The Closed Support Problem over a Complete Intersection Ring

by

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ABSTRACT

The issue of closed support for the local cohomology of Noetherian modules and the related problem of finiteness of the set of associated primes of local cohomology have been intensely studied in the literature. Although it is known that the local cohomology of a hypersurface ring of characteristic $p > 0$ has closed support, it remains an open question whether this property holds for complete intersection rings of higher codimension. For an ideal J generated by a regular sequence of length c in a regular ring R of characteristic $p > 0$, the closed support property for the local cohomology of R/J would follow from known results in the literature if the local cohomology of J itself always had a finite set of associated primes. We give the first example of a module of the form $H^i_I(J)$ with an infinite set of associated primes to demonstrate that this is not necessarily the case. In fact, for $i < 4$ (resp. $i < 5$), we show that if R/J is a domain (resp. factorial), then $\operatorname{Ass} H^i_I(J)$ is finite if and only if $\operatorname{Ass} H^{i-1}_I(R/J)$ is finite. Our proof of this statement involves a novel generalization of an isomorphism of Hellus. To obtain positive results on closed support, in joint work of the author with Eric Canton, we construct a chain complex consisting of direct sums of Frobenius-stable annihilators in the local cohomology module $H^c_J(R)$. We prove that this complex is exact, and using the exactness property, we show that for an ideal I of R such that R/I is Cohen-Macaulay, the module $H^{\mathrm{ht}(I/J)+c}_{I/J}(R/J)$ has closed support.

CHAPTER I

Introduction

Algebra is concerned with structure. We are interested in the similarities and differences between various instances of an abstract structure specified by some list of formal properties. For example, the abstract structure of a *ring* consists of a set R equipped with two binary operations $+$ and $\cdot$, referred to as addition and multiplication, respectively, that together satisfy several properties. We require that addition is commutative and associative, that there exists an element 0 in R satisfying $0+a=a$ for all a, and that for each a, there is an element $-a$ satisfying $-a+a=0$. Multiplication must be associative, must distribute over addition, and we require that there exists an element 1 in R such that $1 \cdot a = a \cdot 1 = a$. If multiplication in R also satisfies the commutative property, $a \cdot b = b \cdot a$ for all a and b, then R is called a *commutative ring.*

Below are several instances of the abstract structure of a commutative ring.

1. Let $\mathbb{Z}$ denote the set of integers, where addition and multiplication are familiar.
2. Let $\mathbb{Q}[X]$ denote the set of polynomials in the variable X with coefficients belonging to the rational numbers $\mathbb{Q}$. Addition and multiplication of polynomials is also quite familiar.
3. Let $\mathbb{C}$ denote the set of complex numbers $a+bi$, where a and b are real numbers.

Addition and multiplication treat i as a variable in a polynomial expression with the additional requirement $i \cdot i = -1$.

4. Let C_{12} denote the set $\{[1], [2], \ldots, [12]\}$ regarded as hour symbols on the face of a clock, with addition and multiplication defined accordingly – for example, $[3] + [11] = [2]$ and $[2] \cdot [8] = [4]$.

5. Let $\mathbb{F}_2$ denote the set of Boolean truth values $\{\texttt{TRUE}, \texttt{FALSE}\}$, with addition defined by logical `XOR`, and multiplication defined by logical `AND`.

6. Let $\mathbb{F}_2[z, w]$ denote the set of polynomials in the variables z and w, with coefficients belonging to $\mathbb{F}_2$. Addition and multiplication are defined in a manner formally similar to in $\mathbb{Q}[X]$, but coefficients are manipulated using the operations of $\mathbb{F}_2$.

7. Let S denote the set of continuous functions from the unit sphere to the real numbers, with addition and multiplication defined pointwise.

While each is an example of a commutative ring, the algebraic differences between these instances are at least as interesting as their similarities. The statement

"*For each nonzero element a of the ring R, there is an element b such that $ab = 1$.*"

is true for $\mathbb{C}$ and $\mathbb{F}_2$, but false for $\mathbb{Z}$, $\mathbb{Q}[X]$, C_{12}, $\mathbb{F}_2[z, w]$, and S. The cancellation property

"*For any b and c in the ring R, if $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$.*"

holds in every example given above except for the ring C_{12} – consider that $[2] \cdot [8] = [2] \cdot [2]$ but $[8] \neq [2]$ – and the ring S. A nonzero commutative ring that satisfies the cancellation property is called an *integral domain* or sometimes just a *domain*. This is equivalent to the condition that $ab = 0$ implies $a = 0$ or $b = 0$.

We may also investigate algebraic properties that have no analogue in familiar structures like the integers or real numbers. For example, the statement

"*For all elements a and b in the ring R, it holds that* $(a+b)^2 = a^2 + b^2$*.*"

is true for $\mathbb{F}_2$ and $\mathbb{F}_2[z,w]$, but false for all other examples given. This strange property has far-reaching implications for the rings in which it holds.

The class of commutative rings is extraordinarily rich and, for this reason, it is extraordinarily difficult to prove nontrivial theorems that hold for all commutative rings. By imposing additional hypotheses, we may obtain interesting classes of commutative rings that are more tractable to work with.

A subset I of a commutative ring R that is closed under addition and that satisfies $ar \in I$ for all $a \in I$ and all $r \in R$ is called an *ideal*. We say that an ideal I is *generated* by a list of elements $a_1, \ldots, a_m \in I$ if for each $b \in I$, there exist $r_1, \ldots, r_t \in R$ such that $b = r_1 a_1 + \ldots r_m a_m$. The ideal generated by $a_1, \ldots, a_m$ is sometimes denotes $Ra_1 + \ldots + Ra_m$ or $(a_1, \ldots, a_m)R$. A ring is called *Noetherian* – named after the algebraist Emmy Noether – if every ideal is generated by a finite list of elements. It is equivalent to require that every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ terminates in the sense that $I_n = I_{n+1}$ for all sufficiently large n. The study of commutative Noetherian rings holds a position of central importance in commutative algebra. The class is rich enough to capture most rings one will encounter in the related fields of algebraic number theory and algebraic geometry, while still imposing a sufficiently high level of control to enable the proof of some truly remarkable theorems.

The rings $\mathbb{Z}$, $\mathbb{Q}[X]$, $\mathbb{C}$, C_{12}, $\mathbb{F}_2$, and $\mathbb{F}_2[z,w]$ are Noetherian. The ring S is not. Almost all[1] rings we study from this point on will be commutative and Noetherian.

[1] with one notable exception, given in Chapter III

Some of the most striking differences between various special classes of Noetherian rings become apparent only when we consider how that ring acts on other algebraic objects. A *module* over a ring R, also called an R-module, is a set M equipped with an addition operation[2] and a *linear action* of R, also called a scalar multiplication, $r \cdot m$ for $r \in R$ and $m \in M$ that distributes over addition: $r \cdot (m+n) = r \cdot m + r \cdot n$ for $r \in R$, $m, n \in M$.

It is often easier to study *finitely generated* R-modules. We say that M is generated by a list of elements $m_1, \ldots, m_t \in M$ if $M = Rm_1 + \cdots + Rm_t$, where this notation is defined similarly to its use in the context of ideals. In fact, an ideal of R is precisely a subset that is also an R-module when equipped with the $+$ and $\cdot$ operations inherited from R. The more general notion of a *submodule* of an R-module M is defined similarly.

For commutative rings like $\mathbb{C}$ and $\mathbb{F}_2$ where every nonzero element has a multiplicative inverse – such rings are called *fields* – there is a particularly simple description of all finitely generated modules, at least up to isomorphism[3]

Theorem. *Let K be a field. Every finitely generated R-module is isomorphic to a direct sum[4] of the form $K^{\oplus n}$. This number n is the rank of M, and two K-modules with different ranks are non-isomorphic.*

The module theory of a field – $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{F}_2$, or otherwise – is precisely the study of vector spaces over that field. For any commutative ring R, we can make sense of the direct sum $R^{\oplus n}$ – called a *free module* of rank n – and it remains true that a free module is determined up to isomorphism by its rank. However, unless R is a field,

[2] Namely, a commutative, associative operation $+$ on M for which there exists an element $0 \in M$ satisfying $0 + m = m$ for all $m \in M$, and such that every element m of M has an additive inverse $-m$ satisfying $-m + m = 0$

[3] Two modules are called *isomorphic* if there is an invertible structure preserving map between them. That is, an invertible map $f : M \to N$ that satisfies $f(m+n) = f(m) + f(n)$ and $f(r \cdot m) = r \cdot f(m)$ for $r \in R$, $m \in M$.

[4] If M is an R-module, $M^{\bigoplus n}$ consists of n-tuples of elements of m with addition and scalar multiplication defined componentwise. The direct sum $M_1 \oplus \cdots \oplus M_t$ is defined analogously.

the free modules alone do not paint a complete picture.

The integers $\mathbb{Z}$ may, at first glance, seem somewhat simpler than the complex numbers $\mathbb{C}$. Upon investigating their corresponding module theories, however, one will find that $\mathbb{Z}$ is structurally somewhat more complicated than $\mathbb{C}$.

Definition I.1. Let M be a $\mathbb{Z}$-module and p be a prime integer. The *p-torsion component* of M, denoted $\Gamma_p(M)$, is the set of elements u in M such that $p^k \cdot u = 0$ for some natural number k.

Theorem I.2. *Every finitely generated $\mathbb{Z}$-module M is isomorphic to a direct sum $\mathbb{Z}^{\oplus n} \oplus \Gamma_{p_1}(M) \oplus \cdots \oplus \Gamma_{p_t}(M)$ for some $n \geq 0$ and some (possibly empty) list $p_1, \ldots, p_t$ of distinct prime integers.*

It is possible to completely classify[5] all finitely generated p-torsion $\mathbb{Z}$-modules, but even a complete understanding of internal structure does not necessarily imply an understanding of mappings between structures.

The situation over a field can be misleading in this respect. If K is a field, V is an n-dimensional vector space over K, and U is an m-dimensional subspace of V, then the quotient V/U is always an $n - m$ dimensional vector space. Without knowing anything whatsoever about the manner in which U is embedded into V, we can immediately classify the quotient of V by U up to isomorphism. This is not the case over $\mathbb{Z}$.

For any prime integer p, the p-torsion components of $\mathbb{Z}$ and its submodules $2\mathbb{Z}$ and $3\mathbb{Z}$ are zero. The quotients $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, however, both have nontrivial torsion. The quotient of $\mathbb{Z}^{\oplus 2}$ by the (free, rank 1) $\mathbb{Z}$-module spanned by $(1, -1)$ is a free module of rank 1 with no torsion components. The quotient of $\mathbb{Z}^{\oplus 2}$ by the (free,

[5] An analogous classification exists for finitely generated $\mathbb{Q}[x]$-modules, but both this classification and the analogue of Theorem I.2 fail even for a ring like $\mathbb{F}_2[z, w]$.

rank 1) $\mathbb{Z}$-span of $(14, 0)$ has a free component of rank 1, a 2-torsion component, and a 7-torsion component. The nature of the embedding significantly affects the structure of the resulting quotient.

That the p-torsion structure of M/N cannot be determined from the p-torsion components of M and N suggests that our understanding of the operation $\Gamma_p(-)$ is incomplete.

1.1 The local cohomology of a commutative ring

From this point onward, we will freely make use of terminology and basic tools of homological algebra. The reader may consult [Wei94] as a reference. For general definitions and results on commutative rings, see [Mat89].

To any commutative ring R and any ideal I of R, we may define a functor $\Gamma_I(-)$ that takes an R-module M to its I-torsion component $\Gamma_I(M)$, defined as the set of elements $u \in M$ such that $I^n u = 0$ for some $n > 0$. Similar to what we observed in our discussion of p-torsion components over $\mathbb{Z}$, one cannot determine $\Gamma_I(M/N)$ given only the modules $\Gamma_I(M)$ and $\Gamma_I(N)$ or even given the map $\Gamma_I(M \to N)$.

The functor $\Gamma_I(-)$ is left exact but not exact. For a short exact sequence

$$0 \to N \to M \to M/N \to 0$$

the corresponding sequence with the rightmost 0 omitted

$$0 \to \Gamma_I(N) \to \Gamma_I(M) \to \Gamma_I(M/N)$$

is exact, but the map $\Gamma_I(M) \to \Gamma_I(M/N)$ is typically not surjective. It is possible to construct a functor $H^1_I(-)$ such that the quotient of $\Gamma_I(M/N)$ by the image of $\Gamma_I(M)$ can be recovered as the kernel of $H^1_I(M \to N)$ – which is reassuring – but we encounter a similar problem if we try to determine $H^1_I(M/N)$ using $H^1_I(M)$ and

$H^1_I(N)$ only. A new functor $H^2_I(-)$ can be built to describe the quotient of $H^1_I(M/N)$ by the image of $H^1_I(M)$ in terms of the modules $H^2_I(M)$ and $H^2_I(N)$. This process may continue (a priori) forever, requiring the construction of an infinite family of functors $\{H^i_I(-)\}$ – the derived functors of $\Gamma_I(-)$, referred to as *local cohomology functors* – resulting in the following long exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_I(N) \arrow[r] & \Gamma_I(M) \arrow[r] & \Gamma_I(M/N) \arrow[dll] \\
 & H^1_I(N) \arrow[r] & H^1_I(M) \arrow[r] & H^1_I(M/N) \arrow[dll] \\
 & H^2_I(N) \arrow[r] & H^2_I(M) \arrow[r] & H^2_I(M/N) \arrow[dll] \\
 & H^3_I(N) \arrow[r] & \cdots &
\end{tikzcd}$$

It is striking that even if our primary interest is the study of finitely generated modules, we are required to consider non-finitely generated modules in order to fully understand the I-torsion components of quotients. This is even true over $\mathbb{Z}$. While the 2-torsion part of the quotient of $\mathbb{Z} \xrightarrow{12} \mathbb{Z}$ cannot be recovered from $\Gamma_{(2)}(\mathbb{Z} \xrightarrow{12} \mathbb{Z})$, it does appear as the kernel of $H^1_{(2)}(\mathbb{Z} \xrightarrow{12} \mathbb{Z})$. To understand how this embedding works would require us to investigate the structure of $H^1_{(2)}(\mathbb{Z})$, which is not finitely generated. It is realized as the quotient $\mathbb{Z}[2^{-1}]/\mathbb{Z}$ and generated by the classes we will denote as $\{\!\{1/2^t\}\!\}$ for $t \geq 1$. Multiplying $\{\!\{1/2^t\}\!\}$ by an element of $\mathbb{Z}$ can only ever decrease t.

The module $H^1_{(x)}(\mathbb{F}_2[x])$, isomorphic to $\mathbb{F}_2[x,x^{-1}]/\mathbb{F}_2[x]$, is generated over $\mathbb{F}_2[x]$ by classes of fractions $\{\!\{1/x^t\}\!\}$ for $t \geq 1$. Consider the effect of the map $F : H^1_{(x)}(\mathbb{F}_2[x]) \to H^1_{(x)}(\mathbb{F}_2[x])$ that sends $\{\!\{a/b\}\!\}$ to $\{\!\{a^2/b^2\}\!\}$. Using the fact that $1+1=0$ in $\mathbb{F}_2$, one may show without too much difficulty that F is an additive map. While multiplying $\{\!\{1/x^t\}\!\}$ by an element of $\mathbb{F}_2[x]$ can only ever decrease t, repeated application of the map F can take $\{\!\{1/x\}\!\}$ to any power $\{\!\{1/x^{2^e}\}\!\}$ for $e \geq 0$. In a sense that we will make precise in Chapter III, the module $H^1_{(x)}(\mathbb{F}_2[x])$ is finitely generated over a (necessarily noncommutative) augmentation of the ring $\mathbb{F}_2[x]$ by the $\mathbb{F}_2$-linear operator F.

It is far less obvious, but nonetheless true[6], that *every* module of the form $H^i_I(\mathbb{F}_p[x_1,\ldots,x_n])$ is finitely generated over the noncommutative augmentation of the ring $\mathbb{F}_p[x_1,\ldots,x_n]$ by an $\mathbb{F}_p$-linear operator F defined in terms of pth powers. Using a different sort of noncommutative enlargement of the ring $\mathbb{C}[x_1,\ldots,x_n]$ involving differential operators, one can make a similar statement for modules of the form $H^i_I(\mathbb{C}[x_1,\ldots,x_n])$ [Lyu93].

Given the fact that at least some local cohomology modules over become finitely generated over various noncommutative enlargements of the base ring R, it is natural to wonder whether they may share any useful properties in common with finitely generated R-modules. Our primary focus shall be on two particular properties. Namely, the fact that any finitely generated R-module has a finite set of associated primes and a Zariski closed support in $\mathrm{Spec}(R)$. When R is a *regular ring* – such as $\mathbb{F}_p[x_1,\ldots,x_n]$ or $\mathbb{C}[x_1,\ldots,x_n]$ – there are a number of cases[7] in which the local cohomology of R has a finite set of associated primes. The class of *complete intersection rings* – rings such as $\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$ or $\mathbb{F}_2[x,y,z,w]/(xz-yw,x^3+y^3+z^3+w^3)$ – are generally more difficult to control. The finiteness of associated primes property, for example, is known to fail for the local cohomology of a number of complete intersection rings. It remains an open question, however, whether the local cohomology of a general Noetherian ring R must have closed support. We refer to this question as the *closed support problem* over R. The closed support problem is generally open for complete intersection rings.

[6]See Theorem III.15, a result of Lyubeznik [Lyu97].

[7]We review what is known about finiteness of associated primes for the local cohomology of a regular ring in Section 2.4.

1.2 Overview of this thesis

In Chapter II, we will state the basic properties of local cohomology modules that will be necessary in the sequel for ease of reference, and we will review what is known in the literature about the support and associated primes of local cohomology. The situation over a ring of prime characteristic $p > 0$ is generally the best understood.

To make substantial statements about the local cohomology of a ring of prime characteristic $p > 0$, we require the notion of an $R\langle F\rangle$*-module* [Bli01] and certain results from the closely related theory of *F-modules* [Lyu97]. The primary goal of Chapter III is to review a number of Lyubeznik's finiteness results on the induced Frobenius action on local cohomology. These results make extensive use of the of the crucial hypothesis of regularity, and suggest the difficulties that we will encounter upon relaxing that hypothesis in subsequent chapters.

Before beginning work on complete intersection rings proper, a few more functorial tools are necessary. For any containment of ideals $I \subseteq I'$, the natural inclusion $\Gamma_{I'}(-) \to \Gamma_I(-)$ induces a family of natural transformations $H^i_{I'}(-) \to H^i_I(-)$ for all $i \geq 0$. In Chapter IV, we present the following original result of the author.

Theorem (IV.4)**.** *Let R be a Noetherian ring and let $I \subseteq R$ be any ideal. Fix $i \geq 0$. There is an ideal $I' \supseteq I$ (resp. $I'' \supseteq I$) of height $ht(I') \geq i-1$ (resp. of height $ht(I'') \geq i$) such that the natural transformation $H^i_{I'}(-) \to H^i_I(-)$ (resp. $H^i_{I''}(-) \to H^i_I(-)$) is an isomorphism of functors (resp. a surjection of functors).*

This result generalizes an isomorphism theorem of Hellus [Hel01, Theorem 3], who gives an isomorphism of modules $H^i_I(R)$ (rather than of functors $H^i_I(-)$) under the hypothesis that R is Cohen-Macaulay and local. In a different direction, for a ring map $R \to S$ and an ideal I of R, we study a family of natural transformations

$h^i_f(-) : S \otimes_R H^i_I(-) \to H^i_I(S \otimes_R -)$ (see Definition III.12) and show that, in the following sense, these transformations are compatible the Frobenius homomorphism of S.

Theorem (IV.10)**.** *Let $R \to S$ be a homomorphism between two Noetherian rings of prime characteristic $p > 0$, fix an ideal $I \subseteq R$ and an index $i \geq 0$, and let M be an $R\langle F\rangle$-module. The natural map*

$$S \otimes_R H^i_I(M) \to H^i_I(S \otimes_R M)$$

is a morphism of $S\langle F\rangle$-modules.

Our investigation of complete intersections begins in Chapter V. We review a result of Hochster and Núñez-Betancourt stating that the local cohomology of a positive characteristic hypersurface ring has closed support. This result is a corollary of their theorem that if R is regular, J is an ideal, and $\operatorname{Ass} H^i_I(J)$ is a finite set, then $\operatorname{Supp} H^{i-1}_I(R/J)$ is closed [HNB17, Theorem 4.12]. Their theorem raises the following question: If R is regular and J is an ideal generated by a regular sequence, must the set $\operatorname{Ass} H^i_I(J)$ be finite? We give the following positive answer in cohomological degree $i = 2$.

Theorem I.3 (V.11)**.** *Let R be a regular ring, and I and J be ideals of R. The set $Ass\, H^2_I(J)$ is finite.*

In cohomological degree $i \geq 3$, the situation is more complicated. We give the first example (Theorem V.5) in the literature of a module of the form $H^3_I(J)$ with an infinite set of associated primes, answering Hochster and Núñez-Betancourt's question in the negative. We prove the following theorem giving conditions under which the finiteness of $\operatorname{Ass} H^i_I(J)$ is in fact equivalent to the finiteness of $\operatorname{Ass} H^{i-1}_I(R/J)$.

Theorem (V.16)**.** *Let R be an LC-finite regular ring, let $J \subseteq R$ be an ideal generated by a regular sequence of length $c \geq 2$, and let $S = R/J$. For an ideal $I \supseteq J$,*

(ii) If the irreducible components of Spec(S) are disjoint (e.g. S is a domain), then $\operatorname{Ass} H^3_I(J)$ is finite if and only if $\operatorname{Ass} H^2_I(S)$ is finite.

(iii) If S is normal and locally almost factorial (e.g. S is a UFD), then $\operatorname{Ass} H^4_I(J)$ is finite if and only if $\operatorname{Ass} H^3_I(S)$ is finite.

This result presents a significant obstacle to the direct generalization of the methods of Hochster and Núñez-Betancourt to complete intersection rings of codimension $c \geq 2$.

Chapter VI and onward deal with joint work of the author and Eric Canton. To a regular sequence $\underline{\mathbf{f}} = f_1, \ldots, f_c$ of length c in a Noetherian ring R, letting $f = \prod_{i=1}^c f_i$, we study the Frobenius action $f^{p-1}F_{\text{nat}}$ on the module $H^c_{\underline{\mathbf{f}}R}(R)$, where F_{nat} denotes the natural action of the Frobenius. The primary goal of Chapter VI is to establish a number of basic properties of the action $f^{p-1}F_{\text{nat}}$ and to motivate its relevance to the closed support problem for $R/\underline{\mathbf{f}}$. By sending $1 \in R/\underline{\mathbf{f}}$ to the Čech cohomology class $\{\!\{1/f\}\!\}$, one obtains an $R\langle F\rangle$-linear embedding $R/\underline{\mathbf{f}}R \hookrightarrow H^c_{\underline{\mathbf{f}}R}(R)$. Our eventual is to use this embedding to construct an alternative complex (given in Chapter VII) to the $R\langle F\rangle$-linear short exact sequence $0 \to \underline{\mathbf{f}}R \to R \to R/\underline{\mathbf{f}}R \to 0$ that forms the basis in Hochster and Núñez-Betancourt's approach to closed support.

In codimension $c \geq 2$, the embedding of $R/\underline{\mathbf{f}}R$ into $H^c_{\underline{\mathbf{f}}R}(R)$ leaves behind a cokernel whose local cohomology is somewhat too complicated to use directly. The purpose of Chapter VII is to arrange a family of annihilator submodules of $H^c_{\underline{\mathbf{f}}R}(R)$ into a complex $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)$ (see Definition VII.2) of length c whose terms are local cohomology modules with respect to various subsequences of $\underline{\mathbf{f}}$. These terms are somewhat more tractable to understand after applying local cohomology functors with respect to an ideal I containing $\underline{\mathbf{f}}$. The main result of this section, representing joint work of the author and Eric Canton, is that the augmentation of the $\mathbb{\Delta}^{\bullet}_{\underline{\mathbf{f}}}(R)$ complex is exact (Theorem VII.7).

Finally, in Chapter VIII, using the complex constructed in the previous chapter, we present our main result, concerning the support of the local cohomology of a positive characteristic complete intersection ring with respect to a Cohen-Macaulay ideal.

Theorem (VIII.1)**.** *Let R be a regular ring of prime characteristic $p > 0$, let $\underline{\mathbf{f}}$ be a permutable regular of length c, and let I be an ideal containing $\underline{\mathbf{f}}$ such that R/I is Cohen-Macaulay. Let h denote the height of $I/\underline{\mathbf{f}}R$ in the ring $S = R/\underline{\mathbf{f}}R$. Then $H^{h+c}_{I/\underline{\mathbf{f}}R}(S)$ has closed support.*

Convention: Throughout this paper, we assume that all given rings are commutative and Noetherian unless stated otherwise.

CHAPTER II

Background on Local Cohomology

Our goal in this chapter is to state some fundamental results in the theory of local cohomology for future reference. We will omit most proofs. The main reference for this material is Brodmann and Sharp [BS12], although we will also be using some terminology and properties of certain classes of rings for which the reader may wish to consult Bruns and Herzog [BH98]. General statements on homological algebra may be found in Weibel [Wei94].

Notational remark: If F and G are functors $\mathfrak{C} \to \mathfrak{D}$, we will write natural transformations from F to G as $\phi(-) : F(-) \to G(-)$, which consists of the data of a map denoted $\phi(A) : F(A) \to G(A)$ for each object A of $\mathfrak{C}$, such that for any $\mathfrak{C}$-morphism $f : A \to B$, $\phi(B) \circ F(f) = G(f) \circ \phi(A)$.

Let Mod_R denote the category of modules over a ring R (not necessarily of prime characteristic $p > 0$), and let Kom_R denote the category of cohomologically indexed complexes of R-modules. Let $H^i : \mathrm{Kom}_R \to \mathrm{Mod}_R$ denote the functor that takes a complex $C^\bullet$ to its ith cohomology module $H^i(C^\bullet)$.

2.1 The I-torsion functor and local cohomology

Definition II.1. Let R be a ring and let I be an ideal of R. The *I-torsion functor*, denoted $\Gamma_I(-)$, is the functor that takes an R-module M to the submodule consisting of all elements $u \in M$ annihilated by some sufficiently high power of I. For a map $f : M \to N$, the map $\Gamma_I(f)$ is the restriction of f to $\Gamma_I(M)$.

Since the annihilator of I^n in M is precisely the set $\mathrm{Hom}_R(R/I^n, M)$, we may identify the functor $\Gamma_I(-)$ with the direct limit $\varinjlim_n \mathrm{Hom}_R(R/I^n, -)$.

The I-torsion functor detects the presence of associated primes in the set $V(I) = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$ in the following sense.

Proposition II.2. *Let R be a ring, I be an ideal, and M be an R-module.*

1. *M has an associated prime in $V(I)$ if and only if $\Gamma_I(M) \neq 0$.*
2. *P is an associated prime of M if and only if $\Gamma_{PR_P}(M_P) \neq 0$*
3. *The support of M is contained in $V(I)$ if and only if $\Gamma_I(M) = M$.*
4. *P is a minimal prime of M if and only if $\Gamma_{PR_P}(M_P) = M_P$*

Given a quotient $Q = M/N$, one may be interested in describing the associated primes of Q in terms of data involving M and (the embedding map of) N only. Item (ii) of the proposition above suggests that we apply the functor $\Gamma_{PR_P}((-)_P)$ to a short exact sequence $0 \to N \to M \to Q \to 0$, but the resulting sequence is only exact if the rightmost map to 0 is dropped: $0 \to \Gamma_{PR_P}(N_P) \to \Gamma_{PR_P}(M_P) \to \Gamma_{PR_P}(Q_P)$.

In general, $\Gamma_I(-)$ is a left-exact functor but is not exact. If each module N, M, and Q in our short exact sequence were replaced with an injection resolution, the functor $\Gamma_I(-)$ would produce a short exact sequence of complexes that generally have nontrivial cohomology.

Definition II.3. Let R be a ring, let I be an ideal, and let M be an R-module. Let $M \to E^\bullet$ be an injective resolution. The *ith local cohomology module of M with respect to I* is the module $H^i(\Gamma_I(E^\bullet))$ which does not depend on the choice of injective resolution $E^\bullet$. The functor that takes M to $H^i_I(M)$, that is, the ith right derived functor of $\Gamma_I(-)$, is denoted $H^i_I(-)$. To any short exact sequence $0 \to N \to M \to Q \to 0$, there is a natural family of connecting homomorphisms $\delta^i : H^i_I(Q) \to H^{i+1}_I(N)$ resulting in a functorial *long exact sequence in local cohomology with respect to I.*

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_I(N) \arrow[r] & \Gamma_I(M) \arrow[r] & \Gamma_I(Q) \arrow[dll, rounded corners] \\
& H^1_I(N) \arrow[r] & H^1_I(M) \arrow[r] & H^1_I(Q) \arrow[dll, rounded corners] \\
& H^2_I(N) \arrow[r] & H^2_I(M) \arrow[r] & H^2_I(Q) \arrow[dll, rounded corners] \\
& H^3_I(N) \arrow[r] & \cdots &
\end{tikzcd}$$

Since $\Gamma_I(-)$ is unchanged if I is replaced by another ideal having the same radical, the same can be said of each functor $H^i_I(-)$. A module of the form $H^i_I(M)$ is automatically I-torsion, and because an I-torsion module has an injective resolution by I-torsion modules [BS12, Corollary 2.1.6], it holds that $H^i_I(M) = 0$ whenever M is I-torsion and $i \geq 1$. From this, it is not hard to see that $H^i_I(M) \simeq H^i_I(M/\Gamma_I(M))$ for all $i \geq 1$.

There is another (very different) kind of functorial long exact sequence involving local cohomology modules that we will make extensive use of in the sequel.

Theorem II.4 (The Mayer-Vietoris Sequence)**.** *Let R be a Noetherian ring and let I and J be ideals. There is a sequence of natural transformations, below, that is exact when $(-)$ is replaced by any R-module M.*

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{I+J}(-) \arrow[r] & \Gamma_I(-) \oplus \Gamma_J(-) \arrow[r] & \Gamma_{I\cap J}(-) \arrow[dll, rounded corners] \\
& H^1_{I+J}(-) \arrow[r] & H^1_I(-) \oplus H^1_J(-) \arrow[r] & H^1_{I\cap J}(-) \arrow[dll, rounded corners] \\
& H^2_{I+J}(-) \arrow[r] & H^2_I(-) \oplus H^2_J(-) \arrow[r] & H^2_{I\cap J}(-) \arrow[dll, rounded corners] \\
& H^3_{I+J}(-) \arrow[r] & \cdots &
\end{tikzcd}$$

For an R-module M, an *M-regular sequence* is a list of elements $\underline{\mathbf{f}} = f_1, \ldots, f_c$ such that $\underline{\mathbf{f}}M \neq M$ and such that f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$. We use the term *regular sequence* instead of *R-regular sequence* when $M = R$. Let R be Noetherian. The *depth of a module M on an ideal I*, denoted $\operatorname{depth}_I(M)$, is the length of a maximal M-regular sequence contained in I (the Noetherianity of R ensures that a regular sequence cannot be extended indefinitely). It is not a priori obvious that depth is well-defined – that is, that all maximal M-regular sequences in I have the same length – but this follows at once from a theorem of Rees.

Theorem II.5 (Rees's Theorem)**.** *Let R be a Noetherian ring, let I be an ideal, let N be a finitely generated I-torsion module, and let M be an arbitrary finitely generated R-module. If I contains a maximal M-regular sequence of length c, then $Ext^i_R(N, M) = 0$ for $i < c$ and $Ext^c_R(N, M) \neq 0$.*

Since $\Gamma_I(-)$ may be identified with $\varinjlim_n \operatorname{Hom}_R(R/I^n, -)$, it is a straightforward exercise of homological algebra to show that $H^i_I(-)$ may be identified with the direct limit $\varinjlim_n \operatorname{Ext}^i_R(R/I^n, -)$ for all i. One may then extend Rees's theorem to the following statement.

Theorem II.6. *Let R be a Noetherian ring, let I be an ideal, and let M be a finitely generated R-module. Then $H^i_I(M) = 0$ for $i < depth_I(M)$ and $H_I^{depth_I(M)}(M) \neq 0$.*

2.2 I-transform functors

For an ideal $I \subseteq R$, the *I-transform functor* is defined by

$$D_I(-) := \varinjlim_t \operatorname{Hom}_R(I^t, -)$$

$D_I(-)$ is a left exact functor whose right derived functors satisfy $\mathscr{R}^i D_I(-) \simeq H_I^{i+1}(-)$. There is a sense in which $D_I(-)$ forces $\operatorname{depth}_I(-) \geq 2$ without modifying higher local cohomology on I. Namely, for any R-module M, $\Gamma_I(D_I(M)) = H^1_I(D_I(M)) = 0$, and $H^i_I(D_I(M)) = H^i_I(M)$ for all $i \geq 2$.

Lemma II.7. *[BS12, Theorem 2.2.4(i)] Let R be a Noetherian ring and fix an ideal $I \subseteq R$. There is a natural transformation $\eta_I(-) : Id \to D_I(-)$ such that, for any R-module M, there is an exact sequence*

$$0 \to \Gamma_I(M) \to M \xrightarrow{\eta_I(M)} D_I(M) \to H^1_I(M) \to 0$$

Lemma II.8. *[BS12, Proposition 2.2.13] Let R be a Noetherian ring, and $I \subseteq R$ be an ideal. Let $e : M \to M'$ be a homomorphism of R-modules such that $\mathrm{Ker} e$ and $\mathrm{Coker} e$ are both I-torsion. Then*

(i) The map $D_I(e) : D_I(M) \to D_I(M')$ is an isomorphism.

(ii) There is a unique R-module homomorphism $\varphi : M' \to D_I(M)$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[dr, "\eta_I(M)"'] & M' \arrow[d, "\varphi"] \\ & D_I(M) \end{tikzcd}$$

commutes. In fact, $\varphi = D_I(e)^{-1} \circ \eta_I(M')$.

(iii) The map φ of (ii) is an isomorphism if and only if $\eta_I(M')$ is an isomorphism, and this is the case if and only if $\Gamma_I(M') = H^1_I(M') = 0$.

The main property of the ideal transform functor that we will require is the following.

Proposition II.9. *Let R be a Noetherian ring, y an element of R, and $I_0 \subseteq R$ an ideal. Let $I = yR \cap I_0$. There is a natural isomorphism of functors $D_{I_0}(-)_y \simeq D_I(-)$*

Proof. Precomposing $\eta_{I_0}(-)_y : (-)_y \to D_{I_0}(-)_y$ with $\mathrm{Id} \to (-)_y$ we obtain a natural transformation $\gamma(-) : \mathrm{Id} \to D_{I_0}(-)_y$. We claim that for any module M, both the kernel and cokernel of $\gamma(M) : M \to D_{I_0}(M)_y$ are $I = yR \cap I_0$-torsion:

- $\mathrm{Ker}(\gamma(M))$ consists of those $m \in M$ such that $m/1 \in \Gamma_{I_0}(M)_y$, or, equivalently, $y^t m \in \Gamma_{I_0}(M)$ for some $t \geq 0$. Let s be such that $I_0^s y^t m = 0$. Then $(yI_0)^{\max(s,t)} m = 0$, so $m \in \Gamma_{yI_0}(M) = \Gamma_{yR \cap I_0}(M)$ (since $\sqrt{yI_0} = \sqrt{yR \cap I_0}$).

- An element of $C = \mathrm{Coker}(\gamma(M))$ can be represented by $c = f/y^t$ for some $f \in D_{I_0}(M)$, $t \geq 0$. $\mathrm{Coker}(\eta_{I_0}(M))$ is I_0-torsion, so there is some s such that $I_0^s f \subseteq \mathrm{Im}\eta_{I_0}(M)$. Since $f = y^t c$, we have $(yI_0)^{\max(s,t)} c \subseteq \mathrm{Im}\gamma(M)$. The element of C represented by c therefore belongs to $\Gamma_{yI_0}(C) = \Gamma_{yR \cap I_0}(C)$.

Lemma II.8(ii) therefore gives a map $\varphi(M) : D_{I_0}(M)_y \to D_I(M)$, specifically $\varphi(M) = D_I(\gamma(M))^{-1} \circ \eta_I(D_{I_0}(M)_y)$. Both of the composite maps come from natural transformations $D_I(\gamma(-))^{-1}$ and $\eta_I(D_{I_0}(-)_y)$, so the result is a natural transformation $\varphi(-) : D_{I_0}(-)_y \to D_I(-)$.

It remains to show that that $\varphi(M)$ is an isomorphism for each M, which is equivalent, by Lemma II.8(iii), to showing that $\Gamma_I(D_{I_0}(M)_y) = H_I^1(D_{I_0}(M)_y) = 0$. This can be done using the Mayer-Vietoris sequence associated with the intersection $yR \cap I_0$. Each module $H^i_{yR+I_0}(D_{I_0}(M)_y)$ vanishes because $y \in yR + I_0$ acts as a unit on $D_{I_0}(M)_y$, and likewise for the modules $\Gamma_{yR}(D_{I_0}(M)_y)$ and $H^1_{yR}(D_{I_0}(M)_y)$. Note that $\mathrm{depth}_{I_0}(D_{I_0}(M)) \geq 2$, and localization can only make depth go up, so, $\Gamma_{I_0}(D_{I_0}(M)_y) = H^1_{I_0}(D_{I_0}(M)_y) = 0$.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \Gamma_{yR\cap I_0}(D_{I_0}(M)_y) \arrow[dll] \\
& 0 \arrow[r] & 0 \arrow[r] & H^1_{yR\cap I_0}(D_{I_0}(M)_y) \arrow[dll] \\
& 0 \arrow[r] & 0 \oplus H^2_{I_0}(D_{I_0}(M)_y) \arrow[r] & H^2_{yR\cap I_0}(D_{I_0}(M)_y)
\end{tikzcd}$$

We can now see that $\Gamma_I(D_{I_0}(M)_y) = H^1_I(D_{I_0}(M)_y) = 0$, as desired. □

Corollary II.10. *Let R be a Noetherian ring, y an element of R, and $I_0 \subseteq R$ an ideal. Let $I = yR \cap I_0$. Then for all $i \geq 2$, there is a natural isomorphism of functors*

$H^i_I(-) \simeq H^i_{I_0}(-)_y$.

Proof. It is equivalent to show that $\mathscr{R}^{i-1}D_I(-) \simeq (\mathscr{R}^{i-1}D_{I_0}(-))_y$. We can calculate $\mathscr{R}^{i-1}D_I(M)$ as $H^{i-1}(D_I(E^\bullet))$ where $M \to E^\bullet$ is an injective resolution, but by Proposition II.9, $D_I(-) \simeq D_{I_0}(-)_y$ where $(-)_y$ commutes with the formation of cohomology. Thus,

$$H^{i-1}(D_I(E^\bullet)) \simeq H^{i-1}(D_{I_0}(E^\bullet))_y = \left(\mathscr{R}^{i-1}D_{I_0}(M)\right)_y.$$

□

2.3 The Čech complex

If f is an element of R and M is an R-module, the sequence

$$0 \to M \to M_f \to 0 = (0 \to R \to R_f \to 0) \otimes_R M$$

is called the *Čech complex on M with respect to f*, denoted $\check{C}^\bullet(f;M)$. Let $\underline{\mathbf{f}} = f_1,\ldots,f_t$ be a sequence of elements of arbitrary length. The *Čech complex on R with respect to $\underline{\mathbf{f}}$*, denoted $\check{C}^\bullet(\underline{\mathbf{f}};R)$, is the total complex of the tensor product

$$\check{C}^\bullet(f_1;R) \otimes \cdots \otimes \check{C}^\bullet(f_t;R)$$

and *Čech complex on M with respect to $\underline{\mathbf{f}}$* is $M \otimes_R \check{C}^\bullet(\underline{\mathbf{f}};R)$, denote $\check{C}^\bullet(\underline{\mathbf{f}};M)$. This complex has the form

$$0 \to M \to \bigoplus_{1\le i\le t} M_{f_i} \to \bigoplus_{1\le i<j\le t} M_{f_if_j} \to \cdots \to M_{f_1\cdots f_t} \to 0$$

Theorem II.11. *[BS12, Theorem 5.1.19] Let R be a Noetherian ring, let I be an ideal, and let $\underline{\mathbf{f}} = f_1,\ldots,f_t$ be any sequence of elements generating I. The functor that takes M to $H^i(\check{C}^\bullet(\underline{\mathbf{f}};M))$ is naturally isomorphic to the ith local cohomology functor $H^i_I(-)$.*

A straightforward but important application of Čech cohomology is the following result.

Theorem II.12. *[BS12, Theorem 4.2.1] Let $R \to S$ be a homomorphism between two Noetherian rings, let I be an ideal of R, and let N be an S-module. The module $H^i_I(N)$ is obtained by restricting scalars to regard N as an R-module, and we may regard $H^i_I(N)$ as an S-module by letting $s \in S$ act as $H^i_I(N \xrightarrow{s} N)$. There is a natural isomorphism of S-modules $H^i_I(N) \simeq H^i_{IS}(N)$.*

The *arithmetic rank* of an ideal I, denoted $\mathrm{ara}(I)$ is the least number of generators of an ideal having the same radical as I. Since the local cohomology functors with respect to I are unchanged if I is replaced by an ideal having the same radical, since local cohomology may be computed using the Čech complex on any set of generators, and since the Čech complex has no nonzero terms in cohomological degree greater than the length of the generating sequence chosen, the following result is clear.

Corollary II.13. *[BS12, Corollary 3.3.3] Let R be a Noetherian ring and I be an ideal. The functor $H^i_I(-)$ is equal to the zero functor for all $i > ara(I)$.*

This is not the only functorial vanishing theorem that we will make use of in the sequel.

Theorem II.14. *[BS12, Theorem 6.1.2] Let $(R, \mathfrak{m})$ be a local ring of dimension n. The functor $H^i_I(-)$ is equal to the zero functor for $i > \dim(R)$.*

Another final key application of the Čech complex is the following manipulation obtained by forming the cohomology of the total complex of a tensor product of two Čech complexes, one of which has a large amoung of vanishing in its cohomology.

Theorem II.15. *Let R be a Noetherian ring, let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence,* *and let I be an ideal containing $\underline{\mathbf{f}}$. There is a natural isomorphism $H^i_I(H^c_{\underline{\mathbf{f}}R}(R)) \simeq H^{i+c}_I(R)$.*

Proof. Let $\underline{\mathbf{g}} = g_1, \ldots, g_t$ be a sequence of elements generating I. This is [Wei94, Theorem 5.5.10] (see also [Wei94, Definitions 5.6.1, 5.6.2]) applied to the double complex $\check{C}^\bullet(\underline{\mathbf{g}}; R) \otimes_R \check{C}^\bullet(\underline{\mathbf{f}}; R)$ using the vanishing $H^i_{\underline{\mathbf{f}}}(\check{C}^\bullet(\underline{\mathbf{f}}; R)) = 0$ for $i \neq c$. □

2.4 A brief review of finiteness properties

In what follows, it will be helpful to refer to the following property.

Definition II.16. Let R be a Noetherian ring. Call an R-module M *LC-finite* if, for any ideal I of R and any $i \geq 0$, the module $H^i_I(M)$ has a finite set of associated primes.

For example, over any Noetherian ring R, the indecomposable injective module $E_R(R/P)$ is LC-finite. Since $\operatorname{Ass} E_R(R/P) = \{P\}$ [BH98, Lemma 3.2.7], we have $\Gamma_I(E_R(R/P)) = 0$ if and only if $I \supseteq P$, and since $H^i_I(E_R(R/P)) = 0$ for $i > 0$ ($E_R(R/P)$ is injective) it holds that $H^i_I(E_R(R/P))$ has as its set of associated primes either $\{P\}$ (if $i = 0$ and $P \supseteq I$) or $\varnothing$ (otherwise). As another example over any Noetherian ring R since $\operatorname{Supp}(H^i_I(M)) \subseteq \operatorname{Supp}(M)$ for any R-module M, a module with finite support is also trivially LC-finite. Any module over a semilocal ring of dimension at most 1 is trivially LC-finite. We shall call a ring LC-finite if it is LC-finite as a module over itself. As we shall soon see, it is not typically the case that all (finitely generated, nor even cyclic) modules over an LC-finite ring are LC-finite.

The class of LC-finite rings is closed under localization. If there is a finite set of maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_t$ of R such that $\operatorname{Spec}(R) - \{\mathfrak{m}_1, \cdots, \mathfrak{m}_t\}$ can be covered by finitely many charts $\operatorname{Spec}(R_f)$, each of which is LC-finite, then R is LC-finite. If R is LC-finite and $A \to R$ is pure (e.g., if A is a direct summand of R), then A is

LC-finite [HNB17, Theorem 3.1(d)].

While there are interesting classes of not-necessarily-regular rings known to have the property of LC-finiteness – for example, F-finite rings of finite F-representation type (FFRT) [HNB17, Theorem 5.7] – much of the existing finiteness literature is concerned primarily with the class of regular rings.

When R is regular of prime characteristic $p > 0$, a celebrated theorem of Huneke and Sharp states that R is LC-finite[1] [HS93, Corollary 2.3]. Lyubeznik proved that LC-finiteness holds for smooth K-algebras when K is a field of characteristic 0 [Lyu93, Remark 3.7(i)] and for any regular local ring containing $\mathbb{Q}$ [Lyu93, Theorem 3.4]. Concerning regular rings of mixed characteristic, unramified regular local rings [Lyu00, Theorem 1], smooth $\mathbb{Z}$-algebras [BBL$^+$14, Theorem 1.2], and regular local rings of dimension ≤ 4 [Mar01, Theorem 2.9] are LC-finite. It is an open question whether there exist non-LC-finite regular rings [Hoc19].

The property of LC-finiteness can fail over a hypersurface ring. The first example of this phenomenon is due to Singh [Sin00], who describes a hypersurface ring S finitely generated over $\mathbb{Z}$,

$$S = \frac{\mathbb{Z}[u, v, w, x, y, z]}{ux + vy + wz}$$

such that for all prime integers p, the module $H^3_{(x,y,z)}(S)$ has a nonzero p-torsion element. Katzman [Kat02] showed that the property is not necessarily recovered by restricting to (graded or local) rings containing a field. The hypersurface ring

$$S = \frac{K[[u, v, w, x, y, z]]}{wu^2x^2 - (w + z)uxvy + zv^2y^2}$$

has a local cohomology module, $H^2_{(x,y)}(S)$, with an infinite set of associated primes. Katzman's hypersurface is not a domain, but Singh and Swanson [SS04] construct ex-

[1]This result was later generalized by Lyubeznik's theory of F-modules [Lyu97]. Lyubeznik in fact shows that the local cohomology modules $H^i_I(R)$ – and in fact, any iterated local cohomology module $H^{i_1}_{I_1}(\cdots(H^{i_t}_{I_t}(R))\cdots)$ is itself LC-finite. See Chapter III.

amples of equicharacteristic local hypersurface rings to demonstrate that $\operatorname{Ass} H^3_I(S)$ can be infinite even if S is a UFD that is simultaneously F-regular ring (in characteristic $p > 0$) or has rational singularities (in equal characteristic 0). In this sense, the presence of even relatively mild singularities can obstruct LC-finiteness.

Controlling the support is sometimes more tractable than controlling the full set of associated primes. If M finitely generated over S, then the set $\operatorname{Supp} H^i_I(M)$ is known to be closed – equivalently, $H^i_I(M)$ is known to have finitely many minimal primes – whenever (i) S has prime characteristic $p > 0$, I is generated by i elements, and $M = S$ [Kat05, Theorem 2.10], (ii) S is standard graded, M is graded, I is the irrelevant ideal, and i is the cohomological dimension of I on M [RŞ05, Theorem 1], (iii) S is local of dimension at most 4 [HKM09, Proposition 3.4], (iv) I has cohomological dimension at most 2 [HKM09, Theorem 2.4], or (v) $M = S$ and $S = R/fR$ for some regular ring R of prime characteristic $p > 0$ and some nonzerodivisor $f \in R$ [HNB17, KZ17]. It is not known how far these results generalize. It is an open question whether $H^i_I(M)$ must be closed for all $i \geq 0$ and all ideals I, where S is Noetherian and M is finitely generated [Hoc19, Question 2]. We restrict our attention to this question of closed support in the case $M = S$, which we will refer to as the closed support problem for S. The closed support problem has a trivially positive answer over an LC-finite ring, so assume S is not LC-finite.

We shall focus in particular on the results of Katzman and Zhang [KZ17] or Hochster and Núñez-Betancourt [HNB17] on the local cohomology of positive characteristic hypersurface[2] rings. It is not known whether this result generalizes to hypersurface rings of characteristic 0, or to positive characteristic complete intersection rings of codimension ≥ 2. Our primary interest shall be in the latter ques-

[2]A hypersurface ring is a complete intersection ring of codimension 1.

tion, though in Chapter V we will investigate the prospect of applying Hochster and Núñez-Betancourt's methods to a ring R/J where R is an arbitrary LC-finite regular ring and J is a complete intersection ideal of codimension $c \geq 2$.

We will require some supplementary results on finiteness of associated primes in the sequel. The following two statements are both well known, and a suitable reference is Hellus [Hel01].

Theorem II.17. *Let R be a Noetherian ring and let M be a finitely generated module. For any ideal I, the module $H^1_I(M)$ has a finite set of associated primes.*

Theorem II.18. *Let R be a Noetherian ring, let I be an ideal, and let M be an R-module. The module $H_I^{depth_I(M)}(M)$ has a finite set of associated primes.*

A stronger version of the latter is given by Brodmann and Lashgari Faghani.

Theorem II.19 (Brodmann, Lashgari Faghani [BLF00])**.** *Let R be a Noetherian ring, let I be an ideal, and let M be an R-module. Let t be the least integer such that $H^t_I(M)$ is not finitely generated. Then $H^t_I(M)$ has a finite set of associated primes.*

CHAPTER III

Frobenius Actions and F-Modules

Introduction

Throughout this chapter, R shall denote a commutative Noetherian ring of *prime characteristic* $p > 0$, that is, a ring for which the kernel of the canonical map $\mathbb{Z} \to R$ is exactly $p\mathbb{Z}$, for some prime integer $p > 0$. Since p divides each binomial coefficient $\binom{p}{i}$ for $0 < i < p$, the *Frobenius map* $r \mapsto r^p$ of R, denoted $F_R : R \to R$, is a ring homomorphism. Some R-modules can be naturally equipped with an additive self-map that formally shares certain properties in common with the Frobenius homomorphism of R. A *Frobenius action* on an R-module M is an additive map $\beta : M \to M$ that satisfies $\beta(rm) = r^p\beta(m)$ for all $r \in R$, $m \in M$.

There are at least two useful equivalent perspectives one may use to study Frobenius actions. The first is the notion of an $R\langle F\rangle$-module, where $R\langle F\rangle$ denotes the noncommutative ring $R\{F\}/(r^pF - Fr \,|\, r \in R)$ presented as a quotient of the ring $R\{F\}$ obtained by freely adjoining a single noncommutative variable F to R. We will be particularly concerned with the property of being finitely generated over $R\langle F\rangle$. The second perspective involves the *structure morphism*[1] of a Frobenius action. When the structure morphism of an action is an isomorphism, the corresponding $R\langle F\rangle$-module is called *unit*, following [Bli01]. Unit $R\langle F\rangle$-modules are precisely the subject

[1] Definition III.2

of Lyubeznik's theory of F-modules [Lyu97], although Lyubeznik did not use this terminology in his paper. Of particular importance are F-finite F-modules, namely, those unit $R\langle F\rangle$-modules that are finitely generated over $R\langle F\rangle$. We will highlight some of the remarkable finiteness properties of finitely generated unit $R\langle F\rangle$-modules. In Section 3.3 we will describe the *natural action* of the Frobenius induced by a local cohomology functor $H^i_I(-)$. If R is regular, Lyubeznik proved that if M is unit and finitely generated over $R\langle F\rangle$, then so is $H^i_I(M)$, when equipped with the natural action induced from M. This result strengthens an earlier result of Huneke and Sharp [HS93] on the associated primes of the local cohomology of a regular ring, and is of fundamental importance to a number of results we will prove in the sequel.

The main reference for this section is Blickle [Bli01] in the setting where R is not necessarily regular and M is not necessarily unit, and Lyubeznik [Lyu97] in the unit setting over a regular ring.

3.1 Notation and teminology

It can sometimes be helpful to have notation that distinguishes the domain and codomain of the Frobenius homomorphism. Let $R^{1/p}$ denote the set of formal symbols $r^{1/p}$ for $r \in R$, with addition $r^{1/p} + s^{1/p} = (r+s)^{1/p}$ and multiplication $r^{1/p}s^{1/p} = (rs)^{1/p}$. It is clear that $R^{1/p}$ is isomorphic to R as an abstract ring, but we regard $R^{1/p}$ as an R-algebra via the Frobenius homomorphism $F_R : R \to R^{1/p}$, sending $r \mapsto (r^p)^{1/p}$. The resulting R-module structure on $R^{1/p}$ has the form $r \cdot s^{1/p} = (r^p s)^{1/p}$ for $r \in R$, $s^{1/p} \in R^{1/p}$. The Frobenius map is injective if and only if R is reduced. In this case, we do no harm in identifying $r \in R$ with its image $(r^p)^{1/p}$ in $R^{1/p}$.

For an R-module M, we define the $R^{1/p}$-module $M^{1/p}$ of formal symbols $m^{1/p}$ for $m \in M$ with addition $m^{1/p} + n^{1/p} = (m+n)^{1/p}$ and $R^{1/p}$ multiplication $r^{1/p}m^{1/p} =$

$(rm)^{1/p}$, for $r \in R$, $m, n \in M$. By restriction of scalars along $F_R : R \to R^{1/p}$, we can also make $M^{1/p}$ into an R-module, with $rm^{1/p} = (r^p m)^{1/p}$. There is an isomorphism of abelian groups $M^{1/p} \to M$ that sends $m^{1/p} \mapsto m$ for all $m \in M$, and we refer to this as the *formal p-th power map*. For any power $q = p^e$, the ring $R^{1/q}$, the module $M^{1/q}$, and the formal q-th power map $M^{1/q} \to M$ may be defined in an analogous manner.

Depending on whether we are denoting the target copy of F_R as R or $R^{1/p}$ in a given context, we may regard base change along the Frobenius homomorphism as either a functor $\mathrm{Mod}_R \to \mathrm{Mod}_R$, in which case we use the notation $M \mapsto \mathcal{F}_R(M)$, or as a functor $\mathrm{Mod}_R \to \mathrm{Mod}_{R^{1/p}}$, where we will use the notation $M \mapsto R^{1/p} \otimes_R M$.

3.2 $R\langle F\rangle$-modules and their structure morphisms

Let $R\langle F\rangle$ denote the ring

$$R\langle F\rangle = \frac{R\{F\}}{(r^p F - Fr \mid r \in R)}$$

where we use $R\{F\}$ to denote the algebra obtained by adjoining a free noncommutative variable F to R, and the denominator is the two-sided ideal of $R\{F\}$ with generators $r^p F - Fr$ for $r \in R$. There is a natural ring homomorphism $R\langle F\rangle \to \mathrm{Hom}_{\mathbb{Z}}(R, R)$ sending $a \in R$ to the multiplication map $r \mapsto ar$ and sending F to the Frobenius homomorphism of R. In general, the homomorphism $R\langle F\rangle \to \mathrm{Hom}_{\mathbb{Z}}(R, R)$ is not injective – for example, $F - 1 \in \mathbb{F}_p\langle F\rangle$ acts as zero on the field $\mathbb{F}_p$, and $F - F^2$ acts as zero on the ring $\mathbb{F}_p[t]/t^2$.

As a left R-module, $R\langle F\rangle$ is free on the generators $1, F, F^2, \ldots$. As a right R-module, $R\langle F\rangle$ is isomorphic to $\bigoplus_{e=0}^{\infty} R^{1/p^e}$. The tensor product $R\langle F\rangle \otimes_R M$ with an R-module M may be understood accordingly.

An $R\langle F\rangle$-module (by which we always mean a left $R\langle F\rangle$-module) is precisely the data of an R-module M equipped with an additive map $\beta : M \to M$ satisfying $\beta(rm) = r^p\beta(m)$ for $r \in R$ and $m \in M$, describing the action of $F \in R\langle F\rangle$. We refer to β as a *Frobenius action* on M. When $M = R$, we refer to the Frobenius homomorphism $F_R : R \to R$ as the *natural action* of R. If M and N are $R\langle F\rangle$-modules with Frobenius actions $\alpha : M \to M$ and $\beta : N \to N$, respectively, then an $R\langle F\rangle$-linear map $h : M \to N$ is precisely the data of an R-linear map that satisfies $h\circ\beta = \alpha\circ h$ – we will call a map *Frobenius stable* if this is the case. Since both $\mathrm{Ker}(h)$ and $\mathrm{Coker}(h)$ are themselves $R\langle F\rangle$-modules, we may refer the *induced actions* of the Frobenius inherited from M and N, respectively. If W is a multiplicative subset of R, and M is an $R\langle F\rangle$-module, then there is a unique Frobenius action on $W^{-1}M$ that makes the natural map $M \to W^{-1}M$ Frobenius stable. This action is described by $F(m/w) = F(m)/w^p$ for $m \in M$, $w \in W$. We may also regard $W^{-1}M$ as an $(W^{-1}R)\langle F\rangle$-module in an obvious way.

Of primary importance to our applications is the fact that some non-finitely generated R-modules, when equipped with a suitable action of the Frobenius, become finitely generated when regarded as modules over the ring $R\langle F\rangle$. For example, if f is a nonunit of R, then R_f is not finitely generated over R. As an $R\langle F\rangle$-module, however, R_f is cyclic with generator $1/f$.

When R is a Noetherian ring, finite generation over $R\langle F\rangle$ implies closed support as an R-module[2].

Theorem III.1. *[HNB17, Lemma 4.5] Let R be a Noetherian ring of prime characteristic $p > 0$, let M be an $R\langle F\rangle$-module, and let N be an R-submodule of M that generates M over $R\langle F\rangle$. Then the support of M, regarded as an R-module, is equal to the support of N. In particular, if M is finitely generated over $R\langle F\rangle$ and N is*

[2]When we speak of the support of an $R\langle F\rangle$-module, we always mean the support of the underlying R-module upon applying the forgetful functor $\mathrm{Mod}_{R\langle F\rangle} \to \mathrm{Mod}_R$.

the R-span of a finite generating set, then Supp $(M) =$ Supp (N) is a Zariski closed subset of Spec(R).

An $R\langle F\rangle$-linear homomorphic image of a finitely generated $R\langle F\rangle$ module is still finitely generated, and thus, still has closed support. However, we caution that $R\langle F\rangle$ is generally neither left nor right Noetherian – consider, for example, $R = \mathbb{F}_p[x]$ and the (left, right, or two-sided) ideal of $R\langle F\rangle$ generated by $xF, xF^2, xF^3, \ldots$. There are additional conditions we may impose on an $R\langle F\rangle$-module to gain better control over submodules of finitely generated modules. These conditions involve the *structure morphism* of the corresponding Frobenius action.

Definition III.2. Let R be a ring of prime characteristic $p > 0$ and let M be an $R\langle F\rangle$-module with Frobenius action $\beta : M \to M$. The map $M \to M^{1/p}$ that sends $m \mapsto (\beta(m))^{1/p}$ is an R-linear map from M to an $R^{1/p}$-module, and therefore induces an $R^{1/p}$-linear map $\theta : R^{1/p} \otimes_R M \to M^{1/p}$, with $r^{1/p} \otimes m \mapsto (r\beta(m))^{1/p}$ for $r \in R$ and $m \in M$. If we are regarding Frobenius as a map $R \to R$ (rather than $R \to R^{1/p}$) with base change functor $\mathcal{F}_R : \mathrm{Mod}_R \to \mathrm{Mod}_R$, then θ may be regarded as an R-linear map $\theta : \mathcal{F}_R(M) \to M$. This map is the *structure morphism* of the $R\langle F\rangle$-module M (or of the Frobenius action β, depending on context).

Given an arbitrary $R^{1/p}$-linear map $\theta : R^{1/p} \otimes_R M \to M^{1/p}$, we can define a corresponding Frobenius action on M by first mapping $M \to R^{1/p} \otimes_R M$ via $m \mapsto 1 \otimes m$, and letting $F(m)$ be the image of $\theta(1 \otimes m)$ under the formal pth power map $M^{1/p} \to M$. An $R\langle F\rangle$-module structure on an R-module M is in this manner equivalent data to specifying an $R^{1/p}$-linear (resp. R-linear) map $R^{1/p} \otimes_R M \to M^{1/p}$ (resp. $\mathcal{F}_R(M) \to M$).

The action of higher powers of the Frobenius, F^e for $e \geq 0$, can be recovered from the structure morphism $\theta : \mathcal{F}_R(M) \to M$ in the following way. Construct a map

$\theta_e : \mathcal{F}_R^e(M) \to M$ by composing $\mathcal{F}_R^t(\theta) : \mathcal{F}_R^{t+1}(M) \to \mathcal{F}_R^t(M)$ for $0 \leq t < e$, as shown below.

$$\theta_e : \mathcal{F}_R^e(M) \xrightarrow{\mathcal{F}_R^{e-1}(\theta)} \mathcal{F}_R^{e-1}(M) \longrightarrow \cdots \longrightarrow \mathcal{F}_R^2(M) \xrightarrow{\mathcal{F}_R(\theta)} \mathcal{F}_R(M) \xrightarrow{\quad\theta\quad} M$$

By convention, we let θ_0 denote the identity $M \to M$. The composition of the natural map $M \to \mathcal{F}_R^e(M)$ with $\theta_e : \mathcal{F}_R^e(M) \to M$ is precisely the action of F^e, and the module $R \cdot F^e(M)$ is exactly the image of θ_e. In $R^{1/p}$ notation, we may take a direct sum over all such maps θ_e to obtain an R-linear map

$$\Theta : \bigoplus_{e=0}^{\infty} \mathcal{F}_R^e(M) \to M \tag{3.1}$$

Recalling the structure of $R\langle F\rangle$ as a right R-module, Θ may be understood as a map $R\langle F\rangle \otimes_R M \to M$. For any R-submodule N of M, the $R\langle F\rangle$-span of N is the image of the composition $R\langle F\rangle \otimes_R N \to R\langle F\rangle \otimes_R M \xrightarrow{\Theta} M$, with N generating M over $R\langle F\rangle$ if and only if $R\langle F\rangle \otimes_R N \to M$ is surjective. To say that N is $R\langle F\rangle$-stable is precisely to say that the image of $R\langle F\rangle \otimes_R N \to M$ is contained in N. It is clearly sufficient to ensure that the image of $\mathcal{F}_R(N) \to \mathcal{F}_R(M) \xrightarrow{\theta} M$ is contained in N.

We gain a particularly fine level of control when the structure morphism of an $R\langle F\rangle$-module is an isomorphism.

Definition III.3. Let R be a ring of prime characteristic $p > 0$. Call an $R\langle F\rangle$-module M *unit*[3] if the corresponding structure morphism $\theta : \mathcal{F}_R(M) \to M$ of M is an isomorphism.

If $\mathcal{F}_R(R)$ is identified with R in the natural way[4], then the structure morphism $\mathcal{F}_R(R) \to R$ of the natural action on R is the identity map $R \xrightarrow{1} R$, so the natural

[3] Lyubeznik [Lyu97] uses the term "F-module" for what we refer to here as a "unit $R\langle F\rangle$-module" and he proves a number of powerful results within the category of F-modules. Our choice of terminology follows Blickle [Bli01] (also [BB05, EK04]) because we will need to consider morphisms between both unit and non-unit $R\langle F\rangle$-modules in the sequel. There are places in the literature where the term "level" is used in place of "unit" [HS77].

[4] As $S \otimes_R R$ may be identified with S for any R-algebra S.

action of R is trivially unit. For an ideal $I \subseteq R$, the Frobenius homomorphism of R/I can be understood as an action either over R or over R/I. In the former case, the structure morphism $\mathcal{F}_R(R/I) = R/I^{[p]} \to R/I$ is the quotient map by $I/I^{[p]}$. In the latter case, the structure morphism $\mathcal{F}_{R/I}(R/I) = R/I \to R/I$ is the identity. In other words, if we regard R/I equipped with its natural action as an $(R/I)\langle F\rangle$-module, it is unit, but unless $I = I^{[p]}$, the $R\langle F\rangle$-module R/I is never unit. To avoid ambiguity, we will refer to the former as the $R\langle F\rangle$ structure morphism and the latter as the $(R/I)\langle F\rangle$ structure morphism.

In general, given a map $R \to S$ and an $R\langle F\rangle$-module M, there is a natural way to endow $S \otimes_R M$ with an $S\langle F\rangle$-module structure in such a way that M being an $R\langle F\rangle$ unit implies that $S \otimes_R M$ is an $S\langle F\rangle$ unit. We will use $R^{1/p}$ notation for clarity. Note that the commutative square of maps

$$\begin{array}{ccc} R^{1/p} & \longrightarrow & S^{1/p} \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

gives a canonical identification of the functors $S^{1/p} \otimes_{R^{1/p}} (R^{1/p} \otimes_R -)$ and $S^{1/p} \otimes_S (S \otimes_R -)$ that take R-modules to $S^{1/p}$-modules.

Definition III.4. Let $R \to S$ be a homomorphism between two rings of prime characteristic $p > 0$ and let M be an $R\langle F\rangle$-module with structure morphism $\theta_R : R^{1/p} \otimes_R M \to M^{1/p}$. Define an $S^{1/p}$-linear map, the *base-changed structure morphism* of M, as follows.

$$\theta_S : S^{1/p} \otimes_S (S \otimes_R M) = S^{1/p} \otimes_{R^{1/p}} (R^{1/p} \otimes_R M) \xrightarrow{1 \otimes \theta_R} S^{1/p} \otimes_{R^{1/p}} M^{1/p} = (S \otimes_R M)^{1/p}$$

When we refer to $S \otimes_R M$ as an $S\langle F\rangle$-module, it is understood that this refers to the structure morphism θ_S.

If θ_R is an isomorphism, it is clear that the same is true of $1 \otimes \theta_R$. If $u_1, \ldots, u_t$

generate M over $R\langle F\rangle$, then $1 \otimes u_1, \ldots, 1 \otimes u_t$ generate $S \otimes_R M$ over $S\langle F\rangle$. The following is now clear.

Proposition III.5. *Let $R \to S$ be a map between two rings of prime characteristic $p > 0$, and let M be an $R\langle F\rangle$-module.*

1. *([Bli01, pp. 20]) If M is unit over $R\langle F\rangle$, then $S \otimes_R M$ is unit over $S\langle F\rangle$.*
2. *([Bli01, Proposition 2.22]) If M is finitely generated over $R\langle F\rangle$, then $S \otimes_R M$ is finitely generated over $S\langle F\rangle$.*

A fundamental observation is that the category of unit $R\langle F\rangle$-modules, as a (full) subcategory of the category of $R\langle F\rangle$-modules, is abelian [Lyu97, pp. 72]. This follows quickly from the proposition below. It is an observation of Blickle [Bli01, pp. 18] that the unit property of the cokernel requires no extra hypotheses on R, but the unit property of the kernel requires the Frobenius homomorphism to be flat. This will be the case for many results that follow. Due to a classic result of Kunz [Kun69], for a Noetherian ring of prime characteristic $p > 0$, flatness of the Frobenius homomorphism is equivalent to the assumption that R is regular.

Proposition III.6. *(see [Lyu97, pp. 72], [Bli01, pp. 18]) Let R be a Noetherian ring of prime characteristic $p > 0$, and let $h : M \to N$ be an $R\langle F\rangle$-linear map between two unit $R\langle F\rangle$-modules. Then Coker(h), equipped with the $R\langle F\rangle$-module structure induced from N, is unit. If R is regular, then Ker(h), equipped with the $R\langle F\rangle$-module structure induced from M, is also unit.*

Proof. Express $h : M \to N$ as the composition of two $R\langle F\rangle$-linear maps $M \twoheadrightarrow V$ and $V \hookrightarrow N$. Let α, β, γ, δ, and ε denote the structure morphisms of M, N, V, Ker(h), and Coker(h), respectively. Since $\mathcal{F}_R(-)$ is exact, we have the following two commutative diagrams whose rows are exact.

$$\begin{array}{ccccccccc} & & \mathcal{F}_R(\mathrm{Ker}(h)) & \longrightarrow & \mathcal{F}_R(M) & \longrightarrow & \mathcal{F}_R(V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \mathrm{Ker}(h) & \longrightarrow & M & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} & & \mathcal{F}_R(V) & \longrightarrow & \mathcal{F}_R(N) & \longrightarrow & \mathcal{F}_R(\mathrm{Coker}(h)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\varepsilon} & & \\ 0 & \longrightarrow & V & \longrightarrow & N & \longrightarrow & \mathrm{Coker}(h) & \longrightarrow & 0 \end{array}$$

The surjectivity of α and β imply the surjectivity of γ and ε, respectively. The snake lemma together with the fact that β is an isomorphism implies that $\mathrm{Ker}(\varepsilon)$ is isomorphic to $\mathrm{Coker}(\gamma)$, which vanishes, so ε is an isomorphism. If R is regular, then the flatness of $\mathcal{F}_R(-)$ implies that the kernels of δ and γ embed into the kernels of α and β, respectively, and therefore vanish. The snake lemma implies that $\mathrm{Coker}(\delta) \simeq \mathrm{Ker}(\gamma) = 0$, so δ is an isomorphism. $\square$

Of particularly importance is Lyubeznik's result that *finitely generated* unit $R\langle F\rangle$-modules *also* form an abelian category. The issue of passing finite generation to unit submodules of finitely generated unit modules is the main source of difficulty. The following theorem of Lyubeznik grants an incredible degree of control over unit the submodules of unit $R\langle F\rangle$-modules.

Theorem III.7. *[Lyu97, Proposition 2.5(b)] Let R be a regular ring and M be an $R\langle F\rangle$-module generated over $R\langle F\rangle$ by the R-submodule N. Let U be a unit $R\langle F\rangle$-submodule of M. Then $U \cap N$ generates U over $R\langle F\rangle$.*

Proof. Since $M = \bigcup_{e=0}^{\infty} R \cdot F^e(N)$ and $U = \bigcup_{e=0}^{\infty} U \cap (R \cdot F^e(N))$, it suffices to show that for each e, $R \cdot F(U \cap (R \cdot F^e(N))) = U \cap (R \cdot F^{e+1}(N))$.

Let θ denote the structure morphism of M and let $\theta_e : \mathcal{F}_R^e(M) \to M$ be defined as in Diagram (3.2). Note that if R is regular, then for any R-submodule V of M, the exactness of $\mathcal{F}_R(-)$ allows us to identify $\mathcal{F}_R^e(V)$ with a submodule of $\mathcal{F}_R^e(M)$. We can therefore make sense of the statement $R \cdot F^e(V) = \theta_e(\mathcal{F}_R^e(V))$. Crucially, given two submodules V_1 and V_2 of M, the exactness of $\mathcal{F}_R(-)$ also gives $\mathcal{F}_R(V_1 \cap V_2) = \mathcal{F}_R(V_1) \cap \mathcal{F}_R(V_2)$. Since M is unit, θ is also compatible with intersections. We proceed to compute

$$\begin{aligned} R \cdot F(U \cap (R \cdot F^e(N))) &= \theta(\mathcal{F}_R(U \cap \theta_e(\mathcal{F}_R^e(N))) \\ &= \theta(\mathcal{F}_R(U)) \cap \theta(\mathcal{F}_R(\theta_e(\mathcal{F}_R^e(N)))) \end{aligned}$$

Since U is unit under the action restricted from M, $\theta(\mathcal{F}_R(U)) = U$. Finally, we have $\theta(\mathcal{F}_R(\theta_e(\mathcal{F}_R^e(N)))) = R \cdot F(R \cdot F^e(N)) = R \cdot F^{e+1}(N)$, as desired. $\square$

Once this method of controlling unit submodules has been established, Lyubeznik obtains the following two statements as essentially as a corollary of Theorem III.7.

Corollary III.8. *Let R be a regular ring and let M be a finitely generated unit $R\langle F\rangle$-module.*

1. *[Lyu97, Proposition 2.7] The set of unit submodules of M satisfies the ascending chain condition.*
2. *[Lyu97, Theorem 2.8] Every unit submodule of M is finitely generated over $R\langle F\rangle$.*

The ascending chain condition in particular is the main ingredient of Lyubeznik's finiteness theorem on the set of associated primes for a unit $R\langle F\rangle$-module. We shall sketch his argument to illustrate how this is the case. For an ideal I of R, the I-torsion submodule $\Gamma_I(M)$ of an $R\langle F\rangle$-module M is clearly Frobenius stable. Using the exactness of $\mathcal{F}_R(-)$, one can directly show that M is unit, then $\Gamma_I(M)$ is unit[5]. If P is

[5]This statement is actually a particular case of a more general phenomenon concerning the preservation of the unit property under the application of local cohomology functors, which we will discuss in more detail in the next section.

a maximal associated prime of M, we obtain produce a unit submodule, $M_1 = \Gamma_P(M)$ with only one associated prime. Repeating the argument on M/M_1 and taking preimages in M, we obtain another unit submodule $M_2 \supseteq M_1$ such that M_2/M_1 has a single associated prime. This procedure results in a chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of unit submodules such that each quotient M_i/M_{i-1} has a single associated prime. The chain terminates after finitely many steps, so $\mathrm{Ass}(M)$ is contained in the union of the sets of associated primes of only finitely many factors in this filtration.

Theorem III.9. *[Lyu97, Theorem 2.12(a)] Let R be a regular ring and let M be a finitely generated unit $R\langle F\rangle$-module. The set $Ass(M)$ is finite.*

3.3 The Natural Action on Local Cohomology

If R is a Noetherian ring and M is an $R\langle F\rangle$-module, then for any ideal I and any $i \geq 0$, the local cohomology modules $H^i_I(M)$ inherit an $R\langle F\rangle$-module structure from M. We describe its structure morphism in terms only of maps induced by the functor $H^i_I(-)$ to show that there is no dependence on the choice of generators for I, but once this independence has been established, we will typically prefer to work in terms of the Čech complex.

Definition III.10. Let R be a Noetherian ring of prime characteristic $p > 0$. Let $j_M : M \to R^{1/p} \otimes_R M$ denote the natural R-linear map $u \mapsto 1 \otimes u$ for $u \in M$. Let I be an ideal of R and fix $i \geq 0$. The R-linear map $H^i_I(j_M) : H^i_I(M) \to H^i_I(R^{1/p} \otimes_R M) = H^i_{IR^{1/p}}(R^{1/p} \otimes_R M)$ has as its target an $R^{1/p}$-module, and therefore induces an $R^{1/p}$-linear map $j^i_{I,M} : R^{1/p} \otimes_R H^i_I(M) \to H^i_{I^{1/p}}(R^{1/p} \otimes_R M)$, where we have identified $H^i_{IR^{1/p}}(-) = H^i_{I^{1/p}}(-)$ since the ideals $IR^{1/p} = (I^{[p]}R)^{1/p}$ and $I^{1/p}$ of $R^{1/p}$ have the same radical.

Define the $R\langle F\rangle$-structure morphism $\theta^i_{I,M}$ of $H^i_I(M)$ as follows.

(3.2)

$$\theta^i_{I,M} : R^{1/p} \otimes_R H^i_I(M) \xrightarrow{j^i_{I,M}} H^i_{I^{1/p}}(R^{1/p} \otimes_R M) \xrightarrow{H^i_{I^{1/p}}(\theta)} H^i_{I^{1/p}}(M^{1/p}) = (H^i_I(M))^{1/p}$$

The corresponding Frobenius action on $H^i_I(M)$ is called the *natural action* induced by M.

If $R^{1/p}$ is flat over R, the map $j^i_{I,M}$ is readily seen to be an isomorphism [BS12, Theorem 4.3.2]. If M is unit, $H^i_{I^{1/p}}(\theta)$ is an isomorphism. The following is now clear.

Proposition III.11. *[Lyu97, Example 1.2 (b)] Let R be a regular ring of prime characteristic $p > 0$ and let M be a unit $R\langle F\rangle$-module. Then for any ideal I of R and any $i \geq 0$, the natural action on $H^i_I(M)$ is unit.*

Let $\underline{\mathbf{f}} = f_1, \ldots, f_t$ be a choice of generators for I, let $C^\bullet$ denote the Čech complex $\check{C}^\bullet(\underline{\mathbf{f}}; R)$, and let $C^\bullet_M = C^\bullet \otimes_R M$. The homomorphisms $j_M : M \to R^{1/p} \otimes_R M$ and $\theta : R^{1/p} \otimes_R M \to M^{1/p}$ induce maps of complexes[6] $J_M : C^\bullet_M \to R^{1/p} \otimes_R C^\bullet_M$ and $\Theta : R^{1/p} \otimes_R C^\bullet_M \to (C^\bullet_M)^{1/p}$. The induced maps on the cohomology of these complexes, $H^i(J_M)$ and $H^i(\Theta)$, are precisely the maps $H^i_I(j_M)$ and $H^i_I(\theta)$ induced by the local cohomology functor $H^i_I(-)$, and therefore, do not depend in any way on the choice of generators for I. We may therefore use these maps $H^i(J_M)$ and $H^i(\Theta)$ to describe the structure morphism of $H^i_I(M)$ in terms of the Čech complex without worrying the result may change given a different choice of generators.

We will make use of the following family of natural transformations both here and in the next chapter.

Definition III.12. If $f : R \to S$ is a ring homomorphism and $C^\bullet$ is an R-complex, then the natural (R-linear) map $C^\bullet \to S \otimes_R C^\bullet$ induces $H^i(C^\bullet) \to H^i(S \otimes_R C^\bullet)$,

[6] We have used the identification $M^{1/p} \otimes C^\bullet = \check{C}(\underline{\mathbf{f}}^{[p]}; M)^{1/p} = \check{C}(\underline{\mathbf{f}}; M)^{1/p} = (C^\bullet_M)^{1/p}$.

which factors uniquely through the natural map $H^i(C^\bullet) \to S \otimes_R H^i(C^\bullet)$ to an S-linear map $S \otimes_R H^i(C^\bullet) \to H^i(S \otimes_R C^\bullet)$. Call this map $h^i_f(C^\bullet)$, and let h^i_f denote the corresponding natural transformation

$$h^i_f(-) : S \otimes_R H^i(-) \longrightarrow H^i(S \otimes_R -)$$

of functors $\mathrm{Kom}_R \to \mathrm{Mod}_S$.

If the homomorphism $f : R \to S$ is understood from context, we will write $h^i_{S/R}(-)$ instead of $h^i_f(-)$. The latter, more precise, notation is reserved for ambiguous cases, such as when $R = S$ has prime characteristic p and f is the Frobenius homomorphism. Note also that S is flat over R if and only if $h^i_{S/R}(-)$ is an isomorphism of functors.

Let $\underline{\mathbf{f}} = f_1, \cdots, f_t$ be a sequence of elements of R, let $C^\bullet == \check{C}^\bullet(\underline{\mathbf{f}}; R)$, and for M an R-module, let $C^\bullet_M := C^\bullet \otimes_R M$. In this context, the map $h^i_{R^{1/p}/R}(C^\bullet_M) : R^{1/p} \otimes_R H^i(C^\bullet_M) \to H^i(R^{1/p} \otimes_R C^\bullet_M)$ is precisely $j^i_{I,M} : R^{1/p} \otimes_R H^i_I(M) \to H^i_{I^{1/p}}(R^{1/p} \otimes_R M)^{1/p}$ from diagram 3.2.

In $\mathcal{F}_R(-)$ notation, note that the complex $\mathcal{F}_R(C^\bullet)$ is canonically identified[7] with $C^\bullet$ itself, and likewise, for any R-module M, $\mathcal{F}_R(C^\bullet_M)$ is canonically identified with $C^\bullet_{\mathcal{F}_R(M)}$. We can therefore understand $C^\bullet \otimes_R -$ applied to θ as a map $\Theta : \mathcal{F}_R(C^\bullet_M) = C^\bullet \otimes_R \mathcal{F}_R(M) \to C^\bullet_M$.

The Čech construction of the structure morphism of $H^i_I(M)$ is as follows.

$$\theta^i_{I,M} : \mathcal{F}_R(H^i(C^\bullet_M)) \xrightarrow{h^i_{F_R}(C^\bullet_M)} H^i(\mathcal{F}_R(C^\bullet_M)) \xrightarrow{H^i(\Theta)} H^i(C^\bullet_M) \tag{3.3}$$

Since $C^\bullet_M = C^\bullet \otimes_R M$, the definition is completely functorial in M, so that the map $H^i_I(M) \to H^i_I(N)$ induced by $N \to M$ with the structure morphisms above is readily seen to be $R\langle F\rangle$-linear. We may therefore regard $H^i_I(-)$ as a functor from

[7] For any element $g \in R$, there is no difference between localization with respect to the multiplicative system $\{1, g, g^2, g^3, \cdots\}$ and the multiplicative system $\{1, g^p, g^{2p}, g^{3p}\}$.

$R\langle F\rangle$-modules to $R\langle F\rangle$-modules. By applying $C^\bullet \otimes_R -$ to a short exact sequence $0 \to N \to M \to Q \to 0$ of $R\langle F\rangle$-modules, the resulting short exact sequence of complexes $0 \to C^\bullet_N \to C^\bullet_M \to C^\bullet_Q \to 0$ makes it clear that all connecting homomorphisms in the corresponding long exact sequence in local cohomology are also $R\langle F\rangle$-linear.

Proposition III.13. *[Lyu97, Example 1.2 (b), (b')[8]] Let R be a Noetherian ring of prime characteristic $p > 0$. Let I be an ideal of R and fix $i \geq 0$. Given an $R\langle F\rangle$-module M, if $H^i_I(M)$ is regarded as an $R\langle F\rangle$-module via the structure morphism* (3.2) *induced from M, then the association $M \mapsto H^i_I(M)$ is a functor from the category of $R\langle F\rangle$-modules to itself. Moreover, for any short exact sequence of $R\langle F\rangle$-modules $0 \to N \to M \to Q \to 0$, the connecting homomorphisms in the long exact sequence induced by $\Gamma_I(-)$ are $R\langle F\rangle$-linear.*

The Čech complex definition of the $R\langle F\rangle$ structure of $H^i_I(M)$ is given in terms of the structure morphism of the complex $C^\bullet \otimes_R M$, and the structure morphism of $C^\bullet \otimes_R M$ is precisely the data of an $R\langle F\rangle$-module structure on each term $C^i \otimes_R M$ such that the differentials of the complex are $R\langle F\rangle$-linear. In the case where $M = R$, each term of $C^\bullet$ is a direct sum of localizations R_f equipped with their natural actions.

Proposition III.14. *[Bli01, Lemma 2.2.4[9]] Let R be a Noetherian ring and let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a sequence of elements. The $R\langle F\rangle$-module $H^c_{\underline{\mathbf{f}}R}(R)$, equipped with its natural action, is unit and finitely generated.*

Proof. Let $f = \prod_{i=1}^c f_i$. In terms of the complex $C^\bullet = \check{C}^\bullet(\underline{\mathbf{f}}; R)$, $H^c_{\underline{\mathbf{f}}}(R)$ is the cokernel of the map $C^{c-1} \to C^c$ where both C^{c-1} and C^c are unit and finitely generated. By

[8]Lyubeznik's argument uses an injective resolution equipped with compatible Frobenius actions on every term, and the construction of this resolution requires that R is regular and M is unit. These hypotheses can be relaxed if the role of the injective resolution in his argument is replaced by the Čech complex, making use of [Wei94, Proposition 1.3.4] on the short exact sequence of complexes $0 \to C^\bullet_N \to C^\bullet_M \to C^\bullet_Q \to 0$.

[9]We are only using the case $M = R$, and the preservation of the unit property and finite generation under homomorphic image, which as Blickle discusses on pp. 18, does not require R to be regular.

III.6, the cokernel of a map between unit $R\langle F\rangle$-modules is unit[10]. Since $C^c = R_f$ is finitely generated (e.g., by $1/f$), so is its $R\langle F\rangle$-homomorphic image, $H^c_{\underline{\mathbf{f}}}(R)$. $\square$

A much stronger statement can be made if R is regular. In this case, if M is a finitely generated unit $R\langle F\rangle$-module, Lyubeznik shows that any localization M_g for $g \in R$ is also unit and finitely generated [Lyu97, Proposition 2.9(b)]. Since Čech complex $\check{C}^\bullet(\underline{\mathbf{f}}; M)$ is a complex of finitely generated unit $R\langle F\rangle$-modules, the kernels of the differentials are themselves unit and finitely generated by Proposition III.6 and Corollary III.8. The same can be said for any $R\langle F\rangle$-homomorphic image of those kernels, and the result below follows at once.

Theorem III.15 (Lyubeznik; Proposition 2.10 [Lyu97])**.** *Let R be a regular ring of prime characteristic $p > 0$. Let M be a finitely generated unit $R\langle F\rangle$-module, let I be an ideal of R, and fix $i \geq 0$. Then $H^i_I(M)$, regarded as an $R\langle F\rangle$-module via the structure induced from M, is finitely generated and unit over $R\langle F\rangle$.*

The following is an immediate consequence of Theorems III.15 and III.9.

Theorem III.16. *Let R be a regular ring of prime characteristic $p > 0$ and let M be a finitely generated unit $R\langle F\rangle$-module. For any ideal I of R and any $i \geq 0$, the module $H^i_I(M)$ has finitely many associated primes.*

---

[10]The hypothesis of regularity is only necessary to ensure the unit property of kernels.

CHAPTER IV

The Hellus Isomorphism and Other Functorial Tools

Let R be a Noetherian ring. To a given local cohomology functor $H^i_I(-)$ with respect to some ideal I of R, we may associate a pair (i, h) consisting of the cohomological degree i and the height of the defining ideal $h = \mathrm{ht}(I)$. In this section we will show that those local cohomology functors for which the pair (i, h) satisfies $i \le h+1$ are fully general in the sense of the following theorem (consider $k = i - h + 1$ if $i > h + 1$). Note that the height of the unit ideal is $\inf(\varnothing) = +\infty$.

Theorem (IV.4)**.** *Let R be a Noetherian ring, let I be an ideal of height h, and fix $k \ge 0$. There is an ideal $I_k \supseteq I$ such that $ht(I_k) \ge h + k$ and such that the natural transformation $H^i_{I_k}(-) \to H^i_I(-)$ is an isomorphism (resp. a surjection) of functor for all $i > h + k$ (resp. $i = h + k$).*

This statement is an result of the author, written up for publication in [Lew19]. Reduction to the case $i \le h + 1$ will drastically reduce the difficulty of certain key proofs in subsequent chapters. It is a generalization of an isomorphism theorem of Hellus that we restate below.

Theorem IV.1. *[Hel01, Theorem 3] Let R be a Cohen-Macaulay local ring, let I be an ideal of height h, and fix $k \ge 0$. There is an ideal $I_k \supseteq I$ such that $ht(I_k) \ge h+k$ and such that the natural map $H^{h+k+1}_{I_k}(R) \to H^{h+k+1}_I(R)$ is an isomorphism. If*

$H_I^{h+k+1}(R) \neq 0$*, then* I_k *can be chosen such that* $ht(I_k) = h + k$*.*

While it is interesting that the Cohen-Macaulay hypothesis can be eliminated, the Cohen-Macaulay case of this isomorphism remains particularly interesting. The modules $H_I^i(R)$ such that $(i, \mathrm{ht}(I))$ satisfies $i \leq \mathrm{ht}(I) + 1$ vanish unless $i = \mathrm{ht}(I)$ or $i = \mathrm{ht}(I) + 1$. The former is fairly straightforward to deal with as far as finiteness questions are concerned, see e.g. Theorem II.18. Effectively, we need only consider the case $i = \mathrm{ht}(I) + 1$ – a drastic simplification.

After proving our generalization of Hellus's isomorphism, we move on to a separate issue involving the compatibility (Theorem IV.9) of the natural transformations $h^i_{I;S/R}(-) : S \otimes_R H^i_I(-) \to H^i_I(S \otimes_R -)$ with the Frobenius functors of R and S. This leads to the following statement

Theorem (IV.10)**.** *Let* $R \to S$ *be a map between two Noetherian rings of prime characteristic* $p > 0$*, and let* M *be an* $R\langle F\rangle$*-module. Then the natural map*

$$h^i_{I;S/R}(M) : S \otimes_R H^i_I(M) \to H^i_I(S \otimes_R M)$$

is a morphism of $S\langle F\rangle$*-modules.*

This will be particularly useful in Section 5.4 where R and S are regular and M is unit. These results of the author are also written up for publication in [Lew19].

4.1 A generalized isomorphism of Hellus

As this situation may arise in a number of proofs in this section, note that, by convention, the intersection of prime ideals of R indexed by the empty set is taken to be $\bigcap_{i\in\varnothing} P_i = R$. Recall also that an ideal is said to have pure height h if all of its minimal primes have height exactly h.

We require a notion of parameters in a global ring to proceed, and the following lemma provides one suitable for use in our main proofs.

Lemma IV.2. *Let R be a Noetherian ring, let I be a proper ideal of height $h \geq 0$, and let $J \subseteq I$ be an ideal of height $j \geq 0$.*

(a) If an ideal of the form $(x_1, \cdots, x_h)R$ has height h, then it has pure height h.

(b) Any sequence $x_1, \cdots, x_j \in J$ generating an ideal of height j (including the empty sequence if $j = 0$) can be extended to a sequence $x_1, \cdots, x_h \in I$ generating an ideal of height h.

(c) There is a sequence $x_1, \cdots, x_h \in I$ such that $(x_1, \cdots, x_h)R$ has (necessarily pure) height h.

Proof. (a) Every minimal prime of a height h ideal has height at least h by definition, and every minimal prime of an h-generated ideal has height at most h by Krull's height theorem [Mat89, Theorem 13.5].

(b) If $j = h$, there is nothing to do, so assume $j < h$. By induction, it is enough to show that we can extend the sequence by one element. Since $j < h$, I is not contained in any minimal prime of $(x_1, \cdots, x_j)R$ (all of which have height j), and so we may choose $x \in I$ avoiding all such primes. A height j prime containing $(x_1, \cdots, x_j)R$ therefore cannot also contain xR. Thus, the minimal primes of $(x, x_1, \cdots, x_j)R$ have height at least $j+1$. By Krull's height theorem, they also have height at most $j+1$.

(c) This follows at once from (b) by taking $J = (0)$. □

Our method of enlarging an ideal to obtain the desired functorial isomorphism will proceed inductively replacing I with $I + yR$ for some suitable choice of $y \in R$. The lemma below describes how we will choose the element y. Note that some of our applications require the resulting isomorphism to have special properties with respect to a sequence of j elements generating an ideal J of height j contained in I, and this needs to be taken into consideration in our choice of the element y.

Lemma IV.3. *Let R be a Noetherian ring, let I be a proper ideal of height h, and let $J \subseteq I$ be an ideal of height $j \leq h$ generated by j elements. There is an element $y \in R$ that satisfies the following properties.*

(i) $ara_R(yR \cap I) = h$

(ii) $ara_{R/J}(y(R/J) \cap (I/J)) = h - j$

(iii) $ht(yR + I) \geq h + 1$

Proof. Write $J = (x_1, \cdots, x_j)R$, and extend this sequence to $x_1, \cdots, x_h \in I$ generating an ideal of height h. I is contained in at least one minimal prime of $(x_1, \cdots, x_h)R$.

Let $P_1, \cdots, P_t$ be the minimal primes of $(x_1, \cdots, x_h)R$ containing I, and $Q_1, \cdots, Q_s$ be the minimal primes of $(x_1, \cdots, x_h)R$ that do not. We may have $s = 0$. Since these primes are pairwise incomparable, there exist elements $y \in Q_1 \cap \cdots \cap Q_s$ that avoid the union $P_1 \cup \cdots \cup P_t$ (if $s = 0$, we can take $y = 1$). For any such y, it holds that

$$yR \cap I \subseteq P_1 \cap \cdots \cap P_t \cap Q_1 \cap \cdots \cap Q_s = \sqrt{(x_1, \cdots, x_h)R}$$

and thus $yR \cap I \subseteq yR \cap \sqrt{(x_1, \cdots, x_h)R}$. Since $(x_1, \cdots, x_h) \subseteq I$, we see that

$$yR \cap (x_1, \cdots, x_h)R \subseteq yR \cap I \subseteq yR \cap \sqrt{(x_1, \cdots, x_h)R}$$

It follows at once that that

$$\sqrt{yR \cap I} = \sqrt{yR \cap (x_1, \cdots, x_h)R} = \sqrt{(yx_1, \cdots, yx_h)R}$$

producing an upper bound on arithmetic rank: $\mathrm{ara}_R(yR \cap I) \leq h$. To obtain the lower bound $\mathrm{ara}_R(yR \cap I) \geq h$, suppose that for some $t < h$ we had a sequence of elements $z_1, \cdots, z_t$ generating an ideal with the same radical as $yR \cap I$. Then

$\sqrt{(z_1, \cdots, z_t)R} = \sqrt{(yx_1, \cdots, yx_h)R}$, and upon localizing at P_1, we would have $\sqrt{(z_1, \cdots, z_t)R_{P_1}} = \sqrt{(x_1, \cdots, x_h)R_{P_1}}$ since y is a unit in R_{P_1}. It would follow that $\sqrt{(x_1, \cdots, x_h)R_{P_1}}$ has height no more than t, which is a contradiction.

Since $yR/J \cap I/J \subseteq \sqrt{(x_{j+1} \cdots, x_h)R/J}$, an identical argument to the above shows that

$$\sqrt{yR/J \cap I/J} = \sqrt{(yx_{j+1}, \cdots, yx_h)R/J}$$

so $\mathrm{ara}_{R/J}(yR/J \cap I/J) \leq h - j$.

We have established (i) and (ii). Concerning (iii), note that that all primes containing $yR+I$ also contain $(x_1, \cdots, x_h)R$, and thus, to show that $\mathrm{ht}(yR+I) \geq h+1$, it is enough to show that none of the height h primes containing $(x_1, \cdots, x_h)R$ appear in $V(yR+I)$. But this is clear, since $\{P \supseteq (x_1, \cdots, x_h)R \mid \mathrm{ht}(P) = h\} = \{P_1, \cdots, P_t, Q_1, \cdots, Q_s\}$. None of the primes P_i contain yR, and none of the primes Q_j contain I. $\square$

Theorem IV.4. *Let R be a Noetherian ring, let I be an ideal of height h, and let $J \subseteq I$ be an ideal of height $j \geq 0$ generated by j elements. For any $k \geq 0$, there is an ideal $I_{k,J} \supseteq I$ such that*

- *$ht(I_{k,J}) \geq ht(I) + k$*
- *The natural transformation $H^i_{I_{k,J}}(-) \to H^i_I(-)$ is an isomorphism on R-modules for all $i > h + k$, and an isomorphism on R/J-modules for all $i > h - j + k$. If $i = h + k$ (resp. $i = h - j + k$) this natural transformation is a surjection on R-modules (resp. R/J-modules).*

Proof. If $k = 0$, choose $I_{0,J} = I$. Fix $k \geq 1$, and suppose that we've chosen the ideal $I_{k-1,J}$ by induction. We must choose $I_{k,J}$. For brevity, we will suppress J from our notation, and write I_{k-1} and I_k for $I_{k-1,J}$ and $I_{k,J}$, respectively.

If $\mathrm{ht}(I_{k-1}) > h+k-1$ we can simply pick $I_k = I_{k-1}$, so assume that $\mathrm{ht}(I_{k-1}) = h+k-1$. By Lemma IV.3 there is an element $y \in R$ such that $\mathrm{ht}(yR + I_{k-1}) \geq (h+k-1)+1$, with

$$\mathrm{ara}_R(yR \cap I_{k-1}) \leq h+k-1 \;\text{ and }\; \mathrm{ara}_{R/J}(y(R/J) \cap I_{k-1}/J) \leq (h+k-1)-j$$

Consider the Mayer-Vietoris sequence on the intersection $yR \cap I_{k-1}$. We use $(-)$ in our notation to mean that the sequence is exact when $-$ is replaced by any R-module M, and that all maps in the sequence are given by natural transformations.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^{i-1}_{yR\cap I_{k-1}}(-) \arrow[dll, out=-10, in=170] \\
H^i_{yR+I_{k-1}}(-) \arrow[r] & H^i_{yR}(-) \oplus H^i_{I_{k-1}}(-) \arrow[r] & H^i_{yR\cap I_{k-1}}(-)
\end{tikzcd}$$

Let $i > h+k$. Since $i-1 > \mathrm{ara}_R(yR \cap I_{k-1})$, we get vanishing $H^{i-1}_{yR\cap I_{k-1}}(-) = H^i_{yR\cap I_{k-1}}(-) = 0$. Since $i \geq h+k+1 \geq 2$, we also have $H^i_{yR}(-) = 0$, and therefore obtain a natural isomorphism $H^i_{yR+I_{k-1}}(-) \xrightarrow{\sim} H^i_{I_{k-1}}(-)$. Notice that if $i = h+k$, then we still have $H^i_{yR\cap I_{k-1}}(-) = 0$, so

$$H^i_{yR+I_{k-1}}(-) \to H^i_{yR}(-) \oplus H^i_{I_{k-1}}(-) \to 0$$

is exact, implying that the component map $H^i_{yR+I_{k-1}}(-) \to H^i_{I_k}(-)$ is surjective. Working with R/J-modules, an identical argument using the fact that

$$\mathrm{ara}_{R/J}(y(R/J) \cap I_{k-1}/J) \leq (h+k-1)-j$$

shows that

$$H^i_{y(R/J)+I_{k-1}/J}(-) \xrightarrow{\sim} H^i_{I_{k-1}/J}(-)$$

when $i > h+k-j$ and

$$H^i_{y(R/J)+I_{k-1}/J}(-) \twoheadrightarrow H^i_{I_{k-1}/J}(-)$$

when $i = h + k - j$. Finally, $\mathrm{ht}(yR + I_{k-1}) \geq h + k$, so we may in fact choose $I_k = yR + I_{k-1}$, which completes the induction. □

Corollary IV.5. *Let R be a Noetherian ring and let $I \subseteq R$ be any ideal. Fix $i \geq 0$. There is an ideal $I' \supseteq I$ (resp. $I'' \supseteq I$) such that*

- *$ht(I') \geq i - 1$ (resp. $ht(I'') \geq i$)*
- *$H^i_{I'}(-) \xrightarrow{\sim} H^i_I(-)$ (resp. $H^i_{I''}(-) \twoheadrightarrow H^i_I(-)$)*

Proof. Let $h = \mathrm{ht}(I)$. If $h \geq i - 1$ (resp. $h \geq i$) simply choose $I' = I$ (resp. $I'' = I$). Otherwise, $h < i - 1$ (resp. $h < i$). Apply Theorem IV.4 in the case $k = i - 1 - h$ (resp. $k' = i - h$) to obtain an ideal $I' \supseteq I$ (resp. $I'' \supseteq I$) satisfying $\mathrm{ht}(I') \geq h + k = i - 1$ (resp. $\mathrm{ht}(I'') \geq h + k' = i$) and $H^i_{I'}(-) \xrightarrow{\sim} H^i_I(-)$, since $i > h + k$ (resp. $H^i_{I''}(-) \twoheadrightarrow H^i_I(-)$, since $i = h + k'$). □

An immediate application of this theorem is to generalize a corollary of Hellus [Hel01, Corollary 2]. This generalization provides a new proof of a result of Marley [Mar01, Proposition 2.3], namely, for any Noetherian ring R, any ideal $I \subseteq R$, and any R-module M, $\{P \in \mathrm{Supp}\, H^i_I(M) \,|\, \mathrm{ht}(P) = i\}$ is a finite set. Since our result comes from a surjection of functors, we will describe it in terms of the "support" of $H^i_I(-)$.

By the *support* of a functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$, we mean the set of primes $P \in \mathrm{Spec}(R)$ such that $F(-)_P$ is not the zero functor. That is to say,

$$\mathrm{Supp}(F) := \{P \in \mathrm{Spec}(R) \,|\, \exists M \in \mathrm{Mod}_R \text{ such that } F(M)_P \neq 0\}$$

For example, if $I \subseteq R$ is an ideal and $i \geq 0$, then $\mathrm{Supp}\, H^i_I(-) \subseteq V(I)$. It is clear (see, e.g., Theorem II.13 or Theorem II.14) that this inclusion need not be sharp. For any $i > \mathrm{ht}(I)$, the following Corollary shows us how to find a closed set containing $\mathrm{Supp}\, H^i_I(-)$ strictly smaller than $V(I)$.

Corollary IV.6. *(cf. Marley [Mar01, Proposition 2.3]) Let R be a Noetherian ring and I be an ideal. Then for all $i \geq 0$, there is an ideal $I'' \supseteq I$ with $ht(I'') \geq i$ such that $Supp\, H^i_I(-) \subseteq V(I'')$. In particular, for any R-module M, the set*

$$\{P \in Supp\, H^i_I(M) \mid ht(P) = i\}$$

is a subset of $Min_R(R/I'')$, and is therefore finite. If R is semilocal and $i \geq \dim(R) - 1$, then $Supp\, H^i_I(M)$ is a finite set.

Proof. Fix $i \geq 0$ and write $h = \mathrm{ht}(I)$. If $i < h$, then because $\mathrm{Supp}\, H^i_I(-) \subseteq V(I)$, we already have $\mathrm{ht}(P) \geq h > i$ for all $P \in \mathrm{Supp}\, H^i_I(-)$ and there is nothing to prove. So assume that $i \geq h$. By Corollary IV.5, there is an ideal $I'' \supseteq I$ such that $\mathrm{ht}(I'') \geq i$ and $H^i_{I''}(-) \twoheadrightarrow H^i_I(-)$. In particular, for any R-module M, $H^i_I(M)$ is I''-torsion, and thus $\mathrm{Supp}\, H^i_I(M) \subseteq V(I'')$. All primes in $V(I'')$ have height at least i. Any primes of height exactly i must be among the minimal primes of I'', of which there are only finitely many. □

4.2 Compatibility and simultaneous base change

Recall the natural transformations $h^i_f(-)$ of Definition III.12, also denoted $h^i_{S/R}(-)$, associated with a ring homomorphism $f : R \to S$. In this section, we will prove a number of compatibility properties for these transformations prior to proving our main result on the $S\langle F\rangle$-linearity of certain base changed maps.

Proposition IV.7. *Let $R \to S \to T$ be ring homomorphisms. The following diagram of functors $Kom_R \to Mod_T$ commutes*[1]*.*

[1] The equalities shown in this diagram come from identifying the functor $T \otimes_S (S \otimes_R -)$ with $T \otimes_R -$

$$
\begin{tikzcd}
T \otimes_R H^i(-) \arrow[r, equal] \arrow[ddrr, "h^i_{T/R}(-)"'] & T \otimes_S (S \otimes_R H^i(-)) \arrow[r, "id_T \otimes_S h^i_{S/R}(-)"] & T \otimes_S H^i(S \otimes_R -) \arrow[d, "h^i_{T/S}(S \otimes_R -)"] \\
 & & H^i(T \otimes_S (S \otimes_R -)) \arrow[d, equal] \\
 & & H^i(T \otimes_R -)
\end{tikzcd}
$$

Proposition IV.8. *Fix a commutative square of ring homomorphisms*

$$
\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R' \arrow[r] & S'
\end{tikzcd}
$$

There is a commutative square of functors $Kom_R \to Mod_{S'}$

$$
\begin{tikzcd}
S' \otimes_S (S \otimes_R H^i(-)) \arrow[r, "id_{S'} \otimes_S h^i_{S/R}(-)"] \arrow[d] & S' \otimes_S H^i(S \otimes_R -) \arrow[d] \\
S' \otimes_{R'} H^i(R' \otimes_R -) \arrow[r, "h^i_{S'/R'}(R' \otimes_R -)"'] & H^i(S' \otimes_{R'} (R' \otimes_R -))
\end{tikzcd}
$$

Proof. Apply Proposition IV.7 to $R \to S \to S'$ in order to see that the upper right corner of the below diagram commutes, and then to $R \to R' \to S'$ to see that the lower left corner commutes as well:

$$
\begin{tikzcd}
S' \otimes_S (S \otimes_R H^i(-)) \arrow[r, "\mathrm{id}_{S'} \otimes_S h^i_{S/R}(-)"] \arrow[d, equal] & S' \otimes_S H^i(S \otimes_R -) \arrow[d, "h^i_{S'/S}(S \otimes_R -)"] \\
S' \otimes_R H^i(-) \arrow[d, equal] \arrow[dr, "h^i_{S'/R}(-)"] & H^i(S' \otimes_S (S \otimes_R -)) \arrow[d, equal] \\
S' \otimes_{R'} (R' \otimes_R H^i(-)) \arrow[d, "\mathrm{id}_{S'} \otimes_{R'} h^i_{R'/R}(-)"'] & H^i(S' \otimes_R -) \arrow[d, equal] \\
S' \otimes_{R'} H^i(R' \otimes_R -) \arrow[r, "h^i_{S'/R'}(R' \otimes_R -)"'] & H^i(S' \otimes_{R'} (R' \otimes_R -))
\end{tikzcd}
$$

□

The main application of the above compatibility statement is when $R \to S$ is a map between two rings of prime characteristic $p > 0$, and $R \to R'$, $S \to S'$ are the Frobenius homomorphisms of R and S, respectively. If M is an $R\langle F\rangle$-module with structure morphism $\theta : \mathcal{F}_R(M) \to M$, note that $S \otimes_R M$ can be regarded as an $S\langle F\rangle$-module via the structure isomorphism $\mathrm{id}_S \otimes \theta : S \otimes_R \mathcal{F}_R(M) \to S \otimes_R M$, where $S \otimes_R \mathcal{F}_R(M)$ and $\mathcal{F}_S(S \otimes_R M)$ are identified in the canonical way[2].

The relevant version of Proposition IV.8 in this setting is as follows.

Corollary IV.9. *Let $R \to S$ be a homomorphism between two rings of prime characteristic $p > 0$. The following diagram commutes,*

$$\begin{array}{ccc} \mathcal{F}_S(S \otimes_R H^i(-)) & \xrightarrow{\mathcal{F}_S(h^i_{S/R}(-))} & \mathcal{F}_S(H^i(S \otimes_R -)) \\ \downarrow & & \downarrow \\ S \otimes_R H^i(\mathcal{F}^e_R(-)) & \xrightarrow[h^i_{S/R}(\mathcal{F}_R(-))]{} & H^i(S \otimes_R \mathcal{F}^e_R(-)) \end{array}$$

For a Noetherian ring R of prime characteristic $p > 0$, M an $R\langle F\rangle$-module, I an ideal of R, and $i \geq 0$, we recall below the construction of the structure morphism of $H^i_I(M)$ as an $R\langle F\rangle$-module in terms of the structure morphism of M in terms of the Čech complex. Let $\underline{\mathbf{f}} = f_1, \cdots, f_t$ be a sequence of elements of R, let $C^\bullet = \check{C}^\bullet(\underline{\mathbf{f}}; R)$, and let $C^\bullet_M = C^\bullet \otimes_R M$. Let $\theta : \mathcal{F}_R(M) \to M$ be the structure morphism of M and let $\Theta : \mathcal{F}_R(C^\bullet_M) \to C^\bullet_M$ be the corresponding map of complexes. From diagram (3.3), the structure morphism of the natural action on $H^i_I(M)$ induced by M is the composition shown below.

$$\mathcal{F}_R(H^i(C^\bullet_M)) \xrightarrow{h^i_{F_R}(C^\bullet_M)} H^i(\mathcal{F}_R(C^\bullet_M)) \xrightarrow{H^i(\Theta)} H^i(C^\bullet_M) \tag{4.1}$$

[2]Using different notation, this is simply identifying $S^{1/p} \otimes_{R^{1/p}} (R^{1/p} \otimes_R M)$ with $S^{1/p} \otimes_S (S \otimes_R M)$ using the equality of the composite maps $R \to S \to S^{1/p}$ and $R \to R^{1/p} \to S^{1/p}$

Theorem IV.10. *Let $R \to S$ be a homomorphism between two Noetherian rings of prime characteristic $p > 0$, fix an ideal $I \subseteq R$ and an index $i \geq 0$, and let M be an $R\langle F\rangle$-module with structure morphism $\theta : \mathcal{F}_R(M) \to M$. The natural map*

$$S \otimes_R H^i_I(M) \to H^i_I(S \otimes_R M)$$

is a morphism of $S\langle F\rangle$-modules.

Proof. Let $C^\bullet_M = \check{C}^\bullet(\underline{\mathbf{f}}; M)$ be the Čech complex on M associated with a sequence of elements $\underline{\mathbf{f}} = f_1, \cdots, f_t$ generating I. It is enough to show that the diagram

$$\begin{array}{ccc}
\mathcal{F}_S(S \otimes_R H^i(C^\bullet_M)) & \xrightarrow{\mathcal{F}_S(h^i_{S/R}(C^\bullet_M))} & \mathcal{F}_S(H^i(S \otimes_R C^\bullet_M)) \\
\downarrow & & \downarrow \\
S \otimes_R H^i(C^\bullet_M) & \xrightarrow{h^i_{S/R}(C^\bullet_M)} & H^i(S \otimes_R C^\bullet_M)
\end{array}$$

commutes, where the vertical arrows are the structure morphisms of $S \otimes_R H^i_I(M)$ and $H^i_I(S \otimes_R M)$ as $S\langle F\rangle$-modules, respectively. Let $\Theta : \mathcal{F}(C^\bullet_M) \to C^\bullet_M$ denote the morphism of complexes induced by θ. Using the decomposition (3.3) of the structure morphism of $H^i_I(M)$, the stated result is equivalent to showing that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{F}_S(S \otimes_R H^i(C^\bullet_M)) & \xrightarrow{\mathcal{F}_S(h^i_{S/R}(C^\bullet_M))} & \mathcal{F}_S(H^i(S \otimes_R C^\bullet_M)) \\
\| & & \downarrow {\scriptstyle h^i_{F_S}(S \otimes_R C^\bullet_M)} \\
S \otimes_R \mathcal{F}_R(H^i(C^\bullet_M)) & & H^i(\mathcal{F}_S(S \otimes_R C^\bullet_M)) \\
{\scriptstyle \mathrm{id}_S \otimes h^i_{F_R}(C^\bullet_M)} \downarrow & & \| \\
S \otimes_R H^i(\mathcal{F}_R(C^\bullet_M)) & \xrightarrow{h^i_{S/R}(\mathcal{F}_R(C^\bullet_M))} & H^i(S \otimes_R \mathcal{F}_R(C^\bullet_M)) \\
{\scriptstyle \mathrm{id}_S \otimes H^i(\Theta)} \downarrow & & \downarrow {\scriptstyle H^i(\mathrm{id}_S \otimes \Theta)} \\
S \otimes_R H^i(C^\bullet_M) & \xrightarrow{h^i_{S/R}(C^\bullet_M)} & H^i(S \otimes_R C^\bullet_M)
\end{array}$$

The commutativity of the rectangle of maps in the top three rows is precisely the content of Corollary IV.9 applied to the complex $C^\bullet_M$. The square of maps in the bottom two rows is induced from the diagram that results from applying $H^i(-)$ to

$$\begin{tikzcd}
\mathcal{F}_R(C^\bullet_M) \arrow[r, "\text{nat}"] \arrow[d, "\Theta"'] & S \otimes_R \mathcal{F}_R(C^\bullet_M) \arrow[d, "(\mathrm{id}_S \otimes \Theta)"] \\
C^\bullet_M \arrow[r, "\text{nat}"] & S \otimes_R C^\bullet_M
\end{tikzcd}$$

Recall that $C^\bullet_M = C^\bullet \otimes_R M$ and $\mathcal{F}_R(C^\bullet_M) = C^\bullet \otimes_R \mathcal{F}_R(M)$, where $C^\bullet = \check{C}^\bullet(\underline{\mathbf{f}}; R)$, so that the above diagram is $C^\bullet \otimes_R -$ applied to the diagram below, which obviously commutes.

$$\begin{tikzcd}
\mathcal{F}_R(M) \arrow[r, "\text{nat}"] \arrow[d, "\theta"'] & S \otimes_R \mathcal{F}_R(M) \arrow[d, "(\mathrm{id}_S \otimes \theta)"] \\
M \arrow[r, "\text{nat}"] & S \otimes_R M
\end{tikzcd}$$

$\square$

CHAPTER V

Parameter Ideals Following Hochster and Núñez-Betancourt

A *complete intersection ring* is a Noetherian ring S such that, for all prime ideals P of S, the completion $\widehat{S_P}$ of the local ring (S_P, PS_P) is the quotient of a regular local ring by an ideal generated by a regular sequence. Our interest in this chapter and those that follow is in those complete intersection rings that are globally presentated as homomorphic images of a regular rings. Namely, given a regular ring R, we are interested in studying the local cohomology of complete intersection rings of the form $S = R/\underline{\mathbf{f}}R$ where $\underline{\mathbf{f}} = f_1, \ldots, f_c$ a regular sequence in R. The length c of this regular sequence is the *codimension* of S. If $c = 1$, we refer to S as a *hypersurface ring.* When $\underline{\mathbf{f}}$ is a regular sequence, we refer to the ideal $\underline{\mathbf{f}}R$ generated by $\underline{\mathbf{f}}$ as a *parameter ideal* regardless of whether R is local.

We will further restrict our focus to the setting in which the regular ring R has the property that $\operatorname{Ass} H^i_I(R)$ is finite for all $i \geq 0$ and all ideals I, which we refer to as LC-finiteness for shorthand (Definition II.16). This property holds, for example, when R has prime characteristic $p > 0$ [Lyu97, HS93], or if R is a smooth algebra over a field of characteristic 0 [Lyu93] or over the integers [BBL$^+$14].

Let R be an LC-finite regular ring, let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence of R, and let $S = R/\underline{\mathbf{f}}R$. That is to say, there is a short exact sequence

$$0 \to \underline{\mathbf{f}}R \to R \to S \to 0$$

that presents S as the homomorphic image of the LC-finite module R. Let I be an ideal of R containing $\underline{\mathbf{f}}$ (corresponding to an arbitrary ideal of S) and fix $i \geq 0$. There is an exact sequence

$$\cdots \to H^i_I(\underline{\mathbf{f}}R) \to H^i_I(R) \to H^i_I(S) \to H^{i+1}_I(\underline{\mathbf{f}}R) \to \cdots \tag{5.1}$$

If we would like to investigate the question of closed support for $H^i_I(S)$, we are naturally lead to ask the following two questions.

1. Does the cokernel of the map $H^i_I(\underline{\mathbf{f}}R) \to H^i_I(R)$ have closed support?
2. Does the kernel of the map $H^{i+1}_I(\underline{\mathbf{f}}R) \to H^{i+1}_I(R)$ have closed support?

A key insight of Hochster and Núñez-Betancourt [HNB17] is that in prime characteristic $p > 0$, we can give an affirmative answer to Question 1 in the following manner. When R and S are equipped with their natural Frobenius actions and $\underline{\mathbf{f}}R$ is equipped with the Frobenius action restricted from that of R, the short exact $0 \to \underline{\mathbf{f}}R \to R \to S \to 0$ can be understood as a short exact sequence of $R\langle F\rangle$-modules. By Proposition III.13, all morphisms in the long exact sequence (5.1) are $R\langle F\rangle$-linear. Since R is unit and finitely generated, Theorem III.15 shows that $H^i_I(R)$ remains unit and finitely generated. Thus, the cokernel of $H^i_I(\underline{\mathbf{f}}R) \to H^i_I(R)$ is an $R\langle F\rangle$-linear homomorphic image of a finitely generated $R\langle F\rangle$-module, and hence, remains finitely generated. By Theorem III.1, that homomorphic image has closed support.

To show that the kernel of $H^{i+1}_I(\underline{\mathbf{f}}R) \to H^{i+1}_I(R)$ has closed support, it would clearly suffice to show that $H^{i+1}_I(\underline{\mathbf{f}}R)$ has finitely many associated primes. We are therefore lead to the following theorem. Note that while our interest is primarily in the case where J is generated by a regular sequence, this hypothesis was not required in the preceding argument – nor was the hypothesis that $I \supseteq J$.

Theorem V.1 (Hochster, Núñez-Betancourt; Theorem 4.12)**.** *Let R be a regular ring of prime characteristic $p > 0$, let J be an ideal of R, and let $S = R/J$. Let I be an ideal of R and fix $i \geq 0$. If $Ass\, H_I^{i+1}(J)$ is finite, then $Supp\, H_I^i(S)$ is closed.*

If J is the principal ideal generated by some nonzerodivisor $f \in R$ – which is to say, $S = R/J$ is the hypersurface ring R/fR – then J is isomorphic to R as an abstract R-module, and in particular, is LC-finite. Hochster and Núñez-Betancourt thereby obtain the following result on hypersurfaces as essentially a corollary of Theorem V.1.

Theorem V.2 (Hochster, Núñez-Betancourt; Corollary 4.13)**.** *Let R be a regular ring of prime characteristic $p > 0$, let $f \in R$ be a nonzerodivisor, and let $S = R/fR$. Let I be an ideal of R and fix $i \geq 0$. The support of $H_I^i(S)$ is Zariski closed in $Spec(S)$.*

It remains an open question whether their theorem on the support of the local cohomology of positive characteristic hypersurface rings generalizes to complete intersection rings of arbitrary codimension, or even whether it generalizes to hypersurface rings of characteristic 0. At least in the positive characteristic setting, by Theorem V.1, it would suffice to show that for any ideal J generated by a regular sequence $\underline{\mathbf{f}}$ of R, the module $H_I^i(J)$ has finitely many associated primes. Despite only having immediate applications in prime characteristic $p > 0$, one may pose the following question for LC-finite regular rings in any characteristic.

Question V.3. *Let R be an LC-finite regular ring and $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence of R. Is the R-module $\underline{\mathbf{f}}R$ LC-finite? In other words, for I an ideal of R, and $i \geq 0$, must $Ass\, H_I^i(\underline{\mathbf{f}}R)$ be a finite set?*

We prove the following positive result in cohomological degree $i = 2$. Our main case of interest for this theorem is when R is regular and J is generated by a regular sequence, but these hypotheses are not necessary for the theorem.

Theorem V.4 (V.11)**.** *Let R be a locally almost factorial (Definition V.6) Noetherian normal ring, and I and J be ideals of R. The set $\operatorname{Ass} H^2_I(J)$ is finite.*

In cohomological degree $i \geq 3$, we show that Question V.3 has a negative answer at the level of generality in which it's stated. In Theorem V.5 we present an example of the author, written up for publication in [Lew19], showing that $\operatorname{Ass} H^3_I(\underline{\mathbf{f}}R)$ can be an infinite set when $\underline{\mathbf{f}} = f, g$ is a regular sequence of length 2.

The counterexample presented in Section 5.1 crucially requires that $H^2_I(R/\underline{\mathbf{f}})$ has an infinite set of associated primes – in fact, $R/\underline{\mathbf{f}}$ in this example is Katzman's hypersurface ring [Kat02]. It is natural to ask whether one can avoid choosing a sequence $\underline{\mathbf{f}}$ with $H^{i-1}_I(R/\underline{\mathbf{f}})$ already having an infinite set of associated primes, but this may not be possible to do. In Section 5.3 we prove the following.

Theorem (V.16)**.** *Let R be an LC-finite regular ring, let $J \subseteq R$ be an ideal generated by a regular sequence of length $c \geq 2$, and let $S = R/J$. For an ideal $I \supseteq J$,*

(ii) If the irreducible components of Spec(S) are disjoint (e.g. S is a domain), then $\operatorname{Ass} H^3_I(J)$ is finite if and only if $\operatorname{Ass} H^2_I(S)$ is finite.

(iii) If S is normal and locally almost factorial (e.g. S is a UFD), then $\operatorname{Ass} H^4_I(J)$ is finite if and only if $\operatorname{Ass} H^3_I(S)$ is finite.

Since we are only interested in Question V.3 in the case where $\operatorname{Ass} H^i_I(S)$ is infinite, the above theorem presents a significant challenge to the prospect of applying Theorem V.1 to the closed support problem in the codimension $c \geq 2$ setting. We will address this issue further in the next chapter.

At the end of the present chapter, in Section 5.4, we show that in the positive characteristic setting, it is possible to impose sufficiently restrictive hypotheses on $R/\underline{\mathbf{f}}$ so as to recover a positive answer to Question V.3. We actually prove a somewhat more general statement.

Theorem (V.18)**.** *Let R be a regular ring of prime characteristic $p > 0$, let M be a finitely generated unit $R\langle F\rangle$-module M, and let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence such that $R/\underline{\mathbf{f}}$ is regular. The module $\underline{\mathbf{f}}M$ is LC-finite.*

5.1 An example in which Ass $H^3_I((f, g)R)$ is an infinite set

The following example demonstrates that Question V.3 has a negative answer.

Theorem V.5. *Let K be a field, let $R = K[u, v, w, x, y, z, t]$, and let $f = wv^2x^2 - (w+z)vxuy + zu^2y^2$ be the defining equation of Katzman's hypersurface ring [Kat02]. The set Ass $H^3_{(t,f,x,y)}((t, f)R)$ is infinite.*

Proof. Let $\underline{\mathbf{f}} = t, f$, a codimension 2 regular sequence in R, and let $A = K[u, v, w, x, y, z]$. Note that $R/\underline{\mathbf{f}}R = A/fA$. Let $I = (t, f, x, y)R$, and observe that $H^i_I(R/\underline{\mathbf{f}}R) = H^i_{(x,y)}(R/\underline{\mathbf{f}}R) = H^i_{(x,y)}(A/f)$ for all i. Since $\operatorname{depth}_I(R) = 3$ (the sequence $t, f, x \in I$ is R-regular), the long exact sequence from applying $\Gamma_I(-)$ to $0 \to \underline{\mathbf{f}}R \to R \to R/\underline{\mathbf{f}}R \to 0$ begins with

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
& H^1_I(\underline{\mathbf{f}}R) \arrow[r] & 0 \arrow[r] & H^1_{(x,y)}(A/fA) \arrow[dll] \\
& H^2_I(\underline{\mathbf{f}}R) \arrow[r] & 0 \arrow[r] & H^2_{(x,y)}(A/fA) \arrow[dll] \\
& H^3_I(\underline{\mathbf{f}}R) \arrow[r] & H^3_I(R) \arrow[r] & 0 \arrow[dll] \\
& H^4_I(\underline{\mathbf{f}}R) \arrow[r] & H^4_I(R) \arrow[r] & 0
\end{tikzcd}$$

From this, we see that $H^2_{(x,y)}(A/fA)$ embeds into $H^3_I(\underline{\mathbf{f}}R)$. By [Kat02, Theorem 1.2], Ass $H^2_{(x,y)}(A/fA)$ is infinite, so Ass $H^3_I(\underline{\mathbf{f}}R)$ is infinite as well. $\square$

Katzman's hypersurface could be replaced with any globally presented complete intersection ring S known to have an ideal I and index $i \geq 0$ such that $H^i_I(S)$ has infinitely many associated primes. Let $S = A/\underline{\mathbf{g}}A$ where A is a regular ring and $\underline{\mathbf{g}} = g_1, \ldots, g_t$ is a regular sequence. Let $R = A[z_1, \cdots, z_n]$ for $n \gg 0$, and let $\underline{\mathbf{f}} = z_1, \cdots, z_n$. We have $S \simeq R/(\underline{\mathbf{f}}, \underline{\mathbf{g}})R$. Let $I' = (\underline{\mathbf{f}}, \underline{\mathbf{g}})R + IR$ and observe that $H^i_I(S) \simeq H^i_{I'}(S)$ for all $i \geq 0$. By choosing n large enough, we can ensure that $\operatorname{depth}_{I'}(R) > \dim(S) + 1$. Using the long exact sequence from applying $\Gamma_{I'}(-)$ to $0 \to (\underline{\mathbf{f}}, \underline{\mathbf{g}})R \to R \to S \to 0$, it follows at once that $H^{i+1}_{I'}((\underline{\mathbf{f}}, \underline{\mathbf{g}})R) \simeq H^i_I(S)$. In sufficiently large cohomological degrees, $\operatorname{Ass} H^j_I((\underline{\mathbf{f}}, \underline{\mathbf{g}})R)$ is isomorphic to $H^j_{I'}(R)$, and in degrees $1 \leq j \leq \dim(S) + 1$, the local cohomology of $(\underline{\mathbf{f}}, \underline{\mathbf{g}})R$ is identical to that of S in one degree lower.

5.2 The Finiteness of $\operatorname{Ass} H^2_I(J)$

While our main interest for the results in this section is the case in which R is regular and LC-finite, we do not require the full strength of those hypotheses here. We need only assume that R is normal and satisfies the following condition.

Definition V.6. A normal domain R is called *almost factorial* if the class group of R is torsion. A normal ring R is called *locally almost factorial* if R_P is almost factorial for all $P \in \operatorname{Spec}(R)$.

For example, if R_P is a UFD for all $P \in \operatorname{Spec}(R)$, then R is locally almost factorial. A regular ring is therefore locally almost factorial. Hellus shows that an almost factorial Cohen-Macaulay local ring of dimension at most four is LC-finite [Hel01, Theorem 5]. Our usage of the almost factorial hypothesis is motivated by its use in [Hel01], although our motivating setting is ultimately the regular case.

Our goal is to show that $\operatorname{Ass} H^2_I(J)$ is finite for any ideals I and J of R. The results of this section no hypotheses on the ideal J of R – we even permit the case where J is the unit ideal.

When R is a domain, the lemma below shows that the main case is $\operatorname{depth}_I(R) = 1$.

Lemma V.7. *Let R be a Noetherian domain, and let $J \subseteq R$ be an ideal. If $I \subseteq R$ is an ideal such that $\operatorname{depth}_I(R) \neq 1$, then $\operatorname{Ass} H^2_I(J)$ is finite.*

Proof. If $I = (0)$ or $I = R$, there is nothing to do, so we assume that I is a nonzero proper ideal. Since R is a domain, this implies that both J and R are I-torsionfree, giving $\operatorname{depth}_I(R) > 0$ and by hypothesis $\operatorname{depth}_I(R) \neq 1$, so we have $\operatorname{depth}_I(R) \geq 2$. The following sequence is exact.

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \Gamma_I(R/J) \arrow[dll, out=-10, in=170] \\ & H^1_I(J) \arrow[r] & 0 \arrow[r] & H^1_I(R/J) \arrow[dll, out=-10, in=170] \\ & H^2_I(J) \arrow[r] & H^2_I(R) \arrow[r] & H^2_I(R/J) \end{tikzcd}$$

Note that $H^1_I(J) \simeq \Gamma_I(R/J)$ is finitely generated, meaning that $H^2_I(J)$ is either finitely generated or the first non-finitely-generated local cohomology module of J on I, and the stated result follows at once from Brodmann and Lashgari Faghani [BLF00, Theorem 2.2]. □

To deal with the case $\operatorname{depth}_I(R) = 1$, our goal is to locally decompose I (up to radicals) as an ideal of the form $yR \cap I_0$ for $y \in R$ a nonzerodivisor and I_0 an ideal such that $\operatorname{depth}_{I_0}(R) \geq 2$. Such a decomposition would enable us to rewrite $H^2_I(J)$ as $H^2_{I_0}(J)_y$ using the functorial isomorphism of Corollary II.10.

The first step in decomposing I is to express $\sqrt{I}$ as the intersection of a depth ≥ 2 component with a component of pure height 1.

Lemma V.8. *Let R be a Noetherian normal ring, and $I \subseteq R$ be an ideal such that $depth_I(R) = 1$. Then $\sqrt{I} = L \cap I_0$ for some ideal L given by the intersection of height one primes, and some ideal $I_0 \subseteq R$ with $depth_{I_0}(R) \geq 2$.*

Proof. First note that for an ideal $\mathfrak{a}$ in a normal ring R, $\operatorname{depth}_{\mathfrak{a}}(R) = 1$ if and only if $\operatorname{ht}(\mathfrak{a}) = 1$. Indeed, if $\operatorname{ht}(\mathfrak{a}) = 1$, then certainly $\operatorname{depth}_{\mathfrak{a}}(R) \leq 1$. R has no embedded primes, so if $\mathfrak{a}$ is not contained in any minimal prime of R, $\mathfrak{a}$ contains a nonzerodivisor, and thus $\operatorname{depth}_{\mathfrak{a}}(R) \geq 1$. If on the other hand, we assume $\operatorname{depth}_{\mathfrak{a}}(R) = 1$, then clearly $\operatorname{ht}(\mathfrak{a}) \geq 1$. Take $x \in \mathfrak{a}$ a nonzerodivisor. Since $\operatorname{depth}_{\mathfrak{a}}(R/xR) = 0$, $\mathfrak{a}$ is contained in an associated prime of xR, all of which have height 1, giving $\operatorname{ht}(\mathfrak{a}) \leq 1$.

Since $\operatorname{ht}(I) = 1$, the radical of I can be written as $L \cap I_0$ where L has pure height one and I_0 is an intersection of primes of height ≥ 2. Since $\operatorname{ht}(I_0) \geq 2$, it must be the case that $\operatorname{depth}_{I_0}(R) \geq 2$. ∎

To proceed, we require the hypothesis of "almost factoriality."

The main property we require is that every height 1 prime of an almost factorial ring is principal up to taking radicals. An ideal of pure height 1 can be expressed up to radicals as the product of height 1 primes, so in an almost factorial ring, any pure height 1 ideal is principal up to radicals. In a locally almost factorial ring, we can cover $\operatorname{Spec}(R)$ with finitely many charts in which this is the case. To show that $\operatorname{Ass} H^2_I(J)$ is finite, it would certainly suffice to show that $\operatorname{Ass} H^2_{IR_f}(JR_f)$ is finite on each chart $\operatorname{Spec}(R_f)$ of a finite cover of $\operatorname{Spec}(R)$.

Lemma V.9. *Let R be a locally almost factorial Noetherian normal ring, and L be an ideal of pure height 1. There is a finite cover of $Spec(R)$ by open charts $Spec(R_{f_1}), \cdots, Spec(R_{f_t})$ such that for each i, the expanded ideal LR_{f_i} has the same radical as a principal ideal.*

Proof. We do no harm in replacing L with $\sqrt{L}$, so assume L is radical. Consider a single point $P \in \operatorname{Spec}(R)$. Since R_P is almost factorial, we can write $LR_P = \sqrt{yR_P}$ for some $y \in R_P$. Up to multiplying by units of R_P, we may assume that y is an element of R. Since $y \in LR_P \cap R$, there is some $u \in R - P$ such that $uy \in L$. Also, since R is Noetherian, there is some $n > 0$ such that $L^n R_P \subseteq yR_P$, hence $L^n \subseteq yR_P \cap R$, and there is some $v \in R - P$ such that $vL^n \subseteq yR$. If $f = uv$, then we see that $y \in LR_f$ and $L^n \subseteq yR_f$, giving $LR_f = \sqrt{yR_f}$.

Our choice of f depends on P. Varying over all $P \in \operatorname{Spec}(R)$, we obtain a collection of open charts $\{\operatorname{Spec}(R_{f_P})\}_{P \in \operatorname{Spec}(R)}$ which cover $\operatorname{Spec}(R)$ such that (the expansion of) L is principal up to radicals on each chart. Since $\operatorname{Spec}(R)$ is quasicompact, finitely many of these charts cover the whole space. $\square$

Corollary V.10. *Let R be a locally almost factorial Noetherian normal ring, and $I \subseteq R$ be an ideal such that $\operatorname{depth}_I(R) = 1$. Then there is an ideal $I_0 \subseteq R$ with $\operatorname{depth}_{I_0}(R) \geq 2$, and a finite cover of Spec(R) by open charts Spec$(R_{f_1}), \cdots,$ Spec(R_{f_t}) such that for each i, $\sqrt{IR_{f_i}} = \sqrt{y_i R_{f_i}} \cap I_0$ for some $y_i \in R$.*

The main result of this section now follows.

Theorem V.11. *Let R be a locally almost factorial Noetherian normal ring, and I, J be ideals of R. The set $\operatorname{Ass} H^2_I(J)$ is finite.*

Proof. R is a product of normal domains $R_1 \times \cdots \times R_k$, and J is a product of ideals $J_1 \times \cdots \times J_k$ with $J_i \subseteq R_i$. It is enough to show that $\operatorname{Ass} H^2_{IR_i}(J_i)$ is finite for all i, so assume that R is a domain. By Lemma V.7, we need only deal with the case in which $\operatorname{depth}_I(R) = 1$. We will show that $\operatorname{Ass} H^2_{IR_f}(J_f)$ is finite for each chart $\operatorname{Spec}(R_f)$ in a finite cover of $\operatorname{Spec}(R)$. By Corollary V.10, working with one chart at a time, and replacing R by R_f and I by an ideal with the same radical, we may assume that I has the form $I = yR \cap I_0$ where $\mathrm{depth}_{I_0}(R) \geq 2$. By Corollary II.10, this decomposition gives $H^2_I(J) \simeq H^2_{I_0}(J)_y$. It is therefore enough to show that $\mathrm{Ass}\, H^2_{I_0}(J)$ is finite. But $\mathrm{depth}_{I_0}(R) \geq 2$, so this follows from Lemma V.7. $\square$

5.3 Finiteness of Ass $H^i_I(J)$ vs finiteness of Ass $H^{i-1}_I(R/J)$

In this section, we concern ourselves with the following question.

Question V.12. *Let R be an LC-finite regular ring, $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence, and I be an ideal of R containing $\underline{\mathbf{f}}$. Does the finiteness of Ass $H^{i-1}_I(R/\underline{\mathbf{f}}R)$ imply the finiteness of Ass $H^i_I(\underline{\mathbf{f}}R)$?*

If $c = 1$, then $\underline{\mathbf{f}}R \simeq R$ as an R-module, and thus $\mathrm{Ass}\, H^i_I(J)$ is finite by hypothesis, and the question has a trivially positive answer. We therefore restrict our attention to the case $c \geq 2$. We think of i as being fixed with I varying. The case $i = 2$ has a positive answer, since $\mathrm{Ass}\, H^1_I(R/\underline{\mathbf{f}}R)$ is finite – as is true of $H^1_I(M)$ for any finitely generated module M, due to Theorem II.17 – and $\mathrm{Ass}\, H^2_I(\underline{\mathbf{f}}R)$ is finite by Theorem V.11. Our goal is to give a partial positive answer to this question when $i = 3$ and when $i = 4$. As i gets larger, our results require increasingly restrictive hypotheses on the ring $R/\underline{\mathbf{f}}R$.

To begin, notice that we can very easily ignore ideals I where the depth of R on I is too large.

Lemma V.13. *Let R be a Noetherian ring, let I and J be ideals of R, and let $S = R/J$. Fix $i \geq 1$ and assume Ass $H^i_I(R)$ is finite. If $\mathrm{depth}_I(R) > i - 1$, then Ass $H^i_I(J)$ is finite if and only if Ass $H^{i-1}_I(R/J)$ is finite.*

Proof. There is a short exact sequence

$$0 \to H^{i-1}_I(R/J) \to H^i_I(J) \to N \to 0$$

where $N \subseteq H^i_I(R)$, so $\operatorname{Ass} N$ is finite. $\square$

We may therefore restrict our focus to the case where $\operatorname{depth}_I(R) \leq i-1$. Using the isomorphism of Theorem IV.4, we may further restrict ourselves to the case $\operatorname{depth}_I(R) = i-1$, as described in the following proposition.

Proposition V.14. *Let R be a Cohen-Macaulay ring, and let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence of length $j \geq 1$. Fix a nonnegative integer i, and let I be an ideal containing $\underline{\mathbf{f}}$ such that depth$_I(R) \leq i-1$. Then there is an ideal $I' \supseteq I$ such that depth$_{I'}(R) \geq i-1$ and such that $H^i_{I'}(\underline{\mathbf{f}}R) \simeq H^i_I(\underline{\mathbf{f}}R)$ and $H^{i-1}_{I'}(R/\underline{\mathbf{f}}R) \simeq H^{i-1}_I(R/\underline{\mathbf{f}}R)$.*

Proof. Write $h = \operatorname{ht}(I)$. Applying Theorem IV.4 with $k = i-1-h$, we obtain an ideal $I' \supseteq I$ such that $\operatorname{depth}_{I'}(R) = \operatorname{ht}(I') \geq i-1$ such that the natural transformation $H^\ell_{I'}(-) \to H^\ell_I(-)$ is an isomorphism on R-modules whenever $\ell > i-1$ and on $R/\underline{\mathbf{f}}R$-modules whenever $\ell > i-1-c$. In particular, we see that $H^i_{I'}(-) \to H^i_I(-)$ is an isomorphism on R-modules $H^{i-1}_{I'}(-) \to H^{i-1}_I(-)$ is an isomorphism on $R/\underline{\mathbf{f}}R$-modules. $\square$

Assume that R is Cohen-Macaulay and $\underline{\mathbf{f}} = f_1, \ldots, f_c$ is a regular sequence of codimension $c \geq 2$. Fix $i \geq 0$ and let I be an ideal of R containing $\underline{\mathbf{f}}$. Write $a = \operatorname{depth}_I(R/\underline{\mathbf{f}}R) = \operatorname{depth}_I(R) - c$. If $a + c \leq i-1$, then by Corollary V.14, we can replace I with a possibly larger ideal I' in order to assume that $a + c \geq i-1$, without affecting $H^i_I(\underline{\mathbf{f}}R)$ and $H^{i-1}_I(R/\underline{\mathbf{f}}R)$. Lemma V.13 gives a positive answer to Question 3 if $a + c > i-1$, so we may assume that $a + c = i-1$. Note in particular that this allows us to ignore all values of i and c for which $c > i-1$. Below is a table illustrating the relevant values of a to consider for various small values of i and c.

	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$c=2$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$
$c=3$	$\varnothing$	$a=0$	$a=1$	$a=2$	$a=3$
$c=4$	$\varnothing$	$\varnothing$	$a=0$	$a=1$	$a=2$
$c=5$	$\varnothing$	$\varnothing$	$\varnothing$	$a=0$	$a=1$
$c=6$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$a=0$
$c=7$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

We will attack the cases $a=0$ and $a=1$ directly in order to deal with cohomological degrees $i=3$ and $i=4$. The next lemma is our main tool in doing so.

Lemma V.15. *Fix $a \geq 0$. Let R be a Noetherian ring, let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence of length $c \geq 2-a$, let I be an ideal containing $\underline{\mathbf{f}}$, and let $S = R/\underline{\mathbf{f}}R$. Suppose that IS can be decomposed (up to radicals) as $yS \cap I_0$ with $\operatorname{depth}_{I_0}(S) > a$. Suppose further that $\operatorname{Ass} H_I^{c+a+1}(R)$ is finite. Then $\operatorname{Ass} H_I^{c+a+1}(\underline{\mathbf{f}}R)$ is finite if and only if $\operatorname{Ass} H_I^{c+a}(S)$ is finite.*

Proof. By Corollary II.10, there is a natural isomorphism $H^i_{I_0}(S)_y \simeq H^i_I(S)$ for all $i \geq 2$, so that in particular, $H_I^{c+a}(S)$ is an S_y-module. The natural map $\psi : H_I^{c+a}(R) \to H_I^{c+a}(S)$ therefore factors through $H_I^{c+a}(R) \to S_y \otimes_R H_I^{c+a}(R)$ to give an S_y-linear map $S_y \otimes_R H_I^{c+a}(R) \to H_I^{c+a}(S)$.

$$\begin{tikzcd} H_I^{c+a}(R) \arrow[r, "\psi"] \arrow[dr] & H_I^{c+a}(S) \\ & S_y \otimes_R H_I^{c+a}(R) \arrow[u] \end{tikzcd}$$

We claim that $\psi = 0$, and for this it suffices to show that $S_y \otimes_R H_I^{c+a}(R) = 0$.

Consider the decomposition of I up to radicals as $yS \cap I_0$ in S. We can replace y by some lift mod $\underline{\mathbf{f}}R$ to assume that $y \in R$, and since I_0 is expanded from R, we can write $I_0 = I_0'S$ for some ideal I_0' of R containing $\underline{\mathbf{f}}$. We therefore have $I = (y, \underline{\mathbf{f}})R \cap I_0'$ in R (after possibly replacing I by an ideal with the same radical). Note that $\operatorname{depth}_{I_0'}(R) > c+a$. We can write $S_y \otimes_R H_I^{c+a}(R) = S_y \otimes_{R_y} H_{IR_y}^{c+a}(R_y)$ where $IR_y = (y, \underline{\mathbf{f}})R_y \cap I_0'R_y = I_0'R_y$, and thus $H_{IR_y}^{c+a}(R_y) = H_{I_0'}^{c+a}(R)_y$. Since $\operatorname{depth}_{I_0'}(R) > c+a$, we have $H_{I_0'}^{c+a}(R) = 0$ and consequently, $\psi = 0$.

We therefore have an exact sequence

$$0 \to H_I^{c+a}(S) \to H_I^{c+a+1}(J) \to H_I^{c+a+1}(R).$$

Since $\operatorname{Ass} H_I^{c+a+1}(R)$ is finite, the claim follows at once. □

We can now prove the main result of this section.

Theorem V.16. *Let R be an LC-finite regular ring, let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence of length $c \geq 2$, and let $S = R/\underline{\mathbf{f}}R$. For any ideal I of R containing $\underline{\mathbf{f}}$,*

(i) $\operatorname{Ass} H_I^i(\underline{\mathbf{f}}R)$ and $\operatorname{Ass} H_I^{i-1}(S)$ are always finite for $i \leq 2$.

(ii) If the irreducible components of Spec(S) are disjoint, then $\operatorname{Ass} H_I^3(\underline{\mathbf{f}}R)$ is finite if and only if $\operatorname{Ass} H_I^2(S)$ is finite.

(iii) If S is normal and locally almost factorial, then $\operatorname{Ass} H_I^4(\underline{\mathbf{f}}R)$ is finite if and only if $\operatorname{Ass} H_I^3(S)$ is finite.

Proof. Concerning (i), it holds that for any finitely generated R-module M, $\operatorname{Ass} H_I^i(M)$ is finite whenever $i \leq 1$ by Theorem II.17. The finiteness of $\operatorname{Ass} H_I^2(\underline{\mathbf{f}}R)$ is the subject of Theorem V.11.

For (ii), we may use Corollary V.14 to replace I with a possibly larger ideal in order to assume that $\operatorname{depth}_I(R) \geq 2$. By Lemma V.13, (ii) is immediate if $\operatorname{depth}_I(R) > 2$, so assume $\operatorname{depth}_I(R) = 2$. Since $\underline{\mathbf{f}} \subseteq I$ and $c \geq 2$, it follows that $c = 2$ and $\operatorname{depth}_I(S) = \operatorname{ht}(IS) = 0$. Let $e_1, \cdots, e_t \in S$ be a complete set of orthogonal idempotents. The minimal primes of S are $\sqrt{(1-e_1)S}, \cdots, \sqrt{(1-e_t)S}$, and thus, every pure height 0 ideal of S has arithmetic rank at most 1. Up to radicals, we can therefore write IS as $yS \cap I_0$ where $\mathrm{ht}(I_0) = \mathrm{depth}_{I_0}(S) > 0$. Since $c \geq 2 - 0$, the claim follows from Lemma V.15 in the case where $a = 0$.

For (iii), again using Corollary V.14 and Lemma V.13, we may assume that $\mathrm{depth}_I(R) = \mathrm{depth}_I(S) + c = 3$. Since $c \geq 2$, this means $\mathrm{depth}_I(S) \leq 1$. If $c = 3$, giving $\mathrm{depth}_I(S) = 0$, then we may argue as in the proof of (ii) (note that S is a product of domains). If $c = 2$, giving $\mathrm{depth}_I(S) = 1$, then by Corollary V.10 there is a finite cover of $\mathrm{Spec}(S)$ by charts $\mathrm{Spec}(S_{f_1}), \cdots, \mathrm{Spec}(S_{f_t})$ such that for each i, we can write (up to radicals) $IS_{f_i} = y_i S_i \cap I_{0,i}$ with $\mathrm{depth}_{I_{0,i}}(S_{f_i}) > 1$. Replace $f_1, \cdots, f_t$ with lifts from $R/\underline{\mathbf{f}}R$ to R in order to assume $f_1, \cdots, f_t \in R$. Lemma V.15 in the case $a = 1$ shows that for each i, $\mathrm{Ass}\, H^4_I(\underline{\mathbf{f}}R)_{f_i}$ is finite if and only if $\mathrm{Ass}\, H^3_I(S)_{f_i}$ is finite. The charts $\mathrm{Spec}(R_{f_1}), \cdots, \mathrm{Spec}(R_{f_t})$ do not necessarily cover $\mathrm{Spec}(R)$, but they do cover the subset $V(\underline{\mathbf{f}}R)$. Since $I \supseteq \underline{\mathbf{f}}$, $\mathrm{Supp}\, H^\ell_I(-) \subseteq V(I) \subseteq V(\underline{\mathbf{f}}R)$ for all ℓ, so showing that $\mathrm{Ass}\, H^4_I(\underline{\mathbf{f}}R)$ is finite is equivalent to showing that $\mathrm{Ass}\, H^4_I(\underline{\mathbf{f}}R)_{f_i}$ is finite for each i. The result we proved on each chart therefore implies $\mathrm{Ass}\, H^4_I(\underline{\mathbf{f}}R)$ is finite if and only if $\mathrm{Ass}\, H^3_I(S)$ is finite. $\square$

Under the hypotheses of (iii), we can give the following partial answer to Question 3 for local rings of sufficiently small dimension.

Corollary V.17. *Let $(R, \mathfrak{m}, K)$ be an LC-finite regular local ring of dimension at most 7, let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence of length $c \geq 2$ such that $S = R/J$ is normal and almost factorial. Let I be any ideal of R containing $\underline{\mathbf{f}}$. Then for all $i \geq 1$, $\mathrm{Ass}\, H^i_I(\underline{\mathbf{f}}R)$ is finite if and only if $\mathrm{Ass}\, H^{i-1}_I(S)$ is finite.*

Proof. The case $i \leq 4$ is the subject of Theorem V.16. We must have $\dim(S) \leq 5$ since $c \geq 2$, so by Corollary IV.6, $\mathrm{Supp}\, H^{i-1}_I(S)$ (and hence $\mathrm{Ass}\, H^{i-1}_I(S)$) is a finite set if $i - 1 \geq 4$. Likewise, for any homomorphic image $H^{i-1}_I(S) \twoheadrightarrow N$, the set $\mathrm{Supp}\,(N)$ is finite. There is an exact sequence $0 \to N \to H^i_I(\underline{\mathbf{f}}R) \to M \to 0$ where N is a homomorphic image of $H^{i-1}_I(S)$ and M is a submodule of $H^i_I(R)$. If $i \geq 5$, both $\mathrm{Ass}\,(N)$ (a subset of $\mathrm{Supp}\,(N)$) and $\mathrm{Ass}\,(M)$ (a subset of $\mathrm{Ass}\, H^i_I(R)$) are finite, so $\mathrm{Ass}\, H^i_I(\underline{\mathbf{f}}R)$ is finite as well. □

5.4 Regular parameter ideals in characteristic $p > 0$

Let $(R, \mathfrak{m}, K)$ be a regular local ring. Recall that a parameter ideal $J \subseteq R$ is called *regular* if it is generated by an R-regular sequence whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over K. Every ideal J such that R/J is regular has this form. If R is complete and contains a field, then by the Cohen Structure Theorem, all examples of regular parameter ideals are isomorphic to an example of the form $R = K[[x_1, \cdots, x_m, z_1, \cdots, z_n]]$ and $J = (x_1, \cdots, x_m)R$ for some $m, n \geq 0$.

In this section, we will show that if R is a regular ring of prime characteristic $p > 0$ and J is an ideal such that R/J is regular, then for any ideal $I \subseteq R$ and any $i \geq 0$, the set $\mathrm{Ass}\, H^i_I(J)$ is finite. This result is a corollary of a stronger result, taking $M = R$ in the theorem below.

Theorem V.18. *Let R be a regular ring of prime characteristic $p > 0$, let $J \subseteq R$ be an ideal such that R/J is regular, and let M be a finitely generated unit $R\langle F\rangle$-module. Then JM is LC-finite. That is, for any ideal $I \subseteq R$ and any $i \geq 0$, the module $H^i_I(JM)$ has finitely many associated primes.*

Proof. By Theorem III.15, $H^i_I(M)$ is unit and finitely generated over $R\langle F\rangle$. By Proposition III.5, $(R/J) \otimes_R H^i_I(M)$ is finitely generated and unit over R/J. Proposition III.5 also shows that $(R/J) \otimes_R M$ is a finitely generated unit $(R/J)\langle F\rangle$-module, so because R/J is regular, Theorem III.15 shows that $H^i_I((R/J) \otimes_R M)$ is finitely generated and unit over $(R/J)\langle F\rangle$ as well.

The natural map $H^i_I(M) \to H^i_I(M/JM)$ factors through the map $H^i_I(M) \to H^i_I(M)/JH^i_I(M)$, and since $R \twoheadrightarrow R/J$ is surjective, the images of $H^i_I(M)$ and $H^i_I(M)/JH^i_I(M)$ inside $H^i_I(M/JM)$ are equal. Thus,

$$\text{Coker}\left(H^i_I(M) \to H^i_I(M/JM)\right) = \text{Coker}\left(H^i_I(M)/JH^i_I(M) \to H^i_I(M/JM)\right)$$

By Theorem IV.10, $H^i_I(M)/JH^i_I(M) \to H^i_I(M/JM)$, which we may write as

$$(R/J) \otimes_R H^i_I(M) \to H^i_I((R/J) \otimes_R M),$$

is an $(R/J)\langle F\rangle$-linear map. We have already recognized both the source and target as finitely generated and unit over $(R/J)\langle F\rangle$. The cokernel of $H^i_I(M)/JH^i_I(M) \to H^i_I(M/JM)$ is therefore itself finitely generated – as is any quotient of a finitely generated $S\langle F\rangle$-module by an $S\langle F\rangle$-submodule – and is unit by Proposition III.6. By Theorem III.9, it must therefore have a finite set of associated primes.

Regarding the claim about the associated primes of $H^i_I(JM)$, apply $\Gamma_I(-)$ to $0 \to JM \to M \to M/JM \to 0$ to obtain the exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^{i-1}_I(M) \arrow[r] & H^{i-1}_I(M/JM) \arrow[dll, out=-10, in=170] \\ H^i_I(JM) \arrow[r] & H^i_I(M) \arrow[r] & \cdots \end{tikzcd}$$

We have a short exact sequence

$$0 \to \text{Coker}\left(H^{i-1}_I(M) \to H^{i-1}_I(M/JM)\right) \to H^i_I(JM) \to N \to 0$$

for some submodule $N \subseteq H^i_I(M)$, and the stated result now follows at once. $\square$

CHAPTER VI

Complete Intersection Rings as Annihilator Submodules

Throughout this chapter and those that follow, we will sometimes write $\underline{\mathbf{f}}$ as shorthand for the ideal generated by $\underline{\mathbf{f}}$, for example, in the notation $H^c_{\underline{\mathbf{f}}}(R)$ or $R/\underline{\mathbf{f}}$. In context, this should not cause any confusion.

Let R be a regular ring of prime characteristic $p > 0$, let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence in R, and let $S = R/\underline{\mathbf{f}}$ denote the corresponding complete intersection ring of codimension c. For an ideal I of R and an index $i \geq 0$, we return to the question of whether the support of $H^i_I(S)$ is Zariski closed.

We recall briefly the approach to the closed support problem discussed in the previous chapter. The corresponding short exact sequence

$$0 \to \underline{\mathbf{f}}R \to R \to R/\underline{\mathbf{f}} \to 0$$

induces a long exact sequence

$$\cdots \to H^i_I(\underline{\mathbf{f}}R) \xrightarrow{\alpha_i} H^i_I(R) \to H^i_I(S) \to H^{i+1}_I(\underline{\mathbf{f}}R) \xrightarrow{\alpha_{i+1}} \cdots$$

and at least in positive characteristic setting, Hochster and Núñez-Betancourt (see Theorem V.1) prove closed support for the cokernel of α_i, thereby reducing the main problem to a matter of controlling the associated primes of the kernel of α_{i+1} – if $\operatorname{Ass} H^{i+1}_I(\underline{\mathbf{f}}R)$ is finite, $\operatorname{Supp} H^i_I(S)$ must be closed. The original question of whether

$H^i_I(S)$ is closed – equivalently, whether Min $H^i_I(S)$ is a finite set – is nontrivial only in the setting where Ass $H^i_I(R/\underline{\mathbf{f}})$ is presumed to be infinite.

In codimension $c \geq 2$, Theorem V.16 shows that certain hypotheses on the ring $R/\underline{\mathbf{f}}$ (e.g., that it is a domain) guarantee a negative answer to the following question even in cohomological degree less than 4.

Question VI.1. *In the notation established above, suppose that Ass $H^i_I(R/\underline{\mathbf{f}})$ is infinite. Can the set Ass $H^{i+1}_I(\underline{\mathbf{f}}R)$ be finite?*

Because this question can have a negative answer, a proof of closed support for complete intersection rings of codimension 2 and higher is therefore unlikely to straightforwardly arise from attempting to control (submodules of) $H^{i+1}_I(\underline{\mathbf{f}}R)$.

From the viewpoint of $R\langle F\rangle$-modules, there are at least two difficulties that stand out. The first is that $\underline{\mathbf{f}}R$ is not unit, so despite being finitely generated, we cannot expect finite generation over $R\langle F\rangle$ to be preserved after applying a local cohomology functor $H^{i+1}_I(-)$. The second difficulty is that, even if $H^{i+1}_I(\underline{\mathbf{f}}R)$ did happen to be finitely generated over $R\langle F\rangle$, the property of finite generation does not generally pass to $R\langle F\rangle$ submodules[1].

Suppose that we were able to embed $R/\underline{\mathbf{f}}$ into an LC-finite module M, say via $0 \to R/\underline{\mathbf{f}} \to M \to Q \to 0$ for some quotient Q. Suppose further that all of these modules are finitely generated over $R\langle F\rangle$ and that all maps in the short exact sequence are $R\langle F\rangle$-linear. In this case, we would consider the long exact sequence

$$\cdots \to H^{i-1}_I(M) \xrightarrow{\beta} H^{i-1}_I(Q) \to H^i_I(R/\underline{\mathbf{f}}) \to H^i_I(M) \to \cdots$$

Since M is LC-finite by hypothesis, every submodule of $H^i_I(M)$ has finitely many associated primes. The question of whether $H^i_I(R/\underline{\mathbf{f}})$ has closed support is reduced,

[1] An exception is when both the original module and the submodule in question happened to be unit.

in this case, to the question of whether the cokernel of β has closed support. Since all maps in the long exact sequence are $R\langle F\rangle$-linear, it would suffice to show that $H_I^{i-1}(Q)$ is finitely generated. Of course, if $H_I^i(R/\underline{\mathbf{f}})$ were finitely generated over $R\langle F\rangle$, we would be finished, so assume that it is not. The analogue of Question VI.1 in this setting is as follows.

Question VI.2. *In the notation established above, suppose that the $R\langle F\rangle$-module $H_I^i(R/\underline{\mathbf{f}})$ is not finitely generated. Can the $R\langle F\rangle$-module $H_I^{i-1}(Q)$ be finitely generated?*

In Chapter VIII, we will describe a short exact sequence

$$0 \to R/\underline{\mathbf{f}} \to M \to Q$$

where M is LC-finite and, under appropriate vanishing conditions on the local cohomology of R, the $R\langle F\rangle$-mdodule $H_I^{i-1}(Q)$ is indeed finitely generated, yielding a novel closed support result on the local cohomology of S. This short exact sequence is part of a longer (exact) complex of $R\langle F\rangle$-modules, whose construction is the main concern of Chapter VII.

Our present goal in the chapter is to motivate and construct the $R\langle F\rangle$-linear embeddings that will be used in the sequel. This and the next two chapters represent joint work of the author and Eric Canton, originally appearing in [CL20].

6.1 The Fedder action

We continue the notation established in the introduction of this chapter.

Regarding the closed support problem over a positive characteristic hypersurface ring, Katzman and Zhang [KZ17] give a proof of Theorem V.2 independent of the methods of Hochster and Núñez-Betancourt, based on explicitly describing the supports of the kernel and cokernel of the map $H_I^i(R) \xrightarrow{f} H_I^i(R)$ for each $i \geq 0$. The short exact sequence $0 \to \underline{\mathbf{f}}R \to R \to S \to 0$ in the case $c = 1$ is isomorphic to $0 \to R \xrightarrow{f} R \to R/f \to 0$, giving the long exact sequence below.

$$\cdots \to H^i_I(R) \xrightarrow{f} H^i_I(R) \to H^i_I(R/f) \to H^{i+1}_I(R) \xrightarrow{f} \cdots \tag{6.1}$$

Recall from Theorem II.15 that if $\underline{\mathbf{f}} = f_1, \ldots, f_c$ is a regular sequence of codimension c, then for any ideal $I \supseteq \underline{\mathbf{f}}$ and any $i \geq 0$, there is a natural isomorphism $H^i_I(H^c_{\underline{\mathbf{f}}}(R)) \cong H^{i+c}_I(R)$. The multiplication map $H^i_I(R) \xrightarrow{f} H^i_I(R)$ is precisely the map induced by $H^{i-1}_I(-)$ on $H^1_f(R) \xrightarrow{f} H^1_f(R)$. Thus, one may consider the long exact sequence 6.1 as instead being induced by applying $\Gamma_I(-)$ to the short exact sequence below.

$$0 \to R/fR \to H^1_f(R) \xrightarrow{f} H^1_f(R) \to 0 \tag{6.2}$$

In this sequence, $1 \in R/fR$ is sent to the Čech cohomology class $\{\!\{1/f\}\!\} \in H^1_f(R)$. If one does not wish to directly compute the supports of the kernel and cokernel of the multiplication-by-f map on $H^i_I(R)$, then we can still obtain the conclusion that Supp $H^i_I(R/f)$ is closed by an $R\langle F\rangle$-module argument analogous to Hochster and Núñez-Betancourt, so long as the short exact sequence (6.2) can be made $R\langle F\rangle$-linear. Indeed, if the right-most copy of $H^1_f(R)$ is equipped with the natural Frobenius action F_{nat} and the middle copy of $H^1_f(R)$ is equipped with the Frobenius action $f^{p-1}F_{\text{nat}}$, then one can readily verify that this is the case.

In the short exact sequence $0 \to R/f \to M \to Q \to 0$ of (6.2), now regarded as a sequence of $R\langle F\rangle$-modules, one might observe that Q is unit and finitely generated by Proposition III.14, answering Question VI.2 in the affirmative. An alternate proof of Theorem V.2 follows immediately from these observations.

The observation that R/f is isomorphic to the annihilator of the ideal fR in the module $H^1_f(R)$, and that the inclusion map of that annihilator submodule can be made $R\langle F\rangle$-linear, generalizes to the higher codimension setting. Fix $c \geq 2$ and let $f = f_1 \cdots f_c$. The Čech cohomology class $\{\!\{1/f\}\!\}$ is precisely the annihilator submodule $(0 :_{H^c_{\underline{\mathbf{f}}}(R)} \underline{\mathbf{f}})$ in $H^c_{\underline{\mathbf{f}}}(R)$. If $H^c_{\underline{\mathbf{f}}}(R)$ is equipped with the Frobenius action $f^{p-1}F_{\text{nat}}$, one may directly verify that the R-submodule spanned by $\{\!\{1/f\}\!\}$ is $R\langle F\rangle$-stable. We will prove a more slightly general statement in Proposition VI.6.

Letting $Q_{\underline{\mathbf{f}}}$ denote the cokernel of $R/\underline{\mathbf{f}} \hookrightarrow H^c_{\underline{\mathbf{f}}}(R)$, we obtain the following short exact sequence of $R\langle F\rangle$-modules whose middle term, $H^c_{\underline{\mathbf{f}}}(R)$, is LC-finite[2].

$$0 \to R/\underline{\mathbf{f}} \to H^c_{\underline{\mathbf{f}}}(R) \to Q_{\underline{\mathbf{f}}} \to 0 \tag{6.3}$$

The $R\langle F\rangle$-module $H^c_{\underline{\mathbf{f}}}(R)$ equipped with the Frobenius action $f^{p-1}F_{\text{nat}}$ is finitely generated – in fact, it is cyclic, generated by the Čech cohomology class $\{\!\{1/f^2\}\!\}$. However, $H^c_{\underline{\mathbf{f}}}(R)_{\text{fed}}$ is never unit and $Q_{\underline{\mathbf{f}}}$ is not unit for $c \geq 2$. In both cases, the structure morphism is a surjective map with a nontrivial kernel that we compute explicitly in Section 6.4. Thus, we cannot necessarily expect finite generation over $R\langle F\rangle$ for $H^i_I(H^c_{\underline{\mathbf{f}}}(R)_{\text{fed}})$ or $H^i_I(Q_{\underline{\mathbf{f}}})$. For this reason, we will eventually require a somewhat more elaborate construction than the short exact sequence 6.3, and this construction is the subject of Chapter VII.

In what follows, we will refer to the $f^{p-1}F_{\text{nat}}$ as the *Fedder action* on $H^c_{\underline{\mathbf{f}}}(R)$. The terminology is motivated by the relationship between this action and a result of Fedder [Fed83] concerning Gorenstein local rings.

6.2 The Fedder action associated with a Gorenstein local ring

In this section, the term *F-finite*[3] to refer to a ring R of prime characteristic $p > 0$ with the property that $R^{1/p}$ is a finitely generated R-module.

[2]Recall that LC-finiteness is a property of R-modules, not $R\langle F\rangle$-modules, so Theorem III.16 using the natural action is sufficient to prove this statement.

[3]This is not to be confused with Lyubeznik's definition of F-finite, discussed in previous chapters, and referring to the property of finite generation for unit $R\langle F\rangle$-modules.

In the following lemma, we refer to the *canonical module* ω_A of a Cohen-Macaulay local ring A. See Chapter 3 of [BH98] for proofs of the assertions made below.

Lemma VI.3. *Let $(S, \mathfrak{m})$ be an F-finite Gorenstein local ring of prime characteristic $p > 0$, and let $q = p^e$. For some non-zero map $T : S^{1/q} \to S$, we have an isomorphism of $S^{1/q}$-modules $Hom_S(S^{1/q}, S) \simeq T \cdot S^{1/q}$*

When $S = R/J$ is Gorenstein for R an F-finite regular local ring, the fact that $\mathrm{Hom}_S(S^{1/q}, S)$ is cyclic leads to the following description of $(J^{[p]} : J)$ [Fed83].

Lemma VI.4. *Let R be a regular local ring of prime characteristic $p > 0$ and let $J \subset R$ is an ideal such that R/J is Gorenstein. For some $g \in R$, we have $(J^{[p]} : J) = gR + J^{[p]}$, and $(J^{[p^e]} : J^{[p^{e-1}]}) = g^{p^{e-1}}R + J^{[p^e]}$ for all $e \geq 1$.*

Proof. Since R is regular, $\mathcal{F}_R(-)$ is exact, and thus, $(J^{[p^e]} :_R J^{[p^{e-1}]}) = (J^{[p]} : J)^{[p^{e-1}]}$ for all $e \geq 1$. Thus, it suffices to prove the case in which $e = 1$. If we fix a generator T for $\mathrm{Hom}_R(R^{1/p}, R)$ over $R^{1/p}$, then the map

$$(J^{[p]} : J)/J^{[p]} \xrightarrow{\sim} \mathrm{Hom}_{R/J}((R/J)^{1/p}, (R/J))$$

sending $(r + J^{[p]})$ to $(T \cdot r) + J$ is an isomorphism [Fed83]. By VI.3, we know $\mathrm{Hom}_{R/J}((R/J)^{1/p}, R/J)$ is cyclic over $(R/J)^{1/p}$, say via $(T \cdot g) + J$ for $g \in (J^{[p]} : J)$. We conclude $(J^{[p]} : J) = gR + J^{[p]}$. $\square$

Let R/J be a Gorenstein quotient with R regular, and fix a generator g for $(J^{[p]} : J)$. Lemma VI.4 allows us to define a directed system

$$0 \longrightarrow R/J \xrightarrow{g} R/J^{[p]} \xrightarrow{g^p} R/J^{[p^2]} \xrightarrow{g^{p^2}} R/J^{[p^3]} \longrightarrow \cdots \tag{6.4}$$

whose transition maps are injective.

Let $M = \varinjlim_e (R/J^{[p^e]}, g^{p^e})$ denote the direct limit of the system. The embedding $R/J \hookrightarrow M$ can be made Frobenius stable with respect to the natural action of R/J. Specifically, let M be equipped with the action $\beta : M \to M$ described by $gF : R/J^{[p^e]} \to R/J^{[p^{e+1}]}$ at the unit of its defining directed system. The compatibility with the natural action $F : R/J \to R/J$ is shown below.

$$\begin{tikzcd}
 & 0 \arrow[r] & R/J \arrow[r, "g"] \arrow[d, "gF"] \arrow[dl, "F"'] & R/J^{[p]} \arrow[r, "g^p"] \arrow[d, "gF"] & R/J^{[p^2]} \arrow[r, "g^{p^2}"] \arrow[d, "gF"] & \cdots \arrow[r] & M \arrow[d, "\beta"] \\
0 \arrow[r] & R/J \arrow[r, "g"] & R/J^{[p]} \arrow[r, "g^p"] & R/J^{[p^2]} \arrow[r, "g^{p^2}"] & R/J^{[p^3]} \arrow[r, "g^{p^3}"] & \cdots \arrow[r] & M
\end{tikzcd}$$

We refer to the resulting action $\beta : M \to M$ on M as the *Fedder action.*

6.3 The Fedder action associated with a regular sequence

For a regular sequence $\underline{\mathbf{f}} = f_1, \ldots, f_c$, by an abuse of notation, we will use $\underline{\mathbf{f}}^{[t]}$ to denote the sequence $f_1^t, \ldots, f_c^t$ – which is still regular [BH98, Exercise 1.1.10] – regardless of whether t is a power of the characteristic. Let $f = \prod_{i=1}^c f_i$.

When the ideal J in the directed system (6.4) is generated by a regular sequence $\underline{\mathbf{f}} = f_1, \ldots, f_c$, the Fedder socle of $R/\underline{\mathbf{f}}^{[p]}$ has a clear choice of generator: f^{p-1}. Moreover, the direct limit (6.4) is identifiable as $H^c_{\underline{\mathbf{f}}R}(R)$. For $q = p^e$, the action sending $r + \underline{\mathbf{f}}^{[q]} \in R/\underline{\mathbf{f}}^{[q]}$ to $f^{p-1}r^p + \underline{\mathbf{f}}^{[qp]} \in R/\underline{\mathbf{f}}^{[qp]}$ at the unit of the directed system $(R/\underline{\mathbf{f}}^{[q]}, f^{qp-q})_{e=0}^{\infty}$ has the form

$$\left\{\!\!\left\{ \frac{r}{f^q} \right\}\!\!\right\} \mapsto \left\{\!\!\left\{ \frac{r^p f^{p-1}}{f^{qp}} \right\}\!\!\right\}$$

in terms of Čech cohomology classes in $H^c_{\underline{\mathbf{f}}}(R)$.

While our primary motivation is the case in which R is a regular ring, the colon properties

$$(\underline{\mathbf{f}}^{[b]} : f^{b-a}) = \underline{\mathbf{f}}^{[a]} \quad \text{and} \quad (\underline{\mathbf{f}}^{[b]} : \underline{\mathbf{f}}^{[a]}) = f^{b-a}R + \underline{\mathbf{f}}^{[b]}$$

for two positive integers $b > a$ hold for an arbitrary regular sequence $\underline{\mathbf{f}}$ in a Noetherian ring. Many properties of the constructions that follow therefore work in greater generality than the motivating setting of the previous section.

Definition VI.5. Let R be a Noetherian ring of prime characteristic $p > 0$, let $\underline{\mathbf{f}} = f_1, \ldots, f_c \in R$ be a regular sequence of codimension c, and let $f = \prod_{i=1}^{c} f_i$. Let F_{nat} denote the natural action of the Frobenius on $H^c_{\underline{\mathbf{f}}}(R)$ (see Definition III.10). The *Fedder action with respect to* $\underline{\mathbf{f}}$ on $H^c_{\underline{\mathbf{f}}}(R)$ – or simply *the Fedder action*, if the sequence $\underline{\mathbf{f}}$ is understood – is defined by $F_{\text{fed}} := f^{p-1} F_{\text{nat}}$. When there is risk of confusion, we write $H^c_{\underline{\mathbf{f}}}(R)_{\text{nat}}$ and $H^c_{\underline{\mathbf{f}}}(R)_{\text{fed}}$ to distinguish the (non-equivalent) $R\langle F\rangle$-modules obtained when $H^c_{\underline{\mathbf{f}}}(R)$ is equipped with the natural action and the Fedder action, respectively.

The $R\langle F\rangle$-module $H^c_{\underline{\mathbf{f}}}(R)_{\text{nat}}$ is cyclic with generator $\{\!\{1/f\}\!\}$. Concerning the Fedder action, $H^c_{\underline{\mathbf{f}}}(R)_{\text{fed}}$ is still cyclic, but we require a different generator. The element $1 \mapsto \{\!\{1/f\}\!\}$ is preserved by the Fedder action, but notice that for each $q = p^e$ we have

$$F_{\text{fed}} : \left\{\!\!\left\{ \frac{r}{f^{q+1}} \right\}\!\!\right\} \mapsto \left\{\!\!\left\{ \frac{r^p}{f^{qp+1}} \right\}\!\!\right\}$$

for all $r \in R$, and thus, for example, the class $\{\!\{1/f^2\}\!\}$ generates.

Perhaps the most useful property of the Fedder action is its compatibility with embeddings of annihilators of subsequences of $\underline{\mathbf{f}}$, in the sense of the following proposition. This compatibility was observed in [Can16], essentially as the result of applying $H^*_{\underline{\mathbf{f}}}(-)$ to the Koszul complex $K^\bullet(\underline{\mathbf{g}}; R)$.

Proposition VI.6. *Let R be a Noetherian ring of prime characteristic $p > 0$, let $g_1, \ldots, g_t, f_1, \ldots, f_c \in R$ be a regular sequence, and write $\underline{\mathbf{g}} = g_1, \ldots, g_t$, $\underline{\mathbf{f}} = f_1, \ldots, f_c,$, $g = \prod_{i=1}^{t} g_i$ and $f = \prod_{i=1}^{c} f_i$. Consider $H^c_{\underline{\mathbf{f}}}(R/\underline{\mathbf{g}})$ and $H^{t+c}_{\underline{\mathbf{g}},\underline{\mathbf{f}}}(R)$ as $R\langle F\rangle$-*

modules via the Fedder actions with respect to $\underline{\mathbf{f}}$ and $\underline{\mathbf{g}}, \underline{\mathbf{f}}$, respectively. There is an $R\langle F\rangle$-linear injection

$$H^c_{\underline{\mathbf{f}}}(R/\underline{\mathbf{g}}) \hookrightarrow H^{t+c}_{\underline{\mathbf{g}},\underline{\mathbf{f}}}(R)$$

whose image is the annihilator $(0 :_{H^{t+c}_{\underline{\mathbf{g}},\underline{\mathbf{f}}}(R)} \underline{\mathbf{g}})$.

Proof. Since $(\underline{\mathbf{g}}^{[a]}, \underline{\mathbf{f}}^{[a]}) : g^{a-1} = (\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[a]})$ for all $a \geq 1$, multiplication by g^{a-1} induces a well-defined injection $\phi_a : R/(\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[a]}) \xrightarrow{g^{a-1}} R/(\underline{\mathbf{g}}^{[a]}, \underline{\mathbf{f}}^{[a]})$, and since $(\underline{\mathbf{g}}^{[a]}, \underline{\mathbf{f}}^{[a]}) : \underline{\mathbf{g}} = g^{a-1}R + (\underline{\mathbf{g}}^{[a]}, \underline{\mathbf{f}}^{[a]})$, the image of ϕ_a is precisely $(0 :_{R/(\underline{\mathbf{g}}^{[a]}, \underline{\mathbf{f}}^{[a]})} \underline{\mathbf{g}})$. The maps ϕ_a form a map of directed systems $(R/(\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[a]}), f)_{a=1}^{\infty}$ to $(R/(\underline{\mathbf{g}}^{[a]}, \underline{\mathbf{f}}^{[a]}), gf)_{a=1}^{\infty}$ via

$$\begin{array}{ccc} R/(\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[a]}) & \xrightarrow{g^{a-1}} & R/(\underline{\mathbf{g}}^{[a]}, \underline{\mathbf{f}}^{[a]}) \\ \downarrow{\scriptstyle f^{b-a}} & & \downarrow{\scriptstyle (gf)^{b-a}} \\ R/(\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[b]}) & \xrightarrow{g^{b-1}} & R/(\underline{\mathbf{g}}^{[b]}, \underline{\mathbf{f}}^{[b]}) \end{array}$$

On the direct limits, this produces an injection $\phi : H^c_{\underline{\mathbf{f}}}(R/\underline{\mathbf{g}}) \hookrightarrow H^{t+c}_{\underline{\mathbf{g}},\underline{\mathbf{f}}}(R)$, whose image is precisely $(0 :_{H^{t+c}_{\underline{\mathbf{g}},\underline{\mathbf{f}}}(R)} \underline{\mathbf{g}})$. For each $q = p^e$, the Fedder action with respect to $\underline{\mathbf{f}}$ sends the class $r + (\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[q]}) \in R/(\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[q]})$ to $f^{p-1}r^p + (\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[qp]}) \in R/(\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[qp]})$, which is sent by ϕ_{qp} to $g^{qp-1}(f^{p-1}r^p + (\underline{\mathbf{g}}^{[qp]}, \underline{\mathbf{f}}^{[qp]}) \in R/(\underline{\mathbf{g}}^{[qp]}, \underline{\mathbf{f}}^{[qp]})$. On the other hand, ϕ_q sends $r + (\underline{\mathbf{g}}, \underline{\mathbf{f}}^{[q]})$ to $g^{q-1}r + (\underline{\mathbf{g}}^{[q]}, \underline{\mathbf{f}}^{[q]})$, and the Fedder action with respect to $\underline{\mathbf{g}}, \underline{\mathbf{f}}$ sends $g^{q-1}r + (\underline{\mathbf{g}}^{[q]}, \underline{\mathbf{f}}^{[q]})$ to

$$(gf)^{p-1}\left(g^{q-1}r\right)^p + (\underline{\mathbf{g}}^{[qp]}, \underline{\mathbf{f}}^{[qp]}) = g^{qp-1}(f^{p-1}r^p) + (\underline{\mathbf{g}}^{[qp]}, \underline{\mathbf{f}}^{[qp]}),$$

as desired. $\square$

6.4 The structure morphism of the Fedder action

Let R be regular, let $\underline{\mathbf{f}} = f_1, \ldots, f_c \in R$ be a regular sequence of codimension c, and let $Q_{\underline{\mathbf{f}}}$ denote the cokernel of the Frobenius stable embedding $R/\underline{\mathbf{f}} \hookrightarrow H^c_{\underline{\mathbf{f}}}(R)_{\text{fed}}$ that sends $1 \mapsto \{\!\{1/f\}\!\}$, equipped with its induced action.

$$0 \to R/\underline{\mathbf{f}} \to H^c_{\underline{\mathbf{f}}}(R)_{\text{fed}} \to Q_{\underline{\mathbf{f}}} \to 0 \tag{6.5}$$

To give a Frobenius action on a complex $A^\bullet$ is precisely to choose a Frobenius action on each term A^i such that the differentials $d^i : A^i \to A^{i+1}$ are Frobenius stable. Analogous to the situation with modules, the data of a Frobenius action on $A^\bullet$ is equivalent to specifying an R-linear map of complexes $\Theta : \mathcal{F}_R(A^\bullet) \to A^\bullet$ – the *structure morphism* of the complex. In this section, we will describe the structure morphism of the three-term complex (6.5).

Theorem VI.7. *Let R be a Noetherian ring of prime characteristic $p > 0$, let $\underline{\mathbf{f}} = f_1, \ldots, f_c \in R$ be a regular sequence, let $f = \prod_{i=1}^{c} f_i$, and let $A^\bullet$ denote the complex described in* (6.5) *corresponding to the Frobenius stable embedding $R/\underline{\mathbf{f}} \to H^c_{\underline{\mathbf{f}}}(R)_{fed}$. Let $\theta_{R/\underline{\mathbf{f}}}$, θ_{fed}, and θ_Q denote the $R\langle F\rangle$ structure morphisms of $R/\underline{\mathbf{f}}$, $H^c_{\underline{\mathbf{f}}}(R)_{fed}$, and $Q_{\underline{\mathbf{f}}}$, respectively. Let Θ denote the structure morphism of $A^\bullet$. There is an exact sequence of complexes*

$$0 \to K^\bullet \to \mathcal{F}_R(A^\bullet) \xrightarrow{\Theta} A^\bullet \to 0$$

described term-by-term in the diagram below with exact[4] *rows and columns, where $\mathcal{F}_R(H^c_{\underline{\mathbf{f}}}(R))$ is identified with $H^c_{\underline{\mathbf{f}}}(R)$ in the natural way*[5].

$$\begin{tikzcd}
0 & & & 0 & 0 & 0 & \\
A^\bullet \arrow[u] & : & 0 \arrow[r] & R/\underline{\mathbf{f}} \arrow[r] \arrow[u] & H^c_{\underline{\mathbf{f}}}(R) \arrow[r] \arrow[u] & Q_{\underline{\mathbf{f}}} \arrow[r] \arrow[u] & 0 \\
\mathcal{F}_R(A^\bullet) \arrow[u, "\Theta"] & : & 0 \arrow[r] & R/\underline{\mathbf{f}}^{[p]} \arrow[r] \arrow[u, "\theta_{R/\underline{\mathbf{f}}}"] & H^c_{\underline{\mathbf{f}}}(R) \arrow[r] \arrow[u, "\theta_{fed}"] & \mathcal{F}_R(Q_{\underline{\mathbf{f}}}) \arrow[r] \arrow[u, "\theta_Q"] & 0 \\
K^\bullet \arrow[u] & : & 0 \arrow[r] & \underline{\mathbf{f}}/\underline{\mathbf{f}}^{[p]} \arrow[r] \arrow[u] & (0 :_{H^c_{\underline{\mathbf{f}}}(R)} f^{p-1}) \arrow[r] \arrow[u] & V_{\underline{\mathbf{f}}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[u] & & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

---

[4]Note that exactness of the second row is a property of regular sequences in any Noetherian ring, and does not require $\mathcal{F}_R(-)$ to be exact.

[5]That is, using the structure isomorphism of the natural action; see Theorem III.14.

The module $V_{\underline{\mathbf{f}}} := Ker(\theta_Q)$ can be described by the direct limit

$$\cdots \to \frac{\underline{\mathbf{f}}^{q+1}}{f^q R + \underline{\mathbf{f}}^{[q+p]}} \xrightarrow{f^{qp-q}} \frac{\underline{\mathbf{f}}^{qp+1}}{f^{qp} R + \underline{\mathbf{f}}^{[qp+p]}} \to \cdots \to V_{\underline{\mathbf{f}}}$$

Proof. We first describe $A^\bullet$ as a direct limit of complexes $\varinjlim_e (A_e^\bullet, \psi_e)$,

$$\begin{tikzcd}
A^\bullet & : & 0 \arrow[r] & R/\underline{\mathbf{f}} \arrow[r] & H^c_{\underline{\mathbf{f}}}(R) \arrow[r] & Q_{\underline{\mathbf{f}}} \arrow[r] & 0 \\
\vdots \arrow[u] & & & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
A^\bullet_{e+1} \arrow[u] & : & 0 \arrow[r] & R/\underline{\mathbf{f}} \arrow[u] \arrow[r, "f^{qp}"] & R/\underline{\mathbf{f}}^{[qp+1]} \arrow[u] \arrow[r] & R/(f^{qp}R + \underline{\mathbf{f}}^{[qp+1]}) \arrow[u] \arrow[r] & 0 \\
A^\bullet_e \arrow[u, "\psi_e"] & : & 0 \arrow[r] & R/\underline{\mathbf{f}} \arrow[u, "1"] \arrow[r, "f^q"] & R/\underline{\mathbf{f}}^{[q+1]} \arrow[u, "f^{qp-q}"] \arrow[r] & R/(f^q R + \underline{\mathbf{f}}^{[q+1]}) \arrow[u, "f^{qp-q}"] \arrow[r] & 0 \\
\vdots \arrow[u] & & & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

In the direct limit, an element $r+\underline{\mathbf{f}}^{[q+1]} \in R/\underline{\mathbf{f}}^{[q+1]}$ maps to the Čech cohomology class $\{\!\{r/f^{q+1}\}\!\}$. To describe the structure morphism $\Theta : \mathcal{F}_R(A^\bullet) \to A^\bullet$, we specify a map $\Theta_{e-1} : \mathcal{F}_R(A^\bullet_{e-1}) \to A^\bullet_e$ for each e by taking the obvious quotient maps term-by-term.

$$\begin{tikzcd}
A^\bullet_e & : & 0 \arrow[r] & R/\underline{\mathbf{f}} \arrow[r, "f^q"] & R/\underline{\mathbf{f}}^{[q+1]} \arrow[r] & R/(f^q R + \underline{\mathbf{f}}^{[q+1]}) \arrow[r] & 0 \\
\mathcal{F}_R(A^\bullet_{e-1}) \arrow[u, "\Theta_{e-1}"] & : & 0 \arrow[r] & R/\underline{\mathbf{f}}^{[p]} \arrow[u, two heads] \arrow[r, "f^q"] & R/\underline{\mathbf{f}}^{[q+p]} \arrow[u, two heads] \arrow[r] & R/(f^q R + \underline{\mathbf{f}}^{[q+p]}) \arrow[u, two heads] \arrow[r] & 0
\end{tikzcd}$$

The exactness of the second row of the above diagram is a property of regular sequences that holds irrespective of whether $\mathcal{F}_R(-)$ is exact. One may verify that the following diagram of complexes commutes

$$\begin{tikzcd}
\mathcal{F}_R(A^\bullet_e) \arrow[r, "\Theta_e"] & A^\bullet_{e+1} \\
\mathcal{F}_R(A^\bullet_{e-1}) \arrow[u, "\mathcal{F}_R(\psi_{e-1})"] \arrow[r, "\Theta_{e-1}"] & A^\bullet_e \arrow[u, "\psi_e"]
\end{tikzcd}$$

so that the $(\Theta_e)_{e=1}^\infty$ induce a well-defined map on the direct limit $\mathcal{F}_R(A^\bullet) \to A^\bullet$. To describe the Frobenius action on the class $r + \underline{\mathbf{f}}^{[q+1]}$, the inclusion $R/\underline{\mathbf{f}}^{[q+1]} \to$

$F_*R \otimes_R (R/\underline{\mathbf{f}}^{[q+1]})$ sends $r + \underline{\mathbf{f}}^{[q+1]}$ to $r^p + \underline{\mathbf{f}}^{[qp+p]}$ once the codomain is identified with $R/\underline{\mathbf{f}}^{[qp+p]}$. The quotient $R/\underline{\mathbf{f}}^{[qp+p]} \twoheadrightarrow R/\underline{\mathbf{f}}^{[qp+1]}$ sends that class to $r^p + \underline{\mathbf{f}}^{[qp+1]}$. On the corresponding Čech cohomology classes in the direct limit, we have

$$\left\{\!\!\left\{ \frac{r}{f^{q+1}} \right\}\!\!\right\} \mapsto \left\{\!\!\left\{ \frac{r^p}{f^{qp+1}} \right\}\!\!\right\} = \left\{\!\!\left\{ \frac{f^{p-1}r^p}{f^{qp+p}} \right\}\!\!\right\} = F_{\mathrm{fed}} \left\{\!\!\left\{ \frac{r}{f^{q+1}} \right\}\!\!\right\}$$

as desired. Concerning the kernel of the structure map, we have for each e a commutative diagram with exact rows and columns.

$$\begin{tikzcd}
0 & & & 0 & 0 & 0 & \\
A_e^\bullet \arrow[u] & : & 0 \arrow[r] & \frac{R}{\underline{\mathbf{f}}} \arrow[r, "f^q"] \arrow[u] & \frac{R}{\underline{\mathbf{f}}^{[q+1]}} \arrow[r] \arrow[u] & \frac{R}{(f^qR+\underline{\mathbf{f}}^{[q+1]})} \arrow[r] \arrow[u] & 0 \\
\mathcal{F}_R(A_{e-1}^\bullet) \arrow[u, "\Theta_{e-1}"] & : & 0 \arrow[r] & \frac{R}{\underline{\mathbf{f}}^{[p]}} \arrow[r, "f^q"] \arrow[u] & \frac{R}{\underline{\mathbf{f}}^{[q+p]}} \arrow[r] \arrow[u] & \frac{R}{(f^qR+\underline{\mathbf{f}}^{[q+p]})} \arrow[r] \arrow[u] & 0 \\
K_e^\bullet \arrow[u] & : & 0 \arrow[r] & \frac{\underline{\mathbf{f}}}{\underline{\mathbf{f}}^{[p]}} \arrow[r, "f^q"] \arrow[u] & \frac{\underline{\mathbf{f}}^{[q+1]}}{\underline{\mathbf{f}}^{[q+p]}} \arrow[r] \arrow[u] & \frac{(f^qR+\underline{\mathbf{f}}^{[q+1]})}{(f^qR+\underline{\mathbf{f}}^{[q+p]})} \arrow[r] \arrow[u] & 0 \\
0 \arrow[u] & & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $\mathcal{F}_R(\psi_{e-1})$ induces a map $K_e^\bullet \to K_{e+1}^\bullet$

$$\begin{tikzcd}
K_{e+1}^\bullet & : & 0 \arrow[r] & \frac{\underline{\mathbf{f}}}{\underline{\mathbf{f}}^{[p]}} \arrow[r, "f^{qp}"] & \frac{\underline{\mathbf{f}}^{[qp+1]}}{\underline{\mathbf{f}}^{[qp+p]}} \arrow[r] & \frac{\underline{\mathbf{f}}^{[qp+1]}}{(f^{qp}R+\underline{\mathbf{f}}^{[qp+p]})} \arrow[r] & 0 \\
K_e^\bullet \arrow[u] & : & 0 \arrow[r] & \frac{\underline{\mathbf{f}}}{\underline{\mathbf{f}}^{[p]}} \arrow[r, "f^q"] \arrow[u, "1"] & \frac{\underline{\mathbf{f}}^{[q+1]}}{\underline{\mathbf{f}}^{[q+p]}} \arrow[r] \arrow[u, "f^{qp-q}"] & \frac{\underline{\mathbf{f}}^{[q+1]}}{(f^qR+\underline{\mathbf{f}}^{[q+p]})} \arrow[r] \arrow[u, "f^{qp-q}"] & 0
\end{tikzcd}$$

compatible with the rest of the directed system. Note that $\underline{\mathbf{f}}^{[q+1]}/\underline{\mathbf{f}}^{[q+p]} = (\underline{\mathbf{f}}^{[q+p]} : f^{p-1})/\underline{\mathbf{f}}^{[q+p]}$ for each $q = p^e$, so that in the direct limit, $\mathrm{Ker}(\theta_{\mathrm{fed}}) = (0 :_{H^c_{\underline{\mathbf{f}}}(R)} f^{p-1})$. $\square$

Note that the kernel $(0 :_{H^c_{\underline{\mathbf{f}}}(R)} f^{p-1})$ of θ_{fed} can be explicitly described as a local cohomology module, namely, it is isomorphic to $H^{c-1}_{\underline{\mathbf{f}}}(R/f^{p-1})$.

Proposition VI.8. *Let R be a Noetherian ring, let $\underline{\mathbf{f}} = f_1, \ldots, f_c \in R$ be a regular* *sequence of codimension c, and let $h \in R$ be a nonzerodivisor. Then*

$$(0 :_{H^c_{\underline{\mathbf{f}}}(R)} h) \cong H^{c-1}_{\underline{\mathbf{f}}}(R/h).$$

Proof. Consider the double complex $(0 \to R \xrightarrow{h} R \to 0) \otimes_R \check{C}^\bullet(\underline{\mathbf{f}}; R)$.

$$\begin{array}{ccccccccccc}
& & 0 & & & & 0 & & 0 & & \\
& & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{C}^1(\underline{\mathbf{f}}; R) & \longrightarrow & \cdots & \longrightarrow & \check{C}^{c-1}(\underline{\mathbf{f}}; R) & \longrightarrow & \check{C}^c(\underline{\mathbf{f}}; R) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & \check{C}^1(\underline{\mathbf{f}}; R) & \longrightarrow & \cdots & \longrightarrow & \check{C}^{c-1}(\underline{\mathbf{f}}; R) & \longrightarrow & \check{C}^c(\underline{\mathbf{f}}; R) & \longrightarrow & 0 \\
& & \downarrow & & & & \downarrow & & \downarrow & & \\
& & 0 & & & & 0 & & 0 & &
\end{array}$$

If we compute cohomology first horizontally and then vertically, we see that the cohomology of the total complex in degree c is $\operatorname{Hom}_R(R/h, H^c_{\underline{\mathbf{f}}}(R))$, since $E_2 = E_\infty$ and the first $c-1$ columns vanish. On the other hand, if we first take cohomology vertically and then horizontally, then $E_2 = E_\infty$ and the first row vanishes, so the cohomology of the total complex in degree c is also isomorphic to $H^{c-1}(R/h \otimes_R \check{C}^\bullet(\underline{\mathbf{f}}; R)) = H^{c-1}_{\underline{\mathbf{f}}}(R/h)$, as desired. $\square$

The kernel of θ_Q has a particularly nice description in codimension 2, where it decomposes as a direct sum of two local cohomology modules, $V_{f,g} \cong H^1_f(R/g^{p-1}) \oplus H^1_g(R/f^{p-1})$.

Proposition VI.9. *Let R be a Noetherian ring of prime characteristic $p > 0$, let $f, g \in R$ be a regular sequence, and let $V_{f,g}$ denote the following direct limit over all $q = p^e$*

$$\cdots \to \frac{(f^{q+1}, g^{q+1})}{((fg)^q, f^{q+p}, g^{q+p})} \xrightarrow{f^{qp-q}} \frac{(f^{qp+1}, g^{qp+1})}{((fg)^{qp}, f^{qp+p}, g^{qp+p})} \to \cdots \to V_{f,g}$$

Then

$$V_{f,g} \cong H^1_g(R/f^{p-1}) \oplus H^1_f(R/g^{p-1})$$

Proof. Note that if $af^{q+1} = bg^{q+1} \bmod ((fg)^q, f^{q+p}, g^{q+p})$ for some $a, b \in R$, then

$$\begin{aligned} a \in ((fg)^q, f^{q+p}, g^{q+1}) : f^{q+1} &= \big((f^{q+p}, g^{q+1}) : (f^p, g)\big) : f^{q+1} \\ &= \big((f^{q+p}, g^{q+1}) : f^{q+1}\big) : (f^p, g) \\ &= \big(f^{p-1}, g^{q+1}\big) : (f^p, g) \\ &= \big(f^{p-1}, g^q\big) \end{aligned}$$

so that $af^{q+1} \in (f^{q+p}, f^{q+1}g^q)$, which is zero mod $((fg)^q, f^{q+p}, g^{q+p})$. Thus, the generators $u_{1,e} := f^{q+1}$ and $u_{2,e} := g^{q+1}$ of $(f^{q+1}, g^{q+1})/((fg)^q, f^{q+p}, g^{q+p})$ have R-spans with an intersection of 0, yielding a direct sum

$$\frac{(f^{q+1}, g^{q+1})}{((fg)^q, f^{q+p}, g^{q+p})} \cong \frac{Ru_{1,e}}{g^q u_{1,e}R + f^{p-1}u_{1,e}R} \oplus \frac{Ru_{2,e}}{f^q u_{2,e}R + g^{p-1}u_{2,e}R}$$

The transition map $(fg)^{qp-q}$ sends $f^{q+1} = u_{1,e}$ to $g^{qp-q}f^{qp+1} = g^{qp-q}u_{1,e+1}$, and likewise, $u_{2,e} \mapsto f^{qp-q}u_{2,e+1}$, breaking into the direct sum of transition maps on the u_1 and u_2 components,

$$\cdots \to \frac{Ru_{1,e}}{g^q u_{1,e}R + f^{p-1}u_{1,e}R} \xrightarrow{g^{qp-q}} \frac{Ru_{1,e+1}}{g^{qp} u_{1,e+1}R + f^{p-1}u_{1,e+1}R} \to \cdots \to H^1_g(R/f^{p-1})$$

and

$$\cdots \to \frac{Ru_{2,e}}{f^q u_{2,e}R + g^{p-1}u_{2,e}R} \xrightarrow{f^{qp-q}} \frac{Ru_{2,e+1}}{f^{qp} u_{2,e+1}R + g^{p-1}u_{2,e+1}R} \to \cdots \to H^1_f(R/g^{p-1})$$

as desired. $\square$

CHAPTER VII

A Complex of Annihilator Submodules

Throughout this chapter, R denotes a Noetherian ring of prime characteristic $p > 0$. For a regular sequence $\underline{\mathbf{f}} = f_1, \ldots, f_c$ in R, let $f = f_1 \cdots f_c$, and let $H^c_{\underline{\mathbf{f}}}(R)_{\mathrm{nat}}$ and $H^c_{\underline{\mathbf{f}}}(R)_{\mathrm{fed}}$ denote the $R\langle F\rangle$-modules obtained when $H^c_{\underline{\mathbf{f}}}(R)$ is equipped with the natural action, F_{nat}, or with the Fedder action, $f^{p-1}F_{\mathrm{nat}}$, respectively.

In codimension $c = 1$, for $f \in R$ a nonzerodivisor, Proposition VI.6 gives an $R\langle F\rangle$-linear map $R/f \to H^1_f(R)_{\mathrm{fed}}$ whose image is the annihilator submodule $(0 :_{H^1_f(R)} f)$. Since multiplication by f is surjective, the cokernel of the inclusion $(0 :_{H^1_f(R)} f) \hookrightarrow H^1_f(R)$ is isomorphic to $H^1_f(R)$. If this copy of $H^1_f(R)$ is equipped with the natural action, we obtain the short exact sequence of $R\langle F\rangle$-linear maps below.

$$0 \to R/f \to H^1_f(R)_{\mathrm{fed}} \xrightarrow{f} H^1_f(R)_{\mathrm{nat}} \to 0 \tag{7.1}$$

The unitness of the cokernel of the embedding $R/f \to H^1_{(f,g)}(R)_{\mathrm{fed}}$ is a significant advantage in this setting.

Consider the case of codimension $c = 2$. Let f, g be a regular sequence of R with the property that g is a nonzerodivisor[1]. Proposition VI.6 provides an $R\langle F\rangle$-linear embedding $R/(f, g) \to H^2_{f,g}(R)_{\mathrm{fed}}$ whose image is the annihilator submodule $(0 :_{H^2_{f,g}(R)} (f, g))$, but the cokernel of this embedding, is not unit – see Theorem VI.7 in general and Proposition VI.9 regarding codimension $c = 2$ in particular.

[1] This turns out to be equivalent to requiring that both f, g and g, f are regular sequences.

It is not impossible to obtain a unit $R\langle F\rangle$-module by taking suitable $R\langle F\rangle$-linear quotients of $H^2_{f,g}(R)_{\text{fed}}$. For example, consider the annihilator ($R\langle F\rangle$-stable) submodule $(0 :_{H^2_{f,g}(R)} fg)$, which is generated over R by those Čech cohomology classes $\{\!\{ f^{-a}g^{-b} \}\!\}$ such that either $a = 1$ or $b = 1$. The cokernel of the embedding $(0 :_{H^2_{f,g}(R)} fg) \to H^2_{f,g}(R)_{\text{fed}}$ is spanned over R by the images of those classes $\{\!\{ f^{-a}g^{-b} \}\!\}$ with $a \geq 2$ or $b \geq 2$, and the map

$$H^2_{f,g}(R)_{\text{fed}}/(0 :_{H^2_{f,g}(R)} fg) \to H^2_{f,g}(R)_{\text{nat}}$$

that sends $\{\!\{ f^{-a}g^{-b} \}\!\}$ in $H^2_{f,g}(R)_{\text{fed}}/(0 :_{H^2_{f,g}(R)} fg)$ to the class $\{\!\{ f^{-a+1}g^{-b+1} \}\!\}$ in $H^2_{f,g}(R)_{\text{nat}}$ is readily verified to be $R\langle F\rangle$-linear. The surjectivity of multiplication by fg on $H^2_{f,g}(R)$ now gives the following exact sequence of $R\langle F\rangle$-modules.

$$0 \to (0 :_{H^2_{f,g}(R)} fg) \to H^2_{f,g}(R)_{\text{fed}} \xrightarrow{fg} H^2_{fg}(R)_{\text{nat}} \to 0 \tag{7.2}$$

Let us extend this sequence further to the left. Our complete intersection rings $R/(f,g)$ appears as $(0 :_{H^2_{f,g}(R)} (f,g))$, which is the intersection of two other annihilator submodules of $R\langle F\rangle$-submodules, $(0 :_{H^2_{f,g}(R)} f)$ and $(0 :_{H^2_{f,g}(R)} g)$, whose sum is all of $(0 :_{H^2_{f,g}(R)} fg)$. The submodule $(0 :_{H^2_{f,g}(R)} f)$ is spanned over R by Čech cohomology classes of the form $\{\!\{ f^{-1}g^{-b} \}\!\}$ for $b \in \mathbb{N}$, and $(0 :_{H^2_{f,g}(R)} g)$ is spanned by those of the form $\{\!\{ f^{-a}g^{-1} \}\!\}$ for $a \in \mathbb{N}$. The maps defined by $\{\!\{ g^{-b} \}\!\} \mapsto \{\!\{ f^{-1}g^{-b} \}\!\}$ $\{\!\{ f^{-a} \}\!\} \mapsto \{\!\{ f^{-a}g^{-1} \}\!\}$ provide $R\langle F\rangle$-linear isomorphisms[2] $H^1_g(R/f) \to (0 :_{H^2_{f,g}(R)} f)$ and $H^1_f(R/g) \to (0 :_{H^2_{f,g}(R)} g)$, respectively, using Frobenius actions $g^{p-1}F_{\text{nat}}$ and $f^{p-1}F_{\text{nat}}$ on $H^1_g(R/f)$ and $H^1_f(R/g)$. Letting $U = (0 :_{H^2_{f,g}(R)} f)$ and $V = (0 :_{H^2_{f,g}(R)} g)$, the $R\langle F\rangle$-linear exact sequence $0 \to U \cap V \to U \oplus V \to U + V \to 0$ may therefore be expressed as follows.

$$0 \to R/(f,g) \to H^1_f(R/g) \oplus H^1_g(R/f) \to (0 :_{H^2_{f,g}(R)} fg) \to 0 \tag{7.3}$$

[2]Both of these embeddings are in fact special cases of the embedding described in Proposition VI.6.

The sequences and together yield a four-term exact sequence of $R\langle F\rangle$-modules.

$$0 \to R/(f,g) \to H^1_f(R/g) \oplus H^1_g(R/f) \to H^2_{f,g}(R)_{\text{fed}} \xrightarrow{fg} H^2_{f,g}(R)_{\text{nat}} \to 0 \tag{7.4}$$

We regard the exact sequences 7.1 and 7.4 as the augmentations of a pair of (cohomologically indexed) complexes of $R\langle F\rangle$-modules

$$\mathbb{A}^\bullet_f(R): \quad R/f \to H^1_f(R)_{\text{fed}}$$

and

$$\mathbb{A}^\bullet_{f,g}(R): \quad R/(f,g) \to H^1_f(R/g) \oplus H^1_g(R/f) \to H^2_{f,g}(R)_{\text{fed}}$$

Both of these complexes, in codimensions $c = 1$ and $c = 2$, satisfy $H^i(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)) = 0$ for $i < c$ and $H^c(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)) \simeq H^c_{\underline{\mathbf{f}}}(R)_{\text{nat}}$, where $\underline{\mathbf{f}}$ denotes either f or f, g, respectively.

The main result of this chapter, which represents original work of the author and Eric Canton appearing in [CL20], is that a complex of $R\langle F\rangle$-modules completely analogous to the $c = 1$ and $c = 2$ complexes above can be constructed for permutable regular sequences of arbitrary length. The higher codimension $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)$ complexes are the main technical tool required for the applications presented in Chapter VIII.

Before we begin the construction, we set some terminology and notation that will be used throughout both this chapter and the next.

Definition VII.1. Let R be a ring and $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a regular sequence in R. Call $\underline{\mathbf{f}}$ *permutable* if $f_{\sigma(1)}, \ldots, f_{\sigma(c)}$ is a regular sequence for all permutations σ on the set $\{1, \ldots, c\}$.

Permutability is automatic for regular sequences in a local ring [BH98, Proposition 1.1.6], or for regular sequences of homogeneous elements in a standard graded ring [BH98, Exercise 1.5.23]. Let $[c] := \{1, \ldots, c\}$, and for a subset $T \subseteq [c]$, let $\underline{\mathbf{f}}_T$ denote the subsequence of $\underline{\mathbf{f}}$ indexed by the elements of T. Let $\widetilde{T} = [c] \setminus T$, with $\underline{\mathbf{f}}_{\widetilde{T}}$ the complementary subsequence to $\underline{\mathbf{f}}_T$ in $\underline{\mathbf{f}}$. The permutability of $\underline{\mathbf{f}}$ is equivalent to the hypothesis that $\underline{\mathbf{f}}_T$ is a regular sequence for all subsets $T \subseteq [c]$ [BH98, Exercise 1.2.21].

7.1 Construction of the $\mathbb{\Delta}$ Complex

Let R be a Noetherian ring, and fix a permutable regular sequence $\underline{\mathbf{f}} = f_1, \ldots, f_c \in R$. For $T \subseteq [c]$, let $f_T = \prod_{i \in T} f_i$, and write $f = f_{[c]}$ for convenience. For $a \geq 1$, recall that we denote $\underline{\mathbf{f}}^{[a]} := f_1^a, \ldots, f_c^a$ regardless of whether a is a power of the characteristic, and the notation $\underline{\mathbf{f}}_T^{[a]}$ denotes the subsequence of $\underline{\mathbf{f}}^{[a]}$ indexed by $T \subseteq [c]$.

Since $\underline{\mathbf{f}}$ is permutable, we can use Proposition VI.6 to obtain identifications

$$H^{c-i}_{\underline{\mathbf{f}}_T}(R/\underline{\mathbf{f}}_{\widetilde{T}}) = (0 :_{H^c_{\underline{\mathbf{f}}}(R)} \underline{\mathbf{f}}_{\widetilde{T}}) \tag{7.5}$$

for each subset $T \subseteq [c]$. Let $M = H^c_{\underline{\mathbf{f}}}(R)$, and observe that there are inclusions $\iota_{T,S} : (0 :_M \underline{\mathbf{f}}_{\widetilde{S}}) \hookrightarrow (0 :_M \underline{\mathbf{f}}_{\widetilde{T}})$ whenever $S \subseteq T$.

Definition VII.2. Let R be a Noetherian ring and let $\underline{\mathbf{f}}$ be a permutable regular sequence of codimension $c \geq 1$. Let $M = H^c_{\underline{\mathbf{f}}}(R)$. The $\mathbb{\Delta}$*-complex of* $\underline{\mathbf{f}}$, denoted $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)$, is the chain complex $(M^\bullet, \partial^\bullet)$ defined as follows.

- $M^i = \bigoplus_{|S|=i} (0 :_M \underline{\mathbf{f}}_{\widetilde{S}})$ for $0 \leq i \leq c$
- $\partial^i = \sum_{j=1}^c (-1)^j d^i_j$ where $d^i_j|_{(0:_M \underline{\mathbf{f}}_{\widetilde{S}})}$ is the direct sum of the inclusion maps $\iota_{S,T} : (0 :_M \underline{\mathbf{f}}_{\widetilde{S}}) \hookrightarrow (0 :_M \underline{\mathbf{f}}_{\widetilde{T}})$ ranging over the sets $T \supseteq S$ of size $|T| = i+1$ such that $T \setminus S$ is the jth element of T, enumerated so that $t_a < t_b$ (as elements of $[c]$) when $a < b$.

The choice of differentials gives $\{M^i\}_{i=0}^c$ the structure of a *semi-cosimplicial* R-module [Wei94, Def. 8.1.9, Ex. 8.1.6], which is to say, $d^{i+1}_k d^i_j = d^{i+1}_j d^i_{k-1}$ for $j < k$. Checking that $\partial^{i+1}\partial^i = 0$ is similar to checking that the chain maps in a Čech complex square to zero, and depends only on the semi-cosimplicial structure [Wei94, Def. 8.2.1].

For each $1 \leq n \leq c$, we may define a quotient complex $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n$ of $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)$ in the following manner. Note that this definition depends on the specific order of elements in the sequence $f_1, \ldots, f_c$. If $\sigma : [c] \to [c]$ is a non-trivial bijection, then setting $g_i = f_{\sigma(i)}$, $1 \leq i \leq c$, $\mathbb{A}$ complex of the regular sequence $g_1, \ldots, g_c$ would have a distinct collection of quotients.

Definition VII.3. Let R be a Noetherian ring and let $\underline{\mathbf{f}}$ be a permutable regular sequence of codimension $c \geq 1$. For fixed n such that $1 \leq n \leq c$, define the complex $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n$ by

$$\mathbb{A}^i_{\underline{\mathbf{f}}}(R)_n := \bigoplus_{S \subset [n],\ |S|=i} (0 :_M \underline{\mathbf{f}}_{\widetilde{S}}) \subseteq \mathbb{A}^i_{\underline{\mathbf{f}}}(R)$$

with differentials $\partial^i_0 = \sum_{j=1}^n (-1)^j d^i_{j,n}$ defined so that the map $d^i_{j,n}|_{(0 :_M \underline{\mathbf{f}}_{\widetilde{S}})}$ for $S \subseteq [n]$ is the direct sum of inclusions $\iota_{S,T} : (0 :_M \underline{\mathbf{f}}_{\widetilde{S}}) \hookrightarrow (0 :_M \underline{\mathbf{f}}_{\widetilde{T}})$ ranging only over the sets T of size $|T| = i+1$ such that $S \subseteq T \subseteq [n]$ and such that $T \setminus S$ is the jth element of T, enumerated so that $t_i < t_j$ (as elements of $[c]$) when $i < j$.

For example, $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n = (0 \to R/\underline{\mathbf{f}} \to 0)$ and $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(A)_c = \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)$. For each n, there is a surjection of complexes $\pi_n : \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n \to \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_{n-1}$ defined term-by-term in the obvious way, and we let $K^\bullet_n$ denote the kernel of π_n

$$0 \longrightarrow K^\bullet_n \longrightarrow \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n \longrightarrow \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_{n-1} \longrightarrow 0$$

For our calculation of the cohomology of $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)$, the key observations are as follows.

Proposition VII.4. *Let R be a Noetherian ring and let $\underline{\mathbf{f}}$ be a permutable regular sequence of codimension $c \geq 1$. For fixed n such that $1 \leq n \leq c$, where all set complements (e.g. $\widetilde{[n]}$) are taken within $[c]$, we have the following.*

1. $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n = \mathbb{A}^\bullet_{\underline{\mathbf{f}}_{[n]}}(R/\underline{\mathbf{f}}_{\widetilde{[n]}})$.

2. $K^\bullet_n = H^1_{f_n}(\mathbb{A}^\bullet_{\underline{\mathbf{f}}_{[n-1]}}(R/\underline{\mathbf{f}}_{\widetilde{[n]}}))[-1]$.

where $[-1]$ *denotes the right-shift operator on cohomologically indexed complexes.*

Proof. Let $M = H^c_{\underline{\mathbf{f}}}(R)$. The module $\mathbb{A}^i_{\underline{\mathbf{f}}}(R)_n$ is the direct sum of annihilators $(0 :_M \underline{\mathbf{f}}_{\widetilde{S}})$ ranging over all subsets $S \subseteq [n]$ of size $|S| = i$. In particular, we have $[c] - [n] \subseteq [c] - S$ for all such S, so that

$$(0 :_M \underline{\mathbf{f}}_{[c]-S}) = (0 :_{(0 :_M \underline{\mathbf{f}}_{[c]-[n]})} \underline{\mathbf{f}}_{[n]-S}) = (0 :_{M_n} \underline{\mathbf{f}}_{[n]-S})$$

where $M_n = H^n_{\underline{\mathbf{f}}_{[n]}}(R/\underline{\mathbf{f}}_{[c]-[n]})$, and thus, $\mathbb{A}^i_{\underline{\mathbf{f}}}(R)_n = \mathbb{A}^i_{\underline{\mathbf{f}}_{[n]}}(R/\underline{\mathbf{f}}_{[c]-[n]})$. The agreement of the differentials in the complexes $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n$ and $\mathbb{A}^\bullet_{\underline{\mathbf{f}}_{[n]}}(R/\underline{\mathbf{f}}_{[c]-[n]})$ is a straightforward consequence of their definitions.

Concerning K^i_n, the kernel of $\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_n \twoheadrightarrow \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R)_{n-1}$ is the direct sum of the annihilators $(0 :_M \underline{\mathbf{f}}_{\widetilde{S}})$ ranging over subsets $S \subseteq [n]$ of size $|S| = i$ such that $n \in S$. We therefore have an isomorphism

$$H^i_{\underline{\mathbf{f}}_S}(R/\underline{\mathbf{f}}_{[c]-S}) = H^1_{f_n}\left(H^{i-1}_{\underline{\mathbf{f}}_{S-\{n\}}}(R/\underline{\mathbf{f}}_{[c]-S})\right) = H^1_{f_n}\left((0 :_{M_n} \underline{\mathbf{f}}_{[n-1]-(S-\{n\})})\right)$$

where, once again, $M_n = H^n_{\underline{\mathbf{f}}_{[n]}}(R/\underline{\mathbf{f}}_{[c]-[n]})$. The sets $S - \{n\}$ for $S \subseteq [n]$ of size $|S| = i$ such that $n \in S$ correspond precisely to the subsets $S' \subseteq [n-1]$ of size $|S'| = i - 1$. Thus, $K^i_n = \mathbb{A}^{i-1}_{\underline{\mathbf{f}}_{[n-1]}}(R/\underline{\mathbf{f}}_{[c]-[n]})$. Confirming agreement of the corresponding differentials is straightforward. □

Example VII.5. Suppose $c = 4$. When $n = 0$, $\mathbb{A}^i_{\underline{\mathbf{f}}}(R)_n = (0 :_M \underline{\mathbf{f}}) \cong R/\underline{\mathbf{f}}$ for $i = 0$ and $\mathbb{A}^i_{\underline{\mathbf{f}}}(R)_n = 0$ for $i > 0$. We show $\mathbb{A}_n$ for $1 \leq n \leq 4$, identifying $(0 :_M \underline{\mathbf{f}}_{\widetilde{S}})$ with $H^i_{\underline{\mathbf{f}}_S}(R/\underline{\mathbf{f}}_{\widetilde{S}})$. Components of ∂^i corresponding to d^i_1, d^i_2, d^i_3, and d^i_4 are indicated in red, blue, dashed red, and dashed blue, with (dashed or solid) red indicating a sign change.

<table>
<tr><th>n</th><th>$\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)_n$</th></tr>
<tr><td>1</td><td>

$$\begin{tikzcd}[column sep=small]
 & H^1_{f_1}(R/\underline{\mathbf{f}}_{\{2,3,4\}}) \\
R/\underline{\mathbf{f}} \arrow[ur, red] &
\end{tikzcd}$$

</td></tr>
<tr><td>2</td><td>

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & H^1_{f_1}(R/\underline{\mathbf{f}}_{\{2,3,4\}}) \arrow[r, red] & H^2_{f_1,f_2}(R/\underline{\mathbf{f}}_{\{3,4\}}) \\
 & \oplus & \\
R/\underline{\mathbf{f}} \arrow[uur, red] \arrow[r, blue] & H^1_{f_2}(R/\underline{\mathbf{f}}_{\{1,3,4\}}) \arrow[uur, red] &
\end{tikzcd}$$

</td></tr>
<tr><td>3</td><td>

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & H^1_{f_1}(R/\underline{\mathbf{f}}_{\{2,3,4\}}) \arrow[r, red] \arrow[ddr, blue] & H^2_{f_1,f_2}(R/\underline{\mathbf{f}}_{\{3,4\}}) \arrow[ddr, red] & \\
 & \oplus & \oplus & \\
R/\underline{\mathbf{f}} \arrow[uur, red] \arrow[r, blue] \arrow[ddr, red, dashed, bend right] & H^1_{f_2}(R/\underline{\mathbf{f}}_{\{1,3,4\}}) \arrow[uur, red] \arrow[ddr, blue] & H^2_{f_1,f_3}(R/\underline{\mathbf{f}}_{\{2,4\}}) \arrow[r, red] & H^3_{f_1,f_2,f_3}(R/f_4) \\
 & \oplus & \oplus & \\
 & H^1_{f_3}(R/\underline{\mathbf{f}}_{\{1,2,4\}}) \arrow[uur, red] \arrow[r, blue] & H^2_{f_2,f_3}(R/\underline{\mathbf{f}}_{\{1,4\}}) \arrow[uur, red] &
\end{tikzcd}$$

</td></tr>
<tr><td>4</td><td>

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & H^1_{f_1}(R/\underline{\mathbf{f}}_{\{2,3,4\}}) \arrow[r, red] \arrow[ddr, blue] \arrow[ddddddr, red, dashed] & H^2_{f_1,f_2}(R/\underline{\mathbf{f}}_{\{3,4\}}) \arrow[ddr, blue] \arrow[ddddddr, blue] & & \\
 & \oplus & \oplus & & \\
R/\underline{\mathbf{f}} \arrow[uur, red] \arrow[r, blue] \arrow[ddr, red, dashed, bend right] \arrow[ddddddr, blue, dashed, bend right] & H^1_{f_2}(R/\underline{\mathbf{f}}_{\{1,3,4\}}) \arrow[uur, red] \arrow[ddr, blue] \arrow[ddddddr, red, dashed] & H^2_{f_1,f_3}(R/\underline{\mathbf{f}}_{\{2,4\}}) \arrow[r, red] \arrow[ddddddr, blue] & H^3_{f_1,f_2,f_3}(R/f_4) \arrow[ddddddr, red] & \\
 & \oplus & \oplus & & \\
 & H^1_{f_3}(R/\underline{\mathbf{f}}_{\{1,2,4\}}) \arrow[uur, red] \arrow[r, blue] \arrow[ddddddr, red, dashed] & H^2_{f_2,f_3}(R/\underline{\mathbf{f}}_{\{1,4\}}) \arrow[uur, red] \arrow[ddddddr, blue] & \oplus & \\
 & & \oplus & & \\
 & \oplus & H^2_{f_1,f_4}(R/\underline{\mathbf{f}}_{\{2,3\}}) \arrow[r, red] \arrow[ddr, blue] & H^3_{f_1,f_2,f_4}(R/f_3) \arrow[ddr, red] & \\
 & & \oplus & & \\
 & H^1_{f_4}(R/\underline{\mathbf{f}}_{\{1,2,3\}}) \arrow[uur, red] \arrow[r, blue] \arrow[ddr, red, dashed, bend right] & H^2_{f_2,f_4}(R/\underline{\mathbf{f}}_{\{1,3\}}) \arrow[uur, red] \arrow[ddr, blue] & H^3_{f_1,f_3,f_4}(R/f_2) \arrow[r, red] & H^4_{\underline{\mathbf{f}}}(R) \\
 & & \oplus & & \\
 & & H^2_{f_3,f_4}(R/\underline{\mathbf{f}}_{\{1,2\}}) \arrow[uur, red] \arrow[r, blue] & H^3_{f_2,f_3,f_4}(R/f_1) \arrow[uur, red] &
\end{tikzcd}$$

</td></tr>
</table>

The subcomplexes $K_2^\bullet$, $K_3^\bullet$, and $K_4^\bullet$ are displayed with terms generally to the lower right. For example, $K_4^\bullet$ consists of the terms in $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)_4$ that involve the local cohomology of quotients $R/\underline{\mathbf{f}}_S$ for subsets $S \subseteq \{1,2,3\}$.

7.2 Computing the Cohomology of the $\mathbb{\Delta}$ Complex

We continue with the notation of the last section. Let R be a Noetherian ring, let $\underline{\mathbf{f}}$ be a permutable regular sequence of codimension $c \geq 1$, and let $f = \prod_{i=1}^{c} f_i$. We denote by $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)^+$ the augmented chain complex equal to $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)$ in degrees $\leq c$, with augmentation $\mathbb{\Delta}_{\underline{\mathbf{f}}}^{c+1}(R)^+ := H^c_{\underline{\mathbf{f}}}(R)$ and differential $\partial^c : \mathbb{\Delta}^c_{\underline{\mathbf{f}}}(R)^+ \to \mathbb{\Delta}^{c+1}_{\underline{\mathbf{f}}}(R)^+$ given by the multiplication by f map $H^c_{\underline{\mathbf{f}}}(R) \to H^c_{\underline{\mathbf{f}}}(R)$.

Lemma VII.6. *Let R be a Noetherian ring, let $\underline{\mathbf{f}}$ be a permutable regular sequence of codimension $c \geq 1$. Suppose $H^i(\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)^+) = 0$ for all $1 \leq i \leq c+1$, and let $h \in R$ extend $\underline{\mathbf{f}}$ to a permutable regular sequence $f_1, \ldots, f_c, h$ of codimension $c+1$. Then $H^i(H^1_h(\mathbb{\Delta}_{\underline{\mathbf{f}}}(R)^+)) = 0$ for all $1 \leq i \leq c+1$.*

Proof. For the sake of notational convenience, write $\mathbb{\Delta}^\bullet = \mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)^+$. We compute H^1_h via the double complex $\check{C}(h;R) \otimes_R \mathbb{\Delta}^\bullet$, i.e. $0 \to \mathbb{\Delta}^\bullet \to (\mathbb{\Delta}^\bullet)_h \to 0$, as shown below.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{\Delta}^0 \arrow[r] \arrow[d] & \mathbb{\Delta}^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{\Delta}^c \arrow[r, "f"] \arrow[d] & \mathbb{\Delta}^{c+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathbb{\Delta}^0)_h \arrow[r] \arrow[d] & (\mathbb{\Delta}^1)_h \arrow[r] \arrow[d] & \cdots \arrow[r] & (\mathbb{\Delta}^c)_h \arrow[r, "f"] \arrow[d] & (\mathbb{\Delta}^{c+1})_h \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 & 0. &
\end{tikzcd}$$

By hypothesis, $H^i(\mathbb{\Delta}^\bullet) = 0$ for all i, so $H^i((\mathbb{\Delta}^\bullet)_h) = 0$ for all i as well. If we first take cohomology horizontally, we therefore obtain $E_1 = E_\infty = 0$. The cohomology of the total complex of the double complex is therefore zero. If, on the other hand, we take vertical cohomology first, then we arrive at a double complex with one nonzero row,

$$0 \longrightarrow H^1_h(\mathbb{\Delta}^0) \longrightarrow \cdots \longrightarrow H^1_h(\mathbb{\Delta}^{c+1}) \longrightarrow 0.$$

The modules $H^i(H^1_{(h)}(\mathbb{\Delta}^\bullet))$ appear now as the horizontal cohomology, with $E_2 = E_\infty$, and since only a single row is nonzero, they yield the cohomology of the total complex, which vanishes. $\square$

We are now ready to prove the main theorem of this chapter.

Theorem VII.7. *Let R be a Noetherian ring, and let $\underline{\mathbf{f}}$ be a permutable regular sequence of codimension $c \geq 1$. Then $H^i(\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)) = 0$ for $0 \leq i < c$, and $H^c(\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)) \cong H^c_{\underline{\mathbf{f}}}(R)$, with augmentation map isomorphic to multiplication by $f := \prod_1^c f_i$.*

If R has prime characteristic $p > 0$, then $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)$ is a complex of $R\langle F\rangle$-modules considered with their Fedder actions, and the induced Frobenius action on the augmentation $H^c_{\underline{\mathbf{f}}}(R) \cong H^c(\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R))$ is the natural action.

Proof. The differentials of $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)$ are direct sums of inclusions of submodules of $M = H^c_{\underline{\mathbf{f}}}(R)$, and by Proposition VI.6 these inclusions are Fedder-action linear in characteristic $p > 0$. Our statement about the induced Frobenius action on $H^c(\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R))$ is proven at the end of this argument; the bulk of this proof is calculation of the cohomology.

We proceed by by induction on c, with base case $c = 1$. With $f = f_1$, the complex $\mathbb{\Delta}^\bullet_f(R)$ is

$$0 \longrightarrow R/f \xrightarrow{(r+fR)\mapsto \{\!\{r/f\}\!\}} H^1_f(R) \longrightarrow 0.$$

The map $R/f \to H^1_f(R)$ shown above is clearly injective, so $H^0(\mathbb{\Delta}^\bullet_f(R)) = 0$. Moreover, the image of $R/f \to H^1_f(R)$ is precisely the kernel of multiplication by f. Multiplication by f on $H^1_f(R)$ is surjective, and the exactness of $0 \to R/f \to H^1_f(R) \xrightarrow{f} H^1_f(R) \to 0$ implies that $H^1(\mathbb{\Delta}^\bullet_f(R)) = H^1_f(R)$.

Now assume the theorem has been proven for any permutable regular sequence of codimension $c \geq 1$ in a Noetherian ring. Let $\underline{\mathbf{f}}, h = f_1, \ldots, f_c, h \in R$ be a permutable regular sequence of codimension $c + 1$. From Proposition VII.4, $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R/h)$ is the quotient complex $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}},h}(R)_c$ of $\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}},h}(R) = \mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R)_{c+1}$. The kernel $K^\bullet_{c+1}$ of this quotient is isomorphic to $H^1_h(\mathbb{\Delta}^\bullet_{\underline{\mathbf{f}}}(R))[-1]$, giving us the short exact sequence of complexes shown below.

$$0 \longrightarrow H^1_h(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R))[-1] \longrightarrow \mathbb{A}^\bullet_{\underline{\mathbf{f}},h}(R) \longrightarrow \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R/h) \longrightarrow 0, \tag{7.6}$$

By Lemma VII.6 and the induction hypothesis, $H^i\left(H^1_h(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R))[-1]\right) = 0$ for $i \leq c$, and that $H^{c+1}\left(H^1_h(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R))[-1]\right) = H^{c+1}(\mathbb{A}_{\underline{\mathbf{f}},h}(R)) = H^c_{\underline{\mathbf{f}}}(R)$. Likewise, we may apply the induction hypothesis to the $\mathbb{A}^\bullet$ complex of the regular sequence $\underline{\mathbf{f}}$ of codimension c in R/h to obtain $H^i(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R/h)) = 0$ for $i < c$ and $H^c(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R/h)) = H^c_{\underline{\mathbf{f}}}(R/h)$.

We now study the long exact sequence in cohomology from the short exact sequence (7.6). To simplify notation, let $\mathbb{A}^\bullet := \mathbb{A}^\bullet_{\underline{\mathbf{f}},h}(R)$, $\mathbb{A}^\bullet_1 := \mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R/h)$, and $K^\bullet := H^1_h(\mathbb{A}^\bullet_{\underline{\mathbf{f}}}(R))[-1]$. We obtain

$$\cdots \longrightarrow H^i(K^\bullet) \longrightarrow H^i(\mathbb{A}^\bullet) \longrightarrow H^i(\mathbb{A}^\bullet_1) \xrightarrow{\delta} H^{i+1}(K^\bullet) \longrightarrow \cdots$$

We immediately see that $H^i(\mathbb{A}^\bullet) = 0$ for $i < c$. Using that $H^c(K^\bullet) = 0$, we have an exact sequence

(7.7)

$$0 \longrightarrow H^c(\mathbb{A}^\bullet) \longrightarrow H^c(\mathbb{A}^\bullet_1) \xrightarrow{\delta} H^{c+1}(K^\bullet) \longrightarrow H^{c+1}(\mathbb{A}^\bullet) \longrightarrow 0.$$

We claim that δ is injective. To see this, we start with recalling the construction of δ. We begin with the map from row c to row $c+1$ in the short exact sequence of complexes.

$$\begin{tikzcd}
0 \arrow[r] & K^c \arrow[r] \arrow[d, "\partial^c_K"] & \mathbb{A}^c \arrow[r] \arrow[d, "\partial^c_{\mathbb{A}}"] & \mathbb{A}^c_1 \arrow[r] \arrow[d, "\partial^c_{\mathbb{A}_1}"] & 0 \\
0 \arrow[r] & K^{c+1} \arrow[r] & \mathbb{A}^{c+1} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Let $M = H^{c+1}_{\underline{\mathbf{f}},h}(R)$. We identify $\mathbb{A}^c_1 = (0 :_M h)$ and $K^c = \bigoplus_{i=1}^c (0 :_M f_i)$. Let $\iota : \mathbb{A}^c_1 \hookrightarrow \mathbb{A}^c$ denote the obvious splitting.

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^c (0 :_M f_i) \arrow[r] \arrow[d, "\partial^c_K"] & (0 :_M h) \oplus \left(\bigoplus_{i=1}^c (0 :_M f_i)\right) \arrow[r, shift right] \arrow[d, "\partial^c_{\mathbb{A}}"] & (0 :_M h) \arrow[r] \arrow[d, "\partial^c_{\mathbb{A}_1}"] \arrow[l, shift right, "\iota"'] & 0 \\
0 \arrow[r] & M \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

A class $\{\!\{\eta\}\!\}_{\mathbb{\Delta}_1} \in H^c(\mathbb{\Delta}_1^\bullet)$ is represented by $\eta \in (0 :_M h)$, where we write a subscript $\{\!\{\cdots\}\!\}_C$ to indicate the complex $C^\bullet$ with respect to which we're taking cohomology. By definition,

$$\delta(\{\!\{\eta\}\!\}_{\mathbb{\Delta}_1}) = \{\!\{\partial^c_{\mathbb{\Delta}}(\iota(\eta))\}\!\}_K \in H^{c+1}(K).$$

If $\delta(\{\!\{\eta\}\!\}_{\mathbb{\Delta}_1}) = 0$, then $\partial^c_{\mathbb{\Delta}}(\iota(\eta))$ is in the image of ∂^c_K, which is $\sum_{j=1}^c (0 :_M f_j)$. Note that $\partial^c_{\mathbb{\Delta}} \circ \iota$ is just the inclusion map $(0 :_M h) \hookrightarrow M$, possibly up to a sign change, so to say that $\partial^c_{\mathbb{\Delta}}(\iota(\eta)) \in \sum_{j=1}^c (0 :_M f_j)$ means precisely that

$$\eta \in (0 :_M h) \cap \left(\sum_{j=1}^{c} (0 :_M f_j)\right) = \operatorname{Im}(\partial^{c-1}_{\mathbb{\Delta}_1}),$$

Thus, $\{\!\{\eta\}\!\}_{\mathbb{\Delta}_1} = 0$, and δ is injective.

We conclude that $H^c(\mathbb{\Delta}^\bullet) = 0$ from the sequence (7.7). Following the reasoning of the last paragraph, the image of $\partial^c_{\mathbb{\Delta}}$ is $(0 :_M h) + \sum_{j=1}^c (0 :_M f_j)$, which is the kernel of multiplication by $hf = h\,(\prod_1^c f_j)$. Thus, the augmentation map $\mathbb{\Delta}^{c+1} \twoheadrightarrow H^{c+1}(\mathbb{\Delta}^\bullet)$ is (by definition) the quotient $M \twoheadrightarrow M/(0 :_M fh)$. Since multiplication by fh is surjective on M, this provides an isomorphism $\varphi : H^{c+1}(\mathbb{\Delta}^\bullet) \to M$,

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}(\partial^c_{\mathbb{\Delta}}) \arrow[r] \arrow[d, equal] & \mathbb{\Delta}^{c+1} \arrow[r, "\text{aug.}"] \arrow[d, equal] & H^{c+1}(\mathbb{\Delta}^\bullet) \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & (0 :_M fh) \arrow[r] & M \arrow[r, "fh"] & M \arrow[r] & 0
\end{tikzcd}$$

So, under the isomorphism φ that identifies $H^{c+1}(\mathbb{\Delta}^\bullet)$ with $M = H^{c+1}_{\underline{\mathbf{f}},h}(R)$, the augmentation map $\mathbb{\Delta}^{c+1} \xrightarrow{\text{aug.}} H^{c+1}(\mathbb{\Delta}^\bullet)$ is isomorphic to the multiplication map $M \xrightarrow{fh} M$, as claimed. Except for the final statement about the induced Frobenius action in characteristic $p > 0$, we have proven the theorem.

The final statement comes down to a direct calculation. For a given representative $\eta \in \mathbb{\Delta}^{c+1} = H^{c+1}_{\underline{\mathbf{f}},h}(R)_{\text{fed}}$, the induced Frobenius action $\overline{F}$ on $H^{c+1}(\mathbb{\Delta}^\bullet)$ sends $\{\!\{\eta\}\!\}_{\mathbb{\Delta}}$ to $\{\!\{F_{\text{fed}}(\eta)\}\!\}_{\mathbb{\Delta}}$. Under the identification $\varphi : H^{c+1}(\mathbb{\Delta}^\bullet) \xrightarrow{\sim} H^{c+1}_{\underline{\mathbf{f}},h}(R)$, the augmentation map $\eta \mapsto \{\!\{\eta\}\!\}_{\mathbb{\Delta}}$ is the multiplication $\eta \mapsto (fh)\eta$. Thus,

$$\overline{F}((fh)\eta) = (fh)F_{\mathrm{fed}}(\eta) = (fh)^p F_{\mathrm{nat}}(\eta) = F_{\mathrm{nat}}((fh)\eta)$$

so $\overline{F} = F_{\mathrm{nat}}$, as desired. $\square$

CHAPTER VIII

Application to Closed Support

Due to the isomorphism in Theorem IV.4, for a Noetherian ring S and an S-module M, every local cohomology module of M is isomorphic to one of the form $H_I^i(M)$ for I an ideal satisfying $i \leq \operatorname{ht}(I) + 1$. When S is Cohen-Macaulay and $M = S$ (cf. [Hel01, Theorem 3]) our attention is therefore restricted to the cases $H_I^{\operatorname{ht}(I)}(S)$ and $H_I^{\operatorname{ht}(I)+1}(S)$. The module $H_I^{\operatorname{ht}(I)}(R)$ has a finite set of associated primes (see Theorem II.18), so questions about the support or associated primes of the local cohomology $H_I^i(S)$ for S a Cohen-Macaulay ring can, without loss of generality, be posed entirely for modules of the form $H_I^{\operatorname{ht}(I)+1}(S)$.

Given a regular ring R and a regular sequence $\underline{\mathbf{f}} = f_1, \ldots, f_c$, the closed support problem for $R/\underline{\mathbf{f}}$ is, by the preceding discussion, determined by the behavior of modules of the form $H_{I/\underline{\mathbf{f}}}^{\operatorname{ht}(I/\underline{\mathbf{f}})+1}(R/\underline{\mathbf{f}})$ where I is an ideal of R containing $\underline{\mathbf{f}}$. Indeed, the hypersurface support theorems of Hochster and Núñez-Betancourt [HNB17, Corollary 4.13] or Katzman and Zhang [KZ17, Theorem 7.1] may be interpreted as the statement that in prime characteristic $p > 0$, all modules of the form $H_{I/f}^{\operatorname{ht}(I/f)+1}(R/f)$ have closed support.

Our main application in this chapter, representing original work of the author and Eric Canton appearing in [CL20], states that if $\underline{\mathbf{f}} = f_1, \ldots, f_c$ is a permutable regular sequence in a regular ring R of prime characteristic $p > 0$, then the module

$H^{\mathrm{ht}(I/\underline{\mathbf{f}}R)+c}_{I/\underline{\mathbf{f}}R}(R/\underline{\mathbf{f}}R)$ has closed support for any ideal I satisfying the vanishing hypothesis $H^i_I(R) = 0$ for $\mathrm{ht}(I) < i < \mathrm{ht}(I) + c$. Note that the hypothesis on I is vacuous if $c = 1$, and is satisfied automatically if R/I is Cohen-Macaulay, due to a result of Peskine and Szpiro [PS73].

To establish notation, assume from this point onward that the regular sequence $\underline{\mathbf{f}} = f_1, \ldots, f_c$ is permutable, and let $f = \prod_{i=1}^{c} f_i$. Given a subset $T \subseteq [c]$, recall that we use $\underline{\mathbf{f}}_T$ to denote the subsequence of $\underline{\mathbf{f}}$ indexed by T, and let $f_T = \prod_{i \in T} f_i$. For any such subset T of size $|T| = b$, let $N^a_{\underline{\mathbf{f}}_T}$ denote the kernel of the ith differential $\partial^a : \mathbb{\Delta}^a_{\underline{\mathbf{f}}_T}(R) \to \mathbb{\Delta}^{a+1}_{\underline{\mathbf{f}}_T}(R)$ when $a < b$, and let $N^b_{\underline{\mathbf{f}}_T}$ denote the kernel of the augmentation map $\mathbb{\Delta}^b_{\underline{\mathbf{f}}_T}(R) \to H^b(\mathbb{\Delta}^b_{\underline{\mathbf{f}}_T}(R))$. By Theorem VII.7, we have the following statements.

- $N^1_{\underline{\mathbf{f}}_T} = R/\underline{\mathbf{f}}_T$.
- $N^b_{\underline{\mathbf{f}}_T}$ fits into an exact sequence

$$0 \to N^b_{\underline{\mathbf{f}}_T} \to H^b_{\underline{\mathbf{f}}_T}(R)_{\mathrm{fed}} \xrightarrow{f_T} H^b_{\underline{\mathbf{f}}_T}(R)_{\mathrm{nat}} \to 0$$

- For all values $1 \leq a < b$, there is an exact sequence of the form

$$0 \to N^a_{\underline{\mathbf{f}}_T} \to \bigoplus_{S \subseteq T,\ |S| = b-a} H^a_{\underline{\mathbf{f}}_{T-S}}(R/\underline{\mathbf{f}}_S) \to N^{a+1}_{\underline{\mathbf{f}}_T} \to 0.$$

The compatibility of the differentials ∂^i with the Fedder actions of each term in the $\mathbb{\Delta}^{\bullet}_{\underline{\mathbf{f}}_T}(R)$ complex implies that each module of the form $N^a_{\underline{\mathbf{f}}_T}$ carries an induced Frobenius action. The short exact sequences displayed above may therefore be understood over $R\langle F\rangle$, and by Proposition III.13, the long exact sequences that result from applying a functor $\Gamma_I(-)$ consist entirely of $R\langle F\rangle$-linear maps.

Note that the vanishing hypotheses of the following theorem are automatically satisfied if R/I is Cohen-Macaulay or if $c = 1$.

Theorem VIII.1. *Let R be a regular ring of prime characteristic $p > 0$, let $\underline{\mathbf{f}} = f_1, \ldots, f_c$ be a permutable regular sequence of codimension $c \geq 1$, and let $I \supseteq \underline{\mathbf{f}}$ be an ideal such that $H^i_I(R) = 0$ for $ht(I) < i < ht(I) + c$. The module $H^{ht(I/\underline{\mathbf{f}}R)+c}_{I/\underline{\mathbf{f}}R}(R/\underline{\mathbf{f}}R)$ has Zariski closed support in Spec$(R/\underline{\mathbf{f}}R)$.*

Proof. For convenience, write $t = \mathrm{ht}(I/\underline{\mathbf{f}}R) = \mathrm{ht}(I) - c$. We will make heavy use of the $N^a_{\underline{\mathbf{f}}_T}$ notation introduced in the preceding discussion.

Our first aim is to show that the following three statements hold for all b such that $1 \leq b \leq c$,

(i) For any subset $T \subseteq [c]$ of size $|T| = b$, and for all a satisfying $\max(1, 3-c+b) \leq a \leq b$ (an empty range of values if $c \leq 2$), it holds that $H^j_I(N^a_{\underline{\mathbf{f}}_T}) = 0$ whenever

$$t + c + 2 - a \leq j \leq t + 2c - 1 - b.$$

(ii) For any subset $T \subseteq [c]$ of size $|T| = b$, and for all a satisfying $\max(1, 2-c+b) \leq a \leq b$ (an empty range if $c = 1$) the module $H^{t+c+1-a}_I(N^a_{\underline{\mathbf{f}}_T})$ is finitely generated over $R\langle F\rangle$.

(iii) For any subset $T \subseteq [c]$ of size $|T| = b$, and for all $\max(1, 2-c+b) \leq a \leq b$ (an empty range of values if $c = 1$), the module $H^{t+2c-b}_I(N^a_{\underline{\mathbf{f}}_T})$ has a finite set of associated primes.

The proof is by induction on b, beginning with the case $b = 1$. We will actually start by showing that the statements hold whenever $b = a$, i.e., for the modules $N^b_{\underline{\mathbf{f}}_T}$ when $|T| = b$. This immediately implies the $b = 1$ case, since $N^1_{f_j} = R/f_j$. So, fix $1 \leq b \leq c$ and let $T \subseteq [c]$ be a subset of size $|T| = b$. Concerning the module $N^b_{\underline{\mathbf{f}}_T}$, we have an exact sequence

$$0 \to N^b_{\underline{\mathbf{f}}_T} \to H^b_{\underline{\mathbf{f}}_T}(R)_{\mathrm{fed}} \xrightarrow{f_T} H^b_{\underline{\mathbf{f}}_T}(R)_{\mathrm{nat}} \to 0$$

From the long exact sequence induced by $\Gamma_I(-)$ along with our vanishing hypothesis $H^i_I(R) = 0$ for $t + c + 1 \leq i \leq t + 2c - 1$, it is readily verified that (i) so long as $c \geq 3$, $H^j_I(N^b_{\underline{\mathbf{f}}_T}) = 0$ for $t + c + 2 - b \leq j \leq t + 2c - 1 - b$. (ii) So long as $c \geq 2$, $H^{t+c+1-b}_I(N^b_{\underline{\mathbf{f}}_T})$ is an $R\langle F\rangle$ homomorphic image of $H^t_I(H^c_{\underline{\mathbf{f}}}(R)_{\text{nat}})$, and is therefore finitely generated over $R\langle F\rangle$ (see Theorem III.15, and recall that $H^c_{\underline{\mathbf{f}}}(R)_{\text{nat}}$ is unit and finitely generated). Finally, (iii) so long as $c \geq 2$, $H^{t+2c-b}_I(R/\underline{\mathbf{f}}_T)$ is isomorphic to a submodule of $H^{t+c}_I(H^c_{\underline{\mathbf{f}}}(R))$, and hence has a finite set of associated primes (see Theorem III.16).

Now take b in the range $2 \leq b \leq c$, since the case $b = 1$ is proven. Suppose that the statements (i)–(iii) have been shown for subsets $S \subseteq [c]$ of size $|S| < b$, and fix $T \subseteq [c]$ any subset of size b. For this set T, we will demonstrate the claims (i)–(iii) about the modules $N^a_{\underline{\mathbf{f}}_T}$ by a decreasing induction on a. The case $a = b$ has already been shown.

Fix $a < b$ and suppose we've proven (i)–(iii) for the modules $N^r_{\underline{\mathbf{f}}_T}$ whenever $a < r \leq b$. We will show that the statements hold for the module $N^a_{\underline{\mathbf{f}}_T}$ using the short exact sequence

$$0 \to N^a_{\underline{\mathbf{f}}_T} \to \bigoplus_{S\subseteq T,\ |S|=b-a} H^a_{\underline{\mathbf{f}}_{T-S}}(R/\underline{\mathbf{f}}_S) \to N^{a+1}_{\underline{\mathbf{f}}_T} \to 0,$$

To show claim (i), fix j in the range $t+c+2-a \leq j \leq t+2c-1-b$ and consider the exact sequence

$$\cdots \to H^{j-1}_I(N^{a+1}_{\underline{\mathbf{f}}_T}) \to H^j_I(N^a_{\underline{\mathbf{f}}_T}) \to \bigoplus_{S\subseteq T,\ |S|=b-a} H^j_I(H^a_{\underline{\mathbf{f}}_{T-S}}(R/\underline{\mathbf{f}}_S)) \to \cdots$$

note that for each subset $S \subseteq T$ of size $|S| = b - a$, we have

$$H^j_I(H^a_{\underline{\mathbf{f}}_{T-S}}(R/\underline{\mathbf{f}}_S)) = H^{j+a}_I(R/\underline{\mathbf{f}}_S)$$

where $R/\underline{\mathbf{f}}_S = N^1_{\underline{\mathbf{f}}_S}$. The inequality $t + c + 1 \leq j + a \leq t + 2c - 1 - (b - a)$ gives us vanishing $H^{j+a}_I(N^1_{\underline{\mathbf{f}}_S}) = 0$ for each subset $S \subseteq T$ of size $b - a$ by induction, since

$|S| < |T|$. Since $t+c+2-(a+1) \leq j-1 \leq t+2c-1-b$, we also have $H_I^{j-1}(N_{\underline{\mathbf{f}}_T}^{a+1}) = 0$ by induction, since $a+1 > a$. The vanishing of $H_I^j(N_{\underline{\mathbf{f}}_T}^a)$ follows at once.

For claim (ii), the relevant exact sequence is

$$\cdots \to H_I^{t+c-a}(N_{\underline{\mathbf{f}}_T}^{a+1}) \to H_I^{t+c+1-a}(N_{\underline{\mathbf{f}}_T}^a) \to \bigoplus_{S \subseteq T,\ |S|=b-a} H_I^{t+c+1}(N_{\underline{\mathbf{f}}_S}^1) \to \cdots$$

Since $a \geq \max(1, 2-c+b)$, we have that $1 \geq 3-c+(b-a)$, so claim (i) for the module $N_{\underline{\mathbf{f}}_S}^1$ implies that $H_I^{t+c+1}(N_{\underline{\mathbf{f}}_S}^1) = 0$. Thus, $H_I^{t+c+1-a}(N_{\underline{\mathbf{f}}_T}^a)$ is a quotient of $H_I^{t+c-a}(N_{\underline{\mathbf{f}}_T}^{a+1})$ by some ($R\langle F\rangle$-stable) submodule. As $R\langle F\rangle$-modules, $H_I^{t+c+1-(a+1)}(N_{\underline{\mathbf{f}}_T}^{a+1})$ is finitely generated by induction on a, so the same is true of its image $H_I^{t+c+1-a}(N_{\underline{\mathbf{f}}_T}^a)$.

To show claim (iii), consider the exact sequence

$$\cdots \to H_I^{t+2c-b-1}(N_{\underline{\mathbf{f}}_T}^{a+1}) \to H_I^{t+2c-b}(N_{\underline{\mathbf{f}}_T}^a) \to \bigoplus_{S \subseteq T,\ |S|=b-a} H_I^{t+2c-b+a}(N_S^1) \to \cdots$$

The condition $a \geq 2-c+b$ implies that $a+1 \geq 3-c+b$, so claim (i) for the module $N_{\underline{\mathbf{f}}_T}^{a+1}$ shows that $H_I^{t+2c-1-b}(N_{\underline{\mathbf{f}}_T}^{a+1}) = 0$. Thus $H_I^{t+2c-b}(N_{\underline{\mathbf{f}}_T}^a)$ is isomorphic to a submodule of a direct sum of modules of the form $H_I^{t+2c-(b-a)}(N_S^1)$, for $S \subseteq T$ a subset of size $b-a$. Since $2-c+(b-a) \leq 1$, each $H_I^{t+2c-(b-a)}(N_S^1)$ has a finite set of associated primes. The induction is complete and the claims (i)–(iii) have been demonstrated.

We are now ready to show that $H_I^{t+c}(R/\underline{\mathbf{f}}R) = H_I^{t+c}(N_{\underline{\mathbf{f}}R}^1)$ has closed support. This is known in the case $c = 1$ (see Theorem V.2). For $c \geq 2$, there is an exact sequence

$$\cdots \to H_I^{t+c-1}(N_{\underline{\mathbf{f}}R}^2) \to H_I^{t+c}(N_{\underline{\mathbf{f}}R}^1) \to \bigoplus_{S \subseteq T,\ |S|=1} H_I^{t+c+1}(N_{\underline{\mathbf{f}}_S}^1) \to \cdots$$

Since $2 = \max(1, 2+c-c)$, the module $H_I^{t+c+1-2}(N_{\underline{\mathbf{f}}R}^2)$ is finitely generated over $R\langle F\rangle$, and thus, any $R\langle F\rangle$ homomorphic image will have closed support by Theorem

III.1. Additionally, $1 \geq 2 - c + (c-1)$, so $H_I^{t+2c-(c-1)}(N^1_{\underline{\mathbf{f}}_S})$ (for each singleton set $S \subseteq T$) has a finite set of associated primes. The claim about the support of $H_I^{t+c}(N^1_{\underline{\mathbf{f}}R})$ follows at once. $\square$

8.0.1 Nesting of Supports

In this subsection, we remark that the support of the local cohomology of a complete intersection cut out by a regular sequence $f_1, \dots, f_c$ has a curious nesting property in relation to the supports of the local cohomologies of the complete intersections defined by subsequences of $f_1, \dots, f_c$.

Theorem VIII.2. *Let R be a Cohen-Macaulay ring of prime characteristic $p > 0$, let $\underline{\mathbf{f}} = f_1, \dots, f_c$ be a permutable regular sequence, and let I be an ideal containing $\underline{\mathbf{f}}R$. For $T \subseteq [c]$, let $\underline{\mathbf{f}}_T$ be the ideal generated by the subsequence of $f_1, \dots, f_c$ indexed by T. For any $\delta \geq 0$,*

$$Supp\, H^{ht(I/\underline{\mathbf{f}}_T)+\delta}_{I/\underline{\mathbf{f}}_T}(R/\underline{\mathbf{f}}_T) \subseteq Supp\, H^{ht(I/\underline{\mathbf{f}}R)+\delta}_{I/\underline{\mathbf{f}}R}(R/\underline{\mathbf{f}}R)$$

In particular, if $\underline{\mathbf{h}} = f_1, \dots, f_c, g_1, \dots, g_t$ is a maximal length regular sequence in I and if $\underline{\mathbf{h}}$ is permutable, then

$$Supp\, H^{ht(I)+\delta}_{I}(R) \subseteq Supp\, H^{ht(I/\underline{\mathbf{f}}R)+\delta}_{I/\underline{\mathbf{f}}R}(R/\underline{\mathbf{f}}R) \subseteq Supp\, H^{\delta}_{I/\underline{\mathbf{h}}}(R/\underline{\mathbf{h}})$$

Proof. Let $\underline{\mathbf{h}} = f_1, \dots, f_c, g_1, \dots, g_t$ be a maximal length regular sequence contained in I. Via the obvious inclusions $T \subseteq [c] \subseteq [c+t]$, we may write $\underline{\mathbf{f}}R = \underline{\mathbf{h}}_{[c]}$ and $\underline{\mathbf{f}}_T = \underline{\mathbf{h}}_T$. Let $b = |T|$. Observe that

$$H^{\mathrm{ht}(I/\underline{\mathbf{f}}_T)+\delta}_{I/\underline{\mathbf{f}}_T}(R/\underline{\mathbf{f}}_T) = H^\delta_I\left(H^{t+c-b}_{\underline{\mathbf{h}}_{[c+t]-T}}(R/\underline{\mathbf{h}}_T)\right),$$

and that

$$H^{\mathrm{ht}(I/\underline{\mathbf{f}}R)+\delta}_{I/\underline{\mathbf{f}}R}(R/\underline{\mathbf{f}}R) = H^\delta_I\left(H^{t}_{\underline{\mathbf{h}}_{[c+t]-[c]}}(R/\underline{\mathbf{h}}_{[c]})\right).$$

Let $A = R/\underline{\mathbf{h}}_T$, and consider the ring $R/\underline{\mathbf{h}}_{[c]}$ as being cut out from A by a regular sequence of length $c - b$ (indexed by the set $[c] - T$). The result follows by a straightforward induction using the following lemma. $\square$

Lemma VIII.3. *Let A be a Noetherian ring, let $f_1, \cdots, f_t, h \in A$ be a permutable regular sequence, and let $\underline{\mathbf{f}} = f_1, \ldots, f_t$. Let I be an ideal containing $\underline{\mathbf{f}}, h$. Then for any $\delta \geq 0$,*

$$\mathit{Supp}\, H^\delta_I(H^{t+1}_{\underline{\mathbf{f}},h}(R)) \subseteq \mathit{Supp}\, H^\delta_I(H^t_{\underline{\mathbf{f}}}(R/h))$$

Proof. Suppose that, after replacing R by R_P for some $P \in \operatorname{Spec}(R)$, we obtain vanishing $H^\delta_I(H^t_{\underline{\mathbf{f}}}(R/h)) = 0$. We would like to show that $H^\delta_I(H^{t+1}_{\underline{\mathbf{f}},h}(R)) = 0$, where we recall that

$$H^{t+1}_{\underline{\mathbf{f}},h}(R) = H^t_{\underline{\mathbf{f}}}(H^1_h(R)) = \varinjlim_n H^t_{\underline{\mathbf{f}}}(R/h^n)$$

It would therefore suffice to show that $H^\delta_I(H^t_{\underline{\mathbf{f}}}(R/h^n)) = 0$ for all $n \geq 1$. By hypothesis, this is true when $n = 1$, so fix $n > 1$ and suppose for the sake of induction that $H^\delta_I(H^t_{\underline{\mathbf{f}}}(R/h^j)) = 0$ for all $j < n$.

Note that $(h^n :_R h) = h^{n-1}R$, i.e., the annihilator of h in R/h^nR is $h^{n-1}R/h^nR$, isomorphic as an R-module to R/hR. Mapping R/hR onto the image of h^{n-1}, we get

$$0 \to R/h \xrightarrow{h^{n-1}} R/h^n \to R/h^{n-1} \to 0$$

inducing the exact sequence

$$\cdots \to H^{t-1}_{\underline{\mathbf{f}}}(R/h^{n-1}) \to H^t_{\underline{\mathbf{f}}}(R/h) \to H^t_{\underline{\mathbf{f}}}(R/h^n) \to H^t_{\underline{\mathbf{f}}}(R/h^{n-1}) \to H^{t+1}_{\underline{\mathbf{f}}}(R/h) \to \cdots$$

The arithmetic rank of $\underline{\mathbf{f}}$ is t, so $H^{t+1}_{\underline{\mathbf{f}}}(R/h) = 0$. Since $\underline{\mathbf{h}}$ is permutable, the sequence $h^{n-1}, f_1, \cdots, f_t$ is a regular, and consequently, $f_1, \ldots, f_t$ is an R/h^{n-1}-regular sequence. Since $\operatorname{depth}_{\underline{\mathbf{f}}}(R/h^{n-1}) = t$, we get $H^{t-1}_{\underline{\mathbf{h}}}(R/h^{n-1}) = 0$. Thus, the sequence

$$0 \to H^t_{\underline{\mathbf{f}}}(R/h) \to H^t_{\underline{\mathbf{f}}}(R/h^n) \to H^t_{\underline{\mathbf{f}}}(R/h^{n-1}) \to 0$$

is exact, and so is

$$\cdots \to H^{\delta}_{I}(H^{t}_{\underline{\mathbf{f}}}(R/h)) \to H^{\delta}_{I}(H^{t}_{\underline{\mathbf{f}}}(R/h^{n})) \to H^{\delta}_{I}(H^{t}_{\underline{\mathbf{f}}}(R/h^{n-1})) \to \cdots .$$

We have $H^{\delta}_{I}(H^{t}_{\underline{\mathbf{f}}}(R/h)) = 0$, and by induction $H^{\delta}_{I}(H^{t}_{\underline{\mathbf{f}}}(R/h^{n-1})) = 0$. It follows that $H^{\delta}_{I}(H^{t}_{\underline{\mathbf{h}}}(R/h^{n})) = 0$. $\square$

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