Local Well-posedness of Free Boundary Relativistic Barotropic Fluid Equation in Minkowski Space-time

by

Yuxin Wang

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Doctoral Committee:

Professor Sijue Wu, Chair
Professor Ivo Dinov
Professor Peter Miller
Professor Jeffrey Rauch

Yuxin Wang

yuxinxw@umich.edu

ORCID iD: 0000-0001-5123-469X

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TABLE OF CONTENTS

ABSTRACT

In this dissertation, we study the motion of a relativistic barotropic fluid with a free boundary. Relativistic barotropic fluids are fluids whose density and pressure are directly related. Such fluids are used as models in a wide range of scientific applications from meteorology to astrophysics. We prove that such a system of equations is locally well-posed, and the unique solution has the same regularity as the initial data. We use two methods to achieve the result. The first method is to construct approximate solutions to the equation by mollifying the nonlinear equation directly in the Lagrangian coordinate, and then passing to the limit so that the approximate solutions converge to the exact solution. The second method is to first consider the equation on a linear level, solve the linear equation using Galerkin approximation, and then solve the nonlinear equation by iterating on the solutions to these linear equations and passing to the limit.

CHAPTER I

Introduction

In this dissertation, we consider the motion of a fluid domain in the relativistic setting. Let $(\mathbb{R}^{1+3}, m)$ be the Minkowski space-time with metric

$$m = \begin{pmatrix} -1 & \mathbf{0}_{3\times 1}^T \\ \mathbf{0}_{3\times 1} & I_{3\times 3} \end{pmatrix}. \tag{1.1}$$

We use $m_{\alpha\beta}$ to denote entries in the metric m, and $m^{\alpha\beta}$ to denote entries in the inverse metric m^{-1}.

Throughout the dissertation, we adopt the Einstein summation convention, and the indices are raised and lowered with respect to the metric. We use Greek letters α, β etc to denote indices $0, 1, 2, 3$, and Latin letters i, j etc to denote indices $1, 2, 3$.

We consider a fluid domain $\Omega \subset \mathbb{R}^{1+3}$, representing a fluid body in the Minkowski space-time surrounded by vacuum. Before introducing the fluid equation, we define a few quantities that are associated with the fluid. We have the *proper energy density*, denoted by ρ, and the *pressure*, denoted by p, both of which are non-negative functions. We also have the *number density of particles*, denoted by n. Let v be the fluid velocity, which is a unit-length, future-directed time-like vector field. That is,

$$m_{\alpha\beta} v^\alpha v^\beta = -1, \quad \text{and} \quad v^0 > 0.$$

The *particle current* is defined as

$$I^\mu = n v^\mu. \tag{1.2}$$

For a *perfect fluid*, the *energy tensor of matter* is given by

$$T^{\mu\nu} = (\rho + p) v^\mu v^\nu + p (m^{-1})^{\mu\nu}. \tag{1.3}$$

We know that the motion of the fluid is governed by the conservation laws

$$\nabla_\mu T^{\mu\nu} = 0, \quad \text{and} \tag{1.4}$$

$$\nabla_\mu I^\mu = 0. \tag{1.5}$$

In this work, we consider a special class of the perfect fluid, namely *barotropic* fluid, where the pressure p is a function of the density ρ:

$$p = f(\rho). \tag{1.6}$$

Under this setting, the authors in [1] derived a system of equation for the motion of barotropic fluid, which we present below.

Assume that the function $f(\rho)$ is strictly increasing, so that it has an inverse $\rho = f^{-1}(p)$. Assume further the following integral converges:

$$F(p) = \int_0^p \frac{dp'}{p' + \rho} = \int_0^p \frac{dp'}{p' + f^{-1}(p')}.$$

Let

$$V := e^F \cdot v,$$

$$\sigma := e^F,$$

$$G := \frac{p + \rho}{\sigma^2}.$$

Remark 1. Here σ determines F, which determines p, which determines ρ through the function $\rho = f^{-1}(p)$. Hence, G is in fact a function of σ^2 alone.

Observe that since the fluid is surrounded by vacuum, $p = 0$ on the time-like boundary of the fluid domain $\partial\Omega$ and $p \geq 0$ in Ω, so in particular we have that

$$\sigma^2 \equiv 1 \quad \text{on } \partial\Omega \tag{1.7}$$

$$\sigma^2 \geq 1 \quad \text{in } \Omega. \tag{1.8}$$

Then projecting the conservation laws (1.4)-(1.5) onto the space spanned by v and the space that is orthogonal to v respectively, we arrive at the following equations of motion:

$$V^\mu \nabla_\mu V^\nu + \frac{1}{2}\nabla^\nu \sigma^2 = 0 \quad \text{in } \Omega \tag{1.9}$$

$$\nabla_\mu (G(\sigma^2) V^\mu) = 0 \quad \text{in } \Omega. \tag{1.10}$$

We consider the free-boundary problem, so the time-like boundary $\partial\Omega$ evolves according to the re-normalized fluid velocity V, and is also part of the unknown. The boundary conditions, which we call the *liquid boundary condition*, are that on the time-like free boundary $\partial\Omega \subset \mathbb{R}^{1+3}$, $\sigma^2 \equiv 1$ and the fluid velocity is tangent to the boundary $\partial\Omega$:

$$\sigma^2 \equiv 1 \quad \text{on } \partial\Omega \tag{1.11}$$

$$V|_{\partial\Omega} \in \mathcal{T}(\partial\Omega). \tag{1.12}$$

We assume that the initial fluid domain Ω_0 and the initial data (V_0, σ_0^2) are given, and they satisfy the following conditions:

$$V_0^0 \geq c_0 > 0 \quad \text{in } \Omega_0 \tag{1.13}$$

$$-(V_0)_\mu (V_0)^\mu = \sigma_0^2 \geq 1 \quad \text{in } \Omega_0 \tag{1.14}$$

$$\sigma_0^2 \equiv 1 \quad \text{on } \partial\Omega_0 \tag{1.15}$$

$$(\nabla_\mu \sigma_0^2)(\nabla^\mu \sigma_0^2) \geq c_0 > 0 \quad \text{on } \partial\Omega_0 \tag{1.16}$$

for some constant $c_0 > 0$.

From now on, we shall consider the system of equations for the re-normalized velocity V, and drop the notation v. We prove the local well-posedness of the system of equations (1.9)-(1.12) with initial conditions satisfying (1.13)-(1.16).

1.1 History of the Problem

One of the earliest work regarding existence of solutions to the relativistic fluid problem is [9], which models the dynamics of a gaseous star and shows the existence of local solutions under certain conditions on the initial data. The existence of a particular class of solutions to the gaseous model was later established in [16]. Other advances on the well-posedness of the free-boundary relativistic fluid problems are considerably more recent.

In the case of the gaseous model, an a priori estimate was obtained in [7] and [6]. An existence result on the unbounded domain was obtained in [19].

In the case of the liquid model, [12] proved existence of solutions in two space dimensions. The author later derived an a priori estimate and an existence result for a similar kind of liquid model in [14], [13] and [15].

For the case of a three-dimensional free-boundary barotropic fluid, [5] proved an a priori estimate assuming that the initial data is small. In [10], the authors studied the free boundary problem with liquid boundary conditions for the hard phase model, which is an irrotational barotropic fluid for which sound speed is equal to the speed of light. This corresponds to the case $G \equiv 0$ in our equation, and furthermore vorticity is constantly zero. In their work, an priori estimate was established for general initial data in Sobolev spaces, and a local well-posedness result was shown using linearization and iteration.

One of the key ideas in [10] is to reduce the equations (1.9)-(1.12), which is a fully nonlinear system on a free domain, to a quasilinear system. The authors of [10] mentioned that they were motivated by the Newtonian counterpart of the problem, i.e., the water wave problem, which considers the motion of an incompressible and irrotational ideal fluid in a free domain. It was shown in [20, 21] that for such a system, taking a material derivative[1] D_t of the Euler equations results in a quasilinear system of equations:

$$\begin{cases} (D_t^2 + \tilde{a}\nabla_{\tilde{n}})\tilde{V} = -\nabla D_t \tilde{p} & \text{on } \partial\tilde{\Omega}_t \\ \Delta \tilde{V} = 0 & \text{in } \tilde{\Omega}_t \end{cases} . \tag{1.17}$$

Here $D_t := \partial_t + \tilde{V} \cdot \nabla$ is the material derivative. The terms $\tilde{a}$ and $\nabla D_t \tilde{p}$ were written as boundary integrals, and the author proved that (1.17) is a quasilinear equation of hyperbolic type, and a local well-posedness result was established.

In the more general case with nonzero vorticity, [2] considered the system of equation

$$\begin{cases} -\nabla \tilde{p} = \nabla_i \tilde{V} \nabla^i \tilde{V} & \text{in } \tilde{\Omega}_t \\ -\nabla D_t \tilde{p} = \partial_t \tilde{p} \nabla \tilde{V}^i + G(\nabla \tilde{V}, \nabla^{(2)} \tilde{p}) & \text{in } \tilde{\Omega}_t \\ \tilde{p} = 0 & \text{on } \partial\tilde{\Omega}_t \\ D_t \tilde{p} = 0 & \text{on } \partial\tilde{\Omega}_t \end{cases} , \tag{1.18}$$

and replaced the analysis on boundary integrals by the elliptic regularity theory. An a priori estimate was obtained in [2] assuming that the Taylor sign condition holds:

$$-\frac{\partial \tilde{p}}{\partial \tilde{n}} \geq \tilde{c}_0 > 0 \quad \text{on } \partial\tilde{\Omega}_t. \tag{1.19}$$

Motivated by the aforementioned work in [20, 21] and [2], the authors of [10] took

[1] The material derivative measures the rate of change of a quantity along the trajectory of a fluid particle, and is equivalent to ∂_t in the Lagrangian coordinate.

a material derivative

$$D_V = V^\mu \nabla_\mu$$

of the system (1.9)-(1.12), and established the local well-posedness result after analyzing the resulting quasilinear system. The authors remarked that although their quasilinear system looked similar to (1.17), there are a few important differences: one is that the Laplacian Δ is replaced by the d'Alembertian $\square$; the second is that the hyperbolic Dirichlet-to-Neumann map $\nabla_{\tilde{n}}$ is not obviously seen to be positive; the third is that it is not clear if the right hand side consists of lower order terms; and finally it is not obvious in what functional space the Cauchy problem can be solved. These issues were addressed in [10].

In this dissertation, we will extend the results in [10] and prove an a priori estimate as well as a well-posedness result for general barotropic fluid with liquid boundary condition. That is, we work with the general case with non-zero vorticity as well as non-constant function G. We use two approaches to tackle the problem, which are summarized in the next section.

1.2 Results and Key Ideas

We prove that the equations (1.9)-(1.12), which govern the motion of a general free-boundary barotropic fluid with liquid boundary condition, are locally well-posed. That is, we assume that the initial domain Ω_0 as well as the initial data (V_0, σ_0^2) are sufficiently smooth, and prove that there is some time T such that the system of equations (1.9)-(1.12) has a solution on the time interval $[0, T]$. Moreover, the time T depends only on the initial data, and the fluid domain Ω_t as well as the solution (V, σ^2) admit the same regularity as their initial conditions. We will give two proofs for this result; the first one uses mollification and works well when the initial domain

Ω_0 is unbounded, and the second one is adapted from [10], and address the case when the domain Ω_0 is bounded. We will discuss the ideas after we state the main Theorem.

Theorem 1.1. *Let $M = 10^2$. Let Ω_0 be the initial domain. If Ω_0 is unbounded, we consider $\mathcal{D} = \mathbb{R}^3_+ = \{(y_1, y_2, y_3) : y_3 \geq 0\}$, and if Ω_0 is bounded, we consider $\mathcal{D} = B = \{(y_1, y_2, y_3) : y_1^2 + y_2^2 + y_3^2 \leq 1\}$.*

Assume that Ω_0 is smooth enough such that there is a map $Y : \Omega_0 \to \mathcal{D}$ with

$$\nabla_x^{(p)} Y \in L^2(\Omega_0) \quad \forall p \leq (M+2)/2. \tag{1.20}$$

Assume further that the initial data (V_0, σ_0^2) satisfy the conditions (1.13)-(1.16) and possess the following regularity:

$$\nabla^{(p)} D_{V_0}^k V_0 \in L^2(\Omega_0) \quad \forall 2p + k \leq M + 2 \text{ and } k \leq M + 1 \tag{1.21}$$

$$D_{V_0}^k V_0 \in L^2(\partial\Omega_0) \quad \forall k \leq M + 1 \tag{1.22}$$

$$\nabla^{(p)} D_{V_0}^{k+1} \sigma_0^2 \in L^2(\Omega_0) \quad \forall 2p + k \leq M + 2 \text{ and } k \leq M + 1 \tag{1.23}$$

$$D_{V_0}^k \sigma_0^2 \in H_0^1(\Omega_0) \quad \forall k \leq M + 1. \tag{1.24}$$

Then there exist a time $T > 0$, a unique domain $\Omega = \cup_{t \in [0,T]} \Omega_t$, and a unique solution (V, σ^2), such that the system (1.9)-(1.12) is satisfied in $t \in [0, T]$. Moreover, the time T depends only on the initial data, and V, σ^2, Ω enjoy the same regularity of their initial data.

To prove this Theorem, our first step is to borrow the idea of [10] and take material derivatives of the equations in order to obtain a quasilinear system of equations. The derivation is shown in Section 1.3.

[2] The quantity M is the total number of material derivatives that we take in the a priori estimate. The number 10 carries no special significance; any large integer would work

After obtaining the quasilinear system of equations, we use two different approaches to establish the local well-posedness result.

Our first approach is novel, and it enables us to tackle the nonlinear problem directly. We mollify the nonlinear equation, and study the mollified problem as a system of ODEs. Note that since we are dealing with a free-boundary problem, the mollification process is most easily done if the domain is fixed. To this end, we rewrite the quasilinear system in Lagrangian coordinates, so that the domain can be fixed. The Lagrangian formulation is presented in Section 1.4.

After obtaining the Lagrangian formulation, we use mollification to convert the nonlinear system into a system of ODEs. One important ingredient in the mollification process is that we will fill the vacuum with a virtual flow, so that the mollifiers are well-defined. This is discussed in Section 1.5.

We will then prove an a priori estimate that is independent of the mollification parameter, and use the solutions to the mollified equations to construct a solution to the original equation by passing to the limit.

This method works well for an unbounded fluid domain, and we will use it to prove Theorem 1.1 for the case when Ω_0 is unbounded. In fact, an advantage of this approach compared to the second is that it tackles the nonlinear equation directly, and we believe that it will be useful in solving a wide range of free-boundary fluid problems. However, we encountered some difficulties when extending the result to a bounded fluid domain. Thus, we will use a second approach to establish the local well-posedness result for the case of a bounded fluid domain, i.e., Theorem 1.1 for the case when Ω_0 is bounded.

Our second approach is an adaptation from the method in [10] to the general setting. We consider the equation on a linear level, solve the linear equations using

Galerkin approximation, and iterate on these solutions to obtain a solution to the nonlinear equation. Note that our approach is *not* to linearize the equation around a solution; rather, we consider the functions that appear in the differential operator, as well as the functions on the right hand side, as given. We will discuss this approach in more details in section 1.8.

Summarizing, our goal is to prove Theorem 1.1 for both the unbounded domain and the bounded domain. Our plan consists of the following steps:

- derive the quasilinear system of equation;
- prove an a priori estimate for the quasilinear system;
- use approach one (i.e. mollification) to obtain the existence of a solution when the fluid domain Ω_0 is unbounded;
- prove the uniqueness of the solution, which shows Theorem 1.1 when Ω_0 is unbounded;
- use approach two (i.e. linearization and Galerkin approximation) to obtain the existence of a solution when the fluid domain Ω_0 is bounded;
- prove the uniqueness of the solution, which proves Theorem 1.1 when Ω_0 is bounded.

Some of the major difficulties in the proof include the following:

1. It is not clear how to mollify the equation, since the fluid boundary $\partial\Omega$ is also part of the unknown. More precisely, let f be a function that is defined in the domain Ω. Recall that the mollified function $J_\epsilon f(x)$ is essentially a weighted average of the values of f that are close to the point x. Thus, when the point

$x \in \Omega$ lies close to the (unknown) boundary $\partial\Omega$, part of the support of J_ϵ may lie outside of the domain, and it is not clear how to extend f outside of the fluid domain, so in this case, $J_\epsilon f$ is not well-defined. To solve this problem, we will use the Lagrangian formulation to straighten the boundary and define a virtual flow outside of the fluid domain, so that the boundary conditions are maintained, and a mollifier can be defined.

2. When a mollifier is defined, it is not clear if the a priori estimate on the unmollified equation still holds for the mollified equation. To tackle this difficulty, we need to mollify the equation appropriately so that an a priori estimate can still be obtained.

3. With the addition of the vorticity w and the non-constant function G, it is no longer clear if the terms involving w and G are lower-order terms. An analysis is indeed necessary in order to control these terms in the a priori estimate.

In the next section, we achieve the first bullet point; that is, we take a material derivative to obtain a quasilinear system of equations from (1.9)-(1.12).

1.3 Derivation of the Fluid Equation

As discussed in Section 1.1, the authors of [10] considered the system of equations (1.9)-(1.12) in the special case where $G \equiv 1$, the fluid is irrotational, and the sound speed is equal to the speed of light. In this dissertation, we shall treat the more general case with a non-zero vorticity, and with G being a sufficiently smooth function with derivatives satisfying a boundedness condition. We will discuss the details of the boundedness condition on G when presenting the Lagrangian formulation.

This section, following [10], derives a quasilinear system by equations by taking the material derivative of the system (1.9)-(1.12). Recall that V is the re-normalized fluid velocity, and $\sigma^2 = \|V\|^2$.

Throughout the dissertation, we use $\Omega_t \subset \mathbb{R}^{1+3}$ to denote the fluid domain at time t, and $\partial\Omega := \cup_t \partial\Omega_t$ to denote the time-like boundary of the fluid domain. That is, for each t, Ω_t is a three-dimensional manifold, and $\partial\Omega_t$ is its (two-dimensional) boundary.

1.3.1 Equations for Fluid Velocity V

In what follows, we define the material derivative as

$$D_V f := V^\mu \nabla_\mu f. \tag{1.25}$$

Following the approach in [10], we take a material derivative of equation (1.9) to obtain

$$\begin{aligned} 0 &= D_V^2 V^\nu + \frac{1}{2} V^\alpha \nabla_\alpha (\nabla^\nu \sigma^2) \\ &= D_V^2 V^\nu + \frac{1}{2} \nabla^\nu (V^\alpha \nabla_\alpha \sigma^2) - \frac{1}{2} (\nabla^\nu V^\alpha)(\nabla_\alpha \sigma^2). \end{aligned} \tag{1.26}$$

Let the space-time normal vector n be the normal vector to $\partial\Omega$ under the metric m. By equation (1.11), we know that $\nabla\sigma^2$ is parallel to n. Let

$$a := \sqrt{(\nabla_\mu \sigma^2)(\nabla^\mu \sigma^2)}, \tag{1.27}$$

then we can write

$$\nabla \sigma^2 = -an.$$

Using this notation, we see that on the boundary $\partial\Omega$, the equation (1.26) can be written as

$$D_V^2 V^\nu + \frac{1}{2} a n_\alpha (\nabla^\nu V^\alpha) = D_V^2 V^\nu - \frac{1}{2} (\nabla_\alpha \sigma^2)(\nabla^\nu V^\alpha) = -\frac{1}{2} \nabla^\nu (D_V \sigma^2).$$

We would like to write the term $an_\alpha(\nabla^\nu V^\alpha)$ as a normal derivative on $\partial\Omega$. To this end, define the vorticity as

$$w^{\alpha\beta} = \nabla^\alpha V^\beta - \nabla^\beta V^\alpha, \tag{1.28}$$

then the preceding equation becomes

$$D_V^2 V^\nu + an_\alpha(\nabla^\alpha V^\nu) = D_V^2 V^\nu - \frac{1}{2}(\nabla_\alpha \sigma^2)(\nabla^\alpha V^\nu) \tag{1.29}$$

$$= -\frac{1}{2}\nabla^\nu(D_V\sigma^2) + \frac{1}{2}w^{\mu\alpha}\nabla_\alpha\sigma^2 \quad \text{on } \partial\Omega. \tag{1.30}$$

This is the boundary equation for V.

The interior equation of V is derived from (1.10). By (1.10), we know that

$$G\nabla_\mu V^\mu + V^\mu \nabla_\mu G = 0. \tag{1.31}$$

Taking one more derivative, we obtain the d'Alembertian operator, so the interior equation of V is the following wave equation with nonlinearity:

$$\begin{aligned} \Box V^\nu &= \nabla_\mu \nabla^\mu V^\nu \\ &= \nabla_\mu(\nabla^\nu V^\mu + w^{\mu\nu}) \\ &= \nabla^\nu\left(-\frac{V^\mu\nabla_\mu G}{G}\right) + \nabla_\mu w^{\mu\nu} \\ &= -\nabla^\nu(D_V \log G) + \nabla_\mu w^{\mu\nu}. \end{aligned} \tag{1.32}$$

Notice that if $G \equiv 1$ and $w^{\alpha\beta} \equiv 0$, our equations coincide with the equations derived in [10].

1.3.2 Equations for σ^2

Recall that the boundary condition for σ^2 is simply $\sigma^2 \equiv 1$ on $\partial\Omega$.

The interior equation for σ^2 follows from (1.9) and (1.31). As with V, the interior equation of σ^2 is also a nonlinear hyperbolic equation, albeit with a different wave operator:

$$\begin{aligned}\nabla_\nu \nabla^\nu \sigma^2 &= -2\nabla_\nu (V^\mu \nabla_\mu V^\nu) \\ &= -2(\nabla_\nu V^\mu)(\nabla_\mu V^\nu) - 2D_V(\nabla_\nu V^\nu) \\ &= -2(\nabla_\nu V^\mu)(\nabla_\mu V^\nu) - 2D_V\left(-\frac{V^\mu \nabla_\mu G}{G}\right) \\ &= -2(\nabla_\nu V^\mu)(\nabla_\mu V^\nu) + 2D_V^2(\log G).\end{aligned}$$

Note that since G is a function of σ^2, the last term on the right hand side also involves two derivatives of σ^2, and is not a lower order term. So the operator on σ^2 is in fact $\Box\sigma^2 - 2D_V^2(\log G)$. We will compute what $-2D_V^2(\log G)$ equates to, in terms of σ^2, when discussing the Lagrangian formulation. Our assumption on the function G is that the operator $\Box\sigma^2 - 2D_V^2(\log G)$ is a hyperbolic operator on σ^2. The precise condition on G will be presented in Section 1.4.

In summary, the equations for V and σ read:

$$\begin{cases} D_V^2 V^\mu + \frac{1}{2} a \cdot \nabla_n V^\mu = \frac{1}{2} w^{\mu\alpha} \nabla_\alpha \sigma^2 - \frac{1}{2}\nabla^\mu (D_V \sigma^2) & \text{on } \partial\Omega \\ \Box V^\mu = \partial_\alpha w^{\alpha\mu} - \partial^\mu D_V \log G & \text{in } \Omega. \end{cases} \tag{1.33}$$

$$\begin{cases} \Box\sigma^2 - 2D_V^2(\log G) = -2\nabla_\mu V^\alpha \nabla_\alpha V^\mu & \text{in } \Omega \\ \sigma^2 \equiv 1 & \text{on } \partial\Omega. \end{cases} \tag{1.34}$$

1.3.3 Equations for $D_V \sigma^2$

For reasons that will become clear in the energy estimate, we will in fact work with $D_V\sigma^2$ rather than σ^2. To this end, we shall take one D_V derivative of 1.34. We use

$$[A, B] = AB - BA$$

to denote the commutator of operators A and B.

On the boundary, since V is tangent to $\partial\Omega$, we know that $D_V\sigma^2 \equiv 0$ on $\partial\Omega$. In the interior, we take one D_V derivative to $\Box\sigma^2$ and obtain

$$\begin{aligned}\Box D_V\sigma^2 =&[\Box, D_V]\sigma^2 + D_V(\Box\sigma^2)\\ =&2(\nabla^\mu V^\nu)(\nabla_\mu\nabla_\nu\sigma^2) + 2D_V^3(\log G) - 2V^\nu\nabla_\nu(\nabla_\mu V^\alpha\nabla_\alpha V^\mu).\end{aligned}$$

The last term, which involves two derivatives of V such as $\nabla_\nu\nabla_\mu V^\alpha$, is troublesome from the energy estimate point of view, since we will see that $D_V\sigma^2$ and V enjoy the same regularity, which means that $\nabla^{(2)}V$ is not of lower order as compared with $\Box D_V\sigma^2$. We will put this term into a nicer form by commuting $V^\nu\nabla_\nu = D_V$ with ∇_μ, and use the relation between σ^2 and V, namely equation (1.9), to convert D_VV into derivatives of σ^2:

$$\begin{aligned}\Box D_V\sigma^2 =&2(\nabla^\mu V^\nu)(\nabla_\mu\nabla_\nu\sigma^2) + 2D_V^3(\log G) + 4(\nabla_\mu V^\nu)(\nabla_\nu V^\alpha)(\nabla_\alpha V^\mu)\\ &- 2(\nabla_\mu(D_V V^\alpha))(\nabla_\alpha V^\mu) - 2(\nabla_\alpha(D_V V^\mu))(\nabla_\mu V^\alpha)\\ =&4(\nabla^\mu V^\nu)(\nabla_\mu\nabla_\nu\sigma^2) + 2D_V^3(\log G) + 4(\nabla_\mu V^\nu)(\nabla_\nu V^\alpha)(\nabla_\alpha V^\mu).\end{aligned}$$

We have converted the terms involving $\nabla^{(2)}V$ into $\nabla^{(2)}\sigma^2$, which is easily seen to be of lower order than $\nabla^{(2)}D_V\sigma^2$. As before, since G involves σ^2, the term $D_V^3\log G$ is in fact not a lower order term, and we will compute the exact formula for $D_V^3(\log G)$ as a function of σ^2 when we describe the Lagrangian formulation of the equation.

In summary, the system of equation for $D_V\sigma^2$ is:

(1.35)

$$\begin{cases}\Box D_V\sigma^2 - 2D_V^3(\log G) =4(\nabla^\mu V^\nu)(\nabla_\mu\nabla_\nu\sigma^2) + 4(\nabla_\mu V^\nu)(\nabla_\nu V^\alpha)(\nabla_\alpha V^\mu) & \text{in } \Omega\\ D_V\sigma^2 \equiv 0 & \text{on } \partial\Omega.\end{cases}$$

1.3.4 Equations for Vorticity w

Finally, we derive the equation of the vorticity $w^{\mu\nu}$:

$$\begin{aligned} D_V w^{\mu\nu} &= D_V\left(\nabla^\mu V^\nu - \nabla^\nu V^\mu\right) \\ &= [D_V, \nabla^\mu]V^\nu - [D_V, \nabla^\nu]V^\mu + \nabla^\mu(D_V V^\nu) - \nabla^\nu(D_V V^\mu). \end{aligned} \tag{1.36}$$

By equation (1.9),

$$\nabla^\mu(D_V V^\nu) = -\frac{1}{2}\nabla^\mu\nabla^\nu\sigma^2 = \nabla^\nu(D_V V^\mu),$$

so the last two terms in (1.36) cancel out. We thus have

$$\begin{aligned} D_V w^{\mu\nu} &= [D_V, \nabla^\mu]V^\nu - [D_V, \nabla^\nu]V^\mu \\ &= -(\nabla^\mu V_\alpha)(\nabla^\alpha V^\nu) + (\nabla^\nu V_\alpha)(\nabla^\alpha V^\mu) \\ &= -(\nabla^\mu V_\alpha)w^{\alpha\nu} - (\nabla^\mu V_\alpha)(\nabla^\nu V^\alpha) + (\nabla^\nu V_\alpha)w^{\alpha\mu} + (\nabla^\nu V_\alpha)(\nabla^\mu V_\alpha) \\ &= -(\nabla^\mu V_\alpha)w^{\alpha\nu} + (\nabla^\nu V_\alpha)w^{\alpha\mu}. \end{aligned}$$

Therefore, the equation for the vorticity w is

$$D_V w^{\mu\nu} = -(\nabla^\mu V_\alpha)w^{\alpha\nu} + (\nabla^\nu V_\alpha)w^{\alpha\mu}. \tag{1.37}$$

1.3.5 Initial Data

We have presented the equations that $V, \sigma^2, D_V\sigma^2, w$ satisfy. The next step in formulating the Cauchy problem is to specify the initial data.

From the formulation of the problem, we see that the interior equations for u and Λ are some hyperbolic equations. Hence, we need a set of prescribed initial data containing V_0 and its time derivative. By equation (1.9), we know that

$$D_V V^\nu = -\frac{1}{2}\nabla^\nu\sigma^2,$$

so in fact, prescribing σ^2 at $t = 0$ is equivalent to prescribing the material derivative V. Thus, the initial data are the initial fluid domain Ω_0 and functions (V_0, σ_0^2) such that

$$\begin{cases} V_0^0 \geq c_0 > 0 & \text{in } \Omega_0 \\ -(V_0)_\mu (V_0)^\mu = \sigma_0^2 \geq 1 & \text{in } \Omega_0 \\ \sigma_0^2 \equiv 1 & \text{on } \partial\Omega_0 \\ (\nabla_\mu \sigma_0^2)(\nabla^\mu \sigma_0^2) \geq c_0 > 0 & \text{on } \partial\Omega_0 \end{cases} \tag{1.38}$$

for some positive constant $c_0 > 0$.

In summary, in this section, we derived the system of equations for $V, \sigma^2, D_V\sigma^2, w$ in the fluid domain Ω and on the boundary $\partial\Omega$. The goal of this dissertation is to establish well-posedness of this system of equations. In particular, to establish the existence of solution, we will appeal to the existence theory of ordinary differential equations (ODE), which requires us to approximate this system of partial differential equations so that the equations become an ODE. We will use mollification in the case of the unbounded domain, and Galerkin approximation in the case of a bounded domain, to convert the partial differential equations to a system of ODE. Both mollification and Galerkin approximation work the best when one has a fixed domain, which motivates us to consider the Lagrangian formulation of the problem, so that for each time t, the fluid domain $\Omega_t \subset \mathbb{R}^{1+3}$ is transformed into a fixed shape. This is the goal of the next section.

1.4 Fluid Equation in Lagrangian Coordinates

In this section, we will derive the system of equations for $V, \sigma^2, D_V\sigma^2$ and w in the Lagrangian coordinates. The resulting equations will be the same for both the unbounded and the bounded domain, so we use $\mathcal{D}$ to denote the domain of the Lagrangian variable. Specifically, when the initial domain Ω_0 is unbounded, we use $\mathcal{D} = \mathbb{R}^3_+ := \{(y_1, y_2, y_3) : y_3 \geq 0\}$; when Ω_0 is bounded, we use $\mathcal{D} = B := \{(y_1, y_2, y_3) : y_1^2 + y_2^2 + y_3^2 \leq 1\}$.

We first present the change of coordinate formula. Assume for now that $[0, T]$ is some time interval in which V exists, and we write $I = [0, T]$. Consider $X : I \times \mathcal{D} \to \Omega$, defined as the solution of

$$\partial_t X^j(t, y) = \frac{V^j}{V^0}(t, X(t, y)). \tag{1.39}$$

Then $X(t, \cdot) : \mathcal{D} \to \Omega_t$, so $X(t, \cdot)^{-1}$ will transform the fluid domain Ω_t at time t into the Lagrangian domain with a fixed shape $\mathcal{D}$. Our next goal is then to apply the change of coordinate formula induced by X, and write the systems of equations (1.33), (1.35), (1.37) in terms of the Lagrangian variable $y \in \mathcal{D}$.

The pullback Minkowski metric on $I \times \mathcal{D}$ is

$$\begin{aligned} g = &-\left(1 - \sum_{i=1}^{3} \frac{(V^i)^2}{(V^0)^2} \circ X\right) dt^2 + 2 \sum_{i,\ell=1}^{3} \left(\frac{V^i}{V^0} \circ X \frac{\partial X^i}{\partial y^\ell}\right) dt dy^\ell \\ &+ \sum_{i,k,\ell=1}^{3} \frac{\partial X^i}{\partial y^k} \frac{\partial X^i}{\partial y^\ell} dy^k dy^\ell. \end{aligned} \tag{1.40}$$

We define $g_{\alpha\beta}$ as the components of g, $g^{\alpha\beta}$ as the components of g^{-1}, and $|g| = -\det g$.

Using the pullback metric g, the corresponding d'Alembertian (similar to the Laplace–Beltrami operator, but having a ∂_t component) is:

$$\Box_g f := \frac{1}{\sqrt{g}} \nabla_\alpha \left(\sqrt{g} g^{\alpha\beta} \nabla_\beta f\right). \tag{1.41}$$

The interior equations of $V \circ X$ and $(D_V \sigma^2) \circ X$ will be in terms of this new wave operator.

We define composition by

$$f \circ X := f(t, X(t, y)).$$

Let $u = V \circ X$, $\Sigma^2 := \sigma^2 \circ X$, and $\Lambda := (D_V\sigma^2) \circ X$. And by a slight abuse of notation, we still denote $w \circ X$ by w. Then, changing variables in (1.33), the equation of $u = V \circ X$ becomes

$$
\begin{cases}
(u_0)^2\partial_t^2 u^\nu - \frac{1}{2}g^{\alpha\beta}(\nabla_\alpha \Sigma^2)\nabla_\beta u^\nu = \frac{1}{2}(u^0)^2 w^\nu_\alpha g^{\alpha\beta}\nabla_\beta(\Sigma^2) & \\
\qquad - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha X^\nu \nabla_\beta \Lambda & \text{on } I \times \partial\mathcal{D} \\
\qquad + 2u^0\partial_t u^0 \partial_t u^\nu & \\
\Box_g u^\nu = g^{\alpha\beta}\nabla_\alpha(X^\mu)\nabla_\beta(w^\nu_\mu) - g^{\alpha\beta}\nabla_\alpha X^\nu \nabla_\beta\left((\log G)'\Lambda\right) & \text{in } I \times \mathcal{D}
\end{cases}
\tag{1.42}
$$

To simplify the notation, we will define the operator on $\partial\mathcal{D}$ as

$$
\mathcal{P}f := (u_0)^2\partial_t^2 f - \frac{1}{2}(\nabla_\alpha \Sigma^2)g^{\alpha\beta}\nabla_\beta f. \tag{1.43}
$$

For the purpose of the energy estimate, we shall work with $\Lambda = (D_V\sigma^2) \circ X$ instead of Σ^2. Recall that $D_V^3(\log G)$ is not a lower order term, so we need to specify the new wave operator for Λ.

We define the metric h to be:

$$
h^{\alpha\beta} := \begin{cases}
g^{00} - 2(u^0)^2(\log G)' & \alpha, \beta = 0 \\
g^{\alpha\beta} & \text{otherwise}
\end{cases}. \tag{1.44}
$$

Upon changing the coordinates, the equation (1.35) becomes

$$
\begin{cases}
\Lambda \equiv 0 & \text{on } I \times \partial\mathcal{D} \\
\Box_h \Lambda = 4g^{\alpha\beta}(\partial_\beta u^\nu)\partial_\alpha\left(m_{\mu\nu}g^{\gamma\delta}(\partial_\delta X^\nu)(\partial_\gamma \Sigma^2)\right) & \\
\qquad + 4m_{\rho\nu}m_{\nu\kappa}g^{\alpha\beta}g^{\gamma\delta}(\partial_\delta X^\kappa)(\partial_\alpha u^\nu)(\partial_\beta u^\mu)(\partial_\gamma u^\rho) & \text{in } I \times \mathcal{D} \\
\qquad + 2\partial_t u^0(\log G)''\Lambda^2 + u^0(\log G)^{(3)}\Lambda^3 & \\
\qquad - \partial_t u^0(\log G)'\Lambda^2 - u^0\partial_t u^0(\log G)'\partial_t\Lambda &
\end{cases}. \tag{1.45}
$$

In order for the system (1.45) to be hyperbolic, we need to assume further that $h^{00} < 0$ initially. That is, the initial data satisfies

$$
-\left(1 - \sum_{i=1}^{3}\frac{(V^i)^2}{(V^0)^2}\right) - 2(V^0)^2 \cdot (\log G)' \leq -2c < 0 \quad \text{at } t = 0 \tag{1.46}
$$

for some constant $c > 0$. The control on higher order derivatives on V will then guarantee that (1.46) is strictly satisfied within some time interval.

Recall that $\Lambda = (D_V \sigma^2) \circ X$, so the equation for Σ^2 is simply

$$\partial_t \Sigma^2 = \frac{1}{u^0} \Lambda \quad \text{in } I \times \mathcal{D}. \tag{1.47}$$

Remark 2. Since $\Lambda \equiv 0$ on $I \times \partial\mathcal{D}$, in particular we have $\Sigma^2 \equiv 1$ on $I \times \partial\mathcal{D}$, so the original boundary condition for Σ^2 is still satisfied.

Finally, the equation for w is

$$\partial_t w^{\mu\nu} = \frac{1}{u^0} \left(-g^{\delta\gamma} (\nabla_\delta X^\mu)(\nabla_\gamma u_\alpha) w^{\alpha\nu} + g^{\delta\gamma} (\nabla_\delta X^\nu)(\nabla_\gamma u_\alpha) w^{\alpha\mu} \right). \tag{1.48}$$

In summary, in this section, we applied a change of coordinate formula to convert the systems of equation (1.33), (1.35), (1.37) into a system in terms of the Lagrangian variable y, which is defined on a fixed domain $y \in \mathcal{D}$. This enables us to either mollify the equation on $\mathcal{D} = \mathbb{R}^3_+$ or apply Galerkin approximation on $\mathcal{D} = B$.

We will describe the mollification process on the unbounded domain $\mathcal{D} = \mathbb{R}^3_+$ in the next section, since the mollifier follows naturally from the Lagrangian formulation. We will postpone the Galerkin approximation process in Chapter IV, since it involves linearization and the weak formulation, which takes a longer section to describe in details.

1.5 Mollified Equation in the Unbounded Domain

Recall that our strategy of proof for the unbounded domain is to consider the mollified equations, so that the system becomes an ODE system. In this section, we derive the mollified version of equations (1.39)-(1.48).

1.5.1 Mollified Operators

To start with, we define the family of mollifiers $(J_\epsilon)_{\epsilon>0}$ to be the frequency cut off at $1/\epsilon$; that is, for $f \in L^2(\mathbb{R}^3)$,

$$J_\epsilon f(x) := \int_{|\xi|<1/\epsilon} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

We define the mollified operators accordingly:

$$\begin{aligned}
&\tilde{\nabla}_0 f = \nabla_0 f = \partial_t f \\
&\tilde{\nabla}_j f = \partial_j J_\epsilon f \\
&\tilde{\nabla}_g^\alpha f = g^{\alpha\beta}\tilde{\nabla}_\beta f \\
&P^\epsilon f = (J_\epsilon u_0)^2 \partial_t^2 f - \frac{1}{2}(\tilde{\nabla}_\alpha \Sigma^2)\cdot J_\epsilon \left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right) \\
&\tilde{\Box}_g f = \tilde{\nabla}_\alpha \left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right) + \frac{1}{2}(\tilde{\nabla}_\alpha \log|g|) g^{\alpha\beta}\nabla_\beta J_\epsilon f.
\end{aligned}$$

It is clear that when $J_\epsilon = \mathrm{Id}$, the operators are equal to their un-mollified counterparts. We will describe the mollified equations using these mollified operators, but before presenting the equations, note that our functions u, Λ etc are defined on $\mathcal{D} = \mathbb{R}^3_+$, whereas J_ϵ operates on functions that are defined on $\mathbb{R}^3$. Hence, in order for the mollified operators to make sense, we first need to extend our functions to be defined on $\mathbb{R}^3$. This is our next topic.

1.5.2 Odd and Even Extensions

Before changing all the operators to the mollifier operators above, we need to extend our functions' domain from $\mathbb{R}^3_+$ to $\mathbb{R}^3$, so that the mollification makes sense. The strategy is to fill the lower half space with a virtual fluid body, whose flow are compatible with the actual fluid domain on the boundary $\partial\mathbb{R}^3_+$. This virtual flow is defined as follows:

$$u(t, y_1, y_2, y_3) := u(t, y_1, y_2, -y_3) \quad \forall y_3 < 0, \tag{1.49}$$

$$\Lambda(t, y_1, y_2, y_3) := -\Lambda(t, y_1, y_2, -y_3) \quad \forall y_3 < 0. \tag{1.50}$$

The preceding equations imply that

$$X(t, y_1, y_2, y_3) = X(t, y_1, y_2, -y_3) \quad \forall y_3 < 0, \tag{1.51}$$

$$\Sigma^2(t, y_1, y_2, y_3) = 2 - \Sigma^2(t, y_1, y_2, -y_3) \quad \forall y_3 < 0. \tag{1.52}$$

In sum, u is extended to be an *even* function with respect to y_3, and Λ is extended to be an *odd* function with respect to y_3. Such a distinction is picked so that the boundary conditions for the virtual and the actual flows are compatible as $y_3 \to 0$.

1.5.3 Mollified Equations on the Unbounded Domain

Now we are ready to present the mollified equations. The idea is, on the right hand side of the equations, we replace the derivative operators with the mollified derivative operators; on the left hand side, however, the mollification is more delicate, since we need to mollify appropriately so that the energy estimate would close. The mollified operators $\mathcal{P}^\epsilon$ and $\tilde{\Box}_g$ are designed so that the energy estimate follows naturally. We will see some insights on the definition of $\mathcal{P}^\epsilon$ and $\tilde{\Box}_g$ as we discuss the toy models later.

Note that we leave the ∂_t^2 derivatives in both $\mathcal{P}^\epsilon$ and $\tilde{\Box}_g$ un-mollified, so that the system remains an ODE system. The precise formulation of the ODE system will be specified in Chapter III.

Now we present the mollified equations.

The equation for X is

$$\partial_t X^j(t, y) = J_\epsilon \frac{J_\epsilon u^j}{J_\epsilon u^0}(t, y). \tag{1.53}$$

The metric g is:

(1.54)

$$g = -\left(1 - \sum_{i=1}^{3} \frac{(J_\epsilon u^i)^2}{(J_\epsilon u^0)^2}\right) dt^2 + 2\sum_{i,\ell=1}^{3} \left(\frac{J_\epsilon u^i}{J_\epsilon u^0}\frac{\partial X^i}{\partial y^\ell}\right) dtdy^\ell + \sum_{i,k,\ell=1}^{3} \left(\frac{\partial X^i}{\partial y^k}\frac{\partial X^i}{\partial y^\ell}\right) dy^k dy^\ell.$$

And the metric h is defined as before:

$$h^{\alpha\beta} := \begin{cases} g^{00} - 2(u^0)^2(\log G)' & \alpha,\beta = 0 \\ g^{\alpha\beta} & \text{otherwise} \end{cases}. \tag{1.55}$$

The equations for u become

(1.56)

$$\begin{cases} \mathcal{P}^\epsilon u^\nu = \frac{1}{2}(u^0)^2 w^\nu_\alpha g^{\alpha\beta}\tilde{\nabla}_\beta(\Sigma^2) \\ \qquad - \frac{1}{2}g^{\alpha\beta}\tilde{\nabla}_\alpha X^\nu \tilde{\nabla}_\beta \Lambda + 2u^0\partial_t u^0 \partial_t u^\nu & \text{on } I\times\partial\mathbb{R}^3_+ \\ \tilde{\Box}_g u^\nu = g^{\alpha\beta}\tilde{\nabla}_\alpha(X^\mu)\tilde{\nabla}_\beta(w^\nu_\mu) \\ \qquad - g^{\alpha\beta}(\tilde{\nabla}_\alpha X^\nu)\left((\log G)''(\tilde{\nabla}_\beta\Sigma^2)\Lambda + (\log G)'(\tilde{\nabla}_\beta\Lambda)\right) & \text{in } I\times\mathbb{R}^3_+. \end{cases}$$

And the equation for Λ is

$$\begin{cases} \Lambda \equiv 0 & \text{on } I\times\partial\mathbb{R}^3_+ \\ \tilde{\Box}_h \Lambda = 4g^{\alpha\beta}(\tilde{\nabla}_\beta u^\nu)\tilde{\nabla}_\alpha\left(m_{\mu\nu}g^{\gamma\delta}(\tilde{\nabla}_\delta X^\mu)(\tilde{\nabla}_\gamma\Sigma^2)\right) \\ \qquad + 4m_{\rho\nu}m_{\nu\kappa}g^{\alpha\beta}g^{\gamma\delta}(\tilde{\nabla}_\delta X^\kappa)(\tilde{\nabla}_\alpha u^\nu)(\tilde{\nabla}_\beta u^\mu)(\tilde{\nabla}_\gamma u^\rho) + & \text{in } I\times\mathbb{R}^3_+ \\ \qquad + 2\partial_t u^0(\log G)''\Lambda^2 + u^0(\log G)^{(3)}\Lambda^3 \\ \qquad - \partial_t u^0(\log G)'\Lambda^2 - u^0\partial_t u^0(\log G)'\partial_t\Lambda \end{cases}. \tag{1.57}$$

The equation for Σ^2 is

$$\partial_t\Sigma^2 = J_\epsilon\left(\frac{1}{J_\epsilon u^0}J_\epsilon\Lambda\right) \qquad \text{in } I\times\mathbb{R}^3_+. \tag{1.58}$$

Remark 3. Observe that J_ϵ is radially symmetric. Since we extend Λ as an odd function across $\partial\mathbb{R}^3_+$, we know that $J_\epsilon\Lambda \equiv 0$ on $I\times\partial\mathbb{R}^3_+$ as well. Thus the boundary condition $\Sigma^2 \equiv 1$ on $I\times\partial\mathbb{R}^3_+$ is still valid.

Finally, the equation for w is

$$\partial_t w^{\mu\nu} = J_\epsilon \left[\frac{1}{J_\epsilon u^0} \left(-g^{\delta\gamma}(\tilde{\nabla}_\delta X^\mu)(\tilde{\nabla}_\gamma u_\alpha) w^{\alpha\nu} + g^{\gamma\delta}(\tilde{\nabla}_\delta X^\nu)(\tilde{\nabla}_\gamma u_\alpha) w^{\alpha\mu} \right) \right]. \tag{1.59}$$

1.6 A Toy Model

Before presenting the strategy of the proof, let us consider a toy model, which motivates the energy that we will use in the a priori estimate.

1.6.1 Toy Model for u

Let us consider the following problem, which is a toy model for the equation of u:

$$\begin{cases} -\partial_t^2 u + \nabla_i \nabla^i J_\epsilon u = f & \text{in } \mathbb{R}^3_+ \\ \partial_t^2 u + n_i J_\epsilon \left(\tilde{\nabla}^i u \right) = g & \text{on } \partial\mathbb{R}^3_+ \end{cases}. \tag{1.60}$$

In what follows, assume that u, f, g are sufficiently smooth and decay fast enough at infinity.

We multiply the interior equation by $\partial_t u$, integrate on $\mathbb{R}^3_+$, and use integration by parts to obtain

$$\begin{aligned} & -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+} (\partial_t u)^2 \, dy \\ =& -\int_{\mathbb{R}^3_+} \nabla_i \nabla^i J_\epsilon u \cdot \partial_t u \, dy + \int_{\mathbb{R}^3_+} (\partial_t u) \cdot f \, dy \\ =& \int_{\mathbb{R}^3_+} \nabla^i J_\epsilon u \cdot \partial_t \nabla_i u \, dy - \int_{\partial\mathbb{R}^3_+} n_i \nabla^i J_\epsilon u \cdot \partial_t u \, dS + \int_{\mathbb{R}^3_+} (\partial_t u) \cdot f \, dy \\ =& \int_{\mathbb{R}^3_+} \nabla^i J_\epsilon u \cdot \partial_t \nabla_i J_\epsilon u \, dy - \int_{\partial\mathbb{R}^3_+} n_i \nabla^i J_\epsilon u \cdot \partial_t u \, dS + \int_{\mathbb{R}^3_+} (\partial_t u) \cdot f \, dy \\ =& \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+} |\nabla_y J_\epsilon u|^2 \, dy - \int_{\partial\mathbb{R}^3_+} (g - \partial_t^2 u) \cdot (\partial_t u) \, dS + \int_{\mathbb{R}^3_+} (\partial_t u) \cdot f \, dy \\ =& \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+} |\nabla_y J_\epsilon u|^2 \, dy + \frac{1}{2}\frac{d}{dt}\int_{\partial\mathbb{R}^3_+} (\partial_t u)^2 \, dS \end{aligned}$$

$$- \int_{\partial\mathbb{R}^3_+} g \cdot (\partial_t u)\, dS + \int_{\mathbb{R}^3_+} (\partial_t u) \cdot f\, dy.$$

Re-organizing the terms, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^3_+} (\partial_t u)^2 + |\nabla_y J_\epsilon u|^2\, dy + \int_{\partial\mathbb{R}^3_+} (\partial_t u)^2\, dS \right) \\
=& \int_{\partial\mathbb{R}^3_+} g \cdot (\partial_t u)\, dS - \int_{\mathbb{R}^3_+} (\partial_t u) \cdot f\, dy.
\end{aligned}$$

Therefore, if we are able to control $\|g\|_{L^2(\partial\mathbb{R}^3_+)}$ and $\|f\|_{L^2(\mathbb{R}^3_+)}$, then the a priori estimate would follow from Gronwall's inequality. This is the fundamental building block of the energy for u. The precise statement of the energy estimate is in Lemma 2.2.

1.6.2 Toy Model for Λ

For Λ, the toy model has a simpler boundary condition:

$$\begin{cases} -\partial_t^2 \Lambda + \nabla_i \nabla^i J_\epsilon \Lambda = f & \text{in } \mathbb{R}^3_+ \\ \Lambda = 0 & \text{on } \partial\mathbb{R}^3_+ \end{cases}. \tag{1.61}$$

Again, we multiply the interior equation by $\partial_t \Lambda$, integrate on $\mathbb{R}^3_+$, and use integration by parts:

$$\begin{aligned}
&- \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3_+} (\partial_t \Lambda)^2\, dy \\
=& \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3_+} |\nabla_y J_\epsilon \Lambda|^2\, dy - \underbrace{\int_{\partial\mathbb{R}^3_+} n_i (\nabla^i J_\epsilon \Lambda)(\partial_t \Lambda)\, dS}_{(*)} + \int_{\mathbb{R}^3_+} (\partial_t \Lambda) \cdot f\, dy.
\end{aligned}$$

Since $\partial_t \Lambda \equiv 0$ on $\partial\mathbb{R}^3_+$, we know that $(*) \equiv 0$, so the energy estimate reads

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^3_+} (\partial_t \Lambda)^2\, dy + \int_{\mathbb{R}^3_+} |\nabla_y J_\epsilon \Lambda|^2\, dy \right) = - \int_{\mathbb{R}^3_+} (\partial_t \Lambda) \cdot f\, dy.$$

However, recall that the boundary equation for u contains a term involving $\nabla\Lambda$, so having only $\int_{\mathbb{R}^3_+} (\partial_t \Lambda)^2 + |\tilde{\nabla}_y \Lambda|^2\, dy$ in the energy does not provide sufficient control on $\|\nabla\Lambda\|_{L^2(\partial\mathbb{R}^3_+)}$. The remedy is to multiply the interior equation not only by $\partial_t\Lambda$, but also by a term $Q^\mu\tilde{\nabla}_\mu\Lambda$ for some vector field Q. The idea of multiplying by a vector field to obtain the a priori estimate was also used in, for instance, [4] and [10]. The details are discussed in Lemma 2.4.

Similarly, to control $\tilde{\nabla}u$ on the boundary, we also need multiply $\tilde{\Box}_g u$ by not only $\partial_t u$, but also some $Q^\mu\tilde{\nabla}_\mu u$. The details are shown in Lemma 2.5.

1.7 Strategy to Control Lower Order Terms

Judging from the equations (1.56) and (1.57), we see that the right hand sides, which are supposed to be lower order terms, are roughly of the form

$$(\tilde{\nabla}^{p_1}\phi_1)\cdots(\tilde{\nabla}^{p_r}\phi_r)\cdot(\tilde{\nabla}^{p}\psi),$$

where $\psi \in \{u, \Lambda\}$ is the term that contains the most derivatives, and $(\tilde{\nabla}^{p_i}\phi_i)$ are of lower order. Our strategy to control this quantity is to bound the lower order terms in L^∞ and highest order term in L^2:

$$\|(\tilde{\nabla}^{p_1}\phi_1)\cdots(\tilde{\nabla}^{p_r}\phi_r)\cdot(\tilde{\nabla}^{p}\psi)\|_{L^2} \leq \|\tilde{\nabla}^{p_1}\phi_1\|_{L^\infty}\cdots\|\tilde{\nabla}^{p_r}\phi_r\|_{L^\infty}\cdot\|\tilde{\nabla}^{p}\psi\|_{L^2}.$$

From the toy models in the preceding section, however, we are only able to control L^2-based norms of u and Λ, so in order to gain control on L^∞ norms, we will use the Sobolev embedding

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^2}.$$

However, in the toy models, we can only obtain a bound on the H^1 norms of u and Λ. Since we are working with a free-boundary problem, directly taking spatial derivatives to the original equations is not feasible. The way to gain H^2 control is to take extra ∂_t derivatives of (1.56) and (1.57). This then calls for a trade-off of derivatives, which roughly says that if u and Λ have sufficiently many time derivatives, then they are also somewhat smooth in the spatial sense. The precise statement is contained in Proposition 2.13, which states that ∂_t^2 is as "costly" as one ∇_y derivative. Intuitively, one can speculate that a result as such is plausible by looking at the structure of the equation. The boundary equation for u is of the form

$$(u^0)^2 \cdot \partial_t^2 u + a\nabla_n u = \text{lower order terms},$$

where, as we will see, u^0 and a are strictly positive and bounded from below. This suggests that ∂_t^2 is as costly as one ∇ derivative.

One main ingredient that we use when proving this trade-off is the elliptic estimate, which roughly says that

$$\|\tilde{\nabla}_y^{(2)} f\|_{L^2} \lesssim \|\Delta f\|_{L^2}$$

with Δ being the Laplacian (or an elliptic operator in general). If f satisfies the wave equation, that is,

$$\Delta f = \Box f + \partial_t^2 f + \text{lower order terms},$$

then we are able to control $\|f\|_{H^2}$ if we have bounds on $\|\Box f\|_{L^2}$ and $\|\partial_t^2 f\|_{L^2}$. This enables us to obtain control on spatial derivatives from the control on time derivatives, which then allows us to bound the lower order terms in L^∞ norm.

1.8 Outline of the Dissertation

Our goal is to establish the local well-posedness result for the system of equations (1.33)-(1.35). As was discussed in Section 1.1, the free boundary problem for an irrotational barotropic fluid where the sound speed is equal to the speed of light was studied in [10], and a well-posedness result was obtained. In this dissertation, we will establish a local well-posedness result for a general barotropic fluid. We mentioned in Section 1.2 that we will present two approaches: the first is to solve the nonlinear equation directly using mollification, which works well on an unbounded fluid domain; the second is adapted from [10] to the case of a general barotropic fluid on a bounded domain.

For the first approach (i.e. mollifying the nonlinear equation directly), we consider the equations in terms of the Lagrangian variable. We then construct a virtual flow outside of the fluid domain, mollify the equations to obtain a system of ODEs, and construct a solution to the original problem from the solutions to the mollified problem.

This mollification method works well when the original fluid domain is unbounded; though in the case when the fluid domain is bounded, we encountered some difficulties in constructing the virtual flow and mollifying. Thus, we will instead use Galerkin method when the fluid domain is bounded, which constitutes our second approach

In the rest of this section, we will discuss the strategy of our proof in more details. The idea of our proof is based on an a priori estimate, which is followed by proof of existence and uniqueness. We will prove the local well-posedness result on both a bounded domain $\mathcal{D} = B = \{(y_1, y_2, y_3) : y_1^2 + y_2^2 + y_3^2 \leq 1\}$ and an unbounded domain $\mathcal{D} = \mathbb{R}^3_+ = \{(y_1, y_2, y_3) : y_3 \geq 0\}$. Some of the parallel arguments in the two cases of obtaining the a priori estimate and the uniqueness of solution are quite similar and will therefore be omitted for brevity. We will mostly focus on the parts that are different between the two cases. The proof for existence of solutions, however, is very different between the two cases.

In Chapter II, we establish an a priori estimate that will be applicable for both the case of the bounded domain and the case of the unbounded domain. We shall only present the proof for the a priori estimate in the unbounded domain though, since the proof for the case of the bounded domain is completely the same by changing the domain name from $\mathcal{D} = \mathbb{R}^3_+$ to $\mathcal{D} = B$. To establish the existence of solution on $\mathcal{D} = \mathbb{R}^3_+$, we will use mollification, which requires an a priori estimate for the mollified equation. Notice, however, that by setting $J_\epsilon = \mathrm{Id}$, we may regard the a priori estimate on the un-mollified equation as a special case for the a priori estimate on the mollified equation. To avoid such redundancy, we only present the proof of a priori estimate for the mollified equation on the unbounded domain. That is, our Theorem 2.1 is the a priori estimate for the mollified equation on $\mathcal{D} = \mathbb{R}^3_+$. To obtain the a priori estimate for the un-mollified equation on $\mathcal{D} = \mathbb{R}^3_+$, one sets $J_\epsilon = \mathrm{Id}$; to obtain the a priori estimate for the un-mollified equation on $\mathcal{D} = B$, one sets $J_\epsilon = \mathrm{Id}$ and furthermore $\mathcal{D} = B$ in the proof for Theorem 2.1.

Next, Chapter III establishes the existence of solution on the unbounded domain $\mathcal{D} = \mathbb{R}^3_+$. We will show that the system of mollified equations is in fact an ODE, with a Lipschitz continuous right hand side in some suitable space $\mathcal{B}$. We then appeal to the existence result for ODE to establish existence of solution to the mollified system of equations. One can then obtain the existence of solution to the un-mollified system of equations by passing to a subsequence as $\epsilon \to 0$, and showing convergence in some lower regularity norm. We will take a sufficiently large number of derivatives so that the convergence is uniform in space and time, showing that the limit in fact solves the (un-mollified) system of equations in the strong sense. One can then observe that the strong limit has to be equal to the weak limit, which shows that the strong limit enjoys the same regularity as the initial data. This will complete our proof for the existence of solution in the unbounded domain $\mathcal{D} = \mathbb{R}^3_+$.

Our next goal is to prove uniqueness of the solution. The same proof works for both $\mathcal{D} = \mathbb{R}^3_+$ and $\mathcal{D} = B$. The idea is to take the difference between two solutions, and use the a priori estimate to show that the difference vanishes. The details are presented in the second half of Chapter III. The local well-posedness result on an unbounded domain, i.e. Theorem 1.1, is thus complete.

For the case of a bounded domain, recall that the a priori estimate was already proven in Chapter II, and we establish existence using the Galerkin approximation method. Since Galerkin method calls for a linear equation, we first consider the linear version of the equations (i.e. where the wave operator $\Box_g$ and $\Box_h$, as well as the right hand side terms, are given).[3] This enables us to project the equation onto some finite dimensional subspaces of $H^1(B)$, which allows us to, again, appeal to the existence and uniqueness result of ODEs. We can then pass to the limit as the dimensions of these subspaces go to infinity to obtain a solution to the linear equation. The details are discussed in Chapter IV.

Our last step is to show that by iteratively solving the linear equations, the solutions to the linear problems converge. We will furthermore show that this limit solves the nonlinear equation, and enjoys the same regularity as the initial data. This is done in Chapter V. The proof regarding uniqueness is completely identical to the case of the unbounded domain by changing $\mathcal{D} = \mathbb{R}^3_+$ to $\mathcal{D} = B$, so we present the statement and refrain from copying the exact same argument. This will complete our proof for Theorem 1.1.

[3]We emphasize again that our approach here is *not* to linearize around a certain solution. Rather, we consider the functions that appear in the differential operator as well as on the right hand side of the equation as given.

CHAPTER II

A priori Estimate on an Unbounded Domain

In this chapter, we prove an a priori estimate, which, in fact, will work for both $\mathcal{D} = \mathbb{R}^3_+$ and $\mathcal{D} = B$. As discussed in Chapter I, the result is a critical building block for the proof for existence and uniqueness, which we achieve using two different methods for the two types of domains. Specifically, we will use mollification for the unbounded domain $\mathcal{D} = \mathbb{R}^3_+$, and will use Galerkin approximation for the bounded domain $\mathcal{D} = B$. Both cases will share the same a priori estimate. However, the energy estimate for the mollified equation will take a similar form as the un-mollified a priori estimate; to minimize redundancy, we shall prove the a priori estimate here for the *mollified equation*, bearing in mind that for the case $\mathcal{D} = B$, we set $\epsilon = 0$, so that the mollifier equals to the identity operator, and thus all commutators with J_ϵ automatically vanish.

2.1 Definition of the Energy

The goal of this section is to define the energy for which we perform the a priori estimate. Let $k \geq 0$ be an integer, representing the number of time derivatives that we take on the variables u and Λ. We define the energy functionals as follows: $E^\epsilon_k[u], E^\epsilon_{\leq k}[u]$ will be the functionals for u and its time derivatives, $\underline{E}^\epsilon_k, \underline{E}^\epsilon_{\leq k}$ will be the functional for Λ and its time derivatives, and $\mathcal{E}_k^\epsilon[u,\Lambda]$ will be the total energy of u and Λ as well as their time derivatives.

$$
\begin{aligned}
E_k^\epsilon[u](t) &= \int_{\mathcal{D}_t} |g^{00}|\cdot|\partial_t^{k+1}u(t,y)|^2 + g^{ij}(\partial_i J_\epsilon \partial_t^k u)(\partial_j J_\epsilon \partial_t^k u)\, dy + \\
&\quad + \int_{\partial\mathcal{D}_t} |g^{00}|\cdot|\partial_t^{k+1}u(t,y)|^2\, dS \\
\underline{E}_k^\epsilon[\Lambda](t) &= \sup_{0\le\tau\le t}\int_{\mathcal{D}_\tau} |g^{00}|\cdot|\partial_t^{k+1}\Lambda(\tau,y)|^2 + g^{ij}(\partial_i J_\epsilon \partial_t^k \Lambda)(\partial_j J_\epsilon \partial_t^k \Lambda)\, dy + \\
&\quad + \int_0^t\int_{\partial\mathcal{D}_\tau} |g^{00}|\cdot|\partial_t^{k+1}\Lambda(\tau,y)|^2 + g^{ij}(\partial_i J_\epsilon \partial_t^k \Lambda)(\partial_j J_\epsilon \partial_t^k \Lambda)\, dS d\tau \\
E_{\le k}^\epsilon[u](t) &= \sum_{j\le k} E_k^\epsilon[u](t) \\
\underline{E}_{\le k}^\epsilon[\Lambda](t) &= \sum_{j\le k} \underline{E}_j^\epsilon[\Lambda](t) \\
\mathcal{E}_k^\epsilon[u,\Lambda](t) &= \left(\sup_{0\le\tau\le t} E_{\le k}^\epsilon[u](\tau)\right) + \underline{E}_{\le k}^\epsilon[\Lambda](t).
\end{aligned}
$$

Remark 4. Note that the energy functionals for u and Λ are different, as they satisfy different boundary conditions. The motivation was discussed in the toy models in Chapter I. The energy $E_k^\epsilon[u](t)$ contains L^2 norm of $\partial_t^{k+1}u$ on $\partial\mathcal{D}$, whereas the energy $\underline{E}_k^\epsilon[\Lambda](t)$ only contains a weighted L^2 norm of $\partial_t^{k+1}\Lambda$ on $\partial\mathcal{D}$. The difference will be evident as we prove the basic energy estimates for u and Λ.

Remark 5. The energy $\mathcal{E}_k^\epsilon$ depends on not only u and Λ, but also the metric $g^{\alpha\beta}$. We did not explicitly write out this dependency in order to simplify the notation.

We will show in Chapter III that the equations (1.53)-(1.59) are a system of ordinary differential equations, and that for each $\epsilon > 0$, there is a unique solution $(u^\epsilon, \Lambda^\epsilon, w^\epsilon, X^\epsilon, (\Sigma^2)^\epsilon)$ on some time interval $[0, T^\epsilon]$. We need to prove that these solutions exist on some time interval $[0,T]$ where $T > 0$ is independent of ϵ. Then we extract a solution to the original set of equations and prove regularity. The first step is to prove the following a priori estimate which is uniform in ϵ.

Theorem 2.1. *Assume that* u^ϵ, Λ^ϵ, $(\Sigma^2)^\epsilon$, w^ϵ *solve (1.56)-(1.59) on* $[0,T]\times\mathcal{D}$, *with*

$$\mathcal{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](T)\le C_1$$

for some integer M *and constant* C_1. *Denote the energy functional in Sobolev spaces by*

(2.1)

$$\mathfrak{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](T):=\mathcal{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](T)+\sup_{0\le t\le T}\sum_{(k,p):k+2p\le M+2}\left\|\partial_t^k u^\epsilon\right\|^2_{H^p(\mathbb{R}^3_{+t})}+\left\|\partial_t^k\Lambda^\epsilon\right\|^2_{H^p(\mathbb{R}^3_{+t})}.$$

Then there is some polynomial P_M *with non-negative coefficients such that if* $T>0$ *is small (depending only on* C_1 *and* $\mathfrak{E}^\epsilon_M(0)$*), then for all* $t\in[0,T]$,

$$\mathfrak{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](t)\le\mathfrak{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](0)+\int_0^t P_M(\mathfrak{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](\tau))\,d\tau \tag{2.2}$$

In particular, (by, say, Lemma 2.15), we know that there is a time interval $[0,T]$, *where* $T>0$ *depends only on the initial data, such that*

$$\mathfrak{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](T)\lesssim\mathfrak{E}^\epsilon_M[u^\epsilon,\Lambda^\epsilon](0). \tag{2.3}$$

Remark 6. We emphasize that the smallness of T only depends on C_1 and the initial data in the preceding Proposition. This enables us to obtain a uniform bound as $\epsilon\to 0$.

Remark 7. To obtain a uniform bound, we shall fix an arbitrary $\epsilon\ge 0$, and consider a solution $(u^\epsilon,\Lambda^\epsilon)$. To simplify the notation, for the rest of this section we shall drop the dependence on ϵ when there is no risk of confusion. For instance, we will write $(u^\epsilon)^0$ as u^0 to avoid the extra superscript ϵ.

Remark 8. Let f and g be two functions. We use

$$f\overset{\epsilon}{\sim}g$$

to mean that

$$\|f - c\cdot g\|_{L^2(\mathcal{D}_t)} \le P(\mathfrak{E}^\epsilon_M[u^\epsilon, \Lambda^\epsilon](t))$$

for some constant c and some polynomial P with non-negative coefficients. Here c and P are independent of ϵ.

We now prove some energy lemmas that will be applicable to generic functions. These will be the building blocks for showing the a priori estimate in Theorem 2.1.

2.2 Fundamental Energy Lemmas

To obtain the a priori estimate, we first prove a few fundamental Lemmas, which will be applied to $\partial_t^k u$ and $\partial_t^k \Lambda$ to prove Theorem 2.1.

Lemma 2.2 will be applied to u and $\partial_t^k u$.

Lemma 2.2. *Assume $g^{00} < 0$, and $f = f(t, y)$ is a function such that $f \in C^2([0,T]\times \mathcal{D}) \cap L^\infty([0,T], H^2(\mathcal{D}))$, $\partial_t f \in C^1([0,T]\times\mathcal{D}) \cap L^\infty([0,T], H^1(\mathcal{D}))$. When $\mathcal{D} = \mathbb{R}^3_+$, we extend f in an even manner, that is $f(t, y_1, y_2, y_3) := f(t, y_1, y_2, -y_3)$ for $y_3 < 0$, so that J_ϵ is well-defined on f. When $\mathcal{D} = B$, we set $J_\epsilon = \mathrm{Id}$ as usual.*

Then we have

$$\begin{aligned}
(2.4)\qquad & \int_{\mathcal{D}} \tilde{\nabla}_\alpha f\left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\cdot a\partial_t f\, dy \\
=&\partial_t\left[-\frac{1}{2}\int_{\mathcal{D}} a|g^{00}|\left(\partial_t f\right)^2 + g^{ij}(\partial_i J_\epsilon f)(\partial_j J_\epsilon f)\, dy - \int_{\partial\mathcal{D}} u_0^2\left(\partial_t f\right)^2\, dS\right] \\
&+2\int_{\partial\mathcal{D}} \partial_t f\cdot \mathcal{P}^\epsilon f\, dS + 2\int_{\partial\mathcal{D}} u_0\partial_t u_0\left(\partial_t f\right)^2\, dS \\
&-\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f\cdot[\tilde{\nabla}_\alpha, a]\partial_t f\, dy + \frac{1}{2}\int_{\mathcal{D}}\partial_t\left(ag^{\alpha\beta}\right)\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\, dy.
\end{aligned}$$

Re-organizing the terms and integrating with respect to time yields

$$(2.5)\qquad \frac{1}{2}\int_{\mathcal{D}_t} a|g^{00}|\left(\partial_t f\right)^2 + g^{ij}\partial_i J_\epsilon f\partial_j J_\epsilon f\, dy + \int_{\partial\mathcal{D}_t} u_0^2\left(\partial_t f\right)^2\, dS$$

$$
\begin{aligned}
=&\frac{1}{2}\int_{\mathcal{D}_0} a|g^{00}|\left(\partial_t f\right)^2 + g^{ij}\partial_i J_\epsilon f \partial_j J_\epsilon f\, dy + \int_{\partial\mathcal{D}_0} u_0^2\left(\partial_t f\right)^2\, dS \\
&-\int_0^t\int_{\mathcal{D}} \tilde{\Box}_g f\cdot a\partial_t f\, dyd\tau + \frac{1}{2}\int_0^t\int_{\mathcal{D}} (\tilde{\nabla}_\alpha \log|g|) g^{\alpha\beta}\tilde{\nabla}_\beta J_\epsilon f\, dyd\tau \\
&+2\int_0^t\int_{\partial\mathcal{D}} \partial_t f\cdot \mathcal{P}^\epsilon f\, dSd\tau + 2\int_0^t\int_{\partial\mathcal{D}} u_0\partial_t u_0\left(\partial_t f\right)^2\, dSd\tau \\
&-\int_0^t\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f\cdot[\tilde{\nabla}_\alpha, a]\partial_t f\, dyd\tau + \frac{1}{2}\int_0^t\int_{\mathcal{D}} \partial_t\left(ag^{\alpha\beta}\right)\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\, dyd\tau.
\end{aligned}
$$

Proof. Let $k\in C^1([0,T]\times\mathbb{R}^3)\cap L^\infty([0,T],H^1(\mathbb{R}^3))$, $\partial_t k\in L^\infty([0,T],L^2(\mathbb{R}^3))$. We have[1]

$$
\begin{aligned}
\int_{\mathcal{D}}\tilde{\nabla}_\alpha\left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\cdot k\, dy = {}&\partial_t\left[\int_{\mathcal{D}} g^{0\beta}(\tilde{\nabla}_\beta f)k\, dy\right] - \int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f\tilde{\nabla}_\alpha k\, dy \\
&+\int_{\partial\mathcal{D}} J_\epsilon\left(g^{j\beta}\tilde{\nabla}_\beta f\right) kn_j\, dS.
\end{aligned} \tag{2.6}
$$

Consider $k=a\partial_t f$. We further compute each term on the right hand side of (2.6). The second term on the right hand side is

$$
\begin{aligned}
&\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f\tilde{\nabla}_\alpha k\, dy \\
=&\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f[\tilde{\nabla}_\alpha, a](\partial_t f)\, dy + \int_{\mathcal{D}} ag^{\alpha\beta}\tilde{\nabla}_\beta f\partial_t\tilde{\nabla}_\alpha f\, dy \\
=&\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f[\tilde{\nabla}_\alpha, a](\partial_t f)\, dy + \frac{1}{2}\partial_t\left[\int_{\mathcal{D}} ag^{\alpha\beta}\tilde{\nabla}_\beta f\tilde{\nabla}_\alpha f\, dy\right] - \frac{1}{2}\int_{\mathcal{D}}\partial_t(ag^{\alpha\beta})\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\, dy.
\end{aligned}
$$

The last term on the right hand side of (2.6) is (noting that $n_0\equiv 0$)

$$
\begin{aligned}
&\int_{\partial\mathcal{D}} n_j J_\epsilon\left(g^{j\beta}\tilde{\nabla}_\beta f\right)(a\partial_t f)\, dS \\
=&\int_{\partial\mathcal{D}} an_\alpha J_\epsilon\left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)(\partial_t f)\, dS \\
=&\int_{\partial\mathcal{D}} (\partial_t f)\cdot 2\left(\mathcal{P}^\epsilon f - (u^0)^2\partial_t^2 f\right)\, dS \\
=&2\int_{\partial\mathcal{D}} (\partial_t f)(\mathcal{P}^\epsilon f)\, dS - \partial_t\left[\int_{\partial\mathcal{D}} (u^0)^2(\partial_t f)^2\, dS\right] + \int_{\partial\mathcal{D}} 2u^0(\partial_t u^0)(\partial_t f)^2\, dS.
\end{aligned}
$$

[1] Recall that $\tilde{\nabla}_0=\partial_t$ and $\tilde{\nabla}_j=J_\epsilon\nabla_j$.

Substituting back into equation (2.6), we have

$$\begin{aligned}
&\int_{\mathcal{D}} \tilde{\nabla}_\alpha \left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)(a\partial_t f)\,dy \\
=&\partial_t\left[\int_{\mathcal{D}} a g^{0\beta}\tilde{\nabla}_\beta f \partial_t f\,dy - \frac{1}{2}\int_{\mathcal{D}} a g^{\alpha\beta}\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\,dy - \int_{\partial\mathcal{D}} (u^0)^2(\partial_t f)^2\,dS\right] \\
&- \int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f[\tilde{\nabla}_\alpha, a](\partial_t f)\,dy + \frac{1}{2}\int_{\mathcal{D}} \partial_t(a g^{\alpha\beta})\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\,dy \\
&+ 2\int_{\partial\mathcal{D}} (\partial_t f)(\mathcal{P}^\epsilon f)\,dS + \int_{\partial\mathcal{D}} 2u^0\partial_t u^0(\partial_t f)^2\,dS.
\end{aligned}$$

Re-organizing the terms in the square bracket, we obtain equation (2.4). Integrating with respect to t, and substituting

$$\tilde{\Box}_g f = \tilde{\nabla}_\alpha\left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right) + \frac{1}{2}\left(\tilde{\nabla}_\alpha \log|g|\right) g^{\alpha\beta}\tilde{\nabla}_\beta J_\epsilon f,$$

we obtain equation (2.5). □

Therefore, we have established the fundamental energy estimate for u and $\partial_t^k u$. The next two Lemmas will be applied to Λ and $\partial_t^k \Lambda$.

Lemma 2.3. *Assume $g^{00} < 0$, $f \in C^2([0,T]\times\mathcal{D}) \cap L^\infty([0,T], H^2(\mathcal{D}))$, $\partial_t f \in C^1([0,T]\times\mathcal{D}) \cap L^\infty([0,T], H^1(\mathcal{D}))$, and $f \equiv 0$ on $[0,T]\times\partial\mathcal{D}$. When $\mathcal{D} = \mathbb{R}^3_+$, we extend f in an odd manner to the lower half space, that is $f(t, y_1, y_2, y_3) := -f(t, y_1, y_2, -y_3)$ for $y_3 < 0$. When $\mathcal{D} = B$, we set $J_\epsilon = \mathrm{Id}$ as usual.*

Then

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{D}_t} (-g^{00})\cdot(\partial_t f)^2 + g^{ij}\tilde{\nabla}_i f\tilde{\nabla}_j f\,dy \\
=&\frac{1}{2}\int_{\mathcal{D}_0} (-g^{00})\cdot(\partial_t f)^2 + g^{ij}\tilde{\nabla}_i f\tilde{\nabla}_j f\,dy - \int_0^t\int_{\mathcal{D}} \tilde{\Box}_g f\cdot\partial_t f\,dyd\tau \\
&+ \frac{1}{2}\int_0^t\int_{\mathcal{D}}\left(\tilde{\nabla}_\alpha\log|g|\right) g^{\alpha\beta}\tilde{\nabla}_\beta J_\epsilon f\,dyd\tau + \frac{1}{2}\int_0^t\int_{\mathcal{D}} (\partial_t g^{\alpha\beta})\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\,dyd\tau.
\end{aligned}$$

Proof. The result follows from multiplying $\tilde{\Box}_g f$ by $\partial_t f$ and integration by parts. The computation is similar to the proof of Lemma 2.2, except the treatment of the boundary terms. Since f is a constant on the boundary, we know that $\partial_t f \equiv 0$ on $[0,T]\times\partial\mathcal{D}$, so

$$\begin{aligned}
&\int_{\mathcal{D}} \tilde{\nabla}_\alpha \left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\cdot(\partial_t f)\,dy\\
=&\partial_t\left[\int_{\mathcal{D}} g^{0\beta}(\tilde{\nabla}_\beta f)(\partial_t f)\,dy\right]-\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f\tilde{\nabla}_\alpha(\partial_t f)\,dy+\underbrace{\int_{\partial\mathcal{D}} J_\epsilon\left(g^{j\beta}\tilde{\nabla}_\beta f\right)(\partial_t f)n_j\,dS}_{=0}\\
=&\partial_t\left[\int_{\mathcal{D}} g^{0\beta}(\tilde{\nabla}_\beta f)(\partial_t f)\,dy-\frac{1}{2}\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f\tilde{\nabla}_\alpha f\,dy\right]+\frac{1}{2}\int_{\mathcal{D}}(\partial_t g^{\alpha\beta})\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\,dy.
\end{aligned}$$

Integrating with respect to t and substituting the formula for $\tilde{\Box}_g$ as before, we obtain the desired result. $\Box$

For reasons that will be clear as we complete the energy estimate, the control on $\partial_t^k\Lambda$ in the interior is not sufficient. We use the following Lemma to control $\partial_t^k\Lambda$ on the boundary.

Lemma 2.4. *Assume $g^{00}<0$, $f\in C^2([0,T]\times\mathcal{D})\cap L^\infty([0,T],H^2(\mathcal{D}))$, $\partial_t f\in C^1([0,T]\times\mathcal{D})\cap L^\infty([0,T],H^1(\mathcal{D}))$, and $f\equiv 0$ on $[0,T]\times\partial\mathcal{D}$. When $\mathcal{D}=\mathbb{R}^3_+$, we extend f in an odd manner to the lower half space, that is $f(t,y_1,y_2,y_3):=-f(t,y_1,y_2,-y_3)$ for $y_3<0$. When $\mathcal{D}=B$, we set $J_\epsilon=\mathrm{Id}$ as usual.*

Then there is a future-directed and time-like vectorfield Q that does not depend on f, such that

$$\begin{aligned}
&\sup_{0\le\tau\le t}\left(\int_{\mathcal{D}_\tau}|\partial_t f|^2+|\tilde{\nabla}_y f|^2\,dy\right)+\int_0^t\int_{\partial\mathcal{D}}\left(\tilde{\nabla}_\alpha f g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\,dS\,d\tau\\
\lesssim&\int_{\mathcal{D}_0}|\partial_t f|^2+|\tilde{\nabla}_y f|^2\,dy+\left|\int_0^t\int_{\mathcal{D}}(\tilde{\Box}_g f)(Q^\mu\tilde{\nabla}_\mu f)\,dyd\tau\right|\\
&+\left(\|\nabla_{t,y}Q\|^2_{L^\infty([0,T]\times\mathcal{D})}+\|\nabla_{t,y}g\|^2_{L^\infty([0,T]\times\mathcal{D})}\right)\cdot\int_0^t\int_{\mathcal{D}_\tau}|\partial_t f|^2+|\tilde{\nabla}_y f|^2\,dyd\tau.
\end{aligned}$$

Proof. As in the previous two Lemmas, we integrate by parts to obtain

$$\int_{\mathcal{D}}\tilde{\nabla}_\alpha\left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)(Q^\mu\tilde{\nabla}_\mu f)\,dy \tag{2.7}$$

$$
\begin{aligned}
&= \int_{\mathcal{D}} \partial_t \left(g^{0\beta}\tilde{\nabla}_\beta f\right)(Q^\mu \tilde{\nabla}_\mu f)\, dy + \int_{\mathcal{D}} \partial_j J_\epsilon \left(g^{j\beta}\tilde{\nabla}_\beta f\right)(Q^\mu \tilde{\nabla}_\mu f)\, dy \\
&= \partial_t \int_{\mathcal{D}} g^{0\beta}(\tilde{\nabla}_\beta f)(Q^\mu \tilde{\nabla}_\mu f)\, dy - \int_{\mathcal{D}} \left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha (Q^\mu \tilde{\nabla}_\mu f)\, dy \\
&\quad + \int_{\partial\mathcal{D}} n_j J_\epsilon \left(g^{j\beta}\tilde{\nabla}_\beta f\right)(Q^\mu \tilde{\nabla}_\mu f)\, dS.
\end{aligned}
$$

We shall put the second and third terms on the right hand side into a form that is easier to work with. For the second term, we have

$$
\begin{aligned}
&\int_{\mathcal{D}} \left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha (Q^\mu \tilde{\nabla}_\mu f)\, dy \\
&= \int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f([\tilde{\nabla}_\alpha, Q^\mu]\tilde{\nabla}_\mu f)\, dy + \underbrace{\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f Q^\mu \tilde{\nabla}_\mu \tilde{\nabla}_\alpha f\, dy}_{(\dagger)}.
\end{aligned}
\tag{2.8}
$$

Now, the term (†) can be further reduced:

$$
\begin{aligned}
(\dagger) &= \int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f Q^0 \partial_t \tilde{\nabla}_\alpha f\, dy + \int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f Q^j \partial_j J_\epsilon \tilde{\nabla}_\alpha f\, dy \\
&= \int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f Q^0 \partial_t \tilde{\nabla}_\alpha f\, dy + \int_{\partial\mathcal{D}} n_j J_\epsilon (Q^j g^{\alpha\beta}\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\, dS \\
&\quad - \int_{\mathcal{D}} \partial_j J_\epsilon \left(Q^j g^{\alpha\beta}\tilde{\nabla}_\beta f\right)(\tilde{\nabla}_\alpha f)\, dy \\
&= \int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f Q^0 \partial_t \tilde{\nabla}_\alpha f\, dy + \int_{\partial\mathcal{D}} n_j J_\epsilon (Q^j g^{\alpha\beta}\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\, dS \\
&\quad - \int_{\mathcal{D}} [\tilde{\nabla}_j, Q^j g^{\alpha\beta}]\tilde{\nabla}_\beta f \tilde{\nabla}_\alpha f\, dy - (\dagger) + \int_{\mathcal{D}} Q^0 g^{\alpha\beta}\partial_t \tilde{\nabla}_\beta f \tilde{\nabla}_\alpha f\, dy.
\end{aligned}
$$

That is,

$$
\begin{aligned}
(\dagger) &= \int_{\mathcal{D}} Q^0 g^{\alpha\beta}\partial_t \tilde{\nabla}_\beta f \tilde{\nabla}_\alpha f\, dy + \frac{1}{2}\int_{\partial\mathcal{D}} n_j Q^j (g^{\alpha\beta}\tilde{\nabla}_\beta f) J_\epsilon \tilde{\nabla}_\alpha f\, dS \\
&\quad - \frac{1}{2}\int_{\mathcal{D}} ([\tilde{\nabla}_j, Q^j g^{\alpha\beta}]\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\, dy \\
&= \frac{1}{2}\int_{\partial\mathcal{D}} n_j J_\epsilon (Q^j g^{\alpha\beta}\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\, dS - \frac{1}{2}\int_{\mathcal{D}} ([\tilde{\nabla}_j, Q^j g^{\alpha\beta}]\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\, dy \\
&\quad + \frac{1}{2}\partial_t \int_{\mathcal{D}} Q^0 g^{\alpha\beta}\tilde{\nabla}_\alpha f \tilde{\nabla}_\beta f\, dy - \frac{1}{2}\int_{\mathcal{D}} \partial_t \left(Q^0 g^{\alpha\beta}\right)\tilde{\nabla}_\alpha f \tilde{\nabla}_\beta f\, dy.
\end{aligned}
$$

Substituting back into equation (2.8) and further equation (2.7), we have

$$
\int_{\mathcal{D}} \tilde{\nabla}_\alpha \left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)(Q^\mu \tilde{\nabla}_\mu f)\, dy
\tag{2.9}
$$

$$
\begin{aligned}
=&\partial_t \left(\int_{\mathcal{D}} \underbrace{g^{0\beta}(\tilde{\nabla}_\beta f)(Q^\mu \tilde{\nabla}_\mu f) - \frac{1}{2} Q^0 g^{\alpha\beta} \tilde{\nabla}_\alpha f \tilde{\nabla}_\beta f}_{\text{(I)}} \, dy \right) \\
&+ \underbrace{\int_{\partial\mathcal{D}} n_j J_\epsilon \left(g^{j\beta} \tilde{\nabla}_\beta f \right) (Q^\mu \tilde{\nabla}_\mu f) \, dS - \frac{1}{2} \int_{\partial\mathcal{D}} n_j J_\epsilon (Q^j g^{\alpha\beta} \tilde{\nabla}_\beta f) \tilde{\nabla}_\alpha f \, dS}_{\text{(II)}} \\
&- \int_{\mathcal{D}} g^{\alpha\beta} \tilde{\nabla}_\beta f ([\tilde{\nabla}_\alpha, Q^\mu] \tilde{\nabla}_\mu f) \, dy + \frac{1}{2} \int_{\mathcal{D}} ([\tilde{\nabla}_j, Q^j g^{\alpha\beta}] \tilde{\nabla}_\beta f) \tilde{\nabla}_\alpha f \, dy \\
&+ \frac{1}{2} \int_{\mathcal{D}} \partial_t \left(Q^0 g^{\alpha\beta} \right) \tilde{\nabla}_\alpha f \tilde{\nabla}_\beta f \, dy.
\end{aligned}
$$

The commutators are, as we will see, of lower order, so we shall focus on the main terms (I) and (II). We will see that (I) is controlled by $\|\tilde{\nabla} f\|_{L^2(\mathcal{D})}$, so now we massage (II) into a nicer form.

Since $f \equiv \text{const}$ on $I \times \partial\mathcal{D}$, we know

$$
n_\alpha = \frac{1}{|\tilde{\nabla} f|} \tilde{\nabla}_\alpha f, \quad \text{and thus} \quad Q^\mu \tilde{\nabla}_\mu f = (Q^\mu n_\mu) |\tilde{\nabla} f|.
$$

Hence, the first term in (II) can be written as

$$
\begin{aligned}
\int_{\partial\mathcal{D}} n_j J_\epsilon \left(g^{j\beta} \tilde{\nabla}_\beta f \right) (Q^\mu \tilde{\nabla}_\mu f) \, dS &= \int_{\partial\mathcal{D}} n_j J_\epsilon \left(g^{j\beta} \tilde{\nabla}_\beta f \right) (Q^\mu n_\mu) |\tilde{\nabla} f| \, dS \\
&= \int_{\partial\mathcal{D}} (\tilde{\nabla}_\alpha f) J_\epsilon \left(g^{\alpha\beta} \tilde{\nabla}_\beta f \right) (n_\mu Q^\mu) \, dS.
\end{aligned}
$$

For the same reason, the second term in (II) can be written as:

$$
\begin{aligned}
&\int_{\partial\mathcal{D}} n_j J_\epsilon (Q^j g^{\alpha\beta} \tilde{\nabla}_\beta f) \tilde{\nabla}_\alpha f \, dS \\
=& \int_{\partial\mathcal{D}} n_j ([J_\epsilon, Q^j](g^{\alpha\beta} \tilde{\nabla}_\beta f)) \tilde{\nabla}_\alpha f \, dS + \int_{\partial\mathcal{D}} (n_\mu Q^\mu) J_\epsilon \left(g^{\alpha\beta} \tilde{\nabla}_\beta f \right) (\tilde{\nabla}_\alpha f) \, dS.
\end{aligned}
$$

Summing up the two preceding equations, we obtain

$$
\begin{aligned}
\text{(II)} &= \frac{1}{2} \int_{\partial\mathcal{D}} (\tilde{\nabla}_\alpha f) J_\epsilon \left(g^{\alpha\beta} \tilde{\nabla}_\beta f \right) (n_\mu Q^\mu) \, dS - \frac{1}{2} \int_{\partial\mathcal{D}} n_j ([J_\epsilon, Q^j](g^{\alpha\beta} \tilde{\nabla}_\beta f)) \tilde{\nabla}_\alpha f \, dS \\
&= \frac{1}{2} \int_{\partial\mathcal{D}} \left(\tilde{\nabla}_\alpha f g^{\alpha\beta} \tilde{\nabla}_\beta f \right) (n_\mu Q^\mu) \, dS + \frac{1}{2} \int_{\partial\mathcal{D}} (\tilde{\nabla}_\alpha f) \left([J_\epsilon, g^{\alpha\beta}] \tilde{\nabla}_\beta f \right) (n_\mu Q^\mu) \, dS
\end{aligned}
$$

$$- \frac{1}{2}\int_{\partial\mathcal{D}} n_j([J_\epsilon, Q^j](g^{\alpha\beta}\tilde{\nabla}_\beta f))\tilde{\nabla}_\alpha f\, dS.$$

Next, we integrate (2.9) with respect to time, and rearrange the terms to obtain

(2.10)

$$\begin{aligned}
&\int_{\mathcal{D}_t - \mathcal{D}_0} (\mathrm{I})\, dy + \underbrace{\frac{1}{2}\int_0^t \int_{\partial\mathcal{D}} \left(\tilde{\nabla}_\alpha f g^{\alpha\beta}\tilde{\nabla}_\beta f\right)(n_\mu Q^\mu)\, dS\, d\tau}_{(*)} \\
&= -\int_0^t \int_{\mathcal{D}_\tau} (\tilde{\Box}_g f)(Q^\mu \tilde{\nabla}_\mu f)\, dy d\tau + \frac{1}{2}\int_0^t \int_{\mathcal{D}_\tau} \left(\tilde{\nabla}_\alpha \log|g|\right) g^{\alpha\beta}\tilde{\nabla}_\beta f\, dy d\tau \\
&\quad - \frac{1}{2}\int_0^t \int_{\partial\mathcal{D}} (\tilde{\nabla}_\alpha f)\left([J_\epsilon, g^{\alpha\beta}]\tilde{\nabla}_\beta f\right)(n_\mu Q^\mu)\, dS d\tau \\
&\quad + \frac{1}{2}\int_0^t \int_{\partial\mathcal{D}} n_j([J_\epsilon, Q^j](g^{\alpha\beta}\tilde{\nabla}_\beta f))\tilde{\nabla}_\alpha f\, dS d\tau \\
&\quad - \int_0^t \int_{\mathcal{D}_\tau} g^{\alpha\beta}\tilde{\nabla}_\beta f([\tilde{\nabla}_\alpha, Q^\mu]\tilde{\nabla}_\mu f)\, dy d\tau - \frac{1}{2}\int_0^t \int_{\mathcal{D}} ([\tilde{\nabla}_j, Q^j g^{\alpha\beta}]\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\, dy d\tau \\
&\quad - \frac{1}{2}\int_0^t \int_{\mathcal{D}} \partial_t\left(Q^0 g^{\alpha\beta}\right)\tilde{\nabla}_\alpha f \tilde{\nabla}_\beta f\, dy d\tau.
\end{aligned}$$

We choose a vector field Q such that $Q^\mu n_\mu \equiv 1$ on $[0,T]\times\partial\mathcal{D}$ and $\nabla Q \in L^\infty([0,T]\times\mathcal{D})$[2]. For the two terms on the left hand side of the preceding equation, we will add $\int_{\mathcal{D}_t-\mathcal{D}_0}(\mathrm{I})\, dy$ to the energy in Lemma 2.4; the term $(*)$ is the main term, and we seek to control the remaining terms with $(*)$ and $\int_{\mathcal{D}_\tau} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy$.

The commutators on the boundary can be estimated using interior H^1-norms:

$$\begin{aligned}
&\int_0^t \left|\int_{\partial\mathcal{D}} (\tilde{\nabla}_\alpha f)\left([J_\epsilon, g^{\alpha\beta}]\tilde{\nabla}_\beta f\right)(n_\mu Q^\mu)\, dS\right| d\tau \\
\leq &\delta \int_0^t \int_{\partial\mathcal{D}} |\tilde{\nabla}_\alpha f|^2\, dS d\tau + \frac{1}{\delta}\int_0^t \int_{\partial\mathcal{D}} |[J_\epsilon, g^{\alpha\beta}]\tilde{\nabla}_\beta f|^2\, dS d\tau \\
\lesssim &\delta\cdot(*) + \frac{1}{\delta}\int_0^t \left\|[J_\epsilon, g^{\alpha\beta}]\tilde{\nabla} f\right\|^2_{H^1(\mathcal{D}_\tau)} d\tau
\end{aligned}$$

[2] In the unbounded domain $\mathcal{D} = \mathbb{R}^3_+$, the space-time normal vector is $n = (0,0,0,-1)$, so the Q vectorfield can be defined by, for instance, $Q = (1,0,0,-1)$. In the case of the bounded domain $\mathcal{D} = B$, a constant Q will not satisfy our requirement, but the same condition can be achieved by setting, for instance, $Q(t,y_1,y_2,y_3) = (1, (y_1^2+y_2^2+y_3^2)y_1, (y_1^2+y_2^2+y_3^2)y_2, (y_1^2+y_2^2+y_3^2)y_3)$. In either case, $\|Q\|_{L^\infty([0,T]\times\mathcal{D})}$ and $\|\nabla Q\|_{L^\infty([0,T]\times\mathcal{D})}$ are finite and are independent of all variables and functions that we are concerned with.

$$\lesssim \delta \cdot (*) + \frac{1}{\delta}\left(\|\nabla g\|^2_{L^\infty([0,T]\times\mathcal{D})} \cdot \int_0^t \|\tilde{\nabla} f\|^2_{L^2(\mathcal{D}_\tau)}\, d\tau \right),$$

and similarly

$$\int_0^t \int_{\partial\mathcal{D}} n_j([J_\epsilon, Q^j](g^{\alpha\beta}\tilde{\nabla}_\beta f))\tilde{\nabla}_\alpha f\, dS d\tau$$
$$\lesssim \delta \cdot (*) + \frac{1}{\delta}\left(\|\nabla Q\|^2_{L^\infty([0,T]\times\mathcal{D})} \cdot \int_0^t \|\tilde{\nabla} f\|^2_{L^2(\mathcal{D}_\tau)}\, d\tau \right).$$

By choosing a constant $\delta > 0$ small enough, we see that $\delta \cdot (*)$ can be absorbed into the left hand side of equation (2.10).

The interior commutators can be controlled using Cauchy-Schwarz Inequality:

$$\left| \int_0^t \int_{\mathcal{D}_\tau} g^{\alpha\beta}\tilde{\nabla}_\beta f([\tilde{\nabla}_\alpha, Q^\mu]\tilde{\nabla}_\mu f)\, dy d\tau \right| + \left| \frac{1}{2}\int_0^t \int_{\mathcal{D}} ([\tilde{\nabla}_j, Q^j g^{\alpha\beta}]\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\, dy d\tau \right|$$
$$+ \left| \frac{1}{2}\int_0^t \int_{\mathcal{D}} \partial_t \left(Q^0 g^{\alpha\beta}\right) \tilde{\nabla}_\alpha f \tilde{\nabla}_\beta f\, dy d\tau \right|$$
$$\lesssim \left(\|\nabla_{t,y} Q\|^2_{L^\infty([0,T]\times\mathcal{D})} + \|\nabla_{t,y} g\|^2_{L^\infty([0,T]\times\mathcal{D})} \right) \cdot \int_0^t \int_{\mathcal{D}_\tau} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy d\tau.$$

Moreover, by Cauchy-Schwarz, we see that (I) can be controlled by the main energy in Lemma 2.4:

$$\left| \int_{\mathcal{D}_t} (\mathrm{I})\, dy \right| \lesssim \left(\|g\|_{L^\infty([0,T]\times\mathcal{D})} + \|Q\|_{L^\infty([0,T]\times\mathcal{D})} \right) \cdot \int_{\mathcal{D}_t} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy.$$

Therefore, adding (2.10) to a sufficiently large constant times $\int_{\mathcal{D}_t} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy$, and using Lemma (2.4), we have

$$\sup_{0\le\tau\le t} \left(\int_{\mathcal{D}_\tau} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy \right) + \int_0^t \int_{\partial\mathcal{D}} \left(\tilde{\nabla}_\alpha f g^{\alpha\beta} \tilde{\nabla}_\beta f \right)\, dS\, d\tau$$
$$\lesssim \int_{\mathcal{D}_0} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy + \left| \int_0^t \int_{\mathcal{D}} (\tilde{\Box}_g f)(Q^\mu \tilde{\nabla}_\mu f)\, dy d\tau \right|$$
$$+ \left(\|\nabla_{t,y} Q\|^2_{L^\infty([0,T]\times\mathcal{D})} + \|\nabla_{t,y} g\|^2_{L^\infty([0,T]\times\mathcal{D})} \right) \cdot \int_0^t \int_{\mathcal{D}_\tau} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy d\tau.$$

This is the desired estimate. □

The next Lemma is very similar to Lemma 2.4, but applies to functions that are not necessarily equal to a constant on $\partial\mathcal{D}$. We will apply this Lemma on $\partial_t^k u$. The purpose is to estimate the time-weighted $L^2(\partial\mathcal{D})$ norm by norms in Lemma 2.2.

Lemma 2.5. *Assume $g^{00} < 0$, $f \in C^2([0,T]\times\mathcal{D}) \cap L^\infty([0,T], H^2(\mathcal{D}))$, $\partial_t f \in C^1([0,T]\times\mathcal{D}) \cap L^\infty([0,T], H^1(\mathcal{D}))$. When $\mathcal{D} = \mathbb{R}^3_+$, we extend f in an even manner to the lower half space, that is $f(t,y_1,y_2,y_3) := f(t,y_1,y_2,-y_3)$ for $y_3 < 0$. When $\mathcal{D} = B$, we set $J_\epsilon = \mathrm{Id}$ as usual.*

Then there is a future-directed and time-like vectorfield Q that does not depend on f, such that

$$
\begin{aligned}
&\sup_{0\le\tau\le t}\left(\int_{\mathcal{D}_\tau} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy\right) + \int_0^t \int_{\partial\mathcal{D}} \left(\tilde{\nabla}_\alpha f g^{\alpha\beta}\tilde{\nabla}_\beta f\right) dS\, d\tau \\
\lesssim &\int_{\mathcal{D}_0} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy \\
&+ \left|\int_0^t \int_{\mathcal{D}} (\tilde{\Box}_g f)(Q^\mu \tilde{\nabla}_\mu f)\, dy d\tau\right| + \int_0^t \int_{\partial\mathcal{D}} |\partial_t f|^2 + |\tilde{\nabla}_n f|^2\, dS d\tau \\
&+ \left(\|\nabla_{t,y}(\tilde{\nabla}\Sigma^2/a)\|^2_{L^\infty([0,T]\times\mathcal{D})} + \|\nabla_{t,y} g\|^2_{L^\infty([0,T]\times\mathcal{D})}\right)\cdot \int_0^t \int_{\mathcal{D}_\tau} |\partial_t f|^2 + |\tilde{\nabla}_y f|^2\, dy d\tau,
\end{aligned}
$$

where $\tilde{\nabla}_n f = n_\alpha J_\epsilon(g^{\alpha\beta}\tilde{\nabla}_\beta f)$. In particular, one may choose $Q^\mu = K\delta^{\mu=1} + g^{\mu\nu}\tilde{\nabla}_\nu\Sigma^2/a$ for some large constant K.

Proof. The proof is very similar to that of Lemma 2.4. The only differences are the treatment of the terms (I) and (II) in equation (2.9). We will omit the derivation before equation (2.9) since it completely overlaps with the proof of the previous Lemma, and focus on the analysis of (I) and (II).

Let $K > 0$ be a large constant to be specified later. We set $Q^\mu = K\delta^{\mu=1} + g^{\mu\nu}\tilde{\nabla}_\nu\Sigma^2/a$, so that $Q^\mu = K\delta^{\mu=1} + g^{\mu\nu}n_\nu$ on $\partial\mathcal{D}$.

Term (I). Recall that (I) was defined as:

$$
(\mathrm{I}) = g^{0\beta}(\tilde{\nabla}_\beta f)(Q^\mu\tilde{\nabla}_\mu f) - \frac{1}{2}Q^0 g^{\alpha\beta}\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f.
$$

We show that in fact, using our definition of Q, (I) is approximately $\frac{1}{2}K(g^{00}(\partial_t f)^2 - g^{ij}\tilde{\nabla}_i f\tilde{\nabla}_j f)$:

$$
\begin{aligned}
\text{(I)} =& g^{0\beta}(\tilde{\nabla}_\beta f)(Q^\mu\tilde{\nabla}_\mu f) - \frac{1}{2}Q^0 g^{\alpha\beta}\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f \\
=& \frac{1}{2}g^{00}Q^0(\partial_t f)^2 + \left(g^{0i}Q^j - \frac{1}{2}Q^0 g^{ij}\right)\tilde{\nabla}_i f\tilde{\nabla}_j f + g^{00}Q^j(\partial_t f)(\tilde{\nabla}_j f) \\
=& \frac{K}{2}\left[g^{00}(\partial_t f)^2 - g^{ij}(\tilde{\nabla}_i f)(\tilde{\nabla}_j f)\right] + g^{0\nu}(\tilde{\nabla}_\nu\Sigma^2/a)g^{00}\cdot(\partial_t f)^2 - g^{00}Q^j(\partial_t f)(\tilde{\nabla}_j f) \\
& - \left(\frac{1}{2}g^{0\mu}(\tilde{\nabla}_\nu\Sigma^2/a)g^{ij} - g^{0i}g^{\nu j}(\tilde{\nabla}_\nu\Sigma^2/a)\right)\tilde{\nabla}_i f\tilde{\nabla}_j f.
\end{aligned}
$$

Since g and $\tilde{\nabla}\Sigma^2/a$ are bounded, we see that indeed, if K is large, then the first term on the right hand side will be the dominant term. To be more precise, we have

$$
\left|\underbrace{\text{(I)} - \frac{K}{2}\left[g^{00}(\partial_t f)^2 - g^{ij}(\tilde{\nabla}_i f)(\tilde{\nabla}_j f)\right]}_{:=R_1}\right| \le \left|\frac{K}{8}\left[g^{00}(\partial_t f)^2 - g^{ij}(\tilde{\nabla}_i f)(\tilde{\nabla}_j f)\right]\right|. \tag{2.11}
$$

Term (II). Recall that (II) was:

$$
\int_{\partial\mathcal{D}} n_j J_\epsilon\left(g^{j\beta}\tilde{\nabla}_\beta f\right)(Q^\mu\tilde{\nabla}_\mu f)\,dS - \frac{1}{2}\int_{\partial\mathcal{D}} n_j J_\epsilon(Q^j g^{\alpha\beta}\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\,dS.
$$

Then, the first term in (II) can be written as

$$
\begin{aligned}
&\int_{\partial\mathcal{D}} n_j J_\epsilon\left(g^{j\beta}\tilde{\nabla}_\beta f\right)(Q^\mu\tilde{\nabla}_\mu f)\,dS \\
=& \int_{\partial\mathcal{D}} K(\tilde{\nabla}_n f)(\partial_t f)\,dS - \int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)J_\epsilon\left(g^{\mu\nu}n_\nu\tilde{\nabla}_\mu f\right)\,dS \\
=& \int_{\partial\mathcal{D}} K(\tilde{\nabla}_n f)(\partial_t f)\,dS - \int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)n_\nu J_\epsilon\left(g^{\mu\nu}\tilde{\nabla}_\mu f\right)\,dS \\
& - \int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)\left([J_\epsilon,\tilde{\nabla}_\nu\Sigma^2/a]g^{\mu\nu}\tilde{\nabla}_\mu f\right)\,dS \\
=& \int_{\partial\mathcal{D}} K(\tilde{\nabla}_n f)(\partial_t f)\,dS - \int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)^2\,dS - \int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)\left([J_\epsilon,\tilde{\nabla}_\nu\Sigma^2/a]g^{\mu\nu}\tilde{\nabla}_\mu f\right)\,dS.
\end{aligned}
$$

We compute

$$
n_j Q^j = n_j(g^{i\nu}n_\nu) = n_\mu g^{\mu\nu}n_\nu =: |n|_g^2.
$$

Using this notation, the second term in (II) can be written as

$$
\begin{aligned}
&-\frac{1}{2}\int_{\partial\mathcal{D}} n_j J_\epsilon\left(Q^j g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha f\, dS\\
=&-\frac{1}{2}\int_{\partial\mathcal{D}} n_j J_\epsilon\left(Q^j g^{\alpha\beta}\tilde{\nabla}_\beta f\right) J_\epsilon\left(\tilde{\nabla}_\alpha f\right)\, dS\\
=&-\frac{1}{2}\int_{\partial\mathcal{D}} n_j\left([J_\epsilon, Q^j] g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha f\, dS-\frac{1}{2}\int_{\partial\mathcal{D}} n_j Q^j J_\epsilon\left(g^{\alpha\beta}\tilde{\nabla}_\beta f\right)(\tilde{\nabla}_\alpha J_\epsilon f)\, dS\\
=&-\frac{1}{2}\int_{\partial\mathcal{D}} n_j\left([J_\epsilon, Q^j] g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha f\, dS-\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2\left([J_\epsilon, g^{\alpha\beta}]\tilde{\nabla}_\beta f\right)(\tilde{\nabla}_\alpha J_\epsilon f)\, dS\\
&-\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2 g^{\alpha\beta}(\tilde{\nabla}_\beta J_\epsilon f)(\tilde{\nabla}_\alpha J_\epsilon f)\, dS\\
=&-\frac{1}{2}\int_{\partial\mathcal{D}} n_j\left([J_\epsilon, Q^j] g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha f\, dS-\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2\left([J_\epsilon, g^{\alpha\beta}]\tilde{\nabla}_\beta f\right)(\tilde{\nabla}_\alpha J_\epsilon f)\, dS\\
&-\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2 g^{00}(\partial_t J_\epsilon f)^2\, dS-\int_{\partial\mathcal{D}} |n|_g^2 g^{0j}(\partial_t J_\epsilon f)(\tilde{\nabla}_j f)\, dS\\
&-\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2 g^{ij}(\tilde{\nabla}_i f)(\tilde{\nabla}_j f)\, dS.
\end{aligned}
$$

Adding up the two terms of II, we have

$$
\begin{aligned}
(\mathrm{II}) =& \int_{\partial\mathcal{D}} K(\tilde{\nabla}_n f)(\partial_t f)\, dS-\int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)^2\, dS-\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2 g^{ij}(\tilde{\nabla}_i f)(\tilde{\nabla}_j f)\, dS\\
&-\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2 g^{00}(\partial_t J_\epsilon f)^2\, dS-\int_{\partial\mathcal{D}} |n|_g^2 g^{0j}(\partial_t J_\epsilon f)(\tilde{\nabla}_j f)\, dS\\
&-\int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)\left([J_\epsilon, \tilde{\nabla}_\nu \Sigma^2/a] g^{\mu\nu}\tilde{\nabla}_\mu f\right)\, dS\\
&-\frac{1}{2}\int_{\partial\mathcal{D}} n_j\left([J_\epsilon, Q^j] g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha f\, dS+\frac{1}{2}\int_{\partial\mathcal{D}} |n|_g^2\left([J_\epsilon, g^{\alpha\beta}]\tilde{\nabla}_\beta f\right)(\tilde{\nabla}_\alpha J_\epsilon f)\, dS.
\end{aligned}
$$

Then, integrating (2.9) with respect to time, we obtain

(2.12)

$$
\begin{aligned}
&\int_{\mathcal{D}_t}\frac{K}{2}\left[-g^{00}(\partial_t f)^2+g^{ij}(\tilde{\nabla}_i f)(\tilde{\nabla}_j f)\right]-R_1\, dy+\frac{1}{2}\int_0^t\int_{\partial\mathcal{D}} |n|_g^2 g^{ij}\tilde{\nabla}_i f\tilde{\nabla}_j f\, dS d\tau\\
=&\int_{\mathcal{D}_0}\frac{K}{2}\left[-g^{00}(\partial_t f)^2+g^{ij}(\tilde{\nabla}_i f)(\tilde{\nabla}_j f)\right]-R_1\, dy\\
&+\int_0^t\int_{\partial\mathcal{D}} K(\tilde{\nabla}_n f)(\partial_t f)-(\tilde{\nabla}_n f)^2\, dS d\tau-\frac{1}{2}\int_0^t\int_{\partial\mathcal{D}} |n|_g^2 g^{00}(\partial_t J_\epsilon f)^2\, dS d\tau\\
&-\int_0^t\int_{\partial\mathcal{D}} |n|_g^2 g^{0j}(\partial_t J_\epsilon f)(\tilde{\nabla}_j f)\, dS d\tau-\int_0^t\int_{\partial\mathcal{D}}(\tilde{\nabla}_n f)\left([J_\epsilon, \tilde{\nabla}_\nu\Sigma^2/a] g^{\mu\nu}\tilde{\nabla}_\mu f\right)\, dS d\tau
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t\int_{\partial\mathcal{D}} n_j\left([J_\epsilon, Q^j]g^{\alpha\beta}\tilde{\nabla}_\beta f\right)\tilde{\nabla}_\alpha f\,dSd\tau\\
&+\frac{1}{2}\int_0^t\int_{\partial\mathcal{D}} |n|_g^2\left([J_\epsilon, g^{\alpha\beta}]\tilde{\nabla}_\beta f\right)(\tilde{\nabla}_\alpha J_\epsilon f)\,dSd\tau\\
&-\int_0^t\int_{\mathcal{D}} g^{\alpha\beta}\tilde{\nabla}_\beta f([\tilde{\nabla}_\alpha, Q^\mu]\tilde{\nabla}_\mu f)\,dyd\tau+\frac{1}{2}\int_0^t\int_{\mathcal{D}}([\tilde{\nabla}_j, Q^j g^{\alpha\beta}]\tilde{\nabla}_\beta f)\tilde{\nabla}_\alpha f\,dyd\tau\\
&+\frac{1}{2}\int_0^t\int_{\mathcal{D}} \partial_t\left(Q^0 g^{\alpha\beta}\right)\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta f\,dyd\tau.
\end{aligned}
$$

The commutators can be estimated as in Lemma 2.4. The only terms that are different from Lemma 2.4 are R_1 and $K(\tilde{\nabla}_n f)(\partial_t f)-(\tilde{\nabla}_n f)^2$. We are able to control R_1 by equation (2.11), and the term $K(\tilde{\nabla}_n f)(\partial_t f)-(\tilde{\nabla}_n f)^2$ using Cauchy-Schwarz Inequality. Therefore, overall, we have obtained the claimed result. □

We also have the elliptic estimate (proven, for instance, in [18]):

Lemma 2.6. *Let f be a compactly supported function in $\mathcal{D}$. Since*

$$\tilde{\Delta}_k f := \tilde{\nabla}_i\left(k^{ij}(\tilde{\nabla}_j f)\right)$$

is an elliptic operator for $k\in\{g,h\}$, we have the following elliptic estimates:

$$
\begin{aligned}
\|\tilde{\nabla}_y^{(2)} f\|_{L^2(\mathcal{D})} &\lesssim \|\tilde{\Delta}_k f\|_{L^2(\mathcal{D})}+\|J_\epsilon f\|_{H^{3/2}(\partial\mathcal{D})}\\
\|\tilde{\nabla}_y^{(2)} f\|_{L^2(\mathcal{D})} &\lesssim \|\tilde{\Delta}_k f\|_{L^2(\mathcal{D})}+\|\nabla_n J_\epsilon f\|_{H^{1/2}(\partial\mathcal{D})}.
\end{aligned}
$$

2.3 Higher Order Equations

Recall that in Chapter I, we mentioned that the strategy to obtain L^∞ controls on the lower order terms was to use Sobolev embedding, which calls for an energy estimate on the terms $\partial_t^k u$ and $\partial_t^k\Lambda$. To use the fundamental energy Lemmas in the previous section, we compute $\mathcal{P}^\epsilon\partial_t^k u$, $\tilde{\Box}_g\partial_t^k u$ and $\tilde{\Box}_h\partial_t^k\Lambda$ in this section.

To derive the higher order equations, we first note a few commutator identities.

Lemma 2.7. *We have the following commutator identities.*

1. *The following are true:*

$$\begin{aligned}
[\partial_\mu, f]\theta =& (\partial_\mu f)\cdot\theta \\
[\partial_\mu, \tilde{\nabla}^\alpha_g]\theta =& \partial_\mu g^{\alpha\beta}\cdot\tilde{\nabla}_\beta\theta \\
[\partial_\mu, \tilde{\Box}_g]\theta =& \tilde{\nabla}_\alpha\left(\partial_\mu g^{\alpha\beta}\tilde{\nabla}_\beta\theta\right) + \frac{1}{2}\partial_\mu\left(\tilde{\nabla}_\alpha\log|g|\cdot g^{\alpha\beta}\right)\cdot\tilde{\nabla}_\beta\theta \\
[\partial_\mu, \mathcal{P}^\epsilon]\theta =& \partial_\mu((J_\epsilon u^0)^2)\partial_t^2\theta - \frac{1}{2}\left(\tilde{\nabla}_\alpha\partial_\mu\Sigma^2\right)J_\epsilon\left(g^{\alpha\beta}\tilde{\nabla}_\beta J_\epsilon\theta\right) \\
& - \frac{1}{2}\left(\tilde{\nabla}_\alpha\Sigma^2\right)J_\epsilon\left(\partial_\mu g^{\alpha\beta}\cdot\tilde{\nabla}_\beta\theta\right).
\end{aligned}$$

2. *For* $k\geq 1$, $[\partial_t^k, \tilde{\Box}_g]\theta$ *is a linear combination of terms of the forms:*

   (a) $J_\epsilon\left((\nabla^{p_1}\partial_t^{k_1}g)(\tilde{\nabla}^{p_2+1}\partial_t^{k_2}\theta)\right)$ *where* $k_1+k_2=k$, $p_1+p_2=1$, *and* $k_2\leq k-1$.

   (b) $(\partial_t^{p_1}\partial_t^{k_1}g)(\tilde{\nabla}\partial_t^{p_2+k_2}\theta)$ *where* $k_1+k_2=k$, $p_1+p_2=1$, *and* $k_2\leq k-1$.

   (c) $\left(\partial_t^{k_1}(\tilde{\nabla}_\alpha\log|g|\cdot g^{\alpha\beta})\right)\left(\tilde{\nabla}_\beta\partial_t^{k_2}J_\epsilon\theta\right)$ *where* $k_1+k_2=k$ *and* $k_1\geq 1$.

3. *For* $k\geq 1$, $[\partial_t^k, \mathcal{P}^\epsilon]\theta$ *is a linear combination of terms of the forms:*

   (a) $(\partial_t^{k_1}(J_\epsilon u^0)^2)(\partial_t^2\partial_t^{k_2}\theta)$ *where* $k_1+k_2=k$ *and* $k_1\geq 1$.

   (b) $\left(\tilde{\nabla}_\alpha\partial_t^{k_1}\Sigma^2\right)J_\epsilon\left(\partial_t^{k_2}g^{\alpha\beta}\cdot\tilde{\nabla}_\beta\partial_t^{k_3}\theta\right)$ *where* $k_1+k_2+k_3=k$ *and* $k_3\leq k-1$.

Proof. We prove each claim.

1. These are obtained via a direct calculation.

2. We prove by induction on k. When $k=1$, by the preceding point, we have

$$\begin{aligned}
[\partial_t, \tilde{\Box}_g]\theta &= \tilde{\nabla}_\alpha\left(\partial_t g^{\alpha\beta}\tilde{\nabla}_\beta\theta\right) + \frac{1}{2}\partial_t\left(\tilde{\nabla}_\alpha\log|g|\cdot g^{\alpha\beta}\right)\cdot\tilde{\nabla}_\beta J_\epsilon\theta \\
&= \nabla_y J_\epsilon\left((\partial_t g)\tilde{\nabla}_\beta\theta\right) + \partial_t\left((\partial_t g)\tilde{\nabla}_\beta\theta\right) + \frac{1}{2}\partial_t\left(\tilde{\nabla}_\alpha\log|g|\cdot g^{\alpha\beta}\right)\cdot\tilde{\nabla}_\beta J_\epsilon\theta.
\end{aligned}$$

   These correspond to terms of types (a), (b) and (c) respectively in our claim for $k=1$. For higher order derivatives, we note

$$[\partial_t^{k+1}, \tilde{\Box}_g]\theta = \partial_t\left([\partial_t^k, \tilde{\Box}_g]\theta\right) + [\partial_t, \tilde{\Box}_g](\partial_t^k\theta).$$

Using the formula for $k=1$, we see that $[\partial_t^{k+1}, \tilde{\Box}_g]\theta$ takes the desired form.

3. Again, we use induction on k. When $k=1$, we have

$$[\partial_t, \mathcal{P}^\epsilon]\theta = \partial_t((J_\epsilon u^0)^2)\partial_t^2\theta - \frac{1}{2}\left(\tilde{\nabla}_\alpha \partial_t \Sigma^2\right) J_\epsilon\left(g^{\alpha\beta}\tilde{\nabla}_\beta\theta\right) - \frac{1}{2}\left(\tilde{\nabla}_\alpha\Sigma^2\right) J_\epsilon\left(\partial_t g^{\alpha\beta}\cdot\tilde{\nabla}_\beta\theta\right).$$

The first term on the right hand side constitutes (a) in our claim, and the second and third terms make (b) in our claim. For higher order derivatives, again we have

$$[\partial_t^{k+1}, \mathcal{P}^\epsilon]\theta = \partial_t\left([\partial_t^k, \mathcal{P}^\epsilon]\theta\right) + [\partial_t, \mathcal{P}^\epsilon](\partial_t^k\theta).$$

and our claim follows from induction.

□

Using these identities, we are able to derive higher order equations for u and Λ. In what follows, since only the number of derivatives matters in closing the energy estimate, we shall suppress the indices (e.g. writing u^ν as u) to simplify the notations. In particular, the notation "g" represents the entries in $g^{\alpha\beta}$.

Lemma 2.8. *For any $k \geq 0$,*

$$\mathcal{P}^\epsilon\partial_t^k u = F_k^\epsilon \tag{2.13}$$

where F_k^ϵ is a linear combination of terms of the forms

1. $(\partial_t^{k_1}(J_\epsilon u^0)^2)(\partial_t^2\partial_t^{k_2}u)$ *where* $k_1+k_2=k$ *and* $k_1\geq 1$.
2. $\left(\tilde{\nabla}_\alpha\partial_t^{k_1}\Sigma^2\right) J_\epsilon\left(\partial_t^{k_2}g^{\alpha\beta}\cdot\tilde{\nabla}_\beta\partial_t^{k_3}u\right)$ *where* $k_1+k_2+k_3=k$ *and* $k_3\leq k-1$.
3. $(\partial_t^{k_1}(u^0)^2)(\partial_t^{k_2}w)(\partial_t^{k_3}g)(\tilde{\nabla}\partial_t^{k_4}\Sigma^2)$ *where* $k_1+k_2+k_3+k_4=k$.

4. $(\partial_t^{k_1} g)(\tilde{\nabla}\partial_t^{k_2} X)(\tilde{\nabla}\partial_t^{k_3}\Lambda)$ *where* $k_1 + k_2 + k_3 = k$.

5. $(\partial_t^{k_1+1}(u^0)^2)(\partial_t^{k_2+1} u)$ *where* $k_1 + k_2 = k$.

Proof. We write

$$\mathcal{P}^\epsilon \partial_t^k u = -[\partial_t^k, \mathcal{P}^\epsilon]u + \partial_t^k(F_0^\epsilon),$$

where F_0^ϵ is given in equation (1.56). Terms of the forms 1 and 2 come from $[\partial_t^k, \mathcal{P}^\epsilon]u$ according to Lemma 2.7. Terms of the form 3, 4, 5 come from taking ∂_t^k to the right hand side of F_0^ϵ. □

Lemma 2.9. *For any* $k \geq 0$,

$$\tilde{\Box}_g \partial_t^k u = G_k^\epsilon \tag{2.14}$$

where G_k^ϵ *is a linear combination of terms of the forms*

1. $J_\epsilon\Big((\nabla^{p_1}\partial_t^{k_1} g)(\tilde{\nabla}^{p_2+1}\partial_t^{k_2} u)\Big)$ *where* $k_1 + k_2 = k$, $p_1 + p_2 = 1$, *and* $k_2 \leq k - 1$.

2. $(\partial_t{}^{p_1}\partial_t^{k_1} g)(\tilde{\nabla}\partial_t^{p_2+k_2} u)$ *where* $k_1 + k_2 = k$, $p_1 + p_2 = 1$, *and* $k_2 \leq k - 1$.

3. $\Big(\partial_t^{k_1}(\tilde{\nabla}_\alpha \log|g| \cdot g^{\alpha\beta})\Big)\Big(\tilde{\nabla}_\beta J_\epsilon \partial_t^{k_2} u\Big)$ *where* $k_1 + k_2 = k$ *and* $k_1 \geq 1$.

4. $(\partial_t^{k_1} g)(\tilde{\nabla}\partial_t^{k_2} X)(\tilde{\nabla}\partial_t^{k_3} w)$ *where* $k_1 + k_2 + k_3 = k$.

5. $(\partial_t^{k_1} g)(\tilde{\nabla}\partial_t^{k_2} X)(\log G)^{(p)}(\partial_t^{k_3}\Sigma^2)\cdots(\partial_t^{k_m}\Sigma^2)(\tilde{\nabla}\partial_t^{k_{m+1}}\Sigma^2)(\partial_t^{k_{m+2}}\Lambda)$ *where* $p \leq k + 2$ *and* $k_1 + \cdots + k_{m+2} = k$.

6. $(\partial_t^{k_1} g)(\tilde{\nabla}\partial_t^{k_2} X)(\log G)^{(p)}(\partial_t^{k_3}\Sigma^2)\cdots(\partial_t^{k_m}\Sigma^2)(\tilde{\nabla}\partial_t^{k_{m+1}}\Lambda)$ *where* $p \leq k+1$ *and* $k_1 + \cdots + k_{m+1} = k$.

Proof. Similar as the previous Lemma, terms of the forms 1, 2, 3 come from $[\partial_t^k, \tilde{\Box}_g]u$. Terms of the forms 4, 5, 6 come from taking ∂_t^k to the formula of G_0^ϵ. □

Lemma 2.10. *For any $k \geq 0$,*

$$\tilde{\Box}_h \partial_t^k \Lambda = H_k^\epsilon \tag{2.15}$$

where H_k^ϵ is a linear combination of terms of the forms

1. $J_\epsilon \left((\nabla^{p_1} \partial_t^{k_1} h)(\tilde{\nabla}^{p_2+1} \partial_t^{k_2} \Lambda) \right)$ *where $k_1 + k_2 = k$, $p_1 + p_2 = 1$, and $k_2 \leq k - 1$.*
2. $(\partial_t{}^{p_1} \partial_t^{k_1} h)(\tilde{\nabla} \partial_t^{p_2+k_2} \Lambda)$ *where $k_1 + k_2 = k$, $p_1 + p_2 = 1$, and $k_2 \leq k - 1$.*
3. $\left(\partial_t^{k_1} (\tilde{\nabla}_\alpha \log |h| \cdot h^{\alpha\beta}) \right) \left(\tilde{\nabla}_\beta J_\epsilon \partial_t^{k_2} \Lambda \right)$ *where $k_1 + k_2 = k$ and $k_1 \geq 1$.*
4. $(\partial_t^{k_1} g)(\tilde{\nabla} \partial_t^{k_2} u)(\nabla^{p_1} \partial_t^{k_3} g)(\nabla^{p_2} \nabla \partial_t^{k_4} X)(\nabla^{p_3} \tilde{\nabla} \partial_t^{k_5} \Sigma^2)$ *where $p_1 + p_2 + p_3 = 1$ and $k_1 + \cdots + k_5 = k$.*
5. $(\partial_t^{k_1} g)(\partial_t^{k_2} g)(\tilde{\nabla} \partial_t^{k_3} X)(\tilde{\nabla} \partial_t^{k_4} u)(\tilde{\nabla} \partial_t^{k_5} u)(\tilde{\nabla} \partial_t^{k_6} u)$ *where $k_1 + \cdots + k_6 = k$.*
6. $(\partial_t^{k_1} u^0)(\partial_t^{k_2} u^0)(\partial_t^{k_3} \Lambda) \cdots (\partial_t^{k_m} \Lambda)(\log G)^{(p)}$ *where $k_1 + \cdots k_m \leq k + 1$ and $p \leq k + 3$.*

Proof. Similar as the previous Lemma, terms of the forms 1, 2, 3 come from $[\partial_t^k, \tilde{\Box}_g]\Lambda$. Terms of the forms 4, 5, 6 come from taking ∂_t^k to the formula of H_0^ϵ. □

Lemma 2.11. *For any $k \geq 0$,*

$$\nabla \partial_t^k X = J_\epsilon \left(\frac{1}{(J_\epsilon u^0)^{2k}} I_k^\epsilon \right) \tag{2.16}$$

where I_k^ϵ is a linear combination of terms of the form

$$(\tilde{\nabla} \partial_t^{k_1} u)(\partial_t^{k_2} J_\epsilon u) \cdots (\partial_t^{k_{2^k}} J_\epsilon u)$$

with $k_1 + \cdots + k_{2^k} = k - 1$.

Proof. Recall that

$$\partial_t X(t,y) = J_\epsilon \frac{J_\epsilon u}{J_\epsilon u^0}(t,y).$$

Taking derivative with respect to y, we have

$$\nabla \partial_t X = J_\epsilon \left(\frac{1}{(J_\epsilon u^0)^2} \left((\tilde{\nabla} u)(J_\epsilon u^0) - J_\epsilon u (\tilde{\nabla} u^0) \right) \right).$$

The claim follows from taking ∂_t^{k-1} derivatives of the preceding equation. □

Lemma 2.12. *For any $k \geq 0$,*

$$\partial_t^k g^{\alpha\beta} = \frac{A_k^\epsilon}{P} \tag{2.17}$$

where A_k^ϵ is a linear combination of terms of the form

$$(\nabla \partial_t^{k_1} X) \cdots (\nabla \partial_t^{k_m} X)$$

with $k_1 + \cdots + k_m = k$, and P is a polynomial in ∇X.

Hence,

$$\nabla \partial_t^k g^{\alpha\beta} = \frac{B_k^\epsilon}{P}$$

where B_k^ϵ is a linear combination of terms of the form

$$(\nabla^{(2)} \partial_t^{k_1} X) \cdots (\nabla \partial_t^{k_m} X)$$

with $k_1 + \cdots + k_m = k$, and P is a polynomial in ∇X (which may be different from the polynomial in $\partial_t^k g^{\alpha\beta}$).

Proof. Recall that

$$g_{ij} = \sum_k \frac{\partial X^k}{\partial y^i} \frac{\partial X^k}{\partial y^j},$$

and g^{ij} is the ij-th entry of the inverse metric g^{-1}. Recall that for a matrix M, $M^{-1} = \frac{1}{\det M}(\operatorname{adj} M)$, so each $g^{\alpha\beta}$ can be written as a rational function of $g_{\alpha\beta}$. Therefore, $g^{\alpha\beta}$ is a rational function of the terms $\frac{\partial X^k}{\partial y^i}$. In particular, the polynomial P is in fact a power of $\det g$. The higher order formulae follow from taking derivatives. □

In this section, we computed the right hand sides of the equations of $\partial_t^k u$ and $\partial_t^k \Lambda$. The controls on $\partial_t^k u$ and $\partial_t^k \Lambda$ will, as we shall see, provide H^2 and hence L^∞ controls on the lower order quantities. An essential component in obtaining the L^∞ bounds is the trade-off between ∂_t and ∇_y derivatives, which says that in some sense, ∂_t^2 is as "costly" as one ∇_y derivative. The precise statement is the topic of the next section.

2.4 Trading Spatial Derivatives with Time Derivatives

In this section, we prove a Proposition that is key to the proof of Theorem 2.1. It enables the control of the H^p norm of u and Λ in terms of the energies E and $\underline{E}$, and is a key ingredient is obtaining L^∞ bounds on lower order terms.

The motivation is as follows: since we can take many ∂_t derivatives to the variables u and Λ, we would like to claim that sufficiently many time derivatives can guarantee some spatial smoothness. In fact, we claim that $\partial_t^2 f$ enjoys approximately the same regularity as $\nabla_y f$ for $f \in \{u, \Lambda\}$. Details are given in the next Proposition.

Proposition 2.13. *Assume that for some integer $M > 0$,*

$$\sum_{k+2p \le M+2} \|\tilde{\nabla}_y^p \partial_t^k u\|_{L^2(\mathcal{D}_t)}^2 + \|\tilde{\nabla}_y^p \partial_t^k \Lambda\|_{L^2(\mathcal{D}_t)}^2 \le C_M < \infty. \tag{2.18}$$

If M is sufficiently large and $T > 0$ is sufficiently small, then under the assumptions of Theorem 2.1, for any $t \in [0, T]$, we have

(2.19)

$$\sum_{k+2p \le M+2} \|\tilde{\nabla}_y^p \partial_t^k u\|_{L^2(\mathcal{D}_t)}^2 + \|\tilde{\nabla}_y^p \partial_t^k \Lambda\|_{L^2(\mathcal{D}_t)}^2$$

$$\lesssim \sup_{0\le\tau\le t} E_M[u](\tau) + \sup_{0\le\tau\le t} \underline{E}_M[\Lambda](\tau) + \sum_{k+2p \le M+2} \|\tilde{\nabla}_y^p \partial_t^k u\|_{L^2(\mathcal{D}_0)}^2 + \|\tilde{\nabla}_y^p \partial_t^k \Lambda\|_{L^2(\mathcal{D}_0)}^2.$$

Before we present the proof, let us first note a few results that will be useful in the proof.

The first result we will use is the *Abstract bootstrap argument*. A proof can be found in, for instance, [17].

Lemma 2.14. *Let $J \ni 0$ be a time interval, such that for each $t \in J$, there are two statements: the "Hypothesis" $H(t)$ and the "conclusion" $C(t)$. Suppose that the following are true:*

1. *Hypothesis implies conclusion: if $H(t)$ is true for some $t \in J$, then $C(t)$ is also true for that t.*

2. *Conclusion is stronger than hypothesis: if $C(t)$ is true for some $t_0 \in J$, then $H(t)$ is true for some neighborhood of t_0.*

3. *Conclusion is closed: if $t_1, t_2, \cdots$ is a sequence of time in J that converges to $t_0 \in J$, and $C(t_i)$ is true for all $i = 1, 2, \cdots$, then $C(t_0)$ is also true.*

4. *Base case: $H(t)$ is true for some $t \in J$.*

Then $C(t)$ is true for all $t \in J$.

Lemma 2.15. *Assume that $E(t)$ is a continuous function satisfying*

$$E(t) \leq E(0) + \int_0^t cE(\tau)^r \, d\tau$$

for some positive integer r and some positive constant c. Assume $E(0) < \infty$. Then there is a time interval $[0, T]$ such that

$$E(t) \leq 2E(0) \quad \forall t \in [0, T],$$

where T only depends on c, r and $E(0)$.

Proof. This is a direct application of Lemma 2.14. Let $H(t)$ be the statement $E(t) \leq 4E(0)$, and $C(t)$ be the statement $E(t) \leq 2E(0)$. Then assumptions 2-4 in Lemma 2.14 are clearly satisfied. We only need to prove assumption 1. Let $T = E(0)/(c(4E(0))^r)$. We have

$$\begin{aligned} E(t) \leq & E(0) + \int_0^t c(4E(0))^r \, d\tau \\ \leq & E(0) + ct \cdot 4^r E(0)^r \\ \leq & 2E(0) \quad \forall t \in [0, T]. \end{aligned}$$

Thus, $C(t)$ holds for all $t \in [0, T]$. □

Corollary 2.16. *Let*

$$E(t) := \sum_{k+2p \leq M+2} \|\tilde{\nabla}_y^p \partial_t^k u\|_{L^2(\mathcal{D}_t)}^2 + \|\tilde{\nabla}_y^p \partial_t^k \Lambda\|_{L^2(\mathcal{D}_t)}^2.$$

Assume

$$E(t) \leq E(0) + \int_0^t E(\tau)^r \, d\tau$$

for some positive integer r. Then for $T > 0$ is small (depending only on $E(0)$), we have

$$\|\tilde{\nabla}^p \partial_t^k u\|_{L^2(\mathcal{D}_t)} + \|\tilde{\nabla}^p \partial_t^k \Lambda\|_{L^2(\mathcal{D}_t)} \lesssim 1 \quad \forall (k+1) + 2p \leq M + 2. \tag{2.20}$$

and

$$\|\tilde{\nabla}^p \partial_t^k u\|_{L^\infty(\mathcal{D}_t)} + \|\tilde{\nabla}^p \partial_t^k \Lambda\|_{L^\infty(\mathcal{D}_t)} \lesssim 1 \quad \forall k + 1 + 2(p+2) \leq M + 2. \tag{2.21}$$

Proof. By Lemma 2.15, we know that there is some $T > 0$ depending only on $E(0)$ such that $E(t) \leq 2E(0)$ for all $t \in [0, T]$. Then for $t \in [0, T]$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{D}_t} (\tilde{\nabla}^{(p)} \partial_t^k u)^2 \, dy = \int_{\mathcal{D}_t} (\tilde{\nabla}^{(p)} \partial_t^k u)(\tilde{\nabla}^{(p)} \partial_t^{k+1} u) \, dy$$

$$\left|\frac{1}{2}\frac{d}{dt}\int_{\mathcal{D}_t}(\tilde{\nabla}^{(p)}\partial_t^k u)^2\,dy\right| \le \|\tilde{\nabla}^{(p)}\partial_t^k u\|_{L^2(\mathcal{D}_t)}\|\tilde{\nabla}^{(p)}\partial_t^{k+1} u\|_{L^2(\mathcal{D}_t)}$$

$$\le E(t) \le 2E(0).$$

Thus,

$$\int_{\mathcal{D}_t}(\tilde{\nabla}^{(p)}\partial_t^k u)^2\,dy \le \int_{\mathcal{D}_0}(\tilde{\nabla}^{(p)}\partial_t^k u)^2\,dy + \left|\frac{d}{dt}\int_{\mathcal{D}_t}(\tilde{\nabla}^{(p)}\partial_t^k u)^2\,dy\right|$$

$$\le 5E(0).$$

The estimate for Λ follows similarly.

To obtain the L^∞ estimate, we use the Sobolev embedding $\|f\|_{L^\infty} \lesssim \|f\|_{H^2}$. The conclusion then follows. □

The next two Corollaries guarantee the strict positiveness of $a, u^0, -g^{00}, -h^{00}$, as well as the positive-definiteness of (g^{ij}).

Corollary 2.17. *Assume the same assumptions as in Proposition 2.13. Assume further that there is a constant $c > 0$ such that*

$$a(t,y) \ge 2c, \quad J_\epsilon u^0(t,y) \ge 2c, \quad -g^{00} \ge 2c, \quad -h^{00} \ge 2c$$

at $t = 0$. Then there is a time $T > 0$, depending only on the initial data, such that for all $t \in [0,T]$ and for any $y \in \mathcal{D}$,

$$a(t,y) \ge c, \quad J_\epsilon u^0(t,y) \ge c, \quad -g^{00} \ge c, \quad -h^{00} \ge c.$$

Proof. All of the estimates follow from control on L^∞ norm of the ∂_t derivatives of the respective quantities, and signs on the initial conditions.

Recall that $a^2 = g^{\alpha\beta}(\tilde{\nabla}_\alpha\Sigma^2)(\tilde{\nabla}_\beta\Sigma^2)$, so

$$a\partial_t a = g^{\alpha\beta}(\tilde{\nabla}_\alpha\partial_t\Sigma^2)(\tilde{\nabla}_\beta\Sigma^2)$$

$$= g^{\alpha\beta}\left(\tilde{\nabla}_\alpha J_\epsilon \frac{J_\epsilon \Lambda}{J_\epsilon u^0}\right)(\tilde{\nabla}_\beta \Sigma^2).$$

By the preceding corollary, we may bound the L^∞ norm of each term, and thus control $\|\partial_t a\|_{L^\infty}$. But then

$$a(t,y) \geq a(0,y) - |t| \cdot \|\partial_t a\|_{L^\infty},$$

so if T is small, then $a(t,y) > c$ for all $t \in [0,T]$. The bounds on u are similar. □

Corollary 2.18. *Assume the same assumptions as in Proposition 2.13. Then if* $T > 0$ *is small (depending only on the initial data), the matrix with* (i,j)*-th entry* $(g^{ij}(t))$ *is strictly positive definite*[3] *on the time interval* $[0,T]$.

Proof. Let $G(t)$ be the matrix with (i,j)-th entry $(g^{ij}(t))$. At $t = 0$, we know that $G(0)$ is positive definite by the definition of the pullback metric. Since each $g^{\alpha\beta}$ is a differentiable function of t, by the standard result on eigenvalues (see, for instance, [8]), there is a time interval $[0,T]$ such that $G(t)$ is positive definite for all $t \in [0,T]$. □

Equipped with the L^∞ bounds on the lower order terms for all $t \in [0,T]$, we are now ready to prove Proposition 2.13.

Proof for Proposition 2.13. We prove by induction. When $p \leq 1$, the claim follows from the definition of E and $\underline{E}$. Now, we impose the inductive assumption that

(2.22)

$$\sum_{q\leq p}\sum_{k+2q\leq M+2} \|\tilde{\nabla}_y^q \partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\tilde{\nabla}_y^q \partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_t)}$$

$$\lesssim \sup_{0\leq\tau\leq t} E_M[u](\tau) + \sup_{0\leq\tau\leq t} \underline{E}_M[\Lambda](\tau) + \sum_{q\leq p}\sum_{k+2q\leq M+2} \|\tilde{\nabla}_y^p \partial_t^k u\|^2_{L^2(\mathcal{D}_0)} + \|\tilde{\nabla}_y^p \partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_0)},$$

[3] Recall that (g^{ij}) is the spatial component of the metric.

and aim at proving that

(2.23)

$$\sum_{k+2(p+1)\le M+2} \|\tilde\nabla_y^{p+1}\partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\tilde\nabla_y^{p+1}\partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_t)}$$

$$\lesssim \sup_{0\le\tau\le t} E_M[u](\tau) + \sup_{0\le\tau\le t} \underline{E}_M[\Lambda](\tau) + \sum_{q\le p+1}\sum_{k+2q\le M+2} \|\tilde\nabla_y^{p}\partial_t^k u\|^2_{L^2(\mathcal{D}_0)} + \|\tilde\nabla_y^{p}\partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_0)},$$

Note that by Corollary 2.16, we are able to control L^∞ norm of the lower order terms given the inductive hypothesis (2.22).

Let us analyze the terms

$$\|\tilde\nabla_y^{p+1}\partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_t)} \quad \text{and} \quad \|\tilde\nabla_y^{p+1}\partial_t^k u\|^2_{L^2(\mathcal{D}_t)}$$

Estimate on the Λ term. We first deal with $\|\tilde\nabla_y^{p+1}\partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_t)}$. Recall that $\Lambda\equiv 0$ on $\partial\mathcal{D}$, so $\partial_t^k\Lambda \equiv 0$ on $\partial\mathcal{D}$ as well. By Lemma (2.6), we have

$$\begin{aligned}
&\|\tilde\nabla_y^{p+1}\partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_t)}\\
\lesssim&\|\tilde\Delta_g\tilde\nabla_y^{p-1}\partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_t)}\\
\le&\|[\tilde\Delta_g,\nabla_y^{p-1}]J_\epsilon\partial_t^k\Lambda\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_y^{p-1}\tilde\Delta_g J_\epsilon\partial_t^k\Lambda\|^2_{L^2(\mathcal{D}_t)}\\
\le&\|[\tilde\Delta_g,\nabla_y^{p-1}]J_\epsilon\partial_t^k\Lambda\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_y^{p-1}\tilde\Box_g J_\epsilon\partial_t^k\Lambda\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_y^{p-1}J_\epsilon\partial_t^{k+2}\Lambda\|^2_{L^2(\mathcal{D}_t)}.
\end{aligned}$$

The last term in the preceding equation is bounded by the right hand side of (2.23) by the inductive hypothesis. We only need to control the first two terms.

The term $[\tilde\Delta_g,\nabla_y^{p-1}]J_\epsilon\partial_t^k\Lambda$ is a sum of terms of the forms:

1. $J_\epsilon\left((\nabla^{(k_1+1)}g)(\tilde\nabla^{(k_2+1)}\partial_t^k\Lambda)\right)$ where $k_1+k_2=p-1$ and $k_2\le p-2$.
2. $J_\epsilon\left((\nabla^{(k_1)}g)(\tilde\nabla^{(k_2+2)}\partial_t^k\Lambda)\right)$ where $k_1+k_2=p-1$ and $k_2\le p-2$.

The highest order term in $[\tilde\Delta_g,\nabla_y^{p-1}]J_\epsilon\partial_t^k\Lambda$ is thus $\tilde\nabla_y^p\partial_t^k\Lambda$, the L^2 norm of which can be controlled by the inductive hypothesis. Thus, $\|[\tilde\Delta_g,\nabla_y^{p-1}]J_\epsilon\partial_t^k\Lambda\|^2_{L^2(\mathcal{D}_t)}$ is bounded by the right hand side of (2.23).

We now analyze $\nabla_y^{p-1}\tilde{\Box}_g J_\epsilon \partial_t^k \Lambda \overset{\epsilon}{\sim} \nabla_y^{p-1} H_k^\epsilon$. Recall that by Lemma 2.10, $\nabla_y^{p-1} H_k^\epsilon$ contains six types of terms. The highest order terms of forms 1, 2, 3 in H_k^ϵ are $\tilde{\nabla}_y^{p+1}\partial_t^{k-1}\Lambda$, which can be controlled by the right hand side of (2.23). The highest order terms in 4 and 5 are $\tilde{\nabla}^p \partial_t^k u$, $\tilde{\nabla}^{p+1}\partial_t^k \Sigma^2 \overset{\epsilon}{\sim} \tilde{\nabla}^{p+1}\partial_t^{k-1}\Lambda$, which are controlled as desired. The highest order term in 6 is $\tilde{\nabla}^{p-1}\partial_t^{k+1}\Lambda$, which is also controlled. Hence, $\|\nabla_y^{p-1}\tilde{\Box}_g J_\epsilon \partial_t^k \Lambda\|^2_{L^2(\mathcal{D})}$ is bounded by the right hand side of (2.23).

Therefore, we have shown that $\|\tilde{\nabla}_y^{p+1}\partial_t^k \Lambda\|^2_{L^2(\mathcal{D}_t)}$ is bounded by the right hand side of (2.23).

Estimate on the u term. Next, we consider $\|\tilde{\nabla}_y^{p+1}\partial_t^k u\|^2_{L^2(\mathcal{D}_t)}$. The interior analysis is very similar to that of Λ, but the boundary term is more complicated. As before, by Lemma (2.6), we have

$$
\begin{aligned}
&\|\tilde{\nabla}_y^{p+1}\partial_t^k u\|^2_{L^2(\mathcal{D}_t)}\\
\lesssim&\|\tilde{\Delta}_g \tilde{\nabla}_y^{p-1}\partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_n \tilde{\nabla}_y^{p-1}\partial_t^k u\|^2_{H^{1/2}(\partial\mathcal{D}_t)}\\
\le&\|[\tilde{\Delta}_g, \nabla_y^{p-1}]J_\epsilon \partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_y^{p-1}\tilde{\Delta}_g J_\epsilon \partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_n \tilde{\nabla}_y^{p-1}\partial_t^k u\|^2_{H^{1/2}(\partial\mathcal{D}_t)}\\
\le&\|[\tilde{\Delta}_g, \nabla_y^{p-1}]J_\epsilon \partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_y^{p-1}\tilde{\Box}_g J_\epsilon \partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_y^{p} J_\epsilon \partial_t^{k+1} u\|^2_{L^2(\mathcal{D}_t)}+\\
&\quad+\|\nabla_y^{p-1} J_\epsilon \partial_t^{k+2} u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_n \tilde{\nabla}_y^{p-1}\partial_t^k u\|^2_{H^{1/2}(\partial\mathcal{D}_t)}\\
=&\|[\tilde{\Delta}_g, \nabla_y^{p-1}]J_\epsilon \partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_y^{p-1}\tilde{\Box}_g J_\epsilon \partial_t^k u\|^2_{L^2(\mathcal{D}_t)} + \|\tilde{\nabla}_y^{p} \partial_t^{k+1} u\|^2_{L^2(\mathcal{D}_t)}+\\
&\quad+\|\tilde{\nabla}_y^{p-1} \partial_t^{k+2} u\|^2_{L^2(\mathcal{D}_t)} + \|\nabla_n \tilde{\nabla}_y^{p-1}\partial_t^k u\|^2_{H^{1/2}(\partial\mathcal{D}_t)}.
\end{aligned}
$$

The third and the fourth terms in the preceding equation are clearly bounded by the right hand side of (2.23) by the inductive hypothesis. We only need to control the first, the second, and the last terms.

As in the case of Λ, the term $[\tilde{\Delta}_g, \nabla_y^{p-1}]J_\epsilon \partial_t^k u$ consists of terms of the form:

1. $J_\epsilon\left((\nabla^{(k_1+1)}g)(\tilde{\nabla}^{(k_2+1)}\partial_t^k u)\right)$ where $k_1 + k_2 = p-1$ and $k_2 \le p-2$.

2. $J_\epsilon\left((\nabla^{(k_1)}g)(\tilde{\nabla}^{(k_2+2)}\partial_t^k u)\right)$ where $k_1+k_2=p-1$ and $k_2 \le p-2$.

The highest order term in $[\tilde{\Delta}_g, \nabla_y^{p-1}]J_\epsilon\partial_t^k u$ is $\tilde{\nabla}_y^p\partial_t^k u$, the L^2 norm of which can be controlled by the inductive hypothesis. Thus, $\|[\tilde{\Delta}_g, \nabla_y^{p-1}]J_\epsilon\partial_t^k u\|^2_{L^2(\mathcal{D}_t)}$ is bounded by the right hand side of (2.23).

We now analyze $\nabla_t^{p-1}\tilde{\Box}_g J_\epsilon\partial_t^k u \overset{\epsilon}{\sim} \nabla_y^{p-1}G_k^\epsilon$. Recall that by Lemma 2.9, G_k^ϵ contains six types of terms. The highest order term of forms 1, 2, 3 in $\nabla^{p-1}G_k^\epsilon$ is $\tilde{\nabla}_y^{p+1}\partial_t^{k-1}u$, which can be controlled by the right hand side of (2.23). The highest order term in 4 is $\tilde{\nabla}^p\partial_k w$, which consists of $\tilde{\nabla}^{p-1}\left(\nabla^2\partial_t^{k-1}u\right) \overset{\epsilon}{\sim} \tilde{\nabla}^{p+1}\partial_t^{k-1}u$. This is also controlled as required. The highest order term in 5 and 6 are $\tilde{\nabla}^p\partial_t^k\Lambda$, which is also controlled in the right form. Hence, $\|\nabla_y^{p-1}\tilde{\Box}_g J_\epsilon\partial_t^k u\|^2_{L^2(\mathcal{D}_t)}$ can be bounded by the right hand side of (2.23).

The last term that we need to estimate is $\|\nabla_n\tilde{\nabla}_y^{p-1}\partial_t^k u\|^2_{H^{1/2}(\partial\mathcal{D}_t)}$. We would like to commute $\mathcal{P}^\epsilon$ with $\tilde{\nabla}_y^{p-1}$, but the commutator requires some special care, since the ∂_t^2 term in $\mathcal{P}^\epsilon$ is not mollified. We compute that

$$\begin{aligned}
\nabla_n\tilde{\nabla}_y^{p-1}\partial_t^k u &= (\tilde{\nabla}_\alpha\Sigma^2)J_\epsilon\left(g^{\alpha\beta}\nabla_\beta J_\epsilon\tilde{\nabla}_y^{p-1}\partial_t^k u\right)\\
&= (\tilde{\nabla}_\alpha\Sigma^2)J_\epsilon\left([g^{\alpha\beta}, \nabla_y^{p-1}\partial_t^k]\nabla_\beta J_\epsilon u\right) + (\tilde{\nabla}_\alpha\Sigma^2)\partial_t^k\nabla_y^{p-1}J_\epsilon\left(g^{\alpha\beta}\nabla_\beta J_\epsilon\partial_t^k u\right)\\
&= (\tilde{\nabla}_\alpha\Sigma^2)J_\epsilon\left([g^{\alpha\beta}, \nabla_y^{p-1}\partial_t^k]\nabla_\beta J_\epsilon u\right) + [(\tilde{\nabla}_\alpha\Sigma^2), \nabla_y^{p-1}\partial_t^k J_\epsilon]J_\epsilon\left(g^{\alpha\beta}\nabla_\beta J_\epsilon u\right)\\
&\quad + \partial_t^k\tilde{\nabla}_y^{p-1}\left((\tilde{\nabla}_\alpha\Sigma^2)J_\epsilon\left(g^{\alpha\beta}\nabla_\beta J_\epsilon u\right)\right)\\
&= (\tilde{\nabla}_\alpha\Sigma^2)J_\epsilon\left([g^{\alpha\beta}, \nabla_y^{p-1}\partial_t^k]\nabla_\beta J_\epsilon u\right) + [(\tilde{\nabla}_\alpha\Sigma^2), \nabla_y^{p-1}\partial_t^k J_\epsilon]J_\epsilon\left(g^{\alpha\beta}\nabla_\beta J_\epsilon u\right)\\
&\quad - 2\tilde{\nabla}_y^{p-1}\partial_t^k\mathcal{P}^\epsilon u + 2\partial_t^k\tilde{\nabla}^{p-1}\partial_t^2 u.
\end{aligned}$$

The last term is readily in the right hand side of (2.23), so we will analyze the first three terms. The first term is a linear combination of terms of the from

$$(\tilde{\nabla}_\alpha\Sigma^2)J_\epsilon\left(\nabla_y^{p_1}\partial_t^{k_1}g^{\alpha\beta}\cdot\nabla_y^{p_2}\nabla_\beta J_\epsilon\partial_t^{k_2}u\right), \quad p_1+p_2=p-1, k_1+k_2=k, p_2+k_2\le k+p-2.$$

As before, the derivatives on g can be bounded by those on u (see, for instance, Lemma 2.20). Thus, the highest order terms are $\nabla_y^{p-1} J_\epsilon \partial_t^k u$ and $\nabla_y^{p-2} J_\epsilon \partial_t^{k+1} u$. Their $H^{1/2}(\partial\mathcal{D})$ norms are bounded by their respective $H^1(\mathcal{D})$ norms, which are controlled by the right hand side of (2.23).

The second term is a linear combination of terms of the forms:

1. $J_\epsilon \left(\nabla^{p_1} \tilde{\nabla}_\alpha \partial_t^{k_1} \Sigma^2 \cdot \tilde{\nabla}^{p_2} \left(g^{\alpha\beta} \nabla_\beta J_\epsilon \partial_t^{k_2} u \right) \right)$ where $k_1 + k_2 = k$, $p_1 + p_2 = p - 1$ and $k_2 + p_2 \leq k + p - 2$.

2. $[J_\epsilon, \nabla^{p_1} \tilde{\nabla}_\alpha \partial_t^{k_1} \Sigma^2] \left(\tilde{\nabla}^{p_2} \left(g^{\alpha\beta} \nabla_\beta J_\epsilon \partial_t^{k_2} u \right) \right)$ where $k_1 + k_2 = k$, $p_1 + p_2 = p - 1$ and $k_2 + p_2 \leq k + p - 1$.

The highest order terms in 1 are $\tilde{\nabla}_y^{p-1} \partial_t^k u$ and $\tilde{\nabla}_y^{p-2} \partial_t^{k+1} u$. Their $H^{1/2}(\partial\mathcal{D})$ norms are bounded by their respective $H^1(\mathcal{D})$ norms, which are controlled by the right hand side of (2.23). To treat terms in 2, note that $\|[J_\epsilon, \theta] \nabla \phi\|_{L^2} \lesssim \|\nabla \theta\|_{H^2} \|\phi\|_{L^2}$, so the highest order terms are the same as those in 1, which have shown to be controlled as required.

The third term is $\tilde{\nabla}_y^{p-1} \partial_t^k (P^\epsilon u)$. By a slight abuse of notation, we shall consider F_0^ϵ as being defined in $\mathcal{D}$, so $\|\tilde{\nabla}_y^{p-1} \partial_t^k (P^\epsilon u)\|_{H^{1/2}(\partial\mathcal{D})} \lesssim \|\nabla \tilde{\nabla}_y^{p-1} \partial_t^k F_0^\epsilon\|_{H^1(\mathcal{D})} \lesssim \|\tilde{\nabla}_y^p \partial_t^k F_0^\epsilon\|_{L^2(\mathcal{D})}$. We shall analyze the terms in F_0^ϵ. The highest order terms in $\tilde{\nabla}^p \partial_t^k F_0^\epsilon$ are $\tilde{\nabla}_y^{p+1} \partial_t^k \Sigma^2$ and $\tilde{\nabla}_y^p \partial_t^{k+1} u$, and $\tilde{\nabla}_y^{p+1} \partial_t^k \Lambda$. The first two are readily controlled, and the third term was shown to be bounded by the right hand side of (2.23), so we have finished controlling $\|\tilde{\nabla}_y^{p+1} \partial_t^k u\|_{L^2(\mathcal{D})}^2$.

The proof is now complete.

□

In this section, we have shown a critical ingredient in the proof for the a priori estimate. Namely, given the energy $\mathcal{E}_M^\epsilon$, we are able to bound the functions u and

Λ, as well as their time and spatial derivatives, in the standard Sobolev spaces. Moreover, we have shown that ∂_t^2 is as "costly" as ∇_y in terms of the Sobolev norms.

We have one more section before closing the a priori estimate. In the next section, we will establish some controls on the terms X, w, g etc, which will be useful when we analyze the highest order terms.

2.5 Controlling the Lower Order Terms

Before we close the a priori estimate, let us first establish some controls on the lower order terms, which utilizes Proposition 2.13. In this section, we assume the same condition as in Theorem 2.1.

Lemma 2.19. *For $k \geq 1$, there is a polynomial R_k with non-negative coefficients such that if $T > 0$ is small, then for all $t \in [0, T]$,*

$$\|\nabla \partial_t^k X\|^2_{L^2(\mathbb{R}^3_{+t})} \lesssim R_k \left(\sup_{0 \leq \tau \leq r} E^\epsilon_{k-1}[u](\tau) \right) + \mathfrak{E}^\epsilon_M[u, \Lambda](0). \tag{2.24}$$

Proof. Recall that by Lemma 2.11, we have

$$\nabla \partial_t^k X = J_\epsilon \left(\frac{I_k^\epsilon}{(J_\epsilon u^0)^{2^k}} \right).$$

The highest order terms in I_k^ϵ are $\tilde{\nabla} \partial_t^{k-1} u$ and $\partial_t^{k-1} u$, and we may control the lower order terms by

$$\begin{aligned} \left\| \tilde{\nabla} \partial_t^j u \right\|^2_{L^\infty(\mathbb{R}^3_{+t})} \lesssim \left\| \tilde{\nabla} \partial_t^j u \right\|^2_{H^2(\mathbb{R}^3_{+t})} &\lesssim \left\| \tilde{\nabla} \partial_t^{j+4} u \right\|^2_{L^2(\mathbb{R}^3_{+t})} + \mathfrak{E}^\epsilon_M[u, \Lambda](0) \\ &\leq \sup_{0 \leq \tau \leq t} E^\epsilon_{j+4}[u](\tau) + \mathfrak{E}^\epsilon_M[u, \Lambda](0). \end{aligned}$$

Moreover, by Corollary 2.17, $\frac{1}{J_\epsilon u^0}$ is strictly positive. Our result thus follows. □

Lemma 2.20. *For $k \geq 1$, there is a polynomial R_k (which might be different from the polynomial in Lemma 2.19) and a polynomial S_k, both with non-negative coefficients,* *such that if $T > 0$ is small, then for all $t \in [0, T]$,*

$$\left\|\partial_t^k g^{\alpha\beta}\right\|^2_{L^2(\mathbb{R}^3_{+t})} \lesssim R_k \left(\sup_{0\le\tau\le r} E^{\epsilon}_{k-1}[u](\tau) \right) + \mathfrak{E}^{\epsilon}_M[u,\Lambda](0), \tag{2.25}$$

$$\left\|\nabla\partial_t^k g^{\alpha\beta}\right\|^2_{L^2(\mathbb{R}^3_{+t})} \lesssim S_k \left(\sup_{0\le\tau\le r} E^{\epsilon}_{k+1}[u](\tau) \right) + \mathfrak{E}^{\epsilon}_M[u,\Lambda](0). \tag{2.26}$$

Proof. Recall that by Lemma 2.12, we have

$$\partial_t^k g^{\alpha\beta} = \frac{A_k^\epsilon}{P}, \quad \nabla\partial_t^k g^{\alpha\beta} = \frac{B_k^\epsilon}{P}.$$

Here P is a power of $\det g$. We know that at $t = 0$, $\det g \ge 2c > 0$ for some constant c, since g is the pull-back metric. But then $\frac{d}{dt}\det g$ is a polynomial in components of $g_{\alpha\beta}$, and thus a polynomial in ∇X. Thus, by Lemma 2.19, we know that there is a time T such that for all $t \in [0, T]$, $\det g \ge c > 0$. Thus $\frac{1}{P}$ is strictly positive.

It then remains to control A_k^ϵ and B_k^ϵ. The highest order term in A_k^ϵ is $\nabla\partial_t^k X$, which can be controlled by Lemma 2.19, and the lower order terms can be controlled in L^∞ norm as we did in the proof of Lemma 2.19.

The highest order term in B_k^ϵ is $\tilde{\nabla}^{(2)}\partial_t^{k-1}u$. By Proposition 2.13, we have

$$\left\|\tilde{\nabla}^{(2)}\partial_t^{k-1}u\right\|^2_{L^2(\mathbb{R}^3_{+t})} \lesssim \sup_{0\le\tau\le t} E^{\epsilon}_{k+1}[u](\tau) + \mathfrak{E}^{\epsilon}_M[u,\Lambda](0).$$

The lower order terms can be controlled as before. Thus we obtain the desired result. □

We are now in a position to close the a priori estimate.

2.6 Closing the a priori Estimate

In this section, we prove the claim in Theorem 2.1. Recall that we need to show that there is some polynomial P_M such that

$$\mathfrak{E}^{\epsilon}_k[u,\Lambda](t) \lesssim \mathfrak{E}^{\epsilon}_k[u,\Lambda](0) + \int_0^t P_M(\mathfrak{E}^{\epsilon}_k[u,\Lambda](\tau))\, d\tau \tag{2.27}$$

2.6.1 Estimate on Λ

By Lemma 2.4, we know that for some integer r,

$$\underline{E}_k^\epsilon[\Lambda](t) \lesssim \underline{E}_k^\epsilon[\Lambda](0) + \left| \int_0^t \int_{\mathcal{D}_\tau} (\tilde{\Box}_h \partial_t^k \Lambda)(Q^\mu \tilde{\nabla}_\mu \partial_t^k \Lambda)\, dy d\tau \right| + \int_0^t \underline{E}_k[\Lambda](\tau) \cdot \mathcal{E}_M^\epsilon[u, \Lambda](\tau)^r \, d\tau. \tag{2.28}$$

The first and the last terms on the right hand side are clearly bounded by the right hand side of (2.27); it remains to analyze the second term. Recall that $\tilde{\Box}_h \partial_t^k \Lambda = H_k^\epsilon$ contains 6 types of terms, and we shall analyze each one of them.

Terms of form 1 in H_k^ϵ. We first deal with terms of form 1 in H_k^ϵ. The highest order terms are $\tilde{\nabla}\partial_t^k h$ and $\tilde{\nabla}^{(2)}\partial_t^{k-1}\Lambda$. We write $\tilde{\nabla}\partial_t^k h = F^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\partial_t^{k-1}u$ where $F^{\alpha\beta}$ is a function such that

$$\sup_{0\le t\le T} \|F^{\alpha\beta}\|_{L^\infty(\mathcal{D}_t)} + \|\nabla F^{\alpha\beta}\|_{L^\infty(\mathcal{D}_t)} \lesssim \mathcal{E}_{k-1}^\epsilon(T)^p$$

for some integer p. Note that if at least one of α, β is 0, then $\|F^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\partial_t^{k-1}u\|_{L^2(\mathcal{D}_t)} \le E_k[u](t)$, so we may simply use Cauchy-Schwarz. Hence, assume henceforth that $\alpha, \beta \neq 0$. We compute the term $(F^{ij}\nabla_i\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)$:

$$\begin{aligned}
&(F^{ij}\nabla_i\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)\\
=&\nabla_i(F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda) - (\nabla_i F^{ij})(\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)\\
=&\nabla_i[(F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)] - (F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)\nabla_i(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)\\
&\quad - (\nabla_i F^{ij})(\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)\\
=&\nabla_i[(F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)] - (F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)(\nabla_i Q^\mu)(\tilde{\nabla}_\mu\partial_t^k\Lambda)\\
&\quad - (F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu)\nabla_i(\tilde{\nabla}_\mu\partial_t^k\Lambda) - (\nabla_i F^{ij})(\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda).
\end{aligned} \tag{2.29}$$

The second and the last terms can be controlled using Cauchy-Schwarz. For the first term, we have

$$\begin{aligned}
&\int_0^t \int_{\mathcal{D}_\tau} \nabla_i[(F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu \tilde{\nabla}_\mu \partial_t^k \Lambda)]\, dy d\tau \\
&= \int_0^t \int_{\partial\mathcal{D}_\tau} n_i (F^{ij}\nabla_j J_\epsilon \partial_t^{k-1}u)(Q^\mu \tilde{\nabla}_\mu \partial_t^k \Lambda)\, dS d\tau \\
&\lesssim \int_0^t \left(\frac{1}{\delta}\int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}\partial_t^{k-1}u|^2\, dS\right) + \left(\delta \int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}\partial_t^k \Lambda|^2\, dS\right) d\tau. \qquad (2.30)
\end{aligned}$$

We will use Lemma 2.5 to estimate the first term on the right hand side. Since we will re-use this result when closing the estimate on u, let us summarize this conclusion in the following Lemma.

Lemma 2.21. *Let k be an integer, and $\eta > 0$ be a fixed number. Then there is a polynomial R_η such that if T is small (depending only on η), then for all $t \in [0,T]$, we have*

$$\int_0^t \int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}\partial_t^k u|^2\, dS d\tau \lesssim E^\epsilon_{\leq k}[u](0) + R_\eta(\mathcal{E}^\epsilon_k[u,\Lambda](t)) + \eta \mathcal{E}^\epsilon_{k+1}[u,\Lambda](t). \qquad (2.31)$$

We will postpone the proof until the end of finishing estimating terms of form 1 in H^ϵ_k. Now, let us use it to close the estimate (2.30). We have

$$\begin{aligned}
&\int_0^t \left(\frac{1}{\delta}\int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}\partial_t^{k-1}u|^2\, dS\right) + \left(\delta \int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}\partial_t^k \Lambda|^2\, dS\right) d\tau \\
\lesssim &\frac{1}{\delta}\left[E^\epsilon_{\leq k-1}[u](0) + R_\eta(\mathcal{E}^\epsilon_{k-2}[u,\Lambda](t)) + \eta \mathcal{E}^\epsilon_{k-1}[u,\Lambda](t)\right] + \delta \cdot \underline{E}^\epsilon_k[\Lambda](t).
\end{aligned}$$

The last term, $\delta \cdot \underline{E}^\epsilon_k[\Lambda](t)$, can be absorbed into the left hand side of equation (2.28). The remaining terms are of lower order. Thus, we have controlled the first term on the right hand side of equation (2.29).

Next, we consider the third term in (2.29).

We pay special attention to the case when $\mu = 0$, since $\nabla_i \tilde{\nabla}_\mu \partial_t^k \Lambda$ is un-mollified. We shall borrow a mollifier from the other terms:

$$\int_0^t \int_{\mathcal{D}_\tau} (\nabla_j J_\epsilon \partial_t^{k-1}u)(F^{ij}Q^0)\nabla_i(\partial_t^{k+1}\Lambda)\, dy d\tau$$

$$= \underbrace{\int_0^t \int_{\mathcal{D}_\tau} (\nabla_j J_\epsilon \partial_t^{k-1} u) \cdot \big([J_\epsilon, F^{ij} Q^0] \nabla_i (\partial_t^{k+1}\Lambda) \big) \, dy d\tau}_{\mathrm{I}}$$

$$+ \underbrace{\int_0^t \int_{\mathcal{D}_\tau} (\nabla_j J_\epsilon \partial_t^{k-1} u)(F^{ij} Q^0) \tilde{\nabla}_i (\partial_t^{k+1}\Lambda) \, dy d\tau}_{\mathrm{II}} .$$

The term, I, being a commutator, can be estimated by Cauchy-Schwarz:

$$\begin{aligned} \mathrm{I} &\le \int_0^t \|\tilde{\nabla} \partial_t^{k-1} u\|^2_{L_2(\mathcal{D}_\tau)} + \|\nabla(F^{ij} Q^0)\|^2_{L^\infty(\partial \mathcal{D}_\tau)} \cdot \|\partial_t^{k+1}\Lambda\|^2_{L^2(\partial\mathcal{D}_\tau)} \, d\tau \\ &\le \int_0^t E^\epsilon_{k-1}[u](\tau) + \|\nabla(F^{ij} Q^0)\|^2_{L^\infty(\partial\mathcal{D}_\tau)} \cdot \underline{E}^\epsilon_k[\Lambda](\tau) \, d\tau. \end{aligned}$$

The term II involves $\tilde{\nabla}\partial_t^{k+1}\Lambda$, which is of higher order than our energy. Thus, we will treat this term by integration by parts, which moves one ∂_t from Λ to u. The procedure is similar to the case when $\mu \neq 0$, so we will discuss the details as we treat the terms with $\mu \neq 0$.

Thus, we have analyzed the case when $\mu = 0$. Henceforth, assume $\mu \neq 0$, so that $\nabla_i \tilde{\nabla}_\mu \partial_t^k \Lambda \overset{\epsilon}{\sim} \tilde{\nabla}_y^{(2)} \partial_t^k \Lambda$. Note that $\tilde{\nabla}^{(2)} \partial_t^k \Lambda$ is of higher order than the energy, while the term $\tilde{\nabla}\partial_t^{k-1} u$ could undertake one more time derivative, so we use integration by parts to transfer one ∂_t onto $\tilde{\nabla}\partial_t^{k-1} u$. The integrals on $\mathcal{D}_0$ and $\mathcal{D}_t$ will show up as we transfer the ∂_t derivative:

$$\begin{aligned} &\int_0^t \int_{\mathcal{D}_\tau} (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(Q^\mu) \nabla_i (\tilde{\nabla}_\mu \partial_t^k \Lambda) \, dy d\tau \\ = &\int_{\mathcal{D}_t} (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(Q^\mu) \nabla_i (\tilde{\nabla}_\mu \partial_t^{k-1} \Lambda) \, dy \\ &- \int_{\mathcal{D}_0} (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(Q^\mu) \nabla_i (\tilde{\nabla}_\mu \partial_t^{k-1} \Lambda) \, dy \\ &- \int_0^t \int_{\mathcal{D}_\tau} (\nabla_j J_\epsilon \partial_t^k u)(F^{ij} Q^\mu) \nabla_i (\tilde{\nabla}_\mu \partial_t^{k-1} \Lambda) \, dy d\tau \\ &- \int_0^t \int_{\mathcal{D}_\tau} (\nabla_j J_\epsilon \partial_t^{k-1} u) \partial_t (F^{ij} Q^\mu) \nabla_i (\tilde{\nabla}_\mu \partial_t^{k-1} \Lambda) \, dy d\tau. \end{aligned} \tag{2.32}$$

By Lemma 2.6, we know that

$$\|\tilde{\nabla}^{(2)} \partial_t^{k-1}\Lambda\|_{L^2(\mathcal{D}_t)} \lesssim \|\tilde{\Box}_h \partial_t^{k-1}\Lambda\|_{L^2(\mathcal{D}_t)} + \|\tilde{\nabla}\partial_t^k \Lambda\|_{L^2(\mathcal{D}_t)} + \|\partial_t^{k+1}\Lambda\|_{L^2(\mathcal{D}_t)}$$

$$\leq \|H^\epsilon_{k-1}\|_{L^2(\mathcal{D}_t)} + \underline{E}_k[\Lambda](t)^{1/2}$$
$$\lesssim E_k[u](t)^{1/2} + \underline{E}_k[\Lambda](t)^{1/2}.$$

We use this observation to control each term on the right hand side of (2.32):

- The first term. We can absorb part of this term into the left hand side of equation (2.28). Indeed, for some $\delta > 0$ to be determined,

$$\begin{aligned}&\left|\int_{\mathcal{D}_t} (F^{ij}\nabla_j J_\epsilon \partial_t^{k-1} u)(Q^\mu)\nabla_i(\tilde{\nabla}_\mu \partial_t^{k-1}\Lambda)\, dy\right|\\ \leq&\frac{1}{\delta}\int_{\mathcal{D}_t} |\tilde{\nabla}\partial_t^{k-1}u|^2\, dy + \delta\int_{\mathcal{D}_t} \|F^{ij}Q^\mu\|^2_{L^\infty(\mathcal{D}_t)} \cdot |\tilde{\nabla}^{(2)}\partial_t^{k-1}\Lambda|^2\, dy\\ \lesssim&\frac{1}{\delta}E^\epsilon_{k-1}[u](t) + \delta\|F^{ij}Q^\mu\|^2_{L^\infty(\mathcal{D}_t)} \cdot \left(E^\epsilon_k[u](t) + \underline{E}^\epsilon_k[\Lambda](t)\right).\end{aligned}$$

  Thus, by choosing δ small enough (independently of ϵ), the term $\delta\|F^{ij}Q^\mu\|^2_{L^\infty(\mathcal{D}_t)}\cdot$ $(E^\epsilon_k[u](t) + \underline{E}^\epsilon_k[\Lambda](t))$ can be absorbed into the left hand side of (2.28), and the term $\frac{1}{\delta}E^\epsilon_{k-1}[u](t)$ is of the desired form as in the right hand side of (2.27).

- The second term. By the same analysis as the previous bullet point, by choosing for instance $\delta = 1$, this term is of the desired form as in the right hand side of (2.27).

- The third term. We have

$$\begin{aligned}&\left|\int_0^t\int_{\mathcal{D}_t} (\nabla_j J_\epsilon \partial_t^k u)(F^{ij}Q^\mu)(\tilde{\nabla}^{(2)}\partial_t^{k-1}\Lambda)\, dyd\tau\right|\\ \lesssim&\int_0^t \|F^{ij}Q^\mu\|_{L^\infty(\mathcal{D}_\tau)} \cdot \left(E_k[u](\tau) + \underline{E}_k[\Lambda](\tau)\right)\, d\tau.\end{aligned}$$

  This is clearly of the form as in the right hand side of (2.27).

- The last term. We have

$$\left|\int_0^t\int_{\mathcal{D}_\tau} (\nabla_j J_\epsilon \partial_t^{k-1} u)\partial_t(F^{ij}Q^\mu)\nabla_i(\tilde{\nabla}_\mu \partial_t^{k-1}\Lambda)\, dyd\tau\right|$$

$$\leq \int_0^t \|\partial_t(F^{ij}Q^\mu)\|_{L^\infty(\mathcal{D}_\tau)} \cdot (E_k^\epsilon[u](\tau) + \underline{E}_k[\Lambda](\tau))\, d\tau.$$

This is also of the form as in the right hand side of (2.27).

Thus, we have controlled every term on the right hand side of (2.32), which completes our analysis of equation (2.29). This closes our energy estimate for terms of form 1 in H_k^ϵ.

Before moving on to estimating terms of forms 2 – 6 in H_k^ϵ, let us first finish the proof we owed in closing terms of form 1 in H_k^ϵ.

Proof of Lemma 2.21. We will use induction on k. By Lemma 2.5, we know

(2.33)

$$\begin{aligned} &\sup_{0\leq\tau\leq t}\left(\int_{\mathcal{D}_\tau} |\tilde{\nabla}_{t,y}\partial_t^k u|^2\, dy\right) + \int_0^t \int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}\partial_t^k u|^2\, dS d\tau \\ \lesssim &\int_{\mathcal{D}_0} |\tilde{\nabla}_{t,y}\partial_t^k u|^2\, dy + \int_0^t \int_{\mathcal{D}_\tau} |\tilde{\Box}_g \partial_t^k u|^2\, dy d\tau + \mathcal{E}_8^\epsilon[u,\Lambda](t)\cdot \int_0^t \int_{\mathcal{D}_\tau} |\tilde{\nabla}\partial_t^k u|^2\, dy d\tau \\ &+ \int_0^t \int_{\partial\mathcal{D}_\tau} |\partial_t^{k+1} u|^2\, dS d\tau + \int_0^t \int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}_n \partial_t^k u|^2\, dS d\tau. \end{aligned}$$

The first four term on the right hand side are easily seen to be bounded by

$$E_{k-1}^\epsilon[u](0) + t\cdot \mathcal{E}_k^\epsilon[u,\Lambda](t)^p$$

for some integer p. It remains to treat the last term. We know that

$$\tilde{\nabla}_n \partial_t^k u = \frac{1}{a}\left(F_k^\epsilon - (u^0)^2 \partial_t^{k+2} u\right).$$

Thus,

$$\begin{aligned} &\int_0^t \int_{\partial\mathcal{D}_\tau} |\tilde{\nabla}_n \partial_t^k u|^2\, dS d\tau \\ \leq &\left\|\frac{1}{a}\right\|_{L^\infty([0,t]\times\partial\mathcal{D})}^2 \cdot \int_0^t \int_{\partial\mathcal{D}_\tau} |F_k^\epsilon|^2\, dS d\tau + \left\|\frac{(u^0)^2}{a}\right\|_{L^\infty([0,t]\times\partial\mathcal{D})}^2 \cdot \int_0^t \int_{\partial\mathcal{D}_\tau} |\partial_t^{k+2} u|^2\, dS d\tau. \end{aligned}$$

Recall that our strategy was to prove by induction, and the previous analyses apply to the case when $k = 0$ as well. To treat F_0^ϵ, we note that the highest order terms are $\tilde{\nabla} X, \tilde{\nabla}\Sigma^2, \tilde{\nabla}\Lambda, \partial_t u$, all of which can be easily bounded by the right hand side in Lemma 2.21. So the base case is true.

Now, for $k \geq 1$, we analyze each term in F_k^ϵ, bearing in mind that by the inductive hypothesis, Lemma 2.21 holds with k replaced by any positive integer smaller than k.

- Terms of form 1. The highest order term is $\partial_t^{k+1} u$, and

$$\int_0^t \int_{\partial \mathcal{D}_\tau} |\partial_t^{k+1} u|^2 \, dS d\tau \leq t \cdot \mathcal{E}_k^\epsilon[u, \Lambda](t).$$

- Terms of form 2. The highest order terms are $\tilde{\nabla}\partial_t^{k+1}\Lambda$, $\partial_t^k g$, and $\tilde{\nabla}\partial_t^{k-1} u$. We have

$$\int_0^t \int_{\partial \mathcal{D}_\tau} |\tilde{\nabla}\partial_t^{k+1}\Lambda|^2 \, dS d\tau \leq \underline{E}_{k+1}[\Lambda](t),$$
$$\int_0^t \int_{\partial \mathcal{D}_\tau} |\partial_t^k g|^2 \, dS d\tau \lesssim \int_0^t \int_{\partial \mathcal{D}_\tau} |\tilde{\nabla}\partial_t^{k-1} u|^2 \, dS d\tau,$$

  and the bound on $\int_0^t \int_{\partial \mathcal{D}_\tau} |\tilde{\nabla}\partial_t^{k-1} u|^2 \, dS d\tau$ follows from the inductive hypothesis.

- Terms of form 3. The highest order terms are $\partial_t^k u$, $\partial_t^k w \overset{\epsilon}{\sim} w\tilde{\nabla}\partial_t^{k-1} u$, $\partial_t^k g \overset{\epsilon}{\sim} \tilde{\nabla}\partial_t^{k-1} u$, $\tilde{\nabla}\partial_t^k \Sigma^2 \overset{\epsilon}{\sim} \tilde{\nabla}\partial_t^{k-1}\Lambda$. Among these terms, the first and the last terms are contained in the energy $\mathcal{E}_{k-1}^\epsilon[u, \Lambda](t)$, and $\tilde{\nabla}\partial_t^{k-1} u$ is controlled by inductive hypothesis.

- Terms of form 4. The highest order terms are $\partial_t^k g$, $\tilde{\nabla}\partial_t^k X \overset{\epsilon}{\sim} \tilde{\nabla}\partial_t^{k-1} u$, $\tilde{\nabla}\partial_t^k \Lambda$. We have shown the bounds on the first two terms, and the last term is contained in $\underline{E}_k^\epsilon[\Lambda](t)$.

- Terms of form 5. The highest order term is $\partial_t^{k+1}u$, which is contained in $\mathcal{E}_k^\epsilon[u,\Lambda](t)$.

Hence, we have shown that every term in (2.33) can be controlled by the claimed formula. The proof is now complete. □

In sum, we finished estimating terms of form 1 in H_k^ϵ. To close the estimate on Λ, we will need to control terms of the forms 2 – 6 in H_k^ϵ.

Terms of forms 2 – 6 in H_k^ϵ.

The highest order terms in form 2 are $\partial_t^{k+1}h \overset{\epsilon}{\sim} \partial_t^{k+1}u + \tilde{\nabla}\partial_t^k u$ and $\tilde{\nabla}\partial_t^l\Lambda$; the highest order terms in form 3 are $\nabla\partial_t^k h$ and $\tilde{\nabla}\partial_t^k\Lambda$; the highest order terms in form 4 are $\nabla\partial_t^k g$, $\tilde{\nabla}\partial_t^k u$, $\nabla^2\partial_t^k X$, and $\tilde{\nabla}^{(2)}\partial_t^k\Sigma^2 \overset{\epsilon}{\sim} \tilde{\nabla}^{(2)}\partial_t^{k-1}\Lambda$; the highest order terms in form 5 are $\partial_t^k g$, $\nabla\partial_t^k X$, and $\tilde{\nabla}\partial_t^k u$; the highest order terms in form 6 are $\partial_t^{k+1}u$ and $\partial_t^{k+1}\Lambda$. When pairing with $\partial_t^{k+1}\Lambda$, all of these terms can be controlled using Cauchy-Schwarz.

Therefore, we have proved

$$\underline{E}_k[\Lambda](t) \lesssim \underline{E}_k[\Lambda](0) + \left|\int_0^t \int_{\mathcal{D}_\tau} (\tilde{\Box}_h\partial_t^k\Lambda)(Q^\mu\tilde{\nabla}_\mu\partial_t^k\Lambda)\,dyd\tau\right| + \int_0^t \underline{E}_k[\Lambda](\tau)\cdot\mathcal{E}_M^\epsilon[u,\Lambda](\tau)^r\,d\tau. \tag{2.34}$$

2.6.2 Estimate on u

The strategy of estimating u is similar to that of Λ, except that we have an additional boundary term.

By Lemma 2.2, we know that there is some integer r such that

(2.35)

$$E_k[u](t) \lesssim E_k[u](0) + \int_0^t E_k[u](\tau)\cdot\mathcal{E}_M^\epsilon[u,\Lambda](\tau)^r\,d\tau +$$

$$+\left|\int_0^t\int_{\mathcal{D}_\tau}(\tilde{\Box}_g\partial_t^k u)(\partial_t^{k+1}u)\,dyd\tau\right|+\left|\int_0^t\int_{\partial\mathcal{D}_\tau}(\mathcal{P}^\epsilon\partial_t^k u)(\partial_t^{k+1}u)\,dSd\tau\right|.$$

We need to bound the last two terms.

Controlling $\left|\int_0^t\int_{\mathcal{D}_\tau}(\tilde{\Box}_g\partial_t^k u)(\partial_t^{k+1}u)\,dyd\tau\right|$

We first analyze terms in $(\tilde{\Box}_g\partial_t^k u)(\partial_t^{k+1}u)=G_k^\epsilon\cdot(\partial_t^{k+1}u)$. As in the case of Λ, G_k^ϵ contains 6 types of terms, and we shall control each one of them.

Terms of form 1 in G_k^ϵ. The highest order terms of form 1 in G_k^ϵ are $\nabla\partial_t^k g$ and $\tilde{\nabla}^{(2)}\partial_t^{k-1}u$. The analysis is very similar to that of the terms of form 1 in H_k^ϵ, so we shall highlight the differences. One major difference is that $\partial_t^k\Lambda\equiv 0$ on $\partial\mathcal{D}$, while $\partial_t^{k+1}u$ is not, leaving us an extra boundary term to tackle. Another difference is that $Q^\mu\tilde{\nabla}_\mu\Lambda$ contains spatial derivatives on Λ, but $\partial_t^{k+1}u$ only contains time derivatives, making this part of the computation simpler. We now present the result.

As in the case of Λ, $\tilde{\nabla}^{(2)}\partial_t^{k-1}u$ is of higher order than the energy, so we integrate by parts to move one $\tilde{\nabla}_y$ derivative onto $\partial_t^{k+1}u$ and one ∂_t derivative onto $\tilde{\nabla}^{(2)}\partial_t^{k-1}u$. The first part of this computation is similar to the case of Λ:

$$\begin{aligned}
&(F^{ij}\nabla_iJ_\epsilon\nabla_jJ_\epsilon\partial_t^{k-1}u)(\partial_t^{k+1}u)\\
=&([F^{ij},\nabla_iJ_\epsilon]\nabla_jJ_\epsilon\partial_t^{k-1}u)(\partial_t^{k+1}u)+\nabla_iJ_\epsilon(F^{ij}\nabla_jJ_\epsilon\partial_t^{k-1}u)(\partial_t^{k+1}u)\\
=&-(\nabla_iF^{ij})(\partial_jJ_\epsilon\partial_t^{k-1}u)(\partial_t^{k+1}u)+([F^{ij},J_\epsilon]\nabla_i\nabla_jJ_\epsilon\partial_t^{k-1}u)(\partial_t^{k+1}u)\\
&\quad+\nabla_iJ_\epsilon(F^{ij}\nabla_jJ_\epsilon\partial_t^{k-1}u)(\partial_t^{k+1}u).
\end{aligned}$$

The first two terms can be controlled by Cauchy-Schwarz. We will focus on the last term. Integrating on $[0,t]\times\mathcal{D}$, we have

$$\begin{aligned}
&\int_0^t\int_{\mathcal{D}_\tau}\nabla_iJ_\epsilon(F^{ij}\nabla_jJ_\epsilon\partial_t^{k-1}u)(\partial_t^{k+1}u)\,dyd\tau\\
=&-\int_0^t\int_{\mathcal{D}_\tau}(F^{ij}\nabla_jJ_\epsilon\partial_t^{k-1}u)\nabla_iJ_\epsilon(\partial_t^{k+1}u)\,dyd\tau
\end{aligned}$$

$$+\int_0^t \int_{\partial \mathcal{D}_\tau} n_i J_\epsilon (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(\partial_t^{k+1} u)\, dSdt.$$

The last term vanishes in the case of Λ. In our case here, we shall tackle it using Lemma 2.21:

$$\begin{aligned}
&\left| \int_0^t \int_{\partial \mathcal{D}_\tau} n_i J_\epsilon (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(\partial_t^{k+1} u)\, dSd\tau \right| \\
\lesssim &\int_0^t \int_{\partial \mathcal{D}_\tau} |\tilde{\nabla} \partial_t^{k-1} u|^2 \, dSd\tau + \int_0^t \int_{\partial \mathcal{D}_\tau} |\partial_t^{k+1} u|^2 \, dSd\tau \\
\lesssim & E^\epsilon_{\leq k-1}[u](0) + R_\eta(\mathcal{E}^\epsilon_{k-1}[u,\Lambda](t)) + \eta \mathcal{E}^\epsilon_k[u,\Lambda](t) + \int_0^t E_k[u](\tau)\, d\tau.
\end{aligned}$$

By choosing $\eta > 0$ small, we can absorb the term $\eta \mathcal{E}^\epsilon_k[u,\Lambda](t)$ into the left hand side of (2.27). All the remaining terms are of the desired form.

The other term, $\int_0^t \int_{\mathcal{D}_\tau} (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u) \nabla_i J_\epsilon (\partial_t^{k+1} u)\, dyd\tau$, can be treated similarly as in the case of Λ, where the idea is to move one ∂_t derivative away from $\tilde{\nabla} \partial_t^{k+1} u$. We have

$$\begin{aligned}
&(F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u) \nabla_i J_\epsilon (\partial_t^{k+1} u) \\
=&(F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u) \partial_t (\nabla_i J_\epsilon \partial_t^k u) \\
=&\partial_t \left[(F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(\nabla_i J_\epsilon \partial_t^k u) \right] - \partial_t (F^{ij})(\nabla_j J_\epsilon \partial_t^{k-1} u)(\nabla_i J_\epsilon \partial_t^k u) \\
&- (F^{ij})(\nabla_j J_\epsilon \partial_t^k u)(\nabla_i J_\epsilon \partial_t^k u).
\end{aligned}$$

Integrating on $[0,t] \times \mathcal{D}$, we have

$$\begin{aligned}
&\int_0^t \int_{\mathcal{D}_\tau} (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u) \nabla_i J_\epsilon (\partial_t^{k+1} u)\, dyd\tau \\
=&\int_{\mathcal{D}_t} (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(\nabla_i J_\epsilon \partial_t^k u)\, dy - \int_{\mathcal{D}_0} (F^{ij} \nabla_j J_\epsilon \partial_t^{k-1} u)(\nabla_i J_\epsilon \partial_t^k u)\, dy \\
&- \int_0^t \int_{\mathcal{D}_\tau} \partial_t (F^{ij})(\nabla_j J_\epsilon \partial_t^{k-1} u)(\nabla_i J_\epsilon \partial_t^k u)\, dyd\tau \\
&- \int_0^t \int_{\mathcal{D}_\tau} (F^{ij})(\nabla_j J_\epsilon \partial_t^k u)(\nabla_i J_\epsilon \partial_t^k u)\, dyd\tau.
\end{aligned}$$

As in the case of Λ, the first term can be absorbed into the left hand side of (2.35) by using Cauchy-Schwarz Inequality and choosing a small δ (which is independent of ϵ); the other two terms can be controlled using Cauchy-Schwarz Inequality directly. The computation is precisely the same as in the case of Λ, so we omit the details. Hence, we have finished controlling

$$\int_0^t \int_{\mathcal{D}_\tau} (\nabla \partial_t^k g)(\partial_t^{k+1} u)\, dy d\tau.$$

The term $(\tilde{\nabla}^{(2)} \partial_t^{k-1} u)(\partial_t^{k+1} u)$ has also been controlled in the analysis. Thus, we have finished estimating the terms in G_k^ϵ that take form 1.

Terms of form 2 – 6 in G_k^ϵ

The highest order terms of form 2 are $\partial_t^{k+1} g$ and $\tilde{\nabla} \partial_t^k u$; the highest order terms of form 3 are $\nabla \partial_t^k g$ and $\tilde{\nabla} \partial_t^{k-1} u$. All are either controlled, or are of lower order than those in 1.

The highest order terms of form 4 are $\partial_t^k g$, $\nabla \partial_t^k X$, and $\tilde{\nabla} \partial_t^k w$. The former two have been estimated in the case of Λ, so we shall focus on the third. Recall that $\partial_t w$ is a linear combination of terms of the form $\frac{1}{J_\epsilon u^0} g(\nabla X)(\tilde{\nabla} u) w$. Thus, $\partial_t^k w$ is a linear combination of terms of the form

$$\frac{1}{(J_\epsilon u^0)^r} (\partial_t^{k_1} J_\epsilon u^0)(\partial_t^{k_2} g)(\nabla \partial_t^{k_3} X)(\tilde{\nabla} \partial_t^{k_4} u)(\partial_t^{k_5} w)$$

where r is a positive integer, and $k_1 + \cdots + k_5 = k - 1$. Therefore, the highest order terms in $\nabla \partial_t^k w$ are: $\tilde{\nabla} \partial_t^{k-1} u$, $\nabla \partial_t^{k-1} g$, $\nabla^{(2)} \partial_t^{k-1} X$, $\tilde{\nabla}^{(2)} \partial_t^{k-1} u$, $\nabla \partial_t^{k-1} w$. We have shown that the first three can be controlled. The last can be controlled by induction. The most difficult term is $\tilde{\nabla}^{(2)} \partial_t^{k-1} u$, but we have shown that when paired with $\partial_t^{k+1} u$, this term can also be bounded. Therefore, $\tilde{\nabla} \partial_t^k w$ is controlled as required.

The highest order terms of form 5 are $\partial_t^k g$, $\nabla \partial_t^k X$, $\tilde{\nabla} \partial_t^k \Sigma^2 \overset{\epsilon}{\sim} \tilde{\nabla} \partial_t^{k-1} \Lambda$, and $\partial_t^k \Lambda$. It is clear that all have their $L^2(\mathcal{D}_t)$ norms controlled.

The highest order terms of form 6 are $\partial_t^k g$, $\nabla\partial_t^k X$, $\partial_t^k\Sigma^2 \overset{\epsilon}{\sim} \partial_t^{k-1}\Lambda$, and $\tilde{\nabla}\partial_t^k\Lambda$. It is again clear that all have their $L^2(\mathcal{D}_t)$ norms controlled.

Therefore, we have shown that

$$\left|\int_0^t\int_{\mathcal{D}_\tau}(\tilde{\Box}_g\partial_t^k u)(\partial_t^{k+1}u)\,dyd\tau\right| \lesssim \mathfrak{E}_k^\epsilon[u,\Lambda](0) + \int_0^t P_M(\mathfrak{E}_k^\epsilon[u,\Lambda](\tau))\,d\tau$$

as claimed in (2.27).

Controlling $\left|\int_0^t\int_{\partial\mathcal{D}_\tau}(\mathcal{P}^\epsilon\partial_t^k u)(\partial_t^{k+1}u)\,dSd\tau\right|$

It remains to treat the boundary term

$$\left|\int_0^t\int_{\partial\mathcal{D}_\tau}(\mathcal{P}^\epsilon\partial_t^k u)(\partial_t^{k+1}u)\,dSd\tau\right|.$$

Recall that F_k^ϵ contains 5 types of terms, and we shall deal with each type.

<u>Terms of form 1 in F_k^ϵ.</u> The highest order term of type 1 in F_k^ϵ is $\partial_t^{k+1}u$, which is clearly in $L^2(\partial\mathcal{D}_t)$.

<u>Terms of form 2 in F_k^ϵ.</u> The terms of type 2 needs some more analysis. The highest order terms are $\tilde{\nabla}\partial_t^k\Sigma^2 \overset{\epsilon}{\sim} \tilde{\nabla}\partial_t^{k-1}\Lambda$, $\partial_t^k g$, $\tilde{\nabla}\partial_t^{k-1}u$. The first type, $\tilde{\nabla}\partial_t^{k-1}\Lambda$, is already in the energy.

We know that $\partial_t^k g = G\tilde{\nabla}\partial_t^{k-1}u$ where G is a function such that

$$\sup_{0\le t\le T}\|G\|_{L^\infty(\mathcal{D}_t)} + \|\nabla G\|_{L^\infty(\mathcal{D}_t)} \lesssim \mathcal{E}_{k-1}^\epsilon(T)^r$$

for some integer r. Therefore, using Lemma 2.21, we obtain

$$\begin{aligned}
&\int_0^t\int_{\partial\mathcal{D}_\tau} G(\tilde{\nabla}\partial_t^{k-1}u)(\partial_t^{k+1}u)\,dSd\tau\\
\le&\int_0^t\int_{\partial\mathcal{D}_\tau}|\tilde{\nabla}\partial_t^{k-1}u|^2\,dSd\tau + \int_0^t\int_{\partial\mathcal{D}_\tau}|\partial_t^{k+1}u|^2\,dSd\tau\\
\lesssim& E_{\le k-1}^\epsilon[u](0) + R_\eta(\mathcal{E}_{k-1}^\epsilon[u,\Lambda](t)) + \eta\mathcal{E}_k^\epsilon[u,\Lambda](t) + \int_0^t E_k[u](\tau)\,d\tau.
\end{aligned}$$

As before, we choose $\eta > 0$ small so that $\eta\mathcal{E}_k^\epsilon[u,\Lambda](t)$ can be absorbed into the left hand side of the energy, and the rest are of the desired form. The only case left for terms of form 2 is then $\tilde{\nabla}\partial_t^{k-1}u$. But the analysis on $\partial_t^k g$ already contains an analysis on this term. Hence, terms of form 2 have been shown to be bounded.

Terms of form $3-5$ in F_k^ϵ. The highest order terms in form 3 are $\partial_t^k u$, $\partial_t^k g$, $\partial_t^k w$, $\tilde{\nabla}\partial_t^k \Sigma^2$. The only term that hasn't been shown to be bounded is $\partial_t^k w$. The highest order term in $\partial_t^k w$ is $\tilde{\nabla}\partial_t^{k-1}u$, and we have bounded its $L^2(\partial\mathcal{D})$ norm.

The highest order terms in form 4 is $\tilde{\nabla}\partial_t^k\Lambda$, and that of form 5 is $\partial_t^{k+1}u$. Both are clearly bounded.

Therefore, we have completed the estimate on u.

Summing the energy estimates on u and Λ, we have thus obtained an energy estimate on u and Λ that is uniform in ϵ.

2.7 Conclusion on the a priori Estimate

In summary, in this section, we proved an a priori estimate for the mollified equations (1.56)-(1.59). One important component is the trade-off between spatial and time derivatives, which enables us to convert the energy $\mathcal{E}$ into energies based on Sobolev norms.

As remarked before, if we set $J_\epsilon = \mathrm{Id}$, then we obtain the a priori estimate for the un-mollified system of equations (1.42)-(1.48). If we set $J_\epsilon = \mathrm{Id}$ and let $\mathcal{D} = B$, then we obtain the a priori estimate for the un-mollified system of equations on a bounded domain.

The a priori estimate plays an important role in the proof for existence and uniqueness of solutions, which is the topic of the next chapter.

CHAPTER III

Existence of Solution on an Unbounded Domain

Having obtained the uniform bound on energy, we are now ready to address the existence and uniqueness of solution to the original equation. In this section, we let M be the same integer that appeared in Theorem 2.1. That is, M is the total number of ∂_t that we commute with $\tilde{\Box}_g$ when deriving the a priori estimate on u. Since we do not aim at achieving the lowest regularity result in this dissertation, we may assume, for instance, $M = 10$.

3.1 Existence of Solution to the Mollified Equations

We first show that for any fixed $\epsilon > 0$, the mollified system of equations (1.56-1.59) has a solution. To do so, we need to prove that the right hand side of these equations are Lipschitz continuous with respect to a norm that we will specify. The precise statement is in Proposition 3.4.

Before stating the proposition, we first describe why mollification makes the system an ODE. This is because for $\epsilon > 0$, mollified derivatives can be annihilated by orders of $1/\epsilon$, as described in the following Lemma.

Lemma 3.1. *Let $k \geq 0$ be an integer. Then for any $\phi \in L^2(\mathbb{R}^3)$, we have*

$$\|\nabla^k J_\epsilon \phi\|_{L^2} \lesssim \frac{1}{\epsilon^k} \cdot \|\phi\|_{L^2}, \tag{3.1}$$

where the constant does not depend on ϕ or ϵ.

In particular, by Sobolev embedding, we also have

$$\|J_\epsilon \phi\|_{L^\infty} \lesssim \frac{1}{\epsilon^2}\|\phi\|_{L^2}. \tag{3.2}$$

Proof. We have

$$\begin{aligned}\|\nabla^k J_\epsilon \phi\|_{L^2}^2 &\lesssim \int_{|\xi|\le 1/\epsilon} |\xi|^{2k}|\hat{\phi}(\xi)|^2\, d\xi \\ &\le \frac{1}{\epsilon^{2k}}\int_{|\xi|\le 1/\epsilon} |\hat{\phi}(\xi)|^2\, d\xi.\end{aligned}$$

This is the desired result. □

Now we are ready to provide the ODE system.

3.2 The ODE System

Definition 3.1. We define the operators on u and Λ. We write

$$\mathbf{\Phi}^\epsilon(u,u')(y) := \begin{cases} \frac{1}{(J_\epsilon u^0)^2}\left(\mathcal{P}^\epsilon u - \frac{1}{2} a n_\alpha J_\epsilon\left(g^{\alpha 0}u' + g^{\alpha j}\tilde{\nabla}_j u\right)\right) & \text{if } y \in \partial\mathbb{R}^3_+ \\ \frac{1}{g^{00}}\Big[-(\partial_t g^{00})u' - (\partial_t g^{0j})\tilde{\nabla}_j u - g^{0j}(\tilde{\nabla}_j u') & \\ \qquad -\tilde{\nabla}_j(g^{0j}u') - \tilde{\nabla}_i(g^{ij}\tilde{\nabla}_j u) + \tilde{\Box}_g u\Big] & \text{if } y \in \mathbb{R}^3_+. \end{cases}$$

$$\mathbf{\Psi}^\epsilon(\Lambda,\Lambda')(y) := \begin{cases} 0 & \text{if } y \in \partial\mathbb{R}^3_+ \\ \frac{1}{h^{00}}\Big[-(\partial_t h^{00})\Lambda' - (\partial_t h^{0j})\tilde{\nabla}_j \Lambda - h^{0j}(\tilde{\nabla}_j \Lambda') & \\ \qquad -\tilde{\nabla}_j(h^{0j}\Lambda') - \tilde{\nabla}_i(h^{ij}\tilde{\nabla}_j \Lambda) + \tilde{\Box}_h \Lambda\Big] & \text{if } y \in \mathbb{R}^3_+. \end{cases}$$

Definition 3.2. To simplify the notation, we suppress the dependence of u, Λ on ϵ. We define the ODE system as follows:

$$\frac{d}{dt}u^{(0)} = u^{(1)} \tag{3.3}$$

$$\frac{d}{dt}u^{(k+1)} = \mathbf{\Phi}^\epsilon(u^{(k)}, u^{(k+1)}) \quad \forall k = 0, \cdots, M \tag{3.4}$$

$$\frac{d}{dt}\Lambda^{(0)} = \Lambda^{(1)} \tag{3.5}$$

$$\frac{d}{dt}\Lambda^{(k+1)} = \mathbf{\Psi}^\epsilon(\Lambda^{(k)}, \Lambda^{(k+1)}) \quad \forall k = 0, \cdots, M \tag{3.6}$$

$$\frac{d}{dt}\Sigma^2 = J_\epsilon\left(\frac{1}{J_\epsilon u^0} J_\epsilon \Lambda\right) \tag{3.7}$$

$$\frac{d}{dt}X^j = J_\epsilon\left(\frac{J_\epsilon u^j}{J_\epsilon u^0}\right) \tag{3.8}$$

$$\frac{d}{dt}w^{\mu\nu} = J_\epsilon\left(\frac{1}{J_\epsilon u^0}\left(-g^{\delta\gamma}(\tilde{\nabla}_\delta X^\mu)(\tilde{\nabla}_\gamma u_\alpha)w^{\alpha\nu} + g^{\gamma\delta}(\tilde{\nabla}_\delta X^\nu)(\tilde{\nabla}_\gamma u_\alpha)w^{\alpha\mu}\right)\right). \tag{3.9}$$

Symbolically, we write $u := (u^{(0)}, \cdots u^{(M+1)})$, $\Lambda := (\Lambda^{(0)}, \cdots, \Lambda^{(M+1)})$, and

$$\frac{d}{dt}\begin{pmatrix} u \\ \Lambda \\ \Sigma^2 \\ X \\ w \end{pmatrix} = \mathbf{F}^\epsilon \begin{pmatrix} u \\ \Lambda \\ \Sigma^2 \\ X \\ w \end{pmatrix}. \tag{3.10}$$

In the definition of $\mathbf{\Phi}^\epsilon$ and $\mathbf{\Psi}^\epsilon$, the formula for $\mathcal{P}^\epsilon u^{(M)}$, $\tilde{\Box}_g u^{(M)}$, and $\tilde{\Box}_h \Lambda^{(M)}$ are replaced by F^ϵ_M, G^ϵ_M, H^ϵ_M respectively.

Remark 9. Note that we did not define higher order derivatives of Σ^2, X, w, so whenever we encounter higher order derivatives of Σ^2, X, w in the formulae of $F^\epsilon_M, G^\epsilon_M, H^\epsilon_M$, we replace them with functions of $u^{(k)}$ and $\Lambda^{(k)}$. For instance, if $\partial_t^k g$ contains a term $\tilde{\nabla}\partial_t^j u$, then we replace it with $\tilde{\nabla} u^{(j)}$.

Remark 10. Note that in fact, $\mathbf{\Phi}^\epsilon(u^{(k)}, u^{k+1})$ and $\mathbf{\Phi}^\epsilon(\Lambda^{(k)}, \Lambda^{(k+1)})$ depends also on the lower order terms $u^{(j)}, \Lambda^{(j)}$ for $j = 0, \cdots, k-1$. We chose not to write the full dependency so that the definition has a clearer format.

Definition 3.3. We consider the space $\mathcal{B}$ with norm

$$\left\|(u, \Lambda, \Sigma^2, X, w)\right\|_{\mathcal{B}} \tag{3.11}$$

$$= \left(\sum_{k=0}^{M} \|u^{(k)}\|_{H^1(\mathbb{R}^3_+)} + \|u^{(k+1)}\|_{L^2(\partial\mathbb{R}^3_+)} + \|\Lambda^{(k)}\|_{H^1(\mathbb{R}^3_+)} + \|\Lambda^{(k+1)}\|_{L^2(\partial\mathbb{R}^3_+)} \right)$$

$$+ \|u^{(M+1)}\|_{L^2(\mathbb{R}^3_+)} + \|\Lambda^{(M+1)}\|_{L^2(\mathbb{R}^3_+)}$$

$$+ \|\Sigma^2\|_{H^1(\mathbb{R}^3_+)} + \|X\|_{H^1(\mathbb{R}^3_+)} + \|w\|_{H^1(\mathbb{R}^3_+)}.$$

We denote the right hand side of (3.3)-(3.9) by $\mathbf{F}^\epsilon$. We need to show that $\mathbf{F}^\epsilon$ is a map from $\mathcal{B}$ to $\mathcal{B}$.

Lemma 3.2. *Let* $(u, \Lambda, \Sigma^2, X, w) \in \mathcal{B}$ *satisfy (3.3)-(3.9). Assume* $(u^0)^{(0)} \geq 2c > 0$ *at* $t = 0$. *Then there is some* $T^\epsilon > 0$ *such that for all* $t \in [0, T^\epsilon]$, *we have*

$$J_\epsilon (u^0)^{(0)} \geq c > 0.$$

Proof. We have

$$J_\epsilon(u^0(t,y))^{(0)} = J_\epsilon(u^0(0,y))^{(0)} + \int_0^t \frac{d}{dt} J_\epsilon(u^0(\tau,y))^{(0)}\, d\tau$$

$$\geq J_\epsilon(u^0(0,y))^{(0)} - t \cdot \left\| J_\epsilon(u^0)^{(1)} \right\|_{L^\infty(\mathbb{R}^3_+)}.$$

By Lemma 3.1,

$$\left\| J_\epsilon(u^0)^{(1)} \right\|_{L^\infty(\mathbb{R}^3_+)} \lesssim \frac{1}{\epsilon} \|u^{(1)}\|_{H^1(\mathbb{R}^3_+)},$$

so for t small, we have the desired result. □

Lemma 3.3. *Let* $(u, \Lambda, \Sigma^2, X, w) \in \mathcal{B}$. *Then*

$$\mathbf{F}^\epsilon(u, \Lambda, \Sigma^2, X, w) \in \mathcal{B}.$$

Proof. The idea is that by Lemma 3.1, we can control higher order derivatives by lower order derivatives (compensated with powers of $1/\epsilon$).

We first estimate X, w, g, h in L^∞ norm. Recall that $\partial_t X^j = J_\epsilon \left(\frac{J_\epsilon(u^j)^{(0)}}{J_\epsilon(u^0)^{(0)}} \right)$. Hence,

$$\left\| \nabla^{(\ell)} \partial_t X \right\|_{L^\infty(\mathbb{R}^3_+)} \lesssim \frac{1}{\epsilon^{\ell+2}} \left\| J_\epsilon u^{(0)} \right\|_{L^2(\mathbb{R}^3_+)}.$$

Similarly,

$$\left\|\nabla^{(\ell)}\partial_t^k X\right\|_{L^\infty(\mathbb{R}^3_+)} \lesssim \frac{1}{\epsilon^{\ell+2}}\left\|J_\epsilon u^{(k)}\right\|_{L^2(\mathbb{R}^3_+)}.$$

Thus, for all $\ell \geq 0$ and for all $k \leq M+1$, we have $\nabla^{(\ell)}\partial_t^k X \in L^\infty$. Similar estimates hold for w, g, h.

Next, we may estimate the right hand side of equations (3.3)-(3.9). The most difficult cases are for $\frac{d}{dt}u^{(M+1)}$ and $\frac{d}{dt}\Lambda^{(M+1)}$, so we shall focus on these two highest order equations. The lower order equations will follow similarly.

Interior terms. We need to prove that $\mathbf{\Phi}^\epsilon(u^{(M)}, u^{(M+1)}) \in L^2(\mathbb{R}^3_+)$. The equation in the interior part of $\mathbf{\Phi}^\epsilon(u^{(M)}, u^{(M+1)})$ contains terms of the forms

$$(\nabla^{\ell_1}\partial_t^{r_1}\phi_1)\cdots(\nabla^{\ell_m}\partial_t^{r_m}\phi_m)\cdot(\tilde{\nabla}^{(p)}f^{(k)})$$

and

$$(\nabla^{\ell_1}\partial_t^{r_1}\phi_1)\cdots(\nabla^{\ell_m}\partial_t^{r_m}\phi_m)\cdot(u^{(M+1)})$$

where $(\nabla^{\ell_1}\partial_t^{r_1}\phi_1)\cdots(\nabla^{\ell_m}\partial_t^{r_m}\phi_m)$ are terms that can be bounded in L^∞ norm by the first paragraph in the proof, and $f \in \{u, \Lambda\}$ represents the highest order term, with $p \leq 2$ and $k \leq M+1$. But by Lemma (3.1),

$$\|\tilde{\nabla}^{(p)}f^{(k)}\|_{L^2(\mathbb{R}^3_+)} \lesssim \frac{1}{\epsilon^2}\|f^{(k)}\|_{L^2(\mathbb{R}^3_+)},$$

so overall, $\mathbf{\Phi}^\epsilon(u^{(M)}, u^{(M+1)}) \in L^2(\mathbb{R}^3_+)$. The estimate for $\mathbf{\Psi}^\epsilon(\Lambda^{(M)}, \Lambda^{(M+1)})$, as well as the right hand side of equations for Σ^2, X, w, are similar.

Boundary terms. The boundary condition for Λ is easy to see, so we focus the discussion on $u^{(M+1)}$. The boundary definition of $\mathbf{\Phi}^\epsilon(u^{(M)}, u^{(M+1)})$ contains three types of terms:

- $\frac{1}{(J_\epsilon(u^{(0)})^0)^2}\mathcal{P}^\epsilon\left(u^{(M)}\right) = \frac{1}{(J_\epsilon(u^{(0)})^0)^2}F_k^\epsilon$. By Lemma 3.2, we are able to control $\frac{1}{(J_\epsilon(u^{(0)})^0)^2}$ in L^∞ norm, so it suffices to analyze the terms in F_k^ϵ. For the mollified terms such as $\tilde{\nabla}_j u^{(k)}$, we can simply use the Trace Theorem (Lemma B):

$$\left\|\tilde{\nabla}_j u^{(k)}\right\|_{L^2(\partial\mathbb{R}^3_+)} \lesssim \left\|\tilde{\nabla}_j u^{(k)}\right\|_{H^1(\mathbb{R}^3_+)} \le \frac{1}{\epsilon}\left\|u^{(k)}\right\|_{H^1(\mathbb{R}^3_+)}.$$

The only un-mollified terms are $\phi \cdot u^{(M+1)}$ where ϕ is a lower order term that can be controlled in L^∞ norm. Then clearly we may bound this term in $L^2(\partial\mathbb{R}^3_+)$.

- $an_\alpha J_\epsilon(g^{\alpha 0}u^{(M+1)})$. Again, the control follows from $\|u^{(M+1)}\|_{L^2(\partial\mathbb{R}^3_+)}$.
- $an_\alpha J_\epsilon(g^{\alpha j}\tilde{\nabla}_j u^{(M)})$. We may bound this term using the Trace Theorem, similar to what we did in the first bullet point.

Therefore, $\mathbf{F}^\epsilon$ is a map from $\mathcal{B}$ to $\mathcal{B}$. □

Next, we shall prove that $\mathbf{F}^\epsilon$ is Lipschitz continuous, which will enable us to prove an existence and uniqueness result of the ODE system.

Proposition 3.4. *Let $\mathcal{O}_r := \{\phi \in \mathcal{B} : \|\phi\|_{\mathcal{B}} \le r\}$. There is a constant $C_2 > 0$ (which depends on ϵ and r), such that for any $\phi, \psi \in \mathcal{O}_r$, we have*

$$\|\mathbf{F}^\epsilon(\phi) - \mathbf{F}^\epsilon(\psi)\|_{\mathcal{B}} \le C_2\|\phi - \psi\|_{\mathcal{B}}. \tag{3.12}$$

Proof. Note that the formula for components of $\mathbf{F}^\epsilon$ are sums of terms of the form

$$(\tilde{\nabla}^{a_1}\phi_1)\cdots(\tilde{\nabla}^{a_p}\phi_p)$$

where $p > 0$ is an integer, $a_1, \cdots, a_p \in \{0, 1, 2\}$, and

$$\phi_1, \cdots, \phi_p \in \{u^{(k)}, \Lambda^{(k)}, \partial_t^k w, \partial_t^k X, \partial_t^k g, \partial_t^k h : k \le M+1\}.$$

Therefore, we may write the difference using the standard triangular trick

$$(\tilde{\nabla}^{a_1}\phi_1)\cdots(\tilde{\nabla}^{a_p}\phi_p) - (\tilde{\nabla}^{a_1}\psi_1)\cdots(\tilde{\nabla}^{a_p}\psi_p)$$

$$= (\tilde{\nabla}^{a_1}(\phi_1 - \psi_1))(\tilde{\nabla}^{a_2}\phi_2)\cdots(\tilde{\nabla}^{a_p}\phi_p) + (\tilde{\nabla}^{a_1}\psi_1)(\tilde{\nabla}^{a_2}(\phi_2 - \psi_2))(\tilde{\nabla}^{a_3}\phi_3)\cdots(\tilde{\nabla}^{a_p}\phi_p)+$$

$$+ \cdots + (\tilde{\nabla}^{a_1}\psi_1)\cdots(\tilde{\nabla}^{a_{p-1}}\psi_{p-1})(\tilde{\nabla}^{a_p}(\phi_p - \psi_p)).$$

By Lemma 3.1, we can again control the terms $\tilde{\nabla}^{a_i}(\phi_i - \psi_i)$ in $\|\cdot\|_{\mathcal{B}}$ and the non-difference terms in L^∞. This gives the desired result. $\square$

Next, we appeal to the standard ODE existence and uniqueness theory (see, for instance, [11]).

Lemma 3.5. *Let X be a Banach space, $\psi_0 \in X$, $B(\psi_0, r) = \{\phi \in X : \|\phi - \psi_0\|_X \leq r\}$. Let $I = [-T, T]$. Consider the ODE system*

$$(*) \qquad \frac{d}{dt}\phi = F(\phi(t)), \quad \phi(0, y) = \phi_0(y).$$

Assume that $F : B(\psi_0, r) \to X$ satisfies the following conditions:

1. *There is a constant $L < \infty$ such that*

$$\|F(\phi) - F(\psi)\|_X \leq L\|\phi - \psi\|_X \quad \forall \phi, \psi \in B(\phi_0, r).$$

2. *There is a constant K such that for all $\phi \in B(\phi_0, r)$,*

$$\|F(\phi)\|_X \leq K.$$

Let $T_0 < \min\{T, r/K\}$. Then the following are true.

1. *For each $\phi_0 \in B(\psi_0, r - KT_0)$, the ODE system $(*)$ has a unique solution on the interval $J = [-T_0, T_0]$.*

2. *The solution ϕ depends continuously on ϕ_0, and ϕ and $\frac{d}{dt}\phi$ are both jointly continuous in (t, y).*

3. Let $\mathcal{U}$ be an open subset of X. Assume that $\|F(\phi)-F(\psi)\|_X \le L\|\phi-\psi\|_X$ for all $\phi,\psi\in\mathcal{U}$. Let $\phi_0\in\mathcal{U}$. Then there is a maximal time of existence $(-a,b)$ (with possibly $a,b=\infty$) such that either

$$\limsup_{t\nearrow b}\|F(\phi(t))\|_X=\infty,$$

or

$$\lim_{t\nearrow b}\phi(t):=\phi(b-)\in X \text{ exists, but } \phi(b-)\notin\mathcal{U}.$$

Proof. 1. Let $Y:=C(I,B(\phi_0,r))$. We define

$$S(\phi)(t):=\phi_0+\int_0^t F(\phi(\tau))\,d\tau.$$

Note that if $\phi\in Y$, then

$$\|S(\phi)-\phi_0\|_Y\le\left|\int_0^t\|F(\phi(\tau))\|_X\,d\tau\right|\le|t|K,$$

so for $t\in[-T_0,T_0]$, we know that $S(\phi)\in B(\phi_0,r)$. Thus, $S:Y\to Y$.

Now, ϕ solves the ODE iff $S(\phi)=\phi$. We define $\phi_0(t,y)\equiv\phi_0(y)$, and $\phi_n:=S(\phi_{n-1})$. Then,

$$\begin{aligned}\|\phi_n(t)-\phi_{n-1}(t)\|_X&=\left\|\int_0^t F(\phi_{n-1}(\tau))-F(\phi_{n-2}(\tau))\,d\tau\right\|_X\\&\le L\int_0^t\|\phi_{n-1}(\tau)-\phi_{n-2}(\tau)\|_X\,d\tau,\end{aligned}$$

so inductively,

$$\begin{aligned}&\|\phi_n(t)-\phi_{n-1}(t)\|_X\\ \le&L^{n-1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-2}}\|\phi_1(t_{n-2})-\phi_0(t_{n-2})\|_X\,dt_{n-2}\cdots dt_1dt\\ \le&\frac{(Lt)^{n-1}}{(n-1)!}\|\phi_1-\phi_0\|_Y.\end{aligned}$$

Now,

$$\|\phi_1 - \phi_0\|_Y = \sup_{t\in I} \left| \int_0^t F(\phi_0)\, d\tau \right|$$
$$\leq T\|F(\phi_0)\|_X \leq KT.$$

Therefore,

$$\|\phi_n - \phi_{n-1}\|_Y \leq \frac{KL^{n-1}T^n}{(n-1)!},$$

and thus

$$\sum_n \|\phi_n - \phi_{n-1}\|_Y < \infty.$$

That is, ϕ_n is a Cauchy sequence in Y, and thus converges to come $\phi \in L^\infty(J, X)$ (uniformly in (t, y)). Hence, ϕ satisfies the integral form of the ODE:

$$\phi(t) = \phi_0 + \int_0^t F(\phi(\tau))\, d\tau.$$

Thus $\phi(t)$ is differentiable in t and satisfies the differential form of the ODE.

To show that ϕ is unique, suppose ϕ and $\tilde{\phi}$ are two solutions, and let $e(t) = \|\phi(t) - \tilde{\phi}(t)\|_X$. Then

$$e(t) \leq K \int_0^t \|\phi(\tau) - \tilde{\phi}(\tau)\|_X\, d\tau$$
$$\leq K \int_0^t e(\tau)\, d\tau.$$

Since $e(0) = 0$, by Grönwall's Inequality (Lemma B.5), we know that $e(t) \equiv 0$. Thus ϕ is unique.

2. The continuity of ϕ follows since it is a uniform limit of continuous functions. The continuity of $\frac{d}{dt}\phi = \phi_0 + F(\phi)$ follows since ϕ_0 is continuous in y and F is Lipschitz continuous.

3. By uniqueness of local solution, it is clear there there is a maximal time of existence for solution to the ODE $(*)$. Assume that

$$M := \limsup_{t \nearrow b} \|F(\phi(t))\|_X < \infty.$$

Then for all t close to b, we know that $\|F(\phi(t))\|_X \leq 2M$, so for $t, \tilde{t}$ close to b, we have

$$\|\phi(t) - \phi(\tilde{t})\|_X \leq |t - \tilde{t}| \cdot 2M.$$

Therefore, $\phi(b-)$ exists. If $\phi(b-) \in \mathcal{U}$, then by part 1, we may solve the system $(*)$ with initial data $\phi(b-)$ and extend the solution on a nonzero interval beyond b. Hence, it must be that $\phi(b-) \notin \mathcal{U}$.

□

Hence, we conclude that for each $\epsilon > 0$, the system of ODE (3.3) - (3.9) has a unique solution on some time interval $[0, T^\epsilon)$, where T^ϵ is the maximal time of existence according to Lemma 3.5. Our next goal is to appeal to Theorem 2.1 to show that the solutions exist on an interval $[0, T]$ where T does not depend on ϵ. To do so, we first show that the solution to the ODE system (3.3)-(3.9) in fact solves the PDE system (1.56)-(1.59).

Lemma 3.6. *Let $(u, \Lambda, \Sigma^2, X, w)$ be the solution to (3.3)-(3.9) on $t \in [0, T]$. Then $u := u^{(0)}$ and $\Lambda := \Lambda^{(0)}$ satisfy (1.56)-(1.59) on $t \in [0, T]$.*

Proof. Let $e := (u^{(0)}, \cdots, u^{(M+1)}, \Lambda^{(0)}, \cdots, \Lambda^{(M+1)}) - (u, \cdots, \partial_t^{M+1} u, \Lambda, \cdots, \partial_t^{M+1} \Lambda)$. Then $e(0, y) \equiv 0$ by the initial condition. Taking time derivative, we know that

$$\frac{d}{dt} e_1 = \frac{d}{dt} u^0 - \partial_t u = u^{(1)} - \partial_t u = e_2$$
$$\vdots$$

$$\begin{aligned}
\frac{d}{dt}e_{M+2} &= \frac{d}{dt}u^{M+1} - \partial_t^{M+1}u = \mathbf{\Phi}^\epsilon\left(u^{(M)}, u^{(M+1)}\right) - \mathbf{\Phi}^\epsilon\left(\partial_t^M u, \partial_t^{M+1}u\right) \\
\frac{d}{dt}e_{M+3} &= \frac{d}{dt}\Lambda^0 - \partial_t\Lambda = \Lambda^{(1)} - \partial_t\Lambda = e_{M+4} \\
&\vdots \\
\frac{d}{dt}e_{2M+4} &= \frac{d}{dt}\Lambda^{M+1} - \partial_t^{M+1}\Lambda = \mathbf{\Psi}^\epsilon\left(\Lambda^{(M)}, \Lambda^{(M+1)}\right) - \mathbf{\Psi}^\epsilon\left(\partial_t^M\Lambda, \partial_t^{M+1}\Lambda\right).
\end{aligned}$$

Recall that in Lemma 3.4, we showed that $\mathbf{F}^\epsilon$ is Lipschitz, so in particular, $\mathbf{\Phi}^\epsilon$ and $\mathbf{\Psi}^\epsilon$ are also Lipschitz, so

$$\frac{d}{dt}\|e\|_{\mathcal{B}} \leq C_2\|e\|_{\mathcal{B}}.$$

Since $\|e(0,\cdot)\|_{\mathcal{B}} = 0$ by the initial condition, $e \equiv 0$ by Gronwall, so u and Λ indeed solve the PDE system on $[0, T]$. □

We are now ready to show that the solutions to the PDE system exist on a uniform time interval, and thus we are able to construct a solution to the original un-mollified PDE system.

3.3 Convergence of Mollified Solutions

We first show that there is some $T > 0$, which does not depend on ϵ, such that the mollified system of ODE has a unique solution for $t \in [0, T]$.

Lemma 3.7. *There is a time $T > 0$ such that the solutions $(u^\epsilon, \Lambda^\epsilon, (\Sigma^2)^\epsilon, X^\epsilon, w^\epsilon)$ are in $C([0,T], \mathcal{B})$ for all ϵ.*

Proof. By the criterion for existence of solution (i.e. part 3 in Lemma 3.5), the solution $(u^\epsilon, \Lambda^\epsilon)$ will cease to exist as $t \nearrow T^\epsilon$ if and only if $\lim_{t\nearrow T^\epsilon} \mathfrak{E}_M[u^\epsilon, \Lambda^\epsilon](t) = \infty$. However, by Theorem 2.1, we know that $\mathfrak{E}_M[u^\epsilon, \Lambda^\epsilon](t)$ depends only on the initial data, which is uniform for all ϵ. Therefore, the time of existence T^ϵ can be taken to be independent of ϵ. □

Thus, we have shown that there is some $T > 0$ independent of ϵ such that $u^\epsilon, \Lambda^\epsilon \in C([0,T], \mathcal{B})$ for all $\epsilon > 0$, and such that there is a constant C_3 so that

$$\mathfrak{E}_M[u^\epsilon, \Lambda^\epsilon](t) < C_3 \quad \forall t \in [0,T]\ \forall \epsilon > 0.$$

This enables us to extract a convergent subsequence from $(u^\epsilon, \Lambda^\epsilon)$. A key ingredient in showing that the subsequence converges is to show that the right hand side of the ODE system converges to the un-mollified system pointwisely as $\epsilon \to 0$. The following Lemma will be useful.

Lemma 3.8. *Let $\alpha > 0$. Assume that there is some $K > 0$ such that $\|f_\epsilon\|_{H^{\alpha+2}} < K$ for all ϵ, and*

$$\lim_{\epsilon \to 0} \|f_\epsilon - f\|_{H^{\alpha+1}} = 0.$$

Then

$$\lim_{\epsilon \to 0} \|\nabla J_\epsilon f_\epsilon - \nabla f\|_{H^\alpha} = 0.$$

In particular, if $\alpha \geq 2$, then by Sobolev embedding,

$$\lim_{\epsilon \to 0} \|\nabla J_\epsilon f_\epsilon - \nabla f\|_{L^\infty} = 0.$$

Proof. We know that

$$\begin{aligned}
\|\nabla J_\epsilon f_\epsilon - \nabla f\|_{H^\alpha} &\leq \|\nabla J_\epsilon f_\epsilon - \nabla f_\epsilon\|_{H^\alpha} + \|\nabla f_\epsilon - \nabla f\|_{H^\alpha} \\
&\leq \|(\mathrm{Id} - J_\epsilon)\nabla f_\epsilon\|_{H^\alpha} + \|f_\epsilon - f\|_{H^{\alpha+1}} \\
&\leq \epsilon \|f_\epsilon\|_{H^{\alpha+2}} + \|f_\epsilon - f\|_{H^{\alpha+1}}.
\end{aligned}$$

The claim then follows from sending $\epsilon \to 0$. □

We will use Arzela-Ascoli Theorem to extract a subsequence from $(u^\epsilon)_\epsilon$. We record the theorem as follows.

Lemma 3.9. *[Arzela-Ascoli Theorem] Let (X, d) be a compact metric space, B be a Banach space, and $C(X, B)$ be the space of bounded continuous functions $f : X \to B$. Let $\mathcal{F} \subset C(X, B)$ be a family of continuous functions satisfying the following properties:*

1. *For each $x \in X$, $\{f(x) : f \in \mathcal{F}\}$ is precompact in B. That is, every sequence in $\{f(x) : f \in \mathcal{F}\}$ has a convergent subsequence.*

2. *$\mathcal{F}$ is equicontinuous for every $x_0 \in X$. That is, for all $\epsilon > 0$, there is some $\delta(x_0, \epsilon) > 0$ such that for all $f \in \mathcal{F}$,*

$$\|x_0 - x\|_X \leq \delta(x_0, \epsilon) \implies \|f(x_0) - f(x)\|_B \leq \epsilon.$$

Then $\mathcal{F}$ is precompact. That is, every sequence $(f_n)_{n \geq 1}$ in $\mathcal{F}$ has a subsequence that converges uniformly in X to a function in $C(X, B)$.

Theorem 3.10. *There is a solution (u, Λ) to (1.42)-(1.45) on an interval $I = [0, T]$ with $T > 0$. Moreover, (u, Λ) possess the same regularity as their initial data.*

Proof. We will prove the claim in a few steps.

Step 1. We first show that there is a solution $(u, \Lambda, \Sigma^2, X, w)$ to the un-mollified system of equations. For every ϵ, we know that

$$\sum_{0 \leq \ell \leq 4} \|\partial_t^\ell u^\epsilon\|_{C(I, H^{M/2+1-\ell})} + \|\partial_t^\ell \Lambda^\epsilon\|_{C(I, H^{M/2+1-\ell})} < C_3. \tag{3.13}$$

We will use Arzela-Ascoli Theorem (Lemma 3.9) to extract a subsequence from $(u^\epsilon)_\epsilon$. To do so, we verify that:

1. $(\partial_t^\ell u^\epsilon)^\epsilon$ are equicontinuous in t for $\ell \leq 2$. This is because

$$\|\partial_t^\ell u^\epsilon(t_1) - \partial_t^\ell u^\epsilon(t_2)\|_{H^{M/2-\ell}} \leq \|\partial_t^{\ell+1} u^\epsilon\|_{L^\infty(I, H^{M/2-\ell})} \cdot |t_1 - t_2| < C_3 |t_1 - t_2|.$$

2. For any t, $(\partial_t^\ell u^\epsilon)^\epsilon$ is precompact in $H^{M/2-\ell}$. This is because $H^{M/2+1-\ell} \subset\subset H^{M/2-\ell}$.

Therefore, by Arzela-Ascoli Theorem, there is a subsequence $(\partial_t^\ell u^{\epsilon_\nu})$ that converges in $C(I, H^{M/2-\ell})$. That is, there are some $u^{(\ell)} \in C(I, H^{M/2-\ell})$ for $\ell = 0, 1, 2$ such that

$$\sum_{0\le\ell\le 2} \|\partial_t^\ell u^{\epsilon_\nu} - u^{(\ell)}\|_{C(I,H^{M/2-\ell})} \to 0 \quad \text{as } \nu \to 0. \tag{3.14}$$

To simplify the notation, we restrict to the convergent subsequence and relabel $\epsilon = \epsilon_\nu$. To prove that (u, Λ) satisfy the un-mollified equation, we first prove that $\partial_t u^{(0)} = u^{(1)}$ and $\partial_t u^{(1)} = u^{(2)}$. The proof for both claims are very similar, so we only show the former one in detail.

Let $\mathcal{B}' = H^{M/2-1}$. We want to show that for any $t \in [0, T)$ and for any $\eta > 0$, there is some $\delta > 0$ such that

$$0 < |\Delta t| < \delta \implies \left\| \frac{1}{\Delta t}\left(u^{(0)}(t+\Delta t) - u^{(0)}(t) - u^{(1)}(t)\right)\right\|_{\mathcal{B}'} < \eta. \tag{3.15}$$

Fix some $\Delta t \neq 0$. We write the difference in a symmetric form:

$$\begin{aligned}
&\left\| \frac{1}{\Delta t}\left(u^{(0)}(t+\Delta t) - u^{(0)}(t)\right) - u^{(1)}(t)\right\|_{\mathcal{B}'} \\
\le &\left\| \frac{1}{\Delta t}\left((u^\epsilon)(t+\Delta t) - (u^\epsilon)(t)\right) - \partial_t u^\epsilon(t)\right\|_{\mathcal{B}'} + \left\|\partial_t u^\epsilon - u^{(1)}\right\|_{\mathcal{B}'} \\
&+ \left\| \frac{1}{\Delta t}\left((u^\epsilon)(t+\Delta t) - u^{(0)}(t+\Delta t)\right)\right\|_{\mathcal{B}'} + \left\| \frac{1}{\Delta t}\left((u^\epsilon)(t) - u^{(0)}(t)\right)\right\|_{\mathcal{B}'} \\
= &I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since $\|\partial_t^2 u^\epsilon\|_{L^\infty(I,\mathcal{B}')} < C_3$ for all ϵ, we know that

$$I_1 \le C_3|\Delta t|,$$

and clearly

$$I_3, I_4 \le \frac{1}{\Delta t}\|u^\epsilon - u^{(0)}\|_{L^\infty(I,\mathcal{B}')}.$$

Thus, by choosing $\epsilon = \epsilon(\Delta t)$ small, we can arrange that

$$I_1 + \cdots + I_4 < \eta.$$

This shows that $\partial_t u^{(0)} = u^{(1)}$ in $\mathcal{B}'$. In particular, since $M/2 - 1 > 2$, we know that the two functions agree pointwisely. The proof for $\partial_t u^{(1)} = u^{(2)}$ is almost identical, so we refrain from repeating ourselves.

Next, we show that the right hand side of the ODE also converges to the unmollified equation. The proof is again very similar for $u^{(0)}, u^{(1)}$ and $u^{(2)}$, so we only prove the most difficult case of $u^{(2)}$. That is, we need to show that

$$\begin{aligned} &\frac{1}{(g^\epsilon)^{00}} \cdot \Big(-(\partial_t (g^\epsilon)^{00}) \partial_t u^\epsilon - (\partial_t (g^\epsilon)^{0j}) \tilde{\nabla}_j u^\epsilon \\ &\quad - (g^\epsilon)^{0j} (\tilde{\nabla}_j \partial_t u^\epsilon) - \tilde{\nabla}_j ((g^\epsilon)^{0j} \partial_t u^\epsilon) - \tilde{\nabla}_i ((g^\epsilon)^{ij} \tilde{\nabla}_j u^\epsilon) \Big) \\ &+ \frac{1}{(g^\epsilon)^{00}} \cdot \Big((g^\epsilon)^{\alpha\beta} \tilde{\nabla}_\alpha ((X^\epsilon)^\mu) \tilde{\nabla}_\beta ((w^\epsilon)^\nu_\mu) \\ &\quad - (g^\epsilon)^{\alpha\beta} \partial_\alpha (X^\epsilon)^\nu \left((\log G)'' (\tilde{\nabla}_\beta (\Sigma^2)^\epsilon) \Lambda^\epsilon + (\log G)' (\tilde{\nabla}_\beta \Lambda^\epsilon) \right) \Big) \end{aligned} \tag{3.16}$$

converges to

$$\begin{aligned} &\frac{1}{g^{00}} \left(-(\partial_t g^{00}) \partial_t u - (\partial_t g^{0j}) \nabla_j u - g^{0j} (\nabla_j \partial_t u) - \nabla_j (g^{0j} \partial_t u) - \nabla_i (g^{ij} \nabla_j u) \right) \\ &+ \frac{1}{g^{00}} \left(g^{\alpha\beta} \nabla_\alpha (X^\mu) \nabla_\beta (w^\nu_\mu) - g^{\alpha\beta} \partial_\alpha X^\nu \left((\log G)'' (\nabla_\beta \Sigma^2) \Lambda + (\log G)' (\nabla_\beta \Lambda) \right) \right) \end{aligned} \tag{3.17}$$

in $L^\infty(I, H^{M/2-1})$ as $\epsilon \to 0$. Here, notice that by Lemma 2.12,

$$g = g(X, u, \nabla X, \nabla u)$$

is a rational function with strictly positive denominator, and thus has bounded derivatives of all order. Thus, we can write the difference using the triangular trick, similar to the proof of Proposition 3.4. By Lemma 3.8, we have estimates that are of the form, for instance,

$$\|\tilde{\nabla} u^\epsilon - \nabla u\|_{H^{M/2-1}} \leq \epsilon \|u^\epsilon\|_{H^{M/2+1}} + \|u^\epsilon - u\|_{H^{M/2}},$$

which, combined with (3.14), proves the convergence.

Thus, the limit (u, Λ) solves the un-mollified problem (1.42)-(1.45) on the interval $[0, T]$.

Step 2. We now prove that (u, Λ) enjoy the same regularity as their initial conditions.

For each t, since $u^\epsilon, \Lambda^\epsilon$ are uniformly bounded in $H^{M/2+1}$, upon passing to a subsequence, they also have a weak limit in $H^{M/2+1}$. But we have shown that they converge to (u, Λ) pointwisely, so the weak limit must be equal to the strong limit, thus showing that for each t, the functions $(u, \Lambda, \Sigma^2, X, w) \in \mathcal{B}$ as well. This establishes the spatial regularity of $(u, \Lambda, \Sigma^2, X, w)$. To show the regularity in time, recall that in Step 1, we showed that $u^{(\ell)} \in C([0, T], H^{M/2-\ell})$ for $\ell \leq 2$. Repeating the same argument, we can establish the time regularity of u up to $\partial_t^{M/2-2} u$; that is, $\partial_t^\ell u \in C([0, T], H^{M/2-\ell})$ for $\ell \leq M/2 - 2$. To establish the highest order regularity, recall from Lemma 2.8 - Lemma 2.10, the highest order terms in $\Box_g \partial_t^{M/2-2} u$ and $\Box_h \partial_t^{M/2-2} \Lambda$ are $\nabla^{(2)} \partial_t^{M/2-3} u$ and $\nabla^{(2)} \partial_t^{M/2-3} \Lambda$ respectively, so $G_{M/2-2}, H_{M/2-2} \in L^2([0, T], H^1) \cap H^1([0, T], L^2)$. By the regularity of solutions to hyperbolic equations[1], we see that $\partial_t^m \partial_t^{M/2-2} u, \partial_t^m \partial_t^{M/2-2} \Lambda \in L^\infty([0, T], H^{2-m})$ for $m = 0, 1, 2$.

Thus, we have shown the regularity of the highest order terms as well. $\Box$

3.4 Uniqueness of Solution

Finally, we prove that the solution, which was shown to exist, is unique. The proof takes the difference between two solutions, and show that the difference is subject to a system of equation that is similar to the original system. The uniqueness thus follows from the a priori estimate applied to the difference of two solutions, as well

[1] shown, for instance, in [3]

as the regularity of each individual solution.

Theorem 3.11. *Assume that* $(u, \Lambda, X, w, \Sigma^2)$ *solves the systems (1.39)-(1.48) on some time interval* $[0,T]$ *ans satisfies* $\mathcal{E}_M[u,\Lambda](T) < \infty$. *Then the solution is unique.*

Proof. We first need to obtain an energy estimate for the solution to the exact equation. We in fact consider the energy $\mathcal{E}_{M-1}[u,\Lambda]$, so that the regularity required in Lemma 2.21 - 2.5 are satisfied.

As was remarked before, by setting $J_\epsilon = \mathrm{Id}$, our a priori estimate in Theorem 2.1 would apply to the solution to the exact equation. Assume that there are two solutions (u,Λ) and $(\tilde{u},\tilde{\Lambda})$ with the same initial condition, and there is some $C_1 > 0$, $T > 0$ such that

$$\mathcal{E}_M[u,\Lambda](T) < C_1, \quad \mathcal{E}_M[\tilde{u},\tilde{\Lambda}](T) < C_1.$$

By Theorem 2.1, we know that there is some $C_2 > 0$ such that

$$\mathfrak{E}_{M-1}[u,\Lambda](T) < C_2, \quad \mathfrak{E}_{M-1}[\tilde{u},\tilde{\Lambda}](T) < C_2.$$

We consider equations (1.42)-(1.45); that is,

$$\partial_t^2 u = \mathbf{\Phi}(u, \partial_t u), \quad \partial_t^2 \tilde{u} = \mathbf{\Phi}(\tilde{u}, \partial_t \tilde{u}).$$

Note that Proposition 3.4 does not apply here since the Lipschitz constant depended on ϵ. However, we can achieve a similar result with a reduction in regularity. Let $e = u - \tilde{u}, \delta = \Lambda - \tilde{\Lambda}$. Then

$$\partial_t^2 e = \mathbf{\Phi}(u, \partial_t u) - \mathbf{\Phi}(\tilde{u}, \partial_t \tilde{u}) := F_u, \quad \partial_t^2 \delta = \mathbf{\Phi}(\Lambda, \partial_t \Lambda) - \mathbf{\Phi}(\tilde{\Lambda}, \partial_t \tilde{\Lambda}) := F_\Lambda.$$

Here, we use the triangular trick to reduce F_u to a function in terms of e. In the interior, F_u is a linear combination of terms of the forms:

1. $\frac{g^{ij}}{g^{00}} \nabla_i \nabla_j e$.

2. $F(\phi_1, \cdots, \phi_m) \cdot (\partial_t^\ell e)$, where F is a rational function with bounded derivatives of all orders, $\phi_i \in \{\partial_t^k \nabla^p f : f = g, \tilde{g}, X, \tilde{X}, \Sigma^2, \tilde{\Sigma}^2, w, \tilde{w}, u, \tilde{u}, \Lambda, \tilde{\Lambda}, 0 \leq k, p \leq 2\}$, $\ell \in \{0, 1\}$.

3. $F(\phi_1, \cdots, \phi_m) \cdot (\nabla e)$, where F is a rational function with bounded derivatives of all orders, $\phi_i \in \{\partial_t^k \nabla^p f : f = g, \tilde{g}, X, \tilde{X}, \Sigma^2, \tilde{\Sigma}^2, w, \tilde{w}, u, \tilde{u}, \Lambda, \tilde{\Lambda}, 0 \leq k, p \leq 2\}$.

On the boundary, F_u is a linear combination of terms of the form:

1. $\frac{1}{2} n_\alpha g^{\alpha\beta} \nabla_\beta e$.

2. $F(\phi_1, \cdots, \phi_m) \cdot (\partial_t^\ell e)$, where F is a rational function with bounded derivatives of all orders, $\phi_i \in \{\partial_t^k \nabla^p f : f = g, \tilde{g}, X, \tilde{X}, \Sigma^2, \tilde{\Sigma}^2, w, \tilde{w}, u, \tilde{u}, \Lambda, \tilde{\Lambda}, 0 \leq k, p \leq 2\}$, $\ell \in \{0, 1\}$.

Therefore, e satisfies an equation of the form

$$\begin{cases} (u^0)^2 \partial_t^2 e + \frac{1}{2} n_\alpha g^{\alpha\beta} \nabla_\beta e = F_1 & \text{on } [0, T] \times \partial\mathbb{R}^3_+ \\ \Box_g e = F_2 & \text{in } [0, T] \times \mathbb{R}^3_+, \end{cases} \tag{3.18}$$

Thus, using Lemma 2.2, and recalling that $e, \partial_t e \equiv 0$ at $t = 0$, we can show that

$$\int_{\mathbb{R}^3_{+t}} |\nabla_{t,y} e|^2 \, dy + \int_{\partial\mathbb{R}^3_{+t}} |\partial_t e|^2 \, dS \lesssim \int_0^t \int_{\mathbb{R}^3_{+\tau}} F_2 \cdot \partial_t e \, dy d\tau + \int_0^t \int_{\partial\mathbb{R}^3_{+\tau}} F_1 \cdot \partial_t e \, dS d\tau.$$

Now, by Lemma B.4, we may bound terms of the form $F(\phi_1, \cdots, \phi_m)$ in the definition of F_1 and F_2 by C_2^r for some integer r. Thus, by Gronwall's inequality,

$$\int_{\mathbb{R}^3_{+t}} |\nabla_{t,y} e|^2 \, dy + \int_{\partial\mathbb{R}^3_{+t}} |\partial_t e|^2 \, dS \equiv 0 \quad \forall t \in [0, T].$$

To estimate F_Λ, we note that δ also satisfies an equation of the form

$$\begin{cases} \delta = 0 & \text{on } I \times \partial\mathbb{R}^3_+ \\ \Box_h \delta = G & \text{in } I \times \mathbb{R}^3_+, \end{cases} \tag{3.19}$$

where G is a linear combination of terms of the forms:

1. $F(\phi_1, \cdots, \phi_m) \cdot (\partial_t^\ell \delta)$, where F is a rational function with bounded derivatives of all orders, $\phi_i \in \{\partial_t^k \nabla^p f : f = g, \tilde{g}, X, \tilde{X}, \Sigma^2, \tilde{\Sigma}^2, w, \tilde{w}, u, \tilde{u}, \Lambda, \tilde{\Lambda}, 0 \le k, p \le 2\}$, $\ell \in \{0, 1\}$.

2. $F(\phi_1, \cdots, \phi_m) \cdot (\nabla \delta)$, where F is a rational function with bounded derivatives of all orders, $\phi_i \in \{\partial_t^k \nabla^p f : f = g, \tilde{g}, X, \tilde{X}, \Sigma^2, \tilde{\Sigma}^2, w, \tilde{w}, u, \tilde{u}, \Lambda, \tilde{\Lambda}, 0 \le k, p \le 2\}$.

Again, using Lemma 2.3, we can show that

$$\int_{\mathbb{R}^3_{+t}} |\nabla_{t,y} \delta|^2 \, dy \lesssim \int_0^t \int_{\mathbb{R}^3_{+\tau}} F \cdot \partial_t \delta \, dt d\tau,$$

and we may bound $\|F(\phi_1, \cdots, \phi_m)\|_{L^\infty(I \times \mathbb{R}^3_+)}$ by C_2^r for some integer r. By Gronwall's inequality again, we can show that

$$\int_{\mathbb{R}^3_{+t}} |\nabla_{t,y} \delta|^2 \, dy \equiv 0 \quad \forall t \in [0, T].$$

This shows the uniqueness of the solution (u, Λ). □

In conclusion, we utilized the a priori estimate in Chapter II to establish the local well-posedness result for equations (1.42)-(1.48) on the unbounded domain $\mathcal{D} = \mathbb{R}^3_+$. We furthermore showed that the solutions enjoy the same regularity as their initial data. This proves Theorem 1.1 with Ω_0 is unbounded.

CHAPTER IV

Linear Equation on a Bounded Domain

We have established the local well-posedness of the system of equations (1.42)-(1.45) when the initial domain Ω_0 is unbounded. In this section, we prove a similar result, but for a bounded domain Ω_0. That is, we consider the Lagrangian coordinate

$$X : \Omega \to [0, T] \times B, \quad X(t, \cdot) : \Omega_t \to B,$$

where $B := \{y \in \mathbb{R}^3 : |y| \leq 1\}$. Our notations will be the same as the unbounded case.

The strategy of the proof will be different. We will establish an a priori estimate for the *un-mollified* system only, and adapt the proof for existence in [10] to our system of equations. That is, we prove existence by considering the linear system, and then use an iteration on the linear system to create a solution to the fully nonlinear system of equations.

Remark 11. Note that by a linear system, we mean the system of equations with given coefficients and known right hand side. That is, we replace the coefficients in $\Box_g$ and $\Box_h$, as well as the functions on the right hand side of the equations, with known functions. We do *not* linearize the equations around a certain solution.

We shall start with an a priori estimate that is very similar to Theorem 2.1.

Proposition 4.1. *Assume that u, Λ, Σ^2, w solve (1.42)-(1.48) on $[0,T]\times B$, with*

$$\mathcal{E}_M[u,\Lambda](T)\leq C_1$$

for some integer M and constant C_1. Denote

$$\mathfrak{E}_M[u,\Lambda](T) := \mathcal{E}_M[u,\Lambda](T) + \sup_{0\leq t\leq T}\sum_{k+2p\leq M+2}\left\|\partial_t^k u\right\|^2_{H^p(\mathbb{R}^3_{+t})} + \left\|\partial_t^k\Lambda\right\|^2_{H^p(\mathbb{R}^3_{+t})}. \tag{4.1}$$

Then there is some polynomial P_M with non-negative coefficients such that if $T>0$ is small (depending only on C_1 and $\mathfrak{E}_M(0)$), then for all $t\in[0,T]$,

$$\mathfrak{E}_M[u,\Lambda](t)\leq \mathfrak{E}_M[u,\Lambda](0) + \int_0^t P_M(\mathfrak{E}_M[u,\Lambda](\tau))\,d\tau \tag{4.2}$$

In particular, (by, say, Lemma 2.15), we know that there is a time interval $[0,T]$, where $T>0$ depends only on the initial data, such that

$$\mathfrak{E}_M[u,\Lambda](T)\lesssim \mathfrak{E}_M[u,\Lambda](0). \tag{4.3}$$

Proof. The proof is identical to that of Theorem 2.1 by replacing J_ϵ with Id and $\mathcal{D}$ with B. We omit the details. □

The next task is then to establish existence. As remarked earlier, we use a strategy that is similar to that in [10]: we consider the linear equation, and show that each linear equation admits a unique solution.

By linear equation, we mean the system (1.42)-(1.45) with g and h replaced by some known function on the left hand side, and $X, w, \Sigma^2, \Lambda, u$ replaced by known functions on the right hand side. We seek to first obtain a weak solution to this system of linear equations. The weak formulation is motivated by [10] and the section on linear hyperbolic equations in [3].

4.1 Existence of Solution u

We will establish the existence of solution u by Galerkin approximation, which calls for a weak formulation of the problem first. This is our next goal.

4.1.1 Weak Formulation

Let $\phi : B \to \mathbb{R}$ be a smooth function, and $\theta \in C^2(B)$ satisfies

$$\begin{cases} (u^0)^2 \partial_t^2 \theta + \frac{1}{2} a n_\alpha g^{\alpha\beta} \nabla_\beta \theta = f & \text{on } \partial B \\ \Box_g \theta = q & \text{in } B. \end{cases}$$

Then

$$\begin{aligned}
&\int_{B_t} (\Box_g \theta) \phi \, dy \\
&= \int_{B_t} \nabla_\alpha (g^{\alpha\beta} \nabla_\beta \theta) \cdot \phi \, dy + \int_{B_t} \frac{1}{2} (\nabla_\alpha \log |g|) g^{\alpha\beta} (\nabla_\beta \theta) \phi \, dy \\
&= \int_{B_t} \partial_t (g^{0\beta} \nabla_\beta \theta) \phi \, dy - \int_{B_t} g^{j\beta} \nabla_\beta \theta \nabla_j \phi \, dy + 2 \int_{\partial B_t} \frac{1}{a} \left(f - (u^0)^2 \partial_t^2 \theta \right) \phi \, dS \\
&\quad + \int_{B_t} \frac{1}{2} (\nabla_\alpha \log |g|) g^{\alpha\beta} (\nabla_\beta \theta) \phi \, dy.
\end{aligned}$$

Regrouping terms involving derivatives of θ and ϕ, we have

(4.4)

$$\begin{aligned}
&\int_{B_t} q\phi \, dy - \int_{\partial B_t} \frac{2}{a} f \phi \, dS \\
&= \int_{B_t} \partial_t^2 \theta \cdot g^{00} \phi \, dy - \int_{\partial B_t} \partial_t^2 \theta \cdot \frac{2(u^0)^2}{a} \phi \, dS \\
&\quad + \int_{B_t} \partial_t \theta \cdot \left[(\partial_t g^{00}) + \frac{1}{2} (\nabla_\alpha \log |g|) g^{0\alpha} - (\nabla_j g^{0j}) \right] \phi \, dy + \int_{B_t} \partial_t \theta \cdot \left(-2 g^{0j} \right) \nabla_j \phi \, dy \\
&\quad + \int_{\partial B_t} \partial_t \theta \cdot (n_j g^{0j}) \phi \, dS \\
&\quad + \int_{B_t} \nabla_j \theta \cdot \left[(\partial_t g^{0j}) + \frac{1}{2} (\nabla_\alpha \log |g|) g^{\alpha j} \right] \phi \, dy + \int_{B_t} \nabla_j \theta \cdot \left(-g^{ij} \right) \nabla_i \phi \, dy.
\end{aligned}$$

In the rest of this section, define[1]

$$\langle\theta,\phi\rangle := \int_B \theta\phi\,dy = \text{ pairing in } L^2(B)$$
$$\langle\!\langle\theta,\phi\rangle\!\rangle := \int_{\partial B} \theta\phi\,dS = \text{ pairing in } L^2(\partial B)$$
$$(\theta,\phi) := \text{ pairing between } (H^1(B))' \text{ and } H^1(B),$$

and

$$\gamma := \frac{2(u^0)^2}{a}.$$

We define the bounded linear map $\Phi : H^1(B) \to (H^1(B))'$:

$$(\Phi(\theta),\phi) := \left\langle -g^{00}\theta,\phi\right\rangle + \langle\!\langle\gamma\theta,\phi\rangle\!\rangle\,. \tag{4.5}$$

Comparing with (4.4), we see that the weak equation actually involves $\Phi(\theta'')$ rather than $\Phi(\theta)''$, so we compute the difference:

$$\begin{aligned}&(\Phi(\theta)'',\phi)\\ =&\partial_t^2\int_B -g^{00}\theta\phi\,dy + \partial_t^2\int_{\partial B}\gamma\theta\phi\,dS\\ =&(\Phi(\theta''),\phi) - \left\langle 2(\partial_t g^{00})\theta',\phi\right\rangle - \left\langle(\partial_t^2 g^{00})\theta,\phi\right\rangle + \langle\!\langle 2(\partial_t\gamma)\theta',\phi\rangle\!\rangle + \left\langle\!\!\left\langle(\partial_t^2\gamma)\theta,\phi\right\rangle\!\!\right\rangle.\end{aligned}$$

Then by (4.4), we know that the weak equation is:

$$(\Phi(\theta)'',\phi) + \mathcal{L}(\theta,\phi) = -\langle q,\phi\rangle + \langle\!\langle(2/a)f,\phi\rangle\!\rangle \quad \forall\phi\in H^1(B), \tag{4.6}$$

where $\mathcal{L}(\theta,\phi)$ represents the weak formulation of the lower order derivatives on θ:

$$\mathcal{L}(\theta,\phi) = \sum_{i=1}^{7}\mathcal{L}_i(\theta,\phi)$$
$$\mathcal{L}_1(\theta,\phi) = \left\langle\theta', 2g^{0j}\nabla_j\phi\right\rangle$$
$$\mathcal{L}_2(\theta,\phi) = \left\langle\theta', \left((\partial_t g^{00}) - \frac{1}{2}(\nabla_\alpha\log|g|)g^{0\alpha} + (\nabla_j g^{0j})\right)\phi\right\rangle$$

[1]Recall that all the functions we consider are real-valued.

$$\mathcal{L}_3(\theta,\phi) = \left\langle \theta, \partial_t^2 g^{00}\phi \right\rangle$$

$$\mathcal{L}_4(\theta,\phi) = \left\langle \nabla_j\theta, g^{ij}\nabla_i\phi \right\rangle$$

$$\mathcal{L}_5(\theta,\phi) = \left\langle \nabla_j\theta, \left(-(\partial_t g^{0j}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{\alpha j}\right)\phi \right\rangle$$

$$\mathcal{L}_6(\theta,\phi) = \left\langle\!\left\langle \theta', (-n_j g^{0j} - 2(\partial_t\gamma))\phi \right\rangle\!\right\rangle$$

$$\mathcal{L}_7(\theta,\phi) = \left\langle\!\left\langle \theta, -(\partial_t^2\gamma)\phi \right\rangle\!\right\rangle.$$

We also need to derive the higher order equations. The difference of these equations with (4.6) come from the analogue of the commutators $[\Box_g, \partial_t^k]$ and $[\mathcal{P}, \partial_t^k]$.

We first derive the first order equation. Assume that θ satisfies (4.6) and is sufficiently regular in t. Then

$$\begin{aligned}
&[(\Phi(\theta)'', \phi) + \mathcal{L}(\theta,\phi)]' - [(\Phi(\theta')'', \phi) + \mathcal{L}(\theta',\phi)] \\
=& -\left\langle \partial_t g^{00}\theta'', \phi \right\rangle + \left\langle\!\left\langle \partial_t\gamma\theta'', \phi \right\rangle\!\right\rangle \\
&+ \left\langle \theta', 2(\partial_t g^{0j})\nabla_j\phi \right\rangle \\
&+ \left\langle \theta', \partial_t\left(-(\partial_t g^{00}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{0\alpha} + (\nabla_j g^{0j})\right)\phi \right\rangle \\
&+ \left\langle \nabla_j\theta, (\partial_t g^{ij})\nabla_i\phi \right\rangle \\
&+ \left\langle \nabla_j\theta, \partial_t\left(-(\partial_t g^{0j}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{\alpha j}\right)\phi \right\rangle \\
&+ \left\langle\!\left\langle \theta', \partial_t\left(-n_j g^{0j}\right)\phi \right\rangle\!\right\rangle.
\end{aligned}$$

Let $\theta_0 := \theta$ and $\theta_1 := \theta_0'$. We record the highest order terms in the preceding equation as

$$\mathcal{C}(\theta_1,\phi) := -\left\langle (\partial_t g^{00})\theta_1', \phi \right\rangle + \left\langle\!\left\langle (\partial_t\gamma)\theta_1', \phi \right\rangle\!\right\rangle, \tag{4.7}$$

and the lower order terms as

$$I_1^j = -(2\partial_t g^{0j})\theta_0' - (\partial_t g^{ij})\nabla_i\theta_0$$

$$I_2 = \left(\partial_t^2 g^{00} + \frac{1}{2}\partial_t((\nabla_\alpha \log|g|)g^{0\alpha}) - \nabla_j \partial_t g^{0j}\right)\theta_0'$$

$$I_3 = \left(\partial_t^2 g^{0j} + \frac{1}{2}\partial_t\left((\nabla_\alpha \log|g|)g^{\alpha j}\right)\right)\nabla_j \theta_0$$

$$B_1 = \partial_t(n_j g^{0j})\theta_0'.$$

Then the equation for θ_1 is:

$$(\Phi(\theta_1)'', \phi) + \mathcal{L}(\theta_1, \phi) + \mathcal{C}(\theta_1, \phi) = -\langle \partial_t q, \phi\rangle + \langle\!\langle \partial_t(2f/a), \phi\rangle\!\rangle + \left\langle I_1^j, \nabla_j \phi\right\rangle + \langle I_2 + I_3, \phi\rangle + \langle\!\langle B_1, \phi\rangle\!\rangle . \tag{4.8}$$

Proceeding inductively, we see that the weak equation for $\theta_k := \theta_0^{(k)}$, where $k \geq 2$, is

$$\begin{aligned}(\Phi(\theta_k)'', \phi) &+ \mathcal{L}(\theta_k, \phi) + k\mathcal{C}(\theta_k, \phi)\\ = -&\left\langle \partial_t^k q, \phi\right\rangle + \left\langle\!\left\langle \partial_t^k(2f/a), \phi\right\rangle\!\right\rangle\\ &+ \left\langle \partial_t^{k-1} I_1^j, \nabla_j\phi\right\rangle + \left\langle \partial_t^{k-1}(I_2 + I_3), \phi\right\rangle + \left\langle\!\left\langle \partial_t^{k-1} B_1, \phi\right\rangle\!\right\rangle\\ &+ \left\langle k(\partial_t^2 g^{00})\theta_{k-1}', \phi\right\rangle - \left\langle\!\left\langle k(\partial_t^2 \gamma)\theta_{k-1}', \phi\right\rangle\!\right\rangle .\end{aligned} \tag{4.9}$$

Assuming sufficient regularity on q, f, a, γ, we shall prove that the equations (4.6)-(4.9) have a unique solution. This is the content of our next subsection.

4.1.2 Proof for Existence

Theorem 4.2. *Let M be the integer in Proposition 4.1, and u, Λ have the regularity as described in Proposition 4.1. Then there is a time interval $[0, T]$ with $T > 0$ in which (4.6) - (4.9) have a unique solution for all $k = 0, \cdots M$. Moreover, denoting the solution to the k-th order equation by θ_k, then the following are true:*

1. *Compatibility: $\theta_{j-1}' = \theta_j$ for $j = 1, \cdots, M$.*

2. *Energy estimate: let*

$$\mathbb{E}_M[\theta](t) = \sum_{k=0}^{M} \int_{B_t} (-g^{00})|\theta_k'|^2 + g^{ij}\nabla_i\theta_k\nabla_j\theta_k \, dy + \int_{\partial B_t} \gamma|\theta_k'|^2 \, dS.$$

Then there is a polynomial P such that for all $t \in [0,T]$,

$$\mathbb{E}_M[\theta](t) \le \mathbb{E}_M[\theta](0) \cdot \big(1 + tP(\mathfrak{E}_M[u,\Lambda](T)) \cdot e^{tP(\mathfrak{E}_M[u,\Lambda](T))}\big)\,.$$

Before we give the proof, we shall first note some results on regularity for a weak solution (assuming its existence). We will use the following result in the proof of Theorem 4.2, very much like the way we used Proposition 2.13 in the proof of Theorem 2.1.

Proposition 4.3. *Assume the same assumption as Theorem 4.2. Suppose θ_k solves the k-th order weak equation for $k = 1, \cdots, M$ on some time interval $I = [0,T]$. Then for $k \le M-2$ we have*

$$\|\theta_k\|_{L^\infty(I\times B)} \lesssim \|\theta'_{k+1}\|_{L^\infty(I,L^2(B))} + \|\theta_{k-1}\|_{L^\infty(I,H^1(B))}. \tag{4.10}$$

Before presenting the proof, we note a standard result on weak solutions to a (second order) elliptic equation, which is similar to Lemma 2.6. This is the same as Lemma 3.5 in [10], and was proven in, for instance, [18].

Lemma 4.4. *Suppose $\phi \in H^1(B)$ satisfies*

$$\big\langle g^{ij}\nabla_i\phi, \nabla_j\psi\big\rangle = \langle\!\langle b,\psi\rangle\!\rangle + \langle W,\psi\rangle \quad \forall\psi \in H^1(B) \tag{4.11}$$

for some $b \in H^{1/2}(\partial B)$, $W \in L^2(B)$. Then $\phi \in H^2(B)$ and

$$\|\phi\|_{H^2(B)} \lesssim \|W\|_{L^2(B)} + \|b\|_{H^{1/2}(\partial B)}. \tag{4.12}$$

We will use Lemma 4.4 very much like the way we used Lemma 2.6 to prove Proposition 2.13.

Proof for Proposition 4.3. By assumption, we know that $\theta_k \in H^1(B)$ for $k \le M$. Then by Lemma 4.4, we know that for $k \le M-2$,

$$\|\theta_k\|_{H^2(B)} \lesssim \|\theta''_k\|_{L^2(B)} + \|\theta_k\|_{H^{1/2}(\partial B)} + \|\theta_k\|_{L^2(B)} + \|\theta_{k-1}\|_{H^1(B)}$$

$$\leq \|\theta'_{k+1}\|_{L^2(B)} + \|\theta_k\|_{H^1(B)} + \|\theta_k\|_{L^2(B)} + \|\theta_{k-1}\|_{H^1(B)}.$$

The claim thus follows. □

We are now in a position to prove Theorem 4.2. The strategy of the proof is somewhat similar to Proposition 3.2 in [10], but there are a few crucial differences. To start with, our system of equations is different, with a non-constant G and a non-zero vorticity. Secondly, we first fix an m, and show that the system of ODEs in $\theta_1, \cdots, \theta_M$ can be solved approximately by projecting onto an m-dimensional subspace of $H^1(B)$. This leads to the approximate solutions $\theta_{1,(m)}, \cdots, \theta_{M,(m)}$. We then send $m \to \infty$ to recover the solution to the original weak problem $\theta_1, \cdots, \theta_M$. This is in contract to the approach [10], which sends $m \to \infty$ for each order k, and deals with the highest order equations separately.

Proof for Theorem 4.2. In this proof, assume that $\{e_\ell\}$ is both an orthogonal basis of $H^1(B)$ and an orthonormal basis of $L^2(B)$. Since $\{e_\ell\}$ is a basis of $H^1(B)$, and $\mathrm{tr} : H^1(B) \to L^2(\partial B)$ is surjective, we know that $\{e_\ell\}$ spans $L^2(\partial B)$ as well.

Existence of approximate solutions. Let $m > 0$ be an integer. Let P_m be the orthogonal projection onto $\mathrm{span}\{e_\ell : \ell \leq m\}$, so the corresponding approximate operator $\Phi_m : H^1(B) \to (H^1(B))'$ is

$$(\Phi_m(\theta), \phi) := (\Phi(\theta), P_m\phi) = (-g^{00}\theta, P_m\phi) + \langle\!\langle \gamma\theta, P_m\phi \rangle\!\rangle \,. \tag{4.13}$$

We look for an approximate solution of the form

$$\theta_{k,(m)}(t, y) := \sum_{\ell=1}^{m} \theta^\ell_{k,m}(t) e_\ell(y), \quad \theta^\ell_{k,m} \in C^2([0, T]) \; \forall \ell = 1, \cdots, m.$$

To define the approximate weak equation, we further consider the projected quantities:

$$I^j_{1,m} = -2(\partial_t g^{0j})\theta'_{0,(m)} - (\partial_t g^{ij})\nabla_i \theta_{0,(m)}$$

$$I_{2,m} = \left(\partial_t^2 g^{00} + \frac{1}{2}\partial_t((\nabla_\alpha \log|g|)g^{0\alpha}) - \nabla_j \partial_t g^{0j}\right)\theta'_{0,(m)}$$

$$I_{3,m} = \left(\partial_t^2 g^{0j} + \frac{1}{2}\partial_t\left((\nabla_\alpha \log|g|)g^{\alpha j}\right)\right)\nabla_j \theta_{0,(m)}$$

$$B_{1,m} = \partial_t(n_j g^{0j})\theta'_{0,(m)}.$$

The m-th approximate weak equation is then

$$\begin{aligned}(4.14)\qquad &\left(\Phi_m(\theta_{k,(m)})'', e_\ell\right) + \mathcal{L}(\theta_{k,(m)}, e_\ell) + k\mathcal{C}(\theta_{k,(m)}, e_\ell) \\ = &- \left\langle \partial_t^k q, e_\ell\right\rangle + \left\langle\!\left\langle \partial_t^k (2f/a), e_\ell\right\rangle\!\right\rangle \\ &+ \left\langle \partial_t^{k-1} I_{1,m}^j, \nabla_j e_\ell\right\rangle + \left\langle \partial_t^{k-1}(I_{2,m} + I_{3,m}), e_\ell\right\rangle + \left\langle\!\left\langle \partial_t^{k-1} B_{1,m}, e_\ell\right\rangle\!\right\rangle \\ &+ \left\langle k(\partial_t^2 g^{00})\theta'_{k-1,(m)}, e_\ell\right\rangle - \left\langle\!\left\langle k(\partial_t^2 \gamma)\theta'_{k-1,(m)}, e_\ell\right\rangle\!\right\rangle.\end{aligned}$$

We claim that (4.14) is a system of linear second order ODE in $\vec{\theta}_{k,m} := (\theta^1_{k,m}, \cdots, \theta^m_{k,m})$, where $\vec{\theta}_m : [0,T] \to \mathbb{R}^m$. To see this, note that (4.14) is of the form

$$(4.15)\qquad A(t)\frac{d^2}{dt^2}\vec{\theta}_{k,m}(t) + B(t)\frac{d}{dt}\vec{\theta}_{k,m} = C(t)\vec{\theta}_{k,m} + d(t)$$

for some matrices $A, B, C : [0,T] \to \mathbb{R}^{m\times m}$ and vector $d : [0,T] \to \mathbb{R}^m$. The matrices B and C are clearly bounded as linear operators on $\mathbb{R}^m$, so to utilize the result on existence and uniqueness of ODE, we only need to show that $A(t)$ is invertible for all t, and $\|A^{-1}(t)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)}$ is bounded for all t. We compute $A(t)$ explicitly:

$$(4.16)\qquad A(t)_{i,j} = \left\langle -g^{00}(t)e_i, e_j\right\rangle + \left\langle\!\left\langle \gamma(t)e_i, e_j\right\rangle\!\right\rangle.$$

Fix any $v \in \mathbb{R}^m$, and let $\psi(y) := \sum_{i=1}^m v_i e_i(y)$. Then

$$\begin{aligned}v^T A(t) v &= \sum_{i,j=1}^m v_i A(t)_{i,j} v_j \\ &= \int_B (-g^{00}(t))(v_i e_i(y))(v_j e_j(y))\,dy + \int_{\partial B} \gamma(t,y)(v_i e_i(y))(v_j e_j(y))\,dS \\ &\geq \alpha \cdot \left(\|\psi\|^2_{L^2(B)} + \|\psi\|^2_{L^2(\partial B)}\right)\end{aligned}$$

$$\geq \alpha\|v\|^2_{L^2(\mathbb{R}^m)}.$$

It is clear that $A(t)$ is a symmetric matrix for all t; therefore, $A(t)$ is symmetric positive-definite, and thus

$$\|A^{-1}(t)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \frac{1}{\alpha} \quad \forall t \in [0,T].$$

Hence, we know that (4.15) has a unique solution for all $t \in [0,T]$. Out next goal is to pass to the limit $m \to \infty$.

The approximate solutions are compatible. We claim that $\theta'_{k-1,(m)} = \theta_{k,(m)}$ in $L^2(B)$. This is because by the definition of (4.14), $\theta'_{k-1,(m)}$ and $\theta_{k,(m)}$ satisfy exactly the same equation with the same initial data, and thus they must agree by the existence and uniqueness theory of ODE.

Uniform bound on $\theta_{k,(m)}$. We multiply (4.14) by $(\theta^\ell_{k,m}(t))'$ and sum with respect to ℓ to obtain the following:

$$\begin{aligned}
&\sum_{\ell=1}^m \big(\Phi(\theta_{k,(m)})'', e_\ell\big)\cdot(\theta^\ell_{k,m}(t))' \\
&= \big(\Phi(\theta_{k,(m)})'', \theta'_{k,(m)}\big) \\
&=\frac{1}{2}\frac{d}{dt}\left[\int_B (-g^{00})|\theta'_{k,(m)}|^2 - (\partial_t^2 g^{00})|\theta_{k,(m)}|^2\,dy + \int_{\partial B}\gamma|\theta'_{k,(m)}|^2 + (\partial_t^2\gamma)|\theta_{k,(m)}|^2\,dS\right] \\
&\quad - \int_B \frac{3}{2}(\partial_t g^{00})|\theta'_{k,(m)}|^2\,dy + \int_{\partial B}\frac{3}{2}(\partial_t\gamma)|\theta'_{k,(m)}|^2\,dS + \frac{1}{2}\int_B \partial_t^3 g^{00}|\theta_{k,(m)}|^2\,dy \\
&\quad - \frac{1}{2}\int_{\partial B}\partial_t^3\gamma|\theta_{k,(m)}|^2\,dS.
\end{aligned}$$

$$\begin{aligned}
&\sum_{\ell=1}^m \mathcal{L}(\theta_{k,(m)}, e_\ell)\cdot(\theta^\ell_{k,m}(t))' \\
&=\mathcal{L}(\theta_{k,(m)}, \theta'_{k,(m)}) \\
&=\int_B \theta'_{k,(m)}(2g^{0j})\nabla_j\theta'_{k,(m)}\,dy + \int_B\left((\partial_t g^{00}) - \frac{1}{2}(\nabla_\alpha\log|g|)g^{0\alpha} + (\nabla_j g^{0j})\right)|\theta'_{k,(m)}|^2\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_B \theta_{k,(m)}(\partial_t^2 g^{00})\theta'_{k,(m)}\,dy+\int_B \nabla_i\theta_{k,(m)}g^{ij}\nabla_j\theta'_{k,(m)}\,dy\\
&+\int_B \nabla_j\theta_{k,(m)}\left(-(\partial_t g^{0j})-\frac{1}{2}(\nabla_\alpha\log|g|)g^{\alpha j}\right)\theta'_{k,(m)}\,dy\\
&+\int_{\partial B}|\theta'_{k,(m)}|^2(-n_j g^{0j}-2(\partial_t\gamma))\,dS\\
&+\int_{\partial B}\theta_{k,(m)}\theta'_{k,(m)}(-\partial_t^2\gamma)\,dS\\
=&\frac{1}{2}\frac{d}{dt}\left[\int_B(\partial_t^2 g^{00})|\theta_{k,(m)}|^2\,dy+\int_B g^{ij}\nabla_i\theta_{k,(m)}\nabla_j\theta_{k,(m)}\,dy+\int_{\partial B}|\theta_{k,(m)}|^2(-\partial_t^2\gamma)\,dS\right]\\
&+\int_B\left((\partial_t g^{00})-\frac{1}{2}(\nabla_\alpha\log|g|)g^{0\alpha}\right)|\theta'_{k,(m)}|^2\,dy-\int_B\frac{1}{2}\partial_t^3 g^{00}|\theta_{k,(m)}|^2\,dy\\
&-\frac{1}{2}\int_B(\partial_t g^{ij})\nabla_i\theta_{k,(m)}\nabla_j\theta_{k,(m)}\,dy\\
&-\int_B\nabla_j\theta_{k,(m)}\left((\partial_t g^{0j})+\frac{1}{2}(\nabla_\alpha\log|g|)g^{\alpha j}\right)\theta'_{k,(m)}\,dy\\
&+\int_{\partial B}|\theta'_{k,(m)}|^2(-2(\partial_t\gamma))\,dS+\frac{1}{2}\int_{\partial B}\partial_t^3\gamma|\theta_{k,(m)}|^2\,dS.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\ell=1}^m \mathcal{C}(\theta_{k,(m)},e_\ell)\cdot(\theta^\ell_{k,m}(t))'\\
=&\mathcal{C}(\theta_{k,(m)},\theta'_{k,(m)})\\
=&-\int_B(\partial_t g^{00})|\theta'_{k,(m)}|^2\,dy+\int_{\partial B}(\partial_t\gamma)|\theta'_{k,(m)}|^2\,dS.
\end{aligned}
$$

Thus, summing up the preceding bullet points, we know that the left hand side of (4.14), after multiplying by $\theta^\ell_{k,m}$ and summing up ℓ, gives:

$$
\begin{aligned}
\mathrm{HLS}=&\sum_{\ell=1}^m\left[\left(\Phi_m(\theta_{k,(m)})'',e_\ell\right)+\mathcal{L}(\theta_{k,(m)},e_\ell)+k\mathcal{C}(\theta_{k,(m)},e_\ell)\right]\cdot\theta^\ell_{k,m}\\
=&\frac{1}{2}\frac{d}{dt}\left[\int_B(-g^{00})|\theta'_{k,(m)}|^2+g^{ij}\nabla_i\theta_{k,(m)}\nabla_j\theta_{k,(m)}\,dy+\int_{\partial B}\gamma|\theta'_{k,(m)}|^2\,dS\right]\\
&-\int_B(k+\frac{1}{2})(\partial_t g^{00})|\theta'_{k,(m)}|^2\,dy+\int_{\partial B}(k+\frac{1}{2})(\partial_t\gamma)|\theta'_{k,(m)}|^2\,dS\\
&+\int_B\left((\partial_t g^{00})-\frac{1}{2}(\nabla_\alpha\log|g|)g^{0\alpha}\right)|\theta'_{k,(m)}|^2\,dy\\
&-\frac{1}{2}\int_B(\partial_t g^{ij})\nabla_i\theta_{k,(m)}\nabla_j\theta_{k,(m)}\,dy
\end{aligned}
$$

$$+ \int_B \nabla_j \theta_{k,(m)} \left(-(\partial_t g^{0j}) - \frac{1}{2} (\nabla_\alpha \log |g|) g^{\alpha j} \right) \theta'_{k,(m)} \, dy$$
$$+ \int_{\partial B} |\theta'_{k,(m)}|^2 (-2(\partial_t \gamma)) \, dS.$$

The right hand side of (4.14), after multiplying by $\theta^\ell_{k,m}$ and summing up ℓ, gives:

$$\begin{aligned}
\text{RHS} = &- \int_B \left(\partial_t^k q + \partial_t^{k-1} (I_{2,m} + I_{3,m}) - k \partial_t^2 g^{00} \theta'_{k-1,(m)} \right) \theta'_{k,(m)} \, dy \\
&- \int_B (\partial_t^{k-1} I_1^j) \nabla_j \theta'_{k,(m)} \, dy \\
&+ \int_{\partial B} \left(\partial_t^k (2f/a) - \partial_t^{k-1} (B_{1,m}) - k (\partial_t^2 \gamma) \theta'_{k-1,(m)} \right) \theta'_{k,(m)} \, dS.
\end{aligned}$$

Let

$$E_m(t) = \int_B (-g^{00}) |\theta'_{k,(m)}|^2 + g^{ij} \nabla_i \theta_{k,(m)} \nabla_j \theta_{k,(m)} \, dy + \int_{\partial B} \gamma |\theta'_{k,(m)}|^2 \, dS$$

Then equating LHS and RHS, and integrating with respect to time, we see that

$$\begin{aligned}
E_m(t) \lesssim & E_m(0) + \int_0^t E_m(\tau) \cdot \left(\sum_{r=0,1} \|\nabla_{t,y}^{(r)} g\|_{L^\infty(B)} + \|\partial_t \gamma\|_{L^\infty(\partial B)} \right) d\tau \\
&+ \int_0^t E_m(\tau) \cdot \Big(\|\partial_t^2 g^{00}\|_{L^\infty(I \times B)} \cdot \|\theta'_{k-1,(m)}\|^2_{L^2(B_\tau)} \\
&\qquad + \|\partial_t^2 \gamma\|_{L^\infty(I \times \partial B)} \cdot \|\theta'_{k-1,(m)}\|^2_{L^2(\partial B_\tau)} \Big) \, d\tau \\
&+ \underbrace{\left| \int_0^t \int_B \left(-\partial_t^k q + \partial_t^{k-1} (I_{2,m} + I_{3,m}) \right) \theta'_{k,(m)} \, dy d\tau \right|}_{:=\mathrm{I}} \\
&+ \underbrace{\left| \int_0^t \int_{\partial B} \left(\partial_t^k (2f/a) - \partial_t^{k-1} (B_{1,m}) \right) \theta'_{k,(m)} \, dS d\tau \right|}_{:=\mathrm{II}} \\
&+ \underbrace{\left| \int_0^t \int_B (\partial_t^{k-1} I_1^j) \nabla_j \theta'_{k,(m)} \, dy d\tau \right|}_{:=\mathrm{III}}.
\end{aligned} \tag{4.17}$$

The first term can be controlled by the assumption on u, Λ. The control on the second term follows from regularity of $\theta_{k-1,(m)}$ by induction, and assumptions on u, Λ. We seek to control the remaining three terms:

- $\|\partial_t^k q\|_{L^2(I,L^2(B))}$. Recall that $q = \Box_g u$, so by Lemma 2.9, $\partial_t^k q$ consists of terms of the forms 4 – 6 in the statement of Lemma 2.9. As before, each term is a product of a few terms, where we seek to control the lower order terms in $L^\infty(B)$ and higher order terms in $L^2(B)$. The highest order terms of form 4 are $\partial_t^k g, \nabla\partial_t^k X, \nabla\partial_t^k w$; the highest order terms of form 5 are $\partial_t^k g, \nabla\partial_t^k X, \nabla\partial_t^k \Sigma^2 \overset{\epsilon}{\sim} \nabla\partial_t^{k-1}\Lambda, \partial_t^k\Lambda$; the highest order terms of form 6 are $\partial_t^k g, \nabla\partial_t^k X, \nabla\partial_t^k\Lambda$. All of these can be controlled by $\mathfrak{E}_M[u,\Lambda]$.

- $\|\partial_t^{k-1}(I_{2,m}+I_{3,m})\|_{L^2(I,L^2(B))}$. In fact, we will not be able to control $\|\partial_t^{k-1}(I_{2,m}+I_{3,m})\|_{L^2(I,L^2(B))}$ when $k = M$. Instead, to treat the highest order case (which also works for lower order equations), we will use a computation that is similar to the F^{ij} manipulation when we estimated $\int_\tau\int_{\mathbb{R}^3_+}(\tilde{\Box}_g\partial_t^k u)(\partial_t^{k+1}u)\,dyd\tau$ in the a priori estimate. More precisely, when ∂_t^{k-1} falls on $\partial_t^2 g$ and $\theta'_{0,(m)}$, we are able to control, for instance,
$$\left|\int_0^t\int_B \partial_t^{k+1}g^{00}\theta'_{0,(m)}\theta'_{k,(m)}\,dyd\tau\right| \le \int_0^t \|\partial_t^{k+1}g^{00}\|^2_{L^2(B)} + \|\theta'_{k,(m)}\|^2_{L^2(B)}\,d\tau.$$
The most difficult term is when ∂_t^{k-1} falls on $\nabla\partial_t g$. That is, we need to control
$$\left|\int_0^t\int_B(\nabla\partial_t^k g)(\theta'_{0,(m)})(\theta'_{k,(m)})\,dyd\tau\right|.$$
To do so, recall that there is some function $F^{\alpha\beta}$ with
$$\sup_{0\le t\le T}\|F^{\alpha\beta}\|_{L^\infty(B_t)} + \|\nabla F^{\alpha\beta}\|_{L^\infty(B_t)} < \infty,$$
such that
$$\nabla\partial_t^k g = F^{\alpha\beta}\nabla_\alpha\nabla_\beta\partial_t^{k-1}u.$$
Then, we seek to move one ∇ derivative to $\theta_{k,(m)}$ and move one ∂_t derivative to g:
$$\int_0^t\int_B(\nabla\partial_t^k g)(\theta'_{0,(m)})(\theta'_{k,(m)})\,dyd\tau$$

$$
\begin{aligned}
&= \int_0^t \int_B (F^{\alpha\beta} \nabla_\alpha \nabla_\beta \partial_t^{k-1} u)(\theta'_{0,(m)})(\theta'_{k,(m)})\, dy d\tau \\
&= \int_0^t \int_{\partial B} n_\alpha F^{\alpha\beta} \nabla_\beta \partial_t^{k-1} u(\theta'_{0,(m)})(\theta'_{k,(m)})\, dS d\tau \\
&\quad - \int_0^t \int_B (\nabla_\beta \partial_t^{k-1} u) \nabla_\alpha \left(F^{\alpha\beta} (\theta'_{0,(m)})(\theta'_{k,(m)}) \right)\, dy d\tau \\
&= \int_0^t \int_{\partial B} n_\alpha F^{\alpha\beta} \nabla_\beta \partial_t^{k-1} u(\theta'_{0,(m)})(\theta'_{k,(m)})\, dS d\tau \\
&\quad - \int_0^t \int_B (\nabla_\beta \partial_t^{k-1} u) \nabla_\alpha \left(F^{\alpha\beta} (\theta'_{0,(m)}) \right) (\theta'_{k,(m)})\, dy d\tau \\
&\quad - \int_0^t \int_B (\nabla_\beta \partial_t^{k-1} u) \left(F^{\alpha\beta} (\theta'_{0,(m)}) \right) \nabla_\alpha (\theta'_{k,(m)})\, dy d\tau.
\end{aligned}
$$

Recall that by Lemma (2.5), we are able to control

$$
\int_0^t \int_{\partial B} |\nabla_\beta \partial_t^{k-1} u|^2 \, dS d\tau,
$$

so the first and second term on the right hand side are controlled. We further work on the last term:

$$
\begin{aligned}
&\left| \int_0^t \int_B (\nabla_\beta \partial_t^{k-1} u) \left(F^{\alpha\beta} (\theta'_{0,(m)}) \right) \nabla_\alpha (\theta'_{k,(m)})\, dy d\tau \right| \\
\le & \left| \int_{B_t} (\nabla_\beta \partial_t^{k-1} u) \left(F^{\alpha\beta} (\theta'_{0,(m)}) \right) \nabla_\alpha (\theta_{k,(m)})\, dy \right| \\
&+ \left| \int_{B_0} (\nabla_\beta \partial_t^{k-1} u) \left(F^{\alpha\beta} (\theta'_{0,(m)}) \right) \nabla_\alpha (\theta_{k,(m)})\, dy \right| \\
&+ \left| \int_0^t \int_B (\nabla_\beta \partial_t^{k} u)(F^{\alpha\beta} \theta'_{0,(m)})(\nabla_\alpha \theta_{k,(m)})\, dy d\tau \right| \\
&+ \left| \int_0^t \int_B (\nabla_\beta \partial_t^{k-1} u) \partial_t (F^{\alpha\beta} \theta'_{0,(m)})(\nabla_\alpha \theta_{k,(m)})\, dy d\tau \right| \\
\le & \delta \int_{B_t} |\nabla_\alpha \theta_{k,(m)}|^2 \, dy + \frac{1}{\delta} \int_{B_t} |F^{\alpha\beta} \theta'_{0,(m)} \nabla_\beta \partial_t^{k-1} u|^2 \, dy \\
&+ \int_{B_0} |\nabla_\alpha \theta_{k,(m)}|^2 \, dy + \int_{B_0} |F^{\alpha\beta} \theta'_{0,(m)} \nabla_\beta \partial_t^{k-1} u|^2 \, dy \\
&+ \int_0^t E_m(\tau) \Big(\|(\nabla_\beta \partial_t^{k} u)(F^{\alpha\beta} \theta'_{0,(m)})\|^2_{L^2(B_\tau)} + \\
&\qquad\qquad + \|(\nabla_\beta \partial_t^{k-1} u) \partial_t (F^{\alpha\beta} \theta'_{0,(m)})\|^2_{L^2(B_\tau)} \Big) \, d\tau.
\end{aligned}
$$

Therefore, by choosing δ small, this term can be absorbed into the left hand side of (4.17), and we have controlled the term I.

- $\|\partial_t^k(f/a)\|_{L^2(I,L^2(\partial B))}$. Recall that $a^2 = g^{\alpha\beta}(\nabla_\alpha\Sigma^2)(\nabla_\beta\Sigma^2)$, so the highest order term in $\partial_t^k a$ is $\nabla\partial_t^k\Sigma^2$, which is controllable by the assumption on Σ^2. Recall also that $f = \mathcal{P}u$, so $\partial_t^k f = \partial_t^k\mathcal{P}u$ is given by terms of the form 3, 4, 5 in Lemma (2.8) with $\epsilon = 0$. The analysis is identical to the computation that we did when estimating $\int_0^t\int_{\partial\mathbb{R}^3_+}(\mathcal{P}^\epsilon\partial_t^k u)(\partial_t^{k+1}u)\,dSd\tau$, except that this case here is simpler because we do not have the commutators of the forms 1, 2 in F_k^0. We refrain from copying the same calculation.

- $\|\partial_t^{k-1}B_{1,m}\|_{L^2(I,L^2(\partial B))}$. The highest order terms in $\partial_t^{k-1}B_{1,m}$ are

$$\partial_t^k\left(\nabla_j\Sigma^2 g^{0j}\right)\theta'_{0,(m)}, \quad \text{and} \quad \partial_t\left(\nabla_j\Sigma^2 g^{0j}\right)\partial_t^{k-1}\theta'_{0,(m)}.$$

The first term is controlled by $\|\nabla\partial_t^{k-1}\Sigma^2\|_{L^2(I,L^2(\partial B))}$ and $\|\nabla\partial_t^{k-1}u\|_{L^2(I,L^2(\partial B))}$ by Lemmas 2.4 and 2.5. The second term can be controlled by $\|\theta'_{k-1}\|_{L^2(I,L^2(\partial B))}$. Thus we are able to control II in (4.17).

- Finally, we analyze III.

$$\begin{aligned}
&\int_0^t\int_B(\partial_t^{k-1}I^j_{1,m})\nabla_j\theta'_{k,(m)}\,dyd\tau\\
&=\int_0^t\frac{d}{dt}\left(\int_B(\partial_t^{k-1}I^j_{1,m})\nabla_j\theta_{k,(m)}\,dy\right)d\tau-\int_0^t\int_B(\partial_t^kI^j_{1,m})\nabla_j\theta_{k,(m)}\,dyd\tau\\
&=\int_{B_t}(\partial_t^{k-1}I^j_{1,m})\nabla_j\theta_{k,(m)}\,dy-\int_{B_0}(\partial_t^{k-1}I^j_{1,m})\nabla_i\theta_{k,(m)}\,dy\\
&\quad-\int_0^t\int_B(\partial_t^kI^j_{1,m})\nabla_j\theta_{k,(m)}\,dyd\tau.
\end{aligned}$$

We compute $\partial_t^{k-1}I^j_{1,m}$, in which the highest order terms are

$$(\partial_t^kg^{0j})\theta'_{0,(m)}, \quad (\partial_t^kg^{ij})\nabla_i\theta_{0,(m)}, \quad (\partial_tg^{0j})\partial_t^{k-1}\theta'_{0,(m)}, \quad (\partial_tg^{ij})\nabla_i\partial_t^{k-1}\theta_{0,(m)}.$$

Thus, $\|\partial_t^{j-1} I^j_{1,m}\|_{L^2(I,L^2(B))}$ can be controlled. The term $\int_{B_t}(\partial_t^{k-1} I^j_{1,m})\nabla_j\theta_{k,(m)}\,dy$ can thus be bounded by

$$\left|\int_{B_t}(\partial_t^{k-1} I^j_{1,m})\nabla_j\theta_{k,(m)}\,dy\right| \le \delta E_m(t) + \frac{1}{\delta}\int_{B_t}|\partial_t^{k-1} I_{3,m}|^2\,dy,$$

which can be absorbed into the left hand side of (4.17) by choosing $\delta > 0$ small. Similarly, the term $\int_{B_0}(\partial_t^{k-1} I^j_{1,m})\nabla_i\theta_{k,(m)}\,dy$ can be bounded by

$$\left|\int_{B_0}(\partial_t^{k-1} I_{3,m})\nabla_i\theta_{k,(m)}\,dy\right| \le E_m(0) + \int_{B_0}|\partial_t^{k-1} I_{3,m}|^2\,dy.$$

We deal with the most difficult term $\int_0^t\int_B(\partial_t^k I^j_{1,(m)})\nabla_j\theta_{k,(m)}\,dyd\tau$. In computing $\partial_t^k I^j_{1,m}$,, when ∂_t^k fall on $\partial_t g$, we can easily bound the result by $\|\partial_t^{k+1} g\|_{L^2(B)}$, so we shall focus on the case when ∂_t^k falls on $\theta'_{0,(m)}$ and $\nabla_i\theta_{0,(m)}$. We have

$$\begin{aligned}
&\left|\int_0^t\int_B(\partial_t g^{0j})(\partial_t^k\theta'_{0,(m)})(\nabla_j\theta_{k,(m)})\,dyd\tau\right|\\
=&\left|\int_0^t\int_B(\partial_t g^{0j})(\theta'_{k,(m)})(\nabla_j\theta_{k,(m)})\,dyd\tau\right|\\
\le&\|\partial_t g\|_{L^\infty(I\times B)}\cdot\left|\int_0^t E_m(\tau)\,d\tau\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^t\int_B(\partial_t g^{ij})(\nabla_i\partial_t^k\theta_{0,(m)})(\nabla_j\theta_{k,(m)})\,dyd\tau\right|\\
=&\left|\int_0^t\int_B(\partial_t g^{ij})(\nabla_i\theta_{k,(m)})(\nabla_j\theta_{k,(m)})\,dyd\tau\right|\\
\le&\|\partial_t g\|_{L^\infty(I\times B)}\cdot\left|\int_0^t E_m(\tau)\,d\tau\right|.
\end{aligned}$$

Thus III has been shown to be bounded. Note that this is the reason why we had to fix m and solve for all orders of the approximate equations before taking the limit $m\to\infty$, for otherwise we will not be able to control $\partial_t^k\theta'_0$.

We have analyzed all terms in (4.17), so by Gronwall's inequality, we have proved that there is some C_4 such that

$$\sup_{t\in[0,T]}\|\theta'_{k,(m)}\|_{L^2(B_t)} + \|\theta_{k,(m)}\|_{H^1(B_t)} + \|\theta'_{k,(m)}\|_{L^2(\partial B_t)} < C_4 < \infty \tag{4.18}$$

for all $m \geq 1$. Here C_4 depends on $\mathfrak{E}_M[u, \Lambda](T)$ only, and does not depend on m.

To finish our uniform bound, we need to estimate $\|\Phi_m(\theta_{k,(m)})''\|_{L^2(I,(H^1(B))')}$. Let $\phi \in H^1(B)$. Then by (4.14) and the estimate (4.18), we know that

$$\big(\Phi_m(\theta_{k,(m)})'', \phi\big) \lesssim C_4 \|\phi\|_{H^1(B)}$$

as well, which, after possibly enlarging C_4, shows that

$$\|\Phi_m(\theta_{k,(m)})''\|_{L^2(I,(H^1(B))')} \leq C_4 < \infty. \tag{4.19}$$

To summarize what we have achieved so far, we have proven that for each fixed integer m, the system of approximate solutions (4.14) has a unique solution on $[0, T]$ for $k = 0, \cdots, M$. And moreover, these solutions satisfy the bounds (4.18) and (4.19) uniformly in m.

Our next goal is to let $m \to \infty$ and construct a solution to the actual weak equation.

Existence of weak solutions. By the uniform boundedness established earlier, we know that there is a subsequence $\theta_{k,(m_n)}$ as well as functions $\theta_k \in L^2(I, H^1(B))$ with $\theta_k' \in L^2(I, L^2(B))$ and $\phi_k \in L^2(U, (H^1(B))')$ such that

$$\begin{aligned} \theta_{k,(m_n)} &\rightharpoonup \theta_k \quad \text{weakly in } L^2(I, H^1(B)) \\ \theta'_{k,(m_n)} &\rightharpoonup \theta'_k \quad \text{weakly in } L^2(I, L^2(B)) \\ \Phi_{m_n}(\theta_{k,(m_n)}) &\rightharpoonup \phi_k \quad \text{weakly in } L^2(I, (H^1(B))'). \end{aligned}$$

We first show a compatibility result. To simplify the notation, in what follows, we will relabel the subsequence and set $m = m_n$.

- $\theta'_{k-1} = \theta_k$ in $L^2(B)$. Let $\psi \in L^2(B)$, then

$$\big\langle \theta'_{k-1}, \psi \big\rangle = \lim_{m\to\infty} \big\langle \theta'_{k-1,(m)}, \psi \big\rangle$$

$$= \lim_{m\to\infty} \left\langle \theta_{k,(m)}, \psi \right\rangle$$

$$= \langle \theta_k, \psi \rangle .$$

Our claim thus follows.

- $\Phi(\theta) = \phi$ in $(H^1(B))'$. To see this, we first compare their action on a dense subset of $H^1(B)$. Let $K > 0$ and $\psi = \sum_{\ell=1}^K \psi^\ell e_\ell$. Then

$$\begin{aligned}(\Phi(\theta), \psi) &= \left\langle -g^{00}\theta, \psi \right\rangle + \langle\!\langle \gamma\theta, \psi \rangle\!\rangle \\ &= \lim_{n\to\infty} \left\langle -g^{00}\theta_{(m_n)}, \psi \right\rangle + \left\langle\!\left\langle \gamma\theta_{(m_n)}, \psi \right\rangle\!\right\rangle \\ &= \lim_{n\to\infty} \left(\Phi_{m_n}(\theta_{(m_n)}), \psi\right) \\ &= (\phi, \psi) .\end{aligned}$$

Since ψ of such forms are dense in $H^1(B)$, we know that $\Phi(\theta) = \phi$ in $(H^1(B))'$, that is,

$$\Phi_{m_n}(\theta_{(m_n)}) \rightharpoonup \Phi(\theta) \quad \text{weakly in } L^2(I, (H^1(B))').$$

Next, we need to show that θ_k is a weak solution to (4.9). To see this, we take a function

$$\psi = \sum_{\ell=1}^{K} d^\ell(t) e_\ell, \quad d^\ell \in C^\infty([0, T]) \ \forall \ell.$$

We multiply (4.14) by $d^\ell(t)$ and sum up with respect to ℓ:

$$\begin{aligned}&\left(\Phi_m(\theta_{k,(m)})'', \psi\right) + \mathcal{L}(\theta_{k,(m)}, \psi) + k\mathcal{C}(\theta_{k,(m)}, \psi) \\ = &- \left\langle \partial_t^k q, \psi \right\rangle + \left\langle\!\left\langle \partial_t^k (2f/a), \psi \right\rangle\!\right\rangle \\ &+ \left\langle \partial_t^{k-1} I_{1,m}^j, \nabla_j \psi \right\rangle + \left\langle \partial_t^{k-1} (I_{2,m} + I_{3,m}), \psi \right\rangle + \left\langle\!\left\langle \partial_t^{k-1} B_{1,m}, \psi \right\rangle\!\right\rangle \\ &+ \left\langle k(\partial_t^2 g^{00})\theta'_{k-1,(m)}, \psi \right\rangle - \left\langle\!\left\langle k(\partial_t^2 \gamma)\theta'_{k-1,(m)}, \psi \right\rangle\!\right\rangle .\end{aligned}$$

Sending $m \to \infty$, we see that θ satisfies equation (4.9) for this particular ψ. But then since such ψ are dense in $H^1(B)$, we know that θ satisfies equation (4.9) for all $\psi \in H^1(B)$, showing that it is a weak solution to the actual weak solution.

Uniqueness of weak solutions. Now we show that the weak solution to (4.9) is unique. Since (4.9) is a linear equation, by taking the difference of two solutions, we may assume that $\theta_k(0, y) \equiv 0$, the right hand side of (4.9) is identically zero, and need to show that $\theta_k(t, \cdot) \equiv 0$ for all $t \in [0, T]$. For this part, we suppress the dependence on k, and our goal is to show that $\theta \equiv 0$ is the only solution to

$$(\Phi(\theta)'', \psi) + \mathcal{L}(\theta, \psi) + k\mathcal{C}(\theta, \psi) = 0 \quad \forall \psi \in H^1(B) \tag{4.20}$$

satisfying $\theta \in L^2(I, H^1(B)), \theta' \in L^2(I, L^2(B)), \Phi(\theta) \in L^2(I, (H^1(B))')$.

To see this, let θ be any solution to (4.20) with the specified regularity. Fix any $s \in (0, T)$. and consider

$$\delta(t) := \begin{cases} -\int_t^s \theta(\tau)\, d\tau & \text{if } 0 \le t \le s \\ 0 & \text{if } s \le t \le T \end{cases},$$

so that $\delta'(t) = \theta(t)$ for $t \in [0, s]$. Clearly $\delta(t) \in H^1(B)$, so substituting $\psi = \delta$ and integrating with respect to time, we have

$$\underbrace{\int_0^s (\Phi(\theta)'', \delta)\ d\tau}_{\text{I}} + \underbrace{\int_0^s \mathcal{L}(\theta, \delta)\, d\tau}_{\text{II}} + k \underbrace{\int_0^s \mathcal{C}(\theta, \delta) d\tau}_{\text{III}} = 0. \tag{4.21}$$

We compute I:

$$\begin{aligned} &\int_0^s \partial_\tau (\Phi(\theta)', \delta)\ d\tau - \int_0^s (\Phi(\theta)', \delta')\ d\tau \\ &= (\Phi(\theta)', \delta)\Big|_0^s - \int_0^s (\Phi(\theta)', \theta)\ d\tau. \end{aligned}$$

The first term on the right hand side vanishes, since $\Phi(\theta)' \equiv 0$ at $\tau = 0$ and $\delta \equiv 0$ at $\tau = s$, and we continue with the second term:

$$\text{I} = -\int_0^s (\Phi(\theta)', \theta)\ d\tau$$

$$
\begin{aligned}
&= -\int_0^s \int_B \partial_t(-g^{00\theta})\theta dy d\tau - \int_0^s \int_{\partial B} \partial_t(\gamma\theta)\theta \, dS d\tau \\
&= \frac{1}{2}\int_{B_s - B_0} g^{00}|\theta|^2 \, dy - \frac{1}{2}\int_{\partial B_s - \partial B_0} \gamma|\theta|^2 \, dS d\tau \\
&\quad + \frac{1}{2}\int_0^s \int_B (\partial_t g^{00})|\theta|^2 \, dy d\tau - \frac{1}{2}\int_0^s \int_{\partial B} (\partial_t \gamma)|\theta|^2 \, dS d\tau.
\end{aligned}
$$

To compute II, we use the fact that $\theta = \delta'$ for $t \in [0, s]$. The exact formula for II is complicated (as we shall see soon), but what is important is that it is equal to some main terms involving $\nabla_y \delta$ at $t = 0$, in addition to some lower order terms involving $\|\delta\|_{H^1(B)}$, $\|\theta\|_{L^2(B)}$, as well as some low order derivatives of g and γ. We remark that several terms in the result will vanish because $\delta(s) \equiv 0$ and $\delta'(0) = \theta(0) \equiv 0$.

$$
\begin{aligned}
\mathrm{II} = &\int_0^s \left\langle \delta'', 2g^{0j}\nabla_j \delta \right\rangle + \left\langle \delta'', \left((\partial_t g^{00}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{0\alpha} + (\nabla_j g^{0j}) \right) \delta \right\rangle \\
&+ \left\langle \delta', \partial_t^2 g^{00} \delta \right\rangle + \left\langle \nabla_j \delta', g^{ij}\nabla_i \delta \right\rangle + \left\langle \nabla_j \delta', \left(-(\partial_t g^{0j}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{\alpha j} \right) \delta \right\rangle \\
&+ \left\langle\!\left\langle \delta'', (-n_j g^{0j} - 2(\partial_t \gamma))\delta \right\rangle\!\right\rangle + \left\langle\!\left\langle \delta', -(\partial_t^2 \gamma)\delta \right\rangle\!\right\rangle \, d\tau \\
= &\underbrace{\int_{B_s - B_0} \frac{1}{2} g^{ij}\nabla_i \delta \nabla_j \delta \, dy}_{-\int_{B_0} \cdots \, dy} + \underbrace{\int_{B_s - B_0} \delta'\delta \left((\partial_t g^{00}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{0\alpha} + (\nabla_j g^{0j}) \right) dy}_{=0} \\
&+ \underbrace{\int_{B_s - B_0} (2g^{0j})\delta' \nabla_j \delta \, dy}_{=0} + \underbrace{\int_{B_s - B_0} |\delta|^2 \nabla_j \left(-(\partial_t g^{0j}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{\alpha j} \right) dy}_{=-\int_{B_0} \cdots \, dy} \\
&+ \frac{1}{2} \underbrace{\int_{\partial B_s - \partial B_0} |\delta|^2 n_j \left(-(\partial_t g^{0j}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{\alpha j} \right) dS d\tau}_{=-\int_{\partial B_0} \cdots \, dS} \\
&+ \underbrace{\int_{\partial B_s - \partial B_0} \delta'\delta(-n_j g^{0j} - 2(\partial_t \gamma)) \, dS}_{=0} + \frac{1}{2} \underbrace{\int_{\partial B_s - \partial B_0} |\delta|^2 (-\partial_t^2 \gamma) \, dS}_{=-\int_{\partial B_0} \cdots \, dS} \\
&- \int_0^s \int_B \delta'\delta \partial_t \left(-\frac{1}{2}(\nabla_\alpha \log|g|)g^{0\alpha} + (\nabla_j g^{0j}) \right) dy d\tau \\
&- \int_0^s \int_B |\delta'|^2 \left((\partial_t g^{00}) - \frac{1}{2}(\nabla_\alpha \log|g|)g^{0\alpha} \right) dy d\tau \\
&- \frac{1}{2}\int_0^s \int_B (\partial_t g^{ij})\nabla_i \delta \nabla_j \delta \, dy d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^s\int_B |\delta|^2\nabla_j\partial_t\left(-(\partial_t g^{0j})-\frac{1}{2}(\nabla_\alpha\log|g|)g^{\alpha j}\right)\,dyd\tau\\
&-\int_0^s\int_B (\nabla_j\delta)\delta'\left((\partial_t g^{0j})-\frac{1}{2}(\nabla_\alpha\log|g|)g^{\alpha j}\right)\,dyd\tau\\
&+\frac{1}{2}\int_0^s\int_{\partial B}|\delta|^2 n_j\partial_t\left((\partial_t g^{0j})+\frac{1}{2}(\nabla_\alpha\log|g|)g^{\alpha j}\right)\,dSd\tau+\int_0^s\int_{\partial B}\frac{1}{2}|\delta|^2\partial_t^3\gamma\,dSd\tau\\
&-\int_0^s\int_{\partial B}\delta'\delta\partial_t(-n_jg^{0j}-2(\partial_t\gamma))\,dSd\tau-\int_0^s\int_{\partial B}|\delta'|^2(-2(\partial_t\gamma))\,dSd\tau.
\end{aligned}
$$

The computation for III is similar, albeit simpler:

$$
\begin{aligned}
\mathrm{III}=&-\underbrace{\int_{B_s-B_0}(\partial_t g^{00})\delta'\delta\,dy}_{=0}+\underbrace{\int_{\partial B_s-\partial B_0}(\partial_t\gamma)\delta'\delta\,dS}_{=0}\\
&+\int_0^s\int_B(\partial_t^2 g^{00})\delta'\delta\,dyd\tau+\int_0^s\int_B(\partial_t g^{00})|\delta'|^2\,dyd\tau\\
&-\int_0^s\int_{\partial B}(\partial_t^2\gamma)\delta'\delta\,dSd\tau-\int_0^s\int_{\partial B}(\partial_t\gamma)|\delta'|^2\,dSd\tau.
\end{aligned}
$$

Thus, by (4.21) and Gronwall's inequality, we have a constant C_5 depending on low order derivatives of g and γ, such that

$$
\begin{aligned}
&\|\delta(0)\|^2_{H^1(B)}+\|\theta(s)\|^2_{L^2(B)}+\|\theta(s)\|^2_{L^2(\partial B)}\\
\le& C_5\int_0^s\|\delta(\tau)\|^2_{H^1(B)}+\|\theta(\tau)\|^2_{L^2(B)}+\|\theta(\tau)\|^2_{L^2(\partial B)}\,d\tau.
\end{aligned}
\tag{4.22}
$$

Next, consider

$$
\kappa(t):=\int_0^t\theta(\tau)\,d\tau,
$$

so for $t\in[0,s]$, we have $\delta(\tau)=\kappa(\tau)-\kappa(s)$ and thus $\delta(0)=-\kappa(s)$. Hence, (4.22) gives

$$
\begin{aligned}
&\|\kappa(s)\|^2_{H^1(B)}+\|\theta(s)\|^2_{L^2(B)}+\|\theta(s)\|^2_{L^2(\partial B)}\\
\le& C_5\int_0^s\|\kappa(\tau)-\kappa(s)\|^2_{H^1(B)}+\|\theta(\tau)\|^2_{L^2(B)}+\|\theta(\tau)\|^2_{L^2(\partial B)}\,d\tau\\
\le& C_5\int_0^s 2\|\kappa(\tau)\|^2_{H^1(B)}+\|\theta(\tau)\|^2_{L^2(B)}+\|\theta(\tau)\|^2_{L^2(\partial B)}\,d\tau+2sC_5\|\kappa(s)\|^2_{H^1(B)},
\end{aligned}
$$

implying

$$
\begin{aligned}
&(1-2sC_5)\|\kappa(s)\|^2_{H^1(B)} + \|\theta(s)\|^2_{L^2(B)} + \|\theta(s)\|^2_{L^2(\partial B)} \\
\leq & C_5 \int_0^s 2\|\kappa(\tau)\|^2_{H^1(B)} + \|\theta(\tau)\|^2_{L^2(B)} + \|\theta(\tau)\|^2_{L^2(\partial B)}\, d\tau.
\end{aligned}
$$

Choose $s < \frac{1}{2C_5}$, then we see that $\theta \equiv 0$ for $t \in [0, s]$. Now, repeat the argument with initial time $t = s$, we know that $\theta \equiv 0$ on $[s, 2s]$; iterating, we see that $\theta \equiv 0$ for all $t \in [0, T]$.

Therefore, we have established the existence and uniqueness of weak solution to (4.9). □

Next, we shall prove that a similar result also holds for the equation on Λ.

4.2 Existence of Solution Λ

This section is largely parallel to the previous, except that the boundary condition on Λ is simpler.

Let $\theta \in C^2(B)$ be a solution satisfying

$$
\begin{cases}
\theta = 0 & \text{on } \partial B \\
\Box_h \theta = q & \text{in } B.
\end{cases}
$$

Let $\phi : B \to \mathbb{R}$ be a smooth function that vanishes on ∂B. Then

$$
\begin{aligned}
&\int_{B_t} (\Box_h \theta)\phi\, dy \\
=&\int_{B_t} \nabla_\alpha(h^{\alpha\beta}\nabla_\beta\theta)\cdot\phi\, dy + \int_{B_t} \frac{1}{2}(\nabla_\alpha \log|h|)h^{\alpha\beta}(\nabla_\beta\theta)\phi\, dy \\
=&\int_{B_t} \partial_t(h^{0\beta}\nabla_\beta\theta)\phi\, dy - \int_{B_t} h^{j\beta}\nabla_\beta\theta\nabla_j\phi\, dy + \int_{B_t} \frac{1}{2}(\nabla_\alpha \log|h|)h^{\alpha\beta}(\nabla_\beta\theta)\phi\, dy.
\end{aligned}
$$

Regrouping terms involving derivatives of θ, we obtain

(4.23)

$$
\int_{B_t} q\phi\, dy
$$

$$
\begin{aligned}
&= \int_{B_t} \partial_t^2 \theta \cdot h^{00} \phi \, dy \\
&\quad + \int_{B_t} \partial_t \theta \cdot \left[(\partial_t h^{00}) + \frac{1}{2} (\nabla_\alpha \log |h|) h^{0\alpha} - (\nabla_j h^{0j}) \right] \phi \, dy + \int_{B_t} \partial_t \theta \cdot \left(-2h^{0j} \right) \nabla_j \phi \, dy \\
&\quad + \int_{B_t} \nabla_j \theta \cdot \left[(\partial_t h^{0j}) + \frac{1}{2} (\nabla_\alpha \log |h|) h^{\alpha j} \right] \phi \, dy + \int_{B_t} \nabla_j \theta \cdot \left(-h^{ij} \right) \nabla_i \phi \, dy.
\end{aligned}
$$

We define the bounded linear map $\Phi : H_0^1(B) \to (H_0^1(B))'$:

$$
(\Phi(\theta), \phi) := \left\langle -h^{00} \theta, \phi \right\rangle . \tag{4.24}
$$

As before, we compute the difference:

$$
(\Phi(\theta)'', \phi) = (\Phi(\theta''), \phi) - \left\langle 2(\partial_t h^{00}) \theta', \phi \right\rangle - \left\langle (\partial_t^2 h^{00}) \theta, \phi \right\rangle .
$$

Then by (4.23), we know that the weak equation is:

$$
(\Phi(\theta)'', \phi) + \mathcal{L}(\theta, \phi) = - \langle q, \phi \rangle \quad \forall \phi \in H_0^1(B), \tag{4.25}
$$

where $\mathcal{L}(\theta, \phi)$ represents the weak formulation of the lower order derivatives on θ:

$$
\begin{aligned}
\mathcal{L}(\theta, \phi) &= \sum_{i=1}^{5} \mathcal{L}_i(\theta, \phi) \\
\mathcal{L}_1(\theta, \phi) &= \left\langle \theta', 2h^{0j} \nabla_j \phi \right\rangle \\
\mathcal{L}_2(\theta, \phi) &= \left\langle \theta', \left((\partial_t h^{00}) - \frac{1}{2} (\nabla_\alpha \log |h|) h^{0\alpha} + (\nabla_j h^{0j}) \right) \phi \right\rangle \\
\mathcal{L}_3(\theta, \phi) &= \left\langle \theta, \partial_t^2 h^{00} \phi \right\rangle \\
\mathcal{L}_4(\theta, \phi) &= \left\langle \nabla_j \theta, h^{ij} \nabla_i \phi \right\rangle \\
\mathcal{L}_5(\theta, \phi) &= \left\langle \nabla_j \theta, \left(-(\partial_t h^{0j}) - \frac{1}{2} (\nabla_\alpha \log |h|) h^{\alpha j} \right) \phi \right\rangle .
\end{aligned}
$$

Let $\theta_0 := \theta$ and $\theta_1 := \theta_0'$. We have as before

$$
\mathcal{C}(\theta_1, \phi) := - \left\langle (\partial_t h^{00}) \theta_1', \phi \right\rangle + \langle\!\langle (\partial_t \gamma) \theta_1', \phi \rangle\!\rangle , \tag{4.26}
$$

and

$$
I_1^j = -(2\partial_t h^{0j}) \theta_0' - (\partial_t h^{ij}) \nabla_i \theta_0
$$

$$I_2 = \left(\partial_t^2 h^{00} + \frac{1}{2}\partial_t((\nabla_\alpha \log|h|)h^{0\alpha}) - \nabla_j \partial_t h^{0j}\right)\theta_0'$$
$$I_3 = \left(\partial_t^2 h^{0j} + \frac{1}{2}\partial_t\left((\nabla_\alpha \log|h|)h^{\alpha j}\right)\right)\nabla_j \theta_0.$$

Then the equation for θ_1 is:

$$(\Phi(\theta_1)'', \phi) + \mathcal{L}(\theta_1, \phi) + \mathcal{C}(\theta_1, \phi) = -\langle \partial_t q, \phi\rangle + \left\langle I_1^j, \nabla_j \phi\right\rangle + \langle I_2 + I_3, \phi\rangle. \tag{4.27}$$

And for $\theta_k := \theta_0^{(k)}$, where $k \geq 2$, the equation is

$$\begin{aligned}(\Phi(\theta_k)'', \phi) + \mathcal{L}(\theta_k, \phi) + k\mathcal{C}(\theta_k, \phi) = &-\left\langle \partial_t^k q, \phi\right\rangle \\ &+ \left\langle \partial_t^{k-1} I_1^j, \nabla_j \phi\right\rangle + \left\langle \partial_t^{k-1}(I_2 + I_3), \phi\right\rangle \\ &+ \left\langle k(\partial_t^2 h^{00})\theta_{k-1}', \phi\right\rangle.\end{aligned} \tag{4.28}$$

Assuming sufficient regularity on q, f, a, γ, we shall prove that the equations (4.25)-(4.28) have a unique solution.

Theorem 4.5. *Let M be the integer in Proposition 4.1, and u, Λ have the regularity as described in Proposition 4.1. Then there is a time interval $[0, T]$ with $T > 0$ in which (4.25)-(4.28) have a unique solution for all $k = 0, \cdots M$. Moreover, denoting the solution to the k-th order equation by θ_k, then the following are true:*

1. *Compatibility: $\theta_{j-1}' = \theta_j$ for $j = 1, \cdots, M$.*

2. *Energy estimate: let*

$$\tilde{\mathbb{E}}_M[\theta](t) = \sum_{k=0}^{M} \int_{B_t} (-g^{00})|\theta_k'|^2 + g^{ij}\nabla_i \theta_k \nabla_j \theta_k \, dy.$$

Then there is a polynomial P such that for all $t \in [0, T]$,

$$\tilde{\mathbb{E}}_M[\theta](t) \leq \tilde{\mathbb{E}}_M[\theta](0) \cdot \left(1 + tP(\mathfrak{E}_M[u, \Lambda](T)) \cdot e^{tP(\mathfrak{E}_M[u,\Lambda](T))}\right).$$

Proof for Theorem 4.5. Again, the proof runs largely in parallel with Theorem 4.2, so we will be brief on the computational details and focus on the parts that are different.

Existence of approximate solutions. Let $\{e_\ell\}$ be an orthogonal basis of $H_0^1(B)$, which is also an orthonormal basis of $L^2(B)^2$. As before, we first construct the approximate solutions. Let $\Phi_m : H_0^1(B) \to (H_0^1(B))'$ be

$$(\Phi_m(\theta), \phi) = \left(-h^{00}\theta, P_m\phi\right),$$

and the projected quantities be

$$\begin{aligned}
I_{1,m}^j &= -2(\partial_t h^{0j})\theta'_{0,(m)} - (\partial_t h^{ij})\nabla_i\theta_{0,(m)} \\
I_{2,m} &= \left(\partial_t^2 h^{00} + \frac{1}{2}\partial_t((\nabla_\alpha \log|h|)h^{0\alpha}) - \nabla_j\partial_t h^{0j}\right)\theta'_{0,(m)} \\
I_{3,m} &= \left(\partial_t^2 h^{0j} + \frac{1}{2}\partial_t\left((\nabla_\alpha \log|h|)h^{\alpha j}\right)\right)\nabla_j\theta_{0,(m)}
\end{aligned}$$

We look for an approximate solution of the form

$$\theta_{k,(m)}(t,y) := \sum_{\ell=1}^m \theta_{k,m}^\ell(t)e_\ell(y), \quad \theta_{k,m}^\ell \in C^2([0,T]) \ \forall \ell = 1, \cdots, m.$$

The m-th approximate weak equation is

(4.29)

$$\begin{aligned}
&\left(\Phi_m(\theta_{k,(m)})'', e_\ell\right) + \mathcal{L}(\theta_{k,(m)}, e_\ell) + k\mathcal{C}(\theta_{k,(m)}, e_\ell) \\
= &-\left\langle \partial_t^k q, e_\ell\right\rangle + \left\langle \partial_t^{k-1} I_{1,m}^j, \nabla_j e_\ell\right\rangle + \left\langle \partial_t^{k-1}(I_{2,m} + I_{3,m}), e_\ell\right\rangle + \left\langle k(\partial_t^2 h^{00})\theta'_{k-1,(m)}, e_\ell\right\rangle.
\end{aligned}$$

As before, (4.29) is a system of linear second order ODE in $\vec{\theta}_{k,m} := (\theta_{k,m}^1, \cdots, \theta_{k,m}^m)$, where $\vec{\theta}_m : [0,T] \to \mathbb{R}^m$, because the matrix coefficient in front of the second order derivative is

$$A(t)_{i,j} = \left\langle -h^{00}(t)e_i, e_j\right\rangle, \tag{4.30}$$

[2]For instance, we may let e_ℓ be the eigenfunctions of the Laplacian Δ with zero Dirichlet boundary condition

which is symmetric positive-definite with a bounded inverse for all $t \in [0,T]$.

The approximate solutions are compatible. As before, $\theta'_{k-1,(m)} = \theta_{k,(m)}$ in $L^2(B)$.

Uniform bound on $\theta_{k,(m)}$. We multiply (4.29) by $(\theta^\ell_{k,m}(t))'$ and sum with respect to ℓ to obtain the estimate:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_B (-h^{00})|\theta'_{k,(m)}|^2 + h^{ij}\nabla_i\theta_{k,(m)}\nabla_j\theta_{k,(m)}\,dy\right]\\
&-\int_B (k+\frac{1}{2})(\partial_t h^{00})|\theta'_{k,(m)}|^2\,dy + \int_B \left((\partial_t h^{00}) - \frac{1}{2}(\nabla_\alpha \log|h|)h^{0\alpha}\right)|\theta'_{k,(m)}|^2\,dy\\
&-\frac{1}{2}\int_B (\partial_t h^{ij})\nabla_i\theta_{k,(m)}\nabla_j\theta_{k,(m)}\,dy\\
&+\int_B \nabla_j\theta_{k,(m)}\left(-(\partial_t h^{0j}) - \frac{1}{2}(\nabla_\alpha\log|h|)h^{\alpha j}\right)\theta'_{k,(m)}\,dy\\
=&-\int_B \left(\partial_t^k q + \partial_t^{k-1}(I_{2,m}+I_{3,m}) - k\partial_t^2 h^{00}\theta'_{k-1,(m)}\right)\theta'_{k,(m)}\,dy - \int_B (\partial_t^{k-1}I_1^j)\nabla_j\theta'_{k,(m)}\,dy.
\end{aligned}$$

Let

$$E_m(t) = \int_B (-h^{00})|\theta'_{k,(m)}|^2 + h^{ij}\nabla_i\theta_{k,(m)}\nabla_j\theta_{k,(m)}\,dy,$$

then

$$\begin{aligned}
E_m(t) \lesssim & E_m(0) + \int_0^t E_m(\tau)\cdot\left(\sum_{r=0,1}\|\nabla^{(r)}_{t,y}g\|_{L^\infty(B)}\right)d\tau\\
&+\int_0^t E_m(\tau)\cdot\left(\|\partial_t^2 h^{00}\|_{L^\infty(I\times B)}\cdot\|\theta'_{k-1,(m)}\|^2_{L^2(B_\tau)}\right)d\tau\\
&+\underbrace{\left|\int_0^t\int_B \left(-\partial_t^k q + \partial_t^{k-1}(I_{2,m}+I_{3,m})\right)\theta'_{k,(m)}\,dyd\tau\right|}_{:=\mathrm{I}}\\
&+\underbrace{\left|\int_0^t\int_B (\partial_t^{k-1}I_1^j)\nabla_j\theta'_{k,(m)}\,dyd\tau\right|}_{:=\mathrm{II}}.
\end{aligned} \tag{4.31}$$

As usual, the first term can be controlled by the assumption on u,Λ. The control on the second term follows from the regularity of $\theta_{k-1,(m)}$ by induction, and assumptions on u,Λ. We seek to control the remaining two terms:

- $\int_0^t \partial_t^k q\cdot\theta'_{k,(m)}\,dyd\tau$. Recall that $q = \Box_h\Lambda$, so by Lemma 2.10, $\partial_t^k h$ consists of terms of the forms 4 – 6 in the statement of Lemma 2.10. The highest order terms are $\nabla\partial_t^k g$ and $\nabla^{(2)}\partial_t^{k-1}\Lambda$. In fact, we won't be able to bound these two terms in $L^2(I, L^2(B))$, but we can analyze the product when they are paired with $\theta'_{k,(m)}$. The analysis here is similar to the one we did when closing the a priori estimate on Λ. We start with $\nabla\partial_t^k h$.

Recall that there is some function $F^{\alpha\beta}$ with

$$\sup_{0\le t\le T} \|F^{\alpha\beta}\|_{L^\infty(B_t)} + \|\nabla F^{\alpha\beta}\|_{L^\infty(B_t)} < \infty,$$

such that

$$\nabla\partial_t^k h \cdot \theta'_{k,(m)} = (F^{\alpha\beta}\nabla_\alpha\nabla_\beta\partial_t^{k-1}u) \cdot \theta'_{k,(m)}.$$

We integrate by part to transfer one ∇ onto $\theta_{k,(m)}$ and one time derivative onto u:

$$\begin{aligned}
&\int_0^t \int_B (F^{\alpha\beta}\nabla_\alpha\nabla_\beta\partial_t^{k-1}u) \cdot \theta'_{k,(m)}\, dyd\tau \\
=& -\int_0^t \int_B (\nabla_\beta\partial_t^{k-1}u)(\nabla_\alpha\theta'_{k,(m)})F^{\alpha\beta}\, dyd\tau \\
&-\int_0^t \int_B (\nabla_\beta\partial_t^{k-1}u)(\nabla_\alpha F^{\alpha\beta})\theta'_{k,(m)}\, dyd\tau \\
=& -\int_{B_t} (\nabla_\beta\partial_t^{k-1}u)(\nabla_\alpha\theta_{k,(m)})F^{\alpha\beta}\, dy + \int_{B_0} (\nabla_\beta\partial_t^{k-1}u)(\nabla_\alpha\theta_{k,(m)})F^{\alpha\beta}\, dy \\
&+\int_0^t \int_B (F^{\alpha\beta}\nabla_\beta\partial_t^k u + \partial_t F^{\alpha\beta}\nabla_\beta\partial_t^{k-1}u)(\nabla_\alpha\theta_{k,(m)})\, dyd\tau \\
&-\int_0^t \int_B (\nabla_\beta\partial_t^{k-1}u)(\nabla_\alpha F^{\alpha\beta})\theta'_{k,(m)}\, dyd\tau.
\end{aligned}$$

Then, we can write

$$\left|\int_0^t \int_B (\nabla_\beta\partial_t^{k-1}u)(\nabla_\alpha\theta'_{k,(m)})F^{\alpha\beta}\, dyd\tau\right| \le \delta\|\nabla\theta_{k,(m)}\|^2_{L^2(B)} + \frac{1}{\delta}\|\nabla\partial_t^{k-1}u\|^2_{L^2(B)}$$

and choose a small δ such that $\delta\|\nabla\theta_{k,(m)}\|^2_{L^2(B)}$ can be absorbed into the $E_m(t)$ on the left hand side of (4.31). The rest can be estimated using Cauchy-Schwarz Inequality. The treatment of $\nabla^{(2)}\partial_t^{k-1}\Lambda$ is virtually the same, after replacing u with Λ. Thus we have bounded $\int_0^t \int_B \partial_t^k q \cdot \theta'_{k,(m)}\, dyd\tau$.

- $\int_0^t \int_B \partial_t^{k-1}(I_{2,m} + I_{3,m}) \cdot \theta'_{k,(m)}\, dyd\tau$. Again, we will not be able to control $\|\partial_t^{k-1}(I_{2,m} + I_{3,m})\|_{L^2(I,L^2(B))}$ when $k = M$, and we use the integration by parts trick that is similar to the previous bullet point, and almost identical to the case of u in Theorem 4.2, to estimate this term. We omit the details. These two estimates bound I.

- The treatment of II is almost identical to the analysis on III in the proof for Theorem 4.2. We refrain from copying the same argument.

We have analyzed all terms in (4.31), so by Gronwall's inequality, we have proved that there is some C_4 such that

$$\sup_{t\in[0,T]} \|\theta'_{k,(m)}\|_{L^2(B_t)} + \|\theta_{k,(m)}\|_{H_0^1(B_t)} < C_4 < \infty \tag{4.32}$$

for all $m \geq 1$. Here C_4 depends on $\mathfrak{E}_M[u, \Lambda](T)$ only, and does not depend on m.

The part when we pass to the limit as $m \to \infty$, as well as uniqueness and compatibility, follows from the same argument as in Theorem 4.2. We omit the details. □

4.3 Conclusion on the Linearized Equations

In summary, in this chapter, we proved an a priori estimate for the system of nonlinear equations on the bounded domain B. We then considered the original system of equations on a linear level, and used Galerkin approximation to find solutions to this linear system of equations.

Our next and final chapter will be devoted to showing the existence and uniqueness of a solution to the system of equations (1.42)-(1.48) on a bounded domain. The strategy is to make use of the linear equations, as well as the associated energy estimates on the linear level, to obtain the solution iteratively.

CHAPTER V

Solution on a Bounded Domain

Equipped with the linear theory, we are now ready to obtain a solution to the actual nonlinear equation by iteratively solving the linear equations. This section is adapted from the iterative scheme in [10]. We provide more details in showing the convergence of the iterative scheme, and moreover state and prove the uniqueness result.

5.1 Equation for $u^{(m)}, \Lambda^{(m)}$

We define the solutions $u^{(m)}$ and $\Lambda^{(m)}$ inductively. To start with, we specify the 0-th iteration.

5.1.1 Definition for the Initial Iteration

We define the initial iteration as follows. Let $X^{(0)}(0,\cdot) : B \to \Omega_0$ be the Lagrangian coordinate at $t = 0$. We require that X is smooth. Recall that $V_0(0,x)$ is the prescribed initial condition for the velocity, and $V_0^0(0,x)$ is the first coordinate of $V_0(0,x)$. For later time, define $X^{(0)}(0,y)$ as

$$X^{(0)}(t,y) = X^{(0)}(0,y) + t \cdot \frac{V_0(0, X^{(0)}(0,y))}{V_0^0(0, X^{(0)}(0,y))}.$$

Let $g^{(0)}$ be the pull-back metric accordingly. We can then define the initial iteration of u and Λ as a polynomial function in t. The definition is not unique, but the idea is to define $u^{(0)}$ and $\Lambda^{(0)}$ so that they are smooth in t, and satisfy the initial condition. One such choice could be:

$$u^{(0)}(t,y) = (t^2+1)\cdot V_0(0,X^{(0)}(0,y)) + t\cdot \frac{(D_V V_0)(0,X^{(0)}(0,y))}{V_0^0(0,X^{(0)}(0,y))}$$
$$\Lambda^{(0)}(t,y) = (t^2+1)\cdot (D_V\sigma_0^2)(0,X^{(0)}(0,y)) + t\cdot \frac{(D_V^2\sigma_0^2)(0,X^{(0)}(0,y))}{V_0^0(0,X^{(0)}(0,y))}.$$

One can easily check that they satisfy the initial conditions, have the desired regularity, and that $\Lambda^{(0)}(t,y)\equiv 0$ for $y\in\partial B$.

5.1.2 The Iteration

Next, we define the further iterations of u and Λ in the natural way. So far, we have taken an initial guess of the coefficients of the linear equation, and upon solving this linear equation, we can use this solution to define the coefficients of the linear equation in the next iteration.

Let $m\geq 0$. Assume $u^{(m)},\Lambda^{(m)},(\Sigma^2)^{(m)},X^{(m)},g^{(m)},h^{(m)},w^{(m)}$ are given. We define the known variables that appear in the weak formulation:

$$\gamma^{(m)} = \frac{2((u^{(m)})^0)^2}{a^{(m)}}$$
$$\left(\Phi^{(m)}(\theta),\phi\right) = \left\langle -(g^{(m)})^{00}\theta,\phi\right\rangle + \left\langle\!\left\langle \gamma^{(m)}\theta,\phi\right\rangle\!\right\rangle,$$

and replace all the variables in the definition of $\mathcal{L},\mathcal{C},q,f,I_1^j,I_2,I_3,B_1$ with the known functions $u^{(m)},\Lambda^{(m)},(\Sigma^2)^{(m)}, X^{(m)},g^{(m)},h^{(m)},w^{(m)}$. Our next goal is to prove that these functions $u^{(m)},\Lambda^{(m)}$ are uniformly bounded for all m.

5.2 Uniform Boundedness

In this section, we will show that $u^{(m)}$ and $\Lambda^{(m)}$ are uniformly bounded in a norm that we next specify. Define[1]

(5.1)

$$\mathbb{E}_M[u,\Lambda](t) = \sup_{0\le\tau\le t}\sum_{k=0}^{M} \|\partial_t^{k+1}u(\tau)\|^2_{L^2(B)} + \|\partial_t^k u(\tau)\|^2_{H^1(B)} + +\|\partial_t^{k+1}u(\tau)\|^2_{L^2(\partial B)}$$
$$+\sum_{k=0}^{M} \|\partial_t^{k+1}\Lambda(\tau)\|^2_{L^2(B)} + \|\partial_t^k\Lambda(\tau)\|^2_{H^1(B)}.$$

Proposition 5.1. *Let M be the integer in Theorems 4.2 and 4.5. Then there is some constant $A<\infty$ and $T>0$ such that*

(5.2)
$$\mathbb{E}_M[u^{(m)},\Lambda^{(m)}](T) < A \quad \forall m = 0,1,\cdots.$$

Proof. Recall that by definition of the weak solutions, all $u^{(m)}$ and $\Lambda^{(m)}$ agree at $t=0$. So we know that

$$\mathbb{E}_M[u^{(m)},\Lambda^{(m)}](0) = E_0 \quad \forall m\ge 0$$

for some constant $0\le E_0<\infty$. By Theorems 4.2 and 4.5, we know that there is an integer $r>1$ (such that when x is large, $x^r>P(x)$ for the polynomials that appear) such that

$$\mathbb{E}_M[u^{(m)},\Lambda^{(m)}](T) \le E_0\left(1+T\mathbb{E}_M[u^{(m-1)},\Lambda^{(m)}](T)^r\cdot e^{T\mathbb{E}_M[u^{(m)},\Lambda^{(m-1)}](T)^r}\right).$$

Denote $\alpha_m := \mathbb{E}_M[u^{(m)},\Lambda^{(m)}](T)$. We claim that for $T>0$ small, there is a constant A such that $\alpha_m<A$ for all m.

To prove this, let

$$f(x) = E_0\cdot\left(1+Tx^r\cdot e^{Tx^r}\right).$$

[1]Here $[u,\Lambda]$ means that u,Λ are arguments of the functional. It does not represent a commutator.

In fact, we will prove that α_m converges to the smaller fixed point of f. It suffices to show that $f(x)$ is concave up and has two fixed points. We compute

$$f''(x) = e^{Tx^r} rTx^{r-2} \left(rT^2 x^{2r} + (3rT - T)x^r + r - 1 \right).$$

It is clear that $f''(x) > 0$ for all $x > 0$, so f is concave up indeed. When $T \to 0$, we see that the equation $f(x) = x$ has at least one solution. The existence of the other solution follows from concavity.

Since α_m converges, we see that is is bounded for all m. □

Remark 12. Note that in order to prove uniform boundedness, the time of existence T might be smaller than the time of existence in Theorems 4.2 and 4.5. This is due to the nonlinear nature of our equation.

Next, we will show that the iteration converges. In fact, we will show that $u^{(m)}, \Lambda^{(m)}$ is a Cauchy sequence in m under a norm which we will specify. The main idea of the proof to use the triangular trick, which is similar to what we did for the case of the unbounded domain.

5.3 Convergence

We prove that the sequence $u^{(m)}, \Lambda^{(m)}$ is a Cauchy sequence in the following norm. We define the norm on the difference between two consecutive terms to be

(5.3)

$$\begin{aligned} e_m(t) = \sup_{0 \le \tau \le t} \sum_{k \le 5} &\left(\int_{B_\tau} |\nabla_{t,y} \partial_t^k (u^{(m+1)} - u^{(m)})|^2 \, dy + \int_{\partial B_\tau} |\partial_t^{k+1} (u^{(m+1)} - u^{(m)})|^2 \, dS \right) \\ &+ \left(\sup_{0 \le \tau \le t} \sum_{k \le 5} \int_{B_\tau} |\nabla_{t,y} \partial_t^k (\Lambda^{(m+1)} - \Lambda^{(m)})|^2 \, dy \right) \\ &+ \left(\sum_{k \le 5} \int_0^t \int_{\partial B_\tau} |\partial_t^{k+1} (\Lambda^{(m+1)} - \Lambda^{(m)})|^2 \, dS d\tau \right). \end{aligned}$$

One could easily see that $e_m(t)$ is motivated from the energy $\mathcal{E}_M^{\epsilon=0}(t)$ that appeared in the a priori estimate; except that we are only taking the first 5 time derivatives and do not utilize the full M time derivatives. We will prove convergence in this lower regularity space.

Proposition 5.2. *There is a constant C such that*

$$\sum_{m=0}^{\infty} e_m(t) \lesssim e^{CT} \quad \forall t \in [0,T]. \tag{5.4}$$

Here the implicit constant only depends on e_0.

Recall that e_m controls the difference between two consecutive iterations of u and Λ, so in particular (5.4) show that $u^{(m)}$ and $\Lambda^{(m)}$ form a Cauchy sequence with norms

$$\|u\|_{\mathcal{B}_u} = \sum_{k\le 5} \|\partial_t^k u\|_{L^\infty([0,T],H^1(B))} + \|\partial_t^{k+1} u\|_{L^\infty([0,T],L^2(B))} + \|\partial_t^{k+1} u\|_{L^\infty([0,T],L^2(\partial B))}$$

$$\|\Lambda\|_{\mathcal{B}_\Lambda} = \sum_{k\le 5} \|\partial_t^k \Lambda\|_{L^\infty([0,T],H^1(B))} + \|\partial_t^{k+1} \Lambda\|_{L^\infty([0,T],L^2(B))} + \|\partial_t^{k+1} \Lambda\|_{L^2([0,T],L^2(\partial B))}$$

respectively.

Proof. Since $M \gg 5$, we know that the equations for u and Λ are in fact satisfied in the strong sense. That is, if we denote

$$\begin{aligned}
F^{(m)} :=& \frac{1}{2}((u^{(m)})^0)^2 (w^{(m)})^\nu_\alpha (g^{(m)})^{\alpha\beta}\partial_\beta((\Sigma^{(m)})^2) \\
&- \frac{1}{2}(g^{(m)})^{\alpha\beta}\partial_\alpha(X^{(m)})^\nu \partial_\beta(\Lambda^{(m)}) + 2(u^{(m)})^0\partial_t(u^{(m)})^0\partial_t(u^{(m)})^\nu \\
G^{(m)} :=& (g^{(m)})^{\alpha\beta}\partial_\alpha((X^{(m)})^\mu)\partial_\beta((w^{(m)})^\nu_\mu) \\
&- (g^{(m)})^{\alpha\beta}\partial_\alpha(X^{(m)})^\nu\partial_\beta\left((\log G)'(\Lambda^{(m)})\right) \\
H^{(m)} :=& 4(g^{(m)})^{\alpha\beta}(\partial_\beta(u^{(m)})^\nu)\partial_\alpha\left(m_{\mu\nu}(g^{(m)})^{\gamma\delta}(\partial_\delta(X^{(m)})^\nu)(\partial_\gamma(\Sigma^{(m)})^2)\right) \\
&+ 4m_{\rho\nu}m_{\nu\kappa}(g^{(m)})^{\alpha\beta}(g^{(m)})^{\gamma\delta}(\partial_\delta(X^{(m)})^\kappa)(\partial_\alpha(u^{(m)})^\nu)(\partial_\beta(u^{(m)})^\mu)(\partial_\gamma(u^{(m)})^\rho) \\
&+ 2\partial_t(u^{(m)})^0(\log G)''(\Lambda^{(m)})^2 + (u^{(m)})^0(\log G)^{(3)}(\Lambda^{(m)})^3
\end{aligned}$$

$$- \partial_t (u^{(m)})^0 (\log G)' (\Lambda^{(m)})^2 - (u^{(m)})^0 \partial_t (X^{(m)})^0 (\log G)' \partial_t (\Lambda^{(m)}),$$

then the equations for $u^{(m+1)}$ and $\Lambda^{(m+1)}$ can be written as:

(5.5)

$$\begin{cases} ((u^{(m)})^0)^2 \partial_t^2 (u^{(m+1)})^\nu - \frac{1}{2} (g^{(m)})^{\alpha\beta} (\partial_\alpha (\Sigma^{(m)})^2) \partial_\beta (u^{(m+1)})^\nu = F^{(m)} & \text{on } [0,T] \times \partial B \\ \Box_{(g^{(m)})} (u^{(m+1)})^\nu = G^{(m)} & \text{in } [0,T] \times B \end{cases}$$

and

$$\begin{cases} (\Lambda^{(m+1)}) \equiv 0 & \text{on } [0,T] \times \partial B \\ \Box_{(h^{(m)})} (\Lambda^{(m+1)}) = H^{(m)} & \text{in } [0,T] \times B \end{cases} . \tag{5.6}$$

And (5.5), (5.6) are satisfied in the strong sense.

The differences $u^{(m+1)} - u^{(m)}$ and $\Lambda^{(m+1)} - \Lambda^{(m)}$ then satisfy equations of a similarly form. On ∂B, the difference $u^{(m+1)} - u^{(m)}$ satisfies

$$\begin{aligned} &(u^0_{(m)})^2 \partial_t^2 \left(u^{(m+1)} - u^{(m)} \right) + a^{(m)} n^{(m)}_\alpha g^{\alpha\beta}_{(m)} \nabla_\beta \left(u^{(m+1)} - u^{(m)} \right) \\ =& G^{(m)} - G^{(m-1)} - \left((u^0_{(m)})^2 - (u^0_{(m-1)})^2 \right) \partial_t^2 u^{(m)} \\ &- \left(a^{(m)} n^{(m)}_\alpha g^{\alpha\beta}_{(m)} - a^{(m-1)} n^{(m-1)}_\alpha g^{\alpha\beta}_{(m-1)} \right) \nabla_\beta u^{(m)} \\ =:& \tilde{G}^{(m)}. \end{aligned}$$

In B, the difference $u^{(m+1)} - u^{(m)}$ satisfies

$$\begin{aligned} \nabla_\alpha \left(g^{\alpha\beta}_{(m)} \nabla_\beta (u^{(m+1)} - u^{(m)}) \right) =& F^{(m)} - F^{(m-1)} - \nabla_\alpha \left((g^{\alpha\beta}_{(m)} - g^{\alpha\beta}_{(m-1)}) \nabla_\beta u^{(m)} \right) \\ =:& \tilde{F}^{(m)}. \end{aligned}$$

It is evident that both equations have exactly the same terms as (5.5) on the left hand side, and the terms on the right hand side depends on $F^{(m)}, F^{(m-1)}, G^{(m)}, G^{(m-1)}$. Similarly, on ∂B, the difference $\Lambda^{(m+1)} - \Lambda^{(m)}$ is identically zero, and in B, the difference $\Lambda^{(m+1)} - \Lambda^{(m)}$ satisfies

$$\Box_{(h^{(m)})} (\Lambda^{(m+1)} - \Lambda^{(m)}) = H^{(m)} - H^{(m-1)} - \nabla_\alpha \left((h^{\alpha\beta}_{(m)} - h^{\alpha\beta}_{(m-1)}) \nabla_\beta \Lambda^{(m)} \right)$$

$$=:\tilde{H}^{(m)}.$$

Inspecting the formulae for $F^{(m)}, G^{(m)}, H^{(m)}$, we see that they are all linear combinations of terms of the form

$$(\nabla_{t,y}^{k_1}\phi_1)\cdots(\nabla_{t,y}^{k_p}\phi_p)$$

where $k_i \leq 2$, and $\phi_i \in \{u^{(m)}, \Lambda^{(m)}, (\Sigma^2)^{(m)}, X^{(m)}, g^{(m)}, h^{(m)}, w^{(m)}\}$. Among these terms, $(\Sigma^2)^{(m)}, X^{(m)}, g^{(m)}, h^{(m)}, w^{(m)}$ are rational functions of $u^{(m)}, \Lambda^{(m)}$ with bounded derivatives. So using the triangular trick again, we may write $\tilde{F}^{(m)}, \tilde{G}^{(m)}, \tilde{H}^{(m)}$ as linear combinations of terms of the form

$$\frac{1}{P}\cdot(\nabla_{t,y}^{k_1}\phi_1^{(m)})\cdots(\nabla_{t,y}^{k_p}\phi_p^{(m)})\cdot(\nabla_{t,y}^{n_1}\psi_1^{(m-1)})\cdots(\nabla_{t,y}^{n_q}\psi_q^{(m-1)})\cdot(\nabla_{t,y}^{r}\delta^{(m)})$$

where P is a polynomial in $u^{(m)}, \Lambda^{(m)}, u^{(m-1)}, \Lambda^{(m-1)}$ such that $1/P$ is bounded, $k_i, n_i, r \leq 5$, $\phi_i, \psi_i \in \{u, \Lambda\}$, and $\delta^{(m)} \in \{u^{(m+1)} - u^{(m)}, \Lambda^{(m+1)} - \Lambda^{(m)}\}$. By the uniform bound in Proposition 5.1, we know that $\frac{1}{P}\cdot(\nabla_{t,y}^{k_1}\phi_1^{(m)})\cdots(\nabla_{t,y}^{k_p}\phi_p^{(m)})\cdot(\nabla_{t,y}^{n_1}\psi_1^{(m-1)})\cdots(\nabla_{t,y}^{n_q}\psi_q^{(m-1)})$ can be bounded in L^∞ norm. Thus, appealing to the energy estimates in Theorems 4.2 and 4.5, we know that there is some constant C such that

$$e_m(t) \leq C\cdot\int_0^t e_{m-1}(\tau)\,d\tau.$$

By induction, we see that

$$\begin{aligned} e_m(t) &\leq C^m\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_m} e_0(t_m)\,dt_m\cdots dt_1 \\ &\leq \frac{C^m t^m}{m!}\sup_{0\leq\tau\leq t} e_0(\tau), \end{aligned}$$

and thus

$$\sum_{m=0}^{\infty} e_m(T) \leq e^{CT}\sup_{0\leq\tau\leq T} e_0(\tau).$$

The statement that $u^{(m)}, \Lambda^{(m)}$ form a Cauchy sequence in the $\|\cdot\|_{\mathcal{B}_u}$ and $\|\cdot\|_{\mathcal{B}_\Lambda}$ norms respectively clearly follows. □

Corollary 5.3. *Assume the same conditions as in Proposition 5.2. Then there is some u, Λ in $\mathcal{B}_u$ and $\mathcal{B}_\Lambda$ respectively, such that*

$$u^{(m)} \to u \quad \text{in } \mathcal{B}_u$$

$$\Lambda^{(m)} \to \Lambda \quad \text{in } \mathcal{B}_\Lambda.$$

Moreover, u, Λ have the same regularity as the initial data.

Proof. The spaces

$$L^\infty([0,T], H^1(B)), L^\infty([0,T], L^2(B)), L^\infty([0,T], L^2(\partial B)), L^2([0,T], L^2(\partial B))$$

are complete, so $\mathcal{B}_u$ and $\mathcal{B}_\Lambda$ are Banach spaces. Thus the limits exist and $\|u\|_{\mathcal{B}_u} < \infty$, $\|\Lambda\|_{\mathcal{B}_\Lambda} < \infty$.

The regularity follows because upon passing to a subsequence, $u^{(m)}$ and $\Lambda^{(m)}$ also has a weak limit with the same regularity, and since $u^{(m)} \to u$ and $\Lambda^{(m)} \to \Lambda$ strongly, the weak limit has to coincide with the strong limit. Thus the strong limit u, Λ have the same regularity as their initial data. □

5.4 Uniqueness of Solution

The last ingredient is the uniqueness of the solution, which we have shown to exist.

Theorem 5.4. *Assume that $(u, \Lambda, X, w, \Sigma^2)$ solve the systems (1.39)-(1.48) on some time interval $[0,T]$, such that $\mathcal{E}_M[u,\Lambda](T) < \infty$. Then the solution is unique.*

Proof. The uniqueness of the solution follows from exactly the same argument as in Theorem 3.11, except we change the domain to $\mathcal{D} = B$. □

5.5 Conclusion

In this chapter, we adapted the iterative scheme in [10] to construct a solution to (1.39)-(1.48) on the bounded domain, and furthermore addressed its regularity and uniqueness. This proves Theorem 1.1 when the domain Ω_0 is bounded.

APPENDICES

Appendices

A Commutators and Identities

Lemma A.1. *We have*

1. $[\partial_t, f]g = (\partial_t f)g$.

2. $[\tilde{\nabla}_j, f]g = J_\epsilon\left((\partial_j f)g\right) + [J_\epsilon, f]\partial_j g$.

Proof. Both follow from direct computations. $\square$

Lemma A.2. *Assume f, g are defined on $\mathbb{R}^3_+$, and extended either oddly or evenly to $\mathbb{R}^3$. Then J_ϵ is self-adjoint. That is,*

$$\int_{\mathbb{R}^3_+} (J_\epsilon f)g\, dx = \int_{\mathbb{R}^3_+} f(J_\epsilon g)\, dx.$$

Proof. For $y \in \mathbb{R}^3$, write $\overline{y} = (y_1, y_2)$, so $y = (\overline{y}, y_3)$. We compute that

$$\begin{aligned}
\int_{\mathbb{R}^3_+} (J_\epsilon f)g\, dx &= \int_{x\in\mathbb{R}^3_+}\int_{\mathbb{R}^3} \eta_\epsilon(x-y)f(y)g(x)\, dydx \\
&= \int_{x\in\mathbb{R}^3_+}\int_{y:y_3>0} \eta_\epsilon(x-y)f(y)g(x)\, dydx \\
&\quad + \int_{x\in\mathbb{R}^3_+}\int_{y:y_3<0} \eta_\epsilon(x-y)f(y)g(x)\, dydx \\
&= \int_{x\in\mathbb{R}^3_+}\int_{y:y_3>0} \eta_\epsilon(x-y)f(y)g(x)\, dydx + \\
&\quad + \int_{x\in\mathbb{R}^3_+}\int_{y:y_3>0} \eta_\epsilon((\overline{x}-\overline{y}, x_3+y_3))f((\overline{y}, -y_3))g(x)\, d\overline{y}dy_3dx \\
&= \int_{x\in\mathbb{R}^3_+}\int_{y:y_3>0} \eta_\epsilon(x-y)f(y)g(x)\, dydx + \\
&\quad \pm \int_{x\in\mathbb{R}^3_+}\int_{y:y_3>0} \eta_\epsilon((\overline{x}-\overline{y}, x_3+y_3))f((\overline{y}, y_3))g(x)\, d\overline{y}dy_3dx.
\end{aligned}$$

In either case, since η is radial, the equation is symmetric in x and y, so the result follows. $\square$

Lemma A.3. *Let* $f, g \in \mathcal{S}(\mathbb{R}^3)$. *Then*

$$\int_{\mathbb{R}^3_+} gf\nabla_j f\,dy = \frac{1}{2}\int_{\partial\mathbb{R}^3_+} n_j g f^2\,dS - \frac{1}{2}\int_{\mathbb{R}^3_+} (\nabla_j g) f^2\,dy.$$

Proof. We have

$$\int_{\mathbb{R}^3_+} gf\nabla_j f\,dy = \int_{\partial\mathbb{R}^3_+} n_j g f^2\,dS - \int_{\mathbb{R}^3_+} (\nabla_j g) f^2\,dy - \int_{\mathbb{R}^3_+} (\nabla_j f) g f\,dy.$$

Rearranging the terms gives the desired result. □

B Common Estimates

We list here the some common estimates that were used in the proof. In this section, let $\epsilon > 0$ be a constant, and J_ϵ be the frequency cut-off:

$$\widehat{J_\epsilon f}(\xi) := \hat{f}(\xi) \cdot \chi_{|\xi|<1/\epsilon}.$$

Let $\hat{\eta}(\xi) = \chi_{|\xi|<1/\epsilon}$, then we may also write

$$J_\epsilon f(x) = f * \eta_\epsilon(x) = \int_{\mathbb{R}^3} \eta_\epsilon(x-y) f(y)\,dy = \int_{\mathbb{R}^3} \frac{1}{\epsilon}\eta\left(\frac{x-y}{\epsilon}\right) f(y)\,dy.$$

Lemma B.1. *Let* $f, g \in \mathcal{S}(\mathbb{R}^3)$. *Then*

1. $\|[J_\epsilon, f]g\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^\infty(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)}$.
2. $\|[J_\epsilon, f]\nabla g\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{H^3(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)}$.
3. $\|\nabla[J_\epsilon, f]g\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{H^3(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)}$.

Proof. We prove each one of the estimates.

1. This is clear since $\|J_\epsilon\|_{\mathcal{L}(L^2,L^2)} \leq 1$.

2. We have

$$\widehat{[J_\epsilon, f]\nabla} g(\xi) = \int_\eta \hat{f}(\xi-\eta)\left(\chi_{|\xi|<1/\epsilon} - \chi_{|\eta|<1/\epsilon}\right) 2\pi i\eta\hat{g}(\eta)\,d\eta.$$

Thus, by Cauchy-Schwarz,

$$
\begin{aligned}
&\int_\xi \left|\widehat{[J_\epsilon, f]\nabla g}(\xi)\right|^2 \, d\xi \\
&\lesssim \int_\xi \left(\int_\eta |\hat{g}(\eta)|^2 \, d\eta\right) \left(\int_\eta |\hat{f}(\xi-\eta) \left(\chi_{|\xi|<1/\epsilon} - \chi_{|\eta|<1/\epsilon}\right) \eta|^2 \, d\eta\right) \, d\xi \\
&= \|g\|_{L^2}^2 \cdot \int_\xi \int_\eta |\hat{f}(\xi-\eta)|^2 |\eta|^2 \left(\chi_{|\xi|<1/\epsilon} - \chi_{|\eta|<1/\epsilon}\right)^2 \, d\eta \, d\xi \\
&= \|g\|_{L^2}^2 \cdot \int_\xi \int_\eta \left(|\hat{f}(\xi-\eta)|^2 (1+|\xi-\eta|)^6\right) \cdot \\
&\qquad\qquad \underbrace{\frac{|\eta|^2}{(1+|\xi-\eta|)^6} \left(\chi_{|\xi|<1/\epsilon} - \chi_{|\eta|<1/\epsilon}\right)^2}_{:=K(\xi,\eta)} \, d\eta \, d\xi.
\end{aligned}
\tag{B.1}
$$

When $|\eta| < 1/\epsilon$, $K(\xi, \eta) \neq 0$ iff $|\xi| > 1/\epsilon$, in which case

$$
|K(\xi, \eta)| \lesssim \frac{1}{1+|\xi|^4}.
$$

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$$

Thus,

$$
\begin{aligned}
&\int_\xi \int_\eta \left(|\hat{f}(\xi-\eta)|^2 (1+|\xi-\eta|)^4\right) K(\xi,\eta) \, d\eta \, d\xi \\
&\lesssim \int_\xi \int_{|\eta|<1/\epsilon} \left(|\hat{f}(\xi-\eta)|^2 (1+|\xi-\eta|)^6\right) \cdot \frac{1}{1+|\xi|^4} \, d\eta d\xi \\
&\qquad + \int_\xi \int_{|\eta|>1/\epsilon} \left(|\hat{f}(\xi-\eta)|^2 (1+|\xi-\eta|)^6\right) \cdot \frac{1}{1+|\eta|^4} \, d\eta d\xi \\
&= \left(\int_\xi \frac{1}{1+|\xi|^4} d\xi\right) \int_{|\eta|<1/\epsilon} \left(|\hat{f}(\eta)|^2 (1+|\eta|)^6\right) \, d\eta \\
&\qquad + \left(\int_{|\eta|>1/\epsilon} \frac{1}{1+|\eta|^4} \, d\eta\right) \int_\xi \left(|\hat{f}(\xi)|^2 (1+|\xi|)^6\right) d\xi \\
&\lesssim \|f\|_{H^3}^2.
\end{aligned}
$$

Substituting back into equation (B.1), we obtain the desired result.

3. We have

$$\nabla[J_\epsilon, f]g = [J_\epsilon, \nabla f]g + [J_\epsilon, f]\nabla g,$$

so the estimate follows from the previous two and Sobolev embedding.

□

Lemma B.2. *Let $0 < k < m$ and $f \in H^m$. Then we have the following interpolation relation*

$$\|f\|_{H^k} \lesssim \|f\|_{H^m}^{k/m} \cdot \|f\|_{L^2}^{1-k/m}.$$

Proof. By Hölder's inequality (with $p = m/k, q = m/(m-k)$),

$$\begin{aligned}\int (1+|\xi|)^{2k}|\hat{f}(\xi)|^2\, d\xi &= \int (1+|\xi|)^{2k}|\hat{f}(\xi)|^{2k/m}|\hat{f}(\xi)|^{2(m-k)/m} d\xi \\ &\le \left(\int (1+|\xi|)^{2m}|\hat{f}(\xi)|^2\, d\xi\right)^{k/m} \cdot \left(\int |\hat{f}(\xi)|^2\, d\xi\right)^{1-m/k}.\end{aligned}$$

This is the desired estimate. □

Lemma B.3. *Suppose $f \in H^k$ and $m \le k$. Then*

$$\|(\mathrm{Id} - J_\epsilon)f\|_{H^{k-m}} \lesssim \epsilon^m \|f\|_{H^k}.$$

Proof. We have

$$\begin{aligned}\int_{|\xi|>1/\epsilon} (1+|\xi|)^{2(k-m)}|\hat{f}(\xi)|^2\, d\xi &= \int_{|\xi|>1/\epsilon} \frac{1}{(1+|\xi|)^{2m}}(1+|\xi|)^{2k}|\hat{f}(\xi)|^2\, d\xi \\ &\lesssim \epsilon^{2m} \int_{|\xi|>1/\epsilon} (1+|\xi|)^{2k}|\hat{f}(\xi)|^2\, d\xi,\end{aligned}$$

which gives the desired estimate. □

Lemma B.4. *Let $F \in C^\infty$ and $u \in H^k \cap L^\infty$. Then*

$$\|F(u)\|_{H^k} \lesssim 1 + \|u\|_{H^k},$$

where the constant only depends on $\|F^j\|_\infty$ for $j = 0, \cdots, k$.

Proof. This is a standard result proven in, say, [18]. □

Lemma B.5. *[Grönwall's Inequality] Let $E(t)$ be a non-negative function on $[0, T]$ satisfying*

$$E(T) \leq C_1 + C_2 \int_0^t E(\tau)\, d\tau \quad \forall t \in [0, T]$$

for some constants $C_1, C_2 \geq 0$. Then

$$E(t) \leq C_1 \cdot \left(1 + C_2 t e^{C_2 t}\right) \quad \forall t \in [0, T]. \tag{B.2}$$

In particular, if $C_1 = 0$, then

$$E(t) \equiv 0 \quad \forall t \in [0, T].$$

Proof. This is a standard result proven in, for instance, [3]. □

Lemma B.6 (Trace Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $\partial\Omega$ is C^1. Then there exists a bounded linear operator*

$$T : H^1(\Omega) \to L^2(\partial\Omega)$$

such that

1. *$Tf = f|_{\partial\Omega}$ is $f \in H^1(\Omega) \cap C(\overline{\Omega})$, and*
2. *for each $f \in H^1(\Omega)$,*

$$\|Tf\|_{L^2(\partial\Omega)} \lesssim \|f\|_{H^1(\Omega)}.$$

Proof. This is a standard result proven in, for instance, [3]. □

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