# Test Elements, Analogues of Tight Closure, and Size for Ideals

by

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# TABLE OF CONTENTS

# ABSTRACT

We give many new results related to the theory of tight closure and its generalizations. Explicitly, we establish a series of results showing that the Jacobian ideal is contained in the test ideal for tight closures both in equal characteristic $p$ and equal characteristic 0 for algebras essentially of finite type over power series rings (they are called *semianalytic algebras*). We move on to introduce and study a new closure called wepf in mixed characteristic, and prove that it is a Dietz closure satisfying the Algebra axiom. This is the first known example of a Dietz closure in mixed characteristic. This is achieved by proving that the epf closure satisfies what we call the $p$-colon-capturing property. We define and study the relationships with properties connected with tight closure. For example, we show that a persistent closure operation that captures colons automatically captures the plus closure, i.e., the contraction of the expansion of an ideal to the absolute integral closure of the ring. We also show that the existence of persistent closure operations between two complete local domains gives us a weakly functorial version of the existence of big Cohen-Macaulay algebras for them. We also develop a new numerical notion for ideals called size using the theory of quasilength, and show that the size of an ideal is always between its height and arithmetic rank. We show under mild conditions that the size is the same as height for one-dimensional primes in a local ring whose completion is a domain. We further study the additive property and the asymptotic additive property of quasilength.

# CHAPTER I

# Introduction

## 1.1   An Overview

We want to explain a little bit about one of the origins of commutative algebra in number theory. Around 1637, Pierre de Fermat wrote the following in the margin of a copy of an ancient Greek text on mathematics called "Arithmetica."

*"It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain." – Fermat*

This is the famous "Fermat's Last Theorem," i.e., that equations of the form $a^n + b^n = c^n$ have no solutions in positive integers if $n$ is an integer greater than $2$. Although Fermat claimed to have a general proof of his conjecture, no proof by him has ever been found. His claim stood unproven for the next three and a half centuries. By analyzing Fermat's equation, people realized that it suffices to show it has no solution when $n$ is an odd prime and $n = 4$. The case $n = 4$ was proved by Fermat, which, interestingly, is the only proof that is found to be written by Fermat. Leonhard Euler proved the case $n = 3$ in 1770. In the nineteenth century Adrien Marie Legendre and P.G. Lejeune Dirichlet independently proved the theorem for $n = 5$.

It took a long time for people to find a practical way to deal systematically with Fermat's equation. In 1847, Gabriel Lamé outlined a proof of Fermat's Last Theorem based on factoring the equation $x^p + y^p = z^p$ in the complex numbers for a prime integer $p$. His proof failed, however, because it assumed a property called "unique factorization" in a context where the property fails. Following Lamé's approach, Ernst Kummer carefully studied the unique factorization property of certain integer rings of cyclotomic fields and proved Fermat's Last Theorem in many cases. The extension of Kummer's ideas to the general case was accomplished independently by Leopold Kronecker and Richard Dedekind during the next forty years. Dedekind created the basics of commutative algebra, i.e., the theory of modules and ideals, which are the main concepts that we will study in this thesis. Before we dive deeper into the theme of commutative algebra, let us finish the story of Fermat's Last Theorem. Around 1955, Japanese mathematicians Goro Shimura and

Yutaka Taniyama came up with a conjecture addressing a possible link between elliptic curves and modular forms. In 1986, Kenneth Ribet proved that the Taniyama–Shimura conjecture implies Fermat's Last Theorem. Finally, in 1994, Sir Andrew Wiles proved Fermat's Last Theorem by proving a form of the Taniyama-Shimura conjecture. A gap was filled by Richard Taylor.

Apart from providing tools for the study of Fermat's Last Theorem, commutative algebra flourished on its own over the last century. With contributions from great mathematicians like David Hilbert, Emmy Noether, Jean-Pierre Serre, Wolfgang Krull, Masayoshi Nagata, Oscar Zariski, and many more, commutative algebra has become an important and interesting subject of modern mathematics.

Commutative algebra can be described as the study of commutative rings and their ideals and modules. A ring is a set with addition, subtraction, and multiplication satisfying certain properties. Some examples of a ring include the integers $\mathbb{Z}$, the complex numbers $\mathbb{C}$, and the polynomials in one variable over the real numbers $\mathbb{R}[x]$. Ideals in rings can be thought of as a generalization of the notion of the set of multiples of a number in the integers. For example, the ideal generated by $\{xy\}$ in the ring $\mathbb{R}[x, y]$ is the set of all polynomials that are products of $xy$ and some other polynomial. We usually write it as $(xy)$. There are also ideals generated by two or more generators. For instance, the ideal $(x, y)$, which consists of polynomials with no constant term, needs two generators. The study of ideals is also closely related to the study of solution sets to polynomial equations. The solution sets form geometric objects called *algebraic sets* (or *varieties* if they are irreducible). For the ideal $(xy)$ in $\mathbb{R}[x, y]$, the algebraic set is the set of points in the $xy$-plane satisfying the equation $xy = 0$, i.e., the union of the $x$-axis and the $y$-axis. Note that there can be many different ideals corresponding to the same algebraic set. For example, we can take the ideal $(x^2 y^2)$ and still get the union of the $x$-axis and the $y$-axis. This is because solving an equation $f(x, y) = 0$ in $\mathbb{R}^2$ is the same as solving the equation $(f(x, y))^2 = 0$, or any higher power of $f(x, y)$ equaling to zero. To remedy this, we can instead consider the *radical* of an ideal $I$, i.e., the set of elements that have some power in $I$. The procedure of taking the radical of an ideal is a special case of taking a closure of an ideal. Roughly speaking, taking a closure of an ideal $I$ is a way to produce a new ideal $J$ that is closely related to $I$. For example, the radical (closure) of $I$ defines the same algebraic set as $I$. By taking the radical in the case of an algebraically closed field, the correspondence between ideals and algebraic sets becomes one-to-one and therefore makes study easier and cleaner.

In the history of commutative algebra, many closure operations have been defined and studied by mathematicians. One example is the radical (closure) we mentioned above. Often, the study of different concepts eventually leads to the study of closure operations. There is a notion called multiplicity, which, roughly speaking, describes the number of times that a geometric object passes through a point. The study of multiplicity is closely connected to the study of a closure operation called integral closure. On the other hand, the further exploration of closure theory produces a great

many fruitful results that can be applied to problems not directly related to the closure operation. One of the most famous examples is the application of tight closure to a network of conjectures in commutative algebra, called the *homological conjectures*.

## 1.2 Tight Closure

In commutative algebra, homological conjectures have generated a tremendous amount of activity in the last 50 years. They concern a number of interrelated (sometimes surprisingly so) conjectures relating various homological properties of a commutative ring to its internal ring structure. [Hoc75] gives a nice summary of a list of these conjectures. Many of these conjectures in positive characteristic are resolved due to the development of tight closure, invented by Mel Hochster and Craig Huneke in their celebrated paper [HH90]. They also used tight closure to prove various remarkable results, e.g., the existence of balanced big Cohen-Macaulay algebras for rings containing a field and a containment theorem for symbolic powers in equal characteristic regular local rings (cf. [Hoc73, HH92, HH94b, HH94a, ELS01, HH02]). Along with the development of tight closure, there are several unexpected applications. For example, by using *descent techniques* and tight closure, one can show that for $n+1$ polynomials in $n$ variables $f_1, \ldots, f_{n+1} \in \mathbb{C}[x_1, \ldots, x_n]$, one has $f_1^n \cdots f_{n+1}^n \in (f_1^{n+1}, \ldots, f_{n+1}^{n+1})$. The statement is elementary, but the proof is by no means obvious, even in the case $n = 2$.

There are several important developments from tight closure theory. We want to list three of them that will be relevant to this thesis.

### 1.2.1 Test element theory

Test ideals were first introduced in the same paper introducing tight closure ([HH90]). Since their invention, they have found applications far beyond their original scope, including Frobenius splittings ([MR85, RR85]) and singularity theory ([HH94a, HH89]). For a good survey on this, we refer to [ST12]. There are various generalizations of test ideals, e.g., to pairs in positive characteristic ([HY03, HT04]) and to pairs in mixed characteristic ([MS18a, MS18b]). Test ideals are also closely related to multiplier ideals in equal characteristic 0 ([Smi00, Har01]).

### 1.2.2 Generalizations to other characteristics

Inspired by the fruitful results of tight closure, Raymond C. Heitmann developed four closure operations, $\mathrm{ep}, \mathrm{r1}, \mathrm{epf}$, and $\mathrm{r1f}$, in the mixed characteristic case ([Hei01]). He also proved one of them, the epf closure, satisfies the (usual) colon-capturing ([Hei02, Theorem 3.7]) for rings of mixed characteristic of dimension at most $3$. Based on this result, he was able to prove the direct summand conjecture in that case ([Hei02]). Recently, due to the development of perfectoid theory ([Sch12]),

3

many homological conjectures in mixed characteristic have been resolved by Yves André, Bhargav Bhatt, Raymond Heitmann, and Linquan Ma ([And18a, And18b, And20, Bha17b, HM18]). With the help of perfectoid techniques, Raymond Heitmann and Linquan Ma were able to prove that epf closure satisfies the (usual) colon-capturing condition ([HM21, Corollary 3.11]).

### 1.2.3  Axiomatization of closures

Geoffrey Dietz and Rebecca R.G. studied the relation between the existence of balanced big Cohen-Macaulay algebras (modules) and closure operations. Dietz introduced seven axioms (see [Die10, Axiom 1.1] and Axiom Set 3.1.4 in Section 3.1). We call a closure operation a *Dietz closure* if it satisfies all of Dietz's axioms. Dietz proved that a local domain $R$ has a Dietz closure if and only if it has a balanced big Cohen-Macaulay module. In [R.G18], R.G. introduced a new axiom called the *Algebra axiom*, and proved that the existence of a Dietz closure satisfying the Algebra axiom is equivalent to the existence of a balanced big Cohen-Macaulay algebra. Recently, the existence of balanced big Cohen-Macaulay algebras in mixed characteristic was completely solved by Yves André using perfectoid techniques ([And18a]).

## 1.3  Results and Outline

In Chapter II, we aim to extend some results about test ideals in equal characteristic $p$, and contribute to the theory of test ideals of tight closures defined in equal characteristic 0 for (semi-/affine-)analytic algebras ([HH99]).

A key property of test ideals is that they multiply the tight closure of any ideal back into that ideal. The theory has been generalized to arbitrary closure operations in any characteristic ([ERG19, PG21]).

Some of the main results of Chapter II are summarized in the following theorems. See Definition 2.1.9 for the definition of semianalytic algebras, Definition 2.1.18 for "equiheight," and Definition 2.2.7 for "absolute reducedness."

**Theorem 1.3.1** (Theorem 2.3.3)**.** *Let $R$ be a semianalytic $K$-algebra that is the localization of an absolutely reduced equiheight affine-analytic $K$-algebra where $\mathrm{char}(K) = p$. Then the Jacobian ideal $\mathcal{J}(R/K)$ is contained in the test ideal of $R$ for $K$-tight closure, and, hence, the test ideals for small equational tight closure.*

*Remark* 1.3.2. The result above implies the result of Theorem 2.3.1, where $R$ is assumed to be an absolutely reduced equidimensional complete $K$-algebra. However, the proof of Theorem 2.3.3 relies on Theorem 2.3.1.

**Theorem 1.3.3** (Corollary 2.5.11). *If $R$ is a semianalytic $K$-algebra that is the localization of a reduced equiheight affine-analytic $K$-algebra, then the Jacobian ideal $\mathcal{J}(R/K)$ is contained in the test ideal. For any flat $K$-algebra morphism $R \to R'$ with geometrically regular fibers, the expansion of the Jacobian ideal $\mathcal{J}(R/K)R'$ is contained in the test ideal of $R'$. Here, both test ideals are for $K$-tight closure, which is contained in the test ideals for small equational tight closure.*

*Remark* 1.3.4. The result above implies the case when $R$ is a reduced equidimensional complete $K$-algebra (Theorem 2.5.6) and the case when $R$ is a reduced equiheight affine-analytic $K$-algebra (Theorem 2.5.7). But the proof of Corollary 2.5.11 relies on Theorem 2.5.6 and Theorem 2.5.7.

The purpose of Chapter III is to develop a new closure operation in mixed characteristic called wepf (Definition 3.3.1), and prove that

**Theorem 1.3.5** (Theorem 3.3.8). *The* wepf *closure is a Dietz closure satisfying the Algebra axiom.*

This gives a new proof of the existence of big Cohen-Macaulay algebras, and hence big Cohen-Macaulay modules. We achieve this by proving a strong property about the epf closure of ideals generated by part of system of parameters (Theorem 3.2.4), which we call *p-colon-capturing* (Definition 3.2.3). This property generalizes some results in [HM21]. We point out that our $p$-colon-capturing property can also be used to prove that r1f is a Dietz closure satisfying the Algebra axiom. So far as we know, the problem of whether epf is a Dietz closure remains open.

We also prove the following result about module-finite extensions. See the discussion right above Construction 3.4.2 for a brief introduction on the notion of phantom extension.

**Theorem 1.3.6** (Theorem 3.4.1). *If $R \to S$ is a module-finite extension of complete local domains of mixed characteristic $p$ with an $F$-finite residue field, then this map is* epf*-phantom.*

This result, together with Heitmann and Ma's result [HM21, Theorem 3.19], implies the direct summand conjecture. We make great use of techniques from Yves André's results in [And18b] and Bhargav Bhatt's results in [Bha17b]. We also prove a property (Theorem 3.5.13) similar to $p$-colon-capturing in the positive characteristic case. This is a completely new phenomenon in tight closure theory.

We want to discuss some related results about closure theory in Chapter IV. We first extend the result [Die18, Theorem 4.8] to a more general setting (Corollary 4.1.10). For this purpose, we also generalize various notions to the non-domain case. Then we define two more axioms, the colon-capturing axiom (Axiom 4.2.2) and the persistence axiom (Axiom 4.2.3), and discuss various related results. In particular, we show the following result.

**Theorem 1.3.7** (Theorem 4.2.11). *If $R \to S$ is a ring map between complete local domains and* cl *is a persistent Dietz closure satisfying the Algebra axiom with respect to the category of complete local*

*domains, then we obtain a weakly functorial version of the existence of their big Cohen-Macaulay algebras, i.e., there exists a big Cohen-Macaulay $R$-algebra $B$, a big Cohen-Macaulay $S$-algebra $C$, and a map $B \to C$ such that*

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

*commutes.*

Finally, we introduce two more closure operations PBCM and BCM in mixed-characteristic, and discuss the containment problems among PBCM, BCM, epf, and wepf.

In Chapter V, we develop a new notion called *size* for an ideal in a ring $R$ (Definition 5.2.1) based on the notion of quasilength introduced by Mel Hochster and Craig Huneke in their joint paper [HH09]. We show that the size of an ideal is a quantity invariant up to radicals (Proposition 5.2.4), and is always between the height and the arithmetic rank of the ideal (Proposition 5.2.5). We also show that the size of an ideal is unchanged when we kill finitely many nilpotent elements (Theorem 5.2.8). Moreover, we show that a finitely generated ideal is of size 0 if and only if it is nilpotent (Proposition 5.2.9). In the case of prime ideals, we have the following result.

**Theorem 1.3.8** (Theorem 5.2.10). *Let $R$ be a local ring and $P$ a prime ideal of $R$ such that $\dim R/P = 1$. Suppose that there is some $c$ such that $P^{(cn)} \subseteq P^n$ for all $n$ (this holds if the completion of $R$ is a domain) and $R/P$ is module-finite over a regular local ring $A$ (this holds if $R/P$ is complete). Then $\mathsf{size}(P) = \mathrm{ht}(P)$.*

Many properties of size are hard to study because the notion of size is based on quasilength. One difficulty with quasilength is that it is not additive, even in the case of the direct sum of two modules ([HZ18, Example 3.5]). We first show additivity of quasilength in a special case (Proposition 5.3.1), and generalize [HZ18, Example 3.5] in Proposition 5.3.3. Then we study the asymptotic behavior of the additive property. More precisely, we prove that

**Theorem 1.3.9** (Theorem 5.3.15). *Suppose that $(R, \mathfrak{m})$ is a noetherian local ring of dimension 1. Then there exists a positive constant $C$ (independent of $M$ and $I$) such that for any ideal $I \subseteq R$ and any finitely generated module $M$, we have*

$$
Cn\mathcal{L}_I(M/I^n M) \leqslant \mathcal{L}_I((M/I^n M)^{\oplus n}) \leqslant n\mathcal{L}_I(M/I^n M)
$$

*for any positive integer $n$.*

## 1.4  Definitions and Notation

While we have a preliminary section in each chapter to discuss definitions and notation related to that chapter, it is convenient to fix some notation that will be used throughout this thesis.

- All rings are commutative associative rings with a multiplicative identity element 1.

- $p$ will always be a positive prime integer.

Let $R$ be a ring. An element $x \in R$ is called a *nonzerodivisor* if for any other element $y \in R$ such that $xy = 0$, we have $y = 0$. $R$ is called a *domain* if all nonzero elements of $R$ are nonzerodivisors. If $R$ is a domain, the *fraction field* $\mathrm{Frac}(R)$ of $R$ is the localization of $R$ at all its nonzero elements, i.e., $\mathrm{Frac}(R) = (R - \{0\})^{-1}R$. The *absolute integral closure* $R^+$ of $R$ is the integral closure of $R$ in an algebraic closure of its fraction field.

By "a local ring $(R, \mathfrak{m}, k)$" we mean a ring in which $\mathfrak{m}$ is the only maximal ideal of $R$ and $k$ is the residue field of $R$, i.e., $k = R/\mathfrak{m}$. Sometimes we omit $k$ in the triple if we do not need to refer to it. Given a $d$-dimensional noetherian local ring $(R, \mathfrak{m})$, by Krull's height theorem, the maximal ideal $\mathfrak{m}$ is a minimal ideal of an ideal generated by $d$ elements $x_1, \ldots, x_d$, and no fewer. The $d$ elements are called a *system of parameters* for the local ring $R$.

**Definition 1.4.1.** Let $R$ be a ring and $M$ an $R$-module. A sequence $x_1, \ldots, x_n$ is called a *regular sequence* on $M$ if the following conditions hold:

- $x_1$ is a nonzerodivisor on $M$.

- $x_i$ is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$, where $2 \leqslant i \leqslant n$.

- $M/(x_1, \ldots, x_n)M \neq 0$.

We can also relate regular sequences to $\mathrm{Ext}$ modules: Let $R$ be a noetherian ring and let $M$ be an $R$-module. Let $\underline{x} = (x_1, \ldots, x_n)$ be a regular sequence on $M$ contained in some ideal $I \subseteq R$. Then $\mathrm{Ext}_R^i(R/I, M) = 0$ for $0 \leqslant i \leqslant n$. If $n$ is the longest possible length of a regular sequence in $I$ on $M$, then $\mathrm{Ext}_R^n(R/I, M) \neq 0$.

A $d$-dimensional noetherian local ring $(R, \mathfrak{m}, k)$ is *Cohen-Macaulay* if one (equivalently, every) system of parameters of $R$ is a regular sequence on $R$. Equivalently, $R$ is Cohen-Macaulay if $\mathrm{Ext}^i(k, R) = 0$ where $1 \leqslant i < d$ and $\mathrm{Ext}^d(k, R) \neq 0$. $R$ is a *Gorenstein* ring if it is Cohen-Macaulay and $\dim_k \mathrm{Ext}^d(k, R) = 1$. $R$ is a regular local ring if the maximal ideal $\mathfrak{m}$ is generated by $d$ elements. Although it is not obvious, regular local rings are Gorenstein, hence, Cohen-Macaulay.

Most rings we study in commutative algebra are not even Cohen-Macaulay. Thus, we have the following definition.

**Definition 1.4.2.** Let $(R, \mathfrak{m})$ be a noetherian local ring. An $R$-module $M$ is a *big Cohen-Macaulay module* if some system of parameters for $R$ is a regular sequence on $M$. It is called a *balanced big Cohen-Macaulay module* if every system of parameters of $R$ is a regular sequence on $M$. An $R$-algebra $S$ is a *(balanced) big Cohen-Macaulay algebra* if it is a (balanced) big Cohen-Macaulay module over $R$.

The difference between a balanced and nonbalanced big Cohen-Macaulay module is very minor. In fact, by a result from [BS83], the $\mathfrak{m}$-adic completion of any (nonbalanced) big Cohen-Macaulay module is a balanced big Cohen-Macaulay module. Hence, from now on, we will omit the word "balanced." By "big Cohen-Macaulay algebra/module" we mean that every system of parameters is a regular sequence on this algebra/module. The terminology "big" here means that we do not require the algebra/module to be finitely generated. A finitely generated Cohen-Macaulay module is called a *small Cohen-Macaulay module*.

# CHAPTER II

## Test Elements for Tight Closures in Equal Characteristic

This chapter is organized as follows: we first collect some preliminaries in Section 2.1 and have a discussion on Jacobian ideals in Section 2.2. Then we prove the new results that the Jacobian ideal is contained in the test ideal for complete $K$-algebras (Theorem 2.3.1), and for reduced semianalytic $K$-algebras that are localizations of equiheight affine-analytic $K$-algebras (Theorem 2.3.3) in characteristic $p$ in Section 2.3. After that, we discuss the definition of tight closure in equal characteristic 0 and prove that the Jacobian ideal is contained in the test ideal for affine $K$-algebras (Theorem 2.4.9) in equal characteristic 0 in Section 2.4. Finally in Section 2.5, we start to discuss descent techniques and prove similar results for complete local $K$-algebras (Theorem 2.5.6). Then we describe a similar descent process and prove similar results for reduced affine-analytic $K$-algebras (Theorem 2.5.7). Based on Theorem 2.5.7, we will establish the same result for reduced semianalytic $K$-algebras that are localizations of equiheight affine-analytic $K$-algebras (Corollary 2.5.11).

### 2.1 Preliminaries

We discuss some preliminary material about semianalytic algebras, the module of Kähler differentials and their relation to regularity. Throughout this section, unless otherwise stated, we do not assume any characteristic constraint. Most material here is covered in [Kun86], but has been reworded in a way that is better adapted to our needs in this thesis. We start with some notation.

**Definition 2.1.1.** Let $R$ be a noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(R)$ a prime ideal of $R$. Let $M$ be an $R$-module and $I \subseteq R$ an ideal of $R$.

  (i) $\mathrm{big\,ht}(I)$ is the big height of $I$, i.e., $\mathrm{big\,ht}(I) = \max\{\mathrm{ht}(Q) \mid Q \in \mathrm{Min}(I)\}$.

  (ii) If $I \subseteq \mathfrak{p}$, then the notion $\mathrm{ht}_{\mathfrak{p}}(I)$ represents the smallest height of a prime $P$ such that $I \subseteq P \subseteq \mathfrak{p}$. Note that this is equal to $\mathrm{ht}(IR_{\mathfrak{p}})$ in $R_{\mathfrak{p}}$.

  (iii) $\dim_{\mathfrak{p}} R$ is the supremum of the lengths of all chains of prime ideals containing $\mathfrak{p}$. We have $\dim_{\mathfrak{p}} R = \dim R/\mathfrak{p} + \dim R_{\mathfrak{p}}$.

(iv) The *regularity defect* of $R$ at $\mathfrak{p}$ is defined to be $\mathrm{rd}_{\mathfrak{p}}(R) \coloneqq \mathrm{edim}(R_{\mathfrak{p}}) - \dim(R_{\mathfrak{p}})$, where $\mathrm{edim}(R_{\mathfrak{p}})$ is the embedded dimension of the noetherian local ring $R_{\mathfrak{p}}$.

(v) The residue field of $\mathfrak{p}$ is denoted by $\kappa_{\mathfrak{p}}(R) \coloneqq R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

(vi) $\mu(M)$ is the minimal number of generators of $M$ and $\mu_{\mathfrak{p}}(M)$ is the minimal number of generators of the localized module $M_{\mathfrak{p}}$.

(vii) If $A$ is a matrix with entries in $R$, then $\mathrm{rank}_{\mathfrak{p}}(A)$ represents the determinantal rank of the matrix $A$ over the residue field $\kappa_{\mathfrak{p}}(R)$ at prime $\mathfrak{p}$.

### 2.1.1   Test ideals

We will use the following definition for test ideals in this chapter.

**Definition 2.1.2.** [PG21, Definition 3.1] Let $R$ be a ring and $\mathsf{cl}$ be a closure operation on $R$-modules. The big test ideal of $R$ associated to $\mathsf{cl}$ is defined as

$$\tau_{\mathsf{cl}}(R) = \bigcap_{N \subseteq M} \left( N : N_M^{\mathsf{cl}} \right)$$

where the intersection runs over any (not necessarily finitely-generated) $R$-modules $N, M$. Similarly, we define the finitistic test ideal of $R$ associated to $\mathsf{cl}$ as

$$\tau_{\mathsf{cl}}^{\mathrm{fg}}(R) = \bigcap_{\substack{N \subseteq M \\ M/N \text{ finitely generated}}} \left( N : N_M^{\mathsf{cl}} \right).$$

There are two subtleties when working with this definition:

1. As we see in Definition 2.1.2, there are two kinds of test ideals (the (big) test ideal and the finitistic test ideal).

2. Test ideals are defined in terms of modules. One can also define them purely in terms of ideals.

For the first point, these two notions associated to tight closure in characteristic $p$ are conjectured to be the same ([ST12, Conjecture 5.14]), and proved to be the same in several cases [LS99, LS01]. Since we only work with tight closure of ideals in noetherian rings, we will stick to the notion of finitistic test ideals. For the second point, test ideals defined in terms of (finitely generated) modules and in terms of all ideals in the ring coincide when the base ring $R$ is approximately Gorenstein ([Hoc77, Definition-Proposition 1.1, Definition 1.3]). See the remark below.

*Remark* 2.1.3. Let cl be a closure operation on $R$ satisfying Semiresiduality and Functoriality (see Axiom (v) and (iv), see also [PG21, Definition 2.1, 2.2]). If $R$ is approximately Gorenstein ([HH90, 8.6]), then the finitistic test ideal for modules defined in [PG21, Definition 3.1] coincides with the test ideal for ideals associated with cl, i.e., $\tau_{\mathrm{cl}}^{\mathrm{fg}}(R) = \bigcap_{I \subseteq R}(I : I^{\mathrm{cl}})$ (The argument in [HH90, Proposition 8.15] works for any general closure satisfying the Semiresiduality and Functoriality axioms).

The condition of being approximately Gorenstein is fairly weak, e.g., when $R$ is either reduced and excellent or when $R$ is of depth 2, $R$ is approximately Gorenstein. In fact, all rings we are working with in this chapter are excellent and reduced, hence, approximately Gorenstein. So we make the following convention throughout the chapter.

*Convention* 2.1.4. By the test ideal, we mean the *finitistic test ideal* associated to tight closure in the sense of Definition 2.1.2 in term of ideals (the case of modules follows from the case of ideals by Remark 2.1.3). We shall also call the elements in the test ideal "test elements." Note that in the literature, test elements are elements in the test ideal which avoids all minimal primes of the ring.

### 2.1.2 Derivations and universal extensions

A good reference here is Chapters 1-3, 11-12, in [Kun86].

**Definition 2.1.5.** Let $R_0$ be a ring, $R$ an $R_0$-algebra and $M$ an $R$-module.

(i) A $R_0$-*derivation* of $R$ in $M$ is an $R_0$-linear map $\delta : R \to M$ that satisfies the Leibniz rule, i.e., for all $a, b \in R$, $\delta(ab) = a\delta b + b\delta a$. In the case $R_0 = \mathbb{Z}$, $\mathbb{Z}$-derivations of $R$ are simply called *(absolute) derivations* of $R$ ([Kun86, 1.1]).

(ii) An $R_0$-derivation $\mathrm{d} : R \to M$ is called *universal* if for any $R_0$-derivation $\delta : R \to N$ there is one and only one $R$-linear map $\ell : M \to N$ such that $\delta = \ell \circ \mathrm{d}$ ([Kun86, 1.18]).

(iii) If $\mathrm{d} : R \to M$ is a universal $R_0$-derivation of $R$ then the $R$-module $M$ is denoted by $\Omega_{R/R_0}$ and is called *the module of (Kähler) differentials* of $R$ over $R_0$. The universal derivation of $R$ over $R_0$ is sometimes denoted by $\mathrm{d}_{R/R_0}$ ([Kun86, 1.20]). It is well-known that the module of Kähler differentials exists. If we denote the kernel of the canonical map $R \otimes_{R_0} R \to R$ by $I$, then $\Omega_{R/R_0} \cong I/I^2$ ([Kun86, 1.7, 1.21]).

(iv) For an $R_0$-derivation $\delta : R \to M$, we shall write $R\delta R$ for the submodule of $M$ generated by $\{\delta r\}_{r \in R}$. If $\mathrm{d}$ is the universal $R_0$-derivation of $R$, then $R\mathrm{d}R = \Omega_{R/R_0}$ ([Kun86, 1.21 (b)]).

Chapters 2 and 3 in [Kun86] describe differential algebras and universal extensions of differential algebras. Here, we work with derivations, which are more relevant to our needs. These two

languages can be translated to and from one another in most cases. The definition below is [Kun86, 1.24].

**Definition 2.1.6.** Let $R$ and $S$ be two $R_0$-algebras and $\rho : R \to S$ an $R_0$-algebra morphism. Let $\delta : R \to M$ be an $R_0$-derivation of $R$ and $\delta' : S \to M'$ an $R_0$-derivation of $S$. Then $\delta'$ is called an *extension* of $\delta$ if there exists an $R$-linear map $\varphi : M \to M'$ such that

$$
\begin{array}{ccc}
R & \xrightarrow{\ \rho\ } & S \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\
M & \xrightarrow{\ \varphi\ } & M'
\end{array}
$$

is commutative. An extension $\delta'$ of $\delta$ is called *universal* if any other extensions $\Delta : S \to N$ of $\delta$ can be uniquely written as a specialization of $\delta'$, i.e., there exists a unique $S$-linear map $\phi : M' \to N$ such that $\Delta = \phi \circ \delta'$.

The most important result about universal extensions of an $R_0$-derivation $\delta : R \to R\delta R$ is that they exist [Kun86, 3.20]. From the definition we see that the universal extension $\delta' : S \to M'$ of $\delta$ is unique up to canonical isomorphism. We write $\Omega_{S/\delta}$ for $M'$ and call $\Omega_{S/\delta}$ the *module of Kähler differentials of $S$ over $\delta$*. If $\delta$ is the trivial derivation, i.e., $R\delta R = 0$, then $\Omega_{S/\delta} = \Omega_{S/R}$ is the usual module of Kähler differentials of $S$ over $R$.

The module of Kähler differentials is finitely generated for affine $R_0$-algebras. Finite generation is important when we define the Jacobian ideal in Section 2.2. However, modules of Kähler differentials are not necessarily finitely generated for power series rings over $R_0$ (or more generally, semianalytic $K$-algebras, see Definition 2.1.9). So we have the following definitions.

**Definition 2.1.7.**  (i) An $R_0$-derivation $\delta : R \to R\delta R$ of $R$ is called *finite* if $R\delta R$ is finitely generated as an $R$-module.

(ii) $\mathrm{d} : R \to M$ is called *universally finite* if $\mathrm{d}$ is finite and each finite $R_0$-derivation $\delta$ of $R$ factors through $\mathrm{d} : R \to M$ with respect to an $R$-homomorphism ([Kun86, 11.1]). If such $\mathrm{d} : R \to M$ exists, then $M$ is unique up to a canonical $R$-isomorphism. We write $\widetilde{\Omega}_{R/R_0}$ for $M$ and called it the *universally finite module of differentials of $R$ over $R_0$*.

(iii) Let $\rho : R \to S$ be a homomorphism of $R_0$-algebras and $\delta : R \to R\delta R$ a derivation of $R/R_0$. An $R_0$-derivation $\mathrm{d} : S \to N$ of $S$ into an $S$-module $N$ is called a *universally finite $\rho$-extension of $\delta$*, if the following hold:

(a) $\mathrm{d}$ is a $\rho$-extension of $\delta$ and finite (i.e., $S\mathrm{d}S$ finitely generated)

(b) If $\Delta : S \to N'$ is an arbitrary finite $\rho$-extension of $\delta$, then there is exactly one $S$-linear map $h : N \to N'$ with $\Delta := h \circ \mathrm{d}$.

If the universally finite $\rho$-extension $\mathrm{d} : S \to N$ of $\delta$ exists, we write $N := \widetilde{\Omega}_{S/\delta}$ and call this the *universally finite module of differentials of $S/\delta$*. In case $\delta$ is the trivial derivation of $R$ we write $\widetilde{\Omega}_{S/R}$ instead of $\widetilde{\Omega}_{S/\delta}$ ([Kun86, 11.4]).

The most important result here is that under mild assumptions, universally finite modules of differentials exist. Explicitly, we have the following theorem ([Kun86, 12.5]).

**Theorem 2.1.8.** *Let $R$ be an $R_0$-algebra and assume that $R$ is noetherian. Let $I$ be an ideal of $R$ and $\widehat{(-)}$ the completion in the $I$-topology.*

(i) *If $\Omega$ is a finite module of differentials of $R$ over $R_0$, then $\widetilde{\Omega}_{\widehat{R}/R_0}$ exists and $\widetilde{\Omega}_{\widehat{R}/R_0} = \widehat{\Omega}$.*

(ii) *If $\Omega_{R/R_0}$ is finite, then $\widetilde{\Omega}_{\widehat{R}/R_0}$ exists and $\widetilde{\Omega}_{\widehat{R}/R_0} = \widehat{\Omega}_{R/R_0}$.*

### 2.1.3 Structure of semianalytic algebras

In the definition below, the terminology in the first and the third parts (analytic and semianalytic) is taken from [Kun86, Chapter 13].

**Definition 2.1.9.** Let $R$ be a $K$-algebra where $K$ is a field.

(i) $R$ is called an *analytic $K$-algebra*, if there is a power series algebra $P = K[\![x_1, \ldots, x_n]\!]$ such that $R$ is module-finite over $P$.

(ii) $R$ is called an *affine-analytic $K$-algebra*, if there is a power series algebra $P = K[\![x_1, \ldots, x_n]\!]$ such that $R$ is of finite type over $P$.

(iii) $R$ is called a *semianalytic $K$-algebra*, if there is a power series algebra $P = K[\![x_1, \ldots, x_n]\!]$ such that $R$ is essentially of finite type over $P$.

The key result here is the following proposition.

**Proposition 2.1.10.** *[Kun86, 13.4] Any reduced semianalytic $K$-algebra $R$ contains a unique maximal analytic $K$-algebra $A$, i.e., all $K$-subalgebras of $R$ that are analytic $K$-algebras are contained in $A$. If $A'$ is an arbitrary analytic $K$-algebra with $A' \subseteq R$ such that $R$ is essentially of finite type over $A'$, then $A$ is the integral closure of $A'$ in $R$.*

For a reduced semianalytic $K$-algebra $R$ we denote by $\mathrm{A}(R)$ the maximal analytic subalgebra of $R$. If $R$ is not reduced, such an algebra need not exist. If $R$ is a domain, then $\mathrm{A}(R)$ is a local domain, because $\mathrm{A}(R)$ is always a direct product of local rings. We also have that the $K$-algebra maps between reduced semianalytic $K$-algebras are compatible with taking the maximal analytic

$K$-subalgebra. That is, if $\varphi : R \to S$ is a homomorphism of reduced semianalytic $K$-algebras, then $\varphi(\mathrm{A}(R)) \subseteq \mathrm{A}(S)$. Hence $\varphi$ induces a $K$-homomorphism $\mathrm{A}(\varphi) : \mathrm{A}(R) \to \mathrm{A}(S)$ ([Kun86, 13.5]).

For each reduced semianalytic $K$-algebra $R$, let $\widetilde{\Omega}_{R/K}$ denote the universal $R$-extension of $\widetilde{\Omega}_{\mathrm{A}(R)/K}$, the universally finite differential module of $\mathrm{A}(R)$ over $K$, whose existence is guaranteed by [Kun86, 12.9]. If $A'$ is an arbitrary analytic $K$-algebra such that $R$ is essentially of finite type over $A'$, then by the transitive law for universal extensions, $\widetilde{\Omega}_{R/K}$ is also the universal $R$-extension of $\widetilde{\Omega}_{A'/K}$.

*Remark* 2.1.11. In [Kun86], Kunz uses the notion $\mathrm{D}_K(R)$ to refer to the universal $R$-extension of $\widetilde{\Omega}_{\mathrm{A}(R)/K}$. He uses $\Omega_{R/\delta}$ for any $\delta : A \to A\delta A$ if $R$ is not reduced and $R$ is essentially of finite type over $A$ (note that in this case $\mathrm{A}(R)$ is not well-defined). Since we are only working with reduced semianalytic algebras, there is no ambiguity in using $\widetilde{\Omega}_{R/K}$.

We also need the definition of "analytic transcendence degree" of field extensions.

**Definition 2.1.12.** Let $K$ be a field and $X_1, \ldots, X_n$ indeterminates over $K$. The field $F :=$ $K(\!(X_1, \ldots, X_n)\!)$ will always denote the fraction field of the power series ring $K[\![X_1, \ldots, X_n]\!]$. Let $L$ be a field extension of $K$.

- $L$ is called *semianalytic extension field* of $K$, if there is a $K$-homomorphism $F \to L$ such that $L$ is finitely generated over $F$. If $L$ is a finite extension of $F$, we call $L$ an *analytic extension field* of $K$.

- Let $L$ be an analytic field extension of $K$. Suppose $L$ is finite over $F \subseteq L$. Then $n$ is called the *analytic transcendence degree* of $L$ over $K$, $n := \mathrm{aTr}\deg(L/K)$, and $\{X_1, \ldots, X_n\}$ is called an *analytic transcendence basis* of $L$ over $K$. The basis $\{X_1, \ldots, X_n\}$ is called *separating*, if $L$ is separable over $F$. $L$ is called *analytically separable* over $K$, if $L$ has a separating analytic transcendence basis over $K$ ([Kun86, 13.7]). Note that the number $n$ above is an invariant of the field extension $L$ over $K$, as it is the Krull dimension of the maximal analytic algebra $\mathrm{A}(L)$.

### 2.1.4 Primes in affine-analytic algebras

Next we want to analyze the primes in a reduced affine-analytic ring $R$. There are basically two types of primes in affine-analytic algebras.

**Definition 2.1.13.** Let $R$ be a reduced affine-analytic $K$-algebra. Let $A := \mathrm{A}(R)$ and let $\mathrm{JRad}(A)$ denote the Jacobson radical of $A$. In this chapter, a prime ideal $Q$ in $R$ is called

- *special* if $Q + \mathrm{JRad}(A)R$ is a proper ideal of $R$;

- *typical* if $Q + \mathrm{JRad}(A)R = R$ is the whole ring.

*Remark* 2.1.14. Under the assumption of Definition 2.1.13, since $\mathrm{A}(R)$ is module-finite over some complete local ring, $\mathrm{A}(R)$ is a product of several complete local rings, i.e., $\mathrm{A}(R) = A_1 \times \cdots \times A_s$. Each $A_i$ is a complete local domain and we call its maximal ideal $\mathfrak{m}_i$. If we have a presentation

$$R = T/I \text{ where } T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m],$$

then each $\mathfrak{m}_i$ is radical of the image of $(x_1, \ldots, x_n)$ in $A_i$. Thus, the Jacobson radical $\mathrm{JRad}(\mathrm{A}(R)) = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_s$ is the radical of the image of $(x_1, \ldots, x_n)$ in $\mathrm{A}(R)$. Since for any prime ideal $Q$, $Q + \mathrm{JRad}(\mathrm{A}(R))R$ is a proper ideal if and only if $Q + (x_1, \ldots, x_n)R$ is a proper ideal, we may use $(x_1, \ldots, x_n)R$ to detect the type of a prime ideal of $R$.

Note that a prime ideal $Q$ is special if it is contained in a special maximal ideal, and special maximal ideals are those ideals containing $\mathfrak{m}R$.

We aim to show that in the ring $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$, special maximal ideals have height equal to $\dim T$ while typical maximal ideals have height one less than $\dim T$. For this purpose, we need the following lemma.

**Lemma 2.1.15.** *If $(\mathcal{R}, \mathfrak{m})$ is an equidimensional local ring of dimension $n$, and $f \in \mathfrak{m}$ is not in any minimal prime, then $\mathcal{R}_f$ has dimension $n - 1$, and $\mathcal{R}_f$ is a Hilbert ring, i.e., every prime (hence, every radical ideal) is an intersection of maximal ideals.*

*If in addition $\mathcal{R}$ is catenary, then all maximal ideals of $\mathcal{R}_f$ have height $n - 1$.*

*Proof.* We first show that $\dim \mathcal{R}_f$ is of dimension $n - 1$. Since $\mathcal{R}_f$ is a localization of $\mathcal{R}$, the dimension cannot go up. So $\dim \mathcal{R}_f \leqslant \dim \mathcal{R}$. If there is a prime chain of length $n$ in $\mathcal{R}_f$, the preimage of it will be a prime chain of length $n$ contained in $\mathfrak{m}$, which will imply that $\mathfrak{m}$ has height $n + 1$. So we have $\dim \mathcal{R} < n$. To see that $\dim \mathcal{R}_f = n - 1$, extend $f$ to be a system of parameters $f, f_1, \ldots, f_{n-1}$ in $\mathcal{R}$. Let $Q$ be a minimal prime of $(f_1, \ldots, f_{n-1})\mathcal{R}$. Then $f \notin Q$ and $Q\mathcal{R}_f$ has height $n - 1$.

To show that $\mathcal{R}_f$ is a Hilbert ring, we prove that any prime $P$ in $\mathcal{R}_f$ is an intersection of maximal ideals in $\mathcal{R}_f$. Equivalently, we show that the intersection of maximal ideals in $\mathcal{R}_f/P$ is zero. Let $\mathcal{P}$ be the preimage of $P$ in $\mathcal{R}$. Then $\mathcal{R}_f/P = (\mathcal{R}/\mathcal{P})_f$. We replace $\mathcal{R}$ by $\mathcal{R}/\mathcal{P}$, and then we aim to show that the intersection of all maximal ideals in $\mathcal{R}_f$ is zero when $\mathcal{R}$ is a domain.

Let $0 \neq g \in \mathcal{R}$ be a non-unit in $\mathcal{R}_f$. Extend $fg$ to a system of parameters $fg, g_1, \ldots, g_{n-1}$ for $\mathcal{R}$. Choose a minimal prime $Q'$ of $(g_1, \ldots, g_{n-1})$. Then $Q'$ does not contain $fg$. Hence $\mathcal{R}/Q'$ has dimension 1. By the argument above, $(\mathcal{R}/Q')_f$ has dimension exactly one less than $\mathcal{R}/Q'$. So $(\mathcal{R}/Q')_f$ is a field. Therefore $\mathcal{R}_f/Q'\mathcal{R}_f$ is a field. So $Q'\mathcal{R}_f$ is maximal, and does not contain $g$. Hence, the intersection of all maximal ideals in $\mathcal{R}_f$ is zero.

15

Given any maximal $\mathfrak{m}'$ ideal of $\mathcal{R}_f$, it contains a minimal prime of $\mathcal{R}$ expanded to $\mathcal{R}_f$. We can kill that minimal prime and assume that we are in the domain case. Let $\mathcal{M}$ be the preimage of $\mathfrak{m}'$ in $\mathcal{R}$. Then $f \neq 0$ in $\mathcal{R}/\mathcal{M}$. Hence $\dim \mathcal{R}/\mathcal{M} \geqslant 1$. $\mathcal{M}$ is one of the maximal ideals that do not contain $f$, so $\dim \mathcal{R}/\mathcal{M} = 1$. If $\mathcal{R}$ is catenary, then $\operatorname{ht} \mathcal{M} = n - 1 \Rightarrow \operatorname{ht} \mathfrak{m}' = n - 1$.

$\square$

*Remark* 2.1.16. By an example in [Nag75, Appendix A], there is a local domain $\mathcal{R}$ of dimension 3 that has a prime $\mathcal{Q}$ such that $\operatorname{ht} \mathcal{Q} = 1$ and $\dim(\mathcal{R}/\mathcal{Q}) = 1$. This provides a local domain of dimension 3 with saturated chains of length 2 and of length 3. Thus $R$ has a prime $\mathcal{Q}'$ such that $\operatorname{ht} \mathcal{Q}' = 2$ and $\dim(\mathcal{R}/\mathcal{Q}') = 1$. So if $f \in \mathfrak{m} - (\mathcal{Q} \cup \mathcal{Q}')$, then $\mathcal{R}_f$ has maximal ideals of height one and of height 2. Therefore the "catenary" condition in Lemma 2.1.15 cannot be omitted.

Now we are ready to characterize the heights of typical and special maximal ideals.

**Proposition 2.1.17.** *Let* $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$. *Then special maximal ideals in* $T$ *have height* $n + m = \dim(T)$, *and typical maximal ideals in* $T$ *have height* $n + m - 1$.

*Proof.* Let $Q$ be a maximal ideal of $T$. If $Q$ is special, then we can kill $\mathfrak{m}$ and $Q/\mathfrak{m}$ is a maximal ideal in $T/\mathfrak{m} = K[z_1, \ldots, z_m]$. Since $T$ is catenary, we have $\operatorname{ht} Q = \operatorname{ht}(Q/\mathfrak{m}) + \operatorname{ht} \mathfrak{m} = n + m$.

If $Q$ is typical, then there is some $f \in \mathfrak{m}$ not in $Q$ such that $fa + 1 \in Q$. Let $\mathcal{Q}$ be the preimage of $Q$ in $P$. Then $T/Q$ is a field finitely generated over $P/\mathcal{Q}$. By the generalized Noether normalization theorem, $T/Q$ is module-finite over a polynomial ring over $(P/\mathcal{Q})_g$. Then $(P/\mathcal{Q})_g$ must be a field and $T/Q$ is a finite extension of it. So $\mathcal{Q}P_g$ is a maximal ideal of $P_g$, which by Lemma 2.1.15, has height $n - 1$. By the dimension formula, we have

$$\operatorname{ht} Q - \operatorname{ht} \mathcal{Q} = \operatorname{tr} \deg(\operatorname{Frac}(T)/\operatorname{Frac}(P)) - \operatorname{tr} \deg((T/Q)/(P_{\mathcal{Q}}/\mathcal{Q}P_{\mathcal{Q}})).$$

Since $T/Q$ is finite extension of $(P/\mathcal{Q})_g$, it is also finite extension of $P_{\mathcal{Q}}/\mathcal{Q}P_{\mathcal{Q}})$. Hence the right-hand side is $m - 0$. So $\operatorname{ht} Q = \operatorname{ht} \mathcal{Q} + m = n - 1 + m$. $\square$

**Definition 2.1.18.** Let $R$ be a reduced affine-analytic $K$-algebra. Let $\{\mathfrak{q}_i\}_{1 \leqslant i \leqslant s}$ be the set of minimal primes of $R$. Then $R$ is called *equiheight* if one of the following conditions occur:

- If only one type of primes occurs (all are special or all are typical, see Definition 2.1.13), then $\dim_{\mathfrak{q}_i} R = \dim R$ for $1 \leqslant i \leqslant s$.

- If both types of primes occur, then

$$\dim_{\mathfrak{q}_i}(R) = \begin{cases} \dim R & \text{if } \mathfrak{q}_i \text{ special} \\ \dim R - 1 & \text{if } \mathfrak{q}_i \text{ typical} \end{cases}$$

for $1 \leqslant i \leqslant s$.

**Proposition 2.1.19.** *Let $R$ be a reduced affine-analytic $K$-algebra. Then $R$ is equiheight if and only if for any presentation $R = T/I$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$ and $I \subseteq T$ an ideal, $I$ is of pure height, i.e., all minimal primes of $I$ have the same height.*

*Proof.* Let $\{\mathfrak{q}_i\}_{1 \leqslant i \leqslant s}$ be the set of all minimal primes of $R$ and let $\{Q_i\}_{1 \leqslant i \leqslant s}$ be the set of primes in $T$ such that $Q_i$ is the preimage of $\mathfrak{q}_i$. Note that a prime in $R$ is special if and only if its preimage in $T$ is special by Remark 2.1.14.

If $Q_i$ (hence, $\mathfrak{q}_i$) is special, then it is contained in some special maximal ideal $\mathfrak{m}_i$ of $T$. Since $T$ is catenary and $\mathrm{ht}(\mathfrak{m}_i) = \dim T = n + m$ by Proposition 2.1.17, we have that $\dim_{\mathfrak{q}_i} R = \dim T - \mathrm{ht}\, Q_i = n + m - \mathrm{ht}\, Q_i$.

If $Q_i$ is typical, then it is only contained in typical maximal ideals. By Proposition 2.1.17, we have $\dim_{\mathfrak{q}_i} R = n + m - 1 - \mathrm{ht}\, Q_i$.

Then both directions follow from these two formulas. $\qquad\qquad\square$

*Remark* 2.1.20. Note that Proposition 2.1.19 also implies that if a reduced affine-analytic $K$-algebra $R$ has a presentation $T/I$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$ and $I$ has pure height in $T$, then for any other presentation $R \cong T'/I'$ where $T' = K[\![x'_1, \ldots, x'_{n'}]\!][z'_1, \ldots, z'_{m'}]$ and $I' \subseteq T'$, the kernel $I'$ has pure height in $T'$ as well. This statement can also be proved directly by comparing two different presentations. We can form the larger ring $S = T[\![x_1, \ldots, x_n, x'_1, \ldots, x'_{n'}]\!][z_1, \ldots, z_m, z'_1, \ldots, z'_{m'}]$ and compare both presentations with the presentation $S \twoheadrightarrow R$. Then we can form $S$ from $T$ by adjoining one variable at a time. So we assume that $S$ has one more variable $y$ than $T$, i.e., either $S = K[\![x_1, \ldots, x_n, y]\!][z_1, \ldots, z_m]$ or $S = T[y]$. Then since $T \twoheadrightarrow R$, we can choose $f \in T$ such that it maps to the same image as $y$. Then the kernel of $S \twoheadrightarrow R$ is generated by $(I, y - f)S$. Since $y - f$ is not contained in any minimal prime of $I$, the new kernel $I' = (I, y - f)$ is of pure height (one larger than the pure height of $I$).

### 2.1.5 Height of ideals in regular rings

First we state Serre's intersection theorem, see [Ser75, Chapitre V, B.6, Théorème 3].

**Theorem 2.1.21** (Serre's intersection theorem)**.** *Let $A$ be a regular ring and $P, Q$ be two prime ideals in $A$ such that $P + Q$ is a proper ideal. Then we have*

$$\mathrm{ht}(P) + \mathrm{ht}(Q) \geqslant \mathrm{ht}(P + Q).$$

We immediately see that

**Corollary 2.1.22.** *Let $I, J$ be two ideals in a regular local ring $A$. Then we have*

$$\operatorname{ht}(I) + \operatorname{ht}(J) \geqslant \operatorname{ht}(I + J).$$

*Proof.* Choose $P$ minimal over $I$ such that $\operatorname{ht}(P) = \operatorname{ht}(I)$ and $Q$ minimal over $J$ such that $\operatorname{ht}(Q) = \operatorname{ht}(J)$. Since we are in the local case, the sum $P + Q$ is contained in the maximal ideal of $A$. By Theorem 2.1.21, $\operatorname{ht}(P) + \operatorname{ht}(Q) \geqslant \operatorname{ht}(P + Q)$. Since $P + Q \supseteq I + J$, we have $\operatorname{ht}(P + Q) \geqslant \operatorname{ht}(I + J)$. So $\operatorname{ht}(I) + \operatorname{ht}(J) \geqslant \operatorname{ht}(I + J)$. $\qquad\qquad\square$

The following theorem, implied by Serre's complete intersection theorem, is well-known to experts. Since we cannot find a solid source, we include a proof here.

**Theorem 2.1.23.** *Let $R$ be a noetherian regular ring and let $I \subseteq R$ be an ideal. Let $h = \operatorname{big}\operatorname{ht}(I)$. Let $R \to S$ be a ring homomorphism between noetherian rings. If $IS$ is a proper ideal, then $\operatorname{ht}(IS) \leqslant h$.*

Before giving the proof of Theorem 2.1.23, we will need the following "Cohen factorization theorem" ([AFH94, Theorem 1.1]).

**Theorem 2.1.24** (Cohen factorization). *Let $\varphi : R \to S$ be a local ring map between noetherian local rings. Then there exists a complete local ring $T$ and two local ring maps $\tau : R \to T$ and $\theta : T \to S$ such that*

   *(i) $\varphi = \theta \circ \tau$ and $\theta : T \twoheadrightarrow S$ is a surjection,*

   *(ii) $\tau$ is flat and $T/\mathfrak{m}_R T$ is regular where $\mathfrak{m}_R$ is the maximal ideal of $R$.*

*Such a decomposition is called a* Cohen factorization.

*Remark* 2.1.25. Since the map $\tau$ in the Theorem 2.1.24 is flat and local, it is, in fact, faithfully flat.

*Proof of Theorem 2.1.23.* Suppose for contradiction that $\operatorname{ht}(IS) > h$. Since $IS \subseteq S$ is proper, there is a minimal prime $Q$ of $IS$ in $S$ such that $\operatorname{ht}(Q) = \operatorname{ht}(IS)$. Then $\operatorname{ht}(IS_Q) = \operatorname{ht}(IS) = \dim(S_Q)$. Since $I$ will generate a $Q$-primary ideal in the completion, we see that $\operatorname{ht}(IS_Q) = \operatorname{ht}(I\widehat{S_Q})$. We can also kill a minimal prime of $\widehat{S_Q}$ and still get the same dimension. Then we may assume without loss of generality that $(S, Q)$ is a complete local domain. The contraction ideal $\mathfrak{m} = Q^c$ of $Q$ in $R$ is a prime ideal containing $I$ as $IS \subseteq Q \Rightarrow I \subseteq (IS)^c \subseteq \mathfrak{m}$. So there is a minimal prime $P$ of $I$ lies in-between, i.e., $I \subseteq P \subseteq \mathfrak{m}$. And we have $\operatorname{ht}(P) \leqslant h$. Localizing at $\mathfrak{m}$ does not change the height of $P$, so we may replace $R$ by $R_\mathfrak{m}$ and $P$ by $PR_\mathfrak{m}$ and assume without loss of generality that $R \to S$ is a local map between local rings.

By Theorem 2.1.24, there exists a map $R \to T \twoheadrightarrow S$ such that $T$ is also regular and $R \to T$ is faithfully flat. Then $\mathrm{ht}(PT) = \mathrm{ht}(P) \leqslant h$. Since $S$ is a domain, the kernel of the map $T \twoheadrightarrow S$ is a prime ideal $P'$. Note that the image of $PT + P'$ is $Q$-primary in $S$. So $\mathrm{ht}(PT + P') - \mathrm{ht}(P') = \mathrm{ht}(Q) > h$. On the other hand, by Theorem 2.1.21, we have $\mathrm{ht}(PT) + \mathrm{ht}(P') \geqslant \mathrm{ht}(PT + P')$. So we have $\mathrm{ht}(PT) > h$, which contradicts the fact that $\mathrm{ht}(PT) \leqslant h$. $\qquad\square$

## 2.2 Regularity and Jacobian Ideals

### 2.2.1 Absolute regularity

In [Kun86], Kunz discusses absolute regularity for analytic algebras. Here we extend the notion to affine-analytic algebras. First we have the following definition.

**Definition 2.2.1.** Every analytic $K$-algebra $A$ is a finite extension of a power series algebra $K[\![X_1, \ldots, X_d]\!] \subseteq A$. We call $K[\![X_1, \ldots, X_d]\!] \hookrightarrow A$ a *Noether normalization* of $A$.

The following is a modification of [Kun86, 14.10].

**Definition 2.2.2.** Let $R$ be a reduced affine-analytic algebra over a field $K$ and let $L$ be a field extension of $K$. A *constant field extension* $R_L$ of $R$ with $L$ is an affine-analytic $L$-algebra $R_L$ for which there is a local $K$-homomorphism $R \to R_L$ satisfying the following universal property: if $\beta : R \to S$ is any local $K$-homomorphism into an affine-analytic $L$-algebra $S$, then there is exactly one $L$-homomorphism $\gamma : R_L \to S$ such that $\beta = \gamma \circ \alpha$.

The existence is easily shown: Let $A := \mathrm{A}(R)$ be the unique maximal analytic $K$-subalgebra of $R$. Then we have a Noether normalization $K[\![X_1, \ldots, X_d]\!] \hookrightarrow A$. For any field extension $L$ of $K$, then tensor product $R_L := L[\![X_1, \ldots, X_d]\!] \otimes_{K[\![X_1, \ldots, X_d]\!]} R$ is the constant field extension of $R$ with $L$. Alternatively, since the constant field extension of analytic algebras is constructed in the paragraph below [Kun86, 14.10], we can form $A_L$ and then $R_L := A_L \otimes_A R$.

We extend results in [Kun86, 14.11] to the case of affine-analytic rings. For this purpose, we need the following lemma

**Lemma 2.2.3.** *Let $D$ be a domain affine over the complete local domain $(C, \mathfrak{m}, K)$ where $K \subseteq C$. Then*

- *If $\mathfrak{m}D \neq D$, then $\dim(D) = \dim(C) + \mathrm{Tr}\deg_{\mathrm{Frac}(C)}(\mathrm{Frac}(D))$;*

- *if $\mathfrak{m}D = D$, then $\dim(D) = \dim(C) + \mathrm{Tr}\deg_{\mathrm{Frac}(C)}(\mathrm{Frac}(D)) - 1$.*

*Proof.* Throughout the proof, we always let $Q$ be a maximal ideal of $C$ and let $P$ be its contraction in $C$. Then by dimension formula,

$$\operatorname{ht}(Q) - \operatorname{ht}(P) = \operatorname{Tr} \deg_{\operatorname{Frac}(C)}(\operatorname{Frac}(D)) - \operatorname{Tr} \deg_{\kappa_P(C)}(\kappa_Q(D)). \qquad (2.2.1)$$

Suppose that $\mathfrak{m}D$ is a proper ideal. Then on the one hand,

$$(2.2.1) \Rightarrow \operatorname{ht}(Q) \leqslant \dim(C) + \operatorname{Tr} \deg_{\operatorname{Frac}(C)}(\operatorname{Frac}(D)).$$

On the other hand, since $\mathfrak{m}D$ is a proper ideal, we can choose $Q$ maximal in $D$ containing $\mathfrak{m}D$. Then $P = \mathfrak{m}$ and $\operatorname{ht}(P) = \dim(C)$. Since $D/Q$ is affine over $C/P \cong K$, the transcendental degree of $\kappa_Q(D)$ over $\kappa_P(C)$ is zero. So by (2.2.1), we have $\operatorname{ht}(Q) = \dim(C) + \operatorname{Tr} \deg_{\operatorname{Frac}(C)}(\operatorname{Frac}(D))$. So the first bullet point is proved.

Now we assume that $\mathfrak{m}D = D$. Then we claim that $P$ is of dimension 1, i.e., of height $\dim(C) - 1$. Since $\mathfrak{m}D = D$, there is some $x$ in $\mathfrak{m}$ that extends to a unit in $D$. So $Q$ avoids this $x$, which implies that $P$ avoids it as well. Therefore $P \neq \mathfrak{m}$. By the generalized Noether normalization theorem, $D/Q$ is module-finite over $(C/P)_f$ for some $f \in C/P$. Since $D/Q$ is a field, we deduce that $(C/P)_f$ is a field. But $C/P$ is not a field. Hence, $P$ has height $\dim C - 1$ and the claim is proved.

By (2.2.1), $\operatorname{ht}(Q) = \dim(C) - 1 + \operatorname{Tr} \deg_{\operatorname{Frac}(C)}(\operatorname{Frac}(D))$. This is true for any maximal ideal in $D$. Hence $\dim D = \dim(C) - 1 + \operatorname{Tr} \deg_{\operatorname{Frac}(C)}(\operatorname{Frac}(D))$. $\qquad \square$

**Proposition 2.2.4.** *Let $R$ be a reduced affine-analytic $K$-algebra and let $A := \operatorname{A}(R)$. Let*

$$K[\![X_1, \ldots, X_d]\!] \hookrightarrow A$$

*be a Noether normalization of $A$, and $L/K$ a field extension.*

  (i) *For any ideal $I$ of $R$, $(R/I)_L = R_L/IR_L$.*
 (ii) *$R_L$ is faithfully flat over $R$.*
(iii) *$\dim R_L = \dim R$.*
 (iv) *If $R$ is equiheight (Definition 2.1.18), so is $R_L$.*
  (v) *For any $\mathfrak{q} \in \operatorname{Spec}(R)$ there is a $\mathfrak{p} \in \operatorname{Spec}(R_L)$ such that $\dim_{\mathfrak{p}} R_L \geqslant \dim_{\mathfrak{q}} R$.*
 (vi) *$\widetilde{\Omega}_{R_L/L} \cong R_L \otimes_R \widetilde{\Omega}_{R/K}$.*

*Proof.* 2.2.4.(i): Let $J = I \cap A$. Then by [Kun86, 14.11(b)], we have $(A/J)_L = A_L/JA_L$. Since $A/J = \operatorname{A}(R/I)$, we have

$$(R/I)_L = (A/J)_L \otimes_{A/J} (R/I) = (A_L/JA_L) \otimes_{A/J} (R/I) = A_L \otimes_A (R/I) = R_L/IR_L.$$

2.2.4.(ii): Since $L[\![X_1, \ldots, X_d]\!]$ is faithfully flat over $K[\![X_1, \ldots, X_d]\!]$, the base-changed map $R \to R_L$ is faithfully flat as well.

2.2.4.(iii): Let $\{\mathfrak{q}_i\}_{1 \leqslant i \leqslant s}$ be the set of minimal primes of $R$. Then $\cap_{j=1}^s \mathfrak{q}_j = 0$. Hence $\cap_{j=1}^s \mathfrak{q}_j R_L = 0$. So the dimension of $R_L$ is the supremum of the dimensions of $R_L/\mathfrak{q}_j R_L \cong (R/\mathfrak{q}_j)_L$. So we can base change to $R/\mathfrak{q}_i$ for some $i$ without changing $\dim R$ and $\dim R_L$. Now we assume that $R$ is a domain, $A_0 = K[\![x_1, \ldots, x_n]\!] \subseteq R$ and $B_0 = L[\![x_1, \ldots, x_n]\!]$. Then $R_L = R \otimes_{A_0} B_0$. By (2), we know that $R \hookrightarrow R_L$ and $\dim R_L \geqslant \dim R$. So we only need to show that $\dim R_L \leqslant \dim R$. Note that nonzerodivisors on $R$ are also nonzerodivisors on $R_L$. Hence $R_L$ is $R$-torsion free. If $\mathfrak{p}$ is a minimal prime of $R_L$, then $\mathfrak{p} \cap R = 0$. Choose $\mathfrak{p}$ minimal such that $\dim R_L = \dim R_L/\mathfrak{p}$. Write $R' = R_L/\mathfrak{p}$ and $C_0 = B_0/(\mathfrak{p} \cap B_0)$. We have $R \hookrightarrow R'$ and $A_0 \hookrightarrow C_0$. Since $R$ generates $R_L$ over $B_0$, it likewise generates $R'$ over $C_0$. So we can choose a (finite) set of elements in $R$ which will be a transcendence basis for $\mathrm{Frac}(R')$ over $\mathrm{Frac}(C_0)$. Then the same set of elements must be algebraically independent over $\mathrm{Frac}(A_0)$. Suppose that this finite set has $t$ elements. Then

$$
\begin{aligned}
\dim(R) &\geqslant \dim(A_0) + t - \varepsilon_{A_0,R} \quad \text{(by Lemma 2.2.3)} \\
&= \dim(B_0) + t - \varepsilon_{A_0,R} \\
&\geqslant \dim(C_0) + t - \varepsilon_{A_0,R} \\
&\geqslant \dim(C_0) + t - \varepsilon_{C_0,R'} = \dim(R')
\end{aligned}
$$

where $\varepsilon_{A_0,R}$ is 1 if $(x_1, \ldots, x_n)R = R$ and 0 otherwise. Note that if $\varepsilon_{A_0,R} = 1$ then $\varepsilon_{C_0,R'} = 1$. So this is proved.

2.2.4.(iv): Write $R = T/I$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$. Then by Proposition 2.1.19, we assume that $I$ has pure height $h$. Therefore by Lemma 2.2.5, $IT'$ has pure height $h$ as well, where $T' = T[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$. Since $R_L = T'/IT'$, we conclude that $R_L$ is equiheight.

2.2.4.(v): Let $\mathfrak{m}$ be a maximal ideal in $R$ such that $\dim_\mathfrak{m} R = \dim_\mathfrak{q} R$. There is a prime ideal $\mathfrak{m}'$ in $R_L$ lying over $\mathfrak{m}$. Since $R_\mathfrak{m} \to (R_L)_{\mathfrak{m}'}$ is still faithfully flat, we have

$$
\dim_{\mathfrak{m}'} R_L \geqslant \dim(R_L)_{\mathfrak{m}'} \geqslant \dim R_\mathfrak{m} = \dim_\mathfrak{m} R = \dim_\mathfrak{q} R.
$$

2.2.4.(vi): Since the universal finite module of Kähler differentials is calculated as the cokernel of the Jacobian matrix, the conclusion follows directly. $\qquad\square$

**Lemma 2.2.5.** *Let $T$ be flat over $S$ and $I \subseteq S$ a proper ideal such that all minimal primes have height $h$. If $IT \neq T$ (this is automatic when $S \to T$ is faithfully flat), then all minimal primes of $IT$ have the same height $h$.*

*Proof.* Let $Q$ be a minimal prime of $IT$ in $T$, and let $P$ be its contraction in $S$. So $S_P \to T_Q$ is a faithfully flat map. Since $Q$ is minimal over $IT$, $T_Q/IT_Q$ has dimension zero. By base change, this

21

ring is faithfully flat over $S_P/IS_P$. So $S_P/IS_P$ has dimension zero. Hence, $P$ is minimal over $I$ and therefore it has height $h$. Since $\dim(T_Q) = \dim(S_P) + \dim(T_Q/PT_Q]$ and $T_Q/IT_Q \twoheadrightarrow T_Q/PT_Q$ have dimension zero, we conclude that $\dim(T_Q) = \dim(S_P) = h$. $\qquad\square$

We generalize the definition of absolute regularity [Kun86, 14.12] to the affine-analytic case.

**Definition 2.2.6.** An affine-analytic $K$-algebra $R$ is called *absolutely regular* at $\mathfrak{q} \in \mathrm{Spec}(R)$, if for any field extension $L/K$ and any $\mathfrak{p} \in \mathrm{Spec}(R_L)$ with $\mathfrak{p} \cap R = \mathfrak{q}$ the local ring $(R_L)_\mathfrak{p}$ is regular.

The notion of absolute regularity is equivalent to the notion of regularity when $K$ is of characteristic 0.

We also define absolute reducedness as follows.

**Definition 2.2.7.** An affine-analytic $K$-algebra $R$ is called *absolutely reduced*, if for any field extension $L/K$, $R_L$ is reduced.

A affine-analytic $K$-algebra $R$ is absolutely reduced if and only if $R_P$ is absolutely regular for any minimal prime $P$.

### 2.2.2 Jacobian ideals

Let $R$ be a ring.

**Definition 2.2.8** (Fitting ideals). For any finitely presented $R$-module $M$, let $R^m \xrightarrow{(a_{ij})} R^n \to M$ be a presentation of $M$. The $i$th *fitting ideal* is the ideal generated by the minors of size $n - i$ of the matrix $(a_{ij})$.

The Fitting ideals do not depend on the choice of generators and relations of $M$. Here, we also use the convention that the $i$th fitting ideal is the whole ring $R$ if $n - i \leqslant 0$, and the zero ideal if $n - i > \min\{n, m\}$. For more about Fitting ideals, [Kun86, Appendix D] is a good reference.

Let $S$ be an $R$-algebra and $\delta : R \to R\delta R$ be a derivation such that $\Omega_{S/\delta}$ is a finitely presented $S$-module. We have the following definition.

**Definition 2.2.9.** [Kun86, 10.1] The $i$th Fitting ideal of $\Omega_{S/\delta}$

$$\mathscr{J}_i(S/\delta) := \mathrm{F}_i(\Omega_{S/\delta})$$

is called $i$th Jacobian ideal of $S/\delta$. In case $\delta$ is the trivial derivation of $R$, we write $\mathscr{J}_i(S/R)$ for $\mathscr{J}_i(S/\delta)$.

Clearly, we have

**Proposition 2.2.10.** *Under the assumption of the definition above*

(i) $\mathscr{J}_0(S/\delta) \subseteq \mathscr{J}_1(S/\delta) \subseteq \mathscr{J}_2(S/\delta) \subseteq \cdots$.

(ii) $\mathscr{J}_i(S/\delta) = S$ *for* $i \geqslant \mu(\Omega_{S/\delta})$.

**Definition 2.2.11.** If $S$ is a finitely generated $K$-algebra where $K$ is a field, then the *Jacobian ideal* $\mathcal{J}_{R/K}$ is defined to be the first nonzero Fitting ideal of $\Omega_{S/K}$, i.e., $\mathcal{J}_{R/K} = \mathscr{J}_r(R/K)$ if $\mathscr{J}_0(R/K) = \cdots = \mathscr{J}_{r-1}(R/K) = (0)$ and $\mathscr{J}_r(R/K) \neq (0)$.

**Proposition 2.2.12.** *Let $R$ be an $A$-algebra that is essentially of finite type over $A$ where $A$ is noetherian and universally catenary. Suppose that we have a presentation $R = W^{-1}(T/I)$ where $T = A[x_1, \ldots, x_n]$, $I \subseteq T$ an ideal and $W$ a multiplicatively closed subset of $T$ disjoint from $I$. For a prime ideal $\mathfrak{q} \in \mathrm{Spec}(R)$, let $Q$ be the preimage of $\mathfrak{q}$ in $T$ (then, $\mathfrak{q} \cap W = \varnothing$) and $q := Q \cap A$. We let $J_{R/A}$ be the Jacobian matrix. Then*

(i) $(\dagger): \mathrm{rank}_{\mathfrak{q}}(J_{R/A}) \leqslant \mathrm{ht}_Q(qT + I) - \mathrm{ht}(q)$. *R is smooth over $A$ at $\mathfrak{q}$ if and only if $A_q \to R_{\mathfrak{q}}$ is flat and equality holds in $(\dagger)$. In this case, we also have*

$$\mathrm{ht}_Q(qT + I) - \mathrm{ht}(q) = \mathrm{rank}_{\mathfrak{q}}(J_{R/A}) = \mathrm{ht}_Q(I) = \mu_Q(I).$$

(ii) *Assume, in addition, that $R$ is reduced, $A$ is regular and $I \cap A = (0)$. Then $\mathrm{rank}_{\mathfrak{q}}(J_{R/A}) \leqslant \mathrm{ht}_Q(I)$. If we assume furthermore that $R$ is generically smooth over $A$, then $\mathcal{J}_{R/A} = \mathscr{J}_{n-\mathrm{big\,ht}(I)}(R/A)$.*

*where notation is from Definition 2.1.1.*

*Proof.* Let $R' = T/I$ and $\mathfrak{q}'$ be the preimage of $\mathfrak{q}$ in $R'$. Then $R'_{\mathfrak{q}'} \cong R_{\mathfrak{q}}$, and any statement about $R_{\mathfrak{q}}$ can be proved using the affine $A$-algebra $R'$ and the prime ideal $\mathfrak{q}'$. Hence, we may replace $R$ by $R'$ and $\mathfrak{q}$ by $\mathfrak{q}'$ without affecting anything.

2.2.12.(i): Since everything here is local, we can work over the field $\kappa_q(A)$. Then [Kun86, 7.14] shows that $\mu_{\mathfrak{q}}(\Omega_{R/A}) \geqslant \dim_{\mathfrak{q}}(\kappa_q(A) \otimes_A R)$. Since $\mathrm{rank}_{\mathfrak{q}}(J_{R/A}) = n - \mu_{\mathfrak{q}}(\Omega_{R/A})$, we have $\mathrm{rank}_{\mathfrak{q}}(J_{R/A}) \leqslant n - \dim_{\mathfrak{q}}(\kappa_q(A) \otimes_A R)$.

Since $\kappa_q(A) \otimes_A R \cong \kappa_A \otimes_A T/I(\kappa_A \otimes_A T)$, we have

$$\begin{aligned}
\dim_{\mathfrak{q}} \kappa_q(A) \otimes_A R &= \dim_Q \kappa_A \otimes_A T/I(\kappa_A \otimes_A T) \\
&= n - \mathrm{ht}_Q\big(I(\kappa_A \otimes_A T)\big) \\
&= n - \mathrm{ht}_Q\big(I(T/qT)\big) \\
&= n - \mathrm{ht}_Q(qT + I) + \mathrm{ht}_Q(qT) \\
&= n - \mathrm{ht}_Q(qT + I) + \mathrm{ht}(q).
\end{aligned}$$

23

So $\operatorname{rank}_{\mathfrak{q}}(J_{R/A}) \leqslant \operatorname{ht}_Q(qT + I) - \operatorname{ht}(q)$.

By [Kun86, 8.1], we know that $R$ is smooth over $A$ at $\mathfrak{q}$ if and only if $A_q \to R_{\mathfrak{q}}$ is flat and $\mu_q(\Omega_{R/A}) \leqslant \dim_{\mathfrak{q}} R_q/qR_q$. If any of these equivalent conditions is satisfied, then $IT_Q$ is generated by a $T_Q$-regular sequence of length $\dim(T_Q/qT_Q) - \dim(R_{\mathfrak{q}}/qR_{\mathfrak{q}})$.

We note that $\dim T_Q/qT_Q = \dim T_Q - \operatorname{ht}(qT_Q) = \operatorname{ht}(Q) - \operatorname{ht}_Q(qT) = \operatorname{ht}(Q) - \operatorname{ht}(q)$. Since $R_{\mathfrak{q}}/qR_{\mathfrak{q}} \cong T_Q/(I+pT)T_Q$, we have $\dim R_{\mathfrak{q}}/qR_{\mathfrak{q}} = \dim T_Q - \operatorname{ht}((qT+I)T_Q) = \operatorname{ht}(Q) - \operatorname{ht}_Q(qT+I)$. So when $R$ is smooth over $A$ at $\mathfrak{q}$, we know that $\mu_Q(I) = \operatorname{ht}_Q(I) = \operatorname{ht}_Q(qT + I) - \operatorname{ht}(q)$. In this case, since $IT_Q$ is generated by a $T_Q$-regular sequence, we have $\operatorname{rank}_{\mathfrak{q}}(J_{R/A}) = \operatorname{ht}_Q(I)$.

2.2.12.(ii): By 2.2.12.(i), we know that $\operatorname{rank}_{\mathfrak{q}}(J_{R/A}) \leqslant \operatorname{ht}_Q(qT + I) - \operatorname{ht}(q)$.

Since $A$ is regular, so is $T = A[x_1, \ldots, x_n]$ and its localization $T_Q$. By Corollary 2.1.22, we have $\operatorname{ht}((qT + I)T_Q) \leqslant \operatorname{ht}(qT_Q) + \operatorname{ht}(IT_Q)$. Note that $qT + I \subseteq Q$, so $\operatorname{ht}((qT + I)T_Q) = \operatorname{ht}_Q(qT + I)$. Since $qT$ is a prime of $T$ contained in $Q$, we have $\operatorname{ht}(qT_Q) = \operatorname{ht}_Q(qT) = \operatorname{ht}(qT) = \operatorname{ht}(q)$. So

$$\operatorname{ht}_Q(qT + I) \leqslant \operatorname{ht}(q) + \operatorname{ht}_Q(I) \Rightarrow \operatorname{ht}_Q(qT + I) - \operatorname{ht}(q) \leqslant \operatorname{ht}_Q(I).$$

Hence, we have $\operatorname{rank}_{\mathfrak{q}}(J_{R/A}) = \operatorname{ht}_Q(qT + I) - \operatorname{ht}(q) \leqslant \operatorname{ht}_Q(I)$.

To prove the second statement, we need to show that the maximal rank, i.e., $\operatorname{big} \operatorname{ht}(I)$ can be achieved. Let $\mathfrak{q}$ be a minimal prime of $R$ such that $\operatorname{ht}_Q(I) = \operatorname{big} \operatorname{ht}(I) = \operatorname{ht}(Q)$ where $Q$ is the preimage of $\mathfrak{q}$ in $T$. Since $I$ is radical, we have $IT_Q = QT_Q$. Let $q := Q \cap A$. Then

$$qA_q = QT_Q \cap A_q = IT_Q \cap A_q = (I \cap A)A_q = (0).$$

Since $A \to R$ is generically smooth, we know that $A_q \to R_{\mathfrak{q}}$ is smooth. Hence, in this case, $\operatorname{rank}_{\mathfrak{q}}(J_{R/A}) = \operatorname{ht}_Q(I) = \operatorname{big} \operatorname{ht}(I)$. Therefore $\mathscr{J}_{n-\operatorname{big} \operatorname{ht}(I)}(R/A) \neq (0)$.

On the other hand, any $i$th Jacobian ideal with $i < n - \operatorname{big} \operatorname{ht}(I)$ must be zero. If there is some $i_0 < n - \operatorname{big} \operatorname{ht}(I)$ such that $\mathscr{J}_{i_0}(R/A) \neq (0)$, then there is some size $n - i_0$ minor nonzero, call it $\Delta$. Since $R$ is reduced, $\Delta$ is not nilpotent. So there is some minimal prime $\mathfrak{q}'$ of $R$ such that $\Delta \notin \mathfrak{q}'$. Then $\operatorname{rank}_{\mathfrak{q}'}(J_{R/A}) \geqslant n - i_0 > \operatorname{big} \operatorname{ht}(I) \geqslant \operatorname{ht}_{Q'}(I)$ where $Q'$ is the preimage of $\mathfrak{q}'$ in $T$, which violates the first inequality. $\square$

**Corollary 2.2.13.** *Let $R$ be a $K$-algebra that is essentially of finite type over $K$, where $K$ is a perfect field. Suppose that we have a presentation $R = W^{-1}T/I$ where $T = K[x_1, \ldots, x_n]$, $I \subseteq T$ an ideal and $W \subseteq T$ a multiplicatively closed subset. For a prime ideal $\mathfrak{q} \in \operatorname{Spec}(R)$, let $Q$ be the preimage of $\mathfrak{q}$ in $T$. We let $J_{R/K}$ be the Jacobian matrix. Then*

*(i) $\operatorname{rank}_{\mathfrak{q}}(J_{R/K}) \leqslant \operatorname{ht}_Q(I)$ and $R$ is smooth over $K$ if and only if the equality holds.*

*(ii) $R$ is regular at $\mathfrak{q}$ if and only if $\mathscr{J}_{\dim T - \operatorname{ht}_Q(I)}(R/K) \nsubseteq \mathfrak{q}$ where $Q$ is the preimage of $\mathfrak{q}$ in $T$.*

*(iii)* *The Jacobian ideal is* $\mathcal{J}_{R/K} = \mathscr{I}_{\dim T - \operatorname{big ht}(I)}(R/K)$. *If $R$ is equidimensional, then*

$$\operatorname{big ht}(I) = \operatorname{ht}(I) \text{ and } \dim R = \dim T - \operatorname{ht}(I).$$

*So the Jacobian ideal $\mathcal{J}_{R/K} = \mathscr{I}_{\dim R}(R/K)$.*

*(iv)* *Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the set of all minimal primes of $R$ and let $Q_i$ be the preimage of $\mathfrak{q}_i$ in $T$ $(1 \leqslant i \leqslant t)$. Then*

$$\operatorname{Sing}(R) = V\left(\prod_{i=1}^{t}(\mathscr{I}_{\dim T - \operatorname{ht}(Q_i)}(R/K) + \mathfrak{q}_i)\right) = V\left(\bigcap_{i=1}^{t}(\mathscr{I}_{\dim T - \operatorname{ht}(Q_i)}(R/K) + \mathfrak{q}_i)\right).$$

*If $R$ is equidimensional, then the right-hand side of the above equality simplifies to $V(\mathcal{J}_{R/K})$, and we have $\operatorname{Sing}(R) = V(\mathcal{J}_{R/K})$.*

*where notation is from Definition 2.1.1.*

*Proof.* For 2.2.13.(i), let $A = K$ in Proposition 2.2.12. Then $q = 0$ and the equality follows.

For 2.2.13.(ii), $R$ is smooth at $\mathfrak{q}$ if and only if $\operatorname{rank}_{\mathfrak{q}}(J_{R/K}) = \operatorname{ht}_Q(I)$ if and only if there is a size $\operatorname{ht}_Q(I)$ minor of $J_{R/K}$ outside $\mathfrak{q}$ if and only if $\mathscr{I}_{n-\operatorname{ht}_Q(I)}(R/K) \not\subseteq \mathfrak{q}$.

For 2.2.13.(iii), the first statement follows directly from Proposition 2.2.12.(ii) and the fact that any field extension of a perfect field is separable, hence, geometrically regular ([Kun86, 5.18, 7.13]). The second statement follows from the first one.

For 2.2.13.(iv), let $\mathfrak{p} \in \operatorname{Sing}(R)$. By 2.2.13.(iii), we have $\mathscr{I}_{n-\operatorname{ht}_P(I)}(R/K) \subseteq \mathfrak{p}$ where $P$ is the preimage of $\mathfrak{p}$ in $T$. Suppose that $\mathfrak{q}_i$ is the minimal prime of $R$ such that $Q_i$ is contained in $P$ and $\operatorname{ht}(Q_i) = \operatorname{ht}_P(I)$. Then $\mathfrak{p}$ contains both $\mathfrak{q}_i$ and $\mathscr{I}_{n-\operatorname{ht}_P(I)}(R/K) = \mathscr{I}_{n-\operatorname{ht}(Q_i)}(R/K)$, which shows that $\mathfrak{q}_i + \mathscr{I}_{n-\operatorname{ht}(Q_i)}(R/K) \subseteq \mathfrak{p}$. So $\subseteq$ is shown.

On the other hand, suppose that $\mathfrak{p}$ is not in the right-hand side of the equality. If $\mathfrak{p}$ contains a minimal prime $\mathfrak{q}_i$, then $\mathfrak{p}$ cannot contain $\mathscr{I}_{n-\operatorname{ht}(Q_i)}(R/K)$. This is true for any minimal prime that $\mathfrak{p}$ contains. So there is some $\mathfrak{q}_i$ such that $\operatorname{ht}(Q_i) = \operatorname{ht}_P(I)$ and $\mathfrak{q}_i \subseteq \mathfrak{p}$. Then $\mathfrak{p}$ does not contain $\mathscr{I}_{n-\operatorname{ht}_P(I)}(R/K)$, so $R$ is regular at $\mathfrak{p}$, which shows the $\supseteq$ direction.

The last statement about equidimensional rings comes down to the following computation: let $h = \operatorname{big ht}(I) = \operatorname{ht}(I)$, then

$$\bigcap_{i=1}^{t}(\mathscr{I}_{\dim T - \operatorname{ht}(Q_i)}(R/K) + \mathfrak{q}_i) = \bigcap_{i=1}^{t}(\mathscr{I}_{n-h}(R/K) + \mathfrak{q}_i)$$

$$= \left(\bigcap_{i=1}^{t}\mathfrak{q}_i\right) + \mathscr{I}_{n-h}(R/K)$$

$$= (0) + \mathcal{J}_{R/K}.$$

$\square$

**Example 2.2.14.** Let $R = K[x, y, z]/(xz, yz)$ as an example. We have $T = K[X, Y, Z]$, $I = (XZ, YZ)$ with $T \to R$ sending $X \mapsto x, Y \mapsto y, Z \mapsto z$. We know that $\mathrm{ht}(I) = \mathrm{ht}((z)) = 1, \mathrm{big\,ht}(I) = \mathrm{ht}((x, y)) = 2$. The Jacobian matrix $J_{R/K}$ is computed to be

$$\begin{pmatrix} z & 0 & y \\ 0 & z & x \end{pmatrix}.$$

Let $\mathfrak{p}_0 = (x, y)R$ and $P_1 = (z)R$ be the minimal primes of $R$ and let $P_0 = (X, Y)T$, $P_1 = (Z)T$ be their preimages in $T$ respectively. Then we have $\mathscr{J}_{3-2}(R/K) = (xz, yz, z^2)R$ and $\mathscr{J}_{3-1}(R/K) = (x, y, z)R$.

The Jacobian ideal, by Corollary 2.2.13.(ii), is $\mathscr{J}_{3-2}(R/K) = (z^2, xz, yz)$, which does not define the singular locus because $(z) \in V(\mathcal{J}_{R/K})$ but $R_{(z)} \cong K(Z)$ is regular. By Corollary 2.2.13.(iii), we have

$$(\mathfrak{p}_0 + \mathscr{J}_1(R/K)) \cap (\mathfrak{p}_1 + \mathscr{J}_2(R/K)) = (x, y, z^2) \cap (x, y, z) = (x, y, z^2).$$

So the singular locus of $R$ is $V((x, y, z^2)) = V((x, y, z)) = \{(x, y, z)\}$.

*Remark* 2.2.15. In fact, when $R$ defined in Corollary 2.2.13 is not equidimensional, we always have a proper containment

$$\mathrm{Sing}(R) \subsetneqq V(\mathcal{J}_{R/K}).$$

The containment part is easy as for any $\mathfrak{p} \in \mathrm{Sing}(R)$, by Corollary 2.2.13.(iv), we have

$$\begin{aligned} \mathfrak{p} &\supseteq \bigcap_{i=1}^{t} (\mathscr{J}_{\dim T - \mathrm{ht}(P_i)}(R/K) + \mathfrak{p}_i) \\ &\supseteq \bigcap_{i=1}^{t} (\mathscr{J}_{\dim T - \mathrm{big\,ht}(I)}(R/K) + \mathfrak{p}_i) = \bigcap_{i=1}^{t} (\mathcal{J}_{R/K} + \mathfrak{p}_i) \\ &\supseteq \mathcal{J}_{R/K}. \end{aligned}$$

On the other hand, there is some minimal prime $\mathfrak{q}$ of $R$ such that $\mathrm{ht}(Q) < \mathrm{big\,ht}(I)$ where $Q$ is the preimage of $\mathfrak{q}$ in $T$. Then since $R$ is regular at $\mathfrak{q}$, we know that $\mu_{\mathfrak{q}}(\Omega_{R/K}) = \dim T - \mathrm{ht}_Q(I) = n - \mathrm{ht}(Q)$. By [Kun86, 10.6], we have $\mathscr{J}_{n-\mathrm{ht}(Q)-1}(R/K) \subseteq \mathfrak{q}$ and $\mathscr{J}_{n-\mathrm{ht}(Q)}(R/K) \nsubseteq \mathfrak{q}$. Since $\mathrm{ht}(Q) < \mathrm{big\,ht}(I) \Rightarrow \mathrm{ht}(Q) + 1 \leqslant \mathrm{big\,ht}(I)$, we have $\mathcal{J}_{R/K} = \mathscr{J}_{n-\mathrm{big\,ht}(I)}(R/K) \subseteq \mathscr{J}_{n-\mathrm{ht}(Q)-1}(R/K)$. So $\mathfrak{q} \in V(\mathcal{J}_{R/K})$ but $R$ is regular at $\mathfrak{q}$.

We generalize [Kun86, 14.13] to the affine-analytic case.

**Theorem 2.2.16.** *Let $R$ be a reduced affine-analytic $K$-algebra where $K$ is a field. Suppose that we have a presentation $R = T/I$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$. Write $\mathcal{A} = K[\![x_1, \ldots, x_n]\!]$.*

*For a prime ideal $\mathfrak{q} \in \operatorname{Spec}(R)$, let $Q$ be the preimage of $\mathfrak{q}$ in $T$. We let $J_{R/K}$ be the Jacobian matrix from $\widetilde{\Omega}_{R/K}$. Then the following are equivalent:*

   *(i) $R$ is absolutely regular at $\mathfrak{q}$.*

   *(ii) $(\widetilde{\Omega}_{R/K})_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module of rank $n + m - \operatorname{ht}_Q(I)$.*

   *(iii) $\mu_{\mathfrak{q}}(\widetilde{\Omega}_{R/K}) \leqslant n + m - \operatorname{ht}_Q(I)$.*

*where notation is from Definition 2.1.1.*

*Proof.* Let $R' = T/I$, and let $\mathfrak{q}'$ be the preimage of $\mathfrak{q}$ in $R'$. Since $R'_{\mathfrak{q}'} \cong R_{\mathfrak{q}}$. Note that $R$ may not be reduced. But $R$ is still a finitely generated affine algebra over $A$. Since both [Kun86, 13.15, 13.16], which we use in the following proof, work for the setting whenever $R$ is essentially of finite type over $A$, we will replace $R$ and $\mathfrak{q}$ by $R'$ and $\mathfrak{q}'$ respectively and still write them as $R$ and $\mathfrak{q}$.

Let $q = Q \cap \mathcal{A}$. We note that $R/\mathfrak{q} \cong T/Q$. Since $T/Q$ is finitely generated over $\mathcal{A}/q$, we know that $A(\kappa_{\mathfrak{q}}(R))$ is module-finite over $\mathcal{A}/q$. Let $F := \operatorname{Frac}(A(\kappa_{\mathfrak{q}}(R)))$. Then $\operatorname{aTr deg}(F/K) = \dim \mathcal{A}/q = n - \operatorname{ht} q$.

Since $T$ is finitely generated over $\mathcal{A}$, by the dimension formula, we have $\operatorname{ht} Q - \operatorname{ht} q = m - \operatorname{Tr deg}(\kappa_{\mathfrak{q}}(R)/F)$ since $\kappa_{\mathfrak{q}}(R) \cong \operatorname{Frac}(T/Q)$.

So we can write

$$
\begin{aligned}
\dim(R_{\mathfrak{q}}) &+ \operatorname{aTr deg}(F/K) + \operatorname{Tr deg}(\kappa(\mathfrak{q})/F) \\
&= \dim R_{\mathfrak{q}} + (n - \operatorname{ht} q) + (m - \operatorname{ht} Q + \operatorname{ht} q) \\
&= n + m + \dim R_{\mathfrak{q}} - \operatorname{ht} Q \\
&= n + m + (\dim T_Q - \operatorname{ht}_Q I) - \dim T_Q \\
&= n + m - \operatorname{ht}_Q(I).
\end{aligned}
$$

To prove the equivalence, we simply make the following modifications to the proof of [Kun86, 14.13]

- The reference to [Kun86, 14.11] is replaced by references to Proposition 2.2.4.

- The reference to [Kun86, 13.15, 13.16] for computing the rank is replaced by the calculation above and the rank is replaced by $n + m - \operatorname{ht}_Q(I)$.

and the same proof works. $\qquad\square$

**Corollary 2.2.17.** *Using the notation and the assumption of Theorem 2.2.16, we have*

   *(i) $\operatorname{rank}_{\mathfrak{q}}(J_{R/K}) \leqslant \operatorname{ht}_Q(I)$ and $R$ is smooth over $K$ if and only if the equality holds.*

(ii) $R$ *is regular at* $\mathfrak{q}$ *if and only if* $\mathscr{J}_{\dim T - \mathrm{ht}_Q(I)}(R/K) \nsubseteq \mathfrak{q}$ *where* $Q$ *is the preimage of* $\mathfrak{q}$ *in* $T$.

(iii) *The Jacobian ideal* $\mathcal{J}_{R/K} = \mathscr{J}_{\dim T - \mathrm{big\,ht}(I)}(R/K)$. *If* $I$ *has pure height* $h$, *then* $\mathrm{big\,ht}(I) = \mathrm{ht}(I) = h$. *So the Jacobian ideal* $\mathcal{J}_{R/K} = \mathscr{J}_{\dim T - h}(R/K)$.

(iv) *If* $I$ *does not have pure height, then* $\mathrm{AbsSing}(R) \subseteq V(\mathcal{J}_{R/K})$. *If* $I$ *has pure height, then* $\mathrm{AbsSing}(R) = V(\mathcal{J}_{R/K})$. *Here* $\mathrm{AbsSing}$ *is the "absolute singular locus", which coincides with the (usual) singular locus if* $\mathrm{char}(K) = 0$.

*Proof.* The proof is similar to the proof of Corollary 2.2.13 with the reference to Proposition 2.2.12 replaced by Theorem 2.2.16. $\qquad\square$

## 2.3   Test Elements in Characteristic $p$

The purpose of this section is to generalize the following theorem ([HH99, Corollary 1.5.5]) to the complete case. We aim to prove the following theorem.

**Theorem 2.3.1.** *Let* $K$ *be a field of characteristic* $p$ *and let* $R$ *be a* $d$-*dimensional complete* $K$-*algebra that is equidimensional and absolutely reduced over* $K$. *Then the Jacobian ideal* $\mathcal{J}(R/K)$ *is contained in the test ideal of* $R$, *and remains so after localization and completion.*

Following [HH02, Theorem 3.4], we want to state a more general version of [HH99, Corollary 1.5.4].

**Proposition 2.3.2.** *Let* $A$ *be a regular domain of characteristic* $p$. *Let* $R$ *be a module-finite extension of* $A$ *such that it is torsion-free and generically étale over* $A$. *Then every element* $c$ *of* $\mathcal{J}_{R/A}$ *is such that* $cR^{1/q} \subseteq A^{1/q}[R]$ *for all* $q = p^e$, *and, in particular,* $cR^\infty \subseteq A^\infty[R]$. *Thus, if* $c \in \mathcal{J}_{R/A} \cap R^\circ$, *it is a completely stable test element.*

*Proof.* If we replace the reference to the usual Lipman-Sathaye theorem with the reference to the "generalized Lipman-Sathaye Jacobian theorem" ([Hoc02b, Theorem 3.1]), the same proof in [HH99, Corollary 1.5.4] works. $\qquad\square$

We are ready to prove the main theorem, Theorem 2.3.1.

*Proof.* Since $R$ is an analytic $K$-algebra, we assume that $R = K[\![x_1, \ldots, x_n]\!]/(f_1, \ldots, f_r)$ is a presentation of $R$ as homomorphic image of a power series ring. Then the $(n-d) \times (n-d)$ minors of the Jacobian matrix $(\partial f_i / \partial x_j)$ generate the Jacobian ideal $\mathcal{J}(R/K)$.

By Proposition 2.2.4, we may assume without loss of generality that $K$ is infinite and perfect. (Take, for example, the algebraic closure of $K$). By Cohen's structure theorem, we can write

28

$R = K[\![X_1, \ldots, X_n]\!]/I$ and let $x_i$ be the image of $X_i$ in $R$. The calculation of the Jacobian ideal is independent of the choice of coordinates, so we are free to let $\mathrm{GL}_n(K)$ acts on the set of variables.

By the discussion [Kun86, 12.14] we know that the universal finite differential module $\Omega^1_{R/K} = \Omega^1_{K[\![X_1, \ldots, X_n]\!]/K}/(I, \mathrm{d}I)$. In particular, these $\mathrm{d}x_i$ generate the differential module. The total quotient ring $Q(R)$ is a finite product of analytically separable field extensions of $K$ by [Kun86, Theorem 13.10]. By the proof of the same theorem, we can modify the generators $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ to get a sequence of elements $x'_1, \ldots, x'_n$ such that $x'_1, \ldots, x'_d$ form a system of parameters of $R$ and $\mathrm{d}x_1, \ldots, \mathrm{d}x_d$ form a basis for $\Omega^1_{R/K}$.

Then there is a Zariski dense open subset $U$ of $\mathrm{GL}_n(K)$ such that if we act on the set of variables by an element from $U$ and choose any $d$ of the (new) indeterminates, then the two conditions listed below:

1. The set of $d$ elements form a system of parameters for $R$.

2. Let $A$ be the complete regular ring generated over $K$ by these $d$ elements. Then $R$ is generically smooth over $A$.

By a general position argument we see that there is a Zariski open subset of $\mathrm{GL}_n(K)$ such that the first condition is satisfied. The second condition is satisfied since the differential of any linear combination of $x'_1, \ldots, x'_n$ is the same linear combination of their differentials.

Now suppose that a suitable change of coordinates has been made. For any choice of $d$ of these elements, say $x'_1, \ldots, x'_d$, let $A$ be the regular local ring $K[\![x'_1, \ldots, x'_d]\!]$. Then $R$ is module-finite over $A$ by the general position argument, and the Jacobian ideal $\mathcal{J}_{R/A}$ is generated by the $(n-d)$ size minors of the remaining $n-d$ variables. Since $R$ is equidimensional and reduced, it is likewise torsion-free over $A$. It is generically étale because of the general position of the variables. Then Proposition 2.3.2 finishes the proof. $\qquad\square$

**Theorem 2.3.3.** *Let $R$ be a semianalytic $K$-algebra that is the localization of a reduced equiheight affine-analytic $K$-algebra where $K$ is a field of characteristic $p$. Then the Jacobian ideal $\mathcal{J}(R/K)$ is contained in the test ideal of $R$.*

*Proof.* We can write $R = W^{-1}T/I$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$ and $I$ is of equiheight in $T$. Suppose that we have a counterexample: there is some element $u \in R$ and some ideal $J \in R$ such that $u \in J^*$ but $\delta u \notin J$ for some $\delta \in \mathcal{J}(R/K)$. We can choose a maximal ideal $\mathfrak{m}$ of $R$ such that both $u \in J^*$ and $\delta u \notin J$ still hold in $R_{\mathfrak{m}}$. Then we continue to have these two hold in $\widehat{R_{\mathfrak{m}}}$, the $\mathfrak{m}$-adic completion of $R_{\mathfrak{m}}$.

Since $R_{\mathfrak{m}}$ is reduced, equidimensional and excellent, so is $\widehat{R_{\mathfrak{m}}}$. Let $\mathfrak{n}$ be the preimage of $\mathfrak{m}$ in $T$. Then $I\widehat{T_{\mathfrak{n}}}$ is of pure height $\dim \widehat{T_{\mathfrak{n}}} - \dim \widehat{R_{\mathfrak{m}}} = \dim T_{\mathfrak{n}} - \dim R_{\mathfrak{m}}$. Since $\Omega_{R_{\mathfrak{m}}/K} = \Omega_{R_{\mathfrak{m}}/K[R^p]}$ is complete. We have $\Omega_{\widehat{R_{\mathfrak{m}}}/K} = \Omega_{R_{\mathfrak{m}}/K} = (\Omega_{R/K})_{\mathfrak{m}}$. Hence the Jacobian ideal $\mathcal{J}(R_{\mathfrak{m}}/K) =$

$\mathcal{J}(R/K)R_{\mathfrak{m}}$ expands to the Jacobian ideal $\mathcal{J}(\widehat{R_{\mathfrak{m}}}/K)$. So $\delta \in \mathcal{J}(\widehat{R_{\mathfrak{m}}}/K)$. Since we also have $u \in (J\widehat{R_{\mathfrak{m}}})^*$, we conclude that $\delta u \in J\widehat{R_{\mathfrak{m}}}$ by Theorem 2.3.1, which is a contradiction! $\qquad\square$

## 2.4 Definition of Tight Closure in Equal Characteristic 0

There are several ways to define tight closure in equal characteristic 0. We focus here on $K$-tight closure and on small equational tight closure, which is the case when $K = \mathbb{Q}$. The tight closure gets larger and the test ideal gets smaller if the field $K$ gets larger. These are the simplest notions to define and there does not appear to be much motivation to use more complicated notions.

In this section, we briefly introduce the definition of $K$-tight closure in equal characteristic 0. We usually omit the reference to $K$ in the definition. We start with affine $K$-algebras, then we pass to noetherian $K$-algebras. In the case $K = \mathbb{Q}$, this is called the small equational tight closure.

### 2.4.1 Tight closure for affine algebras over fields of characteristic 0

Let $R$ be an affine $K$-algebra where $K$ is a field of characteristic 0. Let $N \subseteq M$ be finitely generated $R$-modules and $u \in M$ an element in the module. We want to "descend" the data over $R$, i.e., the quintuple $(K, R, N, M, u)$, to some finitely generated $\mathbb{Z}$-subalgebra $\mathcal{A}$ of $K$. Roughly speaking, the *descent data* for the quintuple from $R$ to $\mathcal{A}$ is also a quintuple $(\mathcal{A}, R_{\mathcal{A}}, N_{\mathcal{A}}, M_{\mathcal{A}}, u_{\mathcal{A}})$ such that when tensored with $K$ over $\mathcal{A}$, we recover the original quintuple. The formal definition below is taken from [HH99, (2.1.2) Descent data].

**Definition 2.4.1.** Let a quintuple $(K, R, M, N, u)$ be defined as above. By *descent data* for this quintuple, we mean a quintuple $(\mathcal{A}, R_{\mathcal{A}}, N_{\mathcal{A}}, M_{\mathcal{A}}, u_{\mathcal{A}})$ satisfying the following conditions:

(i) $\mathcal{A}$ is a finitely generated $\mathbb{Z}$-subalgebra of $K$.

(ii) $R_{\mathcal{A}}$ is a finitely generated $\mathcal{A}$-subalgebra of $R$ such that the inclusion $R_{\mathcal{A}} \subseteq R$ induces an isomorphism of $R_K$ with $R$. Moreover, $R_{\mathcal{A}}$ is $\mathcal{A}$-free.

(iii) $M_{\mathcal{A}}, N_{\mathcal{A}}$ are finitely generated $\mathcal{A}$-submodules of $M, N$ respectively such that $N_{\mathcal{A}} \subseteq M_{\mathcal{A}}$ and all of the modules $M_{\mathcal{A}}, N_{\mathcal{A}}, M_{\mathcal{A}}/N_{\mathcal{A}}$ are $\mathcal{A}$-free. Moreover, the diagram below

$$
\begin{array}{ccc}
N_{\mathcal{A}} \hookrightarrow M_{\mathcal{A}} & \Rightarrow & N_{\mathcal{A}} \otimes_{\mathcal{A}} K \hookrightarrow M_{\mathcal{A}} \otimes_{\mathcal{A}} K \\
\downarrow \quad\quad \downarrow & & \quad\downarrow{\cong} \quad\quad\quad \downarrow{\cong} \\
N \hookrightarrow M & & N \hookrightarrow M
\end{array}
$$

as $R$-modules.

(iv) The element $u \in M$ is in $M_{\mathcal{A}}$ and $u = u_{\mathcal{A}}$.

The most important fact is that descent data do exist ([HH99, (2.1.3) Discussion: the existence of descent data.]), and in fact there are a lot of them. We actually have $R = \varinjlim_B R_B$ where $B$ runs through all finitely generated $\mathbb{Z}$-subalgebras with $\mathcal{A} \subseteq B \subseteq K$ and $R_B$ is the descent of $R$. Similarly we have $M = \varinjlim_B M_B$ and $N = \varinjlim_B N_B$ ([HH99, Proposition 2.1.9]). Let us give an example to illustrate this definition.

**Example 2.4.2.** Let $R = \mathbb{Q}[x, y, z]/(x^2/2 + y^3/3 + z^5/5)$ and $K = \mathbb{Q}$. Then we can take $\mathcal{A}$ to be $\mathbb{Z}[1/2, 1/3, 1/5]$. Therefore $x^2/2 + y^3/3 + z^5/5$ makes sense in $\mathcal{A}[x, y, z]$ and we can form $R_\mathcal{A} = \mathcal{A}[x, y, z]/(x^2/2 + y^3/3 + z^5/5)$. In fact, take $B$ to be any finitely generated $\mathcal{A}$-algebra such that $\mathcal{A} \subseteq B \subseteq \mathbb{Q}$, then $R_B := R_\mathcal{A} \otimes_\mathcal{A} B$ will be a descent of $R$ to $B$.

We give the definition of tight closure on a finitely generated $\mathbb{Z}$-algebra $\mathcal{A}$ below ([HH99, (2.2.2)]).

*Convention* 2.4.3. For a finitely generated $\mathbb{Z}$-algebra $A$, a property P holds for almost all $\mu \in \operatorname{Max} \operatorname{Spec}(\mathcal{A})$ if there is some open dense subset $U$ of $\operatorname{Max} \operatorname{Spec}(\mathcal{A})$ such that P holds for all $\mu \in U$. Let Q be a class of rings (e.g., all domains). By "for almost all rings in Q that $A$ maps to" we mean that "there is an element $a \in A$ such that for all rings in Q that $A_a$ maps to."

**Definition 2.4.4** (Affine case)**.** Let $\mathcal{A}$ be a finitely generated $\mathbb{Z}$-algebra. Let $M_\mathcal{A}$ be an $\mathcal{A}$-module and $N_\mathcal{A} \subseteq M_\mathcal{A}$ a submodule. We say that $u_\mathcal{A} \in M_\mathcal{A}$ is in $(N_\mathcal{A})_{M_\mathcal{A}}^{*/\mathcal{A}}$ if for almost all (Convention 2.4.3) $\mu \in \operatorname{Max} \operatorname{Spec}(\mathcal{A})$, $u_\kappa \in \langle N_\kappa \rangle_{M_\kappa}^*$ where $\kappa = \mathcal{A}/\mu$.

Based on the affine case, we define the tight closure for a finitely generated $K$-algebra $R$ as follows:

**Definition 2.4.5** (Finitely generated $K$-algebra)**.** Let $R$ be a finitely generated $K$-algebra and let $N \subseteq M$ be $R$-modules. We say that $u \in M$ is in the tight closure of $N_M^{*K}$ if there exists descent data $(\mathcal{A}, R_\mathcal{A}, M_\mathcal{A}, N_\mathcal{A}, u_\mathcal{A})$ for $(K, R, M, N, u)$ such that $u_\mathcal{A} \in (N_\mathcal{A})_{M_\mathcal{A}}^{*/\mathcal{A}}$ over $R_\mathcal{A}$ in the sense of Definition 2.4.4.

### 2.4.2 Test elements in the affine case

We want to generalize both Theorem 2.4.9 and Corollary 2.4.10 in [HH99] to the non-domain case.

**Proposition 2.4.6.** *Let $\mathcal{A}$ be a finitely generated $\mathbb{Z}$-domain with fraction field $\mathcal{F}$, and let $R_\mathcal{A}$ containing $\mathcal{A}$ be a finitely generated $\mathcal{A}$-algebra. Suppose that $R_\mathcal{A}$ is module-finite over a regular ring $\mathcal{T}_\mathcal{A}$ and that $R_\mathcal{F}$ is geometrically reduced. Then the nonzero elements in the Jacobian ideal $\mathcal{J}(R_\mathcal{A}/\mathcal{T}_\mathcal{A})$ are universal test elements for $\mathcal{A} \to R_\mathcal{A}$ in the sense of [HH99, Definition 2.4.2].*

*Proof.* Localize at one element of $\mathcal{A}^\circ$ so that $\mathcal{A}$ is regular, $\mathcal{A} \to \mathcal{T}_\mathcal{A}$ is smooth and also so that $R_\mathcal{A}$ is $\mathcal{A}$-free. For almost all (Convention 2.4.3) fields $\mathcal{L}$ to which $\mathcal{A}$ maps, $R_\mathcal{L}$ is geometrically reduced ([HH99, (2.3.6)b]). If follows that for almost all (Convention 2.4.3) regular domains $\Lambda$, $R_\Lambda$ is geometrically reduced and the extension $\mathcal{T}_\Lambda \subseteq R_\Lambda$ is module-finite. Now the result is immediate from Proposition 2.3.2 $\hfill\square$

**Corollary 2.4.7.** *Let $\mathcal{A}$ be a finitely generated $\mathbb{Z}$-domain with fraction field $\mathcal{F}$, and let $R_\mathcal{A}$ containing $\mathcal{A}$ be a finitely generated $\mathcal{A}$-algebra. Suppose that $R_\mathcal{F}$ is geometrically reduced and $d$-dimensional. Then the nonzero elements in the Jacobian ideal $\mathcal{J}(R_\mathcal{A}/\mathcal{A})$ are universal test elements for $\mathcal{A} \to R_\mathcal{A}$ in the sense of [HH99, Definition 2.4.2].*

**Lemma 2.4.8.** *Let $\mathcal{A}$ be a finitely generated $\mathbb{Z}$-domain and let $R_\mathcal{A}$ containing $\mathcal{A}$ be a finitely generated $\mathcal{A}$-algebra. Suppose that $I_\mathcal{A} \subseteq R_\mathcal{A}$ is an ideal and $u_\mathcal{A} \in R_\mathcal{A}$ is an element. If $u_\kappa \in I_\mathcal{A} R_\kappa$ for almost all (Convention 2.4.3) $\kappa \in \mathrm{Max}\,\mathrm{Spec}(A)$, then $u_\mathcal{A} \in I_\mathcal{A}$.*

*Proof.* Assume for contradiction that $u_\mathcal{A} \notin I_\mathcal{A}$. Consider the $R_\mathcal{A}$-module $(I_\mathcal{A} + u_\mathcal{A})/I_\mathcal{A}$. By assumption it is a finitely generated nonzero module. By [HR74, Lemma (8.1)] we can localize at one element of $\mathcal{A}$ to make it free. Hence its rank can be checked by base change to any $\mathcal{A}_\kappa$. Then we conclude that it has rank 0, i.e., it is a zero module. So we have $u_\mathcal{A} \in I_\mathcal{A}$. $\hfill\square$

By Convention 2.1.4, in order to show that some element is a test element, we will show that it multiplies the tight closure of any ideal in the ring back to the ideal.

**Theorem 2.4.9.** *Let $R$ be a reduced finitely generated equidimensional $K$-algebra of Krull dimension $d$. Then $\mathcal{J}(R/K)$ is contained in the test ideal.*

*Proof.* We write $R = K[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then $(f_1, \ldots, f_r)$ has pure height $n - d$, and the Jacobian ideal $\mathcal{J}(R/K)$ is generated by the size $(n - d)$ minors of the Jacobian matrix. Let $\delta$ be one of the minors.

Let $I$ be an ideal of $R$ and let $u \in I^{*K}$. Let $A$ be the finitely generated $\mathbb{Z}$-subalgebra of $K$ such that all polynomials of $f_1, \ldots, f_r$ and all size $(n - d)$ minors of the Jacobian matrix are defined in $A[x_1, \ldots, x_n]$. Localizing at finitely many nonzero elements (equivalently, one nonzero element) of $A$, we can make $(A, R_A = A[x_1, \ldots, x_n]/(f_1, \ldots, f_r), I_A, u_A)$ a descent of $(K, R, I, u)$. We will write $\delta \in R_A$ by abusing notation. Then clearly $\delta \in \mathcal{J}(R_A/A)$ and remains so after any base change $A \to B$ where $A \subseteq B \subseteq K$.

There are descent data over $A_0$ for $(K, R, I, u)$ such that $u_{A_0} \in I_{A_0}^{*/A_0}$. By [HH99, 2.1.6] we can enlarge $A$ to make $R_A$ contain the descent data over $A_0$. Hence, we have $u_A \in I_A^{*/A}$. Hence for all most all $\mu \in \mathrm{Max}\,\mathrm{Spec}(A)$, by Corollary 2.4.7, we know that the image of $\delta$ in $R_\kappa$ is a test element. So we have, in particular, $\delta u_\kappa \in I_A R_\kappa$. By Lemma 2.4.8, we conclude that $\delta u_A \in I_A$, which implies

that $\delta u \in I$ in $R$. Since $\delta$ and $u$ are arbitrarily chosen, we conclude that $\mathcal{J}(R/K)I^{*K} \subseteq I$ for any ideal $I \subseteq R$. $\hfill\square$

### 2.4.3 Affine progenitors

Let $K$ be a field and let $S$ be a noetherian $K$-algebra. Note that $S$ may not necessarily be finitely generated over $K$. Let $N \subseteq M$ be finitely generated $S$-modules, $\underline{u}$ a finite sequence of elements of $M$. We have the following definition of an affine progenitor ([HH99, Definition 3.1.1]).

**Definition 2.4.10.** By an *affine progenitor* for $(S, M, N, \underline{u})$ we shall means a septuple $\mathcal{M} = (R, M_R, N_R, \underline{u}_R, h, \beta, \eta_R)$ where

- $R$ is a finitely generated $K$-algebra.

- $h : R \to S$ a $K$-homomorphism.

- $M_R$ is a finitely generated $R$-module with an $R$-linear map $\beta : M_R \to M$ such that the induced map $\beta_* : S \otimes_R M_R \to M$ is an isomorphism.

- $\underline{u}_R$ is a finite sequence of elements of $M_R$ such that $\beta_*$ maps $\underline{u}_R$ to $\underline{u}$.

- $\eta_R$ is an $R$-linear map from $N_R$ to $M_R$ and the induced map $N_S \to M_S \to M$ maps $N_S$ onto $N$.

We refer to $R$ as the *base ring* of the affine progenitor.

The data of the affine progenitor is captured by the following diagram.
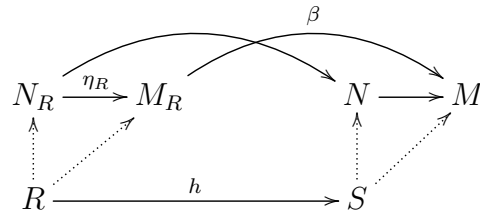


where the dashed arrow $R \dashrightarrow M$ means $M$ is an $R$-module. Note that we do not require $\eta_R$ to be injective, nor do we require that $N_S$ be isomorphic to $N$. Also we do not require $R$ to be a subring of $S$.

### 2.4.4 General noetherian $K$-algebras

We are ready to define a notion of tight closure, called direct tight closure, for a general noetherian $K$-algebra $S$.

**Definition 2.4.11.** Let $S$ be a noetherian $K$-algebra and $N \subseteq M$ be finitely generated $S$-modules. Let $u \in M$. Then we say that $u$ is in the *direct $K$-tight closure $N^{>*K}$* of $N$ in $M$ if there exists an affine progenitor $(R, M_R, N_R, u_R)$ (Definition 2.4.10) for $(S, M, N, u)$ such that $u_R \in \langle N_R \rangle_{M_R}^{*K}$ as in Definition 2.4.5.

We also have a notion of tight closure called "formal $K$-tight closure," defined below.

**Definition 2.4.12.** Let $R$ be a noetherian $K$-algebra. By a complete local domain $B$ (at prime $P$) of $R$ we mean $\widehat{R_P}$ modulo a minimal prime, where $P \subseteq R$ is a prime ideal. We say that $u$ is in the *formal $K$-tight closure $N^{\mathsf{f}*K}$* of $N$ in $M$ if for every complete local domain $B$ of $R$, $1 \otimes u$ is in the direct $K$-tight closure of $\langle B \otimes_R N \rangle$ in $B \otimes_R M$.

We will use the definition of formal $K$-tight closure for the actual definition for all cases. This will not cause any conflicts as we have the following remarkable result.

**Theorem 2.4.13.** *[HH99, Theorem 3.4.1] Let $S$ be a locally excellent noetherian algebra over a field $K$ of characteristic $0$. Let $N \subseteq M$ be finitely generated $S$-modules. Then the following three conditions on an element $u \in M$ are equivalent:*

*(i)* $u \in N_M^{>*K}$.

*(ii) For every maximal ideal $\mathfrak{m}$ of $S$, if $C = \widehat{S_\mathfrak{m}}$ then $u_C \in \langle N_C \rangle_{M_C}^{>*K}$.*

*(iii)* $u \in N_M^{\mathsf{f}*K}$.

*In particular, we have $N_M^{>*K} = N_M^{\mathsf{f}*K}$.*

Consequently, for all three cases we have the following corollary.

**Corollary 2.4.14.** *[HH99, Corollary 3.4.2] Let $R$ be a finitely generated algebra over a field $K$ of characteristic zero. Let $N \subseteq M$ be finitely generated $R$-modules. Then $N_M^{*K} = N_M^{>*K} = N_M^{\mathsf{f}*K}$.*

Finally, we are ready to give the definition of (small) equational tight closure in equal characteristic 0.

**Definition 2.4.15.** [HH99, Definition 3.4.3] Let $R$ be a noetherian $K$-algebra, where $K$ is a field of characteristic zero, and let $N \subseteq M$ be finitely generated $R$-modules.

(i) We define the $K$-tight closure $N^{*K}$ of $N$ in $M$ to be the formal $K$-tight closure of $N$ in $M$.

(ii) Every noetherian ring $R$ of equal characteristic zero is (uniquely) a $\mathbb{Q}$-algebra. When $K = \mathbb{Q}$ we shall refer to the direct $\mathbb{Q}$-tight closure of $N$ in $M$ as the *direct equational tight closure* of $N$ in $M$, and denote it $N_M^{>*\mathsf{eq}}$. we shall refer to the $\mathbb{Q}$-tight closure of $N$ in $M$ as the *equational tight closure* of $N$ in $M$, and denote it $N_M^{*\mathsf{eq}}$.

## 2.5 Test Elements in Characteristic 0

We aim to prove Theorem 2.5.6. Before that, we have to make great use of the Artin-Rotthaus theorem (Theorem 2.5.1) to discuss a series of descent results ((A1) - (A11)).

### 2.5.1 Descent data

We need the following Artin-Rotthaus theorem ([AR88]).

**Theorem 2.5.1** (Artin-Rotthaus). *Let $K$ be a field. Then the power series ring $K[\![x_1,\ldots,x_n]\!]$ is a direct limit of smooth $K[x_1,\ldots,x_n]$-algebras.*

The Artin-Rotthaus theorem is also a consequence of Néron-Popescu desingularization (cf. [Swa98, Theorem 1.1])

**Theorem 2.5.2.** *Let $f : A \to B$ be a ring homomorphism. Then $f$ is flat with geometrically regular fibers if and only if $B$ is a filtered colimit of smooth $A$-algebras.*

We first note that maps in the Artin-Rotthaus theorem are not necessarily injective. Also the Krull dimension of the algebras occurring in the direct limit may be arbitrarily large. So in most cases, we study the height of ideals generated by certain elements, rather than the Krull dimension of the quotient ring.

We will be able to preserve many properties while passing to a larger algebra, i.e., these properties will hold for all algebras occurring after a certain index $\nu$.

Let $R$ be a reduced, equidimensional complete local ring of dimension $d$ with coefficients field $K$, i.e., $R = K[\![x_1,\ldots,x_n]\!]/(f_1,\ldots,f_m)$. We write $A = K[x_1,\ldots,x_n]$. Then $\widehat{A} = K[\![x_1,\ldots,x_n]\!]$, and $f_i \in \widehat{A}$ for each $i$.

(A1) (Eventually injective) Note that for any element $a \in \widehat{A}$, there is some $\nu$ and some $\widetilde{a} \in A_\nu$ mapping to $a$. Note that $A_\nu \to \widehat{A}$ may not be injective. But the kernel is a finitely generated ideal and maps to zero in $\widehat{A}$, hence it must be zero when we map to a large enough algebra $A_\mu$. Moreover, the image of $A_\nu$ in $A_\mu$ maps injectively to $\widehat{A}$, and remains so mapping to any $A_\lambda$ where $\lambda \geqslant \mu$. Therefore, we will denote the image of $\widetilde{a}$ in these $A_\lambda$ by $a$ directly.

(A2) (Descent of elements) We shall write "for all $\mu \gg \nu$" to mean that "there exists some $\lambda > \nu$ and for all $\mu \geqslant \lambda$." By (A1), for any element $a \in \widehat{A}$, there exists some $\nu$ such that $a \in A_\mu$ for all $\mu \gg \nu$. Hence, we can say that there exists some $\mu$ such that $a \in A_\mu$. Of course, for any $A_\lambda$ where $\lambda \geqslant \mu$, we have $a \in A_\lambda$.

(A3) (Notation) We shall frequently use the following notation:

- We write $\mathfrak{m}$ for the maximal ideal of $\widehat{A}$ and let $\mathfrak{m}_\nu$ denote the contraction of $\mathfrak{m}$ in $A_\nu$. Since $A_\nu/\mathfrak{m}_\nu \hookrightarrow \widehat{A}/\mathfrak{m} = K$, and so must be $K$. Hence $\mathfrak{m}_\nu$ is a maximal ideal of $A_\nu$.

- Note that any element $a$ in $A_\nu - \mathfrak{m}_\nu$ maps to an element in $\widehat{A} - \mathfrak{m}$. So its image is necessarily a unit, and we actually have $(A_\nu)_a \to \widehat{A}$. Also, we have such maps for $(A_\nu)_a \to A_\mu$ for all $\mu \gg \nu$. Moreover, we can do this for finitely many such elements by localizing at their product.

- By the bullet point above, the localizations we obtained above are cofinal with the direct system. So we can always assume for each $\mu \gg \nu$, a localization is made, if needed, and we shall indicate this by "for all $\mu \gg_{\text{loc}} \nu$."

(A4) (Descent of ideals) For any ideal $\mathfrak{a} \subseteq \widehat{A}$, if $\mathfrak{a}$ is generated by $a_1, \ldots, a_k$, then there exists a $\nu$ such that all of the $a_i$ are in $A_\nu$. Therefore we have $\mathfrak{a}_\mu := (a_1, \ldots, a_k)A_\mu$ for all $\mu \geqslant \nu$. If in addition, we have an element $u \in \mathfrak{a} \subseteq \widehat{A}$, we can write $u = z_1 a_1 + \cdots + z_k a_k$ for some $z_1, \ldots, z_k \in \widehat{A}$. By choosing a larger $\mu$ we have $u, z_1, \ldots, z_k \in A_\mu$. Then the element $u - \sum_{i=1}^k z_i a_i \in A_\mu$ maps to zero in $\widehat{A}$. By passing to a larger $\mu$ we may assume that this is honestly zero. Therefore for all $\mu \gg \nu$, we have $u \in \mathfrak{a}_\mu$.

(A5) (Description of the maximal ideal) Any element in $\mathfrak{m}_\nu$ is in $(x_1, \ldots, x_n)A_\mu$ for all $\mu \gg \nu$. Let $\widetilde{a} \in \mathfrak{m}_\nu$ be an element. This is trivial if $\widetilde{a}$ maps to zero. If $\widetilde{a}$ maps to a nonzero element $a$ in $\mathfrak{m} = (x_1, \ldots, x_n)\widehat{A}$, by (A4), there exists $\mu$ such that $a \in (x_1, \ldots, x_n)A_\mu$.

(A6) (Descent of radicals) Let $u_1, \ldots, u_n \in \widehat{A}$ be a system of parameters. Then by (A2) and (A5), there exists some $\nu$ such that $u_1, \ldots, u_n \in A_\mu$ and $(u_1, \ldots, u_n)A_\mu \subseteq (x_1, \ldots, x_n)A_\mu$ for all $\mu \geqslant \nu$. Since each $x_i$ has some power in $(u_1, \ldots, u_n)\widehat{A}$, we choose a larger $\mu$ such that each $x_i$ also has a power in $(u_1, \ldots, u_n)A_\mu$. Then this implies that $(u_1, \ldots, u_n)A_\mu$ and $(x_1, \ldots, x_n)A_\mu$ have the same radical.

The next proposition says that we can descend regular sequences in $\widehat{A}$.

**Proposition 2.5.3.** *If elements $u_1, \ldots, u_k$ in $\widehat{A}$ form a regular sequence, then there exists $\nu$ such that they form a regular sequence in $A_\mu$ for all $\mu \gg_{\text{loc}} \nu$.*

*Proof.* We can extend $u_1, \ldots, u_k$ to a full system of parameters $u_1, \ldots, u_n \in \widehat{A}$. If we can show that $u_1, \ldots, u_n$ form a regular sequence in $A_\mu$ for all $\mu \gg_{\text{loc}} \nu$, then the conclusion follows immediately.

By (A6) we can find $\mu$ such that $(u_1, \ldots, u_n)A_\mu \subseteq (x_1, \ldots, x_n)A_\mu$ and they have the same radical. By the construction of $A_\mu$, we know that $x_1, \ldots, x_n$ form a regular sequence on $A_\mu$, which implies that the Koszul homology $\mathrm{H}_i(x_1, \ldots, x_n; A_\mu)$ vanishes for each $i$. Since they have the same radical, we also have $\mathrm{H}_i(u_1, \ldots, u_n; A_\mu)$ vanish. Then $u_1, \ldots, u_n$ form a regular sequence on $(A_\mu)_{\mathfrak{m}_\mu}$.

The modules $\frac{(u_1,\ldots,u_h):_{A_\mu} u_{h+1}}{(u_1,\ldots,u_h)A_\mu}$ are finitely generated and become zero once we localize at $\mathfrak{m}_\mu$. It is clear that we can localize at one element $r$ to make all these modules zero. Thus $u_1,\ldots,u_n$ form a regular sequence on $(A_\mu)_r$. $\qquad\square$

Next we observe that we can descend ideals while preserving their heights, see (A7); if the ideal has pure height, we can preserve that, see (A11). For this purpose, we need the following fact.

**Fact 2.5.4.** *[HH99, Facts 2.3.7] Let $R$ be a noetherian ring and $I$ an ideal of $R$.*

  (i) *If $I$ is proper then $I$ has height at least $h$ if and only if there is a sequence of elements $x_1,\ldots,x_h$ in $I$ such that for all $i$, $0 \leqslant i \leqslant h-1$, $x_{i+1}$ is not in any minimal prime of $(x_1,\ldots,x_i)R$.*

  (ii) *$I$ has height at most $h$ if and only if there exists a proper ideal $J$ containing $I$ and an element $y$ of $R$ not a zerodivisor on $J$ such that $yJ \subseteq \sqrt{I}$ and $yJ$ is contained in the radical of an ideal generated by at most $h$ elements.*

*Proof.* For 2.5.4.(i), since $I$ is not in any minimal prime of $(0)R$, we can choose $x_1 \in I$ avoiding all minimal primes of $(0)R$. Suppose that $x_1,\ldots,x_i$ are chosen. Since $\mathrm{ht}(I) \geqslant h > i$, it is not contained in any minimal primes of $(x_1,\ldots,x_i)R$. Hence, it is not contained in the union of all these minimal primes. So we can choose $x_{i+1} \in I$ avoiding all minimal primes.

For 2.5.4.(ii), If $I \subseteq J$, and $y$ is a nonzerodivisor on $J$, then $y$ avoids all associated primes of $I$ and $J$. So we have $\mathrm{ht}(I) = \mathrm{ht}(IR_y)$ and $\mathrm{ht}(J) = \mathrm{ht}(JR_y)$. Note that in $R_y$, we have $JR_y \subseteq \sqrt{I}R_y \subseteq \sqrt{JR_y}$. Since $JR_y$ is contained in the radical of an ideal generated by at most $h$ elements, it has height at most $h$. So $\mathrm{ht}(I) = \mathrm{ht}(IR_y) = \mathrm{ht}(\sqrt{I}R_y) \leqslant \mathrm{ht}(JR_y) \leqslant h$. Now we assume that $I$ has height at most $h$. Let $P$ be one of the minimal primes of $I$ such that $\mathrm{ht}(P) = \mathrm{ht}(I) \leqslant h$. Then $IR_P$ is $PR_P$-primary. So $PR_P = \sqrt{I}R_P$. We also know that $PR_P$ is the radical of at most $h$ elements (a system of parameters) in $R_P$. By looking at the generators, we can localize at one element $y \in R - P$ such that $PR_y \subseteq \sqrt{I}R_y$ and $PR_y$ is contained in the radical of an ideal generated by at most $h$ elements. Raise $y$ to a power if necessary and let $J = P$, then we have $yJ \subseteq \sqrt{I}$ and $yJ$ is contained in the radical of an ideal generated by at most $h$ elements. $\qquad\square$

(A7) (Preserving height while descending) For any ideal $\mathfrak{a} = (a_1,\ldots,a_n)\widehat{A}$ of height $h$, we have $\mathfrak{a}_\mu$ for all $\mu \gg \nu$. On the one hand, let $x_1,\ldots,x_h$ be a maximal regular sequence in $\mathfrak{a}$. Then we have $(x_1,\ldots,x_h) \subseteq (a_1,\ldots,a_n)$ in $A_\mu$, and $x_1,\ldots,x_h$ continue to be a regular sequence for all $\mu \gg_{\mathrm{loc}} \nu$ by Proposition 2.5.3. So we have $\mathrm{ht}(\mathfrak{a}_\mu) \geqslant h$ for all $\mu \gg_{\mathrm{loc}} \nu$. On the other hand, by Fact 2.5.4, there is some ideal $J$ and an element $y$ in $\widehat{A}$ such that

  • $I \subseteq J$.

  • $yJ \subseteq \sqrt{I}$.

- There is some power $l$ such that $(yJ)^l \subseteq K$ where $K$ is generated by $h$ elements.

- $y$ is a nonzerodivisor on $J$.

All these except the last bullet point can be achieved using (A4). The last one is done by Proposition 2.5.3. Hence for all $\mu \gg_{\mathrm{loc}} \nu$, we also have $\mathrm{ht}(a_\mu) \geqslant h$.

Next we want to prove some results about descending modules and exact sequences by descending their presentations.

(A8) (Descent of finitely generated modules and maps in-between) Let $M$ be a finitely generated $\widehat{A}$-module. Since $\widehat{A}$ is noetherian, we can write $M$ as the cokernel of a matrix $\alpha$. For large enough $\nu$, we have all entries of $\alpha$ in $A_\nu$, so we can form $M_\nu := \mathrm{Coker}(\alpha_\nu)$. For a map $f : M \to N$ between $\widehat{A}$-modules $M, N$, we have lifting of maps

$$
\begin{array}{ccccccc}
F_1 & \twoheadrightarrow & K & \hookrightarrow & F_0 & \twoheadrightarrow & M \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g|_K} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
F_1' & \to & K' & \hookrightarrow & F_0' & \twoheadrightarrow & N
\end{array}
$$

where all $F_1, F_0, F_1', F_0'$ are free $\widehat{A}$-modules. So we have commutative diagram

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\alpha} & F_0 \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\
F_1' & \xrightarrow{\beta} & F_0'
\end{array} \ .
$$

Choose $\nu$ large enough such that all matrices makes sense and the corresponding diagram commutes. Then we get a map from $M_\nu = \mathrm{Coker}(\alpha_\nu)$ to $N_\nu = \mathrm{Coker}(\beta_\nu)$, which recovers $M \to N$ once we tensor with $\widehat{A}$.

(A9) (Descent of finite free resolutions) A finite free resolution of a module $\mathrm{Coker}(\alpha)$ over $\widehat{A}$ descends to a finite free resolution over some $A_\nu$.

*Proof.* We need the Buchsbaum-Eisenbud criterion for acyclicity ([BE73]). Let $F_\bullet = 0 \to F_n \to \cdots \to F_0$ be a complex of finite free $\widehat{A}$-modules and let $\varphi_i : F_i \to F_{i-1}$ be the maps. We can descend each map $\varphi_i$ (as a matrix) to a large enough $A_\nu$. Call the descended complex $F_\bullet'$ and the corresponding maps $\varphi_i'$. Then after passing to some $\mu \geqslant \nu$, we have

- The compositions of consecutive maps are zero.

- The determinantal rank of $\varphi_i'$ is the same as over $\widehat{A}$. So $\mathrm{rank}(\varphi_{i+1}') + \mathrm{rank}(\varphi_i') = \mathrm{rank}(F_i')$.

38

- The ideal generated by the rank size minors at $i$th spot $I_{\mathrm{rank}(\varphi_i')}(\varphi_i')$ is either the whole ring or contains a regular sequence of length $i$.

The first and the second bullet points are achieved by (A2) and the facts that $\varphi_{i+1} \circ \varphi_i = 0, I_{\mathrm{rank}(\varphi_i)+1}(\varphi_i) = 0$ over $\widehat{A}$. For the third bullet point, the proof splits into two cases:

- If $I_{\mathrm{rank}(\varphi_i)}(\varphi_i) = \widehat{A}$, then $I_{\mathrm{rank}(\varphi_i')}(\varphi_i')$ contains some element that maps to a unit in $\widehat{A}$. Hence this element will become invertible after a suitably large index $\nu$.

- If $I_{\mathrm{rank}(\varphi_i)}(\varphi_i)$ contains a regular sequence of length at least $i$ in $\widehat{A}$, by Proposition 2.5.3 we know that this holds in a suitable localization of $A_\mu$ for large enough $\mu$.

So acyclicity follows for $\mu \gg_{\mathrm{loc}} \nu$. $\qquad\square$

(A10) (Descent of short exact sequences) A short exact sequence of finitely generated modules over $\widehat{A}$ descends to a short exact sequence over some $A_\nu$.

*Proof.* We take finite free resolutions of the first and the third modules of the sequence, and fill in maps for the direct sums of the free modules for a given degree to give a free resolution of the middle module in the sequence. Then we can descend the whole resolution and the maps between them by (A9). $\qquad\square$

**Lemma 2.5.5.** *Let $T$ be a Gorenstein local ring. A finitely generated $T$-module $M$ has pure dimension $\dim(T) - h$ if and only if it embeds in a finite direct sum of modules of the form $T/(x_1, \ldots, x_h)T$, where each $x_1, \ldots, x_h$ is a regular sequence in $T$.*

*Proof.* If $M$ embeds into such a direct sum, then $M$ clearly has pure height $h$. Assume that $M$ has pure height $h$, and let $P_1, \ldots, P_n$ be the set of associated primes of $M$. Then each $P_i$ has height $h$ for $1 \leqslant i \leqslant h$. Consider the map $M \to \oplus_{i=1}^h M_{P_i}$. The kernel of this map consists of elements killed by $W = R - \cup_{i=1}^n P_i$. Since $W$ consists of only nonzerodivisors on $M$, the map $M \to \oplus_{i=1}^h M_{P_i}$ is injective. Each $M_{P_i}$ is a module of finite length over $R_{P_i}$. So $M_{P_i}$ embeds into the direct sum of finitely many copies of the injective hull $E_i$ of $R_{P_i}/P_i R_{P_i}$. We can work with the image of $M_{P_i}$ in each copy of the injective hull respectively. So assume without loss of generality that $M_{P_i}$ embeds into one copy of $E_i$. Let $u_1, \ldots, u_h \in P_i$ be part of a system of parameters in $R$ such that they form a system of parameters in $R_{P_i}$. Then $E_i = \varinjlim_s R/(x_1^s, \ldots, x_h^s)R$. Note that $M_{P_i}$ is finitely generated, so it maps to some $R/(x_1^s, \ldots, x_h^s)R$ for some $s$. Since each $R/(x_1^s, \ldots, x_h^s)R$ injects into $E_i$, the map $M_{P_i} \to R/(x_1^s, \ldots, x_h^s)R$ must be injective as well, which proves the claim. $\qquad\square$

(A11) Let $\mathfrak{a} \subseteq \widehat{A}$ be an ideal of pure height $h$, then for $\mu \gg_{\mathrm{loc}} \nu$, the descent ideal $\mathfrak{a}_\mu$ also has pure height $h$.

*Proof.* This is equivalent to $\widehat{A}/\mathfrak{a}$ having pure height $h$. By Lemma 2.5.5, this is equivalent to $\widehat{A}/\mathfrak{a}$ injecting into a finite direct sum of modules obtained by killing a regular sequence in $\widehat{A}$. By Proposition 2.5.3, (A8) and (A10) we can descend the presentation of $\widehat{A}/\mathfrak{a}$ as well as the injection map. Then $\mathfrak{a}_\mu$ will have pure height $h$ as well for all $\mu \gg_{\text{loc}} \nu$. $\qquad\square$

### 2.5.2 The complete local case

We would like to prove the following result which is analogous to [HH99, Corollary 2.4.10].

**Theorem 2.5.6.** *Suppose that $R$ is a reduced, equidimensional, complete local ring of dimension $d$ over $K$. Then the Jacobian ideal $\mathcal{J}(R/K)$ is contained in the test ideal for $K$-tight closure, and, hence, the test ideal for small equational tight closure.*

*Proof.* Suppose that we have a presentation $R = K[\![x_1, \ldots, x_n]\!]/(f_1, \ldots, f_r)$. Then the Jacobian ideal is generated by $(n-d) \times (n-d)$ minors of the Jacobian matrix $(\partial f_i/\partial x_j)$.

Write $A = K[x_1, \ldots, x_n]$ and then $\widehat{A} = K[\![x_1, \ldots, x_n]\!] \twoheadrightarrow R$. Since $R$ is equidimensional, the kernel $(f_1, \ldots, f_r)$ has pure height $h = n - d$. Let $\delta$ be an $h \times h$ minor of $\left(\frac{\partial f_i}{\partial x_j}\right)$. Let $u \in I^*$ where $I \subseteq R$ an ideal.

For each positive integer $N$, we aim to prove that $\delta u \in I + \mathfrak{m}^N$. Fix an $N$. Since each $f_i$ is a power series in $x_i$, we can truncate $f$ at degree $N$, i.e., let $f_i^{\leq N}$ be the sum of terms in $f_i$ of degree at most $N$ and each term in $f_i - f_i^{\leq N}$ is divisible by a $N + 1$ power of $x_i$. So we can write

$$f_i = f_i^{\leq N} + \sum_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} u_{i,\underline{\alpha}}$$

where $\underline{\alpha} \in \mathbb{N}^n$ with $|\underline{\alpha}| = N + 1$, for some $u_{i,\underline{\alpha}} \in \widehat{A}$. Note that each $f_i^{\leq N}$ is in $A$.

By (A2), We can fix an index $\nu_0$ such that for all $\nu \geq \nu_0$, $A_\nu$ contains the generators of $I$, $u$ and all these $u_{i,\underline{\alpha}}$. For each $\nu$, consider a presentation $A[y_1, \ldots, y_s] \twoheadrightarrow A_\nu$. It has a kernel generated by $G_1, \ldots, G_t$, i.e., $A_\nu \cong A[y_1, \ldots, y_s]/(G_1, \ldots, G_t)$. Since all $u_{i,\underline{\alpha}}$ and $x_i$ are in $A_\nu$, we can write $F_i = f_i^{\leq N} + \sum_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} u_{i,\underline{\alpha}}$. Let $R_\nu = A_\nu/(F_1, \ldots, F_r)A_\nu = A[y_1, \ldots, y_s]/(F_1, \ldots, F_r, G_1, \ldots, G_t)$. Since $(F_1, \ldots, F_r)$ is a descent of the ideal $(f_1, \ldots, f_r)$, by (A11), the ideal $(F_1, \ldots, F_r)A_\mu$ also has pure height $h$ for $\mu \gg_{\text{loc}} \nu$. Equivalently, $R_\mu$ is equidimensional for all $\mu \gg_{\text{loc}} \nu$. Now consider the Jacobian matrix of $R_\mu$ over $k$. Let $h + s$ be the height of the ideal $(F_1, \ldots, F_r, G_1, \ldots, G_t)$ in the ring $A_\mu$. Then the Jacobian ideal $\mathcal{J}(R_\mu/K)$ is generated by the $h + s$ minors of the $(r + t) \times (n + s)$ matrix

$$\begin{pmatrix} \left(\frac{\partial F_i}{\partial x_j}\right)_{r \times n} & \left(\frac{\partial F_i}{\partial y_k}\right)_{r \times s} \\ \left(\frac{\partial G_l}{\partial x_j}\right)_{t \times n} & \left(\frac{\partial G_l}{\partial y_k}\right)_{t \times s} \end{pmatrix}. \tag{2.5.1}$$

Since $u \in (IR)^{*K}$, there is some affine progenitor $R'$ such that this holds. We can make $A_\mu$ large enough to contain all the generators of $R'$ over $K$ to get a map $A' \to A_\mu$. Then we also have

40

$u \in (IR_\mu)^*$. By Theorem 2.4.9, the image in $R_\mu$ of the elements in $\mathcal{J}(R_\mu/k)$ multiplies the tight closure back into the ideal itself. We know that $\mathcal{J}(R_\mu/k)u \subseteq IR_\mu$.

Note that in the matrix (2.5.1), the lower-right corner $\left(\frac{\partial G_l}{\partial y_k}\right)_{t \times s}$ is the Jacobian matrix of $A_\mu/A$. Since $A_\mu$ is smooth over $A$, the Jacobian ideal is the unit ideal.

For each $F_i$, we have

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial f_i^{\leqslant N}}{\partial x_j} + \mathfrak{m}^N,$$

$$\frac{\partial F_i}{\partial y_j} = 0 + \sum_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \frac{\partial u_{i,\underline{\alpha}}}{\partial y_k}.$$

So there is some $h \times h$ minor $\widetilde{\delta}$ of $\left(\frac{\partial F_i}{\partial x_j}\right)_{r \times n}$ such that $\widetilde{\delta} - \delta \in \mathfrak{m}^N$. Thinking of the matrix (2.5.1) in the ring $R_\mu/\mathfrak{m}_\mu^N$ where $\mathfrak{m}_\mu$ is the image in $R_\mu$ of the descent of $\mathfrak{m}$ to $A_\mu$ by (A5), we have

$$\begin{pmatrix} \overline{J(R/K)} & 0 \\ * & \mathcal{Q} \end{pmatrix},$$

where $\mathcal{Q}$ is the image of the Jacobian matrix $J(A_\mu/A)$. Hence, the $s \times s$ minors of $\mathcal{Q}$ generate the unit ideal. Since the product of any $h \times h$ minor of $\overline{J(R/K)}$ and $s \times s$ minor of $\mathcal{Q}$ is in $\mathcal{J}(R_\mu/K)$, we have

$$\overline{\delta} \cdot \mathcal{J}(R_\mu/K) \subseteq \mathcal{J}(R_\mu/K)R_\mu/\mathfrak{m}_\mu^N \Rightarrow \widetilde{\delta} \in \mathcal{J}(R_\mu/K)R_\mu/\mathfrak{m}_\mu^N.$$

Therefore, we have $\widetilde{\delta}u \in (I + \mathfrak{m}_\mu^N)R_\mu$, which implies that $\delta u \in I + \mathfrak{m}^N$ in $R$. Since this is true for any $N$, we conclude that $\delta u \in \bigcap_N (I + \mathfrak{m}^N)R = I$. $\qquad \square$

### 2.5.3   The affine-analytic case

We want to prove

**Theorem 2.5.7.** *Suppose that $R$ is a reduced affine-analytic $K$-algebra that is equiheight. Then the Jacobian ideal $\mathcal{J}(R/K)$ is contained in the test ideal for $K$-tight closure, and, hence, the test ideal for small equational tight closure.*

We need to establish results (B1) - (B3) similar to (A1) - (A3). Let us begin with some discussion on the setup. Let $R = T/I$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$ and $I = (f_1, \ldots, f_r)T$. Then by assumption, all minimal primes of $I$ in $T$ are of the same height $h$. Write $\underline{z}$ for the sequence of elements $z_1, \ldots, z_m$ and using the notation from previous section, we write $A = K[x_1, \ldots, x_n]$ and $\widehat{A} = K[\![x_1, \ldots, x_n]\!]$. Then $R = T/I$ where $T = \widehat{A}[\underline{z}]$. By hypothesis, all minimal primes of $I$ in $T$ have the same height and $I$ has no embedded primes. Suppose that $\mathrm{ht}(I) = h$ and $I = (f_1, \ldots, f_r)T$.

Then the complete Jacobian matrix is given by

$$
\begin{pmatrix}
\dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} & \vline & \dfrac{\partial f_1}{\partial z_1} & \cdots & \dfrac{\partial f_1}{\partial z_m} \\
\vdots & \ddots & \vdots & \vline & \vdots & \ddots & \vdots \\
\dfrac{\partial f_r}{\partial x_1} & \cdots & \dfrac{\partial f_r}{\partial x_n} & \vline & \dfrac{\partial f_r}{\partial z_1} & \cdots & \dfrac{\partial f_r}{\partial z_m}
\end{pmatrix}.
$$

We will prove this using the Artin-Rotthaus theorem to approximate $\widehat{A}$ and also $T$. Therefore we will establish (A1)-(A11) as before for $T$. We first establish (A1)-(A5), summarized in the following bullet point.

(B1) Note that any element in $\widehat{A}$ will be in the image of some $A_\nu$ and the map is eventually injective in the sense of (A1), i.e., there exists some $\mu \geqslant \nu$ such that for all $\gamma \geqslant \mu$, the image of $A_\nu$ in $A_\gamma$ maps injectively into $\widehat{A}$. So any polynomial in $T$ will necessarily be in $T_\nu$. In particular, we can define $f_1, \ldots, f_r$ in $T_\nu := A_\nu[\underline{z}]$ and form $R_\nu = T_\nu/(f_1, \ldots, f_r)$.

(B2) The statements in (A4), (A5) and (A6) also hold here: we can descend an ideal by descending its generators. In particular, we can descend the maximal ideal. We can also descend an ideal and its radical. We will write $\mathfrak{m}$ for the maximal ideal of $\widehat{A}$ and $\mathfrak{m}_\nu$ for its contraction back to $A_\nu$. Then one is allowed to localize at any elements in $A_\nu - \mathfrak{m}_\nu$ for any $T_\nu$.

Let us work with a counter-example to Theorem 2.5.7, i.e., there is some ideal $J \subseteq R$ and $u \in R$ such that $u \in J^*$, and some $\delta \in \mathcal{J}(R/K)$ such that $\delta u \notin J$. Let $Q$ be a minimal prime of the proper ideal $J :_R \delta u$. This continues to be a counterexample in $R_Q$. We will make repeated use of the fact that we can localize at finitely many elements (in fact, one by localizing at the product of them) outside $Q$.

(B3) We will localize at finitely many elements in $T_\nu - Q^c T_\nu$ where $Q^c$ is the contraction of $Q$ in $T_\nu$. We will use the notation $\mu \gg_{Q-\mathrm{loc}} \nu$ to indicate this.

We need the following lemma to deal with preserving height while descending.

**Lemma 2.5.8.** *Let $Q \subseteq T$ be fixed and let $I_1, \ldots, I_k$ be finitely many ideals contained in $Q$. Suppose that $J_1, \ldots, J_k$ are ideals in $T_\nu$ such that $J_i T = I_i$, $1 \leqslant i \leqslant k$. Then for all $\mu \gg_{Q-\mathrm{loc}} \nu$, all associated primes of each $J_i T_\mu$ are contained in the contraction of $Q$, and the height of $J_i T_\mu$ is the same as the height of $I_i T_Q$. If $g_1, \ldots, g_h$ form a regular sequence in $QT_Q$, then for all $\mu \gg_{Q-\mathrm{loc}} \nu$, their images in $T_\mu$ also form a regular sequence.*

*Proof.* There are only finitely many ideals $J_1, \ldots, J_k$ and each has finitely many associated primes. For each $J_i$, we can choose an element in all associated primes not contained in $Q^c$ and avoiding associated primes contained in $Q^c$. Then by localizing at these elements (or their product), we assume that all associated primes are contained in $Q^c$.

42

Let $g_1, \ldots, g_h$ be a regular sequence in $QT_Q$. This implies that all associated primes of $(g_1, \ldots, g_i)$ contained in $Q$ have height $i$. For each associated prime $P$ of $(g_1, \ldots, g_i)T_\nu$ in $T_\nu$, since the height of $PT_Q$ is $i$ by Theorem 2.1.23, $P$ has height at least $i$. On the other hand, $P$ cannot have height more than $i$ due to the Krull's height theorem. So $P$ has height $i$. Since $T_\mu$ is Cohen-Macaulay, $g_1, \ldots, g_h$ is a regular sequence.

For preservation of height, we start with prime ideals. Let $P$ be a prime ideal in $T_\nu$. The height of $P$ cannot increase when expand to $T_Q$. Choose a maximal regular sequence in $PT_Q$. For all $\mu \gg_{Q-\mathrm{loc}} \nu$, they will form a regular sequence in $T_\mu$. Hence $\mathrm{ht}(PT_\mu) = \mathrm{ht}(PT_Q)$.

For general ideal $J$, we first choose $T_\mu$ where $\mu \gg_{Q-\mathrm{loc}} \nu$ such that all minimal primes of $JT_\mu$ are contained in $Q^c$. We will also replace $J$ by its radical in $T_\mu$, i.e., the intersection of all minimal primes of $J$. Let $P_1, \ldots, P_l$ be all the minimal primes of $J$. Then

$$P_1 T_Q \cap \cdots \cap P_l T_Q \subseteq JT_Q \subseteq P_i T_Q$$

for each $i$. So we have $\mathrm{ht}(JT_Q) \leqslant \min\{\mathrm{ht}(P_i T_Q)\}$. For any minimal prime of $JT_Q$, it must also contain some $P_i T_Q$. Therefore $\mathrm{ht}(JT_Q) \geqslant \min\{\mathrm{ht}(P_i T_Q)\}$. Hence, $\mathrm{ht}(JT_Q) = \min\{\mathrm{ht}(P_i T_Q)\} = \mathrm{ht}(J)$. $\qquad\square$

*Proof of Theorem 2.5.7.* Write $A = K[x_1, \ldots, x_n]$ and $T = \widehat{A}[z_1, \ldots, z_m]$. Then $T \twoheadrightarrow R$. The kernel $(f_1, \ldots, f_r)$ has pure height $h = n - d$. Let $\delta$ be a $h \times h$ minor of $\left(\frac{\partial f_i}{\partial x_j}\right)$. Let $u \in J^*$ where $J \subseteq R$ an ideal.

We work with a counterexample as before. Let $Q$ be a prime ideal containing $J :_R \delta u$, then $\delta u \notin JR_Q$ and since $\mathfrak{m} \subseteq Q$, we have $\delta u \notin (J + \mathfrak{m}^N)R_Q$ for some $N$. Fix this $N$.

We aim to prove that $\delta u \in I + \mathfrak{m}^N$. Since each $f_i$ is a polynomial in $z$ with coefficients being power series in $x_i$, we can truncate each coefficient of $f$ at degree $N$, i.e., if

$$f_i(z_1, \ldots, z_m) = \sum_{\substack{\text{finitely many } \underline{\beta} \in \mathbb{N}^m}} c_{\underline{\beta}} \underline{z}^{\underline{\beta}}$$

where each $c_{\underline{\beta}}$ is in $\widehat{A}$, then we can form $c_{\underline{\beta}}^{\leqslant N}$ and the difference $c_{\underline{\beta}} - c_{\underline{\beta}}^{\leqslant N}$ is in $\mathfrak{m}^{N+1}T$. So we can write

$$f_i(z_1, \ldots, z_m) = \underbrace{\sum_{\substack{\text{finitely many } \underline{\beta} \in \mathbb{N}^m}} c_{\underline{\beta}}^{\leqslant N} \underline{z}^{\underline{\beta}}}_{f_i^{\leqslant N}} \quad + \quad r \tag{2.5.2}$$

for some $r \in \mathfrak{m}^{N+1}T$. Let $f_i^{\leqslant N}$ be the summation in (2.5.2). The difference $r$ is the polynomial in

$z_1, \ldots, z_m$ with coefficients in $\mathfrak{m}$. So we can write

$$f_i = f_i^{\leqslant N} + \sum_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} u_{i,\underline{\alpha}}(z_1, \ldots, z_m)$$

where $\underline{\alpha} \in \mathbb{N}^n$ with $|\underline{\alpha}| = N + 1$, for some $u_{i,\underline{\alpha}}(z_1, \ldots, z_m) \in T$. Note that each $f_i^{\leqslant N}$ is in $A[z_1, \ldots, z_m]$.

By (B1), We can fix an index $\nu_0$ such that for all $\nu \gg_{Q-\mathrm{loc}} \nu_0$, $T_\nu$ contains the generators of $J$, $u$ and all these $u_{i,\alpha}$. For each $\nu$, consider a presentation $A[y_1, \ldots, y_s] \twoheadrightarrow A_\nu$. It has a kernel generated by $G_1, \ldots, G_t$, i.e., $A_\nu \cong A[y_1, \ldots, y_s]/(G_1, \ldots, G_t)$, and we have $T_\nu = A_\nu[z_1, \ldots, z_m] = A[y_1, \ldots, y_s, z_1, \ldots, z_m]/(G_1, \ldots, G_t)$.

Since all $u_{i,\underline{\alpha}}(z_1, \ldots, z_m)$ and $x_i$ are in $T_\nu$, we can form $F_i = f_i^{\leqslant N} + \sum_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} u_{i,\underline{\alpha}}$. Let $R_\nu = T_\nu/(F_1, \ldots, F_r)T_\nu$. Since $(F_1, \ldots, F_r)$ is a descent of the ideal $(f_1, \ldots, f_r)$, by Lemma 2.5.8, the ideal $(F_1, \ldots, F_r)A_\mu$ also has pure height $h$ for $\mu \gg_{\mathrm{loc}} \nu$. Now consider the Jacobian matrix of $R_\mu$ over $k$. Let $h + s$ be the height of the ideal $(F_1, \ldots, F_r, G_1, \ldots, G_t)$ in the ring $A_\mu$. Then the Jacobian ideals $\mathcal{J}(R_\mu/K)$ are generated by the $h + s$ minors of the $(r + t) \times (n + m + s)$ matrix

$$\begin{pmatrix} \left(\frac{\partial F_i}{\partial x_j}\right)_{r \times n} & \left(\frac{\partial F_i}{\partial z_l}\right)_{r \times m} & \left(\frac{\partial F_i}{\partial y_k}\right)_{r \times s} \\ \left(\frac{\partial G_l}{\partial x_j}\right)_{t \times n} & \left(\frac{\partial G_l}{\partial z_l}\right)_{t \times m} & \left(\frac{\partial G_l}{\partial y_k}\right)_{t \times s} \end{pmatrix}. \tag{2.5.3}$$

Since $u \in (JR)^{*K}$, there is some affine progenitor $R'$ such that $u \in (JR')^{*K}$ holds. We can make $T_\mu$ large enough to contain all the generators of $R'$ over $k$ to get a map $R' \to T_\mu$. Then we also have $u \in (JR_\mu)^*$. By Theorem 2.4.9, the image in $R_\mu$ of the elements in $\mathcal{J}(R_\mu/K)$ multiplies the tight closure of any ideal back into the ideal. We know that $\mathcal{J}(R_\mu/K)u \subseteq JR_\mu$.

Note that in the matrix (2.5.3), the lower-right corner $\left(\frac{\partial G_l}{\partial y_k}\right)_{t \times s}$ is the Jacobian matrix of $A_\mu/A$. Since $A_\mu$ is smooth over $A$, the Jacobian ideal is a unit ideal.

For each $F_i$, we have

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial f_i^{\leqslant N}}{\partial x_j} + \mathfrak{m}^N,$$

$$\frac{\partial F_i}{\partial z_l} = \frac{\partial f_i^{\leqslant N}}{\partial z_l} + \mathfrak{m}^{N+1},$$

$$\frac{\partial F_i}{\partial y_k} = 0 + \sum_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \frac{\partial u_{i,\underline{\alpha}}}{\partial y_k}.$$

So there is some $h \times h$ minor $\widetilde{\delta}$ of $\left(\frac{\partial F_i}{\partial x_j}\right)_{r \times n}$ such that $\widetilde{\delta} - \delta \in \mathfrak{m}^N$. Thinking of the matrix (2.5.3) in the ring $R_\mu/\mathfrak{m}_\mu^N$ where $\mathfrak{m}_\mu$ is the image in $R_\mu$ of the descent of $\mathfrak{m}$ to $A_\mu$ by (B2), we see that the

44

matrix becomes

$$\begin{pmatrix} \overline{J(R/K)} & 0 \\ * & \mathcal{Q} \end{pmatrix},$$

where $\mathcal{Q}$ is the image of the Jacobian matrix $J(A_\mu/A)$. Since the product of any $s \times s$ minor of $\mathcal{Q}$ and any $h \times h$ minor of $\overline{J(R/K)}$ is in $\mathcal{J}(R_\mu/K)$. We have

$$\widetilde{\delta} \cdot \mathcal{J}(A_\mu/A) \subseteq \mathcal{J}(R_\mu/K)R_\mu/\mathfrak{m}_\mu^N \Rightarrow \widetilde{\delta} \in \mathcal{J}(R_\mu/K)R_\mu/\mathfrak{m}_\mu^N.$$

Therefore we have $\widetilde{\delta}u \in (I + \mathfrak{m}_\mu^N)R_\mu$, which implies that $\delta u \in I + \mathfrak{m}^N$ in $R$. So we obtain a contradiction! We conclude that $\delta u \in J$. □

### 2.5.4 The semianalytic case

We want to show:

**Theorem 2.5.9.** *Suppose that we have a flat map with geometrically regular fibers $R \to R'$ where $R$ is a reduced affine-analytic equiheight $K$-algebra. Then the expansion of the Jacobian ideal $\mathcal{J}(R/K)R'$ is contained in the test ideal of $R'$ for $K$-tight closure, and, hence, the test ideal for small equational tight closure.*

Since $R$ is a reduced affine-analytic $K$-algebra, it is approximately Gorenstein. We need to show that $R'$ is also approximately Gorenstein so that Convention 2.1.4 makes sense. Hence, we prove the following proposition.

**Proposition 2.5.10.** *Let $S \to T$ be flat with Gorenstein fibers. If every local ring of $S$ is approximately Gorenstein (this condition holds, for example, if $S$ is excellent and reduced), then $T$ is also approximately Gorenstein.*

*Proof.* Let $\mathfrak{n}$ be a maximal ideal of $T$ and let $\mathfrak{q}$ be its contraction in $S$. Then $S_\mathfrak{q} \to T_\mathfrak{n}$ is local and flat, with Gorenstein fibers. So we reset the notation and assume that $(S, \mathfrak{q}) \to (T, \mathfrak{n})$ is flat local with Gorenstein fibers. By assumption $S$ is approximately Gorenstein. So there exists a sequence of irreducible ideals $I_t$ in $S$ cofinal with powers of $\mathfrak{q}$. Let $x_1, \ldots, x_h$ be a system of parameters in $T/\mathfrak{q}T$. We claim that $I_t T + (x_1^t, \ldots, x_h^t)$ is a sequence of irreducible ideals in $T$ cofinal with powers of $\mathfrak{n}$.

The "cofinal" part is trivial from the construction. For irreduciblity, note that $S/I_t$ is Gorenstein. Hence, $T/I_t T$ is Gorenstein as it is flat local over $S/I_t$ with Gorenstein fibers. By construction, $x_1^t, \ldots, x_h^t$ is a regular sequence on $T/I_t T$. Hence, $T/(I_t T + (x_1^t, \ldots, x_h^t))$ is Gorenstein, which implies that $I_t T + (x_1^t, \ldots, x_h^t)$ is irreducible. □

*Proof of Theorem 2.5.9.* By Proposition 2.5.10 we know that $R'$ is also approximately Gorenstein. By Theorem 2.5.1, $R'$ is the filtered direct limit of smooth $R$-algebras. Since any smooth extension $S$ of $R$ is an affine-analytic $K$-algebra, we only need to show that the $\mathcal{J}(R/K)S \subseteq \mathcal{J}(S/K)$. Then Theorem 2.5.7 will finish the proof.

Since an element is in an ideal if and only if it is so over each connected component, if $\mathrm{Spec}(R)$ has several connected components, we can deal with each component separately. So we assume that $\mathrm{Spec}(R)$ is connected. Let $S$ be a smooth extension of $R$. Again if $\mathrm{Spec}(S)$ has multiple connected components, we can deal with each component separately. So we assume that $\mathrm{Spec}(S)$ is connected as well.

We can find finitely many elements $g_j \in R$, $f_j$ in $S$ such the $f_j g_j$ generate the unit ideal of $S$, the $g_j$ generate the unit ideal of $R$, and each $S_{f_j g_j}$ is a special smooth extension of $R_{g_j}$., i.e., étale over a polynomial extension.

Let us deal with each piece $R_{g_i} \to S_{f_i g_i}$ separately. We write $R_i, S_i$ for $R_{g_i}, S_{f_i g_i}$. Then $S_i$ is standard étale over a polynomial ring $T_i$ over $R_i$. We can write $S_i \cong (T_i[X]/H(X))_G$ where $H(X)$ is monic in $T_i$, and $G$ a multiple of $H'(X)$. If $H(X)$ in any minimal prime of $(T_i[X])_G$, say $\mathfrak{q}$, then $\mathfrak{p} := \mathfrak{q} \cap T_i$ is also a minimal prime and since $\mathfrak{p}(T_i[X])_G$ is a prime. We conclude that $\mathfrak{q} = \mathfrak{p}(T_i[X])_G$. Then $H(X)$ has all its coefficients in $\mathfrak{p}$, which implies that $H'(X) \in \mathfrak{p}(T_i[X])_G = \mathfrak{q}$. But this is a contradiction as $H'(X)$ is a unit in $(T_i[X])_G$.

If we work with a presentation $R = T/I$ of $R$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$, then we can assume that $I = (f_1, \ldots, f_r)$ has pure height $h$ in $T$ and $S = T[y_1, \ldots, y_\ell]/I'$ where $I' = (f_1, \ldots, f_r, g_1, \ldots, g_s)$. Since $\mathrm{Spec}(S)$ is connected, the above argument shows that all minimal primes of $I'$ in $S$ will have the same height, which we denote by $h + t$. Then we can write the Jacobian matrix blockwise as

$$\begin{pmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial f_i}{\partial z_k} & 0 \\ * & * & \frac{\partial g_b}{\partial y_a} \end{pmatrix}.$$

The $s \times s$ minors of the right bottom block $\left(\frac{\partial g_b}{\partial y_a}\right)$ generate $\mathcal{J}(S/R)$, which is the unit ideal. The Jacobian ideal $\mathcal{J}(S/K)$ is generated by $h + s$ minors of this Jacobian matrix. Hence any $h \times h$ minor of the block $\left(\frac{\partial f_i}{\partial x_j} \quad \frac{\partial f_i}{\partial z_k}\right)$, multiplies $\mathcal{J}(S/K)$ into the Jacobian ideal $\mathcal{J}(S/K)$, which implies that $\mathcal{J}(R/K)S \subseteq \mathcal{J}(S/K)$. □

**Corollary 2.5.11.** *If $R$ is a semianalytic $K$-algebra that is the localization of a reduced equiheight affine-analytic $K$-algebra, then the Jacobian ideal $\mathcal{J}(R/K)$ is contained in the test ideal. For any flat $K$-algebra morphism $R \to R'$ with geometrically regular fibers, the expansion of the Jacobian ideal $\mathcal{J}(R/K)R'$ is contained in the test ideal of $R'$. Here, all test ideals are for $K$-tight closure, which are contained in the corresponding test ideals for small equational tight closure.*

*Proof.* Write $R = W^{-1}T/I$ where $T = K[\![x_1, \ldots, x_n]\!][z_1, \ldots, z_m]$, the assumption is that $I$ has pure

height $h$ in $T$. Since $T/I \to R$ is a flat map with geometrically flat fibers, using Theorem 2.5.9, we conclude that the Jacobian ideal expanded $\mathcal{J}((T/I)/K)R$ is contained in the test ideal. Since $R$ is a localization of $T/I$, we have $\mathcal{J}(R/K) = \mathcal{J}((T/I)/K)R$.

The composition of the maps $T/I \to R \to R'$ is still flat. Since $\mathrm{Spec}(R)$ is a subset of $\mathrm{Spec}(T/I)$, the composition map $T/I \to R'$ is of geometrically fibers. Hence by Theorem 2.5.9, the Jacobian ideal expanded $\mathcal{J}((T/I)/K)R'$ is contained in the test ideal of $R'$. Since

$$\mathcal{J}((T/I)/K)R' = \mathcal{J}(R/K)R',$$

the conclusion is proved. □

# CHAPTER III

# WEPF Closure in Mixed-characteristic

This chapter is organized as follows: Section 3.1 collects some preliminaries on basic notations and techniques. In Section 3.2, we prove the $p$-colon-capturing property (Definition 3.2.3) using the perfectoid Abhyankar lemma (Theorem 3.1.11). This is one of our main results (Theorem 3.2.4). In Section 3.3 we introduce our new closure operation wepf, and prove that it is a Dietz closure satisfying the Algebra axiom (Theorem 3.3.4, Theorem 3.3.8). In Section 3.4 we show that module-finite extensions are epf-phantom (Theorem 3.4.1) using techniques different from those in Section 3.2. Finally, in Section 3.5, we study the behaviour of regular sequences on some non-noetherian rings (Theorem 3.5.7) and prove results similar to $p$-colon-capturing (Theorem 3.5.13) in characteristic $p$. Theorem 3.5.7 is needed in the proof.

## 3.1 Preliminaries

A ring of mixed characteristic $p$ is a ring $R$ of characteristic $0$ with $p$ in every maximal ideal of $R$. We will work with a complete local ring of mixed characteristic $p$ in all sections of this chapter except Section 3.5. We will also use the following notation.

*Notation* 3.1.1. Let $R$ be a domain and let $R^+$ be an absolute integral closure of $R$. For any $R$-module $M$, we write $M^+ := R^+ \otimes_R M$. For any submodule $W \subseteq M$, we write $W^+$ for the tensor product $R^+ \otimes_R W$, and $\mathrm{Im}(W^+ \to M^+)$ for the image of the map $R^+ \otimes_R W \to R^+ \otimes_R M$ in $M^+$.

Note that in the literature, the notation $I^+$ means plus closure of $I$, i.e., $IR^+ \cap R$. Since we are using neither the plus closure nor the notation $I^+$ in this chapter, there should be no confusion.

### 3.1.1 Closure operations in mixed characteristic

Let us recall the definition of Heitmann's two closure operations, epf and r1f, below.

**Definition 3.1.2.** Let $R$ be an integral domain of mixed characteristic $p$ and let $I$ be an ideal of $R$. Then an element $x \in R$ is in the (full) extended plus closure of $I$, i.e., $x \in I^{\mathrm{epf}}$, provided there exists $c \in R - \{0\}$ such that for every positive rational number $\varepsilon$ and every positive integer $N$,

$c^\varepsilon x \in (I, p^N)R^+$. The element $x$ is in the (full) rank $1$ closure of $I$, i.e., $x \in I^{\mathsf{r1f}}$, if for every rank one valuation $\nu$ on $R^+$, every positive integer $N$, and every positive rational number $\varepsilon$, there exists $d \in R^+ - \{0\}$ with $\nu(d) < \varepsilon$ such that $dx \in (I, p^N)R^+$.

From the definition above, we immediately see that the epf closure is always contained in the r1f closure, i.e., $I^{\mathsf{epf}} \subseteq I^{\mathsf{r1f}}$ for any ideal $I \subseteq R$. We also note that there is a natural generalization of these definitions to (finitely generated) modules. See also [R.G16, Definition 7.1]. We include the definition below.

**Definition 3.1.3.** Let $R$ be an integral domain of mixed characteristic $p$ and let $W \subseteq M$ be finitely generated $R$-modules. Let $M^+, W^+$ be the notation in Notation 3.1.1, and let $u$ be an element of $M$. Then $u \in M$ is in the epf closure of $W$ if there is some $c \in R - \{0\}$ such that for any $\varepsilon \in \mathbb{Q}^+, N \in \mathbb{N}$ we have

$$c^\varepsilon \otimes u \in \operatorname{Im}(W^+ \to M^+) + p^N M^+.$$

Moreover, $u$ is in the r1f closure of $W$ if for every rank one valuation $\nu$ on $R^+$, every positive integer $N$, and every positive rational number $\varepsilon$, there exists $d \in R^+ - \{0\}$ with $\nu(d) < \varepsilon$ such that

$$d \otimes u \in \operatorname{Im}(W^+ \to M^+) + p^N M^+.$$

### 3.1.2  Closure axioms

Here we present the seven axioms defined by Dietz in [Die10], together with the Algebra axiom defined by R.G. in [R.G18, Axiom 3.1].

**Axiom Set 3.1.4.** Let $(R, \mathfrak{m})$ be a complete local domain possessing a closure operation cl. Let $Q, M$ and $W$ be arbitrary finitely generated $R$-modules with $Q \subseteq M$.

(i) (Extension) $Q_M^{\mathsf{cl}}$ is a submodule of $M$ containing $Q$.
(ii) (Idempotence) $(Q_M^{\mathsf{cl}})_M^{\mathsf{cl}} = Q_M^{\mathsf{cl}}$.
(iii) (Order-preservation) If $Q \subseteq M \subseteq W$, then $Q_W^{\mathsf{cl}} \subseteq M_W^{\mathsf{cl}}$.
(iv) (Functoriality) Let $f : M \to W$ be a homomorphism. Then $f(Q_M^{\mathsf{cl}}) \subseteq (f(Q))_W^{\mathsf{cl}}$.
(v) (Semi-residuality) If $Q_M^{\mathsf{cl}} = Q$, then $0_{M/Q}^{\mathsf{cl}} = 0$.
(vi) (Faithfulness) The maximal ideal $\mathfrak{m}$ and the zero ideal $0$ are closed.
(vii) (Generalized Colon-capturing) Let $x_1, \ldots, x_{k+1}$ be a partial system of parameters for $R$ and let $J = (x_1, \ldots, x_k)$. Suppose that there exists a surjective homomorphism $f : M \to R/J$ such that $f(v) = x_{k+1} + J$. Then

$$(Rv)_M^{\mathsf{cl}} \bigcap \operatorname{Ker} f \subseteq (Jv)_M^{\mathsf{cl}}.$$

(viii) (R.G.'s Algebra Axiom) If $R \overset{\alpha}{\to} M, 1 \mapsto e_1$ is cl-phantom, then the map $\alpha' : R \to \mathrm{sym}^2(M)$ which sends $1 \mapsto e_1 \otimes e_1$ is cl-phantom.

The seventh axiom (the generalized colon-capturing axiom) is also equivalent to the following axiom if the closure operation satisfies the other six Dietz axioms. See [Die10, Lemma 1.3].

**Axiom 3.1.5.** Let $R$ be a complete local domain possessing a closure operation cl. Assume that $\dim R = d$. Let $x_1, \ldots, x_{k+1}$ be a partial system of parameters for $R$ where $0 \leqslant k < d$ and let $J = (x_1, \ldots, x_k)$ ($J = 0$ if $k = 0$). Suppose that there exists a homomorphism $f : M \to R/J$ such that $f(v) = x_{k+1} + J$. Then

$$(Rv)_M^{\mathsf{cl}} \bigcap \mathrm{Ker}\, f \subseteq (Jv)_M^{\mathsf{cl}}.$$

Since both the epf and r1f closures satisfy the first 6 axioms ([R.G16, Section 7]), we have no trouble using this equivalent form.

The axiom (vii) in Axiom Set 3.1.4 and its alternate form Axiom 3.1.5 are rather subtle in comparison with the other axioms. It is not even obvious that tight closure satisfies this condition. Axiom (vii) implies that the closure operation gives closures that are "big enough" without being "too big". In particular, they must be big enough to capture colons while being trivial on regular rings. The notion of "too big" is subtle. Note that integral closure for ideals can be extended to modules such that it satisfies axioms (i) - (vi) and ordinary colon capturing. This closure is "too big" in the sense that it does not satisfy the axiom (vii).

However, generalized colon-capturing is most critical in the proof by Dietz that the existence of a closure operation satisfying axioms (i) - (vii) in Axiom Set 3.1.4 is equivalent to the existence of a balanced big Cohen-Macaulay module. Dietz also proved that the usual notion of colon-capturing follows from it ([Die10, Proposition 1.4]).

### 3.1.3 Phantom extensions

The notion of phantom extensions was first introduced by Hochster and Huneke in [HH94b, Section 5] in order to produce a new proof for the existence of big Cohen-Macaulay modules. In the same paper, they also proved that every module-finite extension of a reduced ring of positive characteristic is a phantom extension ([HH94b, Theorem 5.17]). The generalized notion related to a closure operation was introduced by Dietz ([Die10, Definition 2.2]), which we record below.

An exact sequence $0 \to R \overset{\alpha}{\to} M \to Q \to 0$ determines an element $\epsilon$ in $\mathrm{Ext}_R^1(Q, R)$ via the Yoneda correspondence. If $P_\bullet$ is a projective resolution of $Q$ consisting of finitely generated projective modules $P_i$, then $\epsilon$ is a cocycle element in $\mathrm{Hom}_R(P_1, R)$. We call $\epsilon$ cl-phantom if $\epsilon$ is in $\mathrm{Im}\,(\mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R))_{\mathrm{Hom}_R(P_1, R)}^{\mathsf{cl}}$.

*Remark* 3.1.6. This is different from requiring that $\epsilon$ as an element of $\mathrm{Ext}_R^1(Q,R)$ is in the cl closure of $0$. Because $\mathrm{Ext}_R^1(Q,R)$ is a submodule of $\mathrm{Hom}_R(P_1,R)/\mathrm{Im}\,(\mathrm{Hom}_R(P_0,R) \to \mathrm{Hom}_R(P_1,R))$, the cl-closure of $0$ in the latter one could be potentially larger than the closure in the former one.

We note that this definition is independent of the choice of the resolution of $Q$. For proofs, see [Die10, Discussion 2.3].

### 3.1.4  Almost mathematics

The language of almost mathematics is carefully studied in [GR03]. We will not use the full strength of that. The setup of almost mathematics is given by a ring $A$ together with an $A$-flat ideal $I$ such that $I^2 = I$. The situation where almost mathematics is involved in this thesis usually occurs over an algebra $A$ with an $A$-flat ideal $I = (c^{1/p^\infty})A$, where $(c^{1/p^\infty})A$ means the ideal generated by a compatible system of $p$-power roots of $c$, i.e., $(c^{1/p}, c^{1/p^2}, \dots)A$. This situation can be explained explicitly: let $M$ be an $A$-module. An element $u \in M$ is $I$-almost zero, i.e., $u \stackrel{a}{=} 0$, if and only if $c^{1/p^k}u = 0$ for any $k \in \mathbb{N}$, or, equivalently, $Iu = 0$. An element $u$ is $I$-almost in a submodule $N$ of $M$, i.e., $u \stackrel{a}{\in} N$ if its image in $M/N$ is almost zero. A submodule $N_1$ is $I$-almost in $N_2$, i.e., $N_1 \stackrel{a}{\subseteq} N_2$, if every element in $N_1$ is $I$-almost in $N_2$. Two submodules $N_1, N_2$ of $M$ are $I$-almost equal, i.e., $N_1 \stackrel{a}{=} N_2$, if $N_1 \stackrel{a}{\subseteq} N_2$ and $N_2 \stackrel{a}{\subseteq} N_1$. We will usually focus on ideals rather than submodules.

### 3.1.5  Almost-pro-isomorphisms

Here, we briefly discuss the notion of almost mathematics in the world of pro-objects. See the detailed discussion in [Bha17a, Section 11.3]. We fix a ring $A$ and an $A$-flat ideal $I$ such that $I^2 = I$. Let us consider a simpler setting: all objects are projective systems $\{M_j\}_{j \in J}$ of $A$-modules indexed by the positive integers.

**Definition 3.1.7.** A pro-$A$-module $\{M_j\}_{j \in J}$ of $A$-modules is *almost-pro-zero* if for each $w \in I$ and $j \in J$, there exists some $k \geqslant j$ such that the transition map $M_k \to M_j$ has its image killed by $w$; a map $\{M_j\}_{j \in J} \to \{N_k\}_{k \in K}$ of pro-$A$-modules is called an *almost-pro-isomorphism* if the kernel and cokernel pro-objects are almost-pro-zero.

In particular, we need the following lemma from [Bha17a, Corollary 11.3.5].

**Lemma 3.1.8.** *Let* $\{M_j\}_{j \in J} \to \{N_k\}_{k \in K}$ *be an almost-pro-isomorphism, and let* $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ *be an $R$-linear functor. Then* $\mathrm{R}\varprojlim_j F(M_j) \to \mathrm{R}\varprojlim_k F(N_k)$ *is an almost isomorphism on each cohomology group.*

### 3.1.6 Perfectoid algebras

We will freely use the language of perfectoid spaces ([Sch12]). Throughout this chapter we will always work in the following situation: Let $A$ be a complete unramified regular local ring of mixed characteristic $p$ that has an $F$-finite residue field $k$. By Cohen's structure theorem $A \cong V[\![x_2, \ldots, x_d]\!]$ where $V$ is the coefficient ring of $A$. Let $k^{\mathrm{pc}}$ be the perfect closure of $k$ and $W(k^{\mathrm{pc}})$ be the Witt vectors of $k^{\mathrm{pc}}$. Let $A_0$ be the $p$-adic completion of $A \otimes_V W(k^{\mathrm{pc}})$.

Let $K^\circ$ be the $p$-adic completion of $W(k^{\mathrm{pc}})[p^{1/p^\infty}] := \cup_{i=1}^\infty W(k^{\mathrm{pc}})[p^{1/p^i}]$ and $K = K^\circ[\frac{1}{p}]$. Then $K$ is a perfectoid field with $K^\circ$ its valuation ring. An element $\pi \in K^\circ$ that satisfies $|p| \leqslant |\pi| < 1$ is called a *pseudo-uniformizer*. All theorems we cite in this section work for any choice of pseudo-uniformizer (usually we choose $\pi = p$). Let $A_{\infty,0}$ be the $p$-adic completion of $A_0[p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]$. Then $A_{\infty,0}$ is an integral perfectoid $K^\circ$-algebra, and $A_{\infty,0}[\frac{1}{p}]$ is a perfectoid $K = K^\circ[\frac{1}{p}]$-algebra.

*Remark* 3.1.9. We note that $A_{\infty,0}$ is also referred to as a perfectoid $K^\circ$-algebra (without "integral"). The difference is very technical and will not affect any conclusion in our proofs. Explicitly, a perfectoid $K^\circ$-algebra $\mathcal{A}$ is a $\pi$-adically complete $K^\circ$-algebra flat over $K^\circ$, and the map $\mathcal{A}/\pi^{1/p} \to \mathcal{A}/\pi$ is an isomorphism. Let $\mathcal{A}_*$ be the set of elements in $\mathcal{A}[\frac{1}{\pi}]$ that are $(\pi^{1/p^\infty})$-almost in $\mathcal{A}$, i.e., $\mathcal{A}_* = \{a \in \mathcal{A}[\frac{1}{\pi}] \mid \pi^{1/p^\infty} a \in \mathcal{A}\}$. By definition we have $\mathcal{A} \stackrel{a}{\cong} \mathcal{A}_*$ for any perfectoid $K^\circ$-algebra $\mathcal{A}$. An integral perfectoid $K^\circ$-algebra $\mathcal{A}$ is a perfectoid $K^\circ$-algebra such that $\mathcal{A} \cong \mathcal{A}_*$ (an honest isomorphism). Since we always work in the $\pi^{1/p^\infty}$-almost world, this difference will not affect anything.

We next state a result of André ([And18a, Section 2.5]) in a form we need. See also [Bha17b, Theorem 1.5] or [Bha17a, Theorem 9.4.3].

**Theorem 3.1.10.** *Let $\mathcal{A}^\circ$ be an integral perfectoid $K^\circ$-algebra and let $\pi$ be a pseudo-uniformizer of $K^\circ$. Let $g \in \mathcal{A}^\circ$ be an element. Then there exists a map $\mathcal{A}^\circ \to \mathcal{B}^\circ$ of integral perfectoid $K^\circ$-algebras that is almost faithfully flat modulo $\pi$ such that the element $g$ admits a compatible system of $p$-power roots $g^{1/p^k}$ in $\mathcal{B}^\circ$.*

We need this compatible system of $p$-power roots of $g$ to make use of the following remarkable result of André, which is referred to as the "Perfectoid Abhyankar Lemma" ([And18b, Theorem 0.3.1]). Again, we rephrase it into a form that suits our objectives. Here, for any perfectoid $K$-algebra $\mathcal{A}$, we use $\mathcal{A}^\circ$ to denote its ring of power-bounded elements, i.e., elements whose powers form a bounded subset in $\mathcal{A}$. The ring $\mathcal{A}^\circ$ is a perfectoid $K^\circ$-algebra if $\mathcal{A}$ is a perfectoid $K$-algebra.

**Theorem 3.1.11.** *Let $\mathcal{A}^\circ$ be a perfectoid $K^\circ$-algebra, and $\mathcal{A}$ a perfectoid $K$-algebra. Suppose that $g \in \mathcal{A}^\circ$ is a nonzerodivisor that admits a compatible system of $p$-power roots of $g$. Let $\mathcal{B}'$ be a finite étale $\mathcal{A}[\frac{1}{g}]$-algebra. Then*

*(i)* *There exists a larger perfectoid algebra $\mathcal{B}$ between $\mathcal{A}$ and $\mathcal{B}'$ such that the inclusion $\mathcal{A} \to \mathcal{B}$ is continuous. We have $\mathcal{B}[\frac{1}{g}] = \mathcal{B}'$, and $\mathcal{B}^\circ$ is contained in the integral closure of $\mathcal{A}^\circ$ and this inclusion is a $(pg)^{1/p^\infty}$-almost isomorphism.*

*(ii)* *For any $m \in \mathbb{N}$, $\mathcal{B}^\circ/p^m$ is $(pg)^{1/p^\infty}$-almost finite étale over $\mathcal{A}^\circ/p^m$.*

Typically, one has a complete local domain $R$ module-finite over $A$. One often starts with $R$ and chooses $A$. We can choose $g \in A$ to be a discriminant of $R$ over $A$. Thus $R_g$ is finite étale over $A_g$. We apply Theorem 3.1.10 to $A_{\infty,0}$ and $g \in A$ to obtain an integral perfectoid $K^\circ$-algebra $A_\infty$ that contains a compatible system of $p$-power roots of $g$. Then $R \otimes A_\infty[\frac{1}{g}]$ is finite étale over $A_\infty[\frac{1}{g}]$. The way we use Theorem 3.1.11 is by setting $\mathcal{A}^\circ = A_\infty, \mathcal{A} = A_\infty[\frac{1}{p}], \mathcal{B}' = R \otimes A_\infty[\frac{1}{pg}]$.

## 3.2 $p$-Colon-Capturing

Let $R$ be a $d$-dimensional complete local domain of mixed characteristic $p$. We will define the $p$-colon-capturing property (Definition 3.2.3) and then start to prove that epf satisfies this property (Theorem 3.2.4).

Let us discuss the behavior of the epf closure in $R^+$. For any ideal $I \subseteq R$, the epf closure of $IR^+$ in $R^+$ is the set of elements

$$\left\{ u \in R^+ \mid \exists c \in R - \{0\}, \quad c^\varepsilon u \in IR^+ + p^N R^+, \quad \forall N \in \mathbb{N}, \varepsilon \in \mathbb{Q}^+ \right\}.$$

*Remark* 3.2.1. One can also use $c \in R^+$ instead of $c \in R$, i.e.,

$$(IR^+)^{\mathsf{epf}} = \left\{ u \in R^+ \mid \exists c \in R^+ - \{0\}, \quad c^\varepsilon u \in IR^+ + p^N R^+, \quad \forall N \in \mathbb{N}, \varepsilon \in \mathbb{Q}^+ \right\}.$$

Note that if some element $c \in R^+ \smallsetminus \{0\}$ works, since $c$ is integral over $R$, it has a nonzero multiple $cs \in R$, and $r = cs$ will work as well.

We have an easy observation.

**Lemma 3.2.2.** *Let $R$ be an integral domain of mixed characteristic $p$. Then for any ideal $I \subseteq R$, we have*
$$(IR^+)^{\mathsf{epf}} = \bigcup (IS)^{\mathsf{epf}} \quad \text{for all } S \subseteq R^+ \text{ module-finite over } R.$$

*Proof.* The containment $\supseteq$ is obvious. For the converse direction, suppose $u \in (IR^+)^{\mathsf{epf}}$. Then by definition we have $c^\varepsilon u \in (I, p^N)R^+$ for some $c \in R$. Since $u$ is algebraic over $R$, there is some module-finite extension $S$ of $R$ such that $u \in S$, and then Definition 3.1.2 implies that $u \in (IS)^{\mathsf{epf}}$. $\qquad \square$

We give the definition of our key property, $p$-colon-capturing.

**Definition 3.2.3.** Let $R$ be a $d$-dimensional complete local domain of mixed characteristic $p$. Let $x_1, \ldots, x_n$ be part of a system of parameters of $R$. We say that $x_1, \ldots, x_n$ satisfies $p$-*colon-capturing* if there is some fixed positive integer $N_0$ such that for all integers $N \geqslant N_0$ we have

$$\left(x_1, \ldots, x_{n-1}, p^N\right) :_{R^+} x_n \subseteq \left(\left(x_1, \ldots, x_{n-1}, p^{N-N_0}\right) R^+\right)^{\mathsf{epf}}.$$

The main theorem we aim to prove in this section is stated below.

**Theorem 3.2.4.** *Let $R$ be a complete local domain of mixed characteristic $p$ with an $F$-finite residue field. Then all systems of parameters in $R$ satisfy $p$-colon-capturing.*

In order to prove the main theorem, we need two lemmas.

**Lemma 3.2.5.** *Let $A$ be a regular complete local domain of mixed characteristic $p$. Let $x_1, \ldots, x_d$ be a system of parameters in $A$. Since $A$ is noetherian, we may choose some $N_0$ such that $(x_1, \ldots, x_n)A :_A p^\infty = (x_1, \ldots, x_n)A :_A p^{N_0}$. Let $T$ be a $(pg)^{1/p^\infty}$-almost flat $A$-algebra. Then for all $N \geqslant N_0$ and $1 \leqslant n \leqslant d$, we have*

$$\left(x_1, \ldots, x_{n-1}, p^N\right) T :_T x_n \overset{a}{\subseteq} \left(x_1, \ldots, x_{n-1}, p^{N-N_0}\right) T.$$

*Proof.* Since $T$ is $(pg)^{1/p^\infty}$-almost flat over $A$,

$$(x_1, \ldots, x_n)T :_T p^\infty \overset{a}{=} \left((x_1, \ldots, x_n)A :_A p^\infty\right) T$$
$$= \left((x_1, \ldots, x_n)A :_A p^{N_0}\right) T \overset{a}{=} (x_1, \ldots, x_n)T :_T p^{N_0}.$$

Let $N \geqslant N_0$ be some arbitrary integer and let $u$ be an arbitrary element in $\left(x_1, \ldots, x_{n-1}, p^N\right) T :_T x_n$. We have

$$ux_n - wp^N \in (x_1, \ldots, x_{n-1})T \tag{3.2.1}$$

for some $w \in T$, which implies that $w \in (x_1, \ldots, x_n) T :_T p^N$. Note that since

$$(x_1, \ldots, x_n) T :_T p^N \subseteq (x_1, \ldots, x_n) T :_T p^\infty \overset{a}{=} (x_1, \ldots, x_n) T :_T p^{N_0},$$

we have

$$w \overset{a}{\in} (x_1, \ldots, x_n) T :_T p^{N_0}. \tag{3.2.2}$$

Let $\varepsilon$ be an arbitrary positive rational number. We rewrite (3.2.1) as

$$ux_n - \left(wp^{N_0}\right) p^{N-N_0} \in (x_1, \ldots, x_{n-1})T \tag{3.2.3}$$
$$\Rightarrow (pg)^\varepsilon \, ux_n - \left((pg)^\varepsilon \, wp^{N_0}\right) p^{N-N_0} \in (x_1, \ldots, x_{n-1})T \tag{3.2.4}$$

(3.2.2) shows that for any $\varepsilon$, $(pg)^\varepsilon \, wp^{N_0} \in (x_1, \ldots, x_n) \, T$. So for each $\varepsilon > 0$, there is some $v_\varepsilon \in T$ such that $(pg)^\varepsilon \, wp^{N_0} - v_\varepsilon x_n \in (x_1, \ldots, x_{n-1}) \, T$. Combining this with (3.2.4), we have

$$(pg)^\varepsilon \, ux_n - (v_\varepsilon x_n) p^{N-N_0} \in (x_1, \ldots, x_{n-1}) T$$
$$\Rightarrow (pg)^\varepsilon \, u - v_\varepsilon p^{N-N_0} \in (x_1, \ldots, x_{n-1}) T :_T x_n.$$

Since $T$ is $(pg)^{1/p^\infty}$-almost flat over $A$, we have $\left((x_1, \ldots, x_{n-1}) T :_T x_n\right) \stackrel{a}{\subseteq} (x_1, \ldots, x_{n-1}) T$. So we have

$$\forall \varepsilon, \quad (pg)^\varepsilon \, u \stackrel{a}{\in} \left(x_1, \ldots, x_{n-1}, p^{N-N_0}\right) T.$$

Since this is true for all $\varepsilon$, we conclude that $u \stackrel{a}{\in} (x_1, \ldots, x_{n-1}, p^{N-N_0}) T.$ $\qquad \square$

**Lemma 3.2.6.** *Let $R$ be a complete local domain that is module-finite over a regular complete local domain $A$, where both $R$ and $A$ are of mixed characteristic $p$. Let $y_1, \ldots, y_n$ be part of some system of parameters in $R$. There exists some positive integer $N_0$ such that for any integer $N \geqslant N_0$ and any $R$-algebra $T$ that is $(pg)^{1/p^\infty}$-almost flat over $A$,*

$$\left(y_1, \ldots, y_{n-1}, p^N\right) T :_T y_n \stackrel{a}{\subseteq} \left(y_1, \ldots, y_{n-1}, p^{N-N_0}\right) T.$$

*Proof.* Let $k$ be the number of elements in $\{y_1, \ldots, y_n\}$ that are in $A$. We prove the lemma by induction on $n$ and $n - k$. The base cases $n = k$ for all $1 \leqslant n \leqslant d$ follow from Lemma 3.2.5 with the same $N_0$. Let us choose the same $N_0$ from Lemma 3.2.5. We assume that $n - k > 0$. To simplify the notation, we write $\underline{y}$ for the sequence $y_2, \ldots, y_{n-1}$.

If $y_n \notin A$, then we can choose some $w_n \in (y_1, \underline{y}, y_n) R \cap A$ that is not contained in any minimal prime of $(y_1, \underline{y}) R$ in $R$, and then $y_1, \underline{y}, w_n$ continue to be part of a system of parameters in $R$. There is one more element of $y_1, \underline{y}, w_n$ in $A$ than of $y_1, \underline{y}, y_n$. By the induction hypothesis on $n - k$, there is some $N_0$ such that for any $N \geqslant N_0$ and any $R$-algebra $T$ that is $(pg)^{1/p^\infty}$-almost flat over $A$,

$$\left(y_1, \underline{y}, p^N\right) T :_T w_n \stackrel{a}{\subseteq} \left(y_1, \underline{y}, p^{N-N_0}\right) T. \tag{3.2.5}$$

Note that we have

$$\left(y_1, \underline{y}, p^N\right) T :_T y_n = \left(y_1, \underline{y}, p^N\right) T :_T (y_1, \underline{y}, y_n),$$
$$\left(y_1, \underline{y}, p^N\right) T :_T w_n = \left(y_1, \underline{y}, p^N\right) T :_T (y_1, \underline{y}, w_n),$$

and

$$\left(y_1, \underline{y}, p^N\right) T :_T (y_1, \underline{y}, y_n) \subseteq \left(y_1, \underline{y}, p^N\right) T :_T (y_1, \underline{y}, w_n). \tag{3.2.6}$$

The result now follows from (3.2.5) and (3.2.6).

In the remaining case, we can assume that $y_n \in A$. Without loss of generality, we assume that $y_1 \notin A$. By applying what we have proved above to the sequence $\underline{y}, y_n, y_1$, we know that there is some $N_1$ such that for all $N \geqslant N_1$ and all $R$-algebras $T$ that are $(pg)^{1/p^\infty}$-almost flat over $A$,

$$\left(\underline{y}, y_n, p^N\right) T :_T y_1 \overset{a}{\subseteq} \left(\underline{y}, y_n, p^{N-N_1}\right) T. \tag{3.2.7}$$

Also, by applying the induction hypothesis on $n$ to the shorter sequence $\underline{y}, y_n$, there is some $N_2$ such that for all $N \geqslant N_2$ and all $R$-algebras $T$ that are $(pg)^{1/p^\infty}$-almost flat over $A$,

$$\left(\underline{y}, p^N\right) T :_T y_n \overset{a}{\subseteq} \left(\underline{y}, p^{N-N_2}\right) T. \tag{3.2.8}$$

Let $N \geqslant N_1 + N_2$ be an integer. For any $u \in \left(y_1, \underline{y}, p^N\right) T :_T y_n$, we can write

$$y_n u = u_1 y_1 + \cdots + u_{n-1} y_{n-1} + v p^N \tag{3.2.9}$$

for some $u_1, \ldots, u_{n-1}, v \in T$. Then $u_1 y_1 \in \left(\underline{y}, y_n, p^N\right) T$, which by (3.2.7) implies that

$$u_1 \overset{a}{\in} \left(\underline{y}, y_n, p^{N-N_1}\right) T.$$

For any positive rational number $\varepsilon$, we have $(pg)^\varepsilon u_1 = b_2 y_2 + \cdots + b_n y_n + w p^{N-N_1}$ for some $b_2, \ldots, b_n, w \in T$ which depend on $\varepsilon$. Multiply (3.2.9) by $(pg)^\varepsilon$ and make use of the expression of $(pg)^\varepsilon u_1$. This yields

$$(pg)^\varepsilon y_n u = (pg)^\varepsilon u_1 y_1 + \cdots + (pg)^\varepsilon u_{n-1} y_{n-1} + (pg)^\varepsilon v p^N$$
$$\Rightarrow (pg)^\varepsilon y_n u - b_n y_1 y_n \in \left(\underline{y}, p^{N-N_1}\right) T$$
$$\Rightarrow (pg)^\varepsilon u - b_n y_1 \in \left(\underline{y}, p^{N-N_1}\right) T :_T y_n$$
$$\Rightarrow (pg)^\varepsilon u - b_n y_1 \overset{a}{\in} \left(\underline{y}, p^{N-N_1-N_2}\right) T \quad \text{(by (3.2.8))}$$
$$\Rightarrow (pg)^\varepsilon u \overset{a}{\in} \left(y_1, \underline{y}, p^{N-N_1-N_2}\right) T,$$

and this is true for any positive rational number $\varepsilon$ and any $u \in \left(y_1, \underline{y}, p^N\right) T :_T y_n$. Hence, we can conclude that

$$\left(y_1, \ldots, y_{n-1}, p^N\right) T :_T y_n \overset{a}{\subseteq} \left(y_1, \ldots, y_{n-1}, p^{N-N_1-N_2}\right) T.$$

$\square$

Next we discuss some perfectoid constructions that will be used in the proof of Theorem 3.2.4.

*Construction* 3.2.7. Let $A \to S$, a module-finite map of complete local domains of mixed characteristic $p$, be given where $A$ is regular and unramified with an $F$-finite residue field. We apply the same

construction as in Section 3.1.6 to obtain $A_{\infty,0}$. Since $A \to S$ is generically étale, there is some element $g$ in $A$ such that $(p, g)$ generates a height 2 ideal and $A_{pg} \to S_{pg}$ is finite étale. Let $A_\infty$ be obtained by applying Theorem 3.1.10 to $A_{\infty,0}$ and $g$. Then by Theorem 3.1.11 (where $\mathcal{A} = A_\infty[\frac{1}{p}]$ and $\mathcal{B}' = A_\infty[\frac{1}{pg}] \otimes_A S$), we are able to find an $S$-algebra $\mathcal{B}^\circ$ satisfying the following properties:

- $\mathcal{B}^\circ$ is $(pg)^{1/p^\infty}$-almost flat over $A$.

- There exists a $(pg)^{1/p^\infty}$-almost map from $\mathcal{B}^\circ$ to $S^{pg}$ where $S^{pg}$ is the integral closure of $\widehat{S^+}$ in $\widehat{S^+}[\frac{1}{pg}]$, and $\widehat{S^+}$ is the $p$-adic completion of $S^+$.

For proofs, see [HM21, Lemma 3.8, Lemma 3.9].

A direct consequence of the construction above is the following lemma.

**Lemma 3.2.8.** *With notation as above, let $I \subseteq S$ be an ideal of $S$ and $u \in S$. If $u$ is $(pg)^{1/p^\infty}$-almost in $I\mathcal{B}^\circ$, then $u \in I^{\mathrm{epf}}$.*

*Proof.* Since $\mathcal{B}^\circ$ maps $(pg)^{1/p^\infty}$-almostly to $S^{pg}$, we have $u \overset{a}{\in} IS^{pg}$. By [HM21, Lemma 3.3] we know that $S^{pg}$ is $(pg)^{1/p^\infty}$-almost isomorphic to $\widehat{S^+}$. Hence $u \overset{a}{\in} I\widehat{S^+}$. Then [HM21, Lemma 3.1] finishes the proof. $\qquad\square$

We are ready to prove our main result of this section.

*Proof of Theorem 3.2.4.* Since $R$ is a complete local domain, by Cohen's structure theorem, there is a complete regular local domain $A$ such that $A \to R$ is a module-finite extension. So $A$ has an $F$-finite residue field. We fix this $A$ for the remainder of the proof.

For any $x_1, \ldots, x_n$ that is part of a system of parameters, we want to prove that there is some $N_0$ such that for all $N \geqslant N_0$,

$$\left(y_1, \ldots, y_{n-1}, p^N\right) R^+ :_{R^+} y_n \subseteq \left(\left(y_1, \ldots, y_{n-1}, p^{N-N_0}\right) R^+\right)^{\mathrm{epf}}.$$

We apply Lemma 3.2.6 to the system of parameters $y_1, \ldots, y_n$. We learn that there is some positive integer $N_0$ such that for any $N \geqslant N_0$ and any $R$-algebra $T$ that is $(pg)^{1/p^\infty}$-almost flat over $A$,

$$\left(y_1, \ldots, y_{n-1}, p^N\right) T :_T y_n \overset{a}{\subseteq} \left(y_1, \ldots, y_{n-1}, p^{N-N_0}\right) T. \tag{3.2.10}$$

Let $u$ be an arbitrary element in $\left(y_1, \ldots, y_{n-1}, p^N\right) R^+ :_{R^+} y_n$. Then we have

$$y_n u = y_1 u_1 + \cdots + y_{n-1} u_{n-1} + v p^N \tag{3.2.11}$$

for some $u_1, \ldots, u_{n-1}, v \in R^+$. All elements here are integral over $R$. Hence, there is some module-finite extension $S$ of $R$ such that this relation holds in $S$, i.e., $u \in \left(y_1, \ldots, y_{n-1}, p^N\right) S :_S y_n$.

Applying Construction 3.2.7 to $A \to S$, we obtain an $S$-algebra $\mathcal{B}^\circ$ that is $(pg)^{1/p^\infty}$-almost flat over $A$. $\mathcal{B}^\circ$ is also an $R$-algebra. So we set $T = \mathcal{B}^\circ$ in (3.2.10) and obtain that

$$\left(y_1, \ldots, y_{n-1}, p^N\right) \mathcal{B}^\circ :_{\mathcal{B}^\circ} y_n \overset{a}{\subseteq} \left(y_1, \ldots, y_{n-1}, p^{N-N_0}\right) \mathcal{B}^\circ \tag{3.2.12}$$

for all $N \geqslant N_0$. Since the relation (3.2.11) maps to a relation in $\mathcal{B}^\circ$, we see that $u$ is in the left-hand side of (3.2.12). Hence it is $(pg)^{1/p^\infty}$-almost in the right-hand side of (3.2.12). Then, by Lemma 3.2.8, we know that $u \in \left((x_1, \ldots, x_{n-1}, p^{N-N_0}) S\right)^{\mathsf{epf}}$ for all $N \geqslant N_0$. This completes the proof, by Lemma 3.2.2. $\qquad\square$

### 3.3   Weak epf Closure

In this section, we develop a new closure operation, called "weak epf closure", and denote it by wepf. We prove that it satisfies not only the generalized colon-capturing property (Theorem 3.3.4), but also two stronger colon-capturing properties (Proposition 3.3.6). We also show that wepf is a Dietz closure satisfying the Algebra axiom (Theorem 3.3.8). Let us begin with the definition of wepf.

**Definition 3.3.1.** Let $R$ be a complete local domain of mixed characteristic $p > 0$. Let $I \subseteq R$ be an ideal. Then the *weak* epf *closure* of $I$, denoted by $I^{\mathsf{wepf}}$, is defined to be $I^{\mathsf{wepf}} := \bigcap_{N=1}^\infty (I, p^N)^{\mathsf{epf}}$. Similarly for finitely generated $R$-modules $W \subseteq M$, we define $W^{\mathsf{wepf}} := \bigcap_{N=1}^\infty (W + p^N M)_M^{\mathsf{epf}}$.

*Remark* 3.3.2. It is clear from the definition that $I^{\mathsf{epf}} \subseteq I^{\mathsf{wepf}}$. So the weak epf closure also satisfies the usual colon-capturing. It is not hard to see that $I^{\mathsf{wepf}} \subseteq I^{\mathsf{r1f}}$: Let $u \in I^{\mathsf{wepf}}$. For any rank 1 valuation $\nu$ on $R^+$, any $N \in \mathbb{N}$ and any $\varepsilon \in \mathbb{Q}^+$, since $u \in I^{\mathsf{wepf}} \subseteq (I, p^N)^{\mathsf{epf}}$, there is some $c \in R$ such that $c^\delta u \in (I, p^N) R^+$ for any $\delta \in \mathbb{Q}^+$. We choose $\delta$ small enough such that $\nu(c^\delta) < \varepsilon$. The existence of such a sequence of $c^\delta$ implies that $u \in I^{\mathsf{r1f}}$.

Since the epf closure on complete regular local domains with $F$-finite residue fields is trivial by [HM21, Theorem 3.9], we have

$$I^{\mathsf{wepf}} = \bigcap_{N=1}^\infty \left(I, p^N\right)^{\mathsf{epf}} = \bigcap_{N=1}^\infty \left(I, p^N\right) = I.$$

Therefore we have

**Corollary 3.3.3.** *The* wepf *closure is trivial on complete regular local domains of mixed characteristic with $F$-finite residue fields.*

Next we want to show that the wepf closure satisfies the generalized colon-capturing axiom using $p$-colon-capturing.

**Theorem 3.3.4.** *Let $R$ be a complete local domain of mixed characteristic $p$ with an $F$-finite residue field. Then the* wepf *closure on $R$ satisfies the generalized colon-capturing axiom.*

*Proof.* Let $M$ be an $R$-module and let $x_1, \ldots, x_{n+1}$ be a partial system of parameters of $R$. Let $f : M \to R/I$ be a morphism of $R$-modules where $I = (x_1, \ldots, x_n)R$ and $f(v) = \bar{x}_{n+1}$ in $R/I$. Suppose that $u \in (Rv)_M^{\mathsf{wepf}} \cap \operatorname{Ker} f$. We want to show that $u \in (Iv)_M^{\mathsf{wepf}}$.

If we apply $p$-colon-capturing (Theorem 3.2.4) to the system of parameters $x_1, \ldots, x_{n+1}$, we learn that there is some $N_0$ such that for any $N \geqslant N_0$ we have

$$\left(x_1, \ldots, x_n, p^N\right) R^+ :_{R^+} x_{n+1} \subseteq \left(\left(x_1, \ldots, x_n, p^{N-N_0}\right) R^+\right)^{\mathsf{epf}}. \tag{3.3.1}$$

Since $u \in (Rv)_M^{\mathsf{wepf}}$, we have $u \in \bigcap_{N=1}^{\infty} (Rv + p^N M)_M^{\mathsf{epf}}$, i.e., $u \in (Rv + p^N M)_M^{\mathsf{epf}}$ for all positive integers $N$ or equivalently all $N \geqslant N_0$. Fix $N = N_1 \geqslant N_0$. We know that there is some nonzero element $c \in R$ such that for any $\varepsilon \in \mathbb{Q}^+$, we have $c^\varepsilon \otimes u \in \operatorname{Im}(R^+ \otimes_R v \to M^+) + p^{N_1} M^+$, where $M^+ = R^+ \otimes_R M$ as in Notation 3.1.1. So there is some $a \in R^+$ and $\mu \in M^+$ such that $c^\varepsilon \otimes u = a \otimes v + p^{N_1}\mu$. We apply $1 \otimes_R f$ and note that $u \in \operatorname{Ker} f$. Hence, $ax_{n+1} + p^{N_1}(1 \otimes_R f)(\mu) \in IR^+$, which gives us $a \in (x_1, \ldots, x_n, p^{N_1})R^+ :_{R^+} x_{n+1}$. By (3.3.1), we have $a \in ((x_1, \ldots, x_n, p^{N_1-N_0})R^+)^{\mathsf{epf}}$. Hence, there is some $c'$ (depending on $N_1 - N_0$) such that

$$(c')^\varepsilon a \in (x_1, \ldots, x_n)R^+ + p^{N_1-N_0}R^+.$$

Now everything on the right-hand side of $(c'c)^\varepsilon \otimes u = (c')^\varepsilon a \otimes v + (c')^\varepsilon p^{N_1}\mu$ is in

$$\operatorname{Im}(IR^+ \otimes_R v \to M^+) + p^{N_1-N_0}M^+.$$

We have
$$(c'c)^\varepsilon \otimes u \in \operatorname{Im}(IR^+ \otimes_R v \to M^+) + p^{N_1-N_0}M^+.$$

Therefore, $u \in (Iv, p^{N_1-N_0})_M^{\mathsf{epf}}$ for all $N_1 \geqslant N_0$. We conclude that $u \in (Iv)_M^{\mathsf{wepf}}$. $\qquad\square$

*Remark* 3.3.5. In the proof above, the element $c'$ depends on $N_1 - N_0$. Hence, we can not use the same $c'$ when $N_1$ changes. Therefore, we do not have an obvious way to prove that epf closure satisfies the generalized colon-capturing axiom. So far as we know, this is an open question.

We also prove that the wepf closure satisfies some strong colon-capturing conditions (both versions A and B) defined in [R.G16, Definition 3.9].

**Proposition 3.3.6.** *Let $x_1, \ldots, x_k$ be a partial system of parameters in a complete local ring $R$ of mixed characteristic $p$ with an $F$-finite residue field. Let $t, a$ be two positive integers. Then*

*(i)* $((x_1^t, x_2, \ldots, x_k)R)^{\mathsf{wepf}} :_R x_1^a \subseteq ((x_1^{t-a}, x_2, \ldots, x_k)R)^{\mathsf{wepf}}$ *for all $a < t$;*

*(ii)* $((x_1, \ldots, x_k)R)^{\mathsf{wepf}} :_R x_{k+1} \subseteq ((x_1, \ldots, x_k)R)^{\mathsf{wepf}}$.

*Proof.* 3.3.6.(i): For the first containment, consider an element $u \in ((x_1^t, x_2, \ldots, x_k)R)^{\mathsf{wepf}} :_R x_1^a$. Then

$$ux_1^a \in ((x_1^t, x_2, \ldots, x_k)R)^{\mathsf{wepf}}.$$

For any $N$ there is some $c_N \in R$ such that for any $\varepsilon$ we have $c_N^\varepsilon u x_1^a \in (x_1^t, x_2, \ldots, x_k, p^N) R^+$. So there is some $v \in R^+$ such that $c_N^\varepsilon x_1^a u - x_1^t v \in (x_2, \ldots, x_k, p^N) R^+$. Then we have

$$c_N^\varepsilon u - x_1^{t-a} v \in (x_2, \ldots, x_k, p^N) R^+ :_{R^+} x_1^a.$$

By Theorem 3.2.4 there is some $N_0$ such that for any $N \geqslant N_0$ we have

$$c_N^\varepsilon u - x_1^{t-a} v \in \left( (x_2, \ldots, x_k, p^{N-N_0}) R^+ \right)^{\mathsf{epf}}.$$

So there is another element $d_{N-N_0} \in R^+$ such that for any positive rational number $\delta$, we have

$$d_{N-N_0}^\delta (c_N^\varepsilon u - x_1^{t-a} v) \in \left( x_2, \ldots, x_k, p^{N-N_0} \right) R^+$$
$$\Rightarrow d_{N-N_0}^\delta c_N^\varepsilon u \in \left( x_1^{t-a}, x_2, \ldots, x_k, p^{N-N_0} \right) R^+.$$

We can choose $\delta = \varepsilon$. Hence, we conclude that $u \in ((x_1^{t-a}, x_2, \ldots, x_k, p^{N-N_0})R)^{\mathsf{epf}}$. Since this is true for all $N \geqslant N_0$, we conclude that $u \in ((x_1^{t-a}, x_2, \ldots, x_k)R)^{\mathsf{wepf}}$.

3.3.6.(ii): For the second containment, the proof is similar.

For any $u \in ((x_1, x_2, \ldots, x_k)R)^{\mathsf{wepf}} :_R x_{k+1}$, we have $ux_{k+1} \in ((x_1, x_2, \ldots, x_k)R)^{\mathsf{wepf}}$. So for any $N$ there is some $c_N$ such that for all $\varepsilon$ we have

$$c_N^\varepsilon x_{k+1} u \in (x_1, x_2, \ldots, x_k, p^N) R^+.$$

So we have $c_N^\varepsilon u \in (x_1, x_2, \ldots, x_k, p^N) R^+ :_{R^+} x_{k+1}$. Again, by Theorem 3.2.4, we have some $N_0$ such that for all $N \geqslant N_0$, $c_N^\varepsilon u \in ((x_1, x_2, \ldots, x_k, p^{N-N_0}) R^+)^{\mathsf{epf}}$.

So there is another element $d_{N-N_0}$ such that for any positive rational $\delta$, we have

$$d_{N-N_0}^\delta c_N^\varepsilon u \in \left( x_1, x_2, \ldots, x_k, p^{N-N_0} \right) R^+.$$

For the same reason as in the end of the proof of (1), we conclude that $u \in ((x_1, x_2, \ldots, x_k)R)^{\mathsf{wepf}}$.
$\square$

*Remark* 3.3.7. We point out that the same arguments in both Theorem 3.3.4 and Proposition 3.3.6 work for the r1f closure. Interested readers can work out the details of the proof.

Next, we prove that wepf satisfies all of Dietz's and R.G.'s axioms (Axiom Set 3.1.4). This gives a new proof of the existence of big Cohen-Macaulay algebras. The results in this section are not used in Section 3.4 and Section 3.5.

In [R.G16, Proposition 7.2] R.G. proved that the usual epf closure satisfies the first six axioms. Next we prove

**Theorem 3.3.8.** *The* wepf *closure satisfies all axioms in Axiom Set 3.1.4.*

*Proof.* (i) Since $(Q + p^N M)_M^{\mathsf{epf}}$ is a submodule containing $Q$ for each positive integer $N$, we conclude that the intersection, that is the wepf closure of $Q$, is a submodule of $M$ containing $Q$.

(ii) Since the ambient module is always $M$ here, we omit the subscript and write $Q^{\mathsf{wepf}}$ for $Q_M^{\mathsf{wepf}}$. We need to show that $Q^{\mathsf{wepf}}$ is a wepf closed submodule, i.e., $(Q^{\mathsf{wepf}})^{\mathsf{wepf}} = Q^{\mathsf{wepf}}$. Let $u \in (Q^{\mathsf{wepf}})^{\mathsf{wepf}}$. Then, by definition, for each $N$ there is some $c_N \in R$ such that for any $\varepsilon \in \mathbb{Q}^+$ we have

$$c_N^\varepsilon \otimes u \in \mathrm{Im}((Q^{\mathsf{wepf}})^+ \to M^+) + p^N M^+.$$

So there exist some elements $r_1, \ldots, r_n \in R^+$, $q_1, \ldots, q_n \in Q^{\mathsf{wepf}}$, and $v \in M^+$ such that

$$c_N^\varepsilon \otimes u = \sum_{i=1}^n r_i \otimes q_i + p^N v.$$

Look at one $q_i$. For each positive integer $N_i$, there is some $c_{i,N_i} \in R^+$ such that for any $\varepsilon_i \in \mathbb{Q}^+$ we have $c_{i,N_i}^{\varepsilon_i} q_i \in \mathrm{Im}(Q^+ \to M^+) + p^{N_i} M^+$. Choose $N_i$ to be $N$, and we have

$$(\prod_{i=1}^n c_{i,N}^{\varepsilon_i}) c_N^\varepsilon \otimes u \in \mathrm{Im}(Q^+ \to M^+) + p^N M^+.$$

Choose $\varepsilon_i$ to be $\varepsilon$, this implies that $u \in (Q + p^N M)^{\mathsf{epf}}$. Since this is true for any $N$, we conclude that $u \in Q^{\mathsf{wepf}}$.

(iii) This is true for epf closure, and hence we have $(Q + p^N W)_W^{\mathsf{epf}} \subseteq (M + p^N W)_W^{\mathsf{epf}}$ for all positive integer $N$. Hence $\bigcap_N (Q + p^N W)_W^{\mathsf{epf}} \subseteq \bigcap_N (M + p^N W)_W^{\mathsf{epf}}$, i.e., $Q_W^{\mathsf{wepf}} \subseteq M_W^{\mathsf{wepf}}$.

(iv) Note that

$$f(Q_M^{\mathsf{wepf}}) = f\left(\bigcap_N (Q + p^N M)_M^{\mathsf{epf}}\right) \subseteq \bigcap_N f\left((Q + p^N M)_M^{\mathsf{epf}}\right).$$

For each term we have

$$f\left((Q + p^N M)_M^{\mathsf{epf}}\right) \subseteq \left((f(Q) + p^N f(M))_W^{\mathsf{epf}}\right) \subseteq \left((f(Q) + p^N W)_W^{\mathsf{epf}}\right).$$

We conclude that

$$f(Q_M^{\text{wepf}}) \subseteq \bigcap_N \left( (f(Q) + p^N W)_W^{\text{epf}} \right) = (f(Q))_W^{\text{wepf}} \,.$$

(v) Again, we omit the subscript as the ambient module is always $M$. Assume that $Q$ is wepf-closed. Let $\overline{u}$ be an element in $0_{M/Q}^{\text{wepf}}$. We want to show that $\overline{u} = 0$. For each $N$ we have $c_N^\varepsilon \overline{u} \in p^N (M/Q)^+$. Since we have $(M/Q)^+ \simeq M^+/\text{Im}(Q^+ \to M^+)$, we conclude that $c_N^\varepsilon u \in \text{Im}(Q^+ \to M^+) + p^N M^+$ for any $u$ that is a preimage of $\overline{u}$ in $M$. Therefore, $u \in Q_M^{\text{wepf}} = Q$. Hence $\overline{u} = 0$ in $M/Q$.

(vi) Note that $R$ is of mixed characteristic $p$. So $p \in \mathfrak{m}$ and hence $\mathfrak{m} + p^N R = \mathfrak{m} \Rightarrow \mathfrak{m}^{\text{wepf}} = \mathfrak{m}^{\text{epf}} = \mathfrak{m}$. We prove that $0^{\text{wepf}} = 0$ by citing known results. The same argument in the last part of the proof of [R.G16, Proposition 7.2] works directly for wepf. This argument also works for r1f closure, i.e., $0^{\text{r1f}} = 0$, which implies that $0^{\text{wepf}} = 0$. Dietz pointed out that $0^{\text{cl}} = 0$ follows from the other 5 axioms and generalized colon-capturing in [Die18, Lemma 1.3(e)]. Thus, for the case where the complete local ring has a $F$-finite residue field, we have an alternate proof of $0^{\text{wepf}} = 0$ using the generalized colon-capturing property Theorem 3.3.4.

(vii) See Theorem 3.3.4.

(viii) We also point out that similar arguments to those in [R.G18, Proposition 3.19] work for wepf closure and therefore, wepf also satisfies the Algebra axiom. □

*Remark* 3.3.9. Note that if a closure operation satisfies both the functoriality axiom (axiom (iv)) and the semi-residuality axiom (axiom (v)) in Axiom Set 3.1.4, then the statement in the semi-residuality axiom can be improved to $Q_M^{\text{cl}} = Q$ if and only if $0_{M/Q}^{\text{cl}} = 0$. The "only if" direction is the semi-residuality axiom. The "if" direction comes from the functoriality axiom: consider the map $f : M \to M/Q$, we have $f(Q_M^{\text{cl}}) \subseteq (f(Q))_{M/Q}^{\text{cl}} = 0_{M/Q}^{\text{cl}} = 0$. So $f(Q_M^{\text{cl}}) \subseteq \text{Ker}(f) = Q$.

The following proposition is proved by using standard techniques of reducing the closure problem for submodules to the case of ideals. The proof we include here is basically the same as the proof of [HH90, Proposition 8.7].

**Proposition 3.3.10.** *Let* $(R, \mathfrak{m})$ *be a complete regular local domain of mixed characteristic with* $F$*-finite residue fields. Then every submodule* $W$ *of a finitely generated module* $M$ *is* wepf*-closed.*

*Proof.* We want to show that for any $u \in M$ not in $W$, we have $u \notin W^{\text{wepf}}$. We may replace $W$ by a submodule of $M$ maximal with respect to not containing $u$, and we may replace $M, W$, and $u$ by $M/W, 0$, and $u + W$. Then $u$ is in every nonzero submodule of $M$. We claim that $M$ is now of finite length, i.e., $M$ has only one associated prime $\mathfrak{m}$.

Suppose that $M$ has two associated primes $P, Q$. Let $v_1, v_2 \in M$ be elements such that $\text{Ann}_R(v_1) = P$ and $\text{Ann}_R(v_2) = Q$. Then $Rv_1 \subseteq M$ is isomorphic to $R/P$. So every element

in $Rv_1$ has annihilator $P$. Similarly, every element in $Rv_2$ has annihilator $Q$. Then $P = Q$ as $u$ is in both submodules. Thus $\mathrm{Ass}(M)$ consists of only one prime $P$.

Next we show that $P$ must be the maximal ideal $\mathfrak{m}$. If not, then the image of $(\mathfrak{m}^n + P)/P$ in $R/P \hookrightarrow M$ contains $u$ for every positive integer $n$. But $\bigcap_{n=1}^{\infty}(\mathfrak{m}^n + P) = P$ as $R$ is noetherian. This is impossible, and we have $\mathrm{Ass}(M) = \{\mathfrak{m}\}$.

Since $u$ is in every nonzero submodule of $M$, $u$ spans the socle in $M$, and $M$ is an essential extension of $K = R/\mathfrak{m} \cong Ru$. Since R is regular, there exists an irreducible $\mathfrak{m}$-primary ideal $J \subseteq \mathrm{Ann}_R(M)$. The Artin ring $R/J$ is self-injective, and $M$ is an essential extension of $K$ as an $R/J$-module. It follows that $M$ can be embedded in $R/J$. It will then suffice to show that $0$ is wepf closed in $R/J$, i.e., that $J$ is wepf closed in $R$ by Remark 3.3.9. Then Corollary 3.3.3 finishes the proof. $\qquad\square$

**Corollary 3.3.11.** *Let $R$ be a complete regular local domain of mixed characteristic with $F$-finite residue field. Then every submodule $W$ of a finitely generated module $M$ is* epf*-closed.*

*Proof.* We have that $W^{\mathsf{epf}} \subseteq W^{\mathsf{wepf}} = W$. $\qquad\square$

## 3.4 Phantom Extensions

In [HH94b, Theorem 5.13], Mel Hochster and Craig Huneke proved that in characteristic $p$, all module-finite extensions of complete local rings are phantom in the tight closure sense (this notion is discussed in detail below). We prove a similar result, namely any module-finite extension of a complete local ring of mixed characteristic $p$ with an $F$-finite residue field is epf-phantom (hence wepf- and r1f-phantom).

**Theorem 3.4.1.** *If $R \to S$ is a module-finite extension of complete local domains of mixed characteristic $p$ with an $F$-finite residue field, then this map is* epf*-phantom.*

Let us discuss the definition of phantom extension and introduce some notions. See also [Die10, Discussion 2.4].

Suppose that $(R, \mathfrak{m})$ is a complete local domain. Let $S$ be a module-finite extension of $R$. Then $S/R$ is a finitely generated module over $R$. So it has a minimal $R$ basis $e_1, \ldots, e_{n_0}$. The set of column vectors of the $n_0 \times n_0$ identity matrix form an $R$ basis for $R^{\oplus n_0}$, which we denote by $f_1, \ldots, f_{n_0}$. We can map a free module $R^{\oplus n_0}$ onto $S/R$ by $f_i \mapsto e_i$. This map has a finitely generated kernel. Suppose that it is minimally generated by $n_1$ elements. Then we have a minimal resolution of $S/R$:

$$R^{\oplus n_1} \to R^{\oplus n_0} \to S/R.$$

Suppose that the map $R^{\oplus n_1} \to R^{\oplus n_0}$ is represented by a $n_0 \times n_1$ matrix $\nu$ with all entries in $\mathfrak{m}$. Comparing this resolution with the original exact sequence $0 \to R \to S \to S/R \to 0$, we have the

63

following commutative diagram:

$$0 \longrightarrow R \longrightarrow S \longrightarrow S/R \longrightarrow 0 \; ,$$

$$R^{\oplus n_1} \xrightarrow{\;\nu\;} R^{\oplus n_0} \longrightarrow S/R \longrightarrow 0$$

with $\phi$ on the left vertical arrow.

where $\phi$ is a $1 \times n_1$ matrix with entries in $\mathfrak{m}$. Applying $\operatorname{Hom}_R(-, R)$ to the left square, and using the identification $\operatorname{Hom}_R(R^{\oplus l}, R) \cong R^{\oplus l}$ where $l$ is some finite positive integer, we get

$$R \longleftarrow \operatorname{Hom}_R(S, R) \; .$$

$$R^{\oplus n_1} \xleftarrow{\;\nu^\vee\;} R^{\oplus n_0}$$

with $\phi^\vee$ on the left vertical arrow.

We write $(\nu^\vee) R^{\oplus n_1}$ for the submodule generated by the set of column vectors of $\nu^\vee$. The extension $R \to S$ is epf-phantom if $\phi^\vee \in ((\nu^\vee) R^{\oplus n_1})^{\mathsf{epf}}$ ([Die10, Lemma 2.10]). Note that our diagrams here do not completely match those in [Die10, Discussion 2.4]. However, since whether an element in $\operatorname{Ext}^1_R(S/R, R)$ is phantom or not is independent of the choice of the resolution of $S/R$ ([Die10, Discussion 2.3]), the test given here is also valid.

Next we discuss some perfectoid constructions we need.

*Construction* 3.4.2. Let $K$ be a perfectoid field and let $(T[\frac{1}{t}], T)$ be a perfectoid affinoid $K$-algebra, where $t \in K^\circ$ is some uniformizer that lifts to $K^{\circ\flat}$, i.e., it admits a compatible system of $p$-power roots in $K^\circ$. Suppose that $g \in T$ lifts to $T^\flat$. Let $X$ denote the adic spectrum attached to the pair $(T[\frac{1}{t}], T)$, i.e., $X = \operatorname{Spa}(T[\frac{1}{t}], T)$. Let $U_n$ be the rational subset $X\left(\dfrac{t^n}{g}\right)$. Then $\mathcal{O}_X^+(U_n)$ is $t^{1/p^\infty}$-almost isomorphic to the $t$-adic completion of $T[\left(\dfrac{t^n}{g}\right)^{1/p^\infty}]$. See also [Sch12, Lemma 6.4].

We also need a corollary of the quantitative form of Scholze's Hebbarkeitssatz (the Riemann extension theorem) for perfectoid spaces. First we state the theorem. See [Bha17b, Theorem 4.2], or an alternative description in [Bha17a, Theorem 11.2.1].

**Theorem 3.4.3.** *Let* $(T[\frac{1}{t}], T), g \in T, X, U_n$ *be as above. For each* $m \geqslant 0$, *assume that* $g \in T$ *is a nonzerodivisor in* $T/t^m T$. *Then the projective system of natural maps*

$$\{ f_n : T/g^m \to \mathcal{O}_X^+(U_n)/g^m \}$$

*is an almost-pro-isomorphism. In fact, we have the following more precise pair of assertions:*

*(i) The kernels* $\operatorname{Ker}(f_n)$ *are almost* $0$ *for each* $n \geqslant 0$.

*(ii) For each $k \geqslant 0$ and $c \geqslant p^k m$, the transition map $\mathrm{Coker}(f_{n+c}) \to \mathrm{Coker}(f_n)$ has image almost annihilated by $g^{1/p^k}$.*

What we really need is the following corollary:

**Corollary 3.4.4.** *[Bha17a, Corollary 11.2.2] Let $(T[\frac{1}{t}], T), g \in T, X, U_n$ be as above. Assume that $g \in T$ is a nonzerodivisor in $T/t^m T$ for any $m$. Then for any $T$-complex $Q$, any integer $m \geqslant 0$ and any integer $i$, the natural map*

$$\mathrm{Ext}^i_T(Q, T/t^m T) \to \varprojlim_n \mathrm{Ext}^i_T(Q, \mathcal{O}^+_X(U_n)/t^m \mathcal{O}^+_X(U_n))$$

*has kernel and cokernel annihilated by $(tg)^{1/p^\infty}$.*

The proof uses Theorem 3.4.3 above and Lemma 3.1.8 from Section 3.1.

We are ready to prove the main theorem of this section.

*Proof of Theorem 3.4.1.* Let $\alpha \in \mathrm{Ext}^1_R(S/R, R)$ be the obstruction to split $R \to S$. For any $R$-algebra $T$, we shall write $\alpha_T \in \mathrm{Ext}^1_T((S/R) \otimes_R T, T)$ for the corresponding obstruction to splitting the induced map $T \to T \otimes_R S$.

We know that $R$ is a complete local domain of mixed characteristic $p$ with an $F$-finite residue field. By Cohen's structure theorem, we have a module-finite extension $A \to R$ for some unramified complete regular local ring $A$. Since all maps in $A \to R \to S$ are module-finite extensions, and fraction fields have characteristic 0, there is some $g \in A$ such that $A_{pg} \to R_{pg} \to S_{pg}$ are all finite étale extensions.

Let $A_{\infty,0}$ be constructed as in Section 3.1.6 and let $A_\infty$ be an integral perfectoid algebra containing a system of compatible $p$-power roots of $g$, which exists by Theorem 3.1.10. Since $\widehat{R^+}$ is an integral perfectoid $K^\circ$-algebra admitting a compatible system of $p$-power roots of $g$, there is a map $A_\infty \to \widehat{R^+}$ of perfectoid $K^\circ$-algebras.

Since $A_\infty$ is a perfectoid $K$-algebra, we may apply Construction 3.4.2 to

$$X_A := \mathrm{Spa}(A_\infty[\frac{1}{p}], A_\infty)$$

and write $\mathcal{A}_n := \mathcal{O}^+_{X_A}(X_A(\frac{p^n}{g}))$. Similarly, we may apply the construction to $Y := \mathrm{Spa}(\widehat{R^+}[\frac{1}{p}], \widehat{R^+})$ and write $\mathcal{R}_n := \mathcal{O}^+_Y(Y(\frac{p^n}{g}))$. Since $A_\infty \to \widehat{R^+}$, we have an induced map $Y \to X_A$, which in turn induces maps $\mathcal{A}_n \to \mathcal{R}_n$ of perfectoid $K^\circ$-algebras.

Fix an integer $n$. Since $A_{pg} \to R_{pg}$ is a finite étale extension, the base change map $\mathcal{A}_n[\frac{1}{pg}] \to \mathcal{A}_n \otimes_{A_{pg}} R_{pg}$ is also a finite étale extension. Note that $g$ is already inverted in $\mathcal{A}_n[\frac{1}{p}]$. So $\mathcal{A}_n[\frac{1}{p}] \to \mathcal{A}_n \otimes_{A_p} R[\frac{1}{p}]$ is finite étale. By the almost purity theorem ([Sch12, Theorem 7.9 (iii)]), the integral

65

closure of $\mathcal{A}_n \otimes_A R$ in $\mathcal{A}_n \otimes_{A_p} R[\frac{1}{p}]$, denoted $\mathcal{B}_n$, is almost finite étale over $\mathcal{A}_n$. $\mathcal{B}_n$ is an $R$-algebra. Since both $\mathcal{A}_n$ and $R[\frac{1}{p}]$ map to $\mathcal{R}_n[\frac{1}{p}]$, their tensor product over $A$ also maps to $\mathcal{R}_n[\frac{1}{p}]$. This induces a map between power-bounded elements, that is, $\mathcal{B}_n \to \mathcal{R}_n$. By the same argument applied to $R_{pg} \to S_{pg}$, the map $\mathcal{B}_n[\frac{1}{p}] \to \mathcal{B}_n \otimes_R S[\frac{1}{p}]$ is a finite étale extension. Thus, by the almost purity theorem, there is an $S$-algebra $\mathcal{C}_n$ almost finite étale over $\mathcal{B}_n$ that almost splits. Therefore, the map $\mathcal{B}_n \to \mathcal{B}_n \otimes_R S \to \mathcal{C}_n$ almost splits.

Hence $\mathcal{B}_n \to \mathcal{B}_n \otimes_R S$ almost splits modulo $p^m$ for any $m$. In particular, $\alpha_{\mathcal{B}_n/p^m\mathcal{B}_n}$ is almost zero for all $n$ and $m$. Since we have a $p^{1/p^\infty}$-almost map $\mathcal{B}_n \to \mathcal{R}_n$, we also have that $\alpha_{\mathcal{R}_n/p^m\mathcal{R}_n}$ is $(p^2 g)^{1/p^\infty}$-almost zero, hence, $(pg)^{1/p^\infty}$-almost zero. By Corollary 3.4.4, we learn that $\alpha_{\widehat{R^+}/p^m\widehat{R^+}}$ is annihilated by $(pg)^{1/p^\infty}$ for all $m$.

Consider the construction discussed above:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & S & \longrightarrow & S/R & \longrightarrow & 0 \\
 & & \phi \uparrow & & \uparrow & & \uparrow & & \\
 & & R^{\oplus n_1} & \xrightarrow{\ \nu\ } & R^{\oplus n_0} & \longrightarrow & S/R & \longrightarrow & 0
\end{array}
$$

where $\phi$ is a $1 \times n_1$ matrix with entries in $\mathfrak{m}$ and $\nu$ is a $n_0 \times n_1$ matrix with entries in $\mathfrak{m}$. Then the image of $\phi^\vee$ in $(\widehat{R^+}/p^m\widehat{R^+})^{\oplus n_1}$ is $(pg)^{1/p^\infty}$-almost in the image of $\nu^\vee(\widehat{R^+}/p^m\widehat{R^+})^{\oplus n_1}$. Noting that $\widehat{R^+}/p^m\widehat{R^+} \cong R^+/p^m R^+$, we have

$$(pg)^{1/p^\infty}\phi^\vee \in (\nu^\vee)(R^+)^{\oplus n_1} + p^m(R^+)^{\oplus n_1}$$

for any $m$. By the definition of epf closure (Definition 3.1.3), we have $\phi^\vee \in ((\nu^\vee)R^{\oplus n_1})^{\mathsf{epf}}$. Hence, $R \to S$ is epf-phantom by [Die10, Lemma 2.10]. $\qquad\square$

## 3.5 The Positive Characteristic Case

In this section, we discuss an analogue of Theorem 3.2.4 in positive characteristic. Since $p = 0$, the situation is slightly different. We will use an arbitrary element instead, and the proof techniques are quite different. The main result is Theorem 3.5.13.

### 3.5.1 Intersection of ideals and regular sequences

We begin by investigating the behaviour of intersections of finitely generated ideals in some non-noetherian rings.

**Proposition 3.5.1.** *Let $(R, \mathfrak{m})$ be an excellent local domain of prime characteristic $p$. Let $R^+$ be its absolute integral closure. Suppose that $x_1, \ldots, x_d$ is a system of parameters of $R$. Then for any*

*proper finitely generated ideal $J$ in $R^+$ and $x_1, \ldots, x_n$, part of a system of parameters, we have*

$$\bigcap_{N=1}^{\infty} ((x_1, \ldots, x_n) + J^N) R^+ = (x_1, \ldots, x_n) R^+.$$

*Proof.* Suppose $J$ is generated by $r_1, \ldots, r_h$. For any $u \in \bigcap_{N=1}^{\infty} ((x_1, \ldots, x_n) + J^N) R^+$, let $S \subseteq R^+$ be a module-finite extension domain of $R$ containing $x_1, \ldots, x_n, r_1, \ldots, r_h$ and $u$. Then $S$ is also excellent. Let $J_0 = (r_1, \ldots, r_h)S$. Then $u \in ((x_1, \ldots, x_n)S + J_0^N S)^+$ for any $N$. So $u \in ((x_1, \ldots, x_n)S + J_0^N S)^*$. By a well-known result ([HH94a, Theorem 6.1]), we can find a test element $c$ such that for any $q = p^e$, where $e$ is any positive integer,

$$cu^q \in (x_1^q, \ldots, x_n^q)S + (J_0^N)^{[q]}S.$$

Fix $q$. Since we know that $S/(x_1^q, \ldots, x_n^q)$ is $J_0$-adically separated, we have

$$\bigcap_{N=1}^{\infty} \left( (x_1^q, \ldots, x_n^q)S + (J_0^N)^{[q]} S \right) \subseteq (x_1^q, \ldots, x_n^q)S.$$

So we have
$$cu^q \in (x_1^q, \ldots, x_n^q)S,$$

which shows that $u \in ((x_1, \ldots, x_n)S)^*$. But $((x_1, \ldots, x_n)S)^* = ((x_1, \ldots, x_n)S)^+$ by [Smi94, Theorem 5.1]. So $u \in (x_1, \ldots, x_n)S^+ = (x_1, \ldots, x_n)R^+$. □

In order to have enough different regular sequences, we need to adjoin new variables to $R$ and $R^+$. Hence, we introduce the following notation. Let $\Lambda$ be a possibly infinite index set and let $\{t_\lambda : \lambda \in \Lambda\}$ be a set of new variables. For any quasilocal ring $(T, \mathfrak{m}_T)$, the notion $T(t_\lambda : \lambda \in \Lambda)$ means the localization of the polynomial ring $T[t_\lambda : \lambda \in \Lambda]$ at the ideal $\mathfrak{m}_T T[t_\lambda : \lambda \in \Lambda]$. The natural map $T \to T(t_\lambda : \lambda \in \Lambda)$ is faithfully flat. Next, we prove a stronger version of Proposition 3.5.1.

**Proposition 3.5.2.** *Suppose $(R, \mathfrak{m})$ is a complete local domain of prime characteristic $p$. Let $(R^+, \mathfrak{m}_{R^+})$ be its absolute integral closure. Suppose that $x_1, \ldots, x_d$ is a system of parameters of $R$. Then for any index set $\Lambda$ and the set of new variables $\{t_\lambda : \lambda \in \Lambda\}$, any proper finitely generated ideal $J$ in $R^+$ and $x_1, \ldots, x_n$, part of system of parameters, we have*

$$\bigcap_{N=1}^{\infty} ((x_1, \ldots, x_n) + J^N) T = (x_1, \ldots, x_n) T,$$

*where $T = R^+(t_\lambda : \lambda \in \Lambda)$.*

*Proof.* Write $\mathfrak{a}_N$ for the ideal $((x_1, \ldots, x_n) + J^N) R^+$. Let $u$ be an element in $\bigcap_{N=1}^{\infty} \mathfrak{a}_N T$. Then $u$ is a rational function in $t_\lambda$'s. Clear the denominator and assume that $u$ is a polynomial in

$R^+[t_\lambda : \lambda \in \Lambda]$. For each $N$, we know that $u \in \mathfrak{a}_N T$, and, again by clearing denominators, there is some polynomial $g_N \in R^+[t_\lambda : \lambda \in \Lambda]$ such that $g_N u \in \mathfrak{a}_N R^+[t_\lambda : \lambda \in \Lambda]$. We also know that $g_N$ is a unit in $T$. Hence $g_N \notin \mathfrak{m}_{R^+} R^+[t_\lambda : \lambda \in \Lambda]$, which means that at least one coefficient of $g_N$ is not in $\mathfrak{m}_{R^+} R^+$, i.e., is a unit in $R^+$. Since $\mathfrak{a}_N \subseteq \mathfrak{m}_{R^+}$, that coefficient continues to be a unit in $R^+[t_\lambda : \lambda \in \Lambda]/\mathfrak{a}_N R^+[t_\lambda : \lambda \in \Lambda]$. Hence $g_N$ is actually a nonzerodivisor on $R^+[t_\lambda : \lambda \in \Lambda]/\mathfrak{a}_N R^+[t_\lambda : \lambda \in \Lambda]$. Therefore, we conclude that $u \in \mathfrak{a}_N R^+[t_\lambda : \lambda \in \Lambda]$. Since we are now in the polynomial case, this is equivalent to saying that each coefficient of $u$ is in $\mathfrak{a}_N R^+$. Therefore, each coefficient of $u$ is in $\cap_{N=1}^{\infty} \mathfrak{a}_N R^+ \subseteq (x_1, \ldots, x_n) R^+$ (Proposition 3.5.1). So $u \in (x_1, \ldots, x_n) R^+[t_\lambda : \lambda \in \Lambda] \Rightarrow u \in (x_1, \ldots, x_n) T$. $\qquad\square$

For the proof of the main theorem of this section, we need to study the intersection of ideals containing elements of the form $x_n - t_{\lambda_n} x_{n+1}$ where $x_n, x_{n+1}$ are part of a system of parameters.

**Proposition 3.5.3.** *Let $(B, \mathfrak{m}_B)$ be a quasilocal ring, $x, y \in \mathfrak{m}_B$ a permutable regular sequence on $B$ and $t$ an indeterminate. Then for any positive integer $N$, we have*

*(i) $(x - yt, y^N) B(t) \cap B[t] = (x - yt, y^N) B[t]$;*
*(ii) $(x - yt, y^N) B[t] \cap B = (x, y)^N B$.*

*Proof.* For 3.5.3.(i), note that $(x - yt, y) B(t) = (x, y) B(t)$. Any polynomial $g(t) \in B[t]$ that is invertible in $B(t)$ has a nonzero coefficient that is a unit. Hence $g(t)$ is a nonzerodivisor on $B[t]/(x, y) B[t]$. So $y, x - yt, g(t)$ form a regular sequence on $B[t]$, which implies that $y^N, x - yt, g(t)$ form a regular sequence on $B[t]$. Suppose that $p(t) \in (x - yt, y^N) B(t) \cap B[t]$; then we can clear the denominators to get $p(t) g(t) = a(t) y^N + b(t)(x - yt)$ for some $a(t), b(t), g(t) \in B[t]$ with $g(t)$ invertible in $B(t)$. Then since $y^N, x - yt, g(t)$ form a regular sequence on $B[t]$, we conclude that $p(t) \in (x - yt, y^N) B[t]$. The converse containment is obvious.

For 3.5.3.(ii), suppose that $b \in (x - yt, y^N) B[t] \cap B$. Then $b = \alpha(t)(x - yt) + \beta(t) y^N$ for some polynomial $\alpha(t), \beta(t) \in B[t]$. Assume that $\alpha(t) = c_h t^h + \cdots + c_0$ for elements $c_0, c_1, \ldots, c_h \in B$. By comparing the coefficients of $b = \alpha(t)(x - yt) + \beta(t) y^N$, we have

$$-c_h y \in (y^N) B, \tag{3.5.1}$$

$$-c_{i-1} y + c_i x \in (y^N) B, \quad 1 \leqslant i \leqslant h, \tag{3.5.2}$$

$$b - c_0 x \in (y^N) B. \tag{3.5.3}$$

**Claim.** *For each coefficient, we have $c_{h-i} \in (x, y)^{N-1} B$ for $0 \leqslant i \leqslant h$.*

We prove this by induction on $i$. As we have $c_h \in (y^{N-1}) B$ from (3.5.1) as $y$ is, by assumption, a nonzerodivisor on $B$, the case $i = 0$ is obvious.

Assume that the claim is true for $c_{h-i+1}$, i.e., $c_{h-i+1} \in (x,y)^{N-1}B$. So we write

$$c_{h-i+1} = \gamma_{N-1}x^{N-1} + \gamma_{N-2}x^{N-2}y + \cdots + \gamma_0 y^{N-1}$$

for some $\gamma_0, \gamma_1, \ldots, \gamma_{N-1} \in B$.

By (3.5.2) we have

$$-c_{h-i}y + c_{h-i+1}x = y^N \mu,$$

for some $\mu \in B$. Hence we have

$$c_{h-i}y = \gamma_{N-1}x^N + \gamma_{N-2}x^{N-1}y + \cdots + \gamma_0 xy^{N-1} - \mu y^N$$
$$\Rightarrow \left(c_{h-i} + \mu y^{N-1} - \gamma_0 xy^{N-2} - \gamma_1 x^2 y^{N-3} - \cdots - \gamma_{N-2}x^{N-1}\right)y \in (x^N)B.$$

Using the assumption that $y$ is a nonzerodivisor on $B/(x^N)B$, we conclude that

$$c_{h-i} \in \left(x^N, x^{N-1}, x^{N-2}y, \cdots, xy^{N-2}, y^{N-1}\right)B = \left(y^{N-1}, xy^{N-2}, \ldots, x^{N-1}\right)B = (x,y)^{N-1}B.$$

Therefore, the claim is proved.

By the claim above, we have $c_0 \in (x,y)^{N-1}B \Rightarrow c_0 x \in (x,y)^N B$. So (3.5.3) implies that $b \in (x,y)^N B$.

Conversely, it is trivial that $y^N \in (x - yt, y^N)B[t]$. If $x^k y^{N-k} \in (x - yt, y^N)B[t]$ for some $0 \leqslant k \leqslant N$, then $x^{k+1}y^{N-k-1} = (x - yt)x^k y^{N-k-1} + x^k y^{N-k}t \in (x - yt, y^N)B[t]$. So inductively we have $(x,y)^N \subseteq (x - yt, y^N)B[t]$. Hence they are equal. $\qquad\square$

We next observe that

**Lemma 3.5.4.** *Let $(R, \mathfrak{m}, K)$ be a $d$-dimensional complete local domain of prime characteristic $p$. Let $R^+$ be its absolute integral closure. Let $\Lambda$ be an index set and let $\{t_\lambda : \lambda \in \Lambda\}$ be a set of variables. Let $x_1, \ldots, x_d$ be a system of parameters of $R$. Then $x_1, \ldots, x_n$, where $1 \leqslant n \leqslant d$, is a regular sequence on $T = R^+(t_\lambda : \lambda \in \Lambda)$.*

*Proof.* We know that $R^+$ is a big Cohen-Macaulay $R$-algebra, so that $x_1, \ldots, x_d$ is a regular sequence on $R^+$. Since $R^+ \to T$ is faithfully flat, we know that $x_1$ is a nonzerodivisor in $T$ and

$$(x_1, \ldots, x_{n-1})T :_T x_n = ((x_1, \ldots, x_{n-1})R^+ :_{R^+} x_n)T = (x_1, \ldots, x_{n-1})T.$$

Therefore, $x_1, \ldots, x_d$ is a regular sequence on $T$. $\qquad\square$

The following results are standard facts about regular sequences.

**Lemma 3.5.5.** *Let $B$ be a ring and $x, y \in B$ be a regular sequence on $B$. If $y$ is a nonzerodivisor on $B$, then $y, x$ is also a regular sequence.*

*Proof.* The only thing that needs a proof is that $x$ is a nonzerodivisor on $B/(y)B$. Suppose that $\alpha x = \beta y$ for some $\alpha, \beta \in B$. Then by assumption $\beta \in (x)B$. So we can write $\beta = x\beta'$ for some $\beta' \in B$. Then $x(\alpha - \beta' y) = 0$ and $x$ is a nonzerodivisor. So $\alpha = \beta' y \Rightarrow \alpha \in (y)B$. □

**Corollary 3.5.6.** *Let $B$ be a ring and $y, x_1, \ldots, x_n$ be elements of $B$. If both $x_1, \ldots, x_n$ and $y, x_1, \ldots, x_n$ are regular sequences on $B$, then so is $x_1, \ldots, x_k, y, x_{k+1}, \ldots, x_n$ where $0 \leqslant k \leqslant n$. In particular, when $k = n$, $x_1, \ldots, x_n, y$ form a regular sequence on $B$.*

*Proof.* We prove this by induction on $k$. The base case $k = 0$ is one of the assumptions. Now assume that $x_1, \ldots, x_k, y, x_{k+1}, \ldots, x_n$ is a regular sequence on $B$. Let $A = B/(x_1, \ldots, x_k)$. We know that $y, x_{k+1}$ form a regular sequence, and $x_{k+1}$ is a nonzerodivisor on $A$. By Lemma 3.5.5, $x_{k+1}, y$ also form a regular sequence on $A$. It is obvious that $x_{k+2}, \ldots, x_n$ continue to be a regular sequence on $A/(x_{k+1}, y)$. Therefore $x_1, \ldots, x_{k+1}, y, x_{k+2}, \ldots, x_n$ is a regular sequence on $B$. □

The main result we want to prove is about elements of the form $x_n - t_{\lambda_n} x_{n+1}$ in the ring $T$. In the theorem below, if the index set for a variable is empty, then the variable does not occur.

**Theorem 3.5.7.** *Let $(R, \mathfrak{m}, K)$ be a $d$-dimensional complete local domain of prime characteristic $p$. Let $R^+$ be its absolute integral closure. Let $\Lambda$ be an index set and let $\{t_\lambda : \lambda \in \Lambda\}$ be a set of variables. Let $T = R^+(t_\lambda : \lambda \in \Lambda)$. For any system of parameters $x_1, \ldots, x_d$ of $R$, let $z_i = x_i - t_{\lambda_i} x_{i+1}$ for some $t_{\lambda_i} \in \{t_\lambda : \lambda \in \Lambda\}$ ($\lambda_i \neq \lambda_j$ if $i \neq j$ and $1 \leqslant i < d$) and $\{y_1, \ldots, y_h\}, \{w_1, \ldots, w_l\}$ be two subsets of $X_n := \{x_{n+2}, x_{n+3}, \ldots, x_d\}$ where $0 \leqslant h, l \leqslant d - n - 1$. Then we have*

*(i) $y_1, \ldots, y_h, z_1, \ldots, z_n, x_{n+1}$ form a regular sequence on $T$, and*

*(ii) $\bigcap_{N=1}^{\infty} \left( (y_1, \ldots, y_h, z_1, \ldots, z_n) + (x_{n+1}, w_1, \ldots, w_l)^N \right) T = (y_1, \ldots, y_h, z_1, \ldots, z_n) T$*

*for any $0 \leqslant n \leqslant d - 1$.*

*Proof.* We prove both claims at the same time by induction on $n$. The base case $n = 0$ is trivial: when $n = 0$, (1) is the conclusion of Lemma 3.5.4 and (2) is the conclusion of Proposition 3.5.2.

Assume that both claims are true for $n$. We want to prove the case of $n + 1$. So we let both $\{y_1, \ldots, y_h\}$ and $\{w_1, \ldots, w_l\}$ be subsets of $X_{n+1}$. We first prove (1). Note that $X_{n+1} \subseteq X_n$. So $y_1, \ldots, y_h, z_1, \ldots, z_n, x_{n+1}$ is a regular sequence by the induction hypothesis. Since $(z_1, \ldots, z_n, x_{n+1})T = (x_1, \ldots, x_{n+1})T$, it is trivial that $y_1, \ldots, y_h, z_1, \ldots, z_n, x_{n+1}, x_{n+2}$ form a regular sequence on $T$. Let $S = T/(y_1, \ldots, y_h, z_1, \ldots, z_n)T$. If there is some $\alpha \in S$ such that $\alpha z_{n+1} = 0$, then modulo $x_{n+1}$ we see that $\alpha t_{n+1} x_{n+2} = 0$. Hence $\alpha = x_{n+1}\beta$. Since $x_{n+1}$

70

is a nonzerodivisor and $\beta x_{n+1} z_{n+1} = 0$, we have $\beta z_{n+1} = 0$. Repeating this argument we get $\alpha \in (y_1, \ldots, y_h, z_1, \ldots, z_n, x_{n+1}^N)T$ for all $N$. The induction hypothesis (2) shows that

$$\alpha \in (y_1, \ldots, y_h, z_1, \ldots, z_n)T.$$

Hence $z_{n+1}$ is a nonzerodivisor on $S$.

Let $T' = T/(y_1, \ldots, y_h)T$. Since $x_{n+2} \in X_n$, the induction hypothesis shows that both $x_{n+2}, z_1, \ldots, z_n$ and $z_1, \ldots, z_n$ are regular sequences on $T'$. By Corollary 3.5.6, $z_1, \ldots, z_n, x_{n+2}$ also form a regular sequence on $T'$. From the previous paragraph, we know that $z_1, \ldots, z_n, x_{n+1}, x_{n+2}$ is a regular sequence on $T'$, and, again by Corollary 3.5.6, $z_1, \ldots, z_n, x_{n+2}, x_{n+1}$ form a regular sequence on $T'$. So $z_1, \ldots, z_n, x_{n+2}, x_{n+1} - t_{n+1}x_{n+2}$ is a regular sequence on $T'$. Again, from the previous paragraph, $z_1, \ldots, z_n, z_{n+1}$ is already a regular sequence on $T'$. We conclude that $y_1, \ldots, y_h, z_1, \ldots, z_n, z_{n+1}, x_{n+2}$ form a regular sequence on $T$ by Corollary 3.5.6. This proves (1).

For (2), let $Q = R^+(t_\mu : \mu \in \Lambda, \mu \neq \lambda_{n+1})$. Then $T = Q(t_{\lambda_{n+1}})$. Since $(x_{n+2}^N, w_1^N, \ldots, w_l^N)Q$ is cofinal with $(x_{n+2}, w_1, \ldots, w_l)^N Q$, it suffices to show that

$$\bigcap_{N=1}^{\infty} ((y_1, \ldots, y_n, z_1, \ldots, z_{n+1}) + (x_{n+2}^N, w_1^N, \ldots, w_l^N))Q(t_{\lambda_{n+1}}) \subseteq (y_1, \ldots, y_h, z_1, \ldots, z_{n+1})Q(t_{\lambda_{n+1}}).$$

Since $\{w_1, \ldots, w_l\} \cup \{y_1, \ldots, y_h\} \subseteq X_{n+1}$, the ideal $(y_1, \ldots, y_h, w_1^N, \ldots, w_l^N) \subseteq (x_{n+3}, \ldots, x_d)$ is generated by part of a system of parameters of $R$. Hence $x_{n+1}, x_{n+2}$ form a permutable regular sequence on the quotient ring $P = Q/(y_1, \ldots, y_h, w_1^N, \ldots, w_l^N, z_1, \ldots, z_n)Q$. Applying Proposition 3.5.3 to $B = P, x = x_{n+1}, y = x_{n+2}$, we get

$$(y_1, \ldots, y_h, w_1^N, \ldots, w_l^N, z_1, \ldots, z_n, x_{n+1} - t_{\lambda_{n+1}}x_{n+2}, x_{n+2}^N)T \cap Q$$
$$= (y_1, \ldots, y_h, w_1^N, \ldots, w_l^N, z_1, \ldots, z_n)Q + (x_{n+1}, x_{n+2})^N Q.$$

For any $u \in \cap_{N=1}^{\infty}(y_1, \ldots, y_h, w_1^N, \ldots, w_l^N, z_1, \ldots, z_n, x_{n+1} - t_{\lambda_{n+1}}x_{n+2}, x_{n+2}^N)T$, clear the denominators and assume that $u$ is a polynomial in $t_{\lambda_{n+1}}$ of degree $h$. Then $x_{n+2}^h u$ can be considered as a polynomial in $t_{\lambda_{n+1}}x_{n+2}$ of degree $h$. Therefore, we can divide $x_{n+2}^h u$ by the "monic" polynomial $t_{\lambda_{n+1}}x_{n+2} - x_{n+1}$ in $Q[t_{\lambda_{n+1}}]$ and get a remainder $b$ of degree 0 in $t_{\lambda_{n+1}}$. So $x_{n+2}^h u - b \in (z_{n+1})Q[t_{\lambda_{n+1}}]$ and

$$b \in \bigcap_{N=1}^{\infty} (y_1, \ldots, y_h, w_1^N, \ldots, w_l^N, z_1, \ldots, z_{n+1}, x_{n+2}^N)T \cap Q$$
$$\Rightarrow b \in \bigcap_{N=1}^{\infty} ((y_1, \ldots, y_h, w_1^N, \ldots, w_l^N, z_1, \ldots, z_n) + (x_{n+1}, x_{n+2})^N)Q,$$

which implies that $b \in (y_1, \ldots, y_h, z_1, \ldots, z_n)Q$ by the induction hypothesis. Therefore,

$$x_{n+2}^h u \in (y_1, \ldots, y_h, z_1, \ldots, z_{n+1})Q[t_{\lambda_{n+1}}] \Rightarrow x_{n+2}^h u \in (y_1, \ldots, y_h, z_1, \ldots, z_{n+1})T.$$

Note that $y_1, \ldots, y_h, z_1, \ldots, z_{n+1}, x_{n+2}^h$ is a regular sequence on $T$ by our proof of (1) above. Hence, we do not need the factor of $x_{n+2}^h$ and we have $u \in (y_1, \ldots, y_h, z_1, \ldots, z_{n+1})T$. $\qquad \square$

### 3.5.2 Stabilization of colon ideals

We say that the colon ideal $(x_1, \ldots, x_n)R^+ :_{R^+} y^\infty$ *stabilizes* at a positive integer $N$ if

$$(x_1, \ldots, x_n)R^+ :_{R^+} y^\infty = (x_1, \ldots, x_n)R^+ :_{R^+} y^N.$$

The next few results deal with the stability of colon ideals in non-noetherian rings.

**Lemma 3.5.8.** *Let $(R, \mathfrak{m})$ be a $d$-dimensional local domain of prime characteristic $p$ and $T$ an $R$-algebra. Let $x_1, \ldots, x_d$ be a system of parameters of $R$. Suppose that $x_d$ is a nonzerodivisor on $T/(x_1, \ldots, x_{d-1})T$. Then for any element $y \in R$ and $n = d, d-1$, there is some $N_0$ such that*

$$(x_1, \ldots, x_n)T :_T y^\infty = (x_1, \ldots, x_n)T :_T y^{N_0}.$$

*Proof.* The conclusion is trivial if $y$ is a unit. So we assume that $y \in \mathfrak{m}$. For $n = d$, since $\mathfrak{m}$ is nilpotent on $(x_1, \ldots, x_d)R$, and $y \in \mathfrak{m}$, there is some $N_0$ such that $y^{N_0} \in (x_1, \ldots, x_d)R$. This $N_0$ will suffice.

Look at $n = d-1$ and consider the ring $A = (R/(x_1, \ldots, x_{d-1})R)_{x_d}$. It is an artinian ring, and, hence, it is a product of artinian local rings, i.e., $A = A_1 \times \cdots \times A_h$. The image of $y$ in each component $A_i$ is either a unit or a nilpotent, i.e., $\bar{y} = (y_1, \ldots, y_h)$ where, without loss of generality, $y_1, \ldots, y_k$ are nilpotents and $y_{k+1}, \ldots, y_h$ are units. So there is some positive power $N_0$ such that $\bar{y}^{N_0} = (0, 0, \ldots, 0, y_{k+1}^{N_0}, \ldots, y_h^{N_0})$. There is some $a \in A$ such that

$$\bar{y}^{N_0} = a\bar{y}^{2N_0}. \tag{3.5.4}$$

Since $T$ is an $R$-algebra, we have a map $A \to B = (T/(x_1, \ldots, x_{d-1})T)_{x_d}$. The relation (3.5.4) of $\bar{y}^{N_0}$ and $\bar{y}^{2N_0}$ maps to a relation

$$\bar{y}^{N_0} = b\bar{y}^{2N_0} \tag{3.5.5}$$

for some $b \in B$ and the same $N_0$. We claim that $(x_1, \ldots, x_{d-1})T_{x_d} :_{T_{x_d}} y^\infty = (x_1, \ldots, x_{d-1})T_{x_d} :_{T_{x_d}} y^{N_0}$. Take any $u \in (x_1, \ldots, x_{d-1})T_{x_d} :_{T_{x_d}} y^\infty$. Then the image $\bar{u}$ of $u$ in $B = (T/(x_1, \ldots, x_{d-1})T)_{x_d}$ is in $0 :_B \bar{y}^\infty$. So there is some $N$ such that $\bar{y}^N \bar{u} = 0$ in $B$. It is obvious that any higher power of

72

$\bar{y}$ will kill $\bar{u}$. Hence we may assume without loss of generality that $N = mN_0$ and $m \geqslant 2$. Then $\bar{y}^{mN_0}\bar{u} = 0$. Making use of the relation (3.5.5), we have

$$b\bar{y}^{2N_0}\bar{y}^{(m-2)N_0}\bar{u} = 0$$
$$\Rightarrow \bar{y}^{N_0}\bar{y}^{(m-2)N_0}\bar{u} = 0$$
$$\Rightarrow \bar{y}^{(m-1)N_0}\bar{u} = 0.$$

We can repeat this argument until $m$ reaches 1. So $\bar{y}^{N_0}\bar{u} = 0$ in $B$ implies that $y^{N_0}u \in (x_1, \ldots, x_{d-1})T_{x_d}$, which in turn implies that $u \in (x_1, \ldots, x_{d-1})T_{x_d} :_{T_{x_d}} y^{N_0}$. So we have

$$(x_1, \ldots, x_{d-1})T_{x_d} :_{T_{x_d}} y^{\infty} = (x_1, \ldots, x_{d-1})T_{x_d} :_{T_{x_d}} y^{N_0}.$$

But we also know that $x_d$ is a nonzerodivisor on $T/(x_1, \ldots, x_{d-1})T$. So we have

$$(x_1, \ldots, x_{d-1})T :_T y^{\infty} = (x_1, \ldots, x_{d-1})T :_T y^{N_0}$$

as well. $\qquad\qquad\square$

**Theorem 3.5.9.** *Suppose $(R, \mathfrak{m}, K)$ is a $d$-dimensional complete local domain of prime characteristic $p$. Let $R^+$ be its absolute integral closure. Let $\Lambda$ be an uncountable index set and let $\{t_\lambda : \lambda \in \Lambda\}$ be a set of variables. Suppose $x_1, \ldots, x_d$ is a system of parameters of $R$. Then for any $1 \leqslant n \leqslant d$ and any $y \in R$, there is some $N_0$ such that*

$$(x_1, \ldots, x_n)T :_T y^{\infty} = (x_1, \ldots, x_n)T :_T y^{N_0}$$

*where $T = R(t_\lambda : \lambda \in \Lambda)$.*

*Proof.* By Lemma 3.5.8, the conclusion is true for $n = d, d-1$. We assume that $n \leqslant d-2$. We also assume that $y \in \mathfrak{m}$. Let $z_i = x_i - t_{\lambda_i}x_{i+1}$ $(1 \leqslant i \leqslant d-1)$. Consider the sequence $x_1, \ldots, x_n, z_{n+1}, \ldots, z_{d-1}, x_d$. It is easy to see that

$$(x_1, \ldots, x_n, z_{n+1}, \ldots, z_{d-1}, x_d)R(t_{\lambda_{n+1}}, \ldots, t_{\lambda_{d-1}}) = (x_1, \ldots, x_d)R(t_{\lambda_{n+1}}, \ldots, t_{\lambda_{d-1}}).$$

So by Lemma 3.5.8, for any choice of $\lambda_{n+1}, \ldots, \lambda_{d-1}$, there is some $N_0$ such that

$$(x_1, \ldots, x_n, z_{n+1}, \ldots, z_{d-1})T :_T y^{\infty} = (x_1, \ldots, x_n, z_{n+1}, \ldots, z_{d-1})T :_T y^{N_0}.$$

We want to show that

$$(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T :_T y^\infty = (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T :_T y^{N_0}$$

for $k \geqslant n$. We prove this by reverse induction on $k$. The base case $k = d - 1$ is done. Now suppose that this is true for $k + 1$. Look at ideals $(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\mu x_{k+2})T :_T y^\infty$. The induction hypothesis shows that each ideal stabilizes at some $N$. There are uncountably many $\mu \in \Lambda$. So we can find some $N_0$ such that

$$(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\mu x_{k+2})T :_T y^\infty = (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\mu x_{k+2})T :_T y^{N_0}$$
(3.5.6)

holds for infinitely many choices of $\mu$. In particular, there are countably many $\mu_1, \mu_2, \ldots$ avoiding all $\lambda_{n+1}, \ldots, \lambda_k$ such that (3.5.6) holds.

For any $u \in (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T :_T y^\infty$, there is some $N$ such that

$$y^N u \in (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T.$$

So

$$y^N u \in (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_{\mu_i} x_{k+2})T$$
$$\Rightarrow y^{N_0} u \in (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_{\mu_i} x_{k+2})T$$

for all choices of $\mu_i$.

Hence for any $l$, we have

$$y^{N_0} u \in \bigcap_{i=1}^{l} (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_{\mu_i} x_{k+2})T.$$

Let $a_i = x_{k+1} - t_{\mu_i} x_{k+2}$ and $S = T/(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T$. We claim that

$$\bigcap_{i=1}^{l} (a_i) S = \prod_{i=1}^{l} (a_i) S$$
(3.5.7)

for any $l$.

We first notice that any two elements $a_i$ and $a_j$ where $i \neq j \in \mathbb{N}$ form a regular sequence in $S$: because $(a_i, a_j)S = (x_{n+1}, x_{n+2})S$, they form a regular sequence on $S$. We prove (3.5.7) by induction on $l$. The case $l = 1$ is trivial. Suppose that $u \in (\cap_{i=1}^{l}(a_i)S) \cap (a_{l+1})S = (\prod_{i=1}^{l}(a_i)S) \cap (a_{l+1})S$. Write $c_l = \prod_{i=1}^{l} a_i$. Then $u = \alpha c_l = \beta a_{l+1}$ for some $\alpha, \beta \in S$. Since $a_{l+1}, a_i$ $(i \neq l)$ form a regular sequence, so do $a_{l+1}, c_l$. Hence $\alpha \in (a_{l+1})S \Rightarrow u \in (a_{l+1}c_l)S$ and (3.5.7) is proved.

74

So for any $l$, we have

$$y^{N_0}u \in (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T + \prod_{i=1}^{l}(x_{k+1} - t_{\mu_i}x_{k+2})T.$$

Since

$$\bigcap_{l=1}^{\infty}\left((x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T + \prod_{i=1}^{l}(x_{k+1} - t_{\mu_i}x_{k+2})T\right)$$
$$\subseteq \bigcap_{l=1}^{\infty}\left((x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T + ((x_{k+1}, x_{k+2})T)^l\right),$$

we may apply Proposition 3.5.2 to see that the right-hand side is in $(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T$. We conclude that

$$y^{N_0} \in (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T.$$

So we have

$$(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T :_T y^\infty = (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k)T :_T y^{N_0}.$$

The case $k = n$ is the conclusion of the theorem. $\qquad\square$

*Remark* 3.5.10. In Theorem 3.5.9, one can assume that $\Lambda$ is a countably infinite index set. The proof still works if we make the following modification: we observe that for two different variables $t_\lambda$ and $t_\mu$, the map swapping $t_\lambda$ and $t_\mu$ is an automorphism of $T = R(t_\lambda : \lambda \in \Lambda)$. Hence, if we have

$$(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\lambda x_{k+2})T :_T y^\infty = (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\lambda x_{k+2})T :_T y^{N_0}$$

for some $N_0$, then by applying the automorphism we just described, we have

$$(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\mu x_{k+2})T :_T y^\infty = (x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\mu x_{k+2})T :_T y^{N_0}.$$

So the proof where we show that there are countably many $\mu$ such that (3.5.6) holds can be modified as follows: Look at ideals $(x_1, \ldots, x_n, z_{n+1}, \ldots, z_k, x_{k+1} - t_\mu x_{k+2})T :_T y^\infty$. The induction hypothesis shows that each ideal stabilizes at some $N$. If one such ideal stabilizes at some integer $N_0$, then by permuting the variables $t_\mu$ we see that all such ideals stabilize at the same $N_0$.

**Corollary 3.5.11.** *Suppose $(R, \mathfrak{m}, K)$ is a d-dimensional complete local domain of prime characteristic $p$. Let $R^+$ be its absolute integral closure. Suppose $x_1, \ldots, x_d$ is a system of parameters of*

*R. Then for any $1 \leqslant n \leqslant d$ and any $y \in R$, there is some $N_0$ such that*

$$(x_1, \ldots, x_n)R^+ :_{R^+} y^\infty = (x_1, \ldots, x_n)R^+ :_{R^+} y^{N_0}.$$

*Proof.* Let $\Lambda$ be an uncountable index set and $\{t_\lambda : \lambda \in \Lambda\}$ a set of new variables. Then by Theorem 3.5.9, there is some $N_0$ such that

$$(x_1, \ldots, x_n)R^+(t_\lambda : \lambda \in \Lambda) :_{R^+(t_\lambda : \lambda \in \Lambda)} y^\infty = (x_1, \ldots, x_n)R^+(t_\lambda : \lambda \in \Lambda) :_{R^+(t_\lambda : \lambda \in \Lambda)} y^{N_0}.$$

For any element $u \in (x_1, \ldots, x_n)R^+ :_{R^+} y^\infty$, there is some $N$ such that $y^N u \in (x_1, \ldots, x_n)R^+$. So it is also in $(x_1, \ldots, x_n)R^+(t_\lambda : \lambda \in \Lambda)$. We have $y^{N_0} u \in (x_1, \ldots, x_n)R^+(t_\lambda : \lambda \in \Lambda)$ by equality above. So we have

$$y^{N_0} u \in (x_1, \ldots, x_n)R^+(t_\lambda : \lambda \in \Lambda) \cap R^+ \Rightarrow y^{N_0} u \in (x_1, \ldots, x_n)R^+$$

as the map $R^+ \to R^+(t_\lambda : \lambda \in \Lambda)$ is faithfully flat. Hence $u \in (x_1, \ldots, x_n)R^+ :_{R^+} y^{N_0}$, as desired. $\square$

### 3.5.3 The main theorem

We are almost ready to prove our main theorem of this section. Before that, let us derive a useful corollary from the results on the stability of colon ideals.

**Corollary 3.5.12.** *Suppose $(R, \mathfrak{m})$ is a $d$-dimensional complete local domain of prime characteristic $p$. Let $R^+$ be its absolute integral closure. Suppose $x_1, \ldots, x_d$ is a system of parameters of $R$. Then for any $1 \leqslant n \leqslant d$ and any $y \in R$, there is some $N_0$ such that $(x_1, \ldots, x_n)R^+ = \mathfrak{a} \cap \mathfrak{b}$ where $\mathfrak{a} = (x_1, \ldots, x_n, y^{N_0})R^+$ and $\mathfrak{b} = (x_1, \ldots, x_n) :_{R^+} y^{N_0}$.*

*Proof.* By Corollary 3.5.11, there is some $N_0$ such that

$$(x_1, \ldots, x_n)R^+ :_{R^+} y^\infty = (x_1, \ldots, x_n)R^+ :_{R^+} y^{N_0}.$$

Let $\mathfrak{a} = (x_1, \ldots, x_n, y^{N_0})R^+$ and $\mathfrak{b} = (x_1, \ldots, x_n)R^+ :_{R^+} y^{N_0}$. Then we have $\mathfrak{b} = (x_1, \ldots, x_n)R^+ :_{R^+} y^\infty$. For any $u \in \mathfrak{a} \cap \mathfrak{b}$, we can write $u = a_1 x_1 + \cdots + a_n x_n + b y^{N_0}$ and we know that $y^{N_0} u \in (x_1, \ldots, x_n)R^+$. So we have $b y^{2N_0} \in (x_1, \ldots, x_n)R^+ \Rightarrow b \in \mathfrak{b} \Rightarrow b y^{N_0} \in (x_1, \ldots, x_n)R^+$. So $u \in (x_1, \ldots, x_n)R^+$. The reverse inclusion is trivial. $\square$

Now we prove the main theorem.

**Theorem 3.5.13.** *Suppose $(R, \mathfrak{m})$ is a $d$-dimensional complete local domain of prime characteristic $p$. Let $R^+$ be its absolute integral closure. Suppose $x_1, \ldots, x_d$ is a system of parameters of $R$. Then*

*for any* $1 \leqslant n \leqslant d$ *and any* $y \in R$, *there is some positive integer* $N_0$ *such that for all* $N \geqslant N_0$,

$$(x_1, \ldots, x_n, y^N)R^+ :_{R^+} x_{n+1} \subseteq (x_1, \ldots, x_n, y^{N-N_0})R^+.$$

*Proof.* By applying Corollary 3.5.12 to the system of parameters $x_1, \ldots, x_{n+1}$ and $y$, we know that there is some $N_0$ such that if we write $\mathfrak{a} = (x_1, \ldots, x_{n+1}, y^{N_0})R^+$ and $\mathfrak{b} = (x_1, \ldots, x_{n+1})R^+ :_{R^+} y^{N_0}$, then we have $(x_1, \ldots, x_{n+1})R^+ = \mathfrak{a} \cap \mathfrak{b}$. For any $u \in (x_1, \ldots, x_n, y^N)R^+ :_{R^+} x_{n+1}$, we have $u x_{n+1} = u_1 x_1 + \cdots + u_n x_n + v y^N$ for some $u_1, \ldots, u_n, v \in R^+$. Therefore,

$$v \in (x_1, \ldots, x_{n+1})R^+ :_{R^+} y^N \subseteq \mathfrak{b} = (x_1, \ldots, x_{n+1})R^+ :_{R^+} y^{N_0}.$$

So we can write $v y^{N_0} = v_1 x_1 + \cdots + v_{n+1} x_{n+1}$ for some $v_1, \ldots, v_{n+1} \in R^+$. Hence,

$$(u - v_{n+1} y^{N-N_0}) x_{n+1} = (u_1 + v_1 y^{N-N_0}) x_1 + \cdots + (u_n + v_n y^{N-N_0}) x_n$$
$$\Rightarrow u - v_{n+1} y^{N-N_0} \in (x_1, \ldots, x_n)R^+ :_{R^+} x_{n+1}.$$

Since $R^+$ is a big Cohen-Macaulay algebra of $R$, we have $u - v_{n+1} y^{N-N_0} \in (x_1, \ldots, x_n)R^+$. Therefore, we have $u \in (x_1, \ldots, x_n, y^{N-N_0})R^+$. $\qquad \square$

*Remark* 3.5.14. Since $R$ is complete local, by Cohen's structure theorem, $R$ is module-finite over some complete regular local domain $A$ of characteristic $p$ with respect to the system of parameters $x_1, \ldots, x_d$. If the element $y$ happens to be in $A$, then the same proof (i.e., the proof of Theorem 3.2.4) as in the mixed characteristic case also works.

# CHAPTER IV

# Behavior of Analogues of Tight Closure

This chapter is organized as follows: in Section 4.1 we extend the result [Die18, Theorem 4.8] to a more general setting (Corollary 4.1.10). For this purpose, we need to generalize several notions to the non-domain case. In Section 4.2, we add two more axioms to the set of axioms for closure operations and discuss various related results. We show that the persistence axiom, together with Dietz's axioms and the Algebra axiom, imply a weak functorial version of the existence of big Cohen-Macaulay algebras (Theorem 4.2.11). Finally, we introduce three more closure operations in mixed-characteristic case in Section 4.3, and discuss the containment problem between these closure operations in Section 4.4.

## 4.1  Dietz's Axioms in Non-domain Cases

Dietz defined 7 axioms in [Die10] for finitely generated modules over noetherian domains, and generalized them to non-finitely generated modules in [Die18]. Here we further generalize them to non-domain cases.

*Construction* 4.1.1. Given a closure operation dcl ($d$ for domain) defined only for rings that are domains, there is a natural way to generalize this closure operation to non-domain local rings. Let $R$ be a noetherian local ring and $N \subseteq M$ be $R$-modules. We let the new closure operation cl to be defined as $u \in N_M^{\mathsf{cl}}$ if and only if $x \in \langle N \rangle_{M/PM}^{\mathsf{dcl}}$ for any minimal prime $P$, where $\langle N \rangle_{M/PM}$ is the image of $N$ in $M/PM$. In the case of an ideal $I$, the closure cl of $I$ is the intersection of all preimages of $(IR/P)^{\mathsf{dcl}}$ in $R$ where $P$ runs through all minimal primes of $R$.

*Remark* 4.1.2. One may also want to generalize to the non-local case by requiring that $x \in I^{\mathsf{cl}}$ iff $x \in (IR_{\mathfrak{m}})^{\mathsf{cl}}$ for any maximal ideal $\mathfrak{m}$. We know that this is true for tight closure. However, as pointed out in [Hei01, Remark 2.4], we do not know this for epf closure in complete generality.

We want to investigate the generalization of Dietz's axioms under the Construction 4.1.1. Let $(R, \mathfrak{m})$ be a noetherian local ring. All modules here are arbitrary $R$-modules. The first six axioms require no modification. We also note that all proofs in [Die10, Lemma 1.2] work in this generality.

In order to generalize the "generalized colon-capturing" axiom, we introduce the notion of strong system of parameters.

**Definition 4.1.3.** A *(partial) strong system of parameters* is a system of parameters of $R$ and continues to be a (partial) system of parameters modulo every minimal prime.

Now we change the generalized colon-capturing to following:

**Axiom 4.1.4** (Generalized colon-capturing in non-domain case)**.** Let $x_1, \ldots, x_{k+1}$ be a partial *strong* system of parameters for $R$ and let $J = (x_1, \ldots, x_k)$. Suppose that there exists a surjective homomorphism $f : M \to R/J$ and $v \in M$ such that $f(v) = x_{k+1} + J$. Then

$$(Rv)^{\mathsf{cl}}_M \cap \operatorname{Ker} f \subseteq (Jv)^{\mathsf{cl}}_M.$$

### 4.1.1   Some discussion on strong systems of parameters

Suppose that $(R, \mathfrak{m})$ is a $d$-dimensional noetherian local ring, and $I \subseteq R$ is an ideal. By the nonstandard terminology "truly minimal prime," we mean a minimal prime $P$ of $R$ such that $\dim R/P = d$. A truly minimal prime of $I$ corresponds to a truly minimal prime of $(0)$ in $R/I$.

In an equidimensional local ring, a truly minimal prime is the same as a minimal prime. However, in general, the set of truly minimal primes is a subset of the set of minimal primes. To obtain a system of parameters of $R$ in the usual sense, we start with $x_1$ that avoids all truly minimal primes. Because when one avoids all truly minimal primes, the dimension goes down at least by one, but the dimension goes down by at most 1 by Krull's height theorem (choose $y_1, \ldots, y_s$ in $R/x_1 R$ to be a system of parameters where $s = \dim(R/x_1 R)$, then $\mathfrak{m}$ is nilpotent over $(y_1, \ldots, y_s, x_1)$ which implies that $d \leqslant s + 1$). Next, we choose $x_2$ avoiding all truly minimal primes of $(x_1)R$ and minimal primes of $R$, and so on. However, in choosing a strong system of parameters, we have to avoid all minimal primes.

**Example 4.1.5.** Let $S = k[\![x, y, z]\!]/(xz, yz)$ and $\mathfrak{m} = (x, y, z)$. Let $R = S_{\mathfrak{m}}$ and $\mathfrak{m}R$ is the unique maximal ideal, we have three saturated chains:

$$\begin{aligned} (z) &\subseteq (y, z) \subseteq \mathfrak{m}, \\ (z) &\subseteq (x, z) \subseteq \mathfrak{m}, \\ &\quad (x, y) \subseteq \mathfrak{m}. \end{aligned}$$

So $(z)$ is a truly minimal prime while $(x, y)$ is a (usual) minimal prime. We can choose a (usual) system of parameters to be $x, y + z$, which avoids $(z)$, but not $(x, y)$. In this case, there is no full strong system of parameters. One partial strong system of parameter is just $x + z$.

### 4.1.2 Solidity and phantom extensions

We aim to extend the result [Die18, Theorem 4.8] to a more general setting. Specifically, we want to remove the "$F$-finite" and "domain" assumptions in the theorem.

We first note that the notion of "phantom extension" extends to the non-domain case directly, i.e., [Die18, Definition 2.3] works in this generality. All of the results proved in [Die18, Section 2] do not use the domain condition anywhere. Next, we want to extend the notion of "solid algebra" to the non-domain case.

**Definition 4.1.6.** An $R$-algebra $S$ is a solid $R$-algebra if there is a $R$-module map $\gamma : S \to R$ such that $\gamma(1)$ is a nonzerodivisor in $R$.

We first remove the assumption of "$F$-finiteness," which answers a question in [Die18, Section 5, Question (2)], i.e., we want to show that solid algebra maps $\alpha : R \to S$ are always phantom extensions without the $F$-finite condition in characteristic $p$. For this purpose, we need to discuss briefly the notion of the $\Gamma$-construction. Let $k$ be a field of characteristic $p$. A *p-base* $\Lambda$ for $k$ is a subset of $k$ such that $\{\mathrm{d}\lambda : \lambda \in \Lambda\}$ form a basis of the module of Kähler differentials $\Omega_{k/k^p}$. For any $n$ elements $\lambda_1, \ldots, \lambda_n \in \Lambda$, the field $k[\lambda_1^{1/p^e}, \ldots, \lambda_n^{1/p^e}]$ has degree $p^{ne}$ over $k$. If $R$ is a noetherian complete local domain of characteristic $p$, by Cohen's structure theorem, $R$ is module-finite over $A = k[\![x_1, \ldots, x_n]\!]$. In the sequel, we will fix a $p$-base $\Lambda$ of $k$, and let $\Gamma$ be a subset of $\Lambda$, usually *cofinite in $\Lambda$*, i.e., a subset such that $\Lambda \smallsetminus \Gamma$ is finite. Let $k_e^\Gamma$ denote the field extension $k[\gamma^{1/p^e} : \gamma \in \Gamma]$. Let $A_e^\Gamma := k_e^\Gamma[\![x_1, \ldots, x_n]\!]$, and $R_e^\Gamma := R \otimes_A A_e^\Gamma$. By a $\Gamma$-construction of $R$ we mean the $R$-algebra $R^\Gamma := \cup_e R_e^\Gamma$. See [HH94a, 6.11 Discussion and Notation], [HJ21, Subsection 5.1], and [Mur21, Construction 3.1] for a more detailed discussion. The most important fact here is that $R^\Gamma$ is a faithfully flat purely inseparable extension of $R$, which is $F$-finite ([HJ21, Theorem 5.3 (1)]). Moreover, the maximal ideal of $R$ extends to the maximal ideal of $R^\Gamma$.

**Lemma 4.1.7.** *Let $R$ be a noetherian complete local domain and $R \to R^\Gamma$ a $\Gamma$-construction of $R$. Let $M$ be an $R$-module, $H$ a submodule of $M$, and $u$ an element of $M$. Suppose that $M^\Gamma, H^\Gamma, u^\Gamma$ are images of $M, H, u$ under the base change map $R \to R^\Gamma$ respectively. Then $u^\Gamma \in (H^\Gamma)_{M^\Gamma}^*$ implies that $u \in H_M^*$.*

*Proof.* Since $R$ is noetherian complete local, there is some $c \neq 0$ such that $R_c$ is regular. We also have that $R_c^\Gamma$ is regular. Then $c^3$ serves as a big test element for both rings. Then $u^\Gamma \in (H^\Gamma)_{M^\Gamma}^* \Rightarrow c^3 u^\Gamma \in H^\Gamma$. Since $R^\Gamma$ is faithfully flat, we must have $c^3 u \in H$ as well. Then $u \in H_M^*$. $\qquad\square$

**Proposition 4.1.8.** *Let $R$ be a noetherian complete local domain of characteristic $p$ and $S$ a solid $R$-algebra. Then $S$ is a phantom extension of $R$ via tight closure.*

*Proof.* Since $R$ is a noetherian complete local domain of characteristic $p$, it has a faithfully flat local map $R \to R^\Gamma$, where $R^\Gamma$ is a local, $F$-finite algebra. We can base change to $R^\Gamma$ and get a solid $R^\Gamma$-algebra $R^\Gamma \otimes_R S$ (it is still solid because of faithful flatness). We use [Die18, Theorem 4.8] to conclude that $R^\Gamma \otimes_R S$ is a phantom extension of $R^\Gamma$.

Next we make use of the diagram in [Die18, (2.2)]:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R & \overset{\alpha}{\longrightarrow} & S & \longrightarrow & S/R & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\tilde{\nu}} & & \uparrow & & \uparrow{\scriptstyle\mathrm{id}} & & \\
 & & G & \overset{\nu}{\longrightarrow} & F & \longrightarrow & S/R & \longrightarrow & 0
\end{array}
$$

where $G, F$ are free presentation of $S/R$. Since $R^\Gamma$ is flat, this stays as the same diagram over $R^\Gamma$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R^\Gamma & \overset{\alpha}{\longrightarrow} & S \otimes_R R^\Gamma & \longrightarrow & (S/R) \otimes_R R^\Gamma & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\tilde{\nu}^\Gamma} & & \uparrow & & \uparrow{\scriptstyle\mathrm{id}} & & \\
 & & G^\Gamma & \overset{\nu^\Gamma}{\longrightarrow} & F^\Gamma & \longrightarrow & (S/R) \otimes_R R^\Gamma & \longrightarrow & 0
\end{array}
$$

where $\nu^\Gamma, \tilde{\nu}^\Gamma$ are the images of the representing matrix of $\nu, \tilde{\nu}$ in $R^\Gamma$ respectively. By definition ([Die18, Definition 2.3]), we learn that $\tilde{\nu}^\Gamma \in \left(\mathrm{Im}\,(\nu^\Gamma)^\vee\right)^*_{(G^\Gamma)^\vee}$. Finally by Lemma 4.1.7, we conclude that $\tilde{\nu} \in \left(\mathrm{Im}\,\nu^\vee\right)^*_{G^\vee}$. $\qquad\square$

Next we want to remove the domain condition by first dealing with the complete case, then passing to the non-complete case.

**Corollary 4.1.9.** *Let $R$ be a noetherian complete local ring of characteristic $p$ and $S$ a solid $R$-algebra. Then $S$ is a phantom extension of $R$ via tight closure.*

*Proof.* Suppose that $P_1, \ldots, P_n$ are minimal primes of $R$. By the argument above what we need to show is $\tilde{\nu} \in \left(\mathrm{Im}\,\nu^\vee\right)^*_{G^\vee}$. We can apply base change to $R/P_i$ to the diagram for each $i$, and $S \otimes_R R/P_i = S/P_i S$ is still solid because of our definition. Hence, by Proposition 4.1.8, we know that each $S/P_i S$ is a phantom extension of $R/P_i$. So $\tilde{\nu} \in \left(\mathrm{Im}\,\nu^\vee\right)^*_{G^\vee}$ is true if we use base change to any $R/P_i$. By a well-known result of tight closure theory, we can conclude that $\tilde{\nu} \in \left(\mathrm{Im}\,\nu^\vee\right)^*_{G^\vee}$. $\quad\square$

**Corollary 4.1.10.** *Let $R$ be a noetherian local ring of prime characteristic $p$. Suppose that $R$ is reduced and essentially of finite type over an excellent local ring. If $S$ is a solid $R$-algebra, then $S$ is a phantom extension of $R$.*

*Proof.* There exists a nonzerodivisor $c$ such that $R_c$ is regular and $c$ has a power that is a completely stable big test element. Hence there is some power, say $c^N$, of $c$ that serves as big test element for both $R$ and $\widehat{R}$. Since the map $S \to R$ sends $1$ to a nonzerodivisor and $\widehat{R}$ is faithfully flat over $R$, the

image of 1 remains a nonzerodivisor after base change to $\widehat{R}$. Therefore, $S \otimes_R \widehat{R}$ is a solid $\widehat{R}$-algebra. Hence, it is a phantom extension of $\widehat{R}$. We want to prove that $\tilde{\nu} \in (\operatorname{Im} \nu^\vee)^*_{G^\vee}$. Apply base change to $\widehat{R}$ to this diagram. We learn that $\tilde{\nu} \in (\operatorname{Im} \nu^\vee)^*$ in $\widehat{G}^\vee$. So $c^N \tilde{\nu} \in (\operatorname{Im} \nu^\vee)$ in $\widehat{G}^\vee$. Again, by faithful flatness, we conclude that $c^N \tilde{\nu} \in (\operatorname{Im} \nu^\vee)_{G^\vee} \Rightarrow \tilde{\nu} \in (\operatorname{Im} \nu^\vee)^*_{G^\vee}$. Hence, $S$ is a phantom extension of $R$. $\qquad\qquad\square$

## 4.2 Properties of Closure Operations

### 4.2.1 More closure axioms

We define two more axioms for closure operations on noetherian rings without requiring the local condition. Let $R$ be a noetherian ring, cl a closure operation on $R$.

**Definition 4.2.1.** A sequence of elements $x_1, \ldots, x_n$ in $R$ is a *partial strong system of parameters* if $x_1, \ldots, x_n$ form a partial strong system of parameters in $R_P$ for any prime ideal $P$ containing $x_1, \ldots, x_n$.

The closure operation is said to satisfy the colon-capturing property if the following axiom holds.

**Axiom 4.2.2** (Colon-capturing Axiom)**.** Let cl be a closure operation on $R$. If $x_1, \ldots, x_n$ is a partial strong system of parameters in $R$, then $(x_1, \ldots, x_{n-1}) :_R x_n \subseteq (x_1, \ldots, x_{n-1})^{\text{cl}}$.

**Axiom 4.2.3** (Persistence Axiom)**.** Suppose that $\mathcal{C}$ is a collections of rings and homomorphisms among them. Let cl be a closure defined on each ring in $\mathcal{C}$. If for any homomorphism $R \to S$ in $\mathcal{C}$ and any $R$-module $M$ and a submodule $N$, we have

$$\operatorname{Im}(S \otimes_R N^{\text{cl}}_M \to S \otimes_R M) \subseteq (\operatorname{Im}(S \otimes_R N \to S \otimes_R M))^{\text{cl}}_{S \otimes_R M},$$

then we say that cl is a persistent closure with respect to $\mathcal{C}$.

By a "persistent closure operation" we mean a closure operation satisfying Axiom 4.2.3 with respect to some collection. We want to point out that the tight closure satisfies both axioms under mild hypothesis on the rings.

**Definition 4.2.4.** [HH93, Definition 5.1] Let $N \subseteq M$ be arbitrary modules over a noetherian ring $R$. We shall say that $x \in M$ is in the regular closure of $N$ in $M$, which we denote $N^{\text{reg}}_M$, or simply, $N^{\text{reg}}$, if for every homomorphism of $R$ to a regular ring $S$ which maps $R^\circ$ into $S^\circ$, the image of $x$ in $S \otimes_R M$ is in the image of $S \otimes_R N$ in $S \otimes_R M$.

If a closure operation cl is persistent and trivial in regular rings, then for every ideal $I$, $I^{\mathsf{cl}} \subseteq I^{\mathsf{reg}}$. By the result [HH93, Proposition 5.3], one only needs to check regular closure for all maps from $R$ to regular rings with kernel a minimal prime of $R$. In [HH93, Discussion and example 5.6], Hochster and Huneke proved that in the ring $R = k[x, y, z]/(x^3 + y^3 + z^3)$ where $\mathrm{char}(k) \neq 3$, $z \in (x, y)R^{\mathsf{reg}}$ and $z \notin (x, y)R^*$. Therefore regular closure is strictly larger than tight closure.

### 4.2.2  Persistent colon-capturing closure operations

**Definition 4.2.5.** Let $R$ be a noetherian domain. Let $R^+$ be the absolute integral closure of $R$. Then for any ideal $I$, the *plus closure* $I^+$ of $I$ is defined to be the contraction of the expansion $IR^+$ back to $R$.

Suppose that $\Lambda$ is a noetherian domain. Let $\mathcal{C}$ be some collection of $\Lambda$-algebras containing all of the finitely generated $\Lambda$-algebras. Let cl be a persistent closure operation defined on $\mathcal{C}$ that satisfies the colon-capturing axiom (Axiom 4.2.2). We want to show that cl contains the plus closure.

**Theorem 4.2.6.** *Let $R$ be a local domain in $\mathcal{C}$ as above and $I = (f_1, \ldots, f_n)$ a proper ideal in $R$. Let* cl *be a persistent colon-capturing closure operation on $R$. Then $I^+ \subseteq I^{\mathsf{cl}}$.*

*Proof.* If $b \in I^+$, then $b = \sum_{i=1}^n u_i f_i$ where the $u_i \in R^+$ are integral over $R$. Each $u_i$ satisfies a monic equation $g_i(X) = X^{n_i} - z_{i,1} X^{n_i - 1} - \cdots - z_{i,n_i}$ of degree $n_i$ with coefficients in $R$ for some $n_i$.

Let $T_0 = \Lambda[Z_{i,j}, U_i : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n_i]/(g_i(U_i) : 1 \leqslant i \leqslant n)$ where $U_j, Z_{i,j}$ are indeterminates. Then $T_0$ is a polynomial ring since we may use the $i$th equation to solve for $Z_{i,n_i}$. We set $T = T_0[F_1, \ldots, F_n]$ where $F_i$ are indeterminates. Then $T$ is a domain as well. Let $S = \Lambda[F_i, Z_{i,j}, B, Y, U_1 Y, \cdots, U_n Y]$ be a subring of $T$ where $B := \sum_{i=1}^n U_i F_i$. We want to show that $F_1, \ldots, F_n, Y$ form a partial strong system of parameters in $S$.

First of all, we note that $m$ elements in a domain form a (automatically strong) partial system of parameters if and only if they generate an ideal of height $m$: suppose that $y_1, \ldots, y_m$ generate an ideal of height $m$ in the domain $\mathcal{A}$. If $P$ is a prime ideal of $\mathcal{A}$ containing $y_1, \ldots, y_m$, then $(y_1, \ldots, y_m)\mathcal{A}_P$ also has height $m$. Therefore, since $S$ is a domain, it is equivalent to show that $F_1, \ldots, F_n, Y$ generate an ideal of height $n + 1$. For this purpose, let us consider another ring $S_0 = \Lambda[F_i, Z_{i,j}, Y]$. Since $B, U_i Y$ are all algebraic over $S_0$, $S$ is module-finite over $S_0$. In $S_0$, $F_1, \ldots, F_n, Y$ generates an ideal of height $n + 1$. Since $S_0$ is normal, we have the going down theorem for $S_0 \to S$. Hence, $(F_1, \ldots, F_n, Y)S$ also has height $n + 1$.

Since $YB = \sum_{i=1}^n (U_i Y) F_i \in (F_1, \ldots, F_n)S$, we conclude that $B \in (F_1, \ldots, F_n)S^{\mathsf{cl}}$. We have a map $T \to R$ sending $F_i \mapsto f_i$, $Z_{ij} \mapsto z_{ij}$ and $U_i \mapsto u_i$. Hence, we can restrict the map $T \to R$ to get a map $S \to R$. Then, by Axiom 4.2.3, we have $b \in (f_1, \ldots, f_n)R^{\mathsf{cl}} = I^{\mathsf{cl}}$ in $R$. $\qquad\square$

*Remark* 4.2.7. In equal characteristic, we usually take $\Lambda$ to be $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$.

### 4.2.3 Persistence and weak functoriality

Consider a Dietz closure operation cl satisfying the Algebra axiom defined on a collection of noetherian complete local domains. Assume that cl is persistent with respect to this collection. We want to show a weakly functorial version of the existence of big Cohen-Macaulay algebras of rings in this collection.

**Proposition 4.2.8.** *Let $M$ be a finitely generated $R$-module, and $S$ an $R$-algebra. Suppose that* cl *is a closure operation defined for both $R$ and $S$ and persistent for $R \to S$. If $\alpha : R \to M$ is a* $\text{cl}_R$*-phantom extension, then $S \to M \otimes_R S$ is an* $\text{cl}_S$*-phantom extension.*

Before giving the proof, we need a lemma.

**Lemma 4.2.9.** *Using the same notation as in Proposition 4.2.8, if $\alpha \otimes \text{id}_S$ stays injective, then $\alpha \otimes \text{id}_S$ is* $\text{cl}_S$*-phantom.*

*Proof.* Using the same construction as [Die10, (2.8)], we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\alpha} & M & \longrightarrow & Q & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle\phi} & & \uparrow & & \uparrow & & \\
& & R^m & \xrightarrow{\nu} & R^n & \longrightarrow & Q & \longrightarrow & 0
\end{array} \qquad (4.2.1)$$

Then $\alpha$ is cl-phantom if and only if $\phi^\vee \in N_{R^n}^{\text{cl}}$ where $N$ is the submodule spanned by column vectors of $\nu^\vee$. See [Die10, Lemma 2.10].

Base changed to $S$ preserves right exactness and surjections. It also takes free $R$-modules to free $S$-modules. Since by assumption the injectivity of $\alpha$ is also preserved, we conclude that after tensoring the diagram (4.2.1) with $S$ everything is preserved. Since cl is persistent, we conclude that $\alpha \otimes \text{id}_S$ is cl-phantom as well. $\qquad\square$

*Proof of Proposition 4.2.8.* We can always factor a map $R \to S$ as $R \twoheadrightarrow R/P \hookrightarrow S$. The proof breaks down to two steps. First we show that the induced map $R/P \to M \otimes_R R/P$ is $\text{cl}_{R/P}$-phantom. Then we show that the same is true for the injection map $R/P \hookrightarrow S$.

*Step 1*: Write $\mathfrak{m}$ for the maximal ideal of $R$ and $u = \alpha(1) \in M$. Using [Die10, Lemma 2.11] and the assumption that $\alpha$ is $\text{cl}_R$-phantom, we know that $u \notin \mathfrak{m}M$. So $\{u\}$ can be expanded to a minimal generating set of $M$, say $\{u, u_1, ..., u_k\}$. We claim that

$$Ru \cap PM = Pu.$$

It is clear that the right-hand side is always contained in the left-hand side. Suppose that there are elements $r \in R - P$ and $p, p_1, ..., p_k \in P$ such that $ru = pu + p_1u_1 + \cdots + p_ku_k$. Passing to

$M \otimes_R \mathrm{Frac}(R/P)$, we see that $\overline{ru} = 0$. Since $\overline{r}$ is invertible in $\mathrm{Frac}(R/P)$, we have $\overline{u} = 0$. Let $W = R - P$, then $W^{-1}R$ is a flat $R$-algebra. So $\alpha \otimes \mathrm{id}_{W^{-1}R}$ is an injection. By Lemma 4.2.9 we know that $\alpha \otimes \mathrm{id}_{W^{-1}R}$ is cl-phantom. So $u$ is not in $PW^{-1}M$. Hence $\overline{u}$ is not zero in $M \otimes \mathrm{Frac}(R/P)$, a contradiction!

Next we claim that $\alpha \otimes \mathrm{id}_{R/P}$ is still injective. For any $r \in R/P$ such that $\overline{ru} = 0$ in $M/PM$, we have $ru \in PM$. Passing to $M \otimes_R \mathrm{Frac}(R/P)$, we have $\overline{ru} = 0$. Since $\overline{u} \neq 0$, we have $\overline{r} = 0$. But this shows that $r \in P$, i.e., $\overline{r} = 0$ in $R/P$. Hence, $\alpha \otimes \mathrm{id}_{R/P}$ is injective and, therefore, cl-phantom by Lemma 4.2.9.

*Step* 2: Write $T = R/P$, $N$ for $M \otimes R/P$, and $\beta$ for the map $\mathrm{id}_T \otimes \alpha$. We are in the case where $T \to S$ is an injection of noetherian complete local domains. We would like to show that $S \to N \otimes_T S$ is still injection. Since $\mathrm{Frac}(T)$ is flat over $T$, we have injection $\mathrm{Frac}(T) \to N \otimes_T \mathrm{Frac}(T)$. Let $W = T - \{0\} \subseteq S$ be the multiplicative set of nonzero elements of $T$ in $S$. Then $W^{-1}S$ is a $\mathrm{Frac}(T)$-algebra. Hence it is free over $\mathrm{Frac}(T)$. So in turn we have injection $W^{-1}S \to W^{-1}S \otimes_T N$. Let $v = \beta(1)$, and $1 \otimes v$ be the image of $1 \in S$ in $S \otimes_T N$. Let $I = \mathrm{Ann}_S(1 \otimes v)$ be the annihilator of $1 \otimes v$ in $S$. Then we know that $IW^{-1}S = 0$. But $S$ is a domain, so $S \hookrightarrow W^{-1}S$ and $I = 0$. Hence, $\mathrm{id}_S \otimes \beta$ is injective. Again, by Lemma 4.2.9, the proof is complete. $\qquad \square$

For this purpose, we need to discuss some material on modifications.

*Discussion* 4.2.10. Let $(R, \mathfrak{m})$ be a noetherian complete local domain and let $x_1, \ldots, x_n$ be a system of parameters for $R$. Let $M$ be a finitely generated $R$-module. By *a finite sequence of mixed modifications* of $M$ over $R$, we mean a finite sequence of $R$-modules

$$M = M_0 \to M_1 \to M_2 \to \cdots \to M_k$$

such that each map $M_i \to M_{i+1}$ falls into one of the following two cases:

- There is a relation $x_{l+1}u_{l+1} + x_l u_l + \cdots + x_1 u_1 = 0$ for some elements $u_1, \ldots, u_l$ in $M_i$ and $M_i \to M_{i+1}$ is trivializing this relation, i.e.,

$$M_{i+1} := \frac{M_i \oplus RU_1 \oplus \cdots \oplus RU_l}{R \cdot (u_{l+1}, x_1 U_1, \cdots, x_l U_l)}$$

  where $U_1, \ldots, U_l$ are indeterminates. The map $M_i \to M_{i+1}$ is defined by sending elements in $M_i$ to the first copy in $M_{i+1}$.

- $M_{i+1}$ is the second symmetric power of $M_i$, i.e., $M_{i+1} = \mathrm{Sym}^2(M_i)$.

We will call this sequence *bad* if $M = R$ and the image of 1 goes into $\mathfrak{m}M_k$ under the consecutive maps.

Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism between noetherian complete local domains. Let $M$ be a finitely generated $R$-module. By *a finite double sequence of mixed modifications*, we mean a finite sequence of $R$-modules and $S$-modules

$$M = M_0 \to M_1 \to M_2 \to \cdots \to M_k \to M_k \otimes_R S(=: N_0) \to N_1 \to \cdots \to N_h$$

where $M_0 \to \cdots \to M_k$ is a finite sequence of mixed modifications of $M$ over $R$ and $N_0 \to \cdots \to N_h$ a finite sequence of mixed modifications of $S \otimes M_k$ over $S$. We call this sequence *bad* if $M = R$ and the image of 1 in $R$ goes into $\mathfrak{n} N_h$ under the consecutive maps.

We are ready to prove the following main theorem of this section.

**Theorem 4.2.11.** *If $R \to S$ is a ring map between noetherian complete local domains and* cl *is a persistent Dietz closure satisfying the Algebra axiom with respect to the collection of noetherian complete local domains and ring maps between them, then we obtain a weakly functorial version of the existence of their big Cohen-Macaulay algebras, i.e., there exists a big Cohen-Macaulay $R$-algebra $B$ and a big Cohen-Macaulay $S$-algebra $C$ such that*

$$
\begin{array}{ccc}
B & \longrightarrow & C \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

*commutes.*

*Proof.* We follow the same construction in the proof of [R.G18, Theorem 3.3]. Explicitly, we first construct a big Cohen-Macaulay module $B_1$ of $R$ and then we take the $\mathrm{Sym}(B_1)/(1 - e_1)\mathrm{Sym}(B_1)$ as in [R.G18, Remark 3.2]. We iterate these two steps infinitely many times and take the limit $B$. Then $B$ is a big Cohen-Macaulay $R$-algebra. Then we perform the same operation for the $S$-algebra $B \otimes_R S$ and get an $S$-algebra $C$. If $C$ is not a big Cohen-Macaulay $S$-algebra, then by [HH95, Section 3] we know that there is a bad finite double sequence of mixed modifications

$$R = M_0 \to M_1 \to M_2 \to \cdots \to M_k \to M_k \otimes_R S(=: N_0) \to N_1 \to \cdots \to N_h.$$

Since cl is a Dietz closure satisfying the Algebra axiom, by [Die10, Lemma 2.11] and [Die10, Propsition 3.15], we know that each modification in $R = M_0 \to M_1 \to M_2 \to \cdots \to M_k$ is an cl-phantom extension. By Proposition 4.2.8, the cl-phantomness is preserved after base changed to $S$, so this sequence cannot go bad and we get a contradiction. $\square$

*Remark* 4.2.12. If we have a persistent Dietz closure cl not necessarily satisfying the Algebra axiom, then we can show a weak functorial version of the existence of big Cohen-Macaulay modules over

$R \to S$. Explicitly, Proposition 4.2.8 stills holds. We can modify the argument in [Hoc02a, Theorem 4.2] to give a similar result for a double sequence of module modifications. Then the same argument in Theorem 4.2.11 works.

## 4.3  Big Cohen-Macaulay Algebra Closures

We define two new closure operations.

**Definition 4.3.1.** Let $(R, \mathfrak{m})$ be a noetherian local ring and $I \subseteq R$ a proper ideal of $R$. The *big Cohen-Macaulay algebra closure* $I^{\mathsf{BCM}}$ of $I$ is the smallest ideal $J \supseteq I$ such that for every big Cohen-Macaulay $R^+$-algebra $S$, we have $JS \cap R = J$. If $R$ is a noetherian complete local domain of mixed characteristic, then the *perfectoid big Cohen-Macaulay algebra closure* $I^{\mathsf{PBCM}}$ of $I$ is defined to be the smallest ideal $J \supseteq I$ such that for every integral perfectoid big Cohen-Macaulay $R^+$-algebra $S$, we have $JS \cap R = J$.

From the definition, we immediately see that

**Proposition 4.3.2.** *If $R$ is a noetherian complete local domain of mixed characteristic, then for any ideal $I \subseteq R$, we have $I^{\mathsf{PBCM}} \subseteq I^{\mathsf{BCM}}$.*

It is easy to see that BCM (PBCM) is a closure operation. In fact, we have the following characterizations.

**Lemma 4.3.3.** *Let $R$ be a noetherian local domain (resp., a noetherian complete local domain of mixed characteristic). For any element $u \in R$, if there is a big Cohen-Macaulay algebra (resp., integral perfectoid big Cohen-Macaulay algebra) $B$ such that $u \in IB$, then $u \in I^{\mathsf{BCM}}$ (resp., $u \in I^{\mathsf{PBCM}}$).*

*Proof.* Let $J = I^{\mathsf{BCM}}$, then $JB \cap R = J$. Since $u \in IB \subseteq JB$, we have $u \in JB \cap R \Rightarrow u \in J$. The proof for PBCM is similar. $\qquad\square$

In order to get a more precise description for BCM, we need a temporary definition.

**Definition 4.3.4.** Let $R$ be a noetherian local domain, and let $I \subseteq R$ be an ideal. The notion $I^{\mathsf{BCL}}$ denotes the ideal generated by $IB \cap R$ for all big Cohen-Macaulay $R^+$-algebras $B$, i.e., $I^{\mathsf{BCL}} = (IB \cap R : B)R$.

**Lemma 4.3.5.** *Let $R$ be a noetherian local domain. For any proper ideal $I \subseteq R$, let $J_0 = I$ and $J_{i+1} = J_i^{\mathsf{BCL}}$. Then $J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$ is an ascending sequence of ideals. Hence, it stabilizes at some $J_N$. Then $J_N = I^{\mathsf{BCM}}$.*

*Proof.* For each big Cohen-Macaulay $R^+$-algebra $B$, if $J_i \subseteq I^{\mathsf{BCM}}$, then $(J_i B \cap R) R \subseteq I^{\mathsf{BCM}} B \cap R = I^{\mathsf{BCM}}$. Hence, $J_{i+1} = J_i^{\mathsf{BCL}} = (J_i B \cap R : B) R \subseteq I^{\mathsf{BCM}}$. Therefore, we have that $J_i \subseteq J_{i+1} \subseteq I^{\mathsf{BCM}}$. Suppose that this sequence stabilizes at $N$, i.e., $J_{N+1} = J_N$. Then, we have $J_N = (J_N B \cap R : B) R$. For each big Cohen-Macaulay $R^+$-algebra $B$, we have $J_N \subseteq J_N B \cap R \subseteq (J_N B \cap R : B) R = J_N$, which shows that $J_N = J_N B \cap R$. By the definition of $I^{\mathsf{BCM}}$, we know that $I^{\mathsf{BCM}} = J_N$. $\square$

The same proof of Lemma 4.3.5 works for PBCM. In fact, we have a much simpler description. First, we need a definition similar to Definition 4.3.4.

**Definition 4.3.6.** Let $R$ be a noetherian complete local domain of mixed characteristic, and let $I \subseteq R$ be an ideal. We let $I^{\mathsf{PBCL}}$ be the set

$$I^{\mathsf{PBCL}} = \{u \in R \mid \exists \text{ an integral perfectoid big Cohen-Macaulay } R^+\text{-algebra } B \text{ such that } u \in IB\}.$$

**Lemma 4.3.7.** *Let $R$ be a noetherian complete local domain of mixed characteristic, and let $I \subseteq R$ be an ideal. The set $I^{\mathsf{PBCL}}$ is in fact an ideal and there is some integral perfectoid big Cohen-Macaulay algebra $C$ such that $I^{\mathsf{PBCL}} C = IC$.*

*Proof.* Let $u_1, u_2 \in I^{\mathsf{PBCL}}$. There are two integral perfectoid big Cohen-Macaulay $R^+$-algebras $B_1, B_2$ such that $u_1 \in IB_1, u_2 \in IB_2$. There exists a third integral perfectoid big Cohen-Macaulay $R^+$-algebra $B$ dominates $B_1$ and $B_2$ by [MS18b, Theorem 4.9]. We have $u_1, u_2$ both in $IB$ and so is their sum. Hence $u_1 + u_2 \in I^{\mathsf{PBCL}}$.

Since $R$ is noetherian, $I^{\mathsf{PBCL}}$ is generated by finitely many elements, say, $f_1, \ldots, f_n$. For each $f_i$, there is some perfectoid big Cohen-Macaulay $R^+$-algebra $B_i$ such that $f_i \in IB_i$. Take a perfectoid big Cohen-Macaulay $R^+$-algebra $C$ that dominates all $B_i$. Then $I^{\mathsf{PBCL}} C \subseteq IC \subseteq I^{\mathsf{PBCL}} C$. $\square$

**Proposition 4.3.8.** *Let $R$ be a noetherian complete local domain of mixed characteristic. For any ideal $I \subseteq R$, $I^{\mathsf{PBCL}} = I^{\mathsf{PBCM}}$.*

*Proof.* By Lemma 4.3.3, we know that $I^{\mathsf{PBCL}} \subseteq I^{\mathsf{PBCM}}$. On the other hand, let $J = I^{\mathsf{PBCL}}$. We need to verify that $JB \cap R = J$ for any integral perfectoid big Cohen-Macaulay $R^+$-algebra $B$. Let $D$ be an integral perfectoid big Cohen-Macaulay $R^+$-algebra that dominates both $B$ and $C$ by [MS18b, Theorem 4.9]. Since $JC = IC$, we have $JD = ID$. Therefore, we have

$$J \subseteq JB \cap R \subseteq JD \cap R = ID \cap R \subseteq J.$$

Then by the definition of PBCM, we conclude that $J = I^{\mathsf{PBCM}}$. $\square$

A nice property of both PBCM and BCM is that they naturally capture colons.

## 4.4 Comparison of Closures

We prove some comparison theorems about $\mathrm{BCM}, \mathrm{PBCM}, \mathrm{epf}$ and wepf for ideals in noetherian complete local domains with $F$-finite residue field.

Since both BCM and PBCM are closely related to big Cohen-Macaulay algebras, it is convenient to have the following definition to simplify our arguments (cf. [Die07, Definition 3.1], [MS18b, Definition 4.7]).

**Definition 4.4.1.** Let $(R, \mathfrak{m})$ be a local noetherian ring (resp., a noetherian complete local domain of mixed characteristic). An $R$-algebra $S$ is called a *seed (resp., perfectoid seed)* over $R$ if $S$ maps to a big Cohen-Macaulay $R$-algebra (resp., an integral perfectoid big Cohen-Macaulay $R$-algebra).

We also want to discuss the notion of algebra modifications.

*Discussion* 4.4.2. Let $(R, \mathfrak{m})$ be a noetherian local ring and $B$ an $R$-algebra. By *a sequence of algebra modification of $B$* we mean a sequence of $R$-algebras

$$B = B_0 \to B_1 \to \cdots \to B_k$$

such that each $B_{i+1}$ is constructed to be $B_{i+1} = B_i[Z_1, \ldots, Z_n]/(z_{n+1} - x_1 Z_1 - \cdots - x_n Z_n)$ for a relationship $z_1 x_1 + \cdots + z_{n+1} x_{n+1} = 0$ in $B_i$ where $Z_1, \ldots, Z_n$ are $n$ indeterminates over $B_i$ and $x_1, \ldots, x_n$ are part of a system of parameters for $R$. We also say that the map $B_i \to B_{i+1}$ is trivializing the relation $u_1 x_1 + \cdots + u_{n+1} x_{n+1} = 0$. This sequence is called a *bad* sequence of modifications if for some index $t$, $1$ maps to $\mathfrak{m} B_t$. It is shown in [HH95, Section 3] that $B$ can be mapped to a balanced big Cohen–Macaulay algebra for $R$, i.e., $B$ is a seed over $R$, if and only if $B$ does not possess a bad sequence of modifications.

**Lemma 4.4.3.** *Let $(R, \mathfrak{m})$ be a noetherian complete local domain of mixed characteristic. Let $I$ be an ideal of $R$. We have $\bigcap_N (I + \mathfrak{m}^N)^{\mathsf{PBCM}} \subseteq I^{\mathsf{BCM}}$.*

*Proof.* For any $a \in \bigcap_N (I + \mathfrak{m}^N)^{\mathsf{PBCM}}$, there is a big enough perfectoid big Cohen-Macaulay $B$ such that $a \in (I + \mathfrak{m}^N) B$ for any positive integer $N$. Then we have $a \in IB/\mathfrak{m}^N B$ for any $N$. Suppose that $I$ is generated by $f_1, \ldots, f_h$. If $S = \dfrac{B[X_1, \ldots, X_h]}{u - f_1 X_1 - \cdots - f_h X_h}$, then there is a well-defined map $S \to B/\mathfrak{m}^N B$ for any $N$. Suppose that $x_1, \ldots, x_n$ is a system of parameters in $R$. Since $\mathfrak{m}^N$ is cofinal with $(x_1^{N_1}, \ldots, x_n^{N_n}) R$, we also have maps $S \to B/(x_1^{N_1}, \ldots, x_n^{N_n}) B$ for any $(N_1, \ldots, N_n) \in \mathbb{N}^n$.

By Discussion 4.4.2, we want to show that $S$ does not possess a bad sequence of modifications, which will imply that $S$ is a seed. Suppose for contradiction that we have a bad sequence of algebra modifications $S \to S_1 \to \cdots \to S_t$ such that $1 \mapsto \mathfrak{m} S_t$. Then we show that each $S_i$ maps to a certain quotient of $B$, and we get a contradiction from there. Let $N = t + 1$. We have a map

89

$\alpha_0 : S \to B/(x_1^N, \ldots, x_n^N)B$. Suppose that we have a map $\alpha_i : S_i \to B/(x_1^{N_1}, x_2^{N_2}, \ldots, x_n^{N_n})$. Let $S_{i+1}$ be an algebra modification of $S_i$ trivializing the following relation:

$$x_{k+1}s = x_1 s_1 + \cdots + x_k s_k.$$

This relation also holds in $B/(x_1^{N_1}, x_2^{N_2}, \ldots, x_n^{N_n})$. Hence, we have

$$x_{k+1}s \in (x_1, \ldots, x_k)B/(x_1^{N_1}, x_2^{N_2}, \ldots, x_n^{N_n})$$
$$\Rightarrow x_{k+1}s \in (x_1, \ldots, x_k, x_{k+1}^{N_{k+1}}, \ldots, x_n^{N_n})B$$
$$\Rightarrow s \in (x_1, \ldots, x_k, x_{k+1}^{N_{k+1}-1}, x_{k+2}^{N_{k+2}}, \ldots, x_n^{N_n})B.$$

Therefore, we have $s = x_1 b_1 + \cdots + x_k b_k$ in $B/(x_1^{N_1}, \ldots, x_k^{N_k}, x_{k+1}^{N_{k+1}-1}, x_{k+2}^{N_{k+2}}, \ldots, x_n^{N_n})B$. We can construct a map

$$\alpha_{i+1} : S_{i+1} := \frac{S_i[Y_1, \ldots, Y_k]}{s - Y_1 x_1 - \cdots - Y_k x_k} \to B/(x_1^{N_1}, \ldots, x_k^{N_k}, x_{k+1}^{N_{k+1}-1}, x_{k+2}^{N_{k+2}}, \ldots, x_n^{N_n})B$$

where we map the polynomial in $S_i[Y_1, \ldots, Y_k]$ by applying $\alpha_i$ to its coefficients and sending $Y_i \mapsto b_i$. If this is a bad sequence of modifications, then $1 \in \mathfrak{m}S_t$, which implies that $1 \in \mathfrak{m}B/(x_1^{N_1'}, x_2^{N_2'}, \ldots, x_n^{N_n'})B$. This is equivalent to $1 \in \mathfrak{m}B$, which is a contradiction! $\square$

Next we show that $I^{\mathrm{epf}} \subseteq I^{\mathrm{BCM}}$. First we give a proof using a trick due to Ofer Gabber. Then we present a proof using the $p$-colon-capturing property (Theorem 3.2.4).

*Remark* 4.4.4 (Gabber's trick). Let $(R, \mathfrak{m})$ be a noetherian complete local domain, $I \subseteq R$ an ideal, and $u \in R$ an element. Suppose that there exists some nonzero element $c \in R$ such that $c^\varepsilon u \in IB$ for any $\varepsilon \in \mathbb{Q}^+$, where $B$ is a perfectoid big Cohen-Macaulay $R^+$-algebra. Let $\overline{B} = \prod_{\mathbb{N}} B$ and let $S_c$ be the multiplicatively closed set given by $\{(c^{\varepsilon_1}, c^{\varepsilon_2}, \cdots)\}$ where $\varepsilon_i \to 0$ as $i \to \infty$. Then $S_c^{-1}(\prod_{\mathbb{N}} B)$ is a big Cohen-Macaulay algebra and its $\mathfrak{m}$-adic completion $\widetilde{B}$ is a perfectoid big Cohen-Macaulay algebra. In particular, we have $u \in I\widetilde{B}$. In short, whenever we have an "almost" membership, we can make it an honest membership by passing to a bigger perfectoid big Cohen-Macaulay algebra.

**Proposition 4.4.5.** *Let $(R, \mathfrak{m})$ be a noetherian complete local domain of mixed characteristic $p$ with $F$-finite residue field. If $I \subseteq R$ is a proper ideal, then $I^{\mathrm{epf}} \subseteq I^{\mathrm{BCM}}$.*

*Proof.* Suppose that $u \in I^{\mathrm{epf}}$. There exists some $c \in R - \{0\}$ such that $c^\varepsilon u \in (I, p^N)R^+$. Let $B$ be an integral perfectoid big Cohen-Macaulay $R^+$-algebra. Then $c^\varepsilon u \in (I, p^N)B$. By Remark 4.4.4, there exists an integral perfectoid big Cohen-Macaulay $R^+$-algebra $\widetilde{B}$ such that $u \in (I, p^N)\widetilde{B}$ for all $N$. Hence, $u \in \cap_N (I, p^N)^{\mathrm{PBCM}} \subseteq I^{\mathrm{BCM}}$ by Lemma 4.4.3. $\square$

Next, we show that there is a different proof. For this purpose, we need do discuss the notion of partial algebra modifications (cf. [Hoc02a, Definition 4.1]).

*Discussion* 4.4.6 (Partial algebra modifications). Let $(R, \mathfrak{m})$ be a noetherian local ring. Let $M$ be an $R$-module. A *partial algebra modification* of $M$ is a map $M \to M'$ where $M'$ is an $R$-module obtained as follows: if $x_1, \ldots, x_{k+1}$ are part of a system of parameters for $R$, and we have a relation $x_1 u_1 + \cdots + x_k u_k + x_{k+1} u_{k+1} = 0$ where $u_i \in M$ for some integer $k \geqslant 0$, then we choose indeterminates $U_1, \ldots, U_k$ and an integer $N \geqslant 1$, and let

$$M' = M[U_1, \ldots, U_k]_{\leqslant N}/FM[U_1, \ldots, U_k]_{\leqslant N-1}$$

where $F = u_{k+1} - x_1 U_1 - \cdots - x_k U_k$. This makes sense because $F$ has degree 1 in the $U_j$. We will refer to the integer $N$ as the *degree bound* of the partial algebra modification. Note that if $B$ is an $R$-algebra and one takes the direct limit over $N$ of the $B$ for fixed $k, x_1, \ldots, x_{k+1}, U_1, \ldots, U_k$ and $F$, one obtains an algebra modification of $B$. We can define a sequence of partial algebra modifications of an $R$-module $B$ as in in Discussion 4.4.2, and, when $B$ is an $R$-algebra, we call the sequence *bad* precisely if the image of $1 \in B$ in $M_t$ is in $\mathfrak{m}M_t$ for some $t \geqslant 0$. [Hoc02a, Theorem 4.2] shows that $T$ is a seed if and only if $T$ does not possess a bad sequence of partial algebra modifications.

**Proposition 4.4.7.** *Let $(R, \mathfrak{m})$ be a noetherian complete local domain of mixed characteristic $p$ with $F$-finite residue field. If $I \subseteq R$ is a proper ideal, then $I^{\mathsf{epf}} \subseteq I^{\mathsf{BCM}}$.*

*Proof.* Suppose that $a \in I^{\mathsf{epf}}$. Then there is an element $c_0 \in R$ such that $c_0^{\varepsilon_0} a \in (I, p^{N_0})R^+$ for any positive integer $N_0$ and any positive rational number $\varepsilon_0 \in \mathbb{Q}^+$. Suppose that $I$ is generated by $f_1, \ldots, f_h$ in $R$.

Let $T_0 = R[X_1, \ldots, X_h]_{\leqslant D_0}/(a - X_1 f_1 - \cdots - f_h X_h)R[X_1, \ldots, X_h]_{\leqslant D_0-1}$ be a partial algebra modification of $R$. Then we want to show that $T_0$ is a seed, i.e., $T_0$ maps to a big Cohen-Macaulay algebra $B$. Then $a \in IT_0 \Rightarrow a \in IB$.

We first note that $T_0$ maps to $R_{c_0}^+/p^{N_0}R_{c_0}^+$:

$$c_0^{\varepsilon_0} a \in (I, p^{N_0})R^+$$
$$\Rightarrow c_0^{\varepsilon_0} a = f_1 a_1 + \cdots + f_h a_h + p^{N_0} b$$
$$\Rightarrow a = f_1(c_0^{-\varepsilon_0} a_1) + \cdots + f_h(c_0^{-\varepsilon_0})a_h \in R_{c_0}^+/p^{N_0}R_{c_0}^+$$

where $a_1, \ldots, a_h, b \in R^+$. Call this map $\alpha_0$. Then $\mathrm{Im}(\alpha_0) \subseteq c_0^{-D_0 \varepsilon_0} R^+/p^{N_0} R^+$.

By Discussion 4.4.6, suppose for contradiction that there is a bad sequence of partial algebra modifications of $T_0$:

$$T_0 \to T_1 \to T_2 \to \cdots \to T_t$$

such that each partial algebra modification $T_i$ has degree bound $D_i$ and $1 \in T_0$ maps to some element in $\mathfrak{m}T_t$. Let $N_0, N_1, \ldots, N_t$ be a sequence of decreasing positive integers (which we will specify later). We want to construct following $R$-module maps:

$$\begin{array}{ccccccc}
R_{c_0}^+/p^{N_0} R_{c_0}^+ & \longrightarrow & R_{c_0 c_1}^+/p^{N_1} R_{c_0 c_1}^+ & \dashrightarrow & R_{c_0 c_1 \cdots c_{t-1}}^+/p^{N_{t-1}} R_{c_0 c_1 \cdots c_{t-1}}^+ & \longrightarrow & R_{c_0 c_1 \cdots c_t}^+/p^{N_t} R_{c_0 c_1 \cdots c_t}^+ \\
\Big\uparrow{\scriptstyle \alpha_0} & & \Big\uparrow{\scriptstyle \alpha_1} & & \Big\uparrow{\scriptstyle \alpha_{t-1}} & & \Big\uparrow{\scriptstyle \alpha_t} \\
T_0 & \longrightarrow & T_1 & \dashrightarrow & T_{t-1} & \longrightarrow & T_t
\end{array}$$

such that the image of $\alpha_i$ is contained in $(c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}})^{-1} R^+/p^{N_i}$ where $\nu_{j,i} = (\varepsilon_j D_j) \prod_{k=j+1}^i (D_k + 1)$. Let $\alpha_0$ be the map we constructed above. We will inductively construct the diagram above. Let us assume that

- The map $\alpha_i$ has been constructed.

- $T_i \to T_{i+1}$ is trivializing the relation $x_{k+1} u_{k+1} = x_1 u_1 + \cdots + x_k u_k$ where $u_1, \ldots, u_k, u_{k+1} \in T_i$ and $x_1, \ldots, x_{k+1}$ is part of a system of parameters in $R$, i.e.,

$$T_{i+1} = \frac{T_i[e_1, \cdots, e_k]_{\leqslant D_i}}{(u_{k+1} - x_1 e_1 - \cdots - x_k e_k) T_i[e_1, \cdots, e_k]_{\leqslant D_i - 1}}.$$

Since $\alpha_i$ has been constructed, in $R_{c_0 c_1 \cdots c_i}^+/P^{N_i} R_{c_0 c_1 \cdots c_i}^+$ we have

$$\overline{x}_{k+1} \alpha_i(u_{k+1}) = \overline{x}_1 f_i(u_1) + \cdots + \overline{x}_k f_i(u_k).$$

The image is in $(c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}})^{-1} R^+/p^{N_i}$, so we have that

$$c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}} \overline{x}_{k+1} \alpha_i(u_{k+1}) \in (\overline{x}_1, \cdots, \overline{x}_k) R^+/p^{N_i} R^+.$$

Since $c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}} \alpha_i(u_{k+1}) \in R^+/p^{N_i} R^+$, we can choose some $v_{k+1} \in R^+$ that maps to it. Then $x_{k+1} v_{k+1} \in (x_1, \ldots, x_k, p^{N_i}) R^+$. By $p$-colon-capturing (Theorem 3.2.4), we have

$$v_{k+1} \in (x_1, \ldots, x_k, p^{N_i}) :_{R^+} x_{k+1} \subseteq \left((x_1, \ldots, x_k, p^{N_i - N_i'}) R^+\right)^{\mathsf{epf}}.$$

We know that there is some $N_i' \in \mathbb{N}$ such that there exists some $c_{i+1}$ such that for any $\varepsilon_{i+1} \in \mathbb{Q}^+$, we have $c_{i+1}^{\varepsilon_{i+1}} v_{k+1} \in (x_1, \ldots, x_k, p^{N_i - N_i'}) R^+$. Let $N_{i+1} = N_i - N_i'$. Then we have

$$\overline{c}_{i+1}^{\varepsilon_{i+1}} c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}} \overline{\alpha_i(u_{k+1})} \in (\overline{x_1}, \ldots, \overline{x_k}) R^+/p^{N_{i+1}} R^+,$$

as $v_{k+1}$ and $c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}} \alpha_i(u_{k+1})$ map to the same element in a further quotient. There are elements

$v_1, \ldots, v_k \in R^+/p^{N_{i+1}}R^+$ such that

$$\overline{c}_{i+1}^{\varepsilon_{i+1}} c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}} \overline{\alpha_i(u_{k+1})} = \overline{x_1} v_1 + \cdots + \overline{x_k} v_k.$$

So the map $e_i$ to $(c_{i+1}^{\varepsilon_{i+1}} c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}})^{-1} v_i$ extends $\alpha_i$ and gives us a map

$$\alpha_{i+1} : T_{i+1} \to R^+_{c_0 \cdots c_{i+1}}/p^{N_{i+1}} R^+_{c_0 \cdots c_{i+1}}.$$

Then the worst denominator one can get from $\alpha_{i+1}$ is $(c_{i+1}^{\varepsilon_{i+1}} c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}})^{D_{i+1}} \cdot c_i^{\nu_{i,i}} \cdots c_1^{\nu_{1,i}} c_0^{\nu_{0,i}} = c_{i+1}^{\nu_{i+1,i+1}} \cdots c_1^{\nu_{1,i+1}} c_0^{\nu_{0,i+1}}$. Hence the image of $\alpha_{i+1}$ is contained in $(c_{i+1}^{\nu_{i+1,i+1}} \cdots c_1^{\nu_{1,i+1}} c_0^{\nu_{0,i+1}})^{-1} R^+/p^{N_{i+1}} R^+$. We also note that each $N_i'$ only depends on the all $x_i$ in the relation trivialized at that step. Hence, it is determined before constructing all $\alpha_i$. We can choose $N_0$ larger than the sum of all $N_i'$. Then we have $N_t > 0$.

Since this sequence of modifications is bad, $1 \in R$ maps to $\mathfrak{m} T_t$, i.e.,

$$\alpha_t(1) \in (c_t^{\nu_{t,t}} \cdots c_1^{\nu_{1,t}} c_0^{\nu_{0,t}})^{-1} \mathfrak{m} R^+/p^{N_t} R^+.$$

Therefore we have $c_t^{\nu_{t,t}} \cdots c_1^{\nu_{1,t}} c_0^{\nu_{0,t}} \in \mathfrak{m} R^+/p^{N_t} R^+$ which implies that

$$c_t^{\nu_{t,t}} \cdots c_1^{\nu_{1,t}} c_0^{\nu_{0,t}} \in \mathfrak{m} R^+.$$

Finally, we observe that $D_i$ are determined by the sequence $T_i$ and we are allowed to choose any $\varepsilon_i \in \mathbb{Q}^+$. So each $\nu_{j,i}$ is a constant multiplied by an arbitrarily small rational number. Therefore, we get a sequence of elements of arbitrarily small values in $\mathfrak{m} R^+$. But this is a contradiction, as $\mathfrak{m} R^+$ is finitely generated. $\qquad\square$

If the ideal $I = (x_1, \ldots, x_n)R$ happens to be generated by a partial system of parameters in $R$, where $R$ is a noetherian complete local domain with $F$-finite residue field. We would like to show that $I^{\text{epf}} \subseteq I^{\text{PBCM}}$ and $I^{\text{wepf}} \subseteq I^{\text{BCM}}$. We first need a lemma.

**Lemma 4.4.8.** *Let $I = (x_1, \ldots, x_n)R$ be an ideal generated by partial system of parameters in $R$, where $R$ is a noetherian complete local domain with $F$-finite residue field. Then there exists $N_1, \ldots, N_n$ such that for any $u_1, \ldots, u_n \in R^+$ with*

$$\sum_{i=1}^{n} x_i u_i \in p^N R^+$$

*where $N$ is some integer, we have the following:*
   *there exists $c_1, \ldots, c_n$ such that for any $\varepsilon$ we have $v_j^{(i)} \in R^+$ such that*

$$\left(\prod_{j=k}^{n} c_j\right)^{\varepsilon} u_k - \sum_{i=1}^{k-1} v_i^{(k)} x_i + \left(\sum_{i=k+1}^{n} \left(\prod_{j=k}^{i-1} c_j\right)^{\varepsilon} v_k^{(i)} x_i\right) \in p^{N-\Sigma_{i=k+1}^{n} N_i} R^+, \tag{4.4.1}$$

$$\sum_{i=1}^{k} \left(\left(\prod_{j=k+1}^{n} c_j\right)^{\varepsilon} u_i + \sum_{j=k+1}^{n} \left(\prod_{l=k+1}^{j-1} c_j\right)^{\varepsilon} v_i^{(j)} x_j\right) x_i \in p^{N-\Sigma_{i=k+1}^{n} N_i} R^+. \tag{4.4.2}$$

*In particular, if we let $N' = \sum_{i=1}^{n} N_i$, then both are in $p^{N-N'} R^+$. Thus we can re-write (4.4.1) as*

$$\left(\prod_{j=k}^{n} c_j\right)^{\varepsilon} u_k = \sum_{i=1}^{k-1} v_i^{(k)} x_i - \left(\sum_{i=k+1}^{n} \left(\prod_{j=k}^{i-1} c_j\right)^{\varepsilon} v_k^{(i)} x_i\right) + p^{N-N'} w_k$$

*for some $w_1, \ldots, w_n \in R^+$. If we multiply $\sum_{i=1}^{n} u_i x_i \in p^{N-N'} R^+$ by $\left(\prod_{j=1}^{n} c_j\right)^{\varepsilon}$, then we get*

$$\left(\prod_{j=1}^{n} c_j\right)^{\varepsilon} \left(\sum_{i=1}^{n} u_i x_i\right) = \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} c_j\right)^{\varepsilon} p^{N-N'} w_k x_k \in (p^{N-N'} x_1, \ldots, p^{N-N'} x_n) R^+.$$

*Proof.* We will use $(4.4.1)_m, (4.4.2)_m$ to refer to the case $k = m$ in (4.4.1), (4.4.2), respectively. We will prove these two statements simultaneously by induction on $k$. Note that $(4.4.1)_n$ is true, i.e., $\sum_{i=1}^{n} u_i x_i \in p^N R^+$.

Define

$$V(i, k, n) = \left(\prod_{j=k+1}^{n} c_j\right)^{\varepsilon} u_i + \sum_{j=k+1}^{n} \left(\prod_{l=k+1}^{j-1} c_j\right)^{\varepsilon} v_i^{(j)} x_j.$$

Now assume that $(4.4.2)_k$ is true, we will show that this implies both $(4.4.2)_{k-1}$ and $(4.4.1)_k$. Rewrite $(4.4.2)_k$:

$$\sum_{i=1}^{k-1} V(i, k, n) x_i + V(k, k, n) x_k \in p^{N-\Sigma_{i=k+1}^{n} N_i} R^+$$

$$\Rightarrow V(k, k, n) \in \left(x_1, \ldots, x_{k-1}, p^{N-\Sigma_{i=k+1}^{n} N_i}\right) :_{R^+} x_k.$$

By Theorem 3.2.4, we can find an $N_k$ such that there is some $c_k$ such that for any $\delta$ we have

$$c_k^{\delta} V(k, k, n) \in \left(x_1, \ldots, x_{k-1}, p^{N-\Sigma_{i=k+1}^{n} N_i - N_k}\right) R^+.$$

So we can write

$$c_k^{\delta} V(k, k, n) - \left(\sum_{j=1}^{k-1} v_j^{(k)} x_j\right) \in p^{N-\Sigma_{i=k}^{n} N_i} R^+. \tag{4.4.3}$$

Choose $\delta = \varepsilon$, then,

$$c_k^\varepsilon \left( \left( \prod_{j=k+1}^{n} c_j \right)^\varepsilon u_k + \sum_{j=k+1}^{n} \left( \prod_{l=k+1}^{j-1} c_j \right)^\varepsilon v_k^{(j)} x_j \right) - \left( \sum_{j=1}^{k-1} v_j^{(k)} x_j \right) \in p^{N-\sum_{i=k}^{n} N_i} R^+,$$

which is

$$\left( \prod_{j=k}^{n} c_j \right)^\varepsilon u_k - \left( \sum_{j=1}^{k-1} v_j^{(k)} x_j \right) + \sum_{j=k+1}^{n} \left( \prod_{l=k}^{j-1} c_j \right)^\varepsilon v_k^{(j)} x_j \in p^{N-\sum_{i=k}^{n} N_i} R^+,$$

and this proves $(4.4.1)_k$.

On the other hand, by (4.4.3) we have

$$\sum_{i=1}^{k-1} c_k^\varepsilon V(i, k, n) x_i + \left( \sum_{i=1}^{k-1} v_i^{(k)} x_i \right) x_k \in p^{N-\sum_{i=k}^{n} N_i} R^+$$

$$\Rightarrow \sum_{i=1}^{k-1} \left( c_k^\varepsilon V(i, k, n) + v_i^{(k)} x_k \right) x_i \in p^{N-\sum_{i=k}^{n} N_i} R^+$$

$$\Rightarrow \sum_{i=1}^{k-1} V(i, k-1, n) x_i \in p^{N-\sum_{i=k}^{n} N_i} R^+,$$

which proves $(4.4.2)_{k-1}$. $\qquad\square$

**Lemma 4.4.9.** *Let $I = (x_1, \ldots, x_n)R$ be an ideal generated by partial system of parameters in $R$, where $R$ is a noetherian complete local domain with $F$-finite residue field. If $u \in \cap_{N=1}^{\infty}(I, p^N)R^+$, then there exists an integral perfectoid big Cohen-Macaulay algebra $B$ such that $u \in IB$.*

*Proof.* Suppose that $u \in \cap_{N=1}^{\infty}(I, p^N)R^+$, we have

$$u = x_1 u_{1,1} + \cdots + x_n u_{1,n} + p v_1$$

for $u_{1,1}, \ldots, u_{1,n} \in R^+, v_1 \in R^+$. Then we have $p v_1 \in \cap_{N=1}^{\infty}(I, p^N)R^+$, so we write

$$p v_1 = x_1 u_{2,1} + \cdots + x_n u_{2,n} + p^2 v_2.$$

Inductively we can write

$$p^{k-1} v_{k-1} = x_1 u_{k,1} + \cdots + x_n u_{k,n} + p^k v_k$$

with each $\sum_{i=1}^{n} x_i u_{k,i} \in p^{k-1} R^+$.

Choose $N'$ from Lemma 4.4.8 and let $k \geqslant N'$. Apply Lemma 4.4.8 to see that for each such $k$,

there exists $c$ such that for any $\varepsilon$, we have

$$c^\varepsilon \sum_{i=1}^{n} x_i u_{ki} \in (p^{k-N'} x_1, \dots, p^{k-N'} x_n) R^+.$$

By Remark 4.4.4, we can find an integral perfectoid big Cohen-Macaulay algebra $B_k$ such that

$$\sum_{i=1}^{n} x_i u_{ki} \in (p^{k-N'} x_1, \dots, p^{k-N'} x_n) B_k.$$

By [MS18b, Theorem 4.9], we can choose an integral perfectoid big Cohen-Macaulay algebra $B$ that dominates all these $B_k$. Then we have

$$\sum_{i=1}^{n} x_i u_{ki} \in (p^{k-N'} x_1, \dots, p^{k-N'} x_n) B.$$

In $B$, we have $u = \sum_{i=1}^{\infty} \sum_{j=1}^{n} x_j u'_{i,j}$ where $u'_{i,j} \in p^{i-N'} B$ for all $i \geqslant N'$. So the right-hand side converges in $B$, as $B$ is $p$-adically complete. Therefore, $u \in IB$. □

**Theorem 4.4.10.** *Let $I = (x_1, \dots, x_n) R$ be an ideal generated by partial system of parameters in $R$, where $R$ is a noetherian complete local domain with $F$-finite residue field. Then*

*(i)* $I^{\mathsf{epf}} \subseteq I^{\mathsf{PBCM}}$;

*(ii)* $I^{\mathsf{wepf}} \subseteq I^{\mathsf{BCM}}$.

*Proof.* For 4.4.10.(i), if $u \in I^{\mathsf{epf}}$, then there is some $c \in R \smallsetminus \{0\}$ such that for any $\varepsilon$ and any $N$ such that $c^\varepsilon u \in \cap_N (I, p^N) R^+$. There exists an integral perfectoid big Cohen-Macaulay algebra $B$ such that $c^\varepsilon u \in IB$ by Lemma 4.4.9. By Remark 4.4.4 again, we conclude that there is an integral perfectoid big Cohen-Macaulay algebra $C$ such that $u \in IC$. So $u \in I^{\mathsf{PBCM}}$.

For 4.4.10.(ii), we have

$$I^{\mathsf{wepf}} = \bigcap_N (I, p^N)^{\mathsf{epf}} \subseteq \bigcap_N (I, p^N)^{\mathsf{PBCM}} \subseteq I^{\mathsf{BCM}}.$$

□

# CHAPTER V

# Size and Quasilength

The notion of quasilength was introduced by Mel Hochster and Craig Huneke in their joint paper [HH09]. They used quasilength to define two nonnegative real numbers that are intended heuristically as "measures" of the top local cohomology module $H_I^{\dim(R)}(R)$ ([HH09, Section 2]).

We develop a new notion called *size* for an ideal in a ring $R$ (Definition 5.2.1) based on the notion of quasilength in Section 5.2. It is a quantity invariant up to radicals (Proposition 5.2.4), and is always between the height and the arithmetic rank of the ideal (Proposition 5.2.5). We show that the size of an ideal is unchanged when we kill finitely many nilpotent elements (Theorem 5.2.8). We also show that a finitely generated ideal is of size 0 if and only if it is nilpotent (Proposition 5.2.9). Finally, we show that under mild hypothesis, if $R$ is a local domain and $P$ is a prime of dimension 1, then $\mathsf{size}(P) = \mathrm{ht}(P)$.

In Section 5.3, we first show additivity of quasilength for direct sums of two modules in a special case Proposition 5.3.1, and then generalize an example where the additivity property fails for quasilength in Proposition 5.3.3. After that, we proceed to study the asymptotic bounds for quasilength of large direct sums. In Theorem 5.3.15, we give a result of this type in dimension 1.

## 5.1 Preliminaries

We first recall the definition of quasilength from [HH09].

**Definition 5.1.1.** Let $R$ be a ring, $M$ an $R$-module, and $I$ a finitely generated ideal of $R$. If there is a finite length $h$ filtration of $M$ in which the factors are cyclic modules killed by $I$, then we say that $M$ has finite $I$-quasilength at most $h$. We define $\mathcal{L}_I(M) = h$ if $h$ is the minimum number of factors in such a filtration. If there is no such filtration, then we say $\mathcal{L}_I(M) = +\infty$.

From the definition above, we see that the notion of quasilength is very similar to the notion of length. It recovers the definition of length if $I$ happens to be a maximal ideal. However, quasilength, unlike length, is in general not additive over direct sums. See [HZ18, Example 3.5] for the counterexample and a discussion on it in Section 5.3.2.

However, we have the following proposition ([HH09, Proposition 1.1]).

**Proposition 5.1.2.** *Let $R$ be a ring, $I$ a finitely generated ideal of $R$ and $M$ an $R$-module.*

(i) *$M$ has finite $I$-quasilength if and only if $M$ is finitely generated and killed by a power of $I$. In fact, $\nu(M) \leqslant \mathcal{L}_I(M)$, and $I^{\mathcal{L}_I(M)}$ kills $M$.*

(ii) *Assume that $0 \to M' \to M \to M'' \to 0$ is exact. If $M'$ and $M''$ have finite $I$-quasilength then so does $M$, and $\mathcal{L}_I(M) \leqslant \mathcal{L}_I(M') + \mathcal{L}_I(M'')$. If $M$ has finite $I$-quasilength then $M''$ does as well, and $\mathcal{L}_I(M'') \leqslant \mathcal{L}_I(M)$.*

(iii) *If $S$ is an $R$-algebra then $\mathcal{L}^S_{IS}(S \otimes_R M) \leqslant \mathcal{L}^R_I(M)$.*

We also need the following property from [HH09, Proposition 1.2].

**Proposition 5.1.3.** *Let $R$ be a ring, $I$ a finitely generated ideal of $R$ and $M$ an $R$-module. Suppose that $I = (x_1, \ldots, x_d)$. Let $\mathfrak{A}$ be an ideal generated by a set of monomials in $x_1, \ldots, x_d$ containing a power of every $x_j$, and suppose that the number of monomials in the $x_j$ not formally in $\mathfrak{A}$ is $a$. Let $\mathfrak{B}$ be another such ideal such that the number of monomials not formally in $\mathfrak{B}$ is $b$. Suppose that every generator if $\mathfrak{B}$ is formally in $\mathfrak{A}$. Then $\mathcal{L}_I(\mathfrak{A}M/\mathfrak{B}M) \leqslant (b-a)\nu(M/IM)$.*

It follows from the property above that if $I$ has $n$ generators, then $\mathcal{L}_I(R/I^{t+1}) \leqslant \binom{n+t}{t}$. Here, for any real number $n$ and any nonnegative integer $t$, the notation $\binom{n+t}{t}$ means $\frac{(n+t)(n+t-1)\cdots(n+1)}{t!}$. In fact, we have $\binom{n+t}{t} = \frac{1}{(n+t-1)B(n+1,t+1)}$ where $B(-,-)$ is the *beta function*. When $t$ is large enough, we have $B(n+1,t+1) \sim \Gamma(n+1) \cdot (t+1)^{-(n+1)}$. Therefore $\binom{n+t}{t} \sim \frac{t^{n+1}}{n+t-1} \sim t^n$. All of the following lemmas could be proved using these equivalences, but we provide a more elementary proof.

**Lemma 5.1.4.** *Suppose that $(a_t), (b_t), (c_t)$ are sequences of positive numbers.*

(i) *If $0 \leqslant r < s$, then $\lim\limits_{t \to \infty} \binom{s+t}{t}/\binom{r+t}{t} = \infty$.*

(ii) *Suppose $(a_t), (b_t), (c_t)$ are positive sequences. If $\limsup\limits_{t \to \infty} \frac{a_t}{b_t}$ is finite and $\lim\limits_{t \to \infty} \frac{b_t}{c_t} = 0$, then $\limsup\limits_{t \to \infty} \frac{a_t}{c_t} = 0$.*

(iii) *Suppose $(a_t)$ is a positive sequence. If $\limsup\limits_{t \to \infty} \frac{a_t}{\binom{r+t}{t}}$ is finite, then $\limsup\limits_{t \to \infty} \frac{a_t}{\binom{s+t}{t}} = 0$ for any $s > r$.*

(iv) *Suppose $(a_t), (b_t), (c_t)$ are positive sequences. If $\limsup\limits_{t \to \infty} \frac{a_t}{b_t}$ is a nonzero finite number and $\lim\limits_{t \to \infty} \frac{b_t}{c_t} = \infty$, then $\limsup\limits_{t \to \infty} \frac{a_t}{c_t} = \infty$.*

(v) *Suppose $(a_t)$ is a positive sequence. If $\limsup\limits_{t \to \infty} \frac{a_t}{\binom{r+t}{t}}$ is a nonzero finite number, then*

$$\limsup\limits_{t \to \infty} \frac{a_t}{\binom{s+t}{t}} = \infty$$

*for any $s < r$.*

*Proof.* Proof of 5.1.4.(i). We have

$$\lim_{t\to\infty} \binom{s+t}{t} \Big/ \binom{r+t}{t} = \lim_{t\to\infty} \frac{(s+t)(s+t-1)\cdots(s+1)}{(r+t)(r+t-1)\cdots(r+1)}$$

$$= \lim_{t\to\infty} \prod_{i=1}^{t} (1 + \frac{s-r}{r+i})$$

$$= \exp(\sum_{i=1}^{\infty} \ln(1 + \frac{s-r}{r+i})).$$

Since $\ln(1+x) > \frac{x}{1+x}$ for any $x > 0$, we have $\ln(1 + \frac{s-r}{r+i}) > (\frac{s-r}{r+i})/(1 + \frac{s-r}{r+i}) = \frac{s-r}{s+i}$. So the summation in the exponent is growing like the harmonic series, hence it diverges to $\infty$.

Proof of 5.1.4.(ii). If the $\limsup$ is finite, then it is bounded. The rest follows from a straightforward argument.

Proof of 5.1.4.(iii). This follows from 5.1.4.(ii).

Proof of 5.1.4.(iv). Suppose that $\limsup\limits_{t\to\infty} \frac{a_t}{b_t} = M > 0$. Then there is a subsequence $\frac{a_{t_k}}{b_{t_k}}$ converges to it. In particular, there is some $N$ such that for any $k \geqslant N$ we have $\frac{a_{t_k}}{b_{t_k}} > M/2$. But $\frac{b_{t_k}}{c_{t_k}}$ can be arbitrarily large. Therefore the limit of $\frac{a_{t_k}}{c_{t_k}}$ is infinite, which shows that the $\limsup$ is infinite.

Proof of 5.1.4.(v). This follows from 5.1.4.(iv). □

## 5.2 Size

### 5.2.1 The definition

We are ready to give the definition of the size of an ideal.

**Definition 5.2.1.** We define the *size* of $I$ to be

$$\mathsf{size}_R(I) = \inf\{n \quad | \quad \limsup_{t\to\infty} \frac{\mathcal{L}_I(R/I^t)}{t^n} < \infty\}.$$

We also write $\mathsf{size}(I) = \mathsf{size}_R(I)$ if the ambient ring $R$ is clear in the context.

Since $\mathcal{L}_I(R/I^t) \leqslant \binom{n+t-1}{t-1} \sim (t-1)^n$, we know that the size is bounded above by the number of generators of $I$, i.e., $\mathsf{size}(I) \leqslant \nu(I)$. Also, because of Proposition 5.1.2.(iii), we know that $\mathcal{L}_R(R/I^t) \geqslant \mathcal{L}_{IS}(S/I^tS) \Rightarrow \mathsf{size}_R(I) \geqslant \mathsf{size}_S(IS)$, for every $R$-algebra $S$.

### 5.2.2 Upper bounds

Next we show that the size of an ideal is in fact an invariant of ideals up to radicals.

**Proposition 5.2.2.** *Let* $I, J, K \subseteq R$ *be ideals. Then*

*(i) If J has the same radical of I, we have*

$$\text{size}(I) = \inf\{n \mid \limsup_{t \to \infty} \frac{\mathcal{L}_J(R/I^t)}{t^n} < \infty\}.$$

*(ii) If $I \subseteq J$, then $\text{size}(I) \geqslant \text{size}(J)$.*

*(iii) For any ideal $I$, we have $\text{size}(I^h) = \text{size}(I)$ for any $h \in \mathbb{N}$.*

*(iv) If $I \subseteq J \subseteq K$ are three ideals in $R$ and $\text{size}(I) = \text{size}(K)$, then $\text{size}(J) = \text{size}(I) = \text{size}(K)$.*

*Proof.* 5.2.2.(i) is a corollary of Lemma 5.3.5.

Proof of 5.2.2.(ii). We have $I^t \subseteq J^t$ and, hence, $R/I^t$ surjects onto $R/J^t$. So

$$\mathcal{L}_I(R/I^t) \geqslant \mathcal{L}_I(R/J^t) \Rightarrow \limsup_{t \to \infty} \frac{\mathcal{L}_I(R/I^t)}{t^n} \geqslant \limsup_{t \to \infty} \frac{\mathcal{L}_I(R/J^t)}{t^n}.$$

Since the right-hand side is finite whenever the left-hand side is, we conclude that $\text{size}(J) \leqslant \text{size}(I)$.

Proof of 5.2.2.(iii). Clearly we have $\text{size}(I^h) \geqslant \text{size}(I)$ by Proposition 5.2.2.(ii). When $n = \text{size}(I)$, the limit $\limsup_{t \to \infty} \frac{\mathcal{L}_I(R/I^{ht})}{(ht)^n} = \frac{1}{h^n} \limsup_{t \to \infty} \frac{\mathcal{L}_I(R/I^{ht})}{t^n}$ is finite. We conclude that $\text{size}(I^h) \leqslant \text{size}(I)$.

5.2.2.(iv) follows from the fact that $I^t \subseteq J^t \subseteq K^t$. $\square$

*Remark* 5.2.3. For any ideal $I = (x_1, ..., x_k)$ with specified generators, write $I_t = (x_1^t, ..., x_k^t)R$. By Proposition 5.2.2.(iii), we have $I^{kt} \subseteq I^{k(t-1)+1} \subseteq I_t \subseteq I^t$. Therefore we see that we can use $I_t$ to calculate the size of $I$.

Now we are ready to prove

**Proposition 5.2.4.** *Let $R$ be a noetherian ring and let $I \subseteq R$ be an ideal. The size of $I$ is invariant up to radicals. Hence, the size of an ideal is at most the arithmetic rank, i.e., $\text{size}(I) \leqslant \text{ara}(I)$.*

*Proof.* If $\sqrt{I} = K$, it suffices to show that $\text{size}(I) = \text{size}(K)$. Note that $K^h \subseteq I \subseteq K$ for some $h$. By Proposition 5.2.2, we have $\text{size}(K^h) \geqslant \text{size}(I) \geqslant \text{size}(K)$ and $\text{size}(K^h) = \text{size}(K)$. Therefore, the equality holds. The last statement follows directly. $\square$

### 5.2.3 Lower bounds and nilpotents

**Proposition 5.2.5.** *Let $R$ be a noetherian ring and let $I \subseteq R$ be an ideal. If $P$ is a minimal prime of $I$ with height $h$, then $\text{size}(I) \geqslant h$.*

*Proof.* Consider $IR_P$ as an ideal of $R_P$. We know that $\text{size}(I) \geqslant \text{size}(IR_P)$. But in $R_P$, $I$ is $PR_P$-primary ideal. Hence by Proposition 5.2.2.(i), we can calculate the size of $IR_P$ using its

radical ideal $PR_P$. But then the function $t \mapsto \mathcal{L}_{PR_P}(R_P/I^t R_P)$ is the Hilbert function of $I$, which grows as a polynomial of degree $h$. Hence $\mathrm{size}(IR_P) = h$. $\qquad\square$

The superheight, $\mathrm{superht}(I)$, of an ideal $I$ is defined to be the largest height of $IS$ in any $R$-algebra $S$ such that $IS$ is proper. Clearly we have $\mathrm{size}(I) \geqslant \mathrm{superht}(I)$. Next we show that the size of $I$ does not change if we kill a nilpotent element.

**Lemma 5.2.6.** *Let $f$ be a nilpotent element in $R$ and let $I \subseteq R$ be an ideal. Write $\overline{R}$ for $R/fR$ and $\overline{I}$ for $I\overline{R}$. Then $\mathrm{size}_{\overline{R}}(\overline{I}) = \mathrm{size}_R(I)$.*

*Proof.* Clearly we have $\mathrm{size}_R(I) \geqslant \mathrm{size}_{\overline{R}}(\overline{I})$. Let $n = \mathrm{size}_{\overline{R}}\overline{I}$ and we only need to show that $\mathrm{size}_R(I) \leqslant n$. Let us fix some notations. Suppose that $f^h = 0$ for some positive integer $k$. Suppose that $\overline{R}/\overline{I}^t$ has a filtration of minimal length $a_t$, i.e., there are $a_t$ elements $\overline{r}_1, \ldots, \overline{r}_{a_t}$ in $\overline{R}/\overline{I}^t$ such that if $\overline{J}_i$ is the ideal generated by first $i$ elements, then $0 = \overline{J}_0 \subseteq \overline{J}_1 \subseteq \cdots \subseteq \overline{J}_{a_t} = \overline{R}/\overline{I}^t$ is the desired filtration.

Let $r_i$ be an arbitrary lift of $\overline{r}_i$ in $R/I^t$ for each $i$. Without loss of generality we assume that $\overline{r}_{a_t} = 1$ and $r_{a_t} = 1$. Again, write $J_i$ for the ideal generated by first $i$ elements of $r_1, \ldots, r_{a_t}$. Then we have $J_0 = 0$ and $J_{a_t} = R/I^t$. Since in $\overline{R}$ each factor $\overline{J}_i/\overline{J}_{i-1}$ is killed by $\overline{I}$, we have that $IJ_i \subseteq J_{i-1} + fR/I^t$. Let $J_{k,l} = (f^{l+1})R/I^t + f^l J_k$ where $0 \leqslant k \leqslant a_t, 0 \leqslant l < h$. Then by definition we have $J_{0,l} = (f^{l+1})R/I^t = J_{a_t,l+1}$. Since

$$IJ_{k,l} = If^{l+1}R + If^l J_k \subseteq f^{l+1}R + f^l(J_{k-1} + fR) = f^{l+1}R + f^l J_{k-1} = J_{k-1,l},$$

we conclude that $0 = J_{0,h-1} \subseteq J_{1,h-1} \subseteq \cdots \subseteq J_{a_t,h-1} = J_{0,h-2} \subseteq J_{1,h-2} \subseteq \cdots \subseteq J_{a_t,0} = R/I^t$ is a filtration of $R/I^t$ with factors that are cyclic $(R/I)$-modules. Hence, we have $\mathcal{L}_I(R/I^t) \leqslant a_t h$. Therefore, $\limsup\limits_{t\to\infty} \frac{\mathcal{L}_I(R/I^t)}{t^n} \leqslant \limsup\limits_{t\to\infty} \frac{a_t h}{t^n}$ is finite. We have $\mathrm{size}_R(I) \leqslant n$ as desired. $\qquad\square$

*Remark* 5.2.7. The factors in the filtration constructed in the proof are actually $R/(I,f)R$-modules.

**Theorem 5.2.8.** *Let $R$ be a noetherian ring. Then for any ideal $I$, the size of $I$ does not change when passing to the reduced ring of $R$.*

*Proof.* Let $N \subseteq R$ be the nilradical. Then $N$ is generated by $f_1, \ldots, f_k$. Let $R_h = R/(f_1, \ldots, f_h)R, 1 \leqslant h \leqslant k$. By applying Lemma 5.2.6 repeatedly, we have $\mathrm{size}_R(I) = \mathrm{size}_{R_1}(IR_1) = \cdots = \mathrm{size}_{R_k}(IR_k)$. $\qquad\square$

From Theorem 5.2.8, the size of a nilpotent ideal is necessarily zero. Because when passing to the reduced ring, the nilpotent ideal becomes the zero ideal, which has size 0. Next, we show that an ideal $I$ has size 0 if and only if $I$ is a finitely generated nilpotent ideal.

**Proposition 5.2.9.** *Let $I \subseteq R$ be a finitely generated ideal. Then $I$ has size 0 if and only if $I$ is nilpotent.*

*Proof.* If $I$ is nilpotent, by the discussion above we see that $I$ has size 0. Now suppose that $I$ has size 0, which implies that $\mathcal{L}_I(R/I^n)$ is bounded. Let $P$ be a minimal ideal of $R$, then $\mathcal{L}_{IS}(S/I^nS) \leqslant \mathcal{L}_I(R/I^n)$ is bounded where $S = R/P$. If $IS$ is not zero, then let $Q$ be a minimal prime of $I$ in $S$, and consider $IS_Q$. Again $\mathcal{L}_{IS_Q}(S_Q/I^nS_Q) \geqslant \ell(S_Q/I^nS_Q)$ is bounded. But $\ell(S_Q/I^nS_Q)$ cannot be bounded unless $I^nS_Q = 0$. Since $S$ is a domain, we conclude that $IS = 0$. Therefore $I$ is contained in the intersection of all minimal primes of $R$. So $I$ is nilpotent. $\qquad\square$

### 5.2.4   Lower dimensional cases

Let $R$ be a noetherian local ring and $P$ a prime ideal of $R$. As long as the $P$-adic topology coincides with the $P^{(n)}$ symbolic power topology, we have a linear containment, i.e., there exists some $c$ such that $P^{(cn)} \subseteq P^n$ for all $n$ ([Swa00]). This is true if the completion of $R$ is a domain and $\dim R/P = 1$ ([Har70, Theorem 7.1]).

We want to prove the following theorem.

**Theorem 5.2.10.** *Let $R$ be a noetherian local ring and $P$ a prime ideal of $R$ such that $\dim R/P = 1$. Suppose that there is some $c$ such that $P^{(cn)} \subseteq P^n$ for all $n$ (this holds if the completion of $R$ is a domain) and $R/P$ is module-finite over a regular local ring $A$ (this holds if $R/P$ is complete). Then $\mathsf{size}(P) = \mathrm{ht}(P)$.*

*Proof.* We always have $\mathsf{size}(P) \geqslant \mathrm{ht}(P)$. To show the converse, we need to construct a filtration of $R/P^n$ for $n \gg 0$. Since $P^{(cn)} \subseteq P^n$. We first consider the obvious filtration of $R/P^{(cn)}$:

$$0 \subseteq P^{(cn-1)}/P^{(cn)} \subseteq \cdots \subseteq P^{(2)}/P^{(cn)} \subseteq P/P^{(cn)} \subseteq R/P^{(cn)}.$$

Since each factor is a torsion-free $(R/P)$-module, we can refine the filtration so that each factor is a cyclic $(R/P)$-module, such that there are at most $\sum_{k=1}^{cn} \nu_{R/P}(P^{(k-1)}/P^{(k)})$ factors. If $M$ is a torsion-free $(R/P)$-module, then $\nu_{R/P}(M) \leqslant \mathrm{rank}_A(M)$. Hence, the filtration has at most $\sum_{k=1}^{cn} \mathrm{rank}_A(P^{(k-1)}/P^{(k)})$ factors. Each factor is in fact a free $A$-module as $A$ is a discrete valuation ring. Hence, the number of free copies does not change when we tensor with the fraction field $K$ of $A$. Since $R/P$ is module-finite over $A$, $(R/P) \otimes_A K$ is the fraction field $L$ of $R/P$. Therefore, after tensoring with $K$, the number of factors is $[L : K]\ell_{R_P/PR_P}(R_P/P^{cn}R_P)$. This bounds $\mathcal{L}_P(R/P^{(cn)})$ from above and it grows as $(cn)^{\mathrm{ht}(P)}$. So $\mathcal{L}_P(R/P^n)$ grows as most fast as $n^{\mathrm{ht}(P)}$, which shows that $\mathsf{size}(P) \leqslant \mathrm{ht}(P)$. $\qquad\square$

To see that the size of an ideal can be strictly smaller than its arithmetic rank, consider the calculation in [HZ18, Section 4]. Let $R = A[x, y, u, v]$, where $A$ is any noetherian commutative

ring. The ideal $I = (xu, yv, xv + yu) \subseteq R$ has size 2. From their calculation, we have size$(I) \leqslant 2$. But on the other hand, $IR[\frac{1}{x}] = (u, v)R[\frac{1}{x}]$ has height 2. So size$(I) = 2$. It is shown in [HZ18] that $\mathrm{H}_I^3(R) \neq 0$. Hence, the arithmetic rank of $I$ is 3.

## 5.3   Additivity of Quasilength

In Proposition 5.1.2.(ii), we usually do not have equalities. In fact, quasilength is not additive for short exact sequences even if the exact sequence splits. Before diving more into the counterexample to additivity, we want to show that in a special case, additivity does hold for quasilength.

### 5.3.1   Additivity in a special case

**Proposition 5.3.1.** *Let $P$ be a prime ideal of $R$ such that $R/P$ is a principal ideal domain. Assume that $M$ is a finitely generated $P$-torsion module, i.e., there is a power $P^n$ which kills $M$. Assume also that $\mathrm{H}_{\mathfrak{m}}^0(M) = 0$ for any maximal ideal $\mathfrak{m}$ of $R$. Then $\mathcal{L}_P(M) = \mathcal{L}_{PR_P}(M_P) = \ell_{R_P}(M_P)$, and $\ell_{R_P}$ is additive.*

*Proof.* Clearly we have $\mathcal{L}_P(M) \geqslant \mathcal{L}_{PR_P}(M_P)$. So we only need to show the converse. If $N = \mathrm{Ann}_P M$, then $N$ is a torsion-free $(R/P)$-module, hence, free. So we have $\mathcal{L}_{PR_P}(N_P) = \nu(N_P) = \nu(N)$.

Note that $\mathcal{L}_P(M) \leqslant \mathcal{L}_P(N) + \mathcal{L}_P(M/N)$. Since $\mathcal{L}_P(N) \leqslant \nu(N) = \nu(N_P) = \mathcal{L}_{PR_P}(N) = \ell(N_P)$, we have $\mathcal{L}_P(M) \leqslant \ell(N_P) + \mathcal{L}_P(M/N)$. $M/N$ is again finitely generated and $P$-torsion. If $x \in \mathrm{H}_{\mathfrak{m}}^0(M/N)$, then $\mathfrak{m}^k x \in N \Rightarrow \mathfrak{m}^k I x = 0 \Rightarrow I x = 0 \Rightarrow x = 0$ in $M/N$, so $\mathrm{H}_{\mathfrak{m}}^0(M/N) = 0$. By noetherian induction and the additive property of $\ell$, we conclude that $\mathcal{L}(M) \leqslant \mathcal{L}_{PR_P}(M_P) = \ell(M_P)$. $\qquad\square$

### 5.3.2   Generalization of a counterexample

In this subsection, we always require $R$ to be a local ring as, otherwise, even the function $\nu$ of minimal number of generators of a module is not additive. Note that $\nu$ is the quasilength with respect to the zero ideal. We have the following example. Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 - 1)$ and $M = (x, y - 1)R$. Then $M \oplus M \cong R \oplus R$. So $\nu(M \oplus M) = 2 < 2\nu(M) = 4$.

A counterexample to additivity in the local case can be found in [HZ18, Example 3.5]. We briefly record it here. Let $R = k[\![x]\!], I = (x^2)R, M = R/(x)R$ and $N = R/(x^3)R$. Then $\mathcal{L}_I(M) = 1, \mathcal{L}_I(N) = 2$ but $\mathcal{L}_I(M \oplus N) = 2 \neq 1 + 2$. We will generalize this example in Proposition 5.3.3, where we also give a positive result about additivity. We first note:

**Lemma 5.3.2.** *Let $R = k[\![x]\!], \mathfrak{m} = (x)R, I = (x^d)R$ where $d$ is some positive integer. If $M$ is a finitely generated $R$-module of finite $I$-quasilength, then $M$ is of finite length and $\ell(M) \leqslant d \cdot \mathcal{L}_I(M)$.*

*Proof.* Since $I$ is $\mathfrak{m}$-primary, $M$ is of finite length. Each factor in the filtration of $M$ has length at most $\ell(R/I) = d$. Therefore $\ell(M) \leqslant d \cdot \mathcal{L}_I(M)$. $\qquad\square$

Suppose that $M = R/(x^a)R$ for some nonnegative integer $a$. Then the length of $M$ is $\ell(M) = a$. Therefore $\mathcal{L}_I(M) \geqslant \lceil \frac{a}{d} \rceil$. On the other hand, the sequence $x^{(\lceil \frac{a}{d} \rceil - 1)d}, \cdots, x^{2d}, x^d, 1$ clearly generates a filtration for $M$ with $\lceil \frac{a}{d} \rceil$ factors. Hence $\mathcal{L}_I(M) = \lceil \frac{a}{d} \rceil$.

**Proposition 5.3.3.** *Let $R = k[\![x]\!], \mathfrak{m} = (x)R, I = (x^d)R$ where $d$ is some positive integer. Let $M = R/(x^a)R$ and $N = R/(x^b)R$ where $a, b$ are nonnegative integers. Assume without loss of generality that $a \geqslant b$. Write $a = a_0 + a_1 \cdot d, b = b_0 + b_1 \cdot d$ where $0 < a_0 \leqslant d, 0 < b_0 \leqslant d$.*

- *If $a_0 + b_0 > d$, then $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N)$.*

- *If $a_0 + b_0 \leqslant d$, assume that $a_1 > b_1$, then $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N) - 1$.*

*Proof.* We always have $\mathcal{L}_I(M \oplus N) \leqslant \mathcal{L}_I(M) + \mathcal{L}_I(N)$. By the discussion above we have $\mathcal{L}_I(M) = \lceil \frac{a}{d} \rceil = a_1 + 1$ and $\mathcal{L}_I(N) = \lceil \frac{b}{d} \rceil = b_1 + 1$. So $\mathcal{L}_I(M \oplus N) \leqslant a_1 + b_1 + 2$. By Lemma 5.3.2 we have $\mathcal{L}_I(M \oplus N) \geqslant \lceil \frac{\ell(M) + \ell(N)}{d} \rceil = a_1 + b_1 + \lceil \frac{a_0 + b_0}{d} \rceil$. Since $2 \leqslant a_0 + b_0 \leqslant 2d$, $\mathcal{L}_I(M \oplus N) \geqslant a_1 + b_1 + 1$. Therefore, $\mathcal{L}_I(M \oplus N)$ is either $\mathcal{L}_I(M) + \mathcal{L}_I(N)$ or $\mathcal{L}_I(M) + \mathcal{L}_I(N) - 1$.

If $a_0 + b_0 > d$, then by the inequality above we immediately have $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N)$. Therefore for the remaining cases, we assume that $a_0 + b_0 \leqslant d$ and $a_1 > b_1$. In this case, the equality $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N) - 1$ is proved by constructing an explicit filtration:

Let $u_i = (x^{id + a_0}, x^{id}) \in M \oplus N$ for $0 \leqslant i \leqslant a_1 \in M \oplus N$, and $v_j = (x^{jd}, 0)$ for $0 \leqslant j \leqslant b_1$. Let $Q_i$ be the submodule of $M \oplus N$ generated by $u_i$, then we have

$$0 = Q_{a_1} \subseteq Q_{a_1 - 1} \subseteq \cdots \subseteq Q_1 \subseteq Q_0$$

where $Q_i/Q_{i+1}$ are $I$-cyclic modules. Here $Q_0 = 0$ because we have $a_1 > b_1 \Rightarrow a_1 d \geqslant b_1 d + d \geqslant b$. We let $P_j$ be the submodule generated by $Q_0$ and $v_j$ in $M \oplus N$. Then

$$Q_0 \subseteq P_{b_1} \subseteq P_{b_1 - 1} \subseteq \cdots \subseteq P_1 \subseteq P_0 = M \oplus N$$

where $P_j/P_{j-1}$ are $I$-cyclic modules. Note that $b_1 d + d - a_0 \geqslant b_1 d + b_0 = b$ since $a_0 + b_0 \leqslant d$. So we have $x^{b_1 d + d - a_0} u_0 = x^{b_1 d + d - a_0}(x^{a_0}, 1) = (x^{b_1 d + d}, 0) = x^d v_{b_1}$. Hence, $x^d P_{b_1} = x^d v_{b_1} \oplus x^d Q_0 \subseteq Q_0$. Finally, $P_0 = M \oplus N$ because it contains $u_0 = (x^{a_0}, 1)$ and $v_0 = (1, 0)$.

There are $a_1 + b_1 + 1$ factors. So we have $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N) - 1$. $\qquad\square$

**Conjecture 5.3.4.** *Under the same hypothesis as in Proposition 5.3.3, the case when we have $a_1 = b_1$ and $a_0 \geqslant b_0$ is unknown. We conjecture that $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N)$ in this case. In other words, we conjecture that the conclusion of Proposition 5.3.3 is the following:*

*If $a_1 > b_1$ and $a_0 + b_0 \leqslant d$, then $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N) - 1$, and otherwise we have $\mathcal{L}_I(M \oplus N) = \mathcal{L}_I(M) + \mathcal{L}_I(N)$.*

As seen in Proposition 5.3.3, the strict additive property fails quite often. However, it is not far off the sum. In our case, it can only be off by 1. So we want to investigate the asymptotic behavior of quasilength.

### 5.3.3 Asymptotic behaviour

We want to study in the case of a noetherian ring, the asymptotic behavior of the quasilength of the direct sum of $n$ copies of a module $M$ when $n$ goes to infinity. Since we always have $\mathcal{L}_I(M^{\oplus n}) \leqslant n\mathcal{L}_I(M)$, we ask whether there exists a nonzero constant $C$ that may depend on $I$ but not on $M$ such that $\mathcal{L}_I(M^{\oplus n}) \geqslant Cn\mathcal{L}_I(M)$.

If $R$ is a noetherian local ring, we can easily find a constant $C$ that depends on $M$. For example, we can take $C = \frac{\nu(M)}{\mathcal{L}_I(M)}$. Then

$$\mathcal{L}_I(M^{\oplus n}) \geqslant \nu(M^{\oplus n}) = n \cdot \nu(M) \geqslant \frac{\nu(M)}{\mathcal{L}_I(M)} \cdot n \cdot \mathcal{L}_I(M) = Cn \cdot \mathcal{L}_I(M).$$

The following lemma implies that the asymptotic behavior of quasilength is invariant up to radicals.

**Lemma 5.3.5.** *Let $R$ be a noetherian ring. Suppose that two ideals $I$ and $J$ of $R$ have the same radical ideal, i.e., $\sqrt{I} = \sqrt{J}$. Then there exist positive constants $C_1, C_2$ depending on $I$ and $J$ but independent of $M$ such that for any module $M$ of finite $I$-quasilength, we have*

$$C_1\mathcal{L}_I(M) \leqslant \mathcal{L}_J(M) \leqslant C_2\mathcal{L}_I(M).$$

*Proof.* Let $K = \sqrt{I}$, then we have $K^n \subseteq I \subseteq K$. Hence $M$ has finite $I$-quasilength if and only if it has finite $K$-quasilength. And we clearly have $\mathcal{L}_I(M) \leqslant \mathcal{L}_K(M)$ for the inclusion of $I \subseteq K$. On the other hand, if we have a finite filtration of $M$ with cyclic $(R/I)$-module factors, each factor is also an $(R/K^n)$- cyclic module. Hence each factor has a length $D_1 = \mathcal{L}_K(R/K^n)$ filtration. So $\mathcal{L}_K(M) \leqslant D_1\mathcal{L}_I(M)$. The conclusion follows from symmetry. $\square$

More generally, for any finitely generated module $M$ (not necessarily of finite $I$-quasilength), we want to ask

*Question* 5.3.6. Let $R$ be a noetherian ring, $I$ an ideal of $R$, and $M$ an $R$-module. Do we have

$$\mathcal{L}_I((M/I^nM)^{\oplus n}) \geqslant Cn\mathcal{L}_I(M/I^nM) \tag{5.3.1}$$

for some constant $C$ independent of $n$? Here $C$ may depend on the ring $R$, the ideal $I$ and the module $M$.

Let $M$ be finitely generated $R$-module and $I \subseteq R$ a proper ideal. First of all, we note that in Question 5.3.6 we can use any ideal up to radicals by Lemma 5.3.5. In fact, we can give this question an affirmative answer when $R$ is of dimension 1 (Theorem 5.3.15).

### 5.3.3.1 Discussion of possible reductions

In order to prove the inequality (5.3.1) in dimension one, we develop several reductions that can be used in greater generality. To simplify our language, we adopt the following commonly used notation.

**Definition 5.3.7** (Big O and Big Theta). Let $f(n), g(n)$ be two positive real functions defined for positive integers. One writes $f(n) = O(g(n))$ if $f(n)$ is at most a positive constant multiple of $g(n)$ for all sufficiently large values of $n$. Equivalently, $f(n) = O(g(n))$ if there exists a positive real number $M$ such that $f(n) \leqslant M \cdot g(n)$ for all positive integers $n$. If $f(n)$ and $g(n)$ are bounding each other up to a multiple for all sufficiently large $n$, i.e., $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then we write $f(n) = \Theta(g(n))$ (or, equivalently, $g(n) = \Theta(f(n))$).

We will temporarily use following notations and definitions to simplify our arguments.

**Definition 5.3.8.** Let $R$ be a noetherian ring and let $I$ be an ideal of $R$. Suppose that $M, N$ are two $R$-modules. We write $\psi_n(M) = \mathcal{L}_I(M/I^n M)$ and $\Psi_n(M) = \mathcal{L}_I((M/I^n M)^{\oplus n})$. We say that $M$ and $N$ are *equivalent with respect to $I$*, denoted by $M \sim_I N$, if we have $\psi_n(M) = \Theta(\psi_n(N))$ and $\Psi_n(M) = \Theta(\Psi_n(N))$. We usually omit $I$ and write $M \sim N$ if the ideal $I$ is clear from the context.

**Proposition 5.3.9.** *Let $R$ be a noetherian ring and $I$ an ideal of $R$. Suppose that $M, N, M_1, M_2$ are $R$-modules. We have*

*(i) If $N \subseteq M$ is killed by a power of $I$, then $M$ is equivalent to $M/N$.*

*(ii) If $M_1 \subseteq M_2$ are such that some power of $I$ kills $N = M_2/M_1$, i.e., $I^m \subseteq \mathfrak{a} = \mathrm{Ann}(N)$, then $M_1 \sim M_2$.*

*Proof.* For 5.3.9.(i), it is generally true that $\psi_n(M) \geqslant \psi_n(M/N)$ and $\Psi_n(M) \geqslant \Psi_n(M/N)$. Consider the right exact sequences

$$N/I^n N \to M/I^n M \to (M/N)/I^n(M/N) \to 0,$$
$$(N/I^n N)^{\oplus n} \to (M/I^n M)^{\oplus n} \to ((M/N)/I^n(M/N))^{\oplus n} \to 0.$$

By Proposition 5.1.2.(ii), we have $\psi_n(M) \leqslant \psi_n(N) + \psi_n(M/N)$ and $\Psi_n(M) \leqslant \Psi_n(N) + \Psi_n(M/N)$. Since $N$ is killed by a power of $I$, we have $N/I^n N = N$ when $n$ is large enough. Therefore $\psi_n(N)$ is a constant and $\Psi_n(N)$ grows at most linearly. But $\Psi_n(M/N)$ grows at least linearly by counting generators. Hence $\psi_n(M) \leqslant C_1 \psi_n(M/N)$ and $\Psi_n(M) \leqslant C_2 \Psi_n(M/N)$. Hence $M$ is equivalent to $M/N$.

For 5.3.9.(ii), if $\mathfrak{a}$ has $k$ generators, then we have a surjection $M_2^{\oplus k} \twoheadrightarrow \mathfrak{a}M_2$. Write

$$\mathfrak{a}M_2 \to M_1 \to M_1/\mathfrak{a}M_2 \to 0.$$

Let $W = M_1/\mathfrak{a}M_2$. Then $I^n$ kills $W$ for some integer $n > 0$. So we also have

$$\mathfrak{a}M_2/I^n\mathfrak{a}M_2 \to M_1/I^n M_1 \to W/I^n W = W \to 0$$

for all sufficiently large $n$.

Hence,

$$
\begin{aligned}
\mathcal{L}_I(M_1/I^n M_1) &\leqslant \mathcal{L}_I(\mathfrak{a}M_2/I^n\mathfrak{a}M_2) + \mathcal{L}_I(W) \\
&\leqslant \mathcal{L}_I(M_2^{\oplus k}/I^n M_2^{\oplus k}) + \mathcal{L}_I(W) \\
&\leqslant k\mathcal{L}_I(M_2/I^n M_2) + \mathcal{L}_I(W) \\
\Rightarrow \mathcal{L}_I((M_1/I^n M_1)^{\oplus n}) &\leqslant k\mathcal{L}_I((M_2/I^n M_2)^{\oplus n}) + \mathcal{L}_I(W^{\oplus n}).
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathcal{L}_I(M_2/I^n M_2) &\leqslant \mathcal{L}_I(M_1/I^n M_1) + \mathcal{L}_I(N), \\
\mathcal{L}_I((M_2/I^n M_2)^{\oplus n}) &\leqslant \mathcal{L}_I((M_1/I^n M_1)^{\oplus n}) + \mathcal{L}_I(N^{\oplus n}), \\
\mathcal{L}_I(M_1/I^n M_1) &\leqslant k\mathcal{L}_I(M_2/I^n M_2) + \mathcal{L}_I(W), \\
\mathcal{L}_I((M_1/I^n M_1)^{\oplus n}) &\leqslant k\mathcal{L}_I((M_2/I^n M_2)^{\oplus n}) + \mathcal{L}_I(W^{\oplus n}),
\end{aligned}
$$

where both $N$ and $W$ are killed by $\mathfrak{a}$. Since $\mathcal{L}_I(N)$ is a constant and $\mathcal{L}_I(N^{\oplus n})$ grows at most linearly in $n$, both are controlled by the other term by choosing a large enough coefficient. Hence $M_1 \sim M_2$. $\square$

**Proposition 5.3.10.** *Let $R$ be a noetherian ring and $I$ an ideal of $R$. Suppose that $M_1, M_2$ are $R$-modules. Then $\psi_n(M_1 \oplus M_2) = O(\psi_n(M_1) + \psi_n(M_2))$ and $\Psi_n(M_1 \oplus M_2) = O(\Psi_n(M_1) + \Psi_n(M_2))$.*

*Proof.* Write $M_3 = M_1 \oplus M_2$. Then both $M_2$ and $M_1$ are holomorphic image of $M_3$. Therefore

$\psi_n(M_3) \geqslant \psi_n(M_1)$ and $\psi_n(M_3) \geqslant \psi_n(M_2)$. So we have

$$
\begin{aligned}
2\mathcal{L}_I((M_3/I^n M_3)^{\oplus n}) &\geqslant \mathcal{L}_I((M_2/I^n M_2)^{\oplus n}) + \mathcal{L}_I((M_1/I^n M_1)^{\oplus n}) \\
&\geqslant C_2 \cdot n \cdot \mathcal{L}_I(M_2/I^n M_2) + C_1 \cdot n \cdot \mathcal{L}_I(M_1/I^n M_1) \\
&\geqslant C \cdot n (\mathcal{L}_I(M_1/I^n M_1) + \mathcal{L}_I(M_2/I^n M_2)) \\
&\geqslant C \cdot n \cdot \mathcal{L}_I(M_3/I^n M_3)
\end{aligned}
$$

where $C = \min\{C_1, C_2\}$. $\qquad\square$

Next we show that we can kill nilpotent elements in the ring $R$ without affecting the behavior of $\sim_I$.

**Proposition 5.3.11.** *Let $R$ be a noetherian ring and $I$ an ideal of $R$. Suppose that $M$ is an $R$-module. If $u \in R$ is a nilpotent element, then $M \sim_I M/uM$. Moreover, we have $M \sim_I M \otimes_R R_{\mathrm{red}}$.*

*Proof.* It reduces to show this for $u$ such that $u^2 = 0$. Because if $u^k$ is zero, then $M \sim M/u^{\lceil k/2 \rceil} \sim \cdots \sim M/u^2 M \sim M/uM$. Hence, we assume without loss of generality that $u^2 = 0$.

On the one hand, we have $\psi_n(M) \geqslant \psi_n(M/uM)$ and $\Psi_n(M) \geqslant \Psi_n(M/uM)$. On the other hand, we have

$$
\begin{aligned}
\psi_n(M) &\leqslant \psi_n(M/uM) + \psi_n(uM) \leqslant 2\psi_n(M/uM), \\
\Psi_n(M) &\leqslant \Psi_n(M/uM) + \Psi_n(uM) \leqslant 2\Psi_n(M/uM),
\end{aligned}
$$

due to the exact sequence

$$
0 \to uM \hookrightarrow M \twoheadrightarrow M/uM \to 0
$$

and the fact that $M/uM \overset{\cdot u}{\twoheadrightarrow} uM$ is a surjection.

Since $R$ is noetherian, the nilradical of $R$ is generated by finitely many elements $u_1, \ldots, u_k$. By what we have proved above, we have

$$
M \sim_I M/u_1 M \sim_I M/(u_1, u_2)M \sim_I \cdots \sim_I M/(u_1, \ldots, u_k M) = M \otimes_R R_{\mathrm{red}}.
$$

$\qquad\square$

Next we want to discuss this for finite products of rings.

**Lemma 5.3.12.** *Let $R$ be a noetherian ring, $I$ an ideal of $R$, and $M$ an $R$-module. Suppose that $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ and $I = I_1 \times I_2$ where $M_i$ is an $R_i$-module and $I_i \subseteq R_i$ is an ideal. Assume that both $I_1$ and $I_2$ are proper. If (5.3.1) holds for $(R_1, I_1, M_1)$ and $(R_2, I_2, M_2)$ respectively, then it holds for $(R, I, M)$.*

*Proof.* For any $I$ filtration of $M$, each factor is a cyclic $R/I \simeq R_1/I_1 \times R_2/I_2$ module. If we take the sequence of elements generating the filtration and projects onto $M_i$, then it will generate a $I_i$ filtration on $M_i$ of the same length. On the other hand, any two filtrations for $M_1, M_2$ can be made into a filtration of $M$. So we have $\psi_n(M) = \max(\psi_n(M_1), \psi_n(M_2))$ and $\Psi_n(M) = \max(\Psi_n(M_1), \Psi_n(M_2))$. □

Let us discuss a little bit about the module-finite base changes.

**Proposition 5.3.13.** *Let $R$ be a noetherian ring and let $R \to S$ be a module-finite extension. Let $\mathfrak{a}$ be the ideal generated by the image of the map $S \otimes \mathrm{Hom}_R(S, R) \to R$. Let $I$ be an ideal of $R$ such that $I$ has some power contained in $\mathfrak{a}$. Then for any finitely generated $R$-module $M$, we have $M \sim_I S \otimes M$.*

*Proof.* Since $S$ is finitely generated and $R$ is noetherian, $\mathrm{Hom}_R(S, R)$ is finitely generated. Let $k$ be the least number of generators of $\mathrm{Hom}_R(S, R)$ and let $f_1, \ldots, f_k$ be a set of generators. Then we have a map

$$(S \otimes M)^{\oplus k} \to M$$

where $\quad (\sum s^{(1)} \otimes m^{(1)}, \sum s^{(2)} \otimes m^{(2)}, \cdots, \sum s^{(k)} \otimes m^{(k)}) \mapsto \sum \sum f_i(s^{(i)}) m^{(i)}.$

Let $\mathfrak{a}$ be the image of the map $S \otimes \mathrm{Hom}_R(S, R) \to R$. Then the image of above map is $\mathfrak{a}M$. If $I$ has some power in $\mathfrak{a}$, then by Proposition 5.3.9.(ii) we know $M \sim \mathfrak{a}M$. Hence, $\psi_n(M) \leqslant C_1 \psi_n((S \otimes M)^{\oplus k})$ and $\Psi_n(M) \leqslant C_2 \Psi_n((S \otimes M)^{\oplus k})$.

On the other hand, we have $R^{\oplus l} \twoheadrightarrow S$ because $S$ is module-finite over $R$. Hence $M^{\oplus l} \twoheadrightarrow S \otimes M$. So $\psi_n(M^{\oplus l}) \geqslant C_1' \psi_n(S \otimes M)$ and $\Psi_n(M^{\oplus l}) \geqslant C_2' \Psi_n(S \otimes M)$.

Finally, by Proposition 5.3.10, we know that $M \sim S \otimes M$. □

We also want to discuss Question 5.3.6 when we restrict scalars.

**Proposition 5.3.14.** *Let $R, A$ be two noetherian rings. Suppose that $R$ is module-finite over $A$. Suppose that $I \subseteq A$ is an ideal and $J \subseteq R$ has the same radical as $IR$. Suppose that $M$ is a $R$-module. Then $M$ is an $A$-module via restriction of scalars, and $M$ has finite $I$-quasilength if and only if it has finite $J$-quasilength. Furthermore, there are some positive numbers $C_1, C_2$ depending on $I, J$ but not depending on $M$ such that*

$$C_1 \mathcal{L}_I(M) \leqslant \mathcal{L}_J(M) \leqslant C_2 \mathcal{L}_I(M).$$

*Proof.* Since $\sqrt{IR} = \sqrt{J}$. $M$ has finite $J$-quasilength if and only it has finite $IR$-quasilength.

By Lemma 5.3.5, we know that there is some $C_1, C_2$ such that $C_1 \mathcal{L}_{IR}(M) \leqslant \mathcal{L}_J(M) \leqslant C_2 \mathcal{L}_{IR}(M)$. Suppose that $M$ has finite $(IR)$-quasilength. It suffices to show this for $I$ (over

$A$) and $IR$ (over $R$). If there is a $A/I$ cyclic filtration of $M$, suppose that there is a sequence $u_1, \ldots, u_n$ such that $\{M_i = Au_1 + \cdots + Au_i\}$ gives the filtration, then $\{M_i' = Ru_1 + \cdots + Ru_i\}$ gives a $R/IR$ cyclic filtration of $M$. Hence, $\mathcal{L}_I(M) \geqslant \mathcal{L}_{IR}(M)$, and $M$ has finite $I$-quasilength.

Next we assume that $M$ has finite $I$-quasilength. Suppose that $R$ is generated by $\theta_1, \ldots, \theta_l$ over $A$ as an $A$-module. If $M_0 \subseteq \cdots \subseteq M_n = M$ is a filtration of $M$ over $R$, then each factor is a homomorphic image of $R/IR$. Let $\overline{\theta}_1, \ldots, \overline{\theta}_l$ be the image of $\theta_1, \ldots, \theta_l$ in $R/IR$. Then $R/IR = (A/I)\overline{\theta}_1 + \cdots + (A/I)\overline{\theta}_l$. Therefore, we can refine each factor by giving it a length at most $l$ filtration over $A$. Hence, one has a filtration of length $nl$ over $A$. Therefore $l \cdot \mathcal{L}_{IR}(M) \geqslant \mathcal{L}_I(M)$, and $M$ has finite $IR$-quasilength.

Hence, we have

$$\mathcal{L}_{IR}(M) \leqslant \mathcal{L}_I(M) \leqslant l \cdot \mathcal{L}_I(M)$$

as desired. $\qquad\square$

### 5.3.3.2 One-dimensional case

Now we are ready to prove the following main theorem

**Theorem 5.3.15.** *Suppose that $(R, \mathfrak{m})$ is a noetherian local ring of dimension 1. Then there exists a positive constant $C$ (independent of $M$ and $I$) such that for any ideal $I \subseteq R$ and any finitely generated module $M$, we have*

$$Cn\mathcal{L}_I(M/I^n M) \leqslant \mathcal{L}_I((M/I^n M)^{\oplus n}) \leqslant n\mathcal{L}_I(M/I^n M)$$

*for every positive integer $n$.*

*Proof.* We can assume that $R$ is reduced by Proposition 5.3.11. Let $P_1, \ldots, P_n$ be the collection of minimal primes of $R$. Then we have $R \hookrightarrow \frac{R}{P_1} \times \cdots \times \frac{R}{P_n}$. Look at the ideal $\mathfrak{a}$ generated by the sum of images of all possible $R$-linear maps from $\frac{R}{P_1} \times \cdots \times \frac{R}{P_n}$ back to $R$. We claim that $\mathfrak{a}$ contains all elements in $P_1 \cap \cdots \cap P_{k-1} \cap P_{k+1} \cap \cdots \cap P_n$ for $k = 1, \ldots, n$. This is because the intersection $P_1 \cap \cdots \cap P_{k-1} \cap P_{k+1} \cap \cdots \cap P_n$ is killed by $P_k$. Let $u$ be an element in this intersection. We have a map from $R/P_k$ to $Ru$. Then we have $\frac{R}{P_1} \times \cdots \times \frac{R}{P_n} \to R/P_k \to Ru \subseteq R$ with image containing $u$. $\mathfrak{a}$ cannot be contained in any minimal prime of $R$ by prime avoidance. Since $\dim(R) = 1$, $\mathfrak{a}$ must be $\mathfrak{m}$-primary. Hence any ideal $I$ will have some power in $\mathfrak{a}$. By Proposition 5.3.13, we can work with $M \otimes (\frac{R}{P_1} \times \cdots \times \frac{R}{P_n})$. Then by Lemma 5.3.12, we can work with each component individually.

Therefore we can assume without loss of generality that $R$ is a local complete domain of dimension 1 and $M$ is a finitely generated module. Then any ideal in $R$ is either $0$ or $\mathfrak{m}$-primary, and in both cases we can find a $C$ for (5.3.1). $\qquad\square$

# CHAPTER VI

# Questions and Conjectures

Throughout the research process of this thesis, while many results are proved, many questions remain unsolved. We want to record them here for future references.

## 6.1  Test Elements, Tight Closure and Its Analogues

In Chapter II, we studied the test elements for tight closure. If $\mathrm{char}(R) > 0$, an important feature is that under mild assumptions any nonzero element $c$ in $R$ such that $R_c$ is regular has a power that is a test element for tight closure. Similar phenomenon happens for epf closure. We can modify the proof of [MST$^+$20, Corollary 4.2] to get the following result.

**Corollary 6.1.1.** *Let $(R, \mathfrak{m})$ be a complete normal local domain of residue characteristic $p > 0$ and of dimension $d$. Let $J$ be the defining ideal of the singular locus of $R$. Then there exists an integer $N$ such that $J^N I^{\mathsf{epf}} \subseteq I$ and $J^N I^{\mathsf{wepf}} \subseteq I$ for all $I \subseteq R$.*

*Proof.* From the proof of [MST$^+$20, Corollary 4.2], we have

- $I^{\mathsf{epf}} \subseteq (I, p^n)B \cap R$ for some fixed perfectoid big Cohen-Macaulay $R^+$-algebra $B$ and every $n$;

- There exists some $N$ such that $J^N \subseteq \mathrm{Im}(\mathrm{Hom}_R(B, R) \to R)$.

Then the last paragraph of the proof of [MST$^+$20, Corollary 4.2] works through with $\overline{I^h}$ replaced by $I^{\mathsf{epf}}$. Explicitly, for every $r \in J^N$, there exists $\phi \in \mathrm{Hom}_R(B, R)$ such that $\phi(1) = r$. Applying $\phi$ to $I^{\mathsf{epf}} \subseteq (I, p^n)B \cap R$ we get $rI^{\mathsf{epf}} \subseteq (I, p^n)R$ for every $n$. Hence $J^N I^{\mathsf{epf}} \subseteq \cap_n (I, p^n)R = I$.

The second conclusion comes from the calculation that

$$J^N I^{\mathsf{wepf}} = \bigcap_{N \geqslant 0} J^N (I, p^N)^{\mathsf{epf}} \subseteq \bigcap_{N \geqslant 0} J^N (I, p^N) = J^N I.$$

$\square$

*Remark* 6.1.2. In fact, the same calculation in the last line of the proof above shows that epf closure and wepf closure have the same test ideal.

We want to ask the following question.

*Question* 6.1.3. Let $R$ be a noetherian local ring. Let cl be a closure operation on $R$. What conditions should we impose on cl to make sure that if $R_c$ is regular, then there is some power of $c$ that is a test element for cl?

### 6.1.1  More variations of epf **closure**

Instead of considering one closure operation, we can consider a family of closure operations as follows. Let $(R, \mathfrak{m})$ be a complete local domain of mixed characteristic $p$ and $R^+$ its absolute integral closure. Fix an ideal $J$ that contains $p$ and is contained in $\mathfrak{m}$. An element $u \in R$ is in the closure of $I \subseteq R$ if there is some nonzero element $c \in R$ such that for any $N \in \mathbb{N}, \varepsilon \in \mathbb{Q}^+$, we have

$$c^\varepsilon u \in IR^+ + J^N R^+.$$

In particular, if we choose $J$ to be $pR$, then we recover the usual epf closure.

All these closure operations are potentially larger than the epf closure. So they all satisfy the usual colon-capturing property. Moreover, the proof that the epf closure is trivial on regular local rings ([HM21, Theorem 3.9]) works for this family of closures. Therefore, they are trivial on regular local rings. However, we do not know whether any of these closure operations (including the epf closure) satisfy the generalized colon-capturing axiom (Axiom Set (vii)), so we ask:

*Question* 6.1.4. Do any of the closure operations described above satisfy the generalized colon-capturing axiom?

### 6.1.2  Persistence

It is also important to know if any of these closure operations we considered so far is persistent, i.e., it satisfies the Axiom 4.2.3. We do not know whether any closure operation (including r1f, wepf, PBCM, BCM closures) satisfies the persistence axiom for local morphisms between mixed-characteristic complete local domains. One can also ask the same question when the target ring is a complete local domain of characteristic $p$.

A related question is the following:

*Question* 6.1.5. Let cl be a closure operation on $R, S$ that is persistent for $R \to S$ and let $R \to S$ be faithfully flat. What conditions do we need to ensure $(IS)^{\text{cl}} \cap R = I^{\text{cl}}$ where $I \subseteq R$ is a proper ideal?

## 6.2 Size and Quasilength

### 6.2.1 Fundamental questions about size

Given the studies of size in Chapter V, there are a lot of fundamental mysteries about this notion remain unsolved. For example, we want to ask

- Is the size of any ideal always an integer?

- Is the size of an ideal unchanged by localizing at some maximal ideal?

- Is the size of an ideal over a local ring unchanged when passing to its completion at the maximal ideal?

- Does there exist an example of an ideal $I$ such that $\mathsf{size}(I) > \mathrm{superht}(I)$?

- If $R/P$ has isolated singularities, can we find some $R$-algebra $S$ such that $PS$ has the same size as $P$?

- Suppose we know the sizes of two prime ideals $P$ and $Q$. What can we say about the size of $P \cap Q$?

- For a finitely generated $k$-algebra $R$, suppose that $I \to k[x_1, \ldots, x_n] \to R$ is the defining equation of $R$. Then is $\mathsf{size}(I) - \mathrm{ht}(I)$ an invariant of $R$? Explicitly, for a different presentation of $R$, say $J \to k[y_1, \ldots, y_m] \to R$, do we have $\mathsf{size}(I) - \mathrm{ht}(I) = \mathsf{size}(J) - \mathrm{ht}(J)$? This comes down to comparing the size of an ideal $I$ in $R$ with the size of $(I, x)$ in $R[x]$ where $x$ is an indeterminate.

Since quasilength is notoriously hard to compute, so is size. We want to ask the following concrete seemingly computable questions

- What is the size of the ideal generated by the $t \times t$ minors of $r \times s$ matrix of indeterminates, where $r \geqslant s \geqslant t$? We do not know the answer even when $r = 2, s = 3$ and $t = 2$. The height of the ideal generated by $2 \times 2$ minors is 2 and the arithmetic rank of this ideal is 3.

### 6.2.2 Conjectures related to results

Given the result of Theorem 5.2.10, it is natural to conjecture that

**Conjecture 6.2.1.** *In a regular local ring $R$, any prime ideal $P$ has* $\mathsf{size}(P) = \mathrm{ht}(P)$.

In the regular local case, we have $\text{ht}(P) = \text{superht}(P)$ for any prime ideal $P$. If the regular local ring $R$ has Krull dimension $d$, then Conjecture 6.2.1 is true for primes of height $d - 1$ by Theorem 5.2.10. It is also true for primes of height $1$ (because they are principal). So the first unknown case is a height $2$ prime in a dimension $4$ ring. We can also ask whether Conjecture 6.2.1 holds if we only assume that $R$ regular.

Given Theorem 5.3.15, we conjecture that:

**Conjecture 6.2.2.** *Let $R$ be a noetherian local ring. If $M$ is an $R$-module of finite $I$-quasilength where $I$ is an ideal of $R$, then*

$$\mathcal{L}_I(M^{\oplus n}) = n\mathcal{L}_I(M).$$

We do not know if this is true even if $n = 2$.

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