint_{|w+1|=R} - \oint_{|w+1|=\epsilon} - \oint_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},t)\frac{\hat{q}_z(w)+z^L}{\hat{q}_z(w)}.\end{aligned} \tag{4.25}$$

Since R and ϵ can be arbitrarily large or small, the right hand side of (4.25) as a function in z is analytic for any $|z|>0$. Hence the integral with respect to z in (4.19) is independent of Γ since the integrand has an analytic continuation to $\{|z|>0\}$.

(iii) For $L>x_1-y_N+2$, we have $-x_{N-i+1}+y_{N-j+1}+i-j+L-N-1\ge 0$ for all $1\le i,j\le N$. Hence for any $|z|>0$, the integrand $F_{i,j}(w;\vec{x},\vec{y},t)\frac{\hat{q}_z(w)+z^L}{\hat{q}_z(w)}$ in (4.25) is analytic at $w=-1$, which implies

$$\sum_{w\in\mathcal{S}_z} F_{i,j}(w)J(w) = \oint_{|w+1|=R} - \oint_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w)\frac{\hat{q}_z(w)+z^L}{\hat{q}_z(w)}. \tag{4.26}$$

Now for fixed R large enough and ϵ small enough, the right-hand side of (4.26) is an analytic function in z for $|z|$ sufficiently small such that all the roots of $\hat{q}_z$ are in the region $\{|w+1|\le R\}\backslash\{|pw+1|\le\epsilon\}$. Now by the residue theorem the outer contour integral with respect to z in (4.23) equals the integrand evaluated at $z=0$, which equals

$$\det\left[\oint_{|w+1|=R} - \oint_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},t)\right]_{i,j=1}^{N}.$$

(iv) We remark that this property can be easily understood if we use the probabilistic interpretation since $\vec{x}$ and $\vec{x}'$ (and also $\vec{y}$ and $\vec{y}'$) actually represent the same particle configuration on $\mathbb{Z}$ (we just use particles in different period as representatives), hence the transition probability between $\vec{y}$ and $\vec{x}$ and $\vec{y}'$ and $\vec{x}'$ should be the same. Here however we can not directly use this since we have not proven (4.19). In fact, we will need this fact in our proof of (4.19) hence we give an independent algebraic proof here. It suffices to assume $k=2$ and hence $\vec{x}' = (x_2,\cdots,x_N,x_1-L)$, $\vec{y}' = (y_2,\cdots,y_N,y_1-L)$. Clearly we have $\prod_{i=1}^N(1-p\mathbf{1}_{x_{i-1}-x_i}=1) = \prod_{i=1}^N(1-p\mathbf{1}_{x'_{i-1}-x'_i}=1)$. So it suffices to show

$$\det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\right]_{i,j=1}^N = \det\left[\sum_{w\in\mathcal{S}_z} \tilde{F}_{i,j}(w;\vec{x},\vec{y},t)J(w)\right]_{i,j=1}^N,$$

where $\tilde{F}_{i,j}(w;\vec{x},\vec{y},t) = w^{j-i}(w+1)^{-x'_{N-i+1}+y'_{N-j+1}+i-j-1}(1+pw)^{t+i-j}$. By multilinearity we have

$$\begin{aligned}\det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\right]_{i,j=1}^N &= \sum_{w_1,\cdots,w_N\in\mathcal{S}_z} \det\left[F_{i,j}(w_i;\vec{x},\vec{y},t)J(w_i)\right]_{i,j=1}^N \\ = \sum_{\substack{w_i\in\mathcal{S}_z\\1\le i\le N}}\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\prod_{i=1}^N w_i^{\sigma(i)-i}(w_i+1)^{-x_{N-i+1}+y_{N-\sigma(i)+1}+i-\sigma(i)-1}(1+pw_i)^{t+i-\sigma(i)} \\ = \sum_{\substack{w_i\in\mathcal{S}_z\\1\le i\le N}}\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\prod_{i=1}^N w_i^{\sigma(i)-i}(w_i+1)^{-x'_{N-i}+y'_{N-\sigma(i)}+i-\sigma(i)-1}(1+pw_i)^{t+i-\sigma(i)}.\end{aligned}$$

Here again $x'_0 = x'_N + L = x_1 - L + L = x_1$. Now we fix $\tau = (N\cdots 21)\in S_N$

and set $\tilde{w}_i := w_{\tau(i)}$ for $1 \le i \le N$ and $\tilde{\sigma} := \tau^{-1}\sigma\tau$. Then the last line in the above equation equals

$$\sum_{\tilde{w}_1,\cdots,\tilde{w}_N \in \mathcal{S}_z} \sum_{\tilde{\sigma} \in S_N} \operatorname{sgn}(\tilde{\sigma}) \prod_{i=1}^{N} \tilde{w}_i^{\tau\tilde{\sigma}(i)-\tau(i)} (\tilde{w}_i+1)^{-x'_{N-\tau(i)}+y'_{N-\tau\tilde{\sigma}(i)}+\tau(i)-\tau\tilde{\sigma}(i)-1} (1+p\tilde{w}_i)^{t+\tau(i)-\tau\tilde{\sigma}(i)}. \tag{4.27}$$

We claim that for any fixed $\tilde{\sigma} \in S_N$,

$$\begin{aligned}
&\prod_{i=1}^{N} \tilde{w}_i^{\tau\tilde{\sigma}(i)-\tau(i)} (\tilde{w}_i+1)^{-x'_{N-\tau(i)}+y'_{N-\tau\tilde{\sigma}(i)}+\tau(i)-\tau\tilde{\sigma}(i)-1} (1+p\tilde{w}_i)^{t+\tau(i)-\tau\tilde{\sigma}(i)} \\
&= \prod_{i=1}^{N} \tilde{w}_i^{\tilde{\sigma}(i)-i} (\tilde{w}_i+1)^{-x'_{N-i+1}+y'_{N-\tilde{\sigma}(i)+1}+i-\tilde{\sigma}(i)-1} (1+p\tilde{w}_i)^{t+i-\tilde{\sigma}(i)}.
\end{aligned}$$

This then implies

$$\begin{aligned}
&\det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\right]_{i,j=1}^{N} \\
&= \sum_{\substack{\tilde{w}_i \in \mathcal{S}_z \\ 1\le i\le N}} \sum_{\tilde{\sigma}\in S_N} \operatorname{sgn}(\tilde{\sigma}) \prod_{i=1}^{N} \tilde{w}_i^{\tilde{\sigma}(i)-i} (\tilde{w}_i+1)^{-x'_{N-i+1}+y'_{N-\tilde{\sigma}(i)+1}+i-\tilde{\sigma}(i)-1} (1+p\tilde{w}_i)^{t+i-\tilde{\sigma}(i)} \\
&= \det\left[\sum_{w\in\mathcal{S}_z} \tilde{F}_{i,j}(w;\vec{x},\vec{y},t)J(w)\right]_{i,j=1}^{N}.
\end{aligned}$$

To see the claim note that for $i \neq 1$ we have $\tau(i) = i-1$ and for $\tilde{\sigma}(i) \neq 1$ we have $\tau\tilde{\sigma}(i) = \tilde{\sigma}(i) - 1$. Hence

$$\begin{aligned}
&\tilde{w}_i^{\tau\tilde{\sigma}(i)-\tau(i)} (\tilde{w}_i+1)^{-x'_{N-\tau(i)}+y'_{N-\tau\tilde{\sigma}(i)}+\tau(i)-\tau\tilde{\sigma}(i)-1} (1+p\tilde{w}_i)^{t+\tau(i)-\tau\tilde{\sigma}(i)} \\
&= \tilde{w}_i^{\tilde{\sigma}(i)-i} (\tilde{w}_i+1)^{-x'_{N-i+1}+y'_{N-\tilde{\sigma}(i)+1}+i-\tilde{\sigma}(i)-1} (1+p\tilde{w}_i)^{t+i-\tilde{\sigma}(i)},
\end{aligned}$$

for $i \neq 1$ and $\tilde{\sigma}(i) \neq 1$. For the other situations we split into two cases:

Case 1: $\tilde{\sigma}(1) \neq 1$. Then we have

$$\begin{aligned}
&\tilde{w}_1^{\tau\tilde{\sigma}(1)-\tau(1)}(\tilde{w}_1+1)^{-x'_{N-\tau(1)}+y'_{N-\tau\tilde{\sigma}(1)}+\tau(1)-\tau\tilde{\sigma}(1)-1}(1+p\tilde{w}_1)^{t+\tau(1)-\tau\tilde{\sigma}(1)} \\
&= \tilde{w}_1^{\tilde{\sigma}(1)-1}(\tilde{w}_1+1)^{-x'_N+y'_{N-\tilde{\sigma}(1)+1}-\tilde{\sigma}(1)}(1+p\tilde{w}_i)^{t+1-\tilde{\sigma}(1)} \cdot \frac{(1+p\tilde{w}_1)^N}{(\tilde{w}_1+1)^{L-N}\tilde{w}_1^N} \\
&= \tilde{w}_1^{\tilde{\sigma}(1)-1}(\tilde{w}_1+1)^{-x'_N+y'_{N-\tilde{\sigma}(1)+1}-\tilde{\sigma}(1)}(1+p\tilde{w}_i)^{t+1-\tilde{\sigma}(1)} \cdot z^{-L}.
\end{aligned}$$

Similarly if we set $\tilde{\sigma}^{-1}(1) = j$, then the term involving $\tilde{w}_j$ in (4.27) equals

$$\begin{aligned}
&\tilde{w}_j^{\tau(1)-\tau(j)}(\tilde{w}_j+1)^{-x'_{N-\tau(j)}+y'_{N-\tau(1)}+\tau(j)-\tau(1)-1}(1+p\tilde{w}_j)^{t+\tau(j)-\tau(1)} \\
&= \tilde{w}_j^{\tilde{\sigma}(j)-j}(\tilde{w}_j+1)^{-x'_{N-j+1}+y'_{N-\tilde{\sigma}(j)+1}-\tilde{\sigma}(j)+j-1}(1+p\tilde{w}_j)^{t+j-\tilde{\sigma}(j)} \cdot \frac{(\tilde{w}_j+1)^{L-N}\tilde{w}_j^N}{(1+p\tilde{w}_j)^N} \\
&= \tilde{w}_j^{\tilde{\sigma}(j)-j}(\tilde{w}_j+1)^{-x'_{N-j+1}+y'_{N-\tilde{\sigma}(j)+1}-\tilde{\sigma}(j)+j-1}(1+p\tilde{w}_j)^{t+j-\tilde{\sigma}(j)} \cdot z^{L}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\prod_{i=1}^N \tilde{w}_i^{\tau\tilde{\sigma}(i)-\tau(i)}(\tilde{w}_i+1)^{-x'_{N-\tau(i)}+y'_{N-\tau\tilde{\sigma}(i)}+\tau(i)-\tau\tilde{\sigma}(i)-1}(1+p\tilde{w}_i)^{t+\tau(i)-\tau\tilde{\sigma}(i)} \\
&= z^L \cdot z^{-L} \cdot \prod_{i=1}^N \tilde{w}_i^{\tilde{\sigma}(i)-i}(\tilde{w}_i+1)^{-x'_{N-i+1}+y'_{N-\tilde{\sigma}(i)+1}+i-\tilde{\sigma}(i)-1}(1+p\tilde{w}_i)^{t+i-\tilde{\sigma}(i)} \\
&= \prod_{i=1}^N \tilde{w}_i^{\tilde{\sigma}(i)-i}(\tilde{w}_i+1)^{-x'_{N-i+1}+y'_{N-\tilde{\sigma}(i)+1}+i-\tilde{\sigma}(i)-1}(1+p\tilde{w}_i)^{t+i-\tilde{\sigma}(i)}.
\end{aligned}$$

Case 2: $\tilde{\sigma}(1) = 1$. Then we have

$$\begin{aligned}
&\tilde{w}_1^{\tau\tilde{\sigma}(1)-\tau(1)}(\tilde{w}_1+1)^{-x'_{N-\tau(1)}+y'_{N-\tau\tilde{\sigma}(1)}+\tau(1)-\tau\tilde{\sigma}(1)-1}(1+p\tilde{w}_1)^{t+\tau(1)-\tau\tilde{\sigma}(1)} \\
&= \tilde{w}_1^{\tilde{\sigma}(1)-1}(\tilde{w}_1+1)^{-x'_N+y'_{N-\tilde{\sigma}(1)+1}-\tilde{\sigma}(1)}(1+p\tilde{w}_i)^{t+1-\tilde{\sigma}(1)},
\end{aligned}$$

so the claim follows.

□

Now we turn to the proof of formula (4.19). The proof basically follows the idea of [19, 8] and is very similar as the proof of Proposition 3.3.1. The main extra difficulty here is due to parallel update rule, the stationary distribution for the dynamics is non-uniform. In fact one can check $\mu(\vec{x}) \propto \prod_{i=1}^{N}(1 - p\mathbf{1}_{x_{i-1}-x_i=1})$ is the stationary distribution. As a result it turns out that $\frac{P_t(\vec{y}\to\vec{x})}{\prod_{i=1}^{N}(1-p\mathbf{1}_{x_{i-1}-x_i=1})}$ satisfies a relatively simpler dynamics than $P_t(\vec{y} \to \vec{x})$. In fact we have

Lemma 4.3.5. *Given $\vec{x}, \vec{y} \in \mathbb{Z}^N$. Let $G(\vec{x}, t; \vec{y}, 0)$ be the (unique) solution of the following free evolution equation*

$$G(\vec{x}, t+1; \vec{y}, 0) = \sum_{b_1,\cdots,b_N\in\{0,1\}} \prod_{i=1}^{N} p^{b_i}(1-p)^{1-b_i} G(\vec{x} - \vec{b}, t; \vec{y}, 0), \tag{4.28}$$

together with the boundary conditions: for $1 \le i \le N$

$$\begin{aligned}(1-p)\left[G(\cdots, x_i+1, x_i, x_{i+1}, \cdots, t; \vec{y}, 0) - G(\cdots, x_i, x_i, x_{i+1}, \cdots, t; \vec{y}, 0)\right] \\ = p\left[G(\cdots, x_i, x_i-1, \cdots, t; \vec{y}, 0) - G(\cdots, x_i+1, x_i-1, \cdots, t; \vec{y}, 0)\right],\end{aligned} \tag{4.29}$$

and initial condition:

$$G(\vec{x}, 0; \vec{y}, 0) = \frac{\mathbf{1}_{\vec{x}=\vec{y}}}{\prod_{i=1}^{N}(1 - p\mathbf{1}_{x_{i-1}-x_1=1})}. \tag{4.30}$$

Then

$$G(\vec{x}, t; \vec{y}, 0) = \frac{P_t(\vec{y} \to \vec{x})}{\prod_{i=1}^{N}(1 - p\mathbf{1}_{x_{i-1}-x_1=1})}, \quad \text{for all } \vec{x}, \vec{y} \in \mathcal{X}_N^{(L)}. \tag{4.31}$$

Note that we used the convention $x_0 = x_N + L$ so when $i = 1$, (4.29) should be interpreted as

$$\begin{aligned}(1-p)\left[G(x_1, \cdots, x_1 - L + 1, t; \vec{y}, 0) - G(x_1, \cdots, x_1 - L, t; \vec{y}, 0)\right] \\ = p\left[G(x_1 - 1, \cdots, x_1 - L, t; \vec{y}, 0) - G(x_1 - 1, \cdots, x_1 - L + 1, t; \vec{y}, 0)\right].\end{aligned} \tag{4.32}$$

Proof. Set

$$H(\vec{x},t;\vec{y},0) := \frac{P_t(\vec{y}\to\vec{x})}{\prod_{i=1}^{N}(1-p\mathbf{1}_{x_{i-1}-x_i=1})}. \tag{4.33}$$

It suffices to show that for $\vec{x},\vec{y}\in\mathcal{X}_N^{(L)}$, $H(\vec{x},t;\vec{y},0)$ and $G(\vec{x},t;\vec{y},0)$ satisfy the same evolution equation to conclude that $H(\vec{x},t;\vec{y},0)=G(\vec{x},t;\vec{y},0)$ for all $\vec{x},\vec{y}\in\mathcal{X}_N^{(L)}$ since they have the same initial condition. To better describe the evolution equation for $H(\vec{x},t;\vec{y},0)$ it is convenient to introduce the notion of clusters of a particle configuration $\vec{x}$. Given $\vec{x}=(x_1,\cdots,x_N)\in\mathcal{X}_N^{(L)}$, for $1\le i\le N$ and $1\le k\le N$ we say $[x_i,x_{i-1},\cdots,x_{i-k+1}]$ is a cluster of size k of $\vec{x}$ if

$$x_{i+1}+1<x_i=x_{i-1}-1=\cdots=x_{i-k+1}-k+1<x_{i-k}-k,$$

namely particle i through $i-k+1$ are right next to each other while there are at least one empty site to the left of x_i and right of x_{i-k+1}. Here we abuse notation by allowing the index to exceed $\{1,\cdots,N\}$ and this should be understood with the convention $x_{i+kN}=x_i-kL$ for $1\le i\le N$ and $k\in\mathbb{Z}$. For convenience when $[x_i,x_{i-1},\cdots,x_1,x_0,\cdots x_{-j}]$ is a cluster for some $0\le j<N$, we will also say $[x_i,\cdots,x_1,x_N,\cdots,x_{N-j}]$ forms a cluster so that all the indices appearing will be between 1 and N.

Let $\mathcal{N}_c(\vec{x})$ be the number of clusters in configuration $\vec{x}$ and let x_{c_j}, $1\le j\le\mathcal{N}_c(\vec{x})$ be the locations of the left-most particles in each cluster. Then it is straightforward to check that $H(\vec{x},t;\vec{y},0)$ satisfies

$$H(\vec{x},t+1;\vec{y},0)=\sum_{\substack{b_{c_j}\in\{0,1\},\\ 1\le j\le\mathcal{N}_c(\vec{x})}}\prod_{j=1}^{\mathcal{N}_c(\vec{x})}p^{b_{c_j}}(1-p)^{1-b_{c_j}}H(\vec{x}-\sum_{j=1}^{\mathcal{N}_c(\vec{x})}b_{c_j}\vec{e}_{c_j},t;\vec{y},0), \tag{4.34}$$

where $\vec{e}_{c_j}\in\mathbb{Z}^N$ has 1 in the c_j-th coordinate and 0 in the other coordinates.

We claim that for $\vec{x},\vec{y}\in\mathcal{X}_N^{(L)}$, (4.34) and (4.28) takes the same form (with H

replaced by G) provided $G(\vec{x},t;\vec{y},0)$ satisfies boundary conditions (4.29). Due to the sum of products form of (4.34) and (4.28) it suffices to check

$$\sum_{b_i\in\{0,1\}} p^{b_i}(1-p)^{1-b_i}G(\vec{x}-b_i\vec{e}_i,t;\vec{y},0)$$
$$= \sum_{\substack{b_j\in\{0,1\}\\ j=i-m+1,\cdots,i}} \prod_j p^{b_j}(1-p)^{1-b_j}G(\vec{x}-\sum_j b_j\vec{e}_j,t;\vec{y},0), \tag{4.35}$$

for $\vec{x},\vec{y}\in\mathcal{X}_N^{(L)}$ and a single cluster $[x_i,x_{i-1},\cdots,x_{i-m+1}]$ of size m. We will show the stronger statement: (4.35) actually holds for any $\vec{x}=(x_1,\cdots,x_N)\in\mathbb{Z}^N$ with $x_i,x_{i-1},\cdots,x_{i-m+1}$ merely satisfying $x_i=x_{i-1}-1=\cdots=x_{i-m+1}-m+1$. We do not require empty sites at the left and right ends so they may not form a cluster.

We prove this by induction on m. For $m=1$ this is trivial. Assume the claim is true for any clusters of size $\leq m$. Now let $\vec{x}\in\mathbb{Z}^N$ with $x_i=x_{i-1}-1=\cdots=x_{i-m}-m$ for some $1\leq i\leq N$. Here without loss of generality we can assume $i-m\geq 1$, otherwise replace j by $j+N$ for indices $j\leq 0$. Then

$$\sum_{\substack{b_j\in\{0,1\}\\ j=i-m,\cdots,i}} \prod_j p^{b_j}(1-p)^{1-b_j}G(\vec{x}-\sum_j b_j\vec{e}_j,t)$$
$$= \sum_{b_i\in\{0,1\}} p^{b_i}(1-p)^{1-b_i}\cdot\Big(\sum_{\substack{b_j\in\{0,1\}\\ j=i-m,\cdots,i-1}} \prod_{j=i-m}^{i-1} p^{b_j}(1-p)^{1-b_j}G(\vec{x}-\sum_{j=i-m}^{i} b_j\vec{e}_j,t)\Big)$$
$$= \sum_{b_i\in\{0,1\}} p^{b_i}(1-p)^{1-b_i}\cdot\Big(\sum_{b_{i-1}\in\{0,1\}} p^{b_{i-1}}(1-p)^{1-b_{i-1}}G(\vec{x}-b_i\vec{e}_i-b_{i-1}\vec{e}_{i-1},t)\Big)$$
$$= (1-p)^2G(\vec{x},t)+p(1-p)G(\vec{x}-\vec{e}_i,t)$$
$$+\,p(1-p)G(\vec{x}-\vec{e}_{i-1},t)+p^2G(\vec{x}-\vec{e}_i-\vec{e}_{i-1},t), \tag{4.36}$$

where we used induction hypothesis in the second equality of (4.36) for the sum inside the brackets and we suppress the dependence on $\vec{y}$ for $G(\vec{x},t;\vec{y},0)$ to save space. Now by the boundary conditions (4.29)(possibly (4.32)) we have

$$(1-p)[G(\vec{x},t)-G(\vec{x}-\vec{e}_{i-1},t)] = p[G(\vec{x}-\vec{e}_i-\vec{e}_{i-1},t)-G(\vec{x}-\vec{e}_i,t)]. \tag{4.37}$$

Inserting (4.37) into (4.36) we see the last line of (4.36) simplifies to

$$(1-p)G(\vec{x},t)+pG(\vec{x}-\vec{e}_i,t), \tag{4.38}$$

which is precisely the left hand side of equation (4.35) and this completes the proof of Lemma 4.3.5. □

Proof of Proposition 4.3.1. By Lemma 4.3.5 it suffices to prove

$$G(\vec{x},t;\vec{y},0) = \oint_\Gamma \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\right]_{i,j=1}^N. \tag{4.39}$$

To see this, we check that the right-hand side of (4.39) satisfies free evolution equation (4.28), boundary conditions (4.29) and initial condition (4.30).

For the free evolution equation (4.28) note first that it is straightforward to check

$$\begin{aligned}&\sum_{b_{N-i+1}=0}^{1} p^{b_{N-i+1}}(1-p)^{1-b_{N-i+1}}F_{i,j}(w;\vec{x}-\vec{b},\vec{y};t)\\ &= F_{i,j}(w;\vec{x}-\vec{b}+b_{N-i+1}\vec{e}_{N-i+1},\vec{y};t+1).\end{aligned}$$

Hence by multi-linearity of determinants we have

$$\begin{aligned}
&\sum_{\substack{b_k\in\{0,1\}\\ k=1,\cdots,N}} \prod_{k=1}^{N} p^{b_k}(1-p)^{1-b_k} \det\left[\sum_{w\in\mathcal{R}_z} F_{i,j}(w;\vec{x}-\vec{b},\vec{y},t)J(w)\right]_{i,j=1}^{N} \\
&= \det\left[\sum_{w\in\mathcal{S}_z}\sum_{b_{N-i+1}=0}^{1} p^{b_{N-i+1}}(1-p)^{1-b_{N-i+1}} F_{i,j}(w;\vec{x}-\vec{b},\vec{y},t)J(w)\right]_{i,j=1}^{N} \\
&= \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x}-\vec{b}+b_{N-i+1}\vec{e}_{N-i+1},\vec{y},t+1)J(w)\right]_{i,j=1}^{N} \\
&= \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t+1)J(w)\right]_{i,j=1}^{N}.
\end{aligned}$$

Here $\vec{e}_i \in \mathbb{Z}^N$ has 1 in the i-th entry and 0 for the other entries. In the last equality we used the fact that $F_{i,j}(w;\vec{x},\vec{y},t)$ only depends on the $N-i+1$-th entry of $\vec{x}$. Now (4.28) follows from linearity of integration.

Next we check the boundary conditions (4.29). Given $1 \le k \le N$, let $\vec{x} = (x_1,\cdots,x_N) \in \mathbb{Z}^N$ be satisfying $x_{k-1} = x_k + 1$. Note that when $k=1$ this means $x_1 = x_N + L - 1$. Then the boundary conditions (4.29) can be expressed as

$$(1-p)[G(\vec{x},t) - G(\vec{x}-\vec{e}_{k-1},t)] = p[G(\vec{x}-\vec{e}_{k-1}-\vec{e}_k,t) - G(\vec{x}-\vec{e}_i,t)], \tag{4.40}$$

for $2 \le k \le N$ and

$$(1-p)[G(\vec{x},t) - G(\vec{x}-\vec{e}_N,t)] = p[G(\vec{x}-\vec{e}_N-\vec{e}_1,t) - G(\vec{x}-\vec{e}_1,t)], \tag{4.41}$$

for $k=1$. We prove (4.40) first. Note that for $2 \le k \le N$ with $x_{k-1} = x_k + 1$ we have

$$F_{i,j}(w;\vec{x},\vec{y},t) - F_{i,j}(w;\vec{x}-\vec{e}_{k-1},\vec{y},t) = \begin{cases} 0, & \text{if } i \ne N-k+2, \\ -w\cdot F_{i,j}(w;\vec{x},\vec{y},t), & \text{if } i = N-k+2. \end{cases}$$

Hence by multi-linearity we have

$$\det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\right]_{i,j=1}^{N} - \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x}-\vec{e}_{k-1},\vec{y},t)J(w)\right]_{i,j=1}^{N}$$
$$= \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\cdot(1-(1+w)\mathbf{1}_{i=N-k+2})\right]_{i,j=1}^{N}.$$

Similarly

$$\det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x}-\vec{e}_k,\vec{y},t)J(w)\right]_{i,j=1}^{N} - \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x}-\vec{e}_k-\vec{e}_{k-1},\vec{y},t)J(w)\right]_{i,j=1}^{N}$$
$$= \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x}-\vec{e}_k,\vec{y},t)J(w)\cdot(1-(1+w)\mathbf{1}_{i=N-k+2})\right]_{i,j=1}^{N}.$$

Now since $F_{i,j}(w;\vec{x}-\vec{e}_k,\vec{y},t) = F_{i,j}(w;\vec{x},\vec{y},t)$ for $i \neq N-k+1$ and

$$(1-p)F_{N-k+1,j}(w;\vec{x},\vec{y},t) + pF_{N-k+1,j}(w;\vec{x}-\vec{e}_k,\vec{y},t) = w\cdot F_{N-k+2,j}(w;\vec{x},\vec{y},t),$$

we have

$$\begin{aligned}
&(1-p)\cdot\det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)\cdot(1-(1+w)\mathbf{1}_{i=N-k+2})\right]_{i,j=1}^{N}\\
&+p\cdot\det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x}-\vec{e}_k,\vec{y},t)J(w)\cdot(1-(1+w)\mathbf{1}_{i=N-k+2})\right]_{i,j=1}^{N}\\
&=\det\left[M_{i,j}^{(k)}(\vec{x},\vec{y},t)\right]_{i,j=1}^{N} = 0,
\end{aligned}\tag{4.42}$$

where $M_{i,j}^{(k)}(\vec{x},\vec{y},t) = \sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)$ for $i \neq N-k+1, N-k+2$ and $M_{N-k+1,j}^{(k)}(\vec{x},\vec{y},t) = -M_{N-k+2,j}^{(k)}(\vec{x},\vec{y},t) = \sum_{w\in\mathcal{S}_z} F_{N-k+2,j}(w;\vec{x},\vec{y},t)J(w)\cdot w$. These two rows are proportional so the determinant is 0. Now (4.40) follows from linearity of the integral.

The proof of (4.41) is similar. The only thing changes is when $k=1$, we have

$M_{i,j}^{(1)}(\vec{x},\vec{y},t) = \sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},t)J(w)$ for $i \neq 1, N$ while

$$M_{N,j}^{(1)}(\vec{x},\vec{y},t) = \sum_{w\in\mathcal{S}_z} \frac{(1+pw)^N}{w^N(w+1)^{L-N}} F_{1,j}(w;\vec{x},\vec{y},t)J(w)\cdot w = -z^{-L}M_{1,j}^{(1)}(\vec{x},\vec{y},t).$$

Hence $\det[M_{i,j}^{(1)}(\vec{x},\vec{y},t)] = 0$ since row 1 and N are proportional. Note that in the last equality above we used the fact that $w \in \mathcal{S}_z$.

Finally we check the initial condition (4.30). We need to show

$$\oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},0)J(w)\right]_{i,j=1}^{N} = \frac{\mathbf{1}_{\vec{x}=\vec{y}}}{\prod_{i=1}^{N}(1-p\mathbf{1}_{x_{i-1}-x_1=1})} \tag{4.43}$$

Thanks to the cyclic-shift invariance of both sides of (4.43) (see (iv) of Proposition 4.3.4) we can assume without loss of generality that $\vec{x} = (x_1,\cdots,x_N) \in \mathcal{X}_N^{(L)}$ satisfies $x_1 < x_N + L - 1$. In fact since $N < L$ there is at least one $1 \le i \le N$ such that $x_i < x_{i-1} - 1$ and we can replace $\vec{x}$ and $\vec{y}$ by $\vec{x}' := (x_i, x_{i+1}, \cdots, x_N, x_1 - L, \cdots, x_{i-1} - L)$ and $\vec{y}' := (y_i, y_{i+1}, \cdots, y_N, y_1 - L, \cdots, y_{i-1} - L)$ if necessary since the two sides of (4.43) remain the same. By (4.25) we have

$$\begin{aligned}
&\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},0)J(w) = \oint_{|w+1|=R} - \oint_{|w+1|=\epsilon} - \oint_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0)\frac{\hat{q}_z(w)+z^L}{\hat{q}_z(w)} \\
&= \oint_{|w+1|=R} - \oint_{|w+1|=\epsilon} - \oint_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{w^{j-i+N}(w+1)^{-x_{N-i+1}+y_{N-j+1}+i-j-1+L-N}(1+pw)^{i-j}}{w^N(w+1)^{L-N} - z^L(1+pw)^N} \\
&:= I_1(i,j) - I_2(i,j) - I_3(i,j),
\end{aligned}$$

where $I_1(i,j), I_2(i,j), I_3(i,j)$ are the integrals over the three contours, respectively. Here we recall that R and ϵ are large(small) enough so that $\mathcal{S}_z$ is contained in the region $\{\epsilon < |w+1| < R\} \backslash \{|1+pw| \leq \epsilon\}$. For $I_3(i,j)$ note that

$$I_3(i,j) = \oint\limits_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0) + z^L \oint\limits_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{F_{i,j}(w;\vec{x},\vec{y},0)(1+pw)^N}{q_z(w)}$$
$$= \oint\limits_{|pw+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0),$$

since $F_{i,j}(w;\vec{x},\vec{y},0)(1+pw)^N$ is analytic at $w = -1/p$ for all $1 \leq i,j \leq N$ and $q_z(w) = w^N(w+1)^{L-N} - z^L(1+pw)^N = (1+pw)^N \hat{q}_z(w)$ is nonzero at $w = -1/p$. For the other parts we write

$$I_1(i,j) = \oint\limits_{|w+1|=R} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0) + z^L \oint\limits_{|w+1|=R} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{F_{i,j}(w;\vec{x},\vec{y},0)(1+pw)^N}{q_z(w)}$$
$$:= \oint\limits_{|w+1|=R} \frac{\mathrm{d}w}{2\pi\mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0) + z^L I_1'(i,j),$$

and

$$I_2(i,j) = z^{-L} \oint\limits_{|w+1|=\epsilon} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{w^{j-i+N}(w+1)^{-x_{N-i+1}+y_{N-j+1}+i-j-1+L-N}(1+pw)^{i-j-N}}{z^{-L}w^N(w+1)^{L-N}(1+pw)^{-N} - 1}$$
$$:= z^{-L} I_2'(i,j).$$

Depending on properties of integrands in I_1' and I_2' we split into two cases:

Case 1: $x_1 < y_1$. First note that for any $\vec{x},\vec{y} \in \mathcal{X}_N^{(L)}$ we have for all $1 \leq i,j \leq N$

$$x_1 - L + i \leq x_{N-i+1} \leq x_1 - N + i, \quad \text{and} \quad y_1 - L + j \leq y_{N-j+1} \leq y_1 - N + j. \quad (4.44)$$

Now if $x_1 < y_1$, then for any $1 \leq i,j \leq N$ we have

$$-x_{N-i+1} + y_{N-j+1} + i - j - 1 + L - N \geq y_1 - x_1 - 1 \geq 0.$$

Hence the integrand of $I_2'(i,j)$ is analytic at $w=-1$ so $I_2'(i,j)=0$. Therefore in this case we have

$$\begin{aligned}
&\oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},0)J(w)\right]_{i,j=1}^{N} \\
&= \oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\oint_{\Gamma_{0,-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0) + z^L I_1'(i,j)\right]_{i,j=1}^{N} \\
&= \det\left[\oint_{\Gamma_{0,-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0)\right]_{i,j=1}^{N}.
\end{aligned} \tag{4.45}$$

Here in the last equality of (4.45) we take the outer integral contour Γ to be $|z|=r$ and let $r\to 0$. $\Gamma_{0,-1}$ is any simple closed contour with 0 and -1 inside and $-1/p$ outside.

Case 2: $x_1 \geq y_1$. Write

$$I_1'(i,j) = \oint_{|w+1|=R} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{w^{j-i-N}(w+1)^{-x_{N-i+1}+y_{N-j+1}+i-j-1-L+N}(1+pw)^{i-j+N}}{1-z^L w^{-N}(w+1)^{-L+N}(1+pw)^N}. \tag{4.46}$$

Again by (4.44) we have

$$\begin{aligned}
&-x_{N-i+1}+y_{N-j+1}+i-j-1-L+N \\
&\leq -x_1+L-i+y_1-N+j+i-j-1-L+N \leq -1.
\end{aligned} \tag{4.47}$$

We claim that the first inequality in (4.47) is strict and hence $-x_{N-i+1}+y_{N-j+1}+i-j-1-L+N\leq -2$. This is due to our original assumption that $x_N+L-1>x_1$ and hence $x_{N-i+1}>x_1-L+i$ for all $1\leq i\leq N$. Owing to this fact, the integrand in (4.46) is $O(R^{-2})$ since $w^{j-i-N}(1+pw)^{i-j+N}$ remains bounded. Hence $I_1'(i,j)\to 0$ as $R\to\infty$ but since it is independent of large enough R , we have $I_1'(i,j)=0$ for all R large enough. Now a similar argument as in (4.45) with Γ be large circle $|z|=r$

with $r \to \infty$ implies:

$$\begin{aligned}
&\oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\sum_{w\in\mathcal{S}_z} F_{i,j}(w;\vec{x},\vec{y},0)J(w)\right]_{i,j=1}^{N} \\
&= \oint_{\Gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \det\left[\oint_{\Gamma_{0,-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0) + z^{-L} I_2'(i,j)\right]_{i,j=1}^{N} \\
&= \det\left[\oint_{\Gamma_{0,-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0)\right]_{i,j=1}^{N}.
\end{aligned} \tag{4.48}$$

In conclusion we have reduced checking (4.43) to checking the following:

$$\det\left[\oint_{\Gamma_{0,-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} F_{i,j}(w;\vec{x},\vec{y},0)\right]_{i,j=1}^{N} = \frac{\mathbf{1}_{\vec{x}=\vec{y}}}{\prod_{i=1}^{N}(1-p\mathbf{1}_{x_{i-1}-x_1=1})}, \tag{4.49}$$

for any $\vec{x},\vec{y} \in \mathcal{X}_N^{(L)}$ with $x_1 < x_N + L - 1$. But this is precisely equation (3.30) of [19] which appears in checking the determinantal formula for the transition probability of discrete parallel TASEP on $\mathbb{Z}$ satisfies the proper initial condition. We will not repeat the proof here but just point out that due to assumption $x_1 < x_N + L - 1$ we have $1 - p\mathbf{1}_{x_0-x_1=1} = 1$ so (4.49) is really identical to equation (3.30) of [19]. □

4.4 Finite-time Multi-point joint distribution under general initial conditions

4.4.1 A Toeplitz-like determinant formula

In this section we derive a formula for the finite-time multi-point joint distributions for discrete time parallel periodic TASEP under arbitrary initial condition. The proof basically follows the strategy of [9] by performing a multiple sum of transition probabilities over suitable particle configurations. The main technical part is a Cauchy-type identity for summation of left and right eigenfunctions (see Proposition 4.5.4) which generalizes Proposition 3.4 of [9] and some new difficulties appear.

Theorem 4.4.1 (Multi-point joint distribution for discrete time parallel TASEP in $\mathcal{X}_N^{(L)}$)**.** *Let $\vec{y} \in \mathcal{X}_N^{(L)}$ and $\vec{x}(t) = (x_1(t), \cdots, x_N(t)) \in \mathcal{X}_N^{(L)}$ be particle configurations evolving according to the discrete time parallel TASEP in $\mathcal{X}_N^{(L)}$ at time t with initial configuration $\vec{x}(0) = \vec{y}$. Fix a positive integer m. Let $(k_1, t_1), \cdots, (k_m, t_m) \in \{1, \cdots, N\} \times \mathbb{N}$ be distinct with $0 \le t_1 \le \cdots \le t_m$. Let $a_i \in \mathbb{Z}$ for $1 \le i \le m$. Then*

$$\mathbb{P}_{\vec{y}}^{(L)}\left(\bigcap_{i=1}^{m}\{x_{k_i}(t_i) \ge a_i\}\right) = \oint \cdots \oint \frac{\mathrm{d}z_m}{2\pi \mathrm{i} z_m} \cdots \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1} \mathcal{C}^{(L)}(\vec{z}) \mathcal{D}_{\vec{y}}^{(L)}(\vec{z}), \tag{4.50}$$

where the contours for the integrals are nested circles $0 < |z_m| < \cdots < |z_1|$. Here $\vec{z} = (z_1, \cdots, z_m)$. The functions $\mathcal{C}^{(L)}(\vec{z})$ and $\mathcal{D}_{\vec{y}}^{(L)}(\vec{z})$ are defined by

$$\mathcal{C}^{(L)}(\vec{z}) = (-1)^{(N-k_m)(N-1)} z^{(N-k_1)L} \prod_{\ell=2}^{m} \left[z_\ell^{(k_{\ell-1}-k_\ell)L} \left(\left(\frac{z_\ell}{z_{\ell-1}} \right)^L - 1 \right)^{N-1} \right], \tag{4.51}$$

and

$$\mathcal{D}_{\vec{y}}^{(L)}(\vec{z}) \tag{4.52}$$

$$= \det \left[\sum_{\substack{w_\ell \in \mathcal{S}_{z_\ell} \\ \ell = 1, \cdots, m}} \frac{\left(\frac{1+pw_1}{w_1}\right)^{N-i+1} (1+w_1)^{y_{N-i+1}+N-i+1} \cdot w_m^{-j}}{\prod_{\ell=2}^{m}(w_\ell - w_{\ell-1})} \prod_{\ell=1}^{N} G_\ell(w_\ell) \right]_{i,j=1}^{N},$$

where for $1 \le \ell \le m$

$$G_\ell(w) := \frac{w(w+1)(1+pw)}{N + Lw + p(L-N)w^2} \cdot \frac{w^{k_\ell}(1+w)^{-a_\ell-k_\ell}(1+pw)^{t_\ell-k_\ell}}{w^{k_{\ell-1}}(1+w)^{-a_{\ell-1}-k_{\ell-1}}(1+pw)^{t_{\ell-1}-k_{\ell-1}}}. \tag{4.53}$$

Here $k_0 = t_0 = a_0 := 0$ and we suppress the dependence on a_i, k_i and t_i's in $\mathcal{C}^{(L)}(\vec{z})$

and $\mathcal{D}^{(L)}_{\vec{y}}(\vec{z})$.

Proof. The proof is almost identity to the Proof of Theorem 3.4.1 so we omit it. One needs slightly different summation identities since the eigenfunctions are different. See Section 4.5 below. □

4.4.2 Proof of Theorem 4.2.2

The finite time formula obtained in Theorem 4.4.1 contains a factor $\mathcal{D}^{(L)}_{\vec{y}}(\vec{z})$ inside the integrals which is a Toeplitz-like determinant and is hard to take large-time limits. The procedure of re-expressing it as a Fredholm determinant as in Theorem 4.2.2 is almost identical to the one in the proof of Theorem 3.4.1 so we omit the details. We first apply the Proposition 3.6.9 with

$$p_i(w) = w^{i-1}(1+pw)^{N-i}(1+w)^{y_{N-i+1}+N-i+1}, \quad q_j(w) = w^{N-i},$$

for $1 \le i, j \le N$. Theorem 4.2.2 is proved after some simplifications using the algebraic relations similar as in the proof of Theorem 3.4.1.

4.5 Summation identities of eigenfunctions

4.5.1 Summation over single eigenfunction

In this section we state and prove the summation identities used in computing multi-time joint distribution in Section 4.4. For convenience we recall the left and right eigenfunctions defined in (3.51) and (3.52).

Definition 4.5.1 (Left and right eigenfunctions)**.** *Given* $\vec{x} = (x_1, \cdots, x_N) \in \mathcal{X}_N^{(\infty)}$ *and* $p \in \mathbb{C}$, *we define the functions* $\Psi^{\ell}_{\vec{x}}(\vec{w})$ *and* $\Psi^{r}_{\vec{x}}(\vec{w})$ *for* $\vec{w} \in \mathbb{C}^N$ *as follows:*

$$\Psi^{\ell}_{\vec{x}}(\vec{w}) = \det\left[\left(\frac{w_i}{1+pw_i}\right)^{j-1}(1+w_i)^{x_{N-j+1}-j+1}\right]_{i,j=1}^{N}, \tag{4.54}$$

$$\Psi^r_{\vec{x}}(\vec{w}) = \prod_{i=1}^{N}(1 - p\mathbf{1}_{x_{i-1}-x_i=1}) \cdot \det\left[\left(\frac{w_i}{1+pw_i}\right)^{1-j}(1+w_i)^{-x_{N-j+1}+j-1}\right]_{i,j=1}^{N}. \quad (4.55)$$

We start with a summation identity of $\Psi^r_{\vec{x}}(\vec{w})$:

Proposition 4.5.2 (Summation over a single eigenfunction)**.** *Let $z \in \mathbb{C}$ be nonzero. Let $\Psi^r_{\vec{x}}(\vec{w})$ be as in (3.52) where $\vec{w} = (w_1, \cdots, w_N) \in (\mathcal{S}_z)^N$ such that $\prod_{j=1}^N |w_j + 1| > 1$. Then*

$$\sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N\geq 0\}} \Psi^r_{\vec{x}}(\vec{w}) = \prod_{i=1}^{N}(1+w_i) \cdot \det[w_i^{-j}]_{i,j=1}^N. \quad (4.56)$$

Proof. First we write

$$\begin{aligned}
\sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N\geq 0\}} \Psi^r_{\vec{x}}(\vec{w}) &= \sum_{a=0}^{\infty} \sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N=a\}} \Psi^r_{\vec{x}}(\vec{w}) \\
&= \sum_{a=0}^{\infty}\prod_{j=1}^{N}(1+w_j)^{-a} \sum_{\vec{x}'\in\mathcal{X}_N^{(L)}\cap\{x'_N=0\}} \Psi^r_{\vec{x}'}(\vec{w}),
\end{aligned}$$

where the summation over a converges absolutely for $\prod_{j=1}^N |1 + w_j| > 1$. Here $\vec{x}' = \vec{x} - (a, \cdots, a)$. Now we start with computing the summation over $\vec{x} \in \mathcal{X}_N^{(L)}$ with $x_N = 0$ (this is a finite sum so there is no convergence issue). For this we split the sum according the number of particles to the right of the N-th particle in the cluster containing the N-th particle for a given configuration $\vec{x}$:

$$\sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N=0\}} \Psi^r_{\vec{x}}(\vec{w}) = \sum_{k=0}^{N-1}\sum_{\vec{x}\in\mathcal{X}_N^{(L)}(k)} \Psi^r_{\vec{x}}(\vec{w}) := \sum_{k=0}^{N-1} S_k.$$

Where

$$\mathcal{X}_N^{(L)}(k) := \{\vec{x} \in \mathcal{X}_N^{(L)} : x_N = x_{N-1} - 1 = \cdots x_{N-k} - k = 0, x_{N-k-1} - k - 1 > 0\}.$$

To compute S_k we perform the sum in the order $k + 2 \leq x_{N-k-1} < x_{N-k-2} < \cdots <$

$x_1 \leq L-1$. Note that configurations $\vec{x} \in \mathcal{X}_N^{(L)}(k)$ takes the form $(x_1, \cdots, x_{N-k-1}, k, \cdots, 0)$ where $x_{N-k-1} > k+1$. Hence for $\vec{x} \in \mathcal{X}_N^{(L)}(k)$ we have

$$\Psi_{\vec{x}}^r(\vec{w}) = (1-p)^k \prod_{i=1}^{N-k-1} (1 - p\mathbf{1}_{x_{i-1}-x_i=1}) \det[R_{i,j}^{(k,k+1)}(\vec{w})]_{i,j=1}^N,$$

where

$$R_{i,j}^{(k,k+1)}(\vec{w}) = \begin{cases} w_i^{1-j}(1+pw_i)^{j-1}, & 1 \leq j \leq k+1, \\ w_i^{1-j}(1+pw_i)^{j-1}(1+w_i)^{-x_{N-j+1}+j-1}, & k+2 \leq j \leq N. \end{cases}$$

Note that

$$\begin{aligned} &\sum_{x_{N-k-1}=k+2}^{x_{N-k-2}-1} (1 - p\mathbf{1}_{x_{N-k-2}-x_{N-k-1}=1}) R_{i,k+2}^{(k+1)}(\vec{w}) \\ &= w_i^{-2-k}(1+pw_i)^{1+k} - w_i^{-2-k}(1+pw_i)^{2+k}(1+w_i)^{-x_{N-k-2}+k+2} \\ &= w_i^{-2-k}(1+pw_i)^{1+k} - R_{i,k+3}^{(k,k+1)}(\vec{w}). \end{aligned}$$

Adding the $k+3$-th column to the $k+2$-th column we get

$$\sum_{x_{N-k-1}=k+2}^{x_{N-k-2}-1} \Psi_{\vec{x}}^r(\vec{w}) = (1-p)^k \prod_{i=1}^{N-k-2} (1 - p\mathbf{1}_{x_{i-1}-x_i=1}) \det[R_{i,j}^{(k,k+2)}(\vec{w})]_{i,j=1}^N,$$

where

$$R_{i,j}^{(k,k+2)}(\vec{w}) = \begin{cases} w_i^{1-j}(1+pw_i)^{j-1}, & 1 \leq j \leq k+1, \\ w_i^{-j}(1+pw_i)^{j-1}, & j = k+2, \\ w_i^{1-j}(1+pw_i)^{j-1}(1+w_i)^{-x_{N-j+1}+j-1}, & k+3 \leq j \leq N. \end{cases}$$

Now repeating this procedure and performing the sum over $x_{N-k-2}, \cdots, x_2$ we get

$$\sum_{x_2=N-1}^{x_1-1} \cdots \sum_{x_{N-k-1}=k+2}^{x_{N-k-2}-1} \Psi^r_{\vec{x}}(\vec{w}) = (1-p)^k (1-p\mathbf{1}_{x_0-x_1=1}) \det[R^{(k,N-1)}_{i,j}(\vec{w})]^N_{i,j=1},$$

where

$$R^{(k,N-1)}_{i,j}(\vec{w}) = \begin{cases} w_i^{1-j}(1+pw_i)^{j-1}, & 1 \le j \le k+1, \\ w_i^{-j}(1+pw_i)^{j-1}, & k+2 \le j \le N-1, \\ w_i^{1-j}(1+pw_i)^{j-1}(1+w_i)^{-x_{N-j+1}+j-1}, & j = N. \end{cases}$$

Finally note that

$$\begin{aligned} &\sum_{x_1=N}^{L-1} (1-p\mathbf{1}_{x_0-x_1=1}) R^{(k,N-1)}_{i,N}(\vec{w}) \\ &= w_i^{-N}(1+pw_i)^{N-1} - w_i^{-N}(1+pw_i)^N(1+w_i)^{-L-N} \\ &= w_i^{-2-k}(1+pw_i)^{1+k} - z^{-L} R^{(k,N-1)}_{i,1}(\vec{w}). \end{aligned}$$

Here we used the fact that $w_i \in \mathcal{S}_z$. Multiplying the first column by z^{-L} and adding to last column we get

$$S_k := \sum_{x_1=N}^{L-1} \cdots \sum_{x_{N-k-1}=k+2}^{x_{N-k-2}-1} \Psi^r_{\vec{x}}(\vec{w}) = (1-p)^k \det[R^{(k,N)}_{i,j}(\vec{w})]^N_{i,j=1},$$

where

$$R^{(k,N)}_{i,j}(\vec{w}) = \begin{cases} w_i^{1-j}(1+pw_i)^{j-1}, & 1 \le j \le k+1, \\ w_i^{-j}(1+pw_i)^{j-1}, & k+2 \le j \le N. \end{cases}$$

For the purpose of summing over S_k we further rewrite $R^{(k,N)}_{i,j}(\vec{w})$ slightly. Given $0 \le k \le N-3$, we add the j-th column of $R^{(k,N)}(\vec{w})$ to its $j+1$-th column, $j =$

$N, \cdots, k+2$ so that

$$\det[R_{i,j}^{(k,N)}(\vec{w})]_{i,j=1}^{N} = \det[\hat{R}_{i,j}^{(k,N)}(\vec{w})]_{i,j=1}^{N},$$

where

$$\hat{R}_{i,j}^{(k,N)}(\vec{w}) = \begin{cases} w_i^{1-j}(1+pw_i)^{j-1}, & 1 \le j \le k+1, \\ w_i^{-j}(1+pw_i)^{j-1}, & j = k+2, \\ w_i^{-j}(1+pw_i)^{j-1}(1+w_i), & k+3 \le j \le N. \end{cases}$$

For $k = N-2, N-1$ we just set $\hat{R}^{(k,N)}(\vec{w}) = R^{(k,N)}(\vec{w})$. Now we perform the sum over S_k in the order $S_{N-1} + S_{N-2} + \cdots + S_0$. Note that for each $0 \le k \le N-1$, $\hat{R}_{i,j}^{k,N}(\vec{w}) = \hat{R}_{i,j}^{k+1,N}(\vec{w})$ except for $j = k+1$. Hence by multi-linearity of the determinants we have

$$\begin{aligned} S_{N-1} + S_{N-2} &= (1-p)^{N-2}\left((1-p)\det[\hat{R}_{i,j}^{(N-1,N)}(\vec{w})] + \det[\hat{R}_{i,j}^{(N-2,N)}(\vec{w})]\right) \\ &= (1-p)^{N-2}\det[T_{i,j}^{(N-2)}(\vec{w})]_{i,j=1}^{N}, \end{aligned}$$

where

$$T_{i,j}^{(N-2)}(\vec{w}) = \begin{cases} w_i^{1-j}(1+pw_i)^{j-1}, & 1 \le j \le N-1, \\ w_i^{-j}(1+pw_i)^{j-2}(1+w_i), & j = N. \end{cases}$$

Here to simplify the expression we have multiplied the $(N-1)$-th column by $-p(1-p)$ and added to the sum of the N-th column of $\hat{R}^{(N-1,N)}(\vec{w})$ and $\hat{R}^{(N-2,N)}(\vec{w})$, using the simple fact that

$$[(1-p)w_i + 1]\cdot w_i^{-N}(1+pw_i)^{N-1} = w_i^{-N}(1+pw_i)^{N-2}(1+w_i) + p(1-p)w_i^{2-N}(1+pw_i)^{N-2}.$$

Now repeating this procedure we get

$$\sum_{k=0}^{N-1} S_k = \det[T_{i,j}^{(0)}(\vec{w})]_{i,j=1}^{N},$$

where

$$T^{(0)}_{i,j}(\vec{w}) = \begin{cases} w_i^{1-j}(1+pw_i)^{j-1}, & j=1, \\ w_i^{-j}(1+pw_i)^{j-2}(1+w_i), & 2 \le j \le N. \end{cases}$$

Multiplying the j-th column by $-p$ and adding to the $j+1$-th column, for $j = N-1, \cdots, 2$, we get

$$\sum_{k=0}^{N-1} S_k = \det[\hat{T}^{(0)}_{i,j}(\vec{w})]_{i,j=1}^N = \prod_{i=1}^N (1+w_i) \cdot \det[w_i^{-j}]_{i,j=1}^N - \det[w_i^{-j}]_{i,j=1}^N,$$

where

$$\hat{T}^{(0)}_{i,j}(\vec{w}) = \begin{cases} w_i^{1-j}, & j=1, \\ w_i^{-j}(1+w_j), & 2 \le j \le N. \end{cases}$$

Thus we conclude that

$$\begin{aligned} \sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_N \ge 0\}} \Psi^r_{\vec{x}}(\vec{w}) &= \sum_{a=0}^{\infty} \left[\prod_{i=1}^N (1+w_i)^{-a+1} - \prod_{i=1}^N (1+w_i)^{-a} \right] \det[w_i^{-j}]_{i,j=1}^N \\ &= \prod_{i=1}^N (1+w_i) \cdot \det[w_i^{-j}]_{i,j=1}^N. \end{aligned}$$

This completes the proof. □

The following corollary is a simple consequence of Proposition 4.5.2 and the periodic nature of the form of $\Psi^r_{\vec{x}}(\vec{w})$.

Corollary 4.5.3. *Under the same assumption as in Proposition 4.5.2 we have*

$$\begin{aligned} &\sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_{N-k} \ge a\}} \Psi^r_{\vec{x}}(\vec{w}) \\ &= (-1)^{k(N-1)} z^{kL} \prod_{i=1}^N \left[\left(\frac{1+pw_i}{w_i} \right)^k \cdot (1+w_i)^{-a+k+1} \right] \cdot \det[w_i^{-j}]_{i,j=1}^N, \end{aligned} \tag{4.57}$$

for all $0 \le k \le N-1$ *and* $a \in \mathbb{Z}$.

Proof. For given $\vec{x} = (x_1, \cdots, x_N) \in \mathcal{X}_N^{(L)}$ and $0 \le k \le N-1$, set $\vec{x}' := (x_1', \cdots, x_N') = (x_{N-k+1} + L, \cdots, x_N + L, x_1, \cdots, x_{N-k})$. Then the condition $\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_{N-k} \ge a\}$ is the same as $\vec{x}' \in \mathcal{X}_N^{(L)} \cap \{x_N' \ge a\}$. Now consider

$$\Psi_{\vec{x}}^r(\vec{w}) = \det\left[\left(\frac{w_i}{1+pw_i}\right)^{1-j}(1+w_i)^{-x_{N-j+1}+j-1}\right]_{i,j=1}^N.$$

We move the first k columns of the matrix to the end. The resulting determinant equals $(-1)^{k(N-1)}$ times the determinant of the matrix whose (i,j)-th entry has the form $\left(\frac{w_i}{1+pw_i}\right)^{1-j-k}(1+w_i)^{-x'_{N-j+1}+j+k-1}$ for $1 \le j \le N-k$ and the form $\left(\frac{w_i}{1+pw_i}\right)^{1-j-k+N}(1+w_i)^{-x'_{N-j+1}+j+k-1+L-N}$ for $N-k+1 \le j \le N$. But since $\left(\frac{w_i}{1+pw_i}\right)^N (1+w_i)^{L-N} = z^L$ for all $1 \le i \le N$, we have

$$\begin{aligned}&\left(\frac{w_i}{1+pw_i}\right)^{1-j-k+N}(1+w_i)^{-x'_{N-j+1}+j+k-1+L-N}\\ &= z^L \cdot \left(\frac{w_i}{1+pw_i}\right)^{1-j-k}(1+w_i)^{-x'_{N-j+1}+j+k-1}.\end{aligned}$$

Factoring out the common factor $\left(\frac{1+w_i}{w_i(1+pw_i)}\right)^k$ from each row and z^L from the last k columns we conclude that

$$\Psi_{\vec{x}}^r(\vec{w}) = (-1)^{k(N-1)} z^{kL} \prod_{i=1}^N \left[\left(\frac{1+pw_i}{w_i}\right)^k \cdot (1+w_i)^k\right] \Psi_{\vec{x}'}^r(\vec{w}).$$

Hence

$$\begin{aligned}&\sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_{N-k} \ge a\}} \Psi_{\vec{x}}^r(\vec{w})\\ &= (-1)^{k(N-1)} z^{kL} \prod_{i=1}^N \left[\left(\frac{1+pw_i}{w_i}\right)^k \cdot (1+w_i)^k\right] \cdot \sum_{\vec{x}' \in \Omega_{L,N} \cap \{x_N' \ge a\}} \Psi_{\vec{x}'}^r(\vec{w})\\ &= (-1)^{k(N-1)} z^{kL} \prod_{i=1}^N \left[\left(\frac{1+pw_i}{w_i}\right)^k \cdot (1+w_i)^{-a+k+1}\right] \cdot \det[w_i^{-j}]_{i,j=1}^N.\end{aligned}$$

This completes the proof □

4.5.2 Cacuchy-type summation identity for left and right eigenfunctions

Proposition 4.5.4 (Cauchy-type summation identity over left and right eigenfunctions)**.** *Let $z, z' \in \mathbb{C}$ be nonzero such that $(z')^L \neq z^L$. For $\vec{x} = (x_1, \cdots, x_N) \in \mathcal{X}_N^{(L)}$, let $\Psi_{\vec{x}}^r(\vec{w})$ and $\Psi_{\vec{x}}^\ell(\vec{w}')$ be as in (3.51) and (3.52) where $\vec{w} = (w_1, \cdots, w_N) \in (\mathcal{S}_z)^N$ and $\vec{w}' = (w_1', \cdots, w_N') \in (\mathcal{S}_{z'})^N$ satisfy $\prod_{j=1}^N |w_j + 1| > \prod_{j=1}^N |w_j' + 1|$. Then*

$$\begin{aligned} &\sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_N \geq 0\}} \Psi_{\vec{x}}^r(\vec{w}) \Psi_{\vec{x}}^\ell(\vec{w}') \\ &= \left(1 - \left(\frac{z'}{z}\right)^L\right)^{N-1} \prod_{j=1}^N (w_j + 1) \cdot \det\left[\frac{1}{w_i - w_{i'}'}\right]_{i,i'=1}^N. \end{aligned} \tag{4.58}$$

Similar as in Corollary 4.5.3, we can easily extend Proposition 4.5.4 using periodicity to a summation over $\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_{N-k} \geq a\}$ for any $0 \leq k \leq N-1$ and $a \in \mathbb{Z}$:

Corollary 4.5.5. *Under the same assumption as in Proposition 4.5.4 we have*

$$\begin{aligned} \sum_{\vec{x} \in \mathcal{X}_N^{(L)} \cap \{x_{N-k} \geq a\}} \Psi_{\vec{x}}^r(\vec{w}) \Psi_{\vec{x}}^\ell(\vec{w}') = \left(\frac{z}{z'}\right)^{kL} \left(1 - \left(\frac{z'}{z}\right)^L\right)^{N-1} \cdot \det\left[\frac{1}{w_i - w_{i'}'}\right]_{i,i'=1}^N \\ \cdot \prod_{j=1}^N \frac{w_j^{-k}(1 + pw_j)^k (w_j + 1)^{-a+k+1}}{(w_j')^{-k}(1 + pw_j')^k (w_j' + 1)^{-a+k}} \end{aligned} \tag{4.59}$$

for all $0 \leq k \leq N-1$ and $a \in \mathbb{Z}$.

Proof. Similar as the proof of Corollary 4.5.3. □

Remark 4.5.6. It is interesting to note that the right-hand side of (4.58) does not depend on p explicitly (of course the w_i' should satisfy certain algebraic equations which depend on p). Taking $p \to 0$, Proposition 4.5.4 degenerates to Proposition 3.4 of [9], which can be understood as a (periodic version) of Cauchy identity for the Grothendieck polynomial (and its dual), and can be derived from the deformed Cauchy identity for Grothendieck polynomials obtained in Theorem 5.3 of [85]. For $0 < p < 1$, to the best of our knowledge the corresponding Cauchy-type identity (4.58) has not been discussed in the existing literature, at least for the periodic case. The key point here is instead of summing over all configuration $\vec{x}$ with $0 \leq x_N < x_{N-1} < \cdots < x_1$ as in the usual Cauchy identity, we are only summing over those configurations satisfying the extra constraint $x_1 < x_N + L$. For general spectral parameters $\vec{w}$ and $\vec{w}'$ this sum only gives a deformed or generalized Cauchy determinant. It further reduces to a genuine Cauchy determinant when we impose the conditions as in Proposition 4.5.4 that the spectral parameters satisfy suitable Bethe equations.

The proof of Proposition 4.5.4 is rather lengthy so we divide it into three steps and discuss them one by one in the next few subsections. The proof mainly follows the strategy of Proposition 3.4 of [9] but there are several new technicalities. The non-uniform term $\prod_{i=1}^{N}(1-p\mathbf{1}_{x_{i-1}-x_i=1})$ appearing in $\Psi^r_{\vec{x}}(\vec{w})$ leads to extra difficulty and in Step 1 we overcome this by introducing a different way (and slightly more convenient way in our opinion) of expressing the sum in (4.58) comparing to the proof in [9], see Lemma 4.5.7 for details. In Step 2 we establish a key summation identity (see Lemma 4.5.10) which generalizes Lemma 5.4 in [9] while the computation is more delicate. Finally in step 3 we combine the formula obtained in Step 1 and the summation identity obtained in Step 2 to conclude the final result. Throughout the proof several rank-one perturbation formulas for Cauchy determinants are used frequently so we collect all these elementary formulas in a separate section for convenience, see Section 4.5.6 for details.

4.5.3 Proof of Proposition 4.5.4: Step 1

Similar as in the proof of Proposition 4.5.2 we first write

$$\sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N\geq 0\}} \Psi^r_{\vec{x}}(\vec{w})\Psi^\ell_{\vec{x}}(\vec{w}') = \sum_{a=0}^{\infty}\prod_{j=1}^{N}\left(\frac{1+w'_j}{1+w_j}\right)^a \sum_{\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N=0\}} \Psi^r_{\vec{x}}(\vec{w})\Psi^\ell_{\vec{x}}(\vec{w}')$$
$$:= \sum_{a=0}^{\infty}\prod_{j=1}^{N}\left(\frac{1+w'_j}{1+w_j}\right)^a \cdot \mathcal{H}_N(\vec{w},\vec{w}'),$$

so that it suffices to compute the sum over $\vec{x}\in\mathcal{X}_N^{(L)}\cap\{x_N=0\}$, which is a finite sum so there is no convergence issue. The summation over a converges absolutely again by our assumption that $\prod_{j=1}^N|w_j+1| > \prod_{j=1}^N|w'_j+1|$. Expanding the determinants in $\Psi^\ell_{\vec{x}}(\vec{w})$ and $\Psi^r_{\vec{x}}(\vec{w}')$ we get

$$\mathcal{H}_N(\vec{w},\vec{w}') = \sum_{\sigma,\sigma'\in S_N}\mathrm{sgn}(\sigma\sigma')\prod_{j=1}^{N}\left(\frac{w'_{\sigma'(j)}(1+pw_{\sigma(j)})}{w_{\sigma(j)}(1+pw'_{\sigma'(j)})}\right)^{j-1}\mathcal{H}^{(N)}_{\sigma,\sigma'}(\vec{w},\vec{w}'),$$

where

$$\mathcal{H}^{(N)}_{\sigma,\sigma'}(\vec{w},\vec{w}') = \sum_{L=x_0>x_1>\cdots>x_N=0}\prod_{j=1}^{N}(1-p\mathbf{1}_{x_{j-1}-x_j=1})\left(\frac{w'_{\sigma'(j)}+1}{w_{\sigma(j)}+1}\right)^{x_{N-j+1}-j+1}. \quad (4.60)$$

By Lemma 4.5.7 below we see $\mathcal{H}^{(N)}_{\sigma,\sigma'}(\vec{w},\vec{w}')$ equals

$$\sum_{k=1}^{N}\prod_{i=2}^{k}\left(-p+\frac{1}{1-\prod_{j=i}^{k}\frac{w'_{\sigma'(j)}+1}{w_{\sigma(j)}+1}}\right)$$
$$\cdot\prod_{i=k+1}^{N}\left(\frac{w'_{\sigma'(i)}+1}{w_{\sigma(i)}+1}\right)^{L-N}\cdot\left(-p+\frac{1}{1-\prod_{j=k+1}^{i}\frac{w_{\sigma(j)}+1}{w'_{\sigma'(j)}+1}}\right).$$

Hence

$$\mathcal{H}_N(\vec{w},\vec{w}') = \sum_{k=1}^{N} \sum_{\sigma,\sigma'\in S_N} \operatorname{sgn}(\sigma\sigma') \prod_{j=1}^{N} \left(\frac{w'_{\sigma'(j)}(1+pw_{\sigma(j)})}{w_{\sigma(j)}(1+pw'_{\sigma'(j)})} \right)^{j-1} \cdot \prod_{i=k+1}^{N} \left(\frac{w'_{\sigma'(i)}+1}{w_{\sigma(i)}+1} \right)^{L-N}$$
$$\cdot \left[\prod_{i=2}^{k} \left(-p + \frac{1}{1-\prod_{j=i}^{k} \frac{w'_{\sigma'(j)}+1}{w_{\sigma(j)}+1}} \right) \cdot \prod_{i=k+1}^{N} \left(-p + \frac{1}{1-\prod_{j=k+1}^{i} \frac{w_{\sigma(j)}+1}{w'_{\sigma'(j)}+1}} \right) \right]. \tag{4.61}$$

Lemma 4.5.7. *Let $N \geq 1$ be an integer and $w_1, \cdots, w_N$ and $w'_1, \cdots, w'_N$ be distinct complex numbers not equal to -1. Then for any integer $L > N$ we have*

$$\sum_{L=x_0>x_1>\cdots>x_N=0} \prod_{i=1}^{N} (1-p\mathbf{1}_{x_{i-1}-x_i=1}) \prod_{i=1}^{N} \left(\frac{w'_i+1}{w_i+1} \right)^{x_{N-i+1}-i+1}$$
$$= \sum_{k=1}^{N} \left[\prod_{i=2}^{k} \left(-p + \frac{1}{1-\prod_{j=i}^{k} \frac{w'_j+1}{w_j+1}} \right) \cdot \prod_{i=k+1}^{N} \left(-p + \frac{1}{1-\prod_{j=k+1}^{i} \frac{w_j+1}{w'_j+1}} \right) \right] \tag{4.62}$$
$$\cdot \prod_{i=k+1}^{N} \left(\frac{w'_i+1}{w_i+1} \right)^{L-N}.$$

Here any empty product is set to be 1.

Proof. We use an induction on N. For $N = 1$ the identity is obvious. Assume now $N \geq 2$ and the identity holds for all indices less than N. We split the sum into two sums depending on whether $x_1 = L-1$ or not:

$$\sum_{L=x_0>x_1>\cdots>x_N=0} = \sum_{L-1=x_1>\cdots>x_N=0} + \sum_{L-1>x_1>\cdots>x_N=0} := T_1 + T_2.$$

For T_1 we first relabel the indices so that $(x'_0, x'_1, \cdots, x'_{N-1}) := (x_1, x_2, \cdots, x_N)$ and $L' := L-1$. Then by induction hypothesis we have

$$T_1 = \sum_{k=1}^{N-1} \left[\prod_{i=2}^{k} \left(-p + \frac{1}{1-\prod_{j=i}^{k} \frac{w'_j+1}{w_j+1}} \right) \cdot \prod_{i=k+1}^{N} \left(\frac{w'_i+1}{w_i+1} \right)^{L-N} \cdot T_1^{(k)}(\vec{w},\vec{w}') \right]. \tag{4.63}$$

Where

$$T_1^{(k)}(\vec{w},\vec{w}') := (1-p)\prod_{i=k+1}^{N-1}\left(-p+\frac{1}{1-\prod_{j=k+1}^{i}\frac{w_j+1}{w'_j+1}}\right),$$

for $1 \le k \le N-1$ and we set $T_1^{(N)}(\vec{w},\vec{w}') := 0$ for convenience.

For T_2 we calculate the sum directly in the order $L-1 > x_1 > \cdots > x_N = 0$ using Lemma 4.5.8 below which gives:

$$\begin{aligned} T_2 &= \sum_{x_{N-1}=1}^{L-N-1}\cdots\sum_{x_1=x_2+1}^{L-2}\prod_{i=2}^{N}(1-p\mathbf{1}_{x_{i-1}-x_i=1})\prod_{i=1}^{N}\left(\frac{w'_i+1}{w_i+1}\right)^{x_{N-i+1}-i+1} \\ &= \sum_{k=1}^{N}\left[\prod_{i=2}^{k}\left(-p+\frac{1}{1-\prod_{j=i}^{k}\frac{w'_j+1}{w_j+1}}\right)\cdot\prod_{i=k+1}^{N}\left(\frac{w'_i+1}{w_i+1}\right)^{L-N}\cdot T_2^{(k)}(\vec{w},\vec{w}')\right]. \end{aligned} \tag{4.64}$$

Where

$$T_2^{(k)}(\vec{w},\vec{w}') := \sum_{\ell=1}^{N-k}\sum_{k+1=s_1<\cdots<s_\ell\le N} \prod_{i=1}^{\ell}\left(\frac{1}{\left(\prod_{j=s_i}^{s_{i+1}-1}\frac{w'_j+1}{w_j+1}\right)-1}\cdot\prod_{j=s_i+1}^{s_{i+1}-1}\left(-p+\frac{1}{1-\prod_{m=j}^{s_{i+1}-1}\frac{w'_m+1}{w_m+1}}\right)\right),$$

for $1 \le k \le N-1$ and $T_2^{(N)}(\vec{w},\vec{w}') := 1$. Here we are summing over all possible partitions of $\{k+1,\cdots,N\}$ and $s_{i+1} := N+1$. Now comparing (4.63) and (4.64) with (4.62) we see it suffices to prove

$$T_1^{(k)}(\vec{w},\vec{w}') + T_2^{(k)}(\vec{w},\vec{w}') = \prod_{i=k+1}^{N}\left(-p+\frac{1}{1-\prod_{j=k+1}^{i}\frac{w_j+1}{w'_j+1}}\right), \tag{4.65}$$

for all $1 \le k \le N$. Using the simple identity

$$\frac{1}{1-\prod_{j=m}^{n}\frac{w_j+1}{w'_j+1}} + \frac{1}{1-\prod_{j=m}^{n}\frac{w'_j+1}{w_j+1}} = 1, \tag{4.66}$$

(4.65) is further reduced to showing

$$T_k^{(2)}(\vec{w},\vec{w}') = \prod_{i=k+1}^{N-1}\left(-p+\frac{1}{1-\prod_{j=k+1}^{i}\frac{w_j+1}{w'_j+1}}\right)\cdot\frac{1}{\prod_{j=k+1}^{N}\frac{w'_j+1}{w_j+1}-1}, \tag{4.67}$$

which follows from Lemma 4.5.9 below by taking $z_j = \frac{w'_j+1}{w_j+1}$ and properly shifting the indices. □

Lemma 4.5.8. *For complex numbers f_j, set*

$$F_{m,n} = \prod_{j=m}^{n} f_j \quad \textit{for } 1 \le m \le n. \tag{4.68}$$

Then

$$\begin{aligned}&\sum_{x_{N-1}=1}^{L-N-1}\cdots\sum_{x_1=x_2+1}^{L-2}\prod_{j=2}^{N}(1-p\mathbf{1}_{x_{N-j+1}-x_{N-j+2}=1})(f_j)^{x_{N-j+1}-j+1}\\ &=\sum_{k=1}^{N-1}\left[\prod_{i=2}^{k}\left(-p+\frac{1}{1-F_{i,k}}\right)\right]\cdot(F_{k+1,N})^{L-N}\\ &\cdot\left[\sum_{\ell=1}^{N-k}\sum_{k<s_1<\cdots<s_\ell\le N}\prod_{i=1}^{\ell}\left(\prod_{j=s_i+1}^{s_{i+1}-1}\left(-p+\frac{1}{1-F_{j,s_{i+1}-1}}\right)\cdot\frac{1}{(F_{s_i,s_{i+1}-1})-1}\right)\right].\end{aligned} \tag{4.69}$$

Where we set $s_{\ell+1} = N+1$.

Proof. This lemma is a slightly modified version of Lemma 5.3 of [9]. The proof is elementary and almost identical to the proof in [9] so we omit it. □

Lemma 4.5.9. *Let $n \ge 1$ be an integer and $z_1, \cdots, z_n$ be complex numbers such that $\prod_{j=\ell}^{k} z_j \ne 1$ for all $1 \le \ell \le k \le n$. Then*

$$\begin{aligned}&\sum_{\ell=1}^{n}\sum_{1=s_1<\cdots<s_\ell\le n}\prod_{i=1}^{\ell}\left(\frac{1}{\left(\prod_{j=s_i}^{s_{i+1}-1} z_j\right)-1}\cdot\prod_{j=s_i+1}^{s_{i+1}-1}\left(-p+\frac{1}{1-\prod_{m=j}^{s_{i+1}-1} z_m}\right)\right)\\ &=\prod_{i=1}^{n-1}\left(-p+\frac{1}{1-\prod_{j=1}^{i} z_j^{-1}}\right)\cdot\frac{1}{\left(\prod_{j=1}^{n} z_j\right)-1}.\end{aligned} \tag{4.70}$$

Proof. For a fixed integer $M > n$, consider the following sum:

$$\sum_{M>x_1>\cdots>x_n\geq 0} \prod_{i=2}^{n}(1 - p\mathbf{1}_{x_{i-1}-x_i} = 1) \prod_{j=1}^{n} z_j^{x_{n-j+1}-j+1}.$$

We calculate the sum in two different ways: from right to left or from left to right. Namely we set

$$S_M^{r\to\ell} := \sum_{x_n=0}^{M-n} \cdots \sum_{x_1=x_2+1}^{M-1} \prod_{i=2}^{n}(1 - p\mathbf{1}_{x_{i-1}-x_i} = 1) \prod_{j=1}^{n} z_j^{x_{n-j+1}-j+1},$$

and

$$S_M^{\ell\to r} := \sum_{x_1=n-1}^{M-1} \cdots \sum_{x_n=0}^{x_{n-1}-1} \prod_{i=2}^{n}(1 - p\mathbf{1}_{x_{i-1}-x_i} = 1) \prod_{j=1}^{n} z_j^{x_{n-j+1}-j+1}.$$

Then clearly $S_M^{r\to\ell} = S_M^{\ell\to r}$ since they represent the same sum. Now the two sums are calculated by calculating the (almost) geometric sums one by one either from left to right or vice versa. There are 2^n terms in total for both sums since every single term produces two terms after performing the geometric sum once. Each of the terms contains a factor of the form $\prod_{j=n-k+1}^{n} z_j^{M-n+1}$ for some $0 \leq k \leq n$ where $k = 0$ corresponds to the terms containing no such factor. For each k we combine all the terms with the same factor $\prod_{j=n-k+1}^{n} z_j^{M-n+1}$ and write

$$S_M^{r\to\ell} = \sum_{k=0}^{n} \left[C_k^{r\to\ell}(z_1, \cdots, z_n) \cdot \prod_{j=n-k+1}^{n} z_j^{M-n+1} \right],$$

and

$$S_M^{r\to\ell} = \sum_{k=0}^{n} \left[C_k^{\ell\to r}(z_1, \cdots, z_n) \cdot \prod_{j=n-k+1}^{n} z_j^{M-n+1} \right].$$

Where $C_k^{r\to\ell}(z_1, \cdots, z_n)$'s and $C_k^{\ell\to r}(z_1, \cdots, z_n)$'s are some very explicit functions in $z_1, \cdots, z_n$ independent of M which are analytic for all z_i's satisfying the assumption that $\prod_{j=\ell}^{k} z_j \neq 1$ for all $1 \leq k \leq \ell \leq n$. In particular it is straightforward to check(see

Lemma 4.5.8 for example) that

$$\text{LHS of (4.70)} = C_n^{r\to\ell}(z_1,\cdots,z_n), \quad \text{RHS of (4.70)} = C_n^{\ell\to r}(z_1,\cdots,z_n).$$

We claim that $C_k^{r\to\ell}(z_1,\cdots,z_n) = C_k^{\ell\to r}(z_1,\cdots,z_n)$ for all $0 \le k \le n$. This in particular implies (4.70). Due to analyticity it suffices to check this for the z_j's satisfying $|z_j| < 1$ for all $1 \le j \le n$. In this case by letting $M \to \infty$ in the equality $S_M^{r\to\ell} = S_M^{\ell\to r}$ we see $C_0^{r\to\ell} = C_0^{\ell\to r}$. Similarly

$$C_1^{r\to\ell} = \lim_{M\to\infty} z_n^{-M+n-1} \cdot (S_M^{r\to\ell} - C_0^{r\to\ell}) = \lim_{M\to\infty} z_n^{-M+n-1} \cdot (S_M^{\ell\to r} - C_0^{\ell\to r}) = C_1^{\ell\to r}.$$

Repeating this procedure we see $C_k^{r\to\ell} = C_k^{\ell\to r}$ for all $0 \le k \le n$. □

4.5.4 Proof of Proposition 4.5.4: Step 2

In this section we simplify the sum (4.61). We rewrite the sum further by first choosing two index sets J and J' with $|J| = |J'| = k$ and then expressing the sum in terms of summation over index sets J, J':

$$\mathcal{H}_N(\vec{w},\vec{w}') = \sum_{k=1}^{N} \sum_{\substack{J,J'\subset\{1,\cdots,N\}\\ |J|=|J'|=k}} (-1)^{\#(J,J^c)+\#(J',(J')^c)} \mathscr{H}_1(J,J')\mathscr{H}_2(J^c,(J')^c), \tag{4.71}$$

where $\#(J,J^c) := |\{(m,n) \in (J,J^c) : m > n\}|$ and similar for $\#(J',(J')^c)$. The functions $\mathscr{H}_1(J,J')$ and $\mathscr{H}_2(J^c,(J')^c)$ are defined as follows:

$$\mathscr{H}_1(J,J') = \sum_{\substack{\sigma:\{1,\cdots,k\}\to J\\ \sigma':\{1,\cdots,k\}\to J'}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')$$
$$\prod_{i=2}^{k} \left(\frac{w'_{\sigma'(i)}(1+pw_{\sigma(i)})}{w_{\sigma(i)}(1+pw'_{\sigma'(i)})} \right)^{i-1} \cdot \left(-p + \frac{1}{1-\prod_{j=i}^{k} \frac{w'_{\sigma'(j)}+1}{w_{\sigma(j)}+1}} \right),$$

and

$$\mathscr{H}_2(J^c,(J')^c) = \sum_{\substack{\pi:\{k+1,\cdots,N\}\to J^c \\ \pi':\{k+1,\cdots,N\}\to (J')^c}} \mathrm{sgn}(\pi)\mathrm{sgn}(\pi') \frac{\prod_{j'\in(J')^c}(w'_{j'}+1)^{L-N}}{\prod_{j\in J^c}(w_j+1)^{L-N}} \\ \cdot \prod_{i=k+1}^{N} \left(\frac{w'_{\pi'(i)}(1+pw_{\pi(i)})}{w_{\pi(i)}(1+pw'_{\pi'(i)})} \right)^{i-1} \cdot \left(-p + \frac{1}{1-\prod_{j=k+1}^{i} \frac{w_{\pi(j)}+1}{w'_{\pi'(j)}+1}} \right). \tag{4.72}$$

By Lemma 4.5.10 below (which is of interest on its own) we have

$$\mathscr{H}_1(J,J') = \left[\prod_{j\in J}(w_j+1) - \prod_{j'\in J'}(w'_{j'}+1) \right] \cdot \det \left[\frac{1}{w_j - w'_{j'}} \right]_{j\in J, j'\in J'}. \tag{4.73}$$

To simplify $\mathscr{H}_2(J^c,(J')^c)$ we use the assumption that $\vec{w} \in (\mathcal{S}_z)^N$ and $\vec{w}' \in (\mathcal{S}_{z'})^N$. Namely for all $1 \le i, i' \le N$ we have

$$w_i^N(1+pw_i)^{-N}(1+w_i)^{L-N} = z^L, \quad (w'_{i'})^N(1+pw'_{i'})^{-N}(1+w'_{i'})^{L-N} = (z')^L. \tag{4.74}$$

Inserting (4.74) into (4.72) we get

$$H_2(J^c,(J')^c) = \sum_{\substack{\pi:\{k+1,\cdots,N\}\to J^c \\ \pi':\{k+1,\cdots,N\}\to (J')^c}} \mathrm{sgn}(\pi)\mathrm{sgn}(\pi') \frac{\prod_{j'\in(J')^c}(z')^L}{\prod_{j\in J^c} z^L} \\ \cdot \prod_{i=k+1}^{N} \left(\frac{w_{\pi(i)}(1+pw'_{\pi'(i)})}{w'_{\pi'(i)}(1+pw_{\pi(i)})} \right)^{N-i+1} \cdot \left(-p + \frac{1}{1-\prod_{j=k+1}^{i} \frac{w_{\pi(j)}+1}{w'_{\pi'(j)}+1}} \right).$$

Now in order to apply Lemma 4.5.10 we reflect the permutations by defining $\hat{\pi} : \{1,\cdots,N-k\} \to J^c$ as $\hat{\pi}(j) := \pi(N-j+1)$ for $1 \le j \le N$ and similarly for

$\hat{\pi}' : \{1, \cdots, N-k\} \to (J')^c$. Then

$$\mathscr{H}_2(J^c, (J')^c) = \sum_{\substack{\hat{\pi}:\{1,\cdots,N-k\}\to J^c \\ \hat{\pi}':\{1,\cdots,N-k\}\to (J')^c}} \operatorname{sgn}(\hat{\pi})\operatorname{sgn}(\hat{\pi}') \frac{\prod_{j'\in(J')^c}(z')^L}{\prod_{j\in J^c} z^L} \\ \cdot \prod_{i=1}^{N-k} \left(\frac{w_{\hat{\pi}(i)}(1+pw'_{\hat{\pi}'(i)})}{w'_{\hat{\pi}'(i)}(1+pw_{\hat{\pi}(i)})} \right)^i \cdot \left(-p + \frac{1}{1-\prod_{j=i}^{N-k} \frac{w_{\hat{\pi}(j)}+1}{w'_{\hat{\pi}'(j)}+1}} \right). \tag{4.75}$$

Now apply Lemma 4.5.10 again with the role of $\vec{w}$ and $\vec{w}'$ exchanged we have

$$\mathscr{H}_2(J^c, (J')^c) \\ = \frac{\prod_{j'\in(J')^c}(z')^L}{\prod_{j\in J^c} z^L} \cdot \frac{\prod_{j\in J^c} w_j/(1+pw_j)}{\prod_{j'\in(J')^c} w'_{j'}/(1+pw'_{j'})} \cdot \left(-p + \frac{1}{1-\frac{\prod_{j\in J^c}(w_j+1)}{\prod_{j'\in(J')^c}(w'_{j'}+1)}} \right) \\ \cdot \left[\prod_{j'\in(J')^c}(w'_{j'}+1) - \prod_{j\in J^c}(w_j+1) \right] \cdot \det\left[\frac{1}{-w_j+w'_{j'}} \right]_{j\in J^c, j'\in(J')^c}. \tag{4.76}$$

Here the extra factor in equation (4.76) comes from the fact that in (4.75) the product starts from $i=1$ instead of $i=2$ and the exponent is i instead of $i-1$ comparing to (4.77).

Lemma 4.5.10. *Let $n \in \mathbb{N}$. Given any complex numbers w_i and $w'_{i'}$, $i = 1, \cdots, n$ such that $w_i \neq w_{i'}$, for all $1 \leq i, i' \leq n$. Then for any $0 < p < 1$ we have*

$$\sum_{\sigma,\sigma'\in S_n} \operatorname{sgn}(\sigma\sigma') \prod_{j=1}^{n} \left(\frac{w'_{\sigma'(j)}(1+pw_{\sigma(j)})}{w_{\sigma(j)}(1+pw'_{\sigma'(j)})} \right)^{j-1} \prod_{j=2}^{n} \left[-p + \frac{1}{1-\prod_{i=j}^{n} \frac{w'_{\sigma'(i)}+1}{w_{\sigma(i)}+1}} \right] \\ = \left[\prod_{j=1}^{n}(w_j+1) - \prod_{j=1}^{n}(w'_j+1) \right] \det\left[\frac{1}{w_i - w'_{i'}} \right]_{i,i'=1}^{n}. \tag{4.77}$$

Remark 4.5.11. Equation (4.77) should be understood also as a Cauchy summation identity (simpler version than Proposition 4.5.4, for summation over all partitions $\lambda = (x_1-n+1, x_2-n+2, \cdots, x_n)$ with at most $n-1$ rows) of the symmetric funtions

$\Psi^r_{\vec{x}}(\vec{w})/\Delta(\vec{w})$ and $\Psi^\ell_{\vec{x}}(\vec{w}')/\Delta(\vec{w}')$, where $\Delta(\vec{w})$ is the usual Vandermonde determinant. In fact formally we have

$$\sum_{0=x_n<x_{n-1}<\cdots<x_1} \Psi^r_{\vec{x}}(\vec{w})\Psi^\ell_{\vec{x}}(\vec{w}') = \text{LHS of (4.77)}, \tag{4.78}$$

assuming all the infinite geometric series converge absolutely.

Proof of Lemma 4.5.10. The proof is based on induction on n. The main tools are several rank-one perturbation formulas for Cauchy determinants which we will use several times so we collect them in a separate section, see section 4.5.6. For $n = 1$, (4.77) is trivial. Let $n \geq 2$ and assume (4.77) is true for all indices $\leq n-1$. Given $\sigma, \sigma' \in S_n$ we first fix two indices $\ell = \sigma(1)$ and $\ell' = \sigma'(1)$ and shift the restriction of σ and σ' on $\{2, \cdots, n\}$ by 1 but still denote them by σ and σ'. Then

$$\begin{aligned}\text{LHS of (4.77)} = &\sum_{\ell,\ell'=1}^{n} (-1)^{\ell+\ell'} \frac{w_\ell(1+pw'_{\ell'})}{w'_{\ell'}(1+pw_\ell)} \prod_{k=1}^{n} \frac{w'_k(1+pw_k)}{w_k(1+pw'_k)} \\ &\cdot \left(-p + \frac{1}{1 - \frac{w_\ell+1}{w'_{\ell'}+1} \prod_{k=1}^{n} \frac{w'_k+1}{w_k+1}}\right) \cdot \sum_{\substack{\sigma:\{1,\cdots,n-1\}\to\{1,\cdots,n\}\setminus\{\ell\} \\ \sigma':\{1,\cdots,n-1\}\to\{1,\cdots,n\}\setminus\{\ell'\}}} \text{sgn}(\sigma)\text{sgn}(\sigma') \\ &\prod_{i=2}^{n-1} \left(\frac{w'_{\sigma'(i)}(1+pw_{\sigma(i)})}{w_{\sigma(i)}(1+pw'_{\sigma'(i)})}\right)^{i-1} \cdot \prod_{j=2}^{n-1} \left(-p + \frac{1}{1 - \prod_{i=j}^{n-1} \frac{w'_{\sigma'(i)}+1}{w_{\sigma(i)}+1}}\right).\end{aligned}$$

By the induction hypothesis the sum over shifted permutations above

$$\left[\prod_{1\leq j\leq n, j\neq \ell} (w_j+1) - \prod_{1\leq j\leq n, j\neq \ell'} (w'_j+1)\right] \det\left[\frac{1}{w_i - w'_{i'}}\right]_{\substack{1\leq i,i'\leq n \\ i\neq\ell, i'\neq\ell'}}.$$

Hence

$$\begin{aligned}\text{LHS of (4.77)} = &(1-p)\cdot(-1)^n A(-1)\cdot\frac{B(0)A(-1/p)}{A(0)B(-1/p)}\cdot D_1 \\ &+ p\cdot(-1)^n B(-1)\cdot\frac{B(0)A(-1/p)}{A(0)B(-1/p)}\cdot D_2.\end{aligned} \tag{4.79}$$

Here we set $A(z) := \prod_{i=1}^{n}(z - w_i)$ and $B(z) := \prod_{i=1}^{n}(z - w_i')$ and D_1, D_2 are defined as follows:

$$D_1 = \sum_{\ell,\ell'=1}^{n} (-1)^{\ell+\ell'} \frac{w_\ell(1 + pw'_{\ell'})}{w'_{\ell'}(1 + w_\ell)(1 + pw_\ell)} \det\left[\frac{1}{w_i - w'_{i'}}\right]_{\substack{1\le i,i'\le n\\ i\neq\ell, i'\neq\ell'}},$$
$$D_2 = \sum_{\ell,\ell'=1}^{n} (-1)^{\ell+\ell'} \frac{w_\ell(1 + pw'_{\ell'})}{w'_{\ell'}(1 + w'_{\ell'})(1 + pw_\ell)} \det\left[\frac{1}{w_i - w'_{i'}}\right]_{\substack{1\le i,i'\le n\\ i\neq\ell, i'\neq\ell'}}.$$

Now by equation (4.95) and Lemma 4.5.15 we have

$$D_1 = \frac{1}{1-p} \cdot \left[\frac{A(0)B(-1/p)}{B(0)A(-1/p)} - \frac{B(-1/p)}{A(-1/p)} + p\frac{B(-1)}{A(-1)} - \frac{A(0)B(-1)}{B(0)A(-1)} + 1 - p\right] \cdot \det\left[\frac{1}{w_i - w'_{i'}}\right]_{i,i'=1}^{n}, \tag{4.80}$$

and

$$D_2 = \frac{1}{p} \cdot \left[\frac{A(0)}{B(0)} - \frac{B(-1/p)A(0)}{A(-1/p)B(0)} + (p-1)\frac{A(-1)}{B(-1)} + \frac{A(-1)B(-1/p)}{B(-1)A(-1/p)} - p\right] \cdot \det\left[\frac{1}{w_i - w'_{i'}}\right]_{i,i'=1}^{n}. \tag{4.81}$$

Inserting (4.80) and (4.81) into (4.79), after necessary cancellation we obtain

$$\begin{aligned}\text{LHS of (4.77)} &= [(-1)^n A(-1) - (-1)^n B(-1)] \cdot \det\left[\frac{1}{w_i - w'_{i'}}\right]_{i,i'=1}^{n} \\ &= \left[\prod_{j=1}^{n}(1 + w_j) - \prod_{j=1}^{n}(1 + w'_j)\right] \det\left[\frac{1}{w_i - w'_{i'}}\right]_{i,i'=1}^{n}.\end{aligned} \tag{4.82}$$

This completes the proof of Lemma 4.5.10. □

Finally inserting (4.73) and (4.76) into (4.71) and apply Lemma 4.5.12 below we obtain

$$
\begin{aligned}
\hat{S}_N &= \sum_{\substack{J,J'\subset\{1,\cdots,N\}\\ |J|=|J'|}} (-1)^{\#(J;J^c)+\#(J';(J')^c)} H_1(J,J')H_2(J^c,(J')^c) \\
&= -(1-p)\cdot\prod_{j=1}^{N}(w'_j+1)\left(\det\left[\hat{C}(i,i')\right]_{i,i'=1}^{N} - \det\left[\hat{C}(i,i')+\frac{1}{w'_{i'}+1}\right]_{i,i'=1}^{N}\right) \\
&\quad + p\cdot\prod_{j=1}^{N}(w_j+1)\left(\det\left[\hat{C}(i,i')\right]_{i,i'=1}^{N} - \det\left[\hat{C}(i,i')-\frac{1}{w_i+1}\right]_{i,i'=1}^{N}\right),
\end{aligned}
\tag{4.83}
$$

where

$$
\hat{C}(i,i') = \frac{1}{w_i-w'_{i'}} - \left(\frac{z'}{z}\right)^L \frac{w_i(1+pw'_{i'})}{w'_{i'}(1+pw_i)}\frac{1}{w_i-w'_{i'}}. \tag{4.84}
$$

Note that in the first equality of (4.83) we add an extra term corresponding to $|J| = |J'| = 0$ comparing to (4.71) which is harmless since the summand is 0 in this case.

Lemma 4.5.12 (Lemma 5.9 of [9]). *For two $n\times n$ matrices A and B,*

$$
\begin{aligned}
&\sum_{\substack{J,J'\subset\{1,\cdots,n\}\\ |J|=|J'|}} (-1)^{\#(J,J^c)+\#(J',(J')^c)} \det[A(i,i')]_{i\in J,i'\in J'} \det[B(i,i')]_{i\in J^c,i'\in(J')^c} \\
&= \det[A+B]_{1\le i,i'\le n}.
\end{aligned}
\tag{4.85}
$$

4.5.5 Proof of Proposition 4.5.4: Step 3

In this section we further simplify equation (4.83) to conclude the proof of Proposition 4.5.4. We compute $\det[\hat{C}(i,i')]$ first. For notational convenience we set $(z'/z)^L := \mu$. Note that

$$
\begin{aligned}
\hat{C}(i,i') &= \frac{1}{w_i-w'_{i'}} - \mu\cdot\frac{w_i(1+pw'_{i'})}{w'_{i'}(1+pw_i)}\frac{1}{w_i-w'_{i'}} \\
&= (1-\mu)\,\frac{1}{w_i-w'_{i'}} - \mu\cdot\frac{1}{w'_{i'}(1+pw_i)}.
\end{aligned}
$$

Hence by Lemma 4.5.14 and Lemma 4.5.15 we have

$$\begin{aligned}\det[\hat{C}(i,i')]_{i,i'=1}^N &= (1-\mu)^N \det\left[\frac{1}{w_i - w'_{i'}} - \frac{\mu}{1-\mu}\frac{1}{w'_{i'}(1+pw_i)}\right]_{i,i'=1}^N \\ &= (1-\mu)^N \det[C(i,i')]_{i,i'=1}^N \cdot \left(1 - \frac{\mu}{1-\mu}\left(\frac{B(-1/p)A(0)}{A(-1/p)B(0)} - 1\right)\right) \\ &= (1-\mu)^{N-1} \det[C(i,i')]_{i,i'=1}^N \cdot \left(1 - \mu \cdot \frac{B(-1/p)A(0)}{A(-1/p)B(0)}\right). \qquad (4.86)\end{aligned}$$

Here $C(i,i') = \frac{1}{w_i - w'_{i'}}$ and $A(0)$ is the evaluation at $z = 0$ of the polynomial $A(z) := \prod_{j=1}^N (z - w_j)$. The other terms involving $A(\cdot)$ and $B(\cdot)$ are defined in a similar way with $B(z) := \prod_{j=1}^N (z - w'_j)$. Now by Lemma 4.5.13 we have

$$\det\left[\hat{C}(i,i')\right] - \det\left[\hat{C}(i,i') - \frac{1}{w_i+1}\right] = \sum_{k,\ell=1}^N (-1)^{\ell+k} \det[\hat{C}^{\ell,k}] \cdot \frac{1}{w_\ell + 1}, \qquad (4.87)$$

where $\hat{C}^{\ell,k}$ is the matrix obtained by removing row ℓ and column k from $\hat{C}$. Similarly

$$\det\left[\hat{C}(i,i')\right] - \det\left[\hat{C}(i,i') + \frac{1}{w'_{i'}+1}\right] = -\sum_{k,\ell=1}^N (-1)^{\ell+k} \det[\hat{C}^{\ell,k}] \cdot \frac{1}{w'_k + 1}. \qquad (4.88)$$

Now since $\hat{C}^{\ell,k}$ has the same entries as $\hat{C}$ only omitting row ℓ and column k, by (4.86) we have

$$\det[\hat{C}^{\ell,k}] = (1-\mu)^{N-2} \det[C^{\ell,k}] \cdot \left(1 - \mu \cdot \frac{B(-1/p)A(0)}{A(-1/p)B(0)} \cdot \frac{w'_k(1+pw_\ell)}{w_\ell(1+pw'_k)}\right). \qquad (4.89)$$

Where $C^{\ell,k}$ is obtained from removing row ℓ and column k from the $N \times N$ Cauchy matrix C with $C(i,i') = \frac{1}{w_i - w'_{i'}}$ and $A(z) := \prod_{j=1}^N (z - w_j)$ and $B(z) := \prod_{j=1}^N (z - w'_j)$.

Inserting (4.89) into (4.87) we see

$$\begin{aligned}
&\det\left[\hat{C}(i,i')\right] - \det\left[\hat{C}(i,i') - \frac{1}{w_i+1}\right] \\
&= (1-\mu)^{N-2}\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{1}{w_\ell+1} \\
&-\mu(1-\mu)^{N-2}\cdot\frac{B(-1/p)A(0)}{A(-1/p)B(0)}\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{w'_k(1+pw_\ell)}{w_\ell(1+w_\ell)(1+pw'_k)}.
\end{aligned} \tag{4.90}$$

Similarly

$$\begin{aligned}
&\det\left[\hat{C}(i,i')\right] - \det\left[\hat{C}(i,i') + \frac{1}{w'_{i'}+1}\right] \\
&= -(1-\mu)^{N-2}\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{1}{w'_k+1} \\
&+\mu(1-\mu)^{N-2}\cdot\frac{B(-1/p)A(0)}{A(-1/p)B(0)}\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{w'_k(1+pw_\ell)}{w_\ell(1+w'_{k'})(1+pw'_k)}.
\end{aligned} \tag{4.91}$$

Now by (4.96) and Lemma 4.5.15 we have

$$\begin{aligned}
&\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{1}{w_\ell+1} = \det[C]\cdot\left(1-\frac{B(-1)}{A(-1)}\right), \\
&\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{1}{w'_k+1} = \det[C]\cdot\left(\frac{A(-1)}{B(-1)}-1\right), \\
&\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{w'_k(1+pw_\ell)}{w_\ell(1+w_\ell)(1+pw'_k)} \\
&= -\frac{1}{p}\det[C]\cdot\left[\left(1-\frac{A(-1/p)}{B(-1/p)}\right)\frac{B(0)}{A(0)} + \left(p-1+\frac{A(-1/p)}{B(-1/p)}\right)\frac{B(-1)}{A(-1)} - p\right], \\
&\sum_{k,\ell=1}^{N}(-1)^{\ell+k}\det[C^{\ell,k}]\cdot\frac{w'_k(1+pw_\ell)}{w_\ell(1+w_\ell)(1+pw'_k)} \\
&= -\frac{1}{1-p}\det[C]\cdot\left[\left(\frac{B(0)}{A(0)}-1\right)\frac{A(-1/p)}{B(-1/p)} + \left(p-\frac{B(0)}{A(0)}\right)\frac{A(-1)}{B(-1)} + 1 - p\right].
\end{aligned} \tag{4.92}$$

Inserting (4.92) into (4.90) and (4.91) and combine with (4.83), after some tedious simplification we conclude that

$$\hat{S}_N = (1-\mu)^N \left(\prod_{j=1}^{N} (w_j + 1) - \prod_{j=1}^{N} (w'_j + 1) \right) \cdot \det \left[\frac{1}{w_i - w'_{i'}} \right]_{i,i'=1}^{N}. \tag{4.93}$$

This completes the proof of Proposition 4.5.4.

4.5.6 Perturbation formulas for Cauchy determinants

In this section we collect all the elementary linear algebra facts needed in the proof of Proposition 4.5.4. Some of them have already been discussed in [9]. First we state a general linear algebra lemma on rank-one perturbations:

Lemma 4.5.13. *Let $D = [D_{ij}]_{i,j=1}^{n}$ be an $n \times n$ matrix. Then for any function $f, g : \mathbb{C} \to \mathbb{C}$ and complex numbers $x_1, \cdots, x_n$, $y_1, \cdots, y_n$ we have*

$$\det \left[D_{ij} + f(x_i) g(y_j) \right]_{i,j=1}^{n} = \det[D] + \sum_{k,\ell=1}^{n} (-1)^{\ell+k} \det[D^{\ell,k}] g(y_k) f(x_\ell), \tag{4.94}$$

where $D^{\ell,k}$ is obtained by removing row ℓ and column k from D.

Proof. For D invertible by rank-one property and Cramer's rule we have

$$\begin{aligned} \det \left[D_{ij} + f(x_i) g(y_j) \right]_{i,j=1}^{n} &= \det[D] \left(1 + \sum_{k,\ell=1}^{n} g(y_k) (D^{-1})_{k,\ell} f(x_\ell) \right) \\ &= \det[D] + \sum_{k,\ell=1}^{n} (-1)^{\ell+k} \det[D^{\ell,k}] g(y_k) f(x_\ell). \end{aligned}$$

For general matrix D we pick $\{\epsilon_k\}_{k=1}^{\infty}$ such that $\epsilon_k \to 0$ as $k \to \infty$ and $D + \epsilon_k I_n$ are invertible for all ϵ_k. Now apply the above argument for $D + \epsilon_k I_n$ and let $k \to \infty$. □

Next we specialize to the case of C being a Cauchy matrix when the minors can be explicitly calculated:

Lemma 4.5.14. *Assume further that the matrix C is a Cauchy matrix with (i, j)-th entry $\frac{1}{x_i - y_j}$ for distinct complex numbers $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$. Then we further have*

$$\det \left[C_{ij} + f(x_i)g(y_j)\right]_{i,j=1}^{n} = \det[C] \cdot \left(1 - \sum_{k,\ell=1}^{n} \frac{f(x_\ell)B(x_\ell)A(y_k)g(y_k)}{(x_\ell - y_k)A'(x_\ell)B'(y_k)}\right). \qquad (4.95)$$

Here $A(z) := \prod_{i=1}^{n}(z - x_i)$ and $B(z) := \prod_{i=1}^{n}(z - y_i)$ are monic polynomials with roots at x_i's and y_i's.

Proof. For Cauchy matrix C we have

$$\det[C] = \frac{\prod_{1 \le i < j \le n}(x_i - x_j)(y_i - y_j)}{\prod_{1 \le i,j \le n}(x_i - y_j)}.$$

Note that $C^{\ell,k}$ is also a Cauchy matrix so we have

$$\det[C^{\ell,k}] = \frac{\prod_{\substack{1 \le i < j \le n \\ i \ne \ell, j \ne k}}(x_i - x_j)(y_i - y_j)}{\prod_{\substack{1 \le i,j \le n \\ i \ne \ell, j \ne k}}(x_i - y_j)}.$$

Hence

$$\frac{\det[C^{\ell,k}]}{\det[C]} = (-1)^{\ell+k+1} \frac{B(x_\ell)A(y_k)}{(x_\ell - y_k)A'(x_\ell)B'(y_k)}. \qquad (4.96)$$

Now (4.94) and (4.96) imply (4.95). □

For special choices of f and g, equation (4.95) can be further simplified using the residue theorem. We list here all we need in the proof of Proposition 4.5.4.

Lemma 4.5.15. *Given distinct complex numbers $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$. Let C be the Cauchy matrix with (i, j)-th entry $\frac{1}{x_i - y_j}$ and $A(z) = \prod_{i=1}^{n}(z - x_i)$ and $B(z) = \prod_{i=1}^{n}(z - y_i)$. Then*

1. *For* $f(x) = \frac{x}{(1+x)(1+px)}$ *and* $g(y) = \frac{1+py}{y}$ *we have* $\sum_{k,\ell=1}^{n} \frac{f(x_\ell)B(x_\ell)A(y_k)g(y_k)}{(x_\ell - y_k)A'(x_\ell)B'(y_k)}$ *equals*

$$-\frac{1}{1-p} \cdot \left[\left(\frac{A(0)}{B(0)} - 1 \right) \frac{B(-1/p)}{A(-1/p)} + \left(p - \frac{A(0)}{B(0)} \right) \frac{B(-1)}{A(-1)} + 1 - p \right]. \quad (4.97)$$

2. *For* $f(x) = \frac{x}{1+px}$ *and* $g(y) = \frac{1+py}{y(1+y)}$ *we have* $\sum_{k,\ell=1}^{n} \frac{f(x_\ell)B(x_\ell)A(y_k)g(y_k)}{(x_\ell - y_k)A'(x_\ell)B'(y_k)}$ *equals*

$$-\frac{1}{p} \cdot \left[\left(1 - \frac{B(-1/p)}{A(-1/p)} \right) \frac{A(0)}{B(0)} + \left(p - 1 + \frac{B(-1/p)}{A(-1/p)} \right) \frac{A(-1)}{B(-1)} - p \right]. \quad (4.98)$$

3. *For* $f(x) = \frac{1}{1+x}$ *and* $g(y) = 1$ *we have*

$$\sum_{k,\ell=1}^{n} \frac{f(x_\ell)B(x_\ell)A(y_k)g(y_k)}{(x_\ell - y_k)A'(x_\ell)B'(y_k)} = \frac{B(-1)}{A(-1)} - 1. \quad (4.99)$$

Proof. We will only prove part (1), the arguments for the other parts are similar. For $f(z) = \frac{z}{(1+z)(1+pz)}$ and $g(\xi) = \frac{1+p\xi}{\xi}$ consider the double contour integral

$$\oint_{|z|=R} \oint_{|\xi|=r} \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} f(z)g(\xi) \frac{B(z)A(\xi)}{(z-\xi)A(z)B(\xi)}.$$

Where $R > r$ are both large enough so that all the possible poles of the integrand are inside the integral contours. Now since for fixed r the integrand is of order $O(R^{-2})$, the double integral goes to 0 as $R \to \infty$. Thus for all R large enough the double contour integral equals 0. On the other hand by the residue theorem we have

$$\begin{aligned}
0 &= \oint_{|z|=R}\oint_{|\xi|=r} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\frac{\mathrm{d}z}{2\pi\mathrm{i}} f(z)g(\xi)\frac{B(z)A(\xi)}{(z-\xi)A(z)B(\xi)} \\
&= \oint_{|z|=R} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{f(z)}{z}\frac{B(z)A(0)}{A(z)B(0)} + \sum_{k=1}^{n}\oint_{|z|=R}\frac{\mathrm{d}z}{2\pi\mathrm{i}} f(z)g(y_k)\frac{B(z)A(y_k)}{(z-y_k)A(z)B'(y_k)} \\
&= \frac{1}{1-p}\sum_{k=1}^{n}\frac{A(y_k)}{B'(y_k)}\left(\frac{B(-1)}{A(-1)}\frac{1+py_k}{y_k(1+y_k)} - \frac{B(-1/p)}{A(-1/p)}\frac{1}{y_k}\right) \\
&+ \sum_{k,\ell=1}^{n} f(x_\ell)g(y_k)\frac{B(x_\ell)A(y_k)}{(x_\ell-y_k)A'(x_\ell)B'(y_k)}.
\end{aligned} \tag{4.100}$$

Where in the first equality the first contour integral is $O(R^{-2})$ hence 0 for R large enough. The single sum over $1 \le k \le n$ can be obtained as the residue terms for the following single contour integral:

$$\oint_{|\xi|=r}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\frac{1+p\xi}{\xi(1+\xi)}\frac{A(\xi)}{B(\xi)} = \frac{A(0)}{B(0)} - (1-p)\frac{A(-1)}{B(-1)} + \sum_{k=1}^{n}\frac{A(y_k)}{B'(y_k)}\frac{1+py_k}{y_k(1+y_k)}.$$

Here r is large enough so that $|\xi| = r$ contains all possible poles of the integrand inside. On the other hand by considering the residue at ∞ we have $\oint_{|\xi|=r}\frac{d\xi}{2\pi i}\frac{1+p\xi}{\xi(1+\xi)}\frac{A(\xi)}{B(\xi)} = p$. Hence

$$\sum_{k=1}^{n}\frac{A(y_k)}{B'(y_k)}\frac{1+py_k}{y_k(1+y_k)} = p - \frac{A(0)}{B(0)} + (1-p)\frac{A(-1)}{B(-1)}. \tag{4.101}$$

Similar residue analysis on the contour integral $\oint_{|\xi|=r}\frac{\mathrm{d}\xi}{2\pi\mathrm{i}}\frac{1}{\xi}\frac{A(\xi)}{B(\xi)}$ gives

$$\sum_{k=1}^{n}\frac{A(y_k)}{B'(y_k)}\frac{1}{y_k} = 1 - \frac{A(0)}{B(0)}. \tag{4.102}$$

Inserting (4.101) and (4.102) into (4.100) we obtain

$$\begin{aligned}&\sum_{k,\ell=1}^{n} f(x_\ell)g(y_k)\frac{B(x_\ell)A(y_k)}{(x_\ell-y_k)A'(x_\ell)B'(y_k)}\\&=-\frac{1}{1-p}\cdot\left[\left(p-\frac{A(0)}{B(0)}\right)\frac{B(-1)}{A(-1)}+1-p-\left(1-\frac{A(0)}{B(0)}\right)\frac{B(-1/p)}{A(-1/p)}\right].\end{aligned}\tag{4.103}$$

This completes the proof of part (1). □

4.6 Large-time asymptotics under relaxation time scale

In this section we discuss the large time limit of the multi-point distribution for dpTASEP$(L,N,\vec{y})$ under the relaxation time scale $t=O(L^{3/2})$. In Theorem 4.2.4 we state the limit theorem for general initial condition satisfying certain conditions. Below are the precise assumptions on the initial conditions we need:

4.6.1 Assumptions on the initial condition

We now state the assumptions on the sequence of the initial conditions $\vec{y}(L)$ under which we prove the limit theorem. The conditions are in terms of the global energy function and the characteristic function defined in Definition 4.2.9.

Assumption 4.6.1. We assume that the sequence of the initial profiles $\vec{y}=\vec{y}(L)$ satisfies the following three conditions as $L\to\infty$.

(A) (Convergence of global energy) There exist a constant $r\in(0,1)$ and a non-zero function $E_{\mathrm{ic}}(\mathrm{z})$ such that for every $0<\epsilon<1/2$,

$$\mathcal{E}_{\vec{y}}(z)=E_{\mathrm{ic}}(\mathrm{z})\left(1+O(L^{\epsilon-1/2})\right)$$

uniformly for $|\mathrm{z}|<r$ as $L\to\infty$.

(B) (Convergence of characteristic function) There exist constants $0 < r_1 < r_2 < 1$ and a function $\mathrm{ch}_{\mathrm{ic}}(\eta, \xi; \mathrm{z})$ such that for every $0 < \epsilon < 1/8$,

$$\chi_{\vec{y}}(v, u; z) = \mathrm{ch}_{\mathrm{ic}}(\eta, \xi; \mathrm{z}) + O(L^{4\epsilon - 1/2})$$

uniformly for $r_1 < |\mathrm{z}| < r_2$, $u \in \mathcal{L}_z^{(\epsilon)}$ and $v \in \mathcal{R}_z^{(\epsilon)}$ as $L \to \infty$ where

$$\xi = \mathcal{M}_{L,\mathrm{left}}(u) \in \mathrm{L_z} \quad \text{and} \quad \eta = \mathcal{M}_{L,\mathrm{right}}(v) \in \mathrm{R_z}$$

are the images under the maps defined in Lemma 4.7.4.

(C) (Tail estimates of characteristic function) Let r_1 and r_2 be same as in (B). There are constants $\epsilon'', C' > 0$ such that

$$|\chi_{\vec{y}}(v, u; z)| \leq C' L^{\epsilon''} \tag{4.104}$$

for all $(v, u) \in \mathcal{R}_z \times \mathcal{L}_z$ for all $r_1 < |\mathrm{z}| < r_2$.

4.6.2 Step and flat initial conditions

It turns out that Assumption 4.6.1 is not easy to check in general. Nevertheless we are able to verify them for at least the classical step and flat initial conditions. The following proposition combined with Theorem 4.2.4 gives the corresponding limit theorems for dpTASEP started with step and flat initial conditions.

Proposition 4.6.2. *(i) For step initial condition $\vec{y}_{\mathrm{step}} = (-1, -2, \cdots - N)$, Assumption 4.6.1 holds with*

$$E_{\mathrm{step}}(\mathrm{z}) = 1 \quad \textit{and} \quad \mathrm{ch}_{\mathrm{step}}(\eta, \xi; \mathrm{z}) = 1. \tag{4.105}$$

(ii) For flat initial condition $\vec{y}_{\mathrm{flat}} = (-d, \cdots, -Nd)$, where we assume $d = L/N \in \mathbb{N}$, Assumption 4.6.1 holds with

$$\begin{aligned} E_{\mathrm{flat}}(\mathrm{z}) &= (1-\mathrm{z})^{-1/4} e^{-B(\mathrm{z})}, \\ \mathrm{ch}_{\mathrm{flat}}(\eta, \xi; \mathrm{z}) &= e^{-\mathrm{h}(\xi,\mathrm{z}) - \mathrm{h}(\eta,\mathrm{z})} \eta(\eta - \xi) \mathbf{1}_{\xi=-\eta}, \end{aligned} \tag{4.106}$$

for $0 < |\mathrm{z}| < 1$, where $B(\mathrm{z})$ is defined in equation (4.116), $\mathrm{h}(\zeta, \mathrm{z})$ is defined in equation (4.117) and (4.118).

The step case is trivial since by the discussion in Remark 4.2.10 we have $\mathcal{E}_{\mathrm{step}}(z) = \chi_{\mathrm{step}}(v, u; z) \equiv 1$. The calculation for flat case is a bit more involved and we postpone the proof till Section 4.7.5.

4.6.3 Formula for the limiting distribution

The following formula for the relaxation-time limiting distribution was first obtained in [9] for step initial condition and [10] for more general initial conditions. The formula involves $\mathrm{C}_{\mathrm{ic}}^{\mathrm{per}}(\vec{z})$ which are limits of $\mathscr{C}_{\vec{y}}^{(L)}(\vec{z})$ and operators $\mathrm{K}_1^{\mathrm{per}}$ and $\mathrm{K}_{\mathrm{ic}}^{\mathrm{per}}$ which is a limit of $\mathscr{K}_1^{(L)}$ and $\mathscr{K}_{\vec{y}}^{(L)}$. The operators $\mathrm{K}_1^{\mathrm{per}}$ and $\mathrm{K}_{\mathrm{ic}}^{\mathrm{per}}$ are defined on the sets

$$\mathrm{S}_1 := \mathrm{L}_{\mathrm{z}_1} \cup \mathrm{R}_{\mathrm{z}_2} \cup \mathrm{L}_{\mathrm{z}_3} \cup \cdots \cup \begin{cases} \mathrm{R}_{\mathrm{z}_m}, & \text{if } m \text{ is even}, \\ \mathrm{L}_{\mathrm{z}_m}, & \text{if } m \text{ is odd}, \end{cases} \tag{4.107}$$

and

$$\mathrm{S}_2 := \mathrm{R}_{\mathrm{z}_1} \cup \mathrm{L}_{\mathrm{z}_2} \cup \mathrm{R}_{\mathrm{z}_3} \cup \cdots \cup \begin{cases} \mathrm{L}_{\mathrm{z}_m}, & \text{if } m \text{ is even}, \\ \mathrm{R}_{\mathrm{z}_m}, & \text{if } m \text{ is odd}, \end{cases} \tag{4.108}$$

where L_{z} and R_{z} are the sets defined in Definition 4.6.3. We express the limiting distribution function $\mathbb{F}_{\mathrm{ic}}^{\mathrm{per}}$ in terms of the above terms.

Definition 4.6.3. *Given* $0 < |\mathrm{z}| < 1$, *we define the discrete sets* $\mathrm{S_z} := \mathrm{L_z} \cup \mathrm{R_z}$ *where*

$$\begin{aligned} \mathrm{L_z} &:= \{\xi \in \mathbb{C} : e^{-\xi^2/2} = \mathrm{z}\} \cap \{\mathrm{Re}(\xi) < 0\}, \\ \mathrm{R_z} &:= \{\eta \in \mathbb{C} : e^{-\eta^2/2} = \mathrm{z}\} \cap \{\mathrm{Re}(\eta) > 0\}. \end{aligned} \tag{4.109}$$

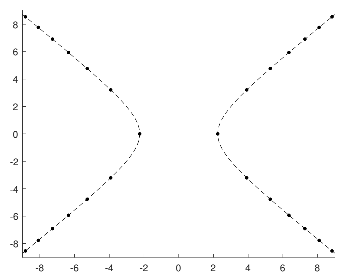

Figure 4.2: The roots for the equation $e^{-\zeta^2/2} = \mathrm{z}$ with $\mathrm{z} = 0.08$, the dashed lines are the corresponding level curve $|e^{-\zeta^2/2}| = |\mathrm{z}|$ for the same z.

Definition 4.6.4 (Limiting function)**.** *Let* $\mathbf{x} = (\mathrm{x}_1, \cdots, \mathrm{x_m})$, $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_m)$, *and* $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_m)$ *be points in* R^m *such that* $\mathrm{p}_j = (\gamma_j, \tau_j) \in [0,1] \times \mathbb{R}_{>0}$. *Assume that*

$$0 < \tau_1 \leq \cdots \leq \tau_m$$

and that $\mathrm{x}_i < \mathrm{x}_{i+1}$ *when* $\tau_i = \tau_{i+1}$ *for* $i = 1, \cdots, m-1$. *Define*

$$\mathbb{F}_{\mathrm{ic}}^{\mathrm{per}}(\mathrm{x}_1, \cdots, \mathrm{x}_m; \mathrm{p}_1, \cdots, \mathrm{p}_m) := \oint \cdots \oint \mathrm{C}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}}) \mathrm{D}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}}) \frac{\mathrm{dz}_m}{2\pi \mathrm{i} \mathrm{z}_m} \cdots \frac{\mathrm{dz}_1}{2\pi \mathrm{i} \mathrm{z}_1}, \tag{4.110}$$

where $\vec{\mathrm{z}} = (\mathrm{z}_1, \cdots, \mathrm{z}_m)$ *and the contours are nested circles satisfying* $0 < |\mathrm{z}_m| < \cdots < |\mathrm{z}_1| < 1$ *and also,* $r_1 < |\mathrm{z}_1| < r_2$ *with* r_1, r_2 *being the constants in Assumption 4.6.1 (B). The first function in the integrand is given by*

$$\mathrm{C}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}}) = E_{\mathrm{ic}}(\mathrm{z}_1) \mathrm{C}_{\mathrm{step}}^{\mathrm{per}}(\vec{\mathrm{z}}). \tag{4.111}$$

The second function is

$$\mathrm{D}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}}) = \det(I - \mathrm{K}_1^{\mathrm{per}}\mathrm{K}_{\mathrm{ic}}^{\mathrm{per}}), \tag{4.112}$$

where $\mathrm{K}_1^{\mathrm{per}} : \ell^2(\mathrm{S}_2) \to \ell^2(\mathrm{S}_1)$ *and* $\mathrm{K}_{\mathrm{ic}}^{\mathrm{per}} : \ell^2(\mathrm{S}_1) \to \ell^2(\mathrm{S}_2)$ *are given by* $\mathrm{K}_1^{\mathrm{per}} = \mathrm{K}_{\mathrm{step},1}^{\mathrm{per}}$ *and*

$$\mathrm{K}_{\mathrm{ic}}^{\mathrm{per}}(\zeta,\zeta') := \begin{cases} \mathrm{ch}_{\mathrm{ic}}(\zeta,\zeta';\mathrm{z}_1)\mathrm{K}_{\mathrm{step},2}^{\mathrm{per}}(\zeta,\zeta'), & \text{if } \zeta \in \mathrm{R}_{\mathrm{z}_1} \text{ and } \zeta' \in \mathrm{L}_{\mathrm{z}_1}, \\ \mathrm{K}_{\mathrm{step},2}^{\mathrm{per}}(\zeta,\zeta'), & \text{otherwise.} \end{cases}$$

The function $\mathrm{C}_{\mathrm{step}}^{\vec{\mathrm{z}}}(\vec{\mathrm{z}})$ and kernels $\mathrm{K}_{\mathrm{step},1}^{\mathrm{per}}$ and $\mathrm{K}_{\mathrm{step},2}^{\mathrm{per}}$ are first obtained in [9] and the definitions will be recalled in the next two sections for completeness.

4.6.4 The factor $\mathrm{C}_{\mathrm{step}}^{\mathrm{per}}(\vec{\mathrm{z}})$

Let $\mathrm{Li}_s(\mathrm{z})$ be the polylogarithm function which is defined by

$$\mathrm{Li}_s(\mathrm{z}) = \sum_{k=1}^{\infty} \frac{\mathrm{z}^k}{k^s} = \frac{\mathrm{z}}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - \mathrm{z}} dt. \tag{4.113}$$

for $|\mathrm{z}| < 1$ and $s \in \mathbb{C}$. Set

$$A_1(\mathrm{z}) = -\frac{1}{\sqrt{2\pi}}\mathrm{Li}_{3/2}(\mathrm{z}) \quad \text{and} \quad A_2(z) = -\frac{1}{\sqrt{2\pi}}\mathrm{Li}_{5/2}(z). \tag{4.114}$$

Let $\log \mathrm{z}$ denote the principal branch of the logarithm function with cut $\mathbb{R}_{\le 0}$. Set

$$\begin{aligned} B(\mathrm{z},\mathrm{z}') &= \frac{\mathrm{z}\mathrm{z}'}{2} \int\int \frac{\eta\xi \log(-\xi+\eta)}{(e^{-\xi^2/2} - \mathrm{z})(e^{-\eta^2/2} - \mathrm{z}')} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \\ &= \frac{1}{4\pi} \sum_{k,k' \ge 1} \frac{\mathrm{z}^k (\mathrm{z}')^{k'}}{(k+k')\sqrt{kk'}} \end{aligned} \tag{4.115}$$

for $0 < |\mathrm{z}|, |\mathrm{z}'| < 1$ where the integral contours are the vertical lines $\mathrm{Re}(\xi) = \mathrm{a}$ and $\mathrm{Re}(\eta) = \mathrm{b}$ with constants a and b satisfying $-\sqrt{-\log|\mathrm{z}|} < \mathrm{a} < 0 < \mathrm{b} < \sqrt{-\log|\mathrm{z}'|}$

oriented from bottom to top. We also set $B(\mathrm{z}) := B(\mathrm{z},\mathrm{z})$. One can check that

$$B(\mathrm{z}) = B(\mathrm{z},\mathrm{z}) = \frac{1}{4\pi}\int_0^{\mathrm{z}} \frac{(\mathrm{Li}_{1/2}(\mathrm{y}))^2}{\mathrm{y}}\mathrm{dy}. \tag{4.116}$$

Definition 4.6.5. *For $\vec{\mathrm{z}} = (\mathrm{z}_1, \cdots, \mathrm{z}_m)$ satisfying $0 < |\mathrm{z}_j| < 1$ and $\mathrm{z}_j \neq \mathrm{z}_{j+1}$ for all j, we define*

$$\mathrm{C}^{\mathrm{per}}_{\mathrm{step}}(\vec{\mathrm{z}}) = \left[\prod_{\ell=1}^{m} \frac{\mathrm{z}_\ell}{\mathrm{z}_\ell - \mathrm{z}_{\ell+1}}\right]\left[\prod_{\ell=1}^{m} \frac{e^{\mathrm{x}_\ell A_1(\mathrm{z}_\ell)+\tau_\ell A_2(\mathrm{z}_\ell)}}{e^{\mathrm{x}_\ell A_1(\mathrm{z}_{\ell+1})+\tau_\ell A_2(\mathrm{z}_{\ell+1})}} e^{2B(\mathrm{z}_\ell)-2B(\mathrm{z}_{\ell+1},\mathrm{z}_\ell)}\right],$$

where we set $\mathrm{z}_{m+1} = 0$.

Note that $\mathrm{C}^{\mathrm{per}}_{\mathrm{step}}(\vec{\mathrm{z}})$, and hence $\mathrm{C}^{\mathrm{per}}_{\mathrm{ic}}(\vec{\mathrm{z}})$, depend on x_i and τ_i, but not the spatial parameters γ_i.

4.6.5 The operators $\mathrm{K}^{\mathrm{per}}_{\mathrm{step},1}$ and $\mathrm{K}^{\mathrm{per}}_{\mathrm{step},2}$

Set

$$\mathrm{h}(\zeta,\mathrm{z}) = \frac{\mathrm{z}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}} \frac{\mathrm{w}\log(\mathrm{w}-\zeta)}{e^{-\mathrm{w}^2/2}-\mathrm{z}}\mathrm{dw} \qquad \text{for } \mathrm{Re}(\zeta) < 0 \text{ and } |\mathrm{z}| < 1 \tag{4.117}$$

and define

$$\mathrm{h}(\zeta,\mathrm{z}) = \mathrm{h}(-\zeta,\mathrm{z}) \qquad \text{for } \mathrm{Re}(\zeta) > 0 \text{ and } |\mathrm{z}| < 1. \tag{4.118}$$

For each i, define

$$\mathrm{f}_i(\zeta) = \begin{cases} e^{-\frac{1}{3}(\tau_i-\tau_{i-1})\zeta^3+\frac{1}{2}(\gamma_i-\gamma_{i-1})\zeta^2+(\mathrm{x}_i-\mathrm{x}_{i-1})\zeta} & \text{for } \mathrm{Re}(\zeta) < 0, \\ e^{\frac{1}{3}(\tau_i-\tau_{i-1})\zeta^3-\frac{1}{2}(\gamma_i-\gamma_{i-1})\zeta^2-(\mathrm{x}_i-\mathrm{x}_{i-1})\zeta} & \text{for } \mathrm{Re}(\zeta) > 0, \end{cases} \tag{4.119}$$

where we set $\tau_0 = \gamma_0 = \mathrm{x}_0 = 0$. We also define

$$\mathrm{Q}_1(j) = 1 - \frac{\mathrm{z}_{j-(-1)^j}}{\mathrm{z}_j} \quad \text{and} \quad \mathrm{Q}_2(j) = 1 - \frac{\mathrm{z}_{j+(-1)^j}}{\mathrm{z}_j},$$

where we set $\mathrm{z}_0 = \mathrm{z}_{m+1} = 0$.

Definition 4.6.6. *Let* S_1 *and* S_2 *be the discrete sets defined in* (4.107) *and* (4.108). *Let*

$$\mathrm{K}^{\mathrm{per}}_{\mathrm{step},1} : \ell^2(\mathrm{S}_2) \to \ell^2(\mathrm{S}_1) \quad \textit{and} \quad \mathrm{K}^{\mathrm{per}}_{\mathrm{step},2} : \ell^2(\mathrm{S}_1) \to \ell^2(\mathrm{S}_2)$$

denote the operators defined by their kernels

$$\mathrm{K}^{\mathrm{per}}_{\mathrm{step},1}(\zeta, \zeta') := (\delta_i(j) + \delta_i(j + (-1)^i)) \frac{\mathrm{f}_i(\zeta) e^{2\mathrm{h}(\zeta, \mathrm{z}_i) - \mathrm{h}(\zeta, \mathrm{z}_{i-(-1)^i}) - \mathrm{h}(\zeta', z_{j-(-1)^j})}}{\zeta(\zeta - \zeta')} \mathrm{Q}_1(j)$$

and

$$\mathrm{K}^{\mathrm{per}}_{\mathrm{step},2}(\zeta', \zeta) := (\delta_j(i) + \delta_j(i - (-1)^j)) \frac{\mathrm{f}_j(\zeta') e^{2\mathrm{h}(\zeta', \mathrm{z}_j) - \mathrm{h}(\zeta', \mathrm{z}_{j+(-1)^j}) - \mathrm{h}(\zeta, z_{i+(-1)^i})}}{\zeta'(\zeta' - \zeta)} \mathrm{Q}_2(i)$$

for

$$\zeta \in (\mathrm{L}_{\mathrm{z}_i} \cup \mathrm{R}_{\mathrm{z}_i}) \cap \mathrm{S}_1 \quad \textit{and} \quad \zeta' \in (\mathrm{L}_{\mathrm{z}_j} \cup \mathrm{R}_{\mathrm{z}_j}) \cap \mathrm{S}_2$$

with $1 \le i, j \le m$. *Here* $\mathrm{L}_{\mathrm{z}_i}$ *and* $\mathrm{R}_{\mathrm{z}_i}$ *are again defined in Definition 4.6.3.*

4.7 Proof of Theorem 4.2.4

In this section we discuss the proof of Theorem 4.2.4. The ideas are similar to the one in [9, 10] so we omit some technical details. Clearly the theorem follows immediately from the following two lemmas, dealing with the asymptotics of $\mathscr{C}^{(L)}_{\vec{y}}(z)$ and $\mathscr{D}^{(L)}_{\vec{y}}(z)$ appearing in the finite-time formula (4.6), respectively.

Lemma 4.7.1 (Asymptotics of $\mathscr{C}^{(L)}_{\vec{y}}(\vec{z})$)**.** *Under the same assumption as in Theorem 4.2.4, we have for fixed* $0 < \epsilon < 1/2$

$$\mathscr{C}^{(L)}_{\vec{y}}(\vec{z}) = \mathrm{C}^{\mathrm{per}}_{\mathrm{ic}}(\vec{\mathrm{z}}) \left(1 + O(L^{\epsilon - 1/2})\right), \quad \text{as } L \to \infty. \tag{4.120}$$

The functions $\mathscr{C}_{\vec{y}}^{(L)}(z)$ *and* $\mathrm{C}_{\mathrm{ic}}^{\mathrm{per}}(\mathrm{z})$ *are defined in* (3.78) *and* (3.47) *respectively, with* $z_i^L = (-1)^N \mathbf{r}_c^L \mathrm{z}_i$*, for* $1 \le i \le m$*.*

Lemma 4.7.2 (Asymptotics of $\mathscr{D}_{\vec{y}}^{(L)}(\vec{z})$)**.** *Under the same assumption as in Theorem 4.2.4, we have the convergence*

$$\lim_{L\to\infty} \mathscr{D}_{\vec{y}}^{(L)}(\vec{z}) = \mathrm{D}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}}), \tag{4.121}$$

where the Fredholm determinants $\mathscr{D}_{\vec{y}}^{(L)}(\vec{z})$ *and* $\mathrm{D}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}})$ *are defined in Section 4.2.4 and* (4.112)*, respectively, and the convergence is locally uniform in* $\vec{z}$*. Here again* z_i *and* z_i *are related by the equation* $z_i^L = (-1)^N \mathbf{r}_c^L \mathrm{z}_i$*.*

The rest of the section is devoted to proving Lemma 4.7.1 and 4.7.2. We start with a discussion on the asymptotic behaviors of the roots of the Bethe polynomial $q_z(w) = w^N(1+w)^{L-N} - z^L(1+pw)^N$ under the critical re-scaling in Section 4.7.1. Then in Section 4.7.2 we list a few lemmas discussing the asymptotics of several products involving these roots under the critical re-scaling. With these preparations we prove Lemma 4.7.1 and Lemma 4.7.2 in Section 4.7.3 and Section 4.7.4 respectively. Finally in Section 4.7.5 we verify the Assumption 4.6.1 for the classical step and flat initial conditions.

4.7.1 Asympotics of the Bethe roots

We assume that the particle density $\varrho := N/L$ stays within a compact subset of $(0,1)$ for all L. It turns out that in the asymptotic analysis for the finite-time formula we have to re-scale the integral parameters z_i so that $|z_i| \to \mathbf{r}_c$ in a certain rate and the main contributions come from Bethe roots within a distance of $O(L^{-1/2})$ to w_c. More precisely for $|z| < \mathbf{r}_c$ we introduced the re-scaled integral parameters z such that:

$$z^L = (-1)^N \mathbf{r}_c^L \mathrm{z},$$

where the assumption $|z| < \mathbf{r}_c$ is equivalent to $|\mathrm{z}| < 1$ and we recall that

$$w_c := -\frac{2\varrho}{1+\sqrt{1-4p\cdot\varrho(1-\varrho)}}, \quad \mathbf{r}_c := \left(\frac{-w_c}{1+pw_c}\right)^{\varrho}(1+w_c)^{1-\varrho}.$$

Under this re-scaling the nesting assumption on the integral parameters $0 < |z_m| < \cdots < |z_1| < \mathbf{r}_c$ in the finite-time m-points formulas becomes $0 < |\mathrm{z}_m| < \cdots < |\mathrm{z}_1| < 1$. From the discussion in Section 4.2.2 we know the level set $\{w \in \mathbb{C} : |w^N(1+w)^{L-N}| = |z^L(1+pw)^N|\}$ consists of two disjoint closed contours for $|z| < \mathbf{r}_c$ so we can define:

Definition 4.7.3. *Given $|z| < \mathbf{r}_c$, we define two closed contours Λ_L and Λ_R by*

$$\begin{aligned}\Lambda_L &:= \{w \in \mathbb{C} : |w^N(1+w)^{L-N}| = |z^L(1+pw)^N|\} \cap \{\mathrm{Re}(w) < w_c\},\\ \Lambda_R &:= \{w \in \mathbb{C} : |w^N(1+w)^{L-N}| = |z^L(1+pw)^N|\} \cap \{\mathrm{Re}(w) > w_c\}.\end{aligned} \tag{4.122}$$

A formal Taylor expansion at $w = w_c$ indicates that as $L \to \infty$, the Bethe equation $w^N(1+w)^{L-N} = z^L(1+pw)^N$ converges to the equation

$$e^{-\zeta^2/2} = \mathrm{z}, \tag{4.123}$$

where $z^L = (-1)^N \mathbf{r}_c^L \mathrm{z}$ and

$$w = w_c + \frac{1+\nu-2\varrho}{1+\nu}\sqrt{\frac{\varrho}{(1-\varrho)\nu}}\zeta L^{-1/2} := w_c + c_0\zeta L^{-1/2}, \tag{4.124}$$

where $\nu := \sqrt{1-4p\cdot\varrho(1-\varrho)}$ and $c_0 := \frac{1+\nu-2\varrho}{1+\nu}\sqrt{\frac{\varrho}{(1-\varrho)\nu}}$. The solution of equation (4.123) is a discrete set given by $\{\pm\sqrt{-2\log \mathrm{z} + 4k\pi i} : k \in \mathbb{Z}\}$ for an arbitrary choice of branches of logarithm and square root, see figure 4.2. Lemma 4.7.4 below precisely quantifies the convergence of Bethe roots near $w = w_c$ to the corresponding roots for the limiting equation.

Lemma 4.7.4. *For any* $0 < \epsilon < 1/8$ *and* $|z| < \mathbf{r}_c$ *fixed, we define*

$$\mathcal{L}_z^{(\epsilon)} := \mathcal{L}_z \cap \mathbb{D}(w_c, c_0 L^{-1/2+\epsilon}),$$

where $\mathbb{D}(a,r)$ *is a disc centered at* a *with radius* r *and* c_0 *is defined in* (4.124)*. Then for the re-scaled parameter* $\mathrm{z} = (-1)^N z^L \mathbf{r}_c^{-L}$ *we have an injective map* $\mathcal{M}_{L,left} : \mathcal{L}_z^{(\epsilon)} \to \mathrm{L_z}$ *satisfying*

$$\left|\mathcal{M}_{L,left}(u) - L^{1/2} c_0^{-1}(u - w_c)\right| \leq L^{-1/2+3\epsilon} \log L,$$

for all $u \in \mathcal{L}_z^{(\epsilon)}$ *and* L *large enough. Furthermore, the map satisfies*

$$\mathrm{L_z} \cap \mathbb{D}(0, L^{\epsilon} - 1) \subset \mathcal{M}_{L,left}(\mathcal{L}_z^{(\epsilon)}) \subset \mathrm{L_z} \cap \mathbb{D}(0, L^{\epsilon} + 1).$$

Similar results hold if we replace $\mathcal{L}_z$ *and* $\mathrm{L_z}$ *by* $\mathcal{R}_z$ *and* $\mathrm{R_z}$*.*

Proof. This lemma is a minor generalization of Lemma 8.1 of [8] by allowing one extra parameter p in the Bethe equation. The proof is almost identical to the one in [8] so we omit the details. □

4.7.2 Asymptotics of various products over Bethe roots

In this section we collect all the results involving limits of products of the Bethe roots appearing in the finite-time formula that are independent of the parameters $\vec{y}$, a_i, t_i and k_i. The starting point is the following simple integral formula for the sums of functions evaluated at the left or right Bethe roots.

Lemma 4.7.5. *Let* $\varphi(w)$ *be a function analytic in the interior and a neighborhood of* Λ_R*. Then*

$$\sum_{v \in \mathcal{R}_z} \varphi(v) = N\varphi(0) + \oint_{\Sigma_{\mathrm{R}}} \frac{dw}{2\pi i} \frac{z^L (1+pw)^N}{q_z(w)} \frac{\varphi(w)}{J(w)}, \tag{4.125}$$

where we recall that $J(w) = \frac{w(w+1)(1+pw)}{N+Lw+p(L-N)w^2}$*. Similarly if* $\varphi(w)$ *be a function analytic* *in the interior and a neighborhood of* Λ_L*, then*

$$\sum_{u\in\mathcal{L}_z}\varphi(u)=(L-N)\varphi(-1)+\oint_{\Sigma_{\mathrm{L}}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{z^L(1+pw)^N}{q_z(w)}\frac{\varphi(w)}{J(w)}. \tag{4.126}$$

Here Σ_{L} *and* Σ_{R} *are simple closed contours lie in the half-plane* $\{w:\mathrm{Re}(w)<w_c\}$ *(respectively* $\{w:\mathrm{Re}(w)>w_c\}$*) with* Λ_{L} *(respectively* Λ_{R}*) inside. Taking the function* φ *to be the constant function* 1 *implies in particular that*

$$\oint_{\Sigma_{\mathrm{L}}}\frac{dw}{2\pi i}\frac{z^L(1+pw)^N}{q_z(w)}\frac{1}{J(w)}=\oint_{\Sigma_{\mathrm{R}}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{z^L(1+pw)^N}{q_z(w)}\frac{1}{J(w)}=0.$$

Proof. A direct differentiation shows

$$q_z'(w)=q_z(w)\cdot\left(\frac{N}{w}+\frac{L-N}{w+1}\right)+z^L(1+pw)^N\cdot\frac{1}{J(w)}.$$

Hence by the residue theorem we know

$$\sum_{v\in\mathcal{R}_z}\varphi(u)=\oint_{\Sigma_R}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{q_z'(w)}{q_z(w)}\varphi(w)=N\varphi(0)+\oint_{\Sigma_R}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{z^L(1+pw)^N}{q_z(w)}\frac{1}{J(w)}\varphi(w).$$

The proof for (4.126) is similar. □

As taking the logarithm transforms products into sums, the following lemma is a direct consequence of Lemma 4.7.5 and the method of steepest descent.

Lemma 4.7.6. *Given* $|z|<\mathbf{r}_c$ *and* $\mathrm{z}=(-1)^N z^L\mathbf{r}_c^{-L}$*. Suppose* $\varrho=N/L$ *stays in a compact subset of* $(0,1)$*, then for every* $0<\epsilon<1/2$ *the following holds for all large enough* L*.*

(i) For $w=w_c+c_0\zeta L^{-1/2}$ *with* $|\zeta|\le L^{\epsilon/4}$*, where* $c_0=\frac{1+\nu-2\varrho}{1+\nu}\sqrt{\frac{\varrho}{(1-\varrho)\nu}}$ *and* $\nu=$

$\sqrt{1-4p\cdot\varrho(1-\varrho)}$,

$$\frac{z^L(1+pw)^N}{q_z(w)} = \frac{\mathrm{z}}{e^{-\zeta^2/2}-\mathrm{z}}\cdot\left(1+O(L^{-1/2+\epsilon})\right). \tag{4.127}$$

(ii) On the other hand if $|w-w_c|\geq C\cdot L^{\epsilon-1/2}$ *for some* $C>0$*, we have for some* $c>0$ *and* $\alpha>0$,

$$\left|\frac{z^L(1+pw)^N}{q_z(w)}\right| \leq e^{-cL^\alpha}. \tag{4.128}$$

(iii) For w *of* $O(1)$ *distance away from* $0,-1$ *and* $-1/p$ *we have* $\left|\frac{1}{J(w)}\right|\leq C\cdot L$ *for some constant* $C>0$*. Furthermore if* $w=w_c+c_0\zeta L^{-1/2}$ *with* $|\zeta|\leq C\cdot L^\epsilon$ *we have*

$$\frac{1}{J(w)} = -c_0^{-1}\zeta L^{1/2}\cdot\left(1+O(L^{\epsilon-1/2})\right). \tag{4.129}$$

(iv) For $w=w_c+c_0\zeta L^{-1/2}$ *with* $|\zeta|\leq L^{\epsilon/4}$*, we have*

$$\begin{aligned}\prod_{u\in\mathcal{L}_z}\sqrt{w-u} &= (\sqrt{w+1})^{L-N}e^{\frac{1}{2}\mathrm{h}(\zeta,\mathrm{z})}\left(1+O(L^{\epsilon-1/2}\log L)\right) \quad \mathrm{Re}(\zeta)\geq 0,\\ \prod_{v\in\mathcal{R}_z}\sqrt{v-w} &= (\sqrt{-w})^{N}e^{\frac{1}{2}\mathrm{h}(\zeta,\mathrm{z})}\left(1+O(L^{\epsilon-1/2}\log L)\right) \quad \mathrm{Re}(\zeta)\leq 0,\end{aligned} \tag{4.130}$$

where $\mathrm{h}(\zeta,\mathrm{z})$ *is the function defined in* (4.117)*. When* $\mathrm{Re}(\zeta)=0$*,* $\mathrm{h}(\zeta,\mathrm{z})$ *is the limit of* $\mathrm{h}(\eta,\mathrm{z})$ *as* $\eta\to\zeta$ *from* $\mathrm{Re}(\eta)>0$ *for the first case and from* $\mathrm{Re}(\eta)<0$ *for the second case.*

(v) For w *of a finite distance* $O(1)$ *away from the trajectory* $\Lambda_{\mathrm{L}}\cup\Lambda_{\mathrm{R}}$*,*

$$\begin{aligned}\prod_{u\in\mathcal{L}_z}\sqrt{w-u} &= (\sqrt{w+1})^{L-N}\left(1+O(L^{\epsilon-1/2})\right) \quad \text{if } \mathrm{Re}(w)>w_c,\\ \prod_{v\in\mathcal{R}_z}\sqrt{v-w} &= (\sqrt{-w})^{N}\left(1+O(L^{\epsilon-1/2})\right) \quad \text{if } \mathrm{Re}(w)<w_c.\end{aligned} \tag{4.131}$$

(vi) There is a constant $C > 0$ such that for every w satisfying $|w - w_c| \geq L^{\epsilon-1/2}$,

$$e^{-CL^{-\epsilon}} \leq \left| \frac{q_{z,\mathrm{L}}(w)}{(w+1)^{L-N}} \right| \leq e^{CL^{-\epsilon}} \quad \text{if } \mathrm{Re}(w) > w_c,$$

and

$$e^{-CL^{-\epsilon}} \leq \left| \frac{q_{z,\mathrm{R}}(w)}{w^N} \right| \leq e^{CL^{-\epsilon}} \qquad \text{if } \mathrm{Re}(w) < w_c.$$

(vii) We have

$$\frac{\prod_{v\in\mathcal{R}_z}\prod_{u\in\mathcal{L}_{z'}}\sqrt{v-u}}{\prod_{u\in\mathcal{L}_{z'}}\left(\sqrt{-u}\right)^N\prod_{v\in\mathcal{R}_z}\left(\sqrt{v+1}\right)^{L-N}} = e^{-B(\mathrm{z},\mathrm{z}')}\left(1+O(L^{\epsilon-1/2})\right), \tag{4.132}$$

where $B(\mathrm{z}, \mathrm{z}')$ is the function defined in (4.115).

Proof. These estimates are again one-parameter generalizations of Lemma 8.2 and Lemma 8.4 of [8]. The main difference is due to the extra parameter p in the Bethe polynomial $q_z(w) = w^N(1+w)^{L-N} - z^L(1+pw)^N$, the proper critical point for steepest descent analysis now is at $w_c = -\frac{2\varrho}{1+\sqrt{1-4p\cdot\varrho(1-\varrho)}}$, which comes from the larger root of the quadratic equation $p(L-N)w^2 + Lw + N = 0$, as opposed to $w = -\varrho$ for the $p = 0$ degeneration discussed in [8]. A standard steepest descent analysis using integral representations obtained in Lemma 4.7.5 with critical point w_c yields all the estimates. We omit the details. □

4.7.3 Asymptotics of $\mathscr{C}^{(L)}_{\vec{y}}(\vec{z})$

Now we are ready to prove Lemma 4.7.1. Recall that $\mathscr{C}^{(L)}_{\vec{y}}(z) = \mathcal{E}_{\vec{y}}(z_1)\mathscr{C}^{(L)}_{\mathrm{step}}(z)$ where $\mathcal{E}_{\vec{y}}(z_1) = E_{\mathrm{ic}}(\mathrm{z}_1)\left(1+O(L^{\epsilon-1/2})\right)$ as $L \to \infty$ due to Assumption 4.6.1. Hence it suffices to prove

$$\mathscr{C}^{(L)}_{\mathrm{step}}(\vec{z}) = \mathrm{C}^{\mathrm{per}}_{\mathrm{step}}(\vec{\mathrm{z}})\left(1+O(L^{\epsilon-1/2})\right), \quad \text{as } L \to \infty \tag{4.133}$$

where $\mathrm{C}^{\mathrm{per}}_{\mathrm{step}}(\vec{\mathrm{z}})$ is defined in Definition 4.6.5 and $\mathscr{C}^{(L)}_{\mathrm{step}}(\vec{z})$ is defined as follows:

$$\mathscr{C}^{(L)}_{\mathrm{step}}(\vec{z}) := \left[\prod_{\ell=1}^{m} \frac{E_\ell(z_\ell)}{E_{\ell-1}(z_\ell)}\right] \left[\prod_{\ell=1}^{m} \frac{\prod_{u\in\mathcal{L}_{z_\ell}}(-u)^N \prod_{v\in\mathcal{R}_{z_\ell}}(v+1)^{L-N}}{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_\ell})}\right] \cdot \left[\prod_{\ell=2}^{m} \frac{z_{\ell-1}^L}{z_{\ell-1}^L - z_\ell^L}\right] \left[\prod_{\ell=2}^{m} \frac{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_{\ell-1}})}{\prod_{u\in\mathcal{L}_{z_{\ell-1}}}(-u)^N \prod_{v\in\mathcal{R}_{z_\ell}}(v+1)^{L-N}}\right].$$

Here $E_\ell(z) := \prod_{u\in\mathcal{L}_z}(-u)^{-k_\ell} \prod_{v\in\mathcal{R}_z}(v+1)^{-a_\ell-k_\ell}(pv+1)^{t_\ell-k_\ell}$ with $E_0(z) := 1$. Under the re-scaling $z_\ell^L = (-1)^N \mathbf{r}_c^L \mathrm{z}_\ell$ we clearly have $\prod_{\ell=2}^{m} \frac{z_{\ell-1}^L}{z_{\ell-1}^L - z_\ell^L} = \prod_{\ell=2}^{N} \frac{\mathrm{z}_{\ell-1}}{\mathrm{z}_{\ell-1}-\mathrm{z}_\ell}$. On the other hand by Lemma 4.7.6 (iv)

$$\frac{\prod_{u\in\mathcal{L}_{z_\ell}}(-u)^N \prod_{v\in\mathcal{R}_{z_\ell}}(v+1)^{L-N}}{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_\ell})} = e^{B(\mathrm{z}_\ell)}\left(1+O(L^{\epsilon-1/2})\right),$$
$$\frac{\Delta(\mathcal{R}_{z_\ell};\mathcal{L}_{z_{\ell-1}})}{\prod_{u\in\mathcal{L}_{z_{\ell-1}}}(-u)^N \prod_{v\in\mathcal{R}_{z_\ell}}(v+1)^{L-N}} = e^{-B(\mathrm{z}_\ell,\mathrm{z}_{\ell-1})}\left(1+O(L^{\epsilon-1/2})\right).$$

Hence (4.133) follows immediately once we establish the following lemma on the asymptotics of $E_\ell(z)$.

Lemma 4.7.7. *Let* $E(z) = E(z;a,k,t) := \prod_{u\in\mathcal{L}_z}(-u)^{-k} \prod_{v\in\mathcal{R}_z}(v+1)^{-a-k}(pv+1)^{t-k}$ *where* $a,k\in\mathbb{Z}$ *and* $t\in\mathbb{N}$ *are given parameters. Then for* $z^L = (-1)^N\mathbf{r}_c^L\mathrm{z}$ *with* $0<|\mathrm{z}|<1$ *and the parameters satisfying*

$$t = c_1\tau L^{3/2}+O(1), \quad a = c_2 t+\gamma L+O(1), \quad k = c_3 t + c_4\gamma L + c_5 \mathrm{x} L^{1/2}+O(1), \tag{4.134}$$

we have for L *large enough and fixed* $0<\epsilon<1/2$

$$E(z) = e^{\mathrm{x}A_1(\mathrm{z})+\tau A_2(\mathrm{z})}\cdot\left(1+O(L^{\epsilon-1/2})\right). \tag{4.135}$$

where $A_1(\mathrm{z})$ *and* $A_2(\mathrm{z})$ *are scaled polylogarithm functions as in* (4.114). *The constants* c_i *are the same as in* (4.10) *and the re-scaled parameters are chosen such that* $\tau>0$, $\gamma\in[0,1]$ *and* $\mathrm{x}\in\mathbb{R}$.

Proof. Write

$$\log E(z) = \sum_{u \in \mathcal{L}_z} (-k) \log(-u) + \sum_{v \in \mathcal{R}_z} \left[(-a-k+1) \log(v+1) + (t-k+1) \log(1+pv) \right].$$

Apply Lemma 4.7.5 to the two sums over left and right Bethe roots and deform both contours Σ_{L} and Σ_{R} to the vertical line with real part $w_c = -\frac{2\varrho}{1+\sqrt{1-4p\varrho(1-\varrho)}}$ we have

$$\log E(z) = \int_{w_c - \mathrm{i}\infty}^{w_c + \mathrm{i}\infty} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{z^L (1+pw)^N}{q_z(w)} \frac{1}{J(w)} \left(G(w) - G(w_c) \right), \tag{4.136}$$

where $G(w) = (-k)\log(-w) + (a+k)\log(w+1) + (-t+k)\log(1+pw)$. Note that in (4.136) we have used the fact discussed in Lemma 4.7.5 that

$$\oint_{\Sigma_{\mathrm{L}}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{z^L(1+pw)^N}{q_z(w)} \frac{1}{J(w)} G(w_c) = \oint_{\Sigma_{\mathrm{R}}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{z^L(1+pw)^N}{q_z(w)} \frac{1}{J(w)} G(w_c) = 0.$$

Now a Taylor expansion at $w = w_c$ shows

$$\begin{aligned} G(w) - G(w_c) = G'(w_c)(w-w_c) + \frac{G''(w_c)}{2}(w-w_c)^2 + \frac{G'''(w_c)}{6}(w-w_c)^3 \\ + O\left(G^{(4)}(w_c) \cdot (w-w_c)^4 \right). \end{aligned}$$

Set $w = w_c + c_0 \zeta L^{-1/2}$ where c_0 as in equation (4.124) and assume the parameters are re-scaled as in (4.134). After a tedious but straightforward calculation we obtain for $|\zeta| \leq L^{\epsilon/4}$ with $0 < \epsilon < 1/2$

$$G(w) - G(w_c) = -\mathrm{x}\zeta - \frac{\gamma}{2}\zeta^2 + \frac{\tau}{3}\zeta^3 + O(L^{\epsilon - 1/2}). \tag{4.137}$$

Splitting the integral representation for $\log E(\mathrm{z})$ into two parts with $|\zeta| \leq L^{\epsilon/4}$ and $|\zeta| > L^{\epsilon/4}$ and using the estimates for $\frac{z^L(1+pw)^N}{q_z(w)}$ and $\frac{1}{J(w)}$ obtained in Lemma 4.7.6

we see

$$\log E(z) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{\mathrm{z}}{e^{-\zeta^2/2} - \mathrm{z}} \cdot \left(\mathrm{x}\zeta^2 + \frac{\gamma}{2}\zeta^3 - \frac{\tau}{3}\zeta^4\right) \cdot \left(1 + O(L^{\epsilon-1/2})\right) + O(e^{-cL^{\alpha}}), \tag{4.138}$$

for some constants $c, \alpha > 0$. Now (4.135) follows from integral representations of polylogarithm (4.113). □

4.7.4 Asymptotics of $\mathscr{D}_{\vec{y}}^{(L)}(\vec{z})$

Next we discuss the asymptotics of the Fredholm determinant part $\mathscr{D}_{\vec{y}}(z)$. Note first that by a standard series expansion of Fredholm determinants we have

$$\mathscr{D}_{\vec{y}}^{(L)}(\mathrm{z}) = \sum_{\vec{n} \in (\mathbb{Z}_{\geq 0})^m} \frac{1}{(n_1! \cdots n_m!)^2} \mathscr{D}_{\vec{y},\vec{n}}^{(L)}(\vec{z}), \tag{4.139}$$

where $\vec{n} = (n_1, \cdots, n_m)$ and

$$\mathscr{D}_{\vec{y},\vec{n}}^{(L)}(\vec{z}) = (-1)^{|\vec{n}|} \sum_{\substack{U^{(\ell)} \in (\mathcal{L}_{z_\ell})^{n_\ell} \\ V^{(\ell)} \in (\mathcal{R}_{z_\ell})^{n_\ell} \\ \ell = 1, \cdots, m}} \det[\mathscr{K}_1^{(L)}(w_i, w_j')]_{i,j=1}^{|\vec{n}|} \det[\mathscr{K}_{\vec{y}}^{(L)}(w_i', w_j)]_{i,j=1}^{|\vec{n}|}, \tag{4.140}$$

where $U^{(\ell)} = (u_1^{(\ell)}, \cdots u_{n_\ell}^{(\ell)})$ and $V^{(\ell)} = (v_1^{(\ell)}, \cdots v_{n_\ell}^{(\ell)})$ and

$$w_i = \begin{cases} u_k^{(\ell)} & \text{if } k = n_1 + \cdots + n_{\ell-1} + k \quad \text{for integer } k \leq n_\ell \text{ with } \ell \text{ odd }, \\ v_k^{(\ell)} & \text{if } k = n_1 + \cdots + n_{\ell-1} + k \quad \text{for integer } k \leq n_\ell \text{ with } \ell \text{ even }, \end{cases} \tag{4.141}$$

and

$$w_i' = \begin{cases} v_k^{(\ell)} & \text{if } k = n_1 + \cdots + n_{\ell-1} + k \quad \text{for integer } k \leq n_\ell \text{ with } \ell \text{ odd }, \\ u_k^{(\ell)} & \text{if } k = n_1 + \cdots + n_{\ell-1} + k \quad \text{for integer } k \leq n_\ell \text{ with } \ell \text{ even }. \end{cases} \tag{4.142}$$

A similar series expansion holds for the limiting Fredholm determinant $\mathrm{D}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}})$ with $\mathcal{L}_{z_\ell}$ and $\mathcal{R}_{z_\ell}$ replaced by the limiting roots $\mathrm{L}_{\mathrm{z}_\ell}$ and $\mathrm{R}_{\mathrm{z}_\ell}$ and the kernels replaced by the limiting kernels $\mathrm{K}_1^{\mathrm{per}}$ and $\mathrm{K}_{\mathrm{ic}}^{\mathrm{per}}$ defined in Definition 4.6.6.

$$\mathrm{D}_{\mathrm{ic}}^{\mathrm{per}}(\vec{\mathrm{z}}) = \sum_{\vec{n}\in(\mathbb{Z}_{\geq 0})^m} \frac{1}{(n_1!\cdots n_m!)^2} \mathrm{D}_{\mathrm{ic},\vec{n}}^{\mathrm{per}}(\vec{\mathrm{z}}). \tag{4.143}$$

We will prove the convergence of each of these $\mathscr{D}_{\vec{y},\vec{n}}^{(L)}(\vec{z})$ as well as some exponential bounds.

Lemma 4.7.8. *Under the same assumption in Theorem 4.2.4, for every fixed $\vec{n} \in (\mathbb{Z}_{\geq 0})^m$, we have*

(i) $\mathscr{D}_{\vec{y},\vec{n}}^{(L)}(\vec{z}) \to \mathrm{D}_{\mathrm{ic},\vec{n}}^{\mathrm{per}}(\mathrm{z})$ *as* $L \to \infty$.

(ii) There exists constant $C > 0$ such that $|\mathscr{D}_{\vec{y},\vec{n}}^{(L)}(\vec{z})| \leq C^{|\vec{n}|}$ for all L large enough.

It is clear that Lemma 4.7.2 follows immediately from Lemma 4.7.8 by dominated convergence theorem. To prove Lemma 4.7.8 we will prove the convergence of the kernels after proper conjugation for points inside the critical region as well as exponential decay estimates for the kernel at points outside the critical region. The conjugation is as follows: we replace $\mathscr{K}_1^{(L)}$ and $\mathscr{K}_{\vec{y}}^{(L)}$ defined in Section 4.2.4 by $\tilde{\mathscr{K}}_1^{(L)}$ and $\tilde{\mathscr{K}}_{\vec{y}}^{(L)}$ where

$$\begin{aligned}
&\tilde{\mathscr{K}}_1^{(L)}(w,w') := \\
&-\left(\delta_i(j)+\delta_i(j+(-1)^i)\right)\frac{J(w)\sqrt{\tilde{f}_i(w)}\sqrt{\tilde{f}_j(w')}(H_{z_i}(w))^2}{H_{z_{i-(-1)^i}}(w)H_{z_{j-(-1)^j}}(w')(w-w')}Q_1(j), \\
&\tilde{\mathscr{K}}_{\vec{y}}^{(L)}(w,w') := \\
&\left(\delta_j(i)+\delta_j(i-(-1)^j)\right)\frac{J(w')\sqrt{\tilde{f}_j(w')}\sqrt{\tilde{f}_i(w)}(H_{z_j}(w'))^2}{H_{z_{j+(-1)^j}}(w')H_{z_{i+(-1)^i}}(w)(w'-w)}Q_2(i)\Lambda(i,w,w'),
\end{aligned} \tag{4.144}$$

for $w \in (\mathcal{L}_{z_i} \cup \mathcal{R}_{z_i}) \cap \mathscr{S}_1$ and $w' \in (\mathcal{L}_{z_j} \cup \mathcal{R}_{z_j}) \cap \mathscr{S}_2$. Here

$$\tilde{f}_i(w) := \begin{cases} \frac{F_i(w)F_{i-1}(w_c)}{F_{i-1}(w)F_i(w_c)} & \text{for } \mathrm{Re}(w) < w_c, \\ \frac{F_i(w)F_{i-1}(w_c)}{F_{i-1}(w)F_i(w_c)} & \text{for } \mathrm{Re}(w) < w_c. \end{cases} \tag{4.145}$$

We define the square root to be $\sqrt{w} = r^{1/2}e^{i\theta/2}$ for $w = re^{i\theta}$ with $-\pi < \theta \leq \pi$. Note that the product of determinants will always be continuous even though the square root function is not since every $(\tilde{f}_i)^{1/2}$ is multiplied twice. We change the limiting kernels in a similar way by replacing $\mathrm{f_i}$ or f_j in the kernels with $\sqrt{\mathrm{f_i}}\sqrt{\mathrm{f_j}}$ and denote them as $\tilde{\mathrm{K}}_1^{\mathrm{per}}$ and $\tilde{\mathrm{K}}_{\mathrm{ic}}^{\mathrm{per}}$. We have the following asymptotics for the conjugated kernels, which easily implies Lemma 4.7.8.

Lemma 4.7.9. *Fix $0 < \epsilon < 1/(1+2m)$. Let*

$$\Omega = \Omega_L := \left\{ w \in \mathbb{C} : |w - w_c| \leq c_0^{-1} L^{-1/2+\epsilon/4} \right\} \tag{4.146}$$

be a disk centered at w_c. Under the same assumption in Theorem 4.2.4 we have

(i) As $L \to \infty$, uniformly for $w \in \mathscr{S}_1 \cap \Omega$ and $w' \in \mathscr{S}_2 \cap \Omega$

$$\begin{aligned} |\tilde{\mathscr{K}}_1^{(L)}(w, w')| &= |\tilde{\mathrm{K}}_1^{\mathrm{per}}(\zeta, \zeta')| + O(L^{\epsilon-1/2} \log L), \\ |\tilde{\mathscr{K}}_{\vec{y}}^{(L)}(w, w')| &= |\tilde{\mathrm{K}}_{\mathrm{ic}}^{\mathrm{per}}(\zeta, \zeta')| + O(L^{\epsilon-1/2} \log L), \end{aligned} \tag{4.147}$$

where $\zeta \in \mathrm{S}_1, \zeta' \in \mathrm{S}_2$ are the image of w, w' under either the map $\mathcal{M}_{L,left}$ or $\mathcal{M}_{L,right}$ in Lemma 4.7.4.

(ii) As $L \to \infty$, for $w_i \in \mathscr{S}_1 \cap \Omega$ and $w_i' \in \mathscr{S}_2 \cap \Omega$,

$$\begin{aligned} \det\left[\tilde{\mathscr{K}}_1^{(L)}(w_i, w_j')\right]_{i,j=1}^{|\vec{n}|} &\to \det\left[\tilde{\mathrm{K}}_1^{\mathrm{per}}(\zeta_i, \zeta_j')\right]_{i,j=1}^{|\vec{n}|}, \\ \det\left[\tilde{\mathscr{K}}_{\vec{y}}^{(L)}(w_i, w_j')\right]_{i,j=1}^{|\vec{n}|} &\to \det\left[\tilde{\mathrm{K}}_{\mathrm{ic}}^{\mathrm{per}}(\zeta_i, \zeta_j')\right]_{i,j=1}^{|\vec{n}|}, \end{aligned}$$

for each $\vec{n} \in (\mathbb{Z}_{\geq 0})^m$*, where* $\zeta \in \mathrm{S}_1, \zeta' \in \mathrm{S}_2$ *are the image of* w, w' *under either the map* $\mathcal{M}_{L,left}$ *or* $\mathcal{M}_{L,right}$ *in Lemma 4.7.4.*

(iii) There are positive constants c *and* α *such that*

$$|\tilde{\mathscr{K}}_1^{(L)}(w,w')| = O(e^{-cL^\alpha}), \qquad |\tilde{\mathscr{K}}_{\vec{y}}^{(L)}(w',w)| = O(e^{-cL^\alpha}) \tag{4.148}$$

as $L \to \infty$*, uniformly for* $w \in \mathscr{S}_1 \cap \Omega^c$ *and* $w' \in \mathscr{S}_2$*, and also for* $w' \in \mathscr{S}_2 \cap \Omega^c$ *and* $w \in \mathscr{S}_1$*.*

Proof. Due to the structure of the kernel (4.144), the lemma is proved once we establish the corresponding asymptotics and tail estimates for the functions $J(w)$, $H_{z_i}(w)$ and $\tilde{f}_i(w)$. For $J(w)$ and $H_z(w)$ these have already been discussed in Lemma 4.7.6 (iii),(iv) and (vi). The needed estimates for $\tilde{f}_i(w)$ is summarized in the following Lemma 4.7.10, the proof of which is similar to Lemma 4.7.7 so we omit the details. $\square$

Lemma 4.7.10. *Under the same assumption as in Theorem 4.2.4 for the parameters* a_i, k_i, t_i*, for* $w = w_c + c_0\zeta L^{-1/2}$ *with* $w \in \mathcal{L}_z \cup \mathcal{R}_z$ *we have for fixed* $0 < \epsilon < 1/2$

$$\tilde{f}_j(w) = \begin{cases} \mathrm{f}_j(\zeta)\left(1 + O(L^{\epsilon-1/2})\right) & \text{if } |\zeta| \leq L^{\epsilon/4} \\ O(e^{-cL^{3\epsilon/4}}) & \text{if } |\zeta| \geq L^{\epsilon/4} \end{cases} \tag{4.149}$$

for $1 \leq j \leq m$*, where* $\mathrm{f}_j(\zeta)$ *is defined in* (4.119)*.*

4.7.5 Proof of Theorem 4.6.2

In this section we verify that the flat initial condition satisfies Assumption 4.6.1 with the limiting functions E_{flat} and ch_{flat} given by (4.106). We start with a product formula for the pre-limit functions $\mathcal{E}_{\text{flat}}(z)$ and $\chi_{\text{flat}}(v,u;z)$.

Lemma 4.7.11. *Recall the global energy function $\mathcal{E}_{\vec{y}}(z)$ and characteristic function $\chi_{\vec{y}}(v,u;z)$ defined in Definition 4.2.9. For the flat initial condition $\vec{y}=(-d,\cdots,-Nd)$ with $d=L/N\in\mathbb{N}$ we have*

(i) With the standard square root function $\sqrt{w}$ with branch cut $\mathbb{R}_{\le 0}$,

$$\begin{aligned}\mathcal{E}_{\text{flat}}(z) = & \frac{\prod_{v\in\mathcal{R}_z}\left(\sqrt{v+1}\right)^{2-d}}{\prod_{v\in\mathcal{R}_z}\sqrt{p(d-1)v^2+dv+1}} \\ & \cdot\left[\frac{\prod_{v\in\mathcal{R}_z}\prod_{u\in\mathcal{L}_z}\sqrt{v-u}}{\prod_{u\in\mathcal{L}_z}\left(\sqrt{-u}\right)^N\prod_{v\in\mathcal{R}_z}\left(\sqrt{v+1}\right)^{L-N}}\right].\end{aligned} \tag{4.150}$$

(ii) For $v\in\mathcal{R}_z$ and $u\in\mathcal{L}_z$,

$$\chi_{\text{flat}}(v,u;z) = \begin{cases}\frac{u-v}{J(v)}\frac{(v+1)^{L-N}u^N}{q_{z,\mathrm{L}}(v)q_{z,\mathrm{R}}(u)}\frac{1+pv}{1+pu} & \text{if } \frac{u(u+1)^{d-1}}{1+pu}=\frac{v(v+1)^{d-1}}{1+pv}, \\ 0 & \text{otherwise},\end{cases} \tag{4.151}$$

where the functions are given by $J(w)=\frac{w(w+1)(1+pw)}{p(L-N)w^2+Lw+N}$, $q_{z,\mathrm{L}}(w):=\prod_{u\in\mathcal{L}_z}(w-u)$ and $q_{z,\mathrm{R}}(w):=\prod_{v\in\mathcal{R}_z}(w-v)$.

Proof. The proof is similar to the proof of Lemma 10.2 and Lemma 10.3 in [10]. The key observation is the existence of a $d-1$ to 1 map $\mathcal{M}$ from $\mathcal{L}_z$ to $\mathcal{R}_z$ satisfying that if $v=\mathcal{M}(u)$ for $u\in\mathcal{L}_z$, then $\frac{u(u+1)^{d-1}}{1+pu}=\frac{v(v+1)^{d-1}}{1+pv}$. Using this relation we can express the global energy and characteristic functions in terms of products over the Bethe roots. We omit the details. □

Combining Lemma 4.7.11 with the asymptotics obtained in Lemma 4.7.6 we can now prove Theorem 4.6.2.

Proof of Theorem 4.6.2. For fixed $0<\epsilon<1/2$ by Lemma 4.7.6 (viii) we have

$$\frac{\prod_{v\in\mathcal{R}_z}\prod_{u\in\mathcal{L}_{z'}}\sqrt{v-u}}{\prod_{u\in\mathcal{L}_{z'}}\left(\sqrt{-u}\right)^N\prod_{v\in\mathcal{R}_z}\left(\sqrt{v+1}\right)^{L-N}} = e^{-B(z,z')}\left(1+O(L^{\epsilon-1/2})\right). \tag{4.152}$$

Similarly by Lemma 4.7.6 (v) we have

$$\prod_{v\in\mathcal{R}_z}\left(\sqrt{v+1}\right)^{2-d}=1+O(L^{\epsilon-1/2}). \tag{4.153}$$

On the other hand we have $p(d-1)v^2+dv+1=p(d-1)(v-w_c)(w-w_c^*)$ where

$$w_c=\frac{-2\varrho}{1+\sqrt{1-4p\varrho(1-\varrho)}}\quad\text{and}\quad w_c^*=\frac{-2\varrho}{1-\sqrt{1-4p\varrho(1-\varrho)}}.$$

Hence by Lemma 4.7.6 (iv) and (v) we have

$$\begin{aligned}
&\prod_{v\in\mathcal{R}_z}\sqrt{p(d-1)v^2+dv+1}\\
&=\prod_{v\in\mathcal{R}_z}\sqrt{v-w_c}\cdot\prod_{v\in\mathcal{R}_z}\sqrt{p(d-1)(w-w_c^*)}\\
&=\left(\sqrt{-w_c}\right)^N e^{\frac{1}{2}h(0,\mathrm{z})}\cdot\left(\sqrt{p(d-1)}\cdot\sqrt{-w_c^*}\right)^N\cdot\left(1+O(L^{\epsilon-1/2})\right)\\
&=(1-\mathrm{z})^{1/4}\left(1+O(L^{\epsilon-1/2}\right),
\end{aligned} \tag{4.154}$$

where we used the fact that $w_c\cdot w_c^*=\frac{1}{p(d-1)}$ and also $e^{h(0,\mathrm{z})}=(1-\mathrm{z})^{1/2}$ which follows from (4.117). Combining (4.152), (4.153) and (4.154) we conclude that $\mathcal{E}_{\text{flat}}(z)=E_{\text{flat}}(\mathrm{z})(1+O(L^{\epsilon-1/2}))$ as $L\to\infty$.

The argument for the characteristic function part is quite similar. To verify part (B) of Assumption 4.6.1 we note that given $0<\epsilon<1/8$, for $u\in\mathcal{L}_z^{(\epsilon)}$ and $v\in\mathcal{R}_z^{(\epsilon)}$ as defined in Lemma 4.7.4 we have

$$\frac{q_{z,\mathrm{L}}(v)}{(1+v)^{L-N}}=e^{h(\eta,\mathrm{z})}\cdot\left(1+O(L^{4\epsilon-1/2})\right),\quad\frac{q_{z,\mathrm{R}}(u)}{u^N}=e^{h(\xi,\mathrm{z})}\cdot\left(1+O(L^{4\epsilon-1/2})\right), \tag{4.155}$$

where $\xi=\mathcal{M}_{L,\text{left}}(u)$ and $\eta=\mathcal{M}_{L,\text{right}}(v)$ with the injective maps $\mathcal{M}_{L,\text{left}}$ and $\mathcal{M}_{L,\text{right}}$ defined in Lemma 4.7.4 satisfying $\left|\xi-c_0^{-1}L^{1/2}(u-w_c)\right|\le L^{-1/2+3\epsilon}\log L$,

$\left|\eta - c_0^{-1}L^{1/2}(v-w_c)\right| \leq L^{-1/2+3\epsilon}\log L$. This then implies that

$$u - v = c_0 L^{-1/2}(\xi - \eta)\cdot\left(1 + O(L^{3\epsilon-1/2}\log L)\right). \tag{4.156}$$

A straightforward Taylor expansion combined with the injectivity of $\mathcal{M}_{L,\text{left}}$ and $\mathcal{M}_{L,\text{right}}$ shows

$$\mathbf{1}_{\frac{u(u+1)^{d-1}}{1+pu}=\frac{v(v+1)^{d-1}}{1+pv}} = \mathbf{1}_{\xi^2=\eta^2} = \mathbf{1}_{\xi=-\eta}. \tag{4.157}$$

Finally by Lemma 4.7.6 (iii) we have

$$\frac{1}{J(v)} = -c_0^{-1}\eta L^{1/2}\cdot(1+O(L^{\epsilon-1/2})). \tag{4.158}$$

Combining (4.155), (4.156), (4.157) and (4.158) we conclude that

$$\chi_{\text{flat}}(v,u;z) = \text{ch}_{\text{flat}}(\eta,\xi;\text{z}) + O(L^{4\epsilon-1/2}), \quad \text{as } L\to\infty. \tag{4.159}$$

Finally part (C) in Assumption 4.6.1 is clearly true since by Lemma 4.7.6 every factor in for $\chi_{\text{flat}}(v,u;z)$ is $O(1)$ except $\frac{1}{J(v)}$ which is $O(L)$. Thus $|\chi_{\text{flat}}(v,u;z)| \leq C\cdot L$ and part (C) of Assumption 4.6.1 is satisfied. □

Part II

Spectral Rigidity of Random Schrödinger Operators

CHAPTER 5

Introduction to Random Schrödinger Operators and Spatial Conditioning of Point Processes

5.1 Random Schrödinger Operators

5.1.1 Schrödinger Operators and Schrödinger Semigroups

Let $U \subset \mathbb{R}^d$ be some subset of the d-dimensional Euclidean space (typically some open subset or discrete lattice). A Schrödinger operator on U, which we denote by H, is of the form

$$Hf(x) := -\frac{1}{2}\Delta f(x) + V(x)f(x), \tag{5.1}$$

where

(i) The domain of H is typically a dense subset of $L^2(U)$ (or ℓ^2(U) when U is discrete)

(ii) $V : U \to \mathbb{R}$ is a deterministic function called the potential

(iii) Δ is a certain Laplacian-type operator on U

Spectral theory of Schrödinger operators is of fundamental interest to mathematical physcists due to their connections to Schrödinger equations in quantum mechanics and also heat-like diffusions. For the latter it is natural to consider the semigroup associated to a Schrödinger operator H, which is formally the one-parameter family of operators $(e^{-tH})_{t>0}$. Of crucial importance to our purposes (probabilistic) is the Feynmann-Kac formula for a Schrödinger semigroup which expresses the operator e^{-tH} through a functional of certain familiar stochastic processes (Brownian motion or other Markov processes on the state space). A typical example (see e.g. [100]) can be stated as follows:

Example 5.1.1. For $U = \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, one has

$$e^{-tH} f(x) = \mathbb{E}^x \left[\exp\left(-\int_0^t V(B(s))\,\mathrm{d}s \right) f(B(t)) \right], \quad x \in \mathbb{R}^d, \tag{5.2}$$

for a d-dimensional standard Brownian motion B and the expectation $\mathbb{E}^x$ is taking with respect to B conditioning on $B(0) = x$.

5.1.2 Multiplicative noise and random Schrödinger operators

In this part of the thesis we are mostly interested in a random perturbation of the operator H introduced in (5.1), namely we consider instead the random Schrödinger operators (RSOs) of the form

$$\hat{H} f(x) := H f(x) + \xi(x) f(x), \tag{5.3}$$

for a certain random functions $\xi : U \to \mathbb{R}$ which is usually called the noise. From the physical perspective ξ typically models the disorder of the underlying quantum models. The spectral theory of RSOs arises naturally in multiple problems in mathematical physics; we refer to [28] for a general introduction to the subject. When the noise ξ is smooth enough, one may expect a similar Feynman-Kac formula as in (5.2) for the perturbed operator (5.3):

$$e^{-t\hat{H}} = \mathbb{E}^x \left[\exp\left(-\int_0^t V(B(s)) + \xi(B(s))\, \mathrm{d}s \right) f(B(t)) \right], \quad x \in \mathbb{R}^d, \tag{5.4}$$

for the $U = \mathbb{R}^d$ example. In general however ξ may be rather singular (only a Schwartz distribution) and the pointwise product $\xi(x)f(x)$ in (5.3) or the function composition $\xi(B(s))$ in (5.4) may not be well-defined. We leave the proper interpretation of the Feynmann-Kac formula for irregular noises to later chapeters and stay at a formal level in this introductory part.

5.1.3 Two main motivating examples

RSOs of the form (5.3) have found applications in the study of random matrices and interacting particle systems, as well as stochastic partial differential equations (SPDEs). The following two examples serve as the main motivating examples for our study on RSOs and build the connection to the first part:

Example 5.1.2 (Stochastic Airy operators)**.** Consider one-dimensional RSO acting on the half-space $U = (0, \infty)$ with Dirichlet boundary condition at 0 of the following form:

$$\hat{\mathcal{H}}_{(0,\infty)} := -\frac{1}{2}\Delta + \frac{x}{2} + \xi_\beta, \tag{5.5}$$

where ξ_β is a white noise with variance $1/\beta$ for some $\beta > 0$. (For more precise definition see Example 6.2.6) Here the potential is a linear function $V(x) = x/2$. The operator $\hat{\mathcal{H}}_{(0,\infty)}$ naturally appears as the operator limit at the edge of the Dumitriu-Edelman tri-diagonal matrix models for the β-Coulomb gases. Their spectra are known as the *Airy-β processes* and describe the soft edge scaling limits of the β-ensembles (see [14, 76, 95]).

Example 5.1.3 (Parabolic Anderson model and stochastic heat equation)**.** In another direction, the study of the solutions of SPDEs of the form (known as the parabolic Anderson model)

$$\partial_t u = \frac{1}{2}\Delta u - \xi u = -\hat{\mathcal{H}}_I u \tag{5.6}$$

is intimately connected to the spectral theory of $\hat{\mathcal{H}}_I$. More specifically, the *localization* of $\hat{\mathcal{H}}_I$'s eigenfunctions is expected to shed light on the geometry of *intermittent peaks* in (5.6) (e.g., [74, Sections 2.2.3–2.2.4] and references therein). We refer to [29, 32, 47, 48] for a few examples of papers where such ideas have been implemented when ξ is a smooth, white, fractional, or otherwise singular noise (see Examples 6.2.6–6.2.9 for definitions of such noises). If in addition one considers a time-dependent space-time white noise $\xi(x, t)$, then (5.6) is known as the stochastic heat equation with multiplicative space-time white noise, and appears naturally as the linearization of the Kardar-Parisi-Zhang equation introduced in [72] through Cole-Hopf transformation.

5.2 Spatial Conditioning and Number Rigidity

Point processes are well-studied objects in probability [37, 69], due to their applications in many disciplines (e.g., [4]). One of the simplest point processes is the Poisson process, which is such that the numbers of points in disjoint sets are independent. In contrast, for point processes with strong correlations, the notion of *spatial conditioning* (i.e., the distribution of points inside a bounded set conditional on the point configuration outside the set) is of interest. Pioneering work on this subject includes the Dobrushin-Lanford-Ruelle (DLR) formalism (e.g., [39, Sections 1.4-2.4]).

In this part, we are interested in a form of spatial conditioning known as *number rigidity* [59]. A point process is said to be number rigid if for every bounded set A, the configuration of points outside of A determines the number of points inside of A. We refer to [2, 64] for examples of early work on this kind of property. In their seminal paper [59] (see also [54]), Ghosh and Peres introduced (among other things) the notion of number rigidity, and studied its occurrence in two classical point processes. Since then, number rigidity has been shown to have many interesting applications in the theory of point processes (e.g., [21, 26, 27, 54, 55, 90]), and has developed into an active field of research. We refer to [12, 13, 23, 24, 30, 46, 56, 59, 88] for other notions related to number rigidity, such as *higher order/linear rigidity*, *hyperuniformity*, *sub-extensivity*, *quasi-invariance/symmetry*, and *tolerance*.

5.2.1 Number Rigidity

Let Λ be a point process on $\mathbb{R}$ (i.e., a random locally finite counting measure on $\mathbb{R}$). Given a Borel set $A \subset \mathbb{R}$, we let $\Lambda(A)$ denote the number of points of Λ that are inside of A, that is,

$$\Lambda(A) := \sum_{\lambda \in \Lambda} \mathbf{1}_{\{\lambda \in A\}}.$$

More generally, for every function $f : \mathbb{R} \to \mathbb{R}$, we use

$$\Lambda(f) := \sum_{\lambda \in \Lambda} f(\lambda)$$

to denote the linear statistic associated with f. For any Borel set $A \subset \mathbb{R}$, we let $\mathscr{F}_\Lambda(A) := \sigma\big\{\Lambda(\bar{A}) : \bar{A} \subset A\big\}$ denote the σ-algebra generated by the configuration of points inside of A.

Definition 5.2.1 ([59]). *We say that Λ is* **number rigid** *if $\Lambda(A)$ is $\mathscr{F}_\Lambda(\mathbb{R} \setminus A)$-measurable for every bounded Borel set $A \subset \mathbb{R}$.*

5.2.2 The Ghosh-Peres Criterion

We have the following simple sufficient condition for number rigidity:

Proposition 5.2.2 ([59]). *Let $A \subset \mathbb{R}$ be a bounded Borel set. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions satisfying the following conditions.*

1. *Almost surely,* $|\Lambda(f_n \mathbf{1}_{\bar{A}})| < \infty$ *for every* $n \in \mathbb{N}$ *and* $\bar{A} \subset \mathbb{R}$.

2. $|f_n - 1| \to 0$ *as* $n \to \infty$ *uniformly on* A.

3. $\mathbf{Var}[\Lambda(f_n)] \to 0$ *as* $n \to \infty$.

Then, $\Lambda(A)$ *is* $\mathscr{F}_\Lambda(\mathbb{R} \setminus A)$*-measurable.*

Though Proposition 5.2.2 is by now standard in the rigidity literature (e.g., [59, Theorem 6.1]), we nevertheless provide its short proof for the reader's convenience:

Proof of Proposition 5.2.2. For every n, we can write

$$\Lambda(A) = \underbrace{\Lambda(f_n) - \mathbf{E}\big[\Lambda(f_n)\big]}_{E_1^{(n)}} + \underbrace{\Lambda\big((1 - f_n)\mathbf{1}_A\big)}_{E_2^{(n)}} - \underbrace{\big(\Lambda\big(f_n \mathbf{1}_{\mathbb{R}\setminus A}\big) - \mathbf{E}\big[\Lambda(f_n)\big]\big)}_{E_3^{(n)}}.$$

Since the variance of $\Lambda(f_n)$ vanishes, we can choose a sparse enough subsequence $(n_k)_{k\in\mathbb{N}}$ along which $E_1^{(n_k)} \to 0$ almost surely as $k \to \infty$. Next, we note that

$$|E_2^{(n_k)}| \leq \Lambda(A) \left(\sup_{x\in A} |f_{n_k}(x) - 1| \right),$$

which vanishes almost surely as $k \to \infty$ because Λ is locally finite and A is bounded. In particular, $E_3^{(n_k)} \to \Lambda(A)$ as $k \to \infty$, which completes the proof since $E_3^{(n)}$ is $\mathscr{F}_\Lambda(\mathbb{R} \setminus A)$-measurable for every n. □

5.2.3 Exponential linear spectral statistics of RSOs

Owing to the Ghosh-Peres criterion, a now standard way of establishing number rigidity is to control variance of certain linear statistics. Several techniques have been used thus far to control the variance of linear statistics for the purpose of proving number rigidity. Prominent examples include determinantal/Pfaffian or other integrable structure [22, 25, 54, 58, 59], translation invariance and hyperuniformity [57], and finite-dimensional approximations [96]. By using such methods, number rigidity has been established for the zeroes of the *planar Gaussian Analytic Function*, the *Ginibre ensemble*, the *Sine-*β *process* (for all $\beta > 0$), the *Airy-*2 *process*, some *Bessel and Gamma point processes*, and more.

The main objects we are focusing on in this part of the thesis is the eigenvalue point processes of random Schrödinger operators. Namely we are concerned with the following main problem:

Problem 5.2.3. For a given random Schödinger operator $\hat{H}$ which almost surely has compact resolvent, so it has pure point spectrum with eigenvalues

$$-\infty < \lambda_1(\hat{H}) \leq \lambda_2(\hat{H}) \leq \cdot \to \infty$$

What can we say about the point processes $\{\lambda_i(\hat{H})\}_{i=1}^{\infty}$? In particular, are they number rigid?

While there are some integrable structures for certain special RSOs (e.g. the stochastic Airy operator with $\beta = 2$), none of these results provide sufficient conditions that can be applied to general RSOs. We propose to study number rigidity in the spectrum of random Schrödinger operators using a new *semigroup method*: Given that the exponential functions $\mathrm{e}_n(z) := \mathrm{e}^{-z/n}$ converge uniformly to 1 on any bounded set as $n \to \infty$, in order to prove number rigidity of any point process, it suffices to prove that $\mathbf{Var}[\int \mathrm{e}_n \, \mathrm{d}\mathcal{X}] \to 0$ (though the requirement that $\int \mathrm{e}_n \, \mathrm{d}\mathcal{X}$ is finite imposes strong conditions on $\mathcal{X}$). If $\mathcal{X}$ happens to be the eigenvalue point process of a random Schrödinger operator H, then $\int \mathrm{e}_n \, \mathrm{d}\mathcal{X}$ is the trace of the operator $\mathrm{e}^{-H/n}$. Thus, in order to prove the number rigidity of the spectrum of any random Schrödinger operator H, it suffices to prove that

$$\lim_{t \to 0} \mathbf{Var}\big[\mathrm{Tr}[\mathrm{e}^{-tH}]\big] = 0.$$

The reason why this is a particularly attractive strategy to prove number rigidity of general random Schrödinger operators is that, thanks to the Feynman-Kac formula, there exists an explicit probabilistic representation of the semigroup $(\mathrm{e}^{-tH})_{t>0}$ in terms of elementary stochastic processes, making the variance $\mathbf{Var}\big[\mathrm{Tr}[\mathrm{e}^{-tH}]\big]$ amenable to computation.

Finally, as mentioned in Section 5.2, for many point processes the understanding of conditional distributions in spatial conditioning is more sophisticated than number rigidity, such as tolerance in [59] or explicit conditional distributions in [21, 26]. It would be interesting to see if similar insights in the conditional configurations of eigenvalues of general RSOs can be obtained. We leave this to future work.

CHAPTER 6

Spectral Rigidity of Continuous Random Schrödinger Operators via Feynman-Kac Formulas

6.1 Introduction

Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded), and let $V : I \to \mathbb{R}$ be a deterministic potential. Let $\xi : I \to \mathbb{R}$ be a centered stationary Gaussian process with a covariance of the form $\mathbb{E}[\xi(x)\xi(y)] = \gamma(x-y)$, where γ is an even function or Schwartz distribution. (We refer to Section 6.2.1 for a formal definition.) In this chapter, we investigate the number rigidity of the eigenvalue point processes of random Schrödinger operators (RSOs) of the form

$$\hat{\mathcal{H}}_I := -\tfrac{1}{2}\Delta + V + \xi, \tag{6.1}$$

where $\hat{\mathcal{H}}_I$ acts on a subset of functions $f : I \to \mathbb{R}$ that satisfy some fixed boundary conditions (if I has a boundary).

6.1.1 Outline of Results and Method of Proof

To the best of our knowledge, the only RSO whose spectrum is known to be number rigid is the operator

$$\hat{\mathcal{H}}_{(0,\infty)} = -\tfrac{1}{2}\Delta + \tfrac{x}{2} + \xi_2$$

with a Dirichlet boundary condition at zero, where ξ_2 is a white noise with variance 1/2. The proof of this [22] relies on the fact that the eigenvalues of this operator generate the Airy-2 process, which is a determinantal point process (see (6.76)). In this context, our main motivation in this chapter is to provide a unified framework to study the number rigidity of the eigenvalues of general RSOs. As a first step in this direction, we develop a new method of proving number rigidity for RSOs by controlling the variance of exponential linear statistics using Feynman-Kac formulas. Informally, our main result is as follows (we point to Theorems 6.2.21 and 6.2.23 for precise statements).

Theorem 6.1.1 (Informal Statement)**.** *Suppose that $\hat{\mathcal{H}}_I$ acts on either the full space $I = \mathbb{R}$, the half-line $I = (0,\infty)$, or the bounded interval $I = (0,b)$, under some general boundary conditions in the latter two cases (Assumption 6.2.10). Assume that the noise ξ and the deterministic potential V satisfy mild technical conditions (Assumptions 6.2.3 and 6.2.11).*

On the one hand, when I is unbounded, $\hat{\mathcal{H}}_I$'s spectrum is number rigid if V has sufficient growth at infinity (i.e., (6.15) *and* (6.18)–(6.21)*). On the other hand, if $I = (0,b)$, then $\hat{\mathcal{H}}_{(0,b)}$'s spectrum is always number rigid.*

Thus, one of the main advantages of the method developed in this chapter is that it applies under very general assumptions on the noise ξ, the domain I, the boundary conditions on I, and the regularity of the deterministic potential V. However, in cases where the domain I is unbounded, our method comes at the cost of growth assumptions on V.

Remark 6.1.2. It is worth noting that our main result does *not* imply rigidity of the Airy-β process for any $\beta > 0$, since our growth condition in the case of white noise requires V to be superlinear (see (6.18)). In fact, we prove that it is not possible to establish the rigidity of the Airy-2 process by using exponential linear functionals (see Proposition 6.2.25). This suggests (at least for white noise) that, while our growth conditions are not necessary for rigidity, they are the optimal conditions that can be obtained with our semigroup method; see Section 6.2.5 for more details.

The key steps in the proof of our main result are as follows.

(i) We state general conditions (see Assumptions 6.2.3, 6.2.10, and 6.2.11; and Proposition 6.2.20) under which exponential functionals e^{-tx} ($t > 0$) of the spectrum of $\hat{\mathcal{H}}_I$ admit a random Feynman-Kac representation. This follows from a combination of classical semigroup theory and the work on Feynman-Kac formulas for RSOs with irregular Gaussian noise [49, 51, 61] pioneered by Gorin and Shkolnikov.

(ii) The Feynman-Kac formulas in **(i)** give an explicit representation of $\hat{\mathcal{H}}_I$'s semigroup in terms of elementary stochastic processes. This allows to reformulate the vanishing of the variance of exponential linear statistics in terms of a corresponding limit for the self-intersection local time of Brownian bridges on $\mathbb{R}$, or reflected Brownian bridges on the half-line or bounded intervals (see (6.24) and Theorem 6.4.1).

(iii) The main tool we use to control the Brownian bridge self-intersection local time consists of large deviations results for the self-intersection local time of unconditioned Brownian motion on $\mathbb{R}$. The latter has been studied extensively; we refer to [31, Chapter 4] and references therein for details. To bridge the gap between the results on the self-intersection local time of Brownian bridges and the unconditioned Brownian motion, we make use of couplings between reflected Brownian motions on different domains, and the absolute continuity of the midpoint of bridge processes with respect to their unconditioned versions.

(iv) By combining **(i)**–**(iii)**, we obtain our main result (Theorem 6.2.21), which consists of general sufficient conditions (see (6.14) and (6.15)) for the number rigidity of $\hat{\mathcal{H}}_I$'s spectrum in terms of Brownian self-intersection times and the growth rate of V. Then, in Theorem 6.2.23 we apply this result to white, fractional, singular, and smooth noises.

Organization of the Chapter

The rest of this chapter is organized as follows. In Section 6.2, we introduce the setup of this chapter (including the Feynman-Kac formulas at the heart of our method), state our main results, and discuss their optimality. Section 6.3 contains estimates on the decay rate (for small time) of self-intersection local times that are crucial in our method of proof. In Section 6.4, we combine the estimates in Section 6.3 with our Feynman-Kac formulas to control the variance of exponential linear statistics, thus proving our main results, Theorems 6.2.21 and 6.2.23. Section 6.5 demonstrates that the variance of exponential linear statistics cannot be used to prove rigidity of the Airy-2 process. Finally, Section 6.6 provides an elementary estimate on stochastic analysis.

6.2 Setup and Main Results

This section is organized as follows. In Section 5.2.1, we give reminders for basic notions regarding number rigidity. In Section 6.2.1, we state our assumptions regarding the random perturbation ξ in (6.1), and we provide concrete examples of noises that satisfy these assumptions. In Section 6.2.2, we discuss the rigorous definition of the operator $\hat{\mathcal{H}}_I$ and its eigenvalue point process. In Section 6.2.3, we introduce the Feynman-Kac formulas with which we study exponential linear statistics of $\hat{\mathcal{H}}_I$'s spectrum, including a statement that the linear statistics in question are finite and well defined. In Section 6.2.4, we state our main results. Finally, we discuss the optimality of our results and related open problems in Section 6.2.5.

6.2.1 Noise

In this section, we describe the noise ξ considered in this chapter. (Much of the notation in this section and Sections 6.2.2 and 6.2.3 are directly inspired from [49].) Let PC_c denote the set of functions $f : \mathbb{R} \mapsto \mathbb{R}$ that are càdlàg and compactly supported. We begin by introducing the covariance functions that characterize the noise ξ.

Definition 6.2.1. *Let γ be an even function on $\mathbb{R}$ or an even Schwartz distribution on PC_c (that is, $\langle f, \gamma \rangle = \langle \tilde{f}, \gamma \rangle$ for every $f \in \mathrm{PC}_c$, where $\tilde{f}(x) := f(-x)$) such that*

$$\langle f, g \rangle_\gamma := \int_{\mathbb{R}^2} f(x)\gamma(x-y)g(y)\,\mathrm{d}x\mathrm{d}y, \qquad f, g \in \mathrm{PC}_c \tag{6.2}$$

is a semi-inner-product on PC_c, that is,

1. (6.2) *is finite and well defined for every $f, g \in \mathrm{PC}_c$;*

2. *$(f, g) \mapsto \langle f, g \rangle_\gamma$ is sesquilinear and symmetric; and*

3. *$\langle f, f \rangle_\gamma \geq 0$ for all $f \in \mathrm{PC}_c$.*

We denote the seminorm induced by $\langle \cdot, \cdot \rangle_\gamma$ as

$$\|f\|_\gamma := \sqrt{\langle f, f \rangle_\gamma}, \qquad f \in \mathrm{PC}_c.$$

We say that γ is **compactly supported** *if there exists a compact set $A \subset \mathbb{R}$ such that $\langle f, \gamma \rangle = 0$ whenever $f(x) = 0$ for every $x \in A$.*

Remark 6.2.2. In cases where γ is not an almost-everywhere-defined function, the integral over $\gamma(x-y)$ in (6.2) may not be well defined. In such cases, we interpret

$$\langle f, g\rangle_\gamma := \langle f, g * \gamma\rangle = \langle f * \tilde{g}, \gamma\rangle = \langle \tilde{f} * g, \gamma\rangle = \langle f * \gamma, g\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product and $*$ the convolution.

Throughout this chapter, we make the following assumption.

Assumption 6.2.3. We assume that there exists a γ as in Definition 6.2.1 such that

$$\|f\|_\gamma^2 \leq c_\gamma\big(\|f\|_{q_1}^2 + \cdots + \|f\|_{q_\ell}^2\big), \qquad f \in \mathrm{PC}_c \tag{6.3}$$

for some constant $c_\gamma > 0$ and $1 \leq q_1, \ldots, q_\ell \leq 2$, $\ell \in \mathbb{N}$, where $\|f\|_q := \big(\int_{\mathbb{R}} |f(x)|^q \,\mathrm{d}x\big)^{1/q}$ denotes the usual L^q norm.

If Assumption 6.2.3 holds, then it can be shown that there exists a centered Gaussian process $\Xi : \mathbb{R} \to \mathbb{R}$ such that

1. almost surely, $\Xi(0) = 0$ and Ξ has continuous sample paths;
2. Ξ has stationary increments; and
3. Ξ's covariance is given by

$$\mathbb{E}[\Xi(x)\Xi(y)] = \begin{cases} \langle \mathbf{1}_{[0,x)}, \mathbf{1}_{[0,y)}\rangle_\gamma & \text{if } x, y \geq 0 \\ \langle \mathbf{1}_{[0,x)}, -\mathbf{1}_{[y,0)}\rangle_\gamma & \text{if } x \geq 0 \geq y \\ \langle -\mathbf{1}_{[x,0)}, \mathbf{1}_{[0,y)}\rangle_\gamma & \text{if } y \geq 0 \geq x \\ \langle \mathbf{1}_{[x,0)}, \mathbf{1}_{[y,0)}\rangle_\gamma & \text{if } 0 \geq x, y. \end{cases} \tag{6.4}$$

Indeed, the existence of a Gaussian process with covariance (6.4) follows from standard existence theorems since $\langle\cdot,\cdot\rangle_\gamma$ is a semi-inner-product; the stationarity of increments follows from the fact that $\langle f, g\rangle_\gamma$ remains unchanged if we replace f and g by their translates $x \mapsto f(x-z)$ and $x \mapsto g(x-z)$ for some $z \in \mathbb{R}$; and a continuous version can be shown to exists thanks to Kolmorogov's classical theorem for path continuity. (We refer to [49, Remark 2.19 and Section 3.3] for the full details of this argument.) We think of the noise ξ as the formal derivative of the continuous stochastic process Ξ. More precisely:

Definition 6.2.4. *For every $f \in \mathrm{PC}_c$, we define*

$$\xi(f) := \int_{\mathbb{R}} f(x)\ \mathrm{d}\Xi(x), \tag{6.5}$$

where $\mathrm{d}\Xi$ denotes stochastic integration with respect to Ξ interpreted in the pathwise sense of Karandikar [71]; we refer to [49, Section 3.2.1] for the details of this construction.

Remark 6.2.5. The properties of ξ as defined above that we need in this paper are that

1. for every realization of Ξ, the map $\xi : \mathrm{PC}_c \to \mathbb{R}$ is measurable with respect to the uniform topology; and
2. $f \mapsto \xi(f)$ is a centered Gaussian process on PC_c with covariance

$$\mathbb{E}[\xi(f)\xi(g)] = \langle f, g\rangle_\gamma. \tag{6.6}$$

A proof that (6.5) satisfies these properties is the subject of [49, Section 3.2.1].

We now present several examples of noises covered by Assumption 6.2.3. We refer to Lemma 6.4.2 in this paper for a proof that the examples below satisfy (6.3).

Example 6.2.6 (**White**). Let $\sigma > 0$ be fixed. We say that ξ is a **white noise** with variance σ^2 if $\gamma = \sigma^2\delta_0$, where δ_0 denotes the delta Dirac distribution. In this case, the covariance is simply the L^2 inner product

$$\mathbb{E}\big[\xi(f)\xi(g)\big] = \sigma^2\langle f, g\rangle,$$

and ξ can be constructed as the pathwise stochastic integral

$$\xi(f) := \sigma \int_{\mathbb{R}} f(x)\ \mathrm{d}W(x)$$

with respect to a two-sided Brownian motion W.

Example 6.2.7 (**Fractional**)**.** Let $H \in (\frac{1}{2}, 1)$ and $\sigma > 0$ be fixed. We say that ξ is a **fractional noise** with Hurst parameter H and variance σ^2 if

$$\gamma(x) := \sigma^2 H(2H-1)|x|^{2H-2},$$

in which case

$$\mathbb{E}\big[\xi(f)\xi(g)\big] = \sigma^2 H(2H-1) \int_{\mathbb{R}^2} \frac{f(x)g(y)}{|x-y|^{2-2H}}\ \mathrm{d}x\mathrm{d}y.$$

This noise can be constructed as the pathwise stochastic integral

$$\xi(f) := \sigma \int_{\mathbb{R}} f(x)\ \mathrm{d}W^H(x),$$

where W^H is a two-sided fractional Brownian motion with Hurst parameter H.

Example 6.2.8 (**L^p-Singular**)**.** Let $1 \leq p < \infty$. We say that ξ is an **L^p-singular noise** if γ can be decomposed as

$$\gamma = \gamma_1 + \gamma_2,$$

where $\gamma_1 \in L^p(\mathbb{R})$, and γ_2 is uniformly bounded. We can view L^p-singular noise as a generalization of fractional noise, as γ_1 may have point singularities, such as $\gamma_1(x) \sim |x|^{-\mathfrak{e}}$ as $x \to 0$ for some $\mathfrak{e} \in (0,1)$, or $\gamma_1(x) \sim (-\log|x|)^{\mathfrak{e}}$ as $x \to 0$ for some $\mathfrak{e} > 0$.

Example 6.2.9 (**Bounded**)**.** We say that ξ is a **bounded noise** if γ is uniformly bounded. In many such cases ξ gives rise to a pointwise-defined Gaussian process on $\mathbb{R}$ with covariance function $\mathbb{E}[\xi(x)\xi(y)] = \gamma(x-y)$, whence we can simply define

$$\xi(f) := \int_{\mathbb{R}} f(x)\xi(x)\,\mathrm{d}x. \tag{6.7}$$

6.2.2 Operator and Eigenvalue Point Process

We now discuss the definition of the operator $\hat{\mathcal{H}}_I$ and its spectrum. We make the following two assumptions on the domain/boundary conditions of the operator, and the deterministic potential V:

Assumption 6.2.10. We consider three types of domains $I \subset \mathbb{R}$ on which $\hat{\mathcal{H}}_I$ acts: the full space $I = \mathbb{R}$ (**Case 1**), the half-line $I = (0,\infty)$ (**Case 2**), and the bounded interval $I = (0,b)$ for some $b > 0$ (**Case 3**).

In **Case 2**, we consider Dirichlet and Robin boundary conditions at the origin:

$$\begin{cases} f(0) = 0 & \text{(Dirichlet)} \\ f'(0) + \alpha f(0) = 0 & \text{(Robin)} \end{cases} \tag{6.8}$$

where $\alpha \in \mathbb{R}$ is fixed.

In **Case 3**, we consider the Dirichlet, Robin, and mixed boundary conditions at the endpoints 0 and b:

$$\begin{cases} f(0) = f(b) = 0 & \text{(Dirichlet)} \\ f'(0) + \alpha f(0) = -f'(b) + \beta f(b) = 0 & \text{(Robin)} \\ f'(0) + \alpha f(0) = f(b) = 0 & \text{(Mixed 1)} \\ f(0) = -f'(b) + \beta f(b) = 0 & \text{(Mixed 2)} \end{cases} \tag{6.9}$$

where $\alpha, \beta \in \mathbb{R}$ are fixed.

Assumption 6.2.11. $V : I \to \mathbb{R}$ is bounded below and locally integrable on I's closure. If I is unbounded (i.e., **Cases 1 & 2**), then we also assume that

$$\lim_{|x|\to\infty} \frac{V(x)}{\log|x|} = \infty.$$

We may now provide the following definition for the operator $\hat{\mathcal{H}}_I$, which is a direct application of [49, Proposition 2.9], and allows for a rigorous interpretation of the deterministic operator $-\frac{1}{2}\Delta + V$ plus noise ξ through sesquilinear forms (see also [14, 45, 84, 95]):

Proposition 6.2.12. *Given a fixed choice of domain I, boundary conditions, and potential V all satisfying Assumptions 6.2.10 and 6.2.11, let $\mathcal{E}$ denote the sesquilinear form of the corresponding deterministic Schrödinger operator $-\frac{1}{2}\Delta + V$, and let $D(\mathcal{E}) \subset L^2(I)$ be the associated form domain. (We refer to [49, Definition 2.6] for a precise statement of these objects in all cases outlined in Assumption 6.2.10 and 6.2.11, and to [101, Section 7.5 and Example 7.5.3] for the standard operator theoretic terminology used here.)*

Suppose that Assumption 6.2.3 holds, and let ξ be as in Definition 6.2.4. With probability one, there exists a unique self-adjoint operator $\hat{\mathcal{H}}_I$ with dense domain $D(\hat{\mathcal{H}}_I) \subset L^2$ such that

1. $D(\hat{\mathcal{H}}_I) \subset D(\mathcal{E})$;

2. $\langle f, \hat{\mathcal{H}}_I g\rangle = \mathcal{E}(f,g) + \xi(fg)$ *for every* $f, g \in D(\hat{\mathcal{H}}_I)$*; and*

3. $\hat{\mathcal{H}}_I$ *has compact resolvent.*

Remark 6.2.13. Implicit in the statement of Proposition 6.2.12 is the claim that the noise ξ can be suitably extended to products of functions in the form domain $D(\mathcal{E})$. As argued in [49, Remark 2.7], this is not a problem.

With this result in hand, we immediately obtain the following definition of $\hat{\mathcal{H}}_I$'s spectrum by the variational principle (e.g., [97, Theorems XIII.2 and XIII.64]):

Corollary 6.2.14. *Under the same hypotheses and notations as Proposition 6.2.12, there exists a random orthonormal basis* $(\Psi_k)_{k\in\mathbb{N}}$ *of* $L^2(I)$ *and a point process* $\Lambda = (\Lambda_k)_{k\in\mathbb{N}}$ *on the real line* $\mathbb{R}$ *such that, almost surely,*

1. $-\infty < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \nearrow +\infty$*; and*

2. *for every* $k \in \mathbb{N}$,

$$\Lambda_k = \inf_{\substack{f\in D(\mathcal{E}),\ \|f\|_2=1\\ f\perp\Psi_1,\ldots\Psi_{k-1}}} \mathcal{E}(f,f) + \xi(f^2),$$

where Ψ_k *achieves the above infimum.*

6.2.3 Semigroup and Feynman-Kac Formula

We now discuss the semigroup theory of the operator defined in Proposition 6.2.12, and argue that exponential statistics of its eigenvalue point process defined in Corollary 6.2.14 can be studied with a Feynman-Kac formula. Before we can do this, we must introduce some stochastic processes that form the basis of the Feynman-Kac formulas that we use:

Definition 6.2.15. *We use* B *to denote a standard Brownian motion taking values in* $\mathbb{R}$, X *to denote a reflected standard Brownian motion taking values in* $(0,\infty)$*, and*

Y to denote a reflected standard Brownian motion taking values in $(0, b)$. Throughout this chapter, we use Z to denote one of these three processes, depending on which case in Assumption 6.2.10 is being considered, that is

$$Z = \begin{cases} B & (\textbf{Case 1}) \\ X & (\textbf{Case 2}) \\ Y & (\textbf{Case 3}). \end{cases} \tag{6.10}$$

For every $t > 0$ and $x, y \in I$, we denote by

$$Z^x := \big(Z | Z(0) = x\big)$$

the process started at x, and we denote the bridge process from x to y in time t by

$$Z_t^{x,y} := \big(Z | Z(0) = x \text{ and } Z(t) = y\big).$$

We sometimes use $\mathbb{E}^x$ and $\mathbb{E}_t^{x,y}$ to denote the expected value with respect to the law of Z^x and $Z_t^{x,y}$, respectively.

We denote the Gaussian kernel by

$$\mathscr{G}_t(x) := \frac{\mathrm{e}^{-x^2/2t}}{\sqrt{2\pi t}}, \qquad t > 0, \ x \in \mathbb{R}. \tag{6.11}$$

We denote the transition kernel of Z by Π_Z, that is, for every $t > 0$ and $x, y \in I$

$$\Pi_Z(t; x, y) := \begin{cases} \mathscr{G}_t(x - y) & (\textbf{Case 1}) \\ \mathscr{G}_t(x - y) + \mathscr{G}_t(x + y) & (\textbf{Case 2}) \\ \sum_{z \in 2b\mathbb{Z} \pm y} \mathscr{G}_t(x - z) & (\textbf{Case 3}). \end{cases}$$

For any $0 \leq s \leq t$, we let $a \mapsto L_{[s,t]}^a(Z)$ $(a \in I)$ denote the continuous version of *the local time of Z (or its conditioned versions) collected on $[s,t]$, i.e.,*

$$\int_s^t f\big(Z(u)\big)\ \mathrm{d}u = \int_I L^a_{[s,t]}(Z) f(a)\ \mathrm{d}a = \langle L_{[s,t]}(Z), f\rangle \tag{6.12}$$

for any measurable function $f: I \to \mathbb{R}$ (see, e.g., [98, Chapter VI, Corollary 1.6 and Theorem 1.7] for the existence and continuity of local times). We use the shorthand $L_t(Z) := L_{[0,t]}(Z)$.

As a matter of convention, if $Z = X$ or Y, then we distinguish the boundary local time from the above, which we denote as

$$\mathfrak{L}^c_{[s,t]}(Z) := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{\{c-\varepsilon<Z(u)<c+\varepsilon\}}\ \mathrm{d}u$$

for $c \in \partial I$ (i.e., $c = 0$ if $Z = X$ or $c \in \{0, b\}$ if $Z = Y$), also with the shorthand $\mathfrak{L}^c_t(Z) := \mathfrak{L}^c_{[0,t]}(Z)$. We refer to [98, Chapter VI, Corollary 1.9] for the relation between this quantity and the local time as defined in (6.12).

We are now finally in a position to state our Feynman-Kac formulas.

Definition 6.2.16. *In* **Cases 2 & 3**, *let us define the quantities $\bar{\alpha}$ and $\bar{\beta}$ as*

$$\bar{\alpha} := \begin{cases} -\infty & (\textbf{Case 2}, \textit{Dirichlet}) \\ \alpha & (\textbf{Case 2}, \textit{Robin}) \end{cases} \qquad (\bar{\alpha}, \bar{\beta}) := \begin{cases} (-\infty, -\infty) & (\textbf{Case 3}, \textit{Dirichlet}) \\ (\alpha, \beta) & (\textbf{Case 3}, \textit{Robin}) \\ (\alpha, -\infty) & (\textbf{Case 3}, \textit{Mixed 1}) \\ (-\infty, \beta) & (\textbf{Case 3}, \textit{Mixed 2}) \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ are as in (6.8) *and* (6.9). *For every $t > 0$ and $x, y \in I$, we define the* *random kernel*

$$\hat{K}(t;x,y) := \begin{cases} \Pi_B(t;x,y)\mathbb{E}_t^{x,y}\left[\mathrm{e}^{-\langle L_t(B),V\rangle-\xi(L_t(B))}\right] & \textit{(}\textbf{Case 1}\textit{)} \\ \Pi_X(t;x,y)\mathbb{E}_t^{x,y}\left[\mathrm{e}^{-\langle L_t(X),V\rangle-\xi(L_t(X))+\bar{\alpha}\mathfrak{L}_t^0(X)}\right] & \textit{(}\textbf{Case 2}\textit{)} \\ \Pi_Y(t;x,y)\mathbb{E}_t^{x,y}\left[\mathrm{e}^{-\langle L_t(Y),V\rangle-\xi(L_t(Y))+\bar{\alpha}\mathfrak{L}_t^0(Y)+\bar{\beta}\mathfrak{L}_t^b(Y)}\right] & \textit{(}\textbf{Case 3}\textit{)} \end{cases}$$

where we assume that the noise ξ is independent of B, X, or Y; hence the expected value $\mathbb{E}_t^{x,y}$ is with respect to $B_t^{x,y}$, $X_t^{x,y}$, or $Y_t^{x,y}$, conditional on ξ. We denote by $\hat{K}(t)$ the random integral operator on $L^2(I)$ with the above kernel.

Remark 6.2.17. If ξ can be realized as a pointwise-defined measurable map on $\mathbb{R}$, then it follows from (6.7) and (6.12) that

$$\langle L_t(Z),V\rangle + \xi\big(L_t(Z)\big) = \int_0^t V\big(Z(s)\big) + \xi\big(Z(s)\big)\ \mathrm{d}s.$$

Thus, in this case $\hat{K}(t)$ corresponds to the Feynman-Kac representation of the semigroup generated by the classically well-defined operator $\hat{\mathcal{H}}_I := -\frac{1}{2}\Delta + V + \xi$ with the appropriate boundary condition (see e.g., [34, 89, 100, 103], or [49, Theorem 5.4] and references therein for a unified statement).

Remark 6.2.18. Since we use the continuous version of Brownian local time, for every $t > 0$, $L_t(Z)$ is an element of PC_c almost surely. Consequently, the term $\xi(L_t(Z))$ in $\hat{K}(t)$'s definition is well defined in the sense of Definition 6.2.4. The facts that the functions $(x,y) \mapsto \hat{K}(t;x,y)$ and $x \mapsto \hat{K}(t;x,x)$ are measurable on $I \times I$ and I respectively and that $\hat{K}(t) \in L^2(I \times I)$ are proved in [49, Theorem 2.23 and Appendix A].

Remark 6.2.19. In cases where $\bar{\alpha}$ or $\bar{\beta}$ are not finite, we use the conventions $\mathrm{e}^{-\infty} := 0$

and

$$-\infty \cdot \mathfrak{L}_t^c(Z) := \begin{cases} 0 & \text{if } \mathfrak{L}_t^c(Z) = 0 \\ -\infty & \text{if } \mathfrak{L}_t^c(Z) > 0. \end{cases}$$

Thus, for any $c \in \partial I$, if we let $\tau_c(Z) := \inf\{t \geq 0 : Z(t) = c\}$ denote the first hitting time of c, then we can interpret $\mathrm{e}^{-\infty \cdot \mathfrak{L}_t^c(Z)} := \mathbf{1}_{\{\tau_c(Z) > t\}}$.

The following result is a direct consequence of [49, Theorem 2.23] (see also [51, 61]).

Proposition 6.2.20. *Suppose that the same hypotheses as Proposition 6.2.12 hold, and let* $\Lambda = (\Lambda_k)_{k\in\mathbb{N}}$ *denote* $\hat{\mathcal{H}}_I$*'s spectrum, as per Corollary 6.2.14. For every* $t > 0$,

$$0 \leq \mathrm{Tr}[\mathrm{e}^{-t\hat{\mathcal{H}}_I}] = \sum_{k=1}^{\infty} \mathrm{e}^{-t\Lambda_k} = \mathrm{Tr}[\hat{K}(t)] = \int_I \hat{K}(t; x, x)\, \mathrm{d}x < \infty \qquad \textit{almost surely.} \tag{6.13}$$

In particular, exponential linear statistics of the form $x \mapsto \mathrm{e}^{-tx}$ *are well defined in the point process* Λ *for all* $t > 0$*, and can be computed explicitly using the kernels in Definition 6.2.16.*

6.2.4 Main Result

Our main result is as follows.

Theorem 6.2.21. *Suppose that Assumptions 6.2.3, 6.2.10, and 6.2.11 are satisfied, and let* $\hat{\mathcal{H}}_I$ *be as in Proposition 6.2.12. In* **Case 3**, $\hat{\mathcal{H}}_{(0,b)}$*'s spectrum is always number rigid. In* **Cases 1 & 2** *(i.e.,* $\hat{\mathcal{H}}_{\mathbb{R}}$ *or* $\hat{\mathcal{H}}_{(0,\infty)}$*), if* $\mathfrak{d} > 1$ *is such that*

$$\limsup_{t\to 0} t^{-\mathfrak{d}} \left(\sup_{x\in I} \mathbb{E}^x \left[\|L_t(Z)\|_\gamma^{2\theta} \right]^{1/\theta} \right) < \infty \tag{6.14}$$

for every positive θ*, then* $\hat{\mathcal{H}}_I$*'s spectrum is number rigid if the following growth con-*

dition on V holds:

$$\begin{cases} \lim_{|x|\to\infty} \dfrac{V(x)}{|x|^{2/(2\mathfrak{d}-1)}} = \infty & \text{(if } \gamma \text{ is compactly supported)} \\ \lim_{|x|\to\infty} \dfrac{V(x)}{|x|^{2/(\mathfrak{d}-1)}} = \infty & \text{(otherwise).} \end{cases} \tag{6.15}$$

Remark 6.2.22. If we assume (6.3), then (6.14) always holds with at least

$$\mathfrak{d} \geq 1 + 1/\max\{q_1, \ldots, q_\ell\}. \tag{6.16}$$

(i.e., combine the bound (6.3) with (6.24); see (6.40) for the details). In particular, under our assumptions, Theorem 6.2.21 always provides a nontrivial sufficient condition for number rigidity in **Cases 1 & 2**. We nevertheless state the general condition (6.14) in Theorem 6.2.21 instead of (6.16), since it is sometimes possible to find $\mathfrak{d} > 1+1/\max\{q_1, \ldots, q_\ell\}$ such that (6.14) holds, and thus prove number rigidity for a larger class of potentials (see, for example, the case of fractional noise in (6.41)).

From this theorem, we obtain the following corollary, which specializes (6.14) and (6.15) to the four examples of noises considered earlier.

Theorem 6.2.23. *Let ξ be one of the four types of noises considered in Examples 6.2.6–6.2.9. Then,* (6.14) *holds with*

$$\mathfrak{d} := \begin{cases} 3/2 & \text{(white noise)} \\ 1+H & \text{(fractional noise with index } H \in (\tfrac{1}{2},1)\text{)} \\ 2-1/2p & (L^p\text{-singular noise with } p \geq 1) \\ 2 & \text{(bounded noise).} \end{cases} \tag{6.17}$$

In particular, under Assumptions 6.2.10 and 6.2.11, in **Cases 1 & 2** *$\hat{\mathcal{H}}_I$'s spectrum is number rigid if the following sufficient conditions on V are satisfied.*

1. (**White**) *If ξ is a white noise, then*

$$\lim_{|x|\to\infty} \frac{V(x)}{|x|} = \infty. \tag{6.18}$$

2. (**Fractional**) *If ξ is a fractional noise with Hurst index $H \in (\frac{1}{2}, 1)$, then*

$$\lim_{|x|\to\infty} \frac{V(x)}{|x|^{2/H}} = \infty. \tag{6.19}$$

3. (**L^p-Singular**) *If ξ is an L^p-singular noise, then*

$$\begin{cases} \displaystyle\lim_{|x|\to\infty} \frac{V(x)}{|x|^{2p/(3p-1)}} = \infty & \text{(if } \gamma \text{ is compactly supported)} \\ \displaystyle\lim_{|x|\to\infty} \frac{V(x)}{|x|^{4p/(2p-1)}} = \infty & \text{(otherwise).} \end{cases} \tag{6.20}$$

4. (**Bounded**) *If ξ is a bounded noise, then*

$$\begin{cases} \displaystyle\lim_{|x|\to\infty} \frac{V(x)}{|x|^{2/3}} = \infty & \text{(if } \gamma \text{ is compactly supported)} \\ \displaystyle\lim_{|x|\to\infty} \frac{V(x)}{|x|^{2}} = \infty & \text{(otherwise).} \end{cases} \tag{6.21}$$

Theorem 6.2.21 is proved in Section 6.4. The main technical ingredient in this proof is Theorem 6.4.1, which provides quantitative upper bounds on the variance of the linear statistic $\sum_k \mathrm{e}^{-t\Lambda_k}$ as $t \to 0$ using the identity (6.13). The result then follows from an application of Proposition 5.2.2 with test functions of the form $f_n(x) = \mathrm{e}^{-t_n x}$ with $t_n \to 0$, by proving that

$$\lim_{n\to\infty} \mathbf{Var}\big[\Lambda(f_n)\big] = \lim_{n\to\infty} \mathbf{Var}\big[\mathrm{Tr}[\hat{K}(t_n)]\big] = 0$$

under the conditions stated in Theorem 6.2.21. Theorem 6.2.23 is proved in Section

6.4.2.

6.2.5 Questions of Optimality

6.2.5.1 Two Examples

The growth conditions (6.15) raise natural questions concerning the optimality of Theorem 6.2.21. For instance, when ξ is a white noise, it is known that the superlinear condition $V(x)/|x| \to \infty$ in Theorem 6.2.23 is not necessary for the number rigidity of Λ.

Proposition 6.2.24 ([22]). *Let ξ_2 be a white noise with variance $1/2$. Let us denote the operator*

$$\hat{\mathcal{H}}^{(2)}_{(0,\infty)} := -\tfrac{1}{2}\Delta + \tfrac{x}{2} + \xi_2, \tag{6.22}$$

with Dirichlet boundary condition at zero. $\hat{\mathcal{H}}^{(2)}_{(0,\infty)}$'s spectrum is number rigid.

Indeed, one may recognize $\hat{\mathcal{H}}^{(2)}_{(0,\infty)}$ as the stochastic Airy operator with parameter $\beta = 2$ (up to a multiple of $1/2$), whose spectrum forms a determinantal point process (e.g., [95, 107]) known as the Airy-2 process. By using this integrable structure, Bufetov showed in [22, Section 3.2] that $\hat{\mathcal{H}}^{(2)}_{(0,\infty)}$'s spectrum is number rigid. In the following proposition (proved in Section 6.5), we demonstrate how exponential linear statistics fail to show the rigidity of the Airy-2 process, and thus (6.18) is the best general sufficient condition for white noise one can obtain with the method of this chapter:

Proposition 6.2.25. *With $\hat{\mathcal{H}}^{(2)}_{(0,\infty)}$ as in (6.22), it holds that*

$$\lim_{t\to 0} \mathbf{Var}\big[\mathrm{Tr}[\mathrm{e}^{-t\hat{\mathcal{H}}^{(2)}_{(0,\infty)}}]\big] = (4\pi)^{-1}.$$

We also note the following simple example, which shows that our superquadratic condition in (6.21) for bounded noise with general γ is optimal, and provides an example of a random Schrödinger operator whose spectrum is not number rigid.

Example 6.2.26. Let g be a standard Gaussian random variable, and suppose that $\xi(x) = g$ for all $x \in \mathbb{R}$. In our terminology, ξ is a bounded noise with non-compactly-supported covariance function $\gamma(x) = 1$ for all $x \in \mathbb{R}$. Consider the operator

$$\hat{\mathcal{H}}_{\mathbb{R}}^{(\mathrm{HO})} f(x) := -\tfrac{1}{2} f''(x) + x^2 f(x) + \xi(x) f(x), \tag{6.23}$$

acting on the whole space $\mathbb{R}$. It is known that the deterministic operator $-\frac{1}{2}\Delta + x^2$, which is usually called the quantum harmonic oscillator, has a spectrum of the form $\{c_1 k + c_2\}_{k\in\mathbb{N}}$ for some constants $c_1, c_2 > 0$ (e.g., [104, Chapter 2, Proposition 2.2 (ii)]). In particular, the spectrum of (6.23) consists of the randomly shifted semilattice $\{c_1 k + c_2 + g\}_{k\in\mathbb{N}}$, which is clearly not number rigid.

6.3 Self-Intersection Local Time

As mentioned in the introduction (see Section 6.1.1) and as evidenced by (6.14), controlling the small-t decay rate of self-intersection local times is a crucial ingredient in the proof of our results. To this effect, in this section, our purpose is to provide one of the main technical ingredient that we use to establish (6.14): Namely, for every $1 \leq q \leq 2$, there exists a nonnegative random variable R_q with finite exponential moments in a neighborhood of zero such that

$$\sup_{x\in I} \|L_t(Z^x)\|_q^2 \leq t^{1+1/q} R_q \qquad \text{for all } t \in (0,1), \tag{6.24}$$

where the inequality in (6.24) is understood in the sense of stochastic domination. (Recall that for any two random variables X and Y, X is said to be *stochastically dominated* by Y if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for any nondecreasing function f. This is equivalent to saying that there exists a random variable Z with the same distribution as Y such that $X \le Z$ almost surely; see, e.g., [70, Theorem 1]). We refer to the proof of Theorem 6.2.23 in Section 6.4.2 for an explanation of how (6.24) is used to prove (6.17).

Proposition 6.3.1. *Define* $\mathcal{L}^{\sup} := \sup_{a\in\mathbb{R}} L_1^a(B^0)$. *Let us denote the maximum and minimum of the Brownian motion* B^x *as*

$$M^x(t) := \sup_{s\in[0,t]} B^x(s) \qquad \text{and} \qquad m^x(t) := \inf_{s\in[0,t]} B^x(s). \tag{6.25}$$

For $q = 1$, *define* $R_q := 1$, *and for* $q \in (1, 2]$, *let*

$$R_q := \begin{cases} 2^{2(q-1)/q}\|L_1(B^0)\|_q^2 & (\textbf{\textit{Cases 1 \& 2}}) \\ c\big(\mathcal{L}^{\sup}\big)^{2(1-1/q)} + c\Big(2(\mathcal{L}^{\sup})^2 + 2\big(M^0(1) - m^0(1)\big)^2\Big)^{2(1-1/q)} & (\textbf{\textit{Case 3}}), \end{cases} \tag{6.26}$$

where $c > 0$ *in* **Case 3** *is a deterministic constant that only depends on the size of the interval* $I = (0, b)$ *and* q. *Then,* (6.24) *holds for all* $q \in [1, 2]$ *with* R_q *shown above.*

Proof. Recall that, thanks to (6.12), $\|L_t(Z)\|_1 = t$. Thus, if $q = 1$, then (6.24) holds trivially with $R_q = 1$.

We therefore only need to prove (6.24) for $q \in (1, 2]$. We argue case by case. Let us begin with **Case 1** which corresponds to $I = \mathbb{R}$. If we couple $B^x = x + B^0$ for all $x \in \mathbb{R}$, then straightforward changes of variables with a Brownian scaling imply that

$$\|L_t(B^x)\|_q^2 = \|L_t(B^0)\|_q^2 \overset{\mathrm{d}}{=} t\left(\int_{\mathbb{R}} L_1^{t^{-1/2}a}(B^0)^q \,\mathrm{d}a\right)^{2/q} = t^{1+1/q}\|L_1(B^0)\|_q^2 \tag{6.27}$$

for every $q > 1$. According to [31, Theorem 4.2.1], for every $q > 1$ there exists some

$c > 0$ such that

$$\mathbb{P}\big[\|L_1(B^0)\|_q^2 > u\big] = \mathrm{e}^{-cu^{q/(q-1)}(1+o(1))}, \qquad u \to \infty. \tag{6.28}$$

This shows $\|L_1(B^0)\|_q^2$ has exponential moments for $1 < q \leq 2$. Thus, in **Case 1** we have (6.24) with $R_q = 2^{2(q-1)/q}\|L_1(B^0)\|_q^2$ since $2^{2(q-1)/q} > 1$ whenever $q > 1$.

Consider now **Case 2** where I is taken to be $(0, \infty)$ and X is a reflected Brownian motion taking values in $(0, \infty)$. By coupling $X^x(t) = |B^x(t)|$ for all $t > 0$, we note that for every $a > 0$, one has $L_t^a(X^x) = L_t^a(|B^x|) = L_t^a(B^x) + L_t^{-a}(B^x)$. Therefore,

$$\begin{aligned}\|L_t(X^x)\|_q^2 &= \left(\int_0^\infty L_t^a(X^x)^q \,\mathrm{d}a\right)^{2/q} \\ &\leq 2^{2(q-1)/q}\left(\int_0^\infty L_t^a(B^x)^q + L_t^{-a}(B^x)^q \,\mathrm{d}a\right)^{2/q} = 2^{2(q-1)/q}\|L_t(B^x)\|_q^2.\end{aligned}$$

By (6.27), the right-hand side of above display is equal in distribution to

$$t^{1+1/q}2^{2(q-1)/q}\|L_1(B^0)\|_q^2.$$

Owing to (6.28), $R_q = 2^{2(q-1)/q}\|L_1(B^0)\|_q^2$ has finite exponential moments for $1 < q \leq 2$, thus the proof of (6.24) in **Case 2** follows.

Finally, consider **Case 3** where I is an interval $(0, b)$ for some $b > 0$ and Y is a reflected Brownian motion taking values in $(0, b)$. We note that we can couple the processes Y^x and B^x in such a way that Y^x is obtained by reflecting the path of B^x on the boundary of $(0, b)$, namely,

$$Y^x(t) = \begin{cases} B^x(t) - 2kb & \text{if } B^x(t) \in [2kb, (2k+1)b], \quad k \in \mathbb{Z}, \\ |B^x(t) - 2kb| & \text{if } B^x(t) \in [(2k-1)b, 2kb], \quad k \in \mathbb{Z}. \end{cases} \tag{6.29}$$

Under this coupling, we observe that for any $z \in (0, b)$, one has

$$L_t^z(Y^x) = \sum_{a \in 2b\mathbb{Z} \pm z} L_t^a(B^x). \tag{6.30}$$

The argument that follows is inspired from the proof of [33, Lemma 2.1] (see also [49, Lemma 5.10]): Under (6.30),

$$\Big(\int_0^b L_t^z(Y^x)^q \, \mathrm{d}z \Big)^{1/q} = \Big(\int_0^b \Big(\sum_{k \in 2b\mathbb{Z}} L_t^{k+z}(B^x) + L_t^{k-z}(B^x) \Big)^q \, \mathrm{d}z \Big)^{1/q}$$
$$\leq 2^{(q-1)/q} \sum_{k \in 2b\mathbb{Z}} \Big(\int_{-b}^b L_t^{k+z}(B^x)^q \, \mathrm{d}z \Big)^{1/q}.$$

Recall that $M^x(t)$ and $m^x(t)$ are the maximum and minimum of B^x in the interval $[0, t]$. In order for $\int_{-b}^b L_t^{k+z}(B^x)^2 \, \mathrm{d}z$ to be nonzero, it must be the case that $M^x(t) \geq k - b$ and $m^x(t) \leq k + b$, or, equivalently, $M^x(t) + b \geq k \geq m^x(t) - b$. Thus, for any $q > 1$,

$$\begin{aligned}
&\sum_{k \in 2b\mathbb{Z}} \Big(\int_{-b}^b L_t^{k+z}(B^x)^q \, \mathrm{d}z \Big)^{1/q} \\
&= \sum_{k \in 2b\mathbb{Z}} \Big(\int_{-b}^b L_t^{k+z}(B^x)^q \, \mathrm{d}z \Big)^{1/q} \mathbf{1}_{\{M^x(t)+b \geq k \geq m^x(t)-b\}} \\
&\leq \Big(\sum_{k \in 2b\mathbb{Z}} \int_{-b}^b L_t^{k+z}(B^x)^q \, \mathrm{d}z \Big)^{1/q} \Big(\sum_{k \in 2b\mathbb{Z}} \mathbf{1}_{\{M^x(t)+b \geq k \geq m^x(t)-b\}} \Big)^{1-1/q} \\
&= \Big(\int_{\mathbb{R}} L_t^a(B^x)^q \, \mathrm{d}a \Big)^{1/q} \Big(\sum_{k \in 2b\mathbb{Z}} \mathbf{1}_{\{M^x(t)+b \geq k \geq m^x(t)-b\}} \Big)^{1-1/q} \\
&\leq c_1 t^{1/q} \Big(\sup_{a \in \mathbb{R}} L_t^a(B^x) \Big)^{1-1/q} \big(M^x(t) - m^x(t) + c_2 \big)^{1-1/q} \\
&\leq c_1 t^{1/q} \left(c_2^{1-1/q} \big(\sup_{a \in \mathbb{R}} L_t^a(B^x) \big)^{1-1/q} + \Big(\sup_{a \in \mathbb{R}} L_t^a(B^x) \cdot \big(M^x(t) - m^x(t) \big) \Big)^{1-1/q} \right)
\end{aligned} \tag{6.31}$$

where $c_1, c_2 > 0$ only depend on b and q: Indeed, the inequality in the third line follows by Hölder's inequality, the equality in the fourth line is obtained by noting that $\sum_{k\in 2b\mathbb{Z}} \int_{-b}^{b} L_t^a(B^x)^q \, \mathrm{d}a$ is equal to $\int_{\mathbb{R}} L_t^a(B^x)^q \, \mathrm{d}a$, we get the inequality in the fifth line by noting that $\int_{\mathbb{R}} L_t^a(B^x)^q \, \mathrm{d}a$ is bounded by $(\sup_{a\in\mathbb{R}} L_t^a(B^x))^{q-1}\|L_t(B^x)\|_1$ where $\|L_t(B^x)\|_1 = t$, and the inequality in the last line follows by bounding $\big(M^x(t) - m^x(t) + c_2\big)^{1-1/q}$ by $\big(M^x(t) - m^x(t)\big)^{1-1/q} + c_2^{1-1/q}$.

Given that the distributions of the supremum of local time of B^x and the range $M^x(t) - m^x(t)$ are independent of the starting point x, by Brownian scaling, we have that

$$t^{1/q}\big(\sup_{a\in\mathbb{R}} L_t^a(B^x)\big)^{1-1/q} \overset{\mathrm{d}}{=} t^{1/2+1/2q}\big(\mathcal{L}^{\sup}\big)^{1-1/q} \tag{6.32}$$

and

$$t^{1/q}\left(\sup_{a\in\mathbb{R}} L_t^a(B^x)\cdot\big(M^x(t) - m^x(t)\big)\right)^{\frac{q-1}{q}} \overset{\mathrm{d}}{=} t\left(\mathcal{L}^{\sup}\cdot\big(M^0(1) - m^0(1)\big)\right)^{\frac{q-1}{q}}. \tag{6.33}$$

Combining (6.31) with (6.32) and (6.33) shows that $\|L_t(Y^x)\|_q^2$ is stochastically dominated by the random variable

$$t^{1+1/q}\left(c\big(\mathcal{L}^{\sup}\big)^{2(1-1/q)} + t^{1-1/q}c\left(2(\mathcal{L}^{\sup})^2 + 2\big(M^0(1) - m^0(1)\big)^2\right)^{2(1-1/q)}\right)$$

where the constant $c > 0$ depends only on b and q. The right-hand side of the above display is bounded by $t^{1+1/q}R_q$ in (6.26) for **Case 3** for all $t \in (0, 1)$. Note that there exists $\theta_0 > 0$ small enough so that

$$\mathbb{E}\left[\exp\left(\theta_0 \sup_{a\in\mathbb{R}} L_1^a(B^0)^2\right)\right], \mathbb{E}\left[\mathrm{e}^{\theta_0(M^0(1)-m^0(1))^2}\right] < \infty \tag{6.34}$$

(e.g., the proof of [33, Lemma 2.1] and references therein). Given that $4(1-1/q) \leq 2$, for $q \in (1,2]$, R_q in **Case 3** has finite exponential moments in a neighborhood of zero. This completes the proof of (6.24) in **Case 3**, and thus the proof of Proposition 6.3.1. $\square$

6.4 Asymptotic Variance Estimates

In this section, we provide the main technical contributions of this chapter, and use the latter to prove our two main theorems. The chief result in this direction consists of the following variance upper bounds for the trace of $\hat K(t)$ as $t \to 0$.

Theorem 6.4.1. *Suppose that Assumptions 6.2.3, 6.2.10, and 6.2.11 hold. Let* $\mathfrak{d} > 1$ *be as in* (6.14)*. In* **Cases 1 & 2***, assume that there exists* $\kappa, \nu, \mathfrak{a} > 0$ *such that*

$$V(x) \geq |\kappa x|^{\mathfrak{a}} - \nu \qquad \textit{for every } x \in I. \tag{6.35}$$

In **Cases 1 & 2***, there exists a finite constant* $C_{\mathfrak{a}} > 0$ *that only depends on* $\mathfrak{a}$ *such that*

$$\begin{cases} \limsup_{t \to 0} \dfrac{\mathbf{Var}\big[\mathrm{Tr}[\hat K(t)]\big]}{t^{\mathfrak{d}-1/2-1/\mathfrak{a}}} \leq \dfrac{C_{\mathfrak{a}}}{\kappa} & \textit{(if } \gamma \textit{ is compactly supported)} \\ \limsup_{t \to 0} \dfrac{\mathbf{Var}\big[\mathrm{Tr}[\hat K(t)]\big]}{t^{\mathfrak{d}-1-2/\mathfrak{a}}} \leq \dfrac{C_{\mathfrak{a}}}{\kappa^2} & \textit{(otherwise).} \end{cases} \tag{6.36}$$

In **Case 3***, one has*

$$\limsup_{t \to 0} \frac{\mathbf{Var}\big[\mathrm{Tr}[\hat K(t)]\big]}{t^{\mathfrak{d}-1}} < \infty. \tag{6.37}$$

The remainder of this section is organized as follows: In Sections 6.4.1 and 6.4.2, we use Theorem 6.4.1 to prove our main results, namely, Theorems 6.2.21 and 6.2.23 respectively. Next, in Section 6.4.3, we prove Theorem 6.4.1. In order to not interrupt the flow of the argument, most of the more technical results used to prove Theorems

6.2.21, 6.2.23, and 6.4.1 are stated without proof in Sections 6.4.1–6.4.3; the technical results in question are then proved Sections 6.4.4 to 6.4.9.

6.4.1 Proof of Theorem 6.2.21

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $t_n \to 0$ as $n \to 0$. For every $n \in \mathbb{N}$, let us define the test function $f_n(x) := \mathrm{e}^{-t_n x}$. This sequence of functions converges to 1 uniformly on compact sets. Moreover, by (6.13),

$$\Lambda(f_n) = \sum_{k=1}^{\infty} \mathrm{e}^{-t_n \Lambda_k} = \mathrm{Tr}[\hat{K}(t_n)] < \infty.$$

Hence, by Proposition 5.2.2, to prove that Λ is number rigid, it suffices to show that

$$\lim_{n\to\infty} \mathbf{Var}\big[\mathrm{Tr}[\hat{K}(t_n)]\big] = 0. \tag{6.38}$$

We now prove that (6.38) holds under the conditions stated in Theorem 6.2.21.

In **Case 3**, (6.38) is an immediate consequence of (6.37) since $\mathfrak{d} > 1$ implies that $O(t^{\mathfrak{d}-1}) = o(1)$ as $t \to 0$. Consider then **Cases 1 & 2**. If we know that $V(x)/|x|^{\mathfrak{a}} \to \infty$, then for every $\kappa > 0$, we can choose $\nu_\kappa > 0$ large enough so that $V(x) \geq |\kappa x|^{\mathfrak{a}} - \nu_\kappa$ for every $x \in I$. As per (6.15), we choose

$$\begin{cases} \mathfrak{a} = 2/(2\mathfrak{d}-1) \iff \mathfrak{d} - 1/2 - 1/\mathfrak{a} = 0 & \text{(if } \gamma \text{ is compactly supported)} \\ \mathfrak{a} = 2/(\mathfrak{d}-1) \iff \mathfrak{d} - 1 - 2/\mathfrak{a} = 0 & \text{(otherwise),} \end{cases}$$

and thus (6.36) yields

$$\limsup_{n\to\infty} \mathbf{Var}\big[\mathrm{Tr}[\hat{K}(t_n)]\big] \leq \begin{cases} C_{\mathfrak{a}}/\kappa & \text{(if } \gamma \text{ is compactly supported)} \\ C_{\mathfrak{a}}/\kappa^2 & \text{(otherwise).} \end{cases}$$

Since $\kappa > 0$ was arbitrary, we then obtain (6.38) in **Cases 1 & 2** by taking $\kappa \to \infty$, thus concluding the proof of Theorem 6.2.21.

6.4.2 Proof of Theorem 6.2.23

We want to prove that (6.14) holds with the choices of $\mathfrak{d} > 1$ in (6.17). Our main tool in proving this is the following lemma.

Lemma 6.4.2. *There exists a constant $c > 0$ (which only depends on γ) such that for every $f \in \mathrm{PC}_c$ and $t > 0$, one has*

$$\|f\|_\gamma^2 \leq \begin{cases} c\|f\|_2^2 & \textit{(white noise)} \\ ct^H\big(t^{-1/2}\|f\|_2^2 + t^{-1}\|f\|_1^2\big) & \textit{(fractional noise with } H \in (\tfrac{1}{2}, 1)\textit{)} \\ c\big(\|f\|_{1/(1-1/2p)}^2 + \|f\|_1^2\big) & \textit{(}L^p\textit{-singular noise with } p \geq 1\textit{)} \\ c\|f\|_1^2 & \textit{(bounded noise).} \end{cases} \tag{6.39}$$

Lemma 6.4.2 is proved in Section 6.4.4, and is a relatively straightforward consequence of applying Young's convolution inequality to the semi-inner-product $\langle f, g\rangle_\gamma$. With (6.39) in hand, the result follows directly from a combination of (6.24) and dominated convergence: On the one hand, if it holds that $\|f\|_\gamma^2 \leq c_\gamma\big(\|f\|_{q_1}^2 + \cdots + \|f\|_{q_\ell}^2\big)$ for some $1 \leq q_i \leq 2$ and $\ell \in \mathbb{N}$, then an application of (6.24) yields

$$\sup_{x\in I} \mathbb{E}^x\big[\|L_t(Z)\|_\gamma^{2\theta}\big]^{1/\theta} = O\left(\sum_{i=1}^{\ell} t^{1+1/q_i}\mathbb{E}[R_{q_i}^\theta]^{1/\theta}\right) = O(t^{1+1/\max\{q_1,\ldots,q_\ell\}}) \tag{6.40}$$

as $t \to 0$. In the case of white, L^p-singular, and bounded noise, this immediately yields (6.17) thanks to (6.39). On the other hand, in the case of fractional noise, an application of (6.24) and (6.39) yields the following asymptotic as $t \to 0$, concluding the proof of (6.17):

$$\sup_{x\in I}\mathbb{E}^x\left[\|L_t(Z)\|_\gamma^{2\theta}\right]^{1/\theta}=O\left(t^H\left(t^{-1/2+3/2}\mathbb{E}[R_2^\theta]^{1/\theta}+t^{-1+2}\mathbb{E}[R_1^\theta]^{1/\theta}\right)\right)=O(t^{1+H}). \tag{6.41}$$

6.4.3 Proof of Theorem 6.4.1

We divide the proof of Theorem 6.4.1 into three steps. In the first step (Section 6.4.3.1), we derive an integral formula of $\mathbf{Var}[\mathrm{Tr}[\hat{K}(t)]]$. The second step (Section 6.4.3.2) provides upper bounds on the different components of the integral formula. Those upper bounds are summarized in few technical lemmas whose proofs are relegated to Section 6.4.5-6.4.9. The third and final step (Section 6.4.3.3) completes the proof of Theorem 6.4.1 by combining the ingredients of Section 6.4.3.2 with the integral formula of Section 6.4.3.1.

6.4.3.1 Step 1. Variance Formula

We begin by introducing some notational shortcuts used throughout this section to improve readability:

Notation 6.4.3. For the remainder of Section 6.4, we use $C,c>0$ to denote constants independent of κ, ν, $\mathfrak{a}$ and t whose precise values may change from one equation to the next, and we use $C_{\mathfrak{a}}>0$ to denote such constants that depend only on $\mathfrak{a}$.

Notation 6.4.4. Let Z be as in (6.10), and let $\bar{Z}$ be an independent copy of Z. For every $t>0$, we define the following random functions: For $(x,y)\in I^2$,

$$\mathcal{A}_t(x,y):=-\langle L_t(Z_t^{x,x})+L_t(\bar{Z}_t^{y,y}),V\rangle,$$

$$\mathcal{B}_t(x,y) := \begin{cases} 0 & (\textbf{Case 1}) \\ \bar{\alpha}\mathfrak{L}^0_t(X^{x,x}_t) + \bar{\alpha}\mathfrak{L}^0_t(\bar{X}^{y,y}_t) & (\textbf{Case 2}) \\ \bar{\alpha}\mathfrak{L}^0_t(Y^{x,x}_t) + \bar{\beta}\mathfrak{L}^b_t(Y^{x,x}_t) + \bar{\alpha}\mathfrak{L}^0_t(\bar{Y}^{y,y}_t) + \bar{\beta}\mathfrak{L}^b_t(\bar{Y}^{y,y}_t) & (\textbf{Case 3}), \end{cases}$$

$$\mathcal{C}_t(x,y) := \frac{\|L_t(Z^{x,x}_t)\|^2_\gamma + \|L_t(\bar{Z}^{y,y}_t)\|^2_\gamma}{2},$$

$$\mathcal{D}_t(x,y) := \langle L_t(Z^{x,x}_t), L_t(\bar{Z}^{y,y}_t)\rangle_\gamma,$$

$$\mathcal{P}_t(x,y) := \Pi_Z(t;x,x)\Pi_Z(t;y,y).$$

Our variance formula is as follows:

Lemma 6.4.5. *Following Notation 6.4.4, it holds that*

$$\mathbf{Var}\big[\mathrm{Tr}[\hat{K}(t)]\big] = \int_{I^2} \mathcal{P}_t(x,y)\,\mathbb{E}\left[\mathrm{e}^{(\mathcal{A}_t+\mathcal{B}_t+\mathcal{C}_t)(x,y)}\left(\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right)\right]\,\mathrm{d}x\mathrm{d}y. \tag{6.42}$$

Lemma 6.4.5 is proved in Section 6.4.5 using the Feynman-Kac formula in Proposition 6.2.20.

6.4.3.2 Step 2. Technical Results

By a combination of applying Hölder's inequality to (6.42) and bounding $\mathcal{P}_t(x,y)$ uniformly in $x,y\in I$ using the right-hand side of (6.79), we obtain the following upper bound for $t\in(0,1]$:

$$\mathbf{Var}\big[\mathrm{Tr}[\hat{K}(t)]\big] \leq Ct^{-1}\int_{I^2}\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4}\mathbb{E}\left[\mathrm{e}^{4\mathcal{B}_t(x,y)}\right]^{1/4} \\ \times\,\mathbb{E}\left[\mathrm{e}^{4\mathcal{C}_t(x,y)}\right]^{1/4}\mathbb{E}\left[\left(\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right)^4\right]^{1/4}\,\mathrm{d}x\mathrm{d}y. \tag{6.43}$$

At this point, the proof of Theorem 6.4.1 is reduced to controlling the $t\to0$ asymptotics of the four terms involving $\mathcal{A}_t$, $\mathcal{B}_t$, $\mathcal{C}_t$, and $\mathcal{D}_t$ on the right-hand side of (6.43).

We now state the technical results we use for this purpose. Our first such result states that the contributions of $\mathcal{B}_t$ and $\mathcal{C}_t$ to (6.43) are uniformly bounded for small t:

Lemma 6.4.6. *For any $\theta > 0$,*

$$\limsup_{t\to 0} \sup_{(x,y)\in I^2} \mathbb{E}\left[\mathrm{e}^{\theta\mathcal{B}_t(x,y)}\right] \leq C, \tag{6.44}$$

$$\limsup_{t\to 0} \sup_{(x,y)\in I^2} \mathbb{E}\left[\mathrm{e}^{\theta\mathcal{C}_t(x,y)}\right] \leq C. \tag{6.45}$$

Lemma 6.4.6 is proved in Section 6.4.6. One of the main technical ingredients in the proof of this result is the estimate (6.24), together with a midpoint sampling trick that allows to extend the latter (which concerns the unconditioned process Z^x) to the bridge processes $Z_t^{x,x}$ (see (6.55)–(6.58) for the details).

Our second and third technical results concern the decay rate of the expectation involving $\mathcal{D}_t$. On the one hand, the following result explains the distinction between general γ and compactly supported γ in Theorem 6.4.1 for **Cases 1 & 2**:

Lemma 6.4.7. *Let $\theta > 0$ be arbitrary. Let $K > 0$ be such that γ is supported on the compact interval $[-K, K]$ (that is, $\langle f, \gamma\rangle = 0$ for every f that vanishes in $[-K, K]$). In* **Case 1***,*

$$\left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^{\theta}\right]\right)^{1/\theta} \leq C\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}} \left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^{2\theta}\right]\right)^{1/2\theta}$$

for all $x, y \in \mathbb{R}$. In **Case 2***, for every $x, y > 0$, one has*

$$\left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^{\theta}\right]\right)^{1/\theta} \leq C\left(\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}} + \mathrm{e}^{-\frac{(|x+y|-K)^2}{2ct}}\right) \left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^{2\theta}\right]\right)^{1/2\theta}.$$

Lemma 6.4.7 is proved in Section 6.4.7, and its proof consists of a formalization of the following simple heuristic: The farther apart x and y are from each other, the more likely it is that the supports of $L_t(Z_t^{x,x})$ and $L_t(\bar{Z}_t^{y,y})$ are separated by a distance of at least $K > 0$, in which case the semi-inner-product $\mathcal{D}_t(x,y) = \langle L_t(Z_t^{x,x}), L_t(\bar{Z}_t^{y,y})\rangle_\gamma$ vanishes if γ is supported in $[-K,K]$. On the other hand, the following result provides an estimate on the decay rate of $\mathrm{e}^{\mathcal{D}_t(x,y)} - 1$ as $t \to 0$, and explains the appearance of the assumption (6.14) in the statement of Theorem 6.2.21:

Lemma 6.4.8. *Let $\mathfrak{d} > 1$ be as in* (6.14). *For any $\theta > 0$,*

$$\limsup_{t\to 0} t^{-\mathfrak{d}} \sup_{(x,y)\in I^2} \left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^\theta\right]\right)^{1/\theta} \leq C.$$

Lemma 6.4.8 is proved in Section 6.4.8. Our final technical result concerns the $t \to 0$ asymptotics of the term involving $\mathcal{A}_t$ in (6.43):

Lemma 6.4.9. *Let $\mathfrak{a}, \kappa > 0$ be as in* (6.35). *One the one hand, it holds that*

$$\limsup_{t\to 0} t^{2/\mathfrak{a}} \int_{I^2} \mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} \,\mathrm{d}x\mathrm{d}y \leq \frac{C_\mathfrak{a}}{\kappa^2} \tag{6.46}$$

in **Cases 1 & 2**. *On the other hand, for every $c, K > 0$, one has*

$$\limsup_{t\to 0} t^{-1/2+1/\mathfrak{a}} \int_{\mathbb{R}^2} \mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} \mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}} \,\mathrm{d}x\mathrm{d}y \leq \frac{C_\mathfrak{a}}{\kappa} \tag{6.47}$$

in **Case 1**; *and in* **Case 2**, *it holds that*

$$\limsup_{t\to 0} t^{-1/2+1/\mathfrak{a}} \int_{(0,\infty)^2} \mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} \left(\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}} + \mathrm{e}^{-\frac{(|x+y|-K)^2}{2ct}}\right) \,\mathrm{d}x\mathrm{d}y \leq \frac{C_\mathfrak{a}}{\kappa}. \tag{6.48}$$

Lemma 6.4.9 is proved in Section 6.4.9, and relies on a formalization of the heuristic that, if we assume (6.35), then we expect that $\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} = O(\mathrm{e}^{2\nu t}\mathrm{e}^{-t(|\kappa x|^\mathfrak{a}+|\kappa y|^\mathfrak{a})})$ as $t \to 0$.

6.4.3.3 Step 3. Conclusion of Proof

We now use the technical lemmas stated in Section 6.4.3.2 to conclude the proof of Theorem 6.4.1. Thanks to (6.43) and Lemma 6.4.6, we have that

$$\mathbf{Var}\big[\mathrm{Tr}[\hat{K}(t)]\big] = O\left(t^{-1}\int_{I^2}\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4}\mathbb{E}\left[\left(\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right)^4\right]^{1/4}\,\mathrm{d}x\mathrm{d}y\right) \tag{6.49}$$

as $t \to 0$, where the constant in O is independent of all parameters. We now control the right-hand-side of (6.49) on a case-by-case basis.

Let us begin with **Case 1**. In the case of general γ (i.e., not necessarily compactly supported), it follows from Lemma 6.4.8 and (6.46) that

$$\begin{aligned}
&\limsup_{t\to 0} t^{-\mathfrak{d}+2/\mathfrak{a}}\int_{\mathbb{R}^2}\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4}\mathbb{E}\left[\left(\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right)^4\right]^{1/4}\,\mathrm{d}x\mathrm{d}y\\
&\leq \limsup_{t\to 0}\left(t^{2/\mathfrak{a}}\int_{\mathbb{R}^2}\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4}\,\mathrm{d}x\mathrm{d}y\right)\left(t^{-\mathfrak{d}}\sup_{(x,y)\in\mathbb{R}^2}\left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right|^4\right]\right)^{1/4}\right)\leq\frac{C_{\mathfrak{a}}}{\kappa^2}.
\end{aligned} \tag{6.50}$$

When combined with (6.49), this yields (6.36) in **Case 1** for general γ. If γ is compactly supported in some interval $[-K,K]$, then it follows from Lemma 6.4.7 that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4}\mathbb{E}\left[\left(\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right)^4\right]^{1/4}\,\mathrm{d}x\mathrm{d}y\\
&\leq\left(\int_{\mathbb{R}^2}\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4}\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}}\,\mathrm{d}x\mathrm{d}y\right)\left(\sup_{(x,y)\in\mathbb{R}^2}\left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right|^8\right]\right)^{1/8}\right).
\end{aligned} \tag{6.51}$$

At this point, by arguing as in (6.50) (except that we replace the estimate (6.46) with

(6.47)), we obtain that

$$\limsup_{t\to 0} t^{-\mathfrak{d}-1/2+1/\mathfrak{a}} \int_{\mathbb{R}^2} \mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} \mathbb{E}\left[\left(\mathrm{e}^{\mathcal{D}_t(x,y)}-1\right)^4\right]^{1/4} \,\mathrm{d}x\mathrm{d}y \leq \frac{C_{\mathfrak{a}}}{\kappa}.$$

Combining this with (6.49) yields (6.36) in **Case 1** for compactly supported γ, concluding the proof of Theorem 6.4.1 in **Case 1**.

The proof of Theorem 6.4.1 in **Case 2** follows from the same steps used in **Case 1**, except that we replace (6.51) with the corresponding bound given by Lemma 6.4.7 in **Case 2**, and that we replace an application of (6.47) with (6.48).

We now conclude the proof of Theorem 6.4.1 with **Case 3**. By Assumption 6.2.11, V is bounded below, i.e., there exists some $c \geq 0$ such that $V(x) \geq -c$ for every x. Thus,

$$\sup_{x,y\in(0,b)} \mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} \leq \mathrm{e}^{2ct} \leq C \tag{6.52}$$

for $t \in (0,1]$. Since the integral in (6.49) is over the bounded domain $I^2 = (0,b)^2$ in **Case 3**, (6.37) then follows from a direct application of Lemma 6.4.8 and (6.52) to (6.49), concluding the proof of Theorem 6.4.1.

6.4.4 Seminorm Bounds: Proof of Lemma 6.4.2

We provide a case-by-case argument. If ξ is a white noise, then up to a constant $\|\cdot\|_\gamma = \|\cdot\|_2$, so the result is immediate.

For fractional noise, up to a constant, we have that

$$\|f\|_\gamma^2 \leq \int_{\mathbb{R}^2} |f(a)\gamma(a-b)f(b)| \,\mathrm{d}a\mathrm{d}b = \int_{\mathbb{R}^2} \frac{|f(a)f(b)|}{|a-b|^{2-2H}} \,\mathrm{d}a\mathrm{d}b.$$

By applying the change of variables $(a,b) \mapsto t^{1/2}(a,b)$ to the right-hand side of this equation, we obtain

$$t \int_{\mathbb{R}^2} \frac{|f(t^{\frac{1}{2}}a)f(t^{\frac{1}{2}}b)|}{|t^{\frac{1}{2}}(a-b)|^{2-2H}} \, \mathrm{d}a\mathrm{d}b = t^H \int_{\mathbb{R}^2} \frac{|f(t^{\frac{1}{2}}a)f(t^{\frac{1}{2}}b)|}{|a-b|^{2-2H}} \, \mathrm{d}a\mathrm{d}b.$$

Next, we write

$$\int_{\mathbb{R}^2} \frac{|f(t^{\frac{1}{2}}a)f(t^{\frac{1}{2}}b)|}{|a-b|^{2-2H}} \, \mathrm{d}a\mathrm{d}b = \Big(\int_{\{|b-a|<1\}} + \int_{\{|b-a|\geq 1\}} \Big) \frac{|f(t^{\frac{1}{2}}a)f(t^{\frac{1}{2}}b)|}{|a-b|^{2-2H}} \, \mathrm{d}a\mathrm{d}b. \tag{6.53}$$

On the one hand, by Young's convolution inequality (e.g., [109]), the first integral (integral over $\{|b-a|<1\}$) in the r.h.s. of (6.53) is bounded above by

$$\left(\int_{-1}^{1} \frac{1}{|z|^{2-2H}} \, \mathrm{d}z \right) \left(\int_{\mathbb{R}} f(t^{\frac{1}{2}}a)^2 \, \mathrm{d}a \right) = \left(\int_{-1}^{1} \frac{1}{|z|^{2-2H}} \, \mathrm{d}z \right) t^{-\frac{1}{2}} \|f\|_2^2,$$

where the right-hand side comes from the change of variables $a \mapsto t^{-\frac{1}{2}}a$. On the other hand, by the same change of variables, the second integral (integral over $\{|b-a| \geq 1\}$) is bounded by

$$\left(\int_{\mathbb{R}} |f(t^{\frac{1}{2}}a)| \, \mathrm{d}a \right)^2 = t^{-1} \|f\|_1^2.$$

Substituting these two bounds in the r.h.s. of (6.53) yields the desired bound on $\|f\|_\gamma^2$ in the case of fractional noise with Hurst parameter $H \in (\frac{1}{2}, 1)$.

Let ξ be an L^p-singular noise with decomposition $\gamma = \gamma_1 + \gamma_2$. Then, the bound on $\|f\|_\gamma^2$ follows from the following use Young's inequality,

$$\begin{aligned} \int_{\mathbb{R}^2} |f(a)\gamma(a-b)f(b)| \, \mathrm{d}a\mathrm{d}b &= \int_{\mathbb{R}^2} |f(a)\gamma_1(a-b)f(b)| \, \mathrm{d}a\mathrm{d}b + \int_{\mathbb{R}^2} |f(a)\gamma_2(a-b)f(b)| \, \mathrm{d}a\mathrm{d}b \\ &\leq \|\gamma_1\|_p \|f\|_q^2 + \|\gamma_2\|_\infty \|f\|_1^2 \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q} + \frac{1}{p} = 2$, or equivalently, $q = 1/(1 - \frac{1}{2p})$.

Finally, if γ is bounded, then

$$\int_{\mathbb{R}^2} |f(a)\gamma(a-b)f(b)| \, \mathrm{d}a\mathrm{d}b \leq \|\gamma\|_\infty \|f\|_1^2,$$

concluding the proof of Lemma 6.4.2, and thus also of Theorem 6.2.23.

6.4.5 Variance Formula: Proof of Lemma 6.4.5

We only prove Lemma 6.4.5 in **Case 1**, since the other cases follow from exactly the same argument. By (6.6), we know that $\mathbb{E}[\mathrm{e}^{-\xi(f)}] = \mathrm{e}^{\frac{1}{2}\|f\|_\gamma^2}$ for all $f \in \mathrm{PC}_c$. Thus, it follows from Fubini's theorem and (6.13) that

$$\begin{aligned}\mathbb{E}\big[\mathrm{Tr}[\hat{K}(t)]\big] &= \int_{\mathbb{R}} \Pi_B(t;x,x)\mathbb{E}_t^{x,x}\left[\mathrm{e}^{-\langle L_t(B),V\rangle}\mathbb{E}_\xi\left[\mathrm{e}^{-\xi(L_t(B))}\right]\right] \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \Pi_B(t;x,x)\mathbb{E}_t^{x,x}\left[\mathrm{e}^{-\langle L_t(B),V\rangle+\frac{1}{2}\|L_t(B)\|_\gamma^2}\right] \, \mathrm{d}x,\end{aligned}$$

where $\mathbb{E}_\xi$ denotes the expectation with respect to ξ, conditional on B. Via another application of Fubini, we get

$$\begin{aligned}\Big(\mathbb{E}\big[\mathrm{Tr}[\hat{K}(t)]\big]\Big)^2 &= \int_{\mathbb{R}^2} \mathcal{P}_t(x,y)\,\mathbb{E}\Big[\mathrm{e}^{-\langle L_t(B_t^{x,x})+L_t(\bar{B}_t^{y,y}),V\rangle+\frac{1}{2}\|L_t(B_t^{x,x})\|_\gamma^2+\frac{1}{2}\|L_t(\bar{B}_t^{y,y})\|_\gamma^2}\Big] \, \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \mathcal{P}_t(x,y)\,\mathbb{E}\Big[\mathrm{e}^{(\mathcal{A}_t+\mathcal{B}_t+\mathcal{C}_t)(x,y)}\Big] \, \mathrm{d}x\mathrm{d}y \qquad (6.54)\end{aligned}$$

where $\bar{B}_t^{y,y}$ is a Brownian bridge independent of $B_t^{x,x}$. A similar computation yields

$$\mathbb{E}\Big[\big(\mathrm{Tr}[\hat{K}(t)]\big)^2\Big] = \int_{\mathbb{R}^2} \mathcal{P}_t(x,y)\,\mathbb{E}\Big[\mathrm{e}^{-\langle L_t(B_t^{x,x})+L_t(\bar{B}_t^{y,y}),V\rangle}\cdot\mathbb{E}_\xi\Big[\mathrm{e}^{-\xi(L_t(B_t^{x,x})+L_t(\bar{B}_t^{y,y}))}\Big]\Big] \, \mathrm{d}x\mathrm{d}y.$$

Given that $\xi(L_t(B_t^{x,x}) + L_t(\bar{B}_t^{y,y}))$ is Gaussian with mean zero and variance

$$\|L_t(B_t^{x,x})\|_\gamma^2 + \|L_t(\bar{B}_t^{y,y})\|_\gamma^2 + 2\langle L_t(B_t^{x,x}), L_t(\bar{B}_t^{y,y})\rangle_\gamma,$$

we may now write

$$\mathbb{E}\Big[\big(\mathrm{Tr}[\hat{K}(t)]\big)^2\Big] = \int_{\mathbb{R}^2} \mathcal{P}_t(x,y)\,\mathbb{E}\Big[\mathrm{e}^{(\mathcal{A}_t+\mathcal{B}_t+\mathcal{C}_t+\mathcal{D}_t)(x,y)}\Big]\;\mathrm{d}x\mathrm{d}y.$$

Finally, the result follows by subtracting (6.54) from $\mathbb{E}\Big[\big(\mathrm{Tr}[\hat{K}(t)]\big)^2\Big]$ in the above display.

6.4.6 Uniformly Bounded Terms: Proof of Lemma 6.4.6

We begin with (6.45). By Independence,

$$\mathbb{E}\left[\mathrm{e}^{\theta\mathcal{C}_t(x,y)}\right] = \mathbb{E}_t^{x,x}\left[\mathrm{e}^{\frac{\theta}{2}\|L_t(Z)\|_\gamma^2}\right]\mathbb{E}_t^{y,y}\left[\mathrm{e}^{\frac{\theta}{2}\|L_t(Z)\|_\gamma^2}\right]$$

As it turns out, (6.45) follows from (6.24). The trick that we use to prove this makes several other appearances in this chapter: Since the exponential function is nonnegative, for every $\theta > 0$, it follows from the tower property and the Doob h-transform that

$$\begin{aligned}\mathbb{E}_t^{x,x}\left[\mathrm{e}^{\theta\|L_t(Z)\|_\gamma^2}\right] &= \mathbb{E}\left[\mathbb{E}_t^{x,x}\left[\mathrm{e}^{\theta\|L_t(Z)\|_\gamma^2}\middle|Z_t^{x,x}(t/2)\right]\right]\\ &= \int_I \mathbb{E}_t^{x,x}\left[\mathrm{e}^{\theta\|L_t(Z)\|_\gamma^2}\middle|Z_t^{x,x}(t/2)=y\right]\frac{\Pi_Z(t/2;x,y)\Pi_Z(t/2;y,x)}{\Pi_Z(t;x,x)}\;\mathrm{d}y. \quad (6.55)\end{aligned}$$

If we condition on $Z_t^{x,x}(t/2) = y$, then the paths $\big(Z_t^{x,x}(s) : 0 \leq s \leq t/2\big)$ and $\big(Z_t^{x,x}(s) : t \leq s \leq t/2\big)$ are independent and have respective distributions $Z_{t/2}^{x,y}$ and $Z_{t/2}^{y,x}$. Since Π_Z is a symmetric kernel, the time-reversed process $s \mapsto$

$Z^{y,x}_{t/2}(t-s)$ $(0 \leq s \leq t)$ is equal in distribution to $Z^{x,y}_{t/2}$. Since local time is additive, $\mathbb{E}^{x,x}_t\Big[\mathrm{e}^{\theta\|L_t(Z)\|^2_\gamma}\big|Z^{x,x}_t(t/2) = y\Big]$ is equal to $\mathbb{E}^{x,x}_t\Big[\mathrm{e}^{\theta\|L_{t/2}(Z)+L_{[t/2,t]}(Z)\|^2_\gamma}\big|Z^{x,x}_t(t/2) = y\Big]$. Moreover,

$$\begin{aligned}
&\mathbb{E}^{x,x}_t\Big[\mathrm{e}^{\theta\|L_{t/2}(Z)+L_{[t/2,t]}(Z)\|^2_\gamma}\big|Z^{x,x}_t(t/2) = y\Big]\\
&\leq \mathbb{E}^{x,x}_t\Big[\mathrm{e}^{2\theta(\|L_{t/2}(Z)\|^2_\gamma+\|L_{[t/2,t]}(Z)\|^2_\gamma)}\big|Z^{x,x}_t(t/2) = y\Big]\\
&= \mathbb{E}^{x,y}_{t/2}\Big[\mathrm{e}^{2\theta\|L_{t/2}(Z)\|^2_\gamma}\Big]^2 \leq \mathbb{E}^{x,y}_{t/2}\Big[\mathrm{e}^{4\theta\|L_{t/2}(Z)\|^2_\gamma}\Big],
\end{aligned} \tag{6.56}$$

where the inequality in the second line follows from a combination of the triangle inequality (since $\|\cdot\|_\gamma$ is a seminorm) and $(z+\bar{z})^2 \leq 2(z^2+\bar{z}^2)$, the first equality in (6.56) follows from the fact that local time is invariant with respect to time reversal, and the second inequality in (6.56) follows from Jensen's inequality.

At this point, if we let

$$\mathfrak{s}(Z) := \sup_{t\in(0,1]}\sup_{x,y\in I}\frac{\Pi_Z(t/2;y,x)}{\Pi_Z(t;x,x)}, \tag{6.57}$$

which we know is finite thanks to (6.79), then, owing to the last inequality of (6.56), we obtain

$$\mathbb{E}^{x,x}_t\left[\mathrm{e}^{\theta\|L_t(Z)\|^2_\gamma}\right] \leq \mathfrak{s}(Z)\int_I \mathbb{E}^{x,y}_{t/2}\left[\mathrm{e}^{4\theta\|L_{t/2}(Z)\|^2_\gamma}\right]\Pi_Z(t/2;x,y)\,\mathrm{d}y = \mathfrak{s}(Z)\,\mathbb{E}^x\left[\mathrm{e}^{4\theta\|L_{t/2}(Z)\|^2_\gamma}\right] \tag{6.58}$$

for every $t \leq 1$. In conclusion, to prove (6.45), it is enough to show that

$$\limsup_{t\to 0}\sup_{x\in I}\mathbb{E}^x\left[\mathrm{e}^{\theta\|L_t(Z)\|^2_\gamma}\right] \leq C.$$

This follows directly from a combination of (6.3), (6.24), and dominated convergence.

We now prove (6.44). In **Case 1** the result is trivial. In **Case 2**, by using essentially the same argument leading up to (6.58), we have that

$$\mathbb{E}_t^{x,x}\left[\mathrm{e}^{\theta\mathfrak{L}_t^0(X)}\right] \leq C\,\mathbb{E}^x\left[\mathrm{e}^{2\theta\mathfrak{L}_{t/2}^0(X)}\right].$$

By coupling $X^x(s) = |B^x(s)|$ for all $s \geq 0$, this yields

$$\mathbb{E}_t^{x,x}\left[\mathrm{e}^{\theta\mathfrak{L}_t^0(X)}\right] \leq C\,\mathbb{E}^x\left[\mathrm{e}^{2\theta\mathfrak{L}_{t/2}^0(B)}\right],$$

where we define

$$\mathfrak{L}_t^a(B) := \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{a-\varepsilon<B(s)<a+\varepsilon\}}\,\mathrm{d}s = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{a-\varepsilon<|B(s)|<a+\varepsilon\}}\,\mathrm{d}s.$$

for any $a \in \mathbb{R}$. Thus, by a straightforward application of Hölder's inequality, it suffices to prove that

$$\limsup_{t\to 0}\sup_{x\in\mathbb{R}}\mathbb{E}^x\left[\mathrm{e}^{\theta\mathfrak{L}_t^0(B)}\right] \leq C.$$

By Brownian scaling, $\mathfrak{L}_t^0(B^x) \stackrel{\mathrm{d}}{=} t^{1/2}\mathfrak{L}_1^0(B^{t^{-1/2}x})$. By repeating the proof of [49, Lemma 5.6] in its entirety, we have that

$$\sup_{x\in\mathbb{R}}\mathbb{E}^x\left[\mathrm{e}^{\theta t^{1/2}\mathfrak{L}_1^0(B)}\right] = \mathbb{E}^0\left[\mathrm{e}^{\theta t^{1/2}\mathfrak{L}_1^0(B)}\right] \leq C,$$

and thus the result follows from dominated convergence.

Consider now **Case 3**. Once again arguing as in (6.58), it suffices to prove that

$$\limsup_{t\to 0}\sup_{x\in(0,b)}\mathbb{E}^x\left[\mathrm{e}^{\theta\mathfrak{L}_t^c(Y)}\right] \leq C, \qquad c \in \{0, b\}. \tag{6.59}$$

Recall the coupling of Y and B in (6.29). Under this coupling, we observe that

$$\mathfrak{L}_t^c(Y^x) = \begin{cases} \sum_{a\in 2b\mathbb{Z}} \mathfrak{L}_t^a(B^x) & (c=0) \\ \sum_{a\in b(2\mathbb{Z}+1)} \mathfrak{L}_t^a(B^x) & (c=b). \end{cases} \tag{6.60}$$

Consider the case $c = 0$. According to (6.60), we see that

$$\mathfrak{L}_t^0(Y^x) \le \sup_{a\in\mathbb{R}} \mathfrak{L}_t^a(B^x) \cdot \mathfrak{n}_t,$$

where $\mathfrak{n}_t$ counts the number of intervals of the form $[kb,(k+1)b]$ $(k \in \mathbb{Z})$ such that

$$\inf_{kb\le a\le (k+1)b} \mathfrak{L}_t^a(B^x) > 0.$$

It is easy to see that there exists constants $c_1, c_2 > 0$ that only depend on b such that for every $t > 0$, one has $\mathfrak{n}_t \le c_1 (M^x(t) - m^x(t) + c_2)$, where we denote M^x and m^x as in (6.25). By Brownian scaling,

$$\sup_{a\in\mathbb{R}} \mathfrak{L}_t^a(B^x) \overset{\mathrm{d}}{=} t^{1/2} \sup_{a\in\mathbb{R}} \mathfrak{L}_1^a(B^0),$$

and

$$\left(\sup_{a\in\mathbb{R}} \mathfrak{L}_t^a(B^x)\right) \big(M^x(t) - m^x(t)\big) \overset{\mathrm{d}}{=} t \left(\sup_{a\in\mathbb{R}} \mathfrak{L}_1^a(B^0)\right) \big(M^0(1) - m^0(1)\big).$$

By combining the fact that these terms are independent of x with (6.34), we obtain (6.59) for $c = 0$. The proof for $c = b$ is nearly identical, thus concluding the proof of (6.44), and therefore the proof of Lemma 6.4.6.

6.4.7 Compactly Supported γ: Proof of Lemma 6.4.7

We begin with the claimed bound in **Case 1**. Since γ is supported in $[-K, K]$, in order for the quantity $\mathcal{D}_t(x,y) = \langle L_t(B^{x,x}_t), L_t(\bar{B}^{y,y}_t)\rangle_\gamma$ to be nonzero, it must be the case that

$$\begin{cases} \max_{0\le s\le t} B^{x,x}_t(s) + K \ge \min_{0\le s\le t} \bar{B}^{y,y}_t(s) & (\text{if } x \le y) \\ \max_{0\le s\le t} \bar{B}^{y,y}_t(s) + K \ge \min_{0\le s\le t} B^{x,x}_t(s) & (\text{if } x \ge y). \end{cases}$$

Looking at the case where $x \le y$, this means that

$$\begin{aligned} &\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^\theta\right]^{1/\theta} \\ &= \mathbb{E}\left[1\!\!1_{\{\max_{0\le s\le t} B^{x,x}_t(s) + K \ge \min_{0\le s\le t} \bar{B}^{y,y}_t(s)\}} \left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^\theta\right]^{1/\theta} \\ &\le \mathbb{P}\left[\max_{0\le s\le t} B^{x,x}_t(s) + K \ge \min_{0\le s\le t} \bar{B}^{y,y}_t(s)\right]^{1/2\theta} \mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^{2\theta}\right]^{1/2\theta} \end{aligned} \tag{6.61}$$

If we apply a Brownian scaling and use the fact that the maxima of brownian bridges have sub-Gaussian tails, then

$$\begin{aligned} &\mathbb{P}\left[\max_{0\le s\le t} B^{x,x}_t(s) + K \ge \min_{0\le s\le t} \bar{B}^{y,y}_t(s)\right]^{1/2\theta} \\ &\qquad = \mathbb{P}\left[\max_{0\le s\le 1} B^{0,0}_1(s) + \max_{0\le s\le 1} \bar{B}^{0,0}_1(s) \ge (y-x-K)/t^{1/2}\right]^{1/2\theta} \le C\mathrm{e}^{-\frac{(y-x-K)^2}{2ct}}. \end{aligned}$$

A similar bound is obtained when $x \ge y$, which, when combined with (6.61), concludes the proof of Lemma 6.4.7 in **Case 1**.

We now provide the proof of Lemma 6.4.7 in **Case 2**. By Hölder's inequality,

$$\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^\theta\right]^{1/\theta} \le \mathbb{P}\big[\langle L_t(X^{x,x}_t), L_t(\bar{X}^{y,y}_t)\rangle_\gamma \ne 0\big]^{1/2\theta} \mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^{2\theta}\right]^{1/2\theta}.$$

Note that we can couple X and B so that $X^x(t) = |B^x(t)|$ for all $t \ge 0$. Then, conditioning on the endpoint corresponds to

$$X_t^{x,x} = \big(|B^x| \,\big|B^x(t) \in \{x, -x\}\big).$$

Using this coupling, it follows from [49, (5.9)] that for any nonnegative path functional F,

$$\mathbb{E}\big[F(X_t^{x,x})\big] \leq 2\mathbb{E}\big[F(|B_t^{x,x}|)\big]. \tag{6.62}$$

Consequently, we get the further upper bound

$$\mathbb{P}\big[\langle L_t(X_t^{x,x}), L_t(\bar{X}_t^{y,y})\rangle_\gamma \neq 0\big]^{1/2\theta} \leq 2^{1/2\theta}\mathbb{P}\big[\langle L_t(|B_t^{x,x}|), L_t(|\bar{B}_t^{y,y}|)\rangle_\gamma \neq 0\big]^{1/2\theta}.$$

Given that $L_t^a(|B_t^{x,x}|) = L_t^a(B_t^{x,x}) + L_t^{-a}(B_t^{x,x})$ for all $a > 0$ and similarly for $\bar{B}_t^{y,y}$, we can expand $\langle L_t(|B_t^{x,x}|), L_t(|\bar{B}_t^{y,y}|)\rangle_\gamma$ as the sum

$$\begin{aligned}&\int_{(0,\infty)^2} L_t^a(B_t^{x,x})\gamma(a-b)L_t^b(\bar{B}_t^{y,y})\,\mathrm{d}a\mathrm{d}b + \int_{(0,\infty)^2} L_t^{-a}(B_t^{x,x})\gamma(a-b)L_t^{-b}(\bar{B}_t^{y,y})\,\mathrm{d}a\mathrm{d}b\\ &+ \int_{(0,\infty)^2} L_t^{-a}(B_t^{x,x})\gamma(a-b)L_t^b(\bar{B}_t^{y,y})\,\mathrm{d}a\mathrm{d}b + \int_{(0,\infty)^2} L_t^a(B_t^{x,x})\gamma(a-b)L_t^{-b}(\bar{B}_t^{y,y})\,\mathrm{d}a\mathrm{d}b.\end{aligned}$$

Let us define the set $\mathcal{S} := (-\infty, 0)^2 \cup (0, \infty)^2$. Since γ is assumed to be even, by a simple change of variables, the first two terms in the above sum add up to

$$\int_{\mathcal{S}} L_t^a(B_t^{x,x})\gamma(a-b)L_t^b(\bar{B}_t^{y,y})\,\mathrm{d}a\mathrm{d}b, \tag{6.63}$$

and the last two terms add up to

$$\int_{\mathcal{S}} L_t^a(B_t^{x,x})\gamma(a-b)L_t^{-b}(\bar{B}_t^{y,y})\,\mathrm{d}a\mathrm{d}b. \tag{6.64}$$

Suppose that $0 < x \leq y$. In order for (6.63) to be nonzero, it must be the case that

$$\max_{0\leq s\leq t} B_t^{x,x}(s) + K \geq \min_{0\leq s\leq t} \bar{B}_t^{y,y}(s),$$

and for (6.64) to be nonzero, it must be the case that

$$-\min_{0\leq s\leq t} \bar{B}_t^{y,y}(s) + K \geq \min_{0\leq s\leq t} B_t^{x,x}(s).$$

Thus, by a union bound, followed by Brownian scaling and the fact that Brownian bridge maxima have sub-Gaussian tails, we see that

$$\begin{aligned}
&\mathbb{P}\big[|\langle L_t(|B_t^{x,x}|), L_t(|\bar{B}_t^{y,y}|)\rangle_\gamma| > 0\big]^{1/2\theta} \\
&\leq \mathbb{P}\left[\max_{0\leq s\leq 1} B_1^{0,0}(s) + \max_{0\leq s\leq 1} \bar{B}_1^{0,0}(s) \geq \frac{y-x-K}{t^{1/2}}\right]^{1/2\theta} \\
&\qquad\qquad + \mathbb{P}\left[\max_{0\leq s\leq 1} B_1^{0,0}(s) + \max_{0\leq s\leq 1} \bar{B}_1^{0,0}(s) \geq \frac{x+y-K}{t^{1/2}}\right]^{1/2\theta} \\
&\leq C\left(\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}} + \mathrm{e}^{-\frac{(|x+y|-K)^2}{2ct}}\right).
\end{aligned}$$

The same bound holds for $y \leq x$, concluding the proof of Lemma 6.4.7 in **Case 2**.

6.4.8 Vanishing Term: Proof of Lemma 6.4.8

By combining the inequality $|\mathrm{e}^z - 1| \leq \mathrm{e}^{|z|} - 1 \leq |z|\mathrm{e}^{|z|}$ $(z \in \mathbb{R})$ with $|\mathcal{D}_t(x,y)| \leq \frac{1}{2}\big(\|L_t(Z_t^{x,x})\|_\gamma^2 + \|L_t(\bar{Z}_t^{y,y})\|_\gamma^2\big)$, and applying the triangle inequality, we see that

$$\begin{aligned}
\left(\mathbb{E}\left[\left|\mathrm{e}^{\mathcal{D}_t(x,y)} - 1\right|^\theta\right]\right)^{1/\theta} &\leq C\Big(\mathbb{E}\left[\|L_t(Z_t^{x,x})\|_\gamma^{2\theta}\mathrm{e}^{(\theta/2)(\|L_t(Z_t^{x,x})\|_\gamma^2 + \|L_t(\bar{Z}_t^{y,y})\|_\gamma^2)}\right]^{1/\theta} \\
&\qquad + \mathbb{E}\left[\|L_t(\bar{Z}_t^{y,y})\|_\gamma^{2\theta}\mathrm{e}^{(\theta/2)(\|L_t(Z_t^{x,x})\|_\gamma^2 + \|L_t(\bar{Z}_t^{y,y})\|_\gamma^2)}\right]^{1/\theta}\Big).
\end{aligned}$$

By using independence of Z and $\bar{Z}$ and applying Hölder's inequality, the right-hand side of the above inequality is bounded by

$$C\Bigg(\mathbb{E}_t^{x,x}\left[\|L_t(Z)\|_\gamma^{4\theta}\right]^{1/2\theta}\mathbb{E}_t^{x,x}\left[\mathrm{e}^{\theta\|L_t(Z)\|_\gamma^2}\right]^{1/2\theta}\mathbb{E}_t^{y,y}\left[\mathrm{e}^{(\theta/2)\|L_t(Z)\|_\gamma^2}\right]^{1/\theta}$$
$$+\mathbb{E}_t^{y,y}\left[\|L_t(Z)\|_\gamma^{4\theta}\right]^{1/2\theta}\mathbb{E}_t^{y,y}\left[\mathrm{e}^{\theta\|L_t(Z)\|_\gamma^2}\right]^{1/2\theta}\mathbb{E}_t^{x,x}\left[\mathrm{e}^{(\theta/2)\|L_t(Z)\|_\gamma^2}\right]^{1/\theta}\Bigg).$$

At this point, thanks to (6.45), the proof of Lemma 6.4.8 will be complete if we show that

$$\limsup_{t\to0}t^{-\vartheta}\sup_{x\in I}\left(\mathbb{E}_t^{x,x}\left[\|L_t(Z)\|_\gamma^{2\theta}\right]\right)^{1/\theta}\leq C. \tag{6.65}$$

We claim that (6.65) is a consequence of (6.14). To see this, we once again condition on the midpoint of $Z_t^{x,x}$: With $\mathfrak{s}(Z)<\infty$ as in (6.57), we obtain that for any $t\in(0,1]$,

$$\begin{aligned}
&\mathbb{E}_t^{x,x}\left[\|L_t(Z)\|_\gamma^{2\theta}\right]\\
&=\int_I\mathbb{E}_t^{x,x}\left[\|L_t(Z)\|_\gamma^{2\theta}\Big|Z_t^{x,x}(t/2)=z\right]\frac{\Pi_Z(t/2;x,z)\Pi_Z(t/2;z,x)}{\Pi_Z(t;x,x)}\,\mathrm{d}z\\
&\leq\mathfrak{s}(Z)\int_I\mathbb{E}_t^{x,x}\left[\|L_{t/2}(Z)+L_{[t/2,t]}(Z)\|_\gamma^{2\theta}\Big|Z_t^{x,x}(t/2)=z\right]\Pi_Z(t/2;x,z)\,\mathrm{d}z\\
&\leq C\mathfrak{s}(Z)\int_I\mathbb{E}_t^{x,z}\left[\|L_{t/2}(Z)\|_\gamma^{2\theta}\right]\Pi_Z(t/2;x,z)\,\mathrm{d}z\\
&=C\,\mathbb{E}^x\left[\|L_{t/2}(Z)\|_\gamma^{2\theta}\right],
\end{aligned}$$

where the equality in the second line follows from the Doob h-transform (see (6.55)), the inequality in the fourth line follows from first applying Minkowski's inequality to bound $\|L_{t/2}(Z)+L_{[t/2,t]}(Z)\|_\gamma^{2\theta}$ by $C(\|L_{t/2}(Z)\|_\gamma^{2\theta}+\|L_{[t/2,t]}(Z)\|_\gamma^{2\theta})$, and then using the fact that, under the conditioning $Z_t^{x,x}(t/2)=z$, the local time processes $L_{t/2}(Z_t^{x,x})$

and $L_{[t/2,t]}(Z_t^{x,x})$ are i.i.d. copies of $L_{t/2}(Z_{t/2}^{x,z})$. (We refer back to the passage following (6.55) for details.)

6.4.9 Final Estimates: Proof of Lemma 6.4.9

6.4.9.1 Proof of (6.46)

We begin by proving (6.46) in **Case 1**. By coupling $B_t^{x,x} := x + B_t^{0,0}$ and $\bar{B}_t^{y,y} := y + \bar{B}_t^{0,0}$, it follows from (6.35) that

$$\mathcal{A}_t(x,y) \leq 2\nu t - \kappa^{\mathfrak{a}} \int_0^t \left(\left| x + B_t^{0,0}(s) \right|^{\mathfrak{a}} + \left| y + \bar{B}_t^{0,0}(s) \right|^{\mathfrak{a}} \right) \mathrm{d}s. \tag{6.66}$$

By the change of variables $s \mapsto st$ and a Brownian scaling, we then obtain

$$\begin{aligned} \text{r.h.s. of (6.66)} &= 2\nu t - \kappa^{\mathfrak{a}} \int_0^1 \left(\left| t^{\frac{1}{\mathfrak{a}}} x + t^{\frac{1}{\mathfrak{a}}} B_t^{0,0}(st) \right|^{\mathfrak{a}} + \left| t^{\frac{1}{\mathfrak{a}}} y + t^{\frac{1}{\mathfrak{a}}} \bar{B}_t^{0,0}(st) \right|^{\mathfrak{a}} \right) \mathrm{d}s \\ &\stackrel{\mathrm{d}}{=} 2\nu t - \kappa^{\mathfrak{a}} \int_0^1 \left(\left| t^{\frac{1}{\mathfrak{a}}} x + t^{\frac{1}{2}+\frac{1}{\mathfrak{a}}} B_1^{0,0}(s) \right|^{\mathfrak{a}} + \left| t^{\frac{1}{\mathfrak{a}}} y + t^{\frac{1}{2}+\frac{1}{\mathfrak{a}}} \bar{B}_1^{0,0}(s) \right|^{\mathfrak{a}} \right) \mathrm{d}s. \end{aligned}$$

Let us introduce the shorthands

$$\mathscr{B}_{t,x}(s) := \left| t^{\frac{1}{\mathfrak{a}}} x + t^{\frac{1}{2}+\frac{1}{\mathfrak{a}}} B_1^{0,0}(s) \right|^{\mathfrak{a}}, \qquad \bar{\mathscr{B}}_{t,y}(s) := \left| t^{\frac{1}{\mathfrak{a}}} y + t^{\frac{1}{2}+\frac{1}{\mathfrak{a}}} \bar{B}_1^{0,0}(s) \right|^{\mathfrak{a}} \tag{6.67}$$

so that, by (6.66), one has

$$\begin{aligned} \int_{\mathbb{R}^2} \mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)} \right]^{\frac{1}{4}} \mathrm{d}x\mathrm{d}y &\leq C\mathrm{e}^{2\nu t} \int_{\mathbb{R}^2} \mathbb{E}\left[\mathrm{e}^{-4\kappa^{\mathfrak{a}} \int_0^1 \left(\mathscr{B}_{t,x}(s) + \bar{\mathscr{B}}_{t,y}(s) \right) \mathrm{d}s} \right]^{\frac{1}{4}} \mathrm{d}x\mathrm{d}y \\ &= C\mathrm{e}^{2\nu t} t^{-2/\mathfrak{a}} \int_{\mathbb{R}^2} \mathbb{E}\left[\mathrm{e}^{-4\kappa^{\mathfrak{a}} \int_0^1 \left(\mathscr{B}_{t,t^{-1/\mathfrak{a}}x}(s) + \bar{\mathscr{B}}_{t,t^{-1/\mathfrak{a}}y}(s) \right) \mathrm{d}s} \right]^{\frac{1}{4}} \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{6.68}$$

where in the second line we applied the change of variables $(x,y) \mapsto t^{-1/\mathfrak{a}}(x,y)$. To alleviate notation, let us henceforth write

$$\mathcal{F}_t(x,y) := \mathrm{e}^{-\kappa^{\mathfrak{a}} \int_0^1 \left(\mathscr{B}_{t,t^{-1/\mathfrak{a}}x}(s)+\bar{\mathscr{B}}_{t,t^{-1/\mathfrak{a}}y}(s)\right)\,\mathrm{d}s}, \tag{6.69}$$

noting that the dependence of $\mathfrak{a}$ and κ are implicit in this notation. For every fixed $x, y \in \mathbb{R}$,

$$\lim_{t\to 0} \mathcal{F}_t(x,y) = \mathrm{e}^{-|\kappa x|^{\mathfrak{a}}-|\kappa y|^{\mathfrak{a}}} \tag{6.70}$$

almost surely. Moreover, for every $z, \bar{z} \in \mathbb{R}$,

$$|z+\bar{z}|^{\mathfrak{a}} \geq |z+\bar{z}|^{\min\{\mathfrak{a},1\}} - 1 \geq |z|^{\min\{\mathfrak{a},1\}} - |\bar{z}|^{\min\{\mathfrak{a},1\}} - 1,$$

and therefore

$$\begin{aligned}\sup_{t\in(0,1]} \mathcal{F}_t(x,y)^4 \leq \exp\Big(&-4|\kappa x|^{\min\{\mathfrak{a},1\}} - 4|\kappa y|^{\min\{\mathfrak{a},1\}}\Big) \\ &\times \exp\left(4\kappa^{\min\{\mathfrak{a},1\}}\big(2 + \sup_{s\in[0,1]} |B_1^{0,0}(s)|^{\min\{\mathfrak{a},1\}} + \sup_{s\in[0,1]} |\bar{B}_1^{0,0}(s)|^{\min\{\mathfrak{a},1\}}\big)\right).\end{aligned} \tag{6.71}$$

We recall that the process $s \mapsto |B_1^{0,0}(s)|$ is a Bessel bridge of dimension one (e.g., [98, Chapter XI]). Thanks to the tail asymptotic in [62, Remark 3.1] (the Bessel bridge is denoted by ϱ in that paper), we know that Bessel bridge maxima have finite exponential moments of all orders. Therefore, since the function $\exp(-|\kappa x|^{\min\{\mathfrak{a},1\}} - |\kappa y|^{\min\{\mathfrak{a},1\}})$ is integrable on $\mathbb{R}^2$, it follows from the dominated convergence theorem that

$$\lim_{t\to 0} \int_{\mathbb{R}^2} \mathbb{E}[\mathcal{F}_t(x,y)^4]^{1/4}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} \mathrm{e}^{-|\kappa x|^{\mathfrak{a}}-|\kappa y|^{\mathfrak{a}}}\,\mathrm{d}x\mathrm{d}y = \left(\frac{2\Gamma(1+1/\mathfrak{a})}{\kappa}\right)^2 = \frac{C_{\mathfrak{a}}}{\kappa^2}. \tag{6.72}$$

Combining (6.68)–(6.72) then yields (6.46) in **Case 1**.

We now conclude the proof of (6.46) by showing that the inequality holds also in **Case 2**. Since $V(x) \geq |\kappa x|^{\mathfrak{a}} - \nu$,

$$\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} \leq \mathrm{e}^{2\nu t}\mathbb{E}\left[\mathrm{e}^{-4\kappa^{\mathfrak{a}}\int_0^t \left(\left|X_t^{x,x}(s)\right|^{\mathfrak{a}} + \left|\bar{X}_t^{y,y}(s)\right|^{\mathfrak{a}}\right)\,\mathrm{d}s}\right]^{1/4}.$$

An application of (6.62) then yields

$$\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4} \leq 2\mathrm{e}^{2\nu t}\mathbb{E}\left[\mathrm{e}^{-4\kappa^{\mathfrak{a}}\int_0^t \left(\left|B_t^{x,x}(s)\right|^{\mathfrak{a}} + \left|\bar{B}_t^{y,y}(s)\right|^{\mathfrak{a}}\right)\,\mathrm{d}s}\right]^{1/4};$$

hence the proof of (6.46) in **Case 2** follows from the same argument used in **Case 1**.

6.4.9.2 Proof of (6.47)

We recall that (6.47) is in the setting of **Case 1**. By controlling $\mathcal{A}_t$ in the same way as (6.66), we obtain the bound

$$\begin{aligned}\int_{\mathbb{R}^2} &\mathbb{E}\left[\mathrm{e}^{4\mathcal{A}_t(x,y)}\right]^{1/4}\,\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}}\,\mathrm{d}x\mathrm{d}y \\ &\leq \mathrm{e}^{2\nu t}\int_{\mathbb{R}^2}\mathbb{E}\left[\mathrm{e}^{-4\kappa^{\mathfrak{a}}\int_0^1(\mathscr{B}_{t,x}(s)+\bar{\mathscr{B}}_{t,y}(s))\,\mathrm{d}s}\right]^{1/4}\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}}\,\mathrm{d}x\mathrm{d}y, \quad (6.73)\end{aligned}$$

where we recall that $\mathscr{B}_{t,x}$ and $\bar{\mathscr{B}}_{t,y}$ are denoted as (6.67). By the change of variables $(x,y) \mapsto t^{-1/\mathfrak{a}}(x,y)$, the integral on the right-hand side of (6.73) is bounded above by

$$\begin{aligned}t^{-2/\mathfrak{a}}&\int_{\mathbb{R}^2}\mathbb{E}[\mathcal{F}_t(x,y)^4]^{1/4}\mathrm{e}^{-(|x-y|-t^{1/\mathfrak{a}}K)^2/2ct^{1+2/\mathfrak{a}}}\,\mathrm{d}x\mathrm{d}y \\ &= \sqrt{2\pi c}\cdot t^{1/2-1/\mathfrak{a}}\int_{\mathbb{R}^2}\mathbb{E}[\mathcal{F}_t(x,y)^4]^{1/4}\frac{\mathrm{e}^{-(|x-y|-t^{1/\mathfrak{a}}K)^2/2ct^{1+2/\mathfrak{a}}}}{\sqrt{2\pi c t^{1+2/\mathfrak{a}}}}\,\mathrm{d}x\mathrm{d}y, \quad (6.74)\end{aligned}$$

where we recall that $\mathcal{F}_t$ is defined as in (6.69). Owing to the inequality

$$(|x-y|-t^{1/\mathfrak{a}}K)^2 \geq \min\{(x-y-t^{1/\mathfrak{a}}K)^2, (x-y+t^{1/\mathfrak{a}}K)^2\},$$

we have

$$\mathrm{e}^{-(|x-y|-t^{1/\mathfrak{a}}K)^2/2ct^{1+2/\mathfrak{a}}} \leq \mathrm{e}^{-(x-y-t^{1/\mathfrak{a}}K)^2/2ct^{1+2/\mathfrak{a}}} + \mathrm{e}^{-(x-y+t^{1/\mathfrak{a}}K)^2/2ct^{1+2/\mathfrak{a}}}$$

which yields

$$\frac{\mathrm{e}^{-(|x-y|-t^{1/\mathfrak{a}}K)^2/2ct^{1+2/\mathfrak{a}}}}{\sqrt{2\pi ct^{1+2/\mathfrak{a}}}} \leq \mathscr{G}_{ct^{1+2/\mathfrak{a}}}(x-y-t^{1/\mathfrak{a}}K) + \mathscr{G}_{ct^{1+2/\mathfrak{a}}}(x-y+t^{1/\mathfrak{a}}K),$$

where we recall that $\mathscr{G}_t$ denotes the Gaussian kernel (6.11). Combining this with (6.71) and substituting into (6.74) then shows that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathbb{E}\Big[\mathrm{e}^{-4\kappa^{\mathfrak{a}}\int_0^1(\mathscr{B}_{t,x}(s)+\bar{\mathscr{B}}_{t,y}(s))\,\mathrm{d}s}\Big]^{1/4}\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}}\,\mathrm{d}x\mathrm{d}y \\
&\qquad\leq C_{\mathfrak{a}}t^{1/2-1/\mathfrak{a}}\Bigg(\int_{\mathbb{R}^2}\mathrm{e}^{-|\kappa x|^{\min\{\mathfrak{a},1\}}-|\kappa y|^{\min\{\mathfrak{a},1\}}}\mathscr{G}_{ct^{1+2/\mathfrak{a}}}(x-y-t^{1/\mathfrak{a}}K)\,\mathrm{d}x\mathrm{d}y \\
&\qquad\qquad+\int_{\mathbb{R}^2}\mathrm{e}^{-|\kappa x|^{\min\{\mathfrak{a},1\}}-|\kappa y|^{\min\{\mathfrak{a},1\}}}\mathscr{G}_{ct^{1+2/\mathfrak{a}}}(x-y+t^{1/\mathfrak{a}}K)\,\mathrm{d}x\mathrm{d}y\Bigg).
\end{aligned}\tag{6.75}$$

Owing to a change of variables and the fact that the Gaussian kernel is an approximate identity, the integrals in the right-hand side of (6.75) have the following limits by dominated convergence:

$$\begin{aligned}
&\lim_{t\to0}\int_{\mathbb{R}}\mathrm{e}^{-|\kappa(x\pm t^{1/\mathfrak{a}}K)|^{\min\{\mathfrak{a},1\}}}\left(\int_{\mathbb{R}}\mathrm{e}^{-|\kappa y|^{\min\{\mathfrak{a},1\}}}\mathscr{G}_{ct^{1+2/\mathfrak{a}}}(x-y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\qquad=\int_{\mathbb{R}}\mathrm{e}^{-2|\kappa x|^{\min\{\mathfrak{a},1\}}}\,\mathrm{d}x = \frac{2^{1-1/\min\{\mathfrak{a},1\}}\Gamma\left(1+1/\min\{\mathfrak{a},1\}\right)}{\kappa} = \frac{C_{\mathfrak{a}}}{\kappa}.
\end{aligned}$$

Combining this last result with (6.73) and (6.75) concludes the proof of (6.47).

6.4.9.3 Proof of (6.48)

We now conclude the proof of Lemma 6.4.9 by establishing the estimate (6.48), which we recall is in the setting of **Case 2**. To prove this, we simply note that for any function F and $x > 0$, we have that

$$\int_0^\infty F(x,y)\Big(\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}} + \mathrm{e}^{-\frac{(|x+y|-K)^2}{2ct}}\Big)\,\mathrm{d}y = \int_{\mathbb{R}} F(x,|y|)\mathrm{e}^{-\frac{(|x-y|-K)^2}{2ct}}\,\mathrm{d}y,$$

and thus (6.48) is an immediate consequence of (6.47). With Lemma 6.4.9 established, along with Lemmas 6.4.5–6.4.8, the proof of Theorem 6.4.1 is now fully complete.

6.5 Airy-2 Process Counterexample

In this section, we prove Proposition 6.2.25. For every $\beta > 0$, let ξ_β be a Gaussian white noise with variance $1/\beta$, and define the operator

$$\hat{\mathcal{H}}^{(\beta)}_{(0,\infty)} := -\tfrac{1}{2}\Delta + \tfrac{x}{2} + \xi_\beta,$$

with a Dirichlet boundary condition at zero. The RSO $2\hat{\mathcal{H}}^{(\beta)}_{(0,\infty)}$ is widely known in the literature as the **Stochastic Airy Operator** (e.g., [43, 95]), and we recall that for every $\beta > 0$, the **Airy-β point process**, which we denote by $\mathfrak{Ai}_\beta$, is defined as the eigenvalue point process of $-2\hat{\mathcal{H}}^{(\beta)}_{(0,\infty)}$.

When $\beta = 2$, the Airy-β process has an alternative integrable interpretation, namely, $\mathfrak{Ai}_2$ is the determinantal point process induced by the **Airy kernel**

$$\mathfrak{K}(x,y) := \begin{cases} \dfrac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y} & \text{if } x \neq y \\[2ex] \mathrm{Ai}'(x)^2 - x\mathrm{Ai}(x)^2 & \text{if } x = y, \end{cases} \tag{6.76}$$

where Ai denotes the Airy function

$$\mathrm{Ai}(x) := \frac{1}{\pi} \lim_{b\to\infty} \int_0^b \cos\left(\frac{u^3}{3} + xu\right) \mathrm{d}u, \qquad x \in \mathbb{R}.$$

Let us denote $f_t(x) := \mathrm{e}^{tx}$ for every $t > 0$. By standard formulas for the variance of linear statistics of determinantal point processes (see e.g., [54, Equation (8)]), we have that[1]

$$\mathbf{Var}\big[\mathrm{Tr}[\mathrm{e}^{-2t\hat{\mathcal{H}}^{(2)}_{(0,\infty)}}]\big] = \mathbf{Var}[\mathfrak{Ai}_2(f_t)] = \frac{1}{2} \int_{\mathbb{R}^2} \left(\mathrm{e}^{tx} - \mathrm{e}^{ty}\right)^2 \mathfrak{K}(x,y)^2 \, \mathrm{d}x\mathrm{d}y. \tag{6.77}$$

By expanding the square and using the identity $\mathfrak{K}(x,x) = \int_{\mathbb{R}^2} \mathfrak{K}(x,y)^2 \, \mathrm{d}y$ (since $\mathfrak{K}$ is a symmetric projection kernel [107, Lemma 2]), we can reformulate this to

$$\mathbf{Var}[\mathfrak{Ai}_2(f_t)] = \int_{\mathbb{R}} \mathrm{e}^{2tx}\, \mathfrak{K}(x,x) \, \mathrm{d}x - \int_{\mathbb{R}^2} \mathrm{e}^{t(x+y)}\, \mathfrak{K}(x,y)^2 \, \mathrm{d}x.$$

The computation that follows is essentially taken from [87]. We provide the full details for the reader's convenience. Rewrite the Airy kernel as

$$\mathfrak{K}(x,y) = \int_0^\infty \mathrm{Ai}(u+x)\mathrm{Ai}(u+y) \, \mathrm{d}u$$

Then, using Fubini's theorem, we can write (6.77) as the difference $E_1(t) - E_2(t)$, where

$$E_1(t) := \int_{\mathbb{R}} \mathrm{e}^{2tx} \left(\int_0^\infty \mathrm{Ai}(u+x)^2 \, \mathrm{d}u\right) \mathrm{d}x = \int_0^\infty \left(\int_{\mathbb{R}} \mathrm{e}^{2tx} \mathrm{Ai}(u+x)^2 \, \mathrm{d}x\right) \mathrm{d}u,$$

[1] We note that the variance formula in question is typically only stated for compactly supported functions. The result can easily be improved to (6.77) by using dominated convergence with standard asymptotics for the Airy function such as [1, 10.4.59–10.4.62].

and

$$E_2(t) := \int_{\mathbb{R}^2} \mathrm{e}^{t(x+y)} \left(\int_0^\infty \int_0^\infty \mathrm{Ai}(u+x)\mathrm{Ai}(u+y)\mathrm{Ai}(v+x)\mathrm{Ai}(v+y) \,\mathrm{d}u\mathrm{d}v \right) \mathrm{d}x\mathrm{d}y$$
$$= \int_0^\infty \int_0^\infty \left(\int_{\mathbb{R}} \mathrm{e}^{tx}\mathrm{Ai}(u+x)\mathrm{Ai}(v+x) \,\mathrm{d}x \right)^2 \mathrm{d}u\mathrm{d}v.$$

We note that the application of Fubini in $E_1(t)$ is justified since the integrand is nonnegative, and in $E_2(t)$ it suffices to check

$$\int_0^\infty \int_0^\infty \left(\int_{\mathbb{R}} \mathrm{e}^{tx} \left|\mathrm{Ai}(u+x)\mathrm{Ai}(v+x)\right| \,\mathrm{d}x \right)^2 \mathrm{d}u\mathrm{d}v < \infty.$$

For this, we recall the formula

$$\int_{\mathbb{R}} \mathrm{e}^{tx}\mathrm{Ai}(x+u)\mathrm{Ai}(x+v) \,\mathrm{d}x = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{t^3}{12} - \frac{u+v}{2}t - \frac{(u-v)^2}{4t} \right) \tag{6.78}$$

from [87, Lemma 2.6], and note that by Cauchy-Schwarz, we have

$$\int_{\mathbb{R}} \mathrm{e}^{tx} \left|\mathrm{Ai}(u+x)\mathrm{Ai}(v+x)\right| dx \leq \left(\int_{\mathbb{R}} \mathrm{e}^{tx}\mathrm{Ai}(u+x)^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \mathrm{e}^{tx}\mathrm{Ai}(v+x)^2 dx \right)^{1/2}$$
$$= \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{t^3}{12} - \frac{u+v}{2}t \right)$$

as desired.

With $E_1(t)$ and $E_2(t)$ established, an application of (6.78) yields

$$E_1(t) = \int_0^\infty \frac{\exp\left(\frac{2t^3}{3} - 2tu\right)}{2\sqrt{2\pi t}} \,\mathrm{d}u = \frac{\mathrm{e}^{\frac{2t^3}{3}}}{4\sqrt{2\pi}t^{3/2}}$$

and

$$E_2(t) = \int_0^\infty \int_0^\infty \frac{\exp\left(\frac{t^3}{6} - (u+v)t - \frac{(u-v)^2}{2t}\right)}{4\pi t} \, \mathrm{d}u \mathrm{d}v = \frac{\mathrm{e}^{\frac{2t^3}{3}}}{4\sqrt{2\pi} t^{3/2}} \left(1 - \operatorname{erf}\left(\frac{t^{3/2}}{\sqrt{2}}\right)\right),$$

where $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-w^2} \, \mathrm{d}w$ denotes the error function. Thus

$$\lim_{t\to 0} E_1(t) - E_2(t) = \lim_{t\to 0} \frac{\mathrm{e}^{\frac{2t^3}{3}}}{4\sqrt{2\pi} t^{3/2}} \operatorname{erf}\left(\frac{t^{3/2}}{\sqrt{2}}\right) = \frac{1}{4\pi},$$

concluding the proof of Proposition 6.2.25.

6.6 Transition Density Bounds

Proposition 6.6.1. *There exist constants* $0 < c < C$ *such that for every* $t \in (0,1]$*,*

$$ct^{-1/2} \le \inf_{x\in I} \Pi_Z(t;x,x) \qquad \text{and} \qquad \sup_{(x,y)\in I^2} \Pi_Z(t;x,y) \le Ct^{-1/2}. \tag{6.79}$$

Proof. In **Case 1**, the result follows directly from the fact that $\Pi_B(t;x,y) \le 1/\sqrt{2\pi t}$ and $\Pi_B(t;x,x) = 1/\sqrt{2\pi t}$ for all x,y and t. A similar argument holds for **Case 2**. Consider now **Case 3**. We recall that, by definition,

$$\Pi_Y(t;x,y) := \sum_{z\in 2b\mathbb{Z}\pm y} \mathscr{G}_t(x-z) = \frac{1}{\sqrt{2\pi t}} \left(\sum_{k\in\mathbb{Z}} \mathrm{e}^{-(x-2bk+y)^2/2t} + \mathrm{e}^{-(x-2bk-y)^2/2t} \right).$$

On the one hand, note that $t \mapsto \mathrm{e}^{-z/t}$ is increasing in $t > 0$ for every $z \ge 0$; hence for every $t \in (0,1]$, one has

$$\sup_{(x,y)\in(0,b)^2} \left(\sum_{k\in\mathbb{Z}} \mathrm{e}^{-(x-2bk+y)^2/2t} + \mathrm{e}^{-(x-2bk-y)^2/2t} \right)$$
$$\le \sup_{(x,y)\in(0,b)^2} \left(\sum_{k\in\mathbb{Z}} \mathrm{e}^{-(x-2bk+y)^2/2} + \mathrm{e}^{-(x-2bk-y)^2/2} \right) < \infty.$$

On the other hand, by isolating the $k = 0$ term in $\sum_{k\in\mathbb{Z}} \mathrm{e}^{-(2bk)^2/2t}$,

$$\inf_{x\in(0,b)} \left(\sum_{k\in\mathbb{Z}} \mathrm{e}^{-(2x-2bk)^2/2t} + \mathrm{e}^{-(2bk)^2/2t} \right) \geq \left(\inf_{x\in(0,b)} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-(2x-2bk)^2/2t} \right) + 1 \geq 1,$$

concluding the proof. $\square$

CHAPTER 7

On Spatial Conditioning of the Spectrum of Discrete Random Schrödinger Operators

7.1 Introduction

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a countably infinite connected graph with uniformly bounded degrees and a distinguished vertex $0 \in \mathscr{V}$, which we call the root. For example, $\mathscr{G}$ could be the integer lattice $\mathbb{Z}^d$, any semiregular tessellation/honeycomb of $\mathbb{R}^d$ that includes the origin, or a much more general graph.

In this chapter, we are interested in the spectral theory of random Schrödinger-type operators of the form

$$Hf(v) = -H_X f(v) + \big(V(v) + \xi(v)\big) f(v), \qquad v \in \mathscr{V},\ f : \mathscr{V} \to \mathbb{R},$$

where we assume that

1. H_X is the infinitesimal generator of some continuous-time Markov process X on $\mathscr{G}$ (which need not be symmetric);
2. $\xi : \mathscr{V} \to \mathbb{R}$ is a random noise (which may have long-range dependence); and
3. $V : \mathscr{V} \to \mathbb{R} \cup \{\infty\}$ is a deterministic potential with sufficient growth at infinity

(as measured by the size of $V(v)$ as v grows farther away from the root), ensuring that H has a purely discrete spectrum.

More specifically, we are interested in studying the *spatial conditioning* of the spectrum of H, i.e., understanding the random configuration of H's eigenvalues in some domain $B \subset \mathbb{C}$ conditional on the configuration of eigenvalues outside of B. As a first step in this direction, we establish that under general assumptions on H_X, ξ, and V, H's spectrum is *number rigid* in the sense of Ghosh and Peres [59]; that is, the number of eigenvalues of H in bounded domains $B \subset \mathbb{C}$ is a measurable function of the configuration of H's eigenvalues outside of B (we point to Definition 7.3.3 for a precise definition). To the best of our knowledge, ours is the first work to study the occurrence of such a phenomenon in the spectrum of random Schrödinger operators acting on discrete spaces.

7.1.1 Organization

In the remainder of this introduction, we provide an outline of our main results and proof strategy, we compare the results in this chapter to previous investigations in a similar vein, and we discuss a few natural open questions raised by our work.

In Section 7.2, we provide a high-level outline of the proof of our main results. We take this opportunity to explain how our technical assumptions arise from our computations. In Section 7.3, we state our assumptions and main results in full details, namely, Assumptions 7.3.8 and 7.3.12 and Theorems 7.3.16, 7.3.17, and 7.3.18. Then, we prove Theorem 7.3.16 in Section 7.4, we prove Theorem 7.3.17 in Sections 7.5 and 7.6, and we prove Theorem 7.3.18 in Section 7.7.

7.1.2 Outline of Main Results

Let d denote the graph distance on $\mathscr{G}$. For every $v \in \mathscr{V}$, we use $\mathsf{c}_n(v)$, $n \geq 0$, to denote v's coordination sequence in $\mathscr{G}$; that is, for every $n \in \mathbb{N}$, $\mathsf{c}_n(v)$ is the number of vertices $u \in \mathscr{V}$ such that $\mathsf{d}(u, v) = n$. Stated informally, our main result is as follows:

Theorem 7.1.1 (Informal Statement)**.** *Suppose that there exists $d \geq 1$ such that*

$$\sup_{v \in \mathscr{V}} \mathsf{c}_n(v) = O(n^{d-1}) \qquad \text{as } n \to \infty. \tag{7.1}$$

Under mild technical assumptions on the Markov process X and the noise ξ, there exists a constant $d/2 \leq \alpha \leq d$ (which, apart from d, depends on the the range of the covariance in ξ) such that if $V(v)$ grows faster than $\mathsf{d}(0, v)^\alpha$ as $\mathsf{d}(0, v) \to \infty$, then H's eigenvalue point process is number rigid.

See Theorems 7.3.16 and 7.3.17 for a formal statement. Our technical assumptions are stated in Assumptions 7.3.8 and 7.3.12; roughly speaking, our assumptions are that

1. the jump rates of X (which may be site-dependent) are uniformly bounded; and

2. the tails of ξ are not worse than exponential.

In particular, our assumptions allow for X to be non-symmetric (hence, the operator H need not be self-adjoint) and for ξ to have a variety of covariance structures, including long-range dependence.

Remark 7.1.2. The constant d in (7.1), which quantifies the growth rate of the number of vertices, can be thought of as the *dimension* of $\mathscr{G}$ (or, at least, an upper bound of the dimension). To illustrate this, if $\mathscr{G}$ is for example $\mathbb{Z}^d$ or a semiregular tessellation of $\mathbb{R}^d$, then it is easy to see that $cn^{d-1} \leq \mathsf{c}_n(v) \leq Cn^{d-1}$ for some $C, c > 0$. More generally, the constant d is closely related to the *intrinsic dimension* of $\mathscr{G}$, which is the minimal number k such that $\mathscr{G}$ can be embedded in $\mathbb{Z}^k$. We refer to, e.g., [75, 80] for more details.

Remark 7.1.3. In Theorem 7.3.18, we provide concrete examples showing that the growth lower bound of $\mathsf{d}(0,v)^\alpha$ that we impose on V to get rigidity is the best general sufficient condition that can be obtained with our proof method.

7.1.3 Proof Strategy and Previous Results

Despite the fact that the general strategy of proof used in the present chapter is the same as in Chapter 6, the differences between the two settings are such that virtually none of the work carried out there can be directly extended to the present setup. For example:

1. Since we consider operators acting on general graphs $\mathscr{G}$, the treatment of the geometry of the space on which our operators are defined requires a much more careful analysis than that carried out in Chapter 6. In particular (as per Remark 7.1.2), in this chapter we uncover that the dimension of the space plays an important role in the proof of rigidity using the semigroup method.

2. In Chapter 6, we only consider Schrödinger operators whose kinetic energy operator is the standard Laplacian and whose noise is a Gaussian process. As a result, the operators considered therein are all self-adjoint and upper bounds of $\mathbf{Var}\big[\mathrm{Tr}[\mathrm{e}^{-tH}]\big]$ can mostly be reduced to the analysis of self-intersection local times of standard Brownian motion. In contrast, in this chapter we allow for much more general generators H_X and noises ξ. Most notably, the assumptions of this chapter allow for non-self-adjoint operators, which increases the technical difficulties involved (e.g., Sections 7.5 and 7.6).

7.2 Proof Outline

In this section, we present a sketch of the proof of our main theorem in two simple special cases. We take this opportunity to explain how our technical assumptions arise in our computations. For simplicity of exposition, we assume in this outline that $\mathscr{G}$ is the integer lattice $\mathbb{Z}^d$ (i.e., $(u,v) \in \mathscr{E}$ if and only if $\|u-v\|_\infty = 1$, where $\|\cdot\|_\infty$ denotes the usual ℓ^∞ norm), X is the simple symmetric random walk on $\mathbb{Z}^d$, and ξ is a centered stationary Gaussian process with covariance function

$$\gamma(v) := \mathbb{E}[\xi(v)\xi(0)], \qquad v \in \mathbb{Z}^d.$$

As alluded to in Section 6.1.1 in Chapter 6 (and proved in Section 7.6), to prove that H's eigenvalue point process is number rigid, it suffices to show that $\mathrm{Tr}[\mathrm{e}^{-tH}]$'s variance vanishes as $t \to 0$. According to the Feynman-Kac formula, we have that

$$\mathrm{Tr}[\mathrm{e}^{-tH}] = \sum_{v \in \mathbb{Z}^d} \mathbb{E}_X \left[\exp\left(\int_0^t V\big(X(s)\big) + \xi\big(X(s)\big)\, \mathrm{d}s \right) \mathbf{1}_{\{X(t)=X(0)\}} \middle| X(0) = v \right],$$

where $\mathbb{E}_X$ means that we are only averaging with respect to the randomness in the path of X, and we assume that X is independent of the noise ξ. In order to ensure that e^{-tH} is trace class (or even bounded) in the general case, we assume that $\mathscr{G}$ has uniformly bounded degrees; see Section 7.6.1 for more details.

Our first step in the analysis of $\mathrm{Tr}[\mathrm{e}^{-tH}]$ is to note that if t is small, then the probability that there exists some $0 \le s \le t$ such that $X(s) \neq X(0)$ is close to zero (i.e., $1 - \mathrm{e}^{-t} \sim t$). Thus, by working only with the complement of this event, we have that

$$\mathrm{Tr}[\mathrm{e}^{-tH}] \approx \sum_{v \in \mathbb{Z}^d} \mathrm{e}^{-tV(v) - t\xi(v)}. \tag{7.2}$$

A rigorous version of this heuristic is carried out in the proof of Lemma 7.4.6. The latter relies on controlling how far X can travel from its initial value $X(0)$ after a small time (e.g., the tail bound (7.39)), which itself depends on the assumptions that the jump rates of X are uniformly bounded.

Our second step is to identify the leading order asymptotics in the variance of the expression on the right-hand side of (7.2). In the special case where ξ is a stationary Gaussian process with covariance γ, an application of Tonelli's theorem yields

$$\begin{aligned}\mathbf{Var}\left[\sum_{v\in\mathbb{Z}^d}\mathrm{e}^{-tV(v)-t\xi(v)}\right]&=\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-tV(u)-tV(v)}\mathbf{Cov}[\mathrm{e}^{-t\xi(u)},\mathrm{e}^{-t\xi(v)}]\\&=\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-tV(u)-tV(v)}\mathrm{e}^{t^2\gamma(0)}\left(\mathrm{e}^{t^2\gamma(u-v)}-1\right)\\&\approx t^2\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-tV(u)-tV(v)}\gamma(u-v),\end{aligned}\tag{7.3}$$

where the last line follows from a Taylor expansion. A bound of this type can be achieved in the general case thanks to our assumption that ξ's tails are not worse than exponential. We refer to Proposition 7.4.2 for the general form of the variance formula. See Lemmas 7.4.3 and 7.4.4 for quantitative bounds on the vanishing of the covariance of the exponential random field $\mathrm{e}^{-t\xi}$ as $t\to 0$ in terms of the strength of ξ's covariance.

Our third and final step is to identify conditions such that the quantity

$$\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-tV(u)-tV(v)}\gamma(u-v)\tag{7.4}$$

does not blow up at a faster rate than t^{-2} as $t\to 0$. As advertised in our informal statement, this depends on the growth rate of the potential V and the decay rate (if any) of the covariance γ at infinity. To give an illustration of how this is carried out in this chapter, we consider the two simplest (and most extreme) cases of covariance structure:

1. $\big(\xi(v)\big)_{v\in\mathbb{Z}^d}$ are i.i.d., i.e., $\gamma(v)=0$ whenever $v\neq 0$; and

2. $\big(\xi(v)\big)_{v\in\mathbb{Z}^d}$ are all equal to each other, i.e., $\gamma(v)=\gamma(0)$ for all $v\in\mathscr{V}$.

The quantity (7.4) then becomes

$$\sum_{u,v\in\mathbb{Z}^d} \mathrm{e}^{-tV(u)-tV(v)}\gamma(u-v) = \begin{cases} \gamma(0)\sum_{v\in\mathbb{Z}^d} \mathrm{e}^{-2tV(v)} & \text{i.i.d. case,} \\ \gamma(0)\left(\sum_{v\in\mathbb{Z}^d} \mathrm{e}^{-tV(v)}\right)^2 & \text{all equal case.} \end{cases}$$

If we assume that $V(v) \gg \mathsf{d}(0,v)^\alpha$ for some $\alpha > 0$, then for any $\theta > 0$ we have that

$$\sum_{v\in\mathbb{Z}^d} \mathrm{e}^{-\theta tV(v)} \ll \sum_{v\in\mathbb{Z}^d} \mathrm{e}^{-\theta t\mathsf{d}(0,v)^\alpha} = \sum_{n\in\mathbb{N}\cup\{0\}} \mathsf{c}_n(0)\mathrm{e}^{-\theta tn^\alpha}, \tag{7.5}$$

where we recall that $\mathsf{c}_n(0)$ denotes for every $n \in \mathbb{N}$ the number of vertices in $\mathscr{G}$ such that $\mathsf{d}(0,v) = n$. For the d-dimensional integer lattice $\mathbb{Z}^d$, it is easy to check that there exists a constant $C > 0$ such that $\mathsf{c}_n(0) \leq Cn^{d-1}$ for every $n \in \mathbb{N}$, whence (7.5) yields

$$\sum_{v\in\mathbb{Z}^d} \mathrm{e}^{-\theta tV(v)} \ll \sum_{n\in\mathbb{N}\cup\{0\}} n^{d-1}\mathrm{e}^{-\theta tn^\alpha} \approx \int_0^\infty x^{d-1}\mathrm{e}^{-\theta tx^\alpha}\,\mathrm{d}x = O(t^{-d/\alpha}). \tag{7.6}$$

Summarizing our argument so far in (7.2)–(7.6), we are led to the $t \to 0$ asymptotic

$$\mathbf{Var}\big[\mathrm{Tr}[\mathrm{e}^{-tH}]\big] \ll \begin{cases} t^{2-d/\alpha} & \text{i.i.d. case,} \\ t^{2-2d/\alpha} & \text{all equal case.} \end{cases}$$

Thus, H's eigenvalue point process is proved to be number rigid if $V(v) \gg \mathsf{d}(0,v)^{d/2}$ in the i.i.d case and $V(v) \gg \mathsf{d}(0,v)^d$ in the all equal case. If γ has a less extreme decay rate (such as $\gamma(v) = O(\mathsf{d}(0,v)^{-\beta})$ as $\mathsf{d}(0,v) \to \infty$ for some $\beta > 0$), then H's eigenvalue point process is number rigid if $V(v) \gg \mathsf{d}(0,v)^\alpha$ for some $d/2 \leq \alpha \leq d$, where the exact value of α depends on γ's decay rate. We refer to Theorems 7.3.16 and 7.3.17 for the details.

7.3 Main Results

7.3.1 Basic Definitions and Notations

We begin by introducing basic/standard notations that will be used throughout the chapter.

Notation 7.3.1 (Function Spaces)**.** We use $\ell^p(\mathscr{V})$ to denote the space of real-valued absolutely p-summable (or bounded if $p = \infty$) functions on $\mathscr{V}$; we denote the associated norm by $\|\cdot\|_p$. We use $\langle\cdot,\cdot\rangle$ to denote the inner product on $\ell^2(\mathscr{V})$.

Notation 7.3.2 (Operator Theory)**.** Given a linear operator T on $\ell^2(\mathscr{V})$ (or a dense domain $D(T) \subset \ell^2(\mathscr{V})$), we use $\sigma(T)$ to denote its spectrum, and $\sigma_p(T) \subset \sigma(T)$ to denote its point spectrum. If T is bounded, we denote its operator norm by

$$\|T\|_{\mathrm{op}} := \sup_{\|f\|_2=1} \|Tf\|_2.$$

We use $\mathfrak{R}(z,T) := (T-z)^{-1}$ to denote the resolvent of T for all $z \in \mathbb{C} \setminus \sigma(T)$. If λ is an isolated eigenvalue of T, then we let

$$m_a(\lambda, T) := \dim\left(\mathrm{rg}\left(\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_\lambda} \mathfrak{R}(z,T)\,\mathrm{d}z\right)\right)$$

denote λ's algebraic multiplicity, where dim denotes the dimension of a linear space, rg denotes the range of an operator, and Γ_λ denotes a Jordan curve that encloses λ and excludes the remainder of T's spectrum.

Definition 7.3.3 (Rigidity)**.** *Let $\mathscr{X} = \sum_{k\in\mathbb{N}} \delta_{\lambda_k}$ be an infinite point process on $\mathbb{C}$. We say that $\mathscr{X}$ is real-bounded below by a random variable $\omega \in \mathbb{R}$ if $\mathrm{Re}(\lambda_k) \geq \omega$ almost surely for every $k \in \mathbb{N}$. We say that such a point process is number rigid if for every Borel set $B \subset \mathbb{C}$ such that $B \subset (-\infty,\delta] + \mathrm{i}[-\tilde{\delta},\tilde{\delta}]$ for some $\delta,\tilde{\delta} > 0$, the random* *variable $\mathscr{X}(B)$ is measurable with respect to the sigma algebra generated by the set*

$$\big\{\mathscr{X}(A) : A \subset \mathbb{C} \text{ is Borel and } B \cap A = \varnothing\big\}.$$

Remark 7.3.4. In previous works in the literature (and also in Chapter 6), it is most common to define number rigidity as the requirement that $\mathscr{X}(B)$ is measurable with respect to the configuration in $\mathbb{C} \setminus B$ for every bounded Borel set B. This is in part due to the fact that most point processes that have been proved to be number rigid thus far are such that $\mathscr{X}(B) = \infty$ almost surely whenever B is unbounded.

That being said, the fact that we are considering the spectrum of Schrödinger operators whose potentials have a strong growth at infinity means that we are considering eigenvalue point processes that are real-bounded below, in which case a more general notion of number rigidity makes sense. We note that a similarly generalized notion of rigidity appeared in the work of Bufetov on the stochastic Airy operator in [22, Proposition 3.2].

7.3.2 Markov Process

Next, we introduce the Markov processes on the graph $\mathscr{G}$ that generate our random operators, as well as some of the notions we need to describe them. We recall that $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is a countably infinite connected graph with uniformly bounded degrees and a root $0 \in \mathscr{V}$.

Definition 7.3.5 (Markov Process)**.** *Let $\Pi : \mathscr{V} \times \mathscr{V} \to [0,1]$ be a matrix such that*

1. Π *is stochastic, that is, for every $u \in \mathscr{V}$,*

$$\sum_{v \in \mathscr{V}} \Pi(u,v) = 1;$$

2. $\Pi(v,v) = 0$ *for all $v \in \mathscr{V}$; and*

3. *If* $(u,v)\notin\mathscr{E}$*, then* $\Pi(u,v)=\Pi(v,u)=0$.

Let $q:\mathscr{V}\to(0,\infty)$ *be a positive vector. We use* $X:[0,\infty)\to\mathscr{V}$ *to denote the continuous-time Markov process on* $\mathscr{V}$ *defined as follows: If* X *is in state* $u\in\mathscr{V}$*, it waits for a random time with an exponential distribution with rate* $q(u)$*, and then jumps to another state* $v\neq u$ *with probability* $\Pi(u,v)$*, independently of the wait time. Once at the new state,* X *repeats this procedure independently of all previous jumps.*

Remark 7.3.6. We note that condition (3) in the above definition implies that X is a Markov process on the graph $\mathscr{G}$, in the sense that jumps can only occur between vertices that are connected by edges.

Notation 7.3.7. For every $v\in\mathscr{V}$, we use X^v to denote the process X conditioned on the starting point $X(0)=v$. We use $\mathbb{P}^v$ to denote the law of X^v, and $\mathbb{E}^v$ to denote expectation with respect to $\mathbb{P}^v$.

We assume throughout that the Markov process X and the graph $\mathscr{G}$ satisfy the following.

Assumption 7.3.8 (Graph Geometry and Jump Rates)**.** The following two conditions hold:

1. There exists constants $d\geq 1$ and $\mathfrak{c}>0$ such that

$$\sup_{v\in\mathscr{V}}\mathsf{c}_n(v):=\sup_{v\in\mathscr{V}}|\{u\in\mathscr{V}:\mathsf{d}(u,v)=n\}|\leq\mathfrak{c}\,n^{d-1}\qquad\text{for all }n\in\mathbb{N}\cup\{0\},\tag{7.7}$$

   recalling that d is the graph distance in $\mathscr{G}$, that is, $\mathsf{d}(u,v)$ is the length of the shortest path (in terms of number of edges) connecting u and v, and with the convention that $\mathsf{d}(v,v)=0$ for all $v\in\mathscr{V}$.

2. X has uniformly bounded jump rates, that is,

$$\mathsf{q} := \sup_{v \in \mathscr{V}} q(v) < \infty.$$

Remark 7.3.9. We note that the assumption (7.7) simultaneously takes care of the requirement that $\mathscr{G}$ has uniformly bounded degrees (since $\mathsf{c}_1(v) = \deg(v)$) and of the asymptotic growth rate (7.1) stated in our informal theorem.

7.3.3 Feynman-Kac Kernel

We are now in a position to introduce the central objects of study of this chapter, namely, the Feynman-Kac semigroups of the Schrödinger operators we are interested in.

Notation 7.3.10 (Local Time)**.** For every $t \geq 0$, we let $L_t : \mathscr{V} \to [0, t]$ denote X's local time:

$$L_t(v) := \int_0^t \mathbf{1}_{\{X(s)=v\}} \, \mathrm{d}s, \qquad v \in \mathscr{V}.$$

Definition 7.3.11 (Potential and Noise)**.** *Let $V : \mathscr{V} \to \mathbb{R} \cup \{\infty\}$ be a deterministic function, and let $\xi : \mathscr{V} \to \mathbb{R}$ be a random function, where $\mathbb{E}[\xi(v)] = 0$. We denote the set*

$$\mathscr{Z} := \{v \in \mathscr{V} : V(v) = \infty\}, \tag{7.8}$$

Throughout, we make the following assumptions on the noise and potential.

Assumption 7.3.12 (Potential Growth and Noise Tails)**.** There exists $\alpha > 0$ such that

$$\liminf_{\mathsf{d}(0,v)\to\infty} \frac{V(v)}{\mathsf{d}(0,v)^\alpha} = \infty. \tag{7.9}$$

Moreover, ξ satisfies the following conditions:

1. $\mathbb{E}[\xi(v)] = 0$ for every $v \in \mathscr{V}$.

2. There exists $\mathfrak{m} > 0$ such that for every $p \in \mathbb{N}$,

$$\sup_{v \in \mathscr{V}} \mathbb{E}[|\xi(v)|^p] \leq p!\mathfrak{m}^p. \tag{7.10}$$

In the sequel, it will be useful to characterize noises in terms of the decay rate of their covariances. For this purpose, we make the following definition.

Definition 7.3.13 (covariance decay)**.** *We say that ξ has covariance decay of order (at least) $\beta > 0$ if there exists a constant $\mathfrak{C} > 0$ such that*

$$\big|\mathbb{E}[\xi(u)\xi(v)]\big| \leq \mathfrak{C}\left(\mathsf{d}(u,v)+1\right)^{-\beta} \tag{7.11}$$

for every $u, v \in \mathscr{V}$, and such that

$$\big|\mathbb{E}[\xi(u)\xi(v)\xi(w)]\big| \leq \mathfrak{C} \min_{a,b \in \{u,v,w\}} \left(\mathsf{d}(a,b)+1\right)^{-\beta} \tag{7.12}$$

for every $u, v, w \in \mathscr{V}$.

Definition 7.3.14 (Feynman-Kac Kernel)**.** *Define the Feynman-Kac kernel*

$$K_t(u,v) := \mathbb{E}^u\left[\mathrm{e}^{-\langle L_t, V+\xi\rangle}\mathbf{1}_{\{X(t)=v\}}\right], \qquad u, v \in \mathscr{V}, \tag{7.13}$$

where we assume that X is independent of ξ, and that $\mathbb{E}^v$ denotes the expectation with respect to the Markov process X^v, conditional on ξ. We denote the trace of K_t as

$$\mathrm{Tr}[K_t] := \sum_{v \in \mathscr{V}} K_t(v,v).$$

Remark 7.3.15. In the above definition, we use the convention that $\mathrm{e}^{-\infty} := 0$ whenever $V(v) = \infty$, in particular, $K_t(u, v) = 0$ whenever $u \in \mathscr{Z}$ or $v \in \mathscr{Z}$.

7.3.4 Main Results: Variance Upper Bound and Rigidity

We now state our main results. First, we have the following sufficient condition for the vanishing of the variance of the trace of K_t as $t \to 0$:

Theorem 7.3.16. *Suppose that Assumptions 7.3.8 and 7.3.12 hold. In order to have*

$$\lim_{t \to 0} \mathbf{Var}\big[\mathrm{Tr}[K_t]\big] = 0,$$

it is sufficient that the constant α in (7.9) satisfies the following:

1. *if ξ has covariance decay of order $\beta > 0$, then*

$$\alpha \begin{cases} \geq d/2 & \text{when } \beta > d, \\ > d/2 & \text{when } \beta = d, \\ \geq d - \beta/2 & \text{when } \beta < d; \end{cases} \tag{7.14}$$

2. *otherwise, $\alpha \geq d$.*

As a consequence of the above theorem, we have the following result, which states some properties of K_t's infinitesimal generator, including number rigidity.

Theorem 7.3.17. *Suppose that Assumptions 7.3.8 and 7.3.12 hold, and that we take the constant α in (7.9) as in Theorem 7.3.16. The following conditions hold almost surely:*

1. *For every $t > 0$, K_t is a trace class linear operator on $\ell^2(\mathscr{V})$. There exists a random variable $\omega \leq 0$ such that $\|K_t\|_{\mathrm{op}} \leq \mathrm{e}^{-\omega t}$ for all $t > 0$.*

2. *The family of operators $(K_t)_{t>0}$ is a strongly continuous semigroup on $\ell^2(\mathscr{V})$.*

3. The infinitesimal generator

$$H := \lim_{t \to 0} \frac{K_0 - K_t}{t} \tag{7.15}$$

is closed on some dense domain $D(H) \subset \ell^2(\mathscr{V})$, and its action on functions is given by the following matrix:

$$H(u,v) := \begin{cases} -q(u)\Pi(u,v) & \text{if } u \neq v \text{ and } u,v \notin \mathscr{Z}, \\ q(u) + V(u) + \xi(u) & \text{if } u = v \text{ and } u \notin \mathscr{Z}, \\ \infty & \text{if } u \in \mathscr{Z} \text{ or } v \in \mathscr{Z}. \end{cases} \tag{7.16}$$

(In particular, if $f \in D(H)$, then $f(v) = 0$ for every $v \in \mathscr{Z}$.)

In particular, almost surely, H has a pure point spectrum without accumulation point, and the eigenvalue point process (counting algebraic multiplicities)

$$\mathcal{X}_H := \sum_{\lambda \in \sigma(H)} m_a(\lambda, H)\, \delta_\lambda \tag{7.17}$$

is real-bounded below by ω and number rigid in the sense of Definition 7.3.3.

7.3.5 Questions of Optimality

In this section, we study the optimality of the growth assumptions we make on V in Theorem 7.3.16 by considering three counterexamples.

Theorem 7.3.18. *Suppose that X is the nearest-neighbor symmetric random walk on the integer lattice $\mathbb{Z}^d$, that $V(v) := \mathsf{d}(0,v)^\delta$ for some $\delta > 0$, and that ξ is a centered stationary Gaussian process whose covariance function $\gamma(v) := \mathbb{E}[\xi(v)\xi(0)]$ is nonnegative. If one of the following conditions hold:*

1. $\delta \leq d/2$ and $\gamma(v) = \mathbf{1}_{\{v=0\}}$;

2. $\delta \leq d - \beta/2$ *for some* $0 < \beta < d$*, and there exists a constant* $\mathfrak{L} > 0$ *such that* $\gamma(v) \geq \mathfrak{L}\big(\mathsf{d}(0,v)+1\big)^{-\beta}$ *for every* $v \in \mathscr{V}$*; or*

3. $\delta \leq d$ *and* $\inf_{v\in\mathbb{Z}^d} \gamma(v) > \mathfrak{L}$ *for some constant* $\mathfrak{L} > 0$*;*

then we have the variance lower bound

$$\liminf_{t\to 0} \mathbf{Var}\big[\mathrm{Tr}[K_t]\big] > 0.$$

Thus, given that $\mathsf{c}_n(v) \asymp n^{d-1}$ as $n \to \infty$ on $\mathbb{Z}^d$, if one is interested in providing a general sufficient condition for number rigidity on graphs using semigroups, then Theorem 7.3.16 is essentially the optimal result one could hope for.

Remark 7.3.19. An examination of the proof of Theorem 7.3.18 reveals that similar lower bounds can be proved for more general examples with little effort; we restrict our attention to this elementary setting for simplicity of exposition.

7.4 Proof of Theorem 7.3.16

Throughout this section, we assume that Assumptions 7.3.8 and 7.3.12 hold. This section is organized as follows: In Section 7.4.1, we outline the main steps of the proof of Theorem 7.3.16. That is, we state a number of technical propositions and lemmas, which we then use to prove Theorem 7.3.16. Then, in Sections 7.4.2–7.4.6, we prove the technical results stated Section 7.4.1, thus wrapping-up the proof of Theorem 7.3.16.

7.4.1 Proof Outline

7.4.1.1 Step 1. Variance Formula and First Bound

We begin with some notation.

Notation 7.4.1. Let us denote by $(\Omega_\xi, \mathbb{P}_\xi)$ the probability space on which ξ is defined. Let Y be any random element that is independent of ξ, and let F be any measurable function. We denote the random variable

$$\mathbb{E}_\xi\big[F(\xi, Y)\big] := \int_{\Omega_\xi} F(x, Y)\ \mathrm{d}\mathbb{P}_\xi(x);$$

that is, $\mathbb{E}_\xi$ is the conditional expectation with respect to ξ, given Y. Then, for measurable functions F and G, we denote the random variable

$$\mathbf{Cov}_\xi\big[F(\xi, Y), G(\xi, Y)\big] := \mathbb{E}_\xi\big[F(\xi, Y)G(\xi, Y)\big] - \mathbb{E}_\xi\big[F(\xi, Y)\big]\mathbb{E}_\xi\big[G(\xi, Y)\big].$$

Our main tool in the proof of Theorem 7.3.16 is the following variance formula:

Proposition 7.4.2. *For every $u, v \in \mathscr{V}$, we let X^u and $\tilde{X}^v$ be independent copies of the Markov process X started from u and v respectively. We assume that X^u and $\tilde{X}^v$ are independent of the noise ξ, and we denote their local times as*

$$L_t^u(w) := \int_0^t \mathbf{1}_{\{X^u(s)=w\}}\ \mathrm{d}s \qquad \text{and} \qquad \tilde{L}_t^v(w) := \int_0^t \mathbf{1}_{\{\tilde{X}^v(s)=w\}}\ \mathrm{d}s$$

for all $w \in \mathscr{V}$. It holds that

$$\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] = \sum_{u,v\in\mathscr{V}} \mathbb{E}\left[\mathrm{e}^{-\langle L_t^u + \tilde{L}_t^v, V\rangle}\mathbf{Cov}_\xi\left[\mathrm{e}^{-\langle L_t^u, \xi\rangle}, \mathrm{e}^{-\langle \tilde{L}_t^v, \xi\rangle}\right]\mathbf{1}_{\{X^u(t)=u, \tilde{X}^v(t)=v\}}\right].$$

The proof of this proposition, which we provide in Section 7.4.2 below, is essentially a direct consequence of the definition of K_t in (7.13). In order to find sufficient conditions for $\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \to 0$ as $t \to 0$ using this formula, it is convenient to control the contributions coming from V and ξ separately. To this end, we use Hölder's inequality, as well as the elementary fact that $\mathbf{1}_E \leq 1$ for every event E, which yields

$$\mathbb{E}\Big[\mathrm{e}^{-\langle L_t^u+\tilde{L}_t^v,V\rangle}\mathbf{Cov}_\xi\big[\mathrm{e}^{-\langle L_t^u,\xi\rangle},\mathrm{e}^{-\langle\tilde{L}_t^v,\xi\rangle}\big]\mathbf{1}_{\{X^u(t)=u,\tilde{X}^v(t)=v\}}\Big]$$
$$\leq\mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2}\mathbb{E}\Big[\mathbf{Cov}_\xi\Big[\mathrm{e}^{-\langle L_t^u,\xi\rangle},\mathrm{e}^{-\langle\tilde{L}_t^v,\xi\rangle}\Big]^2\Big]^{1/2}$$

for every fixed $u,v\in\mathscr{V}$. Then, by summing both sides of the above inequality over $u,v\in\mathscr{V}$, we obtain our first upper bound for the variance:

$$\mathbf{Var}\big[\mathrm{Tr}[K_t]\big]\leq\sum_{u,v\in\mathscr{V}}\mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2}\mathbb{E}\Big[\mathbf{Cov}_\xi\Big[\mathrm{e}^{-\langle L_t^u,\xi\rangle},\mathrm{e}^{-\langle\tilde{L}_t^v,\xi\rangle}\Big]^2\Big]^{1/2}. \tag{7.18}$$

7.4.1.2 Step 2. Controlling the Contributions from ξ and V

We now state the technical results that we use to control the right-hand side of (7.18). Our first such result is as follows:

Lemma 7.4.3. *Recall the definition of the constant $\mathfrak{m}>0$ in* (7.10)*. There exists a constant $C_1>0$ (which only depends on $\mathfrak{m}$) such that for every $t<1/C_1$, one has*

$$\sup_{u,v\in\mathscr{V}}\mathbb{E}\Big[\mathbf{Cov}_\xi\big[\mathrm{e}^{-\langle L_t^u,\xi\rangle},\mathrm{e}^{-\langle\tilde{L}_t^v,\xi\rangle}\big]^2\Big]^{1/2}\leq C_1t^2.$$

The proof of Lemma 7.4.3, which we provide in Section 7.4.4, follows from estimating expectations of the form $\mathbb{E}^v\big[\mathrm{e}^{-\theta\langle L_t,\xi\rangle}\big]$ using our assumption that ξ's tails are not worse than exponential (i.e., (7.10)). Next, we have the following result, which provides a tighter decay rate in the case where ξ has covariance decay:

Lemma 7.4.4. *Suppose that ξ has covariance decay of order β, as per Definition 7.3.13. Recall the definitions of the constants $\mathfrak{q}$, $\mathfrak{m}$, and $\mathfrak{C}$ in Assumption 7.3.8 (3),* (7.10), (7.11)*, and* (7.12)*. There exists a constant $C_2>0$ (which only depends on $\mathfrak{q}$, $\mathfrak{m}$, $\mathfrak{C}$, and β) such that for every $t<1/C_2$ and $u,v\in\mathscr{V}$, one has*

$$\mathbb{E}\left[\mathbf{Cov}_\xi\left[\mathrm{e}^{-\langle L_t^u,\xi\rangle},\mathrm{e}^{-\langle\tilde{L}_t^v,\xi\rangle}\right]^2\right]^{1/2}\leq C_2\left(t^2\big(\mathsf{d}(u,v)+1\big)^{-\beta}+t^4\right).$$

Lemma 7.4.4 is proved in Section 7.4.5. The proof of this lemma is rather more subtle than that of Lemma 7.4.3, and depends on a careful control of how much X^u and $\tilde{X}^v$ deviate from their respective starting points u and v. We note that the uniform upper bound on X's jump rates in Assumption 7.3.8 (3) is crucial for this lemma.

Remark 7.4.5. The proofs of Lemmas 7.4.3 and 7.4.4 both rely on some elementary formulas and estimates of the moment generating functions of the noises and their covariances, which will be stated and proved in Section 7.4.3.

With Lemmas 7.4.3 and 7.4.4 in hand, it now only remains to control the contribution of the potential V in (7.18). For this, we have the following result:

Lemma 7.4.6. *Recall the definition of $d \geq 1$ and $\mathfrak{c} > 0$ in (7.7). Suppose that we can find some constants $\kappa, \mu > 0$ such that*

$$V(v) \geq \big(\kappa\, \mathsf{d}(0,v)\big)^\alpha - \mu, \qquad v \in \mathscr{V}. \tag{7.19}$$

Then, there exists a constant $C_3 > 0$ (which only depends on α, β, d, and $\mathfrak{c}$) such that

$$\limsup_{t\to 0} t^{2d/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L^u_t+\tilde{L}^v_t,V\rangle}\Big]^{1/2} \leq C_3\kappa^{-2d}; \tag{7.20}$$

$$\limsup_{t\to 0} t^{(2d-\beta)/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L^u_t+\tilde{L}^v_t,V\rangle}\Big]^{1/2} \big(\mathsf{d}(u,v)+1\big)^{-\beta} \leq C_3\kappa^{-2d+\beta} \tag{7.21}$$

for every $0 < \beta < d$; and

$$\limsup_{t\to 0} t^{d/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L^u_t+\tilde{L}^v_t,V\rangle}\Big]^{1/2} \big(\mathsf{d}(u,v)+1\big)^{-\beta} \leq C_3\kappa^{-d} \tag{7.22}$$

for every $\beta > d$.

Lemma 7.4.6, which is proved in Section 7.4.6, follows the strategy outlined in (7.5) and (7.6): The first step of the proof of Lemma 7.4.6 relies on a rigorous implementation of the intuition that, for very small $t > 0$, one expects that

$$\mathbb{E}\big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\big]^{1/2} \approx \mathrm{e}^{-tV(u)-tV(v)}. \tag{7.23}$$

This once again relies on controlling how much X^u and $\tilde{X}^v$ deviate from their starting points. Once a quantitative version of (7.23) is established, we can then use (7.19), which allows to control $\mathbb{E}\big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\big]^{1/2}$ in terms of quantities that only depend on the geometry of $\mathscr{G}$ (more precisely, the graph distance). We then wrap up the proof of the lemma by using the upper bound on the coordination sequences in (7.7), in similar fashion to (7.6).

7.4.1.3 Step 3. Conclusion of Proof

We now combine the technical results stated above to conclude the proof of Theorem 7.3.16. By applying Lemmas 7.4.3 and 7.4.4 to our upper bound (7.18), we get that for every $t < 1/C_1$, one has

$$\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \leq C_1 t^2 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2}, \tag{7.24}$$

and if ξ has covariance decay of order $\beta > 0$, then for every $t < 1/C_2$, one has

$$\begin{aligned}\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \leq C_2 t^2 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2}\big(\mathsf{d}(u,v)+1\big)^{-\beta}\\ + C_2 t^4 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2}.\end{aligned} \tag{7.25}$$

Thanks to our growth assumption in (7.9), for any choice of $\kappa > 0$, we know that there exists a large enough $\mu > 0$ so that (7.19) holds. We may then complete the proof of Theorem 7.3.16 by an application of Lemma 7.4.6. We do this on a case-by-case basis:

Suppose first that ξ has covariance decay of order $0 < \beta < d$ and that $\alpha \geq d - \beta/2 > d/2$. Then, the fact that $2 - (2d-\beta)/\alpha \geq 0$ implies by (7.21) that

$$\begin{aligned}&\limsup_{t\to 0} t^2 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u + \tilde{L}_t^v, V\rangle}\Big]^{1/2}\big(\mathsf{d}(u,v)+1\big)^{-\beta}\\&= \limsup_{t\to 0} t^{2-(2d-\beta)/\alpha} t^{(2d-\beta)/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u + \tilde{L}_t^v, V\rangle}\Big]^{1/2}\big(\mathsf{d}(u,v)+1\big)^{-\beta} \leq C_3 \kappa^{-2d+\beta};\end{aligned}$$

and the fact that $4 - 2d/\alpha > 0$ implies by (7.20) that

$$\begin{aligned}&\limsup_{t\to 0} t^4 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u + \tilde{L}_t^v, V\rangle}\Big]^{1/2}\\&\qquad= \limsup_{t\to 0} t^{4-2d/\alpha} t^{2d/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u + \tilde{L}_t^v, V\rangle}\Big]^{1/2} = 0. \quad (7.26)\end{aligned}$$

Combining this with (7.25) implies that

$$\limsup_{t\to 0} \mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \leq C_2 C_3 \kappa^{-2d+\beta},$$

where we recall that $C_2, C_3 > 0$ do not depend on κ or μ. Since (7.19) holds for any choice of $\kappa > 0$, we can take $\kappa \to \infty$, which then yields $\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \to 0$ as $t \to 0$.

Next, suppose that ξ has covariance decay of order $\beta = d$ and that $\alpha > d/2$. We note that this implies that ξ also has correlation decay of order $\tilde{\beta}$ for any choice of $0 < \tilde{\beta} < d$. Since $\alpha > d/2$ implies that $2d - 2\alpha < d$, we can choose $\tilde{\beta}$ close enough to d so that $2d - 2\alpha < \tilde{\beta}$, which we can rearrange into $2 > (2d - \tilde{\beta})/\alpha$. Thus, (7.21) implies that

$$\limsup_{t\to 0} t^2 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u + \tilde{L}_t^v, V\rangle}\Big]^{1/2}\big(\mathsf{d}(u,v)+1\big)^{-\beta}$$

$$= \limsup_{t\to 0} t^{2-(2d-\tilde{\beta})/\alpha} t^{(2d-\tilde{\beta})/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2} \big(\mathsf{d}(u,v)+1\big)^{-\tilde{\beta}} = 0.$$

Combining this with (7.26), we directly prove that $\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \to 0$ as $t \to 0$ in this case.

Suppose now that ξ has covariance decay of order $\beta > d$ and that $\alpha \geq d/2$. Then, the fact that $2 - d/\alpha \geq 0$ implies by (7.22) that

$$\limsup_{t\to 0} t^2 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2} \big(\mathsf{d}(u,v)+1\big)^{-\beta}$$
$$= \limsup_{t\to 0} t^{2-d/\alpha} t^{d/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2} \big(\mathsf{d}(u,v)+1\big)^{-\beta} \leq C_3\kappa^{-d};$$

and the fact that $4 - 2d/\alpha \geq 0$ implies by (7.20) that

$$\limsup_{t\to 0} t^4 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2}$$
$$= \limsup_{t\to 0} t^{4-2d/\alpha} t^{2d/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2} \leq C_3\kappa^{-2d}. \quad (7.27)$$

Combining this with (7.25) and taking $\kappa \to \infty$ then implies that $\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \to 0$ as $t \to 0$.

Finally, consider the general case where we simply assume that $\alpha \geq d$. Then, $2 - 2d/\alpha \geq 0$, and thus (7.20) implies that

$$\limsup_{t\to 0} t^2 \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2}$$
$$= \limsup_{t\to 0} t^{2-2d/\alpha} t^{2d/\alpha} \sum_{u,v\in\mathscr{V}} \mathbb{E}\Big[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}\Big]^{1/2} \leq C_3\kappa^{-2d}.$$

Since the constants $C_1, C_3 > 0$ are independent of κ and μ, combining this with (7.24) and taking $\kappa \to \infty$ then implies that $\mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \to 0$ as $t \to 0$ in this case. This then completes the proof of Theorem 7.3.16.

7.4.2 Proof of Proposition 7.4.2

Since the random walk X is assumed independent of ξ, by applying Fubini's theorem to the definition of K_t in (7.13), we have that

$$\mathbb{E}\big[\mathrm{Tr}[K_t]\big] = \sum_{v\in\mathscr{V}} \mathbb{E}^v\left[\mathrm{e}^{-\langle L_t,V\rangle}\mathbb{E}_\xi\left[\mathrm{e}^{-\langle L_t,\xi\rangle}\right]\mathbf{1}_{\{X(t)=v\}}\right],$$

where we recall the definition of $\mathbb{E}_\xi$ in Notation 7.4.1. Taking the square of this expression, we then get once again by Fubini's theorem that

$$\mathbb{E}\big[\mathrm{Tr}[K_t]\big]^2 = \sum_{u,v\in\mathscr{V}} \mathbb{E}\left[\mathrm{e}^{-\langle L^u_t+\tilde{L}^v_t,V\rangle}\mathbb{E}_\xi\left[\mathrm{e}^{-\langle L^u_t,\xi\rangle}\right]\mathbb{E}_\xi\left[\mathrm{e}^{-\langle \tilde{L}^v_t,\xi\rangle}\right]\mathbf{1}_{\{X^u(t)=u,\tilde{X}^v(t)=v\}}\right].$$

Thanks to (7.13), it is easy to check that

$$\mathrm{Tr}[K_t]^2 = \sum_{u,v\in\mathscr{V}} \mathbb{E}_\xi\left[\mathrm{e}^{-\langle L^u_t+\tilde{L}^v_t,V+\xi\rangle}\mathbf{1}_{\{X^u(t)=u,\tilde{X}^v(t)=v\}}\right].$$

Taking the expectation of this expression using Fubini's theorem then leads to

$$\mathbb{E}\big[\mathrm{Tr}[K_t]^2\big] = \sum_{u,v\in\mathscr{V}} \mathbb{E}\left[\mathrm{e}^{-\langle L^u_t+\tilde{L}^v_t,V\rangle}\mathbb{E}_\xi\left[\mathrm{e}^{-\langle L^u_t+\tilde{L}^v_t,\xi\rangle}\right]\mathbf{1}_{\{X^u(t)=u,\tilde{X}^v(t)=v\}}\right].$$

The proof of Proposition 7.4.2 is then simply a matter of subtracting $\mathbb{E}\big[\mathrm{Tr}[K_t]\big]^2$ from the above expression for $\mathbb{E}\big[\mathrm{Tr}[K_t]^2\big]$, and using the definition of $\mathbf{Cov}_\xi$ in Notation 7.4.1.

7.4.3 Auxiliary results on estimates of moment generating functions

Before discussing the proofs of Lemma 7.4.3 and Lemma 7.4.4 in the next two subsections, we list here two simple propositions concerning the tail behaviors of the moment generating functions of the noises and their covariances. The first result is a straightforward consequence of Taylor expansions and Assumption 7.3.12 on the tails of the noises.

Proposition 7.4.7. *Under Assumption 7.3.12, for every pair of finitely-supported deterministic functions $f, g : \mathscr{V} \to \mathbb{R}$ such that $\|f+g\|_1, \|f\|_1, \|g\|_1 \leq 1/2\mathfrak{m}$, it holds that*

$$\left|\mathbb{E}\left[\mathrm{e}^{\langle f,\xi\rangle}\right] - 1\right| \leq 2\mathfrak{m}^2\|f\|_1^2 \tag{7.28}$$

and

$$\left|\mathbf{Cov}\left[\mathrm{e}^{\langle f,\xi\rangle}, \mathrm{e}^{\langle g,\xi\rangle}\right]\right| \leq 2\mathfrak{m}^2\big(\|f+g\|_1^2 + \|f\|_1^2 + \|g\|_1^2\big) + 4\mathfrak{m}^4\|f\|_1^2\|g\|_1^2. \tag{7.29}$$

Proof. For every deterministic function $f : \mathscr{V} \to \mathbb{R}$, it follows from a straightforward Taylor expansion of the exponential that

$$\mathbb{E}\left[\mathrm{e}^{\langle f,\xi\rangle}\right] = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1,\ldots,v_p\in\mathscr{V}} \mathbb{E}[\xi(v_1)\cdots\xi(v_p)]f(v_1)\cdots f(v_p), \tag{7.30}$$

with the convention that the term with $p = 0$ above is equal to one. Firstly, since $\mathbb{E}[\xi(v)] = 0$ for all v, the term corresponding to $p = 1$ in (7.30) is zero. Secondly, thanks to our moment growth assumption $\mathbb{E}[|\xi(v)|^p] \leq p!\mathfrak{m}^p$, for every $p \geq 2$ we have that

$$\begin{aligned}&\left|\sum_{v_1,\ldots,v_p\in\mathscr{V}} \mathbb{E}[\xi(v_1)\cdots\xi(v_p)]f(v_1)\cdots f(v_p)\right| \\ &\qquad \leq \sum_{v_1,\ldots,v_p\in\mathscr{V}} \mathbb{E}[|\xi(v_1)|^p]^{1/p}\cdots\mathbb{E}[|\xi(v_p)|^p]^{1/p}|f(v_1)|\cdots|f(v_p)| \leq p!\big(\mathfrak{m}\|f\|_1\big)^p.\end{aligned}$$

Thus, if $\|f\|_1 \leq 1/2\mathfrak{m}$, then we have that

$$\left|\mathbb{E}\left[\mathrm{e}^{\langle f,\xi\rangle}\right]-1\right| \leq \sum_{p=2}^{\infty}(\mathfrak{m}\|f\|_1)^p = \frac{(\mathfrak{m}\|f\|_1)^2}{1-\mathfrak{m}\|f\|_1} \leq 2(\mathfrak{m}\|f\|_1)^2.$$

As for the claim regarding the covariance, for any two random variables Y and Z, we have by the triangle inequality that

$$\begin{aligned}|\mathbf{Cov}[Y,Z]| &= |\mathbb{E}[YZ]-\mathbb{E}[Y]\mathbb{E}[Z]| \\ &\leq |\mathbb{E}[YZ]-1| - |\mathbb{E}[Y]-1||\mathbb{E}[Z]-1| + |1-\mathbb{E}[Y]| + |1-\mathbb{E}[Z]|\end{aligned}$$

Thus, whenever $\|f+g\|_1, \|f\|_1, \|g\|_1 \leq 1/2\mathfrak{m}$, it follows from (7.28) that

$$\left|\mathbf{Cov}\left[\mathrm{e}^{\langle f,\xi\rangle},\mathrm{e}^{\langle g,\xi\rangle}\right]\right| \leq 2\mathfrak{m}^2\big(\|f+g\|_1^2+\|f\|_1^2+\|g\|_1^2\big)+4\mathfrak{m}^4\|f\|_1^2\|g\|_1^2,$$

as desired. □

In cases where we need a more precise control on the covariance, we have the following power series expansion:

Proposition 7.4.8. *Suppose that Assumption 7.3.12 holds. For any two finitely supported deterministic functions* $f, g : \mathscr{V} \to \mathbb{R}$, *one has*

$$\mathbf{Cov}\left[\mathrm{e}^{\langle f,\xi\rangle},\mathrm{e}^{\langle g,\xi\rangle}\right] = \sum_{p=2}^{\infty}\frac{\mathcal{A}_p(f,g)}{p!},$$

where, for every $p \geq 2$, *we denote*

$$\begin{aligned}\mathcal{A}_p(f,g) := \sum_{v_1,\ldots,v_p\in\mathscr{V}}\Bigg(\sum_{m=1}^{p-1}\binom{p}{m}&\mathbf{Cov}[\xi(v_1)\cdots\xi(v_m),\xi(v_{m+1})\cdots\xi(v_p)] \\ &\cdot f(v_1)\cdots f(v_m)g(v_{m+1})\cdots g(v_p)\Bigg). \qquad (7.31)\end{aligned}$$

Proof. Using the same Taylor expansion as in (7.30), we get, on the one hand,

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\langle f+g,\xi\rangle}\right] \\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1,\ldots,v_p\in\mathscr{V}} \mathbb{E}[\xi(v_1)\cdots\xi(v_p)]\big(f(v_1)+g(v_1)\big)\cdots\big(f(v_p)+g(v_p)\big) \\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1,\ldots,v_p\in\mathscr{V}} \sum_{m=0}^{p} \binom{p}{m} \mathbb{E}[\xi(v_1)\cdots\xi(v_p)] f(v_1)\cdots f(v_m) g(v_{m+1})\cdots g(v_p),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\langle f,\xi\rangle}\right]\mathbb{E}\left[\mathrm{e}^{\langle g,\xi\rangle}\right] \\
&= \sum_{m_1,m_2=0}^{\infty} \frac{1}{m_1!m_2!} \Bigg(\sum_{v_1,\ldots,v_{m_1+m_2}\in\mathscr{V}} \mathbb{E}[\xi(v_1)\cdots\xi(v_{m_1})]\mathbb{E}[\xi(v_{m_1+1})\cdots\xi(v_{m_1+m_2})] \\
&\qquad\qquad \cdot f(v_1)\cdots f(v_{m_1}) g(v_{m_1+1})\cdots g(v_{m_1+m_2}) \Bigg) \\
&= \sum_{p=0}^{\infty}\sum_{m=0}^{p} \frac{1}{m!(p-m)!} \Bigg(\sum_{v_1,\ldots,v_p\in\mathscr{V}} \mathbb{E}[\xi(v_1)\cdots\xi(v_m)]\mathbb{E}[\xi(v_{m+1})\cdots\xi(v_p)] \\
&\qquad\qquad \cdot f(v_1)\cdots f(v_m) g(v_{m+1})\cdots g(v_p) \Bigg) \\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1,\ldots,v_p\in\mathscr{V}} \Bigg(\sum_{m=0}^{p} \binom{p}{m} \mathbb{E}[\xi(v_1)\cdots\xi(v_m)]\mathbb{E}[\xi(v_{m+1})\cdots\xi(v_p)] \\
&\qquad\qquad \cdot f(v_1)\cdots f(v_m) g(v_{m+1})\cdots g(v_p) \Bigg).
\end{aligned}$$

We then get the result by subtracting these two expressions. □

7.4.4 Proof of Lemma 7.4.3

By definition of local time, $\|L_t^u\|_1 = \|\tilde{L}_t^v\|_1 = t$, as well as $\|L_t^u + \tilde{L}_t^v\|_1 = 2t$. Thus, by (7.29) in Proposition 7.4.7, if $t < 1/4\mathfrak{m}$, then we have for any $u, v \in \mathscr{V}$ that

$$\left|\mathbf{Cov}_\xi\left[\mathrm{e}^{-\langle L_t^u,\xi\rangle}, \mathrm{e}^{-\langle \tilde{L}_t^v,\xi\rangle}\right]\right| \le 2\mathfrak{m}^2\big(4t^2+t^2+t^2\big) + 4\mathfrak{m}^4t^4 = 12\mathfrak{m}^2t^2 + 4\mathfrak{m}^4t^4.$$

Since the right-hand side of this inequality is not random, the result then follows by noting that $t^4 \le t^2$ when $t \ge 1$ and taking $C_1 := \max\{1, 4\mathfrak{m}, 12\mathfrak{m}^2, 4\mathfrak{m}^4\}$.

7.4.5 Proof of Lemma 7.4.4

For every $u, v \in \mathscr{V}$ and $t > 0$, let us denote by

$$\mathfrak{D}_t^{u,v} := \min_{\substack{a,b \in \mathscr{V} \\ L_t^u(a), \tilde{L}_t^v(b) \neq 0}} \mathsf{d}(a,b)$$

the distance between the ranges of X^u and $\tilde{X}^v$ up to time t. In Section 7.4.5.1 below we prove the following crude version of Lemma 7.4.4: For every $t < \min\{1, 1/4\mathfrak{m}\}$ and $u, v \in \mathscr{V}$,

$$\left| \mathbf{Cov}_\xi \left[\mathrm{e}^{-\langle L_t^u, \xi \rangle}, \mathrm{e}^{-\langle \tilde{L}_t^v, \xi \rangle} \right] \right| \le 2\mathfrak{C} t^2 (\mathfrak{D}_t^{u,v} + 1)^{-\beta} + 64\mathfrak{m}^4 t^4. \tag{7.32}$$

With this in hand, by Minkowski's inequality, we have that

$$\mathbb{E}\left[\mathbf{Cov}_\xi \left[\mathrm{e}^{-\langle L_t^u, \xi \rangle}, \mathrm{e}^{-\langle \tilde{L}_t^v, \xi \rangle} \right]^2 \right]^{1/2} \le 2\mathfrak{C} t^2 \mathbb{E}\big[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta} \big]^{1/2} + 64\mathfrak{m}^4 t^4 \tag{7.33}$$

for every $t < \min\{1, 1/4\mathfrak{m}\}$ and $u, v \in \mathscr{V}$.

Next, we control $\mathfrak{D}_t^{u,v}$ in terms of $\mathsf{d}(u,v)$. We do this in two cases. Suppose first that $\mathsf{d}(u,v) < 16$. In this case, we have the trivial bound

$$\mathbb{E}\big[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta} \big]^{1/2} \le 1 \le 17^\beta \big(\mathsf{d}(u,v) + 1 \big)^{-\beta},$$

which, when combined with (7.33), yields

$$\mathbb{E}\left[\mathbf{Cov}_\xi \left[\mathrm{e}^{-\langle L_t^u, \xi \rangle}, \mathrm{e}^{-\langle \tilde{L}_t^v, \xi \rangle} \right]^2 \right]^{1/2} \le 2 \cdot 17^\beta \mathfrak{C} t^2 \big(\mathsf{d}(u,v) + 1 \big)^{-\beta} + 64\mathfrak{m}^4 t^4 \tag{7.34}$$

for every $t < \min\{1, 1/4\mathfrak{m}\}$ and $u, v \in \mathscr{V}$ such that $\mathsf{d}(u, v) < 16$.

Suppose then that $\mathsf{d}(u, v) \geq 16$. For any $u, v \in \mathscr{V}$ and $t > 0$, we introduce the event

$$E_t^{u,v} := \left\{ \sup_{0 \leq s \leq t} \mathsf{d}\big(X^u(s), u\big) \leq \frac{\mathsf{d}(u, v)}{4} \quad \text{and} \quad \sup_{0 \leq s \leq t} \mathsf{d}\big(\tilde{X}^v(s), v\big) \leq \frac{\mathsf{d}(u, v)}{4} \right\}.$$

With this in hand, given that $(\mathfrak{D}_t^{u,v}+1)^{-\beta} \leq 1$ and $\sqrt{x+y} \leq \sqrt{x}+\sqrt{y}$ for all $x, y \geq 0$,

$$\mathbb{E}\big[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta}\big]^{1/2} \leq \mathbb{E}\big[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta}\mathbf{1}_{E_t^{u,v}}\big]^{1/2} + \mathbb{P}\big[(E_t^{u,v})^c\big]^{1/2}.$$

For any outcome in the event $E_t^{u,v}$, we have by the triangle inequality that

$$\mathsf{d}\big(X^u(s), \tilde{X}^v(\tilde{s})\big) \geq \mathsf{d}(u, v) - \mathsf{d}\big(X^u(s), u\big) - \mathsf{d}\big(\tilde{X}^v(\tilde{s}), v\big) \geq \frac{\mathsf{d}(u, v)}{4}$$

for every $0 \leq s, \tilde{s} \leq t$. In particular, this means that $\mathfrak{D}_t^{u,v}\mathbf{1}_{E_t^{u,v}} \geq \mathsf{d}(u, v)/4$. In Section 7.4.5.2 below, we prove that if $t < \min\{4/\mathfrak{q}, 1/4\mathfrak{q}\mathrm{e}\}$ and $\mathsf{d}(u, v) \geq 16$, then

$$\mathbb{P}\big[(E_t^{u,v})^c\big]^{1/2} \leq \frac{\sqrt{2}\,\mathfrak{q}^2\mathrm{e}^2t^2}{16}. \tag{7.35}$$

Combining these bounds with (7.33), we are led to

$$\mathbb{E}\left[\mathbf{Cov}_\xi\left[\mathrm{e}^{-\langle L_t^u,\xi\rangle}, \mathrm{e}^{-\langle \tilde{L}_t^v,\xi\rangle}\right]^2\right]^{1/2} \\ \leq 2 \cdot 4^\beta \mathfrak{C} t^2\big(\mathsf{d}(u, v) + 1\big)^{-\beta} + \left(\frac{\sqrt{2}\,\mathfrak{q}^2\mathrm{e}^2t^2}{8} + 64\mathfrak{m}^4\right)t^4 \tag{7.36}$$

for all $t < \min\{1, 1/4\mathfrak{m}, 4/\mathfrak{q}, 1/4\mathfrak{q}\mathrm{e}\}$ and $u, v \in \mathscr{V}$ such that $\mathsf{d}(u, v) \geq 16$.

With (7.34) and (7.36) in hand, in order to prove Lemma 7.4.4, it only remains to establish (7.32) and (7.35). We do this in the next two subsections.

7.4.5.1 Proof of (7.32)

Our main tool to prove (7.32) consists of the power series expansion proved in Proposition 7.31:

$$\mathbf{Cov}_\xi\left[\mathrm{e}^{-\langle L_t^u,\xi\rangle},\mathrm{e}^{-\langle \tilde{L}_t^v,\xi\rangle}\right]=\sum_{p=2}^\infty\frac{\mathcal{A}_p(-L_t^u,-\tilde{L}_t^v)}{p!},\tag{7.37}$$

where the terms $\mathcal{A}_p$ are defined in (7.31). Thanks to our moment growth assumptions in (7.10), for every $p\geq 4$ and $1\leq m\leq p-1$, we have that

$$\begin{aligned}&\big|\mathbf{Cov}[\xi(v_1)\cdots\xi(v_m),\xi(v_{m+1})\cdots\xi(v_p)]\big|\\&\leq\big|\mathbb{E}[\xi(v_1)\cdots\xi(v_p)]\big|+\big|\mathbb{E}[\xi(v_1)\cdots\xi(v_m)]\mathbb{E}[\xi(v_{m+1})\cdots\xi(v_p)]\big|\\&\leq\mathbb{E}[|\xi(v_1)|^p]^{1/p}\cdots\mathbb{E}[|\xi(v_p)|^p]^{1/p}\\&\qquad+\mathbb{E}[|\xi(v_1)|^m]^{1/m}\cdots\mathbb{E}[|\xi(v_m)|^m]^{1/m}\mathbb{E}[|\xi(v_{m+1})|^{p-m}]^{1/(p-m)}\cdots\mathbb{E}[|\xi(v_p)|^{p-m}]^{1/(p-m)}\\&\leq p!\mathfrak{m}^p+m!(p-m)!\mathfrak{m}^p\\&\leq 2p!\mathfrak{m}^p.\end{aligned}$$

Therefore, by combining (7.31) with the fact that $\sum_{m=0}^p\binom{p}{m}=2^p$, one has

$$\frac{|\mathcal{A}_p(-L_t^u,-\tilde{L}_t^v)|}{p!}\leq 2\mathfrak{m}^p\sum_{m=1}^{p-1}\binom{p}{m}\|L_t^u\|_1^m\|\tilde{L}_t^v\|_1^{p-m}\leq 2(2\mathfrak{m}t)^p.$$

Next, if ξ has covariance decay of order β, then (7.11) implies that

$$\begin{aligned}|\mathcal{A}_2(-L_t^u,-\tilde{L}_t^v)|&\leq\sum_{w_1,w_2\in\mathscr{V}}\big|\mathbf{Cov}[\xi(w_1),\xi(w_2)]\big|L_t^u(w_1)L_t^u(w_2)\\&=\mathfrak{C}(\mathfrak{D}_t^{u,v}+1)^{-\beta}\|L_t^u\|_1\|\tilde{L}_t^v\|_1\leq\mathfrak{C}t^2(\mathfrak{D}_t^{u,v}+1)^{-\beta}.\end{aligned}$$

and similarly (7.12) implies that

$$|\mathcal{A}_3(-L_t^u, -\tilde{L}_t^v)| \leq \mathfrak{C}t^3(\mathfrak{D}_t^{u,v}+1)^{-\beta}.$$

At this point if we take $t < \min\{1, 1/4\mathfrak{m}\}$, then $t^3 \leq t^2$, and thus it follows from the expansion (7.37) and the estimates above that

$$\begin{aligned}\left|\mathbf{Cov}_\xi\left[\mathrm{e}^{-\langle L_t^u,\xi\rangle}, \mathrm{e}^{-\langle \tilde{L}_t^v,\xi\rangle}\right]\right| &\leq 2\mathfrak{C}t^2(\mathfrak{D}_t^{u,v}+1)^{-\beta} + 2\sum_{p=4}^{\infty}(2\mathfrak{m}t)^p \\ &= 2\mathfrak{C}t^2(\mathfrak{D}_t^{u,v}+1)^{-\beta} + \frac{32\mathfrak{m}^4t^4}{1-2\mathfrak{m}t} \leq 2\mathfrak{C}t^2(\mathfrak{D}_t^{u,v}+1)^{-\beta} + 64\mathfrak{m}^4t^4.\end{aligned}$$

7.4.5.2 Proof of (7.35)

Let us denote by $\mathcal{S}_t(X)$ the number of jumps that X makes in the time interval $[0,t]$. For every $x > 0$ and $v \in \mathscr{V}$, it is easy to see that

$$\mathbb{P}^v\left[\max_{0\leq s\leq t}\mathsf{d}\big(v, X(s)\big) \geq x\right] \leq \mathbb{P}^v\big[\mathcal{S}_t(X) \geq x\big]. \tag{7.38}$$

For every $v \in \mathscr{V}$ and $t \geq 0$, the number of jumps $\mathcal{S}_t(X)$ is stochastically dominated by a poisson random variable with parameter $t\mathfrak{q}$. Therefore, applying the Chernoff bound for the tails of Poisson random variables, we obtain that

$$\sup_{v\in\mathscr{V}}\mathbb{P}^v\left[\max_{0\leq s\leq t}\mathsf{d}\big(v, X(s)\big) \geq x\right] \leq \sup_{v\in\mathscr{V}}\mathbb{P}^v\big[\mathcal{S}_t(X) \geq x\big] \leq \mathrm{e}^{-\mathfrak{q}t}\left(\frac{\mathfrak{q}\mathrm{e}t}{x}\right)^x \tag{7.39}$$

for every $x > \mathfrak{q}t$. In order to specialize this to (7.35), we use the parameter $x := \mathsf{d}(u,v)/4$. If $t < \min\{4/\mathfrak{q}, 1/4\mathfrak{q}\mathrm{e}\}$ and $\mathsf{d}(u,v) \geq 16$, then we have that $4\mathfrak{q}\mathrm{e}t < 1$ and $x > \mathfrak{q}t$, and thus it follows by a union bound that

$$\mathbb{P}\big[(E_t^{u,v})^c\big]^{1/2} \leq \left(\mathbb{P}^u\left[\mathcal{S}_t(X) \geq \frac{\mathsf{d}(u,v)}{4}\right] + \mathbb{P}^v\left[\mathcal{S}_t(X) \geq \frac{\mathsf{d}(u,v)}{4}\right]\right)^{1/2}$$

$$\leq \sqrt{2}\mathrm{e}^{-\mathfrak{q}t/2}\left(\frac{4\mathfrak{q}\mathrm{e}t}{\mathsf{d}(u,v)}\right)^{\mathsf{d}(u,v)/8} \leq \frac{\sqrt{2}\,\mathfrak{q}^2\mathrm{e}^2t^2}{16},$$

as desired.

7.4.6 Proof of Lemma 7.4.6

Notation 7.4.9. Throughout this proof, we use $C > 0$ to denote a constant whose exact value may change from one display to the next. If $C > 0$ depends on some other parameters, this will be explicitly stated.

7.4.6.1 Step 1. General Upper Bound

Our first step in this proof is to provide a general upper bound for $\mathbb{E}[\mathrm{e}^{-2\langle L_t^u+\tilde{L}_t^v,V\rangle}]^{1/2}$ that formalizes the intuition (7.23). To this effect, we claim that if (7.19) holds, then

$$-\langle L_t^u, V\rangle \leq -\big(\kappa t^{1/\alpha}\mathsf{d}(0,u)\big)^{\min\{\alpha,1\}} + \max_{0\leq s\leq t}\Big(\kappa t^{1/\alpha}\mathsf{d}\big(u,X^u(s)\big)\Big)^{\min\{\alpha,1\}} - 1 + \mu t \tag{7.40}$$

for every $u \in \mathscr{V}$ and $t > 0$, and similarly for $-\langle \tilde{L}_t^v, V\rangle$. To see this, we note that

$$\begin{aligned}
-\langle L_t^u, V\rangle &\leq -\int_0^t \Big(\kappa\,\mathsf{d}\big(0,X^u(s)\big)\Big)^\alpha\,\mathrm{d}s + \mu t\\
&= -\int_0^t \Big|\kappa\Big(\mathsf{d}(0,u) - \mathsf{d}(0,u) + \mathsf{d}\big(0,X^u(s)\big)\Big)\Big|^\alpha\,\mathrm{d}s + \mu t\\
&= -\int_0^1 \Big|\kappa t^{1/\alpha}\Big(\mathsf{d}(0,u) - \mathsf{d}(0,u) + \mathsf{d}\big(0,X^u(ut)\big)\Big)\Big|^\alpha\,\mathrm{d}u + \mu t,
\end{aligned} \tag{7.41}$$

where the first line follows directly from (7.19), and the last line follows from a change of variables. For any $x, y \in \mathbb{R}$, the triangle inequality implies that

$$|x-y|^\alpha \geq |x-y|^{\min\{\alpha,1\}} - 1 \geq |x|^{\min\{\alpha,1\}} - |y|^{\min\{\alpha,1\}} - 1.$$

Applying this to (7.41) yields

$$-\langle L_t^u, V\rangle \leq -\big(\kappa t^{1/\alpha}\mathsf{d}(0,u)\big)^{\min\{\alpha,1\}} + \max_{0\leq s\leq t}\left|\kappa t^{1/\alpha}\Big(\mathsf{d}\big(0,X^u(s)\big)-\mathsf{d}(0,u)\Big)\right|^{\min\{\alpha,1\}} - 1 + \mu t.$$

We then obtain (7.40) by combining the fact that $x \mapsto x^{\min\{\alpha,1\}}$ is increasing for $x > 0$ with the reverse triangle inequality $\big|\mathsf{d}\big(0,X^u(s)\big) - \mathsf{d}(0,u)\big| \leq \mathsf{d}\big(u,X^u(s)\big)$.

With (7.40) in hand, we see that $\mathbb{E}[\mathrm{e}^{-2\langle L_t^u+\tilde L_t^v,V\rangle}]^{1/2}$ is bounded above by

$$\begin{aligned}
&\mathrm{e}^{2(\mu t-1)-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}\\
&\cdot\mathbb{E}\left[\exp\left(\max_{0\leq s\leq t}\Big(\kappa t^{1/\alpha}\mathsf{d}\big(u,X^u(s)\big)\Big)^{\min\{\alpha,1\}}+\max_{0\leq s\leq t}\Big(\kappa t^{1/\alpha}\mathsf{d}\big(v,\tilde X^v(s)\big)\Big)^{\min\{\alpha,1\}}\right)\right]^{1/2}.
\end{aligned} \tag{7.42}$$

On the one hand, $\mathrm{e}^{2(\mu t-1)} \to \mathrm{e}^{-2}$ as $t \to 0$ for any choice of $\mu > 0$. On the other hand, thanks to the tail bound (7.39), we know that for every $\theta, \kappa > 0$, one has

$$\limsup_{t\to 0}\sup_{u\in\mathscr{V}}\mathbb{E}\left[\exp\left(\theta\max_{0\leq s\leq t}\Big(\kappa t^{1/\alpha}\mathsf{d}\big(u,X^u(s)\big)\Big)^{\min\{\alpha,1\}}\right)\right] = 1,$$

and similarly for $\tilde X$. Therefore, by a straightforward application of Hölder's inequality on the second line of (7.42), in order to prove Lemma 7.4.6, it suffices to prove that there exists a constant $C > 0$ (which only depends on α, β, d, and $\mathfrak{c}$) such that

$$\limsup_{t\to 0} t^{2d/\alpha}\sum_{u,v\in\mathscr{V}}\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}} \leq C\kappa^{-2d}; \tag{7.43}$$

$$\limsup_{t\to 0} t^{(2d-\beta)/\alpha} \sum_{u,v\in\mathscr{V}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \leq C\kappa^{-2d+\beta} \tag{7.44}$$

for every $0<\beta<d$; and

$$\limsup_{t\to 0} t^{d/\alpha} \sum_{u,v\in\mathscr{V}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \leq C\kappa^{-d} \tag{7.45}$$

for every $\beta>d$. We now prove these claims in two steps.

7.4.6.2 Step 2. Proof of (7.43)

Recalling the definition and upper bound of $\mathscr{G}$'s coordination sequences $\mathsf{c}_n(v)$ in (7.7), we have that

$$\begin{aligned}
&\sum_{u,v\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}} = \left(\sum_{v\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}\right)^2\\
&= \left(\sum_{n\in\mathbb{N}\cup\{0\}} \mathsf{c}_n(0)\,\mathrm{e}^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}}\right)^2 \leq \mathfrak{c}^2\left(\sum_{n\in\mathbb{N}\cup\{0\}} n^{d-1}\mathrm{e}^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}}\right)^2\\
&= \mathfrak{c}^2 t^{(-2d+2)/\alpha}\left(\sum_{n\in t^{1/\alpha}\mathbb{N}\cup\{0\}} n^{d-1}\mathrm{e}^{-(\kappa n)^{\min\{\alpha,1\}}}\right)^2.
\end{aligned} \tag{7.46}$$

By a Riemann sum, we have that

$$\begin{aligned}
&\lim_{t\to\infty} t^{2/\alpha}\left(\sum_{n\in t^{1/\alpha}\mathbb{N}\cup\{0\}} n^{d-1}\mathrm{e}^{-(\kappa n)^{\min\{\alpha,1\}}}\right)^2\\
&\qquad = \left(\int_0^\infty x^{d-1}\mathrm{e}^{-(\kappa x)^{\min\{\alpha,1\}}}\,\mathrm{d}x\right)^2 = \frac{\kappa^{-2d}\Gamma\left(\frac{d}{\min\{1,\alpha\}}\right)^2}{\min\{1,\alpha^2\}}.
\end{aligned} \tag{7.47}$$

Combining this limit with (7.46) yields (7.43), where, as shown on the right-hand side of (7.47), the constant $C>0$ only depends on the parameters α, d, and $\mathfrak{c}$.

7.4.6.3 Step 3. Proof of (7.44) and (7.45)

We now conclude the proof of Lemma 7.4.6 by establishing (7.44) and (7.45). We separate the analysis of the sum on the left-hand sides of (7.44) and (7.45) into two parts, namely, the terms $u, v \in \mathscr{V}$ such that $\mathsf{d}(u,v) > \kappa^{-1}t^{-1/\alpha}$, and those such that $\mathsf{d}(u,v) \leq \kappa^{-1}t^{-1/\alpha}$.

We first consider the terms such that $\mathsf{d}(u,v) > \kappa^{-1}t^{-1/\alpha}$. For these, we have the sequence of upper bounds

$$\begin{aligned}
&\sum_{\substack{u,v\in\mathscr{V}\\ \mathsf{d}(u,v)>\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}}\\
&\leq \sum_{\substack{u,v\in\mathscr{V}\\ \mathsf{d}(u,v)>\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\mathsf{d}(u,v)^{\beta}}\\
&< \kappa^{\beta}t^{\beta/\alpha} \sum_{\substack{u,v\in\mathscr{V}\\ \mathsf{d}(u,v)>\kappa^{-1}t^{-1/\alpha}}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}\\
&\leq \kappa^{\beta}t^{\beta/\alpha}\left(\sum_{v\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}\right)^{2}.
\end{aligned}$$

At this point, by replicating the arguments in Section 7.4.6.2, we get that there exists a constant $C > 0$ that only depends on α, d, and $\mathfrak{c}$, and such that

$$\limsup_{t\to 0} t^{(2d-\beta)/\alpha} \sum_{\substack{u,v\in\mathscr{V}\\ \mathsf{d}(u,v)>\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \leq C\kappa^{-2d+\beta} \tag{7.48}$$

if $0 < \beta < d$; and

$$\lim_{t\to 0} t^{d/\alpha} \sum_{\substack{u,v\in\mathscr{V}\\ \mathsf{d}(u,v)>\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} = 0 \tag{7.49}$$

if $\beta > d$.

We now consider the terms such that $\mathsf{d}(u,v) \le \kappa^{-1}t^{-1/\alpha}$. For those terms, we can reformulate the summands as follows:

$$\sum_{\substack{u,v\in\mathscr{V}\\ \mathsf{d}(u,v)\le\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \tag{7.50}$$

$$= \sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}} \left(\sum_{\substack{v\in\mathscr{V}\\ \mathsf{d}(u,v)\le\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \right)$$

$$= \sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}} \left(\sum_{\substack{v\in\mathscr{V}\\ \mathsf{d}(u,v)\le\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}(\mathsf{d}(u,v)+\mathsf{d}(0,v)-\mathsf{d}(u,v)))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \right).$$

For every every $u,v\in\mathscr{V}$ such that $\mathsf{d}(u,v)\le\kappa^{-1}t^{-1/\alpha}$, the fact that $\mathsf{d}(0,v)\ge 0$ gives the upper bound $\mathrm{e}^{-(\kappa t^{1/\alpha}(\mathsf{d}(0,v)-\mathsf{d}(u,v)))^{\min\{\alpha,1\}}} \le \mathrm{e}$. Putting this into the above equation, we then obtain that

$$(7.50) \le \mathrm{e}\sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}} \left(\sum_{\substack{v\in\mathscr{V}\\ \mathsf{d}(u,v)\le\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(u,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \right)$$

$$\le \mathrm{e}\sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}} \left(\sum_{n=0}^{\kappa^{-1}t^{-1/\alpha}} \frac{\mathsf{c}_n(u)\,\mathrm{e}^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}}}{\big(n+1\big)^{\beta}} \right).$$

Thanks to the uniform bound in (7.7), we then have that

$$(7.50) \le \mathrm{e}\mathfrak{c} \left(\sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}} \right) \left(\sum_{n=0}^{\kappa^{-1}t^{-1/\alpha}} \frac{n^{d-1}\mathrm{e}^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}}}{\big(n+1\big)^{\beta}} \right)$$

$$\le \mathrm{e}^{1+(\kappa t^{1/\alpha})^{\min\{\alpha,1\}}}\mathfrak{c} \left(\sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}} \right)$$

$$\cdot \left(\sum_{n\in\mathbb{N}\cup\{0\}} (n+1)^{d-1-\beta}\mathrm{e}^{-(\kappa t^{1/\alpha}(n+1))^{\min\{\alpha,1\}}} \right)$$

$$= \mathrm{e}^{1+o(1)}\mathfrak{c}\left(\sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}}\right)\left(\sum_{n\in\mathbb{N}} n^{d-1-\beta}\mathrm{e}^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}}\right). \tag{7.51}$$

We now analyze the two sums on the right-hand side of (7.51). Looking at the first term, the same analysis carried out in Section 7.4.6.2 implies that

$$\limsup_{t\to 0} t^{d/\alpha}\sum_{u\in\mathscr{V}} \mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}} \leq C\kappa^{-d}$$

for some C that only depends on α, d, and $\mathfrak{c}$. Next, the second sum in (7.51) is analyzed differently depending on whether $0 < \beta < d$ or $\beta > d$: On the one hand, if $\beta < d$, then by a Riemann sum we have that

$$\begin{aligned}\lim_{t\to 0} t^{(d-\beta)/\alpha}\sum_{n\in\mathbb{N}} n^{d-1-\beta}\mathrm{e}^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}} &= \lim_{t\to 0} t^{1/\alpha}\sum_{n\in t^{1/\alpha}\mathbb{N}} n^{d-1-\beta}\mathrm{e}^{-(\kappa n)^{\min\{\alpha,1\}}} \\ &= \int_0^\infty x^{d-1-\beta}\mathrm{e}^{-(\kappa x)^{\min\{\alpha,1\}}}\,\mathrm{d}x = \frac{\kappa^{-d+\beta}\Gamma\left(\frac{d-\beta}{\min\{\alpha,1\}}\right)}{\min\{\alpha,1\}}.\end{aligned}$$

On the other hand, if $\beta > d$, then we have by dominated convergence that

$$\lim_{t\to 0}\sum_{n\in\mathbb{N}} n^{d-1-\beta}\mathrm{e}^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}} = \sum_{n\in\mathbb{N}} n^{d-1-\beta};$$

we know that the sum on the right-hand side is convergent since $\beta > d$.

Putting these two limits back into (7.51), we then get that there exists a constant $C > 0$ (which only depends on α, d, β, and $\mathfrak{c}$) such that

$$\limsup_{t\to 0} t^{(2d-\beta)/\alpha}\sum_{\substack{u,v\in\mathscr{V}\\ \mathsf{d}(u,v)\leq\kappa^{-1}t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha}\mathsf{d}(0,u))^{\min\{\alpha,1\}}-(\kappa t^{1/\alpha}\mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \leq C\kappa^{-2d+\beta}$$

when $\beta < d$, and such that

$$\limsup_{t \to 0} t^{d/\alpha} \sum_{\substack{u,v \in \mathscr{V} \\ \mathsf{d}(u,v) \leq \kappa^{-1} t^{-1/\alpha}}} \frac{\mathrm{e}^{-(\kappa t^{1/\alpha} \mathsf{d}(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha} \mathsf{d}(0,v))^{\min\{\alpha,1\}}}}{\big(\mathsf{d}(u,v)+1\big)^{\beta}} \leq C\kappa^{-d}$$

when $\beta > d$. Combining this with (7.48) and (7.49) concludes the proof of (7.44) and (7.45). With this in hand, we have now completed the proof of Lemma 7.4.6.

7.5 Spectral Mapping and Multiplicity

A crucial aspect of the proof of Theorem 7.3.17 is the ability to relate exponential linear statistics of the eigenvalue point process (7.17) to the trace of K_t via the identities

$$\mathrm{Tr}[K_t] = \sum_{\mu \in \sigma(K_t) \setminus \{0\}} m_a(\mu, K_t)\, \mu = \sum_{\lambda \in \sigma(H)} m_a(\lambda, H)\, \mathrm{e}^{-t\lambda} \in (0, \infty). \tag{7.52}$$

Though we expect that such a result is known (or at least folklore) in the operator theory community, we were not able to locate any reference that contains all of the precise statements that we need to prove (7.52). (This is especially so since the level of generality in this chapter allows for non-self-adjoint operators.) As such, our purpose in this section is to provide a general criterion for an identity of the form (7.52) to hold (as well as a few more properties), which we then use in Section 7.6 to wrap up the proof of Theorem 7.3.17.

We begin this section with a definition:

Definition 7.5.1. *We say that a linear operator T on $\ell^2(\mathscr{V})$ is finite-dimensional if there exists a finite set $\mathscr{U} \subset \mathscr{V}$ such that $T(u,v) = 0$ whenever $(u,v) \notin \mathscr{U} \times \mathscr{U}$. In particular, if we enumerate the set $\mathscr{U} = \{u_1, \ldots, u_{|\mathscr{U}|}\}$, then T has the same spectrum* *as the* $|\mathscr{U}| \times |\mathscr{U}|$ *matrix* M_T *with entries*

$$M_T(i,j) := T(u_i, u_j), \qquad 1 \leq i, j \leq |\mathscr{U}|. \tag{7.53}$$

The result that we prove in this section is as follows:

Proposition 7.5.2. *Let* $(T_t)_{t>0}$ *be a strongly continuous semigroup of trace class operators on* $\ell^2(\mathscr{V})$ *such that* $\|T_t\|_{\mathrm{op}} \leq \mathrm{e}^{-\omega t}$ *for some* $\omega < 0$*, and let* G *be its infinitesimal generator. The following holds:*

1. G *is closed and densely defined on* $\ell^2(\mathscr{V})$*.*

2. $\sigma(G) = \sigma_p(G)$*, and* $\mathrm{Re}(\lambda) \geq \omega$ *for all* $\lambda \in \sigma(G)$*.*

3. *For every* $t > 0$*,* $\sigma(T_t) \setminus \{0\} = \{\mathrm{e}^{-t\lambda} : \lambda \in \sigma(G)\}$*.*

Moreover, if there exists a sequence of finite-dimensional operators $(G_n)_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} \|\mathfrak{R}(z, G_n) - \mathfrak{R}(z, G)\|_{\mathrm{op}} = 0 \tag{7.54}$$

for at least one $z \in \mathbb{C} \setminus \sigma(G)$ *and such that*

$$\lim_{n\to\infty} \|\mathrm{e}^{-tG_n} - T_t\|_{\mathrm{op}} = 0, \tag{7.55}$$

then for every $t > 0$ *and* $\mu \in \sigma(T_t) \setminus \{0\}$*,*

$$m_a(\mu, T_t) = \sum_{\lambda\in\sigma(G):\ \mathrm{e}^{-t\lambda}=\mu} m_a(\lambda, G). \tag{7.56}$$

As a direct consequence of the above proposition, we have that

$$\mathrm{Tr}[T_t] = \sum_{\mu\in\sigma(T_t)\setminus\{0\}} m_a(\mu, T_t)\,\mu = \sum_{\lambda\in\sigma(G)} m_a(\lambda, G)\,\mathrm{e}^{-t\lambda} \in \mathbb{C}$$

for all $t > 0$, which is precisely the kind of statement that we are looking for. The remainder of this section is now devoted to the proof of Proposition 7.5.2.

7.5.1 Step 1. Closed Generator and Spectral Mapping

We begin with the more straightforward aspects of the statement of Proposition 7.5.2, namely, items (1)–(3). Since $(T_t)_{t>0}$ is strongly continuous and $\|T_t\|_{\mathrm{op}} \leq \mathrm{e}^{-\omega t}$, it follows from the Hille-Yosida theorem (e.g., [44, Chapter II, Corollary 3.6]) that G is closed and densely defined on $\ell^2(\mathscr{V})$. Moreover, $\mathrm{Re}(\lambda) \geq \omega$ for every $\lambda \in \sigma(G)$. Given that the T_t are trace class, we know that $\sigma(T_t) = \sigma_p(T_t)$ and that

$$\mathrm{Tr}[T_t] = \sum_{\mu \in \sigma(T_t) \setminus \{0\}} m_a(\mu, T_t)\mu \in \mathbb{C}$$

by Lidskii's theorem (e.g., [101, Sections 3.6 and 3.12]). Next, by the spectral mapping theorem (e.g., [44, Chapter IV, (3.7) and (3.16)]), we know that for every $t > 0$,

$$\big\{\mathrm{e}^{-t\lambda} : \lambda \in \sigma(G)\big\} \subset \sigma(T_t) \qquad \text{and} \qquad \big\{\mathrm{e}^{-t\lambda} : \lambda \in \sigma_p(G)\big\} = \sigma_p(T_t) \setminus \{0\}. \quad (7.57)$$

In particular, $\sigma(G) = \sigma_p(G)$, concluding the proof of Proposition 7.5.2 (1)–(3).

7.5.2 Step 2. Multiplicities in Finite Dimensions

It now remains to prove (7.56). Before we prove this result, we first prove the corresponding statement in finite dimensions, namely:

Lemma 7.5.3. *Let T be a finite-dimensional linear operator on $\ell^2(\mathscr{V})$ and $F : \mathbb{C} \to \mathbb{C}$ be an analytic function. For every $\mu \in \sigma\big(F(T)\big) = F\big(\sigma(T)\big)$, one has*

$$m_a\big(\mu, F(T)\big) = \sum_{\lambda \in \sigma(T):\ F(\lambda) = \mu} m_a(\lambda, T).$$

Applying this to the exponential map and the operators G_n, we are led to the fact that for every $n \in \mathbb{N}$, $t > 0$, and $\mu \in \sigma(G_n)$ one has

$$m_a(\mu, \mathrm{e}^{-tG_n}) = \sum_{\lambda \in \sigma(G_n):\ \mathrm{e}^{-t\lambda} = \mu} m_a(\lambda, G_n). \tag{7.58}$$

Proof of Lemma 7.5.3. It suffices to prove the result with T replaced by M_T and $F(T)$ replaced by $F(M_T)$, where M_T is the matrix defined in (7.53). Let $M_T = PJP^{-1}$ be M_T's Jordan canonical form. That is, J is the direct sum of M_T's Jordan blocks, and in particular the number of times any $\lambda \in \mathbb{C}$ appears on J's diagonal is equal to $m_a(\lambda, M_T)$. By the standard analytic functional calculus for matrices, we know that $F(M_T) = PF(J)P^{-1}$, where $F(J)$ is the direct sum of M_T's transformed Jordan blocks, wherein any $k \times k$ Jordan block of the form

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}$$

is transformed into the upper triangular matrix

$$\begin{bmatrix} F(\lambda) & F'(\lambda) & F''(\lambda)/2 & \cdots & F^{(k-1)}(\lambda)/(k-1)! \\ & F(\lambda) & F'(\lambda) & \cdots & F^{(k-2)}(\lambda)/(k-2)! \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & F'(\lambda) \\ & & & & F(\lambda) \end{bmatrix}.$$

Given that the characteristic polynomial of $F(M_T)$ is the same as that of $F(J)$, this readily implies the result. $\square$

7.5.3 Step 3. Passing to the Limit

We now complete the proof of Proposition 7.5.2 by arguing that the identity (7.58) persists in the large n limit. Thanks to (7.54) and (7.55), we know that we have the convergences $G_n \to G$ and $\mathrm{e}^{-tG_n} \to T_t$ for every $t > 0$ in the generalized sense of Kato (see [73, Chapter IV, (2.9), (2.20) and p. 206] for a definition of convergence in the generalized sense, and [73, Chapter IV, Theorems 2.23 a) and 2.25] for a proof that norm-resolvent and norm convergence implies convergence in the generalized sense). As shown in [73, Chapter IV, Theorem 3.16] (see also [73, Chapter IV, Section 5] for a discussion specific to the context of isolated eigenvalues), convergence in the generalized sense implies the following spectral continuity results:

Notation 7.5.4. In what follows, we use $B(z,r)$ to denote the closed ball in the complex plane centered at $z \in \mathbb{Z}$ and with raduis $r > 0$.

Corollary 7.5.5. *For every $\lambda \in \sigma(G)$, if $\varepsilon > 0$ is such that $\sigma(G) \cap B(\lambda,\varepsilon) = \{\lambda\}$, then there exists $N \in \mathbb{N}$ large enough so that*

$$\sum_{\tilde{\lambda} \in \sigma(G_n) \cap B(\lambda,\varepsilon)} m_a(\tilde{\lambda}, G_n) = m_a(\lambda, G) \tag{7.59}$$

whenever $n \geq N$.

Conversely, for every $t > 0$ and $\mu \in \sigma(T_t) \setminus \{0\}$, if $\varepsilon > 0$ is such that $\sigma(T_t) \cap B(\mu,\varepsilon) = \{\mu\}$, then there exists $N \in \mathbb{N}$ large enough so that

$$\sum_{\tilde{\mu} \in (\mathrm{e}^{-tG_n}) \cap B(\mu,\varepsilon)} m_a(\tilde{\mu}, \mathrm{e}^{-tG_n}) = m_a(\mu, T_t) \tag{7.60}$$

whenever $n \geq N$.

We are now ready to prove (7.53). We first show that for every $t > 0$ and $\mu \in \sigma(T_t) \setminus \{0\}$, the set $\{\lambda \in \sigma(G) : \ \mathrm{e}^{-t\lambda} = \mu\}$ is finite. Suppose by contradiction that this is not the case. Then, for any integer $M > 0$, we can find at least M distinct eigenvalues $\lambda_1, \ldots, \lambda_M \in \sigma(G)$ such that $\mathrm{e}^{-t\lambda_i} = \mu$. By taking a small enough $\varepsilon > 0$ and large enough $N \in \mathbb{N}$, a combination of (7.58) and (7.60) yields

$$m_a(\mu, T_t) = \sum_{\tilde{\mu} \in \sigma(\mathrm{e}^{-tG_N}) \cap B(\mu, \varepsilon)} m_a(\tilde{\mu}, \mathrm{e}^{-tG_N}) = \sum_{\tilde{\lambda} \in \sigma(G_N):\ \mathrm{e}^{-t\tilde{\lambda}} \in B(\mu,\varepsilon)} m_a(\tilde{\lambda}, G_N). \quad (7.61)$$

Since $z \mapsto \mathrm{e}^{-tz}$ is continuous, we can take $\delta > 0$ small enough so that

1. if $\tilde{\lambda} \in B(\lambda_i, \delta)$ for some $1 \leq i \leq M$, then $\mathrm{e}^{-t\tilde{\lambda}} \in B(\mu, \varepsilon)$; and

2. $\sigma(G) \cap B(\lambda_i, \delta) = \{\lambda_i\}$ for every $1 \leq i \leq M$.

Thus, up to increasing the value of N if necessary, an application of (7.59) to the right-hand side of (7.61) then gives

$$m_a(\mu, T_t) \geq \sum_{i=1}^{M} \sum_{\tilde{\lambda} \in \sigma(G_N) \cap B(\lambda_i, \delta)} m_a(\tilde{\lambda}, G_N) = \sum_{i=1}^{M} m_a(\lambda_i, G) \geq M. \quad (7.62)$$

Since M was arbitrary, this implies that $m_a(\mu, T_t) = \infty$. Since T_t is trace class this cannot be the case, hence we conclude that $\{\lambda \in \sigma(G) :\ \mathrm{e}^{-t\lambda} = \mu\}$ is finite.

By repeating the argument leading up to (7.62), but this time letting M be equal to the number of eigenvalues in the set $\{\lambda \in \sigma(G) :\ \mathrm{e}^{-t\lambda} = \mu\}$, we obtain that

$$m_a(\mu, T_t) \geq \sum_{\lambda \in \sigma(G):\ \mathrm{e}^{-t\lambda} = \mu} m_a(\lambda, G).$$

We now proceed to prove the reverse inequality. Recall that $\{\lambda \in \sigma(G) :\ \mathrm{e}^{-t\lambda} = \mu\}$ contains finitely many elements. Denote them by $\lambda_1, \ldots, \lambda_M$ for some $M \in \mathbb{N}$. Thanks to (7.59), we can find a small enough $\varepsilon > 0$ and large enough $N \in \mathbb{N}$ such that

$$\sum_{i=1}^{M} m_a(\lambda_i, G) = \sum_{\tilde{\lambda} \in \cup_{i=1}^{M} \sigma(G_N) \cap B(\lambda_i, \varepsilon)} m_a(\tilde{\lambda}, G_N) = \sum_{\tilde{\lambda} \in \sigma(G_N) \cap \left(\cup_{i=1}^{M} B(\lambda_i, \varepsilon)\right)} m_a(\tilde{\lambda}, G_N).$$

Then, by (7.58), one has

$$\sum_{\tilde{\lambda}\in\sigma(G_N)\cap\left(\cup_{i=1}^M B(\lambda_i,\varepsilon)\right)} m_a(\tilde{\lambda},G_N)=\sum_{\substack{\tilde{\mu}\in\sigma(e^{-tG_N})\\ \tilde{\mu}\in e^{-t}(\cup_{i=1}^M B(\lambda_i,\varepsilon))}} m(\tilde{\mu},e^{-tG_N}),\qquad (7.63)$$

where we use $\mathrm{e}^{-t}(B)$ to denote the image of a set $B\subset\mathbb{C}$ through the exponential map $z\mapsto \mathrm{e}^{-tz}$. Since the exponential map is open and $\mathrm{e}^{-t\lambda_i}=\mu$ for all $1\leq i\leq M$, we can find a small enough $\delta>0$ such that $B(\mu,\delta)\subset e^{-t}(\cup_{i=1}^M B(\lambda_i,\varepsilon))$ and $\sigma(T_t)\cap B(\mu,\delta)=\{\mu\}$. As a result we get

$$\sum_{i=1}^M m_a(\lambda_i,G)\geq \text{r.h.s. of (7.63)}\geq \sum_{\tilde{\mu}\in\sigma(e^{-tG_N})\cap B(\mu,\delta)} m_a(\tilde{\mu},e^{-tG_N}).\qquad (7.64)$$

At this point, up to increasing N if necessary an application of (7.60) then yields

$$\sum_{i=1}^M m_a(\lambda_i,G)\geq \sum_{\tilde{\mu}\in\sigma(e^{-tG_N})\cap B(\mu,\delta)} m_a(\tilde{\mu},e^{-tG_N})=m_a(\mu,T_t),$$

thus concluding the proof of (7.56) and Proposition 7.5.2.

7.6 Proof of Theorem 7.3.17

In this section, we prove Theorem 7.3.17. We assume throughout that Assumptions 7.3.8 and 7.3.12 hold. We begin with a notation:

Notation 7.6.1. Throughout this proof, we denote X's transition semigroup by

$$\Pi_t(u,v)=\mathbb{P}^u[X(t)=v],\qquad t\geq 0,\ u,v\in\mathscr{V}.$$

7.6.1 Step 1. Boundedness

Our first step in the proof is to show that, almost surely, K_t is a bounded linear operator on $\ell^2(\mathscr{V})$ with $\|K_t\|_{\mathrm{op}} \leq \mathrm{e}^{\omega t}$ for every $t > 0$ for some $\omega < 0$. As is typical in Schrödinger semigroup theory, this relies on controlling the minimum of the random potential $V + \xi$. To this end, we have the following result:

Lemma 7.6.2. *Define the random variable*

$$\omega_0 := \inf_{v \in \mathscr{V}} \big(V(v) + \xi(v)\big). \tag{7.65}$$

$\omega_0 > -\infty$ *almost surely.*

Proof. Thanks to (7.9), it suffices to prove that

$$\liminf_{n \to \infty} \left(\inf_{v \in \mathscr{V}:\ \mathsf{d}(0,v) \leq n} \frac{\xi(v)}{\log n} \right) > -\infty \qquad \text{almost surely.} \tag{7.66}$$

By a union bound and Markov's inequality, for every $\theta, \lambda > 0$,

$$\mathbb{P}\Big(\inf_{v \in \mathscr{V}:\ \mathsf{d}(0,v) \leq n} \xi(v) \leq -\lambda \Big) \leq \sum_{v \in \mathscr{V}:\ \mathsf{d}(0,v) \leq n} \mathrm{e}^{-\theta\lambda} \mathbb{E}\big[\mathrm{e}^{-\theta\xi(v)}\big].$$

On the one hand, thanks to (7.7), we have that

$$|\{v \in \mathscr{V} : \mathsf{d}(0,v) \leq n\}| \leq \mathfrak{c} \sum_{m=1}^{n} m^{d-1} \leq \mathfrak{c} + \mathfrak{c} \int_1^n x^{d-1}\,\mathrm{d}x \leq C n^d$$

for some constant $C > 0$. On the other hand, thanks to the moment bound (7.10), there exists a $\theta > 0$ small enough so that

$$\sup_{v \in \mathscr{V}} \mathbb{E}\big[\mathrm{e}^{-\theta\xi(v)}\big] < \infty.$$

Combining these two observations, we conclude that there exists $\tilde{C}, \theta > 0$ such that

$$\mathbb{P}\Big(\inf_{v \in \mathscr{V}:\ \mathsf{d}(0,v) \leq n} \xi(v) \leq -\lambda \Big) \leq \tilde{C} n^d \mathrm{e}^{-\theta\lambda}, \qquad \lambda > 0.$$

If we take $\lambda = \lambda(n) = c \log n$ for large enough $c > 0$, then $\sum_{n \in \mathbb{N}} \tilde{C} n^d \mathrm{e}^{-\theta\lambda(n)} < \infty$; hence (7.66) holds by the Borel-Cantelli lemma. $\square$

As a direct application of Lemma 7.6.2, we have the inequality $K_t(u,v) \leq \mathrm{e}^{-\omega_0 t}\Pi_t(u,v)$ for every $u, v \in \mathscr{V}$, where we take ω_0 as in (7.65). In particular, $\|K_t\|_{\mathrm{op}} \leq \mathrm{e}^{-\omega_0 t}\|\Pi_t\|_{\mathrm{op}}$. Given that $\omega_0 > -\infty$ almost surely by Lemma 7.6.2, it suffices to prove that Π_t is bounded with $\|\Pi_t\|_{\mathrm{op}} \leq \mathrm{e}^{-t\omega_1}$ for some constant $\omega_1 \leq 0$. We now prove this.

Note that for every $f \in \ell^2(\mathscr{V})$, we have by Jensen's inequality that

$$\|\Pi_t f\|_2^2 = \sum_{v \in \mathscr{V}} \mathbf{E}^v\big[f\big(X(t)\big)\big]^2 \leq \sum_{v \in \mathscr{V}} \mathbf{E}^v\big[f\big(X(t)\big)^2\big] = \sum_{u,v \in \mathscr{V}} \Pi_t(v,u) f(u)^2,$$

from which we conclude that

$$\|\Pi_t\|_{\mathrm{op}} \leq \sqrt{\sup_{u \in \mathscr{V}} \sum_{v \in \mathscr{V}} \Pi_t(v,u)}.$$

If we define the matrix

$$H_X(u,v) := \begin{cases} -q(u)\Pi(u,v) & \text{if } u \neq v \\ q(u) & \text{if } u = v \end{cases}, \qquad u, v \in \mathscr{V}$$

(i.e., the Markov generator of X), then we can write

$$\sum_{v \in \mathscr{V}} \Pi_t(v,u) = \sum_{v \in \mathscr{V}} \sum_{n=0}^{\infty} \frac{(-t)^n H_X^n(v,u)}{n!} \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{v \in \mathscr{V}} |H_X^n(v,u)|.$$

Noting that

$$\sup_{u,v\in\mathscr{V}} |H_X^n(u,v)| \leq \|H_X^n\|_{\mathrm{op}} \leq \|H_X\|_{\mathrm{op}}^n,$$

for every $u,v\in\mathscr{V}$, we have the bound

$$|H_X^n(v,u)| \leq \|H_X\|_{\mathrm{op}}^n \mathbf{1}_{\{\mathsf{d}(u,v)\leq n\}}.$$

By (7.7), for any $u\in\mathscr{V}$, the number of $v\in\mathscr{V}$ such that (u,v) is an edge is bounded by $\mathfrak{c}$. Thus, the number of $v\in\mathscr{V}$ such that $\mathsf{d}(u,v)\leq n$ is crudely bounded by $\mathfrak{c}^n$. Consequently,

$$\|\Pi_t\|_{\mathrm{op}}^2 \leq \sup_{u\in\mathscr{V}} \sum_{v\in\mathscr{V}} \Pi_t(v,u) \leq \sum_{n=0}^{\infty} \frac{(t\mathfrak{c}\|H_X\|_{\mathrm{op}})^n}{n!} = \mathrm{e}^{\mathfrak{c}\|H_X\|_{\mathrm{op}}t}.$$

Thus, it now suffices to prove that $\|H_X\|_{\mathrm{op}}<\infty$.

Recall that, by assumption, $\mathfrak{q}:=\sup_{u\in\mathscr{V}} q(u)<\infty$. For every $f\in\ell^2(\mathscr{V})$,

$$\|H_X f\|_2^2 \leq \mathfrak{q}^2 \sum_{u\in\mathscr{V}} \left(\sum_{v\in\mathscr{V}} \mathbf{1}_{\{(u,v)\in\mathscr{E}\}} f(v)\right)^2 \leq \mathfrak{q}^2 2^{\mathfrak{c}} \sum_{u,v\in\mathscr{V}} \mathbf{1}_{\{(u,v)\in\mathscr{E}\}} f(v)^2,$$

where the last inequality comes from the fact that

$$(x_1+\cdots+x_{\mathfrak{c}})^2 \leq 2^{\mathfrak{c}}(x_1^2+\cdots+x_{\mathfrak{c}}^2), \qquad x_i\in\mathbb{R},$$

and that, by (7.7), for every $v\in\mathscr{V}$ there are at most $\mathfrak{c}$ vertices u such that $(u,v)\in\mathscr{E}$. Using once again this last observation, we have that

$$\sum_{u,v\in\mathscr{V}} \mathbf{1}_{\{(u,v)\in\mathscr{E}\}} f(v)^2 \leq \mathfrak{c}\|f\|_2^2,$$

from which we conclude that $\|H_X\|_{\mathrm{op}}^2 \leq \mathfrak{q}^2 2^{\mathfrak{c}}\mathfrak{c}$, as desired.

7.6.2 Step 2. Continuity of the Semigroup

We now prove the almost-sure strong continuity and semigroup property. Since X is Markov and local time is additive, the semigroup property is trivial. We now prove strong continuity. Let $C_0(\mathscr{V})$ denote the set of functions $f : \mathscr{V} \to \mathbb{R}$ that are finitely supported. Since $C_0(\mathscr{V})$ is dense in $\ell^2(\mathscr{V})$ and a semigroup of bounded linear operators is strongly continuous if and only if it is weakly continuous (e.g., [44, Chapter I, Theorem 5.8]), it suffices to prove that $\langle f, K_t g - g\rangle \to 0$ as $t \to 0$ for every $f, g \in C_0(\mathscr{V})$. For every $g \in C_0(\mathscr{V})$, we know that

$$\lim_{t\to 0} g\big(X(t)\big)\mathrm{e}^{-\langle L_t, V+\xi\rangle} = g\big(X(0)\big) \qquad \text{almost surely.}$$

By the definition of ω_0, it follows that $\langle L_t, V+\xi\rangle \geq \omega_0 t$ which implies that

$$\big|g\big(X(t)\big)\mathrm{e}^{-\langle L_t, V+\xi\rangle}\big| \leq \|g\|_{\ell^\infty}\mathrm{e}^{-\omega_0 t}.$$

Since the right-hand side of this inequality is independent of X, it follows from dominated convergence that

$$\lim_{t\to 0} K_t g(v) = \lim_{t\to 0} \mathbb{E}^v\left[g\big(X(t)\big)\mathrm{e}^{-\langle L_t, V+\xi\rangle}\right] = g(v) \qquad \text{almost surely}$$

for every $v \in \mathscr{V}$. Finally, given that for every $v \in \mathscr{V}$, we have

$$\big|f(v)\big(K_t g(v) - g(v)\big)\big| \leq \|f\|_{\ell^\infty}\|g\|_{\ell^\infty}(\mathrm{e}^{-\omega_0 t}+1)\mathbf{1}_{\{f(v)\neq 0\}},$$

which is summable in v whenever $f \in C_0(\mathscr{V})$, we obtain $\langle f, K_t g - g\rangle \to 0$ as $t \to 0$ by dominated convergence.

7.6.3 Step 3. Trace Class

By the semigroup property, for every $t > 0$, we can write K_t as the product $K_{t/2}K_{t/2}$. Thus, given that the product of any two Hilbert-Schmidt operators is trace class (e.g., [101, Theorem 3.7.4]), it suffices to prove that, almost surely, K_t is Hilbert-Schmidt for all $t > 0$, that is,

$$\sum_{u,v\in\mathscr{V}} K_t(u,v)^2 < \infty.$$

By (7.66), there exists finite random variables $\kappa, \mu > 0$ that only depend on ξ such that

$$V(v) + \xi(v) \geq \big(\kappa\mathsf{d}(0,v)\big)^\alpha - \mu, \qquad v \in \mathscr{V}$$

almost surely. Therefore, it suffices to prove the result with K_t replaced by the kernel

$$\tilde{K}_t(u,v) := \mathrm{e}^{\mu t}\mathbb{E}^u\left[\mathrm{e}^{-\langle L_t,(\kappa\mathsf{d}(0,\cdot))^\alpha\rangle}\mathbf{1}_{\{X(t)=v\}}\right], \qquad u,v \in \mathscr{V}.$$

By Jensen's inequality,

$$\begin{aligned}\sum_{u,v\in\mathscr{V}} \tilde{K}_t(u,v)^2 &\leq \mathrm{e}^{2\mu t}\sum_{u,v\in\mathscr{V}} \mathbb{E}^u\left[\mathrm{e}^{-2\langle L_t,(\kappa\mathsf{d}(0,\cdot))^\alpha\rangle}\mathbf{1}_{\{X(t)=v\}}\right]\\ &= \mathrm{e}^{2\mu t}\sum_{u\in\mathscr{V}} \mathbb{E}^u\left[\mathrm{e}^{-2\langle L_t,(\kappa\mathsf{d}(0,\cdot))^\alpha\rangle}\right].\end{aligned}$$

At this point, the same argument used in (7.39), (7.41), and (7.42) implies that there exists some finite constant $C_{\kappa,t} > 0$ (which depends on κ and t) such that

$$\sum_{u,v\in\mathscr{V}} \tilde{K}_t(u,v)^2 \leq C_{\kappa,t}\mathrm{e}^{2\mu t}\sum_{u\in\mathscr{V}} \mathrm{e}^{-2t(\kappa\mathsf{d}(0,u))^\alpha}.$$

Then, writing the above sum as

$$\sum_{u\in\mathscr{V}} \mathrm{e}^{-2t(\kappa \mathsf{d}(0,u))^\alpha} = \sum_{n\in\mathbb{N}} \mathsf{c}_n(0)\mathrm{e}^{-2t(\kappa n)^\alpha},$$

this is easily seen to be finite for all $t > 0$ by (7.7).

7.6.4 Step 4. Infinitesimal Generator

We now prove the properties of the generator H, except for number rigidity of its spectrum, which is relegated to the next (and final) step of the proof. That K_t's generator is of the form (7.16) follows from the straightforward computation that for every $u, v \in \mathscr{V}$,

$$\lim_{t\to 0} \frac{\mathbf{1}_{\{u=v\}} - K_t(u,v)}{t} = H(u,v) \qquad \text{almost surely}$$

(indeed, recall that by definition of the process X, $\Pi_t(u,v) = q(u)\Pi(u,v)t + o(t)$ as $t \to 0$ whenever $u \neq v$, and that $K_t(u,v) = 0$ if $u \in \mathscr{Z}$ or $v \in \mathscr{Z}$).

Almost surely, $(K_t)_{t>0}$ is a strongly continuous semigroup of trace class operators and $\|K_t\|_{\mathrm{op}} \leq \mathrm{e}^{-\omega t}$. Therefore, by Proposition 7.5.2 (1)–(3), the following holds almost surely:

1. H is closed and densely defined on $\ell^2(\mathscr{V})$.

2. $\sigma(H) = \sigma_p(H)$, and $\mathrm{Re}(\lambda) \geq \omega$ for all $\lambda \in \sigma(H)$.

3. For every $t > 0$, $\sigma(K_t) \setminus \{0\} = \{\mathrm{e}^{-t\lambda} : \lambda \in \sigma(H)\}$.

It now remains to establish the trace identity (7.52), which is crucial in our proof of rigidity. The fact that $\mathrm{Tr}[K_t]$ is a positive real number follows from the fact that

$$\mathrm{Tr}[K_t] = \sum_{v\in\mathscr{V}} K_t(v,v)$$

and that $K_t(u,v) \in [0,\infty)$ for all $u,v \in \mathscr{V}$. To prove the remainder of (7.52), as per Proposition 7.5.2, we need to find a sequence of finite-dimensional operators that converge to H and K_t in the sense of (7.54) and (7.55).

To this end, for every $n \in \mathbb{N}$, let us denote the subset

$$\mathscr{V}_n := \{v \in \mathscr{V} : \mathsf{d}(0,v) \leq n\} \subset \mathscr{V}.$$

Given that $\mathscr{G}$ has uniformly bounded degrees, this must be finite. Thus, the operators

$$H_n(u,v) := H(u,v)\mathbf{1}_{\{(u,v)\in\mathscr{V}_n\}}, \qquad u,v \in \mathscr{V}$$

are finite-dimensional in the sense of Definition 7.5.1. More specifically, H_n is the restriction of H to the set $\mathscr{V}_n$ with Dirichlet boundary on $\mathscr{V} \setminus \mathscr{V}_n$. In particular, if for every $n \in \mathbb{N}$ we denote the hitting time

$$\tau_n := \inf_{t\geq 0}\big\{t \geq 0 : X(t) \notin \mathscr{V}_n\big\},$$

Then e^{-tH_n} is the integral operator on $\ell^2(\mathscr{V})$ with kernel

$$\mathrm{e}^{-tH_n}(u,v) = \mathbb{E}^u\left[\mathrm{e}^{-\langle L_t, V+\xi\rangle}\mathbf{1}_{\{X(t)=v\}}\mathbf{1}_{\{\tau_n>t\}}\right]. \tag{7.67}$$

The proof of (7.52) is now a matter of establishing the following result:

Lemma 7.6.3. *Almost surely, it holds that*

$$\lim_{n\to\infty}\|\mathfrak{R}(z,H_n) - \mathfrak{R}(z,H)\|_{\mathrm{op}} = 0 \tag{7.68}$$

for every $z \in \mathbb{C}$ *such that* $\mathrm{Re}(z) < \omega$ *and*

$$\lim_{n\to\infty}\|\mathrm{e}^{-tG_n} - K_t\|_{\mathrm{op}} = 0 \tag{7.69}$$

for every $t > 0$.

Proof. Given that $0 \leq \mathrm{e}^{-tH_n}(u,v) \leq K_t(u,v)$ for all $u, v \in \mathscr{V}$, it is easy to see that $\|\mathrm{e}^{-tH_n}\|_{\mathrm{op}} \leq \|K_t\|_{\mathrm{op}} \leq \mathrm{e}^{-\omega t}$ for all $t > 0$ almost surely. In particular, any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) < \omega$ is in the resolvent set of H_n and H for all n. Consequently, it follows from [44, Chapter II, Theorem 1.10] that

$$\|\mathfrak{R}(z, H_n) - \mathfrak{R}(z, H)\|_{\mathrm{op}} = \left\| \int_0^\infty \mathrm{e}^{tz}(\mathrm{e}^{-tG_n} - K_t)\,\mathrm{d}t \right\|_{\mathrm{op}} \leq \int_0^\infty \mathrm{e}^{tz}\|\mathrm{e}^{-tG_n} - K_t\|_{\mathrm{op}}\,\mathrm{d}t,$$

where the last inequality follows from [41, Chapter II, Theorem 4 (ii)]. Given that

$$\int_0^\infty \mathrm{e}^{tz}\|\mathrm{e}^{-tG_n} - K_t\|_{\mathrm{op}}\,\mathrm{d}t \leq \int_0^\infty \mathrm{e}^{tz}\big(\|\mathrm{e}^{-tG_n}\|_{\mathrm{op}} + \|K_t\|_{\mathrm{op}}\big)\,\mathrm{d}t \leq 2\int_0^\infty \mathrm{e}^{t(z-\omega)}\,\mathrm{d}t < \infty$$

whenever $\mathrm{Re}(z) < \omega$, we get that (7.68) is a consequence of (7.69) by an application of the dominated convergence theorem.

Let us then prove (7.69). Since the Hilbert-Schmidt norm dominates the operator norm, it suffices to prove that

$$\sum_{u,v\in\mathscr{V}} \big(\mathrm{e}^{-tG_n}(u,v) - K_t(u,v)\big)^2 = \sum_{u,v\in\mathscr{V}} \mathbb{E}^u\left[\mathrm{e}^{-\langle L_t, V+\xi\rangle}\mathbf{1}_{\{X(t)=v\}}\mathbf{1}_{\{\tau_n\leq t\}}\right]^2 \tag{7.70}$$

vanishes as $n \to \infty$ for all $t > 0$ almost surely. By Hölder's inequality, the right-hand side of (7.70) is bounded above by

$$\sum_{u,v\in\mathscr{V}} \mathbb{E}^u\left[\mathrm{e}^{-2\langle L_t, V+\xi\rangle}\mathbf{1}_{\{X(t)=v\}}\right]\mathbb{P}^u[\tau_n \leq t].$$

By mimicking our proof that K_t is trace class, we know that

$$\sum_{u,v\in\mathscr{V}} \mathbb{E}^u\left[\mathrm{e}^{-2\langle L_t, V+\xi\rangle}\mathbf{1}_{\{X(t)=v\}}\right] < \infty$$

for every $t > 0$ almost surely. Thus, by dominated convergence, it suffices to prove that

$$\lim_{n\to\infty} \mathbb{P}^u[\tau_n \le t] = 0$$

for every $u \in \mathscr{V}$ and $t > 0$. Noting that

$$\mathbb{P}^u\left[\max_{0\le s\le t} \mathsf{d}\big(0, X(s)\big) > n\right] \le \mathbb{P}^u\left[\max_{0\le s\le t} \mathsf{d}\big(u, X(s)\big) > n - \mathsf{d}(0,u)\right]$$

for all $n \in \mathbb{N}$ by the triangle inequality, this follows directly from the tail bound (7.39). ☐

7.6.5 Step 5. Rigidity

It now only remains to prove that the point process (7.17) is number rigid in the sense of Definition 7.3.3. The proof of this amounts to a minor modification of the argument in [59, Theorem 6.1] (see also [50, Proposition 2.2]).

Let $B \subset \mathbb{C}$ be a Borel set such that $B \subset (-\infty, \delta] + \mathrm{i}[-\tilde{\delta}, \tilde{\delta}]$ for some $\delta, \tilde{\delta} > 0$. Thanks to the trace identity (7.52), almost surely, we can write

$$\mathcal{X}_H(B) = \sum_{\lambda\in\sigma(H)\cap B} m_a(\lambda, H)$$

as the sum of the following three terms:

$$\sum_{\lambda\in\sigma(H)} m_a(\lambda, H)\,\mathrm{e}^{-t\lambda} - \mathbb{E}\left[\sum_{\lambda\in\sigma(H)} m_a(\lambda, H)\,\mathrm{e}^{-t\lambda}\right] = \mathrm{Tr}[K_t] - \mathbb{E}\big[\mathrm{Tr}[K_t]\big], \tag{7.71}$$

$$\sum_{\lambda\in\sigma(H)\cap B} m_a(\lambda, H)\left(1 - \mathrm{e}^{-t\lambda}\right), \tag{7.72}$$

$$\mathbb{E}\left[\sum_{\lambda\in\sigma(H)} m_a(\lambda, H)\,\mathrm{e}^{-t\lambda}\right] - \sum_{\lambda\in\sigma(H)\setminus B} m_a(\lambda, H)\,\mathrm{e}^{-t\lambda}. \tag{7.73}$$

Since we choose the exponent α in the same way as Theorem 7.3.16, (7.71) converges to zero as $t \to 0$ almost surely along a subsequence. Next, we have that (7.72) is bounded above in absolute value by

$$\mathcal{X}_H(B) \sup_{\zeta \in [\omega,\delta]+\mathrm{i}[\alpha,\beta]} |1 - \mathrm{e}^{-t\zeta}|,$$

where we recall that ω is the random lower bound on the real part of the points in $\mathcal{X}_H$. Since $\mathcal{X}_H$ is real-bounded below and $B \subset (-\infty, \delta] + \mathrm{i}[-\tilde{\delta}, \tilde{\delta}]$, $\mathcal{X}_H(B) < \infty$ almost surely. Thus, (7.72) converges to zero almost surely as $t \to 0$. Thus, $\mathcal{X}_H(B)$ is the almost sure limit of (7.73) as $t \to 0$, along a subsequence. Given that (7.73) is measurable with respect to the configuration of points outside of B for every t and that the limit of measurable functions is measurable, we conclude that $\mathcal{X}_H(B)$ is measurable with respect to the configuration outside of B. This then concludes the proof of number rigidity, and thus of Theorem 7.3.17.

7.7 Proof of Theorem 7.3.18

7.7.1 Step 1. General Lower Bound

We begin by providing a lower bound for $\mathbf{Var}\big[\mathrm{Tr}[K_t]\big]$ in the general setting of the statement of Theorem 7.3.18. This bound will then be shown to remain positive as $t \to 0$ in the cases labelled (1)–(3).

Recalling that γ is the positive definite covariance function of ξ, if we denote the semi-inner-product

$$\langle f, g \rangle_\gamma := \sum_{u,v \in \mathbb{Z}^d} f(u)\gamma(u-v)g(v), \qquad f, g : \mathbb{Z}^d \to \mathbb{R},$$

then our assumption that γ is nonnegative implies that $\langle f, g \rangle_\gamma \geq 0$ whenever f and

g are nonnegative. In particular, we have that

$$\mathbf{Cov}_\xi\left[\mathrm{e}^{-\langle L_t^u,\xi\rangle},\mathrm{e}^{-\langle \tilde L_t^v,\xi\rangle}\right]=\mathrm{e}^{\frac12\langle L_t^u,L_t^u\rangle_\gamma+\frac12\langle \tilde L_t^v,\tilde L_t^v\rangle_\gamma}\left(\mathrm{e}^{\langle L_t^u,\tilde L_t^v\rangle_\gamma}-1\right)\geq 0. \tag{7.74}$$

For every $u,v\in\mathbb{Z}^d$ and $t>0$, denote the event $J_t(u,v):=\{L_t^u=t\mathbf{1}_u \text{ and } \tilde L_t^v=t\mathbf{1}_v\}$. Clearly, $J_t(u,v)\subset\{X^u(t)=u,\tilde X^v(t)=v\}$, and by independence of X^u and $\tilde X^v$,

$$\inf_{u,v\in\mathbb{Z}^d}\mathbb{P}[J_t(u,v)]=\inf_{v\in\mathbb{Z}^d}\mathbb{P}^v[X(s)=v \text{ for every } s\leq t]^2\geq \mathrm{e}^{-2t}. \tag{7.75}$$

We now combine (7.74) and (7.75) to lower bound the variance of $\mathrm{Tr}[K_t]$: By Proposition 7.4.2, we may write

$$\begin{aligned}\mathbf{Var}\big[\mathrm{Tr}[K_t]\big]&\geq\sum_{u,v\in\mathbb{Z}^d}\mathbb{E}\left[\mathrm{e}^{-\langle L_t^u+\tilde L_t^v,V\rangle}\mathrm{e}^{\frac12\langle L_t^u,L_t^u\rangle_\gamma+\frac12\langle\tilde L_t^v,\tilde L_t^v\rangle_\gamma}\left(\mathrm{e}^{\langle L_t^u,\tilde L_t^v\rangle_\gamma}-1\right)\mathbf{1}_{J_t(u,v)}\right]\\&=\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-tV(u)-tV(v)}\mathrm{e}^{t^2\gamma(0)}\left(\mathrm{e}^{t^2\gamma(u-v)}-1\right)\mathbb{P}[J_t(u,v)]\\&\geq\mathrm{e}^{-2t+t^2\gamma(0)}\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-tV(u)-tV(v)}\left(\mathrm{e}^{t^2\gamma(u-v)}-1\right)\\&=\mathrm{e}^{-2t+t^2\gamma(0)}\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-t\mathsf{d}(0,u)^\delta-t\mathsf{d}(0,v)^\delta}\left(\mathrm{e}^{t^2\gamma(u-v)}-1\right),\end{aligned} \tag{7.76}$$

where the first line comes from (7.74) and the fact that $\mathbb{E}[Y]\geq\mathbb{E}[Y\mathbf{1}_E]$ for any nonnegative random variable Y and event E, the second line comes from the definition of the event $J_t(u,v)$, the third line comes from (7.75), and the last line comes from the assumption on V stated in Theorem 7.3.18. As $\mathrm{e}^{-2t+t^2\gamma(0)}\to1$ as $t\to0$, we obtain our general lower bound:

$$\liminf_{t\to0}\mathbf{Var}\big[\mathrm{Tr}[K_t]\big]\geq\liminf_{t\to0}\sum_{u,v\in\mathbb{Z}^d}\mathrm{e}^{-t\mathsf{d}(0,u)^\delta-t\mathsf{d}(0,v)^\delta}\left(\mathrm{e}^{t^2\gamma(u-v)}-1\right). \tag{7.77}$$

We now prove that the right-hand side of (7.77) is positive in cases (1)–(3).

7.7.2 Step 2. Three Examples

Suppose first that $\delta \leq d/2$ and $\gamma(v) = \mathbf{1}_{\{v=0\}}$. On the integer lattice $\mathbb{Z}^d$, it is easy to see that there exists a constant $C > 0$ such that $\mathsf{c}_n(0) \geq Cn^{d-1}$. Therefore, by an application of (7.77), followed by the inequality $\mathrm{e}^x - 1 \geq x$ for all $x \geq 0$ and a Riemann sum, we have that

$$\liminf_{t\to 0} \mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \geq \liminf_{t\to 0} \left(\mathrm{e}^{t^2} - 1\right) \sum_{v\in\mathbb{Z}^d} \mathrm{e}^{-2t\mathsf{d}(0,v)^\delta} \geq \liminf_{t\to 0} t^2 \sum_{n\in\mathbb{N}\cup\{0\}} \mathsf{c}_n(0)\mathrm{e}^{-2tn^\delta}$$
$$\geq C \liminf_{t\to 0} t^{2-d/\delta} t^{1/\delta} \sum_{n\in t^{1/\delta}\mathbb{N}\cup\{0\}} n^{d-1}\mathrm{e}^{-2n} \geq C \int_0^\infty x^{d-1}\mathrm{e}^{-2x}\,\mathrm{d}x > 0.$$

Next, suppose that $\delta \leq d - \beta/2$ and that $\gamma(v) \geq \mathcal{L}\big(\mathsf{d}(0,v)+1\big)^{-\beta}$ for some $0 < \beta < d$ and $\mathcal{L} > 0$. Then, (7.77), the triangle inequality, and the same arguments as in the previous case yield

$$\begin{aligned}
&\liminf_{t\to 0} \mathbf{Var}\big[\mathrm{Tr}[K_t]\big]\\
&\geq \liminf_{t\to 0} \sum_{u,v\in\mathbb{Z}^d} \mathrm{e}^{-t\mathsf{d}(0,u)^\delta - t\mathsf{d}(0,v)^\delta} \left(\mathrm{e}^{\mathcal{L}t^2(\mathsf{d}(u,v)+1)^{-\beta}} - 1\right)\\
&\geq \mathcal{L} \liminf_{t\to 0} t^2 \sum_{u,v\in\mathbb{Z}^d} \mathrm{e}^{-t\mathsf{d}(0,u)^\delta - t\mathsf{d}(0,v)^\delta} \big(\mathsf{d}(0,u) + \mathsf{d}(0,v) + 1\big)^{-\beta}\\
&= \mathcal{L} \liminf_{t\to 0} t^2 \sum_{m,n\in\mathbb{N}\cup\{0\}} \mathsf{c}_m(0)\mathsf{c}_n(0)\,\mathrm{e}^{-tm^\delta - tn^\delta}(m+n+1)^{-\beta}\\
&\geq \mathcal{L}C^2 \liminf_{t\to 0} t^{2-2(d-1)/\delta+\beta/\delta} \sum_{m,n\in t^{1/\delta}\mathbb{N}\cup\{0\}} (mn)^{d-1}\mathrm{e}^{-m^\delta - n^\delta}(m+n+t^\delta)^{-\beta}\\
&= \mathcal{L}C^2 \liminf_{t\to 0} t^{2-2(d-\beta/2)/\delta} \int_0^\infty\int_0^\infty \frac{(xy)^{d-1}}{(x+y)^\beta}\mathrm{e}^{-x^\delta - y^\delta}\,\mathrm{d}x\mathrm{d}y > 0.
\end{aligned}$$

Finally, suppose that $\delta \leq d$ and $\inf_{v\in\mathbb{Z}^d}\gamma(v) > \mathcal{L} > 0$. In this case we obtain that

$$\liminf_{t\to 0} \mathbf{Var}\big[\mathrm{Tr}[K_t]\big] \geq \liminf_{t\to 0} \left(\mathrm{e}^{\mathcal{L}t^2} - 1\right) \sum_{u,v\in\mathbb{Z}^d} \mathrm{e}^{-t\mathsf{d}(0,u)^\delta - t\mathsf{d}(0,v)^\delta}$$

$$\geq \mathcal{L}C^2 \liminf_{t\to 0} t^2 \left(\sum_{n\in\mathbb{N}} n^{d-1} \mathrm{e}^{-2tn^\delta} \right)^2 = \mathcal{L}C^2 \liminf_{t\to 0} t^{2-2d/\delta} \left(\int_0^\infty x^{d-1} \mathrm{e}^{-2x} \, \mathrm{d}x \right)^2 > 0,$$

thus concluding the proof.

APPENDICES

APPENDIX A

Proof of a Cauchy-like Summation Identity by Zhipeng Liu

The original motivation of thise section is to provide proof for one conjectured identity, Corollary 3.5.4 by Yuchen Liao. Sometime later after the proof, we were told that one lemma in our proof, Proposition 3.5.1 now, was also conjectured by Yuchen earlier. So we restructured the proof to fit the main text. The proof of Proposition 3.5.1 and Corollary 3.5.4 are given in Sections A.2 and A.3 respectively.

A.1 Lemmas on perturbations of Cauchy determinants

We will need the following linear algebraic lemmas.

Lemma A.1.1. *Suppose* $\{x_i : 1 \leq i \leq n\}$ *and* $\{y_i : 1 \leq i \leq n\}$ *are two sets of distinct complex numbers, and* f *is a polynomial of degree* $\leq n-1$. *Then*

$$\det\left(-\frac{1}{x_i - y_j} + \frac{1}{x_i - y_j}\frac{f(x_i)}{f(y_j)}\right)_{i,j=1}^{n} = 0.$$

Where we assume that $f(y_j) \neq 0$ *for all* $1 \leq j \leq n$.

Proof. The conclusion is equivalent to

$$\det\left(\frac{f(x_i)-f(y_j)}{x_i-y_j}\right)_{i,j=1}^{n}=0.$$

Since f is a polynomial of degree at most $n-1$, we could write

$$\frac{f(x_i)-f(z)}{x_i-z}=c_{n-2}(z)x_i^{n-2}+c_{n-2}(z)x_i^{n-3}+\cdots+c_0(z),$$

and

$$\det\left(\frac{f(x_i)-g(y_j)}{x_i-y_j}\right)_{i,j=1}^{n}=\det\left(\sum_{0\le\ell_j\le n-2}c_{\ell_j}(y_j)x_i^{\ell_j}\right)_{i,j=1}^{n}$$
$$=\sum_{0\le\ell_1,\cdots,\ell_n\le n-2}\prod_{j=1}^{n}c_{\ell_j}(y_j)\det\left(x_i^{\ell_j}\right)_{i,j=1}^{n}=0,$$

where we used the fact that $\det(x_i^{\ell_j})_{i,j=1}^{n}=0$ since at least two of ℓ_j's are equal. □

Lemma A.1.2. *Suppose $\{x_i : 1\le i\le n\}$ and $\{y_i : 1\le i\le n\}$ are two disjoint sets of complex numbers. Let $p,q\in\mathbb{C}$ be complex numbers such that $x_i\ne -q$ and $y_j\ne -p$ for all $1\le i,j\le n$. Then*

$$\sum_{1\le a,b\le n}(-1)^{a+b}\det\left(\frac{1}{x_i-y_j}\right)_{\substack{1\le i,j\le n\\ i\ne a,j\ne b}}\cdot\frac{x_a-p}{(y_b-p)(x_a+q)}$$
$$=\prod_i\frac{x_i-p}{y_i-p}\cdot\left(1-\prod_i\frac{y_i+q}{x_i+q}\right)\cdot\det\left(\frac{1}{x_i-y_j}\right)_{i,j=1}^{n},$$

and

$$\det\left(\frac{1}{x_i-y_j}+\frac{x_i-p}{(y_j-p)(x_i+q)}\right)_{i,j=1}^{n}$$
$$=\left(1+\prod_i\frac{x_i-p}{y_i-p}\cdot\left(1-\prod_i\frac{y_i+q}{x_i+q}\right)\right)\det\left(\frac{1}{x_i-y_j}\right)_{i,j=1}^{n}.$$

Proof. When $p = 0$ and $q = 1$, these identities were proved in [9], see Lemma 5.5 and equation (5.36). The general case follows from a re-scaling and translation for x_i and y_i on the case with $p = 0$ and $q = 1$. □

Lemma A.1.3. *Suppose $\{x_i : 1 \le i \le n\}$ and $\{y_i : 1 \le i \le n\}$ are two sets of distinct complex numbers. Assume $p, q, r \in \mathbb{C}$. Then*

$$\begin{aligned} &\det\left(\frac{1}{x_i - y_j} + \frac{r}{(y_j - p)(x_i + q)}\right)_{i,j=1}^{n} \\ &= \left(1 - \frac{r}{p+q} \cdot \left(1 - \prod_i \frac{(x_i - p)(y_i + q)}{(y_i - p)(x_i + q)}\right)\right) \det\left(\frac{1}{x_i - y_j}\right)_{i,j=1}^{n}, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} &\det\left(\frac{1}{x_i - y_j} + \frac{r}{y_j - p}\right)_{i,j=1}^{n} \\ &= \left(1 - r \cdot \left(1 - \prod_i \frac{x_i - p}{y_i - p}\right)\right) \det\left(\frac{1}{x_i - y_j}\right)_{i,j=1}^{n}. \end{aligned} \tag{A.2}$$

Proof. Note that if we replace r by rq and let $q \to \infty$ in the first identity, we obtain the second identity immediately. Thus it is sufficient to prove (A.1).

Note that both sides of (A.1) are linear functions of r. (To see this for the left hand side of (A.1) one can use for example Lemma 4.5.13). Hence it is sufficient to verify this identity for two different values of r. Obviously it is true when $r = 0$. Thus we only need to prove the case when $r = p + q$, that is

$$\det\left(\frac{1}{x_i - y_j} + \frac{p+q}{(y_j - p)(x_i + q)}\right)_{i,j=1}^{n} = \prod_i \frac{(x_i - p)(y_i + q)}{(y_i - p)(x_i + q)} \cdot \det\left(\frac{1}{x_i - y_j}\right)_{i,j=1}^{n}. \tag{A.3}$$

Below we prove (A.3) by induction. It is obviously true when $n = 1$. Now we assume that the identity holds for smaller n and want to show it for n.

We view (A.3) as an identity of x_n. Observe that both sides go to zero as $x_n \to \infty$, and only have simple poles at $x_n = y_k, 1 \le k \le n$ and $x_n = -q$. We write the two sides as

$$\frac{C_0}{x_n+q}+\sum_{k=1}^{n}\frac{C_k}{x_n-y_k}, \qquad \frac{C_0'}{x_n+q}+\sum_{k=1}^{n}\frac{C_k'}{x_n-y_k}$$

where C_i, C_i' $(0 \le i \le n)$ are independent of x_n. We first check $C_k = C_k'$ for $1 \le k \le n$. By evaluating the residues at $x_n = y_k$, we get

$$C_k = (-1)^{k+n}\det\left(\frac{1}{x_i-y_j}+\frac{p+q}{(y_j-p)(x_i+q)}\right)_{\substack{1\le i,j\le n\\ i\neq n, j\neq k}}$$

and

$$\begin{aligned} C_k' &= (-1)^{n+k}\prod_i \frac{(x_i-p)(y_i+q)}{(y_i-p)(x_i+q)}\cdot\det\left(\frac{1}{x_i-y_j}\right)_{\substack{1\le i,j\le n\\ i\neq n, j\neq k}} \\ &= (-1)^{n+k}\prod_{i\neq n}\frac{x_i-p}{x_i+q}\cdot\prod_{j\neq k}\frac{y_j+q}{y_j-p}\cdot\det\left(\frac{1}{x_i-y_j}\right)_{\substack{1\le i,j\le n\\ i\neq n, j\neq k}} \end{aligned}$$

where we used the fact that $x_n = y_k$ in the last equation. By using induction we obtain that $C_k = C_k'$.

Finally, instead of showing $C_0 = C_0'$ directly, we want to verify (A.3) for one specific value of x_n: $x_n = p$. If it holds, we have

$$\frac{C_0}{p+q}+\sum_{k=1}^{n}\frac{C_k}{p-y_k}=\frac{C_0'}{p+q}+\sum_{k=1}^{n}\frac{C_k'}{p-y_k}$$

and hence $C_0 = C_0'$.

When $x_n = p$, the right hand side of (A.3) is zero. On the other hand, the entries of the n-th row of the matrix on the left hand side of (A.3) are

$$\frac{1}{p-y_j}+\frac{p+q}{(y_j-p)(p+q)}=0.$$

Thus the left hand side is also zero. We conclude that (A.3) holds when $x_n = p$. Recall the argument above. We finish the induction of (A.3). □

Lemma A.1.4 (Lemma 5.9 of [9])**.** *For two $n\times n$ matrices P and Q,*

$$\sum_{\substack{J,J'\subset\{1,\cdots,n\}\\|J|=|J'|}}(-1)^{\#(J^c;J)+\#((J')^c;J')}\det[P(i,i')]_{i\in J,i'\in J'}\det[Q(i,i')]_{i\in J^c,i'\in (J')^c}$$
$$=\det[P+Q]_{1\le i,i'\le n}.$$

Here we recall that for I,J disjoint subsets of $\{1,\cdots,n\}$, the number of inversions $\#(I;J)$ is defined as

$$\#(I;J):=\left|\{(i,j)\in I\times J: i>j\}\right|. \tag{A.4}$$

A.2 Proof of Proposition 3.5.1

We denote the two sides of equation (3.64) L_n and R_n respectively. Expanding the determinants out we have

$$\mathrm{L}_n := \text{LHS of (3.64)}$$
$$=\sum_{\sigma,\sigma'\in S_n}\operatorname{sgn}(\sigma\sigma')\sum_{A\ge\lambda_1\ge\cdots\ge\lambda_n\ge B}\prod_{j=1}^{n}\left(\prod_{\ell=j+1}^{n}\frac{w'_{\sigma'(j)}-\pi_\ell}{w_{\sigma(j)}-\pi_\ell}\prod_{\ell=1}^{\lambda_j}\frac{w'_{\sigma'(j)}+\hat{\pi}_\ell}{w_{\sigma(j)}+\hat{\pi}_\ell}\cdot\frac{1}{w_{\sigma(j)}+\hat{\pi}_{\lambda_j+1}}\right)$$

On the other hand applying Lemma A.1.4 with $P=(P(i,i'))_{1\le i,i'\le n}$ and $Q=(Q(i,i'))_{1\le i,i'\le n}$ where

$$P(i,i')=\frac{1}{w_i-w'_{i'}}\prod_{\ell=1}^{B}\frac{w'_{i'}+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell},\quad Q(i,i')=\frac{1}{-w_i+w'_{i'}}\prod_{\ell=2}^{n}\frac{w'_{i'}-\pi_\ell}{w_i-\pi_\ell}\cdot\prod_{\ell=1}^{A+1}\frac{w'_{i'}+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell},$$

we have

$$\mathrm{R}_n := \text{RHS of (3.64)}$$

$$= \sum_{\substack{S,S'\subset[1;n]\\|S|=|S'|}} (-1)^{\#(S^c;S)+\#((S')^c;S')} \cdot \det\left(\frac{1}{w_i - w'_{i'}}\right)_{\substack{i\in S\\ i'\in S'}} \cdot \prod_{\ell=1}^{B} \frac{\prod_{i'\in S'}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in S}(w_i+\hat{\pi}_\ell)}$$

$$\cdot \det\left(\frac{1}{-w_i + w'_{i'}}\right)_{\substack{i\in S^c\\ i'\in(S')^c}} \cdot \prod_{\ell=2}^{n} \frac{\prod_{i'\in(S')^c}(w'_{i'}-\pi_\ell)}{\prod_{i\in S^c}(w_i-\pi_\ell)} \cdot \prod_{\ell=1}^{A+1} \frac{\prod_{i'\in(S')^c}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in S^c}(w_i+\hat{\pi}_\ell)}$$

Here the inversion number $\#(I;J)$ is defined in (A.4). Now we prove that $\mathrm{L}_n = \mathrm{R}_n$ by induction on n. For $n = 1$ we have

$$\mathrm{L}_1 = \sum_{\lambda_1=B}^{A} \prod_{\ell=1}^{\lambda_1} \frac{w'_1+\hat{\pi}_\ell}{w_1+\hat{\pi}_\ell} \cdot \frac{1}{w_1+\hat{\pi}_{\lambda_1+1}}$$

$$= \frac{1}{w_1 - w'_1}\left(\prod_{\ell=1}^{B} \frac{w'_1+\hat{\pi}_\ell}{w_1+\hat{\pi}_\ell} - \prod_{\ell=1}^{A+1} \frac{w'_1+\hat{\pi}_\ell}{w_1+\hat{\pi}_\ell}\right) = R_1.$$

Here we used the fact that there are only two terms in the sum in R_1 corresponding to $S = \emptyset$ or $S = \{1\}$ and we used the identity (3.63). Now assume $\mathrm{L}_{n-1} = \mathrm{R}_{n-1}$ holds for some $n \geq 2$. We will first fix $1 \leq a, a' \leq n$ such that $\sigma(n) = a$ and $\sigma'(n) = a'$, sum over $\lambda_1, \cdots, \lambda_{n-1}$ and apply induction hypothesis, and finally sum over a and a'. In this way we have

$$\mathrm{L}_n = \sum_{\lambda_n=B}^{A} \sum_{1\leq a,a'\leq n} (-1)^{a+a'} \prod_{j=1}^{n} \frac{w'_j-\pi_n}{w_j-\pi_n} \cdot \frac{w_a-\pi_n}{w'_{a'}-\pi_n} \cdot \prod_{\ell=1}^{\lambda_n} \frac{w'_{a'}+\hat{\pi}_\ell}{w_a+\hat{\pi}_\ell} \cdot \frac{1}{w_a+\hat{\pi}_{\lambda_n+1}}$$

$$\cdot \sum_{\substack{\sigma:[1;n-1]\to[1;n]\setminus\{a\}\\ \sigma':[1;n-1]\to[1;n]\setminus\{a'\}}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma') \sum_{A\geq\lambda_1\geq\cdots\geq\lambda_{n-1}\geq\lambda_n}$$

$$\prod_{j=1}^{n-1}\left(\prod_{\ell=j+1}^{n-1} \frac{w'_{\sigma'(j)}-\pi_\ell}{w_{\sigma(j)}-\pi_\ell} \prod_{\ell=1}^{\lambda_j} \frac{w'_{\sigma'(j)}+\hat{\pi}_\ell}{w_{\sigma(j)}+\hat{\pi}_\ell} \cdot \frac{1}{w_{\sigma(j)}+\hat{\pi}_{\lambda_j+1}}\right).$$

Applying the induction hypothesis to the last sum above we have

$$\mathrm{L}_n = \sum_{\lambda_n=B}^{A} \sum_{1\leq a,a'\leq n} (-1)^{a+a'} \prod_{j=1}^{n} \frac{w'_j-\pi_n}{w_j-\pi_n} \cdot \frac{w_a-\pi_n}{w'_{a'}-\pi_n} \cdot \prod_{\ell=1}^{\lambda_n} \frac{w'_{a'}+\hat{\pi}_\ell}{w_a+\hat{\pi}_\ell} \cdot \frac{1}{w_a+\hat{\pi}_{\lambda_n+1}}$$

$$\cdot \sum_{\substack{S\subset[1;n]\setminus\{a\}\\ S'\subset[1;n]\setminus\{a'\}\\ |S|=|S'|}} (-1)^{\#(S^c\setminus\{a\};S)+\#((S')^c\setminus\{a'\},S')} \cdot \det\left(\frac{1}{w_i - w'_{i'}}\right)_{\substack{i\in S\\ i'\in S'}} \cdot \prod_{\ell=1}^{\lambda_n} \frac{\prod_{i'\in S'}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in S}(w_i+\hat{\pi}_\ell)}$$

$$\cdot \det\left(\frac{1}{-w_i + w'_{i'}}\right)_{\substack{i\in S^c\setminus\{a\}\\ i'\in(S')^c\setminus\{a'\}}} \cdot \prod_{\ell=2}^{n-1} \frac{\prod_{i'\in(S')^c\setminus\{a'\}}(w'_{i'}-\pi_\ell)}{\prod_{i\in S^c\setminus\{a\}}(w_i-\pi_\ell)} \cdot \prod_{\ell=1}^{A+1} \frac{\prod_{i'\in(S')^c\setminus\{a'\}}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in S^c\setminus\{a\}}(w_i+\hat{\pi}_\ell)}.$$

Now we set $T = S\cup\{a\}$ and $T' = S'\cup\{a'\}$ and rewrite the sum above as summation over T and T' :

$$\mathrm{L}_n = \prod_{j=1}^{n} \frac{w'_j - \pi_n}{w_j - \pi_n} \sum_{\lambda_n=B}^{A} \sum_{\substack{T\subset[1;n]\\ T'\subset[1;n]\\ |T|=|T'|\geq 1}} (-1)^{\#(T^c;T)+\#((T')^c;T')} \cdot \det\left(\frac{1}{-w_i + w'_{i'}}\right)_{\substack{i\in T^c\\ i'\in(T')^c}}$$

$$\cdot \prod_{\ell=2}^{n-1} \frac{\prod_{i'\in(T')^c}(w'_{i'}-\pi_\ell)}{\prod_{i\in T^c}(w_i-\pi_\ell)} \cdot \prod_{\ell=1}^{A+1} \frac{\prod_{i'\in(T')^c}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T^c}(w_i+\hat{\pi}_\ell)} \cdot \prod_{\ell=1}^{\lambda_n} \frac{\prod_{i'\in T'}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T}(w_i+\hat{\pi}_\ell)}$$

$$\cdot \sum_{\substack{a\in T\\ a'\in T'}} (-1)^{\#(T\setminus\{a\};\{a\})+\#(T'\setminus\{a'\},\{a'\})} \cdot \det\left(\frac{1}{w_i - w'_{i'}}\right)_{\substack{i\in T\setminus\{a\}\\ i'\in T'\setminus\{a'\}}} \cdot \frac{w_a - \pi_n}{(w'_{a'}-\pi_n)(w_a+\hat{\pi}_{\lambda_n+1})}.$$

By Lemma A.1.2 the sum in the last row above equals

$$\frac{\prod_{i\in T}(w_i-\pi_n)}{\prod_{i'\in T'}(w'_{i'}-\pi_n)} \cdot \left(1 - \frac{\prod_{i'\in T'}(w'_{i'}+\hat{\pi}_{\lambda_n+1})}{\prod_{i\in T}(w_i+\hat{\pi}_{\lambda_n+1})}\right) \cdot \det\left(\frac{1}{w_i - w'_{i'}}\right)_{\substack{i\in T\\ i'\in T'}}.$$

Hence

$$\mathrm{L}_n = \sum_{\lambda_n=B}^{A} \sum_{\substack{T\subset[1;n]\\ T'\subset[1;n]\\ |T|=|T'|\geq 1}} (-1)^{\#(T^c;T)+\#((T')^c;T')} \cdot \det\left(\frac{1}{-w_i + w'_{i'}}\right)_{\substack{i\in T^c\\ i'\in(T')^c}} \cdot \det\left(\frac{1}{w_i - w'_{i'}}\right)_{\substack{i\in T\\ i'\in T'}}$$

$$\cdot \prod_{\ell=2}^{n} \frac{\prod_{i'\in(T')^c}(w'_{i'}-\pi_\ell)}{\prod_{i\in T^c}(w_i-\pi_\ell)} \cdot \prod_{\ell=1}^{A+1} \frac{\prod_{i'\in(T')^c}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T^c}(w_i+\hat{\pi}_\ell)}$$

$$\cdot \left(\prod_{\ell=1}^{\lambda_n} \frac{\prod_{i'\in T'}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T}(w_i+\hat{\pi}_\ell)} - \prod_{\ell=1}^{\lambda_n+1} \frac{\prod_{i'\in T'}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T}(w_i+\hat{\pi}_\ell)}\right).$$

Interchanging the order of summations and summing over λ_n first we see

$$
\begin{aligned}
\mathrm{L}_n = & \sum_{\substack{T\subset[1;n]\\ T'\subset[1;n]\\ |T|=|T'|}} (-1)^{\#(T^c;T)+\#((T')^c;T')} \cdot \det\left(\frac{1}{-w_i+w'_{i'}}\right)_{\substack{i\in T^c\\ i'\in(T')^c}} \cdot \det\left(\frac{1}{w_i-w'_{i'}}\right)_{\substack{i\in T\\ i'\in T'}} \\
& \cdot \prod_{\ell=2}^{n} \frac{\prod_{i'\in(T')^c}(w'_{i'}-\pi_\ell)}{\prod_{i\in T^c}(w_i-\pi_\ell)} \cdot \prod_{\ell=1}^{A+1} \frac{\prod_{i'\in(T')^c}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T^c}(w_i+\hat{\pi}_\ell)} \\
& \cdot \left(\prod_{\ell=1}^{B} \frac{\prod_{i'\in T'}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T}(w_i+\hat{\pi}_\ell)} - \prod_{\ell=1}^{A+1} \frac{\prod_{i'\in T'}(w'_{i'}+\hat{\pi}_\ell)}{\prod_{i\in T}(w_i+\hat{\pi}_\ell)}\right) \\
& = \mathrm{R}_n - Q_n.
\end{aligned}
$$

Note that we have added the term corresponding to $T = T' = \emptyset$ to the sum which does not harm since the term equals zero. Here

$$
\begin{aligned}
Q_n := & \sum_{\substack{T\subset[1;n]\\ T'\subset[1;n]\\ |T|=|T'|}} (-1)^{\#(T^c;T)+\#((T')^c;T')} \cdot \det\left(\frac{1}{-w_i+w'_{i'}}\right)_{\substack{i\in T^c\\ i'\in(T')^c}} \cdot \det\left(\frac{1}{w_i-w'_{i'}}\right)_{\substack{i\in T\\ i'\in T'}} \\
& \cdot \prod_{\ell=2}^{n} \frac{\prod_{i'\in(T')^c}(w'_{i'}-\pi_\ell)}{\prod_{i\in T^c}(w_i-\pi_\ell)} \cdot \prod_{i=1}^{n}\prod_{\ell=1}^{A+1} \frac{(w'_i+\hat{\pi}_\ell)}{(w_i+\hat{\pi}_\ell)} \\
& = \prod_{i=1}^{n}\prod_{\ell=1}^{A+1} \frac{(w'_i+\hat{\pi}_\ell)}{(w_i+\hat{\pi}_\ell)} \cdot \det\left(\frac{1}{w_i-w'_{i'}} \cdot \left(1-\prod_{\ell=2}^{n}\frac{w'_{i'}-\hat{\pi}_\ell}{w_i-\hat{\pi}_\ell}\right)\right)_{i,i'=1}^{n}.
\end{aligned}
$$

The last determinant is zero by Lemma A.1.1. Hence $\mathrm{L}_n = \mathrm{R}_n$ and this finishes the proof.

A.3 Proof of Corollary 3.5.4

For notational convenience we set

$$
\Psi_\lambda^\ell(\vec{w}) = \det\left(\prod_{\ell=j+1}^{N}\frac{1}{w_i-\pi_\ell}\prod_{\ell=1}^{\lambda_j+1}\frac{1}{w_i+\hat{\pi}_\ell}\right)_{i,j=1}^{N},
$$

$$\Psi_\lambda^r(\vec{w}') = \det\left(\prod_{\ell=j+1}^{N}(w_i' - \pi_\ell)\prod_{\ell=1}^{\lambda_j}(w_i' + \hat{\pi}_\ell)\right)_{i,j=1}^{N}.$$

Then

$$\sum_{\lambda_N+L-N\geq\lambda_1\geq\cdots\geq\lambda_N\geq A}\Psi_\lambda^\ell(\vec{w})\Psi_\lambda^r(\vec{w}') = \sum_{B=A}^{\infty}\sum_{B+L-N\geq\lambda_1\geq\cdots\geq\lambda_N=B}\Psi_\lambda^\ell(\vec{w})\Psi_\lambda^r(\vec{w}').$$

Hence the corollary follows immediately from the following identity after a telescoping summation:

$$\sum_{B+L-N\geq\lambda_1\geq\cdots\geq\lambda_N=B}\Psi_\lambda^\ell(\vec{w})\Psi_\lambda^r(\vec{w}') = \left(1-\left(\frac{z'}{z}\right)^L\right)^{N-1}\cdot\det\left(\frac{1}{w_i - w_{i'}'}\right)_{i,j=1}^{N} \cdot\left(\prod_{i=1}^{N}\prod_{\ell=1}^{B}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell} - \prod_{i=1}^{N}\prod_{\ell=1}^{B+1}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\right). \tag{A.5}$$

To see (A.5) we write

$$\begin{aligned}&\sum_{B+L-N\geq\lambda_1\geq\cdots\geq\lambda_N=B}\Psi_\lambda^\ell(\vec{w})\Psi_\lambda^r(\vec{w}')\\ &= \sum_{B+L-N\geq\lambda_1\geq\cdots\geq\lambda_N\geq B}\Psi_\lambda^\ell(\vec{w})\Psi_\lambda^r(\vec{w}') - \sum_{B+L-N\geq\lambda_1\geq\cdots\geq\lambda_N\geq B+1}\Psi_\lambda^\ell(\vec{w})\Psi_\lambda^r(\vec{w}')\\ &:= J_1 - J_2.\end{aligned}$$

By Proposition 3.5.1 we have

$$\begin{aligned}J_1 &= \prod_{i=1}^{N}\prod_{\ell=1}^{B}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\cdot\det\left(\frac{1}{w_i-w_{i'}'}\cdot\left(1-\prod_{\ell=2}^{N}\frac{w_{i'}'-\pi_\ell}{w_i-\pi_\ell}\prod_{\ell=B+1}^{B+L-N+1}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\right)\right)_{i,j=1}^{N}\\ &= \prod_{i=1}^{N}\prod_{\ell=1}^{B}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\cdot\det\left(\frac{1}{w_i-w_{i'}'}\cdot\left(1-\left(\frac{z'}{z}\right)^L\frac{w_i-\pi_1}{w_{i'}'-\pi_1}\frac{w_i'+\hat{\pi}_{B+1}}{w_i+\hat{\pi}_{B+1}}\right)\right)_{i,j=1}^{N},\end{aligned}$$

where in the second equality we used the periodicity of the parameters $\{\hat{\pi}_i\}_{i\in\mathbb{Z}}$ and the fact that w_i's satisfy the Bethe equations (3.66). Similarly

$$J_2 = \prod_{i=1}^{N}\prod_{\ell=1}^{B+1}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\cdot\det\left(\frac{1}{w_i-w_{i'}'}\cdot\left(1-\left(\frac{z'}{z}\right)^L\frac{w_i-\pi_1}{w_{i'}'-\pi_1}\right)\right)_{i,j=1}^{N}.$$

Set $\mu := (z'/z)^L$. By Lemma A.1.3 above we have

$$\begin{aligned}
J_1 &= \prod_{i=1}^{N}\prod_{\ell=1}^{B}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\cdot\det\left(\frac{1}{w_i-w_{i'}'}\cdot\left(1-\mu\frac{w_i-\pi_1}{w_{i'}'-\pi_1}\frac{w_i'+\hat{\pi}_{B+1}}{w_i+\hat{\pi}_{B+1}}\right)\right)_{i,j=1}^{N}\\
&= \prod_{i=1}^{N}\prod_{\ell=1}^{B}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\cdot(1-\mu)^N\cdot\det\left(\frac{1}{w_i-w_{i'}'}-\frac{\mu}{1-\mu}\frac{\pi_1+\hat{\pi}_{B+1}}{(w_{i'}'-\pi_1)(w_i+\hat{\pi}_{B+1})}\right)_{i,j=1}^{N}\\
&= \prod_{i=1}^{N}\prod_{\ell=1}^{B}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\cdot(1-\mu)^N\cdot\det\left(\frac{1}{w_i-w_{i'}'}\right)_{i,i'=1}^{N}\\
&\cdot\left(1+\frac{\mu}{1-\mu}\left(1-\prod_{i=1}^{N}\frac{w_i-\pi_1}{w_{i'}'-\pi_1}\frac{w_i'+\hat{\pi}_{B+1}}{w_i+\hat{\pi}_{B+1}}\right)\right).
\end{aligned}$$

Similarly

$$J_2 = \prod_{i=1}^{N}\prod_{\ell=1}^{B+1}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\cdot(1-\mu)^N\cdot\det\left(\frac{1}{w_i-w_{i'}'}\right)_{i,i'=1}^{N}\cdot\left(1+\frac{\mu}{1-\mu}\left(1-\prod_{i=1}^{N}\frac{w_i-\pi_1}{w_{i'}'-\pi_1}\right)\right).$$

Hence

$$J_1-J_2 = (1-\mu)^{N-1}\cdot\det\left(\frac{1}{w_i-w_{i'}'}\right)_{i,i'=1}^{N}\cdot\left(\prod_{i=1}^{N}\prod_{\ell=1}^{B}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}-\prod_{i=1}^{N}\prod_{\ell=1}^{B+1}\frac{w_i'+\hat{\pi}_\ell}{w_i+\hat{\pi}_\ell}\right).$$

This proves (A.5) and the corollary then follows.

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