Cohomology of Non-Smooth Rigid Analytic Spaces

by

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de Rham

étale

derived de Rham cohomology

$R\Gamma(\widehat{dR}^{an}_{X/\mathbb{C}_p})$

$\downarrow$

infinitesimal / crystalline cohomology

$R\Gamma_{inf}(X/\mathbb{C}_p)$

$\downarrow$

éh de Rham / Deligne - Du Bois cohomology

$R\Gamma(\underline{\Omega}^{\bullet}_{X/\mathbb{C}_p})$

$\xleftarrow[\text{mod } \xi]{}$ B_{dR}^+ - linear de Rham cohomology theories

period sheaves via derived de Rham complex

de Rham comparison

$\simeq$

(invert ξ).

proétale cohomology

$R\Gamma_{pro\acute{e}t}(\mathbb{B}_{dR}^+)$

$\wr\!\wr$ primitive comparison

étale cohomology

$R\Gamma_{\acute{e}t}(\mathbb{Q}_p) \otimes B_{dR}^+$

Hodge-Tate cohomology

$\bigoplus_{i=0}^{\dim X} R\Gamma(\underline{\Omega}^i_{X/\mathbb{C}_p})(-i)$

Hodge-Tate decomposition

$\simeq$

pro-étale cohomology

$R\Gamma_{pro\acute{e}t}(\hat{\mathcal{O}}_X)$

$\wr\!\wr$ primitive comparison

étale cohomology

$R\Gamma_{\acute{e}t}(\mathbb{Q}_p) \otimes \mathbb{C}_p$

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To my parents
and my grandmother

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ABSTRACT

This thesis considers the cohomology theories of rigid analytic spaces, with a focus on spaces that might be singular. Analogous to the complex algebraic geometry, we generalize derived de Rham cohomology, infinitesimal cohomology, and Deligne–Du Bois cohomology to their rigid analytic counterparts, and study their relations. We also consider the p-adic étale cohomology of rigid analytic spaces, extending the Hodge–Tate decomposition theorem of Faltings and Scholze to non-smooth rigid analytic spaces. The strategy to the latter is the simplicial method and the resolution of singularities. Furthermore, joint with Shizhang Li, we reproduce the period sheaves and the p-adic Poincaré sequence in p-adic Hodge theory, using the derived de Rham complex.

CHAPTER 1

Introduction

In 1960s, Tate [Tat71] introduced the notion of rigid analytic spaces, as a non-archimedean analogue of the complex analytic spaces. Since then, rigid analytic spaces have been developed systemically by a number of mathematicians, and extended into several variants due to the work of Raynaud, Berkovich, and Huber. The theory and its variants are not only subjects of interest themselves in algebraic geometry, but have been used in a wide scope of mathematically areas, including number theory, representation theory, and mathematical physics. The purpose of this thesis is to study the cohomology theories of rigid analytic spaces, with a focus on those spaces that might be singular. More specifically, we consider:

1. various de Rham cohomology theories for singular rigid analytic spaces;

2. p-adic étale cohomology and its decomposition for singular rigid analytic spaces.

In this chapter, we first give a brief introduction about the background of various subjects mentioned above, together with an overview of this thesis.

1.1 Background: Complex algebraic variety and its cohomology

We start with a brief overview on two types of cohomology theories for complex algebraic varieties, namely singular cohomology and algebraic de Rham cohomology.

1.1.1 Singular cohomology

In complex algebraic geometry, one of the most powerful tools is the analytic method, which considers the underlying analytic structure of an algebraic variety over the field of complex numbers $\mathbb{C}$. Through analysis and topology, the analytic method provides us with many invariants of the given algebraic object, and is crucial for the study of geometric properties and the classification of spaces.

More precisely, let us consider a proper algebraic variety X over the field of complex numbers $\mathbb{C}$. The set of complex points of X admits a natural analytic structure of complex analytic space, and we can consider the *singular cohomology group* $\mathrm{H}^n_{\mathrm{Sing}}(X,\mathbb{C})$, which is one of the most important topological invariants of X defined using algebraic topology. For instance, when X is a smooth complex algebraic curve, the underlying analytic space of X can be visualized as a surface "in real life"[1] that have the shape of a donut with several holes on it. In this case, the number of holes on this surface can be read from the dimension of the singular cohomology group of X.

When X is smooth and projective, the underlying complex analytic space of X is a compact Kähler manifold. In this situation, thanks to the *de Rham Theorem*, we can compute those cohomology groups using differential topology:

Theorem 1.1.1.1 (de Rham)**.** *There exists a natural isomorphism*

$$\mathrm{H}^n_{\mathrm{Sing}}(X,\mathbb{C}) \cong \mathrm{H}^n_{\mathrm{dR}}(X).$$

Here the *analytic de Rham cohomology group* $\mathrm{H}^n_{\mathrm{dR}}(X)$ is defined using C^∞-differential forms and measures the failure of integrations. Moreover, de Rham cohomology admits a natural descending filtration, called the *Hodge filtration*. The *Hodge decomposition theorem* then tells us that this filtration splits into a direct sum of subspaces represented by harmonic forms:

Theorem 1.1.1.2 (Hodge decomposition)**.** *(i) There exists a natural decomposition of the de Rham cohomology*

$$\mathrm{H}^n_{\mathrm{dR}}(X) = \bigoplus_{i+j=n} \mathrm{H}^{i,j}(X),$$

where $\mathrm{H}^{i,j}(X)$ *is the subspace of harmonic* (i,j)*-forms.*

(ii) The complex conjugation acts naturally on the de Rham cohomology, and induces the identities

$$\overline{\mathrm{H}^{i,j}(X)} = \mathrm{H}^{j,i}(X).$$

Furthermore, with the help of GAGA theorem, the cohomology of the Hodge filtered analytic de Rham complex is isomorphic to the cohomology of algebraic de Rham complex with its algebraic Hodge filtration, and we have

$$\mathrm{H}^{i,j}(X) \cong \mathrm{H}^j(X,\Omega^i).$$

As a consequence, the obtained graded pieces for the Hodge filtration can be understood algebraically:

[1]Here we mean a surface over the field of real numbers $\mathbb{R}$. This is because as any complex number has two real variables, a complex curve can be regarded as a two dimensional space over $\mathbb{R}$.

Corollary 1.1.1.3. *There is a natural functorial decomposition of the singular cohomology of X as follows:*

$$\mathrm{H}^n_{\mathrm{Sing}}(X, \mathbb{C}) \cong \bigoplus_{i+j=n} \mathrm{H}^j(X, \Omega^i).$$

1.1.2 Algebraic de Rham cohomology and its variants

As a complex algebraic variety is defined algebraically, it is natural to ask if there is a pruely algebraic cohomology theory that also computes singular cohomology.

When X is a smooth algebraic variety over $\mathbb{C}$, the singular cohomology of X is isomorphic to its algebraic de Rham cohomology. The *algebraic de Rham cohomology* is the sheaf cohomology for the algebraic de Rham complex

$$\Omega^\bullet_{X/\mathbb{C}} := \left(0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{C}} \longrightarrow \cdots \longrightarrow \Omega^{\dim X}_{X/\mathbb{C}} \longrightarrow 0 \right),$$

where each $\Omega^i_{X/\mathbb{C}}$ is the sheaf of i-th algebraic Kähler differentials over the variety X, and is coherent over the algebraic structure sheaf $\mathcal{O}_X$. Then we have the following result of Grothendieck ([Gro66]):

Theorem 1.1.2.1 (Grothendieck [Gro66])**.** *Assume X is a smooth complex algebraic variety. Then there is a canonical isomorphism of cohomology groups*

$$\mathrm{H}^i_{\mathrm{Sing}}(X, \mathbb{C}) \cong \mathrm{H}^i(X, \Omega^\bullet_{X/\mathbb{C}}).$$

As an upshot, we get a purely algebraic way to compute the singular cohomology group.

However, if X is non-smooth, cohomology of the usual algebraic de Rham complex may fail to compute its singular cohomology ([AK11]). To get the correct answer, in particular to get an algebraic cohomology theory which computes singular cohomology, there are several methods generalizing algebraic de Rham cohomology to the non-smooth setting:

(1) In [Har75], Hartshorne discovered that if X admits a closed immersion into a smooth variety Y, then the formal completion $\widehat{\Omega^\bullet_{Y/\mathbb{C}}}$ of the de Rham complex $\Omega^\bullet_{Y/\mathbb{C}}$ along $X \to Y$ computes the singular cohomology of X. The result was obtained independently by Deligne (unpublished), and by Herrera–Lieberman [HL71].

In the general case when X is not necessarily embeddable, there exists a ringed *infinitesimal site* $(X/\mathbb{C}_{\mathrm{inf}}, \mathcal{O}_{X/\mathbb{C}})$ (or the crystalline cohomology in characteristic zero) introduced by Grothendieck [Gro68]. It can be shown that its cohomology $\mathrm{H}^i(X/\mathbb{C}_{\mathrm{inf}}, \mathcal{O}_{X/\mathbb{C}})$ coincides with $\mathrm{H}^i(X, \widehat{\Omega^\bullet_{Y/\mathbb{C}}})$ whenever $X \to Y$ is a closed immersion into a smooth variety as above. In particular we obtain a conceptual cohomology theory that is independent of immersions.

Moreover, the method allows us to compute cohomology with nontrivial coefficients, where we could replace $\mathcal{O}_{X/\mathbb{C}}$ by vector bundles with flat connections (or in other words *crystals*).

(2) Another theory, generalizing the de Rham complex of polynomial rings via the simplicial extension, is the *(Hodge-completed) derived de Rham complex* introduced by Illusie [Ill71]. To any scheme X over $\mathbb{C}$, we can associate a filtered derived algebra $\widehat{\mathrm{dR}}_{X/\mathbb{C}}$ to it. It was shown by Illusie in loc. cit. the cohomology of the derived de Rham complex $\widehat{\mathrm{dR}}_{X/\mathbb{C}}$ is isomorphic to the Hartshorne's cohomology, assuming X is of local complete intersection. Later on, using the Adams completion in algebraic topology, Bhatt [Bha12a] showed that the comparison is true for any finite type scheme in characteristic zero, without the l.c.i condition. In particular, for an arbitrary variety $X/\mathbb{C}$, we get the isomorphism

$$\mathrm{H}^i_{\mathrm{Sing}}(X, \mathbb{C}) \cong \mathrm{H}^i(X, \widehat{\mathrm{dR}}_{X/\mathbb{C}}).$$

Here we mention that the first graded piece of $\widehat{\mathrm{dR}_{X/\mathbb{C}}}$ is the cotangent complex $\mathbb{L}_{X/\mathbb{C}}$, which plays an important role in the deformation theory of schemes.

(3) Furthermore, there exists a theory of *Deligne–Du Bois complex* for $X/\mathbb{C}$, introduced by Deligne and studied by Du Bois ([DB81]), that also gives us the correct answer. The Deligne–Du Bois complex is defined via the cohomological descent for resolution of singularities. It could be shown that the singular cohomology of X is isomorphic to the cohomology of the Deligne–Du Bois complex. Moreover, Deligne–Du Bois complex admits a finite *Hodge–Deligne filtration* where each graded piece is a bounded complex of coherent sheaves in the derived category. The induced filtration on cohomology is in fact the Hodge filtration for the mixed Hodge structure, whose associated spectral sequence degenerates at the first page when X is proper. Furthermore, Deligne–Du Bois complex together with its filtration also admits a site theoretical interpretation via the h-topology, where the latter is introduced by Voevodsky in [Voe96]. The theory of h-cohomology of X is studied in [HJ14] and [Lee07].

1.2 Rigid analytic spaces and its cohomology

1.2.1 Rigid analytic spaces

In complex algebraic geometry, one of the most important ingredients of the analytic method is the fact that the fields $\mathbb{R}$ and $\mathbb{C}$ are equipped with the natural archimedean metrics, which makes it possible to measure the distance of points over a complex analytic space. On the other hand, the archimedean metrics over the fields $\mathbb{R}$ and $\mathbb{C}$ can be obtained from the one over the field of rational numbers $\mathbb{Q}$. It is then natural to ask if there are other metrics over the field $\mathbb{Q}$, and if we can furthermore develop a theory of analytic geometry based on them.

In the early 1900s, K. Hensel discovered that apart from the above archimedean metric, there exists a non-trivial *non-archimedean metric* over $\mathbb{Q}$ whose associated completion yields the *field of p-adic numers* $\mathbb{Q}_p$, where p is a prime number. This makes it possible to consider analytic geometry over p-adic numbers. But as the field $\mathbb{Q}_p$ is totally disconnected, the usual construction of manifolds over $\mathbb{Q}_p$ is not meaningful. In 1961, J. Tate ([Tat71]) discovered a new category of analytic-algebraic objects over p-adic numbers, called *rigid analytic spaces*. A rigid (analytic) space is defined as a ringed space that is locally isomorphic to the vanishing locus of finitely many convergent power series in a polydisc of K, where K is a p-adic field. As a consequence, we can use methods from algebraic geometry and complex analytic geometry to study rigid spaces. Here similar to the complex theories, examples of rigid spaces include analytifications of algebraic varieties over p-adic fields. Moreover, there are various notions like smoothness and properness for rigid spaces, which are compatible with the ones for algebraic varieties under the analytification.

1.2.2 p-adic étale cohomology

Let K be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$, $\overline{K}$ be its algebraic closure, and $\mathbb{C}_p$ be the completion of $\overline{K}$. As the field of p-adic numbers and its extensions are totally disconnected, singular cohomology of a rigid space X is not a meaningful invariant. Instead, the correct analogue of singular cohomology in non-archimedean geometry is *p-adic étale cohomology* $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Faltings proved that p-adic étale cohomology of a proper smooth algebraic variety over $\mathbb{Q}_p$ admits a natural Galois equivariant decomposition into a direct sum of Hodge cohomology (with appropriate Tate twists), after the base change to $\mathbb{C}_p$. Precisely, we have the following:

Theorem 1.2.2.1 ([Fal88]). *Let X be a smooth proper algebraic variety over K. Then there exists a natural $\mathrm{Gal}(\overline{K}/K)$-equivariant decomposition*

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} \mathrm{H}^i(X, \Omega^j_{X/K}) \otimes_K \mathbb{C}_p(-j).$$

This miraculous theorem, which has the name *Hodge–Tate decomposition*, not only gives a decomposition of the étale cohomology that is analogous to the Hodge decomposition (Corollary 1.1.1.3) for complex Kähler manifolds, but also encodes the information of its structure of p-adic Galois representation.

For rigid analytic spaces, the analogous decomposition of p-adic étale cohomology was conjectured by Tate. This was then proved for proper smooth rigid spaces by Scholze:

Theorem 1.2.2.2 ([Sch13a]). *Theorem 1.2.2.1 holds for smooth proper rigid spaces over K.*

In order to prove the above theorem, Scholze introduced the *pro-étale topology* $X_{\text{proét}}$ that is locally *perfectoid*, for a given rigid space X. Building on Scholze's construction, in this thesis we extend the result to general proper rigid spaces that are not necessarily smooth:

Theorem 1.2.2.3 ([Guo19], Theorem 1.1.3)**.** *Let X be a proper rigid space over K. There exists a natural $\mathrm{Gal}(\overline{K}/K)$-equivariant decomposition*

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} \mathrm{H}^i(X, \underline{\Omega}^j_X) \otimes_K \mathbb{C}_p(-j).$$

where $\mathrm{H}^i(X, \underline{\Omega}^j_X)$ has the trivial Galois action, and $\mathbb{C}_p(-j)$ is the Tate twist of weight j.

Here in the statement, $\mathrm{H}^i(X, \underline{\Omega}^j_X)$ is the sheaf cohomology of a naturally defined bounded complex of coherent sheaves $\underline{\Omega}^j_X$ over X, which generalizes the j-th graded piece of the classical Deligne–Du Bois cohomology to rigid spaces (see Subsection 1.1.2, part (3), and the rigid analytic version in Theorem 1.2.3.1).

Remark 1.2.2.4. The idea of the proof is the following. In [Sch13a], Scholze showed that the p-adic étale cohomology $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p$ is naturally isomorphic to *pro-étale cohomology* $\mathrm{H}^n_{\text{proét}}(X_{\overline{K}}, \widehat{\mathcal{O}}_X)$, where X is an arbitrary proper rigid space, and $\widehat{\mathcal{O}}_X$ is the pro-étale structure sheaf. Notice that there is a natural map of Grothendieck topologies $\nu : X_{\text{proét}} \to X_{\text{ét}}$. So the study of p-adic étale cohomology can be broken down into two steps: first study the derived direct image $R\nu_* \widehat{\mathcal{O}}_X$ and then consider its sheaf cohomology. The proof of Theorem 1.2.2.3 is then essentially about a systematic study of $R\nu_* \widehat{\mathcal{O}}_X$ for a proper rigid space X, where we show that the derived direct image $R\nu_* \widehat{\mathcal{O}}_X$ admits a decomposition into a direct sum of $\underline{\Omega}^j_X(-j)[-j]$ in the derived category, and each of $\underline{\Omega}^j_X$ lives in an expected range of cohomological degree.

Remark 1.2.2.5. In fact, in this thesis we show that the above results could be extended to proper rigid spaces over $\mathbb{C}_p$ that are not necessarily defined over a discretely valued subfield K, where the decomposition and the cohomological bounds in the derived category in Remark 1.2.2.4 still hold but would not be canonical or Galois equivariant in general (cf. [Guo19, Theorem 7.4.9]).

When X is a proper smooth rigid space over K, the Hodge–Tate decomposition can be obtained from the *de Rham Comparison Theorem*, proved by Scholze in [Sch13a]. The comparison theorem states that there is a filtered isomorphism between the étale cohomology $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ and the de Rham cohomology $\mathrm{H}^n_{\mathrm{dR}}(X/K) \otimes_K \mathrm{B}_{\mathrm{dR}}$ (see the beginning of Subsection 1.2.3 about de Rham cohomology of smooth rigid spaces, and the beginning of Subsection 1.2.4 about the period ring B_{dR}). Moreover, by taking the zero-th graded piece of this comparison, we recover the Hodge–Tate decomposition for proper smooth rigid spaces.

In this thesis, building on de Rham comparison theorem of the smooth case by Scholze ([Sch13a]), we extend the comparison to non-smooth proper rigid spaces, using the simplicial method and Deligne–Du Bois complexes.

Theorem 1.2.2.6 ([Guo19], Theorem 1.1.4)**.** *Let X be a proper rigid space over K. Then there exists a natural Galois equivariant filtered isomorphism of Galois representations*

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} = \mathrm{H}^n(X, \underline{\Omega}^\bullet_X) \otimes_K \mathrm{B}_{\mathrm{dR}},$$

whose zero-th graded piece induces the Hodge–Tate decomposition in Theorem 1.2.2.3.

Here the Deligne–Du Bois complex $\underline{\Omega}^\bullet_X$ and its cohomology for the rigid space X is the rigid analogue of the algebraic one as in Subsection 1.1.2 part (3), and will be introduced in Theorem 1.2.3.1. Similar to the smooth case, the filtration on the left side is defined by the one on the de Rham period ring B_{dR}, and the filtration on the right side is the tensor product filtration produced by that of Deligne–Du Bois cohomology and the natural filtration on the period ring B_{dR}.

The above result produces a comparison between étale cohomology and cohomology of Deligne–Du Bois complexes for non-smooth proper rigid spaces. As a consequence, this generalizes the classical picture of the singular–de Rham comparison to the non-smooth non-archimedean world.

1.2.3 de Rham cohomology

Let K be a finite extension of $\mathbb{Q}_p$ as before. Similar to complex algebraic varieties, rigid spaces also admit the notion of de Rham cohomology, by taking the sheaf cohomology of the *continuous de Rham complex* $\Omega^\bullet_{X/K}$ as below

$$\mathrm{H}^i_{\mathrm{dR}}(X/K) := \mathrm{H}^i(X, \Omega^\bullet_{X/K}),$$

where the differential operators of the complex are continuous with respect to the natural p-adic topology for analytic functions over X. When X is proper smooth over K, it can be shown that de Rham cohomology satisfies the finiteness and the expected cohomological boundedness within the cohomological degrees $[0, 2\dim(X)]$. Moreover, assuming X is the analyfication of a proper smooth algebraic variety over K, de Rham cohomology of the rigid space X is then canonically isomorphic to algebraic de Rham cohomology of the original variety. The latter in particular implies that de Rham cohomology produces correct Betti numbers when X comes from an algebraic variety.

However, similar to the case for complex algebraic varieties, de Rham cohomology of a singular rigid space X does not always give correct Betti numbers. As we mentioned in Subsection 1.1.2, there are several ways to adjust the usual algebraic de Rham cohomology in order to recover singular cohomology. It is then natural to ask if those modifications of de Rham cohomology theories for singular algebraic varieties can be extended to the non-archimedean world.

The next main result of this thesis gives a positive answer to the question, generalizing the theories in Subsection 1.1.2 to rigid spaces.

Theorem 1.2.3.1 ([Guo20], Theorem 1.2.1)**.** *There are three naturally defined cohomology theories, for rigid spaces X over K, in filtered derived category of K-vector spaces:*

$$R\Gamma(X, \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/K}) \longrightarrow R\Gamma_{\mathrm{inf}}(X/K) \longrightarrow R\Gamma(X, \underline{\Omega}^{\bullet}_{X}),$$

such that:

(i) the above cohomology theories generalize derived de Rham cohomology, infinitesimal cohomology, and Deligne–Du Bois cohomology of algebraic varieties in characteristic zero;

(ii) the above maps induce isomorphisms on their underlying complexes, and are all filtered isomorphic to Hodge-filtered de Rham cohomology when X is smooth over K;

(iii) when X is proper over K, the underlying complex of any of these cohomology is a perfect K-complex that lives within cohomology degree $[0, 2\dim(X)]$.

The constructions of the infinitesimal cohomology and the Deligne–Du Bois complexes are similar to their analogues for algebraic varieties, where the first one is defined as the cohomology of the infinitesimal thickenings of the given rigid space X, and the second one is defined using simplicial resolution of singularities guaranteed by Temkin [Tem12]. The schematic derived de Rham complex however does not work directly for rigid spaces; instead, to equip it with a p-adic topology, we need to apply a p-adic completion integrally on the schematic derived de Rham complex, and then consider its filtered completion after inverting p. The obtained filtered algebra is then complete under its filtration, whose graded pieces are wedge powers of the continuous cotangent complex for X over K, where the latter is introduced by Gabber–Ramero in [GR03].

1.2.4 Period sheaves and the derived de Rham complex

In complex geometry, as the *periods* of various integrations have complex values, we need to take the complex coefficients in order to compare singular cohomology with de Rham cohomology. The question becomes much subtler in p-adic geometry if one wants to compare étale cohomology with de Rham cohomology. In fact, this was first conjectured by Grothendieck and was studied in depth by Fontaine, who introduced various period rings in p-adic Hodge theory, including the de Rham ring B_{dR} in the statement of Theorem 1.2.2.6. Moreover, the construction of the period rings were carried into many geometric situations, including for example Scholze's construction of de

Rham period sheaves $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ ([Sch13a]), which was introduced for p-adic smooth formal schemes by Brinon ([Bri08]).

The construction of those period rings and period sheaves are quite complicated, and people have been trying to understand them using differentials. For example, Colmez related the construction of $\mathrm{B}_{\mathrm{dR}}^+$ to Kähler differentials for $\overline{\mathbb{Z}}_p/\mathbb{Z}_p$ (see [Fon94, Appendix]), and later on Beilinson gave another construction of the ring $\mathrm{B}_{\mathrm{dR}}^+$ using the derived de Rham complex ([Bei12]). In my joint work with Shizhang Li, we gave a new construction of period sheaves of Scholze, using the analytic derived de Rham complexes.

Theorem 1.2.4.1 ([GL20], Theorem 1.1, 1.4)**.** *Let X be a smooth rigid space over K. We have natural filtered isomorphisms:*

$$\mathbb{B}_{\mathrm{dR}}^+ \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/K} \text{ and } \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}.$$

Here similar to the construction of the analytic derived de Rham complexes for rigid spaces as in Theorem 1.2.3.1, the topological structure of the period sheaves are obtained via an integral p-adic completion and a rational filtered completion, on the classical derived de Rham complex.

Under the new reformulation, we are able to reinterpret the Poincaré lemma of period sheaves using the derived de Rham complexes ([GL20, Theorem 1.2, 1.4]). In fact, the Poincaré sequence can be naturally obtained via the Gauss–Manin connection of the derived de Rham complexes for the triple of sites $X_{\mathrm{pro\acute{e}t}} \to X \to \mathrm{Spa}(K)$. Moreover, in the rational case, our construction can be generalized to non-smooth rigid spaces that are locally complete intersection. These in particular provide a conceptual construction of the period sheaves and their Poincaré lemma using the differential and the Gauss–Manin connection.

1.3 Outline

The thesis is divided into three parts. The Part I consists of five chapters from Chapter 1 to Chapter 5, and is aiming to generalize the de Rham cohomology theories of non-smooth algebraic varieties to the rigid analytic geometry, as in Theorem 1.2.3.1. In Chapter 2, we develop the foundations of the analytic derived de Rham complex for rigid spaces, which is the p-adic analogue of the algebraic derived de Rham complex introduced by Illusie [Ill72]. In Chapter 3, we generalize the infinitesimal/crystalline cohomology of complex algebraic varieties to rigid spaces, and develop the corresponding theory of the crystal. In Chapter 4, we consider the éh cohomology for rigid spaces, which is a site-theoretical construction of the Deligne–Du Bois cohomology using the cohomological descent in the non-archimedean world. Roughly speaking, the éh cohomology is defined as the sheafification of the usual de Rham cohomology using the resolution of singularity.

In Chapter 5, analogous to the algebraic theory, we show the aforementioned three cohomology theories have the isomorphic underlying complexes for general proper rigid spaces. At last, in Chapter 6 we extend the previous results to their $\mathrm{B}_{\mathrm{dR}}^{+}$-linear analogues, and relate them with the pro-étale cohomology of the de Rham period sheaf by Scholze [Sch13a].

In the Part II, we study the p-adic étale and pro-étale cohomology for rigid spaces, as the non-archimedean analogues of the singular cohomology for complex algebraic varieties. Our aim in this part is to prove the Hodge–Tate decomposition mentioned in Theorem 1.2.2.3. In Chapter 7, we recall the basics of the pro-étale topology and the v-topology. Chapter 8 is then devoted to the proof of the degeneracy theorem of the pro-étale cohomology and the Hodge–Tate decomposition. Here we use the simplicial method and the deformation theory for rigid spaces, and extend the known decomposition theorem of Scholze to the non-smooth rigid spaces and the derived level. We also give an application in Section 8.6 on the vanishing of the Deligne–Du Bois complex.

The Part III is devoted to an understanding of the period sheaves using the analytic derived de Rham complex as in Theorem 1.2.4.1. It is the joint work of the author with Shizhang Li as in [GL20]. This part consists of the Chapter 9 on the integral theory and Chapter 10 on the rational theory, together with an Appendix 1 on the local complete intersection morphism in rigid geometry.

1.4 Conventions and notations

Throughout the thesis, we use the language of the adic space, and refer the reader to Huber's book [Hub96] for basic results about it. We will use mildly the language of the ∞-category in Chapter 2 and Part III, following the conventions in the foundational work [Lur09] and [Lur17]. The symbol K is always denoted as a complete non-archimedean extension of $\mathbb{Q}_p$ where p is a fixed prime number. Unless we mentioned specifically, K is always assumed to be algebraically closed.

Part I

de Rham Cohomology Theories

This part contains five chapters, where we introduce analytic derived de Rham cohomology (Chapter 2), infinitesimal cohomology (Chapter 3), and éh cohomology (Chapter 4) for non-smooth rigid spaces by order, prove their comparison theorems (Chapter 5), and extend those cohomology to the $\mathrm{B}_{\mathrm{dR}}^{+}$-coefficient (Chapter 6).

CHAPTER 2

Derived de Rham Cohomology

In this chapter, we introduce the analytic derived de Rham complex for rigid analytic spaces, as the rigid analytic analogue of the algebraic derived de Rham complex, where the latter was first introduced by Illusie in [Ill72]. The results in this chapter first appeared in the preprint [Guo20, Section 5] by the author.

As we will be working with topologically of finite type adic spaces over the de Rham period ring $\mathrm{B}_{\mathrm{dR,e}}^+ = \mathrm{B}_{\mathrm{dR}}^+/\xi^e$ and p-adic fields, we start this chapter with a brief review in Section 2.1 about basics on topologically finite type algebras over $\mathrm{A}_{\mathrm{inf,e}}$ and $\mathrm{B}_{\mathrm{dR,e}}^+$, analogous to the treatment in [GR03, Chapter 7]. We generalize the notion of the analytic cotangent complex for topologically of finite type algebras over $\mathrm{B}_{\mathrm{dR,e}}^+$ in Section 2.2. Here following [GR03], the analytic cotangent complex is defined by applying a derived p-adic completion at the classical algebraic cotangent complex integrally, and then inverting by p. Next we introduce the notion of the *analytic derived de Rham complex* for rigid spaces over $\mathrm{B}_{\mathrm{dR,e}}^+$ in Section 2.3. Similar to the cotangent complex, in order to incorporate the p-adic topology of affinoid algebras, we need to apply a derived p-adic completion integrally on the algebraic derived de Rham complex, invert by p, and then apply a filtered completion. The obtained object, which is a filtered algebra in the derived category with graded pieces being the wedge algebra of the analytic cotangent complex, generalizes the Hodge-filtered continuous de Rham complex for smooth affinoid algebras. Finally, by checking the sheafy condition, we globalize the previous analytic constructions to general rigid spaces over $\mathrm{B}_{\mathrm{dR,e}}^+$ in Section Section 2.4.

Before we start, we want to mention that we will use the language of ∞-category throughout this chapter. This helps to globalize the affinoid constructions and get a good theory of "sheaf of derived objects", using the ∞-categorical cohomological descent. We will recall the notions of the ∞-category as we start.

Remark 2.0.0.1. The construction of the analytic derived de Rham complex in this chapter can be applied to more general class of analytic Huber rings, which includes for example rigid spaces over an arbitrary p-adic non-archimedean field, and perfectoid spaces, which we will discuss in Part III of this thesis. Moreover, the results of this chapter hold true for rigid spaces over a general p-adic fields.

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2.0.1 Derived ∞-category and filtered ∞-category

We first setup the convention of derived ∞-category and its filtered version in this chapter.

Let $\mathcal{A}$ be a Grothendieck abelian category ([Sta18], Tag 079A). We can associate $\mathcal{A}$ a natural ∞-category $\mathcal{D}(\mathcal{A})$, called the *derived ∞-category of $\mathcal{A}$* ([Lur17], 1.3.5). This is the ∞-categorical enhancement of the classical derived ∞-category, and the homotopy category $\mathrm{hCh}(\mathcal{A})$ of $\mathrm{Ch}(\mathcal{A})$ is the usual derived category $D(\mathcal{A})$. Here we want to mention that the derived ∞-category $\mathcal{D}(\mathcal{A})$ is a stable presentable ∞-category. In the special case when $\mathcal{A}$ is the category of modules over an ring R, we use $\mathcal{D}(R)$ to denote $\mathcal{D}(\mathcal{A})$, which is equipped with a symmetric monoidal structure by the derived tensor product of complexes. As a convention in this chapter, we will call $\mathcal{D}(R)$ the derived category.

For a presentable ∞-category $\mathcal{C}$, we recall the *filtered ∞-category in* $\mathcal{C}$ is defined as the ∞-category

$$\mathrm{DF}(\mathcal{C}) := \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{C}).$$

Moreover, $\mathrm{DF}(\mathcal{C})$ admits a full sub-∞-category $\widehat{\mathrm{DF}}(\mathcal{C})$, called *filtered complete ∞-category in* $\mathcal{C}$, consisting of objects $C_\bullet$ such that $\lim C_\bullet \cong 0$. The natural inclusion functor $\widehat{\mathrm{DF}}(\mathcal{C}) \to \mathrm{DF}(\mathcal{C})$ admits a left adjoint, called the *filtered completion*. When $\mathcal{C} = \mathcal{D}(R)$ is the derived ∞-category of R-modules, we use $\mathrm{DF}(R)$ and $\widehat{\mathrm{DF}}(R)$ to denote $\mathrm{DF}(\mathcal{C})$ and $\widehat{\mathrm{DF}}(R)$ separately. Here we note that by to their homotopy categories (and induced functors), we recover the ordinary filtered derived category.

2.0.2 Hypersheaves

We then give a quick review about sheaves in ∞-category.

Let X be a site, and let $\mathcal{C}$ be a presentable ∞-category. The ∞-category of presheaves in $\mathcal{C}$, denoted as $\mathrm{PSh}(X, \mathcal{C})$, is defined to be the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$ of contravariant functors from X to $\mathcal{C}$. The ∞-category $\mathrm{PSh}(X, \mathcal{C})$ admits a full sub-∞-category $\mathrm{Sh}(X, \mathcal{C})$ of *(infinity) sheaves in* $\mathcal{C}$, consisting of functors $\mathcal{F} : X^{\mathrm{op}} \to \mathcal{C}$ that send coproducts to products and satisfy the descent along Čech nerves: for any covering $U' \to U$ in X, the natural morphism to the limit below is required to be a weak equivalence

$$\mathcal{F}(U) \longrightarrow \lim_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(U'_n), \tag{$*$}$$

where $U'_\bullet \to U$ is the Čech nerve associated with the covering $U' \to U$. Here we note that this is the ∞-categorical analogue of the classical sheaf condition in ordinary categories.

There is a stronger descent condition which requires $(*)$ above to hold with respect to all *hypercovers* $U'_\bullet \to U$ in the site X. Sheaves satisfying such stronger condition are called *hypersheaves*. For example, given any bounded below complex C of ordinary sheaves on a site X, the assignment $U \mapsto \mathrm{R}\Gamma(U, C)$ gives rise to a hypersheaf. The collection of hypersheaves in $\mathcal{C}$ forms a full sub-∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C})$ inside $\mathrm{Sh}(X, \mathcal{C})$.

Let $\mathcal{C} = \mathcal{D}(R)$ be the derived ∞-category of R-modules. Then the ∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C})$ of hypersheaves over X is in fact equivalent to the derived ∞-category $\mathcal{D}(X, R)$ of classical sheaves of R-modules over X, by [Lur18, Corollary 2.1.2.3]. As an upshot, the underlying homotopy category of $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C})$ is the classical derived category of sheaves of R-modules over X. In particular, given a hypersheaf $\mathcal{F}$ of R-modules over X, we can always represent it by an actual complex of sheaves of R-modules.

2.0.3 de Rham period rings

As a setup, we recall the basics of the de Rham period ring. A more detailed introduction of the de Rham period ring could be found in [Fon94].

Let K be a p-adic valuation extension of $\mathbb{Q}_p$ that is complete and algebraically closed. Denote by $\mathcal{O}_K$ to be the ring of integers of K. Then we can define the p-adic ring $\mathrm{A}_{\mathrm{inf}}(\mathcal{O}_K)$ as

$$\mathrm{A}_{\mathrm{inf}} := W(\varprojlim_{x \mapsto x^p} \mathcal{O}_K).$$

There exists a canonical continuous surjection $\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$, where the kernel $\ker(\theta)$ is principal. Fix a compatible system of p^n-th root of unity $\{\zeta_{p^n}\}_n$ in K. Then the element $\xi := \frac{[\epsilon]-1}{[\epsilon]^{\frac{1}{p}}-1}$ generates the ideal $\ker(\theta)$, where $[\epsilon]$ is the Teichmüller lift of the element $(\zeta_1, \zeta_p, \ldots)$ in $\mathrm{A}_{\mathrm{inf}}$.

The *de Rham period ring* $\mathrm{B}_{\mathrm{dR}}^+$ is defined as the ξ-adic completion of the ring $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$. By abuse of the notation, we write $\theta : \mathrm{B}_{\mathrm{dR}}^+ \to K$ as the canonical continuous surjection induced from $\mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$. Note that for each $n \in \mathbb{N}$, we have

$$\mathrm{B}_{\mathrm{dR}}^+/\xi^n = \mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^n,$$

which is a p-adic Tate ring with a canonical ring of definition $\mathrm{A}_{\mathrm{inf}}/\xi^n$ in it. So we can form a Huber pair $(\mathrm{B}_{\mathrm{dR}}^+/\xi^n, (\mathrm{B}_{\mathrm{dR}}^+/\xi^n)^\circ)$ over $(\mathbb{Q}_p, \mathbb{Z}_p)$ for $n \in \mathbb{N}$. The p-adic adic space $\Sigma_n := \mathrm{Spa}(\mathrm{B}_{\mathrm{dR}}^+/\xi^n, (\mathrm{B}_{\mathrm{dR}}^+/\xi^n)^\circ)$ is a nilpotent extension of $\mathrm{Spa}(K, \mathcal{O}_K)$.

In the rest of the thesis, we often use $\mathrm{A}_{\mathrm{inf,e}}$ and $\mathrm{B}_{\mathrm{dR,e}}^+$ to denote quotient rings $\mathrm{A}_{\mathrm{inf}}/\xi^e$ and $\mathrm{B}_{\mathrm{dR}}^+/\xi^e$ separately, in order to simplify the notations.

2.1 Topological algebras over $\mathrm{A}_{\mathrm{inf,e}}$

As a preparation, we first setup basics around the topologically finite type algebras over $\mathrm{A}_{\mathrm{inf,e}}$, and the construction of the analytic cotangent complex, generalizing the discussion for $e = 1$ in [GR03] Section 7.

In this section only, we make the convention that $M^\wedge$ is the classical p-adic completion of M, where M is a $\mathbb{Z}_p$-module.

Definition 2.1.0.1. *Let R be an $\mathrm{A}_{\mathrm{inf,e}}$-algebra.*

(i) We call R is topologically finite type over $\mathrm{A}_{\mathrm{inf,e}}$ *if there exists a surjection of $\mathrm{A}_{\mathrm{inf,e}}$-algebras $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle \to R$ for some $m \in \mathbb{N}$.*

(ii) We call R is topologically of finite presentation over $\mathrm{A}_{\mathrm{inf,e}}$ *if R admits a surjection from $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle \to R$ with kernel being a finitely generated ideal.*

We denote $\mathrm{Alg}_{\mathrm{tfp,e}}$ to be the category of p-adically complete p-torsion free algebras R over $\mathrm{A}_{\mathrm{inf,e}}$ that is of topologically finite presentation.

Similarly, we can extend these notions to the relative situation, replacing $\mathrm{A}_{\mathrm{inf,e}}$ by any $\mathrm{A}_{\mathrm{inf,e}}$-algebra.

Here we list some basic properties about modules over a given $R \in \mathrm{Alg}_{\mathrm{tfp},e}$.

Lemma 2.1.0.2 (cf. [GR03], 7.1.1)**.** *Let R be an algebra in $\mathrm{Alg}_{\mathrm{tfp},e}$. Then we have*

(i) Every finitely generated p-torsion free R-module is finitely presented.

(ii) The ring R is coherent.

(iii) Let N be a finitely generated R-module, $N' \subset N$ a submodule. Then there exists an integer $c \geq 0$, such that

$$p^k N \cap N' \subset p^{k-c} N'$$

for every $k \geq c$. In particular, the subspace topology on N' induced from the p-adic topology on N agrees with the p-adic topology of N'.

(iv) Every finitely generated R-module M is p-adically complete and separated; namely every such M is isomorphic to its p-adic completion $M^\wedge$.

(v) Every submodule of a finite type free R-module F is closed for the p-adic topology of F.

Proof. (i) This is proved in the proof of [BMS18], 13.4. (iii.b); for completeness, we record it here. We do this by induction, and note that for $n = 1$ the case is given in [BL93] 1.2.

Let $\overline{M}$ be the image of M in $M/\xi[\frac{1}{p}]$, and let N be the kernel of $M \to \overline{M}$. The image $\overline{M}$ is finitely generated p-torsion free R/ξ-module, which by induction is a finitely presented R/ξ-module. Note that this also implies the R/ξ-module $\overline{M}$ is a finitely presented R-module. So by [Sta18], Tag 0519, N is finitely generated over R, and to show the finite presented-ness of M it suffices to show the finite presentedness of N. But note that for $x \in N$, there exists some $k \in \mathbb{N}$ such that $p^k x \in \xi M$. This implies that $p^k \xi^{n-1} x = 0$ in M as the element is contained in $\xi^n M = 0$, and by the p-torsion freeness of M we have $\xi^{n-1} x = 0$. So N is a finitely generated p-torsion free R/ξ^{n-1}-module, and by induction we get the result.

(ii) By definition, a ring R is coherent if every finitely generated ideal of R is finitely presented. So by the p-torsion freeness of R and (i), we get the result.

(iii) Let M be the kernel of the map $N \to N/N'[\frac{1}{p}]$. Namely we have the following short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow N/N'[\frac{1}{p}].$$

Then since the image of N in $N/N'[\frac{1}{p}]$ is finitely generated and p-torsion free, by (i) we know the image is finitely presented, and thus M is finitely generated ([Sta18], Tag 0519). Note that the quotient M/N' is p^∞-torsion, so by the finitely generatedness there exists some $c \in \mathbb{N}$ such that $p^c M \subset N'$. Besides, for $x \in N$ such that $p^k x \in M$, the image of x in $N/N'[\frac{1}{p}]$ is also zero. So the definition of M implies that $x \in M$ and $p^k x \in p^k M$. In this way, for $k \geq c$, we have

$$\begin{aligned} p^k N \cap N' &\subset p^k N \cap M \\ &\subset p^k M \\ &\subset p^{k-c} N'. \end{aligned}$$

(iv) We can fit M into the following short exact sequence of R-modules,

$$0 \longrightarrow N \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0,$$

We apply the p-adic completion to the sequence. Then note that since the subspace topology on N is the isomorphic to the p-adic topology by (iii), while the quotient topology on M is the same as the p-adic topology, by [Mat86] Theorem 8.1, we get an exact sequence of

p-adically complete R-modules with continuous maps

$$0 \longrightarrow N^\wedge \longrightarrow R^{\oplus n} \longrightarrow M^\wedge \longrightarrow 0.$$

Compare with the above two exact sequences, we see the natural map $N \to N^\wedge$ is injective while $M \to M^\wedge$ is surjective.

We then assume the R-module M is finitely presented. By the [Sta18] Tag 0519, we know N is finitely generated. In this way, since the surjection of $M \to M^\wedge$ is true for any finitely generated R-module, we see $N \to N^\wedge$ is an isomorphism. In particular, we get $M \cong M^\wedge$. This finishes the (iv) for M being finitely presented over R.

In general, let M be any finitely generated module over R. Take $\overline{M}$ to be the image of M in $M[\frac{1}{p}]$. Then since $\overline{M}$ is finitely generated and p-torsion free, we know $\overline{M}$ is finitely presented and hence the kernel $N = \ker(M \to \overline{M})$ is finitely generated by loc. cit. Notice that by definition N is p^∞-torsion. So there exists some $m \in \mathbb{N}$ such that $p^m N = 0$. Now by the p-torsion freeness of $\overline{M}$, the base change of the exact sequence $0 \to N \to M \to \overline{M} \to 0$ along $R \to R/p^s$ is exact. Moreover, since the inverse system $\{N \otimes_R R/p^s\}_s$ is essentially constant, the inverse limit of the short exact sequence of inverse system is exact, and we get

$$0 \longrightarrow N^\wedge = N \longrightarrow M^\wedge \longrightarrow \overline{M}^\wedge \longrightarrow 0,$$

which by the isomorphism $\overline{M} \cong \overline{M}^\wedge$ we get the result

$$M \cong M^\wedge.$$

So we are done.

(v) Let N be a submodule of a finite free R-module F, and let $M := F/N$ be the quotient. By (iv), since M is finitely generated, we have the canonical isomorphism $M \cong M^\wedge$. As in the proof of (iv), the p-adic completion induces the following short exact sequence

$$0 \longrightarrow N^\wedge \longrightarrow F \longrightarrow M^\wedge \cong M \longrightarrow 0.$$

Hence we get the isomorphism $N \cong N^\wedge$. In particular, since N is complete and its p-adic topology is isomorphic to its subspace topology, we get the closedness of N in F by standard topological argument.

□

Corollary 2.1.0.3. *Let R be a topologically finite type algebra over* $\mathrm{A}_{\mathrm{inf,e}}$.

(i) The ring R is p-adically complete and separated.

(ii) The ring R is topologically finitely presented over $\mathrm{A}_{\mathrm{inf,e}}$ if it is p-torsion free.

(iii) Assume R is in $\mathrm{Alg}_{\mathrm{tfp},e}$, and I is an ideal of R. Then I is finitely presented over R if R/I is p-torsion free.

Proof. (i) Note that R is the quotient of $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle$ for some m, with the latter being in $\mathrm{Alg}_{\mathrm{tfp},e}$. In particular, R is a finitely generated $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle$-module. So the result follows from Lemma 2.1.0.2 (iv).

(ii) By (i), we know R is p-adically complete and p-torsion free. So it suffice to check that for a surjection $\mathrm{A}_{\mathrm{inf,e}}\langle T_1, \ldots, T_m\rangle \to R$, the kernel is finitely generated. This then follows from Lemma 2.1.0.2 (i), since R is a finitely generated $\mathrm{A}_{\mathrm{inf,e}}\langle T_i\rangle$ module that is p-torsion free.

(iii) This follows again from Lemma 2.1.0.2 (i) for the R-module R/I, and the p-torsion free assumption of R.

□

Lemma 2.1.0.4. *Let R be in $\mathrm{Alg}_{\mathrm{tfp},e}$, and F be a flat R-module*

(i) The functor $M \mapsto (M \otimes_R F)^\wedge$ is exact on the category of finitely presented R-modules.

(ii) Given a finitely presented R-module M, the following canonical map is an isomorphism

$$M \otimes_R F^\wedge \longrightarrow (M \otimes_R F)^\wedge.$$

(iii) The R-module $F^\wedge$ is flat over R, and is p-torsion free.

Proof. (i) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely presented R-modules. By assumption, the tensor product with F over R is exact, so it suffices to show that the p-adic completion is flat on $0 \to M' \otimes F \to M \otimes F \to M'' \otimes F \to 0$. By Lemma 2.1.0.2 (iii), there exists an integer $c \geq 0$ such that $p^k M \cap M' \subset p^{k-c} M'$. Applying this inclusion with the tensor product functor $- \otimes_R F$, and notice that the flatness of F implies $(p^k M \cap M') \otimes F = (p^k M \otimes F) \cap (M' \otimes F)$, we see the p-adic topology on $M' \otimes F$ is isomorphic to the subspace topology induced from $M \otimes F$. In particular, by [Mat86] Theorem 8.1, we get the exactness

$$0 \longrightarrow (M' \otimes F)^\wedge \longrightarrow (M \otimes F)^\wedge \longrightarrow (M'' \otimes F)^\wedge \longrightarrow 0.$$

(ii) Assume M has the following presentation

$$R^{\oplus n} \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0.$$

The tensor product of this with F gives

$$F^{\oplus n} \longrightarrow F^{\oplus m} \longrightarrow M \otimes F \longrightarrow 0.$$

We then take the p-adic completion. By (i), we get an exact sequence,

$$(F^\wedge)^{\oplus n} \longrightarrow (F^\wedge)^{\oplus m} \longrightarrow (M \otimes F)^\wedge \longrightarrow 0. \tag{1}$$

On the other hand, we replace F by $F^\wedge$ in the second exact sequence above, and get

$$(F^\wedge)^{\oplus n} \longrightarrow (F^\wedge)^{\oplus m} \longrightarrow M \otimes F^\wedge \longrightarrow 0. \tag{2}$$

The canonical map from (2) to (1) are identities on $F^{\wedge,\oplus n}$ and $F^{\wedge,\oplus m}$. Thus we get the isomorphism.

(iii) It suffices to show that for any injective map of finitely presented modules $M' \to M$, the tensor product with $F^\wedge$ is still injective. This then follows from (ii) and (i).

□

Corollary 2.1.0.5. *Let $f : A \to B$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$. Then the kernel of any surjective A-homomorphism $\rho : A\langle T_i \rangle \to B$ is finitely generated over A. In particular, B is a topologically finitely presented A-algebra.*

Proof. By assumption, we can write A as $\mathrm{A}_{\mathrm{inf,e}}\langle U_j \rangle / I$ for some finitely presented ideal I. Then the surjection ρ can be rewritten as $\mathrm{A}_{\mathrm{inf,e}}\langle U_j, T_i \rangle / I \to B$. By the Corollary 2.1.0.3 (iii), since B is p-torsion free, it suffices to show that the ring $\mathrm{A}_{\mathrm{inf,e}}\langle U_j, T_i \rangle / I$ is in $\mathrm{Alg}_{\mathrm{tfp},e}$; namely it is topologically finite type over $\mathrm{A}_{\mathrm{inf,e}}$ and is p-torsion free, by Corollary 2.1.0.3 (ii). To finish this, we only need to notice that the ring $\mathrm{A}_{\mathrm{inf,e}}\langle U_j \rangle / I = A\langle T_i \rangle$ is the p-adic completion of the A-module $A[T_i]$, while the latter is flat over A. So the result follows from Lemma 2.1.0.4 (iii). □

2.2 Analytic cotangent complex: affinoid case

We then introduce the analytic cotangent complex, for algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$ and affinoid rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$ in this section.

2.2.1 Derived p-adic completion

We recall the basics of derived p-adic completion.

Let R be a $\mathbb{Z}_p$-algebra. For a complex $C = C^\bullet$ of R-modules, recall that the *derived p-adic completion* of C is defined as

$$\mathrm{R}\varprojlim_{m\in\mathbb{N}}(C \otimes^L_R (R \xrightarrow{p^m} R)),$$

as an object in the derived category $\mathcal{D}(R)$ of R-modules. Here the object $(R \xrightarrow{p^m} R)$ is the cone of the map $p^m : R \to R$. An object $C \in \mathcal{D}(R)$ is called *derived p-complete* if C is isomorphic to its derived p-adic completion. The subcategory $\mathcal{D}_p(R)$ of derived p-complete objects is a full subcategory ([Sta18] Tag 091U) of $\mathcal{D}(R)$, and the derived p-adic completion forms a left adjoint functor to the inclusion functor $\mathcal{D}_p(R) \to \mathcal{D}(R)$ ([Sta18] Tag 091V).

There exists a natural isomorphism of complexes of R-modules

$$R \otimes^L_{\mathbb{Z}_p} (\mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p) \cong (R \xrightarrow{p^m} R).$$

From this, the derived functor $C \mapsto C \otimes^L_R (R \xrightarrow{p^m} R)$ in $\mathcal{D}(R)$ can be rewritten as

$$\begin{aligned} C \longmapsto & C \otimes^L_R R \otimes^L_{\mathbb{Z}_p} (\mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p) \\ \cong & C \otimes^L_{\mathbb{Z}_p} (\mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p). \end{aligned}$$

Here we note that since $\mathbb{Z}_p$ is p-torsion free, the complex $\mathbb{Z}_p \xrightarrow{p^m} \mathbb{Z}_p$ is isomorphic to the $\mathbb{Z}_p$-module $\mathbb{Z}_p/p^m[0]$ living at the degree 0. In the case when C is a p-torsion free R-module, by the flatness of C over $\mathbb{Z}_p$, its derived p-adic completion is exactly its classical p-adic completion $\varprojlim_m C/p^m$. This in fact holds true in full generality for complexes as follows.

Lemma 2.2.1.1. *Let C be a cochain complex of p-torsion free $\mathbb{Z}_p$-modules. Then the derived p-completion of C can be represented by the actual complex $\widetilde{C}$, which is obtained by the term-wise classical p-completion of C.*

Proof. We first notice that $\widetilde{C}$ is derived p-complete, as the derived p-completeness can be checked by cohomology ([Sta18, Tag 091N]) and each $\mathrm{H}^i(\widetilde{C})$ is derived p-complete.

When C is bounded to the right, the derived tensor product $C \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$ is represented by the actual complex C/p^m, obtained via term-wise quotient by p^m. In this case, the claim follows from via [Sta18, Tag 09AU] as $\widetilde{C} \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m = C/p^m$.

In general, consider the naive truncation

$$\sigma^{>n}C \longrightarrow C \longrightarrow \sigma^{\leq n}C.$$

By the cohomological finiteness of the functor $-\otimes^L_{\mathbb{Z}_p}\mathbb{Z}/p^m$, we have

$$C\otimes^L_{\mathbb{Z}_p}\mathbb{Z}/p^m \cong R\varprojlim_n\left((\sigma^{\leq n}C)\otimes^L_{\mathbb{Z}_p}\mathbb{Z}/p^m\right).$$

Hence we have

$$\begin{aligned}
R\varprojlim_m C\otimes^L_{\mathbb{Z}_p}\mathbb{Z}/p^m &\cong R\varprojlim_m R\varprojlim_n\left((\sigma^{\leq n}C)\otimes^L_{\mathbb{Z}_p}\mathbb{Z}/p^m\right)\\
&\cong R\varprojlim_m R\varprojlim_n\left((\sigma^{\leq n}C)/p^m\right)\\
&\cong R\varprojlim_n R\varprojlim_m\left((\sigma^{\leq n}C)/p^m\right)\\
&\cong R\varprojlim_n \sigma^{\leq n}\widetilde{C}\\
&= \widetilde{C}.
\end{aligned}$$

□

2.2.2 Analytic cotangent complex for affine formal schemes

Now we introduce the definition and the basic properties of analytic cotangent complexes, for a map of algebras over $\mathrm{A}_{\mathrm{inf,e}}$. The analogous discussion for topologically finite type algebras over K can be found in [GR03, Section 7.1].

Construction 2.2.2.1. Let $f: A\to B$ ba a map of $\mathrm{A}_{\mathrm{inf,e}}$-algebras in $\mathrm{Alg}_{\mathrm{tfp,e}}$. Namely both A and B are p-adically complete p-torsion free algebras over $\mathrm{A}_{\mathrm{inf,e}}$ that are quotients of $\mathrm{A}_{\mathrm{inf,e}}\langle T_1,\ldots,T_m\rangle$ for some $m\in\mathbb{N}$. As an A-algebra, the ring B admits a standard simplicial resolution

$$P_\bullet\to B,$$

where each P_i is a polynomial over A ([Sta18] Tag 08PM). This allows us to give a simplicial $P_\bullet$-modules $\Omega^1_{P_\bullet/A}$, where each $\Omega^1_{P_i/A}$ is the algebraic differential of P_i over A. Recall that the *algebraic cotangent complex* $\mathbb{L}_{B/A}$ is the image of the cochain complex $\Omega^1_{P_\bullet/A}\otimes_{P_\bullet}B$ in the derived category over A. The *analytic cotangent complexes* $\mathbb{L}^{\mathrm{an}}_{B/A}$ *for the* $\mathrm{A}_{\mathrm{inf,e}}$*-algebras* $B\to A$ is then defined as the image of the derived p-adic completion of the $\Omega^1_{P_\bullet/A}\otimes_{P_\bullet}B$, in the derived category of A-modules.

Remark 2.2.2.2. As the polynomial resolution is functorial with respect to the pair (A, B), by Lemma 2.2.1.1 the analytic cotangent complex $\mathbb{L}_{B/A}^{\mathrm{an}}$ can be represented functorially by the actual complex of B-modules, produced by the term-wise p-adic completion of $\Omega^1_{P_\bullet/A} \otimes_{P_\bullet} B$.

There exists a canonical map from the algebraic cotangent complex $\mathbb{L}_{B/A}$ to the analytic cotangent complex $\mathbb{L}_{B/A}^{\mathrm{an}}$. This is given by the counit map of the adjoint pair for the derived p-completion and the inclusion functor $\mathcal{D}_p(A) \to \mathcal{D}(A)$.

Here are some useful results for the analytic cotangent complex of $\mathrm{A}_{\mathrm{inf,e}}$-algebras.

Proposition 2.2.2.3. *Let $f : A \to B$ be a map of $\mathrm{A}_{\mathrm{inf,e}}$-algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$. Assume f is formally smooth. Then we have a canonical isomorphism*

$$\mathbb{L}_{B/A}^{\mathrm{an}} \longrightarrow \Omega_{B/A}^{1,\mathrm{an}}[0],$$

where the right side is the module of (p-adic) continuous differential forms.

Proof. Since f is formally smooth, by Elkik's algebraization result of formally smooth adic algebras, B is isomorphic to the p-adic completion of a smooth A-algebra. In particular, f is flat (Lemma 2.1.0.4) and $f_n := A/p^n \to B/p^n$ is smooth. So by the derived base change formula for the algebraic cotangent complex ([Sta18], Tag 08QQ), since $B/p^n = B \otimes_A^L A/p^n$, we have

$$\mathbb{L}_{B/A} \otimes_A^L A/p^n = \mathbb{L}_{(B/p^n)/(A/p^n)}.$$

Moreover, the smoothness of f_n gives a canonical isomorphism

$$\mathbb{L}_{(B/p^n)/(A/p^n)} \cong \Omega^1_{(B/p^n)/(A/p^n)}[0] = \Omega^1_{B/A} \otimes_A A/p^n[0].$$

In this way, by taking the derived p-adic completion of $\mathbb{L}_{B/A}$ and notice the p-torsion freeness of B and A, we get

$$\begin{aligned}
\mathbb{L}_{B/A}^{\mathrm{an}} =& \mathrm{R}\varprojlim_{n\in\mathbb{N}} \mathbb{L}_{B/A} \otimes_A^L A/p^n \\
\cong& \mathrm{R}\varprojlim_{n\in\mathbb{N}} \Omega^1_{B/A}/p^n[0] \\
=& \Omega_{B/A}^{1,\mathrm{an}}[0].
\end{aligned}$$

□

In the next result, we show that the analytic cotangent complex for a finite morphism coincides with the associated algebraic cotangent complex. Recall that for an object $L^\bullet$ in the derived category of R-modules, it is called *pseudo-coherent* if it is isomorphic to a upper-bounded complex of finite free R-modules.

Proposition 2.2.2.4. *Let $A \to B$ be a map of two topologically finitely presented $\mathrm{A}_{\mathrm{inf,e}}$-algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$, such that B is a finitely presented A-module. Then the algebraic cotangent complex $\mathbb{L}_{B/A}$ is pseudo-coherent. In particular, $\mathbb{L}_{B/A}$ is derived p-complete and we have a canonical isomorphism*

$$\mathbb{L}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A}.$$

Proof. We first show that it suffices to assume $A \to B$ is a surjection. To see this, we first pick a polynomial algebra $A[x_1, \ldots, x_r]$ that maps surjectively onto B. By the finite presentedness assumption of B over A, each x_i satisfies a monic polynomial $f_i(x_i)$ of x_i in A, and the induced map $B' = A[x_1, \ldots, x_r]/(f_1, \ldots, f_r) \to B$ is also surjective. Here we note that the ring B', as a finite algebra over A that is p-torsion free, is automatically p-complete and is also in $\mathrm{Alg}_{\mathrm{tfp,e}}$. Moreover, notice that since the sequence $\{f_1, \ldots, f_r\}$ is a regular sequence in $A[x_i]$, by the distinguished triangle of algebraic cotangent complexes for $A \to B' \to B$ we get

$$B^{\oplus r}[1] \cong \mathbb{L}_{B'/A} \otimes^L_{B'} B \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/B'}.$$

Thus to show the pseudo-coherence of $\mathbb{L}_{B/A}$, it suffices to show this for $\mathbb{L}_{B/B'}$, where $B' \to B$ is a surjective map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$.

Recall that by assumption B is a finite A-module that is p-torsion free. So Lemma 2.1.0.2 implies that B is a finitely presented A-module and there exists an exact sequence of A-modules as below

$$A^{\oplus m} \xrightarrow{f} A^{\oplus r} \longrightarrow B \longrightarrow 0.$$

Moreover, as the image of f is a submodule of $A^{\oplus r}$, which by Lemma 2.1.0.2 (i) is finitely presented, we know $\ker(f)$ is also finitely generated (hence finitely presented as it is inside of $A^{\oplus m}$). This procedure allows to give a finite free A-module resolution of B. In particular, this shows that B is pseudo-coherent over A.

We then take P to be a simplicial polynomial resolution of B over A, and let J be the kernel of the map $P \otimes_A B \to B$. Then by the finite presentedness of B over A, the simplicial A-algebra P is also pseudo-coherent over A. So by taking a base change along $A \to B$, we see $P \otimes_A B$ is a simplicial B-algebra that is pseudo-coherent over B. Moreover, since the map $P \otimes_A B \to B$ has a natural section, the kernel J is also pseudo-coherent ([Sta18] Tag 064X). Notice that the cotangent complex $\mathbb{L}_{B/A}$ fits into the distinguished triangle

$$J \longrightarrow \mathbb{L}_{B/A} \longrightarrow J^2[1].$$

To show the pseudo-coherence of $\mathbb{L}_{B/A}$, by [Sta18] Tag 064U it suffices to show by decreasing induction that $\mathbb{L}_{B/A}$ is n-pseudo-coherent for each $n \leq 1$. When $n = 1$, the result is clear as $\mathbb{L}_{B/A}$

has cohomological degree ≤ 0. Suppose the result is true for $n \leq 0$. Since $A \to B$ is a surjection, the induced surjective map $P \otimes_A B \to B$ is isomorphic on π_0 with kernel living in cohomological degree ≤ -1. Thus by [Ill71] Chap III, 3.3, we have

$$\mathrm{H}^m(J^i) = 0,\ for\ i > -m.$$

This implies that when $i > -n$, J^i is n-pseudo-coherent. On the other hand, by [Ill71] Chap III, 3.3.2, there exists an isomorphism.

$$J^j/J^{j+1} \longrightarrow L\mathrm{Sym}^j_B(\mathbb{L}_{B/A}),\ j \geq 0.$$

The derived symmetric product preserves the n-pseudo-coherence ([GR03] 7.1.18), so J^j/J^{j+1} is n-pseudo-coherent for any $j \geq 0$. Thus the fiber sequence for the quotient $J^i \to J^i/J^{i+1}$ allows us to deduce the n-pseudo-coherent of every J^i. In particular, when $i = 2$, by taking the cohomological twist we know $J^2[1]$ is $(n-1)$-pseudo-coherent. So combining with the quasi-coherence of J, we get the $(n-1)$-quasi-coherence of $I/I^2 = \mathbb{L}_{B/A}$ by [Sta18] Tag 064V.

□

Corollary 2.2.2.5. *Let A be a topologically finitely presented $\mathrm{A}_{\mathrm{inf,e}}$-algebra, and I be a finitely generated regular ideal in A such that $B := A/I$ is p-torsion free. Then we have a canonical isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow I/I^2[1].$$

Proof. Since B is p-torsion free, by the Corollary 2.1.0.3 we know B is in $\mathrm{Alg}_{\mathrm{tfp},e}$. So the result follows from Proposition 2.2.2.4 and the case for algebraic cotangent complex. □

Here is another useful result about the distinguished triangles for triples:

Proposition 2.2.2.6. *Let $A \to B \to C$ be maps of topologically finitely presented $\mathrm{A}_{\mathrm{inf,e}}$ algebras. Then we have*

(i) The analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ is a bounded above pseudo-coherent object in the derived category of B-modules.

(ii) there exists a natural distinguished triangle of pseudo-coherent bounded above objects in the derived category of C-modules

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B C \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{C/B}.$$

Before the proof, we make the following claim:

Lemma 2.2.2.7. *Let $A \to B$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$. Let K be a bounded above complex of A-modules, and K' be its derived p-completion. Then the derived p-completion of $K \otimes_A^L B$ is isomorphic to the derived p-completion of $K' \otimes_A^L B$.*

Proof of the Claim. It suffices to check by the derived tensor product $- \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$, which is then clear as $K \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p \cong K' \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^m$. □

Proof. (i) By the Corollary 2.1.0.5, we may write B as the quotient P/I, where $P = A\langle T_i \rangle$ is a convergent power series ring over A, and I is a finitely generated ideal by the Corollary 2.1.0.3 (iii). We take the distinguished triangle of algebraic cotangent complexes for $A \to P \to B$, and get

$$\mathbb{L}_{P/A} \otimes^L_P B \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/P}.$$

Note that $\mathbb{L}^{\mathrm{an}}_{P/A}$ is isomorphic to the finite free P-module $\Omega^{1,\mathrm{an}}_{P/A}$ (Proposition 2.2.2.3), so after applying the derived p-completion and use the lemma above, we get a distinguished triangle

$$\Omega^{1,\mathrm{an}}_{P/A} \otimes_P B[0] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/P}.$$

Here the $\mathbb{L}^{\mathrm{an}}_{B/P}$ is pseudo-coherent by Proposition 2.2.2.4. Thus we are done.

(ii) For (ii), take the distinguished triangle for algebraic cotangent complexes, we get

$$\mathbb{L}_{B/A} \otimes^L_B C \longrightarrow \mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{C/B}.$$

So the result follows from the lemma above and the pseudo-coherence of each analytic cotangent complex.

□

2.2.3 Analytic cotangent complex for affinoid rigid spaces

We then introduce the basics of the analytic cotangent complex for a map of affinoid algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, using the integral construction given in the last subsection. The analogous discussion for topologically finite type algebras over K can be found in [GR03, Section 7.2].

Construction 2.2.3.1. Let $f : A \to B$ be a map of topologically finite type affinoid algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Namely both A and B are quotients of $K\langle T_1, \ldots, T_m \rangle$ for some $m \in \mathbb{N}$. Denote by $\mathcal{C}_{B/A}$ to be the category of pairs of rings (B_0, A_0), where A_0 and B_0 are rings of definition of A and B separately, such that both of them are in $\mathrm{Alg}_{\mathrm{tfp},e}$, and $f(A_0) \subset B_0$. The morphism among pairs is defined by inclusion maps on each entry separately.

Assume (B_0, A_0) is an object of $\mathcal{C}_{B/A}$. By the construction of the last subsection, we can construct the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$ for B_0/A_0, as the derived p-completion of the algebraic cotangent complex $\mathbb{L}_{B_0/A_0}$.

Definition 2.2.3.2. *The* analytic cotangent complexes for affinoid algebras B/A *is defined as the colimit*

$$\mathbb{L}^{\mathrm{an}}_{B/A} := \operatorname*{colim}_{(B_0,A_0)\in\mathcal{C}_{B/A}} (\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]),$$

as an object in the derived category of B-modules.

Remark 2.2.3.3. As there exists a canonical actual complex representing $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$ (by the term-wise p-adic completion of $\Omega^1_{P/A_0} \otimes_P B$, where P is the standard polynomial resolution of B_0 over A_0), the analytic cotangent complexes can also be represented by a canonical actual complex, defined by taking the colimit of the actual term-wise complete complexes and then invert by p.

Here we note that as there exists a canonical map from $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$ to the algebraic cotangent complex $\mathbb{L}_{B_0/A_0}$ induced by the adjoint pair for derived completion, by inverting p we also have a canonical map from the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ to the algebraic cotangent complex $\mathbb{L}_{B/A}$ for $A \to B$.

We then give a simple description of the analytic cotangent complex for a smooth morphism.

Proposition 2.2.3.4. *Let $A_0 \to B_0$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$, and let $A = A_0[\frac{1}{p}]$ and $B = B_0[\frac{1}{p}]$ be their generic fibers separately, with the induced map $f : A \to B$. Assume the corresponding map of affinoid rigid spaces* $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$ *is smooth. Then we have a natural isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}] \longrightarrow (\Omega^{1,\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]).$$

Proof. By the Corollary 2.1.0.5, B_0 is a topologically finitely presented A_0-algebra. So we can write B_0 as the quotient ring of the relative convergent power series ring $P_0 = A_0\langle T_1, \ldots, T_m\rangle$, by some finitely generated ideal $I_0 \subset P_0$. Denote by P and I to be the ring $P_0[\frac{1}{p}]$ and the ideal $I_0[\frac{1}{p}]$ separately. Then the surjection $P \to B$ induces a closed immersion of $\mathrm{Spa}(B)$ into the m-dimensional unit disc $\mathrm{Spa}(P)$ over $\mathrm{Spa}(A)$. Since both $\mathrm{Spa}(B)$ and $\mathrm{Spa}(P)$ are smooth over $\mathrm{Spa}(A)$, by the Jacobian criterion for the smoothness of adic spaces ([Hub96], 1.6.9), for each maximal ideal $\mathfrak{P}$ of $\mathrm{Spa}(P)$ that contains I, we can always find generators $s_1, \ldots, s_l$ of $I_{\mathfrak{P}}$ such that their derivatives $ds_1, \ldots ds_l$ can be extended to a basis of the continuous differential $\Omega^1_{P/A,\mathfrak{P}}$ at $\mathfrak{P}$. We denote by $\mathfrak{p}$ to be the intersection $\mathfrak{P} \cap A$. Then the above implies that the image of s_i in $P_{\mathfrak{P}} \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ forms a regular sequence. So by the flatness of $P_{\mathfrak{P}}$ over $A_{\mathfrak{p}}$ and the Proposition 15.1.16 in [Gro67] Chap 0, s_i forms a regular sequence in $A_{\mathfrak{P}}$. Since this is true for every maximal ideal $\mathfrak{P}$ of P containing I, we see B is a local complete intersection of P.

Now thanks to the surjectivity of $P_0 \to B_0$, Proposition 2.2.2.4 implies that

$$\mathbb{L}^{\mathrm{an}}_{B_0/P_0} \cong \mathbb{L}_{B_0/P_0}.$$

Moreover, by the flat base change there exits a canonical isomorphism of algebraic cotangent complexes

$$\mathbb{L}_{B_0/P_0}[\frac{1}{p}] \cong \mathbb{L}_{B/P},$$

which by the local complete intersection of $P \to B$, is isomorphic to $I/I^2[1]$. On the other hand, by Proposition 2.2.2.6 and Proposition 2.2.2.3, we have a natural distinguished triangle

$$(\Omega^{1,\mathrm{an}}_{P_0/A_0} \otimes_{P_0} B_0[\frac{1}{p}]) \longrightarrow \mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B_0/P_0}[\frac{1}{p}].$$

Replace the right side by the ideal $I/I^2[1]$ in degree 1, we get an isomorphism

$$\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}] \cong (I/I^2 \longrightarrow \Omega^{1,\mathrm{an}}_{P_0/A_0})[1],$$

Note that I/I^2 is the generic fiber of the B_0-module I_0/I_0^2, so the right side is exactly equals to

$$(\Omega^{1,\mathrm{an}}_{B_0/A_0}[\frac{1}{p}])[0].$$

So we are done. $\square$

As a quite useful upshot, to compute the analytic cotangent complex for affinoid rings, it suffices to use one single pair of rings of definition.

Proposition 2.2.3.5. *Let $A_0 \to B_0$ be a map of algebras in $\mathrm{Alg}_{\mathrm{tfp},e}$, and let $A = A_0[\frac{1}{p}]$ and $B = B_0[\frac{1}{p}]$ be their generic fibers separately, with the induced map $f : A \to B$. Then the map below is an isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A}.$$

Proof. It suffices to show that for any commutative diagram of topologically finitely presented rings of definition

$$\begin{tikzcd} A_0' \arrow[r] & B_0' \\ A_0 \arrow[u] \arrow[r] & B_0 \arrow[u] \end{tikzcd},$$

the induced morphism $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}] \to \mathbb{L}^{\mathrm{an}}_{B_0'/A_0'}[\frac{1}{p}]$ is an isomorphism. Moreover, using the distinguished triangles for $A_0 \to A_0' \to B_0'$ and $A_0 \to B_0 \to B_0'$ separately, we can reduce to show that

$$\mathbb{L}^{\mathrm{an}}_{A_0'/A_0}[\frac{1}{p}] = 0.$$

Then we notice that since $A_0 \to A_0'$ is an isomorphism after inverting by p, this satisfies the assumption of Proposition 2.2.3.4. So it suffices to prove that $\Omega^{1,\mathrm{an}}_{A_0'/A_0}[\frac{1}{p}] = 0$. By the Corollary 2.1.0.5, A_0' is a topologically finitely presented algebra over A_0. We pick a set of generators x_i of A_0' over A_0. Then by $A_0'[\frac{1}{p}] = A_0[\frac{1}{p}]$, there exists a positive integer N such that $p^N x_i \in A_0$. Note that the continuous differential forms $\Omega^{1,\mathrm{an}}_{A_0'/A_0}$ is generated by the dx_i. So we get

$$p^N dx_i = d(p^N x_i) = 0.$$

This implies that $\Omega^{1,\mathrm{an}}_{A_0'/A_0}$ is p^∞-torsion. In particular, we have

$$\Omega^{1,\mathrm{an}}_{A_0'/A_0}[\frac{1}{p}] = 0.$$

So we are done.

□

Here are some applications of the above result.

Corollary 2.2.3.6. *Let $f : A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$ is smooth. Then the projection onto the zero-th homotopy group induces natural isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow \Omega^1_{B/A}[0],$$

where the right side is the modules of continuous differential forms.

Proof. This follows from Proposition 2.2.3.5 and Proposition 2.2.3.4. □

Similar to the integral case, when B is a quotient ring of A, the analytic cotangent complex coincides with the algebraic cotangent complex.

Corollary 2.2.3.7. *Let $f : A \to B$ ba a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$ such that B is a finite A-module. Then the natural map from the analytic cotangent complex to the algebraic cotangent complex below is an isomorphism*

$$\mathbb{L}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A}.$$

Proof. Pick a ring of definition A_0 of A that is topologically finitely presented over $\mathrm{A}_{\mathrm{inf,e}}$. We first notice that under the assumption of $A \to B$, we can find a ring of definition B_0 of B that contains

$f(A_0)$ and is finite over A_0. To find such B_0, we pick s set of A-module generators x_i of B over A. Then each x_i satisfies a monic polynomial $f_i(X) = \sum_{j=0}^{r_i} a_{ij} X^j$ with coefficients in A. Since $A = A_0[\frac{1}{p}]$, we can pick a common integer $N \in \mathbb{N}$, such that coefficients $p^N a_{ij}$ are inside of A_0 for each i and j. From this, we see the element $p^N x_i$ satisfies a monic polynomial with coefficient in A_0. In other words, the subring $B_0 = f(A_0)[p^N x_i]$ of B is finite over A_0.

Now the corollary follows easily from Proposition 2.2.3.4 and Proposition 2.2.2.4, since $\mathbb{L}^{\mathrm{an}}_{B/A}$ is isomorphic to $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]$, while the latter is computed by inverting p at the algebraic cotangent complex $\mathbb{L}_{B_0/A_0}$. Notice that thanks to the flat base change of the algebraic cotangent complex, $\mathbb{L}_{B_0/A_0}[\frac{1}{p}]$ is exactly the algebraic cotangent complex of B over A. So we get the result.

□

As expected, we have the following simple description of the analytic cotangent complex for regular immersion.

Corollary 2.2.3.8. *Let $f : A \to B$ ba a surjective map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that the kernel I is a regular ideal in A. Then we have a natural isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow I/I^2[1].$$

Another quick upshot is the pseudo-coherence of the analytic cotangent complex.

Corollary 2.2.3.9. *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ is a pseudo-coherent complex of bounded above B-modules.*

Proof. This follows from Proposition 2.2.3.5 and the integral version of the result Proposition 2.2.2.6. □

We also obtain the distinguished triangle for triples as follow.

Corollary 2.2.3.10. *Let $A \to B \to C$ be maps of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a distinguished triangle of analytic cotangent complexes of affinoid algebras*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B C \longrightarrow \mathbb{L}^{\mathrm{an}}_{C/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{C/B}.$$

Proof. Let $A_0 \to B_0 \to C_0$ be an arbitrary choice of rings of definition of $A \to B \to C$ that are topologically finitely presented over $\mathrm{A}_{\mathrm{inf,e}}$. By Proposition 2.2.2.6, we have a distinguished triangle

$$\mathbb{L}^{\mathrm{an}}_{B_0/A_0} \otimes^L_{B_0} C_0 \longrightarrow \mathbb{L}^{\mathrm{an}}_{C_0/A_0} \longrightarrow \mathbb{L}^{\mathrm{an}}_{C_0/B_0}.$$

Note that for the first term above we have the equality

$$
\begin{aligned}
(\mathbb{L}^{\mathrm{an}}_{B_0/A_0} \otimes_{B_0} C_0) \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p &\cong (\mathbb{L}^{\mathrm{an}}_{B_0/A_0} \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p) \otimes^L_{B_0 \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p} (C_0 \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p) \\
&\cong \mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B C.
\end{aligned}
$$

So the corollary follows from Proposition 2.2.3.5 by the base change along $\mathbb{Z}_p \to \mathbb{Q}_p$. $\square$

A simple upshot is the following change of base equality.

Corollary 2.2.3.11. *Let $A \to A' \to B$ be maps of topologically finite algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, with $A \to A'$ being étale. Then the natural map below is an isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B/A'}.$$

Proof. By the distinguished triangle of the triple in the Corollary 2.2.3.10, it suffices to show that $\mathbb{L}^{\mathrm{an}}_{A'/A}$ vanishes. So this follows from the assumption and the Corollary 2.2.3.6. $\square$

As last, we have the étale base change formula as below.

Corollary 2.2.3.12. *Let $A \to B \to B'$ be maps of topologically finite algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, with B'/B being étale. Then we have the following natural isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B' \longrightarrow \mathbb{L}^{\mathrm{an}}_{B'/A}.$$

In particular, when there exists an étale morphism $A \to A'$ such that $B' = A' \otimes_A B$, we get the base change formula

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_A A' \cong \mathbb{L}^{\mathrm{an}}_{B'/A'}.$$

Proof. The first isomorphism follows from the distinguished triangle (Corollary 2.2.3.10) for $A \to B \to B'$, and the étaleness of B'/B (Corollary 2.2.3.6). The second isomorphism follows from the Corollary 2.2.3.11 and the equality

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B' = \mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B \otimes_A A'.$$

$\square$

2.3 Derived de Rham complex: affinoid case

In this section, we construct the derived de Rham cohomology for topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$.

As a preparatory step, we first construct the rational analytic derived de Rham complex for a map $A_0 \to B_0$ of $\mathrm{A}_{\mathrm{inf}\,e}$-algebras, which is a filtered complete complex over $A_0[\frac{1}{p}]$, namely an object in $\widehat{\mathrm{DF}}(A_0[\frac{1}{p}])$.

Construction 2.3.0.1. Let A_0 be a $\mathrm{A}_{\mathrm{inf,e}}$-algebra in $\mathrm{Alg}_{\mathrm{tfp},e}$. We want to build a functor F : ${\mathrm{Alg}_{\mathrm{tfp},e}}_{A_0/} \to \widehat{\mathrm{DF}}(\mathrm{B}^+_{\mathrm{dR,e}})$, sending $A_0 \to B_0$ to a filtered complete derived algebra over $A_0[\frac{1}{p}]$.

Step 1 Let P be the standard polynomial resolution of B_0 over A_0. The de Rham complex $\Omega^\bullet_{P/A_0}$ of P over A_0 is then a simplicial complex of P-modules. Moreover, the (direct sum) totalization $\mathrm{Tot}(\Omega^\bullet_{P/A_0})$ is a cochain complex of A_0-modules that comes with a canonical decreasing filtration, defined by $\mathrm{Fil}^i = \mathrm{Tot}(\Omega^{\geq i}_{P/A_0})$.

Step 2 Now we take the derived p-adic completion of the filtered cochain complex $(\mathrm{Tot}(\Omega^\bullet_{P/A_0}), \mathrm{Fil}^i))$, to get an object $(E, \mathrm{Fil}^i E)$ in the filtered derived category, such that E and $\mathrm{Fil}^i E$ are all derived p-adic complete. Then we invert p at $(E, \mathrm{Fil}^i E)$, to get an object $(E[\frac{1}{p}], \mathrm{Fil}^i E[\frac{1}{p}])$ in the filtered derived category of $A_0[\frac{1}{p}]$-modules.

Step 3 At last, we denote $F(B_0/A_0)$ to be the filtered completion of $(E[\frac{1}{p}], \mathrm{Fil}^i E[\frac{1}{p}])$. Thus we get a functor from maps in $\mathrm{Alg}_{\mathrm{tfp},e}$ to the filtered complete derived category of $\mathrm{B}^+_{\mathrm{dR,e}}$-modules ($A_0[\frac{1}{p}]$-modules), sending $A_0 \to B_0$ to $F(B_0/A_0)$.

Remark 2.3.0.2. From the construction above, it is clear that the i-th graded piece of $F(B_0/A_0)$ is isomorphic to

$$L \wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}])[-i]$$

.

Remark 2.3.0.3. Recall given a complex of $\mathbb{Z}_p$-modules C, it admits a natural map onto its derived p-adic completion $\widetilde{C}$. Apply this to the construction above (Step 2), we see there exists a natural filtered map from the algebraic derived de Rham complex $\widehat{\mathrm{dR}}_{B_0[\frac{1}{p}]/A_0[\frac{1}{p}]}$ to $F(B_0, A_0)$.

Remark 2.3.0.4. From the construction above, the natural map from P to B_0 induces a filtered map from $F(B_0, A_0)$ to the continuous de Rham complex $\Omega^{\bullet,\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]$, which is compatible with the differentials. Here the filtration on the latter is the usual Hodge filtration.

Remark 2.3.0.5. As the de Rham complex is equipped with a structure of commutative differential graded algebra, and the above constructions are all lax-symmetric monoidal, the filtered complex $F(B_0/A_0)$ is also naturally a filtered E_∞-algebra in $\mathrm{B}^+_{\mathrm{dR,e}}$-module.

Now we consider the constructions for affinoid rigid spaces. Let $f : A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Recall that the category $\mathcal{C}_{B/A}$ is defined as pairs of rings (B_0, A_0), where A_0 and B_0 are rings of definition of A and B separately, such that both of them are topologically finitely presented over $\mathrm{A}_{\mathrm{inf,e}}$, and $f(A_0) \subset B_0$. The morphism among pairs is defined by the natural inclusion map of pairs. Here we note that by the Corollary 2.1.0.5, B_0 is a topologically finitely presented algebra over A_0 automatically.

Definition 2.3.0.6. *Let $f : A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, and let $\mathcal{C}_{B/A}$ be the category of pairs of their rings of definitions as above. The* analytic derived de Rham complex of B over A, *denoted by $\mathrm{dR}_{B/A}$, is an object $\widehat{\mathrm{DF}}(\mathrm{B}^+_{\mathrm{dR,e}})$, defined as*

$$\mathrm{dR}_{B/A} := filtered\ completion\ of \operatorname*{colim}_{(B_0,A_0)\in\mathcal{C}_{B/A}} F(B_0/A_0).$$

The filtration of $\mathrm{dR}_{B/A}$ is called the *algebraic Hodge filtration*. If we forget the filtered structure, we get the *underlying complex of* $\mathrm{dR}_{B/A}$. It is denoted as $\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{B/A}$, and is defined as the 0-th filtration of $\mathrm{dR}_{B/A}$, which is the image under the natural projection functor

$$\begin{aligned} \widehat{\mathrm{DF}}(\mathrm{B}^+_{\mathrm{dR,e}}) \subset \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{D}(\mathrm{B}^+_{\mathrm{dR,e}})) &\longrightarrow \mathcal{D}(\mathrm{B}^+_{\mathrm{dR,e}}); \\ C_\bullet &\longmapsto C_0. \end{aligned}$$

Corollary 2.3.0.7. *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then the i-th graded piece of $\mathrm{dR}_{B/A}$ is naturally isomorphic to $L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i]$. In particular, for any pair of rings of definition $(B_0, A_0) \in \mathcal{C}_{B/A}$, the natural map below is a filtered isomorphism*

$$F(B_0, A_0) \longrightarrow \mathrm{dR}_{B/A}.$$

Proof. By the construction above, the algebraic Hodge filtration $\mathrm{Fil}^i\mathrm{dR}_{B/A}$ has the graded factor

$$\mathrm{gr}^i\mathrm{dR}_{B/A} = \operatorname*{colim}_{(B_0,A_0)\in\mathcal{C}_{B/A}} L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}][-i].$$

By Proposition 2.2.3.4 and the assumption on (B_0, A_0), each $L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}]$ is isomorphic to the i-th derived wedge product of the analytic cotangent complex for the affinoid algebras B/A. In particular, the transition maps in the colimit above are all isomorphisms, and we can replace them by one single term. Thus we get

$$\mathrm{gr}^i\mathrm{dR}_{B/A} \cong L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i].$$

As an upshot, since the filtered isomorphism can be checked by the graded pieces, the natural map $F(B_0, A_0) \longrightarrow \mathrm{dR}_{B/A}$ is a filtered isomorphism.

$\square$

Remark 2.3.0.8. By taking the colimit for the natural filtered map

$$\widehat{\mathrm{dR}}_{B_0[\frac{1}{p}]/A_0[\frac{1}{p}]} \longrightarrow F(B_0, A_0),$$

we get a canonical filtered map from the algebraic derived de Rham complex $\widehat{\mathrm{dR}}_{B/A}$ to the analytic derived de Rham complex $\mathrm{dR}_{B/A}$. This is compatible with the canonical map of each graded factor

$$L \wedge^i \mathbb{L}_{B/A} \longrightarrow L \wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}.$$

Remark 2.3.0.9. As the colimit is a lax-symmetric monoidal functor, the analytic derived de Rham complex $\mathrm{dR}_{B/A}$ is naturally a filtered E_∞-algebra in $\mathcal{D}(\mathrm{B}^+_{\mathrm{dR,e}})$.

Here we provide a simple description of the analytic derived de Rham complex for two special cases: the smooth case and the complete intersections.

Proposition 2.3.0.10. *Let $A \to B$ be a smooth map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then the natural morphism below from the analytic derived de Rham complex to the continuous de Rham complex is a filtered isomorphism:*

$$\mathrm{dR}_{B/A} \longrightarrow \Omega^\bullet_{B/A}.$$

Proof. By the Remark 2.3.0.4, there exists a natural filtered map from $\mathrm{dR}_{B/A}$ to the continuous de Rham complex $\Omega^\bullet_{B/A}$, which is compatible with the differential maps. By the assumption and Corollary 2.2.3.6, the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ is isomorphic to the module of continuous differential forms $\Omega^1_{B/A}[0]$, which is a free B-module whose rank is equal to the relative dimension $\dim_A(B)$. On the other hand, the de Rham complex of affinoid algebras B over A is bounded above by the relative dimension and is thus complete under the Hodge filtration. The derived wedge product $L \wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}$ is isomorphic to $\wedge^i \Omega^1_{B/A}[0] = \Omega^i_{B/A}[0]$, which vanishes when $i > \dim_A(B)$. So by the Construction 2.3.0.1 above, the natural map from the analytic derived de Rham complex to the de Rham complex of B/A induces an isomorphism from the i-th graded factor $\mathrm{gr}^i \mathrm{dR}_{B/A} = L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i]$ to the i-th continuous differential $\Omega^i_{B/A}[-i]$. In this way, we get a filtered isomorphism from $\mathrm{dR}_{B/A}$ to the de Rham complex $\Omega^\bullet_{B/A}$. $\square$

Proposition 2.3.0.11. *Let $A \to B$ be a surjective map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then the canonical map below is a filtered isomorphism*

$$\widehat{\mathrm{dR}}_{B/A} \longrightarrow \mathrm{dR}_{B/A}.$$

As an upshot, the underlying complex $\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{B/A}$ is isomorphic to the formal completion $\widehat{A}$ for the surjection $A \to B$.

Proof. As both $\mathrm{dR}_{B/A}$ and $\widehat{\mathrm{dR}}_{B/A}$ are filtered complete, it suffices to show that the induced map on each graded factor is an isomorphism. For each $i \in \mathbb{N}$, the induced map $\mathrm{gr}^i\widehat{\mathrm{dR}}_{B/A} \to \mathrm{gr}^i\mathrm{dR}_{B/A}$ is exactly the natural map induced from the derived p-completion integrally (Construction 2.2.3.1). So the first claim follows from the assumption and the Corollary 2.2.3.7.

For the second claim, it follows from the isomorphism between the underlying complex $\underline{\mathrm{dR}}_{B/A}$ of the algebraic derived de Rham complex and the formal completion $\widehat{A}$, which is the main result in [Bha12a] (see [Bha12a, 4.14, 4.16]). $\square$

Corollary 2.3.0.12. *Let $A \to B$ be a surjective map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that the kernel ideal I is regular in A. Then for each $i \in \mathbb{N}$, we have a natural isomorphism*

$$A/I^i[0] \longrightarrow \mathrm{dR}_{B/A}/\mathrm{Fil}^i.$$

In particular, by taking the derived limit, we get a filtered isomorphism of algebras

$$\widehat{A} \cong \mathrm{dR}_{B/A},$$

where the left side is the (classical) I-adic completion of A.

Proof. This follows from Proposition 2.3.0.11, and the case for algebraic cotangent complex explained in the Example 4.5 in [Bha12a], originally proved in [Ill72, Theorem 2.2.6]. $\square$

2.4 Global constructions

In this section, we construct the global analytic cotangent complexes and the global analytic derived de Rham complexes. Our strategy is to show that the affinoid constructions satisfy the hyperdescent for the rigid topology, thus can be extended to a complex of sheaves over the given rigid space.

2.4.1 Unfolding

We first recall the unfolding of a sheaf in ∞-category.

Let X be a site that admits fiber products, and let $\mathcal{B}$ be a basis of X, namely $\mathcal{B}$ is a subcategory of X such that for each object U in X, there exists an object U' in $\mathcal{B}$ covering U. So any hypercovering of an object in X can be refined to a hypercovering with each term in $\mathcal{B}$.

Let $\mathcal{C}$ be a presentable ∞-category. Consider a hypersheaf $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C})$ over $\mathcal{B}$. We can then *unfold* the sheaf $\mathcal{F}$ to a hypersheaf $\mathcal{F}'$ on X, such that its evaluation at any $V \in X$ is given by

$$\mathcal{F}'(V) = \operatorname*{colim}_{U'_\bullet \to V} \varprojlim_{[n] \in \Delta^{\mathrm{op}}} \mathcal{F}(U'_n),$$

where the colimit is indexed over all hypercoverings $U'_\bullet \to V$ with $U'_n \in \mathcal{B}$ for all n. It can be shown that one hypercovering suffices to compute the value of $\mathcal{F}'(V)$ in the above formula: actually for a hypercovering $U'_\bullet \to V$ with each U'_n in the basis $\mathcal{B}$, we have a natural weak-equivalence

$$R \lim_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(U'_n) \longrightarrow \mathcal{F}'(V).$$

In particular for any $U \in \mathcal{B}$, the natural map $\mathcal{F}(U) \longrightarrow \mathcal{F}'(U)$ is a weak-equivalence.

The above construction is functorial with respect to $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C})$, and we get a natural unfolding functor

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C}),$$

which is in fact an equivalence, with the inverse given by the restriction functor $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{C}) \to \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathcal{C})$.

Recall in the special case when $\mathcal{C} = \mathcal{D}(R)$ is the derived ∞-category of R-modules, we have a natural equivalence

$$\begin{aligned} \mathcal{D}(X, R) &\longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X, \mathcal{D}(R)); \\ C &\longmapsto (V \mapsto R\Gamma(V, C)). \end{aligned}$$

As an upshot, to define a complex of sheaves of R-modules over X, it suffices to specify a contravariant functor from the basis $\mathcal{B}$ to $\mathcal{D}(R)$, such that it satisfies the hyperdescent condition within $\mathcal{B}$.

2.4.2 Hyperdescent of $\mathbb{L}^{\mathrm{an}}_{B/A}$ and $\mathrm{dR}_{B/A}$

We first consider the analytic cotangent complex.

Proposition 2.4.2.1. *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, and let $B \to B_\bullet$ be a map from B to a cosimplicial algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that the associated map of rigid spaces $\mathrm{Spa}(B_\bullet) \to \mathrm{Spa}(B)$ is a rigid open hypercovering. Then the induced map below is an isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow R \lim_{[n]\in\Delta^{\mathrm{op}}} \mathbb{L}^{\mathrm{an}}_{B_n/A}.$$

Proof. We first notice that by the étaleness of the map $B \to B_n$ and the Corollary 2.2.3.12, $\mathbb{L}^{\mathrm{an}}_{B_n/A}$ is naturally isomorphic to the base change $\mathbb{L}_{B/A} \otimes^L_B B_n$. So it suffices to show the map below is an isomorphism

$$\mathbb{L}^{\mathrm{an}}_{B/A} \longrightarrow R \lim_{[n]\in\Delta^{\mathrm{op}}} \mathbb{L}^{\mathrm{an}}_{B/A} \otimes^L_B B_n.$$

Note that since each $\mathrm{Spa}(B_n) \to \mathrm{Spa}(B)$ is an open covering of the rigid space $\mathrm{Spa}(B)$, the induced map $B \to B_n$ is flat on structure sheaves ([Hub96, Proposition 1.7.6]). In this way, by the surjectivity of $\mathrm{Spa}(B_n) \to \mathrm{Spa}(B)$, we see the map of affine schemes $\mathrm{Spec}(B_\bullet) \to \mathrm{Spec}(B)$ is a fpqc hypercover, and the isomorphism above follows from the fpqc hyperdescent of quasi-coherent sheaves over the affine scheme $\mathrm{Spec}(B)$.

$\square$

Using the unfolding technique in Subsection 2.4.1, we can extend the affinoid construction of the analytic cotangent complex to the global case.

Corollary 2.4.2.2. *Let $X \to Y = \mathrm{Spa}(A)$ be a map of rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a complex of sheaves of A-modules $\mathbb{L}^{\mathrm{an}}_{X/Y}$ over X, such that for any affinoid open subset $U = \mathrm{Spa}(B)$ of X, we have a natural isomorphism*

$$R\Gamma(U, \mathbb{L}^{\mathrm{an}}_{X/Y}) = \mathbb{L}^{\mathrm{an}}_{B/A}.$$

The complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is called the analytic cotangent complex of X over Y.

Similarly, we could unfold the construction of the analytic derived de Rham complex to an arbitrary rigid space.

Proposition 2.4.2.3. *Let $A \to B$ be a map of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$, and let $B \to B_\bullet$ be a map from B to a cosimplicial algebra over $\mathrm{B}^+_{\mathrm{dR,e}}$, such that the associated map of rigid spaces $\mathrm{Spa}(B_\bullet) \to \mathrm{Spa}(B)$ is a rigid open hypercovering. Then the induced filtered map below is an isomorphism*

$$\mathrm{dR}_{B/A} \longrightarrow R \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{dR}_{B_n/A}.$$

Proof. As a limit functor preserves the filtered completeness, $R\lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{dR}_{B_n/A}$ is an object in $\widehat{\mathrm{DF}}(A)$, and checking the isomorphism above can be reduced to their graded pieces. Moreover, notice that the graded piece functor commutes with small limits and colimits (cf. [BMS19, Lemma 5.2]). Thus we get

$$\begin{aligned}
\mathrm{gr}^i \mathrm{dR}_{B/A} = L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A}[-i] &\longrightarrow \mathrm{gr}^i (R \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{dR}_{B_n/A}) \\
&\cong R \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{gr}^i \mathrm{dR}_{B_n/A} \\
&= R \lim_{[n]\in\Delta^{\mathrm{op}}} L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_n/A}[-i].
\end{aligned}$$

Notice that the wedge product functor commutes with the tensor product, and for each $n \in \mathbb{N}$ we have

$$\begin{aligned} L\wedge^i \mathbb{L}^{\mathrm{an}}_{B_n/A} &\cong L\wedge^i (\mathbb{L}^{\mathrm{an}}_{B/A}\otimes^L_B B_n)\\ &\cong (L\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})\otimes^L_B B_n, \end{aligned}$$

In this way, the natural map of graded pieces above is an isomorphism, by the similar fpqc hyperdescent for $B\to B_\bullet$ as in the proof of Proposition 2.4.2.1. So we are done.

□

Corollary 2.4.2.4. *Let $X \to Y = \mathrm{Spa}(A)$ be a map of rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a complex of sheaves of A-modules $\mathrm{dR}_{X/Y}$ over X, such that for any affinoid open subset $U=\mathrm{Spa}(B)$ of X, we have a natural isomorphism*

$$R\Gamma(U,\mathrm{dR}_{X/Y}) = \mathrm{dR}_{B/A}.$$

The complex $\mathrm{dR}_{X/Y}$ *is called the* analytic derived de Rham complex of X over Y.

CHAPTER 3

Infinitesimal Cohomology

In this chapter, we introduce the infinitesimal cohomology of rigid spaces over $\mathrm{B}_{\mathrm{dR,e}}^{+}$, generalizing the infinitesimal/crystalline cohomology of complex algebraic varieties introduced by Grothendieck [Gro68]. The results in this chapter first appeared in [Guo20, Section 2-4].

We start this chapter by introducing the basics around the infinitesimal site for rigid spaces over the de Rham period ring $\mathrm{B}_{\mathrm{dR,e}}^{+}$ in Section 3.1, including the definition, the notion of the envelope, the relation with the rigid analytic topology, and the functoriality of the infinitesimal topos. Analogous to the crystalline cohomology, we have the notion of the *coherent crystal* as the coefficient, which will be developed in Section 3.2. This is the rigid analytic analogue of the *integrable connection* for complex analytic spaces. Precisely, analogous to the crystalline theory in [BO78], we prove that the category of coherent crystals is equivalent to the category of integrable connections over the envelope in Theorem 3.2.3.1. The rest of the chapter, which is Section 3.3, is devoted to the rigid analytic version of [BdJ11]: we show that the cohomology of a crystal over a given space X can be computed by the cohomology of the de Rham complex for the envelope (Theorem 3.3.2.2). Here we mention that the proof follows the one in [BdJ11], except that by a more careful strudy of filtrations we show the above isomorphism preserves their infinitesimal filtrations.

3.1 Infinitesimal geometry over $\mathrm{B}^+_{\mathrm{dR,e}}$

In this section, we introduce the basics around the infinitesimal geometry over the de Rham period ring $\mathrm{B}^+_{\mathrm{dR,e}} := \mathrm{B}^+_{\mathrm{dR}}/\xi^e$.

3.1.1 Infinitesimal sites

We first introduce the big and the small infinitesimal sites of a rigid space over them, and study two natural maps between their topoi.

Infinitesimal topology

Definition 3.1.1.1. *Let e be a positive integer. A* rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$ *is defined as an adic space of topological finite presentation over Σ_e. Namely X can be covered by affinoid open subspaces which are of the form*

$$\mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR,e}}\langle t_1, \dots, t_n\rangle/I),$$

where I is a (finitely generated) ideal in $\mathrm{B}^+_{\mathrm{dR,e}}\langle t_1, \dots, t_n\rangle$.

The category of rigid spaces over Σ_e is denoted by Rig_{Σ_e}.

Recall that for a map of rigid spaces $f : U \to T$, it is called a *nil closed immersion* if f is a closed immersion (defined by the vanishing of a coherent ideal $\mathcal{I}$ in $\mathcal{O}_T$), such that T admits an open covering $\{T_i,\ i\}$ with $\mathcal{I}|_{T_i}$ being nilpotent.

Note that a nilpotent closed immersion, for which the defining ideal is nilpotent globally, is always a nil closed immersion. The converse is true locally or assuming the quasi-compactness of the target space.

Definition 3.1.1.2. *(a) Let X be a rigid space over Σ_e. The* (small) infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$ *is the Grothendieck topology defined as follows:*

- *The underlying category of $X/\Sigma_{e\,\mathrm{inf}}$ is the collection of pairs (U, T), called* infinitesimal thickenings*, where T is a rigid space over Σ_e, U is an open subspace of X and a closed analytic subspace of T, such that $U \to T$ is a nil closed immersion.*

  Here morphisms between (U_1, T_1) and (U_2, T_2) are defined as maps of pairs over Σ_e such that $U_1 \to U_2$ is the open immersion inside X.

- *A collection of morphism $(U_i, T_i) \to (U, T)$ in $X/\Sigma_{e\,\mathrm{inf}}$ is a covering if both $\{T_i \to T,\ i\}$ and $\{U_i \to U,\ i\}$ are open coverings for the rigid spaces T and U separately.*

(b) The big infinitesimal site $\mathrm{Rig}_{\Sigma_e,\mathrm{INF}}$ *over Σ_e is defined on the category of all of the pairs (U, T) for $U \to T$ being a nil closed immersion of rigid spaces over Σ_e, with the same covering structure as above.*

(c) *The* big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$ of X *is defined as the localization* $\mathrm{Rig}_{\Sigma_e,\mathrm{INF}}|_X$ *of the big site* $\mathrm{Rig}_{\Sigma_e,\mathrm{INF}}$ *at* X*. Namely it is defined on the category of all of the tuples* $\{(U,T), f: U \to X\}$*, where* (U,T) *is an object in* $\mathrm{Rig}_{\Sigma_e,\mathrm{INF}}$*, and* $f: U \to X$ *is a map of rigid spaces over* Σ_e*. The covering structure is induced from that of* $\mathrm{Rig}_{\Sigma_e,\mathrm{INF}}$*.*

To give a sheaf $\mathcal{F}$ over the infinitesimal site, it is equivalent to give a sheaf $\mathcal{F}_T$ over each rigid space T for $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$, such that for each morphism $(i,g): (U_1,T_1) \to (U_2,T_2)$ in $X/\Sigma_{e\,\mathrm{inf}}$, there exists a map of sheaves over T_1

$$g^{-1}\mathcal{F}_{T_2} \longrightarrow \mathcal{F}_{T_1},$$

which satisfies the natural transition compatibility. The same holds for a sheaf over the big infinitesimal site. We call the category of sheaves on $X/\Sigma_{e\,\mathrm{inf}}$ (or $X/\Sigma_{e\mathrm{INF}}$) the *infinitesimal topos*, and denote it by $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ (or $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$).

For two abelian sheaves $\mathcal{F}$ and $\mathcal{G}$ over the infinitesimal site, we sometimes use the notation $\mathcal{F}(\mathcal{G})$ to denote the group of homomorphisms

$$\mathrm{Hom}(\mathcal{G},\mathcal{F}).$$

In the case when $\mathcal{G}$ is a representable sheaf h_T for an infinitesimal thickening (U,T), the above hom group is the group of sections

$$\mathrm{Hom}(h_T,\mathcal{F}) = \mathcal{F}(U,T).$$

There is a natural *structure sheaf* $\mathcal{O}_{X/\Sigma_e}$ over the big or small infinitesimal site, which is defined as

$$\mathcal{O}_{X/\Sigma_e}(U,T) := \mathcal{O}_T(T),\ (U,T) \in X/\Sigma_{e\,\mathrm{inf}}.$$

Remark 3.1.1.3. It is clear from the above definition that the infinitesimal site can be defined for any pair of analytic adic spaces $X \to Z$, not just $X \to \Sigma_e$. In particular, when $Z = \mathrm{Spa}(K_0)$ is a discretely valued field, and X is a rigid space over K_0, we get the analogous version of the infinitesimal site of X over K_0. Moreover, there exists a natural map of sites $X_K/K_{\mathrm{inf}} \to X/K_{0,\mathrm{inf}}$, defined by the base field extension.

Here are some basic properties of the infinitesimal sites.

Lemma 3.1.1.4. *Let* X *be a rigid space over* Σ_e*. Then we have*

(i) *The fiber product exists in the big and the small infinitesimal site of* X *over* Σ_e*, and is compatible with the inclusion functor between the big and the small sites.*

(ii) The equalizer exists in the big and the small infinitesimal site of X over Σ_e, and is compatible with the inclusion functor.

(iii) The finite product is ind-representable in the big and the small infinitesimal site of X over Σ_e, and is compatible with the inclusion functor.

Proof. (i) Let (V_i, T_i) for $i = 0, 1, 2$ be three objects in the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$, with arrows $g_i : (V_i, T_i) \to (V_0, T_0)$ for $i = 1, 2$. Namely each V_i admits a map to X, and $V_i \to T_i$ is a closed immersion that has a nil defining ideal. Then we can form the fiber products of rigid spaces $V_3 := V_1 \times_{V_0} V_2$ and $T_3 := T_1 \times_{T_0} T_2$ over Σ_e, together with a natural map $V_3 \to T_3$. Any infinitesimal thickening (V, T) that admits a compatible family of maps $(V, T) \to (V_i, T_i)$ for $i = 0, 1, 2$ would produce a commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & T \arrow[d] \\ V_3 \arrow[r] & T_3. \end{tikzcd}$$

So it is left to show that $V_3 \to T_3$ is a nil immersion, which can be checked locally by choosing affinoid open subsets of $T_i, i = 01, 2$.

Moreover, in the special case when (V_1, T_i) comes from the small site for $i = 0, 1, 2$ (namely the map $V_i \to X$ is an open immersion for $i = 0, 1, 2$), then the fiber product $V_3 = V_1 \times_{V_0} V_2$ is also open in X. In particular, the fiber product in this case is lying in the small site $X/\Sigma_{e\,\mathrm{inf}}$.

(ii) For the equalizer, consider the two arrows $\alpha, \beta : (V_1, T_1) \rightrightarrows (V_2, T_2)$ in $X/\Sigma_{e\mathrm{INF}}$. Here both V_1 and V_2 admits a map to X, and $V_i \to T_i$ are nil closed immersions. We can first form the equalizer V_3 of $V_1 \rightrightarrows V_2$ and T_3 of $T_1 \rightrightarrows T_2$ in the category of rigid spaces over Σ_e, by the pullback diagram

$$\begin{tikzcd} V_3 \arrow[r] \arrow[d] & V_1 \arrow[d] & T_3 \arrow[r] \arrow[d] & T_1 \arrow[d] \\ V_2 \arrow[r] & V_2 \times_{\Sigma_e} V_2, & T_2 \arrow[r] & T_2 \times_{\Sigma_e} T_2, \end{tikzcd}$$

where the left vertical maps in both diagrams are diagonal embeddings. The left diagram admits a natural map to the right. Moreover, we notice that $V_3 \to T_3$ is a nil immersion, as all of other three terms in the diagram of V_3 are nil immersed into the diagram of T_3. Furthermore, as the map $V_1 \to V_2 \times_{\Sigma_e} V_2$ factors through $V_2 \times_X V_2 \to V_2 \times_{\Sigma_e} V_2$, the pullback V_3 is also isomorphic to the equalizer of $V_1 \rightrightarrows V_2$ in the category of rigid spaces over X. In this way, the object $(V_3, T_3) \in X/\Sigma_{e\mathrm{INF}}$ obtained above forms the equalizer of α, β in the category.

We at last note that the case when α, β comes from the small site is exactly when both of the arrows $V_1 \rightrightarrows V_2$ are open immersions (hence they are the same), where the obtained base change $V_3 \cong V_2 \underset{V_2 \times_X V_2}{\times} V_1 \cong V_2 \times_{V_2} V_1 = V_1$ is also open in X. Thus the construction of the equalizer is compatible with the one in the small site.

(iii) Let (V_i, T_i) for $i = 1, 2$ be two objects in the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$. Then we can form the fiber product $V_3 := V_1 \times_X V_2$ over X, and the fiber product $T_1 \times_{\Sigma_e} T_2$ over Σ_e together with a natural map from V_3, such that any object $(V', T') \in X/\Sigma_{e\mathrm{INF}}$ that admits a map to (V_i, T_i) for $i = 1, 2$ will admit a unique map onto the pair of rigid spaces $(V_3, T_1 \times_{\Sigma_e} T_2)$.

Now the only problem is that the pair $(V_3, T_1 \times_{\Sigma_e} T_2)$ is almost never a pair of infinitesimal thickening. However, notice that the map $V_3 \to T_1 \times_{\Sigma_e} T_2$ can be written as the composition

$$V_3 = V_1 \times_X V_2 \longrightarrow V_1 \times_{\Sigma_e} V_2 \longrightarrow T_1 \times_{\Sigma_e} T_2,$$

where the first map is a locally closed immersion (a composition of a closed immersion and an open immersion) and second map is a nil immersion. This allows us to form the direct limit $\varinjlim_m Y_m$ of all infinitesimal neighborhoods of V_3 into $T_1 \times_{\Sigma_e} T_2$, where each Y_m is the m-th infinitesimal neighborhood of V_3 inside of $T_1 \times_{\Sigma_e} T_2$. In this way, the fiber product of (V_1, T_1) and (V_2, T_2) is ind-represented by the colimit of (V_3, Y_m), for locally each map from an object (V', T') onto the pair $(V_3, T_1 \times_{\Sigma_e} T_2)$ factors through some (V_3, Y_m) (as (V', T') is locally nilpotent).

At last, we note that the construction is independent of big or small infinitesimal sites. Moreover, when V_1 and V_2 are open in X, from the construction above the rigid space V_3 is also open in X. Thus the finite product is compatible between the big and the small sites.

□

Remark 3.1.1.5. In fact, the ind-representable sheaf for the directed limit $\varinjlim_m Y_m$ is the *envelope* of the immersion $V_3 \to T_1 \times_{\Sigma_e} T_2$, which we will introduce in Definition 3.1.2.1 soon.

Relation between big and small sites/topoi Given a rigid space X over Σ_e, there are two natural morphisms of topoi between the big infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ and the small infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ of X. To see this, we first notice that by constructions, there exists a natural inclusion functor

$$X/\Sigma_{e\,\mathrm{inf}} \longrightarrow X/\Sigma_{e\mathrm{INF}}.$$

The inclusion functor is *continuous* in the sense of [Sta18, Tag 00WV], and thus induces two functors between their topoi ([Sta18, Tag 00WU])

- For a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\inf})$ over the small site, there exists a preimage functor μ^{-1} with $\mu^{-1}\mathcal{F}$ being the sheaf associated with the presheaf

$$X/\Sigma_{e\mathrm{INF}} \ni (V,S) \longmapsto \varinjlim_{\substack{(V,S)\to(U,T)\\(U,T)\in X/\Sigma_{e\inf}}} \mathcal{F}(U,T).$$

  The functor μ^{-1} commutes with finite limits.

- The direct image functor μ_*, which is the right adjoint of μ^{-1} and is computed by the restriction. Namely for a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ over the big site, we have $\mu_*\mathcal{G}(U,T) = \mathcal{G}(U,T)$.

This pair of adjoint functors in fact forms a morphism of topoi

$$\mu : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\inf}).$$

To see this, we claim the following:

Lemma 3.1.1.6. *The left adjoint functor $\mu^{-1} : \mathrm{Sh}(X/\Sigma_{e\inf}) \to \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ commutes with any nonempty finite limit.*

Proof. To see this, we first notice that as a left adjoint functor commutes with any small colimit, it suffices to show this for a finite diagram of representable sheaves. Moreover, as a nonempty finite limit can be formed by finite many of finite products and equalizers ([Sta18, Tag 04AS]), it suffices to show that μ^{-1} commutes with finite products and equalizers of representable sheaves, which is given by Lemma 3.1.1.4. So we are done. $\square$

This by definition means that the left adjoint μ^{-1} is exact, so we get a morphism of topoi ([Sta18, Tag 00X1])

$$\mu : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\inf}).$$

On the other hand, the inclusion functor is *cocontinuous* in the sense of [Sta18, Tag 00XJ]. This is because if a collection of thickenings $\{(U_i,T_i)\} \subset X/\Sigma_{e\mathrm{INF}}$ covers a given $(U,T) \in X/\Sigma_{e\inf}$ in the big site, then each (U_i,T_i) is also an object in the small site which together forms a covering of (U,T). So by [Sta18, Tag 00XO], the inclusion functor induces another map of topoi

$$\iota : \mathrm{Sh}(X/\Sigma_{e\inf}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}),$$

consists of the following adjoint pairs of functors

- The functor $\iota^{-1} = \mu_* : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \to \mathrm{Sh}(X/\Sigma_{e\inf})$ is the restriction functor, which commutes with any finite limits.

- The functor $\iota_* : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$, which is the right adjoint of the functor ι^{-1}, sending a sheaf $\mathcal{F}$ over the small site to the sheaf $\iota_*\mathcal{F}$ with the equality

$$X/\Sigma_{e\mathrm{INF}} \ni (V,S) \longmapsto \varprojlim_{\substack{(V,S)\to(U,T)\\ (U,T)\in X/\Sigma_{e\,\mathrm{inf}}}} \mathcal{F}(U,T).$$

Here we notice that when the thickening (V,S) is an object coming from the small site $X/\Sigma_{e\,\mathrm{inf}}$ (namely $V \to X$ is an open immersion), from the description above we then have

$$(\iota_*\mathcal{F})(V,S) = \mathcal{F}(V,S).$$

Furthermore, notice that given an arrow $(V_1,T_1) \to (V_2,T_2)$ in the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$, the associated morphism of rigid spaces $V_1 \to V_2$ is a X-morphism. This in particular implies that the inclusion functor $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e\mathrm{INF}}$ is fully faithful, as when (V_1,T_1) and (V_2,T_2) come from the small site, the only X-morphism between V_1 and V_2 is the open immersion. So by [Sta18, Tag 00XS, Tag 00XT] and Lemma 3.1.1.4, we have [1]

- The functor μ^{-1} commutes with fiber products and equalizers (so with all finite connected limits).

- The canonical natural transformations below are isomorphisms of functors:

$$\mathrm{id} \longrightarrow \mu_* \circ \mu^{-1};\ \iota^{-1} \circ \iota_* = \mu_* \circ \iota_* \longrightarrow \mathrm{id}.$$

3.1.2 Envelopes

Analogous to the infinitesimal theory of complex varieties in [Gro68] and the crystalline theory of schemes in positive characteristic in [BO78], we can define the envelope for a locally closed immersion $X \to Y$ of rigid spaces.

Definition 3.1.2.1. *Let Y be a rigid space over Σ_e, and X be a locally closed analytic subspace in Y, defined by a coherent ideal I in $\mathcal{O}_U$ for U an open subset inside of Y. We denote by Y_n to be the n-th infinitesimal neighborhood of X in Y, which form an object (X,Y_n) in $X/\Sigma_{e\,\mathrm{inf}}$ and is defined by the ideal I^{n+1}.*

The envelope $D_X(Y)$ of X in Y, *is an object in the infinitesimal topos* $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$*, defined by the colimit of the direct system of representable sheaves h_{Y_n} of (X,Y_n) in* $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$*:*

$$D_X(Y) := \varinjlim_{n\in\mathbb{N}} h_{Y_n}.$$

[1] In the notation of [Sta18, Tag 00UZ], the functor μ^{-1} is equal to the functor $\iota_!$.

Note that the definition also works for the big infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$, and under the natural inclusion functor $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e\mathrm{INF}}$ the notions of the envelopes coincide.

Remark 3.1.2.2. In many situations, it is convenient to regard $D_X(Y)$ as an actual locally ringed space, instead of a direct limit of representable sheaves in the infinitesimal topos. Here the associated ringed space structure of the envelope $D_X(Y)$ has the same topological space as the adic space X, and the structure sheaf $\mathcal{D} = \varprojlim_n \mathcal{O}_{Y_n}$ is the inverse limit of structure sheaves of infinitesimal neighborhoods Y_n.

Remark 3.1.2.3. The existence of the colimit in the topos is guaranteed by [Sta18, Tag 00WI].

Remark 3.1.2.4. Here we want to mention that different from the crystalline theory of a scheme over $\mathbb{Z}_p/p^e$, the envelope is almost *never* representable. In the mixed characteristic case, the divided-power structure enforces the defining ideal for a divided power thickening to be nilpotent. However, in equal characteristic zero such a condition is lost and the envelope is not an infinitesimal thickening. This in particular appears when we consider the crystalline theory of a scheme over $\mathbb{C}$.

Though the envelope fails to be representable, we do have a description of an envelope similar to a representable sheaf:

Lemma 3.1.2.5. *For a closed immersion $X \to Y$, the envelope $D_X(Y)$ is isomorphic to the sheaf on $X/\Sigma_{e\,\mathrm{inf}}$ (and $X/\Sigma_{e\mathrm{INF}}$), defined by*

$$(U,T) \longmapsto \mathrm{Hom}((U,T),(X,Y)),$$

where $\mathrm{Hom}((U,T),(X,Y))$ is the set of commutative diagrams of Σ_e-rigid spaces

$$\begin{tikzcd} T \arrow[r] & Y \\ U \arrow[u] \arrow[r] & X \arrow[u] \end{tikzcd},$$

with $U \to X$ being the structure morphism for the object (U,T).

Proof. We first notice that we have a natural map

$$D_X(Y)((U,T)) = \varprojlim_{n\in\mathbb{N}} \mathrm{Hom}((U,T),(X,Y_n)) \longrightarrow \mathrm{Hom}((U,T),(X,Y)),$$

induced by closed immersions $Y_n \to Y_{n+1} \to Y$. So it suffices to check that for a pair of affinoid rigid spaces $(U,T) = (\mathrm{Spa}(R/J), \mathrm{Spa}(R))$ in the infinitesimal site, the above is an equality.

For the surjection, we notice that since (U,T) is affinoid rigid space over Σ_e, the ring R is noetherian and J is nilpotent. In particular, there exists an $n \in \mathbb{N}$, such that $J^{n+1} = 0$. So the map $\mathrm{Spa}(R) \to Y$ factors through a map $\mathrm{Spa}(R) \to Y_n$.

For the injection, assume there are two maps $\alpha, \beta : T \rightrightarrows Y_n$ of rigid spaces over Σ_e such that after the composition with $Y_n \to Y$ they are equal. Then since $Y_n \to Y$ is a closed immersion, by restricting to an affinoid open covering of Y (thus Y_n) it is reduce to the equality of the following two compositions

$$A \to A/I^{n+1} \rightrightarrows R,$$

which implies $A/I^{n+1} \rightrightarrows R$ are equal. So we are done.

□

The following simple observations justify this name of the envelope:

Lemma 3.1.2.6. *Assume Y is smooth over Σ_e. Then the envelope $D_X(Y)$ for a closed immersion of X in Y covers the final object in the infinitesimal topoi $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ and $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$. In other words, the map from $D_X(Y)$ onto the final object in the infinitesimal topoi is an epimorphism of sheaves.*

Proof. We denote by 1 to be the final object in $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ or $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$. Then to show the surjection of the map of sheaves

$$D_X(Y) \longrightarrow 1,$$

it suffices to show that any object (U,T) in the infinitesimal site locally admits a morphism to $D_X(Y)$.

For an affinoid thickening $(U,T) = (\mathrm{Spa}(R/I), \mathrm{Spa}(R))$ with a map $U \to X$, since $U \to T$ is a nil immersion and R is noetherian, there exists an integer m such that I^{m+1} in R. By assumption that Y is smooth, locally there exists a morphism from $\mathrm{Spa}(R)$ to Y that makes the following diagram commute

$$\begin{tikzcd} \mathrm{Spa}(R/I) \arrow[r] \arrow[d] & \mathrm{Spa}(R) \arrow[d] \\ X \arrow[r] & Y. \end{tikzcd}$$

By the nilpotence, the map $\mathrm{Spa}(R) \to Y$ factors canonically through Y_n for $n \geq m$. Thus the map $\mathrm{Spa}(R) \to Y$ factors through the direct limit $D_X(Y) = \varinjlim_{n \in \mathbb{N}} h_{Y_n} \to Y$.

□

The above allows us to give a very general formula to compute the cohomology over the infinitesimal site, using the Čech nerve for an envelope.

Proposition 3.1.2.7. *Let $X \to Y$ be a closed immersion into a smooth rigid space Y over Σ_e. For $n \in \Delta$, we denote $D(n)$ to be the simplicial space where each $D(n)$ is the envelope of X in $Y(n) := Y^{\times_{\Sigma_e} n+1}$. Then there is a natural equivalence of derived functors on the derived category of sheaves over the (big or small) infinitesimal sites*

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}) \longrightarrow R \lim_{[n]\in\Delta^{\mathrm{op}}} R\Gamma(D(n), \mathcal{F}).$$

Here we want to mention that for each $n \in \Delta$, the derived section functor $R\Gamma(D(n), \mathcal{F})$ is computed via the inverse limit

$$R \varprojlim_{m\in\mathbb{N}} R\Gamma((X, Y(n)_m), \mathcal{F}),$$

where each $Y(n)_m$ is the m-th infinitesimal neighborhood of X in $Y(n)$.

Proof. We first notice that $D(n)$ is in fact the $(n+1)$-fold self-product of $D_X(Y)$ in the infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ (or $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ respectively). This is because by Lemma 3.1.2.5, we know

$$D(n) = \mathrm{Hom}(-, (X, Y^{n+1})),$$

which is the same as the contravariant functor $\mathrm{Hom}(-, (X, Y))^{n+1}$ on the infinitesimal site. So the simplicial object $D(\bullet)$ is in fact the coskeleton $\mathrm{cosk}_0(D_X(Y))$ over the final object (in other words, the Čech nerve for the map of sheaves $D_X(Y) \to 1$). In this way, since $D_X(Y) \to 1$ is an effective epimorphism (Lemma 3.1.2.6), by the [Sta18, Tag 09VU], $D(\bullet) \to 1$ is a hypercovering, and we get a natural equivalence of derived functors

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, -) \cong R\Gamma(D(\bullet), -) = R \lim_{[n]\in\Delta^{\mathrm{op}}} R\Gamma(D(n), -).$$

□

As an upshot, we see the restriction functor from the big infinitesimal topos to the small one preserves the cohomology.

Corollary 3.1.2.8. *Let $\mathcal{F}$ be an object in the derived category of sheaves over $X/\Sigma_{e\mathrm{INF}}$. Then the restriction functor $\iota^{-1} = \mu_* : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \to \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ (cf. Paragraph 3.1.1) induces the following isomorphism*

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mu_*\mathcal{F}) \longrightarrow R\Gamma(X/\Sigma_{e\mathrm{INF}}, \mathcal{F}).$$

Proof. We first assume X admits a closed immersion into a smooth rigid space over Σ_e. The claim in this case then follows from Proposition 3.1.2.7, as an envelope is an direct limit $\varinjlim_{m\in\mathbb{N}} h_{Y(n)_m}$ of representable objects in the big and the small sites, and the restriction functor produces the natural equivalence

$$R\Gamma(D(n), \mathcal{F}) \longrightarrow R\Gamma(D(n), \mu_*\mathcal{F}).$$

In general, we may take a hypercovering by affinoid open spaces of X first to reduce to the above special cases. $\square$

3.1.3 Infinitesimal and rigid topology

In this subsection, we relate the infinitesimal topos and the rigid topos together.

Let X be a rigid space over Σ_e. Recall that there is a Grothendieck topology X_{rig} on the category of open subsets in X, called the *rigid site* X_{rig}.

Consider the following two functors:

$$\begin{aligned} u_{X/\Sigma_e *} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) &\longrightarrow \mathrm{Sh}(X_{\mathrm{rig}}); \\ \mathcal{F} &\longmapsto (U \mapsto \Gamma(U/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}|_{U/\Sigma_{e\,\mathrm{inf}}})). \\ u^{-1}_{X/\Sigma_e} : \mathrm{Sh}(X_{\mathrm{rig}}) &\longrightarrow \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}); \\ \mathcal{E} &\longmapsto ((U,T) \mapsto \mathcal{E}(U)). \end{aligned}$$

Since $(u^{-1}_{X/\Sigma_e}\mathcal{E})_T$ is the sheaf $\mathcal{E}|_U$ on $U_{\mathrm{rig}} \cong T_{\mathrm{rig}}$, the functor u^{-1}_{X/Σ_e} commutes with the finite inverse limit. Notice that the pair $(u^{-1}_{X/\Sigma_e},\ u_{X/\Sigma_e *})$ is adjoint. Thus we get a morphism of topoi ([Sta18, Tag 00XA])

$$u_{X/\Sigma_e} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X_{\mathrm{rig}}),$$

which we follow [BO78] and call the *projection morphism*.

The projection morphism u_{X/Σ_e} admits a section. Consider the functor $X/\Sigma_{e\,\mathrm{inf}} \to X_{\mathrm{rig}}$, sending (U,T) onto the open subset U of X. By the definition of $X/\Sigma_{e\,\mathrm{inf}}$, a covering of (U,T) is mapped onto a covering of U. In particular, the map of sites is continuous in the sense of [Sta18]. So we get a morphism of sites

$$i_{X/\Sigma_e} : X_{\mathrm{rig}} \longrightarrow X/\Sigma_{e\,\mathrm{inf}}.$$

The morphisms induces a map of topoi, in a way that for $\mathcal{E} \in \mathrm{Sh}(X_{\mathrm{rig}})$,

$$i_{X/\Sigma_e *}\mathcal{E}(U,T) = \mathcal{E}(U),$$

and for $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, we have

$$i_{X/\Sigma_e}^{-1}\mathcal{F}(U) = \varinjlim_{(U,U)\to(V,T)} \mathcal{F}(V,T) = \mathcal{F}(U,U).$$

From the description, we see the functor i_{X/Σ_e}^{-1} is the restriction functor sending a sheaf $\mathcal{F}$ over $X/\Sigma_{e\,\mathrm{inf}}$ to its restriction $\mathcal{F}_X$ onto the rigid space X.

Remark 3.1.3.1. By the construction of i_{X/Σ_e} and u_{X/Σ_e}, on the rigid topos $\mathrm{Sh}(X_{\mathrm{rig}})$ we have

$$u_{X/\Sigma_e *} \circ i_{X/\Sigma_e *} = id,\ i_{X/\Sigma_e}^{-1} \circ u_{X/\Sigma_e}^{-1} = id,$$

which implies that those morphisms of topoi satisfy

$$u_{X/\Sigma_e} \circ i_{X/\Sigma_e} = id.$$

This justify the name of the projection morphism.

Remark 3.1.3.2. The construction here naturally generalizes to two morphisms between big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$ and the big rigid site $\mathrm{Rig}_{\Sigma_e}|_X$, for a given rigid space over X.

3.1.4 Functoriality

In this subsection, we introduce natural maps of infinitesimal topoi associated with a map of rigid spaces, similar to the construction in [Sta18, Tag 07IC, 07IK].

Let $f : X \to Y$ be a map of rigid spaces over $\Sigma_{e'}$, and assume the structure map $X \to \Sigma_{e'}$ factors through Σ_e for non-negative integers $e \leq e'$. By the construction of the big infinitesimal site, the map f induces a natural functor between $X/\Sigma_{e\mathrm{INF}}$ and $Y/\Sigma_{e'\mathrm{INF}}$, satisfying

$$X/\Sigma_{e\mathrm{INF}} \ni ((U,T), U \to X) \longmapsto ((U,T), U \to X \to Y),$$

where the map $U \to X \to Y$ is the composition of the map f with the structure map of $(U,T) \in X/\Sigma_{e\mathrm{INF}}$. Then it is easy to check that this functor is both continuous and cocontinuous, and commutes withe fiber products and equalizers (cf. Lemma 3.1.1.4). This in particular implies that the functor above induces a morphism of topoi ([Sta18, Tag 00XN, 00XR])

$$f_{\mathrm{INF}} : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \longrightarrow \mathrm{Sh}(Y/\Sigma_{e'\mathrm{INF}}),$$

such that

- The inverse image functor f_{INF}^{-1} commutes with arbitrary limits and colimits, such that for a sheaf $\mathcal{G}$ over $Y/\Sigma_{e'\mathrm{INF}}$, we have

$$f_{\mathrm{INF}}^{-1}\mathcal{G}(U,T) = \mathcal{G}(U,T),$$

  where the second (U,T) is regarded as an object in $Y/\Sigma_{e'\mathrm{INF}}$ by $U \to X \to Y$.

- The direct image functor $f_{\mathrm{INF}*}$, which is the right adjoint to the functor f_{INF}^{-1}, sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ to the sheaf $f_{\mathrm{INF}*}\mathcal{F}$ such that the section is given by

$$Y/\Sigma_{e'\mathrm{INF}} \ni (V,S) \longmapsto \varprojlim_{\substack{(V,S)\to(U,T)\\ (U,T)\in X/\Sigma_{e\mathrm{INF}},\\ V\to U \ \textit{compatible with}\ f}} \mathcal{F}(U,T).$$

Now we consider the small topoi $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ and $\mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$. Analogous to [Sta18, Tag 07IK], we use the map of big topoi to connect them. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \arrow[r, "f_{\mathrm{INF}}"] & \mathrm{Sh}(Y/\Sigma_{e'\mathrm{INF}}) \arrow[d, "\mu_Y"] \\
\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \arrow[u, "\iota_X"] \arrow[r, dotted, "f_{\mathrm{inf}}"'] & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}).
\end{tikzcd}$$

Here we define the morphism of topoi $f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$ to be the composition

$$f_{\mathrm{inf}} = \mu_Y \circ f_{\mathrm{INF}} \circ \iota_X.$$

Then by the definition of those functors, we have

- For a sheaf $\mathcal{G} \in \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$, the inverse image $f_{\mathrm{inf}}^{-1}\mathcal{G}$ is given by the "restriction" of $\mu_Y^{-1}\mathcal{G}$ to the category $X/\Sigma_{e\,\mathrm{inf}}$ via the map f, and is equal to the sheaf associated with the presheaf

$$X/\Sigma_{e\,\mathrm{inf}} \ni (U,T) \longmapsto \varinjlim_{\substack{(U,T)\to(V,S)\\ (V,S)\in Y/\Sigma_{e'\,\mathrm{inf}},\\ U\to V \ \textit{compatible with}\ f}} \mathcal{G}(V,S).$$

- The direct image functor $f_{\mathrm{inf}\,*}$ sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ to the sheaf

$$f_{\mathrm{inf}\,*}\mathcal{F}(V,S) = \varprojlim_{\substack{(U,T)\to(V,S)\\ (U,T)\in X/\Sigma_{e\mathrm{INF}}\\ U\to V \ \textit{compatible with}\ f}} \mathcal{F}(U,T).$$

Remark 3.1.4.1. In the special case when $\mathcal{G} = h_S$ is the representable sheaf of $(V, S) \in Y/\Sigma_{e'\,\mathrm{inf}}$, its inverse image $f_{\mathrm{inf}}^{-1} h_S$ has a simpler formula by

$$f_{\mathrm{inf}}^{-1} h_S(U, T) = \mathrm{Hom}_Y((U, T), (V, S)) := \{\textit{commutative diagrams}\ \begin{tikzcd} X \arrow[d, "f"'] & U \arrow[l] \arrow[r, "nil"] \arrow[d, dotted] & T \arrow[d, dotted] \\ Y & V \arrow[l] \arrow[r, "nil"] & S \end{tikzcd} \}.$$

Here in the diagram above $T \to S$ is a map over $\Sigma_{e'}$.

Remark 3.1.4.2. The functoriality of infinitesimal topoi is compatible with the projection morphism to the rigid topos and its section. Namely the following two diagrams are commutative

$$\begin{tikzcd} \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \arrow[r, "f_{\mathrm{inf}}"] \arrow[d, "u_{X/\Sigma_e}"'] & \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}}) \arrow[d, "u_{Y/\Sigma_{e'}}"] \\ \mathrm{Sh}(X_{\mathrm{rig}}) \arrow[r] & \mathrm{Sh}(Y_{\mathrm{rig}}) \end{tikzcd}, \qquad \begin{tikzcd} \mathrm{Sh}(X_{\mathrm{rig}}) \arrow[r] \arrow[d, "i_{X/\Sigma_e}"'] & \mathrm{Sh}(Y_{\mathrm{rig}}) \arrow[d, "i_{Y/\Sigma_{e'}}"] \\ \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \arrow[r, "f_{\mathrm{inf}}"] & \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}}). \end{tikzcd}$$

We also want to mention that f_{inf} is naturally a map of ringed topoi under the infinitesimal structure sheaves, and we could define the *pullback functor* f_{inf}^* on the category of $\mathcal{O}_{Y/\Sigma_e'}$-sheaves, similar to the scheme theory. Here given a sheaf of $\mathcal{O}_{Y/\Sigma_e'}$-module $\mathcal{G}$ and an object $(U, T) \in X/\Sigma_{e\,\mathrm{inf}}$, the restriction of the pullback $f_{\mathrm{inf}}^* \mathcal{G}$ at the rigid space T is equal to the colimit

$$\varinjlim_{\substack{h:(U,T)\to(V,S) \\ (V,S)\in Y/\Sigma_{e'\,\mathrm{inf}}}} h^*(\mathcal{G}_S).$$

Remark 3.1.4.3. Here we remark that when $f : X \to X$ is the identity map but e is strictly smaller than e', the transition morphism $f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e'\,\mathrm{inf}})$ is induced from the map of sites $f_{\mathrm{inf}} : X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e'\,\mathrm{inf}}$, where the corresponding functor sends $(V, S) \in X/\Sigma_{e'\,\mathrm{inf}}$ onto the thickening $(V, S \times_{\Sigma_{e'}} \Sigma_e)$.

3.2 Crystals

In this section, we study the coherent crystal and its canonical connection.

Before we start, we mention that though stated for rigid spaces over $\mathrm{B}_{\mathrm{dR},e}^+$, the results and proofs in this section hold for rigid spaces over arbitrary p-adic fields.

3.2.1 Crystals and their connections

We first introduce the coherent crystal and a canonical connection associated with it.

Sheaf of differentials

Definition 3.2.1.1. *The* infinitesimal sheaf of differentials $\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}$ *is a sheaf of $\mathcal{O}_{X/\Sigma_e}$-module on $X/\Sigma_{e\,\mathrm{inf}}$ defined as*

$$\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(U,T) := \Omega^{i,cont}_{T/\Sigma_e}(T),$$

locally given by the continuous differentials over Σ_e.

Similarly we could define the infinitesimal sheaf of differentials $\Omega^i_{X/\Sigma_{e\mathrm{INF}}}$ over the big site $X/\Sigma_{e\mathrm{INF}}$. It can be checked easily that the restriction $\mu^{-1}\Omega^i_{X/\Sigma_{e\mathrm{INF}}}$ at the small site is equal to $\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}$.

Here we recall the definition of the sheaf of continuous differentials as follows. Let T be a rigid space over Σ_e, and $T(m)$ be the $m+1$-th self product of T over Σ_e, which is equipped with $m+1$ projection maps onto T and the diagonal map from T. For each $m \in \mathbb{N}$, we denote $T(n)_m$ as the m-th infinitesimal neighborhood of T in $T(n)$. Then each infinitesimal thickening $(T, T(n)_m)$ is an object in $X\backslash\Sigma_{e\,\mathrm{inf}}$.

Let I_T be the coherent sheaf of ideals in $\mathcal{O}_{T(1)}$, defined as the kernel of the map $\mathcal{O}_{T(1)} \to \mathcal{O}_T$ given by the diagonal $T \to T(1)_1$. Then the sheaf of continuous differentials $\Omega^{1,cont}_{T/\Sigma_e}$ is the coherent sheaf I_T/I_T^2 over T. It can be checked that the sheaf of continuous differentials satisfy the universal property among continuous $\mathrm{B}^+_{\mathrm{dR,e}}$-linear derivatives. Without mentioning, we will use Ω^i_{T/Σ_e} to denote the i-th continuous differentials to simplify the notation.

Crystals

Definition 3.2.1.2. *Let $\mathcal{F}$ be a coherent sheaf over $X/\Sigma_{e\,\mathrm{inf}}$ or $X/\Sigma_{e\mathrm{INF}}$. Namely $\mathcal{F}$ is a sheaf on the infinitesimal site such that $\mathcal{F}_T$ is a coherent $\mathcal{O}_T$ module for each infinitesimal thickening (U,T). We call $\mathcal{F}$ a* coherent crystal *if for each morphism of thickenings $(i,g) : (U_1,T_1) \to (U_2,T_2)$ in the infinitesimal site, the natural map*

$$g^*\mathcal{F}_{T_2} = \mathcal{O}_{T_1} \otimes_{g^{-1}\mathcal{O}_{T_2}} g^{-1}\mathcal{F}_{T_2} \longrightarrow \mathcal{F}_{T_1}$$

is an isomorphism of $\mathcal{O}_{T_1}$ modules.

Example 3.2.1.3. The easiest example of coherent crystal is the infinitesimal structure sheaf $\mathcal{O}_{X/\Sigma_e}$, defined either over the small or big infinitesimal sites of X over Σ_e.

Remark 3.2.1.4. The infinitesimal sheaf of differential is not a crystal in general, though it is a coherent sheaf over $\mathcal{O}_{X/\Sigma_e}$.

Here it is not hard to see that the pullback of a coherent crystal is a crystal.

Lemma 3.2.1.5. *Let $f : X \to Y$ be a map of rigid spaces over $\Sigma_{e'}$, and assume the structure map $X \to \Sigma_{e'}$ factors through Σ_e for non-negative integers $e \le e'$. Let $\mathcal{G}$ be a coherent crystal over $Y/\Sigma_{e'\,\mathrm{inf}}$. We denote f_{inf} to be the functoriality map of infinitesimal topoi $f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$.*

(i) Let $(g,h) : (U,T) \to (V,S)$ be a map of thickenings for $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$ and $(V,S) \in Y/\Sigma_{e'\,\mathrm{inf}}$ separately such that $g : U \to V$ is compatible with $f : X \to Y$. Then the restriction of $f_{\mathrm{inf}}^*\mathcal{G}$ at T is naturally isomorphic to the pullback $h^*(\mathcal{G}_S)$ of the coherent sheaf $\mathcal{G}_S$ over S along the map of rigid spaces $h : T \to S$.

(ii) The pullback $f_{\mathrm{inf}}^*\mathcal{G}$ is a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$.

(iii) Both (i) and (ii) hold true for $f_{\mathrm{INF}} : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \to \mathrm{Sh}(Y/\Sigma_{e'\mathrm{INF}})$ and a coherent crystal $\mathcal{G}$ over big infinitesimal sites.

Proof. Let (U,T) be an object in the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$. By the construction, we know the restriction of the pullback $f_{\mathrm{inf}}^*\mathcal{G}$ at the rigid space T is equal to the colimit

$$\varinjlim_{\substack{h:(U,T)\to(V,S)\\ (V,S)\in Y/\Sigma_{e'\,\mathrm{inf}}}} h^*(\mathcal{G}_S),$$

where $h : T \to S$ is the map of rigid spaces over Σ_e'. On the one hand, by the definition of the coherent crystal, for a commutative diagram of infinitesimal thickenings

$$\begin{tikzcd} & (V,S) \arrow[dr, "h''"] & \\ (U,T) \arrow[ur, "h"] \arrow[rr, "h'"'] & & (V',S') \end{tikzcd}$$

that is compatible with $f : X \to Y$, the pullback $g^*(\mathcal{G}_{S'})$ is equal to the coherent sheaf $\mathcal{G}_S$ over S. On the other hand, as in Lemma 3.1.1.4 the finite products are ind-representable in the small site $Y/\Sigma_{e'\,\mathrm{inf}}$. In particular, given two maps of thickenings $h_i : (U,T) \rightrightarrows (V,S)$ where $U \to V$ is the compatible with f, both h_i locally factor through a thickening $(V, S(1)_m)$ for an infinitesimal neighborhood $S(1)_m$ of V in $S(1) = S \times_{\Sigma_{e'}} S$. As an upshot, the pullback $h^*\mathcal{G}_S$ is independent of the map h. In this way, the restriction of $f_{\mathrm{inf}}^*\mathcal{G}$ at T, which is equal to the colimit above, is naturally isomorphic to the coherent sheaf $h^*(\mathcal{G}_S)$ over T for any map of thickenings $h : (U,T) \to (V,S)$, where $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$ and $(V,S) \in Y/\Sigma_{e'\,\mathrm{inf}}$. This finishes the proof of (i).

To check the crystal condition of $f_{\mathrm{inf}}^*\mathcal{G}$, it suffices to note that given a map of objects $g : (U_1,T_1) \to (U_2,T_2)$ in $X/\Sigma_{e\,\mathrm{inf}}$ and a compatible map of infinitesimal thickenings $h : (U_2,T_2) \to$

(V,S) for $(V,S) \in Y/\Sigma_{e' \inf}$, we have

$$(f_{\inf}^* \mathcal{G})_{T_1} \cong g^* h^* (\mathcal{G}_S) \cong g^* (f_{\inf}^* \mathcal{G})_{T_2}.$$

At last, notice that the proof is applicable no matter whether the structure maps $U \to X$ and $V \to Y$ are open immersions. So we are done. $\square$

Example 3.2.1.6. An example of a coherent crystal over the big site is the pullback of a crystal from the small site

$$\mu^* \mathcal{F} = \mu^{-1} \mathcal{F} \bigotimes_{\mu^{-1} \mathcal{O}_{X/\Sigma_{e \inf}}} \mathcal{O}_{X/\Sigma_{e\mathrm{INF}}},$$

where $\mu : \mathrm{Sh}(X/\Sigma_{e\mathrm{INF}}) \to \mathrm{Sh}(X/\Sigma_{e \inf})$ is the canonical map from the big topos to the small topos, as in Subsection 3.1.1. Here the proof is identical to that of Lemma 3.2.1.5.

In particular, the pullback $\mu^* \mathcal{F}$ locally satisfies the same formula as in Lemma 3.2.1.5, (i). For a crystal $\mathcal{F}$ over the small site $X/\Sigma_{e \inf}$ and a thickening $(U,T) \in X/\Sigma_{e\mathrm{INF}}$ in the big site, the restriction of the infinitesimal sheaf $\mu^* \mathcal{F}$ on T is naturally isomorphic to the pullback $g^*(\mathcal{F}_S)$. Here the map $g : T \to S$ of rigid spaces comes from an arbitrary commutative diagram of objects $(i,g) : (U,T) \to (V,S)$ in $X/\Sigma_{e\mathrm{INF}}$ that is compatible with their structure maps $U \to X$ and $V \to X$, such that $V \to X$ is an open immersion.

In fact, we have the following results about crystals over big and small infinitesimal sites.

Proposition 3.2.1.7. *Let X be a rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. There exists a natural equivalence as below*

$$\begin{aligned} \{\textit{coherent crystals over } X/\Sigma_{e \inf}\} &\Longleftrightarrow \{\textit{coherent crystals over } X/\Sigma_{e\mathrm{INF}}\}; \\ \mathcal{F} &\longmapsto \mu^* \mathcal{F}; \\ \mu_* \mathcal{G} &\longleftarrow \mathcal{G}. \end{aligned}$$

Here we recall from Paragraph 3.1.1 that the functor μ_* is the restriction functor from $\mathrm{Sh}(X/\Sigma_{e\mathrm{INF}})$ to

$$\mathrm{Sh}(X/\Sigma_{e \inf}).$$

Proof. It suffices to show the compositions are equivalences. Given a coherent crystal $\mathcal{F}$ over the small infinitesimal site and a thickening $(U,T) \in X/\Sigma_{e \inf}$, we have

$$\begin{aligned} (\mu_* \mu^* \mathcal{F})_T &= (\mu^* \mathcal{F})_T \\ &= \mathcal{F}_T, \end{aligned}$$

where the second equality follows from the Example 3.2.1.6 for the identity map $(U,T)\to(U,T)$.

Conversely, let $\mathcal{G}$ be a coherent crystal over the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$. For any object $(V,S)\in X/\Sigma_{e\mathrm{INF}}$, it can always be covered by open affinoid subsets (V_i,S_i) such that each (V_i,S_i) admits a map to a thickening $(U,T)\in X/\Sigma_{e\inf}$. [2] We denote $g:S_i\to T$ to be the associated map of rigid spaces. Then by the crystal condition of $\mathcal{G}$, we have $\mathcal{G}_{S_i}=g^*(\mathcal{G}_T)$. As an upshot, by the Example 3.2.1.6 again we get the following equalities

$$\begin{aligned}(\mu^*\mu_*\mathcal{G})_{S_i}&=g^*\left((\mu_*\mathcal{G})_T\right)\\&=g^*(\mathcal{G}_T)\\&=\mathcal{G}_{S_i}.\end{aligned}$$

So we are done. □

Connection Recall the definition of general connections for a coherent sheaf.

Definition 3.2.1.8. *Let $\mathcal{F}$ be a coherent sheaf over $X/\Sigma_{e\inf}$. A* connection *of $\mathcal{F}$ is an $\mathrm{B}^+_{\mathrm{dR,e}}$-linear morphisms of sheaves*

$$\nabla:\mathcal{F}\longrightarrow\mathcal{F}\otimes_{\mathcal{O}_{X/\Sigma_{e\inf}}}\Omega^1_{X/\Sigma_{e\inf}},$$

such that ∇ sends $f\cdot x$ onto $f\nabla(x)+x\otimes df$, for f and x being local sections of $\mathcal{O}_{X/\Sigma_{e\inf}}$ and $\mathcal{F}$ separately.

Here we want to mention that similarly we can define the connection for coherent sheaves over the big infinitesimal site.

Now let $\mathcal{F}$ be a coherent crystal on $X/\Sigma_{e\inf}$, and let (U,T) be an object in $X/\Sigma_{e\inf}$. Then by the definition of crystals, the two projection maps $p_0,p_1:T(1)_1\to T$ induce an isomorphism of $\mathcal{O}_{T(1)_1}$-modules:

$$\varepsilon_T:p_0^*\mathcal{F}_T\cong\mathcal{F}_{T(1)_1}\cong p_1^*\mathcal{F}_T.$$

This induces a morphism of $\mathcal{O}_T$-modules given by

$$\begin{aligned}\mathcal{F}_T&\longrightarrow\mathcal{O}_{T(1)_1}\otimes_{\mathcal{O}_T}\mathcal{F}_T\xrightarrow{\varepsilon_T}\mathcal{F}_T\otimes_{\mathcal{O}_T}\mathcal{O}_{T(1)_1},\\x&\longmapsto 1\otimes x\longmapsto\varepsilon_T(1\otimes x).\end{aligned}$$

[2] To see this, we may assume the structure map $V_i\to V\to X$ maps into an open affinoid subset U of X, where U admits a closed immersion into a smooth rigid space Y. Then since $V_i\to S_i$ is a nilpotent thickening, by the smoothness of Y, the map $V_i\to U$ induces a map S_i to some Y_m, where Y_m is an infinitesimal neighborhood of U in Y. Thus the claim follows as (U,Y_m) is in the small site $X/\Sigma_{e\inf}$.

Here we identify the sheaf of $\mathcal{O}_{T(1)_1}$-module $p_1^*\mathcal{F}_T$ as $\mathcal{O}_{T(1)_1} \otimes_{\mathcal{O}_T} \mathcal{F}_T$ (similarly for $p_0^*\mathcal{F}_T = \mathcal{F}_T \otimes_{\mathcal{O}_T} \mathcal{O}_{T(1)_1}$). Besides, the pullback of the above sequence along the diagonal map $T \to T(1)_1$ is the identity, so the image of $\varepsilon_T(1 \otimes x)$ under this pullback map is exactly x.

The map in fact defines a *canonical connection* structure on the sheaf of the $\mathcal{O}_T$-module $\mathcal{F}_T$, by

$$\nabla_T : \mathcal{F}_T \longrightarrow \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^1_{T/\Sigma_e}.$$

$$x \longmapsto \varepsilon_T(1 \otimes x) - x \otimes 1.$$

Here $\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^1_{T/\Sigma_e} = \mathcal{F}_T \otimes_{\mathcal{O}_T} I_T/I_T^2$ can be identified as a subsheaf of $\mathcal{F}_T \otimes_{\mathcal{O}_T} \mathcal{O}_{T(1)_1}$, since $\mathcal{O}_{T(1)_1}$ decomposes into the direct sum $\mathcal{O}_T \oplus \Omega^1_{T/\Sigma_e}$ as a left $\mathcal{O}_T$-module. Note that the map satisfies the axiom for the connection, in the sense that for a section f of $\mathcal{O}_T$ and x of $\mathcal{F}_T$, we have

$$\nabla_T(f \cdot x) = f\nabla_T(x) + x \otimes df,$$

where $df = 1 \otimes f - f \otimes 1$ is in Ω^1_{T/Σ_e}.

At last, we notice that the above is functorial with respect to $(U, T) \in X/\Sigma_{e\,\mathrm{inf}}$, in the sense that for a morphism $(i, g) : (U_1, T_1) \to (U_2, T_2)$, we have the following commutative diagram

$$\begin{tikzcd} g^*(\mathcal{F}_{T_2}) \arrow[r, "g^*\nabla_{T_2}"] \arrow[d] & g^*(\mathcal{F}_{T_2} \otimes_{\mathcal{O}_{T_2}} \Omega^1_{T_2/\Sigma_e}) \arrow[d] \\ \mathcal{F}_{T_1} \arrow[r, "\nabla_{T_1}"] & \mathcal{F}_{T_1} \otimes_{\mathcal{O}_{T_1}} \Omega^1_{T_1/\Sigma_e}. \end{tikzcd}$$

In particular, the functoriality leads to the morphism of sheaves over infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^1_{X/\Sigma_{e\,\mathrm{inf}}}.$$

Definition 3.2.1.9. *Let $\mathcal{F}$ be a coherent crystal over the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$. The* canonical connection *of $\mathcal{F}$ is defined as the morphism as above*

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^1_{X/\Sigma_{e\,\mathrm{inf}}}.$$

de Rham complex of a crystal Similar to the flat connection over schemes ([Ber74], Chapter II, Section 3.2), we can associate a natural de Rham complex to a coherent crystal over the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$ or $X/\Sigma_{e\mathrm{INF}}$, by the integrability of the canonical connection.

Let $\mathcal{F}$ be a coherent sheaf over $\mathcal{O}_{X/\Sigma_e}$ with a connection ∇. For each $k \in \mathbb{N}_+$ we can associate an $\mathcal{O}_{X/\Sigma_e}$-linear morphism

$$\nabla^k : \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^k_{X/\Sigma_e} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^{k+1}_{X/\Sigma_{e\,\mathrm{inf}}},$$

locally given by

$$x \otimes \omega \longmapsto \nabla(x) \wedge \omega + x \otimes d\omega.$$

This produces a chain of maps

$$(\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}, \nabla) := \; 0 \longrightarrow \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^1_{X/\Sigma_{e\,\mathrm{inf}}} \xrightarrow{\nabla^1} \cdots .$$

The connection is called *integrable* if the composition $\nabla^1 \circ \nabla$ is zero, under which assumption we call the above complex the *de Rham complex of $\mathcal{F}$*.

The following proposition justifies the name:

Proposition 3.2.1.10. *Let $\mathcal{F}$ be a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$, and let ∇ be its canonical connection defined in last subsection. Then for each $k \in \mathbb{N}$, we have*

$$\nabla^{k+1} \circ \nabla^k = 0.$$

In particular, the de Rham complex of $\mathcal{F}$ is in fact a complex.

Proof. The proof is identical to that for a crystal over the crystalline site of a scheme, and we refer the reader to[Sta18, Tag 07J6]. □

3.2.2 A criterion for crystal in vector bundles

Given a coherent crystal $\mathcal{F}$ over the infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$, we say $\mathcal{F}$ is a *crystal in vector bundles* if the restriction $\mathcal{F}_T$ is locally free of finite rank over $\mathcal{O}_T$, for every object $(U, T) \in X/\Sigma_{e\,\mathrm{inf}}$. In this subsection, we provides a simple criterion when a coherent crystal is a crystal in vector bundles.

Definition 3.2.2.1. *A coherent crystal $\mathcal{F}$ is* flat over $\mathrm{B}^+_{\mathrm{dR,e}}$ *if for any thickening (U, T) in the infinitesimal site with T being flat over $\mathrm{B}^+_{\mathrm{dR,e}}$, the restriction $\mathcal{F}_T$ at T is also flat over $\mathrm{B}^+_{\mathrm{dR,e}}$.*

Theorem 3.2.2.2. *Let $\mathcal{F}$ be a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$, and let $\mathcal{F}$ be a coherent crystal over that is flat over $\mathrm{B}^+_{\mathrm{dR,e}}$ in the sense of Definition 3.2.2.1. Then $\mathcal{F}$ is a crystal in vector bundles.*

Proof.

Step 1 We first consider the case when $e = 1$. Namely let us assume X is defined over K and $\mathcal{F}$ is a coherent crystal over X/K_{inf} or X/K_{INF}, where the flatness of $\mathcal{F}$ over $\mathrm{B}^+_{\mathrm{dR,e}} = K$ is automatic.

For $m \in \mathbb{N}$, let $T(1)_m$ be the m-th infinitesimal neighborhood of T in $T \times_K T$. The projection map pr_0 to the first factor induces a map from $T(1)_m$ to T, via

$$h:\ T(1)_m \longrightarrow T \times_K T \xrightarrow{\mathrm{pr}_0} T.$$

Moreover, as $T = T(1)_0$ admits a closed immersion into $T(1)_m$, we can form the following non-commutative diagram of thickenings

$$\begin{tikzcd} & T \arrow[dr, hook] & & \\ T(1)_m \arrow[ur, "h"] \arrow[rr, "\mathrm{id}"'] & & T(1)_m \arrow[r, "h"] & T. \end{tikzcd} \tag{$*$}$$

Here we notice that the composition $T \hookrightarrow T(1)_m \xrightarrow{h} T$ above is the identity. We denote the composition of the map $h : T(1)_m \to T$ and the closed immersion $T \to T(1)_m$ and by $g : T(1)_m \to T(1)_m$. Then we get two maps of thickening of U as follows

$$\begin{tikzcd} T(1)_m \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "g"'] & T(1)_m. \end{tikzcd}$$

Then by the definition of the coherent crystal, pulling back along the above two arrows induces an isomorphism of coherent sheaves over $T(1)_m$

$$g^*\mathcal{F}_{T(1)_m} \longrightarrow \mathcal{F}_{T(1)_m}.$$

Moreover, by the assumption on g, we have

$$g^*\mathcal{F}_{T(1)_m} = h^*\mathcal{F}_T. \tag{$**$}$$

Now we base change the diagram $(*)$ above along the closed immersion $t : \mathrm{Spa}(K) \to T$ of any K-point t of T. By the construction of the map $h : T(1)_m \to T$, we get a non-commutative diagram

$$\begin{tikzcd}
 & t = \mathrm{Spa}(K) \arrow[dr, hook] & & \\
t_m \arrow[ur] \arrow[rr, "\mathrm{id}"'] & & t_m \arrow[r] & t,
\end{tikzcd}$$

where t_m is the m-th infinitesimal neighborhood of t in T, and the base changed map $h : t_m \to t = \mathrm{Spa}(K)$ is the structure map of t_m over $\mathrm{Spa}(K)$. Furthermore, after the base change, the isomorphism $g^*\mathcal{F}_{T(1)_m} \cong \mathcal{F}_{T(1)_m}$ in $(**)$ becomes the following isomorphism of torsion sheaves over T that are supported at t

$$\mathcal{F}_T \otimes t_m \cong h^*(\mathcal{F}_T \otimes t).$$

Notice that the fiber $\mathcal{F}_T \otimes t$ is flat and finitely generated over K (i.e it is a finite dimensional vector space over K) and the pullback $h^*(\mathcal{F}_T \otimes t)$ is flat over t_m. Thus by the equality above the base change $\mathcal{F}_T \otimes t_m$ is also flat and finitely generated over t_m.

At last, we take the inverse limit of $\mathcal{F}_T \otimes t_m$ with respect to m. Then $\mathcal{F}_T \otimes_{\mathcal{O}_T} \widehat{\mathcal{O}}_{T,t}$ is flat over the formal completion $\widehat{\mathcal{O}}_{T,t}$ of the rigid space T at the K-point t. Since T is locally noetherian, the formal completion $\widehat{\mathcal{O}}_{T,t}$ is isomorphic to the completion $\widehat{\mathcal{O}}_{T,\bar{t}}$, where the latter is the formal completion of T along its reduced K-valued point $\bar{t}$. In this way, by the faithful flatness of $\widehat{\mathcal{O}}_{T,t}$ over $\mathcal{O}_{T,t}$, the stalk of the coherent sheaf $\mathcal{F}_T$ at t is flat and finitely generated over the local ring, thus projective. Hence by the density of K-points in T, we get the local freeness of $\mathcal{F}_T$.

Step 2 For general $e \in \mathbb{N}$, we make the following claim.

Claim 3.2.2.3. Let A be a flat topologically finite type algebra over $\mathrm{B}_{\mathrm{dR,e}}^+$, and let M be a finite A-module that is flat over $\mathrm{B}_{\mathrm{dR,e}}^+$. Suppose M/ξ is free over A/ξ. Then M is free over A.

Proof of Claim. We prove the claim by induction on e. When $e = 1$, there is nothing to prove. Suppose $e \geq 2$. We choose a map of A-modules $f : A^{\oplus r} \to M$ whose reduction mod ξ is an isomorphism. Then as f is a map of flat $\mathrm{B}_{\mathrm{dR,e}}^+$-modules, the short exact sequence

$0 \to \mathrm{B}^+_{\mathrm{dR},e-1} \to \mathrm{B}^+_{\mathrm{dR,e}} \to K \to 0$ induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^{\oplus r} \otimes_{\mathrm{B}^+_{\mathrm{dR,e}}} \mathrm{B}^+_{\mathrm{dR},e-1} \arrow[r] \arrow[d] & A^{\oplus r} \arrow[r] \arrow[d] & A/\xi^{\oplus r} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes_{\mathrm{B}^+_{\mathrm{dR,e}}} \mathrm{B}^+_{\mathrm{dR},e-1} \arrow[r] & M \arrow[r] & M/\xi \arrow[r] & 0.
\end{tikzcd}$$

Hence the map f is an isomorphism of A-modules by induction. □

Let X' be the base change $X \times_{\Sigma_e} \mathrm{Spa}(K)$. We notice that the pullback $\mathcal{F}/\xi$ of $\mathcal{F}$ along $\mathrm{Sh}(X'/K_{\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ is then a coherent crystal over X'/K_{inf}. Now let (U,T) be a thickening in $X/\Sigma_{e\,\mathrm{inf}}$ such that T is flat over $\mathrm{B}^+_{\mathrm{dR,e}}$, so $(U',T') = (U \times_{\Sigma_e} \mathrm{Spa}(K), T \times_{\Sigma_e} \mathrm{Spa}(K))$ is a thickening in X'/K_{inf}. Then the restriction of $\mathcal{F}/\xi$ on T' for the thickening $(U',T') \in X'/K_{\mathrm{inf}}$ is equal to $\mathcal{F}_T/\xi$, which is crystal over $\mathcal{O}_{T'} = \mathcal{O}_T/\xi$ by the Step 1. So the flatness assumption of $\mathcal{F}$ over $\mathrm{B}^+_{\mathrm{dR,e}}$ implies that $\mathcal{F}_T$ is vector bundle over T whenever T is flat over $\mathrm{B}^+_{\mathrm{dR,e}}$.

At last, we note the following small observation on thickenings.

Claim 3.2.2.4. For any K-point $\bar{t}$ of T, there exists a nilpotent closed immersion $i : T \to T'$ such that $\bar{t}$ admits a $\mathrm{B}^+_{\mathrm{dR,e}}$-lift in T'.

Granting the claim, as (U,T') is also a thickening and $\mathcal{F}_{T'}$ is locally free around the lift of $\bar{t}$, we get the local freeness of $\mathcal{F}_T$ around $\bar{t}$ via the pullback equality

$$\mathcal{F}_T = i^*\mathcal{F}_{T'}.$$

As this is true for any K-point of T, by the density of K-points in the adic spectrum we get the local freeness of $\mathcal{F}_T$ for general T. So we are done.

Proof of the Claim. We now settle the claim. As the statement is local over T, we may assume $T = \mathrm{Spa}(A)$ is affinoid and admits a closed immerion into a polydisc $Y = \mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR,e}}\langle U_i \rangle)$ with the defining ideal (f_j). Assume the image of U_i in K along the closed immersion $\bar{t} \to T \to Y$ is a_i, and let $\widetilde{a}_i$ be a lift of a_i in $\mathrm{B}^+_{\mathrm{dR,e}}$. We then note that the evaluation $f_j(\widetilde{a}_i)$ is contained in the nilpotent ideal $\xi\mathrm{B}^+_{\mathrm{dR,e}}$, for the defining equations f_j of T. Thus we can define $T' = \mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR,e}}\langle U_i \rangle/(f_j)^e$, which is a nil extension of T such that the closed immersion $\bar{t} \to T \to T'$ admits a natural lift to a $\mathrm{B}^+_{\mathrm{dR,e}}$-point via $\widetilde{a}_i$. So we are done.

□

$\square$

Corollary 3.2.2.5. *Any coherent crystal over the infinitesimal site X/K_{inf} or X/K_{INF} is a crystal in vector bundles.*

3.2.3 Integrable connections over envelope

As in the crystalline theory of schemes, there exists an equivalence between the category of coherent crystals over $X/\Sigma_{e\,\text{inf}}$ and the category of coherent sheaves with integrable connections over the envelope.

Before we state the result, we recall from Remark 3.1.2.2 that given an envelope $D = \varinjlim_{n\in\mathbb{N}} Y_n$ of a locally closed immersion $X \to Y$, we can regard the envelope as a locally ringed space over the adic space X. The structure sheaf $\mathcal{D}$ of the envelope is defined as the inverse limit $\varprojlim_{m\in\mathbb{N}} \mathcal{O}_{Y_m}$ over D.

Theorem 3.2.3.1. *Let $X \to Y$ be a closed closed immersion of rigid spaces with Y being smooth over Σ_e, and let $D = D_X(Y)$ be the envelope of X in Y. Then we have a natural equivalence of categories:*

$$\{coherent\ cyrstals\ over\ \mathcal{O}_{X/\Sigma_e}\} \longrightarrow \{(M,\nabla)|\ M \in \mathrm{Coh}(\mathcal{D}),\ \nabla\ integrable\ connection\}$$
$$\mathcal{F} \longmapsto (\mathcal{F}_D, \nabla_D).$$

Here crystals are over either the big infinitesimal site or small infinitesimal site.

Corollary 3.2.3.2. *Let $X \to Y$ be a closed immersion of rigid spaces with Y being smooth over Σ_e, and let $D = D_X(Y)$ be the envelope of X in Y. Then the equivalence above induces the bijections of the following three categories:*

- $\{coherent\ cyrstals\ that\ is\ flat\ over\ \mathrm{B}^+_{\mathrm{dR,e}}\}$.
- $\{crystals\ in\ vector\ bundles\}$.
- $\{(M,\nabla)|\ M \in \mathrm{Vec}(\mathcal{D}),\ \nabla\ an\ integrable\ connection\}$.

Before the proof, we first give a description of the sheaf of differentials over the envelope.

Lemma 3.2.3.3. *Let $X = \mathrm{Spa}(A) \to Y = \mathrm{Spa}(P)$ be a closed immersion of affinoid rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$, with P a smooth affinoid algebra over $\mathrm{B}^+_{\mathrm{dR,e}}$.*

Then we have the following canonical isomorphism

$$\Omega^1_D := \Omega^1_{X/\Sigma_{e\,\text{inf}}}(D) \cong \Omega^1_{P/\Sigma_e} \otimes_P \mathcal{D},$$

which is induced from the map $P \to \mathcal{D}$.

Moreover, the result is true for $\Omega^1_{X/\Sigma_{e\mathrm{INF}}}$ over the big infinitesimal site.

Proof. Recall that Ω^1_D is defined as

$$\Omega^1_{X/\Sigma_{e\inf}}(\varinjlim_{m\in\mathbb{N}} Y_m),$$

which is equal to the inverse limit of the continuous differentials

$$\Gamma(X, \varprojlim_{m\in\mathbb{N}} \Omega^1_{Y_m/\Sigma_e}).$$

Denote t_i to be the étale coordinate of P. This is guaranteed locally by the Jacobian criterion of smoothness, as in [Hub96, 1.6.9]. Let $J = (f_1, \ldots, f_s)$ be the kernel of the surjection $P \to A$, with f_i being its generator. Then we have

$$\mathcal{O}(Y_m) = P/J^{m+1},\ \Omega^1_{Y_m/\Sigma_e} = (\bigoplus_i \mathcal{O}(Y_m)dt_i / \sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df)\tilde{.}$$

So we get

$$\mathcal{D} = \varprojlim_{m\in\mathbb{N}} P/J^{m+1},\ \Omega^1_D = \varprojlim_{m\in\mathbb{N}} (\bigoplus \mathcal{O}(Y_m)dt_i / \sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df),$$

Now consider the natural map $\Omega^1_{P/\Sigma_e} \otimes_P \mathcal{D} \to \Omega^1_D$, sending the generator dt_i of Ω^1_{P/Σ_e} onto the dt_i in the limit.

- We first consider the injectivity. By writing each $f \in J^{m+1}$ as a finite sum of $af_{j_1}\cdots f_{j_{m+1}}$ for $1 \leq j_l \leq r$, each such df is contained in $\sum_j J^m\mathcal{O}(Y_m)df_j$. In particular, the submodule $\sum_{f\in J^{m+1}} \mathcal{O}(Y_m)df$ of $\oplus\,\mathcal{O}(Y_m)dt_i$ is contained the submodule $\sum_j J^m\mathcal{O}(Y_m)df_j$. So it suffices to show the injectivity of

$$\Pi : \bigoplus \mathcal{D}dt_i \longrightarrow \varprojlim_{m\in\mathbb{N}} (\bigoplus \mathcal{O}(Y_m)dt_i / \sum_j J^m\mathcal{O}(Y_m)df_j).$$

  However, the kernel of each $\oplus\,\mathcal{D}dt_i \to \oplus\,\mathcal{O}(Y_m)dt_i/\sum_j J^m\mathcal{O}(Y_m)df_j$ is equals to

$$\sum J^{m+1}\mathcal{D}dt_i + \sum_j J^m df_j,$$

  which is contained in $\oplus\, J^m\mathcal{D}dt_i$. In particular, any element $\sum_i g_i dt_i$ in the $\ker(\Pi)$ is contained in the ideal

$$\bigcap_m \bigoplus J^m \mathcal{D} dt_i.$$

But note that $\mathcal{D}$ is defined as the J-adic completion of P, which implies the above ideal is zero. So we get the injectivity.

- we can write $\Omega^1_{P/\Sigma_e} \otimes_P \mathcal{D}$ as the limit $\bigoplus \mathcal{D} dt_i = \varprojlim_{m\in\mathbb{N}} (\bigoplus \mathcal{O}(Y_m) dt_i)$. Then for each m the map $\bigoplus \mathcal{O}(Y_m) dt_i \to \bigoplus_i \mathcal{O}(Y_m) dt_i / \sum_{f\in J^{m+1}} \mathcal{O}(Y_m) df$ is surjective. For each m, he kernel of the map is $M_m := \sum_{f\in J^{m+1}} \mathcal{O}(Y_m) df$, whose image in M_{m-1} is zero. Thus we get the surjectivity, by the pro-acyclicity of the kernel.

$\square$

The local freeness of the differential sheaf over the envelope allows us to give a more explicit description of the connection associated with a crystal. We assume $X = \mathrm{Spa}(A) \to \mathrm{Spa}(P) = Y$ be a closed immersion of affinoid rigid spaces over Σ_e, such that Y is smooth over Σ_e with a local coordinates $\{t_i\}$. Let M be a coherent sheaf over $\mathcal{D}$ together with a connection ∇ over $\mathrm{B}^+_{\mathrm{dR,e}}$. By Lemma 3.2.3.3 above, the restriction of the infinitesimal differential over $D = D_X(Y)$ is free over $\mathcal{D} = \mathcal{O}_{X/\Sigma_e}(D) = \varprojlim \mathcal{O}(Y_m)$ with a basis dt_i. So for any section $x \in M$, we have

$$\nabla(x) = \sum_i \nabla_i(x) \otimes dt_i,$$

where $\nabla_i : M \to M$ is an $\mathrm{B}^+_{\mathrm{dR,e}}$-linear derivation map.

Now we assume (M, ∇) is integrable. We compose ∇ with ∇^1, and get

$$\begin{aligned}
\nabla^1(\nabla(x)) &= \sum_i \nabla^1(\nabla_i(x) \otimes dt_i) \\
&= \sum_j \sum_i \nabla_j(\nabla_i(x)) \otimes dt_j \wedge dt_i + \sum_i \nabla_i(x) \otimes d(dt_i) \\
&= \sum_j \sum_i \nabla_j(\nabla_i(x)) \otimes dt_j \wedge dt_i.
\end{aligned}$$

By the local freeness of Ω^1_D, the element $dt_j \wedge dt_i$ for $j < i$ forms a basis of Ω^2_D. So we can rewrite the above as

$$\nabla^1(\nabla(x)) = \sum_{j<i} (\nabla_j(\nabla_i(x)) - \nabla_i(\nabla_j(x))) \otimes dt_j \wedge dt_i.$$

By the integrability condition of ∇, the above vanishes for any $x \in \mathcal{F}_D$. So we obtain the following equalities

$$\nabla_i \circ \nabla_j = \nabla_j \circ \nabla_i.$$

Here we note that the commutativity allows us to write the composition of a finite amount of ∇_i as

$$\prod_{E=(e_i)} \nabla_i^{e_i},$$

where $E = (e_i)$ is a tuple of non negative integers parametrized by i.

Now we are ready for the proof of Theorem 3.2.3.1.

Proof of Theorem 3.2.3.1. For a crystal $\mathcal{F}$ over $\mathcal{O}_{X/\Sigma_e}$, we can equip it with its canonical connection $\mathcal{F}$, which is integrable by Proposition 3.2.1.10. So by taking the associated coherent sheaf of $\mathcal{F}$ over D, we get a coherent sheaf $\mathcal{F}_D$ together with an integrable connection ∇_D.

Conversely, let M be a coherent sheaf over $\mathcal{D}$ with an integrable connection ∇. By the smoothness of Y over Σ_e, any object in $X/\Sigma_{e\,\mathrm{inf}}$ can be covered by an open affinoid covering where each piece admits a map to (X, Y). We assume (U, T) is an affinoid thickening with the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & T \arrow[d, "g"] \\ X \arrow[r] & Y \end{tikzcd}$$

Since T is an nilpotent extension of U, the map $g : T \to Y$ factors through the envelope $D = \varinjlim_{m\in\mathbb{N}} Y_m$ of X in Y. We denote this map by $f : T \to D$. Then we get a coherent sheaf $f^*M = M \otimes_{\mathcal{D}} \mathcal{O}_T$ over T.

Now we make the following claim:

Claim 3.2.3.4. The pullback f^*M over T is independent of the choice of $f : T \to D$.

More precisely, let $f_1, f_2, f_3 : T \to D$ be any three maps induced produced as above. Then there exists natural isomorphisms of coherent sheaves $h_{ij} : f_i^*M \to f_j^*M$ over $\mathcal{O}_T$ such that

$$h_{23} \circ h_{12} = h_{13}.$$

We first grant the claim. For each thickening (U, T), we pick an arbitrary covering (U_i, T_i) of (U, T), where (U_i, T_i) admits a map to (X, Y). Then we get the collection of coherent sheaves f_i^*M over each T_i. The claim allows us to produce a transition isomorphism for each restriction of f_i^*M on $T_i \cap T_j$, and they satisfy the cocycle condition when restricted at $T_i \cap T_j \cap T_k$. Hence by gluing them together, we get a coherent sheaf $\mathcal{F}_T$ over (U, T). This produces a sheaf $\mathcal{F}$ over the infinitesimal site. Moreover, the coherent sheaf $\mathcal{F}$ is in fact a coherent crystal, namely the pullback $g^*\mathcal{F}_{T_2} \cong \mathcal{F}_{T_1}$ for any map $(i, g) : (U_1, T_1) \to (U_2, T_2)$ in $X/\Sigma_{e\,\mathrm{inf}}$. This comes from the independence in the claim again, by taking a composition with a map $T_2 \to D$. So we are done.

Proof of the Claim. We at last deal with the Claim. Let $\varphi_j : \mathcal{D} \to \mathcal{O}_T$ be maps of structure sheaves induced from $f_j : T \to D$. We define h_{jk} to be the $\mathcal{O}_T$-linear map given by

$$x \otimes 1 \longmapsto \sum_{E=(e_i)} (\prod_i \nabla_i^{e_i})(x) \otimes \frac{(\varphi_j(t_i) - \varphi_k(t_i))^{e_i}}{e_i!}.$$

Since T is a nilpotent extension of U, for each $t \in \mathcal{D}$, the difference $\varphi_j(t) - \varphi_k(t)$ is nilpotent in $\mathcal{O}_T$. In particular, the above sum is only finite. At last, by the general equality

$$\sum_{n=0}^{N} \frac{u^n}{n!} \cdot \frac{v^{N-n}}{(N-n)!} = \frac{(u+v)^N}{N!},$$

we have $h_{23} \circ h_{12} = h_{13}$. □

□

3.3 Cohomology over $\mathrm{B}^+_{\mathrm{dR,e}}$

In this section, we compute the cohomology of crystals over $X/\Sigma_{e\,\mathrm{inf}}$ using the de Rham complex over the envelope. Our strategy is to construct a double complex computing the Čech-Alexander complex and the de Rham complex in two separate directions, as in [BdJ11].

Remark 3.3.0.1. Before we start, we mention that though our focus is rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$, the discussion in this section works alphabetically for cohomology of crystals over $X/K_{0,\mathrm{inf}}$, where K_0 is an arbitrary p-adic complete non-archimedean field and X is a rigid space over K_0.

3.3.1 Cohomology of crystals over affinoid spaces

We first compute the cohomology of crystals over $X/\Sigma_{e\,\mathrm{inf}}$, for X being an affinoid rigid space over Σ_e.

Let $X = \mathrm{Spa}(A)$ be an affinoid rigid space over Σ_e, together with a closed immersion $X \to Y = \mathrm{Spa}(P)$ for a smooth affinoid rigid space Y over $\mathrm{B}^+_{\mathrm{dR,e}}$. Denote by D to be the envelope of X in Y (Definition 3.1.2.1), $\mathcal{D}$ to be its structure sheaf $\varprojlim_m \mathcal{O}_{Y_m}$ (where Y_m is the m-th infinitesimal neighborhood of X in Y), and J to be the kernel ideal for $\mathcal{D} \to \mathcal{O}_X$. We write Ω^i_D as the group of differentials $\Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}(D)$, which is equal to the inverse limit of continuous differentials

$$\varprojlim_m \Omega^i_{Y_m/\Sigma_e}.$$

By Lemma 3.2.3.3, Ω^i_D is isomorphic to the tensor product $\Omega^i_{Y/\Sigma_e} \otimes_{\mathcal{O}_Y} \mathcal{D}$, and is in particular locally free over $\mathcal{D}$.

We then take the section of the infinitesimal de Rham complex $(\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^\bullet_{X/\Sigma_{e\,\inf}}, \nabla)$ at D, and get a chain complex of $\mathrm{B}^+_{\mathrm{dR,e}}$-modules

$$(M \otimes \Omega^\bullet_D, \nabla_D) := \; 0 \longrightarrow M \xrightarrow{\nabla} M \otimes_{\mathcal{D}} \Omega^1_D \xrightarrow{\nabla^1} \cdots ,$$

where M is the evaluation $\mathcal{F}(D)$ of $\mathcal{F}$ at the envelope D. The complex is naturally filtered by the infinitesimal filtration, whose i-th filtration is the subcomplex

$$0 \longrightarrow J^i M \longrightarrow J^{i-1} M \otimes_{\mathcal{D}} \Omega^1_D \longrightarrow J^{i-2} M \otimes_{\mathcal{D}} \Omega^2_D \longrightarrow \cdots .$$

Our main theorem in this subsection is the following:

Theorem 3.3.1.1. *Let X, Y, $\mathcal{F}$ and M be as above. Then we have a natural filtered isomorphism in the filtered derived category of abelian groups:*

$$R\Gamma(X/\Sigma_{e\,\inf}, \mathcal{F}) \longrightarrow (M \otimes \Omega^\bullet_D, \nabla_D),$$

where the left side is filtered by the infinitesimal filtration.

Remark 3.3.1.2. Note that by the Corollary 3.1.2.8, the above is also isomorphic to the cohomology of the crystal $\mathcal{G}$ over the big infinitesimal site, when $\mathcal{F} = \mu_* \mathcal{G}$ is the restriction of $\mathcal{G}$ defined over the big site $X/\Sigma_{e\mathrm{INF}}$.

The rest of this subsection will be devoted to the proof of the theorem.

Let us first fix some notations for this section.

Denote by $D(n)$ to be the envelope of X in the $(n+1)$-fold self product of Y over Σ_e. When $n = 0$, we write $D(0)$ as D. The simplicial object $D(\bullet)$ forms a hypercovering of the final object in $\mathrm{Sh}(X/\Sigma_{e\,\inf})$, as in Proposition 3.1.2.7.

We fix a coherent crystal $\mathcal{F}$ on $X/\Sigma_{e\,\inf}$. Denote $M(n)$ to be the group of sections $\mathcal{F}(D(n))$ of $\mathcal{F}$ at $D(n)$, $\mathcal{D}(n)$ to be $\mathcal{O}_{X/\Sigma_e}(D(n))$, $J(n)$ to be the kernel for $\mathcal{D}(n) \to \mathcal{O}_X$, and $\Omega^i_{D(n)}$ to be $\Omega^i_{X/\Sigma_{e\,\inf}}(D(n))$. When $n = 0$, we use M and Ω^i_D to abbreviate $M(0)$ and $\Omega^i_{D(0)}$. Here we recall that $\Omega^i_{D(n)} = \Omega^i_{X/\Sigma_{e\,\inf}}(D(n))$ is isomorphic to the tensor product $\Omega^i_{Y(n)/\Sigma_e} \otimes_{\mathcal{O}_{Y(n)}} \mathcal{D}(n)$ (Lemma 3.2.3.3), and is in particular locally free over $\mathcal{D}(n)$.

Čech-Alexander complex First we introduce the Čech-Alexander complex of a coherent $\mathcal{O}_{X/\Sigma_e}$ sheaf $\mathcal{F}$ (not necessarily to be a crystal).

We define $M(\bullet)$ to be the filtered cosimplicial cochain complex

$$M(\bullet) := (\mathcal{F}(D(0)) \longrightarrow \mathcal{F}(D(1)) \longrightarrow \cdots),$$

where the coboundary map is given by the alternating sum of degeneracy maps, and the filtration is the infinitesimal filtration whose i-th filtration at $D(n)$ is $J(n)^i \cdot \mathcal{F}(D(n))$. It is called the *Čech-Alexander complex of $\mathcal{F}$*.

Proposition 3.3.1.3. *Let $\mathcal{F}$ be a coherent infinitesimal sheaf of $\mathcal{O}_{X/\Sigma_e}$-modules as above. Then the Čech-Alexander complex of $\mathcal{F}$ with its filtration computes the cohomology of $\mathcal{F}$. Namely, we have a functorial filtered isomorphism*

$$R\Gamma(X/\Sigma_{e\,\inf}, \mathcal{F}) \cong M(\bullet),$$

in the filtered derived category of abelian groups.

Proof. We first notice that by Proposition 3.1.2.7 about the envelope, we have

$$R\Gamma(X/\Sigma_{e\,\inf}, -) = R\Gamma(D(\bullet), -) = R \lim_{[n]\in\Delta^{\mathrm{op}}} R\Gamma(D(n), -).$$

Moreover, replacing $R\Gamma(X/\Sigma_{e\,\inf}, -)$ by its filtered analogue, the same equality holds on filtered sheaves over $X/\Sigma_{e\,\inf}$.

Denote by $Y(n)_m$ to be the m-th infinitesimal neighborhood of X in $Y(n)$. Since X is the common closed analytic subspace of every $Y(n)_m$, $Y(\bullet)_m$ forms a simplicial object in $X/\Sigma_{e\,\inf}$ with $D(\bullet) = \varinjlim_{m\in\mathbb{N}} h_{Y(\bullet)_m}$. This leads to the equality

$$R\Gamma(D(\bullet), -) = R \varprojlim_{m\in\mathbb{N}} R\Gamma(Y(\bullet)_m, -).$$

Notice that since $Y(n)_m$ is affinoid for each n, by the vanishing of cohomology for coherent sheaves over affinoid rigid spaces, we know

$$R\Gamma(Y(\bullet)_m, \mathcal{F}) = \Gamma(Y(\bullet)_m, \mathcal{F}).$$

Furthermore, by the coherence of $\mathcal{F}$ and the noetherian of $\mathcal{O}(Y(n)_m)$, for each $n \in \mathbb{N}$ the inverse system $\Gamma(Y(n)_m, \mathcal{F})$ satisfies the Mittag-Leffler condition. In this way, we get

$$\begin{aligned} R\varprojlim_{m\in\mathbb{N}} R\Gamma(Y(\bullet)_m, \mathcal{F}) &= R\varprojlim_{m\in\mathbb{N}} \Gamma(Y(\bullet)_m, \mathcal{F}) \\ &= \varprojlim_{m\in\mathbb{N}} \Gamma(Y(\bullet)_m, \mathcal{F}) \\ &= \Gamma(\varinjlim_{m\in\mathbb{N}} Y(\bullet)_m, \mathcal{F}) \\ &= M(\bullet). \end{aligned}$$

□

Čech-Alexander and the de Rham We then connect the Čech-Alexander complex with the de Rham complex together.

Consider the section of the de Rham complex $(\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^\bullet_{X/\Sigma_e \inf}, \nabla)$ at the simplicial space $D(n)$:

$$(M(n) \otimes_{\mathcal{D}(n)} \Omega^\bullet_{D(n)}, \nabla).$$

This produces a double complex $M^{n,m} = M(n) \otimes_{\mathcal{D}(n)} \Omega^m_{D(n)}$ in the first quadrant, with the horizontal coboundary map given by the alternating sum of degeneracy maps for simplicial space $D(\bullet)$, and the vertical coboundary map being the de Rham differential ∇^m. Note that the first column $M^{0,\bullet}$ of this double complex is the de Rham complex $M \otimes_{\mathcal{D}} \Omega^\bullet_D$, while the first row $M^{\bullet,0}$ is the Čech-Alexander complex $M(\bullet)$. So this provides a natural framework for those two types of complexes that we care about.

Moreover, the double complex is naturally filtered via the infinitesimal filtration $\mathcal{O}_{X/\Sigma_e} \supset \mathcal{J}_{X/\Sigma_e} \supset \mathcal{J}^2_{X/\Sigma_e} \cdots$. This is a descending filtration on the double complex, compatible with the cosimplicial structure, such that the i-th filtration on the n-th column is the differential complex

$$J(n)^i \longrightarrow J(n)^{i-1}\Omega^1_{D(n)} \longrightarrow \cdots \longrightarrow J(n)^0\Omega^i_{D(n)} \longrightarrow \Omega^{i+1}_{D(n)} \longrightarrow \cdots,$$

as a subcomplex of $\Omega^\bullet_{D(n)}$. Here we recall $J(n)$ is the kernel ideal of the surjection $\mathcal{D}(n) \to \mathcal{O}_X$, defined as $\mathcal{J}_{X/\Sigma_e}(D(n))$. Note that when $X = Y$ are smooth over $\mathrm{B}^+_{\mathrm{dR,e}}$, the filtration on $\Omega^\bullet_D = \Omega^\bullet_{X/\mathrm{B}^+_{\mathrm{dR,e}}}$ is the usual Hodge filtration.

Furthermore, there are two canonical E_1 spectral sequences associated with the double complex $M^{n,m}$ ([Sta18], Tag 0130), with the formations given by

$$\begin{aligned} {}'E_1^{p,q} &= \mathrm{H}^q(M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}); \\ {}''E_1^{p,q} &= \mathrm{H}^q(M(\bullet) \otimes_{\mathcal{D}(\bullet)} \Omega^p_{D(\bullet)}). \end{aligned}$$

Both of those two spectral sequences converge to the hypercohomology of the total complex ([Sta18], Tag 0132). The same applies when we replace the double complex by its i-th infinitesimal filtration.

Now we make the following two Lemmas about degeneracy of those two spectral sequences:

Lemma 3.3.1.4. *For each $p > 0$, the filtered cochain complex associated with the cosimplicial complex with its infinitesimal filtration*

$$M(\bullet) \otimes_{\mathcal{D}(\bullet)} \Omega^p_{D(\bullet)}$$

is filtered acyclic.

Lemma 3.3.1.5. *Any degeneracy map $D(p) \to D$ induces an filtered quasi-isomorphism of the following two de Rham complexes*

$$M \otimes_{\mathcal{D}} \Omega^\bullet_D \longrightarrow M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)},$$

that is functorial with respect to the crystal $\mathcal{F}$.

We first assume those two lemmas above. By Lemma 3.3.1.4, the spectral sequence $''E_1^{p,q}$ is filtered degenerated in its first page and is convergent to the cohomology of the Čech-Alexander complex $M(\bullet)$ with its infinitesimal filtration.

On the other hand, Lemma 3.3.1.5 implies that the horizontal coboundary map of $'E_1^{p,q}$ is given by

$$\mathrm{H}^q(M \otimes_{\mathcal{D}} \Omega^\bullet_D) \xrightarrow{0} \mathrm{H}^q(M \otimes_{\mathcal{D}} \Omega^\bullet_D) \xrightarrow{1} \mathrm{H}^q(M \otimes_{\mathcal{D}} \Omega^\bullet_D) \xrightarrow{0} \cdots .$$

From this, the second page of $'E_1^{p,q}$ vanishes everywhere except for the column $''E_2^{0,\bullet}$, which is exactly the infinitesimal filtered de Rham complex $M \otimes_{\mathcal{D}} \Omega^\bullet_D$.

In this way, since both of those two spectral sequences are convergent to the total complex in the filtered derived category, we get the filtered isomorphism between the de Rham complex $M \otimes_{\mathcal{D}} \Omega^\bullet_D$ and the Čech-Alexander complex $M(\bullet)$. So by Proposition 3.3.1.3, we get Theorem 3.3.1.1. Here the functoriality follows from that of Lemma 3.3.1.5 and Proposition 3.3.1.3.

Proof of Lemma 3.3.1.4 To complete the proof of Theorem 3.3.1.1, we first prove Lemma 3.3.1.4 in this paragraph.

We first give a proof for the special case when $\mathcal{F}$ is the structure sheaf and $p = 1$.

Lemma 3.3.1.6. *The cosimplicial complex*

$$\Omega^1_D \longrightarrow \Omega^1_{D(1)} \longrightarrow \Omega^1_{D(2)} \longrightarrow \cdots \tag{$*$}$$

is locally cosimplicial homotopic to zero, as filtered cosimplicial abelian groups.

Before the proof of this Lemma, we first recall that a *cosimplicial homotopic equivalence* of two maps $f, g : U \to V$ is defined as a cosimplicial morphism

$$h : U \to \mathrm{Hom}([1], V),$$

such that

$$h \circ s_0 = f,\ h \circ s_1 = g,$$

where $s_i : [0] \to [1]$ are two co-face maps.

A cosimplicial object U is called *cosimplicial homotopic to zero* if its identity map is cosimplicial homotopic to the zero map. Here we note that any additive functor F that sends cosimplicial objects to cosimplicial objects will preserve the cosimplicial homotopic equivalence.

We refer the reader to [Sta18, Tag 019U] for the discussion about cosimplicial homotopic equivalence.

Proof. We first recall that since $D(n)$ is the envelope of $X = \mathrm{Spa}(A)$ in the $n+1$-folded self product of Y over Σ_e, by Lemma 3.2.3.3 above, we have

$$\Omega^1_{D(n)} = \Omega^1_{P^{\otimes n+1}/\Sigma_e} \otimes_{P^{\otimes n+1}} \mathcal{D}(n).$$

Besides, any cosimplicial boundaries map $P^{n+1} \to P^{l+1}$ induces a map $\Omega^1_{D(n)} \to \Omega^1_{D(l)}$. So the cosimplicial complex $(*)$ is the tensor product of the cosimplicial complex $\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}$ along the cosimplicial ring homomorphism

$$P^{\otimes \bullet+1} \longrightarrow \mathcal{D}(\bullet).$$

Moreover, the i-th filtration of the cosimplicial complex $(*)$ is

$$J^{i-1}\Omega^1_D \longrightarrow J(1)^{i-1}\Omega^1_{D(1)} \longrightarrow J(2)^{i-1}\Omega^1_{D(2)} \longrightarrow \cdots,$$

which is isomorphic to the fiber of a map between cosimplicial tensor products

$$\left(\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}\right) \otimes_{P^{\otimes \bullet+1}} \mathcal{D}(\bullet) \longrightarrow \left(\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}\right) \otimes_{P^{\otimes \bullet+1}} \left(\mathcal{D}(\bullet)/J(\bullet)^{i-1}\right).$$

Thus to show the filtered acyclicity, it suffices to show that the cosimplicial module $\Omega^1_{P^{\otimes \bullet+1}/\Sigma_e}$ is homotopic equivalent to zero. Here we notice that when $P = \mathrm{B}^+_{\mathrm{dR,e}}\langle x_i\rangle$, each $P^{\otimes n+1}$ is a ring of convergent power series over $\mathrm{B}^+_{\mathrm{dR,e}}$, and the proof is totally identical to the case for polynomial rings, which is done in [BdJ11], Example 2.16. In general, when P is smooth over $\mathrm{B}^+_{\mathrm{dR,e}}$, it locally admits an étale morphism to an $\mathrm{B}^+_{\mathrm{dR,e}}\langle x_i\rangle$. So the exactness is true locally, hence globally by a Čech-complex argument associated with a covering.

□

End of the proof for Lemma 3.3.1.4. Consider the filtered complex $(*)$ as below:

$$\Omega^1_D \longrightarrow \Omega^1_{D(1)} \longrightarrow \Omega^1_{D(2)} \longrightarrow \cdots . \tag{$*$}$$

As the statement is local, by shrinking to an open subsets of X and Y if necessary, we could assume the complex $(*)$ is filtered cosimplicial homotopic to zero as in Lemma 3.3.1.6. Then we apply the p-th wedge product functor, and the tensor product functor $M(\bullet) \otimes_{\mathcal{D}(\bullet)} -$ successively to the cosimplicial complex $(*)$, then the resulted cosimplicial complex is exactly the one in Lemma 3.3.1.4. But note that since any additive cosimplicial functor preserves the cosimplicial homotopic equivalence, the resulted complex is also filtered homotopic to zero. So we are done. □

Proof of Lemma 3.3.1.5 In this paragraph, we prove Lemma 3.3.1.5.

We first provide the following simpler description of the envelope $\mathcal{D}(p)$:

Lemma 3.3.1.7. *Assume the $\mathrm{B}^+_{\mathrm{dR,e}}$-algebra P admits an étale map from a ring of convergent power series $\mathrm{B}^+_{\mathrm{dR,e}}\langle x_1, \ldots, x_r\rangle$. Then the map of global sections of structure sheaves $\mathcal{D} \to \mathcal{D}(p)$ associated with the degeneracy map $D(p) \to D$ induces an isomorphism*

$$\mathcal{D}(p) \cong \mathcal{D}[[\delta_{i,j}, 1 \leq i \leq p, 1 \leq j \leq r]],$$

where the right side is a ring of formal power series over the topological ring $\mathcal{D}$.

The notation is explained as follows. The projection map $Y(p) \to Y$ of the $p+1$-th self product onto the first copy induces the zero-th degeneracy map $D(p) \to D$. Then we can rewrite $P^{\otimes p+1}$ as $P\langle \delta_{i,j},\ 1 \leq i \leq p,\ 1 \leq j \leq r\rangle$, where $\delta_{i,j}$ is defined as $x_j \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes x_j \otimes \cdots \otimes 1$, with x_j being in the i-th copy of P in the second term.

Proof. We first consider the case when P is equal to the convergent power series ring.

Denote by J to be the kernel of the surjection $P \to A$, and let I be the kernel of the map $P^{\otimes n+1} \to P$. By construction, the ring of sections $\mathcal{D}(p) = \mathcal{O}(D(p))$ is equal to the inverse limit

$$\varprojlim_{m\in\mathbb{N}} P^{\otimes p+1}/(J \otimes \cdots \otimes 1, I)^m,$$

while $\mathcal{D} = \mathcal{O}(D)$ is $\varprojlim_{m\in\mathbb{N}} P/J^m$. So to prove the lemma, it suffices to notice that the above inverse limit is the same as the inverse limit

$$\mathcal{D}(p) = \varprojlim_{n\in\mathbb{N}} (\varprojlim_{m\in\mathbb{N}} P^{\otimes p+1}/(J\otimes\cdots\otimes 1)^m)/\overline{I}^n,$$

where $\overline{I}$ is the image of I along the map $P^{\otimes p+1} \to \varprojlim_{m\in\mathbb{N}} P^{\otimes p+1}/(J\otimes\cdots\otimes 1)^m$.

In fact, we have the following more general result:

Claim 3.3.1.8. Let R be a noetherian ring, and $I,\ J$ be two ideals in R. Then we have a canonical isomorphism

$$\varprojlim_{m\in\mathbb{N}} (\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m \longrightarrow \varprojlim_{m\in\mathbb{N}} R/(I,J)^m,$$

where $\overline{J}$ is the ideal generated by the image of J along the map $R \to \varprojlim_{m\in\mathbb{N}} R/I^n$.

Proof of the Claim. First notice that the sequence of ideals $\{(I,J)^m\}$ and $\{(I^m,J^m)\}$ are cofinal to each other, since

$$(I^{2m},J^{2m}) \subset (I,J)^{2m} = (I^iJ^{2m-i},\ 0\le i\le 2m) \subset (I^m,J^m).$$

So the right side $\varprojlim_{m\in\mathbb{N}} R/(I,J)^m$ can be replaced by $\varprojlim_{m\in\mathbb{N}} R/(I^m,J^m)$.

Then we notice that the R-algebra $A := \varprojlim_{m\in\mathbb{N}}(\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m$ is (I,J)-adic complete over R: To show this, by the [Sta18] Tag 0DYC, it suffices to show that the ring $(\varprojlim_{n\in\mathbb{N}} R/I^n)/J$ is I-adic complete. We then note that $(\varprojlim_{n\in\mathbb{N}} R/I^n)/J = \widehat{R}\otimes_R R/J$, where $\widehat{R}$ is the I-adic completion of R. Since R/J is a finitely generated module over R, the tensor product $\widehat{R}\otimes_R R/J$ is the same as I-adic completion of R/J. Thus the R-algebra A is (I,J)-adic complete. In particular, we have

$$A = \varprojlim_{l\in\mathbb{N}} A/(I^l,J^l).$$

At last, the quotient ring $A/(I^m,J^m)$ is given as

$$\begin{aligned}
A/(I^l,J^l) &= (\varprojlim_{m\in\mathbb{N}} (\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m)/(I^l,J^l) \\
&= (\varprojlim_{n\in\mathbb{N}} R/I^n)/(\overline{I}^l,\overline{J}^l) \\
&= R/(I^l,J^l).
\end{aligned}$$

So we get

$$\begin{aligned}\varprojlim_{m\in\mathbb{N}}(\varprojlim_{n\in\mathbb{N}} R/I^n)/\overline{J}^m &=: A \\ &\cong \varprojlim_{l\in\mathbb{N}} A/(I^l, J^l) \\ &= \varprojlim_{l\in\mathbb{N}} R/(I^l, J^l).\end{aligned}$$

□

At last, let us assume P is a smooth affinoid algebra that admits an étale map to the ring of convergent power series P_0. By the claim above, the lemma is reduced to show that the formal completion $\mathcal{D}(p)$ for $P^{\otimes p+1} \to P$ is isomorphic to $P[[\delta_{i,j}]]$, which is proved in [BMS18, Section 13]. □

Our next observation is about the Euler sequence for the degeneracy map $D(p) \to D$. Denote by $\Omega^1_{D(p)/D}$ to be the module of continuous differentials of $\mathcal{D}(p)$ over $\mathcal{D}$ under the $(\Delta(p))$-adic topology, where $\Delta(p)$ is the kernel ideal for the diagonal map $\mathcal{D}(p) \to \mathcal{D}$. Then we have

Lemma 3.3.1.9. *The Euler sequence for the projection map $Y(p) \to Y$ over Σ_e induces a natural exact sequence of free $\mathcal{D}(p)$-module:*

$$0 \longrightarrow \Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p) \longrightarrow \Omega^1_{D(p)} \longrightarrow \Omega^1_{D(p)/D} \longrightarrow 0,$$

where the map $\Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p) \to \Omega^1_{D(p)}$ sends $dx_i \otimes 1$ to dx_i.

Proof. We consider the inverse limit of the Euler sequences of differentials for the triple $Y(p)_m \to Y_m \to \Sigma_e$, with $m \in \mathbb{N}$. Then by Lemma 3.2.3.3, we see the inverse limit $\varprojlim_{m\in\mathbb{N}}(\Omega^1_{Y_m/\Sigma_e} \otimes_{\mathcal{O}(Y_m)} \mathcal{O}(Y(p)_m))$ is isomorphic to the $\mathcal{D}(p)$-module $\Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p)$. Similarly the inverse limit

$$\varprojlim_{m\in\mathbb{N}} \Omega^1_{Y(p)_m/Y_m}$$

is isomorphic to $\Omega^1_{D(p)/D}$. In particular, we get the following sequence of $\mathcal{D}(p)$-modules

$$0 \longrightarrow \Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p) \longrightarrow \Omega^1_{D(p)} \longrightarrow \Omega^1_{D(p)/D} \longrightarrow 0.$$

To show the sequence is an exact sequence, we may assume P admits an étale map from the ring of convergent power series $\mathrm{B}^+_{\mathrm{dR},e}\langle x_1, \dots, x_r\rangle$. We apply Lemma 3.2.3.3 to the immersion $X \to Y$

and $X \to Y(p) = Y \times \cdots \times Y$ separately. Then we get an description of differentials as follows

$$\Omega^1_D = \bigoplus_{j=1}^{r} \mathcal{D} dx_j,\ \Omega^1_{D(p)} = (\bigoplus_{j=1}^{r} \mathcal{D}(p) dx_j) \oplus (\bigoplus_{\substack{1 \le i \le p \\ 1 \le j \le r}} \mathcal{D}(p) d\delta_{i,j}),$$

Here the projection map $D(p) \to D$ induced from $Y(p) \to Y$ produces the natural monomorphism

$$\Omega^1_D \longrightarrow \Omega^1_{D(p)},$$

sending the generator dx_j onto dx_j in $\Omega^1_{D(p)}$. This gives the injectivity from $\Omega^1_D \otimes_{\mathcal{D}} \mathcal{D}(p)$ into $\Omega^1_{D(p)}$.

Moreover, by the explicit formula in Lemma 3.3.1.7 for ring of convergent power series, the $(\delta_{i,j})$-adic continuous differential of $\mathcal{D}(p)$ over $\mathcal{D}$ is the free $\mathcal{D}(p)$-module generated by $d\delta_{i,j}$, for $1 \le i \le p$ and $1 \le j \le r$. This is exactly the cokernel of the injection above and is the free $\mathcal{D}(p)$-module generated by $d\delta_{i,j}$. Thus we get the short exact sequence as expected.

□

We can construct the relative de Rham complex of $D(p)$ over D, by taking wedge products of $\Omega^1_{D(p)/D}$ and considering the relative differential operator. Then we have the following filtered version of the Poincaré Lemma for infinitesimal differentials:

Lemma 3.3.1.10 (Poincaré Lemma)**.** *There exists a natural quasi-isomorphism to the relative de Rham complex*

$$\mathcal{D} \longrightarrow \Omega^\bullet_{D(p)/D}.$$

Moreover, for each $m \in \mathbb{N}$, the natural induced map below is a quasi-isomorphism

$$\mathcal{D} \to \Omega^\bullet_{D(p)/D}/\Delta(p)^{m+1-\bullet}.$$

Proof. We first assume Y is a unit disc, and by Lemma 3.3.1.7 the ring $\mathcal{D}(p)$ is equal to the ring of formal power series over $\mathcal{D}$ with coordinates $\delta_{i,j}$. For the first argument, it suffices to show that the augmented complex

$$0 \to \mathcal{D} \to \mathcal{D}(p) \to \Omega^1_{D(p)/D} \to \Omega^2_{D(p)/D} \to \cdots \to \Omega^N_{D(p)/D} \to 0 \tag{$*$}$$

is homotopic to 0, where $N = pr$. Using the coordinate interpretation, the complex $(*)$ is an N-th completed tensor product of the complex

$$0 \to \mathcal{D} \to \mathcal{D}[[x]] \to \mathcal{D}[[x]]dx \to 0$$

over $\mathcal{D}$, where the map $\mathcal{D}[[x]] \to \mathcal{D}[[x]]dx$ is the $\mathcal{D}$-linear relative differential. But since $\mathcal{D}$ contains

$\mathbb{Q}$, the relative differential is surjective with kernel being $\mathcal{D}$, which proves the first statement in this case. Moreover, notice that by writing down the differentials $\Omega^i_{D(p)/D}$ in terms of coordinates $\delta_{i,j}$ by Lemma 3.3.1.7, the differential in the complex $(*)$ preserves the degree. In this way, since the quotient $\Omega^\bullet_{D(p)/D}/\Delta(p)^{m+1-\bullet}$ kills exactly elements of degrees higher than m, we get the statement about the quotient complex in this case.

In general, as the statement is étale local with respect to the smooth rigid space $Y = \mathrm{Spa}(P)$, we may assume Y admits an étale morphism to an unit disc. Then the claim follows from a term-wise base change formula in the complex $(*)$, thanks to Lemma 3.2.3.3. $\square$

Here is another observation which we will need in order to compute the cohomology of infinitesimal filtration.

Lemma 3.3.1.11. *Let $D(p) \to D$ be the degeneracy map of envelopes as before, and let $J(p)$, J and $\Delta(p)$ be the kernel ideals for surjections $\mathcal{O}_{D(p)} \to \mathcal{O}_X$, $\mathcal{O}_D \to \mathcal{O}_X$ and $\mathcal{O}_{D(p)} \to \mathcal{O}_D$ separately. Then for $j \leq m$ in $\mathbb{N}$, the natural map below is an isomorphism of $\mathcal{O}_X$-modules*

$$\begin{aligned}&J^{m-j}/J^{m-j+1} \cdot \Delta(p)^j/\Delta(p)^{j+1} \longrightarrow \\ &\left(J^m, J^{m-1}\Delta(p), \cdots, J^{m-j}\Delta(p)^j, J(p)^{m+1}\right) / \left(J^m, \cdots, J^{m-j+1}\Delta(p)^{j-1}, J(p)^{m+1}\right).\end{aligned}$$

Proof. As the statement is local with respect to Y, let us first assume $Y = \mathrm{Spa}(P)$ admits an étale map to a ring of convergent power series. By Lemma 3.3.1.7, $\mathcal{D}(p)$ is the formal power series ring $\mathcal{D}[[\delta_{i,j}]]$, and the ideal $\Delta(p)$ is generated by variables $(\delta_{i,j})$. Notice that as the map $\mathcal{D}[\delta_{i,j}] \to \mathcal{D}[[\delta_{i,j}]]$ is flat and the quotient ideals in the statement can be defined over $\mathcal{D}[\delta_{i,j}]$, it suffices to show the analogous statement for the polynomial ring $\mathcal{D}[\delta_{i,j}]$.

Then as the ring $\mathcal{D}[\delta_{i,j}]$ is a free module over $\mathcal{D}$ with a basis given by monomials of $\delta_{i,j}$, we could express elements x in $\mathcal{D}[\delta_{i,j}] \cap (J^m, J^{m-1}\Delta(p), \cdots, J^{m-j}\Delta(p)^j, J(p)^{m+1})$ using the coordinates as below

$$\begin{aligned}x = a_{r_{l_0}} &+ \sum_{|r_{l_1}|=1} a_{r_{l_1}} \cdot \delta^{r_{l_1}} + \sum_{|r_{l_2}|=2} a_{r_{l_2}} \cdot \delta^{r_{l_2}} + \cdots, \\ a_{r_{l_n}} &\in J^{m-n}, \ \mathit{for}\ 0 \leq n \leq j; \\ a_{r_{l_n}} &\in J^{m+1-n}, \ \mathit{for}\ j < n \leq m+1; \\ a_{r_{l_n}} &\in \mathcal{D}, \ \mathit{for}\ j > m+1.\end{aligned}$$

Here $\delta^{r_{l_n}}$ are monomials in $\delta_{i,j}$ with multi-indexes. Similarly we could do this for elements in $\mathcal{D}[\delta_{i,j}] \cap (J^m, \cdots, J^{m-j+1}\Delta(p)^{j-1}, J(p)^{m+1})$, where in the obtained formula we replace j above by $j-1$. Compare those expressions, we see the statement in the lemma holds for $\mathcal{D}[\delta_{i,j}]$. So by extending this along the flat map $\mathcal{D}[\delta_{i,j}] \to \mathcal{D}[[\delta_{i,j}]]$, we get the result for $\mathcal{D}(p) \cong \mathcal{D}[[\delta_{i,j}]]$.

$\square$

Now we are ready to prove the Lemma 3.3.1.5.

Proof for Lemma 3.3.1.5.

Step 1 We first deal with the underlying complexes and forget the infinitesimal filtration. Our goal is to show that the natural map of complexes below is a quasi-isomorphism

$$M \otimes_{\mathcal{D}} \Omega^\bullet_D \longrightarrow M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}.$$

The de Rham complex $\Omega^\bullet_D$ is equipped with its Hodge filtration, defined by $F^i\Omega^\bullet_D = \sigma^{\geq i}\Omega^\bullet_D$. By the Euler sequence in Lemma 3.3.1.9, the Hodge filtration of $\Omega^\bullet_D$ induces a natural descending filtration on the relative de Rham complex $\Omega^\bullet_{D(p)}$, whose graded piece is $gr^i\Omega^\bullet_{D(p)} = \Omega^i_D \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)/D}$.

Now consider the de Rham complex $(M \otimes_{\mathcal{D}} \Omega^\bullet_D, \nabla_D)$ and $(M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}, \nabla_{D(p)})$ of the crystal $\mathcal{F}$ at D and $D(p)$. The projection $D(p) \to D$ induces a map of complexes

$$M \otimes \Omega^\bullet_D \longrightarrow M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}.$$

By the crystal condition, the base change of M along the map $D(p) \to D$ is exactly $M(p)$. Moreover, by the compatibility of the de Rham complexes, the filtration on $\Omega^\bullet_{D(p)}$ induces a filtration on $M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)} = M \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)}$, given by

$$F^i(M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}) = M \otimes_{\mathcal{D}} F^i\Omega^\bullet_{D(p)}.$$

Each $F^i(M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)})$ is a subcomplex of $M \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)}$, since ∇^i sends elements in M into $M \otimes \Omega^1_D \subset M(p) \otimes \Omega^1_{D(p)}$. Moreover, the i-th graded factor of this filtration is

$$M \otimes_{\mathcal{D}} \Omega^i_D \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)/D},$$

which by Lemma 3.3.1.10 is isomorphic to the $M \otimes_{\mathcal{D}} \Omega^i_D$ via the degeneracy map. In this way, the projection $D(p) \to D$ induces a map of filtered complexes

$$M \otimes_{\mathcal{D}} \Omega^\bullet_D \longrightarrow M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}, \tag{$*$}$$

which on each graded factor is an isomorphism . Hence the map $(*)$ itself is an isomorphism, by the spectral sequence associated with a finite filtration as in [Sta18, Tag 012K] .

Step 2 We then show that the above quasi-isomorphism is filtered under the infinitesimal filtration. More precisely, we claim the graded pieces of the following map in Step 1 is a filtered quasi-isomorphism under their infinitesimal filtrations:

$$M \otimes_{\mathcal{D}} \Omega^i_D \longrightarrow M \otimes_{\mathcal{D}} \Omega^i_D \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)/D}.$$

Consider the $(m+i)$-th graded pieces for $m \in \mathbb{N}$. On the one hand, the $(m+i)$-th graded pieces for the infinitesimal filtration on $M \otimes_{\mathcal{D}} \Omega^\bullet_D$ induces a subquotient $J^m \cdot M \otimes \Omega^i_D / J^{m+1}$ of the left hand side of the above. On the other hand, the $(m+i)$-th graded pieces for infinitesimal filtration on $M(p) \otimes_{\mathcal{D}(p)} \Omega^\bullet_{D(p)}$ induces the following subquotient of the right hand side:

$$J(p)^{m-\bullet} \cdot M \otimes_{\mathcal{D}} \Omega^i_D \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)/D} / J(p)^{m+1-\bullet}.$$

So we get the map of graded pieces as below

$$J^m \cdot M \otimes \Omega^i_D / J^{m+1} \longrightarrow J(p)^{m-\bullet} \cdot M \otimes_{\mathcal{D}} \Omega^i_D \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)/D} / J(p)^{m+1-\bullet}. \qquad (**)$$

Here we note that as the ideal J maps into $J(p)$, the right hand side is an $\mathcal{O}_D/J = \mathcal{O}_X$-linear complex.

To show $(**)$ is a quasi-isomorphism, we need to subdivide the right hand side in a finer way. We introduce a finite increasing filtration on the right hand side of $(**)$, whose j-th filtration is

$$\begin{aligned}
&\left(J^{m-\bullet}, J^{m-1-\bullet}\Delta(p), \cdots J^{m-j}\Delta(p)^{j-\bullet}, J(p)^{m+1-\bullet}\right) \cdot M \otimes_{\mathcal{D}} \Omega^i_{\mathcal{D}} \otimes_{\mathcal{D}} \Omega^\bullet_{D(p)/D} / J(p)^{m+1-\bullet} \\
&= complex \Big(\left(J^m, \cdots, J^{m-j}\Delta(p)^j, J(p)^{m+1}\right) \cdot M \otimes \Omega^i_{\mathcal{D}} \otimes \mathcal{O}_{D(p)} / J(p)^{m+1} \\
&\longrightarrow \left(J^{m-1}, \cdots, J^{m-j}\Delta(p)^{j-1}, J(p)^m\right) \cdot M \otimes \Omega^i_{\mathcal{D}} \otimes \Omega^1_{D(p)/D} / J(p)^m \\
&\longrightarrow \cdots \\
&\longrightarrow \left(J^{m-j}, J(p)^{m+1-j}\right) \cdot M \otimes \Omega^i_{\mathcal{D}} \otimes \Omega^j_{D(p)/D} / J(p)^{m+1-j} \Big).
\end{aligned}$$

The graded pieces of this filtration is the $\mathcal{O}_X$-linear complex

$$\begin{gathered}
\left(J^{m-\bullet}, \cdots, J^{m-j}\Delta(p)^{j-\bullet}, J(p)^{m+1-\bullet}\right) \cdot M \otimes \Omega^i_{\mathcal{D}} \otimes \\
\Omega^\bullet_{D(p)/D} / \left(J^{m-\bullet}, \cdots, J^{m-j+1}\Delta(p)^{j-1-\bullet}, J(p)^{m+1-\bullet}\right)
\end{gathered}$$

We apply the Lemma 3.3.1.11 to this complexes, then the graded piece above can be rewritten as

$$\begin{aligned}
&J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Delta(p)^{j-\bullet}\Omega^\bullet_{D(p)/D}/\Delta(p)^{j-\bullet+1} \\
&= complex\Big(J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Delta(p)^j/\Delta(p)^{j+1} \\
&\longrightarrow J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Delta(p)^{j-1}\Omega^1_{D(p)/D}/\Delta(p)^j \\
&\longrightarrow \cdots \\
&\longrightarrow J^{m-j}M \otimes \Omega^i_D/J^{m-j+1} \otimes_{\mathcal{D}} \Omega^j_{D(p)/D}/\Delta(p)\Big) \\
&\cong \left(J^{m-j}M \otimes \Omega^i_D/J^{m-j+1}\right) \otimes_{\mathcal{D}} \left(\Delta(p)^{j-\bullet}\Omega^\bullet_{D(p)/D}/\Delta(p)^{j+1-\bullet}\right).
\end{aligned}$$

At last, by the graded version of relative Poincaré Lemma in Lemma 3.3.1.10, we have

$$\Delta(p)^{j-\bullet}\Omega^\bullet_{D(p)/D}/\Delta(p)^{j+1-\bullet} \cong \begin{cases} 0, \ j \geq 1; \\ \mathcal{D}, \ j = 0. \end{cases}$$

In this way, the graded pieces of the right hand side of $(**)$ are zero, except for its zero-th graded piece which is naturally isomorphic to $J^m M \otimes \Omega^i_D/J^{m+1}$. Hence $(**)$ is an isomorphism, and we finish the proof.

□

Remark 3.3.1.12. Here we mention that the same study of the infinitesimal filtration works with minor changes for schemes. In particular, the schematic analogue of the proof in [BdJ11, Theorem 2.12] can be improved into a filtered version, and we thus obtain the expected filtered isomorphism in the crystalline theory, which is proved in different methods in [BO78, Theorem 7.23].

3.3.2 Global result

Now we generalize the computation of cohomology to the global situation, without assuming X is affinoid.

The very first result is about the global vanishing.

Proposition 3.3.2.1. *Let X be a rigid space over Σ_e, and let $\mathcal{F}$ be a coherent crystal over $X/\Sigma_{e\,\mathrm{inf}}$. Then for each $i > 0$ and $j \in \mathbb{N}$, we have*

$$Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^i_{X/\Sigma_{e\,\mathrm{inf}}}) = 0.$$

In particular, after applying the derived direct image $Ru_{X/\Sigma_e *}$, the truncation map of the de Rham *complex induces a filtered quasi-ismorphism:*

$$Ru_{X/\Sigma_e *}(\mathcal{F} \otimes \Omega^\bullet_{X/\Sigma_{e\,\inf}}) \longrightarrow Ru_{X/\Sigma_e *}\mathcal{F}.$$

Proof. Recall from Subsection 3.1.3 that $\Gamma(U, u_{X/\Sigma_{e\,\inf}*}\mathcal{G})$ is defined as the 0-th cohomology $\Gamma(U/\Sigma_{e\,\inf}, \mathcal{G})$, and similarly for its filtered analogue. So to show the vanishing of

$$Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F} \otimes \Omega^i_{X/\Sigma_{e\,\inf}}),$$

it suffices to do this locally and assume X is affinoid, together with a closed immersion into a smooth rigid space Y over $\Sigma_{e\,\inf}$. We then notice that by Proposition 3.3.1.3, $Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F} \otimes \Omega^i_{X/\Sigma_{e\,\inf}})$ is computed by the following cosimplicial complex:

$$J^j\mathcal{F} \otimes \Omega^i_{X/\Sigma_{e\,\inf}}(D) \longrightarrow J(1)^j\mathcal{F} \otimes \Omega^i_{X/\Sigma_{e\,\inf}}(D(1)) \longrightarrow J(2)^j\mathcal{F} \otimes \Omega^i_{X/\Sigma_{e\,\inf}}(D(2)) \longrightarrow \cdots,$$

which by Lemma 3.3.1.4 is homotopic to zero when $i > 0$. So we get the vanishing of

$$Ru_{X/\Sigma_e *}(\mathcal{J}^j_{X/\Sigma_e}\mathcal{F} \otimes \Omega^i_{X/\Sigma_{e\,\inf}}).$$

$\square$

Now we can generalize Theorem 3.3.1.1 to the global case, without assuming the affinoid condition:

Theorem 3.3.2.2. *Let $X \to Y$ be a closed immersion of X into a smooth rigid space Y over Σ_e. Let $\mathcal{F}$ be a coherent crystal over $\mathcal{O}_{X/\Sigma_e}$, and let $\mathcal{F}_D = \varprojlim_{m\in\mathbb{N}} \mathcal{F}_{Y_m}$ be the restriction of $\mathcal{F}$ at the envelope $D = D_X(Y) = \varinjlim_{m\in\mathbb{N}} Y_m$, with its de Rham complex $\mathcal{F}_D \otimes \Omega^\bullet_D$. Then there exists a natural isomorphism in the filtered derived category of sheaves of abelian groups over X*

$$Ru_{X/\Sigma_e *}\mathcal{F} \longrightarrow \mathcal{F}_D \otimes \Omega^\bullet_D.$$

Before the proof, we want to mention that the strategy is to produce a natural map between those two complexes of sheaves of abelian groups, where the isomorphism will follow from the affinoid computation.

Proof. By Proposition 3.3.2.1, the truncation map of the de Rham complex $\mathcal{F} \otimes_{\mathcal{O}_{X/\Sigma_e}} \Omega^\bullet_{X/\Sigma_{e\,\inf}} \to \mathcal{F}[0]$ produces a canonical filtered isomorphism in the derived category of $\mathcal{O}_X$-modules

$$Ru_{X/\Sigma_e *}(\mathcal{F} \otimes \Omega^\bullet_{X/\Sigma_{e\,\inf}}) \to Ru_{X/\Sigma_e *}\mathcal{F}.$$

On the other hand, we recall that the envelope $D = D_X(Y)$ is defined as the direct limit $\varinjlim_{m\in\mathbb{N}} h_{Y_m}$ of representable sheaves, where Y_m is the m-th infinitesimal neighborhood of X into Y. In the infinitesimal topos $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, the map from the envelope D to the final object 1 induces a map of derived functors

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, -) \longrightarrow R\Gamma(D, -) = R\varprojlim_{m} R\Gamma(Y_m, -).$$

Similarly for its filtered analogue.

We apply the natural transformation to the filtered de Rham complex $\mathcal{F} \otimes \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}$, and get

$$\begin{aligned} R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{F} \otimes \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}) &\to R\varprojlim_{m} R\Gamma(Y_m, \mathcal{F}_{Y_m} \otimes_{\mathcal{O}_{Y_m}} \Omega^\bullet_{Y_m/\Sigma_e}) \\ &= R\Gamma(X, R\varprojlim_{m\in\mathbb{N}} \mathcal{F}_{Y_m} \otimes_{\mathcal{O}_{Y_m}} \Omega^\bullet_{Y_m/\Sigma_e}) \\ &= R\Gamma(X, \mathcal{F}_D \otimes_{\mathcal{O}_D} \Omega^\bullet_D), \end{aligned}$$

where the last equality follows from the observation that the inverse system $\{\mathcal{F}_{Y_m} \otimes_{\mathcal{O}_{Y_m}} \Omega^\bullet_{Y_m/\Sigma_e}\}_m$ admits a finite filtration, where each graded piece $\{\mathcal{F}_{Y_m} \otimes_{\mathcal{O}_{Y_m}} \Omega^i_{Y_m/\Sigma_e}\}_m$ is a pro-coherent system satisfying the sheaf vertion Mittag-Leffler condition ([BO78, Lemma 7.20]). Similarly for the subcomplex $\mathcal{J}^{m-\bullet}_{X/\Sigma_e}\mathcal{F} \otimes \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}$. Notice that the map is functorial with respect to all locally closed immersions (X, Y) into smooth rigid spaces. In particular, by varying X among all open subsets U of X and considering the above map for locally closed immersions (U, Y), we could enhance the above into the sheaf version filtered morphism

$$Ru_{X/\Sigma_e *}(\mathcal{F} \otimes \Omega^\bullet_{X/\Sigma_{e\,\mathrm{inf}}}) \longrightarrow \mathcal{F}_D \otimes_{\mathcal{D}} \Omega^\bullet_D.$$

Thus by composing with (the inverse of) the filtered isomorphism at the beginning, we get a natural map in the filtered derived category of sheaves of abelian groups over X:

$$Ru_{X/\Sigma_e *}\mathcal{F} \longrightarrow \mathcal{F}_D \otimes_{\mathcal{D}} \Omega^\bullet_D.$$

At last, to show the filtered isomorphism, it suffices to check this by applying $R\Gamma(U, -)$ for all open affinoid subspaces U in X, which we know by Theorem 3.3.1.1. Thus we are done.

□

As a consequence, we get a change of bases formula quite easily.

Proposition 3.3.2.3. *Let X be a rigid space over Σ_e, and $e' \geq e$ be an integer. Let $\mathcal{F}'$ be a crystal in vector bundles over $X/\Sigma_{e'\,\mathrm{inf}}$, and $\mathcal{F}$ be the pullback of $\mathcal{F}'$ along the map of infinitesimal topoi*

$\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e'\,\mathrm{inf}})$. *Then there exists a natural isomorphism of complexes of sheaves of* $\mathrm{B}^+_{\mathrm{dR},e'}$*-modules as below*

$$(Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} \longrightarrow Ru_{X/\Sigma_e*}\mathcal{F}.$$

Proof. We first notice that the natural morphism of infinitesimal sites $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e'\,\mathrm{inf}}$ induces a canonical map in the derived category of sheaves over X

$$Ru_{X/\Sigma_{e'}*}\mathcal{F}' \longrightarrow Ru_{X/\Sigma_e*}\mathcal{F}.$$

Moreover, as the target is $\mathrm{B}^+_{\mathrm{dR,e}}$-linear, by the adjunction for the forgetful functor (from $\mathrm{B}^+_{\mathrm{dR,e}}$-modules to $\mathrm{B}^+_{\mathrm{dR},e'}$-modules) we get a natural map of complexes

$$(Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} \longrightarrow Ru_{X/\Sigma_e*}\mathcal{F}.$$

So it suffices to show this adjunction map is an isomorphism.

Let us first assume there exists a closed immersion $X \to Y'$ of X into a smooth rigid space over $\Sigma_{e'}$. By Theorem 3.3.2.2, we have the following natural isomorphisms

$$\begin{aligned} Ru_{X/\Sigma_{e'}*}\mathcal{F}' &\longrightarrow \mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'};\\ Ru_{X/\Sigma_e*}\mathcal{F} &\longrightarrow \mathcal{F}_D \otimes \Omega^\bullet_D, \end{aligned}$$

where D' is the envelope of X in Y', and D is the envelope of X in $Y = Y' \times_{\Sigma_{e'}} \Sigma_e$ where the latter is smooth over Σ_e. Notice that $\mathcal{O}_{Y'}$ is flat over $\Sigma_{e'}$, and the structure sheaves $\mathcal{O}_{D'}$ is flat over $\mathcal{O}_{Y'}$ (for it is defined as the formal completion of $\mathcal{O}_{Y'}$ along $X \to Y'$). In this way, by the assumption that $\mathcal{F}'$ is a crystal in vector bundles, the complex $\mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'}$ is a $\mathrm{B}^+_{\mathrm{dR},e'}$-linear bounded complex of sheaves of flat $\mathrm{B}^+_{\mathrm{dR},e'}$-modules. Thus we get the equalities

$$\begin{aligned} (Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} &\cong (\mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'}) \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}}\\ &\cong (\mathcal{F}'_{D'} \otimes \Omega^\bullet_{D'})/\xi^e, \end{aligned}$$

which is then isomorphic to the complex $\mathcal{F}_D \otimes \Omega^\bullet_D$ as the envelope $D = D_X(Y)$ is equal to the pullback of $D' = D_X(Y')$ along the surjection $\mathrm{B}^+_{\mathrm{dR},e'} \to \mathrm{B}^+_{\mathrm{dR,e}} = \mathrm{B}^+_{\mathrm{dR},e'}/\xi^e$.

In general, let us denote X_{aff} to be the basis of X_{rig} consisting of affinoid open subsets, which is equipped with the rigid topology. The natural inclusion functor $X_{\mathrm{aff}} \subset X_{\mathrm{rig}}$ then induces an equivalence of their topoi and derived categories of sheaves of abelian groups

$$D(X_{\mathrm{rig}}) \cong D(X_{\mathrm{aff}}).$$

Now we notice that as an object in $D(X_{\mathrm{aff}})$, the derived tensor product $(Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}}$ is equal to the derived sheafification of the functor

$$X_{\mathrm{aff}} \ni U \longmapsto \left(R\Gamma(U/\Sigma_{e'\,\mathrm{inf}}, \mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}}\right).$$

Since U is an affinoid open subset, by the similar argument in the last paragraph the natural adjunction map below is an isomorphism

$$R\Gamma(U/\Sigma_{e'\,\mathrm{inf}}, \mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}} \longrightarrow R\Gamma(U/\Sigma_{e\,\mathrm{inf}}, \mathcal{F}).$$

Thus the derived tensor product $(Ru_{X/\Sigma_{e'}*}\mathcal{F}') \otimes^L_{\mathrm{B}^+_{\mathrm{dR},e'}} \mathrm{B}^+_{\mathrm{dR,e}}$ is naturally isomorphic to the derived sheafification of the functor $U \mapsto R\Gamma(U/\Sigma_{e\,\mathrm{inf}}, \mathcal{F})$, which is exactly $Ru_{X/\Sigma_e*}\mathcal{F}$. So we are done by the equivalence of the derived categories above. $\square$

CHAPTER 4

Deligne–Du Bois Cohomology

In this chapter, we introduce the rigid analytic analogue of the Deligne–Du Bois complex, namely the *éh de Rham complex*, which is the sheafification of the continuous de Rham complex over the *éh topology* of a rigid space X, where the notion for algebraic varieties was first by Geisser [Gei06], modified by the *h topology* of Voevodsky [Voe96]. The results of this chapter first appeared in [Guo19, Section 2, 5, 6].

We start by introducing the notion of the éh topology for a rigid space X in Section 4.1. It is defined on the category of all rigid spaces over X, and the coverings are generated by étale morphisms, universal homeomorphisms, and blowups. Thanks to the desingularization of rigid spaces by Temkin ([Tem12]), we in particular show that the éh topology of X is always *locally smooth* (Corollary 4.1.4.8): X can always be covered by smooth rigid spaces. We then consider the éh sheaf of continuous differentials in Section 4.2, which is defined as the sheafification of the sheaf of continuous Kähler differentials under the éh topology. Our main result is Theorem 4.2.1.1, where we show that the cohomology of éh differentials coincides with the cohomology of usual continuous Kähler differentials when X is smooth. At the end of the chapter, we prove some finiteness and cohomological boundedness for the cohomology of éh differential, using the aforementioned result and the cohomological descent.

4.1 éh-topology

In this section, we introduce the éh-topology and study its local structure.

4.1.1 Rigid spaces

We first give a quick review about rigid spaces, following [Hub96].

Let K be a complete non-archimedean extension of $\mathbb{Q}_p$. Denote by Rig_K the category of *rigid spaces over* $\mathrm{Spa}(K)$; namely its objects consist of adic spaces that are locally of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$. Then for any $X \in \mathrm{Rig}_K$, it can be covered by affinoid open subspaces, where each of them is of the form $\mathrm{Spa}(A, A^+)$ with A being a quotient of the convergent power series ring $K\langle T_1, \dots, T_n\rangle$ for some n. Here A^+ is an integrally closed open subring of A that is of topologically finite type over $\mathcal{O}_K$, and A is complete with respect to the p-adic topology on K. By the finite type condition it can be showed that any such A^+ is equal to A°, the subring consisting of all power-bounded elements in A ([Hub94], 4.4). So to simplify the notation we abbreviate $\mathrm{Spa}(A, A^\circ)$ as $\mathrm{Spa}(A)$ in this setting. Unless otherwise mentioned, in the following discussion we always assume X to be a rigid space over $\mathrm{Spa}(K)$.

For each adic space X, we can define two presheaves: $\mathcal{O}_X$ and $\mathcal{O}_X^+$, such that when the affinoid space $U = \mathrm{Spa}(B, B^+) \subset X$ is open and B is complete, we have

$$\mathcal{O}_X(U) = B,\ \mathcal{O}_X^+(U) = B^+.$$

It is known that for any $X \in \mathrm{Rig}_K$, both $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves. We could also define the coherent sheaves over rigid spaces, in a way that locally the category $\mathrm{Coh}(\mathrm{Spa}(B))$ of coherent sheaves over $\mathrm{Spa}(B)$ is equivalent to the category $\mathrm{Mod}_{fp}(B)$ of finitely presented B-modules ([KL16, Theorem 2.3.3]). An important example of coherent sheaves are *continuous differentials* $\Omega^i_{X/Y}$ for a map of rigid spaces $X \to Y$, which is a coherent sheaf of $\mathcal{O}_X$-module over X ([Hub96, Section 1.6]). [1] Locally for a map of affinoid algebras $A \to B$, it could be defined by taking the p-adic completion and inverting p at the algebraic differential module of B_0 over A_0, where $A_0 \to B_0$ is a map of topologically of finite type rings of definition over $\mathcal{O}_K$.

Recall that a *coherent ideal* is defined as a subsheaf $\mathcal{I}$ of ideals in $\mathcal{O}_X$ that is locally of finite presentation over $\mathcal{O}_X$. It is known when $X = \mathrm{Spa}(A) \in \mathrm{Rig}_K$, there is a bijection between coherent

[1] To simplify notations, we always use Ω^i to denote the continuous differential sheaves in our article, instead of algebraic ones. We will explicitly mention it when the latter comes up.

ideals $\mathcal{I}$ of X and ideals of A, given by

$$\mathcal{I} \longmapsto \mathcal{I}(X);$$
$$\widetilde{I} \longleftarrow I.$$

Here $\widetilde{I}$ is the sheaf of $\mathcal{O}_X$ module associated to

$$U \mapsto I \otimes_A \mathcal{O}_X(U).$$

For a coherent ideal $\mathcal{I}$, we can define an $analytic\ closed\ subset$ of X, by taking

$$Z := \{x \in X \mid \mathcal{O}_{X,x} \neq \mathcal{I}_{X,x}\} = \{x \in X \mid |f(x)| = 0, \forall f \in \mathcal{I}\}.$$

The subset Z has a canonical adic space structure such that when $X = \mathrm{Spa}(A)$ and $\mathcal{I} = \widetilde{I}$, we have $Z = \mathrm{Spa}(A/I, (A/I)^\circ) =: V(I)$.

4.1.2 Blowups

Before we introduce the éh-topology on Rig_K, we first recall the construction of blowup in rigid spaces, following [Con06], 4.1.

Let $X \in \mathrm{Rig}_K$ be a rigid space, and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal. Then for any $U = \mathrm{Spa}(B) \subset X$ open, $\mathcal{I}(U) \subseteq B$ is a finitely generated ideal of B. Let $Z = V(\mathcal{I})$ be the closed rigid subspace defined by $\mathcal{O}_X/\mathcal{I}$, where as a closed subset $V(\mathcal{I}) = \{x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x}\}$ is the support of $\mathcal{O}_X/\mathcal{I}$. Then following Conrad [Con06], 2.3 and 4.1, we define the blowup of X along Z as follows:

Definition 4.1.2.1. *The* blowup $\mathrm{Bl}_Z(X)$ *of X along Z is the X-rigid space*

$$\mathrm{Proj}^{\mathrm{an}}(\bigoplus_{n\in\mathbb{N}} \mathcal{I}^n),$$

which is the relatively analytified Proj *of the graded algebra $\bigoplus_{n\in\mathbb{N}} \mathcal{I}^n$ over the rigid space X (see [Con06], 2.3).*

It is called a smooth blowup *if the blowup center Z is a smooth rigid space over K.*

Remark 4.1.2.2. As a warning, our definition for the smooth blowup is different from some existing contexts, where both X and Z are required to be smooth.

When $X = \mathrm{Spa}(A)$ is affinoid, the blowup of rigid space is in fact the pullback of the schematic blowup $\mathrm{Bl}_I(\mathrm{Spec}(A))$ of $\mathrm{Spec}(A)$ at the ideal I along the map $\mathrm{Spa}(A) \to \mathrm{Spec}(A)$ of locally ringed spaces. Namely, the following natural diagram is cartesian

$$\begin{tikzcd}
\mathrm{Bl}_{V(I)}(\mathrm{Spa}(A)) \arrow[r] \arrow[d] & \mathrm{Bl}_I(\mathrm{Spec}(A)) \arrow[d] \\
\mathrm{Spa}(A) \arrow[r] & \mathrm{Spec}(A).
\end{tikzcd}$$

This follows from the universal property of the relative analytification functor as in [Con06] 2.2.5 and 2.2.3: for a given rigid space Y, there exists a functorial bijection between the collection of morphisms $h : Y \to \mathrm{Bl}_{V(I)}(\mathrm{Spa}(A))$ of rigid spaces over $\mathrm{Spa}(A)$, and the collection of the following commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "g"'] & \mathrm{Bl}_I(\mathrm{Spec}(A)) \arrow[d] \\
\mathrm{Spa}(A) \arrow[r] & \mathrm{Spec}(A),
\end{tikzcd}$$

where f is a map of locally ringed spaces and g is a morphism of rigid spaces.

As what happens in the scheme theory, $\mathrm{Bl}_Z(X)$ satisfies the universal property (see [Con06] after Definition 4.1.1): for any $f : Y \to X$ in Rig_K such that the pullback $f^*\mathcal{I}$ is invertible, it factors uniquely through $\mathrm{Bl}_Z(X) \to X$. This leads to the isomorphism of the blowup map when it is restricted to the open complement $X \backslash Z$. Besides, it can be showed by universal property that rigid blowup is compatible with flat base change and analytification of schematic blowup (see [Con06], 2.3.8). Precisely, for a flat map of rigid spaces $g : Y \to X$ (i.e. $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,x}$ for any $y \in Y$ over $x \in X$), we have

$$\mathrm{Bl}_{g^*\mathcal{I}}(Y) = \mathrm{Bl}_Z(X) \times_X Y.$$

When $X = X_0^{\mathrm{an}}$ is an analytification of a scheme X_0 of finite type over K, with Z being defined by an ideal sheaf $\mathcal{I}_0$ of $\mathcal{O}_{X_0}$, we have

$$\mathrm{Bl}_Z(X) = \mathrm{Bl}_{\mathcal{I}_0}(X_0)^{\mathrm{an}}.$$

We also note that the blowup map $\mathrm{Bl}_Z(X) \to X$ is proper. This is because by the coherence of $\mathcal{I}$, locally $\mathcal{I}$ can be written as a quotient of a finite free module, which (locally) produces a closed immersion of $\mathrm{Bl}_Z(X)$ into a projective space over X, thus is proper over X. Moreover, if both the center Z and the ambient space X are smooth over K, then the blowup itself $\mathrm{Bl}_Z(X)$ is also smooth.

4.1.3 Universal homeomorphisms

Another type of morphisms that will be used later is the universal homeomorphism.

Definition 4.1.3.1. *Let $f : X' \to X$ is a morphism of rigid spaces over K. It is called a* universal homeomorphism *if for any morphism of rigid spaces $g : Y \to X$, the base change of f to $X' \times_X Y \to Y$ is a homeomorphism.*

The following proposition gives a criterion of universal homeomorphisms of rigid spaces:

Proposition 4.1.3.2. *Let $f : X \to Y$ be a morphism of rigid spaces over $\mathrm{Spa}(K)$. Then it is a universal homeomorphism if and only if the following two conditions hold*

(i) f is a finite morphism of rigid spaces.

(ii) For any pair of affinoid open subsets $V = \mathrm{Spa}(A) \subset Y$ and $U = f^{-1}(V) = \mathrm{Spa}(B)$, the corresponding map of schemes

$$\widetilde{f} : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$

is a universal homeomorphism of schemes.

Proof. Assume f is a universal homeomorphism. Let $x \in X$ be a rigid point. Since the map f is quasi-finite, by [Hub96, 1.5.4] there exists open neighborhoods $U \subset X$ of x and V of $f(x)$ such that $f(U) \subset V$ and $f : U \to V$ is finite. We may assume both U and V are connected. On the one hand, the finiteness of $f : U \to V$ implies the image of U is closed. On the other hand, as f is a homeomorphism, $f(U)$ is open in Y, and thus open in V. Combine both of those, we see V is exactly equal to $f(U)$ with $U = f^{-1}(V)$. So by the density of the rigid points $x \in X$, there exists an open covering V_i of Y such that $f^{-1}(V_i)$ is finite over V_i. Hence we get the finiteness of f.

To check the universal homeomorphism for corresponding map of affine schemes, we recall from the Stack Project [Sta18, Tag 04DC], that $\widetilde{f} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism of schemes if and only if it is integral, universally injective and surjective. Since both A and B are K-algebras, where K is an extension over $\mathbb{Q}_p$, it suffices to show the following claim.

Lemma 4.1.3.3. *Let $f : \mathrm{Spa}(B) \to \mathrm{Spa}(A)$ be a universal homeomorphism of affinoid rigid spaces. Then the induced map of affine schemes $\widetilde{f} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is integral, bijective, and induces isomorphisms on their residues fields.*

Proof of Lemma. As we just showed above, the map of affinoid algebras $A \to B$ is finite, thus $\widetilde{f}$ is a finite (hence integral) map of schemes.

For the rest of the claim, we first consider its restrction on closed points. Let $\widetilde{y}$ be a closed point of the scheme $\mathrm{Spec}(A)$, whose residue field $\kappa(\widetilde{y})$ is a finite extension of the p-adic field K.

The defining ideal of $\widetilde{y}$ in the scheme $\mathrm{Spec}(A)$ induces a unique rigid point y of the rigid space $Y = \mathrm{Spa}(A)$, whose residue field is equal to $\kappa(\widetilde{y})$. By assumption, the base change of the universal homeomorphism f along the closed immersion $\{y\} \to Y$ induces a universal homeomorphism $X_y := \mathrm{Spa}(\kappa(\widetilde{y})) \times_{\mathrm{Spa}(A)} \mathrm{Spa}(B) \to \mathrm{Spa}(\kappa(\widetilde{y}))$, whose natural map to $X = \mathrm{Spa}(B)$ is a closed immersion. This implies that the reduced subspace of X_y is a rigid point in $\mathrm{Spa}(B)$, and the corresponding closed subscheme inside of $\mathrm{Spec}(B)$ is supported at a unique closed point. Here we also notice that the residue field of X_y is a finite separatble extension of $\kappa(\widetilde{y})$. Moreover, applying the universal homeomorphism at the base change $X_y \times_{\mathrm{Spa}(\kappa(\widetilde{y}))} X_y \to X_y$, we see the residue field of X_y is isomorphic to $k(\widetilde{y})$. As a consequence, the map $\widetilde{f}$ induces a bijection and isomorphisms of residues fields when restricted to their closed points.

To finish the proof, it suffices to extend the claim for non-closed points. The bijection of $\widetilde{f} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ follows from the density of closed points. To see this, we may assume $\mathrm{Spec}(A)$ is irreducible. Then as $\widetilde{f}$ is a finite morphism whose image contains all closed points, we get the surjectivity of $\widetilde{f}$. For the injectivity, by the homeomorphism between $\mathrm{Spa}(B)$ and $\mathrm{Spa}(A)$, the scheme $\mathrm{Spec}(B)$ admits a unique irreducible component (hence a unique generic point), and has the same dimension as $\mathrm{Spec}(A)$. At last, as the induced map of $\widetilde{f}$ on the generic fields is finite and separable, its isomorphism follows from the bijection of points. So we are done.

$\square$

Conversely, assume f satisfies the two conditions as in the statement. We first notice that both items in the statement are invariant under any base change of rigid soaces. We let $V = \mathrm{Spa}(A)$ and $f^{-1}(V) = \mathrm{Spa}(B)$ be two open affinoid open subsets of Y and X separately. Note that since f is finite, for a morphism of affinoid rigid spaces $\mathrm{Spa}(C) \to \mathrm{Spa}(A)$, the base change $\mathrm{Spa}(C) \times_{\mathrm{Spa}(A)} \mathrm{Spa}(B)$ is exactly $\mathrm{Spa}(B \otimes_A C)$ ([Hub96, 1.4.2]). In particular, we see $\mathrm{Spa}(C) \times_{\mathrm{Spa}(A)} \mathrm{Spa}(B) \to \mathrm{Spa}(C)$ is a finite morphism of rigid spaces, with the underlying map of schemes being a universal homeomorphism. As a consequence, both of the two conditions in the statement above are base change invariant, and to show $f : X \to Y$ is a universal homeomorphism of rigid spaces, it suffices to show that f itself is a homeomorphism. Moreover, by the finiteness, as the map f is both closed and continuous, we are only left to show the bijectivity of f, as a map of rigid spaces.

Now we pick any point $y \in Y$, and consider the completed residue field with its valuation ring $(k(y), k(y)^+)$ of y. We take an open affinoid neighborhood $V = \mathrm{Spa}(A)$ of y with $f^{-1}(V) = \mathrm{Spa}(B)$. Then the base change of the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of schemes gives

$$\mathrm{Spec}(B \otimes_A k(y)) \longrightarrow \mathrm{Spec}(k(y)),$$

which is a universal homeomorphism by assumption. Here the target has exactly one point, and by finiteness we have $B \otimes_A k(y) = B \widehat{\otimes}_A k(y)$. So by assumption the reduced subscheme

$\mathrm{Spec}(B\widehat{\otimes}k(y))_{\mathrm{red}}$ is equal to $k(y)$ (since they are of characteristic 0). We then note that the adic spectrum

$$\mathrm{Spa}(B\widehat{\otimes}_A k(y), B^\circ \otimes_{A^\circ} k(y)^+)$$

is exactly the preimage of y in the rigid space X along the morphism f. Notice that the integral closure of $k(y)^+$ in $k(y)$ is contained in the quotient ring of $B^\circ \otimes_{A^\circ} k(y)^+$ by its nilpotent elements, which has to be $k(y)^+$ itself (as the integral closure is contained in the field $k(y)$ and is finite over $k(y)^+$). In this way, the preimage $f^{-1}(y)$ has exactly one point x whose residue field with valuation is equal to $(k(y), k(y)^+)$. Hence f is bijective, and thus a homeomorphism. $\square$

At last, when the target is assumed to be a smooth rigid space, there is no nontrivial universal homeomorphisms:

Proposition 4.1.3.4. *Let X be a smooth rigid space, and X' be a reduced rigid space. Then any universal homeomorphism $f : X' \to X$ is an isomorphism.*

Proof. By the Proposition 4.1.3.2, every universal homeomorphism $f : X' \to X$ can be covered by morphisms of affinoid spaces $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$, where the underlying morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism. So it suffices to show that when $X = \mathrm{Spa}(A)$ is a smooth affinoid rigid space over $\mathrm{Spa}(K)$, A is a seminormal ring (so any universal homeomorphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ from a reduced scheme is an isomorphism). But note that by the smoothness of X, A is a regular ring (by [Hub96] 1.6.10, locally X is étale over the adic spectrum of Tate algebras $K\langle T_i\rangle$, which is regular). So A is normal, and thus seminormal. $\square$

4.1.4 éh-topology and its structure

Now we can introduce the éh-topology on Rig_K.

Definition 4.1.4.1. *The éh-topology on the category* Rig_K *is the Grothendieck topology such that the covering families are generated by the following types of morphisms:*

- *étale coverings;*
- *universal homeomorphisms;*
- *coverings associated to blowups:* $\mathrm{Bl}_Z(X) \sqcup Z \to X$*, where Z is a closed analytic subset of X.*

In the sense of Grothendieck pretopology in [SGA72] Exposé II.1, this means that a family of maps $\{X_\alpha \to X\}$ is in the set $Cov(X)$ if $\{X_\alpha \to X\}$ can be refined by a finitely many compositions of the three classes of maps above.

We denote by $\mathrm{Rig}_{K,éh}$ *the* big éh site *on* Rig_K *given by the éh-topology. For a given rigid space X over K, we define $X_{\text{éh}}$ as the localization of* $\mathrm{Rig}_{K,éh}$ *on X (in the sense of [Sta18, Tag 00XZ], i.e. it is defined on the category of K-rigid spaces over X with the éh-topology.*

Remark 4.1.4.2.

1. We notice that a covering associated to a blowup $\mathrm{Bl}_Z(X) \sqcup Z \to X$ is always surjective: by the discussion in the Subsection 4.1.2, $\mathrm{Bl}_Z(X) \to X$ is an isomorphism when restricted to $X \backslash Z$.

2. Among the three classes of maps above, a covering associated to a blowup is not base change invariant in general. But note that for any morphism $Y \to X$, the pullback of the blowup $X' = \mathrm{Bl}_Z(X) \to X$ along $Y \to X$ can be refined by the blowup

$$\begin{tikzcd}
\mathrm{Bl}_{Y\times_X Z}(Y)\coprod Y\times_X Z \arrow[dr] \arrow[ddr] & & \\
& Y\times_X \mathrm{Bl}_Z(X)\coprod Y\times_X Z \arrow[r] \arrow[d] & \mathrm{Bl}_Z(X)\coprod Z \arrow[d] \\
& Y \arrow[r] & X.
\end{tikzcd}$$

   We call $\mathrm{Bl}_{Y\times_X Z}(Y)\coprod Y\times_X Z$ the $canonical\ refinement$ for the base change of the blow up.

3. Though denoted as $X_{\text{éh}}$, this site is still a big site. As an extreme case, when $X = \mathrm{Spa}(K)$, the site $X_{\text{éh}}$ is identical to $\mathrm{Rig}_{K,éh}$.

Remark 4.1.4.3. Here we note that our definition of éh-topology is different from h-topology. One of the main differences is that the éh-topology excludes the ramified covering.

For example, consider the n-folded cover map of the unit disc to itself $f : \mathbb{B}^1 \to \mathbb{B}^1$, which sends the coordinate T to T^n. Then f is a finite surjective map that is relatively smooth at all the other rigid points except at $T = 0$, where it is ramified. If f is an éh-covering, by the Theorem 4.1.4.11 which we will prove later, f can be refined by finite many compositions of coverings associated to smooth blowups and étale coverings. Notice that étale coverings are unramified maps that preserve the smoothness and dimensions. Moreover, smooth blowups of a one dimensional smooth rigid space are isomorphic to itself. In this way, such a finite composition will not produce a covering that is ramified at any rigid points, and we get a contradiction.

Example 4.1.4.4. Let X be a rigid space. We take $X' = X_{\mathrm{red}}$ to be the reduced subspace of X. Then $X' \to X$ is a universal homeomorphism, which is then an éh-covering. So in the éh-topology, every space locally is reduced.

Proposition 4.1.4.5. *Let X be a quasi-compact quasi-separated rigid space over K. Assume X_i for $i = 1, \dots, n$ are irreducible components of X (see [Con99]). Then the map*

$$\coprod_{i=1}^{n} X_i \to X$$

is an éh-covering.

Proof. We first claim that the canonical map $\pi : \mathrm{Bl}_{X_1}(X) \to X$ factors through $\bigcup_{i>1} X_i \to X$; in other words, the image of π is disjoint with $X_1 \backslash (\bigcup_{i>1} X_i)$.

Let $x \in X_1 \backslash (\bigcup_{i>1} X_i)$ be any point. Take any open neighborhood $U \subset X_1 \backslash (\bigcup_{i>1} X_i)$ that contains x. Then the base change of π along the open immersion $U \to X$ becomes

$$\mathrm{Bl}_{U \bigcap X_1}(U) \to U,$$

by the flatness of $U \to X$ and the discussion in the Subsection 4.1.2. But by our choice of U, the intersection $U \bigcap X_1$ is exactly the whole space U, which by definition leads to the emptiness of $\mathrm{Bl}_{U \bigcap X_1}(U)$. Thus the intersection of $\mathrm{Bl}_{X_1}(X)$ with $p^{-1}(U)$ is empty, and the point x is not contained in the image of π.

At last, note that the claim leads to the following commutative diagram

$$\begin{tikzcd} \mathrm{Bl}_{X_1}(X) \coprod X_1 \arrow[rr] \arrow[dr] & & X , \\ & (\bigcup_{i>1} X_i) \coprod X_1 \arrow[ur] & \end{tikzcd}$$

which shows that the map $(\bigcup_{i>1} X_i) \coprod X_1 \to X$ is also an éh-covering. Thus by induction on the number of components n, we get the result. $\square$

Here we define a specific types of éh-covering.

Definition 4.1.4.6. *For an éh-covering $f : X' \to X$ of rigid spaces, we say it is a* proper éh-covering *if f is proper, and there exists a nowhere dense analytic closed subset $Z_{\mathrm{red}} \subset X_{\mathrm{red}}$ such that*

$$f|_{f^{-1}(X \backslash Z)_{\mathrm{red}}} : f^{-1}(X_{\mathrm{red}} \backslash Z_{\mathrm{red}}) \longrightarrow X_{\mathrm{red}} \backslash Z_{\mathrm{red}}$$

is an isomorphism.

As an example, a covering associated to a blowup for the center being nowhere dense is a proper éh-covering.

The idea of allowing blowups in the definition of the éh site is to make all rigid spaces éh-locally smooth. To make this explicit, we recall the Temkin's non-embedded disingularization:

Theorem 4.1.4.7 ([Tem12], 1.2.1, 5.2.2)**.** *Let X be a generically reduced, quasi-compact rigid space over* $\mathrm{Spa}(K)$*. Then there exists a composition of finitely many smooth blowups $X_n \to X_{n-1} \to \cdots \to X_0 = X$, such that X_n is smooth.*

Corollary 4.1.4.8 (Local smoothness)**.** *For any quasi-compact rigid space X, there exists a proper éh-covering $f : X' \to X_{\mathrm{red}}$ such that X' is a smooth rigid space over* $\mathrm{Spa}(K)$*. Moreover, f is a composition of finitely many coverings associated to smooth blowups.*

Proof. By the Temkin's result, we may let $X_n \to \cdots \to X_0 = X_{\mathrm{red}}$ be the blowup in that Theorem, such that the center of each $p_i : X_i \to X_{i-1}$ is a smooth analytic subset Z_{i-1} of X_{i-1}. Then by taking the composition of the covering associated to the blowup associated to each p_i, the map

$$X' := X_n \coprod (\sqcup_{i=0}^{n-1} Z_i) \to X_{\mathrm{red}}$$

is a proper éh-covering, such that X' is smooth. So we get the result. □

At last, we give a useful result about the structure of the éh-covering. In order to do this, we need the embedded strong desingularization by Temkin:

Theorem 4.1.4.9 (Embedded desingularization. Temkin [Tem18], 1.1.9, 1.1.13)**.** *Let X be a quasi-compact smooth rigid space over* $\mathrm{Spa}(K)$*, and $Z \subset X$ be an analytic closed subspace. Then there exists a finite sequence of smooth blowups $X' = X_n \to \cdots X_0 = X$, such that the strict transform of Z is also smooth.*

Corollary 4.1.4.10. *Any blowup $f : Y \to X$ over a smooth quasi-compact rigid space X can be refined by a composition of finitely many smooth blowups.*

Proof. Assume Y is given by $\mathrm{Bl}_Z(X)$, where $Z \subset X$ is a closed analytic subspace. Then by the Embedded desingularization, we could find $g : X' \to X$ to be a composition of finitely many smooth blowups such that the strict transform Z' of Z is smooth over K. Here the total transform of Z is $g^{-1}(Z) = Z' \cup E_Z$, where E_Z is a divisor. Next we could blowup Z' in X' and get $h : X'' \to X'$. Note that h itself is a smooth blowup. In this way, the composition $h \circ g$ is a composition of finitely many smooth blowups that factorizes through $f : Y \to X$, by the universal property of f and the observation that the preimage of Z along $h \circ g$ is the divisor

$$h^{-1}(Z') \cup h^{-1}(E_Z).$$

□

Theorem 4.1.4.11. *Let $X \in \mathrm{Rig}_K$ be a quasi-compact smooth rigid space and $f : X' \to X$ be an éh-covering. Then f can be refined by a composition of finitely many étale coverings and coverings associated to smooth blowups over X.*

Proof. By the definition of the éh-topology, a given éh-covering f could be refined by a finitely many compositions of étale coverings, universal homeomorphisms, and coverings associated to blowups. So up to a refinement we may write f as $f : X' = X_n \to X_{n-1} \to \cdots \to X_0 = X$, where each transition map $f_i : X_i \to X_{i-1}$ is one of the above three types of morphisms.

Now we produce a refinement we want, by doing the following operations on f starting from $i = 1$:

- If $X_1 \to X_0$ is an étale morphism, then we are done for this $i = 1$.
- If $X_1 \to X_0$ is a universal homeomorphism, then by the Proposition 4.1.3.4 we may take the reduced subspace of X_1, which is isomorphic to X_0 and thus is smooth.
- If $X_1 \to X_0$ is a covering associated to a blowup, then by the Proposition 4.1.4.10, the associated blowup can be refined by finitely many compositions of smooth blowups. We let $X_1' \to X_1$ be the disjoint union of that refinement with all of the centers. Then we take the base change of $X_n \to \cdots \to X_1$ along $X_1' \to X_1$ and get a new coverings $X_n \times_{X_1} X_1' \to \cdots \to X_1' \to X_0 = X$, i.e.

$$\begin{tikzcd}
X_n \times_{X_1} X_1' \arrow[r] & \cdots \arrow[r] & X_1' \arrow[d] \arrow[dr] & \\
X_n \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X_0 = X.
\end{tikzcd}$$

  Furthermore, starting at $j = 2$, we do the following operation and increase j by 1 each time: If $X_j \to X_{j-1}$ is a covering associated to a blowup, we refine the map $X_j \times_{X_1} X_1' \to X_{j-1} \times_{X_1} X_1'$ by its canonical refinement $X_j' \to X_{j-1} \times X_1'$ (see the Remark 4.1.4.2), and take the base change of the chain $X_n \times_{X_1} X_1' \to \cdots \to X_j \times_{X_1} X_1'$ along $X_j' \to X_j \times_{X_1} X_1'$.[2]

After the discussion of the above three possibilities, $X_n \to \cdots \to X_0$ is refined by finitely many compositions $X_n' \to \cdots X_1' \to X_0$ such that

- $X_1' \to X_0$ is a composition of finitely many étale coverings and coverings associated to smooth blowups;

[2]The covering associated to a blowup is not preserved under the base change, thus we need to adjust all of the maps in $X_n \times_{X_1} X_1' \to \cdots \to X_1'$ so that they will then become exactly those three types of morphisms.

- $X'_n \to X_1$ is a composition of $n-1$ éh-coverings by étale coverings, coverings associated to blowups, or universal homeomorphisms.

In this way, we could do the above operation for $X'_i \to X'_{i-1}$ and $i = 2, \ldots$, each time get a new chain of coverings $X''_n \to \cdots X_0$ such that $X''_i \to X_0$ is a finite compositions of smooth blowups and étale coverings, and $X''_n \to X''_i$ is a composition of $n-i$ coverings of three generating classes. Hence after finitely many operations, we are done.

□

Corollary 4.1.4.12. *Any éh-covering of a quasi-compact rigid space X can be refined by a composition*

$$X_2 \to X_1 \to X_0 = X,$$

where $X_1 = X_{\mathrm{red}}$, the map $X_2 \to X_1$ is equal to finitely many compositions of étale-coverings and coverings associated to smooth blowups, and X_2 is smooth over K.

Proof. Let $X' \to X$ be a given éh-covering. By the Example 4.1.4.4, $X_1 := X_{\mathrm{red}} \to X_0$ is an éh-covering. And by the local smoothness of éh-topology (Corollary 4.1.4.8), there exists a composition of finitely many coverings associated to smooth blowups $Y_1 \to X_1$, such that Y_1 is smooth. So the base change of $X' \times_X X_1 \to X_1$ along $Y_1 \to X_1$ becomes an éh-covering whose target is smooth and quasi-compact. Hence by the Theorem 4.1.4.11 above, we could refine $X' \times Y_1 \to Y_1$ by $X_2 \to Y_1$, where the latter is a finitely many composition of étale coverings and coverings associated to smooth blowups. At last, notice that an étale map or a smooth blowup will not change the smoothness. Hence the composition $X_2 \to X_1 \to X_0$ satisfies the requirement. □

4.2 éh-descent for the differentials

In this section, we prove the descent for the éh-differential of a smooth rigid space $X \in \mathrm{Rig}_K$, where K is any p-adic field (not necessarily algebraically closed). At the end of the section, we apply the éh-descent to the case when X is coming from an algebraic variety, to relate the éh cohomology to the Deligne-Du Bois complex (cf. [DB81], [PS08, Section 7]).

4.2.1 éh-descent

We will follow the idea in [Gei06], showing the vanishing of the cone C for $\Omega^j_{X/K} \to R\pi_*\Omega^j_{\mathrm{\acute{e}h}}$ by comparing the étale cohomology and éh cohomology.

Our main theorem is the following.

Theorem 4.2.1.1 (éh-descent)**.** *Assume $X \in \mathrm{Rig}_K$ is a smooth rigid space over $\mathrm{Spa}(K)$. Then for each $j \in \mathbb{N}$, we have*

$$R\pi_{X*}\Omega^j_{eh} = R^0\pi_{X*}\Omega^j_{eh}[0] = \Omega^j_{X/K}.$$

Remark 4.2.1.2. When $i = j = 0$, the section $\mathcal{O}_{\text{éh}}(X)$ of the éh-structure sheaf on any rigid space X is $\mathcal{O}(X^{sn})$, where X^{sn} is the semi-normalization of X_{red}. In other words, $\mathcal{O}_{\text{éh}} = \mathcal{O}^{sn}$. This follows from [SW20], 10.2.4.

We first show the long exact sequence of continuous differentials for coverings associated to blowups.

Proposition 4.2.1.3. *Let $f : X' \to X$ be a blowup of a smooth rigid space X along a smooth and nowhere dense closed analytic subset $i : Y \subset X$, with the pullback $g : Y' = X' \times_X Y \to Y$. Then the functoriality of Kähler differentials induces the following distinguished triangle in the derived category of X:*

$$\Omega^j_{X/K} \longrightarrow Rf_*\Omega^j_{X'/K} \bigoplus i_*\Omega^j_{Y/K} \longrightarrow i_*Rg_*\Omega^j_{Y'/K}. \qquad (*)$$

Proof. We first note that since the argument is local on X, it suffices to show for any given rigid point $x \in X$, there exists a small open neighborhood of x such that the result is true over that. So we may assume $X = \mathrm{Spa}(A)$ is affinoid, admitting an étale morphism to $\mathbb{B}^n_K = \mathrm{Spa}(K\langle x_1, \ldots, x_n\rangle)$ by [Hub96] 1.6.10, and Y is of dimension r, given by the $\mathrm{Spa}(A/I)$ for an ideal I of A. Moreover, by refining X to a smaller open neighborhood of x if necessary, we could choose a collection of local parameters $f_1, \ldots, f_r$ and $g_1, \ldots, g_{n-r}$ at x, such that $\{g_l\}$ locally generates the ideal defining Y in X. In this way, by the differential criterion for étaleness (see [Hub96], 1.6.9), we could assume $Y \to X$ is an étale base change of the closed immersion

$$\mathbb{B}^r_K \longrightarrow \mathbb{B}^n_K.$$

In particular, the blowup diagram for $\mathrm{Bl}_Y(X) \to X$ locally is the étale base change of $\mathrm{Bl}_{\mathbb{B}^r_K}(\mathbb{B}^n_K)$ along $X \to \mathbb{B}^n_K$.

Then we notice that the blowup of $\mathbb{B}^n$ along $\mathbb{B}^r$ is equivalent to the generic fiber of the p-adic (formal) completion of the blowup

$$\mathbb{A}^r_{\mathcal{O}_K} \longrightarrow \mathbb{A}^n_{\mathcal{O}_K}.$$

Furthermore, as proved in [Gro85, IV. Theorem 1.2.1], there exists a natural distinguished triangle as follows

$$\Omega^j_{\mathbb{A}^n/\mathcal{O}_K} \longrightarrow Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r}(\mathbb{A}^n)/\mathcal{O}_K} \bigoplus i_*\Omega^j_{\mathbb{A}^r/\mathcal{O}_K} \longrightarrow i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r}(\mathbb{A}^n)\times\mathbb{A}^r/\mathcal{O}_K}. \qquad (**)$$

Now we make the following claim:

Claim 4.2.1.4. The sequence $(*)$ for $(X, Y) = (\mathbb{B}^n_K, \mathbb{B}^r_K)$ can be given by the generic base change of the derived p-adic completion of the distinguished triangle $(**)$.

Granting the Claim, since both derived completion and the generic base change are exact functors, we are done.

Proof of the Claim. We first notice since $\mathbb{A}^n_{\mathcal{O}_K} = \mathrm{Spec}(\mathcal{O}_K[T_1, \ldots, T_n])$ is p-torsion free (thus flat over $\mathcal{O}_K$), by [Sta18] Tag 0923, for a complex $C \in D(\mathbb{A}^n_{\mathcal{O}_K})$ its p-adic derived completion is given by

$$R\varprojlim(C \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/p^m\mathcal{O}_K).$$

Moreover, note that differentials of $\mathbb{A}^n_{\mathcal{O}_K}, \mathbb{A}^r_{\mathcal{O}_K}, \mathrm{Bl}_{\mathbb{A}^r_{\mathcal{O}_K}}(\mathbb{A}^n_{\mathcal{O}_K})$ and $\mathrm{Bl}_{\mathbb{A}^r_{\mathcal{O}_K}}(\mathbb{A}^n_{\mathcal{O}_K}) \times_{\mathbb{A}^n_{\mathcal{O}_K}} \mathbb{A}^r_{\mathcal{O}_K}$ over $\mathcal{O}_K$ are all flat over $\mathcal{O}_K$. We use the notations $\mathbb{A}^n_m$ and $\mathbb{A}^r_m$ to abbreviate the schemes $\mathbb{A}^n_{\mathcal{O}_K/p^m}$ and $\mathbb{A}^r_{\mathcal{O}_K/p^m}$ separately. Then the derived base change of $(**)$ along $\mathcal{O}_K \to \mathcal{O}_K/p^m$ can be written as the following

$$\Omega^j_{\mathbb{A}^n_m/(\mathcal{O}_K/p^m)} \longrightarrow Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)} \bigoplus i_*\Omega^j_{\mathbb{A}^r_m/(\mathcal{O}_K/p^m)} \longrightarrow i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)\times_{\mathbb{A}^n_m}\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}. \tag{$***$}$$

Here we use the formula $\Omega^j_{Y/\mathcal{O}_K} \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/p^m = \Omega^j_{Y_m/(\mathcal{O}_K/p^m)}$ for a smooth $\mathcal{O}_K$-scheme Y, together with the derived base change formula for a proper morphism ([Sta18, Tag 07VK]). Hence the derived p-adic completion of $(**)$ is then computed by the derived limit of $(***)$ for $m \in \mathbb{N}$.

At last we discuss those derived limits term by term. For $C_m = \Omega^j_{\mathbb{A}^n_m/(\mathcal{O}_K/p^m)}$ or $C_m = i_*\Omega^j_{\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}$, since their transition maps are surjective, the derived limit has no higher cohomology and we have

$$R\varprojlim \Omega^j_{\mathbb{A}^n_m/(\mathcal{O}_K/p^m)} = \Omega^j_{\widehat{\mathbb{A}}^n/\mathcal{O}_K},\ R\varprojlim i_*\Omega^j_{\mathbb{A}^r_m/(\mathcal{O}_K/p^m)} = \Omega^j_{\widehat{\mathbb{A}}^r/\mathcal{O}_K}.$$

For $C_m = Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)}$ or $i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)\times\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}$, recall we have the formula of the derived functors

$$Rf_*R\varprojlim_m = R\varprojlim_m Rf_*.$$

Apply the formula, we get

$$\begin{aligned} R\varprojlim_m Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)} &\cong Rf_*R\varprojlim_m \Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)} \\ &= Rf_*\Omega^j_{\mathrm{Bl}_{\widehat{\mathbb{A}}^r}(\widehat{\mathbb{A}}^n)/\mathcal{O}_K}. \end{aligned}$$

The analogous holds for $C_m = i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)\times_{\mathbb{A}^n_m}\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}$.

In this way, the derived limit of $(***)$ is isomorphic to

$$\Omega^j_{\widehat{\mathbb{A}}^n/\mathcal{O}_K} \longrightarrow Rf_*\Omega^j_{\mathrm{Bl}_{\widehat{\mathbb{A}}^r}(\widehat{\mathbb{A}}^n)/\mathcal{O}_K} \bigoplus i_*\Omega^j_{\widehat{\mathbb{A}}^r/\mathcal{O}_K} \longrightarrow i_*Rg_*\Omega^j_{\mathrm{Bl}_{\widehat{\mathbb{A}}^r}(\widehat{\mathbb{A}}^n)\times\widehat{\mathbb{A}}^r/\mathcal{O}_K}.$$

At last, we take the base change of this distinguished triangle along $\mathbb{Z}_p \to \mathbb{Q}_p$, then we get $(*)$ for the pair of discs.

$\square$

$\square$

Corollary 4.2.1.5. *Under the above notation for the smooth blowup, we get a natural long exact sequence of étale cohomology of continuous differentials:*

$$\cdots \longrightarrow \mathrm{H}^j(X_{\text{ét}}, \Omega^i_{X/K}) \longrightarrow \mathrm{H}^j(X'_{\text{ét}}, \Omega^i_{X'/K}) \bigoplus \mathrm{H}^j(Y_{\text{ét}}, \Omega^i_{Y/K}) \longrightarrow \mathrm{H}^j(Y'_{\text{ét}}, \Omega^i_{Y'/K}) \longrightarrow \cdots .$$

Similarly there exists a long exact sequence of the covering associated to a blowup for the éh-cohomologies:

Proposition 4.2.1.6. *Let $f : X' \to X$ be a morphism of rigid spaces over $\mathrm{Spa}(K)$, $Y \subset X$ be a nowhere dense analytic closed subspace, and $Y' = Y \times_X X'$ be the pullback. Let X be separated. Assume they satisfy one of the following two conditions:*

(i) $X' \to X$ is a blowup along Y.

(ii) X is quasi-compact, Y is an irreducible component of X, and X' is the union of all the other irreducible components of X

Then the functoriality of differentials induces a natural long exact sequence of cohomologies:

$$\cdots \longrightarrow \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}) \longrightarrow \mathrm{H}^j(X'_{\text{éh}}, \Omega^i_{\text{éh}}) \bigoplus \mathrm{H}^j(Y_{\text{éh}}, \Omega^i_{\text{éh}}) \longrightarrow \mathrm{H}^j(Y'_{\text{éh}}, \Omega^i_{\text{éh}}) \longrightarrow \cdots ,$$

where $\Omega^i_{\text{éh}}$ is the éh-*sheafification of the i-th continuous differential forms.*

Proof. For the rigid space $Z \in \mathrm{Rig}_K$, we denote by h_Z the abelianization of the éh-sheafification of the representable presheaf

$$W \mapsto \mathrm{Hom}_{\mathrm{Spa}(K)}(W, Z).$$

Then for an éh sheaf of abelian group $\mathcal{F}$, we have

$$\mathrm{H}^j(Z_{\text{éh}}, \mathcal{F}) = \mathrm{Ext}^j_{\text{éh}}(h_Z, \mathcal{F}),$$

since h_Z is the final object in the category of sheaves of abelian groups over $Z_{\text{éh}}$. So back to the question, it suffices to prove the exact sequence of éh-sheaves

$$0 \longrightarrow h_{Y'} \longrightarrow h_{X'} \bigoplus h_Y \longrightarrow h_X \longrightarrow 0$$

for above two conditions.

Assume $\alpha : Z \to X$ is a K-morphism. Then since $X' \coprod Y \to X$ is an éh-covering (see 4.1.4.1, 4.1.4.5), the element $\alpha \in h_X(Z)$ is locally given by a map $Z \times_X (X' \coprod Y) \to (X' \coprod Y)$, which is an element in $h_{X'}(Z \times_X X') \bigoplus h_Y(Z \times_X Y)$, so we get the surjectivity.

Now assume $(\sum n_r\beta_r, \sum m_s\gamma_s)$ is an element in $h_{X'}(Z) \bigoplus h_Y(Z)$ whose image in 0 is $h_X(Z)$. After refining Z by an admissible covering of quasi-compact affinoid open subsets if necessary, we may assume Z is quasi-compact affinoid. By taking a further éh-covering of Z, we may also assume Z is smooth and connected, given by $Z = \mathrm{Spa}(A)$ for A integral.

Then we look at the composition of those maps with $(f, i) : X' \coprod Y \to X$.

- Assume $f \circ \beta_1 = f \circ \beta_2$ for some elements β_i.

  In the first setting of the Proposition, since $X' \to X$ is a blowup along a nowhere dense (Zariski) closed subset, the restriction of β_1 and β_2 on the open subset $Z \backslash f^{-1}(Y)$ coincides. So by the assumption that Z is integral (thus equal-dimensional), we see either the closed analytic subset $f^{-1}(Y)$ is the whole Z and both β_1 and β_2 comes from $Z \to Y \times_X X' = Y'$, or $f^{-1}(Y)$ is nowhere dense analytic in Z. If $f^{-1}(Y)$ is nowhere dense in Z, β_1 and β_2 agrees on an Zariski-open dense subset of Z. So by looking at open affinoid subsets of X', the separatedness assumption implies that $\beta_1 = \beta_2$ (see [Har77], Chap. II Exercise 4.2).

  In the second setting, note that $X' \to X$ is a closed immersion. So $f \circ \beta_1 = f \circ \beta_2$ implies $\beta_1 = \beta_2$.

- Assume $i \circ \gamma_1 = i \circ \gamma_2$ for some elements γ_i. Then we get the identity of γ_1 and γ_2 again by the injectivity of the closed immersion $i : Y \to X$.

- Assume there exists a equality $f \circ \beta_i = i \circ \gamma_j$. Since the composition $f \circ \beta_i$ is mapped inside of the analytic subset $Y \subset X$, the map $\beta_i : Z \to X'$ factors through $Z \to X' \times_X Y = Y'$. So β_i comes from $h_{Y'}(Z)$, and by the injectivity of $Y \to X$ again γ_j comes from $h_{Y'}(Z)$.

In this way, by combining all of those identical β_i and γ_j and canceling the coefficients, the rest of $(\sum n_r\beta_r, \sum m_s\gamma_s)$ are all coming from $h_{Y'}(Z)$, thus the middle of the short sequence is exact.

At last, injectivity of $h_{Y'} \to h_{X'} \bigoplus h_Y$ follows from the closed immersion of $Y' \to X'$. So we are done. $\square$

Remark 4.2.1.7. The part (ii) of the Proposition 4.2.1.6 can be regarded as an éh-version Mayer-Vietories sequence.

Proof of the Theorem 4.2.1.1. Now we prove the descent for the éh-differential.

Let $\mathrm{Rig}_{K,ét}$ be the big étale site of rigid spaces over K. Namely it consists of rigid spaces over K, and its topology is defined by étale coverings. Then there exists a natural map of big sites $\pi : \mathrm{Rig}_{K,éh} \to \mathrm{Rig}_{K,ét}$ that fits into the diagram

$$\begin{tikzcd} \mathrm{Rig}_{K,ét} \arrow[r, "\iota_X"] & X_{\text{ét}} \\ \mathrm{Rig}_{K,éh} \arrow[u, "\pi"] \arrow[r] & X_{\text{éh}}. \arrow[u, "\pi_X"'] \end{tikzcd}$$

The sheaf $\Omega^i_{\text{éh}}$ on $\mathrm{Rig}_{K,éh}$ is defined as the éh-sheafification of the continuous differential, which leads to the equality

$$\Omega^i_{\text{éh}} = \pi^{-1}\Omega^i_{/K},$$

where $\Omega^i_{/K}$ is the i-th continuous differential on $\mathrm{Rig}_{K,ét}$. Besides, for any $Y \in \mathrm{Rig}_K$, the direct image along $\mathrm{Rig}_{K,ét} \to Y_{\text{ét}}$ and $\mathrm{Rig}_{K,éh} \to Y_{\text{éh}}$ are exact. So it is safe to use $\mathcal{F}|_{Y_{\text{ét}}}$ ($\mathcal{F}|_{Y_{\text{éh}}}$) to denote the direct image of a sheaf on $\mathrm{Rig}_{K,ét}$ ($\mathrm{Rig}_{K,éh}$) along those restriction maps, either in the derived or non-derived cases.

Let C be a cone of the adjunction map $\Omega^i_{/K} \to R\pi_*\pi^{-1}\Omega^i_{/K} = R\pi_*\Omega^i_{\text{éh}}$. It suffices to show the vanishing of C when restricted to a smooth X; in other words, for each X smooth over K, we want

$$\mathcal{H}^j(C)|_{X_{\text{ét}}} = 0, \forall j.$$

We also note that as both $\Omega^i_{/K}$ and $R\pi_*\pi^*\Omega^i_{/K}$ has trivial cohomology of negative degrees, we have $\mathcal{H}^j(C)|_{X_{\text{ét}}} = 0$ for $j \leq -2$. In particular, C is left bounded.

Now we prove the above statement by contradiction. Assume C is not always acyclic when restricted to the small site $X_{\text{ét}}$ for some smooth rigid space X over K. By the left boundedness of C, we let j be the smallest degree such that $\mathcal{H}^j(C)|_{X_{\text{ét}}} \neq 0$ for some smooth X. Then $\mathcal{H}^{j-l}(C)|_{Y_{\text{ét}}} = 0$ for any $l > 0$ and any smooth Y over K. As this is a local statement, we fix an X to be a smooth, connected, quasi-compact quasi-separated rigid space of the smallest possible dimension such that $\mathcal{H}^j(C)|_{X_{\text{ét}}} \neq 0$. So by our assumption, there exists a nonzero element e in the cohomology group

$$\mathrm{H}^0(X_{\text{ét}}, \mathcal{H}^j(C)) = \mathrm{H}^j(X_{\text{ét}}, C).$$

Here the equality of those two cohomologies follows from the vanishing assumption for $\mathcal{H}^{j-l}(C)|_{X_{\text{ét}}}$ for $l > 0$.

We apply the preimage functor π^{-1} to the triangle

$$\Omega^i_{/K} \longrightarrow R\pi_*\pi^{-1}\Omega^i_{/K} \longrightarrow C,$$

and get a distinguished triangle on $D(\mathrm{Rig}_{K,éh})$

$$\pi^{-1}\Omega^i_{/K} \longrightarrow \pi^{-1}R\pi_*\pi^{-1}\Omega^i_{/K} \longrightarrow \pi^{-1}C.$$

Note that since π^{-1} is exact and the adjoint map $\pi^{-1} \to \pi^{-1} \circ \pi_* \circ \pi^{-1}$ is an isomorphism, by taking the associated derived functors we get a canonical isomorphism

$$\pi^{-1}\Omega^i_{/K} \cong \pi^{-1}R\pi_*\pi^{-1}\Omega^i_{/K}.$$

So $\pi^{-1}C$ is quasi-isomorphic to 0, and there exists an éh-covering $X' \to X$ such that e will vanish when pullback to X'.

Next we use the covering structure of the éh-topology (Proposition 4.1.4.11). By taking a refinement of $X' \to X$ if necessary, we assume $X' \to X$ is the composition

$$X' = X_m \to X_{m-1} \to \cdots \to X_0 = X,$$

where $X_l \to X_{l-1}$ is either a covering associated to a smooth blowup or an étale covering.

Now we discuss the vanishing of the nonzero element e along those pullbacks $X' = X_m \to \cdots \to X$. Assume $e|_{X_{l-1,\text{ét}}}$ is not equal to 0 (which is ture when $l = 1$). If $X_l \to X_{l-1}$ is an étale covering, then since $e|_{X_{l-1,\text{ét}}} \in \mathrm{H}^0(X_{l-1,\text{ét}}, \mathcal{H}^j(C))$ is a global section of a nonzero étale sheaf $\mathcal{H}^j(C)$ on $X_{l-1,\text{ét}}$, the restriction of e onto this étale covering will not be zero by the sheaf axioms. If $X_l \to X_{l-1}$ is a covering associated to a smooth blowup, we then make the following claim:

Claim 4.2.1.8. Under the above assumption, the restriction $e|_{X_{l,\text{ét}}}$ in

$$\mathrm{H}^0(X_{l,\text{ét}}, \mathcal{H}^j(C)) = \mathrm{H}^j(X_{l,\text{ét}}, C)$$

is not equal to 0.

Granting the Claim, since $X' \to X$ is a finite composition of those two types of coverings, the pullback of e to the cohomology group $\mathrm{H}^0(X'_{\text{ét}}, \mathcal{H}^j(C))$ cannot be 0, and we get a contradiction. Hence $C|_{X_{\text{ét}}}$ must vanish in the derived category $D(X_{\text{ét}})$ for smooth quasi-compact rigid space X, and we get the natural isomorphism

$$\Omega^i_{X/K} \to R\pi_{X*}\Omega^i_{\text{éh}},\ \forall i.$$

Proof of the Claim. By assumption, since $e|_{X_{l-1,\text{ét}}}$ is nonzero, it suffices to show that the map of cohomology groups

$$\mathrm{H}^j(X_{l-1,\text{ét}}, C) \longrightarrow \mathrm{H}^j(X_{l,\text{ét}}, C)$$

is injective.

To simplify the notation, we let $X = X_{l-1}$, and $X' = X_l$ be the covering $\mathrm{Bl}_Y(X) \sqcup Y \to X$ associated to the blowup at the smooth center $Y \subset X$. We let Y' be the pullback of Y along $\mathrm{Bl}_Y(X) \to X$. Since $X' \to X$ is a covering associated to a blowup along a smooth subspace Y of smaller dimension, by the two long exact sequence of cohomologies for differentials (Proposition 4.2.1.5, 4.2.1.6), we get

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \mathrm{H}^j(X_{\text{ét}}, \Omega^i_{X/K}) \arrow[r] \arrow[d, "\tau_x"] & \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{ét}}, \Omega^i_{\mathrm{Bl}_Y(X)/K}) \bigoplus \mathrm{H}^j(Y_{\text{ét}}, \Omega^i_{Y/K}) \arrow[r] \arrow[d, "\tau_{\mathrm{Bl}_Y(X)}"', "\tau_Y"] & \mathrm{H}^j(Y'_{\text{ét}}, \Omega^i_{Y'/K}) \arrow[r] \arrow[d, "\tau_{Y'}"] & {} \\
{} \arrow[r] & \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}) \arrow[r] \arrow[d] & \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{éh}}, \Omega^i_{\text{éh}}) \bigoplus \mathrm{H}^j(Y_{\text{éh}}, \Omega^i_{\text{éh}}) \arrow[r] \arrow[d] & \mathrm{H}^j(Y'_{\text{éh}}, \Omega^i_{\text{éh}}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{H}^j(X_{\text{ét}}, C) \arrow[r] \arrow[d] & \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{ét}}, C) \bigoplus \mathrm{H}^j(Y_{\text{ét}}, C) \arrow[r] \arrow[d] & \mathrm{H}^j(Y'_{\text{ét}}, C) \arrow[r] \arrow[d] & . \\
 & {} & {} & {} &
\end{tikzcd}$$

By the assumption of the j, since the cohomology sheaf $\mathcal{H}^{j-l}(C)|_{Y',\text{ét}} = 0$ for $l > 0$, we have

$$\mathrm{H}^{j-1}(Y'_{\text{ét}}, C) = 0.$$

Besides, note that $\dim(X)$ is the smallest dimension such that $C|_{X_{\text{ét}}}$ is not quasi-isomorphic to 0. So both of $\mathrm{H}^j(Y_{\text{ét}}, C)$ and $\mathrm{H}^j(Y'_{\text{ét}}, C)$ are zero. In this way, the third row above becomes an isomorphism between the following two cohomologies

$$\mathrm{H}^j(X_{\text{ét}}, C) \to \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{ét}}, C) \bigoplus 0 = \mathrm{H}^j(X'_{\text{ét}}, C),$$

and we get the injection. □

□

Remark 4.2.1.9. In fact, the proof above works in a coarser topology, generated by rigid topology, universal homeomorphisms and coverings associated to blowups. This is because all we need are the local smoothness and the distinguished triangles for cohomology of differentials, which is a coherent cohomology theory. Moreover, results here can be deduced from the pullback of this coarser topology to the éh topology.

4.2.2 Application to algebraic varieties

Let K be the field $\mathbb{C}_p$ of p-adic complex numbers. We fix an abstract isomorphism of fields between $\mathbb{C}_p$ and $\mathbb{C}$. Our goal in this subsection is to relate the éh cohomology to the singular cohomology, when the rigid space comes from an algebraic variety.

More precisely, we have:

Theorem 4.2.2.1. *Let Y be a proper algebraic variety over $K = \mathbb{C}_p$, and let $X = Y^{\mathrm{an}}$ be its analytification, as a rigid space over K. Then there exists a functorial isomorphism*

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong \mathrm{gr}^j \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C}),$$

where $\mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}))$ is the i-th singular cohomology of the complex manifold $Y(\mathbb{C})$ equipped with the Hodge filtration.

Proof. Let $\rho : Y_\bullet \to Y$ be a map from a simplicial smooth proper algebraic varieties over K onto Y, such that each $Y_n \to (\mathrm{cosk}_n Y_{\leq n})_{n+1}$ is a finite compositions of éh coverings associated to smooth blowups (but with algebraic varieties instead of rigid spaces in the Definition 4.1.4.1). Then the analytification $\rho^{\mathrm{an}} : X_\bullet \to X$ is an éh hypercovering of X by smooth proper rigid spaces $X_n = Y_n^{\mathrm{an}}$. Moreover, the continuous differential sheaves $\Omega^j_{X_n/K}$ of X_n, which is a vector bundle over X_n, is canonically isomorphic to the sheafification of the differential sheaves $\Omega^j_{Y_n/K}$ of the algebraic variety Y_n over K.

Next we apply the cohomological descent, and get the following natural quasi-isomorphism

$$R\pi_{X*}\Omega^j_{\text{éh}} \cong R\rho^{\mathrm{an}}_* R\pi_{X_\bullet *}\Omega^j_{\text{éh}}.$$

As each X_n is smooth over K, by the Theorem 4.2.1.1 we have

$$R\pi_{X_n *}\Omega^j_{\text{éh}} \cong \Omega^j_{X_n/K}.$$

In particular, the derived pushforward $R\pi_{X*}\Omega^j_{\text{éh}}$ can be computed as

$$\begin{aligned} R\pi_{X*}\Omega^j_{\text{éh}} &\cong R\rho^{\mathrm{an}}_* \Omega^j_{X_\bullet/K} \\ &\cong R\rho^{\mathrm{an}}_* (\Omega^j_{Y_\bullet/K})^{\mathrm{an}}. \end{aligned}$$

We then take the derived global section, to get

$$R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong R\Gamma(Y^{\mathrm{an}}, R\rho^{\mathrm{an}}_*(\Omega^j_{Y_\bullet/K})^{\mathrm{an}}).$$

As all of the algebraic varieties Y and Y_n are proper over K with each $\Omega^j_{Y_n/K}$ being coherent, by the rigid GAGA theorem ([Con06, Appendix A1]), we obtain a natural isomorphism

$$R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong R\Gamma(Y, R\rho_* \Omega^j_{Y_\bullet/K}).$$

Now by the construction, the map from the simplicial varieties $Y_\bullet \to Y$ is a *smooth h-hypercovering* in the sense of [HJ14]. In particular, as proved in [HJ14, Theorem 7.12], the complex $R\rho_*\Omega^j_{Y_\bullet/K}$ is naturally isomorphic to the j-th graded piece of the Hodge filtration of the Deligne-Du Bois complex $\underline{\Omega}^\bullet_Y$. So we may replace the derived pushforward, to get the isomorphism of cohomology groups as below

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong \mathrm{H}^i(Y, \underline{\Omega}^j_Y).$$

In this way, as the right side is isomorphic to the j-th graded piece $\mathrm{gr}^j \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C})$ of the Hodge filtration of the singular cohomology ([PS08, 7.3.1]), we get the isomorphism

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong \mathrm{gr}^j \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C}).$$

□

We note that in the proof above, the comparison is compatible with the differential maps on both side. So the above leads to a comparison between the éh de Rham cohomology and the singular cohomology, when X is coming from an algebraic variety.

Corollary 4.2.2.2. *Let Y be a proper algebraic variety over $K = \mathbb{C}_p$, and let $X = Y^{\mathrm{an}}$ be its analytification, as a proper rigid space over K. Then there exists a functorial filtered isomorphism*

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^\bullet_{\text{éh}}) \cong \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C}),$$

where $\mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C})$ is the i-th singular cohomology of the complex manifold $Y(\mathbb{C})$, equipped with the Hodge filtration.

Remark 4.2.2.3. Let $X = Y^{\mathrm{an}}$ be the analytification of a proper algebraic variety Y over $\mathbb{C}_p$ as above. The proof of the Theorem 4.2.2.1 in fact implies that the éh cohomology $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ of $\Omega^j_{\text{éh}}$ is isomorphic to the h cohomology $\mathrm{H}^i(Y_h, \Omega^j_h)$ (via [HJ14, Corollary 6.16]), for the h cohomology of the scheme Y introduced in [HJ14]. So every computation for proper algebraic variety Y in [HJ14] can be used to compute the éh cohomology of the rigid space Y^{an}.

4.3 Finiteness

In this section, we prove a finiteness result about the $R\pi_*\Omega^j_{\text{éh}}$ for X being a rigid space, namely the coherence and the cohomological boundedness of $R\pi_*\Omega^j_{\text{éh}}$, where K is an arbitrary p-adic field.

Assume X is a rigid space over K.

Proposition 4.3.0.1 (Coherence)**.** *The sheaf of $\mathcal{O}_X$-module $R^n\pi_*\Omega^j_{\text{éh}}$ is coherent, for each $n,\ j \in \mathbb{N}$.*

Proof. We first assume X is reduced, since the direct image along $X_{\text{red}} \to X$ preserves the coherence of modules. Then by the local smoothness (Proposition 4.1.4.8), there exists an éh-hypercover $s : X_\bullet \to X$, such that each X_k is smooth and the map $s_k : X_k \to X$ is proper. Here we notice that each $R\pi_{X_k*}\Omega^j_{\text{éh}} = \Omega^j_{X_k/K}$ is coherent on X_k by Theorem 4.2.1.1. So the properness of $s_k : X_k \to X$ implies that each $R^q s_{k*}\Omega^j_{X_k/K}$ is coherent over $\mathcal{O}_X$. On the other hand, thanks to the cohomological descent, the derived direct image $Rs_*R\pi_{X_\bullet *}\Omega^j_{\text{éh}}$ along the éh-hypercover $X_\bullet \to X$ is quasi-isomorphic to the $R\pi_{X*}\Omega^j_{\text{éh}}$. In this way, the E_1-spectral sequence associated to the simplicial object $s : X_\bullet \to X$ (see [Con03], 6.12) provides

$$\begin{aligned} E_1^{p,q} = R^q s_{p*}\Omega^j_{X_p/K} \Rightarrow \mathcal{H}^{p+q}&(Rs_*R\pi_{X_\bullet *}\Omega^j_{\text{éh}}) \\ &= R^{p+q}\pi_{X*}\Omega^j_{\text{éh}}, \end{aligned}$$

where each term on the left side is coherent over X. Hence the sheaf $R^{p+q}\pi_{X*}\Omega^j_{\text{éh}}$ is coherent on X. $\square$

Next we consider the cohomological boundedness of the derived direct image.

Theorem 4.3.0.2 (Cohomological boundedness)**.** *For a quasi-compact rigid space X, the cohomology*

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$$

vanishes except $0 \le i, j \le \dim(X)$.

Remark 4.3.0.3. The analogous statement about the boundedness of the Hodge numbers for varieties over the complex number $\mathbb{C}$ is proved by Deligne ([Del74] Theorem 8.2.4).

Proof. We do this by induction on the dimension of the X. When X is of dimension 0, the reduced subspace X_{red} is a finite disjoint union of $\mathrm{Spa}(K')$ with K'/K finite, which is smooth over $\mathrm{Spa}(K)$. So by the local reducedness of the éh-topology and the vanishing of the higher direct image of $\iota : X_{\text{red}} \to X$, the case of dimension 0 is done by the Theorem 4.2.1.1.

We then assume the result is true for all quasi-compact rigid spaces of dimensions strictly smaller than $\dim(X)$. By the local smoothness (Proposition 4.1.4.8) and the vanishing of the higher direct image along $X_{\mathrm{red}} \to X$ again, we may assume X is reduced and there exists a composition of finitely many blowups at smooth centers

$$X' = X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X,$$

such that $X' = X_n$ is smooth over $\mathrm{Spa}(K)$. Observe that by the property of the éh differential for smooth spaces, the sheaf $R^i\pi_{X_n*}\Omega^j_{\text{éh}}$ is zero except when $i = 0$ and $0 \leq j \leq \dim X_n = \dim X$. So the claim is true for X_n as $\mathrm{H}^i(X_{n,\text{éh}}, \Omega^j_{\text{éh}}) = \mathrm{H}^i(X_n, \Omega^j_{X_n/K})$. Moreover, to prove the claim for X, it suffices to show that if the result is true for X_{l+1}, then it is true for X_l, where $X_{l+1} \to X_l$ is the l-th blowup at a nowhere dense analytic subspace.

To simplify the notation, we let $X' = X_{l+1}$, $X = X_l$, $f : X' \to X$ be the blowup, $i : Y \to X$ be the inclusion map of the blowup center, and Y' be the preimage $X' \times_X Y$ with the map $g : Y' \to X'$. By the assumption, $\mathrm{H}^i(X'_{\text{éh}}, \Omega^j_{\text{éh}})$ vanishes unless $i, j \leq \dim(X) = \dim(X')$. Furthermore, thanks to the induction hypothesis we have $\mathrm{H}^i(Y'_{\text{éh}}, \Omega^j_{\text{éh}}) = 0$ unless $i, j \leq \dim(Y') < \dim(X)$. Now we consider the distinguished triangle of the éh-cohomology (Proposition 4.2.1.6)

$$\cdots \longrightarrow \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \longrightarrow \mathrm{H}^i(X'_{\text{éh}}, \Omega^j_{\text{éh}}) \bigoplus \mathrm{H}^i(Y_{\text{éh}}, \Omega^j_{\text{éh}}) \longrightarrow \mathrm{H}^i(Y'_{\text{éh}}, \Omega^j_{\text{éh}}) \longrightarrow \cdots.$$

We discuss all possible cases:

- If $j > \dim(X)$, then since the blowup center Y is nowhere dense in X, we have $j \dim(X) > \dim(Y')$. So by induction hypothesis on dimensions, both $\mathrm{H}^i(Y_{\text{éh}}, \Omega^j_{\text{éh}})$ and $\mathrm{H}^{i-1}(Y'_{\text{éh}}, \Omega^j_{\text{éh}})$ vanish for every i. Moreover by the assumption on X', we know the vanishing of $\mathrm{H}^i(X'_{\text{éh}}, \Omega^j_{\text{éh}})$, $i \in \mathbb{N}$. So the long exact sequence leads to the vanishing for $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ if $j > \dim(X)$.

- If $i > \dim(X)$, then since $i - 1 > \dim(X) - 1 \geq \dim(Y')$, by induction hypothesis on dimensions again we have $\mathrm{H}^{i-1}(Y'_{\text{éh}}, \Omega^j_{\text{éh}})$ and $\mathrm{H}^i(Y_{\text{éh}}, \Omega^j_{\text{éh}})$ are zero. Similarly we have the vanishing of the $\mathrm{H}^i(X'_{\text{éh}}, \Omega^j_{\text{éh}})$ by the assumption on X'. In this way, the long exact sequence implies that the cohomology $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ is zero for $i > \dim(X)$ and any $j \in \mathbb{N}$.

$\square$

Corollary 4.3.0.4. *Let X be a rigid space over K. Then unless $0 \leq i, j \leq \dim(X)$, the higher direct image $R^i\pi_{X*}\Omega^j_{\text{éh}}$ vanishes.*

Proof. We only need to note that the sheaf $R^i\pi_{X*}\Omega^j_{\text{éh}}$ is the coherent sheaf on X associated to the presheaf

$$U \longmapsto \mathrm{H}^i(U_{\text{éh}}, \Omega^j_{\text{éh}}),$$

for $U \subseteq X$ open and quasi-compact.

$\square$

We will improve the above corollary for locally compactifiable rigid spaces in the Proposition 8.6.0.2 and the Proposition 8.6.0.6, using the degeneracy result developed in the next section and the almost purity theorem in [BS19].

CHAPTER 5

Comparisons of de Rham Cohomology theories

In this chapter, we compare the three de Rham cohomology theories for non-smooth rigid spaces. Our goal is to show that there are filtered morphisms among the three filtered complexes as below:

$$R\Gamma(X, \mathrm{dR}_{X/K}) \longrightarrow R\Gamma_{\mathrm{inf}}(X/K) \longrightarrow R\Gamma(X_{\text{éh}}, \Omega^{\bullet}_{\text{éh}}),$$

which induces quasi-isomorphisms over their underlying complexes. The results in this chapter first appeared in [Guo20, Section 5, 6].

The chapter is divided into three parts. In Subsection 5.1, we produce a natural filtered morphism from the cohomology of the analytic derived de Rham complex to infinitesimal cohomology, which induces a quasi-isomorphism of the underlying complexes, as in Theorem 5.1.2.3. Then in Subsection 5.2, we follow the strategy of Hartshorne [Har75] to show that infinitesimal cohomology satisfies the descent along a blowup square (Theorem 5.2.1.2), and thus an éh sheaf of complexes (see Theorem 5.2.2.2 for K-linear coefficients and Theorem 5.2.2.5 for general $\mathrm{B}^+_{\mathrm{dR,e}}$-linear coefficients). This in particular implies the underlying complexes of infinitesimal cohomology and éh-de Rham cohomology coincide. Using the éh-descent, we could easily get the finiteness and cohomological boundedness of cohomology. At last, in Subsection 5.3, we compare the analytic and algebraic infinitesimal cohomology, showing that those two are filtered-isomorphic for (rigid spaces associated to) proper algebraic varieties (Theorem 5.3.0.1).

5.1 Infinitesimal cohomology and derived de Rham complex

In this subsection, we give a comparison theorem between the infinitesimal cohomology and (the underlying complex of) the analytic derived de Rham complex of a rigid space X over $\mathrm{B}^+_{\mathrm{dR,e}}$. We will use mildly the language of the ∞-category , following the conventions in Chapter 2.

5.1.1 Affinoid comparison

We first consider the affinoid case. Our tool is the Ćech-Alexander complex for infinitesimal cohomology, and the structure of the analytic derived de Rham complex for closed immersions.

Theorem 5.1.1.1. *Let $X = \mathrm{Spa}(A)$ be an affinoid rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a natural isomorphism as below*

$$\widehat{\underline{\mathrm{dR}}}^{\mathrm{an}}_{A/\mathrm{B}^+_{\mathrm{dR,e}}} \longrightarrow R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}).$$

Here $\widehat{\underline{\mathrm{dR}}}^{\mathrm{an}}_{A/\mathrm{B}^+_{\mathrm{dR,e}}}$ is the underlying complex of the analytic derived de Rham complex.

Before the proof, we want to mention that in the proof below, we will see the isomorphism in the statement is induced from a chosen closed immersion $X \to Y$, where Y is a smooth rigid space. Later on, we will use this observation to globalize a general comparison.

Proof. Let $P = \mathrm{B}^+_{\mathrm{dR,e}}\langle T_1, \ldots, T_m\rangle \to A$ be a surjection of topologically finite type algebras over $\mathrm{B}^+_{\mathrm{dR,e}}$. By Proposition 3.3.1.3 for the crystal $\mathcal{F} = \mathcal{O}_{X/\Sigma_e}$, the infinitesimal cohomology of $X/\Sigma_{e\,\mathrm{inf}}$ can be computed by the cosimplicial cochain complex

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}) \cong (\mathcal{O}_{X/\Sigma_e}(D(0)) \longrightarrow \mathcal{O}_{X/\Sigma_e}(D(1)) \longrightarrow \cdots),$$

where $D(\bullet)$ is the cosimplicial object of sheaves over the infinitesimal site, produced by the envelope of A in $P^{\widehat{\otimes}_{\mathrm{B}^+_{\mathrm{dR,e}}}\bullet+1}$ (see the discussion before Theorem 3.3.1.1). Here we recall that by the definition of envelope (cf. Definition 3.1.2.1), the sheaf $D(m)$ is the direct limit of all infinitesimal neighborhoods of $\mathrm{Spa}(A)$ in $\mathrm{Spa}(P^{\widehat{\otimes}m+1})$. In particular, we have the following equality

$$\mathcal{O}_{X/\Sigma_e}(D(m)) = \varprojlim_i P^{\widehat{\otimes}m+1}/I(m)^i,$$

where $I(m)$ is the kernel of the surjection $P^{\widehat{\otimes}_{\mathrm{B}^+_{\mathrm{dR,e}}}m+1} \to A$.

Now by Proposition 2.3.0.11, there exists a natural filtered morphism inducing an isomorphism of their underlying complexes

$$\mathrm{dR}_{A/P^{\widehat{\otimes} m+1}} \longrightarrow \varprojlim_i P^{\widehat{\otimes} m+1}/I(m)^i.$$

By taking the cosimplicial version of the above isomorphism, we get isomorphisms of cosimplicial complexes

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}) \longrightarrow \mathcal{O}_{X/\Sigma_e}(D(\bullet)) \longleftarrow \widehat{\underline{\mathrm{dR}}}^{\mathrm{an}}_{A/P^{\widehat{\otimes}\bullet+1}}.$$

So in order to prove the theorem, it suffices to show that the natural map $\mathrm{B}^+_{\mathrm{dR,e}} \to P^{\widehat{\otimes}_{\mathrm{B}^+_{\mathrm{dR,e}}}\bullet+1} \to A$ induces an isomorphism on analytic derived de Rham complexes. Moreover, by the filtered completeness, it reduces to show the isomorphism

$$\mathbb{L}^{\mathrm{an}}_{A/\mathrm{B}^+_{\mathrm{dR,e}}} \longrightarrow \mathbb{L}^{\mathrm{an}}_{A/P^{\widehat{\otimes}\bullet+1}}.$$

Note that as both A and $P^{\widehat{\otimes}\bullet+1}$ are term-wise topologically finite type over $\mathrm{B}^+_{\mathrm{dR,e}}$, by the distinguished triangle of cotangent complexes for triples (Proposition 2.2.3.10), it suffices to show the vanishing of the following

$$\mathbb{L}^{\mathrm{an}}_{P^{\widehat{\otimes}\bullet+1}/\mathrm{B}^+_{\mathrm{dR,e}}}.$$

At last, we notice that by Proposition 2.2.3.5, the complex $\mathbb{L}^{\mathrm{an}}_{P^{\widehat{\otimes}\bullet+1}/\mathrm{B}^+_{\mathrm{dR,e}}}$ can be computed by inverting p at the term-wise derived p-completion of the algebraic cotangent complex of the Čech nerve $\check{\mathrm{C}}\mathrm{ech}(\mathrm{A}_{\mathrm{inf,e}} \to \mathrm{A}_{\mathrm{inf,e}}[T_1, \ldots, T_r])$. So the vanishing we want follows from the vanishing of the algebraic cotangent complex $\mathbb{L}_{\check{\mathrm{C}}\mathrm{ech}(\mathrm{A}_{\mathrm{inf,e}} \to \mathrm{A}_{\mathrm{inf,e}}[T_i])/(\mathrm{A}_{\mathrm{inf,e}})}$, which is proved in the first part of the Corollary 2.7 in [Bha12a]. $\square$

In the special case when A is a complete intersection, the above can be improved into a filtered isomorphism. Here we recall that the filtration structure on the infinitesimal cohomology, which is called *infinitesimal filtration*, is defined via $R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{J}^\bullet_{X/\Sigma_e})$, where $\mathcal{J}_{X/\Sigma_e}$ is the kernel of the surjection $\mathcal{O}_{X/\Sigma_e} \to \mathcal{O}_X$ over the infinitesimal site.

Corollary 5.1.1.2. *Let $X = \mathrm{Spa}(A)$ be an affinoid rigid space that admits a regular closed immersion into a smooth affinoid rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a natural filtered isomorphism as below*

$$R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e}) \longrightarrow \mathrm{dR}_{A/\mathrm{B}^+_{\mathrm{dR,e}}}.$$

Proof. The proof is identical to the proof of Theorem 5.1.1.1, with the use of the Corollary 2.3.0.12. $\square$

5.1.2 Comparison in general

We are now ready to prove the comparison between the infinitesimal cohomology and (the underlying complex of) the analytic derived de Rham complex, for a general rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$.

We first introduce the category of smooth immersions.

Definition 5.1.2.1. *Let X be a rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. We define the* site SE_X of smooth immersions of X *where:*

- *objects of SE_X consist of tuples $(U, Z, i : U \to Z)$ with U being an affinoid open subset of X, Z a smooth affinoid rigid space over Σ_e, and $i : U \to Z$ is a closed immersion;*
- *morphisms from $(U_1, Z_1, i_1 : U_1 \to Z_1)$ and $(U_2, Z_2, i_2 : U_2 \to Z_2)$ consist of commutative diagrams as below*

$$\begin{tikzcd} U_1 \arrow[r, "i_1"] \arrow[d] & Z_1 \arrow[d] \\ U_2 \arrow[r, "i_2"'] & Z_2, \end{tikzcd}$$

  where $U_1 \to U_2$ is an open immersion over X.
- *A collection of maps $\{(U_j, Z_j, i_j) \to (U, Z, i)\}$ is a covering if $\{U_j \to U\}$ and $\{Z_j \to Z\}$ are coverings of rigid spaces separately.*

There exists a natural projection functor from SE_X to the category of affinoid open subsets X_{aff} of X, by sending an object $(U, Z, i : U \to Z)$ to the open subset U in X. This functor is continuous under their topology. Here the associated *push-forward functor* π_* is the constant functor; namely for an ordinary sheaf $\mathcal{F}$ in the topos $\mathrm{Sh}(X_{\mathrm{aff}})$, the push-forward $\pi_*\mathcal{F}$ satisfies

$$(\pi_*\mathcal{F})(U, Z, i) = \mathcal{F}(U).$$

The *pullback functor* π^{-1} sends a sheaf $\mathcal{G}$ in $\mathrm{Sh}(\mathrm{SE}_X)$ to the sheaf associated with the presheaf

$$(\pi^{-1}\mathcal{G})(U) = \operatorname*{colim}_{(U,Z,i)} \mathcal{G}(U, Z, i),\ U \in X_{\mathrm{rig}}.$$

The colimit above is a filtered (sifted) colimit, as given any two closed immersions $i_1 : U \to Z_1$ and $i_2 : U \to Z_2$, we can find a common refinement of them by $i : U \to Z_1 \times_{\Sigma_e} Z_2$, with natural projection maps $Z_1 \times_{\Sigma_e} Z_2 \to Z_j$ for $j = 1, 2$. In particular, the colimit (thus the inverse image functor π^{-1}) is exact. So, by translating this into the language of sites ([Sta18, Tag 00X1]), we get a natural morphism of sites

$$\pi : X_{\mathrm{aff}} \longrightarrow \mathrm{SE}_X.$$

Before we prove the main theorem, we first notice that to check two objects in $\mathcal{D}(X_{\mathrm{aff}}) = \mathcal{D}(X_{\mathrm{aff}}, \mathbb{Z})$ are isomorphic, it suffices to do so by pulling back to the category of smooth immersions. Precisely, we have the following general lemma.

Lemma 5.1.2.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two objects in the derived category $\mathcal{D}(X_{\mathrm{aff}})$ of sheaves of abelian groups over the site X_{aff}. Assume $f : \pi^{-1}\mathcal{F} \to \pi^{-1}\mathcal{G}$ is an isomorphism in $\mathcal{D}(\mathrm{SE}_X)$ over SE_X. Then the following natural arrows are all isomorphisms*

$$\begin{tikzcd} \pi^{-1}R\pi_*\mathcal{F} \arrow[r, "\widetilde{f}"] \arrow[d] & \mathcal{G} \\ \mathcal{F} & , \end{tikzcd}$$

where both arrows are counit maps associated with $f : \mathcal{F} \to \mathcal{G}$ and the identity $\mathcal{F} \to \mathcal{F}$ separately. In particular, there exists a natural isomorphism $\mathcal{F} \to \mathcal{G}$ in the derived category $\mathcal{D}(X_{\mathrm{aff}})$.

Proof. For each $U \in X_{\mathrm{aff}}^{\omega}$, we have

$$\begin{aligned} R\Gamma(U, \pi^{-1}R\pi_*\mathcal{F}) &= \operatorname*{colim}_{(U,Z,i)} R\Gamma((U,Z,i), R\pi_*\mathcal{F}) \\ &= \operatorname*{colim}_{(U,Z,i)} R\Gamma(U, \mathcal{F}) \\ &\cong R\Gamma(U, \mathcal{F}), \end{aligned}$$

where the last map is an isomorphism as the colimit above is filtered (thus the geometric realization of the index set is contractible). In particular, this implies that the counit maps $\pi^{-1}R\pi_*\mathcal{F} \to \mathcal{F}$ and $\pi^{-1}R\pi_*\mathcal{G}$ are isomorphisms. Thus the claim follows from the following diagram of natural isomorphisms

$$\begin{tikzcd} \pi^{-1}R\pi_*\mathcal{F} \arrow[r, "\pi^{-1}f"] \arrow[d] & \pi^{-1}R\pi_*\mathcal{G} \arrow[d] \\ \mathcal{F} \arrow[r, dotted, "\widetilde{f}"'] & \mathcal{G}. \end{tikzcd}$$

□

Now we are able to prove the comparison theorem.

Theorem 5.1.2.3. *Let X be a rigid space over $\mathrm{B}^+_{\mathrm{dR,e}}$. Then there exists a natural filtered morphism from the analytic derived de Rham complex to the Hodge-filtered infinitesimal cohomology sheaf as below*

$$\mathrm{dR}_{X/\Sigma_e} \longrightarrow (Ru_{X/\Sigma_e *}\mathcal{O}_{X/\Sigma_e},\ Ru_{X/\Sigma_e *}\mathcal{J}_{X/\Sigma_e},\ Ru_{X/\Sigma_e *}\mathcal{J}^2_{X/\Sigma_e},\ \ldots),$$

Moroever, the induced map between their underlying complexes is an isomorphism

$$\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma_e} \longrightarrow Ru_{X/\Sigma_e *}\mathcal{O}_{X/\Sigma_e}.$$

Proof. We first notice that it suffices to show the comparison at the site X_{aff} of affinoid open subsets of X.

To see this, we recall the equivalence of ∞-categories $\mathcal{D}(X,R) \cong \mathrm{Sh}^{\mathrm{hyp}}(X,R)$ (cf. Paragraph 2.0.2), where the right side is the full sub-∞-category of contravariant functors from $U \in X_{\mathrm{rig}}$ to the derived category $\mathcal{D}(R)$. As X_{aff} is a basis of the rigid site X_{rig}, we may use the equivalence $\mathrm{Sh}^{\mathrm{hyp}}(X,R) = \mathrm{Sh}^{\mathrm{hyp}}(X_{\mathrm{aff}},R)$ and regard objects in $\mathcal{D}(X,R)$ as contravariant functors on affinoid open subsets of X. So a map or an isomorphism of sheaves of complexes can be constructed via their evaluations at X_{aff}. Moreover, by Lemma 5.1.2.2, it suffices to show it over the site of smooth immersions SE_X of X, by applying the constant functor π^{-1} to both objects in the sequence $(*)$.

Now for each smooth immersion $i : U = \mathrm{Spa}(B) \to Z = \mathrm{Spa}(P)$ for affinoid open subset $U \subset X$, as in the proof of Theorem 5.1.1.1 we have the following natural map of cosimplicial objects in $\widehat{\mathrm{DF}}(\mathrm{B}^+_{\mathrm{dR,e}})$

$$\mathrm{dR}_{B/\mathrm{B}^+_{\mathrm{dR,e}}} \longrightarrow \mathrm{dR}_{B/P^{\widehat{\otimes}\bullet+1}} \longrightarrow \mathcal{O}_{D(\bullet)} \longleftarrow R\Gamma(U/\Sigma_{e\,\mathrm{inf}}, \mathcal{O}_{X/\Sigma_e})$$

Here $D(n)$ is the envelope of the surjection

$$P^{\widehat{\otimes}(n+1)} = P^{\widehat{\otimes}_{\mathrm{B}^+_{\mathrm{dR,e}}}(n+1)} \to B.$$

Moreover, the induced maps of their underlying complexes above are all isomorphisms, by the proof of the affinoid comparison in Theorem 5.1.1.1.

At last, note that the above maps are functorial with respect to smooth immersions $i : U \to Z$, so we can improve the above map into the level of sheaves over SE_X

$$R\pi_*\mathrm{dR}_{X/\Sigma_e} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{\mathrm{SE}_X} \longrightarrow \mathcal{O}_{\mathrm{SE}_X} \longleftarrow R\pi_* Ru_{X/\Sigma_e *}, \mathcal{O}_{X/\Sigma_e}$$

Here where $\widehat{\mathrm{dR}}^{\mathrm{an}}_{\mathrm{SE}_X}$ is the (cosimplicial) sheaf sending $i : U \to Z$ to the filtered complex $\mathrm{dR}_{B/P^{\widehat{\otimes}\bullet+1}}$, and $\mathcal{O}_{\mathrm{SE}_X}$ is the (cosimplicial) sheaf sending $i : U \to Z$ to the structure sheaves $\mathcal{O}_{D(\bullet)}$ of envelopes $D(\bullet)$. In this way, the isomorphism of the underlying complexes over the site SE_X of smooth immersions

$$R\pi_*\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{X/\mathrm{B}^+_{\mathrm{dR,e}}} \longrightarrow R\pi_* Ru_{X/\Sigma_e *}, \mathcal{O}_{X/\Sigma_e}$$

follows from the above computation at the sections $i : U \to Z$, and thus we get the result by Lemma

5.1.2.2. So we are done.

□

Remark 5.1.2.4. The above comparison map, though functorial with respect to the rigid space X, is constructed in an indirect way. It is natural to ask if we can directly produce a natural morphism from the analytic derived de Rham complex to the infinitesimal cohomology sheaf. Here we want to mention that for algebraic schemes over $\mathbb{C}$, this could be achieved by the universal property of the derived de Rham complex in the ∞-category of filtered E_∞-algebras.

5.2 éh-descent

In this section, we prove the éh hyperdescent of the infinitesimal cohomology of crystals in vector bundles over X/K and $X/\mathrm{B}^+_{\mathrm{dR},\mathrm{e}_{\mathrm{inf}}}$ for a rigid space X. Our goal is to show the comparison between the éh-de Rham cohomology and the infinitesimal cohomology of a crystal.

In order to extend a crystal to any space that maps to X but is not necessarily an open immersion, we will work mostly with coherent crystals over the big site. Here we note that this is only for the technical convenience, as crystals and their cohomology over $X/\Sigma_{e\,\mathrm{inf}}$ or $X/\Sigma_{e\mathrm{INF}}$ are equivalent via pullback and restrictions (Proposition 3.2.1.7, Proposition 3.3.1.2).

5.2.1 Descent for blowup coverings

We first deal with the descent for the blowup covering over the base $\Sigma_1 = \mathrm{Spa}(K)$, for an arbitrary complete non-archimedean p-adic field K, not necessarily algebraically closed. The essential idea follows from Hartshorne's proof for algebraic de Rham cohomology [Har75, Chap II, Section 4], where he provides a long exact sequence of the algebraic de Rham cohomology for a blowup square.

We first give a Mayer-Vietories sequence for infinitesimal cohomology:

Proposition 5.2.1.1 (Mayer-Vietories sequence)**.** *Let X be a union of two closed analytic subspaces X_1 and X_2 over K, and let $\mathcal{F}$ be a coherent crystal over X/K_{INF}. Then the map of rigid spaces $X_1 \cap X_2 \to X_1 \cup X_2 \to X$ induces a natural distinguished triangle as below*

$$Ru_{X/K*}\mathcal{F} \longrightarrow Ru_{X_1/K*}\mathcal{F} \bigoplus Ru_{X_2/K*}\mathcal{F} \longrightarrow Ru_{X_1\cap X_2/K*}\mathcal{F}.$$

Proof. As the functor $u_{X/K*}$ is the sheaf-version of the global section functor $\Gamma(X/K_{\mathrm{INF}}, -)$ (see Subsection 3.1.3), it suffices to show that the maps in the statement above produce a natural distinguished triangle after applying $R\Gamma(U,-)$, for every $U \subset X$ open affinoid. So we may assume there exists a smooth affinoid rigid space $Z = \mathrm{Spa}(P)$ over Σ_e, together with a closed immersion of $X = \mathrm{Spa}(P/I)$ into Z, where I is the defining ideal. Let X_i be the closed analytic subspace defined by the ideal I_i in P. Then by assumption, X is defined by the ideal $I_1 \cap I_2$, and the intersection $X_3 := X_1 \cap X_2$ is defined by $I_3 := I_1 + I_2$. We denote by D and D_i to be the envelope of X and X_i in Z separately. Here we regard D and D_i to be the ringed spaces, where the underlying topological spaces are X and X_i, and their (global sections of) structure sheaves are $\mathcal{O}_D = \varprojlim_n P/I^n$ and $\mathcal{O}_{D_i} = \varprojlim P/I_i^n$ separately (cf. Remark 3.1.2.2).

Now by Theorem 3.3.2.2, the infinitesimal cohomology of the coherent crystal can be functorially identified as the derived global sections of the following

$$\mathcal{F}_D \otimes \Omega_D^\bullet \longrightarrow \mathcal{F}_{D_1} \otimes \Omega_{D_1}^\bullet \bigoplus \mathcal{F}_{D_2} \otimes \Omega_{D_2}^\bullet \longrightarrow \mathcal{F}_{D_3} \otimes \Omega_{D_3}^\bullet,$$

where $\mathcal{F}_D \otimes \Omega_D^\bullet$ (resp. $\mathcal{F}_{D_i} \otimes \Omega_{D_i}^\bullet$) is the restriction of the de Rham complex of the coherent crystal $\mathcal{F}$ onto the envelope $D_X(Z)$ along the closed immersion $X \to Z$ (resp. $D_{X_i}(Z)$ along the closed immersions $X_i \to Z$) separately. Moreover, by the crystal condition, the coherent $\mathcal{O}_{D_i}$-module $\mathcal{F}_{D_i}$ is equal to the tensor product $\mathcal{F}_D \otimes_{\mathcal{O}_D} \mathcal{O}_{D_i}$, and each term of the de Rham complex $\mathcal{F}_{D_i} \otimes \Omega_{D_i}^\bullet$ is equal to the base change of terms of $\mathcal{F}_D \otimes \Omega_D^\bullet$ along $D_i \to D$.[1] In this way, by the compatibility of their Hodge filtrations, to show the sequence above is distinguished, it suffices to show the following short exact sequence of the rings

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D_1} \bigoplus \mathcal{O}_{D_2} \longrightarrow \mathcal{O}_{D_3} \longrightarrow 0.$$

And since each structure sheaf of envelopes are given by the formal completions of the ring P, we reduce the question to show that the following sequence of inverse systems is exact

$$0 \longrightarrow \{P/(I_1 \cap I_2)^n\}_n \longrightarrow \{P/I_1^n\}_n \bigoplus \{P/I_2^n\}_n \longrightarrow \{P/(I_1 + I_2)^n\}_n \longrightarrow 0.$$

Notice that for fixed n, we always have the following short exact sequence

$$0 \longrightarrow P/(I_1^n \cap I_2^n) \longrightarrow P/I_1^n \bigoplus P/I_2^n \longrightarrow P/(I_1^n + I_2^n) \longrightarrow 0.$$

The proof thus follows since the ring P is noetherian, and the inverse systems below are canonically isomorphic

$$\{P/(I_1^n + I_2^n)\}_n \longrightarrow \{P/(I_1 + I_2)^n\}_n, \ \{P/(I_1 \cap I_2)^n\}_n \longrightarrow \{P/(I_1^n \cap I_2^n)\}_n.$$

□

[1] We want to mention that the sheaf Ω_D^j, defined as the inverse limit $\varprojlim_n \Omega^j_{X_n/\Sigma_e}$ for X_n being the n-th infinitesimal neighborhood of X into Z, is naturally isomorphic to the tensor product $\Omega^j_{Z/\Sigma_e} \otimes \mathcal{O}_D$ by Lemma 3.2.3.3. This works similarly for $\Omega^j_{D_i}$, and in particular we get $\Omega^j_{D_i} = \Omega^j_D \otimes_{\mathcal{O}_D} \mathcal{O}_{D_i}$.

Here is our main theorem in this subsection.

Theorem 5.2.1.2. *Let X be a rigid space over K, and let Y be a smooth analytic closed subset of X over K, with the blowup map $f : X' := \mathrm{Bl}_X(Y) \to X$ and the preimage $Y' := Y \times_X X'$ in X'. Then for any coherent crystal $\mathcal{F}$ over X/K_{INF}, the blowup square for $X' \to X$ induces the following distinguished triangle*

$$Ru_{X/K*}\mathcal{F} \longrightarrow Rf_*Ru_{X'/K*}\mathcal{F} \bigoplus Ru_{Y/K*}\mathcal{F} \longrightarrow Rf_*Ru_{Y'/K*}\mathcal{F}. \qquad (*)$$

In particular, by taking the derived global sections at X, we get a distinguished triangle of infinitesimal cohomology

$$R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R\Gamma(X'/K_{\mathrm{INF}}, \mathcal{F}) \bigoplus R\Gamma(Y/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R\Gamma(Y'/K_{\mathrm{INF}}, \mathcal{F}).$$

Before we prove the result, first we recall the formal function theorem for a proper map of rigid spaces.

Theorem 5.2.1.3 ([Kie67], Theorem 3.7)**.** *Let $f : X' \to X$ be a proper map of rigid space over K, and let $\mathcal{I}$ be a sheaf of ideal over X, with $Y = \mathrm{Spa}(A/I)$ being the analytic closed subset of X defined by $\mathcal{I}$. Assume $\mathcal{G}$ is a coherent sheaf over X'. Then the following natural map is an isomorphism:*

$$(R^j f_*\mathcal{G})^\wedge \longrightarrow R^j f_*(\widehat{\mathcal{G}}).$$

Here $(-)^\wedge$ is the classical completion of a sheaf of $\mathcal{O}_X$-modules (or $\mathcal{O}_{X'}$-modules) with respect to the ideal $\mathcal{I}$ (resp. $f^{-1}\mathcal{I} \cdot \mathcal{O}_{X'}$).

Remark 5.2.1.4. Here we note that we may get a more derived version of the above theorem using the derived completion, as in [Sta18, Tag 0A0H]. For our uses, we do not jump into this generality.

The rest of this subsection is devoted to prove Theorem 5.2.1.2.

Special case: X is smooth First we deal with the special case, assuming X itself is smooth over K

When X is smooth, as the blowup center Y is assumed to be smooth, we know the blowup X' is also smooth over K. By Theorem 3.3.2.2, the derived direct image of the coherent crystal over X/K_{inf} and X'/K_{inf} can be computed by their de Rham complexes $\mathcal{F}_X \otimes \Omega^\bullet_{X/K}$ and $\mathcal{F}_{X'} \otimes \Omega^\bullet_{X'/K} = f^*\mathcal{F}_X \otimes \Omega^\bullet_{X'/K}$ separately. On the other hand, the derived direct image $Ru_{Y/K*}\mathcal{F}$ and $Ru_{Y'/K*}\mathcal{F}$ are naturally isomorphic to the de Rham complex over the envelopes $D_Y(X)$ and $D_{Y'}(X')$; namely the complexes

$$\mathcal{F}_D \otimes \Omega^\bullet_{D_Y(X)},\ \mathcal{F}_{D'} \otimes \Omega^\bullet_{D_{Y'}(X')},$$

which are compatible with the Hodge filtrations of $\Omega^\bullet_{X/K}$ and $\Omega^\bullet_{X'/K}$. So the sequence $(*)$ in Theorem 5.2.1.2 is naturally isomorphic to the following sequence of de Rham complexes

$$\mathcal{F}_X\otimes\Omega^\bullet_{X/K}\longrightarrow Rf_*(f^*\mathcal{F}_X\otimes\Omega^\bullet_{X'/K})\bigoplus\mathcal{F}_D\otimes\Omega^\bullet_{D_Y(X)}\longrightarrow Rf_*(\mathcal{F}_{D'}\otimes\Omega^\bullet_{D_{Y'}(X')}).$$

In fact, we want to show the following more general statement:

Proposition 5.2.1.5. *Let $f:X'\to X$ be a proper morphism of smooth rigid spaces over K, and let Y be a closed analytic subset of X, with $Y'=f^{-1}(Y)$. Let $\mathcal{G}'$ and $\mathcal{G}$ be coherent sheaves over X' and X separately, such that $f:X'\to X$ induces an injective map of $\mathcal{O}_X$-modules $\mathcal{G}\to f_*\mathcal{G}'$. Assume f induces an isomorphism between open subsets $X'\backslash Y'$ and $X\backslash Y$, and also induces an isomorphism of the restriction of $\mathcal{G}\to f_*\mathcal{G}'$ on $X\backslash Y$. Then the following natural sequence is a distinguished triangle*

$$\mathcal{G}\otimes\Omega^i_{X/K}\longrightarrow Rf_*(\mathcal{G}'\otimes\Omega^i_{X'/K})\bigoplus\mathcal{G}\otimes\Omega^i_{D_Y(X)/K}\longrightarrow Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K}),\ i\in\mathbb{N}.$$

Proof. Let $\mathcal{M}$ be the coherent sheaf $\mathcal{G}\otimes\Omega^i_{X/K}$ over X, and let $\mathcal{M}'$ be the coherent sheaf $\mathcal{G}'\otimes\Omega^i_{X'/K}$ over X'. By the assumption of smoothness, both $\Omega^i_{X/K}$ and $\Omega^i_{X'/K}$ are locally free, and the natural map $\mathcal{M}\to f_*\mathcal{M}'$ is injective. Furthermore, the restriction of the map $\mathcal{M}\to f_*\mathcal{M}'$ on the open subset $X\backslash Y$ is an isomorphism.

Recall the i-th differential sheaf $\Omega^i_{D_Y(X)/K}$ of $D_Y(X)$, as a sheaf over X, is defined as the inverse limit $\varprojlim_m\Omega^i_{X_m/K}$, where X_m is the m-th infinitesimal neighborhood of Y in X. As is shown in Lemma 3.2.3.3, the sheaf $\Omega^i_{D_Y(X)/K}$ is naturally isomorphic to the formal completion of the coherent sheaf $\Omega^i_{X/K}$ along $Y\to X$, which is also equal to the tensor product $\Omega^i_{X/K}\otimes\mathcal{O}_{D_Y(X)}$. Moreover, since $\mathcal{G}$ is coherent over X, the tensor product $\mathcal{G}\otimes\Omega^i_{D_Y(X)}$ is isomorphic to the formal completion $\widehat{\mathcal{M}}$ of $\mathcal{M}=\mathcal{G}\otimes\Omega^i_{X/K}$ along $Y\to X$. The same also holds for X',Y' and $\mathcal{M}'$.

We denote by C_1 and C_2 to be cones of the map $\mathcal{M}\to Rf_*\mathcal{M}$ and $\widehat{\mathcal{M}}\to Rf_*\widehat{\mathcal{M}}'$ separately. Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & Rf_*\mathcal{M} \arrow[r] \arrow[d] & C_1 \arrow[d] \\
\widehat{\mathcal{M}} \arrow[r] & Rf_*\widehat{\mathcal{M}}' \arrow[r] & C_2.
\end{tikzcd}$$

Here both rows are distinguished.

Now we make the following claim.

Claim 5.2.1.6. The natural map $C_1\to C_2$ of cones above is an isomorphism.

Proof of the Claim. First we notice that since the map $f:X'\to X$ is isomorphic on the open subsets $X'\backslash Y' \to X\backslash Y$, while both X' and X are smooth, the sheaves of differentials $\Omega^i_{X/K}$ and $\Omega^i_{X'/K}$ are vector bundles, and the induced map $\Omega^i_{X/K} \to f_*\Omega^i_{X'/k}$ is injective. On the other hand, the map $\mathcal{G} \to f_*\mathcal{G}'$ is assumed to be an injective map of coherent sheaves in the Proposition. Combine the above two conditions, we see the map $\mathcal{M} \to f_*\mathcal{M}'$ is injective, and the cone lives in cohomological degree $[0, +\infty)$.

By the Formal Function Theorem 5.2.1.3, the cohomology sheaf $R^j f_*\widehat{\mathcal{M}'}$ is naturally isomorphic to the formal completion $(R^j f_*\mathcal{M}')^\wedge$ of $R^j f_*\mathcal{M}'$ along $Y \to X$. Moreover, by the exactness of the formal completion on coherent sheaves, the natural map $\widehat{\mathcal{M}} \to (f_*\mathcal{M}')^\wedge$ is injective, and we have a short exact sequence

$$0 \longrightarrow \widehat{\mathcal{M}} \longrightarrow (f_*\mathcal{M}')^\wedge \longrightarrow \mathcal{H}^0(C_2) \longrightarrow 0.$$

This implies that C_2 lives also in cohomological degree no smaller than zero. Furthermore, by the exactness of the above sequence, the cohomology sheaf $\mathcal{H}^0(C_2)$ is isomorphic to the formal completion of $\mathcal{H}^0(C_1)$ at Y. But since $\mathcal{H}^0(C_1)$ is coherent and is already supported at Y, we have

$$\mathcal{H}^0(C_1) = \mathcal{H}^0(C_1)^\wedge \cong \mathcal{H}^0(C_2).$$

This finishes the degree zero part.

For the higher cohomology, we consider the following diagram of cohomologies

$$\begin{tikzcd} R^j f_*\mathcal{M}' \arrow[r] \arrow[d] & \mathcal{H}^j(C_1) \arrow[d] \\ (R^j f_*\mathcal{M}')^\wedge \arrow[r] & \mathcal{H}^j(C_2). \end{tikzcd}$$

As $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are living in cohomological degree zero, the horizontal maps above are isomorphisms, and it suffices to show for each $i > 0$, the left vertical map above is an isomorphism. But notice that since f induces an isomorphism between $\mathcal{M}$ and $\mathcal{M}'$ over $X\backslash Y$, the higher cohomology sheaf $R^j f_*\mathcal{M}'$ is coherent and is supported over Y. In particular, the formal completion of $R^j f_*\mathcal{M}'$ along $Y \to X$ is equal to itself; namely the natural map below is an isomorphism

$$R^j f_*\mathcal{M}' \longrightarrow (R^j f_*\mathcal{M}')^\wedge.$$

This leads to the isomorphism

$$\mathcal{H}^j(C_1) \cong \mathcal{H}^j(C_2),\ \forall j \geq 1,$$

and we finish the isomorphism between C_1 and C_2.

□

We change the notation back to the Proposition. Then we get two rows of distinguished triangles

$$\begin{tikzcd} \mathcal{G}\otimes\Omega^i_{X/K} \arrow[r] \arrow[d] & Rf_*(\mathcal{G}'\otimes\Omega^i_{X'/K}) \arrow[r] \arrow[d] & C_1 \arrow[d] \\ \mathcal{G}\otimes\Omega^i_{D_Y(X)/K} \arrow[r] & Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K}) \arrow[r] & C_2, \end{tikzcd}$$

the third vertical map is an isomorphism.

Finally, we consider the following two maps,

$$\phi: \ \mathcal{G}\otimes\Omega^i_{X/K} \xrightarrow{(+,+)} Rf_*(\mathcal{G}'\otimes\Omega^i_{X'/K}) \bigoplus (\mathcal{G}\otimes\Omega^i_{D_Y(X)/K});$$

$$\psi: \ Rf_*(\mathcal{G}'\otimes\Omega^i_{X'/K}) \bigoplus (\mathcal{G}\otimes\Omega^i_{D_Y(X)/K}) \xrightarrow{(-,+)} Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K}),$$

where $+$ and $-$ are indicating the signs of the map. As the composition of the above two maps is equal to zero, the map ψ factors though a morphism $\mathrm{Cone}(\phi)\to Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K})$ ([Sta18, Tag 08RI]). In this way, by chasing diagrams and the Claim above, the map $\mathrm{Cone}(\phi)\to Rf_*(\mathcal{G}'\otimes\Omega^i_{D_{Y'}(X')/K})$ is an isomorphism, and we get the distinguished triangle we want.

□

General case We then deal with the general case of Theorem 5.2.1.2.

Proof of Theorem 5.2.1.2. As the theorem is a local statement, by passing to an open covering if necessary it suffices to assume X is affinoid and admits a closed immersion into a smooth affinoid rigid space Z. Moreover, as any coherent crystal over X/K_{INF} is a crystal in vector bundles (Corollary 3.2.2.5), by taking further open subsets we may assume $\mathcal{F}_X\cong\mathcal{O}_X^{\oplus m}$ is trivial over X.

As Y is smooth over K, the blowup $Z'=\mathrm{Bl}_Z(Y)$ of the smooth rigid space Z at the center Y is also smooth. Moreover, as $X\to Z$ is a closed immersion while X' is the blowup of X at Y, the natural map $X'\to Z'$ is also a closed immersion, which is equal to the preimage of X along the blowup map $f: Z'\to Z$. So we get the following commutative diagrams of rigid spaces over K with both of the square being cartesian

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & Z' \arrow[d] \\ Y \arrow[r] & X \arrow[r] & Z. \end{tikzcd}$$

As the restriction $\mathcal{F}_{D_X(Z)}$ is a vector bundle over $\mathcal{O}_{D_X(Z)}$ whose pullback along $X \to D_X(Z)$ is trivial, by Nakayama's lemma, we may let $\mathcal{G}$ be a trivial vector bundle over Z such that the tensor product $\mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{O}_{D_X(Z)}$ is equal to $\mathcal{F}_{D_X(Z)}$. Let $\mathcal{G}'$ be the pullback $f^*\mathcal{G}$, as a trivial bundle over Z'. Then by the crystal condition of $\mathcal{F}$ we have

$$\mathcal{G}' \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{D_{X'}(Z')} = \mathcal{F}_{D_{X'}(Z')}.$$

Here we note that by our choices and the diagram above, the map $\mathcal{G} \to f_*\mathcal{G}'$ is injective and is an isomorphism when restricted to open subsets Z/Y and Z/X. Similar to the proof of Proposition 5.2.1.5, we let $\mathcal{M}$ and $\mathcal{M}'$ be the tensor products $\mathcal{G} \otimes \Omega^i_{Z/K}$ and $\mathcal{G}' \otimes \Omega^i_{Z'/K}$ over Z and Z' separately.

Now by the proof of Proposition 5.2.1.5, we have the following natural commutative diagrams, with each row being distinguished

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & Rf_*\mathcal{M}' \arrow[r] \arrow[d] & C_1 \arrow[d] \\
\widehat{\mathcal{M}_{/X}} \arrow[r] \arrow[d] & Rf_*\widehat{\mathcal{M}'_{/X'}} \arrow[r] \arrow[d] & C_2 \arrow[d] \\
\widehat{\mathcal{M}_{/Y}} \arrow[r] & Rf_*\widehat{\mathcal{M}'_{/Y'}} \arrow[r] & C_3.
\end{tikzcd}$$

Here the sheaf $\widehat{\mathcal{M}_{/X}}$ and similar for the others is denoted to be the formal completion of $\mathcal{M}$ along $X \to Z$. Thanks to Claim 5.2.1.6, the map $C_1 \to C_2$ and the $C_1 \to C_3$ are isomorphisms. In particular, the map $C_2 \to C_3$ is an isomorphism. In this way, as in the last part of the proof for Proposition 5.2.1.5, the second and the third rows above produces the following distinguished triangle

$$\widehat{\mathcal{M}_{/X}} \xrightarrow{(+,+)} Rf_*\widehat{\mathcal{M}'_{/X'}} \bigoplus \widehat{\mathcal{M}_{/Y}} \xrightarrow{(-,+)} Rf_*\widehat{\mathcal{M}'_{/Y'}}.$$

Hence by Lemma 3.2.3.3, we may replace those formal completions by their corresponding sheaves over envelopes, and obtain the distinguished triangle below

$$\begin{aligned}
&\mathcal{F}_{D_X(Z)} \otimes \Omega^i_{D_X(Z)} \longrightarrow Rf_*(\mathcal{F}_{D_{X'}(Z')} \otimes \Omega^i_{D_{X'}(Z')}) \bigoplus \mathcal{F}_{D_Y(Z)} \otimes \Omega^i_{D_Y(Z)} \longrightarrow \\
&Rf_*(\mathcal{F}_{D_{Y'}(Z')} \otimes \Omega^i_{D_{Y'}(Z')}),
\end{aligned}$$

which implies the theorem by taking different i and Theorem 3.3.2.2.

□

Remark 5.2.1.7. With the help of the blowup triangle in Theorem 5.2.1.2, we could show the following: for a universal homeomorphism of rigid spaces $f : X' \to X$ over K and a coherent crystal $\mathcal{F}$ over X/K_{INF}, there exists a natural isomorphism of cohomology sheaves as below

$$Ru_{X/K*}\mathcal{F} \longrightarrow Rf_* Ru_{X'/K*}\mathcal{F}.$$

5.2.2 éh-hyperdescent in general

Now we are ready to prove the éh-descent for a crystal over the infinitesimal site. We first deal with the case for infinitesimal cohomology over an arbitrary p-adic field K, where the strategy is to use the blowup square for the éh-topology in 4 and the descents in the first subsection. After that, we generalize to the case over $\mathrm{B}^+_{\mathrm{dR,e}}$.

Recall that the éh site $X_{\text{éh}}$ is defined over the category $\mathrm{Rig}_K|_X$ of all K-rigid spaces over X and is equipped with the éh-topology (cf. 4). For an object $X' \to X$ in $\mathrm{Rig}_K|_X$, we denote by $\pi_{X'} : X_{\text{éh}} \to X'_{\mathrm{rig}}$ to be the map from the éh site of X to the small rigid site of X'.

We first associate an infinitesimal crystal together with its de Rham complex an analogous construction over the éh topology.

Construction 5.2.2.1 (éh-de Rham complex)**.** Let $\mathcal{F}$ be an infinitesimal sheaf of $\mathcal{O}_{X/K}$-modules over the big infinitesimal site X/K_{INF}. We then associate a sheaf $\mathcal{F}_{\mathrm{rig}} := i_{X/K}^{-1}\mathcal{F}$ of $\mathcal{O}_X = i_{X/K}^{-1}\mathcal{O}_{X/K}$-modules over the big rigid site $\mathrm{Rig}_K|_X$, where $i_{X/K} : \mathrm{Sh}(\mathrm{Rig}_K|_X) \to \mathrm{Sh}(X/K_{\mathrm{INF}})$ is the morphism of topoi as in Subsection 3.1.3. Here the section of $\mathcal{F}_{\mathrm{rig}}$ at an object $f : X' \to X$ in $\mathrm{Rig}_K|_X$ is the $\mathcal{O}_{X'}(X')$-module

$$\mathcal{F}(X', X'),$$

where $(X', X') \in X/\Sigma_{e\mathrm{INF}}$ is the trivial thickening of X'. We could then sheafify it with the éh-topology, and thus get an éh-sheaf $\mathcal{F}_{\text{éh}}$ over the éh site $X_{\text{éh}}$.

Now we specify $\mathcal{F}$ to be a coherent crystal over big the infinitesimal site. As in the discussion of Paragraph 3.2.1, we could associate $\mathcal{F}$ its de Rham complex over the big infinitesimal site

$$\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}.$$

This allows us to get a complex of sheaves over $\mathrm{Rig}_K|_X$ and $X_{\text{éh}}$ separately

$$\begin{aligned} \mathcal{F}_{\mathrm{rig}} \longrightarrow \mathcal{F}_{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathrm{rig}}} \Omega^1_{\mathrm{rig}} \longrightarrow \mathcal{F}_{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathrm{rig}}} \Omega^2_{\mathrm{rig}} \longrightarrow \cdots ; \\ \mathcal{F}_{\text{éh}} \longrightarrow \mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^1_{\text{éh}} \longrightarrow \mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^2_{\text{éh}} \longrightarrow \cdots . \end{aligned}$$

Here the sheaf Ω^i_{rig} is the usual continuous Kähler differential sheaf over the rigid site, and $\Omega^i_{\text{éh}}$ is the éh continuous Kähler differential, which is equal to the éh sheafification of the usual continuous differential. The complex $\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}}$ is called *éh-de Rham complex associated with the crystal $\mathcal{F}$.*

The Construction 5.2.2.1 produces two maps of objects in the derived category of the big rigid topos:

$$Ru_{X/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow \mathcal{F}_{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathrm{rig}}} \Omega^\bullet_{\mathrm{rig}} \longrightarrow R\pi_*(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}}).$$

Formally the first map is given by the sheafified version of the natural transformation

$$\mathrm{Rig}_K|_X \ni X' \longmapsto (R\Gamma(X'/\Sigma_{e\mathrm{INF}}, -) \to R\Gamma((X', X'), -))$$

at the complex of infinitesimal sheaves $\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}})$. The second map above comes from the counit morphism for the adjoint pair (π^{-1}, π_*), where π^{-1} is the éh-sheafification functor. Moreover, the above map can be improved into the filtered derived category, where the left side is equipped with the infinitesimal filtration, and the rest two complexes are equipped with their Hodge filtrations.

We also note that the natural map from the de Rham complex $\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}$ to the crystal $\mathcal{F}$ itself induces a natural isomorphism as below

$$Ru_{X/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow Ru_{X/K*}\mathcal{F},$$

which is proved in Proposition 3.3.2.1.

Now we can state the descent result.

Theorem 5.2.2.2. *Let X be a rigid space over K, and let $\mathcal{F}$ be a coherent crystal over the big infinitesimal site X/K_{INF}. Then the natural map of K-linear complexes below is an isomorphism.*

$$Ru_{X/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow R\pi_{X*}(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}}).$$

In particular, the infinitesimal cohomology of the coherent crystal $\mathcal{F}$ satisfies the éh-hyperdescent.

Proof.

Step 1 In the Step 1, we show that by restricting to a smooth rigid space X' over K that admits a map to X, the morphism in the statement is an isomorphism. Namely the natural morphism below is an isomorphism

$$Ru_{X'/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow R\pi_{X'*}(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}}).$$

We first apply $R\Gamma(X', -)$ for the smooth rigid space X'. On the one hand, we apply Theorem

3.3.2.2 to the trivial closed immersion $X' \to X'$ to get

$$\begin{aligned} R\Gamma(X'/K_{\mathrm{INF}}, \mathcal{F}) &\cong R\Gamma(X'/K_{\mathrm{INF}}, \mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \\ &\cong R\Gamma(X', \mathcal{F}_D \otimes \Omega^\bullet_D) \\ &= R\Gamma(X'_{\mathrm{rig}}, \mathcal{F}_{X'} \otimes \Omega^\bullet_{X'/K}), \end{aligned}$$

where the envelope D for the trivial closed immersion $X' \to X'$ is just X' itself.

On the other hand, the éh-cohomology $R\Gamma(X'_{\text{éh}}, \mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}})$ is in fact isomorphic to the cohomology of the de Rham complex of $\mathcal{F}_{X'}$ given by the restriction of $\mathcal{F}_{\mathrm{rig}} \otimes \Omega^\bullet_{\mathrm{rig}}$ at X'. To see this, we notice that as the natural map of complexes $\mathcal{F}_{\mathrm{rig}} \otimes \Omega^\bullet_{\mathrm{rig}} \to R\pi_{X'*}(\mathcal{F}_{\text{éh}} \otimes \Omega^\bullet_{\text{éh}})$ is filtered with respect to the Hodge filtrations, it suffices to show for each $i \in \mathbb{N}$ the following map is an isomorphism

$$R\Gamma(X'_{\mathrm{rig}}, \mathcal{F}_{X'} \otimes \Omega^i_{X'/K}) \longrightarrow R\Gamma(X'_{\text{éh}}, \mathcal{F}_{\text{éh}} \otimes \Omega^i_{\text{éh}}).$$

Here the restriction $\mathcal{F}_{\text{éh}}|_{X'}$ at the éh site of X' can be given by the éh-sheafification of the rigid sheaf $\mathcal{F}_{\mathrm{rig}}|_{X'}$ over the big site $\mathrm{Rig}_K|_{X'}$. Moreover, as $\mathcal{F}$ is a crystal in vector bundles (Corollary 3.2.2.5) and the statement is local on X' (namely both of the above two complexes satisfies the hyperdescent for rigid topology), by passing to an open rigid subspace of X' if necessary we may assume the restriction $\mathcal{F}_{\mathrm{rig}}|_{X'}$ of $\mathcal{F}$ at X' is isomorphic to the direct sum $\mathcal{O}^m_{X'}$ of structure sheaves. Thus we reduce to show that the natural map of cohomology of differentials below for a smooth K-rigid space X' is an isomorphism

$$R\Gamma(X'_{\mathrm{rig}}, \Omega^i_{X'/K}) \longrightarrow R\Gamma(X'_{\text{éh}}, \Omega^i_{\text{éh}}),$$

which is proved in Theorem 4.2.1.1.

In this way, as the map in the statement is given by the composition

$$R\Gamma(X'/K_{\mathrm{INF}}, \mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow R\Gamma(X'_{\mathrm{rig}}, \mathcal{F}_{X'} \otimes \Omega^\bullet_{X'/K}) \longrightarrow R\Gamma(X'_{\text{éh}}, \mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}}),$$

we see both maps above are isomorphisms when X' is smooth over K.

At last, notice that the cofiber C of the map $Ru_{X'/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^\bullet_{X/K_{\mathrm{INF}}}) \longrightarrow R\pi_{X'*}(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^\bullet_{\text{éh}})$ is a bounded below complex of sheaves over the small rigid site X'_{rig}. If C is not acyclic, then there would exist an open subspace U of X' such that the cohomology $R\Gamma(U_{\mathrm{rig}}, C)$ does not vanishes, which contradicts to the computation above. So we get the isomorphism for smooth K-rigid space X' that admits a map to X:

$$Ru_{X'/K*}(\mathcal{F} \otimes_{\mathcal{O}_{X/K}} \Omega^{\bullet}_{X/K_{\mathrm{INF}}}) \longrightarrow R\pi_{X'*}(\mathcal{F}_{\text{éh}} \otimes_{\mathcal{O}_{\text{éh}}} \Omega^{\bullet}_{\text{éh}}).$$

Step 2 Now we prove the isomorphism of the derived push-forwards as in the statement.

Let $i : X_{\mathrm{red}} \to X$ be the closed immersion by the reduced sub rigid space of X. We first notice that the natural map below is an isomorphism

$$Ru_{X/K*}\mathcal{F} \longrightarrow i_* Ru_{X_{\mathrm{red}}/K*}(\mathcal{F}),$$

This is because locally both of them are computed using the de Rham complex $\mathcal{F}_D \otimes \Omega^{\bullet}_D$, where D is the envelope of X in a smooth rigid spaces (Theorem 3.3.2.2). The same isomorphism holds for the derived direct image of the infinitesimal de Rham complex $\mathcal{F} \otimes \Omega^{\bullet}_{X_{\mathrm{red}}/K_{\mathrm{INF}}}$ by Proposition 3.3.2.1. On the other hand, We notice that as the closed immersion $i : X_{\mathrm{red}} \to X$ is an éh-covering (4.1), which forms a constant éh-hypercovering as the product $X_{\mathrm{red}} \times_X X_{\mathrm{red}}$ is equal to X_{red} itself, we get

$$R\pi_{X*}(\mathcal{F}_{\text{éh}} \otimes \Omega^{\bullet}_{\text{éh}}) \cong i_* R\pi_{X_{\mathrm{red}}*}(\mathcal{F}_{\text{éh}} \otimes \Omega^{\bullet}_{\text{éh}}).$$

Thus the above two isomorphisms allow us to assume X is generically reduced, and by passing to an open subset if necessary we may assume X is quasi-compact.

Now we can do the induction on the dimension of X. When $\dim(X)$ is of dimension zero, as X is quasi-compact and generically reduced, it is then equal to a disjoint union of finite points and in particular is smooth over K, where the statement follows from the Step 1.

In general, by the Temkin's resolution of singularities for rigid spaces ([Tem12]), we can find a finite compositions $X_n \to \cdots X_1 \to X_0 = X$, where each $X_i \to X_{i-1}$ is a blowup at a smooth nowhere dense closed subspace $Y_i \subset X_{i-1}$, such that in the last step X_n is smooth over K. We denote by Y_i' to be the preimage $Y_i \times_{X_{i-1}} X_i$ in X_i, which is of dimension strictly smaller than $\dim(X_i) = \dim(X)$, and we let f_i be the blowup map $X_i \to X_{i-1}$. Then for each $1 \leq i \leq n$, we get a natural distinguished triangle by Theorem 5.2.1.2

$$\begin{aligned} &Ru_{X_{i-1}/K*}(\mathcal{F} \otimes \Omega^{\bullet}_{\mathrm{INF}}) \longrightarrow Rf_{i*}Ru_{X_i/K*}(\mathcal{F} \otimes \Omega^{\bullet}_{\mathrm{INF}}) \bigoplus Ru_{Y_i/K*}(\mathcal{F} \otimes \Omega^{\bullet}_{\mathrm{INF}}) \longrightarrow \\ &Rf_{i*}Ru_{Y_i'/K*}(\mathcal{F} \otimes \Omega^{\bullet}_{\mathrm{INF}}). \end{aligned} \tag{$*$}$$

Again here we use Proposition 3.3.2.1 to replace the $\mathcal{F}$ by its de Rham complex. On the other hand, by the sheafified version of the blowup square in the éh-topology (4.2.1.5), we have a natural distinguished triangle

$$R\pi_{X_{i-1}*}(\mathcal{F}_{\text{éh}} \otimes \Omega^{\bullet}_{\text{éh}}) \longrightarrow R\pi_{X_i*}(\mathcal{F}_{\text{éh}} \otimes \Omega^{\bullet}_{\text{éh}}) \bigoplus R\pi_{Y_i*}(\mathcal{F}_{\text{éh}} \otimes \Omega^{\bullet}_{\text{éh}}) \longrightarrow R\pi_{Y'_i*}(\mathcal{F}_{\text{éh}} \otimes \Omega^{\bullet}_{\text{éh}}). \tag{$**$}$$

The functoriality of the map in the statement allows us to produce a map from the triangle $(*)$ to the triangle $(**)$. Moreover, by the dimension assumption and the induction assumption, we know the statement in the Theorem is true for X_n and all of Y_i and Y'_i. In this way, since $X_0 = X$, by a finite step of descending inductions via comparing the above two triangles $(*)$ and $(**)$, the natural map below is then an isomorphism

$$Ru_{X/K*}(\mathcal{F} \otimes \Omega^{\bullet}_{X/K_{\text{INF}}}) \longrightarrow R\pi_{X*}(\mathcal{F}_{\text{éh}} \otimes \Omega^{\bullet}_{\text{éh}}).$$

So we are done.

□

Remark 5.2.2.3. By the construction of the map in Theorem 5.2.2.2, we see the infinitesimal cohomology of a coherent crystal $\mathcal{F}$ over X/K_{INF} is a direct summand of the cohomology $R\Gamma(X_{\text{rig}}, \mathcal{F}_X \otimes \Omega^{\bullet}_{X/K})$ of the usual de Rham complex over X_{rig}.

Remark 5.2.2.4. The isomorphism in the Theorem above cannot always be improved into a filtered isomorphism. The discrepancy already appears in the schematic theory (see [Bha12a, Example 5.6]).

Now we are able to generalize the éh-hyperdescent to coherent crystals over $X/\Sigma_{e\text{INF}}$ for general e, not just K-linear crystals. We assume K is complete and algebraically closed in the next Theorem, so $\mathrm{B}^+_{\mathrm{dR,e}}$ is well-defined for K.

Theorem 5.2.2.5. *Let X be a rigid space over X, and let $\mathcal{F}$ be a crystal in vector bundles over the big infinitesimal site $X/\Sigma_{e\text{INF}}$. Then the infinitesimal cohomology of $\mathcal{F}$ over $X/\Sigma_{e\text{INF}}$ satisfies the éh-hyperdescent. Namely for an éh-hypercovering $X'_{\bullet} \to X'$ of K-rigid spaces over X, the following natural map is an isomorphism*

$$R\Gamma(X'/\Sigma_{e\text{INF}}, \mathcal{F}) \longrightarrow R\lim_{\Delta^{\text{op}}} \left(R\Gamma(X'_{\bullet}/\Sigma_{e\text{INF}}, \mathcal{F})\right).$$

Proof. We prove the result by induction on e. For $e = 1$, it is Theorem 5.2.2.2. In general, we take the derived tensor product of short exact sequence $0 \to K \to \mathrm{B}^+_{\mathrm{dR,e}} \to \mathrm{B}^+_{\mathrm{dR},e-1} \to 0$ with the complex of sheaves $Ru_{X/\Sigma_e*}\mathcal{F}$. By the big site version of the base change formula in Proposition 3.3.2.3 (cf. Corollary 3.1.2.8), we get a natural distinguished triangle

$$Ru_{X/K*}\mathcal{F}_1 \longrightarrow Ru_{X/\Sigma_e*}\mathcal{F} \longrightarrow Ru_{X/\Sigma_{e-1}*}\mathcal{F}_{e-1},$$

where $\mathcal{F}_1$ and $\mathcal{F}_{e-1}$ are pullbacks of $\mathcal{F}$ along maps of sites $X/K_{\mathrm{INF}} \to X/\Sigma_{e\mathrm{INF}}$ and $X/\Sigma_{e-1,\mathrm{INF}} \to X/\Sigma_{e\mathrm{INF}}$ separately. In this way, applying the natural transformation

$$R\Gamma(X', -) \to R\lim_{\Delta^{\mathrm{op}}} R\Gamma(X'_\bullet, -)$$

to the above triangle, we get the result by induction.

□

5.2.3 Finiteness

With the use of the éh-hyperdescent, we show in this subsection the finiteness and cohomological boundedness of infinitesimal cohomology, assuming the properness of the rigid space

Theorem 5.2.3.1. *Let X be a proper rigid space over K, and let $\mathcal{F}$ be a crystal in vector bundles over the big infinitesimal site $X/\Sigma_{e\mathrm{INF}}$. Then the infinitesimal cohomology $R\Gamma(X/\Sigma_{e\mathrm{INF}}, \mathcal{F})$ is a bounded complex supported in the cohomological degrees $[0, 2n]$, where each cohomology is a finite $\mathrm{B}^+_{\mathrm{dR,e}}$-module.*

Proof.

- We first notice that when X is smooth, the infinitesimal cohomology $R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F})$ is computed by the cohomology of the de Rham complex $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/K}$ via Theorem 3.3.2.2; namely we have
$$R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F}) \cong R\Gamma(X, \mathcal{F} \otimes \Omega^\bullet_{X/K}).$$
Moreover, each term $\mathcal{F} \otimes \Omega^i_{X/K}$ of the de Rham complex is a coherent sheaf over X. Notice that the cohomology of a coherent sheaf over a quasi-compact rigid space vanishes when the degree is above the dimension ([dJvdP96, Proposition 2.5.8]). Thus by the Hodge–de Rham spectral sequence for $\mathcal{F} \otimes \Omega^\bullet_{X/K}$, we get the result for $R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F})$ with smooth proper X.

  In general, we use the base change formula in Proposition 3.3.2.3. By taking the derived tensor product of $Ru_{X/\Sigma_e *}\mathcal{F}$ with the short exact sequence $0 \to K \to \mathrm{B}^+_{\mathrm{dR,e}} \to \mathrm{B}_{\mathrm{dR},e-1} \to 0$, we get a distinguished triangle
$$R\Gamma(X/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R\Gamma(X/\Sigma_{e\mathrm{INF}}, \mathcal{F}) \longrightarrow R\Gamma(X/\Sigma_{e-1\mathrm{INF}}, \mathcal{F}).$$
In this way, the claim for smooth proper X follows from the induction on e.

- In general, we prove by induction on the dimension of X. When X is of dimension zero, it is equal to a nilpotent extension of several points $\mathrm{Spa}(K)$. So the result follows from the éh-hyperdescent along closed immersions by reduced subspaces in Theorem 5.2.2.5 (in other words, we apply to the Čech nerve at closed immersion of the reduced subspace).

Now assume X is reduced of dimension n, and the claim is true for any rigid space of smaller dimension. By the resolution of singularities of rigid space in [Tem18], there exists a finite sequence of maps $X_m \to \cdots X_1 \to X_0 = X$, where X_m is smooth, and each map $X_i \to X_{i-1}$ is a blowup at a closed analytic subspace Y_{i-1} of X_{i-1}, such that each Y_{i-1} is nowhere dense in X_{i-1}. We denote E_i to be the exceptional locus $Y_{i-1} \times_{X_{i-1}} X_i$ of the i-th blowup. We could then apply the éh-hyperdescent in Theorem 5.2.2.5 to the Čech nerve associated with the blowup covering $X_i \coprod Y_{i-1} \to X_{i-1}$. The limit $R\lim_{\Delta^{\mathrm{op}}}$ of the infinitesimal cohomology for the hypercovering is isomorphic to the fiber of the blowup square

$$R\Gamma(X_i/K_{\mathrm{INF}}, \mathcal{F}) \bigoplus R\Gamma(Y_{i-1}/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R\Gamma(Y_i/K_{\mathrm{INF}}, \mathcal{F}),$$

and thus we get a long exact sequence

$$\begin{aligned} &\cdots \longrightarrow \mathrm{H}^j(X_{i-1}/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow \mathrm{H}^j(X_i/K_{\mathrm{INF}}, \mathcal{F}) \bigoplus \mathrm{H}^j(Y_{i-1}/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow \\ &\mathrm{H}^j(Y_i/K_{\mathrm{INF}}, \mathcal{F}) \longrightarrow \cdots . \end{aligned}$$

In this way, with the help of the induction assumption for all Y_i, a further descending induction from X_m to $X_0 = X$ finishes the proof.

□

5.3 Algebraic and analytic infinitesimal cohomology

At the end of the chapter, we prove the comparison between the algebraic infinitesimal cohomology and the analytic infinitesimal cohomology, for a proper algebraic variety.

Recall that for an algebraic variety[2] $\mathcal{X}$ over a p-adic field K, we can define its *(algebraic) infinitesimal site* $\mathcal{X}/K_{\mathrm{inf}}$, whose objects are schematic infinitesimal thickenings $(\mathcal{U}, \mathcal{T})$, where $\mathcal{U}$ is an Zariski open subset of $\mathcal{X}$. The infinitesimal site $\mathcal{X}/K_{\mathrm{inf}}$ is equipped with a structure sheaf $\mathcal{O}_{\mathcal{X}/K}$, and its cohomology is called the *algebraic infinitesimal cohomology*. Moreover, the infinitesimal structure sheaf admits a surjection $\mathcal{O}_{\mathcal{X}/K} \to \mathcal{O}_{\mathcal{X}}$ to the Zariski structure sheaf, whose kernel $\mathcal{J}_{\mathcal{X}/K}$ defines a natural filtration on $R\Gamma(\mathcal{X}/K_{\mathrm{inf}}, \mathcal{O}_{\mathcal{X}/K})$. Similar to the analytic theory, we call this filtration the *(algebraic) infinitesimal filtration*.

Let $X = \mathcal{X}^{\mathrm{an}}$ be the rigid space over K defined as the analytification of a variety $\mathcal{X}$. As the analytification functor $\mathrm{Sch}_K \to \mathrm{Rig}_K$ preserves open and closed immersions, it induces a natural

[2]For our purpose, a *variety* is defined to be a locally of finite type scheme over a field in the article.

map of ringed sites

$$(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K}) \longrightarrow (\mathfrak{X}/K_{\mathrm{inf}}, \mathcal{O}_{\mathfrak{X}/K}).$$

Moreover, as the surjection $\mathcal{O}_{\mathfrak{X}/K} \to \mathcal{O}_{\mathfrak{X}}$ is compatible with $\mathcal{O}_{X/K} \to \mathcal{O}_X$, the natural map of infinitesimal structure sheaves above is then a filtered map. As a consequence, by passing to their cohomology, we get a natural filtered morphism in derived category

$$R\Gamma(\mathfrak{X}/K_{\mathrm{inf}}, \mathcal{O}_{\mathfrak{X}/K}) \longrightarrow R\Gamma(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K}).$$

Our main result in this subsection is the following.

Theorem 5.3.0.1. *Let $\mathfrak{X}$ be a proper algebraic variety over K, and let X be its analytifiation. Then the analytification functor induces a filtered isomorphism of infinitesimal cohomology*

$$R\Gamma(\mathfrak{X}/K_{\mathrm{inf}}, \mathcal{O}_{\mathfrak{X}/K}) \longrightarrow R\Gamma(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K}).$$

Before the proof, we first recall that there is a natural map of ringed sites $(X_{\mathrm{rig}}, \mathcal{O}_X) \to (\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}})$. Here the rigid structure sheaf is flat over the Zariski structure sheaf, and the pullback along the map induces a fully faithful functor from coherent $\mathcal{O}_{\mathfrak{X}}$-modules to coherent $\mathcal{O}_X$-modules.

Moreover, the above map of sites is compatible with the infinitesimal topos. Recall that there exists a natural map of topoi

$$\begin{aligned} u_{\mathfrak{X}/K} : \mathrm{Sh}(\mathfrak{X}/K_{\mathrm{inf}}) &\longrightarrow \mathrm{Sh}(\mathfrak{X}_{\mathrm{Zar}}); \\ \mathcal{F} &\longmapsto (\mathfrak{U} \mapsto \Gamma(\mathfrak{U}/K_{\mathrm{inf}}, \mathcal{F}|_{\mathfrak{U}})). \end{aligned}$$

By construction, this functor is compatible with its rigid version $u_{X/K} : \mathrm{Sh}(X/K_{\mathrm{inf}}) \to \mathrm{Sh}(X_{\mathrm{rig}})$ (cf. Subsection 3.1.3). Namely, the following diagram is commutative

$$\begin{tikzcd} \mathrm{Sh}(X/K_{\mathrm{inf}}) \arrow[r] \arrow[d, "u_{X/K}"'] & \mathrm{Sh}(\mathfrak{X}/K_{\mathrm{inf}}) \arrow[d, "u_{\mathfrak{X}/K}"] \\ \mathrm{Sh}(X_{\mathrm{rig}}) \arrow[r] & \mathrm{Sh}(\mathfrak{X}_{\mathrm{Zar}}). \end{tikzcd}$$

We then claim the following result.

Proposition 5.3.0.2. *Let $\mathfrak{X}$ be an algebraic variety over K, and let X be its analytifcation. Then the complex of coherent $\mathcal{O}_X$-modules $Ru_{X/K*}(\mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K})$ is naturally isomorphic to the analytification of the complex of coherent $\mathcal{O}_{\mathfrak{X}}$-modules $Ru_{\mathfrak{X}/K*}(\mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K})$.*

Proof. We denote the complex of coherent $\mathcal{O}_{\mathfrak{X}}$-modules $Ru_{\mathfrak{X}/K*}(\mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K})$ by C, and we denote the complex $Ru_{X/K*}(\mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K})$ by C'. Then it suffices to show that the natural map below induced from the pullback from $\mathfrak{X}_{\mathrm{Zar}}$ to X_{rig} is an isomorphism of complexes of $\mathcal{O}_X$-modules

$$C \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_X \longrightarrow C'.$$

As the result is a local statement for $\mathfrak{X}$, let us assume $\mathfrak{X} = \mathrm{Spec}(A)$ is a finite type affine scheme over K and $\mathfrak{X} \to \mathfrak{Y} = \mathrm{Spec}(A')$ be a closed immersion into an affine space over K. Moreover, notice that the isomorphism could be checked locally on X, so we may take an open affinoid disc of certain radius $\mathrm{Spa}(B')$ in $\mathfrak{Y}^{\mathrm{an}}$, with the open subset $X \cap \mathrm{Spa}(B') = \mathrm{Spa}(B)$ in X. From our choices, we get a cartesian diagram as below, where horizontal maps are surjective and vertical maps are flat

$$\begin{tikzcd} B' \arrow[r, two heads] & B \\ A' \arrow[u] \arrow[r, two heads] & A \arrow[u] \end{tikzcd}$$

So it suffices to show that

$$R\Gamma(\mathrm{Spec}(A)/K_{\mathrm{inf}}, \mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K}) \otimes_A B \cong R\Gamma(\mathrm{Spa}(B)/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}).$$

We then recall from [BdJ11, Section 2] that the algebraic infinitesimal cohomology can be computed by the Čech-Alexander complex as below

$$D \longrightarrow D(1) \longrightarrow D(2) \longrightarrow \cdots,$$

where $D(m)$ is the formal completion of $A'(m) := A'^{\otimes m+1}_K$ along the surjection $A'(m) \to A$. We take the n-th graded piece for the algebraic infinitesimal filtration, then the cohomology group $R\Gamma(\mathrm{Spec}(A)/K_{\mathrm{inf}}, \mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K})$ is isomorphic to the following map of A-linear cosimplicial complexes

$$J^n_D/J^{n+1}_D \longrightarrow J^n_{D(1)}/J^{n+1}_{D(n)} \longrightarrow J^n_{D(2)}/J^{n+1}_{D(2)} \longrightarrow \cdots,$$

where $J_{D(m)}$ is the kernel of the surjection $A'(m) \to A$. On the other hand, by the Čech-Alexander complex for rigid spaces in Proposition 3.3.1.3, we have

$$R\Gamma(\mathrm{Spa}(B)/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}) \cong \left(J^n_{\mathcal{D}}/J^{n+1}_{\mathcal{D}} \longrightarrow J^n_{\mathcal{D}(1)}/J^{n+1}_{\mathcal{D}(1)} \longrightarrow J^n_{\mathcal{D}(2)}/J^{n+1}_{\mathcal{D}(2)} \longrightarrow \cdots \right),$$

where $\mathcal{D}(m)$ is the formal completion for the surjection $B'(m) := B'^{\widehat{\otimes} m+1}_K \to B$, and $J_{\mathcal{D}(m)}$ is the kernel of the map $B'(m) \to B$. Thus we are left to show the quasi-isomorphism for the canonical map of B-linear cosimplicial complexes below

$$\left(J_D^n/J_D^{n+1} \longrightarrow J_{D(1)}^n/J_{D(n)}^{n+1} \longrightarrow J_{D(2)}^n/J_{D(2)}^{n+1} \longrightarrow \cdots\right) \otimes_A B \longrightarrow$$
$$\left(J_{\mathcal{D}}^n/J_{\mathcal{D}}^{n+1} \longrightarrow J_{\mathcal{D}(1)}^n/J_{\mathcal{D}(1)}^{n+1} \longrightarrow J_{\mathcal{D}(2)}^n/J_{\mathcal{D}(2)}^{n+1} \longrightarrow \cdots\right).$$

At last notice that by our choices, the rigid space $\mathrm{Spa}(B')$ is an open disc of some radius in the affine space $\mathrm{Spec}(A')^{\mathrm{an}}$. In particular, the following map of rings is a cartesian diagram such that vertical maps are flat

$$\begin{tikzcd} B'(m) \arrow[r, two heads] & B' \\ A'(m) \arrow[r, two heads] \arrow[u] & A' \arrow[u] \end{tikzcd}$$

In this way, combining this with the cartesian diagram in the first paragraph, we see the kernel $J_{\mathcal{D}(m)}$ of the surjection $B'(m) \to B$ is equal to the base change of $J_{D(m)}$ along the flat map $A'(m) \to B'(m)$. Hence we get the natural equalities

$$J_{D(m)}B'(m) = J_{D(m)} \otimes_{A'(m)} B'(m) = J_{\mathcal{D}(m)};$$
$$(J_{D(m)}^n/J_{D(m)}^{n+1}) \otimes_A B = (J_{D(m)}^n/J_{D(m)}^{n+1}) \otimes_{A'(m)} B'(m) = J_{\mathcal{D}(m)}^n/J_{\mathcal{D}(m)}^{n+1}.$$

So we are done.

□

At last, we finish the proof of Theorem 5.3.0.1.

Proof of Theorem 5.3.0.1. To show the natural map in the statement is a filtered isomorphism, it suffices to show the isomorphisms for their underlying complexes and each graded pieces separately, as both of them are filtered complete.

For the underlying complexes, this follows from the éh descent. To see this, we first notice that when $\mathcal{X}$ is smooth and proper over K, then the algebraic and analytic infinitesimal cohomology are isomorphic to the algebraic and analytic de Rham cohomology separately ([Gro68], Theorem 3.3.2.2), which are isomorphic to each other by applying the GAGA theorem to their Hodge-filtrations (cf. [Con06, Appendix A.1]). In general, we may assume $\mathcal{X}_\bullet \to \mathcal{X}$ is a simplicial smooth varieties by resolving singularities. Then its analytification $X_\bullet \to X$ is an éh-hypercovering by smooth rigid spaces, and we get the isomorphism

$$\begin{aligned}
R\Gamma(\mathfrak{X}/K_{\mathrm{inf}}, \mathcal{O}_{\mathfrak{X}/K}) &\cong R\lim_{[n]\in\Delta^{\mathrm{op}}} R\Gamma(\mathfrak{X}_n/K_{\mathrm{inf}}, \mathcal{O}_{\mathfrak{X}_n/K}) \\
&\cong R\lim_{[n]\in\Delta^{\mathrm{op}}} R\Gamma(X_n/K_{\mathrm{inf}}, \mathcal{O}_{X_n/K}) \\
&\cong R\Gamma(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K}),
\end{aligned}$$

where the first equality is the h-hyperdescent of algebraic de Rham cohomology for blowups in [Har75], and the last is the éh-hyperdescent for analytic infinitesimal cohomology in 5.2.2.2.

For the graded pieces, by Proposition 5.3.0.2 we have

$$\begin{aligned}
&R\Gamma(X/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}) = R\Gamma(X_{\mathrm{rig}}, Ru_{X/K*}(\mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K})) \cong \\
&R\Gamma(X_{\mathrm{rig}}, \left(Ru_{\mathfrak{X}/K*}(\mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K})\right)^{\mathrm{an}}).
\end{aligned}$$

We denote C to be the bounded below complex of coherent $\mathcal{O}_X$-modules $Ru_{\mathfrak{X}/K*}(\mathcal{J}^n_{\mathfrak{X}/K}/\mathcal{J}^{n+1}_{\mathfrak{X}/K})$. As $R\Gamma(X_{\mathrm{Zar}}, \tau^{>n}C)$ lives in cohomological degree larger than n, we have the natural equalities

$$R\Gamma(X_{\mathrm{Zar}}, C) = \mathrm{colim}_n R\Gamma(X_{\mathrm{Zar}}, \tau^{\leq n}C).$$

Similarly we have

$$R\Gamma(X_{\mathrm{rig}}, C^{\mathrm{an}}) = \mathrm{colim}_n R\Gamma(X_{\mathrm{rig}}, \tau^{\leq n}(C^{\mathrm{an}})).$$

On the other hand, as the rigid structure sheaf $\mathcal{O}_X$ is flat over $\mathcal{O}_{\mathfrak{X}}$, the analytification functor $(-)^{\mathrm{an}} = - \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_X$ on coherent complexes is an exact functor. So for each $n \in \mathbb{N}$, there exists a natural equality

$$\tau^{\leq n}C^{\mathrm{an}} = (\tau^{\leq n}C)^{\mathrm{an}}.$$

Notice that for each bounded complex $\tau^{\leq n}C$ of coherent sheaves, by rigid GAGA theorem ([Con06, Appendix A.1]) we have

$$R\Gamma(X_{\mathrm{Zar}}, \tau^{\leq n}C) \cong R\Gamma(X_{\mathrm{rig}}, (\tau^{\leq n}C)^{\mathrm{an}}).$$

In this way, combining all of the isomorphisms above, we get

$$\begin{aligned} R\Gamma(X_{\mathrm{Zar}}, C) &\cong \mathrm{colim}_n R\Gamma(X_{\mathrm{Zar}}, \tau^{\leq n} C) \\ &\cong \mathrm{colim}_n R\Gamma(X_{\mathrm{rig}}, (\tau^{\leq n} C)^{\mathrm{an}}) \\ &= \mathrm{colim}_n R\Gamma(X_{\mathrm{rig}}, \tau^{\leq n}(C^{\mathrm{an}})) \\ &\cong R\Gamma(X_{\mathrm{rig}}, C^{\mathrm{an}}). \end{aligned}$$

At last, substituting back the definition of C and Proposition 5.3.0.2, we then obtain the formula for graded piece of infinitesimal filtrations:

$$R\Gamma(\mathcal{X}/K_{\mathrm{inf}}, \mathcal{J}^n_{\mathcal{X}/K}/\mathcal{J}^{n+1}_{\mathcal{X}/K}) \cong R\Gamma(X/K_{\mathrm{inf}}, \mathcal{J}^n_{X/K}/\mathcal{J}^{n+1}_{X/K}).$$

□

As an application, we get the comparison with singular cohomology when K is abstractly isomorphic to the field of complex numbers.

Corollary 5.3.0.3. *Assume there exists an abstract isomorphism of fields $K \to \mathbb{C}$. Then for any proper algebraic variety $\mathcal{X}/K$ with its analytification X, there exists a filtered isomorphism of cohomology*

$$\mathrm{H}^i(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K}) \cong \mathrm{H}^i_{\mathrm{Sing}}(\mathcal{X}(\mathbb{C}), \mathbb{C}),$$

where the singular cohomology of $\mathcal{X}(\mathbb{C})$ is filtered by the algebraic infinitesimal filtration.

Proof. This follows from Theorem 5.3.0.1 and the classical result of Hartshorne in [Har75]. □

Using the same idea of the proof for Proposition 5.3.0.2 and Theorem 5.3.0.1, we can prove the base extension formula for infinitesimal cohomology.

Corollary 5.3.0.4. *Let K_0 be a complete p-adic extension of $\mathbb{Q}_p$, and let K be a complete extension of K_0. Assume X is a proper rigid space over K_0, and let $\mathcal{F}$ be a coherent crystal over $X/K_{0,\mathrm{inf}}$. Then the following natural map of filtered complexes is an isomorphism*

$$R\Gamma(X/K_{0,\mathrm{inf}}, \mathcal{F}) \otimes_{K_0} K \longrightarrow R\Gamma(X_K/K_{\mathrm{inf}}, \mathcal{F}_K).$$

CHAPTER 6

Cohomology over $\mathrm{B}_{\mathrm{dR}}^{+}$

Let X be a rigid space over $\mathbb{C}_p$. In this chapter, we generalize the three de Rham cohomology theories in previous chapters to the coefficient over the de Rham period ring $\mathrm{B}_{\mathrm{dR}}^{+}$, and compare them with the pro-étale cohomology of the de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}$. The results in this chapter appear in [Guo20, Section 7].

Analogous to the infinitesimal theory over $\mathrm{B}_{\mathrm{dR,e}}^{+}$, we will introduce the basics of the infinitesimal site of X over $\mathrm{B}_{\mathrm{dR}}^{+}$ together with its relation to $X/\mathrm{B}_{\mathrm{dR,e_{inf}}}^{+}$, as in Section 6.1. Here the infinitesimal site of X over $\mathrm{B}_{\mathrm{dR}}^{+}$ could be thought as the union of all of the sites $X/\mathrm{B}_{\mathrm{dR,e_{inf}}}^{+}$, for $e \in \mathbb{N}$. In Section 6.2, by taking the inverse limit, we extend the comparisons in Chapter 5 to $\mathrm{B}_{\mathrm{dR}}^{+}$-linear coefficients. At last, in Section 6.3 we use the cohomological descent and compare the B_{dR}-linear cohomology theories with the pro-étale cohomology of the de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}$ in [Sch13a], extending the comparison for proper smooth rigid spaces in [BMS18] to all proper rigid spaces (Theorem 6.3.1.2) Together with the Primitive Comparison Theorem ([Sch13b]), we are then able to show the ξ-torsionfreeness of the $\mathrm{B}_{\mathrm{dR}}^{+}$-infinitesimal cohomology, and the degeneracy of the Hodge–de Rham spectral sequence for éh-de Rham cohomology, as in Theorem 6.3.2.1.

6.1 Infinitesimal sites and topoi over $\mathrm{B}_{\mathrm{dR}}^{+}$

We fix a complete algebraic closed p-adic field K. Let X be a rigid space over $\mathrm{B}_{\mathrm{dR}}^{+}/\xi^r$ for some fixed $r \in \mathbb{N}$. To build an infinitesimal cohomology theory with the coefficient being $\mathrm{B}_{\mathrm{dR}}^{+} = \varprojlim_{e\in\mathbb{N}} \mathrm{B}_{\mathrm{dR,e}}^{+}$, we construct an infinitesimal site X/Σ_{inf} as a union of all $X/\Sigma_{e\,\mathrm{inf}}$ for $e \in \mathbb{N}_{\geq r}$, and consider its relation to each infinitesimal site $X/\Sigma_{e\,\mathrm{inf}}$.

6.1.1 The site X/Σ_{inf}

We first give the definition of the infinitesimal site over $\Sigma = \varinjlim_{e\in\mathbb{N}} \Sigma_e$.

Definition 6.1.1.1. *Let X be a rigid space over $\Sigma_r = \mathrm{Spa}(\mathrm{B}_{\mathrm{dR}}^{+}/\xi^r)$, for some fixed $r \in \mathbb{N}$. The* infinitesimal site X/Σ_{inf} over $\mathrm{B}_{\mathrm{dR}}^{+}$ *is defined as follows:*

- *The underlying category of X/Σ_{inf} is the category of pairs (U,T), for (U,T) being an thickening in $X/\Sigma_{e\,\mathrm{inf}}$ for some $e \geq r$.*

  A morphism between (U_1,T_1) and (U_2,T_2) is a morphism of objects in $X/\Sigma_{e\,\mathrm{inf}}$, for e large enough such that both pairs are objects in $X/\Sigma_{e\,\mathrm{inf}}$.

- *A collection of morphism $(U_i,T_i) \to (U,T)$ in X/Σ_{inf} is a covering if $\{T_i \to T\}$ is an open covering for the rigid space T.*

As a category, X/Σ_{inf} is the union of $X/\Sigma_{e\,\mathrm{inf}}$ for all $e \geq r$. It is clear that the topology is locally rigid over each object in X/Σ_{inf}. Thus the description of a sheaf over X/Σ_{inf} is similar to that of a sheaf over $X/\Sigma_{e\,\mathrm{inf}}$ as in Section 3.1.

Remark 6.1.1.2. Similarly to the Discussion in Section 3.1, we could define the big version infinitesimal site X/Σ_{INF}, where the objects are infinitesimal thickenings (U,T) for U being a rigid space over X and $U \to T$ a nil-extension over $\mathrm{B}_{\mathrm{dR}}^+$. The relation between the big infinitesimal sites X/Σ_{INF} and the small one X/Σ_{inf}, including the constructions in the rest of the section, are exactly identical to the case over $\mathrm{B}_{\mathrm{dR,e}}^+$ in Paragraph 3.1.1, and we will not duplicate again here.

6.1.2 Functoriality of $\mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$

The infinitesimal topos $\mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$ is functorial with respect to the rigid space X. Namely, for a map of $\mathrm{B}_{\mathrm{dR}}^+$-rigid space $f : X \to Y$ where ξ is nilpotent, we have a natural map of topoi

$$f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \longrightarrow \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}).$$

The corresponding adjoint pair of functors are given by the following:

- For a sheaf $\mathcal{G} \in \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}})$, the inverse image $f_{\mathrm{inf}}^{-1}\mathcal{G}$ is given by the restriction of $\mu_Y^{-1}\mathcal{G}$ to the category X/Σ_{inf} along the map f, and is equal to the sheaf associated with the presheaf

$$X/\Sigma_{\mathrm{inf}} \ni (U,T) \longmapsto \varinjlim_{\substack{(U,T)\to(V,S) \\ (V,S)\in Y/\Sigma_{\mathrm{inf}}, \\ U\to V \text{ compatible with } f}} \mathcal{G}(V,S).$$

- The direct image functor $f_{\mathrm{inf}\,*}$ sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$ to the sheaf

$$f_{\mathrm{inf}\,*}\mathcal{F}(V,S) = \varprojlim_{\substack{(U,T)\to(V,S) \\ (U,T)\in X/\Sigma_{\mathrm{INF}} \\ U\to V \text{ compatible with } f}} \mathcal{F}(U,T).$$

We want to remind the reader that the construction of those two functors are identical with the construction of the functoriality morphism $\mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$ for the map of rigid spaces

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \Sigma_e \arrow[r] & \Sigma_{e'}, \end{tikzcd}$$

as in Subsection 3.1.4.

6.1.3 Relation with $X/\Sigma_{e\,\mathrm{inf}}$

Topologically, the infinitesimal site X/Σ_{inf} is the limit of $X/\Sigma_{e\,\mathrm{inf}}$ for $e \geq r$. To make this precise, we consider the following morphism of sites:

$$u_e : X/\Sigma_{\mathrm{inf}} \longrightarrow X/\Sigma_{e\,\mathrm{inf}},$$

whose corresponding functor is the canonical inclusion functor that sends $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$ to the object $(U,T) \in X/\Sigma_{\mathrm{inf}}$. Note that by construction, this cocontinuous functor is a fully faithful embedding.

This morphism induces an adjoint pair of functors (u_e^{-1}, u_{e*}) given as follows:

- The functor u_{e*} is the restriction functor, in a way that for a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$ we have

$$(u_{e*}\mathcal{F})_T = \mathcal{F}_T.$$

- For a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$, the sheaf $u_e^{-1}\mathcal{G}$ is the sheaf associated with the presheaf

$$\begin{aligned}(V,S) \mapsto & \varinjlim_{\substack{(V,S)\to(U,T)\\(U,T)\in X/\Sigma_{e\,\mathrm{inf}}}} \mathcal{G}(U,T) \\ &= \begin{cases} \varnothing, \ S \notin \mathrm{Rig}_{\Sigma_e}; \\ \mathcal{G}(V,S), \ S \in \mathrm{Rig}_{\Sigma_e}. \end{cases}\end{aligned}$$

So by the definition of the site X/Σ_{inf}, the restriction of $u_e^{-1}\mathcal{G}$ at (V,S) is

$$(u_e^{-1}\mathcal{G})_S = \begin{cases} \varnothing, \ S \notin \mathrm{Rig}_{\Sigma_e}; \\ \mathcal{G}_S, \ S \in \mathrm{Rig}_{\Sigma_e}. \end{cases}$$

Here we notice that when $\mathcal{G} = h_{(U,T)}$ is the representable sheaf for some object $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$, the inverse image $u_e^{-1}h_{(U,T)}$ is nothing but the representable sheaf $h_{(U,T)}$ in $\mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$.

The morphism of site $u_e : X/\Sigma_{\mathrm{inf}} \to X/\Sigma_{e\,\mathrm{inf}}$ induces a map of topoi

$$u_e : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}).$$

It admits a section $i_e : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$, where the corresponding adjoint pair of functors is given as follows:

- For a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{inf})$, the inverse image $i_e^{-1}\mathcal{G}$ is the sheaf associated with the presheaf

$$X/\Sigma_{e\,\mathrm{inf}} \ni (U,T) \longmapsto \varinjlim_{\substack{(U,T)\to(U,S)\\(U,S)\in X/\Sigma_{\mathrm{inf}}}} \mathcal{G}(U,S) = \mathcal{G}(U,T).$$

  Namely, $i_e^{-1} = u_{e*}$ is the restriction functor.

- The direct image functor i_{e*} sends a sheaf $\mathcal{F} \in \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ to the sheaf

$$i_{e*}\mathcal{F}(V,S) = \varprojlim_{\substack{(V,T)\to(V,S)\\(V,T)\in X/\Sigma_{e\,\mathrm{inf}}}} \mathcal{F}(V,T) = \mathcal{F}(V, S\times_\Sigma \Sigma_e).$$

It is clear that the composition $u_e \circ i_e$ is equal to the identity. We also note that the above functors are functorial with respect to e.

Remark 6.1.3.1. Here we notice that the map i_e is in fact induced from a natural map of sites

$$\begin{aligned} i_e : X/\Sigma_{e\,\mathrm{inf}} &\longrightarrow X/\Sigma_{\mathrm{inf}}; \\ (U, T\times_\Sigma \Sigma_e) &\longleftarrow\!\dashv (U,T). \end{aligned}$$

This is analogous to the nilpotent bases situation, as in the Remark 3.1.4.3

Remark 6.1.3.2. We also want to reminder the reader that the construction of map i_e could be regarded as the functoriality morphism of infinitesimal topoi associated with the following diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d] \\ \Sigma_e \arrow[r] & \Sigma. \end{tikzcd}$$

Remark 6.1.3.3. The construction of u_e and i_e is compatible with the functoriality morphism of infinitesimal topoi $f_{\mathrm{inf}} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}})$ for a map of rigid spaces $f : X/\Sigma_e \to$

$Y/\Sigma_{e'}$. Namely we have the following commutative diagrams among infinitesimal topoi

$$\begin{tikzcd} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \arrow[r, "u_e"] \arrow[d, "f_{\mathrm{inf}}"'] & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \arrow[d, "f_{\mathrm{inf}}"] \\ \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}) \arrow[r, "u_{e'}"'] & \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}}); \end{tikzcd} \quad \begin{tikzcd} \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \arrow[r, "i_e"] \arrow[d, "f_{\mathrm{inf}}"'] & \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \arrow[d, "f_{\mathrm{inf}}"] \\ \mathrm{Sh}(Y/\Sigma_{e'\,\mathrm{inf}}) \arrow[r, "i_{e'}"'] & \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}). \end{tikzcd}$$

6.1.4 Relation to the rigid topos $\mathrm{Sh}(X_{\mathrm{rig}})$

Analogous to Subsection 3.1.3, there exists a natural map of topoi to the rigid site X_{rig} as below

$$u_{X/\Sigma} : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \longrightarrow \mathrm{Sh}(X_{\mathrm{rig}}).$$

The corresponding preimage and the direct image functors are given as below

- $$\begin{aligned} u_{X/\Sigma *} : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) &\longrightarrow \mathrm{Sh}(X_{\mathrm{rig}}); \\ \mathcal{F} &\longmapsto (U \mapsto \Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{F})). \end{aligned}$$

- $$\begin{aligned} u_{X/\Sigma}^{-1} : \mathrm{Sh}(X_{\mathrm{rig}}) &\longrightarrow \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}); \\ \mathcal{E} &\longmapsto ((U,T) \mapsto \mathcal{E}(U)). \end{aligned}$$

Namely the push-forward functor $u_{X/\Sigma *}$ is the sheafified version of the infinitesimal global section functor.

Remark 6.1.4.1. The functor $u_{X/\Sigma}$ is functorial with respect to the rigid space X. Precisely, given a map of rigid spaces $f : X \to Y$ over $\mathrm{B}_{\mathrm{dR}}^+$ where ξ is nilpotent, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \arrow[r, "u_{X/\Sigma}"] \arrow[d, "f_{\mathrm{inf}}"'] & \mathrm{Sh}(X_{\mathrm{rig}}) \arrow[d, "f"] \\ \mathrm{Sh}(Y/\Sigma_{\mathrm{inf}}) \arrow[r, "u_{Y/\Sigma}"'] & \mathrm{Sh}(Y_{\mathrm{rig}}). \end{tikzcd}$$

Remark 6.1.4.2. The functor $u_{X/\Sigma}$ is also compatible with $u_e : \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}})$ and

$i_e : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X/\Sigma_{\mathrm{inf}})$. Namely, the following diagrams commute:

$$\begin{tikzcd} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \arrow[rr, "u_{X/\Sigma}"] \arrow[dr, "u_e"'] & & \mathrm{Sh}(X_{\mathrm{rig}}) \\ & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \arrow[ur, "u_{X/\Sigma_e}"'] & \end{tikzcd} ; \quad \begin{tikzcd} \mathrm{Sh}(X/\Sigma_{\mathrm{inf}}) \arrow[rr, "u_{X/\Sigma}"] & & \mathrm{Sh}(X_{\mathrm{rig}}) \\ & \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \arrow[ul, "i_e"] \arrow[ur, "u_{X/\Sigma_e}"'] & \end{tikzcd} .$$

Here $u_{X/\Sigma_e} : \mathrm{Sh}(X/\Sigma_{e\,\mathrm{inf}}) \to \mathrm{Sh}(X_{\mathrm{rig}})$ is the analogous functor of $u_{X/\Sigma}$ onto the rigid site defined in Subsection 3.1.3.

6.2 Cohomology of crystals over X/Σ_{inf}

In this section, we consider the cohomology of a crystal $\mathcal{F}$ over the infinitesimal site X/Σ_{inf}. Our strategy is to interpret the cohomology of $\mathcal{F}$ as the derived inverse limit of the cohomology of the pullback $i_e^*\mathcal{F}$, where $i_e^*\mathcal{F}$ is a crystal over the site $X/\Sigma_{e\,\mathrm{inf}}$.

To start with, we first describe a crystal over the infinitesimal site X/Σ_{inf}.

Definition 6.2.0.1. *Let X be a rigid space over Σ_{inf} where ξ is nilpotent.*

(i) The infinitesimal structure sheaf *over X/Σ_{inf}, denoted as $\mathcal{O}_{X/\Sigma}$, is a sheaf over X/Σ_{inf} sending a thickening $(U,T) \in X/\Sigma_{\mathrm{inf}}$ onto the global section of $\mathcal{O}_T$ at T as below*

$$\mathcal{O}_{X/\Sigma} : (U,T) \longmapsto \mathcal{O}_T(T).$$

(ii) A coherent crystal *over X/Σ_{inf} is a $\mathcal{O}_{X/\Sigma}$-coherent sheaf $\mathcal{F}$ over X/Σ_{inf} satisfies the crystal condition as in Definition 3.2.1.2. It is called* a crystal in vector bundle *if the restriction $\mathcal{F}_T$ at each infinitesimal thickening $(U,T) \in X/\Sigma_{\mathrm{inf}}$ is a vector bundle over $\mathcal{O}_T$.*

Here we mention that similar to Proposition 3.2.1.7, it can be shown the categories crystals over big and small sites are equivalent.

We notice that the morphism of sites $i_e : X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{\mathrm{inf}}$ in the last section is naturally a morphism of ringed sites for their structure sheaves. Moreover, since the preimage functor i_e^{-1} is equal to the restriction functor onto the subcategory $X/\Sigma_{e\,\mathrm{inf}}$, we get

$$i_e^{-1}\mathcal{O}_{X/\Sigma} = \mathcal{O}_{X/\Sigma_e}.$$

So we can define the *pullback functor* $i_e^*\mathcal{F} := i_e^{-1}\mathcal{F} \otimes_{i_e^{-1}\mathcal{O}_{X/\Sigma}} \mathcal{O}_{X/\Sigma_e}$, which is the same as the restriction functor $i_e^{-1}\mathcal{F}$ itself; namely for an infinitesimal thickening $(U,T) \in X/\Sigma_{e\,\mathrm{inf}}$, we have

$$(i_e^*\mathcal{F})_T = (i_e^{-1}\mathcal{F})_T = \mathcal{F}_T.$$

Here we want to remark that the pullback functor $i_e^* = i_e^{-1}$ is compatible with the pullback functor $f_{\inf}^*$ of the morphism $f_{\inf} : \mathrm{Sh}(X/\Sigma_{e\inf}) \to \mathrm{Sh}(Y/\Sigma_{e'\inf})$ for a map of rigid spaces $f : X/\Sigma_e \to Y/\Sigma_{e'}$.

The main tool of the section is the following lemma, relating a coherent crystal over $X/\Sigma_{\inf}$ with those over $X/\Sigma_{e\inf}$ of ξ-nilpotent coefficients.

Lemma 6.2.0.2. *Let $\mathcal{F}$ be a coherent crystal over the infinitesimal site $X/\Sigma_{\inf}$ (resp. X/Σ_{INF}), and let X be defined over Σ_r for some $r \in \mathbb{N}$. Then we have the following.*

(i) The pullback $i_e^*\mathcal{F}$ for each $e \in \mathbb{N}_{\geq r}$ is a crystal over $X/\Sigma_{e\inf}$. When $\mathcal{F}$ is a crystal in vector bundles, so is $\mathcal{F}$ over $X/\Sigma_{e\inf}$.

(ii) The counit map for the adjoint pairs (i_e^*, i_{e*}) induces the following isomorphism

$$\mathcal{F}/\xi^e \longrightarrow Ri_{e*}i_e^*\mathcal{F}.$$

In pariturclar, we have the natural equivalences as below

$$\mathcal{F} \longrightarrow R\varprojlim_{e\geq r} \mathcal{F}/\xi^e \longrightarrow R\varprojlim_{e\geq r} Ri_{e*}i_e^*\mathcal{F}.$$

Here the transition maps in the last limit are given by the map of infinitesimal sites $X/\Sigma_{e\inf} \to X/\Sigma_{e+1\inf}$ (resp. $X/\Sigma_{e\mathrm{INF}} \to X/\Sigma_{e+1\mathrm{INF}}$) for the closed immersions of bases.

Proof.

(i) The proof of the (i) follows from the definition of the crystal condition.

(ii) We recall from the last section that the push-forward functor $i_{e*}\mathcal{G}$ is given by

$$(i_{e*}\mathcal{G})(U,T) = \mathcal{G}(U, T\times_\Sigma \Sigma_e),$$

for a sheaf $\mathcal{G} \in \mathrm{Sh}(X/\Sigma_{e\inf})$. We denote the fiber product $T\times_\Sigma \Sigma_e$ by T_e, which is an infinitesimal thickening of U that is defined over Σ_e. Apply the above to the pullback $\mathcal{G} = i_e^*\mathcal{F}$ of the crystal $\mathcal{F}$, and notice that i_e^* is the restriction functor, we get

$$\begin{aligned}(Ri_{e*}i_e^*\mathcal{F})(U,T) &= R\Gamma((U,T_e),\mathcal{F}) \\ &= R\Gamma(T_e, \mathcal{F}_{T_e}) \\ &= R\Gamma(T_e, \mathcal{F}_T/\xi^e) \\ &= R\Gamma(T, \mathcal{F}_T/\xi^e),\end{aligned}$$

where the last equality follows from the observation that $T_e \to T$ has the same underlying topological spaces. Hence the cone of $\mathcal{F}/\xi^e \to Ri_{e*}i_e^*\mathcal{F}$, which is bounded below and has no cohomology, vanishes in the derived category.

At last, notice that for a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_{X/\Sigma}$-modules over X/Σ_{inf}, we always have

$$\mathcal{F} \cong R\varprojlim_{e \geq r} \mathcal{F}/\xi^e \cong \varprojlim_{e \geq r} \mathcal{F}/\xi^e.$$

So the last claim in (ii) follows.

□

Now we are able to give the main result about the cohomology of crystals over the infinitesimal site X/Σ_{inf}.

Theorem 6.2.0.3. *Let X be a rigid space over some Σ_r, and let $\mathcal{F}$ be a coherent crystal over X/Σ_{inf}.*

(i) There exists a natural isomorphism of complexes of sheaves of $\mathrm{B}_{\mathrm{dR}}^+$-modules as below:

$$Ru_{X/\Sigma *}\mathcal{F} \longrightarrow R\varprojlim_{e \geq r} Ru_{X/\Sigma_e *}(i_e^*\mathcal{F}).$$

In particular, by applying the derived global section functor, we get

$$R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F}) \cong R\varprojlim_{e \geq r} R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, i_e^*\mathcal{F}).$$

(ii) Let $\{Y_e\}_{e \geq r}$ be a direct system of rigid spaces over Σ_e, such that each Y_e is smooth over Σ_e with $Y_{e+1} \times_{\Sigma_{e+1}} \Sigma_e \cong Y_e$. Assume X admits a closed immersions into Y_r. Then we have natural isomorphisms of complexes of sheaves of $\mathrm{B}_{\mathrm{dR}}^+$-modules as below

$$Ru_{X/\Sigma *}\mathcal{F} \longrightarrow \mathcal{F}_D \otimes \Omega_D^\bullet \cong R\varprojlim_{e \geq r}(\mathcal{F}_{D_X(Y_e)} \otimes \Omega^\bullet_{D_X(Y_e)}),$$

where $D = \varinjlim_{e \geq r} D_X(Y_e)$ is the colimit of envelopes, and $\mathcal{F}_D \otimes \Omega_D^\bullet$ is the de Rham complex of $\mathcal{F}$ over D.

(iii) Suppose $\mathcal{F}$ is a crystal in vector bundles over X/Σ_{inf}. For each $e \geq r$, the natural maps

below are isomorphisms

$$\begin{aligned}(Ru_{X/\Sigma *}\mathcal{F}) \otimes^{L}_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}} &\longrightarrow Ru_{X/\Sigma_e *}(i_e^*\mathcal{F});\\ Ru_{X/\Sigma *}\mathcal{F} &\longrightarrow R\varprojlim_{e} \left((Ru_{X/\Sigma *}\mathcal{F}) \otimes^{L}_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}\right).\end{aligned}$$

In particular, when X is quasi-compact quasi-separated, by applying the derived global section functor we obtain the following canonical equivalences

$$\begin{aligned}R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F}) \otimes^{L}_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}} &\cong R\Gamma(X/\Sigma_{e\,\mathrm{inf}}, i_e^*\mathcal{F});\\ R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F}) &\cong R\varprojlim_{e} \left(R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{F}) \otimes_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}\right).\end{aligned}$$

Before we prove, we want to remark that the result for crystals over the big site X/Σ_{INF} are true and the proof is identical to the small site case.

Proof.

(i) This follows from applying $Ru_{X/\Sigma *}$ to the equivalences $\mathcal{F} \longrightarrow R\varprojlim_{e\geq r} Ri_{e*}i_e^*\mathcal{F}$ in Lemma 6.2.0.2. Here we use the identity of maps of topoi in the last section

$$u_{X/\Sigma} \circ i_e = u_{X/\Sigma_e}.$$

(ii) For each $e \geq r$, by Theorem 3.3.2.2 there exists a natural isomorphism of complexes of sheaves of $\mathrm{B}^+_{\mathrm{dR,e}}$-modules

$$Ru_{X/\Sigma_e *}i_e^*\mathcal{F} \longrightarrow \mathcal{F}_{D_X(Y_e)} \otimes \Omega^\bullet_{D_X(Y_e)}.$$

So the map of ringed sites $X/\Sigma_{e\,\mathrm{inf}} \to X/\Sigma_{e+1\,\mathrm{inf}}$ induced from the closed immersion of the bases $\Sigma_e \to \Sigma_{e+1}$ together with (i) produces the inverse limits

$$\begin{aligned}Ru_{X/\Sigma *}\mathcal{F} &\cong R\varprojlim_{e\geq r}(\mathcal{F}_{D_X(Y_e)} \otimes \Omega^\bullet_{D_X(Y_e)})\\ &\cong \mathcal{F}_D \otimes \Omega^\bullet_D,\end{aligned}$$

where we use the compatibility of the de Rham complexes $\mathcal{F}_{D_X(Y_e)} \otimes \Omega^\bullet_{D_X(Y_e)}$ for different e, by our choices of the direct system of smooth rigid spaces $\{Y_e\}_e$.

(iii) We first notice that the second half of the statement follows from the sheaf version, by the following equality

$$R\Gamma(U, (Ru_{X/\Sigma *}\mathcal{F}) \otimes^L_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}) \cong R\Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{F}) \otimes_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}.$$

Here the equivalence follows from applying $R\Gamma(U, -)$ to the distinguished triangle resolving $\mathrm{B}^+_{\mathrm{dR,e}}$ over $\mathrm{B}^+_{\mathrm{dR}}$ as below

$$Ru_{X/\Sigma *}\mathcal{F} \xrightarrow{\cdot \xi^e} Ru_{X/\Sigma *}\mathcal{F} \longrightarrow (Ru_{X/\Sigma *}\mathcal{F}) \otimes^L_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{B}^+_{\mathrm{dR,e}}.$$

Moreover the above equality shows that the sheaf version isomorphism is local with respect to X, so it suffices to assume that X admits a closed immersion into a direct system of smooth rigid spaces $\{Y_e\}_e$ over Σ_e, where the results follow from the item (ii) and Theorem 3.3.2.2. So we are done.

□

Remark 6.2.0.4. Recall that for a smooth affinoid rigid space $X = \mathrm{Spa}(R)$ over K, the crystalline cohomology of X over $\mathrm{B}^+_{\mathrm{dR}}$, introduced in [BMS18, Section 13], is defined as the inverse limit

$$\varprojlim_{e \in \mathbb{N}} \Omega^\bullet_{D_X(Y_e)},$$

where $X \to Y_e = \mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR,e}}\langle T_i^{\pm 1}\rangle)$ is a closed immersion. So Theorem 6.2.0.3 implies that the infinitesimal cohomology $R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})$ coincides with the crystalline cohomology of X over $\mathrm{B}^+_{\mathrm{dR}}$ in the sense of [BMS18].

With the help of Theorem 6.2.0.3, we can compare the infinitesimal cohomology of X over $\mathrm{B}^+_{\mathrm{dR}}$ with the derived de Rham complex.

Definition 6.2.0.5. *Let X be a rigid space over Σ_r. Then the* analytic derived de Rham complex of X over $\mathrm{B}^+_{\mathrm{dR}}$, *denoted as $\widehat{\underline{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma}$, is defined to be the derived inverse limit*

$$\widehat{\underline{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma} := R\varprojlim_{e \geq r} \widehat{\underline{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma_e},$$

where $\widehat{\underline{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma_e}$ is the underlying complex of the filtered complex dR_{X/Σ_e}.

Apply Theorem 6.2.0.3 (i) to the infinitesimal structure sheaf $\mathcal{O}_{X/\Sigma}$ and the comparison in Theorem 5.1.2.3, we get the following.

Corollary 6.2.0.6. *Let X be a rigid space over Σ_r. There exists a natural isomorphism between the analytic derived de Rham complex and the infinitesimal cohomology sheaves*

$$\underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma} \longrightarrow Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma}.$$

In particular, applying the derived global section, we get the following comparison of cohomology

$$R\Gamma(X, \underline{\widehat{\mathrm{dR}}}^{\mathrm{an}}_{X/\Sigma}) \cong R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma}).$$

The next result concerns the éh-descent for cohomology of crystals over the big infinitesimal site X/Σ_{INF}, where X is a rigid space over K.

Proposition 6.2.0.7. *Let X be a rigid space over K, and let $\mathcal{F}$ be a crystal in vector bundles over the big infiniteismal site X/Σ_{INF}. Then the cohomology sheaf $Ru_{X/\Sigma *}\mathcal{F}$ satisfies the éh-hyperdescent. Namely for an éh-hypercovering $X'_\bullet \to X'$ of K-rigid spaces over X, the following natural map is an isomorphism*

$$R\Gamma(X'/\Sigma_{\mathrm{INF}}, \mathcal{F}) \longrightarrow R\lim_{[n]\in\Delta^{\mathrm{op}}} \left(R\Gamma(X'_n/\Sigma_{\mathrm{INF}}, \mathcal{F})\right).$$

Proof. By Lemma 6.2.0.2, (i), the pullback $i_e^*\mathcal{F}$ over $X/\Sigma_{e\mathrm{INF}}$ is a crystal in vector bundles. Thanks to Theorem 5.2.2.5, we know the natural map $X'_\bullet \to X'$ induces a natural isomorphism as below

$$R\Gamma(X'/\Sigma_{\mathrm{INF}}, i_e^*\mathcal{F}) \longrightarrow R\lim_{[n]\in\Delta^{\mathrm{op}}} \left(R\Gamma(X'_n/\Sigma_{\mathrm{INF}}, i_e^*\mathcal{F})\right).$$

Thus the result we want follows from taking the derived limit over all e, by Theorem 6.2.0.3 (i). $\square$

We want to mention that thanks to the Corollary 3.1.2.8, it is safe to replace the cohomology of $\mathcal{F}$ over the big infinitesimal site by the cohomology $R\Gamma(X'/\Sigma_{\mathrm{inf}}, \iota^{-1}\mathcal{F})$ of the restriction $\iota^{-1}\mathcal{F}$ over the small infinitesimal site X/Σ_{inf}. In particular, by applying the above result to the infinitesimal structure sheaf $\mathcal{O}_{X/\Sigma}$, we see the infinitesimal cohomology over $\mathrm{B}^+_{\mathrm{dR}}$ satisfies the éh-hyperdescent.

Corollary 6.2.0.8. *Let X be a rigid space over K. Then the infinitesimal cohomology*

$$R\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})$$

satisfies the éh-hyperdescent.

Another quick upshot is the finiteness of infinitesimal cohomology for a proper rigid space X.

Proposition 6.2.0.9. *Let X be a proper rigid space of dimension n over K, and let $\mathcal{F}$ be a coherent crystal. The infinitesimal cohomology $R\Gamma(X/\Sigma_{\mathrm{INF}}, \mathcal{F})$ is then a perfect $\mathrm{B}_{\mathrm{dR}}^+$-complex in cohomological degrees $[0, 2n]$.*

Proof. Thanks to Theorem 6.2.0.3 (i), we can write $R\Gamma(X/\Sigma_{\mathrm{INF}}, \mathcal{F})$ as the derived limit of $R\Gamma(X\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})$. Here each $R\Gamma(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})$ is a bounded complex in cohomological degree $[0, 2n]$ such that each cohomology group is finite over $\mathrm{B}_{\mathrm{dR,e}}^+$ (Proposition 5.2.3.1). So the result then follows from the short exact sequence

$$0 \longrightarrow R^1 \varprojlim_e \mathrm{H}^{i-1}(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F}) \longrightarrow \mathrm{H}^i(X/\Sigma_{\mathrm{INF}}, \mathcal{F}) \longrightarrow \varprojlim_e \mathrm{H}^i(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F}) \longrightarrow 0.$$

Here we note that the inverse system $\{\mathrm{H}^{2n}(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})\}_e$ satisfies the Mittag-Leffler condition, by the finiteness of each $\mathrm{H}^{2n}(X/\Sigma_{e\mathrm{INF}}, i_e^*\mathcal{F})$ over $\mathrm{B}_{\mathrm{dR,e}}^+$. $\square$

6.3 Comparison with pro-étale cohomology

In this section, we compare the infinitesimal cohomology of X/Σ_{inf} with the pro-étale cohomology of the de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}$. As an application, we show the degeneracy of the Hodge–de Rham spectral sequence, together with a torsionfreeness of infinitesimal cohomology $\mathrm{H}^i(X/\Sigma, \mathcal{O}_{X/\Sigma_{\mathrm{inf}}})$ over $\mathrm{B}_{\mathrm{dR}}^+$.

Throughout the section, we will assume the basics of the pro-étale topology defined in [Sch13a].

6.3.1 Comparison theorem

Let X be a rigid space over K, and let $X_{\mathrm{proét}}$ be the pro-étale site of X. The pro-étale site admits a basis, which consists of affinoid adic spaces $U = \mathrm{Spa}(B, B^+)$ that are pro-étale over X and are *affinoid perfectoid* (namely, the Huber pair (B, B^+) is a perfectoid algebra over K). Over the pro-étale site, we can associate the complete structure sheaf $\widehat{\mathcal{O}}_X$, whose section at an affinoid perfectoid space $U = \mathrm{Spa}(B, B^+)$ is the K-algebra B. Denote $\nu : X_{\mathrm{proét}} \to X_{\mathrm{rig}}$ to be the canonical morphism from the pro-étale site to the rigid site of X.

We recall from [Sch13a] that the *de Rham period sheaf* $\mathbb{B}_{\mathrm{dR}}^+$, defined as a sheaf of $\mathrm{B}_{\mathrm{dR}}^+$-algebras over $X_{\mathrm{proét}}$, sending an affinoid perfectoid space $U = \mathrm{Spa}(B, B^+)$ onto the ring

$$\mathbb{B}_{\mathrm{dR}}^+(B, B^+) := \varprojlim_m \left(W(\varprojlim_{x \mapsto x^p} B^+/p)[\frac{1}{p}]/\xi^m \right).$$

The sheaf $\mathbb{B}_{\mathrm{dR}}^+$ admits a canonical surjection $\theta : \mathbb{B}_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}}_X$ that are compatible with the surjection map $\theta : \mathrm{B}_{\mathrm{dR}}^+ \to K$ for the period ring $\mathrm{B}_{\mathrm{dR}}^+$. It can be shown that ξ is a nonzero-divisor in $\mathbb{B}_{\mathrm{dR}}^+$, and the ideal $\ker(\theta) \subset \mathbb{B}_{\mathrm{dR}}^+$ is generated by $\xi \in \mathrm{B}_{\mathrm{dR}}^+$. So we could invert the element ξ to get a sheaf of $\mathrm{B}_{\mathrm{dR}} = \mathrm{B}_{\mathrm{dR}}^+[\frac{1}{\xi}]$-algebras over $X_{\mathrm{pro\acute{e}t}}$, which we denote by $\mathbb{B}_{\mathrm{dR}}$. The sheaf of rings $\mathbb{B}_{\mathrm{dR}}$ then admits a natural descending filtration defined by $\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}} := \xi^i\mathbb{B}_{\mathrm{dR}}^+$, where each graded piece $\mathrm{gr}^i\mathbb{B}_{\mathrm{dR}}$, which is equal to $\widehat{\mathcal{O}}_X \cdot \xi^i$, is canonically isomorphic to the pro-étale structure sheaf.

We first recall the comparison between the infinitesimal cohomology and the pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}$ for smooth rigid spaces.

Theorem 6.3.1.1 ([BMS18], Theorem 13.1)**.** *Let X be a smooth rigid space over K. Then there exists a natural map of complexes of sheaves of $\mathrm{B}_{\mathrm{dR}}^+$-modules over X*

$$Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma} \longrightarrow R\nu_*\mathbb{B}_{\mathrm{dR}}^+.$$

It is an isomorphism after inverting ξ.

Proof. This is essentially proved in the [BMS18], Theorem 13.1, and we explain here the relation of their result with our statement.

Let X be a smooth rigid space over K of dimension d. Assume $U = \mathrm{Spa}(R)$ is a *very small* affinoid open subset in X; namely it admits an étale morphism onto a torus $\mathbb{T}_K^d$, where the map can be extended to a closed immersion into a larger torus $\mathbb{T}^n = \mathrm{Spa}(K\langle T_i^{\pm 1}\rangle)$. For any such closed immersion, we could associate the torus $\mathbb{T}^n$ an affinoid perfectoid space $\mathbb{T}^{n,\infty} = \mathrm{Spa}(K\langle T_i^{\pm\frac{1}{p^\infty}}\rangle)$. The canonical map $\mathbb{T}^{n,\infty} \to \mathbb{T}^n$ is pro-étale, and its pullback along $U \to \mathbb{T}^n$ produces a pro-étale morphism from an affinoid perfectoid space $\mathrm{Spa}(R_\infty, R_\infty^+)$ over $U = \mathrm{Spa}(R)$.

We denote by D to be the envelope of U inside of the direct system $\{\mathbb{T}^n_{\mathrm{B}_{\mathrm{dR,e}}^+}\}_e$ of tori over $\{\mathrm{B}_{\mathrm{dR,e}}^+\}_e$. Then for any such choice of morphisms $(U \to \mathbb{T}^d \to \mathbb{T}^n)$, we could construct two $\mathrm{B}_{\mathrm{dR}}^+$-linear complexes

- The de Rham complex $\Omega_D^\bullet$ of U in $\{\mathbb{T}^n_{\mathrm{B}_{\mathrm{dR,e}}^+}\}_e$, that computes the infinitesimal cohomology $R\Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})$ by Theorem 6.2.0.3.

- The Koszul complex $K_{\mathbb{B}_{\mathrm{dR}}^+(R_\infty)} = K_{\mathbb{B}_{\mathrm{dR}}^+(R_\infty)}(\gamma_{u_i}-1)$, that computes the pro-étale cohomology $R\Gamma(U_{\mathrm{pro\acute{e}t}}, \mathbb{B}_{\mathrm{dR}}^+)$.

As in the proof of the [BMS18, Theorem 13.1], for any choice of $(U \to \mathbb{T}^d \to \mathbb{T}^n)$, there exists a natural map of actual complexes

$$\Omega_D^\bullet \longrightarrow K_{\mathbb{B}_{\mathrm{dR}}^+(R_\infty)},$$

which is functorial with respect to the choices of triples, such that it becomes an isomorphism after inverting ξ. Notice that the set of triples for a fixed U is filtered, and the transition map of both complexes are isomorphisms. In this way, the induced isomorphism

$$R\Gamma(U/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})[\frac{1}{\xi}] \to R\Gamma(U_{\mathrm{pro\acute{e}t}}, \mathbb{B}_{\mathrm{dR}})$$

is independent of the triples $(U \to \mathbb{T}^d \to \mathbb{T}^n)$. Since the collection of very small open subsets of X form a basis in the rigid topology, we could then get a natural isomorphism as below

$$Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma}[\frac{1}{\xi}] \longrightarrow R\nu_*\mathbb{B}_{\mathrm{dR}}.$$

$\square$

Using the éh-hyperdescent, we could improve the above result into the non-smooth situations.

Theorem 6.3.1.2. *Let X be a rigid space over K. Then there exists a natural map of complexes of sheaves of $\mathrm{B}_{\mathrm{dR}}^+$-modules over X as below*

$$Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma} \longrightarrow R\nu_*\mathbb{B}_{\mathrm{dR}}^+.$$

It is an isomorphism after inverting ξ.

Proof. In the proof, we use $\nu_X : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{rig}}$ to denote the natural map of sites associated with the rigid space X.

We first notice that the pro-étale cohomology sheaf $R\nu_{X*}\mathbb{B}_{\mathrm{dR}}^+$ and $R\nu_{X*}\mathbb{B}_{\mathrm{dR}}$ satisfy the éh-hyperdescent. To see this, we recall from [Guo19, Section 4] that the derived push-forward $R\nu_{X*}\widehat{\mathcal{O}}_X$ is naturally isomorphic to $R\pi_{X*}C$, where $C = R\alpha_*\widehat{\mathcal{O}}_v \in D^{\geq 0}(X_{\mathrm{\acute{e}h}})$ is the derived push-forward of the completed v-structure sheaf (see [Guo19, Section 3.2]), and $\pi_X : X_{\mathrm{\acute{e}h}} \to X_{\mathrm{rig}}$ is the natural map of sites. As an upshot, since C is a bounded below complex of éh-sheaves, its direct image $R\pi_{X*}C$ in the rigid site naturally satisfies the éh-hyperdescent; namely for an éh-hypercovering $\rho : X_\bullet \to X$ over K, the induced map below is an isomorphism

$$R\pi_{X*}C \longrightarrow R\rho_*R\pi_{X_\bullet *}C.$$

We could then replace the above by the derived push-forward of the pro-étale structure sheaf to get a natural isomorphism

$$R\nu_{X*}\widehat{\mathcal{O}}_X \longrightarrow R\rho_*R\nu_{X_\bullet *}\widehat{\mathcal{O}}_{X_\bullet}.$$

On the other hand, notice the de Rham period sheaf $\mathbb{B}_{\mathrm{dR}}^+$ is completed under the ξ-adic topology such that the i-th graded piece is equal to the complete structure sheaf $\widehat{\mathcal{O}}_X \cdot \xi^i$ up to a twist. In this way, by the hyperdescent for graded pieces and the induction on e, we get

$$\begin{aligned} R\nu_{X*}\mathbb{B}_{\mathrm{dR}}^+ &= R\varprojlim_{e\in\mathbb{N}} R\nu_{X*}\mathbb{B}_{\mathrm{dR}}^+/\xi^e \\ &\cong R\varprojlim_{e\in\mathbb{N}} R\rho_* R\nu_{X_\bullet *}\mathbb{B}_{\mathrm{dR}}^+/\xi^e \\ &\cong R\rho_* R\varprojlim_{e\in\mathbb{N}} R\nu_{X_\bullet *}\mathbb{B}_{\mathrm{dR}}^+/\xi^e \\ &= R\rho_* R\nu_{X_\bullet *}\mathbb{B}_{\mathrm{dR}}^+. \end{aligned}$$

Namely the pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}^+$ hence $\mathbb{B}_{\mathrm{dR}} = \mathbb{B}_{\mathrm{dR}}^+[\frac{1}{\xi}]$ satisfies the éh-hyperdescent.

At last, notice that the collection of maps $f : X' \to X$ for smooth rigid spaces X' form a basis of the éh-site $X_{\text{éh}}$. In this way, the natural comparison map $Ru_{X'/\Sigma *}\mathcal{O}_{X'/\Sigma} \to R\nu_{X'*}\mathbb{B}_{\mathrm{dR}}^+$ for smooth X' extends to a map for X via the éh-hyperdescent (for infinitesimal cohomology sheaf, this is Theorem 6.2.0.7), and by inverting ξ we get the isomorphism

$$Ru_{X/\Sigma *}\mathcal{O}_{X/\Sigma}[\frac{1}{\xi}] \longrightarrow R\nu_*\mathbb{B}_{\mathrm{dR}}.$$

□

Remark 6.3.1.3. The morphism between the infinitesimal cohomology and the pro-étale cohomology is constructed in an indirect way. In fact, by enlarging the infinitesimal site X/Σ_{inf} to a bigger site that allows all (adic spectra of) complete Huber rings as in [Yao19, Construction 5.11], the de Rham period ring $\mathbb{B}_{\mathrm{dR}}^+(R_\infty)$ for a perfectoid algebra R_∞ can be then regarded as a pro-thickening in this enlarged category. In this way, the arrow from the associated ind-object to the final object in the enlarged infinitesimal topos will induce a map on their cohomology, and it can be checked via computations in smooth case and the éh-hyperdescent that this coincides with our morphism.

Consider a special case when X comes from a small subfield below. Precisely, let K_0 be a discretely valued subfield of K such that the residue field of K_0 is perfect. Assume Y is a proper rigid space over K_0, and $X = Y \times_{K_0} K$ is the base field extension of X_0. We recall from [Guo19, Theorem 8.2.2] that there exists a $\mathrm{Gal}(K/K_0)$-equivariant filtered comparison between the pro-étale cohomology $R\Gamma(X_{\text{proét}}, \mathbb{B}_{\mathrm{dR}})$ and the tensor product

$$R\Gamma(Y_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}.$$

Here $\Omega^i_{\text{éh},/K_0}$ is the éh-differential for rigid spaces over K_0, and the filtration is defined by the product filtration, where the éh-de Rham cohomology $R\Gamma(Y_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0})$ is equipped with a natural descending filtration by $\mathrm{Fil}^i = R\Gamma(Y_{\text{éh}}, \Omega^{\geq i}_{\text{éh},/K_0})$. Moreover, by taking the zero-th graded pieces, we get

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) \cong \bigoplus_i R\Gamma(Y_{\text{éh}}, \Omega^i_{\text{éh},/K_0}) \otimes_{K_0} K(-i).$$

From this, we get the following:

Corollary 6.3.1.4. *Let Y be a proper rigid space over the discretely valued subfield K_0 of K as above, and let X be its base extension to K. Then we have a canonical isomorphism*

$$R\Gamma(X/\Sigma_{\text{inf}}, \mathcal{O}_{X/\Sigma})[\frac{1}{\xi}] \cong R\Gamma(Y_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}.$$

In particular, the infinitesimal cohomology of $Y \times_{K_0} K$ over B_{dR} admits a $\mathrm{Gal}(K/K_0)$-equivariant filtration such that the zero-th graded factor is equal to

$$\bigoplus_i R\Gamma(Y_{\text{éh}}, \Omega^i_{\text{éh},/K_0}) \otimes_{K_0} K(-i).$$

6.3.2 Torsionfreeness and Hodge–de Rham degeneracy

For the rest of the section, we prove infinitesimal cohomology $\mathrm{H}^n(X/\Sigma_{\text{inf}}, \mathcal{O}_{X/\Sigma})$ is torsion free over $\mathrm{B}^+_{\mathrm{dR}}$, and show the degeneracy for the éh Hodge–de Rham spectral sequence.

Theorem 6.3.2.1. *Let X be a proper rigid space over K. Then we have the following.*

(i) infinitesimal cohomology $\mathrm{H}^n(X/\Sigma_{\text{inf}}, \mathcal{O}_{X/\Sigma})$ is torsion free over $\mathrm{B}^+_{\mathrm{dR}}$, for each $n \in \mathbb{N}$.

(ii) the éh Hodge–de Rham spectral sequence below degenerates at its first page:

$$E_1^{i,j} = \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{éh}}, \Omega^\bullet_{\text{éh}}).$$

Remark 6.3.2.2. Note that the part (ii) generalizes the degeneracy result in [Guo19, Proposition 8.0.7], where the latter needs the assumption for X to be defined over a discretely valued subfield.

Proof. We first notice that pro-étale cohomology $\mathrm{H}^n(X_{\text{proét}}, \mathbb{B}^+_{\mathrm{dR}})$ is finite free over $\mathrm{B}^+_{\mathrm{dR}}$. Recall the Primitive Comparison Theorem over $\mathrm{B}^+_{\mathrm{dR}}$ as below ([Sch13b])

$$\mathrm{H}^n(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\mathrm{dR}} \cong \mathrm{H}^n(X_{\text{proét}}, \mathbb{B}^+_{\mathrm{dR}}).$$

As the étale cohomology $\mathrm{H}^n(X_{\text{éh}}, \mathbb{Q}_p)$ is a finite dimensional vector space over $\mathbb{Q}_p$, this in particular implies the finite freeness of $\mathrm{H}^n(X_{\text{proét}}, \mathbb{B}^+_{\mathrm{dR}})$ over $\mathrm{B}^+_{\mathrm{dR}}$. In particular, by Theorem 6.3.1.2 and the finiteness in Proposition 6.2.0.9, we get the following relations

$$
\begin{aligned}
\dim_{\mathbb{Q}_p} \mathrm{H}^n(X_{\text{ét}}, \mathbb{Q}_p) &= \operatorname{rank}_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{H}^n(X_{\text{proét}}, \mathbb{B}^+_{\mathrm{dR}}) \\
&= \dim_{\mathrm{B}_{\mathrm{dR}}} \mathrm{H}^n(X_{\text{proét}}, \mathbb{B}_{\mathrm{dR}}) \\
&= \dim_{\mathrm{B}_{\mathrm{dR}}} \mathrm{H}^n(X/\Sigma_{\inf}, \mathcal{O}_{X/\Sigma})[\frac{1}{\xi}] \\
&= \operatorname{rank}_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{H}^n(X/\Sigma_{\inf}, \mathcal{O}_{X/\Sigma})/torsion \\
&\leq \dim_K \mathrm{H}^n(X/\Sigma_{\inf}, \mathcal{O}_{X/\Sigma})/\xi.
\end{aligned}
$$

On the other hand, by the base change formula in Theorem 6.2.0.3, (iii), we get the following short exact sequence of K-vector spaces

$$0 \longrightarrow \mathrm{H}^{n+1}(X/\Sigma_{\inf}, \mathcal{O}_{X/\Sigma})[\xi] \longrightarrow \mathrm{H}^n(X/K_{\inf}, \mathcal{O}_{X/K}) \longrightarrow \mathrm{H}^n(X/\Sigma_{\inf}, \mathcal{O}_{X/\Sigma})/\xi.$$

This implies the inequalities

$$\dim_K \mathrm{H}^n(X/\Sigma_{\inf}, \mathcal{O}_{X/\Sigma})/\xi \leq \dim_K \mathrm{H}^n(X/K_{\inf}, \mathcal{O}_{X/K}).$$

Now using the comparison between infinitesimal cohomology and éh de Rham cohomology for the trivial crystal $\mathcal{F} = \mathcal{O}_{X/K}$ in Theorem 5.2.2.2, we have

$$\dim_K \mathrm{H}^n(X/K_{\inf}, \mathcal{O}_{X/K}) = \dim_K \mathrm{H}^n(X_{\text{éh}}, \Omega^\bullet_{\text{éh}}).$$

Note that the natural Hodge filtration on the éh de Rham complex $\Omega^\bullet_{\text{éh}}$ induces the E_1 spectral sequence

$$E_1^{i,j} = \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{éh}}, \Omega^\bullet_{\text{éh}}).$$

As a consequence, we get

$$\dim_K \mathrm{H}^n(X/K_{\inf}, \mathcal{O}_{X/K}) \leq \sum_{i+j=n} \dim_K \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}).$$

However, by Hodge–Tate decomposition in [Guo19, Theorem 1.1.3], we have

$$\dim_{\mathbb{Q}_p} \mathrm{H}^n(X_{\text{ét}}, \mathbb{Q}_p) = \sum_{i+j=n} \dim_K \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}).$$

Hence combining all of the relations of dimensions above, we see all of the inequalities should be equalities. In this way, the E_1 spectral sequence degenerates at the first page, and for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathrm{H}^{n+1}(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})[\xi] &= 0;\\ \mathrm{H}^{n}(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma})/\xi &= 0.\end{aligned}$$

$\square$

Part II

Pro-étale Cohomology

In this part, we study pro-étale cohomology of a rigid space. Our main goal is to show the *Hodge–Tate decomposition* of p-adic étale cohomology for a proper rigid space over $\mathbb{C}_p$. We will setup the basics about the pro-étale topology and v-topology in Chapter 7, and prove the decomposition theorem in Chapter 8. Results in this part first appeared in [Guo19, Section 3, 4, 7, 9].

CHAPTER 7

Pro-étale topology and v-topology

In this chapter, we recall the basics around the pro-étale topology and v-topology over a given rigid space, in order to build the bridge between the pro-étale topology and the éh-topology. We follow mostly Scholze's foundational work [Sch13a] and [Sch17], together with the Berkeley's lecture notes [SW20] by Scholze and Weinstein.

More precisely, we recall the notion of the *small v-sheaf* in Section 7.1. Here a small v-sheaf is defined over the category of perfectoid spaces in characteristic p, and an important type of examples comes from analytic adic spaces, as in Proposition 7.1.0.4. In Section 7.2, we recall a comparison theorem between the cohomology of the v-strcture sheaves and the pro-étale structure sheaves in Proposition 7.2.0.4, which we will use later in Chapter 8.

7.1 Small v-sheaves

Let Perfd be the category of perfectoid spaces. They are adic spaces that have an open affinoid covering $\{\mathrm{Spa}(A_i, A_i^+), i\}$ such that each A_i is a perfectoid algebra. Since many of our constructions are large, we need to avoid the set-theoretical issue. Following Section 4 in [Sch17], we fix an uncountable cardinality κ with some conditions, and only consider those perfectoid spaces, morphisms, and algebras that are "κ-small". We refer to Scholze's paper for details, and will follow this convention throughout the section.

We first recall the v-topology defined on the category Perfd.

Definition 7.1.0.1 ([Sch17], 8.1)**.** *The* big v-site Perfd_v *is the Grothendieck topology on the category* Perfd*, for which a collection $\{f_i : X_i \to X,\ i \in I\}$ of morphisms is a covering family if for each quasi-compact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasi-compact open $V_i \subset X_i$, such that $|U| = \cup_{i\in J} f(|V_i|)$.*

Here the index category I is assumed to be κ-small.

It is known that the v-site Perfd_v are subcanonical; namely the presheaf represented by any $X \in \mathrm{Perfd}$ is an v-sheaf. Moreover, both integral and rational completed structure sheaves $\widehat{\mathcal{O}}^+ : X \mapsto \widehat{\mathcal{O}}^+_X(X)$ and $\widehat{\mathcal{O}} : X \mapsto \widehat{\mathcal{O}}_X(X)$ are v-sheaves on Perfd ([Sch17], 8.6, 8.7).

We then introduce a special class of v-sheaves that admits a geometric structure, generalizing the perfectoid spaces. Consider the subcategory Perf of the category Perfd consisting of perfectoid spaces of characteristic p. We can equip Perf with the pro-étale topology and the v-topology to get two sites $\mathrm{Perf}_{\text{proét}}$ and Perf_v separately.

Definition 7.1.0.2 ([Sch17], 12.1)**.** *A* small v-sheaf *is a sheaf Y on* Perf_v *such that there is a surjective map of v-sheaves $X \to Y$, where X is a representabile sheaf of some κ-small perfectoid space in characteristic p.*

By the definition and the subcanonicality of the v-topology, any perfectoid space X in characteristic p produces a small v-sheaf.

Here is a non-trivial example.

Example 7.1.0.3 ([SW20], 9.4)**.** Let K be a p-adic extension of $\mathbb{Q}_p$. Then we can produce a presheaf $\mathrm{Spd}(K)$ on Perf, such that for each $Y \in \mathrm{Perf}$, we take

$$\mathrm{Spd}(K)(Y) := \{isomorphism\ classes\ of\ pairs\ (Y^\sharp, \iota : (Y^\sharp)^\flat \to Y)\},$$

where $Y^\sharp$ is a perfectoid space (of characteristic 0) over K, and ι is an isomorphism of perfectoid spaces identifying $Y^\sharp$ as an untilt of Y. It can be showed that $\mathrm{Spd}(K)$ is in fact a small v-sheaf.

By the tilting correspondence, it can be showed that there is an equivalence between the category Perfd_K of perfectoid spaces over K, and the category of perfectoid spaces Y in characteristic p with a structure morphism $Y \to \mathrm{Spd}(K)$ (See [SW20], 9.4.4).

One of the main reasons we introduce small v-sheaves is that it brings both perfectoid spaces and rigid spaces into a single framework. More precisely, we have the following fact:

Proposition 7.1.0.4 ([Sch17],15.5; [SW20], 10.2.3)**.** *Let K be a p-adic extension of $\mathbb{Q}_p$. There is a functor*

$$\{analytic\ adic\ spaces\ over\ \mathrm{Spa}(K)\} \longrightarrow \{small\ v-sheaves\ over\ \mathrm{Spd}(K)\};$$
$$X \longmapsto X^{\diamond},$$

such that when X is a perfectoid space over $\mathrm{Spa}(K)$, the small v-sheaf $X^{\diamond}$ coincides with the representable sheaf for the tilt $X^{\flat}$.

Moreover, the restriction of this functor to the subcategory of seminormal rigid spaces gives a fully faithful embedding:

$$\{seminormal\ rigid\ spaces\ over\ \mathrm{Spa}(K)\} \longrightarrow \{small\ v-sheaves\ over\ \mathrm{Spd}(K)\}.$$

Here we remark that every perfectoid space is seminormal.

We can also define the "topological structure on $X^{\diamond}$": in [Sch17], 10.1, Scholze defines the concept of being open, étale and finite étale for a morphism of pro-étale sheaves over Perfd. In particular, for each small v-sheaf $X^{\diamond}$ coming from an adic space, we can define its small étale site $X^{\diamond}_{\text{ét}}$. Those morphisms between small v-sheaves are compatible with maps of adic spaces, and we have

Proposition 7.1.0.5 ([Sch17], 15.6)**.** *For each $X \in \mathrm{Rig}_K$, the functor $Y \mapsto Y^{\diamond}$ induces an equivalence of small étale sites:*

$$X_{\text{ét}} \cong X^{\diamond}_{\text{ét}},$$

where the site on the left is the small étale site of the rigid space X defined in [Hub96].

This generalizes the tilting correspondence of perfectoid spaces between characteristic 0 and characteristic p.

7.2 Pro-étale and v-topoi over X

In this section, we recall the small pro-étale site and the v-site associated to a given rigid space $X \in \mathrm{Rig}_K$, for K being a p-adic field. Our goal is to produce a topology over X that is large enough to include both pro-étale topology and éh-topology together, and study the relation between their cohomologies.

We start by recalling basic concepts around the topology of small v-sheaves.

First recall that for a perfectoid space X, it is called *quasi-compact* if every open covering admits a finite refinement; and it is called *quasi-separated* if for any pair of quasi-compact perfectoid spaces Y, Z over X, the fiber product $Y \times_X Z$ is also quasi-compact.

The concept of quasi-compactness and quasi-separatedness can be generalized to the pro-étale sheaves and small v-sheaves. A small v-sheaf $\mathcal{F}$ is called *quasi-compact* if for any family of morphisms $f_i : X_i \to \mathcal{F},\ i \in I$ such that $\coprod_{i\in I} X_i \to \mathcal{F}$ is surjective and I is κ-small, it admits a finite subcollection $J \subset I$ such that $\coprod_{j\in J} X_j \to \mathcal{F}$ is surjective. Here X_i are (pro-étale sheaves that are representable by) affinoid perfectoid spaces, The *quasi-separatedness* for small v-sheaves is defined similarly as perfectoid spaces.

We remark that the generalized quasi-compactness and quasi-separatedness are compatible with the definition for perfectoid spaces when a map of small v-sheaves is of the form $X^\diamond \to Y^\diamond$, for $X \to Y$ being a map of analytic adic spaces.

Now we are able to define the two topoi over a given rigid space X.

Definition 7.2.0.1. *Let $X \in \mathrm{Rig}_K$ be a rigid space over the p-adic field K.*

(i) The small pro-étale site *over X, denoted by $X_{\mathrm{pro\acute{e}t}}$, is the Grothendieck topology on the full subcategory of pro-objects in $X_{\mathrm{\acute{e}t}}$ that are pro-étale over X, in the sense of [Sch13a, Section 3]. Its covering families are defined as those jointly surjective pro-étale morphisms $\{f_i : Y_i \to Y,\ i \in I\}$ such that for any quasi-compact open immersion $U \to Y$, there exists a finite subset $J \subset I$ and quasi-compact open $V_j \subset Y_j$ for $j \in J$, satisfying $|U| = \cup_{j\in J} f_j(|V_j|)$. We call its topos the* pro-étale topos *over X, denoted by* $\mathrm{Sh}(X_{\mathrm{pro\acute{e}t}})$.

(ii) The v-site over X *is defined as the site $\mathrm{Perf}_v|_{X^\diamond}$ of perfectoid spaces in characteristic p over $X^\diamond$, with the covering structure given by the v-topology. Namely it is the v-site over the category of pairs $(Y, f : Y \to X^\diamond)$, where Y is a perfectoid space in characteristic p, and $f : Y \to X^\diamond$ is a map of v-sheaves over Perf_v. We call its topos $\mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond})$ the* v-topos over X.

Remark 7.2.0.2. By the [Sta18] Tag 04GY, the v-topos over X is isomorphic to the localization $\mathrm{Sh}(\mathrm{Perf}_v)|_{X^\diamond}$ of v-topos $\mathrm{Sh}(\mathrm{Perf}_v)$ at the small v-sheaf $X^\diamond$.

Remark 7.2.0.3. Given a rigid space X, we can also form the characteristic zero analogue of the v-site $\mathrm{Perfd}_v|_X$, on the category of perfectoid spaces over X (cf. Definition 7.2.0.1 (ii)). The tilting correspondence and the definition of $X^\diamond$ induces a natural equivalence between the v-sites $\mathrm{Perf}_v|_{X^\diamond}$ and $\mathrm{Perfd}_v|_X$, sending an affinoid perfectoid space $Z \to X^\diamond$ onto the associated tilt $Z^\sharp \to X$.

Let $X \in \mathrm{Rig}_K$ be a rigid space. Then there is a natural morphism of topoi $\lambda = (\lambda^{-1}, \lambda_*)$: $\mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}) \to \mathrm{Sh}(X_{\text{proét}})$. The inverse functor λ^{-1} is computed via the functor $(-)^\diamond$ as in Proposition 7.1.0.4. Precisely, when $Y \in X_{\text{proét}}$ is affinoid perfectoid whose associated complete adic space is $\hat{Y}$, the inverse $\lambda^{-1}(Y)$ is the small v-sheaf $\hat{Y}^\diamond$ over $X^\diamond$, representable by the tilt $\hat{Y}^\flat$. As affinoid perfectoid objects form a basis in $X_{\text{proét}}$ ([Sch13a, Proposition 4.8]), this allows us to extend λ^{-1} to the whole category $X_{\text{proét}}$. In particular, by using the Galois descent as in [Sch17, Proposition 15.4], for a rigid space X' that is finite étale over X, we have $\lambda^{-1}(X') = X'^\diamond$. Here we remark that by the loc. cit. the functor λ^{-1} realizes a pro-étale presentation into an actual limit of v-sheaves: when Y is affinoid perfectoid with a pro-étale presentation $\{Y_i\}$ over X, we have $Y^\diamond \cong \varprojlim Y_i^\diamond$.

When $\{Y_i \to Y\}$ is a pro-étale covering of affinoid perfectoid objects over X, the inverse image $\{\hat{Y}_i^\diamond \to \hat{Y}^\diamond\}$ forms a v-covering of representable v-sheaves over $X^\diamond$. For a general pro-étale sheaf $\mathcal{F}$ over $X_{\text{proét}}$, the functor λ^{-1} sends $\mathcal{F}$ to the v-sheaf associated to the presheaf

$$Z \longmapsto \varinjlim_{\substack{Z \to \hat{W}^\diamond \ in\ \mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}),\\ \text{affinoid perfectoid } W \in X_{\text{proét}}}} \mathcal{F}(W).$$

Here we note that when Z is equal to the small v-sheaf $\hat{Y}^\diamond$ for $\hat{Y}$ a perfectoid space underlying a pro-étale object Y over X, the above direct limit is $\mathcal{F}(Y)$. On the other hand, the functor λ_* is the direct image functor, given by

$$\lambda_*\mathcal{G}(Y) = \mathcal{G}(\hat{Y}^\flat), \text{ affinoid perfectoid } Y \in X_{\text{proét}}.$$

We define the *untilted completed structure sheaves* $\widehat{\mathcal{O}}_v$ and $\widehat{\mathcal{O}}_v^+$ on $\mathrm{Perf}_v|_{X^\diamond}$, by sending $Z \to X^\diamond$ to the following

$$\begin{aligned} \widehat{\mathcal{O}}_v(Z) &:= \widehat{\mathcal{O}}(Z^\sharp), \\ \widehat{\mathcal{O}}_v^+(Z) &:= \widehat{\mathcal{O}}^+(Z^\sharp), \end{aligned}$$

where $Z^\sharp$ is the untilt of Z given by the map $Z \to X^\diamond \to \mathrm{Spd}(K)$, as in Proposition 7.1.0.4. By [Sch17, Theorem 8.7], both of them are sheaves on $\mathrm{Perf}_v|_{X^\diamond}$. Here we notice that under the (tilting) equivalence in Remark 7.2.0.3, the sheaves $\widehat{\mathcal{O}}_v$ and $\widehat{\mathcal{O}}_v^+$ are sent to the completed structure sheaves $\widehat{\mathcal{O}}$ and $\widehat{\mathcal{O}}^+$ over $\mathrm{Perfd}_v|_X$ in characteristic zero.

Furthermore, we have the following comparison result on completed pro-étale structure sheaves.

Proposition 7.2.0.4 ((Pro-étale)-v comparison)**.** *The direct image map induces the following canon-*

ical isomorphism of sheaves on $X_{\text{proét}}$:

$$\lambda_*\widehat{\mathcal{O}}_v^+ \longrightarrow \widehat{\mathcal{O}}_X^+.$$

Moreover, for $i > 0$ the sheaf $R^i\lambda_*\widehat{\mathcal{O}}_v^+$ is almost zero.

By inverting p, the similar results hold for $\lambda_*\widehat{\mathcal{O}}_v$ and $R^i\lambda_*\widehat{\mathcal{O}}_v$. In particular, the pro-étale cohomology of $\widehat{\mathcal{O}}_X$ satisfies the v-hyperdescent.

Here we follow the convention of the almost mathematics as in [Sch17, Section 3].

Proof. We first recall that for any quasi-compact analytic adic space Y over K, there exists a pro-étale covering of Y by perfectoid spaces ([Sch17, Lemma 15.3]). In particular, the pro-étale site $X_{\text{proét}}$ admits a basis given by affinoid perfectoid spaces that are pro-étale over X. So it suffices to check the above isomorphism and vanishing condition for $Y \in X_{\text{proét}}$ that are affinoid perfectoid.

The direct image of the untilted integral complete structure sheaf is the pro-étale sheaf associated to

$$Y \longmapsto \Gamma(Y,\ \lambda_*\widehat{\mathcal{O}}_v^+) = \Gamma(\hat{Y}^\diamond,\ \widehat{\mathcal{O}}_v^+),$$

where $Y \in X_{\text{proét}}$ is affinoid perfectoid. But note that since $\hat{Y}^\diamond \cong \hat{Y}^\flat$ is the representable sheaf of an affinoid perfectoid space over $X^\diamond$, by construction of $\widehat{\mathcal{O}}_v^+$ we have

$$\Gamma(\hat{Y}^\diamond, \widehat{\mathcal{O}}_v^+) = \Gamma((\hat{Y}^\flat)^\sharp, \widehat{\mathcal{O}}^+).$$

Here $\hat{Y}$ is the perfectoid space associated to the object $Y \in X_{\text{proét}}$, and $\hat{Y}^\flat$ is the tilt of $\hat{Y}$. So by the isomorphism of perfectoid spaces $(\hat{Y}^\flat)^\sharp \cong \hat{Y}$, we see the $\lambda_*\widehat{\mathcal{O}}_v^+$ is the pro-étale sheaf associated to the presheaf

$$Y \longmapsto \Gamma(Y, \widehat{\mathcal{O}}^+),$$

which is exactly the completed pro-étale structure sheaf over $X_{\text{proét}}$. Thus we get the equality.

For the higher direct image, we first note that $R^i\lambda_*\widehat{\mathcal{O}}_v^+$ is the pro-étale sheaf on $X_{\text{proét}}$ associated to the presheaf

$$Y \longmapsto \mathrm{H}_v^i(\hat{Y}^\flat, \widehat{\mathcal{O}}_v^+)$$

for Y being affinoid perfectoid in $X_{\text{proét}}$. By the construction of $\widehat{\mathcal{O}}_v^+$, the tilting correspondence $\mathrm{Perf}_v|_{X^\diamond} \cong \mathrm{Perfd}_v|_X$ in Remark 7.2.0.3 identifies the sheaf $\widehat{\mathcal{O}}_v^+$ over $\mathrm{Perf}_v|_{X^\diamond}$ with $\widehat{\mathcal{O}}^+$ over $\mathrm{Perfd}_v|_X$. In particular, we have the natural isomorphism of cohomology

$$\mathrm{H}_v^i(\hat{Y}^\flat, \widehat{\mathcal{O}}_v^+) \cong \mathrm{H}_v^i((\hat{Y}^\flat)^\sharp, \widehat{\mathcal{O}}^+) \cong \mathrm{H}_v^i(\hat{Y}, \widehat{\mathcal{O}}^+),$$

which is almost zero by [Sch17, Propositino 8.8] and the assumption on Y. So we are done.

CHAPTER 8

Degeneracy Theorems

In this chapter, we aim to show the degeneracy of the éh-proét spectral sequence

$$R^i\pi_{X*}\Omega^j_{\text{éh}}(-j) \Rightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X$$

and the Hodge–Tate decomposition for rigid spaces.

As a setup, we first introduce the *éh-proét spectral sequence* $R^i\pi_{X*}\Omega^j_{\text{éh}}(-j) \Rightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X$ in Section 8.1. We then recall the relation between the analytic cotangent complex and the liftability of adic spaces in Section 8.2, analogous to the use of the cotangent complex in the deformation theory of schemes. In Section 8.3, by relating the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ with the analytic cotangent complex (Theorem 8.3.1.1), we show for a smooth rigid space X that lifts to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$, the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ splits into the direct sum $\bigoplus_{i\geq 0}\Omega^i_{X/K}(-i)[-i]$ in the derived category (Proposition 8.3.1.3). This implies the degeneracy of the éh-proét spectral sequence for liftable smooth rigid spaces over $\mathbb{C}_p$. As a preparation to the general degeneracy theorem, Section 8.4 is devoted to generalize the first two sections into a (truncated) simplicial diagram of liftablie smooth rigid spaces. With all of the previous ingredients, in Section 8.5 we use the cohomological descent to prove the degeneracy theorem for general proper rigid spaces over $\mathbb{C}_p$ (Theorem 8.5.3.1). Namely for a proper rigid space X over $\mathbb{C}_p$, there is a quasi-isomorphism in the derived category of $\mathcal{O}_X$-modules as below

$$R\nu_*\widehat{\mathcal{O}}_X \longrightarrow \bigoplus_{i=0}^{n} R\pi_{X*}(\Omega^i_{\text{éh}}(-i)[-i]).$$

Here we remark that our degeneracy result can be applied to the more general class of rigid spaces X that are *strongly liftable*, and is compatible with the Galois representation structure on cohomology when X is defined over a discretely valued subfield (see Subsection 8.5.1 for the discussions).

At the end of this chapter, we give two applications of the degeneracy theorem. The first application is a vanishing result on cohomology of éh differential sheaves (Proposition 8.6.0.6), which is the rigid analytic analogue of the vanishing theorem by Guillen-Navarro Aznar-Puerta-Steenbrink for complex varieties. Our proof makes use of the degeneracy together with the *almost* *purity theorem* of the p-adic Hodge theory by the recent advance of Bhatt–Scholze [BS19], while the classical proof uses the mixed Hodge structure in Hodge theory (cf. [PS08, Theorem 7.29]). The second application is the promised Hodge–Tate decomposition for proper rigid spaces, as in Theorem 8.7.0.1.

8.1 éh-proét spectral sequence

In this section, we first connect all of the topologies we defined together, and consider the éh-proét spectral sequence.

Let X be a rigid space over K, for K a complete and algebraically closed p-adic field. We denote by $X_{\text{éh}}$ the localization of the big éh site $\mathrm{Rig}_{K,éh}$ at X. Then the functor $Y \mapsto Y^\diamond$ induces a morphism of topoi

$$\alpha : \mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}) \longrightarrow \mathrm{Sh}(X_{\text{éh}}),$$

where $\alpha^{-1}Y = Y^\diamond$ for $Y \to X$ being a representable sheaf of an adic space. We let $X_{\text{ét}}$ be the small étale site of X, consisting of rigid spaces that are étale over X.

Consider the following commutative diagram of topoi over X

$$\begin{tikzcd}
\mathrm{Sh}(X_{\text{proét}}) \arrow[r, "\nu"] & \mathrm{Sh}(X_{\text{ét}}) \\
\mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}) \arrow[u, "\lambda"] \arrow[r, "\alpha"'] & \mathrm{Sh}(X_{\text{éh}}). \arrow[u, "\pi_X"']
\end{tikzcd}$$

Here we note that the diagram is functorial with respect to X. In particular when $X = X_0 \times_{K_0} K$ is a pullback of X_0 along a non-archimedean field extension K/K_0, the diagram is then equipped with a continuous action of $\mathrm{Aut}(K/K_0)$.

Now by the proét-v comparison (Proposition 7.2.0.4), we have

$$\begin{aligned} R\nu_*\widehat{\mathcal{O}}_X &= R\nu_* R\lambda_* \widehat{\mathcal{O}}_v \\ &= R\pi_* R\alpha_* \widehat{\mathcal{O}}_v. \end{aligned}$$

This induces an (E_2) Leray spectral sequence

$$R^i\pi_* R^j\alpha_*\widehat{\mathcal{O}}_v \Rightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X.$$

We then notice that by the above comparison again, the sheaf $R^j\alpha_*\widehat{\mathcal{O}}_v$ is the éh-sheafification of the presheaf

$$\begin{aligned} X_{\text{éh}} \ni Y \longmapsto & \mathrm{H}^j(Y_v^\diamond, \widehat{\mathcal{O}}_v) \\ =& \mathrm{H}^j(Y_{\text{proét}}, \widehat{\mathcal{O}}_Y). \end{aligned}$$

When Y is smooth, it is in fact the i-th continuous differential:

Theorem 8.1.0.1 ([Sch13b], 3.23)**.** *Let Y be a smooth affinoid rigid space over K. Then we have a* *canonical isomorphism:*

$$\mathrm{H}^j(Y_{\text{proét}}, \widehat{\mathcal{O}}_Y) = \Omega^j_{Y/K}(Y)(-j),$$

where the $\Omega^j_{Y/K}$ *is the sheaf of the j-th continuous differential forms. Here the* "$(-j)$" *means that the cohomology is equipped with an action of the Galois group* $\mathrm{Gal}(K/K_0)$ *by the Tate twist of weight* j*, when* $Y = Y_0 \times_{K_0} K$ *is a base change of a smooth rigid space* Y_0 *over a complete discretely valued field* K_0 *whose residue field is perfect, and* K *is complete and algebraically closed.*

In this way, by the local smoothness of the éh-topology (Proposition 4.1.4.8) and the functoriality, the sheaf $R^j\alpha_*\widehat{\mathcal{O}}_v$ on $X_{\text{éh}}$ is the éh-sheaf associated to

$$smooth\, Y \longmapsto \Omega^j_{Y/K}(Y)(-j),$$

which is exactly the Tate twist of the j-th éh sheaf of differential $\Omega^j_{\text{éh}}(-j)$, introduced in Chapter 4. So substitute this into the spectral sequence above, we get the *éh-proét spectral sequence*

$$R^i\pi_{X*}\Omega^j_{eh}(-j) \Rightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X.$$

8.2 Cotangent complex and liftability

In this section, we first recall basics about the cotangent complex for general adic spaces, together with its use in the deformation theory. We remind the reader that various basic properties of the analytic cotangent complexes for rigid spaces over $\mathrm{B}^+_{\mathrm{dR,e}}$ can be found in Chapter 2.

Let R_0 be a p-adically complete ring; namely there exists a continuous morphism of adic rings $\mathbb{Z}_p \to R_0$ with R_0 being p-adically complete. Recall for a map of complete R_0-algebras $A \to B$ that are p-torsion free, we can define its complete cotangent complex $\widehat{\mathbb{L}}_{B/A}$ as the term-wise p-adic completion of the usual cotangent complex $\mathbb{L}_{B/A}$. Here $\mathbb{L}_{B/A}$ is given by the corresponding complex of the simplicial B-module $\Omega^1_{P_\bullet(B)/A} \otimes_{P_\bullet(B)} B$, where $P_\bullet(B)$ is the standard A-polynomial resolution of B. The image of $\widehat{\mathbb{L}}_{B/A}$ in the derived category of B-modules is the p-adic derived completion of $\mathbb{L}_{B/A}$, which lives in cohomological degrees ≤ 0 such that

$$\mathrm{H}^0(\widehat{\mathbb{L}}_{B/A}) = \widehat{\Omega}^1_{B/A},$$

where $\widehat{\Omega}^1_{B/A}$ is the continuous differential of B over A and is defined as the p-adic completion of the algebraic Kähler differential $\Omega^1_{B/A}$. We note that the construction of the complex $\widehat{\mathbb{L}}_{B/A}$ is functorial with respect to complete R_0-algebras $A \to B$. So when $\mathcal{X} \to \mathrm{Spf}(R_0)$ is an R_0-formal scheme that is p-torsion free, we can construct a complex of presheaves, which assigns the complex $\widehat{\mathbb{L}}_{B/R_0}$ to an affinoid open subset $\mathrm{Spf}(B)$ in $\mathcal{X}$. The *complete cotangent complex* $\widehat{\mathbb{L}}_{\mathcal{X}/R_0}$ for a p-torsion free

R_0-formal scheme $\mathcal{X}$ is the actual complex of sheaves defined by sheafifying the above complex of presheaves term-wisely.

Now following the construction as in [GR03, Section 7.2], for a map of p-adic affinoid Huber pairs $(A, A^+) \to (B, B^+)$, we define its *analytic cotangent complex* $\mathbb{L}^{\mathrm{an}}_{(B,B^+)/(A,A^+)}$ as the colimit

$$\operatorname*{colim}_{\substack{A_0 \to B_0 \\ A_0, B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/A_0}[\frac{1}{p}],$$

where the colimit is indexed over the set of all maps of rings of definition $A_0 \to B_0$ in $A^+ \to B^+$, and $\widehat{\mathbb{L}}_{B_0/A_0}$ is the complete cotangent complex for a map of p-complete rings as above. We often use the notation $\widehat{\mathbb{L}}^{\mathrm{an}}_{B/A}$ instead of $\mathbb{L}^{\mathrm{an}}_{(B,B^+)/(A,A^+)}$ to simplify the notation, when the choice of the rings A^+ and B^+ is clear from the context. The construction is functorial with respect to the pair $(A, A^+) \to (B, B^+)$, and we can sheafify it to define the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ for a map of adic spaces $X \to Y$. Here the complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is a complex of sheaves of $\mathcal{O}_X$-modules that lives in non-positive cohomological degrees, such that

$$\mathrm{H}^0(\mathbb{L}^{\mathrm{an}}_{X/Y}) = \Omega^1_{X/Y},$$

with the latter $\Omega^1_{X/Y}$ is the continuous differential for the map of rigid spaces $X \to Y$.

Remark 8.2.0.1. In many cases where the base ring is fixed, the colimit in the construction above can be simplified.

For example, let (R, R^+) be either a reduced topologically finite type algebra over a p-adic field, or $(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}], \mathrm{A}_{\mathrm{inf}})$, and let R_0 be the fixed ring of definition R^+ (this is guaranteed by the reducedness of A, and the boundedness of R° by for example [BGR84, 6.2.4/1]) or $\mathrm{A}_{\mathrm{inf}}$ separately. Then for an affinoid R-algebra (B, B^+), we have the following natural quasi-isomorphism

$$\operatorname*{colim}_{\substack{R_0 \to B_0 \\ B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/R_0}[\frac{1}{p}] \longrightarrow \widehat{\mathbb{L}}^{\mathrm{an}}_{B/R},$$

where the colimit ranges only among rings of definition of (B, B^+). This is because in both cases the ring R_0 is the largest ring of definition, so the index systems of colimits are cofinal to the one in the original definition.

Moreover, if in addition the integral subring B^+ of the Huber pair (B, B^+) is bounded, then the above colimit can be further simplified into one single complex by the following quasi-isomorphism

$$\widehat{\mathbb{L}}_{B^+/R_0}[\frac{1}{p}] \longrightarrow \widehat{\mathbb{L}}^{\mathrm{an}}_{B/R},$$

which follows from the same reason about the index system.

Remark 8.2.0.2. The construction of the analytic cotangent complexes here are slightly different from the one used in [GR03] and Section 2.2.2 the colimit in the definition of $\mathbb{L}^{\mathrm{an}}_{(B,B^+)/(A,A^+)}$ above is over the set of *all* rings of definitions, while the ones in loc. cit. are over the set of *topologically of finite type* rings of definitions. The reason we include all rings of definition here is to extend the construction to perfectoid algebras, which are almost never topologically of finite type.

To see those two constructions of analytic cotangent complexes for topologically finite type algebras A over $\mathrm{B}^+_{\mathrm{dR}}/\xi^e$ coincide, it suffices to notice that any ring of definitions A_0 of A is contained in a ring of definition A_1 that is topologically finite type over $\mathrm{A}_{\mathrm{inf}}/\xi^e$. When A is reduced (hence is topologically of finite type over K), the subring of power-bounded elements A° is the largest ring of definition which is topologically of finite type over $\mathcal{O}_K$ (apply [BGR84, 6.4.1/5] at a surjection $K\langle T_i\rangle \to A$).

For the general case when A is not necessarily reduced, this can be seen as follows: Let A_0 be the given ring of definition, I_0 be the nilpotent radical of A_0, and A_1 be a ring of definition that is topologically of finite type over $\mathrm{A}_{\mathrm{inf}}/\xi^e$ whose quotient by its nilpotent radical I_1 is $(A_{\mathrm{red}})^\circ$. Here we note that by the p-torsion-freeness of A_1/I_1 and [BMS18, Lemma 13.4, (iii, b)], the ideal I_1 is finitely generated, and we can assume the ideal I_1 is generated by a finite set of elements g_j, $1 \leq j \leq m$. Moreover, the subring A_0 of A is contained in the union of open subrings $\bigcup_{n\in\mathbb{N}} A_1[\frac{1}{p^n}I_1]$, as the latter is the preimage of $(A_{\mathrm{red}})^\circ$ in A along the surjection $A \to A_{\mathrm{red}}$, and $(A_0)_{\mathrm{red}} \subseteq (A_{\mathrm{red}})^\circ$ by the last subsection for reduced rings. By assumption the subring A_0 is bounded, so we could choose an integer n large enough such that $A_0 \subset A_1[\frac{1}{p^n}I_1]$. Therefore the claim follows as the ring of definition $A_1[\frac{1}{p^n}I_1]$ admits a surjection from $\mathrm{A}_{\mathrm{inf}}/\xi^e\langle T_i, S_j\rangle$, where the map extends a surjection $\mathrm{A}_{\mathrm{inf}}/\xi^e\langle T_1, \ldots, T_l\rangle \to A_1$ and sends S_j onto $\frac{1}{p^n}g_j$. Here we remind the reader that the construction makes sense as each $\frac{1}{p^n}g_j$ is nilpotent and in particular is topologically nilpotent.

8.2.1 Lifting obstruction

One of the most important applications of the cotangent complex is the deformation problem.

Let (R, R^+) be a p-adically complete Huber pair over $\mathbb{Q}_p$. Assume I is a closed ideal in R^+. We define S as the adic space $\mathrm{Spa}(R/I, \widetilde{R^+/I})$, and S' as the adic space $\mathrm{Spa}(R/I^2, \widetilde{R^+/I^2})$, where $\widetilde{R^+/I}$ and $\widetilde{R/I^2}$ are integral closures. Let X be a flat S-adic space. Then a *deformation* of X along $S \to S'$ is defined as a closed immersion $i : X \to X'$ of S'-adic spaces with X' being flat over S, such that the defining ideal is i^*I. Namely we have the following cartesian diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X' \arrow[d] \\ S \arrow[r] & S'. \end{tikzcd}$$

We now focus on the case where the coefficient (R, R^+) is specified as below. Assume K is a complete and algebraically closed p-adic field, and let X be a quasi-compact rigid space over $\mathrm{Spa}(K)$. Recall that the ring $\mathrm{A}_{\mathrm{inf}}$ is defined as the ring of the Witt vectors $W(\varprojlim_{x\mapsto x^p} \mathcal{O}_K)$. There is a canonical surjective continuous map $\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$, with kernel being a principal ideal $\ker(\theta) = (\xi)$ for some fixed element $\xi \in \mathrm{A}_{\mathrm{inf}}$. We then recall that the de Rham period ring $\mathrm{B}_{\mathrm{dR}}^+$ is defined to be the ξ-adic completion of $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$. Here we abuse the notation and denote by $\theta : \mathrm{B}_{\mathrm{dR}}^+ \to K$ the canonical surjection, which is induced from $\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$ as above. Note that for $n \geq 1$, we have $\mathrm{B}_{\mathrm{dR}}^+/(\xi)^n = \mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/(\xi)^n$, which is a complete Tate ring over $\mathbb{Q}_p$ ([Hub96, Section 1.1]). In particular the deformation of any rigid space X/K along $(\mathrm{B}_{\mathrm{dR}}^+/\xi^n, \mathrm{A}_{\mathrm{inf}}/\xi^n) \to (K, \mathcal{O}_K)$ is the same as the deformation along $(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^n, \mathrm{A}_{\mathrm{inf}}/\xi^n) \to (K, \mathcal{O}_K)$.

We then note that the deformation theory along $(\mathrm{B}_{\mathrm{dR}}^+, \mathrm{A}_{\mathrm{inf}}) \to (K, \mathcal{O}_K)$ only depends on the p-adic topology. Precisely, we have the following observation:

Lemma 8.2.1.1. *Let X be a topologically of finite type, p-torsion free formal scheme over $\mathrm{A}_{\mathrm{inf}}/\xi^N$ for some $N \in \mathbb{N}$. Let X_n be the base change of X along $\mathrm{A}_{\mathrm{inf}} \to \mathrm{A}_{\mathrm{inf}}/\xi^{n+1}$. Then we have the following quasi-isomorphism*

$$\widehat{\mathbb{L}}_{X/\mathrm{A}_{\mathrm{inf}}} \longrightarrow R\varprojlim_n \widehat{\mathbb{L}}_{X_n/(\mathrm{A}_{\mathrm{inf}}/\xi^{n+1})},$$

where $\widehat{\mathbb{L}}$ is denoted by the p-adic complete cotangent complex given at the beginning of this section.

Proof. We may assume X is affinoid. Let T_n be the p-adic formal scheme $\mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}/\xi^{n+1})$, and let T be the p-adic formal scheme $\mathrm{Spf}(\mathrm{A}_{\mathrm{inf}})$. Consider the following sequence of p-adic formal schemes

$$X_n \longrightarrow T_n \longrightarrow T.$$

Then by taking the distinguished triangle of transitivity for usual cotangent complexes, we get

$$\mathbb{L}_{T_n/T} \otimes^L_{\mathcal{O}_{T_n}} \mathcal{O}_{X_n} \longrightarrow \mathbb{L}_{X_n/T} \longrightarrow \mathbb{L}_{X_n/T_n}. \tag{$*_n$}$$

Here we note that the triangle remains distinguished in $D(\mathcal{O}_{X_n})$ after the derived p-adic completion.

We then take the derived inverse limit (with respect to n) of the p-adic derived completion of $(*_n)$. When $n \geq N$, we have $\widehat{\mathbb{L}}_{X_n/T} = \widehat{\mathbb{L}}_{X/T}$. Besides, since X_n is the base change of X along

$T_n \to T$, as complexes we have the equality $\widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_{T_n}} \mathcal{O}_{X_n} = \widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_T} \mathcal{O}_X$. So by taking the inverse limit with respect to n, we get

$$R \varprojlim_n (\widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_T} \mathcal{O}_X) \longrightarrow \widehat{\mathbb{L}}_{X/T} \longrightarrow R \varprojlim_n \widehat{\mathbb{L}}_{X_n/T_n}.$$

But note that the inverse system $\{\widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_T} \mathcal{O}_X\}_n$ is in fact acyclic. This is because the cotangent complex $\widehat{\mathbb{L}}_{T_n/T}$ is isomorphic to $(\xi)^n/(\xi^{2n})[1]$, while the composition of transition maps

$$(\xi)^{2n}/(\xi^{4n})[1] \longrightarrow (\xi)^{2n-1}/(\xi^{4n-2})[1] \longrightarrow \cdots \longrightarrow (\xi)^n/(\xi^{2n})[1]$$

is 0. In this way, by the vanishing of its $R\varprojlim_n$, we get the quasi-isomorphism we need. $\square$

This Lemma allows us to forget the complicated natural topology on $\mathrm{B}_{\mathrm{dR}}^+$ when we look at the deformation along $\mathrm{B}_{\mathrm{dR}}^+ \to K$. So throughout the article, we will consider the adic space $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}], \mathrm{A}_{\mathrm{inf}})$ that is only equipped with the p-adic topology, and any cotangent complex that has $\mathrm{A}_{\mathrm{inf}}$ or $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$ as the base will be considered p-adically.

Let S and S' be the adic space $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi)$ and $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^2)$ separately. Here we note that S is also equals to $\mathrm{Spa}(K)$. Denote by i the map $X \to \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}], \mathrm{A}_{\mathrm{inf}})$. We let $\mathcal{O}_X(1)$ be the free $\mathcal{O}_X$ module of rank one, defined by

$$i^*(\xi) = \mathcal{O}_X \otimes_{\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \xi \mathrm{A}_{\mathrm{inf}}[\frac{1}{p}] = \xi/\xi^2 \mathcal{O}_X.$$

When X is defined over a discretely valued subfield, it has the Hodge-Tate Galois action of the weight minus one.

Our first result is about the relation between the deformation of X and the splitting of the cotangent complex.

Proposition 8.2.1.2. *Let X be a rigid space over $S = \mathrm{Spa}(K)$. Then a flat lifting X' of X along $S \to S'$ induces a section s_X of $\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X \to \mathbb{L}^{\mathrm{an}}_{X/S'}$ in the distinguished triangle for the transitivity*

$$\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/S'} \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/S}.$$

Moreover, assume $X' \to Y'$ is a map of flat adic spaces over S', which lifts the map $f : X \to Y$ of rigid spaces over K. Then the induced sections above are functorial, in the sense that the *following natural diagram of sections commute:*

$$\begin{tikzcd}
\mathbb{L}^{\mathrm{an}}_{Y/S'} \arrow[r, "s_Y"] \arrow[d] & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{S'} \mathcal{O}_Y \arrow[d] \\
Rf_*\mathbb{L}^{\mathrm{an}}_{X/S'} \arrow[r, "Rf_*(s_X)"] & Rf_*(\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X).
\end{tikzcd}$$

Proof. The base change diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d] \\
S \arrow[r] & S'
\end{tikzcd}$$

induces the following two sequences of maps

$$X \to S \to S',$$
$$X \to X' \to S'.$$

We take their distinguished triangles of transitivity Corollary 2.2.3.10, and get

$$\begin{tikzcd}
& \mathbb{L}^{\mathrm{an}}_{X/S} & \\
\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \arrow[r] & \mathbb{L}^{\mathrm{an}}_{X/S'} \arrow[r] \arrow[u] & \mathbb{L}^{\mathrm{an}}_{X/X'} \\
& \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X \arrow[u] &
\end{tikzcd}$$

where both the vertical and the horizontal are distinguished.

Following [Sta18], Tag 09D8, we could extend the above to a bigger diagram

$$\begin{tikzcd}
\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathbb{L}^{\mathrm{an}}_{X/S} \arrow[r] & E \\
\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \arrow[r] & \mathbb{L}^{\mathrm{an}}_{X/S'} \arrow[r, "\alpha_X"] \arrow[u] & \mathbb{L}^{\mathrm{an}}_{X/X'} \arrow[u] \\
& \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X \arrow[r, equal] \arrow[u] & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X, \arrow[u, "\beta_X"']
\end{tikzcd} \tag{$*$}$$

where E is the cone of

$$(\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \oplus \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_X) \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/S'},$$

which fits into the diagram such that all of the vertical and horizontal triangles are distinguished.

We then make the following claim.

Claim 8.2.1.3. The cone E is isomorphic to 0 in the derived category.

Proof of the Claim. By construction, since the right vertical triangle above is distinguished, it suffices to show that

$$\beta_X : \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/X'}$$

is a quasi-isomorphism.

We may assume $X = \mathrm{Spa}(B, B^+)$ and $X' = \mathrm{Spa}(A, A^+)$ is affinoid, such that $A \otimes_{\mathrm{A_{inf}}/\xi^2} \mathcal{O}_K = A/\xi = B$. Then by construction, the above map can be rewrite as

$$\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_K B \longrightarrow \operatorname*{colim}_{\substack{A_0 \to B_0 \\ A_0, B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/A_0}[\frac{1}{p}],$$

for $A_0 \to B_0$ being all pairs of the rings of definition of (A, A^+) and (B, B^+) separately.

We then note that for a single pair $A_0 \to B_0$ such that $B_0 \cong A_0/\xi$, the map

$$\rho : \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_K B \longrightarrow \widehat{\mathbb{L}}_{A_0/B_0}[\frac{1}{p}].$$

is a quasi-isomorphism: by the surjectivity of $\mathrm{A_{inf}}/\xi^2 \longrightarrow \mathrm{A_{inf}}/\xi = \mathcal{O}_K$ and $B_0 \to A_0$, applying [GR03, 7.1.31] [1], we have

$$\widehat{\mathbb{L}}_{\mathcal{O}_K/(\mathrm{A_{inf}}/\xi^2)} \cong \mathbb{L}_{\mathcal{O}_K/(\mathrm{A_{inf}}/\xi^2)};$$
$$\widehat{\mathbb{L}}_{A_0/B_0} \cong \mathbb{L}_{A_0/B_0}.$$

So under the choice of A_0 and B_0 the map ρ can be rewritten as a map of usual cotangent complexes

$$\rho : \mathbb{L}_{\mathcal{O}_K/(\mathrm{A_{inf}}/\xi^2)} \otimes_{\mathcal{O}_K} B_0[\frac{1}{p}] \longrightarrow \mathbb{L}_{A_0/B_0}[\frac{1}{p}].$$

But by assumption, $A = A_0[\frac{1}{p}]$ is flat over $\mathrm{A_{inf}}/\xi^2[\frac{1}{p}]$, while $B = B_0[\frac{1}{p}]$ is given by $A \otimes_{\mathrm{A_{inf}}/\xi^2} \mathcal{O}_K$. Hence by the flatness of inverting p and the flat base change of the usual cotangent complexes, we see ρ is a quasi-isomorphism.

At last, we only need to note that the collection of rings of definition $\{A_0 \to B_0\}$ such that $B_0 = A_0/\xi$ is cofinal with the collection of all of the $A_0 \to B_0$ (since any given B_0 is a subring of B that is topologically finite type over $\mathcal{O}_K$, we can pick the generators and lift them to A along the

[1] Though the statement there is for topologically finite type algebras over $\mathcal{O}_K$, the proof works for topologically finite type, p-torsion free algebras over $\mathrm{A_{inf}}/\xi^n$ as well.

surjection $A \to B$). So we get

$$\operatorname*{colim}_{\substack{A_0 \to B_0 \\ A_0, B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/A_0}[\frac{1}{p}] \cong \operatorname*{colim}_{A_0 \text{ open bounded}} \widehat{\mathbb{L}}_{(A_0/\xi)/A_0}[\frac{1}{p}],$$

and the latter is quasi-isomorphic to $\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_K B$. So we are done. $\square$

In this way, since E is constructed so that the top horizontal and the right vertical triangles in $(*)$ are distinguished, we see under the assumption, E is quasi-isomorphic to 0. This allows us to get the section

$$s_X : \mathbb{L}^{\mathrm{an}}_{X/S'} \longrightarrow \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X,$$

defined as the composition of the α_X and the β_X^{-1} in $(*)$.

At last, we check the functoriality. Consider the map between two lifts

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ S' \arrow[r, equal] & S'. \end{tikzcd}$$

Then since each term in the big diagram $(*)$ is functorial with respect to $X \to X'$, the map of lifts induces a commutative diagram from the $(*)$ for Y to the derived direct image of $(*)$ for X along $f : X \to Y$. In particular, this implies the commutativity of the following:

$$\begin{tikzcd} \mathbb{L}^{\mathrm{an}}_{Y/S'} \arrow[r, "\alpha_Y"] \arrow[d] & \mathbb{L}^{\mathrm{an}}_{Y/Y'} \arrow[r, "\beta_Y^{-1}"] \arrow[d] & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes \mathcal{O}_Y \arrow[d] \\ Rf_*\mathbb{L}^{\mathrm{an}}_{X/S'} \arrow[r, "Rf_*(\alpha_X)"] & Rf_*\mathbb{L}^{\mathrm{an}}_{X/X'} \arrow[r, "Rf_*(\beta_X^{-1})"] & Rf_*(\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes \mathcal{O}_X). \end{tikzcd}$$

So by combining them, we get the map from s_Y to $Rf_*(s_X)$.

$\square$

At last, we note the following relation between $\mathbb{L}^{\mathrm{an}}_{X/S'}$ and $\mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}}$.

Lemma 8.2.1.4. *Let X be a smooth rigid space over $\mathrm{Spa}(K)$, and $\mathbf{S} = \mathrm{Spa}(\mathrm{A_{inf}}[\frac{1}{p}], \mathrm{A_{inf}})$ be the p-adic complete adic space. Then the sequence of maps $X \to S' \to \mathbf{S}$ induces the quasi-isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{X/\mathrm{A_{inf}}[\frac{1}{p}]} \longrightarrow \tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{X/(\mathrm{A_{inf}}[\frac{1}{p}]/\xi^2)} = \tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{X/S'}.$$

This is functorial with respect to X.

Proof. By taking the distinguished triangle for the transitivity, we get

$$\mathbb{L}^{\mathrm{an}}_{S'/\mathbf{S}} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_X \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}} \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/S'}. \tag{$*$}$$

Since $S' = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^2)$ is the closed subspace of $\mathbf{S} = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})$ that is defined by the regular ideal (ξ^2), we have

$$\mathbb{L}^{\mathrm{an}}_{S'/\mathbf{S}} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_X = (\xi^2)/(\xi^4) \otimes^L_{\mathcal{O}_{S'}} \mathcal{O}_X[1].$$

But note that by the distinguished triangle for $X \to S \to \mathbf{S}$, we have

$$\mathbb{L}^{\mathrm{an}}_{S/\mathbf{S}} \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}} \longrightarrow \mathbb{L}^{\mathrm{an}}_{X/S},$$

where $\mathbb{L}^{\mathrm{an}}_{S/\mathbf{S}} \otimes_{\mathcal{O}_S} \mathcal{O}_X = (\xi)/(\xi^2) \otimes_K \mathcal{O}_X[1] = \xi/\xi^2\mathcal{O}_X[1]$, and $\mathbb{L}^{\mathrm{an}}_{X/S} = \Omega^1_{X/K}[0]$ by the smoothness assumption ([GR03, Theorem 7.2.42]). In this way, since $\mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}}$ lives in cohomological degree -1 and 0 and is killed by ξ^2, the image of $\mathbb{L}^{\mathrm{an}}_{S'/\mathbf{S}} \otimes^L_{\mathcal{O}_{S'}} \mathcal{O}_X = (\xi^2)/(\xi^4) \otimes^L_{\mathcal{O}_{S'}} \mathcal{O}_X[1]$ in $\mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}}$ is 0. Hence the sequence $(*)$ induces the quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{X/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \longrightarrow \tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{X/(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^2)},$$

that lives in degree -1 and 0.

At last, note that since those two distinguished triangles are functorial with respect to X, so is the quasi-isomorphism. ☐

8.3 Degeneracy in the smooth setting

After the basics around the cotangent complex and the lifting criterion, we are going to show the degeneracy theorem for smooth rigid spaces, assuming the liftable condition to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. We fix a complete and algebraically closed p-adic field K as before.

We first prove a simple result about the cotangent complex over the $\mathrm{A}_{\mathrm{inf}}$.

Proposition 8.3.0.1. *Let A be an $\mathrm{A}_{\mathrm{inf}}$-algebra. Then the following natural map of complete cotangent complexes is a quasi-isomorphism*

$$\widehat{\mathbb{L}}_{A/\mathbb{Z}_p} \longrightarrow \widehat{\mathbb{L}}_{A/\mathrm{A}_{\mathrm{inf}}}.$$

Proof. Consider the sequence of maps

$$\mathbb{Z}_p \longrightarrow \mathrm{A}_{\mathrm{inf}} \longrightarrow A.$$

By basic properties of the usual cotangent complex of rings, we get a distinguished triangle in $D^{-}(A)$:

$$\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A \longrightarrow \mathbb{L}_{A/\mathrm{A_{inf}}} \longrightarrow \mathbb{L}_{A/\mathbb{Z}_p}.$$

Apply the derived p-completion, we then get the following distinguished triangle

$$(\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A)^{\wedge} \longrightarrow \widehat{\mathbb{L}}_{A/\mathrm{A_{inf}}} \longrightarrow \widehat{\mathbb{L}}_{A/\mathbb{Z}_p}.$$

By the derived Nakayama's lemma and the equality

$$(\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A)^{\wedge} = (\widehat{\mathbb{L}}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A)^{\wedge},$$

it suffices to prove that $\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p$ (thus $\widehat{\mathbb{L}}_{\mathrm{A_{inf}}/\mathbb{Z}_p}$) is quasi-isomorphic to 0. But note that $\mathrm{A_{inf}} = W(\mathcal{O}_K^\flat)$, where $\mathcal{O}_K^\flat$ is a perfect ring in characteristic p. In this way, since the cotangent complex of a perfect ring is quasi-isomorphic to zero ([BCKW19], Chapter 3, 3.1.6), we get

$$\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p = \mathbb{L}_{\mathcal{O}_K^\flat/\mathbb{F}_p} \cong 0.$$

Hence we obtain the vanishing of the p-adic completion of $\widehat{\mathbb{L}}_{\mathrm{A_{inf}}/\mathbb{Z}_p}$, and so is the quasi-isomorphism we want.

$\square$

Corollary 8.3.0.2. *Let X be an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$, then the sequence of maps $\mathbb{Q}_p \to \mathrm{A_{inf}}[\frac{1}{p}] \to \mathcal{O}_X$ induces a functorial quasi-isomorphism between analytic cotangent complexes*

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathbb{Q}_p} \cong \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathrm{A_{inf}}[\frac{1}{p}]}.$$

8.3.1 Cotangent complex and derived direct image

Now we are able to connect the cotangent complex with the $R\nu_*\widehat{\mathcal{O}}_X$. Our first result is about the truncation of $R\nu_*\widehat{\mathcal{O}}_X$:

Theorem 8.3.1.1. *Let X be a smooth rigid space over $\mathrm{Spa}(K)$. Then there exists a functorial quasi-isomorphism in the derived category of $\mathcal{O}_X$-modules:*

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathrm{A_{inf}}[\frac{1}{p}]}(-1)[-1] \cong \tau^{\leq 1} R\nu_*\widehat{\mathcal{O}}_X,$$

Proof. In order to construct the isomorphism above, we will need the analytic cotangent complex for the complete pro-étale structure sheaf $\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R}$, where (R, R^+) are either $(\mathrm{A_{inf}}[\frac{1}{p}], \mathrm{A_{inf}})$ or $(\mathbb{Q}_p, \mathbb{Z}_p)$.

We will first work at the presheaf level and do the construction for affinoid perfectoid rings, and then show that the cotangent complex is in fact a twist of the complete structure sheaf.

Step 1 (Calculation at affinoid perfectoid)

Denote by X_{ind} the indiscrete site on the category of affinoid perfectoid objects in $X_{\mathrm{pro\acute{e}t}}$. Namely the category X_{ind} is the collection of affinoid perfectoid objects in $X_{\mathrm{pro\acute{e}t}}$, and the topology is the one such that every presheaf on X_{ind} is a sheaf. Then there exists a canonical map of sites $\delta : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{ind}}$. We note that the inverse image functor δ^{-1} is an exact functor on abelian sheaves defined by the sheafification, and we have $L\delta^{-1} = \delta^{-1}$.

Then we can define the completed structure sheaf $\widehat{\mathcal{O}}^+_{\mathrm{ind}}$, such that for $U \in X_{\mathrm{ind}}$ with its underlying perfectoid space $\mathrm{Spa}(A, A^+)$, we have

$$\widehat{\mathcal{O}}^+_{\mathrm{ind}}(U) = A^+, \widehat{\mathcal{O}}_{\mathrm{ind}}(U) = A.$$

Similarly we can define the cotangent complex as

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}^+_X/R^+}(U) = \varinjlim_{\substack{A_0 \subset A^+ \\ A_0 \textit{ open bounded}}} \widehat{\mathbb{L}}_{A_0/R^+},\ \mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}_X/R}(U) = \mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}^+_X/R^+}[\frac{1}{p}].$$

Here the cotangent complex for formal rings (adic rings) are the one introduced at the beginning of the section. We note that by the fact that a perfectoid algebra (A, A^+) is uniform, we know A° is bounded in A ([Sch12], 1.6). In particular, the open subring A^+ of A° is also open bounded, and as complexes we have the equality

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}^+_X/R^+}(U) = \widehat{\mathbb{L}}_{A^+/R^+},\ \mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}_X/R}(U) = \mathbb{L}^{\mathrm{an}}_{A/R}.$$

We also note that by the Proposition 8.3.0.1, the sequence of sheaves $\mathbb{Z}_p \to \mathrm{A}_{\mathrm{inf}} \to \widehat{\mathcal{O}}^+_{\mathrm{ind}}$ induces a quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}_X/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \cong \mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}_X/\mathbb{Q}_p}.$$

Here to check this quasi-isomorphism it suffices to check sections at $U \in X_{\mathrm{ind}}$, since X_{ind} has only trivial coverings.

Moreover, the map of rings $\mathbb{Z}_p \to \mathcal{O}_K \to \widehat{\mathcal{O}}^+_{\mathrm{ind}}$ provides the natural distinguished triangle

$$\widehat{\mathbb{L}}_{\mathcal{O}_K/\mathbb{Z}_p} \widehat{\otimes}_{\mathcal{O}_K} \widehat{\mathcal{O}}^+_{\mathrm{ind}} \to \widehat{\mathbb{L}}_{\mathrm{ind},\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p} \to \widehat{\mathbb{L}}_{\mathrm{ind},\widehat{\mathcal{O}}^+_X/\mathcal{O}_K}.$$

Since the mod p reduction of $\mathcal{O}_K \to \widehat{\mathcal{O}}^+_{\mathrm{ind}}$ is relatively perfect, and $\widehat{\mathbb{L}}_{\mathcal{O}_K/\mathbb{Z}_p}$ is isomorphic to the Breuil-Kisin twist $\mathcal{O}_K\{1\}[1]$ of weight -1, we have the quasi-isomorphism

$$\widehat{\mathbb{L}}_{\mathrm{ind},\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p} \cong \widehat{\mathcal{O}}^+_{\mathrm{ind}}\{1\}[1].$$

So by inverting p, we have

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}_X/\mathbb{Q}_p} \cong \widehat{\mathcal{O}}_{\mathrm{ind}}(1)[1]. \tag{$*$}$$

Here the same is true when we replace $\mathbb{Q}_p$ by $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$.

Step 2 (pro-étale cotangent complex)

Now we go back to the pro-étale topology. As above, let (R, R^+) be either $(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}], \mathrm{A}_{\mathrm{inf}})$ or $(\mathbb{Q}_p, \mathbb{Z}_p)$. We first observe that the definition of (integral) analytic cotangent complex can be extended to the whole pro-étale site $X_{\mathrm{pro\acute{e}t}}$, which is the complex of sheaves given by sheafifying the complex of presheaves that assigns each object U with its underlying perfectoid space $\mathrm{Spa}(A, A^+)$ to

$$\varinjlim_{\substack{A_0 \subset A^+ \\ A_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{A_0/R^+}, \qquad \varinjlim_{\substack{A_0 \subset A^+ \\ A_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{A_0/R^+}[\frac{1}{p}],$$

We denote those two as

$$\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/R^+},\ \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R}.$$

Here we note that the definition is compatible with the one for rigid spaces (see the discussion at the beginning of the Section 8.2). In particular by the functoriality of the construction, the canonical map of ringed sites $\nu : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ induces a natural map

$$\mathbb{L}^{\mathrm{an}}_{X/R} \longrightarrow R\nu_*\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R}.$$

Moreover, as the collection of affinoid perfectoid open subsets form a base for $X_{\mathrm{pro\acute{e}t}}$, the pro-étale cotangent complex is equal to the inverse image of indiscrete cotangent complex along $\delta : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{ind}}$, i.e.

$$\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R} = \delta^{-1}\mathbb{L}^{\mathrm{an}}_{\mathrm{ind},\widehat{\mathcal{O}}_X/R}.$$

Now we take the (derived) inverse image δ^{-1} for the quasi-isomorphism $(*)$ to get the quasi-isomorphism

Step 3 (Comparison)

At last we consider the statement in the Theorem. The map between ringed sites ν :

$(X_{\text{proét}}, \widehat{\mathcal{O}}_X) \to (X, \mathcal{O}_X)$ induces a morphism of cotangent complexes

$$\mathbb{L}^{\text{an}}_{\mathcal{O}_X/\mathrm{A}_{\text{inf}}} \longrightarrow R\nu_* \mathbb{L}^{\text{an}}_{\widehat{\mathcal{O}}_X/\mathrm{A}_{\text{inf}}}.$$

By the Corollary 8.3.0.2 and the natural quasi-isomorphism $\mathbb{L}^{\text{an}}_{\widehat{\mathcal{O}}_X/\mathrm{A}_{\text{inf}}} = \widehat{\mathcal{O}}_X(1)[1]$ in the Step 2, the map above is isomorphic to the following

$$\mathbb{L}^{\text{an}}_{\mathcal{O}_X/\mathbb{Q}_p} \longrightarrow R\nu_* \widehat{\mathcal{O}}_X(1)[1].$$

So in order to show the quasi-isomorphism in the Theorem 8.3.1.1, it suffices to show the quasi-isomorphism of the the map

$$\mathbb{L}^{\text{an}}_{\mathcal{O}_X/\mathbb{Q}_p}(-1)[-1] \longrightarrow \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X. \qquad (***)$$

Now, since the statement is local on X, we may assume X is affinoid, admitting an étale morphism to $\mathbb{T}^n$. Then we note that both sides of the above are invariant under the étale base change: the right side is a complex of étale coherent sheaf, while the base change of the left side is given by the vanishing of the relative cotangent complex for an étale map. [2] So it suffices to show the case when $X = \mathbb{T}^n = \mathrm{Spa}(K\langle T_i^{\pm 1}\rangle, \mathcal{O}_K\langle T_i^{\pm 1}\rangle)$. But notice that the map $(***)$ can be given by inverting p at the sequence

$$\widehat{\mathbb{L}}_{\mathcal{O}_K\langle T_i^{\pm 1}\rangle/\mathbb{Z}_p}\{-1\}[-1] \longrightarrow \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}^+_{\mathbb{T}^n},$$

here $\{-1\}$ is the Breuil-Kisin twist of the weight 1. In this way, by the local computation in [BMS18, 8.15], the map above induces a quasi-isomorphism

$$\widehat{\mathbb{L}}_{\mathcal{O}_K\langle T_i^{\pm 1}\rangle/\mathbb{Z}_p}\{-1\}[-1] \longrightarrow \tau^{\leq 1} L\eta_{\zeta_p - 1} R\nu_* \widehat{\mathcal{O}}^+_{\mathbb{T}^n},$$

[2] This follows from the distinguished triangle $Lf^*\mathbb{L}^{\text{an}}_{\mathbb{T}^n/\mathbb{Q}_p} \to \mathbb{L}^{\text{an}}_{X/\mathbb{Q}_p} \to \mathbb{L}^{\text{an}}_{X/\mathbb{T}^n}$ and the vanishing of $\mathbb{L}^{\text{an}}_{X/\mathbb{T}^n}$ ([GR03, Theorem 7.2.42]), where $f : X \to \mathbb{T}^n$ is an étale morphism. Here we note that as neither X or $\mathbb{T}^n$ is topologically of finite type over $\mathbb{Q}_p$, we cannot apply [GR03] to get the triangle directly. To see the triangle, we first notice that as $X = \mathrm{Spa}(B, B^\circ)$ and $\mathbb{T} = \mathrm{Spa}(A, A^\circ)$ are reduced and topologically of finite type over K, by Remark 8.2.0.1 the analytic cotangent complex can be naturally computed as follows

$$\mathbb{L}^{\text{an}}_{X/\mathbb{Q}_p} = \widehat{\mathbb{L}}_{B^\circ/\mathbb{Z}_p}[\frac{1}{p}],\ \mathbb{L}^{\text{an}}_{\mathbb{T}^n/\mathbb{Q}_p} = \widehat{\mathbb{L}}_{A^\circ/\mathbb{Z}_p}[\frac{1}{p}],\ \mathbb{L}^{\text{an}}_{X/\mathbb{T}^n} = \widehat{\mathbb{L}}_{B^\circ/A^\circ}[\frac{1}{p}].$$

Moreover, the pullback $Lf^*\mathbb{L}^{\text{an}}_{\mathbb{T}^n/\mathbb{Q}_p}$, which is equal to $B \otimes^L_A (\widehat{\mathbb{L}}_{A^\circ/\mathbb{Z}_p}[\frac{1}{p}])$ is naturally isomorphic to $(B^\circ \otimes^L_{A^\circ} \mathbb{L}_{A^\circ/\mathbb{Z}_p})^\wedge[\frac{1}{p}]$. So the distinguished triangle we want can be given by taking the derived p-completion and then inverting p at the distinguished triangle for the algebraic cotangent complex of $\mathbb{Z}_p \to A^\circ \to B^\circ$.

which after inverting p induces the quasi-isomorphism of analytic cotangent complexes

$$\mathbb{L}^{\mathrm{an}}_{\mathbb{T}^n/\mathbb{Q}_p}(-1)[-1] \longrightarrow \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_{\mathbb{T}^n}.$$

Hence we are done.

□

Corollary 8.3.1.2. *Assume X is a smooth rigid space over K that admits a flat lift X' along $\mathrm{B}_{\mathrm{dR}}^+/\xi^2 \to K$. Then the lift X' induces a splitting of $\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X$ into a direct sum of its cohomology sheaves in the derived category.*

Moreover, the splitting is functorial with respect to the lift X'.

Proof. By the above Theorem 8.3.1.1, we have the functorial quasi-isomorphism

$$\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X = \mathbb{L}^{\mathrm{an}}_{X/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}(-1)[-1].$$

Moreover, the Lemma 8.2.1.4 about the truncation provides us with a functorial quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{X/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}(-1)[-1] = (\tau^{\geq -1}(\mathbb{L}^{\mathrm{an}}_{X/(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^2)}))(-1)[-1].$$

At last, by the Proposition 8.2.1.2, the right side splits into the direct sum of its cohomology sheaves if X can be lifted into a flat adic space over $S' = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^2, \mathrm{A}_{\mathrm{inf}}/\xi^2) = \mathrm{Spa}(\mathrm{B}_{\mathrm{dR}}^+/\xi^2, \mathrm{A}_{\mathrm{inf}}/\xi^2)$, such that the splitting in Proposition 8.2.1.2 is functorial with respect to the lift. So we get the result. □

We then notice that the splitting of the derived direct image is in fact true without the truncation.

Proposition 8.3.1.3. *Assume X is a smooth rigid space over K that admits a flat lift X' over $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$. Then the lift X' induces the derived direct image $R\nu_* \widehat{\mathcal{O}}_X$ to split as $\bigoplus_{i\geq 0} \Omega^i_{X/K}(-i)[-i]$ in the derived category.*

Here the isomorphism is functorial with respect to lifts, in the sense that when $f' : X' \to Y'$ is an $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$ morphism between lifts of two smooth rigid spaces $f : X \to Y$ over K, then the induced map $R\nu_* \widehat{\mathcal{O}}_Y \to Rf_* R\nu_* \widehat{\mathcal{O}}_X$ is compatible with the map between the direct sum of differentials.

Proof. By the Corollary 8.3.1.2 above, the given lift to $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$ induces an $\mathcal{O}_X$-linear quasi-isomorphism

$$\mathcal{O}_X[0] \oplus \Omega^1_{X/K}(-1)[-1] \longrightarrow \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X.$$

It is functorial in the sense that if $f' : X' \to Y'$ is $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$-morphism between lifts of a map of two smooth rigid spaces $f : X \to Y$ over K, then the induced map

$$\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_Y \longrightarrow Rf_* \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X$$

is compatible with the section map

$$\begin{tikzcd}
\mathcal{O}_Y[0] \arrow[r] & Rf_*\mathcal{O}_X[0] \\
\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_Y \arrow[u, "s_Y"] \arrow[r] & Rf^* \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X, \arrow[u, "Rf_*(s_X)"']
\end{tikzcd}$$

which are induced by the functoriality in the Proposition 8.2.1.2, Lemma 8.2.1.4, and the Theorem 8.3.1.1.

We compose the decomposition with $\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X \to R\nu_* \widehat{\mathcal{O}}_X$, and get

$$\Omega^1_{X/K}(-1)[-1] \longrightarrow R\nu_* \widehat{\mathcal{O}}_X.$$

Here $R\nu_* \widehat{\mathcal{O}}_X$ is a commutative algebra object in the derived category $D(\mathcal{O}_X)$. Moreover, as in [DI87], the above map can be lifted to a canonical map of commutative algebra objects in the derived category

$$\bigoplus_{i \geq 0} \Omega^i_{X/K}(-i)[-i] \longrightarrow R\nu_* \widehat{\mathcal{O}}_X.$$

This can be constructed as follows: For each $i \geq 1$, the quotient map $(\Omega^1_{X/K})^{\otimes i} \to \Omega^i_{X/K}$ admits a canonical $\mathcal{O}_X$-linear section s_i, by

$$\omega_1 \wedge \cdots \wedge \omega_i \longmapsto \frac{1}{n!} \sum_{\sigma \in S_i} \mathrm{sgn}(\sigma) \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)}.$$

This allows us to give a canonical $\mathcal{O}_X$-linear map from $\Omega^i_{X/K}(-i)[-i]$ to $R\nu_* \widehat{\mathcal{O}}_X$, by the diagram

$$\begin{tikzcd}
(\Omega^1_{X/K}(-1)[-1])^{\otimes^L_{\mathcal{O}_X} i} \arrow[r] & (R\nu_* \widehat{\mathcal{O}}_X)^{\otimes^L_{\mathcal{O}_X} i} \arrow[d] \\
\Omega^i_{X/K}(-i)[-i] \arrow[u, "s_i"] \arrow[r, dotted] & R\nu_* \widehat{\mathcal{O}}_X.
\end{tikzcd}$$

Here the right vertical map is the multiplication induced from that of $\widehat{\mathcal{O}}_X$. We note that since X is smooth over K, the derived tensor product of $\Omega^1_{X/K}$ over $\mathcal{O}_X$ degenerates into the usual tensor product. Moreover, by construction the total map $\oplus_i s_i$ is multiplicative under the wedge products.

Finally, it suffices to show that the isomorphism for the truncation $\tau^{\leq 1}$ can extend to the map above. When X is of dimension one, since $\Omega^i_{X/K}$ is zero for $i \geq 2$, we are done. For the general case, it follows from the functoriality of the section map and the Künneth formula, which is done in [BMS18, 8.14].

□

8.4 Simplicial generalizations

We now generalize results in the past two sections to simplicial cases.

8.4.1 Simplicial sites and cohomology

First we recall briefly the simplicial sites. The general discussion can be found in [Sta18], Chapter 09VI.[3]

Consider a non-augemented simplicial object of sites $\{Y_n\}$. Namely for each nondecreasing map $\phi : [n] \to [l]$ in Δ, where $[n]$ (resp. $[l]$) is the totally ordered set of $n+1$ (resp. $l+1$) elements, there exists a morphism of sites $u_\phi : Y_l \to Y_n$ satisfying the commutativity of diagrams induced from Δ. Then we can define its associated *non-augmented simplicial site* $Y_\bullet$, following the definition of $\mathcal{C}_{total}$ in [Sta18] Tag 09WC. An object of $Y_\bullet$ is defined as an object $U_n \in Y_n$ for some $n \in \mathbb{N}$, and a morphism $(\phi, f) : U_l \to V_n$ is given by a map $\phi : [n] \to [l]$ together with a map of objects $f : U_l \to u_\phi^{-1}(V_n)$ in Y_l. To give a covering of $U \in Y_n$, it means that to specify a collection of $V_i \in Y_n$, such that $\{V_i \to U\}$ is a covering in the site Y_n. It can be checked that the definition satisfies axioms of being a Grothedieck topology. Moreover, by allowing n to include the number -1, we can define the *augmented simplicial site* $Y_\bullet$. Here we remark that unless mentioned specifically, a simplicial site or a simplicial object in our article is always assumed to be non-augmented. Similarly, by replacing Δ by the finite category $\Delta_{\leq m}$ and assume $n \leq m$, we get the definition of the *m-truncated simplicial site* $Y_\bullet$.

From the definition above, in order to give a (pre)sheaf on $Y_\bullet$, it is equivalent to give a collection of (pre)sheaves $\mathcal{F}^n$ on each Y_n together with the data that for any map $\phi : [n] \to [l]$ in the index category, we specify a map of sheaves $\mathcal{F}^n \to u_{\phi *}\mathcal{F}^l$ over Y_n that is compatible with arrows in Δ. This allows us to define the derived category $D(Y_\bullet)$ of abelian sheaves on $Y_\bullet$.

We could also define the concept of simplicial ringed sites, which consists of pairs $(Y_\bullet, \mathcal{O}_{Y_\bullet})$, for $Y_\bullet$ being a simplicial site and $\mathcal{O}_{Y_\bullet}$ being a sheaf of rings on $Y_\bullet$, assuming $u_\phi : (Y_l, \mathcal{O}_{Y_l}) \to (Y_n, \mathcal{O}_{Y_n})$ being maps of ringed sites.

[3]We want to mention that the discussion of the simplicial sites and cohomology in our article might become simpler if we use the language of the infinity categories, as the latter behaves better than the ordinary derived category when we consider a diagram of derived objects.

Remark 8.4.1.1. In the level of the derived category, the category $D(Y_\bullet)$ is not equivalent to the category where objects are given by specifying one in each $D(Y_n)$ together with natural boundary maps, unless we replace derived categories by derived infinity categories and also consider the higher morphisms. This is the main reason why we need to reconstruct many objects in the simplicial level in this section, instead of using the known results for single site or space directly. The essential difference is that an object in the simplicial level has much stronger functoriality than a collection of objects over each individual space.

From the construction above, it is clear that there exists a map of sites $Y_\bullet \to Y_n$. The pushforward functor along this map is restriction functor, sending the collection of sheaves $(\mathcal{F}_l)_l$ to its n-th component $\mathcal{F}_n$ and is exact ([Sta18, Tag 09WG]) for sheaves of abelian groups. Here is a useful Lemma about the vanishing criterion of objects in the derived category of a simplicial site $D^+(Y_\bullet)$.

Lemma 8.4.1.2. *Let K be an object in the derived category $D^+(Y_\bullet)$ of a m-truncated simplicial site $Y_\bullet$ for $m \in \mathbb{N} \cup \{\infty\}$. Then K is acyclic if and only if for each $n \leq m$, the restriction $K|_{Y_n}$ in $D^+(Y_n)$ is acyclic.*

Proof. If K is acyclic, then since the restriction functor is an exact functor, we see $K|_{Y_n}$ is also acyclic.

Conversely, assume $K|_{Y_n}$ is acyclic for each integer $n \leq m$. If K is not acyclic, then by the assumption that K lives in $D^+(Y_\bullet)$, we may assume i is the least integer such that the i-th cohomology sheaf $\mathcal{H}^i(K) \in \mathrm{Ab}(Y_\bullet)$ is nonvanishing. Then by definition, there exists some n such that $\mathcal{H}^i(K)|_{Y_n}$ is nonzero. But again by the exactness of the restriction, we have the equality

$$\mathcal{H}^i(K)|_{Y_n} = \mathcal{H}^i(K|_{Y_n}),$$

where the latter is zero by assumption. So we get a contradiction, and hence K is acyclic. $\square$

As a small upshot, we have

Lemma 8.4.1.3. *Let $\lambda_\bullet : X_\bullet \to Y_\bullet$ be a morphism of two m-truncated simplicial sites for $m \in \mathbb{N} \cup \{\infty\}$, such that for each integer $n \leq m$, the map $\lambda_n : X_n \to Y_n$ is of cohomological descent. Namely the canonical map by the adjoint pair*

$$\mathcal{F} \longrightarrow R\lambda_{n*}\lambda_n^{-1}\mathcal{F}$$

is a quasi-isomorphism for any $\mathcal{F} \in \mathrm{Ab}(Y_n)$. Then $\lambda_\bullet$ is also of cohomological descent, namely for any abelian sheaf $\mathcal{F}^\bullet$ on $Y_\bullet$, the counit map of this adjoint pair is a quasi-isomorphism

$$\mathcal{F}^\bullet \longrightarrow R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet.$$

Proof. Let $\mathcal{C}$ be a cone of the map $\mathcal{F}^\bullet \to R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet$. It suffices to show the vanishing of the cone in the derived category $D(Y_\bullet)$. Then by the exactness of the restriction functor, for any integer $n \leq m$ the image $C|_{Y_n}$ in $D^+(Y_n)$ is also a cone of

$$\mathcal{F}^n \longrightarrow (R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet)|_{Y_n} = R\lambda_{n*}\lambda_n^{-1}\mathcal{F}^n,$$

which vanishes by assumption. Since both $\mathcal{F}^\bullet$ and $R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet$ are lower bounded, the cone $\mathcal{C}$ is also in $D^+(Y_\bullet)$ and we can use the Lemma 8.4.1.2 above. So we get the result.

□

8.4.2 Derived direct image for smooth simplicial spaces

Next, we use simplicial tools above to generalize results of cotangent complexes and derived direct image to their simplicial versions.

Assume $f : X_\bullet \to Y_\bullet$ is a morphism of (m-truncated) simplicial quasi-compact adic spaces over a p-adic Huber pair. Then we could define the *simplicial analytic cotangent complex* $\mathbb{L}^{\mathrm{an}}_{X_\bullet/Y_\bullet}$ as an actual complex of sheaves on the simplicial site $X_\bullet$ such that the n-th term on the adic space X_n is the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_n/Y_n}$, defined as in Section 8.2. In our applications, we will always assume $Y_\bullet$ to be a constant simplicial spaces associated to $T = \mathrm{Spa}(R, R^+)$, for some p-adic Huber pair (R, R^+). We will use the notation $\mathbb{L}^{\mathrm{an}}_{X_\bullet/R}$ or $\mathbb{L}^{\mathrm{an}}_{X_\bullet/T}$ to indicate when the case is constant.

Here we emphasize that as in the definition of the analytic cotangent complex for X_n/Y_n above, the complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/Y_\bullet}$ is actual, namely it is defined in the category of complexes of abelian sheaves on $X_\bullet$, not just an object in the derived category.

Now let $X_\bullet$ be a (m-truncated) simplicial rigid space over $\mathrm{Spa}(K)$. Then we can form the cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}$ over $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$. Moreover we can define the *simplicial differential sheaf* $\Omega^i_{X_\bullet/K}$ on $X_\bullet$ in the way that on each X_n, the component of the sheaf is $\Omega^i_{X_n/K}$.

We first generalize the result about the obstruction of the lifting to the simplicial case:

Proposition 8.4.2.1. *Let $X_\bullet$ be a (m-truncated) smooth quasi-compact simplicial rigid spaces over* $\mathrm{Spa}(K)$. *Then a flat lift $X'_\bullet$ of $X_\bullet$ along $\mathrm{B}^+_{\mathrm{dR}}/\xi^2 \to K$ induces a splitting of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}$ into the direct sum of its cohomological sheaves $\mathcal{O}_{X_\bullet}(1)[1] \bigoplus \Omega^1_{X_\bullet/K}[0]$ in the derived category.*

The quasi-isomorphism is functorial with respect to $X'_\bullet$.

Proof. This is the combination of the Proposition 8.2.1.2 and the Lemma 8.2.1.4. We first notice that the sequence of maps

$$X_\bullet \longrightarrow S'_\bullet = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^2)_\bullet \longrightarrow \mathbf{S}_\bullet = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}])_\bullet$$

induces a map

$$\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \longrightarrow \mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}.$$

But by the proof of the Lemma 8.2.1.4 and the vanishing Lemma 8.4.1.2, the complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}$ lives only in degree -1 and 0, which is isomorphic to the truncation of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}$ at $\tau^{\geq -1}$. So we reduce to consider the splitting of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}$.

Now assume $X_\bullet$ admits a flat lift $X'_\bullet$ over S' The lift leads to the cartesian diagram

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d] & X'_\bullet \arrow[d] \\ S_\bullet \arrow[r] & S'_\bullet, \end{tikzcd}$$

which induces the simplicial version of the diagram $(*)$ as in the proof of the Proposition 8.2.1.2:

$$\begin{tikzcd} \mathbb{L}^{\mathrm{an}}_{X'_\bullet/S'} \otimes_{\mathcal{O}_{X'_\bullet}} \mathcal{O}_{X_\bullet} \arrow[r] \arrow[d, equal] & \mathbb{L}^{\mathrm{an}}_{X_\bullet/S} \arrow[r] & E \\ \mathbb{L}^{\mathrm{an}}_{X'_\bullet/S'} \otimes_{\mathcal{O}_{X'_\bullet}} \mathcal{O}_{X_\bullet} \arrow[r] & \mathbb{L}^{\mathrm{an}}_{X_\bullet/S'} \arrow[u] \arrow[r, "\alpha_\bullet"] & L^{\mathrm{an}}_{X_\bullet/X'_\bullet} \arrow[u] \\ & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet} \arrow[u] \arrow[r, equal] & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet}, \arrow[u, "\beta_\bullet"'] \end{tikzcd}$$

The vanishing of E comes down to the vanishing of $E|_{X_n}$ by the Lemma 8.4.1.2, which is true by assumption and the Proposition 8.2.1.2. So we get a section map $\beta_\bullet^{-1} \circ \alpha_\bullet$, which splits $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}$ into the direct sum of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S}$ and $\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet}$ in the derived category. Note that since $X_\bullet$ is smooth, the cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S}$ is $\Omega^1_{X_\bullet/K}[0]$, while the truncation $\tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet}$ is $\mathcal{O}_{X_\bullet}(1)[1]$. Thus we get the result.

At last, the quasi-isomorphism is functorial with respect to $X'_\bullet$, since the big diagram above is functorial with respect to lifts, as in the proof of the Proposition 8.2.1.2. $\square$

We then try to connect the simplicial version of cotangent complex with the derived direct image of the completed structure sheaves.

Let $X_\bullet$ be a (m-truncated) simplicial quasi-compact rigid spaces over K. Then this induces the following commutative diagram of topoi of simplicial sites, as a simplicial version of the diagram in Section 8.1.

$$\begin{tikzcd} \mathrm{Sh}(X_{\bullet\mathrm{pro\acute{e}t}}) \arrow[r, "\nu_\bullet"] & \mathrm{Sh}(X_{\bullet\mathrm{\acute{e}t}}) \\ \mathrm{Sh}(\mathrm{Perf}_v|_{X_\bullet^\diamond}) \arrow[u, "\lambda_\bullet"] \arrow[r, "\alpha_\bullet"'] & \mathrm{Sh}(X_{\bullet\mathrm{\acute{e}h}}). \arrow[u, "\pi_\bullet"'] \end{tikzcd}$$

We then define the complete pro-étale structure sheaf $\widehat{\mathcal{O}}_{X_\bullet}$ on the pro-étale simplicial site $X_{\bullet\text{proét}}$, by assigning $\widehat{\mathcal{O}}_{X_n}$ on the pro-étale site $X_{n\,\text{proét}}$. Similarly we define the untilted complete structure sheaf $\widehat{\mathcal{O}}_v$ on the site $\mathrm{Perf}_v|_{X_\bullet^\diamond}$. Here we notice the sheaf $\widehat{\mathcal{O}}_v$ satisfies the cohomological descent along the canonical map $\lambda_\bullet : \mathrm{Sh}(\mathrm{Perf}_v|_{X_\bullet^\diamond}) \to \mathrm{Sh}(X_{\bullet\text{proét}})$, by the Lemma 8.4.1.3 and comparison results (Proposition 7.2.0.4). This leads to the equality

$$R\nu_{\bullet*}\widehat{\mathcal{O}}_{X_\bullet} = R\pi_{\bullet*}R\alpha_{\bullet*}\widehat{\mathcal{O}}_v.$$

The restriction of this equality on each X_n is the one in Section 8.1.

Define simplicial éh-differential sheaves $\Omega^i_{\text{éh}\bullet}$ on $X_{\bullet\text{éh}}$ such that on each $X_{n\,\text{éh}}$, the component of the sheaf is $\Omega^i_{\text{éh}}$. It is by the exactness of the restriction functor and the discussion in Section 8.1 that

$$R^j\alpha_{\bullet*}\widehat{\mathcal{O}}_v = \Omega^j_{\bullet\text{éh}}(-j).$$

When X_n is smooth over K for each n, we have

$$R^j\nu_*\widehat{\mathcal{O}}_{X_\bullet} = \Omega^j_{X_\bullet/K}(-j),$$

with

$$R^i\pi_{\bullet*}R^j\alpha_{\bullet*} = \begin{cases} 0, \ i>0; \\ \Omega^j_{X_\bullet/K}(-j), \ i=0. \end{cases}$$

These are consequences of the éh differentials for smooth rigid spaces (Theorem 4.2.1.1).

Proposition 8.4.2.2. *Let $X_\bullet$ be a m-truncated simplicial smooth quasi-compact rigid spaces over* $\mathrm{Spa}(K)$. *Then there exists a canonical quasi-isomorphism*

$$\tau^{\leq 1}R\nu_{\bullet*}\widehat{\mathcal{O}}_{X_\bullet} \cong \mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}(-1)[-1].$$

The quasi-isomorphism is functorial with respect to $X_\bullet$.

Proof. The Proposition is the simplicial version of the Theorem 8.3.1.1.

We first notice that the map of simplicial sites $\nu_\bullet : X_{\text{proét}} \to X_{\bullet\text{ét}}$ induces a map of analytic cotangent complex

$$\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \longrightarrow R\nu_{\bullet*}\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}.$$

Meanwhile, the triple $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}] \to K \to \widehat{\mathcal{O}}_{X_\bullet}$ provides us with a distinguished transitivity triangles

$$\mathbb{L}^{\mathrm{an}}_{K/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \otimes_K \widehat{\mathcal{O}}_{X_\bullet} \longrightarrow \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \longrightarrow \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/K},$$

where the vanishing of $\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/K}$ follows from the Lemma 8.4.1.2 and the proof of the Step 3 in Theorem 8.3.1.1. So by combining the above two, we get the map

$$\Pi : \mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \longrightarrow R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}(1)[1].$$

Then we consider the induced map of the i-th cohomology sheaves $\mathcal{H}^i$. By the exactness of the restriction functor, the restricted map becomes

$$\mathcal{H}^i(\mathbb{L}^{\mathrm{an}}_{X_n/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]}) \longrightarrow \mathcal{H}^i(R\nu_{n*}\widehat{\mathcal{O}}_{X_n}(1)[1]),$$

which is an isomorphism for $i = 0, -1$ by the Theorem 8.3.1.1, and $\mathcal{H}^i(\mathbb{L}^{\mathrm{an}}_{X_n/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]})$ is zero except $i = 0, -1$. So by the vanishing of the cone, Π induces a quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]} \longrightarrow \tau^{\geq -1}(R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}(1)[1])$$

which leads to the result by a twist.

□

Combining the Proposition 8.4.2.2 and the Proposition 8.4.2.1, we get the simplicial version of the splitting for the truncated derived direct image:

Corollary 8.4.2.3. *Assume $X_\bullet$ is a (m-truncated) smooth quasi-compact simplicial rigid space over K, which admits a flat simplicial lift $X'_\bullet$ to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. Then the lift $X'_\bullet$ induces $\tau^{\leq 1}R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}$ to split into the direct sum of its cohomology sheaves $\mathcal{O}_{X_\bullet}[0] \bigoplus \Omega^1_{X_\bullet/K}(-1)[-1]$ in $D(X_\bullet)$.*

Here the splitting is functorial with respect to the lift $X'_\bullet$.

Moreover, similar to the Proposition 8.3.1.3, the splitting can be extended without derived truncations.

Corollary 8.4.2.4. *Assume $X_\bullet$ is a (m-truncated) smooth quasi-compact simplicial rigid space over K that admits a flat lift $X'_\bullet$ to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. Then the lift $X'_\bullet$ induces the derived direct image $R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}$ to split into*

$$\bigoplus_{i\geq 0} \Omega^i_{X_\bullet/K}(-i)[-i]$$

in the derived category, which is also isomorphic to

$$\bigoplus_{i\geq 0} R\pi_{X_\bullet *}(\Omega^i_{\bullet\mathrm{\acute{e}h}}(-i)[-i]).$$

Proof. By the above Corollary, we have a quasi-isomorphism

$$\mathcal{O}_{X_\bullet}[0] \oplus \Omega^1_{X_\bullet/K}(-1)[-1] \longrightarrow \tau^{\leq 1} R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}.$$

Then by composing with $\tau^{\leq 1} R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet} \to R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}$, similar to the proof of the Proposition 8.3.1.3 we may construct the map below

$$\bigoplus_{i\geq 0} \Omega^i_{X_\bullet/K}(-i)[-i] \longrightarrow R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet},$$

whose restriction on each X_n is exactly the quasi-isomorphism in the Proposition 8.3.1.3. Thus by the vanishing of the restriction of the cone, we see the above map is a quasi-isomorphism.

At last, notice that by the smoothness of $X_\bullet$, we get the second direct sum expression. □

8.5 Degeneracy in general

We then generalize the splitting of the derived direct image to the general case, without assuming the smoothness. Our main tools are the cohomological descent and the simplicial generalizations in the last section.

8.5.1 Strong liftability

Before we prove the general degeneracy, we need to introduce a stronger version of the lifting condition, in order to make use of the cohomological descent.

We first give the definition.

Definition 8.5.1.1. *Let X be a quasi-compact rigid space over K. We say X is* strongly liftable *if for each non-negative integer n, there exists an n-truncated augmented simplicial map of adic spaces $X'_{\leq n} \to X'$ over $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$, where X' and each X'_i are flat and topologically of finite type over $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$, such that the pullback along $\mathrm{B}^+_{\mathrm{dR}}/\xi^2 \to K$ induces an n-truncated smooth éh-hypercovering of X over K.*

We call any such augmented $X'_{\leq n} \to X'$ simplicial rigid space (or $X'_{\leq n}$ in short) a strong lift of length *n.*

Example 8.5.1.2. Let $k = \mathcal{O}_K/\mathfrak{m}_K$ be the residue field of $\mathcal{O}_K$, and we fix a section $i : k \to \mathcal{O}_K/p$ for the canonical surjection $\mathcal{O}_K/p \to k$ (whose existence is guaranteed by the formal smoothness of the perfect field k over $\mathbb{F}_p$ ([Sta18, Tag 031Z])). Note that this induces an injection of fields from $W(k)[\frac{1}{p}]$ to K by the universal property of the Witt ring. Let K_0 be a subfield of K that is finite over $W(k)[\frac{1}{p}]$, and let X be a rigid space defined over K_0. We then claim that X is strongly liftable.

To see this, we first notice that as the resolution of singularities holds for rigid spaces over K_0, it suffices to show that any such field K_0 admits a map $K_0 \to \mathrm{B}_{\mathrm{dR}}^+/\xi^2$ compatible with the inclusion $K_0 \to K$ above. Recall the ring $\mathrm{A}_{\mathrm{inf}}$ is defined as $W(\mathcal{O}_{K^\flat})$, where $\mathcal{O}_K^\flat$ is the inverse limit $\varprojlim_{x \mapsto x^p} \mathcal{O}_K/p$. By the construction of $\mathcal{O}_{K^\flat}$ and the functoriality for the inverse limit and for Frobenius maps, the section $i : k \to \mathcal{O}_K/p$ induces a homomorphism $k \to \mathcal{O}_{K^\flat}$, where the latter is a section to the canonical surjection $\mathcal{O}_{K^\flat} \to k$. In this way, thanks to the functoriality of the Witt vector functor, we can lift the section map to $W(k) \to \mathrm{A}_{\mathrm{inf}} = W(\mathcal{O}_{K^\flat})$. As an upshot, we get the following composition

$$W(k)[\frac{1}{p}] \to \mathrm{A}_{\mathrm{inf}}[\frac{1}{p}] \to \mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]/\xi^2 = \mathrm{B}_{\mathrm{dR}}^+/\xi^2,$$

which lifts the map $W(k)[\frac{1}{p}] \to \mathcal{O}_K$ that we started with. At last, note that any finite field extension K_0 of $W(k)[\frac{1}{p}]$ is étale over $W(k)[\frac{1}{p}]$, while $\mathrm{B}_{\mathrm{dR}}^+/\xi^2 \to \mathcal{O}_K$ is an nilpotent extension of $W(k)[\frac{1}{p}]$ algebras. Hence K_0 admits a map to $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$ by the étaleness.

Here we also note that this implies the case when X is defined over a discretely valued subfield $L_0 \subset K$ that is of perfect residue field κ, since any such L_0 is finite over $W(\kappa)[\frac{1}{p}]$, while the latter is contained in $W(k)[\frac{1}{p}]$.

Example 8.5.1.3. Another example is the analytification of a finite type algebraic variety, by the spreading out technique.

Let Y be a finitely presented scheme over K. By [Gro67], 8.9.1, there exists a finitely generated $\mathbb{Q}$-subalgebra A in K together with a finitely presented A-scheme Y_0, such that $Y_0 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) = Y$. As the map $A \to K$ factors through the fraction field of A, we may assume A is a finitely generated field extension of $\mathbb{Q}$ and Y_0 is defined over A. Notice that the transcendental degree of $\mathbb{Q}_p$ over $\mathbb{Q}$ is infinite. So by embedding a transcendental basis of A over $\mathbb{Q}$ into $\mathbb{Q}_p$, we may find a finite extension K_0 of $\mathbb{Q}_p$ such that A can be embedded into K_0. In this way, we reduce the case to Example 8.5.1.2, as Y^{an} can be defined over a discrete valued subfield K_0 of K that has a perfect residue field.

By the upcoming work of the spreading out of rigid spaces by Conrad-Gabber [CG], it turns out that X is strongly liftable if it is a proper rigid space over K.

Proposition 8.5.1.4. *Let X be a proper rigid space over K. Then it is strongly liftable.*

Proof. We follow the proof for the spreading out technique for rigid spaces by Bhatt-Morrow-Scholze in [BMS18] and study the structure of the deformation ring. However, instead of working on one rigid space, we need to work with a finite diagram of proper rigid spaces. Similar to Example 8.5.1.2, we fix a section $i : k = \mathcal{O}_K/\mathfrak{m}_K \to \mathcal{O}_K/p$ to the canonical surjection $\mathcal{O}_K/p \to k$, which induces an inclusion of p-adic fields $W(k)[\frac{1}{p}] \to K$.

Let n be any non-negative integer. By the resolution of singularity (Theorem 4.1.4.7), we can always construct a (n-truncated) smooth éh-hypercovering $X_{\leq n} \to X$ over K, where each X_i is proper over K ([Con03], section 4). Then it suffices to show that there exists a proper éh hypercovering $X_{\leq n} \to X$, together with a smooth rigid space $\mathcal{S}$ over a subfield $K_0 = W(k)[\frac{1}{p}]$ of K, such that the n-truncated simplicial diagram $X_{\leq n} \to X$ can be lifted to a diagram of proper K_0-rigid spaces $\mathcal{X}_{\leq n} \to \mathcal{X}$ over $\mathcal{S}$. This is because the nilpotent extension $\mathrm{B}_{\mathrm{dR}}^+/\xi^2 \to K$ is K_0 linear, so by the smoothness of $\mathcal{S}$, the map $\mathrm{Spa}(K) \to \mathcal{S}$ can be lifted to a map $\mathrm{Spa}(\mathrm{B}_{\mathrm{dR}}^+/\xi^2) \to \mathcal{S}$. Thus the base change of $\mathcal{X}_{\leq n}$ along this lifting does the job.

Now we prove the statement, imitating the proof of Proposition 13.15 and Corollary 13.16 in [BMS18]. We first deal with the formal lifting over the integral base. Let $W = W(k)$ be the ring of the Witt vector for the residue field of K, and $\mathcal{C}_W$ be the category of artinian, complete local W-rings with the same residue field k. We first make the following claim:

Claim 8.5.1.5. There exists an n-truncated smooth éh-hypercovering $X_{\leq n} \to X$ over K, such that it admits a lift to an n-truncated simplicial diagram of p-adically complete, topologically finite type $\mathcal{O}_K$-formal schemes:

$$X_{\leq n}^+ \to X^+.$$

Proof. Fix an $\mathcal{O}_K$-integral model X^+ of X, whose existence is guaranteed by Raynaud's result on the relation between rigid spaces over K and admissible formal schemes over $\mathcal{O}_K$. We now construct inductively the required covering and the integral lift, following the idea of split hypercoverings (see [Con03] Section 4, or [Sta18] Tag 094J for discussions).

By the local smoothness of the éh-topology (Corollary 4.1.4.8), pick $X_0 \to X$ to be a smooth éh covering that is proper over X. By Raynaud's result, there exists a morphism $X_0^+ \to X^+$ of $\mathcal{O}_K$-formal schemes that lifts the $X_0 \to X$. This is the lift of the face map of the simplicial object at the degree 0.

Assume we already have an n-truncated smooth proper éh-hypercovering $X_{\leq n} \to X$ together with the integral lift $X_{\leq n}^+ \to X^+$ over $\mathcal{O}_K$. Then recall from [Con03, 4.12, 4.14] that in order to extend $X_{\leq n} \to X$ to a smooth proper $n+1$-truncated hypercovering whose n-truncation is the same as $X_{\leq n}$, it is equivalent to find a smooth proper éh covering of rigid spaces

$$N \to (\mathrm{cosk}_n X_{\leq n})_{n+1}.$$

Under the construction, the degree $n+1$-term of the resulting $(n+1)$-truncated hypercovering will be

$$X_{n+1} := N \coprod N',$$

for N' being some finite disjoint union of irreducible components of $X_i (0 \leq i \leq n)$ (which is also smooth and proper over K). Such a smooth proper éh-covering N exists by the local smoothness of $X_{\text{éh}}$. Furthermore, while we form this $n+1$-hypercovering of the rigid spaces, we also want to find the integral lift

$$N^+ \to (\mathrm{cosk}_n X^+_{\leq n})_{n+1}$$

of the morphism $N \to (\mathrm{cosk}_n X_{\leq n})_{n+1}$. To do this, we use [Con03, 4.12] and do the same formal construction for N^+ and $X^+_{\leq n} \to X^+$ as above, and extends the latter to an $n+1$-truncated simplical formal schemes

$$X^+_{\leq n+1} \to X^+,$$

where $X^+_{n+1} = N^+ \coprod N'^+$ is an $\mathcal{O}_K$-model of X_{n+1}. In this way, the generic fiber of $X^+_{\leq n+1} \to X^+$ is a $(n+1)$-simplicial object over X whose n-truncation is $X_{\leq n} \to X$, and whose $(n+1)$-th term is $X_{n+1} = N \coprod N'$, which is in fact a smooth proper éh-covering of $(\mathrm{cosk}_n X_{\leq n})_{n+1}$. Hence by the induction hypothesis we are done.

□

We then fix such an éh-hypercovering $X_{\leq n} \to X$ with its integral model $X^+_{\leq n} \to X^+$ as in the claim. Define the functor of deformations of the special fiber $X^+_{\leq n,k} := X^+_{\leq n} \times_W k$

$$Def : \mathcal{C}_W \longrightarrow \mathrm{Set},$$

which assigns each $R \in \mathcal{C}_W$ to the isomorphism classes of lifts of the digrams $X^+_{\leq n,k} \to X^+_k$ along $R \to k$, such that each lifted rigid space is proper and flat over R. This functor is a deformation functor, and admits a versal deformation: to see this, we first note that as in [Sta18], Tag 0E3U, the functor Def satisfies the Rim-Schlessinger condition ([Sta18], Tag 06J2). Then we made the following claim

Claim 8.5.1.6. The tangent space $TDef := Def(k[\epsilon]/\epsilon^2)$ of the deformation functor is of finite dimension.

Proof of the Claim. Notice that there is a natural (forgetful) functor from Def to the deformation functor of the morphism $Def_{Y_s \to Y_t}$, where Y_s is the disjoint union of all the sources of arrows in the diagram $X^+_{\leq n,k}$, and Y_t is the disjoint union of all of those targets. This induces a map between tangent spaces

$$TDef \longrightarrow TDef_{Y_s \to Y_t}.$$

By the construction, both Y_s and Y_t are finite disjoint unions of proper rigid spaces, which are then proper over k. So from [Sta18], Tag 0E3W, we know the tangent space $TDef_{Y_s \to Y_t}$ is finite dimensional. Furthermore, assume D_1 and D_2 are two lifted diagrams over $k[\epsilon]$. Then the difference of D_1 and D_2 is the collection of k-derivations $\mathcal{O}_{X^+_{t(\alpha),k}} \to \alpha_* \mathcal{O}_{X^+_{s(\alpha),k}}$, satisfying certain k-linear relation so that those arrows in D_1 and D_2 commute. In particular, this consists of a subspace of $Der_k(\mathcal{O}_{Y_t}, u_*\mathcal{O}_{Y_s}) = \mathrm{Hom}_{Y_t}(\Omega^1_{Y_t/k}, u_*\mathcal{O}_{Y_s})$, which by the properness again is finite dimensional. In this way, both the kernel and the target of the map $TDef \to TDef_{Y_s \to Y_t}$ are of finite dimensions, thus so is the $TDef$. $\square$

By the above claim and [Sta18, Tag 06IW], the deformation functor Def admits a versal object. In other words, there exists a complete artinian local W-algebra R with the residue field k, and a diagram $\mathcal{X}_{R,\leq n} \to \mathcal{X}_R$ of proper flat formal R schemes deforming $X^+_{\leq n,k} \to X^+_k$, such that the induced classifying map

$$h_R := \mathrm{Hom}_W(R, -) \longrightarrow Def$$

is formally smooth. Moreover, by the proof of Proposition 13.15 in [BMS18], we can take the ind-completion of $\mathcal{C}_W$ and extend Def to a bigger category, which consists of local zero-dimensional W-algebras with residue field k (not necessary to be noetherian). The category includes $\mathcal{O}_K/p^m$, and since $X^+_{\leq n} \to X^+$ is an $\mathcal{O}_K$-lifting of $X^+_{\leq n,k} \to X^+_k$, we see the diagram can be obtained by the base change of the universal family $\mathcal{X}_{R,\leq n} \to \mathcal{X}_R$ along $R \to \mathcal{O}_K = \varprojlim_m \mathcal{O}_K/p^m$.

At last, we invert p at the diagram

$$\begin{tikzcd} X^+_{\leq n} \arrow[r] \arrow[d] & \mathrm{Spf}(\mathcal{O}_K) \arrow[d] \\ \mathcal{X}_{R,\leq n} \arrow[r] & \mathrm{Spf}(R). \end{tikzcd}$$

The diagram $X_{\leq n} \to X$ then can be obtained from a truncated simplicial diagram $\mathcal{X}_{R,\leq n}[\frac{1}{p}] \to \mathcal{X}[\frac{1}{p}]$ of proper K_0-rigid spaces that are flat over $\mathcal{S} = \mathrm{Spa}(R[\frac{1}{p}], R)$. By shrinking S to a suitable locally closed subset if necessary, we may assume S is smooth over K_0. So we are done.

$\square$

8.5.2 Cohomological desent

Another preparation we need is the hypercovering and the cohomological descent.

Assume we have a non-augmented simplicial site $Y_\bullet$ (truncated or not) and another site S. Let $\{a_n : Y_n \to S\}$ for $n \geq 0$ be a collection of morphisms to S such that it is compatible with face maps and degeneracy maps in $Y_\bullet$. Then we can define the *augmentation* morphism $a : \mathrm{Sh}(Y_\bullet) \to \mathrm{Sh}(S)$ between the topoi of $Y_\bullet$ and S, such that for an abelian sheaf $\mathcal{F}^\bullet$ on $Y_\bullet$, we have

$$a_*\mathcal{F}^\bullet = \ker(a_{0*}\mathcal{F}^0 \rightrightarrows a_{1*}\mathcal{F}^1).$$

It can be checked that the derived direct image Ra_* can be written as the composition

$$Ra_* = \mathbf{s} \circ Ra_{\bullet *},$$

where $a_\bullet : Y_\bullet \to S_\bullet$ is the morphism from $Y_\bullet$ to the constant simplicial site $S_\bullet$ associated to S, and s is the exact functor that takes a simplicial complex to its associated cochain complex of abelian groups. Here we call the augmentation $a = \{Y_n \to S\}$ is of $cohomological\ descent$ if the counit map induced by the adjoint pair (a^{-1}, a_*) is a quasi-isomorphism

$$id \longrightarrow Ra_* a^{-1}.$$

The augmentation allows us to compute the cohomology of sheaves on S by the spectral sequence associated to the simplicial site.

Lemma 8.5.2.1 ([Sta18], 0D7A)**.** *Let $Y_\bullet$ be a simplicial site, or a m-truncated simplicial site for $m \geq 0$, and let $a = \{a_n : Y_n \to S\}$ be an augmentation. Then for $K \in D^+(Y_\bullet)$, there exists a natural spectral sequence*

$$E_1^{p,q} = R^q a_{p*}(K|_{X_p}) \Longrightarrow R^{p+q} a_* K,$$

which is functorial with respect to $Y_\bullet \to S$ and K.

Moreover if we assume the $Y_\bullet$ is non-truncated, the augmentation a is of cohomological descent, and $L \in D^+(S)$, then by applying the spectral sequence to $K = a^{-1}L$ we get a natural spectral sequence

$$E_1^{p,q} = R^q a_{p*} a_p^{-1} L \Longrightarrow \mathcal{H}^{p+q}(L).$$

We need another variant of this Lemma in order to use truncated hypercoverings to approximate the cohomology of S.

Proposition 8.5.2.2. *Let $\rho : Y_{\leq m} \to S$ be a m-truncated simplicial hypercovering of sites for $m \in \mathbb{N}$. Then for any $\mathcal{F} \in \mathrm{Ab}(S)$, the cone for the natural adjunction map*

$$\mathcal{F} \to R\rho_* \rho^{-1} \mathcal{F}$$

lives in the cohomological degree $\geq m - 1$.

Proof. Let $\widetilde{\rho} : \mathrm{cosk}_m Y_{\leq m} \to S$ be the m-th coskeleton of $\rho : Y_{\leq m} \to S$. We use the same symbols $\mathrm{cosk}_m Y_{\leq m}$ and $Y_{\leq m}$ to denote their associated simplicial sites. Then there exists a natural map of sites

$$\iota : \mathrm{cosk}_m Y_{\leq m} \longrightarrow Y_{\leq m}.$$

Those two augmentations induce maps of topoi

$$\widetilde{\rho} : \mathrm{Sh}(\mathrm{cosk}_m Y_{\le m}) \to \mathrm{Sh}(S),\ \rho : \mathrm{Sh}(Y_{\le m}) \to \mathrm{Sh}(S).$$

By construction, as maps of topoi we have

$$\widetilde{\rho} = \iota \circ \rho.$$

So from this, for $\mathcal{F} \in \mathrm{Ab}(S)$, we get the following commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[rr] \arrow[dr] & & R\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F} \\
& R\rho_*\rho^{-1}\mathcal{F} \arrow[ur] &
\end{tikzcd}.$$

Now we let $\mathcal{C}$ be a cone of $\mathcal{F} \to R\rho_*\rho^{-1}\mathcal{F}$, and let $\widetilde{\mathcal{C}}$ be a cone of $\mathcal{F} \to R\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F}$. Then the above diagram induces the following commutative diagram of long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{H}^i(\mathcal{F}) \arrow[r] \arrow[d] & R^i\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{H}^i(\widetilde{\mathcal{C}}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{H}^i(\mathcal{F}) \arrow[r] & R^i\rho_*\rho^{-1}\mathcal{F} \arrow[r] & \mathcal{H}^i(\mathcal{C}) \arrow[r] & \cdots
\end{tikzcd}.$$

By the Lemma 8.5.2.1 above and the commutative diagram, we have a map of E_1 spectral sequences

$$\begin{tikzcd}
R^q\widetilde{\rho}_{p*}\widetilde{\rho}_p^{-1}\mathcal{F} \arrow[r, Rightarrow] \arrow[d] & R^{p+q}\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F} \arrow[d] \\
R^q\rho_{p*}\rho_p^{-1}\mathcal{F} \arrow[r, Rightarrow] & R^{p+q}\rho_*\rho^{-1}\mathcal{F}.
\end{tikzcd}$$

But note that since $\widetilde{\rho}$ is the m-coskeleton of ρ, when $p+q \le m$ the formation $R^q\widetilde{\rho}_{p*}\widetilde{\rho}_p^{-1}$ is the same as $R^q\rho_{p*}\rho_p^{-1}$. So we get the isomorphism

$$R^{p+q}\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F} \cong R^{p+q}\rho_*\rho^{-1}\mathcal{F},\ p+q \le m.$$

Besides, since ρ is a m-truncated hypercovering, by Deligne the augmentation $\widetilde{\rho}$ of the coskeleton satisfies the cohomological descent. So the map $\mathcal{F} \to R\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F}$ is a quasi-isomorphism. In this way, the map $\mathcal{H}^i(\mathcal{F}) \to R^i\rho_*\rho^{-1}\mathcal{F}$ is an isomorphism for $i \le m$, and hence $\mathcal{C}$ lives in $D^{\ge m-1}(S)$.

□

8.5.3 The degeneracy theorem

Now we are able to state and prove our main theorem about the degeneracy.

Theorem 8.5.3.1. *Let X be a quasi-compact, strongly liftable rigid space of dimension n over K, and let the augmented truncated simplicial spaces $X'_{\leq m}$ be a strong lift of X of length $m \geq 2n+2$. Then the strong lift $X'_{\leq m}$ induces a quasi-isomorphism*

$$\Pi_{X'_{\leq m}} : R\nu_*\widehat{\mathcal{O}}_X \longrightarrow \bigoplus_{i=0}^{n} R\pi_{X*}(\Omega^i_{\text{éh}}(-i)[-i]).$$

The quasi-isomorphism $\Pi_{X'_{\leq m}}$ is functorial among strong lifts $X'_{\leq m}$ of rigid spaces of length $m \geq 2n+2$, in the sense that a map of m-truncated strong lifts $X'_{\leq m} \to Y'_{\leq m}$ of $f : X \to Y$ will induce the following commutative diagram in the derived category

$$\begin{tikzcd}
Rf_*R\nu_*\widehat{\mathcal{O}}_X \arrow[r, "Rf_*(\Pi_{X'_{\leq m}})"] & \bigoplus_{i=0}^{\dim(X)} Rf_*R\pi_{X*}(\Omega^i_{\text{éh}}(-i)[-i]) \\
R\nu_*\widehat{\mathcal{O}}_Y \arrow[u] \arrow[r, "\Pi_{Y'_{\leq m}}"'] & \bigoplus_{i=0}^{\dim(Y)} R\pi_{Y*}(\Omega^i_{\text{éh}}(-i)[-i]) \arrow[u]
\end{tikzcd}$$

where the right vertical map is induced by the functoriality of the K ahler differential.

Proof. By assumption, we may assume $X_{\leq m}$ is a m-truncated smooth proper éh-hypercovering of X that admits a lift $X'_{\leq m}$ to a simplicial flat adic spaces over $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. Denote by $\rho : X_\bullet \to X$ the augmentation map. Then $X_{\leq m}$ is also an m-truncated v-hypercovering, and we have a natural map

$$\widehat{\mathcal{O}}_v \longrightarrow R\rho_{v*}\rho^{-1}\widehat{\mathcal{O}}_v \cong R\rho_{v*}\widehat{\mathcal{O}}_{\bullet v},$$

whose cone has cohomological degree $m-1 \geq 2n+1$ by the Proposition 8.5.2.2.

We then apply derived direct image functors, and get a natural map

$$\begin{aligned}
R\nu_*\widehat{\mathcal{O}}_X = R\pi_{X*}R\alpha_*\widehat{\mathcal{O}}_v &\to R\pi_{X*}R\alpha_*R\rho_{v*}\widehat{\mathcal{O}}_{\bullet v} \\
&= R\rho_*R\pi_{\bullet*}R\alpha_{\bullet*}\widehat{\mathcal{O}}_{\bullet v} \\
&= R\rho_*R\nu_{\bullet*}\widehat{\mathcal{O}}_{X_\bullet}.
\end{aligned}$$

Here the cone of the map lives in degree $\geq m-1 \geq 2n+1$.

Moreover, by the Corollary 8.4.2.4, the strong lift $X'_{\leq m}$ induces a functorial (among strong lifts) quasi-isomorphism

$$R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet} \longrightarrow \bigoplus_{i\geq 0} R\pi_{\bullet *}(\Omega^i_{\bullet \text{éh}}(-i)[-i]).$$

So we get the following distinguished triangle

$$R\nu_*\widehat{\mathcal{O}}_X \longrightarrow R\rho_* R\pi_{\bullet *}(\Omega^i_{\bullet \text{éh}}(-i)[-i]) \longrightarrow \mathcal{C}_1, \tag{1}$$

where $\mathcal{C}_1 \in D^{\geq 2n+1}(X)$.

Besides, by the Corollary 8.4.2.4 and the Proposition 8.5.2.2 again the truncated éh-hypercovering ρ induces a natural map

$$\bigoplus_{i\geq 0} R\pi_*(\Omega^i_{\text{éh}}(-i)[-i]) \longrightarrow \bigoplus_{i\geq 0} R\pi_*(R\rho_{\text{éh}*}\rho^{-1}_{\text{éh}}\Omega^i_{\text{éh}}(-i)[-i]) = \bigoplus_{i\geq 0} R\rho_* R\pi_{\bullet *}(\Omega^i_{\bullet \text{éh}}(-i)[-i]), \tag{2}$$

whose cone $\mathcal{C}_2$ lives in degree $\geq m-1 \geq 2n+1$.

At last, by combining (1) and (2), we get the following diagram that is functorial with respect to $X'_{\leq m}$, with both horizontal and vertical being distinguished:

$$\begin{tikzcd}
 & \mathcal{C}_2 & \\
R\nu_*\widehat{\mathcal{O}}_X \arrow[r] & R\rho_* R\pi_{\bullet *}(\Omega^i_{\bullet \text{éh}}(-i)[-i]) \arrow[r] \arrow[u] & \mathcal{C}_1 \\
 & \bigoplus_{i\geq 0} R\pi_*(\Omega^i_{\text{éh}}(-i)[-i]) \arrow[u] &
\end{tikzcd}$$

But note that since $\dim(X) = n$, by the cohomological boundedness (the Corollary 4.3.0.4 and the éh-proét spectral sequence Section 8.1), both $R\nu_*\widehat{\mathcal{O}}_X$ and $\bigoplus_{i\geq 0} R\pi_*(\Omega^i_{\text{éh}}(-i)[-i])$ live in degree $\leq 2n$. Thus by taking the truncation $\tau^{\leq 2n}$, we get the quasi-isomorphism

$$R\nu_*\widehat{\mathcal{O}}_X \longrightarrow \tau^{\leq 2n}(R\rho_* R\pi_{\bullet *}(\Omega^i_{\bullet \text{éh}}(-i)[-i])) \xleftarrow{\sim} \bigoplus_{i\geq 0}^n R\pi_*(\Omega^i_{\text{éh}}(-i)[-i]). \tag{3}$$

In this way, by taking $\Pi_{X'_{\leq m}}$ to be the quasi-isomorphism induced from (3), we are done.

□

Corollary 8.5.3.2. *Assume X is a quasi-compact rigid space over K that is either defined over a discretely valued subfield K_0 of perfect residue field, or proper over K. Then we have a non-*

canonical decomposition

$$R\nu_*\widehat{\mathcal{O}}_X \cong \bigoplus_{i=0}^{\dim(X)} R\pi_{X*}(\Omega^i_{\text{éh}}(-i)[-i]).$$

In particular, the éh-proét spectral sequence in Section 8.1 degenerates at E_2.

8.6 Finiteness revisited

In this section, we use the degeneracy of the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ to improve the cohomological boundedness results in Section 4.3.

We first recall the recent work on the perfection and the almost purity in [BS19].

Theorem 8.6.0.1 ([BS19], Proposition 8.5, Theorem 10.8)**.** *Let A be a perfectoid ring, B a finitely presented finite A-algebra, such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is finite étale over an open subset. Then there exists a perfectoid ring B_{perfd} together with a map of A-algebras $B \to B_{\mathrm{perfd}}$, such that it is initial among all of the A-algebra maps $B \to B'$ for B' being perfectoid.*

Proposition 8.6.0.2. *Let X be a rigid space over K. Then $R^n\nu_*\widehat{\mathcal{O}}_X$ vanishes for $n > \dim(X)$.*

Before we prove the statement, we want to mention that the proof of this Proposition will not need the éh-proét spectral sequence developed above.

Proof. Since this is an étale local statement, and any étale covering of X does not change the dimension, by passing X to its open subsets if necessary, we may assume X admits a finite surjective map onto a torus of the same dimension. [4]

We give them some notations. Denote by $X = \mathrm{Spa}(R, R^+)$ an affinoid rigid space over $\mathrm{Spa}(K)$. Assume there exists a finite surjective map $X \to \mathbb{T}_n = \mathrm{Spa}(K\langle T_i\rangle, \mathcal{O}_K\langle T_i\rangle)$ onto the torus of dimension n. Let $\mathbb{T}_n^\infty$ be the natural pro-étale cover of $\mathbb{T}_n$ by extracting all p^n-th roots of T_i, and let $\hat{\mathbb{T}}_n^\infty = \mathrm{Spa}(K\langle T_i^{\frac{1}{p^\infty}}\rangle, \mathcal{O}_K\langle T_i^{\frac{1}{p^\infty}}\rangle)$ be the underlying affinoid perfectoid space. Then the base change of $\mathbb{T}_n^\infty$ along the map $X \to \mathbb{T}_n$ produces a pro-étale cover $X^\infty \to X$ of X. Note that $\mathbb{T}_n^\infty \to \mathbb{T}_n$ is an $\mathbb{Z}_p(1)^n$-torsor, so we have

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^n, R\Gamma(X^\infty_{\text{proét}}, \widehat{\mathcal{O}}_X)).$$

[4]To see the existence of such surjections, we may argue as follows: as a unit disc is covered by finite many tori, it suffices to find a finite map from an affinoid rigid space $X = \mathrm{Spa}(A, A^+)$ onto a unit disc of the same dimension. Let A_0 be a ring of definition of (A, A^+) that is topologically finite type over $\mathcal{O}_K$. Since $A_0/\mathfrak{m}_K$ is a finite type algebra over the residue field $k = \mathcal{O}_K/\mathfrak{m}_K$, by Noether's normalization lemma we could find a subalgebra $k[x_i]$ of $A_0/\mathfrak{m}_K$ such that $A_0/\mathfrak{m}_K$ is finite over $k[x_i]$. In this way, by lifting the map to a morphism $\mathcal{O}_K\langle x_i\rangle \to A_0$, we get a finite surjective morphism from X to a disc.

Thanks to the (pro-étale)-v comparison (Proposition 7.2.0.4), the above is given by

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^n, R\Gamma_v(\hat{X}^{\infty,\diamond}, \widehat{\mathcal{O}}_v)),$$

where $\hat{X}^{\infty,\diamond}$ is the small v-sheaf associated to the analytic adic space $\hat{X}^{\infty}$ as in the Proposition 7.1.0.4. Here we note that since $\hat{\mathbb{T}}_n^{\infty}, \mathbb{T}_n$, and X are all affinoid, we can write $\hat{X}^{\infty}$ as $\mathrm{Spa}(B[\frac{1}{p}], B)$ for some p-adic complete $\mathcal{O}_K$-algebra B.

We then recall that for a perfectoid space Y of characteristic p with a structure map to the v-sheaf $\mathrm{Spd}(K)$, and any K-analytic adic space Z, we have the following bijection (cf. [SW20] 10.2.4):

$$\mathrm{Hom}_{\mathrm{Spa}(K)}(Y^{\sharp}, Z) = \mathrm{Hom}_{\mathrm{Spd}(K)}(Y, Z^{\diamond}),$$

where $Y^{\sharp}$ is the unique untilt (as a perfectoid space over $\mathrm{Spa}(K)$) of Y associated to the structure map $Y \to \mathrm{Spd}(K)$ (Example 7.1.0.3). The bijection implies that as v-sheaves over the site Perf_v, the small v-sheaf $\hat{X}^{\infty,\diamond}$ associated to the adic space $\hat{X}^{\infty}$ is the pullback of the representable v-sheaf $\hat{\mathbb{T}}_n^{\infty,\flat}$ along the map $X^{\diamond} \to \mathbb{T}_n^{\diamond}$. On the other hand, given a perfectoid space Y over $\mathrm{Spd}(K)$ together with the following commutative map

$$\begin{tikzcd} Y^{\sharp} \arrow[r] \arrow[d] & \hat{\mathbb{T}}_n^{\infty} \arrow[d] \\ X \arrow[r] & \mathbb{T}_n, \end{tikzcd}$$

since $X \to \mathbb{T}_n$ is finite surjective of the same dimensions, the Theorem 8.6.0.1 implies that there exits a unique map of adic spaces $Y^{\sharp} \to X_{\text{perfd}}^{\infty} = \mathrm{Spa}(B_{\text{perfd}}[\frac{1}{p}], B_{\text{perfd}})$ that fits into the commutative diagram:

$$\begin{tikzcd} Y^{\sharp} \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X_{\text{perfd}}^{\infty} \arrow[r] \arrow[d] & \hat{\mathbb{T}}_n^{\infty} \arrow[d] \\ & X \arrow[r] & \mathbb{T}_n. \end{tikzcd}$$

Compare the pullback $\hat{X}^{\infty,\diamond}$ with the universal affinoid perfectoid space $X_{\text{perfd}}^{\infty}$, we see the v-sheaf $\hat{X}^{\infty,\diamond}$ is isomorphic to the representable v-sheaf $X_{\text{perfd}}^{\infty,\flat}$, where the latter is given by the tilt of the perfectoid space $X_{\text{perfd}}^{\infty}$. In particular, we get the equality

$$R\Gamma_v(\hat{X}^{\infty,\diamond}, \widehat{\mathcal{O}}_v) = R\Gamma_v(X_{\text{perfd}}^{\infty,\flat}, \widehat{\mathcal{O}}_v).$$

Since the higher v (pro-étale) cohomology of the completed structure sheaf on affinoid perfectoid space vanishes, by combining equalities above we get

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}) = R\Gamma_{\text{cont}}(\mathbb{Z}_p(1)^n, B_{\text{perfd}}[\frac{1}{p}]).$$

At last we note that the above object lives in the cohomological degree $[0, n]$ in the derived category of abelian groups, for the continuous group cohomology of $\mathbb{Z}_p(1)^n$ can be computed by the Koszul complex of length n ([BMS18] Section 7). Thus we are done. □

Remark 8.6.0.3. Here we want to mention that the cohomological bound given here is stronger than the one from Corollary 4.3.0.4 using the éh-proét spectral sequence.

Remark 8.6.0.4. In the proof above, the continuous group cohomology computing $R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X)$ can be defined concretely as $\left(R\varprojlim_m R\Gamma_{\text{disc}}(\mathbb{Z}^n, R\Gamma(X^{\infty}_{\text{proét}}, \widehat{\mathcal{O}}^+_X/p^m))\right)[\frac{1}{p}]$, where $R\Gamma_{disc}(\mathbb{Z}^n, -)$ denotes the discrete group cohomology of $\mathbb{Z}^n$. This is because as X is affinoid (thus quasi-compact and quasi-separated), we have $R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = \left(R\varprojlim_m R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_X/p^m)\right)[\frac{1}{p}]$. Moreover, to compute each $R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}^+_X/p^m)$ we could use the Čech complex of $\widehat{\mathcal{O}}^+_X/p^m$ for the pro-étale covering $X^{\infty} \to X$. We at last note that as the covering is an $\mathbb{Z}_p(1)^n$-torsor, the Čech complex is equivalent to the discrete group cohomology $R\Gamma_{\text{disc}}(\mathbb{Z}^n, R\Gamma(X^{\infty}_{\text{proét}}, \widehat{\mathcal{O}}^+_X/p^m))$, by the isomorphism in [BMS18, Lemma 7.3] for $\Gamma = \mathbb{Z}_p(1)^n$.

Definition 8.6.0.5. *Let X be a rigid space over K. We say X is* locally compactifiable *if there exists an open covering $\{U_i \to X\}_i$ of X, such that each U_i admits an open immersion into a proper rigid space Y_i over K.*

By definition, any proper rigid space over K is locally compactifiable. Moreover, by Nagata's compactification in algebraic geometry, any finite type scheme over K admits an open immersion in a proper scheme over K. So the analytification of any finite type scheme over K is a locally compactifiable rigid space.

Proposition 8.6.0.6. *Let X be a locally compactifiable rigid space over K. Then the higher direct image $R^i\pi_{X*}\Omega^j_{\text{éh}}$ vanishes when $i + j > \dim(X)$.*

Proof. Since the vanishing of the higher direct image is a local statement, by taking an open covering, it suffices to assume X admits an open immersion $f : X \to X'$ for X' being proper over K. Moreover, by dropping the irreducible components of X' that have higher dimensions, we may assume $\dim(X')$ is the same as $\dim(X)$. This is allowed as the dimension of an irreducible rigid space is not changed when we pass to its open subsets (see the discussion before 2.2.3 in[Con03]).

We then notice that the result is true for X': by the Proposition 8.6.0.2, we know $R^n\nu_{X'*}\widehat{\mathcal{O}}_{X'}$ vanishes for $n > \dim(X')$. On the other hand, by the degeneracy in the Corollary 8.5.3.2, each

$R^i\pi_{X'*}\Omega^j_{\text{éh}}(-j)$ is a direct summand of $R^{i+j}\nu_{X'*}\widehat{\mathcal{O}}_{X'}$. This implies that when $i+j>\dim(X')$, the cohomology sheaf $R^i\pi_{X'*}\Omega^j_{\text{éh}}$ vanishes.

Finally, note that by the coherence proved in Section 4.3, since $R^i\pi_{X*}\Omega^j_{\text{éh}}$ is the sheaf associated to the presheaf $U\mapsto \mathrm{H}^i(U_{\text{éh}},\Omega^j_{\text{éh}})$ for open subsets U inside of X, the preimage of $R^i\pi_{X'*}\Omega^j_{\text{éh}}$ along f is exactly $R^i\pi_{X*}\Omega^j_{\text{éh}}$. In this way, by the equality of dimensions $\dim(X)=\dim(X')$, the vanishing of $R^i\pi_{X'*}\Omega^j_{\text{éh}}$ for $i+j>\dim(X')$ implies the vanishing of $R^i\pi_{X*}\Omega^j_{\text{éh}}$ for $i+j>\dim(X)$. So we get the result. $\square$

8.7 Hodge-Tate decomposition for non-smooth spaces

At last of this chapter, we give the application of our results to the Hodge-Tate decomposition for non-smooth spaces, as mentioned in the introduction. Throughout the section, let X be a proper rigid space over a complete algebraically closed non-archimedean field K over $\mathbb{Q}_p$.

Recall that by the Primitive Comparison ([Sch13b], Theorem 3.17), we have

$$\mathrm{H}^n(X_{\text{ét}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}K=\mathrm{H}^n(X_{\text{proét}},\widehat{\mathcal{O}}_X).$$

The equality enables us to compute the p-adic étale cohomology by studying the pro-étale cohomology. In particular, by taking the associated derived version, the right side above can be obtained by

$$R\Gamma(X_{\text{proét}},\widehat{\mathcal{O}}_X)=R\Gamma(X_{\text{ét}},R\nu_*\widehat{\mathcal{O}}_X).$$

Then we recall the following diagram of topoi associated to X in Section 8.1:

$$\begin{tikzcd}
\mathrm{Sh}(X_{\text{proét}}) \arrow[r, "\nu"] & \mathrm{Sh}(X_{\text{ét}}) \\
\mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}) \arrow[u, "\lambda"] \arrow[r, "\alpha"'] & \mathrm{Sh}(X_{\text{éh}}). \arrow[u, "\pi_X"']
\end{tikzcd}$$

The (pro-étale)-v comparison (see the Proposition 7.2.0.4) allows us to replace $R\nu_*\widehat{\mathcal{O}}_X$ by the derived direct image $R\pi_{X*}R\alpha_*\widehat{\mathcal{O}}_v$ of the untilted complete v-structure sheaf. So we have

$$\begin{aligned}R\Gamma(X_{\text{proét}},\widehat{\mathcal{O}}_X)&=R\Gamma(X_{\text{ét}},R\pi_{X*}R\alpha_*\widehat{\mathcal{O}}_v)\\&=R\Gamma(X_{\text{éh}},R\alpha_*\widehat{\mathcal{O}}_v).\end{aligned}$$

By the discussion in Section 8.1, we have

$$R^j\alpha_*\widehat{\mathcal{O}}_v = \Omega^j_{\text{éh}}(-j).$$

So by replacing the above equality into the Leray spectral sequence for the composition of derived functors, we get

$$E_2^{i,j} = \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{proét}}, \widehat{\mathcal{O}}_X).$$

This together with the Primitive Comparison leads to the *Hodge-Tate spectral sequence* for proper rigid space X

$$E_2^{i,j} = \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K.$$

The name is justified by the special case of the éh differential in the Theorem 4.2.1.1: when X is smooth, the higher direct image of the éh-differential vanishes, and the spectral sequence degenerates into

$$E_2^{i,j} = \mathrm{H}^i(X, \Omega^j_{X/K})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K,$$

with each $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ identified with $\mathrm{H}^i(X, \Omega^j_{X/K})$.

Now by the strong liftability of X (Proposition 8.5.1.4) and the Degeneracy Theorem 8.5.3.1, the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ is non-canonically quasi-isomorphic to the direct sum

$$\bigoplus_{i=0}^{\dim(X)} R\pi_{X*}(\Omega^j_{\text{éh}}(-j)[-j]).$$

Replace the $R\nu_*\widehat{\mathcal{O}}_X$ by this direct sum, we have

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = \bigoplus_{i=0}^{\dim(X)} R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}(-j)[-j]).$$

So after taking the n-th cohomology, we see the Hodge-Tate spectral sequence degenerates at its E_2-page.

Theorem 8.7.0.1 (Hodge-Tate spectral sequence)**.** *Let X be a proper rigid space over a complete algebraically closed non-archimedean field K of characteristic 0. Then there exists a natural E_2 spectral sequence to its p-adic étale cohomology*

$$E_2^{i,j} = \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K.$$

Here the spectral sequence degenerates at its second page, and $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ is a finite dimensional K-vector space that vanishes unless $0 \leq i, j \leq n$.

When X is a smooth rigid space, $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ is isomorphic to $\mathrm{H}^i(X, \Omega^j_{X/K})(-j)$, and the spectral sequence is the same as the Hodge-Tate spectral sequence for smooth proper rigid space (in the sense of [Sch13b]).

Proof. The cohomological boundedness of $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ is given by the Theorem 4.3.0.2. The finite dimensionality is given by the properness of X, the coherence of the $R^i\pi_{X*}\Omega^j_{\text{éh}}$ (the Proposition 4.3.0.1), and the following equality

$$R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}) = R\Gamma(X_{\text{ét}}, R\pi_{X*}\Omega^j_{\text{éh}}).$$

Moreover when X is smooth, the isomorphism between $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ and $\mathrm{H}^i(X, \Omega^j_{X/K})(-j)$ follows from the éh-decent of differential by the Theorem 4.2.1.1 $\square$

At last, when X is defined over a discretely valued subfield K_0 of K that has a perfect residue field, the above spectral sequence is Galois equivariant. In particular, since $K(-j)^{\mathrm{Gal}(K/K_0)} = 0$ for $j \neq 0$, the boundary map from $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ to $\mathrm{H}^{i+2}(X_{\text{éh}}, \Omega^{j-1}_{\text{éh}})(-j+1)$ is zero. In this way, the Hodge-Tate spectral sequence degenerates canonically, and the p-adic étale cohomology splits into the direct sum of distinct Hodge-Tate weights

$$\mathrm{H}^n(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = \bigoplus_{i+j=n} \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j).$$

This canonical (Galois equivariant) decomposition is functorial with respect to rigid spaces defined over K_0.

Theorem 8.7.0.2 (Hodge-Tate decomposition)**.** *Let Y be a proper rigid space over a discretely valued subfield K_0 of K that has a perfect residue field. Then the E_2 spectral sequence above degenerates at its second page. In fact, we have a Galois equivariant isomorphism*

$$\mathrm{H}^n(Y_{K\,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = \bigoplus_{i+j=n} \mathrm{H}^i(Y_{\text{éh}}, \Omega^j_{\text{éh},/K_0}) \otimes_{K_0} K(-j).$$

The isomorphism is functorial with respect to rigid spaces Y over K_0.

Part III

Period Sheaves via Derived de Rham Complex

This part of the thesis is from the joint work of the author with Shizhang Li in [GL20].

In this part, we apply the construction of the analytic derived de Rham complex to pro-étale structure sheaves, and identify the derived de Rham complexes with period sheaves in [Sch13a] and [TT19], originally introduced for smooth formal schemes by Brinon ([Bri08]).

Throughout the Part III, we use the convention of the ∞-category as in Chapter 2.

CHAPTER 9

Integral Period Sheaves

In this chapter, we consider the analytic derived de Rham complex for integral pro-étale structure sheaves. We start with the construction of the integral analytic cotangent complex and the derived de Rham complex for a map of p-adic algebras, as in Section 9.1. In Section 9.2, we generalize the *Katz–Oda filtration* to the context of the derived de Rham complex for a triple of algebras, and prove a base change formula for the derived de Rham complex (Proposition 9.2.0.4). At last, we apply constructions and results in the first two sections to the integral pro-étale structure sheaf in Section 9.3, and show that these recover the crystalline period sheaves together with their Poincaré sequence of [TT19] in Theorem 9.4.0.1.

Results in this chapter appeared in [GL20, Section 3].

9.1 Affine construction

In this section we define analytic cotangent complex and analytic derived de Rham complex for a morphism of p-adic algebras. We refer readers to [Bha12b, Sections 2 and 3] for general background of the derived de Rham complex in a p-adic situation.

Construction 9.1.0.1 (Integral constructions)**.** Let $A_0 \to B_0$ be a map of p-adically complete algebras over $\mathcal{O}_k$, and P be the standard polynomial resolution of B_0 over A_0.

We define the *analytic cotangent complex of* $A_0 \to B_0$*,* denoted as $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$, to be the derived p-completion of the complex $\Omega^1_{P/A_0} \otimes_P B_0$ of B_0-modules.

Next we denote $(|\Omega^*_{P/A_0}|, \mathrm{Fil}^*)$ the direct sum totalization of the simplicial complex Ω^*_{P/A_0} together with its Hodge filtration, as an object in $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Ch}(A_0))$. As the de Rham complex of a simplicial ring admits a commutative differential graded algebra structure, we may regard $|\Omega^*_{P/A_0}|$ with its Hodge filtration as an object in $\mathrm{CAlg}(\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Ch}(A_0)))$. Then the *analytic derived de Rham complex of* B_0/A_0, denoted as $\mathrm{dR}^{\mathrm{an}}_{B_0/A_0}$ in the $\mathrm{CAlg}(\mathrm{DF}(A_0))$, is defined as the derived p-completion of the filtered cdga $(|\Omega^*_{P/A_0}|, \mathrm{Fil}^*)$.

Remark 9.1.0.2. By construction, the graded pieces of the derived Hodge filtrations of $\mathrm{dR}^{\mathrm{an}}_{B_0/A_0}$ are given by

$$\mathrm{gr}^i(\mathrm{dR}^{\mathrm{an}}_{B/A}) \cong (\mathrm{L} \wedge^i \mathbb{L}_{B/A})^{\mathrm{an}}[-i],$$

where $\mathrm{L}\wedge^i$ denotes the i-th left derived wedge product, c.f. [Bha12a, Construction 4.1].

Let us establish some properties of this construction before discussing any example.

Lemma 9.1.0.3. *Let $A \to B \to C$ be a triple of rings, then we have a commutative diagram of filtered E_∞ algebras:*

$$\begin{tikzcd}
\mathrm{dR}_{B/A} \arrow[r] \arrow[d] & \mathrm{dR}_{C/A} \arrow[d] \\
B \arrow[r] & \mathrm{dR}_{C/B},
\end{tikzcd}$$

where the left arrow is the projection to 0-th graded piece of the derived Hodge filtration, and the other three arrows come from functoriality of the construction of derived de Rham complex.

Proof. This follows from left Kan extension of the case when B is a polynomial A-algebra and C is a polynomial B-algebra. $\square$

The following is the key ingredient in understanding the analytic derived de Rham complex in situations that are interesting to us.

Theorem 9.1.0.4. *Let $A \to B \to C$ be ring homomorphisms of p-completely flat $\mathbb{Z}_p$-algebras, such that $A/p \to B/p$ is relatively perfect (see [Bha12b, Definition 3.6]). Then we have*

1. $\mathbb{L}^{\mathrm{an}}_{B/A} = 0$, *and* $\mathrm{dR}^{\mathrm{an}}_{B/A} = B$*;*

2. *The natural map* $\mathrm{dR}^{\mathrm{an}}_{C/A} \to \mathrm{dR}^{\mathrm{an}}_{C/B}$ *is an isomorphism;*

3. *We have a commutative diagram:*

$$\begin{tikzcd} \mathrm{dR}^{\mathrm{an}}_{B/A} \arrow[r] \arrow[d, "\cong"] & \mathrm{dR}^{\mathrm{an}}_{C/A} \arrow[d, "\cong"] \\ B \arrow[r] & \mathrm{dR}^{\mathrm{an}}_{C/B}. \end{tikzcd}$$

4. *Assume furthermore that* $B \to C$ *is surjective with kernel* I *and* $B/p \to C/p$ *is a local complete intersection, then the natural map* $B \to \mathrm{dR}^{\mathrm{an}}_{C/B}$ *exhibits the latter as* $D_B(I)^{\mathrm{an}}$*, the* p*-adic completion of the PD envelope of* B *along* I*. Moreover the* p*-adic completion of the PD filtrations* $\mathrm{Fil}^r = I^{[r],\mathrm{an}}$ *are identified with the* r*-th Hodge filtration.*

Note that by [Bha12b, Lemma 3.38] $D_B(I)^{\mathrm{an}}$ is a p-complete flat $\mathbb{Z}_p$-algebra. Hence $I^{[r],\mathrm{an}}$, being submodules of a flat $\mathbb{Z}_p$-module, are also p-torsionfree for all r.

Proof. (1) and (2) follow from the proof of [Bha12b, Corollary 3.8]: one immediately reduces modulo p and appeals to the conjugate filtration. (3) follows from Lemma 9.1.0.3 by taking the derived p-completion.

As for (4), we first apply [Bha12b, Proposition 3.25] and [Ber74, Théorème V.2.3.2] to see that there is a natural filtered map $\mathcal{C}omp_{C/B} \colon \mathrm{dR}^{\mathrm{an}}_{C/B} \to D_B(I)^{\mathrm{an}}$ such that precomposing with $B \to \mathrm{dR}^{\mathrm{an}}_{C/B}$ gives the natural map $B = B^{\mathrm{an}} \to D_B(I)^{\mathrm{an}}$. By [Bha12b, Theorem 3.27] we see that $\mathcal{C}omp_{C/B}$ is an isomorphism for the underlying algebra. To show the same holds for filtrations, it suffices to show that the induced map on graded pieces are isomorphisms as the map is compatible with filtrations. To that end, by a standard spread out technique, we may reduce to the case where B is the p-adic completion of a finite type $\mathbb{Z}_p$ algebra, in particular it is Noetherian, in which case the identification of graded pieces via this natural map follows from a result of Illusie [Ill72, Corollaire VIII.2.2.8]. ☐

Now we are ready to do some examples. An inspiring arithmetic example is worked out by Bhatt.

Example 9.1.0.5 ([Bha12b, Proposition 9.9])**.** There is a filtered isomorphism:

$$A_{\mathrm{crys}} \cong \mathrm{dR}^{\mathrm{an}}_{\mathbb{Z}_p/\mathbb{Z}_p}.$$

Let us work out a geometric example below.

Example 9.1.0.6. Let n be a positive integer. Let $R = \mathbb{Z}_p\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle$, and

$$R_\infty = \mathbb{Z}_p\langle T_1^{\pm 1/p^\infty}, \ldots, T_n^{\pm 1/p^\infty}\rangle = R\langle S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle/(T_i - S_i; 1 \leq i \leq n).$$

Applying (derived p-completion of) the fundamental triangle of cotangent complexes to

$$\mathbb{Z}_p \to R \to R_\infty,$$

one yields that $\mathbb{L}^{\mathrm{an}}_{R_\infty/R} = R_\infty \cdot \{dT_1, \ldots, dT_n\}[1]$.

On the other hand, the fundamental triangle associated with

$$R \to R\langle S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle \to R_\infty$$

gives us $\mathbb{L}^{\mathrm{an}}_{R_\infty/R} = R_\infty \cdot \{T_i - S_i; 1 \leq i \leq n\}[1]$.

The relation between these two presentations of $\mathbb{L}^{\mathrm{an}}_{R_\infty/R}$ is that

$$T_i - S_i = dT_i$$

in $\mathrm{H}_1(\mathbb{L}^{\mathrm{an}}_{R_\infty/R})$, as $\frac{\partial}{\partial T_i}(T_i - S_i) = 1$.[1]

Following the above notation, we describe $\mathrm{dR}^{\mathrm{an}}_{R_\infty/R}$.

Example 9.1.0.7. Applying Theorem 9.1.0.4 to $A = R, B = R\langle S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle$ and $I = (T_1 - S_1, \ldots, T_n - S_n)$, we see that $\mathrm{dR}^{\mathrm{an}}_{R_\infty/R} = \left(\mathrm{D}_{\mathbb{Z}_p\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}, S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle}(I)\right)^{\mathrm{an}}$ is the p-adic completion of the PD envelope of $R\langle S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle$ along I (notice that the PD envelope is p-torsion free, hence derived completion agrees with classical completion), and the Hodge filtrations are (p-adically) generated by divided powers of $\{T_i - S_i\}$. Example 9.1.0.6 shows that the image of $(T_i - S_i)$ in $\mathrm{gr}^1 = \mathbb{L}^{\mathrm{an}}_{R_\infty/R}[-1] = R_\infty \otimes_R \Omega^{1,\mathrm{an}}_{R/\mathbb{Z}_p}$ is identified with $1 \otimes dT_i$. This precise identification will be used later (see Example 10.1.0.7 and the proof of Theorem 10.4.0.1) when we compare certain rational version of the analytic derived de Rham complex with Scholze's period sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$.

9.2 Derived de Rham complex for a triple

Given a pair of smooth morphisms $A \to B \to C$, there is a natural Gauss–Manin connection $\mathrm{dR}_{C/B} \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \Omega^1_{B/A}$, such that $\mathrm{dR}_{C/A}$ is naturally identified with the "totalization" of the following sequence:

$$\mathrm{dR}_{C/B} \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \Omega^1_{B/A} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \Omega^{\dim_{B/A}}_{B/A}.$$

[1] Here we follow the sign conventions in the Stacks Project, see [Sta18, Tag 07MC footnote 1]

Katz and Oda [KO68] observed that this can be explained by a filtration on $\mathrm{dR}_{C/A}$. In this section we shall show how to generalize this to the context of derived de Rham complex for a pair of arbitrary morphisms $A \to B \to C$.

We first need to introduce a way to attach filtration on a tensor product of filtered modules over a filtered E_∞-algebra. The following fact about Bar resolution is well-known, and we thank Bhargav Bhatt for teaching us in this generality.

Lemma 9.2.0.1. *Let A be an ordinary ring, let R be an E_∞-algebra over A, and let M and N be two objects in $\mathcal{D}(R)$. Then the following augmented simplicial object in $\mathcal{D}(A)$*

$$\Big(\cdots \mathrel{\substack{\to\\\to\\\to\\\to}} M \otimes_A R \otimes_A R \otimes_A N \mathrel{\substack{\to\\\to\\\to}} M \otimes_A R \otimes_A N \rightrightarrows M \otimes_A N \Big) \longrightarrow M \otimes_R N$$

displays $M \otimes_R N$ as the colimit of the simplicial objects in $\mathcal{D}(A)$. Here the arrows are given by "multiplying two factors together".

Proof. Since the ∞-category $\mathcal{D}(R)$ is generated by shifts of R [Lur17, 7.1.2.1], commuting tensor with colimit, we may assume that both of M and N are just R. In this case, the statement holds for merely E_1-algebras, as we have a null homotopy $R^{\otimes_A n} \to R^{\otimes_A (n+1)}$ given by tensoring $R^{\otimes_A n}$ with the natural map $A \to R$. $\square$

Construction 9.2.0.2. Let A be an ordinary ring, let R be a filtered E_∞ algebra over A, and let M and N be two filtered R-modules with filtrations compatible with that on R. Then we regard $M \otimes_R N$ as an object in $\mathrm{DF}(A)$ via the Bar resolution in Lemma 9.2.0.1, with

$$\begin{aligned}&\mathrm{Fil}^i(M \otimes_R N) := \mathrm{colim}_{\Delta^{\mathrm{op}}} \\ &\Big(\cdots \mathrel{\substack{\to\\\to\\\to\\\to}} \mathrm{Fil}^i(M \otimes_A R \otimes_A R \otimes_A N) \mathrel{\substack{\to\\\to\\\to}} \mathrm{Fil}^i(M \otimes_A R \otimes_A N) \rightrightarrows \mathrm{Fil}^i(M \otimes_A N) \Big),\end{aligned}$$

where the filtrations on $M \otimes_A R \otimes_A \cdots \otimes_A R \otimes_A N$ are given by the usual Day involution.

Lemma 9.2.0.3. *Let A, R, M, N be as in Construction 9.2.0.2. Then we have*

$$\mathrm{gr}^*(M \otimes_R N) \cong \mathrm{gr}^*(M) \otimes_{\mathrm{gr}^*(R)} \mathrm{gr}^*(N).$$

Proof. We have

$$\begin{aligned}&\mathrm{gr}^*(M \otimes_R N) \\ &\cong \Big(\cdots \mathrel{\substack{\to\\\to\\\to\\\to}} \mathrm{gr}^*(M \otimes_A R \otimes_A R \otimes_A N) \mathrel{\substack{\to\\\to\\\to}} \mathrm{gr}^*(M \otimes_A R \otimes_A N) \rightrightarrows \mathrm{gr}^*(M \otimes_A N) \Big) \\ &\cong \operatorname*{colim}_{\Delta^{\mathrm{op}}} \Big(\cdots \mathrel{\substack{\to\\\to\\\to}} \mathrm{gr}^*(M) \otimes_A \mathrm{gr}^*(R) \otimes_A \mathrm{gr}^*(N) \rightrightarrows \mathrm{gr}^*(M) \otimes_A \mathrm{gr}^*(N) \Big) \\ &\cong \mathrm{gr}^*(M) \otimes_{\mathrm{gr}^*(R)} \mathrm{gr}^*(N).\end{aligned}$$

□

Proposition 9.2.0.4. *Let $A \to B \to C$ be a triple of rings, then the diagram of filtered E_∞-algebras in Lemma 9.1.0.3 induces a filtered isomorphism of filtered E_∞-algebras over B:*

$$\mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} B \cong \mathrm{dR}_{C/B}.$$

Here the left hand side is equipped with the filtration in Construction 9.2.0.2 with the Hodge filtrations on $\mathrm{dR}_{C/A}$ and $\mathrm{dR}_{B/A}$, and $\mathrm{Fil}^i(B) = 0$ for $i \geq 1$. The right hand side is equipped with the Hodge filtration. Denote $\Omega^*_{B/A} := \oplus_i \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}_{B/A})[-i]$ the graded algebra associated with the Hodge filtration.

Proof. After cofibrant replacing B by a simplicial polynomial A-algebra and C by a simplicial polynomial B-algebra, we reduce the statement to the case where B is a polynomial A-algebra and C is a polynomial B-algebra. One verifies directly that in this case we have

$$\mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} B \cong \mathrm{dR}_{C/B} \text{ and } \Omega^*_{C/A} \otimes_{\Omega^*_{B/A}} B \cong \Omega^*_{C/B}.$$

Now we finish proof by recalling that a filtered morphism with isomorphic underlying object is a filtered isomorphism if and only if the induced morphisms of graded pieces are isomorphisms. □

Construction 9.2.0.5. Let $A \to B \to C$ be a triple of rings, then we put a filtration on $\mathrm{dR}_{C/A}$ by the following: $L(i) = \mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} \mathrm{Fil}^i_{\mathrm{H}}(\mathrm{dR}_{B/A})$, viewed as a commutative algebra object in $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{DF}(A)) = \mathrm{Fun}((\mathbb{N} \times \mathbb{N})^{\mathrm{op}}, \mathcal{D}(A))$, where the filtration on $L(i)$ is as in Construction 9.2.0.2 with each factor being equipped with its own Hodge filtrations. We have $L(0) \cong \mathrm{dR}_{C/A}$, and we call $L(i)$ *the i-th Katz–Oda filtration on* $\mathrm{dR}_{C/A}$, and we shall denote it by $\mathrm{Fil}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A})$.

We caution readers that each $\mathrm{Fil}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A})$ is equipped with yet another filtration, we shall still call it the Hodge filtration, the index is often denoted by j. The graded pieces of the Katz–Oda filtration when both arrows in $A \to B \to C$ are smooth were studied by Katz–Oda [KO68], although in a different language, hence the name.

Lemma 9.2.0.6. *Let $A \to B \to C$ be a triple of rings, then*

1. *We have a filtered isomorphism*

$$\mathrm{gr}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A}) \cong \mathrm{dR}_{C/B} \otimes_B \mathrm{st}_i((\mathrm{L}\wedge^i \mathbb{L}_{B/A})[-i]).$$

2. *Under the above filtered isomorphism, the Katz–Oda filtration on $\mathrm{dR}_{C/A}$ witnesses the*

following sequence:

$$\mathrm{dR}_{C/A} \to \mathrm{dR}_{C/B} \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \mathrm{st}_1(\mathbb{L}_{B/A}) \xrightarrow{\nabla} \cdots$$

Here ∇ denotes connecting homomorphisms, which is $\mathrm{dR}_{C/A}$-linear and satisfies Newton–Leibniz rule.

3. *The induced Katz–Oda filtration on $\mathrm{gr}^j_{\mathrm{H}}(\mathrm{dR}_{C/A})$ is complete. In fact $\mathrm{Fil}^i_{\mathrm{KO}}\mathrm{gr}^j_{\mathrm{H}}(\mathrm{dR}_{C/A}) = 0$ whenever $i > j$.*

4. *If $A \to B$ is smooth of equidimension d, then $\mathrm{Fil}^i_{\mathrm{KO}}\mathrm{Fil}^j_{\mathrm{H}}(\mathrm{dR}_{C/A}) \cong 0$ for any $i > d$. In particular, combining with the previous point, we get that in this situation the Katz–Oda filtration is strict exact in the sense of* **??**.

Proof. For (1): we have

$$\mathrm{gr}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A}) \cong \mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}_{B/A})[-i] \cong (\mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} B) \otimes_B \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}_{B/A})[-i],$$

and by Proposition 9.2.0.4 the right hand side can be identified with $\mathrm{dR}_{C/B} \otimes_B \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}_{B/A})[-i]$.

For (2): we just need to show the properties of these ∇'s. With any multiplicative filtration on an E_∞-algebra R, we get a natural filtered map $\mathrm{Fil}^i \otimes_R \mathrm{Fil}^j \to \mathrm{Fil}^{i+j}(R)$ where the left hand side is equipped with the Day convolution filtration (over the underlying algebra R). Now we look at the following commutative diagram:

$$\begin{tikzcd}
(\mathrm{gr}^i \otimes_R \mathrm{gr}^{j+1}) \oplus (\mathrm{gr}^{i+1} \otimes_R \mathrm{gr}^j) \arrow[r] \arrow[d] & \mathrm{Fil}^{i+j}/\mathrm{Fil}^{i+j+2}(\mathrm{Fil}^i \otimes_R \mathrm{Fil}^j) \arrow[r] \arrow[d] & \mathrm{gr}^i \otimes_R \mathrm{gr}^j \arrow[r, "+1"] \arrow[d] & {} \\
\mathrm{gr}^{i+j+1} \arrow[r] & \mathrm{Fil}^{i+j}/\mathrm{Fil}^{i+j+2}(R) \arrow[r] & \mathrm{gr}^{i+j} \arrow[r, "+1"] & {}
\end{tikzcd}$$

to conclude that the connecting morphisms are R-linear and satisfy Newton–Leibniz rule. Since $\mathrm{Fil}^i_{\mathrm{KO}}$ is a multiplicative filtration on $\mathrm{dR}_{C/A}$, we get the desired properties of ∇.

(3) follows from the distinguished triangle of cotangent complexes and their exterior powers.

(4) follows from the definition of the Katz–Oda filtration in Construction 9.2.0.5 and the fact that $\mathrm{Fil}^i_{\mathrm{H}}(\mathrm{dR}_{B/A}) = 0$ whenever $i > d$. □

We do not need the following construction in this paper, but mention it for the sake of completeness of our discussion.

Construction 9.2.0.7. We denote the graded algebra associated with the Hodge filtration on derived de Rham complex by $\mathrm{L}\Omega^*_{-/-}$.[2] Let $A \to B \to C$ be a triple of rings. Note that $\mathrm{L}\Omega^*_{C/A} \cong \mathrm{L}\wedge^*_C(\mathrm{st}_1(\mathbb{L}_{C/A}))[-*]$, and we have a functorial filtration $\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A}$ with quotient being $\mathbb{L}_{C/B}$. Hence there is a functorial multiplicative exhaustive increasing filtration on $\mathrm{L}\Omega^*_{C/A}$, called the *vertical filtration* and denoted by Fil^v_i, consisting of graded-$\mathrm{L}\Omega^*_{B/A}$-submodules with graded pieces given by $\mathrm{gr}^v_i = \mathrm{L}\Omega^*_{B/A} \otimes_B \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}_{C/B})[-i]$.

We refer the reader to [GL20] for a summary of filtrations.

Specializing to the p-adic setting, we get the following.

Lemma 9.2.0.8. *Let $A \to B \to C$ be a triangle of p-complete flat $\mathbb{Z}_p$-algebras. Suppose B/p is smooth over A/p of relative equidimension n. Then we have a p-adic Katz–Oda filtration on $\mathrm{dR}_{C/A}$ which is strict exact and witnesses the following sequence:*

$$0 \to \mathrm{dR}^{\mathrm{an}}_{C/A} \to \mathrm{dR}^{\mathrm{an}}_{C/B} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{C/B} \otimes_B \mathrm{st}_1(\Omega^{1,\mathrm{an}}_{B/A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{C/B} \otimes_B \mathrm{st}_n(\Omega^{n,\mathrm{an}}_{B/A}) \to 0.$$

Recall that the superscript $(-)^{\mathrm{an}}$ denotes the derived p-completion of the corresponding objects. Note that since $\Omega^{i,\mathrm{an}}_{B/A}$ are all finite flat B-modules by assumption and $\mathrm{dR}^{\mathrm{an}}_{C/B}$ is p-complete, the tensor products showing above are already p-complete.

Proof. Take the derived p-completion of the Katz–Oda filtration on $\mathrm{dR}_{C/A}$, we get such a strict exact filtration by Lemma 9.2.0.6. $\square$

9.3 Integral de Rham sheaves

For the rest of this section, we focus on the situation spelled out by the following:

Notation

Let κ be a perfect field in characteristic $p > 0$, and let $k = W(\kappa)[\frac{1}{p}]$ be the absolutely unramified discretely valued p-adic field with the ring of integers $\mathcal{O}_k = W(\kappa)$. Fix a separated formally smooth p-adic formal schemes $\mathfrak{X}$ over $\mathcal{O}_k$. Denote by X its generic fiber, viewed as an adic space over the Huber pair $(k, \mathcal{O}_k)$.

In this situation, there is a natural map of ringed sites

$$w\colon (X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_X) \longrightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

[2] We warn readers that this is not a standard notation, in other literature the symbol $\mathrm{L}\Omega$ is often used to denote the derived de Rham complex.

which sends an open subset $\mathfrak{U} \subset \mathfrak{X}$ to the open subset $U \in X_{\text{proét}}$, where U is the generic fiber of $\mathfrak{U}$. This allows us to define inverse image $w^{-1}\mathcal{O}_{\mathfrak{X}}$ of the integral structure sheaf $\mathcal{O}_{\mathfrak{X}}$, as a sheaf on the pro-étale site $X_{\text{proét}}$.

On the pro-étale site of X, we have a morphism of sheaves of p-complete $\mathcal{O}_k$-algebras:

$$\mathcal{O}_k \longrightarrow w^{-1}\mathcal{O}_{\mathfrak{X}} \longrightarrow \widehat{\mathcal{O}}_X^+. \tag{1}$$

We refer readers to [Sch13a, Sections 3 and 4] for a detailed discussion surrounding the pro-étale site of a rigid space and structure sheaves on it. There is a subcategory $X^{\omega}_{\text{proét}/\mathfrak{X}} \subset X_{\text{proét}}$ consisting of affinoid perfectoid objects $U = \mathrm{Spa}(B, B^+) \in X_{\text{proét}}$ whose image in X is contained in $w^{-1}(\mathrm{Spf}(A_0))$, the generic fiber of an affine open $\mathrm{Spf}(A_0) \subset \mathfrak{X}$. The class of such objects form a basis for the pro-étale topology by (the proof of) [Sch13a, Proposition 4.8]. We first study the behavior of derived de Rham complex for the triangle eq. (1) on $X^{\omega}_{\text{proét}/\mathfrak{X}}$.

Proposition 9.3.0.1. *Let $U = \mathrm{Spa}(B, B^+) \in X_{\text{proét}}$ be an object in $X^{\omega}_{\text{proét}/\mathfrak{X}}$, choose $\mathrm{Spf}(A_0) \subset \mathfrak{X}$ such that the image of U in X is contained in $w^{-1}(\mathrm{Spf}(A_0))$. Then*

1. *the natural surjection $\theta\colon A_{inf}(B^+) \twoheadrightarrow B^+$ exhibits $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathcal{O}_k} = A_{\mathrm{crys}}(B^+)$, the p-completion of the divided envelope of $A_{inf}(B^+)$ along $\ker(\theta)$;*
2. *the natural surjection $w^\sharp \otimes \theta\colon A_0 \hat{\otimes}_{\mathcal{O}_k} A_{inf}(B^+) \twoheadrightarrow B^+$ exhibits $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}$ as the p-completion of the divided envelope of $A_0 \hat{\otimes}_{\mathcal{O}_k} A_{inf}(B^+)$ along $\ker(w^\sharp \otimes \theta)$;*
3. *in both cases, the Hodge filtrations are identified as the p-completion of PD filtrations;*
4. *the filtered algebra $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}$ is independent of the choice of A_0. We denote it as $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathfrak{X}}$.*

Remark 9.3.0.2. In particular, (1) and (2) tells us that these derived de Rham complexes are actually quasi-isomorphic to an honest algebra viewed as a complex supported on cohomological degree 0; (4) tells us that sending $U = \mathrm{Spa}(B, B^+) \in X^{\omega}_{\text{proét}/\mathfrak{X}}$ to $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathfrak{X}}$ gives a well-defined presheaf on $X^{\omega}_{\text{proét}/\mathfrak{X}}$.

Proof of Proposition 9.3.0.1. Applying Theorem 9.1.0.4.(4) to the triangles

$$\mathcal{O}_k \to A_{inf}(B^+) \to B^+ \text{ and } A_0 \to A_0 \hat{\otimes}_{\mathcal{O}_k} A_{inf}(B^+) \to B^+$$

proves (1) and (2) respectively and (3)[3]. As for (4), using separatedness of $\mathfrak{X}$, we reduce to the situation where image of U in X is in a smaller open $w^{-1}(\mathrm{Spf}(A_1)) \subset w^{-1}(\mathrm{Spf}(A_0))$. It suffices to show the natural map $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0} \to \mathrm{dR}^{\mathrm{an}}_{B^+/A_1}$ is a filtered isomorphism, which follows from Lemma 9.2.0.8 as $A_0/p \to A_1/p$ is étale. $\square$

[3] Here we use the unramifiedness of $\mathcal{O}_k$ to verify the relatively perfectness assumption in Theorem 9.1.0.4.

Recall that the subcategory $X^{\omega}_{\text{proét}/\mathfrak{X}} \subset X_{\text{proét}}$ gives a basis for the topology on $X_{\text{proét}}$. Hence any presheaf on $X^{\omega}_{\text{proét}/\mathfrak{X}}$ can be sheafified to a sheaf on $X_{\text{proét}}$.

We define the analytic de Rham sheaf for $\widehat{\mathcal{O}}^+_X$ over $\mathcal{O}_k$ and $w^{-1}\mathcal{O}_{\mathfrak{X}}$ as follows:

Construction 9.3.0.3 ($\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}$ and $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}}$)**.** The *analytic de Rham sheaf of* $\widehat{\mathcal{O}}^+_X/\mathcal{O}_k$, denoted as $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}$, is the p-adic completion of the unfolding of the presheaf on $X^{\omega}_{\text{proét}/\mathfrak{X}}$ which assigns each $U = \mathrm{Spa}(B, B^+)$ the algebra $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathcal{O}_k}$. We equip it with the decreasing Hodge filtration Fil^r_{H} given by the image of p-completion of the unfolding of the presheaf assigning each $U = \mathrm{Spa}(B, B^+)$ the r-th Hodge filtration in $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathcal{O}_k}$.

The *analytic de Rham sheaf of* $\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}$, denoted as $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}}$, is the p-adic completion of the unfolding of the presheaf on $X^{\omega}_{\text{proét}/\mathfrak{X}}$ which assigns each $U = \mathrm{Spa}(B, B^+)$ the filtered algebra $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathfrak{X}}$. Similarly we equip it with the decreasing Hodge filtration Fil^r_H given by the image of p-completion of the unfolding of the presheaf whose value on each $U = \mathrm{Spa}(B, B^+)$ is the r-th Hodge filtration in $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathfrak{X}}$.

The fact that these definitions/constructions make sense follows from Proposition 9.3.0.1 and Remark 9.3.0.2.

One may also define the corresponding mod p^n version of these sheaves. Since sheafifying commutes with arbitrary colimit, the p-adic completion of the sheafification of a presheaf F is the same as the inverse limit over n of the sheafification of presheaves F/p^n. Therefore we have $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}/p^n$ is the same as the sheafification of the presheaf $\mathrm{dR}_{B^+/\mathcal{O}_k}/p^n$. Its r-th Hodge filtration agrees with the sheafification of the presheaf $\mathrm{Fil}^r_H(\mathrm{dR}_{B^+/\mathcal{O}_k}/p^n)$, as sheafifying is an exact functor. Similar statements can be made for the mod p^n version of $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}}$ and its Hodge filtrations.

Now the strict exact Katz–Oda filtration obtained in the Lemma 9.2.0.8 gives us the following:

Corollary 9.3.0.4 (Crystalline Poincaré lemma)**.** *There is a functorial* $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}$*-linear strict exact sequence of filtered sheaves on* $X_{\text{proét}}$*:*

$$0 \to \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \to \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \otimes_{w^{-1}\mathcal{O}_{\mathfrak{X}}} \mathrm{st}_1(w^{-1}\Omega^{1,\mathrm{an}}_{\mathfrak{X}}) \xrightarrow{\nabla} \cdots$$

$$\cdots \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \otimes_{w^{-1}\mathcal{O}_{\mathfrak{X}}} \mathrm{st}_d(w^{-1}\Omega^{d,\mathrm{an}}_{\mathfrak{X}}) \to 0,$$

where d is the relative dimension of $\mathfrak{X}/\mathcal{O}_k$.

Proof. Using the discussion before this Corollary, we reduce to checking this at the level of presheaves on $X^{\omega}_{\text{proét}/\mathfrak{X}}$. Since now everything in sight are supported cohomologically in degree 0 with filtrations given by submodules because of Proposition 9.3.0.1, the strict exact Katz–Oda filtration in Lemma 9.2.0.8 implies what we want. $\square$

Remark 9.3.0.5. We can drop the separatedness assumption on $\mathfrak{X}$ as follows. Since any formal scheme is covered by affine ones, and affine formal schemes are automatically separated, we may define all these de Rham sheaves on each slice subcategory of the pro-étale site of the rigid generic fiber of affine opens of $\mathfrak{X}$. Similar to the proof of Proposition 9.3.0.1.(4), we can show these de Rham sheaves satisfy the base change formula with respect to maps of affine opens of $\mathfrak{X}$ (by appealing to Lemma 9.2.0.8 again), hence these sheaves on the slice subcategories glue to a global one. The Crystalline Poincaré lemma obtained above holds verbatim as exactness of a sequence of sheaves may be checked locally.

9.4 Comparing with Tan–Tong's crystalline period sheaves

Lastly we shall identify the two de Rham sheaves defined above with two period sheaves that show up in the work of Tan–Tong [TT19]. We refer readers to Definitions 2.1. and 2.9. of loc. cit. for the meaning of period sheaves $\mathbb{A}_{\mathrm{crys}}$ and $\mathcal{O}\mathbb{A}_{\mathrm{crys}}$ and their PD filtrations.

We look at the triangle of sheaves of rings:

$$\mathcal{O}_k \to w^{-1}(\mathcal{O}_{\mathfrak{X}})\hat{\otimes}_{\mathcal{O}_k}\mathbb{A}_{inf} \xrightarrow{w^\sharp \hat{\otimes}\theta} \widehat{\mathcal{O}}^+_X.$$

Theorem 9.4.0.1. *The triangle above induces a filtered isomorphism of sheaves:* $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \cong \mathbb{A}_{\mathrm{crys}}$ *and* $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \cong \mathcal{O}\mathbb{A}_{\mathrm{crys}}$.

Moreover, under this identification, the Crystalline Poincaré sequence in Corollary 9.3.0.4 agrees with the one obtained in [TT19, Corollary 2.17].

Proof. We check these isomorphisms modulo p^n for any n. For both cases, the de Rham sheaf and the crystalline period sheaf are both unfoldings of the same PD envelope presheaf (with its PD filtrations) on $X^{\omega}_{\mathrm{pro\acute{e}t}/\mathfrak{X}}$: for the de Rham sheaves this statement follows from Proposition 9.3.0.1 and base change formula of PD envelope (note that taking PD envelope is a left adjoint functor, hence commutes with colimit, in particular, it commutes with modulo p^n for any n), for the crystalline period sheaf this follows from the definition (note that although the $\mathcal{O}\mathbb{A}_{inf}$ defined in Tan–Tong's work uses uncompleted tensor of $w^{-1}(\mathcal{O}_{\mathfrak{X}})$ and $\mathbb{A}_{inf}$ instead of the completed tensors we are using here, the difference goes away when we modulo any power of p and restricts to the basis of affinoid perfectoid objects).

Therefore for both cases, we have natural isomorphisms modulo p^n for any n, taking inverse limit gives the result we want as all sheaves are p-adic completion of their modulo p^n versions.

The claim about matching Poincaré sequences follows by unwinding definitions. Indeed we need to check that ∇ defined in these two sequences agree, but since ∇ is linear over $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \cong \mathbb{A}_{\mathrm{crys}}$, it suffices to check that ∇ agrees on u_i which is the image of $T_i - S_i$ (notation from loc. cot. and Example 9.1.0.7 respectively) by functoriality of the Poincaré sequence. One checks that in both cases their image under ∇ is $1 \otimes dT_i$. $\square$

CHAPTER 10
Rational Period Sheaves

In this chapter, we consider the rational analogue of the integral derived de Rham complexes and period sheaves from Chapter 9. Our goal is to recover the de Rham period sheaves and their p-adic Poincaré sequence of Brinon ([Bri08]) and Scholze ([Sch13a]), using the analytic derived de Rham complex.

We first recall the properties we need about the analytic cotangent complex and the analytic derived de Rham complex in Section 10.1. Here the results are compatible with the ones in Chapter 2 and Section 8.2. We then consider the Poincaré sequence for a map of Huber rings in Section 10.2. In Section 10.3 and Section 10.4, we apply the construction of the analytic derived de Rham complex to the rational pro-étale structure sheaves, and show that these recover Scholze's constructions (Theorem 10.4.0.1). At last, we compute an example of the analytic derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$, where A is an artinian local ring over a p-adic field and B is a perfectoid algebra over A, as in Section 10.5.

The results in this chapter appears in [GL20, Section 4].

To start, let us spell out the setup by recalling the following notation: k is a p-adic field with ring of integers denoted by $\mathcal{O}_k$ and X is a separated[1] rigid space over k which we view as an adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$.

10.1 Affinoid construction

In this section, we recall the construction of the analytic cotangent complex and give the construction of the analytic derived de Rham complex, for a map of Huber rings over a k. For a detailed discussion of the analytic cotangent complex (for topological finite type algebras), we refer readers to [GR03, Section 7.1-7.3].

Let $f\colon (A, A^+) \to (B, B^+)$ be a map of complete Huber rings over k. Denote by $\mathcal{C}_{B/A}$ the filtered category of pairs (A_0, B_0), where A_0 and B_0 are rings of definition of (A, A^+) and (B, B^+) separately, such that $f(A_0) \subset B_0$.

Construction 10.1.0.1 (Analytic cotangent complex, affinoid)**.** For each $(A_0, B_0) \in \mathcal{C}_{B/A}$, denote by $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$ the integral analytic cotangent complex of $A_0 \to B_0$ as in the Construction 9.1.0.1. The *analytic cotangent complex of* $f\colon (A, A^+) \to (B, B^+)$, denoted by $\mathbb{L}^{\mathrm{an}}_{B/A}$, is defined as the filtered colimit

$$\mathbb{L}^{\mathrm{an}}_{B/A} := \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} \mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}].$$

For the convenience of readers, let us list a few properties of analytic cotangent complex for a morphism of rigid affinoid algebras obtained by Gabber–Romero.

Theorem 10.1.0.2. *Let $A \to B$ be a morphism of k-affinoid algebras, then we have:*

1. *[GR03, Theorem 7.1.33.(i)] $\mathbb{L}^{\mathrm{an}}_{B/A}$ is in $\mathcal{D}^{\leq 0}(B)$ and is pseudo-coherent over B;*
2. *[GR03, Lemma 7.1.27.(iii) and Equation 7.2.36] the 0-th cohomology of the analytic cotangent complex is given by the analytic relative differential: $\mathrm{H}_0(\mathbb{L}^{\mathrm{an}}_{B/A}) \simeq \Omega^{\mathrm{an}}_{B/A}$;*
3. *[GR03, Theorem 7.2.42.(ii)] if $A \to B$ is smooth, then $\mathbb{L}^{\mathrm{an}}_{B/A} \simeq \Omega^{\mathrm{an}}_{B/A}[0]$;*
4. *[GR03, Lemma 7.2.46.(ii)] if $A \to B$ is surjective, then the analytic cotangent complex agrees with the classical cotangent complex: $\mathbb{L}_{B/A} \simeq \mathbb{L}^{\mathrm{an}}_{B/A}$.*

Construction 10.1.0.3 (Analytic derived de Rham complex, affinoid)**.** Let $f\colon (A, A^+) \to (B, B+)$ be a map of complete Huber rings over k. For each $(A_0, B_0) \in \mathcal{C}_{B/A}$, by the Construction 9.1.0.1 we could define the integral analytic derived de Rham complex $\mathrm{dR}^{\mathrm{an}}_{B_0/A_0}$, as an object in $\mathrm{CAlg}(\mathrm{DF}(A_0))$.

[1]Just like Remark 9.3.0.5 suggests, we can remove the separatedness assumption in the end.

Then the *analytic derived de Rham complex* $\mathrm{dR}^{\mathrm{an}}_{B/A}$ of (B, B^+) over (A, A^+), as an object in $\mathrm{CAlg}(\mathrm{DF}(A))$, is defined to be the filtered colimit

$$\mathrm{dR}^{\mathrm{an}}_{B/A} := \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} \mathrm{dR}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}].$$

Moreover, the *(Hodge) completed analytic derived de Rham complex* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ of (B, B^+) over (A, A^+), as an object in $\mathrm{CAlg}(\widehat{\mathrm{DF}}(A))$, is defined as the derived filtered completion of $\mathrm{dR}^{\mathrm{an}}_{B/A}$.

By the construction, the graded pieces of the filtered complete A-complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ is given by

$$\begin{aligned} \mathrm{gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) &\cong \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} \mathrm{gr}^i(G(A_0, B_0)) \\ &\cong \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} (\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}[\frac{1}{p}])[-i] \\ &\cong (\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i], \end{aligned} \tag{2}$$

due to the fact that the functor gr^i preserves filtered colimits.

Remark 10.1.0.4 (Complexity of the construction)**.** The two rational constructions above involve colimits among all rings of definitions and seem to be very complicated. A naive attempt would be taking the usual cotangent/derived de Rham complex of $A^+ \to B^+$, apply the derived p-adic completion and invert p (and do the filtered completion, for the derived de Rham complex case) directly. This would *not* give us the expected answer in general, which is essentially due to the possible existence of nilpotent elements in (A, A^+) and (B, B^+).

Take the map $(k, \mathcal{O}_k) \to (B, B^+)$ for $B = k\langle\epsilon\rangle/(\epsilon^2)$ as an example. Then a ring of definition B_0 of B could be $\mathcal{O}_k\langle\epsilon\rangle/(\epsilon^2)$, while there is only one open integral subring of B that contains $\mathcal{O}_k$, namely $\mathcal{O}_k \oplus k\cdot\epsilon$. In this case, it is easy to see that the derived p-completion of cotangent complexes $\mathbb{L}_{B^+/\mathcal{O}_k}$ and $\mathbb{L}_{B_0/\mathcal{O}_k}$ are different, and remain so after inverting p.

Remark 10.1.0.5 (Simplified construction for uniform Huber pairs)**.** Assume both of the Huber pairs $(A, A^+) \to (B, B^+)$ are *uniform*; namely the subrings of power bounded elements A° and B° are bounded in A and B separately. Then both A^+ and B^+ are rings of definition of A and B separately. In particular, the Construction 10.1.0.1 and the Construction 10.1.0.3 can be simplified as follows:

$$\begin{aligned} \mathbb{L}^{\mathrm{an}}_{B/A} &= \mathbb{L}^{\mathrm{an}}_{B^+/A^+}[\frac{1}{p}], \\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} &= filtered\ completion\ of\ ((derived\, p-completion\ of\ \mathrm{dR}_{B^+/A^+})[\frac{1}{p}]), \end{aligned}$$

where we recall that $\mathbb{L}^{\mathrm{an}}_{B^+/A^+}$ is the derived p-completion of the classical cotangent complex $\mathbb{L}_{B^+/A^+}$, and dR_{B^+/A^+} is the classical derived de Rham complex of B^+/A^+, as in [BMS19, Examples 5.11-5.12].

Examples of uniform Huber pairs include reduced affinoid algebras over discretely valued or algebraically closed non-Archimedean fields [FvdP04, Theorem 3.5.6], and perfectoid affinoid algebras [Sch13b, Theorem 6.3].

An arithmetic example of the Hodge-completed analytic derived de Rham complex has been worked out by Beilinson.

Example 10.1.0.6 ([Bei12, Proposition 1.5])**.** We have a filtered isomorphism:

$$B^+_{\mathrm{dR}} \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{\overline{\mathbb{Q}_p}/\mathbb{Q}_p}.$$

Next we work out a geometric example. Let us compute the Hodge-completed analytic derived de Rham complex of a perfectoid torus over a rigid analytic torus. Following the notation in Example 9.1.0.6, let $R = \mathbb{Z}_p\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle$, and $R_\infty = \mathbb{Z}_p\langle T_1^{\pm 1/p^\infty}, \ldots, T_n^{\pm 1/p^\infty}\rangle = R\langle S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle/(T_i - S_i; 1 \leq i \leq n)$.

Example 10.1.0.7. Continue with Example 9.1.0.7. After inverting p and completing along Hodge filtrations, we see that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty[1/p]/R[1/p]}$ is given by the completion of $\mathbb{Q}_p\langle T_i^{\pm 1}, S_i^{1/p^\infty}\rangle$ along $\{T_i - S_i; 1 \leq i \leq n\}$. Here we use Remark 10.1.0.5 to relate $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty/R}$ and $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty[1/p]/R[1/p]}$. A more explicit presentation is

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty[1/p]/R[1/p]} = \mathbb{Q}_p\langle S_1^{\pm 1/p^\infty}, \ldots, S_n^{\pm 1/p^\infty}\rangle[\![X_1, \ldots, X_n]\!]$$

via change of variable $T_i = X_i + S_i$ (hence $T_i^{-1} = S_i^{-1} \cdot (1 + S_i^{-1}X_i)^{-1}$), c.f. the notation before [Sch13a, Proposition 6.10].

We need to understand the output of these constructions for general perfectoid affinoid algebras relative to affinoid algebras. The following tells us that in this situation, the Hodge completed analytic derived de Rham complex can be computed with any ring of definition inside the affinoid algebra.

Lemma 10.1.0.8. *Let (A, A^+) be a topologically finite type complete Tate ring over $(k, \mathcal{O}_k)$, with $A_0 \subset A^+$ being a ring of definition. Let (B, B^+) be a perfectoid algebra over (A, A^+). Then we have:*

1. *The analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A} \cong \mathbb{L}^{\mathrm{an}}_{B^+/A_0}[1/p]$.*

2. *The Hodge completed analytic derived de Rham complex* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \cong \widehat{\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}[1/p]}$*, where the latter is the Hodge completion of* $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}[1/p]$.

In the proof below we will show a stronger statement: the transition morphisms of the colimit process computing left hand side in Construction 10.1.0.1 and Construction 10.1.0.3 are all isomorphisms.

Proof. Let $A_0' \subset A^+$ be another ring of definition containing A_0. It suffices to show that $\mathbb{L}^{\mathrm{an}}_{B^+/A_0}[1/p] \cong \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}[1/p]$ and similarly for their Hodge completed analytic derived de Rham complexes. Since Hodge completed analytic derived de Rham complex of both sides are derived complete with respect to the Hodge filtration, whose graded pieces, by Equation (2), are derived wedge product of relevant analytic cotangent complexes, we see that the statement about Hodge completed analytic derived de Rham complex follows from the statement about analytic cotangent complex.

To show $\mathbb{L}^{\mathrm{an}}_{B^+/A_0}[1/p] \cong \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}[1/p]$, we appeal to the fundamental triangle of (analytic) cotangent complexes:

$$\mathbb{L}^{\mathrm{an}}_{A_0'/A_0} \otimes_{A_0'} B^+ \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}.$$

Here the tensor product does not need an extra p-completion as $\mathbb{L}_{A_0'/A_0}$ is pseudo-coherent, see [GR03, Theorem 7.1.33]. By [GR03, Theorem 7.2.42], the p-complete cotangent complex $\mathbb{L}^{\mathrm{an}}_{A_0'/A_0}$ satisfies

$$\mathbb{L}^{\mathrm{an}}_{A_0'/A_0}[\frac{1}{p}] = \Omega^{1,\mathrm{an}}_{A_0'[\frac{1}{p}]/A_0[\frac{1}{p}]},$$

which vanishes as $A_0'[\frac{1}{p}]$ and $A_0[\frac{1}{p}]$ are both equal to A. Therefore the natural map

$$\mathbb{L}^{\mathrm{an}}_{B^+/A_0}[\frac{1}{p}] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}[\frac{1}{p}]$$

induced by $A_0 \to A_0'$ is a quasi-isomorphism. $\square$

We can understand the associated graded algebra of analytic de Rham complex of perfectoid affinoid algebras over affinoid algebras via the following Theorem 10.1.0.9. Let K be a perfectoid field extension of k that contains p^n-roots of unity for all $n \in \mathbb{N}$.

Theorem 10.1.0.9. *Let* (A, A^+) *be a topologically finite type complete Tate ring over* $(k, \mathcal{O}_k)$. *Assume* (B, B^+) *is a perfectoid algebra containing both* $(K, \mathcal{O}_K)$ *and* (A, A^+)*. Then the graded algebra* $\mathrm{gr}^*(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$ *admits a natural graded quasi-isomorphism to the derived divided power algebra* $\mathrm{L}\Gamma^*_B(\mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}))$*, where the first graded piece fits into a distinguished triangle:*

$$B(1) \longrightarrow \mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathbb{L}^{\mathrm{an}}_{B/A}[-1] \longrightarrow B \otimes_A \mathbb{L}^{\mathrm{an}}_{A/k},$$

which is functorial in $(B, B^+)/(A, A^+)$. In particular, the graded pieces are B-pseudo-coherent.

Here $B(1)$ denote $\ker(\theta)/\ker(\theta)^2$ where $\theta\colon A_{inf}(B^+)[1/p] \twoheadrightarrow B$ is Fontaine's θ map. Our assumption of (B, B^+) containing $(K, \mathcal{O}_K)$ ensures that this is (non-canonically) isomorphic to B itself, see [Sch13a, Lemma 6.3]. After sheafifying everything, it corresponds to a suitable Tate twist of B.

Proof. The identification $\mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathbb{L}^{\mathrm{an}}_{B/A}[-1]$ is already spelled out by Equation (2).

Let us fix a single choice of pair of rings of definition (A_0, B^+) in $\mathcal{C}_{B/A}$. Here A_0 is topologically finitely presented over $\mathcal{O}_k$, and B^+ contains $\mathcal{O}_K$ for K a perfectoid field containing all p^n-th roots of unity.

Consider the following triple: $\mathcal{O}_k \longrightarrow A_0 \longrightarrow B^+$, it induces the following triangle

$$\mathbb{L}^{\mathrm{an}}_{A_0/\mathcal{O}_k} \otimes_{A_0} B^+ \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/\mathcal{O}_k} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0}.$$

Here we again have used the pseudo-coherence [GR03, Theorem 7.1.33] of $\mathbb{L}^{\mathrm{an}}_{A_0/\mathcal{O}_k}$. We need to show $\mathbb{L}^{\mathrm{an}}_{B^+/\mathcal{O}_k}[1/p] \cong B(1)[1]$. To that end, let W be the Witt ring of the residue field of $\mathcal{O}_k$. By looking at the triple $W \to \mathcal{O}_k \to B^+$, we get another sequence

$$\mathbb{L}^{\mathrm{an}}_{B^+/W} \cong B^+(1)[1] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/\mathcal{O}_k} \longrightarrow \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_k/W} \otimes_{\mathcal{O}_k} B^+[1],$$

where the first identification follows from Proposition 9.3.0.1, and the tensor product does not an extra completion again by coherence of $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_k/W}$. Since $k/W[1/p]$ is finite étale, we conclude that $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_k/W}[1/p] = 0$ by [GR03, Theorem 7.2.42]. This ends the proof of the structure of $\mathbb{L}^{\mathrm{an}}_{B/A}$.

Now we turn to the higher graded piece. The i-th graded pieces $\mathrm{gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$ is quasi-isomorphic to $(\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i]$, which by rewriting in terms of the first graded piece is

$$(\mathrm{L}\wedge^i (\mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})[1]))[-i].$$

So by the relation between the derived wedge product and the derived divided power funcotr (with bounded above input, see [Ill71, V.4.3.5]), we get

$$\mathrm{gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathrm{L}\Gamma^i_B(\mathrm{gr}^1(\widehat{\mathrm{dR}}_{B/A})),$$

and we get the divided power algebra structure of the graded algebra $\mathrm{gr}^*(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$. $\square$

Consequently we get cohomological bounds for perfectoid affinoid algebras over various types of affinoid algebras. The notion of local complete intersection and embedded codimension (in the situation that we are working with) is discussed in the Appendix.

Corollary 10.1.0.10. *Let $(B, B^+)/(A, A^+)$ be as in the statement of Theorem 10.1.0.9. Then we have*

1. $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \in \mathcal{D}^{\leq 0}(A)$*;*
2. *if A/k is smooth, then $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \in \mathcal{D}^{[0,0]}(A)$;*
3. *if A/k is local complete intersection with embedded codimension c, then $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \in \mathcal{D}^{[-c,0]}(A)$.*

Proof. Since the out put of $\widehat{\mathrm{dR}}^{\mathrm{an}}$ is always derived complete with respect to its Hodge filtration, it suffices to show these statements for the graded pieces of Hodge filtration.

For (1), this follows from the fact that $\mathbb{L}^{\mathrm{an}}_{B/A} \in \mathcal{D}^{\leq 0}(B)$. (2) follows from (3) as smooth affinoid algebra has embedded codimension 0.

As for (3), we check the graded pieces of Hodge filtration in this case is in $\mathcal{D}^{[-c,0]}$. In fact, we shall show that the graded pieces, as objects in $\mathcal{D}(B)$, have Tor amplitude $[-c, 0]$. First since B contains $\mathbb{Q}$, we have

$$\mathrm{gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathrm{L}\Gamma^i_B(\mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})) \cong \mathrm{LSym}^i_B(\mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})).$$

Using the triangle in Theorem 10.1.0.9, it suffices to show $\mathrm{LSym}^j_B(B \otimes_A \mathbb{L}^{\mathrm{an}}_{A/k})$ have Tor amplitude $[-c, 0]$ for all j. Since $\mathrm{LSym}^j_B(B \otimes_A \mathbb{L}^{\mathrm{an}}_{A/k}) \cong B \otimes_A \mathrm{LSym}^j_A(\mathbb{L}^{\mathrm{an}}_{A/k})$, we are done by Proposition 1.0.0.7. □

10.2 Poincaré sequence

In this section we explain the Poincaré sequence for Hodge completed de Rham complexes.

Lemma 10.2.0.1. *Let $B \to C$ be an A-algebra morphism. Then for every $j \in \mathbb{N}$, the Katz–Oda filtration on $\mathrm{dR}_{C/A}$ induces a functorial strict exact filtration on $\mathrm{dR}_{C/A}/\mathrm{Fil}^j_{\mathrm{H}}$, witnessing the following sequence:*

$$\begin{aligned}&\mathrm{dR}_{C/A}/\mathrm{Fil}^j \to \mathrm{dR}_{C/B}/\mathrm{Fil}^j \xrightarrow{\nabla} \mathrm{dR}_{C/B}/\mathrm{Fil}^{j-1} \otimes_B \mathrm{st}_1(\mathbb{L}_{B/A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \\ &\mathrm{dR}_{C/B}/\mathrm{Fil}^1 \otimes_B \mathrm{st}_{j-1}(\mathrm{L} \wedge^{j-1} \mathbb{L}_{B/A}).\end{aligned}$$

Here $\mathrm{dR}_{C/A}$ and $\mathrm{dR}_{C/B}$ are equipped with Hodge filtrations.

Moreover $\mathrm{Fil}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A}/\mathrm{Fil}^j_{\mathrm{H}}) = 0$ whenever $i > j$.

Proof. We consider the induced Katz–Oda filtration on $\mathrm{dR}_{C/A}/\mathrm{Fil}^i_{\mathrm{H}}$. Since we have mod out Hodge filtration, the Lemma 9.2.0.6 (3) implies the desired vanishing of the $\mathrm{Fil}^i_{\mathrm{KO}}$ when $i > j$, and this in turn implies the strict exactness of these filtrations. □

Specializing to the p-adic situation, we get the following:

Lemma 10.2.0.2. *Let $(A, A^+) \to (B, B^+) \to (C, C^+)$ be a triangle of complete Huber rings over k. Then for each $j \in \mathbb{N}$, we have a functorial strict exact filtration on $\mathrm{dR}^{\mathrm{an}}_{C/A}/\mathrm{Fil}^j$, still denoted by $\mathrm{Fil}^i_{\mathrm{KO}}$, witnessing the following sequence:*

$$\mathrm{dR}^{\mathrm{an}}_{C/A}/\mathrm{Fil}^j \to \mathrm{dR}^{\mathrm{an}}_{C/B}/\mathrm{Fil}^j \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{C/B}/\mathrm{Fil}^{j-1}\widehat{\otimes}_B \mathrm{st}_1(\mathbb{L}^{\mathrm{an}}_{B/A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla}$$
$$\mathrm{dR}^{\mathrm{an}}_{C/B}/\mathrm{Fil}^1\widehat{\otimes}_B \mathrm{st}_{j-1}(\mathrm{L}\wedge^{j-1}\mathbb{L}^{\mathrm{an}}_{B/A}).$$

Here $\mathrm{dR}^{\mathrm{an}}_{C/A}/\mathrm{Fil}^j$ and $\mathrm{dR}^{\mathrm{an}}_{C/B}/\mathrm{Fil}^j$ are equipped with Hodge filtrations.

Moreover $\mathrm{Fil}^i_{\mathrm{KO}}(\mathrm{dR}^{\mathrm{an}}_{C/A}/\mathrm{Fil}^j) = 0$ whenever $i > j$.

Proof. For any triangle of rings of definition $A_0 \to B_0 \to C_0$, we p-complete the filtration from Lemma 10.2.0.1 and invert p, then we take the colimit over all triangles of such triples of rings of definition to get the filtration sought after. Since all the operations involved are (derived-)exact, the resulting filtration still has vanishing: $\mathrm{Fil}^i_{\mathrm{KO}} = 0$ whenever $i > j$, and this again implies the strict exactness. $\square$

In the setting of the above Lemma, after taking limit with j going to ∞, we get the following:

Corollary 10.2.0.3 (Poincare Lemma)**.** *Let $(A, A^+) \to (B, B^+) \to (C, C^+)$ be a triangle of complete Huber rings over k. Then there is a functorial strict exact filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A}$ witnessing the following sequence*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{C/B} \xrightarrow{\nabla} \widehat{\mathrm{dR}}^{\mathrm{an}}_{C/B}\hat{\otimes}_B \mathrm{st}_1(\mathbb{L}^{\mathrm{an}}_{B/A}) \to \cdots. \tag{3}$$

The ∇'s are $\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A}$-linear and satisfy Newton–Leibniz rule.

Proof. Take limit in j of the Katz–Oda filtrations on $\mathrm{dR}^{\mathrm{an}}_{C/A}/\mathrm{Fil}^j$ in Lemma 10.2.0.2 gives the desired filtration. Indeed, inverse limit of complete filtrations is again complete. Moreover we have

$$\mathrm{gr}^i_{\mathrm{KO}}(\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A}) \cong \lim_j \mathrm{gr}^i_{\mathrm{KO}}(\mathrm{dR}^{\mathrm{an}}_{C/A}/\mathrm{Fil}^j) \cong \lim_j \left(\mathrm{dR}^{\mathrm{an}}_{C/B}/\mathrm{Fil}^{j-i}\widehat{\otimes}_B \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i]\right)$$
$$\cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{C/B}\widehat{\otimes}_B \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i],$$

so we get the statement about the sequence that this filtration is witnessing.

Lastly the statement about ∇ is the consequence of a general statement about multiplicative filtrations on E_∞-algebras, see the proof of Lemma 9.2.0.6 (2). $\square$

Remark 10.2.0.4. In fact, the discussion of the Poincar'e sequence above could be obtained via a product formula

$$\widehat{\mathrm{dR}}_{C/A}\widehat{\otimes}_{\widehat{\mathrm{dR}}_{B/A}} B \cong \widehat{\mathrm{dR}}_{C/B},$$

similar to the discussion in section Section 9.2. Here the formula can be obtained via a filtered completion, by p-completing the formula in Proposition 9.2.0.4 and inverting p.

We mention that this formula could also be proved by applying the symmetric monoidal functor gr^* and checking the graded pieces, where the claim is reduced to the distinguished triangle of analytic cotangent complexes for a triple of Huber pairs.

10.3 Rational de Rham sheaves

In this section, we shall apply the construction of the (Hodge completed) analytic derived de Rham complexes to the triangle of sheaves of Huber rings $(k, \mathcal{O}_k) \to (\nu^{-1}\mathcal{O}_X, \nu^{-1}\mathcal{O}_X^+) \to (\widehat{\mathcal{O}}_X, \widehat{\mathcal{O}}_X^+)$ on the pro-étale site, where $\nu\colon X_{\text{proét}} \to X$ is the standard map of sites. The procedure is similar to what we did in Section 9.3, except now we allow X to be locally complete intersection [2] over k, and we shall use the unfolding as discussed in Section 2.4.1.

Let K be a perfectoid field extension of k that contains p^n-roots of unity for all $n \in \mathbb{N}$. There is a subcategory $X^{\omega}_{\text{proét}} \subset X_{\text{proét}}$ consisting of affinoid perfectoid objects $U = \mathrm{Spa}(B, B^+) \in X_{K,\text{proét}}$ whose image in X is contained in an affinoid open $\mathrm{Spa}(A, A^+) \subset X$. The class of such objects form a basis for the pro-étale topology by (the proof of) [Sch13a, Proposition 4.8].

Proposition 10.3.0.1. *Let $U = \mathrm{Spa}(B, B^+) \in X^{\omega}_{\text{proét}}$, choose $\mathrm{Spa}(A, A^+) \subset X$ such that the image of U in X is contained in $\mathrm{Spa}(A, A^+)$. Then*

1. *the natural surjection $\theta\colon A_{inf}(B^+)[1/p] \twoheadrightarrow B$ exhibits $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} = B^+_{\mathrm{dR}}(B)$, and the Hodge filtrations are identified with the $\ker(\theta)$-adic filtrations;*

2. *the presheaf defined by sending U to $\mathrm{gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k})$ is a hypersheaf;*

3. *the assignment sending U to $\mathrm{dR}^{\mathrm{an}}_{B/A}/\mathrm{Fil}^n$ is independent of the choice of $\mathrm{Spa}(A, A^+)$, hence so is the assignment sending U to $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$, we denote it as $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$;*

4. *assuming X/k is a local complete intersection, then the presheaf assigning U to $\mathrm{gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X})$ is a hypersheaf.*

Proof. (1) and (3) follows from the same proof of Proposition 9.3.0.1 (1) and (4) respectively.

Now we prove (2). The i-th graded piece of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}$ is isomorphic to $B(i)$ by Theorem 10.1.0.9 (with (A, A^+) there being $(k, \mathcal{O}_k)$). These are hypersheaves as they are supported in cohomological degree 0 and satisfy higher acyclicity by [Sch13a, Lemma 4.10].

Lastly we we turn to (4). The graded pieces of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$, by (2), is the same as $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ for any choice of A. Notice that, by Theorem 10.1.0.9, the $\mathrm{gr}^i(\mathrm{dR}^{\mathrm{an}}_{B/A})$ has a finite step filtration with graded

[2] See Appendix for the notion of local complete intersection that we are using here.

pieces given by $(\mathrm{L}\wedge^j \mathbb{L}^{\mathrm{an}}_{A/k})\otimes_A B(i-j)$. Since hypersheaf property satisfies two-out-of-three principle in a triangle, it suffices to show that the assignment sending

$$\mathrm{Spa}(B,B^+) = U \mapsto \left(\mathrm{L}\wedge^j \mathbb{L}^{\mathrm{an}}_{A/k}\right)\otimes_A B(i-j)$$

is a hypersheaf. This follows from the fact that $\mathbb{L}^{\mathrm{an}}_{A/k}$ is a perfect complex (as X is assumed to be a local complete intersection over k) and, again, that sending U to $B(m)$ is a hypersheaf for any $m\in\mathbb{Z}$. $\square$

In particular, Proposition 10.3.0.1 tells us that the presheaves given by

$$\mathrm{Spa}(B,B^+) = U \in X^{\omega}_{\text{proét}} \mapsto \begin{cases} \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}/\mathrm{Fil}^n \text{ or} \\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \text{ or} \\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}/\mathrm{Fil}^n \text{ or} \\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X} \end{cases},$$

are all hypersheaves on $X^{\omega}_{\text{proét}}$ (assuming X/k is a local complete intersection for the latter two), using the fact that the hypersheaf property is preserved under taking limit, so we may unfold them to get a hypersheaf on $X_{\text{proét}}$.

The authors believe that the conclusion of Proposition 10.3.0.1 (4) (or a variant) should still hold for general rigid spaces instead of only the local complete intersection ones. Hence we ask the following:

Question 10.3.0.2. Given any rigid space X/k, is it true that the presheaf assigning U to $\mathrm{gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X})$ is always a hypersheaf?

The subtlety is that a pro-étale map of affinoid perfectoid algebras need not be flat.

Now we are ready to define the hypersheaf version of the relative de Rham cohomology.

Definition 10.3.0.3. *The* Hodge-completed analytic derived de Rham complex *of $X_{\text{proét}}$ over k, denoted by $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$, is defined to be the unfolding of the hypersheaf on $X^{\omega}_{\text{proét}}$ whose value at $U=\mathrm{Spa}(B,B^+)\in X^{\omega}_{\text{proét}}$ is $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}$.*

Similarly we define a filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ by unfolding the Hodge filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}$. Since values of unfolding are computed by derived limits, we see immediately that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ is derived complete with respect to the filtration.

This construction is related to Scholze's period sheaf $\mathbb{B}^+_{\mathrm{dR}}$ (see [Sch13a, Definition 6.1.(ii)]) by the following:

Proposition 10.3.0.4. *On $X^{\omega}_{\text{proét}}$ we have a filtered isomorphism $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$ of hypersheaves. Consequently, the 0-th cohomology sheaf of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ is identified with the sheaf $\mathbb{B}^+_{\mathrm{dR}}$ as filtered sheaves on $X_{\text{proét}}$.*

Before the proof, we want to mention that under the equivalence $\mathcal{D}(X,k) \cong \mathrm{Sh}^{\mathrm{hyp}}(X,k)$ and its filtered version (c.f. Subsection 2.0.2), this Proposition implies that the derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ is represented by the ordinary sheaf $\mathbb{B}^+_{\mathrm{dR}}$. Here the induced filtration on $\mathcal{H}^0(\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k})$ is given by $\mathcal{H}^0(\mathrm{Fil}^*\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k})$.

Proof. The first sentence follows from Proposition 10.3.0.1 (1).

Given a hypersheaf F supported in cohomological degree 0 on a basis of a site S, it also defines an ordinary sheaf on S (by taking the 0-th cohomology). The unfolding of F is a hypersheaf in $\mathcal{D}^{\geq 0}$, and its 0-th cohomological sheaf is the ordinary sheaf one obtains.

In our situation, we have the basis $X^{\omega}_{\text{proét}}$ of the site $X_{\text{proét}}$, and Scholze's $\mathbb{B}^+_{\mathrm{dR}}$ (and its filtrations) are defined as the ordinary sheaf obtained from $\mathbb{B}^+_{\mathrm{dR}}(\hat{\mathcal{O}}^+_X)$ (and its $\ker(\theta)$-adic filtrations). Now previous paragraph and the first statement give us the second statement. $\square$

Definition 10.3.0.5. *Let X be a local complete intersection rigid space over k. Then the* Hodge-completed analytic derived de Rham complex *of $X_{\text{proét}}$ over X, denoted by $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$, is defined to be the unfolding of the hypersheaf on $X^{\omega}_{\text{proét}}$ whose value at $U = \mathrm{Spa}(B, B^+) \in X^{\omega}_{\text{proét}}$ is $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$.*

Similarly we define a filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ by unfolding the Hodge filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$. So $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ is also derived complete with respect to the filtration.

If X is a local complete intersection rigid space over k with embedded codimension c. Then by Corollary 10.1.0.10 (3), we see that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ lives in $\mathrm{Sh}^{\mathrm{hyp}}(X_{\text{proét}}, \mathcal{D}^{\geq -c}(k))$.

The Poincaré Lemma obtained in the previous section now immediately yields the following:

Theorem 10.3.0.6. *Let X be a local complete intersection rigid space over k. Then there is a functorial strict exact filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ witnessing the following:*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \mathrm{st}_1(\nu^{-1}(\mathbb{L}^{\mathrm{an}}_{X/k})) \xrightarrow{\nabla} \cdots$$

If X is further assumed to be smooth over k of equidimension d, then the following $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$-linear sequence

$$\begin{aligned} 0 \to \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \mathrm{st}_1(\nu^{-1}(\mathbb{L}^{\mathrm{an}}_{X/k})) \xrightarrow{\nabla} \cdots \\ \cdots \xrightarrow{\nabla} \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \mathrm{st}_d(\nu^{-1}(\mathrm{L} \wedge^d \mathbb{L}^{\mathrm{an}}_{X/k})) \to 0 \end{aligned}$$

is strict exact.

Note that as X/k is assumed to be local complete intersection, these wedge powers of the analytic cotangent complex are (locally) perfect complexes, hence the completed tensor is the same as just tensor.

Proof. Since both unfolding and taking gr^i commute with taking limits, the above follows from unfolding Corollary 10.2.0.3, and the fact that the completed tensor in Corollary 10.2.0.3 is the same as tensor for local complete intersections X/k.

When X is smooth over k, everything in sight (on the basis of affinoid perfectoids in $X^{\omega}_{\text{proét}}$) are supported cohomologically in degree 0 with filtrations given by submodules because of Theorem 10.1.0.9, Corollary 10.1.0.10, and Proposition 10.3.0.1, the strict exact Katz–Oda filtration gives what we want. □

10.4 Comparing with Scholze's de Rham period sheaf

In this section we show that when X is smooth, the de Rham sheaf $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ defined above is related to Scholze's de Rham period sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$. We refer readers to [Sch16, part (3)] for the its definition. Following notation of loc. cit., let $\mathrm{Spa}(R_i, R_i^+)$ be an affinoid perfectoid in $X_{\text{proét}}$ with $\mathrm{Spa}(R_0, R_)^+)$ an affinoid open in X. Then for any i, we have maps

$$R_i^+ \to \mathrm{dR}^{\mathrm{an}}_{R^+/R_i^+} \text{ and } \mathbb{A}_{inf}(R^+) = \mathrm{dR}^{\mathrm{an}}_{R^+/W(\kappa)} \to \mathrm{dR}^{\mathrm{an}}_{R^+/R_i^+},$$

which is compatible with maps to R^+, here κ denotes the residue field of k. The equality above is deduced from Theorem 9.1.0.4 (1). Therefore we get an induced map

$$R_i^+ \hat{\otimes}_{W(\kappa)} \mathbb{A}_{inf}(R^+) \to \mathrm{dR}^{\mathrm{an}}_{R^+/R_i^+} \to \widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_i}.$$

Taking the composition map above, inverting p and completing along the kernel of the surjection onto R (note that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_i}$ lives in cohomological degree 0 by Corollary 10.1.0.10 (2) and is already complete with respect to this filtration), we get a natural arrow:

$$\big((R_i^+ \hat{\otimes}_{W(\kappa)} \mathbb{A}_{inf}(R^+))[1/p]\big)^{\wedge} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_i} \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_0},$$

here we apply Corollary 10.2.0.3 to $(R_0, R_0^+) \to (R_i, R_i^+) \to (R, R^+)$ to see the filtered isomorphism above. This arrow is compatible with index i, hence after taking colimit, we get the following map of sheaves on $X^{\omega}_{\text{proét}}$ (see the discussion before Proposition 10.3.0.1 for the meaning of $X^{\omega}_{\text{proét}}$):

$$f \colon \mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \mid_{X^{\omega}_{\text{proét}}} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X},$$

which is compatible with maps to $\hat{\mathcal{O}}_{X_{\text{proét}}}$ and maps from $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$.

Theorem 10.4.0.1. *The map f above induces a filtered isomorphism of sheaves on $X^{\omega}_{\text{proét}}$. Hence we get that $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ is the 0-th cohomology sheaf of the hypersheaf $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ on $X_{\text{proét}}$.*

Similar to Proposition 10.3.0.4, under the equivalence $\mathcal{D}(X, k) \cong \mathrm{Sh}^{\mathrm{hyp}}(X, k)$ and its filtered version (c.f. Subsection 2.0.2), this Theorem implies that the derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ is represented by the ordinary sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$.

Proof. The second sentence follows from the first sentence, due to the same argument in the proof of the second statement of Proposition 10.3.0.4. So it suffices to show the first statement.

On both sheaves, there are natural filtrations: on $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ we have the $\ker(\theta)$-adic filtration where $\theta\colon \mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \to \hat{\mathcal{O}}_{X_{\text{proét}}}$ and on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ we have the Hodge filtration with the first Hodge filtration being kernel of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \twoheadrightarrow \hat{\mathcal{O}}_{X_{\text{proét}}}$. Since f is compatible with maps to $\hat{\mathcal{O}}_{X_{\text{proét}}}$ and the Hodge filtration is multiplicative, it suffices to show that f induces an isomorphism on their graded pieces. Now locally on $X^{\omega}_{\text{proét}}$, we have that $\mathrm{gr}^*(\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}) \cong \mathrm{Sym}^*_{\hat{\mathcal{O}}_{X_{\text{proét}}}}(\mathrm{gr}^1\mathcal{O}\mathbb{B}^+_{\mathrm{dR}})$ by [Sch13a, Proposition 6.10] and similarly $\mathrm{gr}^*(\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}) \cong \mathrm{Sym}^*_{\hat{\mathcal{O}}_{X_{\text{proét}}}}(\mathrm{gr}^1\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X})$ by Theorem 10.1.0.9 (note that in characteristic 0 divided powers are the same as symmetric powers). Therefore we have reduced ourselves to showing that f induces an isomorphism on the first graded pieces. Their first graded pieces admits a common submodule given by the first graded pieces of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$ which is $\hat{\mathcal{O}}_{X_{\text{proét}}}(1)$.

Now we get the following diagram:

$$\begin{tikzcd}
\hat{\mathcal{O}}_{X_{\text{proét}}}(1) \arrow[r] \arrow[d, "\cong"] & \mathrm{gr}^1\mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \arrow[r] \arrow[d, "\mathrm{gr}^1 f"] & \hat{\mathcal{O}}_{X_{\text{proét}}} \otimes_{\mathcal{O}_X} \Omega^{\mathrm{an}}_X \arrow[d, "g"] \\
\hat{\mathcal{O}}_{X_{\text{proét}}}(1) \arrow[r] & \mathrm{gr}^1\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \arrow[r] & \hat{\mathcal{O}}_{X_{\text{proét}}} \otimes_{\mathcal{O}_X} \Omega^{\mathrm{an}}_X
\end{tikzcd}$$

with both rows being short exact (by [Sch13a, Corollary 6.14] and Theorem 10.1.0.9 respectively) and the left vertical arrow being an isomorphism as f is compatible with the maps from $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$, which is why we get the induced arrow g. Moreover f is linear over $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$, which implies that g is linear over $\hat{\mathcal{O}}_{X_{\text{proét}}}$. Therefore it suffices to show that g induces an isomorphism.

As the statement is étale local, we may assume that

$$X = \mathbb{T}^n = \mathrm{Spa}(k\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle, \mathcal{O}_k\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle).$$

Denote $\mathbb{T}^n_\infty$ the pro-finite-étale tower above $\mathbb{T}^n$ given by adjoining p-power roots of the coordinates

T_i. We have the following diagram

$$\begin{tikzcd}
\mathbb{Z}_p\langle T_i^{\pm 1}, S_i^{1/p^\infty}\rangle = \mathbb{Z}_p\langle T_i^{\pm 1}\rangle\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p\langle S_i^{1/p^\infty}\rangle \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \mid_{\mathbb{T}^n_\infty} \arrow[d, "f"] \\
\mathbb{Q}_p\langle S_i^{\pm 1/p^\infty}\rangle[\![X_i]\!] \arrow[r, "\gamma"] & \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X} \mid_{\mathbb{T}^n_\infty} .
\end{tikzcd}$$

Here the arrow β is given by sending T_i to $X_i + S_i$, and S_i is sent to $1 \otimes [T_i^\flat]$ under α. The element $\alpha(T_i - S_i)$ is $u_i \in \mathrm{Fil}^1\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ whose image in $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}} \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{an}}$ is $1 \otimes dT_i$, see the discussion before [Sch13a, Proposition 6.10]. On the other hand, the element $\beta(T_i - S_i)$ is X_i, and the image of $\gamma(X_i)$ in $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}} \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{an}}$ is also $1 \otimes dT_i$ by Example 10.1.0.7, Example 9.1.0.6 and Example 9.1.0.7. Therefore we get that $g(1 \otimes dT_i) = 1 \otimes dT_i$, since g is linear over $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}$ and Ω_X^{an} is generated by dT_i's, we see that g is an isomorphism, hence finishes the proof. $\square$

Remark 10.4.0.2. In the process of the proof above, we also see that under the identification in Proposition 10.3.0.4 and Theorem 10.4.0.1, the Poincaré sequence obtained in Theorem 10.3.0.6 and the one in Scholze's paper [Sch13a, Corollary 6.13] matches, c.f. proof of the second statement of Theorem 9.4.0.1.

Also the Faltings' extension (see [Sch13a, Corollary 6.14] and Theorem 10.1.0.9), being the first graded pieces of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \cong \mathcal{H}^0(\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X})$, is matched up. In some sense, our proof above reduces to identifying the Faltings' extension, and this is a well-known fact to experts. In fact, this project was initiated after Bhargav Bhatt explained to us how to get Faltings' extension from the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}$.

10.5 An example

In this complementary section, we would like to compute the Hodge-completed analytic derived de Rham complex of a perfectoid algebra over a 0-dimensional k-affinoid algebra. Surprisingly, the underlying algebra (forgetting its filtration) one get always lives in cohomological degree 0, which leads us to the Question 10.5.0.3.

Without loss of generality, let (K, K^+) be a perfectoid field over k, containing all p-power roots of unity, and let A be an artinian local finite k-algebra with residue field being k as well. Let (B, B^+) be a perfectoid affinoid algebra containing (K, K^+) and let $A \to B$ be a morphism of k-algebras. Since perfectoid affinoid algebras are reduced, we get a sequence of maps $k \to A \to k \to B$.

By the above sequence, we get natural filtered k-linear maps:

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \text{ and } \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

This induces a filtered map:

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \otimes_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$$

where the filtration on the source comes from the symmetric monoidal structure on $\mathrm{DF}(k)$. Since this map is compatible with the filtration and the target is complete with respect to its filtration, we get an induced map:

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

Proposition 10.5.0.1. *The map $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ above is a filtered isomorphism.*

Proof. Since both are complete with respect to their filtrations, it suffices to show the map induces an isomorphism on the graded pieces. The graded algebra of both sides are the symmetric algebra (over B) on their first graded pieces, hence it suffices to check $\mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A}) \longrightarrow \mathrm{gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$ being an isomorphism. This follows from the decomposition of analytic cotangent complexes

$$\mathbb{L}^{\mathrm{an}}_{B/A} \cong \mathbb{L}^{\mathrm{an}}_{B/k} \oplus (\mathbb{L}^{\mathrm{an}}_{A/k} \otimes_A B)$$

which is deduced from contemplating the sequence $k \to A \to k \to B$. $\square$

We know that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \cong \mathbb{B}^+_{\mathrm{dR}}(B)$, a result of Bhatt tells us the underlying algebra of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong A$, explanined below. Since $A \to k$ is a surjection, the analytic cotangent complex agrees with the classical cotangent complex, hence we have a filtered isomorphism

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}_{k/A}.$$

Now [Bha12a, Theorem 4.10] implies the underlying algebra $\widehat{\mathrm{dR}}_{k/A}$ is isomorphic to the completion of A along the surjection $A \to k$. Since A is an artinian local ring, this completion is simply A itself. Therefore we get a map of the underlying algebras:

$$\mathbb{B}^+_{\mathrm{dR}}(B) \otimes_k A \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A}$$

Proposition 10.5.0.2. *The map $\mathbb{B}^+_{\mathrm{dR}}(B) \otimes_k A \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A}$ above is an isomorphism. Consequently we have an isomorphism*

$$\mathbb{B}^+_{\mathrm{dR}}(B) \otimes_k A \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

Proof. By definition, we have

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong \lim_n \lim_m \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m,$$

here we have used the (filtered) identification $\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong \widehat{\mathrm{dR}}_{k/A}$ spelled out before this Proposition.

We claim that for any given n, we have an isomorphism

$$\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k A \cong \lim_m \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m.$$

Indeed for each $i \in \mathbb{Z}$, we have the following short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{R}^1 \lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{H}^{i-1}(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)\right) \arrow[dl] \\
\mathrm{H}^i(\lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m\right)) \arrow[r] & \lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{H}^i(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)\right) \arrow[dl] \\
0 &
\end{tikzcd}$$

Since for each m and i, the vector space $\mathrm{H}^{i-1}(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)$ is finite dimensional over k, we see that the inverse system $\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{H}^{i-1}(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)$ satisfies Mittag-Leffler condition, hence the $\mathrm{R}^1 \lim$ term vanishes. By [Bha12a, Theorem 4.10], we have that the inverse system $\{\mathrm{H}^i(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)\}_m$ is pro-isomorphic to 0 if $i \neq 0$ and is pro-isomorphic to A (since A is finite dimensional over k) if $i = 0$, therefore the above short exact sequence becomes

$$\mathrm{H}^i(\lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m\right)) \cong \begin{cases} 0;\ i \neq 0 \\ \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k A;\ i = 0. \end{cases}$$

This gives us the claim above.

Now we have

$$\begin{aligned}&\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong \lim_n (\lim_m \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m) \cong \lim_n (\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k A) \\ &\cong \mathbb{B}^+_{\mathrm{dR}}(B) \otimes_k A\end{aligned}$$

as desired, where the last identification follows from the fact that A is finite over k. $\square$

If one contemplates the example $A = k[\epsilon]/(\epsilon^2)$, one sees that $\mathrm{dR}^{\mathrm{an}}_{B/A}/\mathrm{Fil}^i$ does not live in cohomological degree 0 alone for any $i \geq 2$.

As a consequence of the above Proposition, for the $X = \mathrm{Spa}(A)$ we have an equality of presheaves on $X^{\omega}_{\text{proét}}$:

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \cong \mathbb{B}^+_{\mathrm{dR}} \otimes_k \nu^{-1}\mathcal{O}_X,$$

in particular the underlying algebra of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ pro-étale locally lives in cohomological degree 0.

Motivated by this computation and results in [Bha12a], we end this chapter by asking the following:

Question 10.5.0.3. In what generality shall we expect $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}|_{X^{\omega}_{\mathrm{pro\acute{e}t}}}$ to live in cohomological degree 0? And when that happens, can we re-interpret the underlying algebra via some construction similar to Scholze's $\mathcal{O}\mathbb{B}^{+}_{\mathrm{dR}}$ as in [Sch13a] and [Sch16]?

APPENDIX 1

Local complete intersections in rigid geometry

In this appendix we make a primitive discussion of local complete intersection morphisms in rigid geometry. We remark that the results recorded here hold verbatim with k being a general complete non-Archimedean field. This appendix is taken from the one in [GL20]

In order to talk about local complete intersections, we need to understand how being of finite Tor dimension[1] behaves under base change in rigid geometry.

Lemma 1.0.0.1. *Let A and B be two affinoid k-algebras, and $A \to B$ a morphism of Tor dimension m. Let $P := A\langle T_1, \ldots, T_n\rangle \twoheadrightarrow B$ be a surjection, then we have*

$$\operatorname{Tor}\dim_P(B) \le m + n.$$

The following proof is suggested to us by Johan de Jong.

Proof. Choose a resolution of B by finite free P-modules

$$\ldots \xrightarrow{d_i} M_i \xrightarrow{d_{i-1}} M_{i-1} \ldots \xrightarrow{d_0} M_0 \twoheadrightarrow B.$$

Since P is flat over A, we see that $M := \operatorname{Coker}(d_m)$ is flat over A as $A \to B$ is assumed to be of Tor dimension m [Sta18, Tag 0653]. Moreover M is finitely generated over P since P is Noetherian. Now we use [Li19, Lemma 6.3] to see that M admits a projective resolution over P of length n. Therefore we get that B has a projective resolution over P of length $m + n$. $\square$

Lemma 1.0.0.2. *Let A and B be two affinoid k-algebras, and $A \to B$ a morphism of finite Tor dimension. Let C be any affinoid A-algebra, then the base change (in the realm of rigid geometry) $C \to B\hat{\otimes}_A C$ is also of finite Tor dimension.*

Proof. Choose a surjection $A\langle T_1, \ldots, T_n\rangle \twoheadrightarrow B$, which is of finite Tor dimension by Lemma 1.0.0.1. Then we have a factorization:

$$C \to C\langle T_1, \ldots, T_n\rangle \to B \otimes_{A\langle T_1,\ldots,T_n\rangle} C\langle T_1, \ldots, T_n\rangle \cong B\hat{\otimes}_A C.$$

[1] In classical literature such as [Avr99] this corresponds to the notion of having finite flat dimension.

Since the first arrow is flat and the second arrow, being base change of an arrow of finite Tor dimension, is of finite Tor dimension, we conclude that the composition is of finite Tor dimension [Sta18, Tag 066J]. □

Proposition 1.0.0.3. *Let $A \to B$ a morphism of k-affinoid algebras. Then the following are equivalent:*

1. *any surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ is a local complete intersection;*
2. *there exists a surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ which is a local complete intersection;*
3. *$A \to B$ is of finite Tor dimension and the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ is a perfect B-complex.*

Moreover, any of these three equivalent conditions implies that $\mathbb{L}^{\mathrm{an}}_{B/A}$ is a perfect complex with Tor amplitude in $[-1, 0]$.

Proof. It is easy to see that (1) implies (2).

To see (2) implies (3), first of all $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ is a local complete intersection implies that it is of finite Tor dimension. Since $A \to A\langle T_1, \dots, T_n\rangle$ is flat, we see that $A \to B$ is also finite Tor dimension by [Sta18, Tag 0653]. Next we look at the triangle $A \to A\langle T_1, \dots, T_n\rangle \to B$, which gives rise to a triangle of analytic cotangent complexes:

$$\mathbb{L}^{\mathrm{an}}_{A\langle T_1,\dots,T_n\rangle/A} \otimes_A B \to \mathbb{L}^{\mathrm{an}}_{B/A} \to \mathbb{L}^{\mathrm{an}}_{B/A\langle T_1,\dots,T_n\rangle}.$$

Now Theorem 10.1.0.2 (3) gives that the first term is a perfect complex with Tor amplitude in $[0, 0]$, while condition (2) and Theorem 10.1.0.2.(4) implies that the third term is a perfect complex with Tor amplitude in $[-1, -1]$, hence we see that (2) implies (3) and gives the last sentence as well.

Lastly we need to show that (3) implies (1). To that end we apply Avramov's solution of Quillen's conjecture [Avr99]. As $A \to B$ is of finite Tor dimension, we see that any surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ has finite Tor dimension by Lemma 1.0.0.1. The previous paragraph shows that $\mathbb{L}^{\mathrm{an}}_{B/A}$ being a perfect complex is equivalent to the classical cotangent complex $\mathbb{L}_{B/A\langle T_1,\dots,T_n\rangle}$ being a perfect complex. Now we use Avramov's result [Avr99, Theorem 1.4] to conclude that $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ is a local complete intersection. □

Definition 1.0.0.4. *Let $A \to B$ be a morphism of k-affinoid algebras. The morphism $A \to B$ of k-affinoid algebras is called* a local complete intersection *if one of the three equivalent conditions in Proposition 1.0.0.3 is satisfied.*

Let $Y \to X$ be a morphism of rigid spaces over k. Then this morphism is called a local complete intersection *if for any pair of affinoid domains U and V in X and Y, such that the image of V is contained in U, the induced map of k-affinoid algebras is a local complete intersection.*

We leave it as an exercise (using Theorem 10.1.0.2) that a morphism being a local complete intersection may be checked locally on the source and target. We caution readers that there is a notion of local complete intersection morphism between Noetherian rings, while the notion we define here should (clearly) only be considered in the situation of rigid geometry. These two notions agree when the morphism considered is a surjection. We hope this slight abuse of notion will not cause any confusion. But as a sanity check, let us show here that this notion matches the corresponding notion in classical algebraic geometry under rigid-analytification. The following is suggested to us by David Hansen.

Proposition 1.0.0.5. *Let $f\colon X \to Y$ be a morphism of schemes locally of finite type over a k-affinoid algebra A with rigid-analytification $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$. Then f is a local complete intersection (in the classical sense) if and only if f^{an} is a local complete intersection (in the sense of Definition 1.0.0.4).*

Proof. We first reduce to the case where both of X and Y are affine. Then we may check this after fiber product Y with an affine space so that f is a closed embedding. In this situation, we have identification of ringed sites $X^{\mathrm{an}} \cong X \times_Y Y^{\mathrm{an}}$ and an identification of cotangent complexes:

$$\iota^*\mathbb{L}_{X/Y} \simeq \mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}/Y^{\mathrm{an}}},$$

where $\iota\colon X^{\mathrm{an}} \to X$ is the natural map of ringed sites.

Now we use the fact that classical Tate points on X^{an} is in bijection with closed points on X, and for any such point x, the map $\iota^\sharp\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ of local rings is faithfully flat. Therefore we can check $\mathbb{L}_{X/Y}$ being perfect by pulling back along ι, hence $\mathbb{L}_{X/Y}$ is perfect if and only if $\mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}/Y^{\mathrm{an}}}$ is perfect, and this finishes the proof. $\square$

Next we turn to understand the localization of analytic cotangent complexes for a local complete intersection morphism.

Let us introduce some notions:

Definition 1.0.0.6. *Let $A \to B$ be a morphism of k-affinoid algebras. Let $\mathfrak{m} \subset B$ be a maximal ideal, the* embedded dimension of B/A at $\mathfrak{m}$ *is defined to be the following*

$$\dim_{B/A,\mathfrak{m}} := \dim_{\kappa(\mathfrak{m})}(\Omega^{\mathrm{an}}_{B/A} \otimes_B B/\mathfrak{m}).$$

Let $\mathfrak{n}$ be the preimage of $\mathfrak{m}$ in A (which is also a maximal ideal), we define the embedded codimension of B/A at $\mathfrak{m}$ *to be*

$$\dim_{B/A,\mathfrak{m}} + \dim(A_{\mathfrak{n}}) - \dim(B_{\mathfrak{m}}).$$

The embedded codimension of B/A *is the supremum of that at all maximal ideals* $\mathfrak{m} \subset B$.

Proposition 1.0.0.7. *Let* $A \to B$ *be a local complete intersection morphism of* k*-affinoid algebras. Then at any maximal ideal* $\mathfrak{m} \subset B$*, there is a presentation of the analytic cotangent complex*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B_{\mathfrak{m}} \simeq \left[B_{\mathfrak{m}}^{\oplus c(\mathfrak{m})} \to B_{\mathfrak{m}}^{\oplus d(\mathfrak{m})}\right]$$

where $c(\mathfrak{m})$ *is the embedded codimension of* B/A *at* $\mathfrak{m}$ *and* $d(\mathfrak{m})$ *is the embedded dimension of* B/A *at* $\mathfrak{m}$*. Here* $B_{\mathfrak{m}}^{\oplus d(\mathfrak{m})}$ *is put in degree* 0*.*

In particular the Tor amplitude of $\mathrm{LSym}^i \mathbb{L}^{\mathrm{an}}_{B/A}$ *is always in* $[-\min\{c, i\}, 0]$ *where* c *is the embedded codimension of* B/A*.*

Proof. We may always replace B by a rational domain containing the point $\mathfrak{m}$ (viewed as a classical Tate point on the associated adic space), so we can assume there are power bounded elements $f_1, \dots, f_{d(\mathfrak{m})}$ whose differentials generate the stalk of $\Omega^{\mathrm{an}}_{B/A}$ at $\mathfrak{m}$. Thus we have a map $A' := A\langle T_1, \dots, T_{d(\mathfrak{m})}\rangle \to B$ which is unramified at $\mathfrak{m}$, see [Hub96, Section 1.6]. By Proposition 1.6.8 of loc. cit. we can factortize the map $A' \to B$ as $A' \xrightarrow{h} C \xrightarrow{g} B$ where h is étale and g is surjective.

One checks that the étaleness of h guarantees that the surjection $C \xrightarrow{g} B$ has finite Tor dimension. Moreover Theorem 10.1.0.2 implies that $\mathbb{L}_{B/C}$ is a perfect complex because of the triangle

$$\mathbb{L}^{\mathrm{an}}_{C/A} \otimes_C B \to \mathbb{L}^{\mathrm{an}}_{B/A} \to \mathbb{L}_{B/C}.$$

Hence $C \to B$ is a surjective local complete intersection. Hence the kernel of $C \to B$ around $\mathfrak{m}$ is generated by a length $c(\mathfrak{m})$ regular sequence. This in turn implies that $\mathbb{L}_{B/C} \otimes_B B_{\mathfrak{m}} \simeq B_{\mathfrak{m}}^{\oplus c(\mathfrak{m})}[1]$, which together with the triangle above gives the local presentation we want in the statement.

The statement concerning Tor amplitude can be checked at every maximal ideal which, by our presentation, follows from the formula $\mathrm{LSym}^i(C[1]) \simeq \mathrm{L}\wedge^i(C)[i]$, see [Ill71, V.4.3.4]. $\square$

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