A Spectral Exploration of the Leray Transform in Two Different Settings in $\mathbb{C}^2$

by

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I dedicate this dissertation to my grandfather Hezi Shelah, who pushed me to appreciate not only mathematical rigor, but also trial and error even in the face of difficulties. He is still with me in spirit.

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My close family have always been there to support me from afar, which I don't take for granted. Finally, I'd like to thank my friend Iris for listening to me talk about topics that must have sounded bizarre, at least initially. I've often needed her to remind me that minor obstacles are just temporary.

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Abstract

We are concerned with the Leray transform (referred to as Cauchy-Leray or Leray-Aizenberg by some authors), a skew projection acting on the L^2 space of functions defined on the boundary of a suitable domain $D \subset \mathbb{C}^n$, mapping onto the subspace of boundary values of holomorphic functions in D. We will focus on the case $n = 2$ in two settings, namely convex Reinhardt domains (based on the results of D. Barrett and L. Lanzani) and the so-called rigid Hartogs domains. The starting point is the spectrum of the Leray transform, which depends on the boundary measure. We will extract information about the norm for l_p balls, and more generally about the essential norm and a particular variant. Then we will explore our ability to "hear" a convex Reinhardt domain based on its Leray spectrum. To what extent can we recover the domain?

In the rigid Hartogs setting, the computation of the Leray spectrum will be done for a family of measures. Then we will be able to explore similar topics to the above.

Chapter I

Introduction

1.0.1 Leray Transform

The Leray transform is one possible generalization of the Cauchy transform from complex analysis in one variable to several variables. Unlike the orthogonal Szegö projection, the Leray transform is a skew projection acting on the L^2 space of functions defined on the boundary (endowed with a suitable measure) of a suitable domain $D \subset \mathbb{C}^n$ (the class of which will be determined later depending on the context). To ensure that the kernel is well-defined (let alone the transform), first we need a definition:

Definition I.1. A domain D is called $\mathbb{C}$-*linearly convex* (or lineally convex) if it is the complement of a union of complex hyperplanes.

Now, given a $\mathbb{C}$-linearly convex domain $D \subset \mathbb{C}^n$ with a defining function ρ and a sufficiently smooth boundary S, we can define the Leray transform (for $z \in D$) as

$$\mathbb{L}(f)(z) = \int_{\zeta \in S} f(\zeta) L(\zeta, z), \tag{1.1}$$

where

$$\begin{aligned} L(\zeta, z) &= \frac{1}{(2\pi i)^n} \frac{j^*(\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1})(\zeta)}{< \partial\rho, \zeta - z >^n}, \\ < \partial\rho(\zeta), \zeta - z > &= \sum_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j). \end{aligned} \tag{1.2}$$

and j^* is the pullback of the inclusion $j : S \to \mathbb{C}^n$ acting on $(2n-1)$-forms. We call

$$d\lambda = \frac{j^*(\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1})}{(2\pi i)^n}$$

a *Leray-Levi measure* with respect to ρ (even for a given ρ, any multiple of λ is also called a Leray-Levi measure). The kernel is independent of ρ but does depend on D (there is no universal kernel unless $n = 1$). As in the one variable case, the integral formula returns a holomorphic function $\mathbb{L}f$ defined on the domain if $f \in L^1(S, \lambda)$, and moreover it reproduces holomorphic functions that extend continuously to the boundary (along with their first order partial derivatives) from their boundary values (see Theorem 3.4 in [3] for convex C^2 domains, and more generally Proposition 4.2 in [1]). Finally, the Leray transform stands out from other generalizations of the Cauchy transform in that it transforms nicely under projective automorphisms.

Note that if the boundary S is C^1, the above definition is equivalent to the complex tangent hyperplanes all lying outside D (not just locally). Thus, for a $\mathbb{C}$-linearly convex domain with a C^1 boundary, the denominator of the Leray kernel is well-defined. We still need to address $\overline{\partial}\partial\rho$. For that, we need a property that is close to C^2 smoothness of S:

Definition I.2. We say that D is $C^{1,1}$ if there exists a defining function ρ such that $\nabla\rho \neq 0$ on S and the partial derivatives are Lipschitz functions, i.e.

$$\exists a > 0 \quad \forall z, w \in S \quad |\nabla\rho(z) - \nabla\rho(w)| \leq a|z - w|.$$

The interpretation of $\overline{\partial}\partial\rho$ when S is $C^{1,1}$ but not C^2, is not crucial for this thesis. Suffice to say that the second order partial derivatives exist in the distributional sense and a second order Taylor approximation holds almost everywhere with respect to λ (or equivalently the surface measure σ). Refer to Section 2 of the Lanzani-Stein paper [1] for details. A natural way to ensure that the integral in the definition for the Leray transform converges for all $z \in D$ is that the domain also be bounded. In order for the Leray transform to have a bounded extension from $L^p(S, \lambda)$ to itself for all $1 < p < \infty$ (we still denote the extension by $\mathbb{L}$), it turns out that we need a stronger convexity condition than just $\mathbb{C}$-linear convexity. We stress that this definition is due to L. Lanzani and E. Stein, and other authors might assign a different meaning to the term.

Definition I.3. Let D be a domain in $\mathbb{C}^n$ with a C^1 boundary S and defining function ρ. We call D *strongly* $\mathbb{C}$*-linearly convex* (in the Lanzani-Stein sense) if the two equivalent conditions hold

- $\exists c > 0 \quad \forall z \in D, \zeta \in S \quad | < \partial\rho(\zeta), \zeta - z > | \geq c|\zeta - z|^2.$
- $\exists c' > 0 \quad \forall z \in D, \zeta \in S \quad d^E(z, \zeta + T_\zeta^{\mathbb{C}}) \geq c'|\zeta - z|^2,$

where d^E denotes the Euclidean distance from z to the affine subspace $\zeta + T_\zeta^{\mathbb{C}}$, which is the geometric realization of the complex tangent space $T_\zeta^{\mathbb{C}}$ to S at ζ.

Remark I.4. If S is C^2, then the above definition implies *strong* $\mathbb{C}$*-convexity*, that is

$$\exists c'' > 0 \quad \forall \zeta \in S, h \in T_\zeta^{\mathbb{C}} \quad H_\rho(\zeta; h) \geq c''|h|^2,$$

where the Hessian of ρ is given by

$$H_\rho(\zeta; h) := 2\mathrm{Re}\left(\sum_{1 \leq j,k \leq n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(\zeta) h_j h_k \right) + 2 \sum_{1 \leq j,k \leq n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(\zeta) h_j \overline{h_k}.$$

Strong $\mathbb{C}$-linear convexity can be thought of as a global version of strong $\mathbb{C}$-convexity (the latter implies the former for small $|\zeta - z|$). The latter is also a weaker condition than *strong convexity*, which is analogous using $h \in T_\zeta^{\mathbb{R}}$. Finally, strong convexity also implies strong $\mathbb{C}$-linear convexity, while strong $\mathbb{C}$-convexity implies *strong* *pseudoconvexity*, i.e.

$$\exists c''' > 0 \quad \forall \zeta \in S, h \in T_\zeta^{\mathbb{C}} \quad L_\rho(\zeta; h) := \sum_{1 \le j,k \le n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(\zeta) h_j \overline{h_k} \ge c'''|h|^2.$$

To summarize, for a domain with a C^2 boundary we have

$$\text{strongly convex} \implies \text{strongly}\, \mathbb{C} - \text{linearly convex} \implies \text{strongly}\, \mathbb{C} - \text{convex}$$
$$\implies \text{strongly pseudoconvex}$$

Theorem I.5. *If D is a bounded, strongly $\mathbb{C}$-linearly convex and $C^{1,1}$, then $\mathbb{L}$ is $L^p(S, \lambda)$ bounded for all $1 < p < \infty$. The measure λ may be exchanged with the surface measure σ (induced by the Lebesgue measure) as they are equivalent.*

The proof is quite long (homogeneous space theory is involved) and can be found in the aforementioned Lanzani-Stein paper. Note that these three conditions aren't necessary as discussed in the next chapter (focusing on l_p balls, which apart from boundedness, don't satisfy the other two conditions simultaneously except when $p = 2$).

1.0.2 Leray spectrum for convex Reinhardt domains

Reinhardt domains in $\mathbb{C}^n$ have rotational symmetry with respect to each variable. More formally:

Definition I.6. Let $D \subset \mathbb{C}^n$ be a domain. D is said to be a *Reinhardt domain* if for some $c \in D$ (the center), we have

$$\forall 1 \le j \le n \quad \forall \theta \in \mathbb{R} \quad R_j(D) = D,$$

where $R_j(z) = (z_1, ..., c_j + (z_j - c_j)e^{i\theta}, ..., z_n)$.
We say that D is a *complete Reinhardt domain* if it contains any polydisk spanned by the center c and any $z \in D$. Certainly, a convex Reinhardt domain is complete.

We will be interested in convex Reinhardt domains. This is due to the following fact:

Theorem I.7. *A $\mathbb{C}$-linearly convex complete Reinhardt domain is convex.*

For the short proof (with some projective dual setup), see Example 2.2.4 in [4].

We are ready to follow the Lanzani-Barrett paper [2], which serves as the basis of the first two chapters in this thesis. First, let $\tilde{\mathcal{R}}$ be the collection of C^1 convex Reinhardt domains in $\mathbb{C}^2$ centered at $(0,0)$, strongly convex and C^2 away from the axes. It turns out that each domain in $\tilde{\mathcal{R}}$ has an osculating (to second order) dilated l_p ball at every point away from the axes, where by a dilated l_p ball we mean

$$B_{p,a,b} = \{(z_1, z_2) \in \mathbb{C}^2 : a|z_1|^p + b|z_2|^p < 1\},$$

where $p > 1, a, b > 0$ (independent dilation factors). In more detail, let's first parametrize the boundary curve $\gamma := S \cap \mathbb{R}^2_{\geq 0}$ by a certain parameter

$$s = -\frac{r_1\phi'(r_1)}{\phi(r_1) - r_1\phi'(r_1)} \tag{1.3}$$

where $r_2 = \phi(r_1)$ defines γ. It can be worked out for which conditions an osculating dilated l_p ball exists even at points on the axes, and how the exponent p depends on s. Imposing these conditions, we get the subclass $\mathcal{R}$. For our purposes, it suffices to note that we can generate all domains in $\mathcal{R}$ via

$$r_1(s) = b_1 \exp(-\int_s^1 \frac{dt}{tp(t)}), \quad r_2(s) = b_2 \exp(-\int_0^s \frac{dt}{(1-t)p(t)}), \tag{1.4}$$

where $b_1, b_2 > 0$ are constants and $p : [0,1] \to (1,\infty)$ is continuous and for which

$$\int_0^1 \frac{p(s) - p(0)}{s} ds, \quad \int_0^1 \frac{p(s) - p(1)}{1-s} ds$$

converge. Since we're interested in boundedness of $\mathbb{L}$, we need to pinpoint suitable rotation-invariant measures on S. Parametrizing S by $\zeta = (r_1(s)e^{i\theta_1}, r_2(s)e^{i\theta_2})$, it turns out that the most natural rotation-invariant measure is given by

$$d\mu_0(\zeta) = \frac{1}{4\pi^2} ds \wedge d\theta_1 \wedge d\theta_2.$$

It is has the advantage of offering boundedness for as big a class of domains as possible, at least relative to a family of similar measures given by

$$d\mu_q(\zeta) = |\mathcal{L}(\zeta)|^{1-q}\tilde{\omega}(s)d\sigma(\zeta) = \omega(s)dsd\theta_1 d\theta_2,$$

where $q \in \mathbb{R}$ is called the order of the measure, σ is the surface measure, $\tilde{\omega}$ is a positive continuous function on $[0, 1]$, and the Euclidean norm of the Levi form (in dimension $n = 2$) is given by

$$|\mathcal{L}(\zeta)| = -\frac{j^*(\partial\rho \wedge \overline{\partial}\partial\rho)(\zeta)}{|\nabla\rho|^2 d\sigma}.$$

This family includes the special cases of the surface measure (order 1) and Fefferman measure (order $\frac{2}{3}$; see page 259 in [9] for its definition and properties). Finally, writing $p_1 = p(0), p_2 = p(1)$, it can be shown (as done in [2]) that there exists a positive continuous function φ such that

$$\omega(s) = \varphi(s)s^{q(\frac{1}{p_1} - \frac{1}{p_1*})}(1 - s)^{q(\frac{1}{p_2} - \frac{1}{p_2*})}.$$

Note that μ_0 has order 0. The factor $\tilde{\omega}$ (equivalently φ) does not affect whether or not $\mathbb{L}$ is bounded on $L^2(S, \mu_q)$, and has no effect on the asymptotic behavior of the eigenvalues of $\mathbb{L}^*_{\mu_q}\mathbb{L}$. Thus it is justified to omit it as a subscript. Using Fourier decomposition (separately for each variable), we can write $\mathbb{L} = \bigoplus_{n,m\in\mathbb{Z}} \mathbb{L}_{n,m}$, where each $\mathbb{L}_{n,m}$ acts on functions of the form $g(s)e^{i(n\theta_1 + m\theta_2)}$. It can be shown that

$$(\mathbb{L}_{n,m})^*_{\mu_q}\mathbb{L}_{n,m}f =< f, \tau_{n,m} > \kappa_{n,m},$$

where

$$\kappa_{n,m} = \left(\frac{s}{r_1(s)}\right)^n \left(\frac{1-s}{r_2(s)}\right)^m e^{i(n\theta_1+m\theta_2)},$$

$$\tau_{n,m} = (r_1(s))^n (r_2(s))^m e^{i(n\theta_1+m\theta_2)}.$$

The corresponding eigenvalues are given by

$$\lambda_{n,m} = \left(\frac{(n+m+1)!}{n!m!}\right)^2 \int_0^1 r_1^{2n}(s) r_2^{2m}(s)\omega(s)ds \int_0^1 \left(\frac{s}{r_1(s)}\right)^{2n} \left(\frac{1-s}{r_2(s)}\right)^{2m} \frac{1}{\omega(s)}ds \tag{1.5}$$

for $n, m \in \mathbb{Z}_{\geq 0}$. Note that we can get a basis for $L^2(S, \mu_q)$ by adjoining $\{\kappa_{n,m}\}_{n,m\in\mathbb{Z}_{\geq 0}}$ to some basis for the (infinite-dimensional) kernel. Hence we have

Theorem I.8. $\mathbb{L}_{\mu_q}^* \mathbb{L}_{\mu_q}$ *is diagonalizable for all* $q \in \mathbb{R}$. *In particular, its set of eigenvalues is dense in its spectrum.*

Now recall a general definition:

Definition I.9. The *essential spectrum* of a self-adjoint operator $T : V \to V$ (V is a Hilbert space) is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not a Fredholm operator (an operator is called Fredholm if both its kernel and cokernel are finite-dimensional). Equivalently (due to Weyl's criterion), the essential spectrum consists exactly of limit values of eigenvalues, in addition to the continuous spectrum and isolated eigenvalues of infinite multiplicity.

In our case, 0 is an isolated eigenvalue of infinite multiplicity (there may be others) and the non-zero part of the essential spectrum is exactly the set of limit values of the non-zero eignvalues of $\mathbb{L}_{\mu_q}^* \mathbb{L}$, which can be split into three parts:

1. Limit values of (1.5) that correspond to $\min\{n, m\} \to \infty$ with $\frac{n}{m} \to x \in [0, \infty]$.

Asymptotic analysis shows that this is the interval given by the image of

$$\phi_D : [0,\infty] \to [1,\infty), \quad \phi_D(x) = \frac{\sqrt{p(\frac{x}{1+x})p^*(\frac{x}{1+x})}}{2}, \tag{1.6}$$

where $p^* = \frac{p}{p-1}$ is the Hölder conjugate of p. We call these slope-based limit values.

2. Limit points that correspond to horizontal lines $m = m_0$ with $n \to \infty$. In this case, the limits points are discrete and given by

$$G_{p_2,q}(m_0) = \frac{\Gamma(\frac{2m_0}{p_2} + q(\frac{1}{p_2} - \frac{1}{p_2^*}) + 1)\Gamma(\frac{2m_0}{p_2^*} + q(\frac{1}{p_2^*} - \frac{1}{p_2}) + 1)}{\Gamma^2(m_0+1)(\frac{2}{p_2})^{\frac{2m_0}{p_2} + q(\frac{1}{p_2} - \frac{1}{p_2^*}) + 1}(\frac{2}{p_2^*})^{\frac{2m_0}{p_2^*} + q(\frac{1}{p_2^*} - \frac{1}{p_2}) + 1}}, \tag{1.7}$$

wherever these expressions are well-defined (i.e. the arguments of Γ are positive). We call these horizontal limit values, and visualize them as points on the vertical line at infinity $n = \infty$.

3. Same as above for vertical lines $n = n_0$. Just swap m_0 with n_0 and p_2 with p_1, giving us the limits values $G_{p_1,q}(n_0)$. We call these vertical limit values, as they correspond to the horizontal line at infinity $m = \infty$.

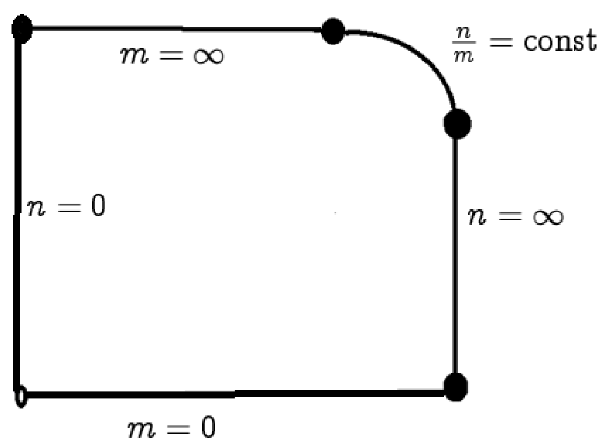

Figure 1.1: The limit values can be visualized as function values corresponding to discrete points on two lines at infinity, as well as the entire connecting arc for the slope-based limit values.

Part 1 relies on a variant of the Laplace method, while the other two parts rely on

Watson's lemma. These tools will be discussed in a different setting. Now note that for all $p > 1, q \in \mathbb{R}$ the function $G_{p,q}(x)$ is well-defined for sufficiently large $x \geq 0$ according to the conditions

$$\frac{x}{p_2} + q(\frac{1}{p_2} - \frac{1}{p_2^*}) + 1 > 0, \quad \frac{x}{p_2^*} + q(\frac{1}{p_2^*} - \frac{1}{p_2}) + 1 > 0.$$

We need the above to hold for all $x \geq 0$ for boundedness, which boils down to considering just $x = 0$. Finally, we need to consider the limit as $x \to 0$. Stirling's formula yields

$$\lim_{x \to \infty} G_{p,q}(x) = \frac{\sqrt{pp^*}}{2} < \infty.$$

In conclusion, we get

Theorem I.10. *For a domain $D \in \mathcal{R}$, $\mathbb{L}$ is bounded on $L^2(S, \mu_q)$ if and only if*

$$|q| < \frac{1}{t(p_1, p_2)}, \tag{1.8}$$

where $t(p_1, p_2) = \max_{j=1,2} |\frac{1}{p_j} - \frac{1}{p_j^*}|$. This condition is vacuous when $p_1 = p_2 = 2$, which is the case when S is C^2 at the axes. Moreover, $q = 0$ satisfies the condition for all domains in $\mathcal{R}$.

See Theorem 1 (and Corollary 18 for the C^2 case) in the Lanzani-Barrett paper [2] for a full proof.

Chapter II

A Norm Approach to Convex Reinhardt Domains in $\mathbf{C}^2$

2.0.1 Leray norm for l_p balls

In the context of convex Reinhardt domains, l_p balls are not only natural due to the construction, but they also seem to be the simplest to analyze in terms of the spectrum of $\mathbb{L}^*_{\mu_0}\mathbb{L}$ (the simplest case is $p = 2$ for which all eigenvalues are 1 if the measure is μ_0). Note that

$$r_1(s) = s^{1/p}, \quad r_2(s) = (1-s)^{1/p}.$$

Plugging the above into (1.5) and using the beta function

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

we get

$$\lambda_p(n, m) = \frac{B[1 + \frac{2n}{p}, 1 + \frac{2m}{p}]B[1 + \frac{2n}{p*}, 1 + \frac{2m}{p*}]}{B[n+1, m+1]^2}.$$

Strategy to calculate the Leray norm

1. $\lambda_p(n, m)$ is symmetric in n, m and extends to $\mathbb{R}^2_{\geq 0}$. Thus, it suffices to consider the region $\Delta := \{(n, m) \in \mathbb{R}^2_{\geq 0} : n \leq m\}$ and hope that $\sup_\Delta \lambda_p(n, m) = \sup_{\mathbb{Z}^2_{\geq 0}} \lambda_p(n, m)$.
2. We can think of Δ as an infinite triangle where λ_p has a continuous extension to the boundary (because the slope-based part of the essential spectrum shrinks to one value). Then this becomes a non-trivial calculus problem.

3. The line $m = 0$ is the easiest as the restriction of λ_p is an increasing rational function.

4. The line $n = \infty$ is handled by observing that $\frac{\partial \lambda_p}{\partial m} < 0$ as follows from a convexity condition on another function to be defined.

5. For the interior, it can be shown that the partial derivatives don't vanish simultaneously.

6. The diagonal lends itself to similar treatment as the line at infinity. The function is increasing along it.

We proceed step by step.

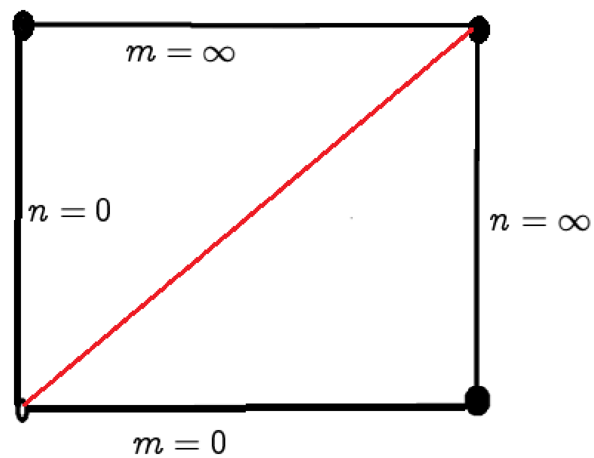

Figure 2.1: We will show that the maximum corresponds to the two vertices that are off the diagonal, i.e. $(\infty, 0)$ and $(0, \infty)$.

The line $m = 0$

Using the identities $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $\quad \Gamma(x + 1) = x\Gamma(x)$, we get

$$\lambda_p(n, 0) = (n + 1)^2 \frac{\Gamma(1 + \frac{2n}{p})\Gamma(1 + \frac{2n}{p^*})}{\Gamma(2 + \frac{2n}{p})\Gamma(2 + \frac{2n}{p^*})} = \frac{(n + 1)^2}{(1 + \frac{2n}{p})(1 + \frac{2n}{p^*})}.$$

Taking the derivative and simplifying gives

$$\lambda_p'(n, 0) = \frac{2n(n + 1)(1 - \frac{4}{pp^*})}{(1 + \frac{2n}{p})^2(1 + \frac{2n}{p^*})^2} \geq 0,$$

which follows from the inequality $pp^* \geq 4$ for all $p > 1$ (the minimum is attained at $p = 2$).

The line $n = \infty$

Define

$$G_p(m) := \frac{\Gamma(\frac{2m}{p}+1)\Gamma(\frac{2m}{p^*}+1)}{\Gamma^2(m+1)(\frac{2}{p})^{\frac{2m}{p}+1}(\frac{2}{p^*})^{\frac{2m}{p^*}+1}}.$$

Taking the logarithmic derivative, a calculation of the logarithmic derivative (to be revisited in more detail in the next subsection for μ_q) shows that $G_p'(m) \leq 0$ if and only if

$$\frac{1}{p}(\psi(\frac{2m}{p}) + \log(p)) + \frac{1}{p^*}(\psi(\frac{2m}{p^*} + \log(p^*)) \leq \psi(m) + \log(2),$$

where $\psi(x) = (\log(\Gamma))'(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. Looking carefully, you can see a concavity condition emerge for the function $h_m(x) := \psi(\frac{2m}{x}) + \log(x)$ (note that $p\frac{1}{p} + p^*\frac{1}{p^*} = 2$). We need to show that $h_m''(x) < 0$ for all $x > 0$. A calculation of the second derivative (again, see the next subsection) followed by a substitution $v = \frac{2m}{x}$ reduces this to proving that

$$v^2\psi''(v) + 2v\psi'(v) < 1 \tag{2.1}$$

for all $v > 0$.

The final key is using the integral representation of the polygamma functions (i.e. derivatives of digamma) and using integration by parts (once or twice). Credit for this idea goes to Prof. Iosif Pinelis who suggested it. First, the integral representation for $k \geq 1$ (see section 6.4 in [5]) is

$$\psi^{(k)}(v) = (-1)^{k+1}\int_0^\infty \frac{z^k e^{-vz}}{1-e^{-z}}dz. \tag{2.2}$$

Plugging this into the LHS of (2.1) and using integration by parts once, where the derivative factor is the numerator (the boundary terms vanish), we get

$$
\begin{aligned}
v^2\psi''(v) + 2v\psi'(v) &= \int_0^\infty \frac{(2vz - v^2z^2)e^{-vz}}{1-e^{-z}} dz \\
&= v\int_0^\infty \frac{z^2e^{-z}}{(1-e^{-z})^2} e^{-vz} dz.
\end{aligned}
$$

It can be shown that $\frac{z^2e^{-z}}{(1-e^{-z})^2} < 1$ for all $z > 0$, which gives us the estimate we seek:

$$
v^2\psi''(v) + 2v\psi'(v) < v\int_0^\infty e^{-vz}dz = 1.
$$

For completeness, we have

$$
\frac{z^2e^{-z}}{(1-e^{-z})^2} < 1 \Leftrightarrow z^2 < e^z - 2 + e^{-z} = z^2 + \sum_{j=2}^\infty \frac{2}{(2j)!} z^{2j},
$$

which is certainly true for $z > 0$ as then the summands in the series are positive.

The interior

We need to consider the set of equations $\frac{\partial\lambda_p}{\partial n} = 0 = \frac{\partial\lambda_p}{\partial m}$. In fact, we will show that $\frac{\partial\lambda_p}{\partial n} = \frac{\partial\lambda_p}{\partial m}$ implies $n = m$, which means that there are no local extrema in the interior of Δ. Using the identities

$$
\begin{aligned}
\frac{\partial}{\partial x}\log(B[x,y]) &= \psi(x) - \psi(x+y), \\
\psi(z+1) &= \psi(z) + \frac{1}{z},
\end{aligned}
$$

the calculation of the logarithmic derivatives yields

$$
\frac{\partial\lambda_p}{\partial n}(n,m) - \frac{\partial\lambda_p}{\partial m}(n,m) = 2\lambda_p(n,m)\left(\frac{1}{p}h_{2n,2m}(p) + \frac{1}{p^*}h_{2n,2m}(p^*) - h_{2n,2m}(2)\right),
$$

where $h_{2n,2m}(x) = \psi(\frac{2n}{x}) - \psi(\frac{2m}{x})$. Now, this function is convex for $m < n$ due to a more general lemma proved by I. Pinelis (using the integral representation and integration by parts twice).

Lemma II.1. *: The function $h_{a,b} = \psi(\frac{a}{x}) - \psi(\frac{b}{x})$ is convex for $a > b > 0$ and $x > 0$.*

See Appendix A for the proof.

The diagonal

It suffices to show that $\frac{\partial \lambda_p}{\partial n} > 0$ as $\frac{\partial \lambda_p}{\partial n} = \frac{\partial \lambda_p}{\partial m}$ on the diagonal by symmetry. Even here we see a convexity condition emerge in the calculation, which is in turn equivalent to

$$\forall x > 0 \quad 4x\psi''(2x) + 4\psi'(2x) - x\psi''(x) - 2\psi'(x) < \frac{4}{(2x+1)^3}.$$

Using the identity $\psi(2x) = \frac{1}{2}\psi(x) + \frac{1}{2}\psi(x+\frac{1}{2}) + \log 2$, the integral representation (2.2), a change of variable $w = xz$ and some simplification, we get

$$\begin{aligned}
\text{LHS} &= \frac{x}{2}(\psi''(x) + \psi''(x+\frac{1}{2})) + (\psi'(x) + \psi'(x+\frac{1}{2})) - x\psi''(x) - 2\psi'(x) \\
&= -\frac{x}{2}\psi''(x) + \frac{x}{2}\psi''(x+\frac{1}{2}) - \psi'(x) + \psi'(x+\frac{1}{2}) \\
&= \frac{1}{x^2}\int_0^\infty \frac{we^{-w}}{1+e^{-\frac{w}{2x}}}(\frac{w}{2} - 1)dw.
\end{aligned}$$

Now we can integrate by parts (with $we^{-w}(\frac{w}{2} - 1)$ as the derivative factor) to get

$$\text{LHS} = \frac{1}{4x^3}\int_0^\infty \frac{e^{-\frac{w}{2x}}}{(1+e^{-\frac{w}{2x}})^2} w^2 e^{-x} dw.$$

All that remains is to bound this integrand using the simple estimate $\frac{1}{(1+e^{-\frac{w}{2x}})^2} < 1$ for $w, x > 0$. Then we finally get

$$\text{LHS} < \frac{1}{4x^3}\int_0^\infty w^2 e^{-(1+\frac{1}{2x})w} dw = \frac{4}{(2x+1)^3},$$

which is exactly the needed inequality.

Conclusion

All of this implies that the norm corresponds to a limit point (along the line $m = 0$ or equivalently $n = 0$). Plugging this into the formula for the vertical/horizontal part of the essential spectrum and taking the square root, we get the following:

Theorem II.2. *For all $p > 1$, denote the unit l_p sphere by S_p. Then we have*

$$\|\mathbb{L}\|_{L^2(S_p,\mu_0)} = \frac{\sqrt{pp^*}}{2}.$$

Remark II.3. 1. Note that among l_p balls, two distinct balls have the same norm if they are the same up to duality (replace p by p^*) or scaling. In those cases, they are in fact isospectral.

2. Other measures are much harder to investigate (especially as the eigenvalues depend on ω), but perhaps this infinite triangle picture (or rather square in the absence of symmetry) isn't that special. Likewise, other domains in $\mathcal{R}$ may be tractable (maybe balls of mixed exponents), but we need to add the slope-based part of the spectrum to the picture (this can be visualized as a square whose corner at infinity is a quarter circle corresponding to all non-negative slopes). However, the calculations for l_p balls heavily rely on properties of the gamma and digamma functions, which don't appear naturally for other domains. Luckily, a fair amount can be said if we look at the essential norm and a certain variant.

2.0.2 Leray essential norm and its variant

First, recall the definition of the essential norm.

Definition II.4. Let V be a Banach space. For a bounded operator $T : V \to V$, the *essential norm* is its distance to the subspace of compact operators $\mathcal{K}(V)$, i.e.

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(V)\}.$$

In our case, since $\mathbb{L}^*_{\mu_q}\mathbb{L}$ is diagonalizable, the essential norm $\|\mathbb{L}\|_{e,q}$ corresponding to μ_q is exactly the supremum of the essential spectrum. It is useful to define a variant that singles out just the slope-based part of the essential spectrum, i.e. the maximum of the function $\phi_D(x)$ defined in (1.6). This leads to the following definition (specific to this setting):

Definition II.5. For $\mathbb{L}$ corresponding to $D \in \mathcal{R}$, the *min-based norm* is defined via

$$\|\mathbb{L}\|_{\min} = \lim_{k\to\infty} \sup_{\min\{n,m\}>k} \lambda_{n,m} = \max_{x\in[0,\infty]} \phi_D(x).$$

There is no dependence on the measure (which is assumed to be some μ_q), hence the lack of subscript. Note that this definition only makes sense for operators that can be diagonalized simultaneously with $\mathbb{L}$.

What can be said about the Leray spectrum for l_p balls and μ_q? For μ_0, we've seen that the essential norm coincides with the norm and has a simple formula $\frac{\sqrt{pp^*}}{2}$. In general, the eigenvalues become much more difficult to work with, and even the symmetry with respect to the swap $n \leftrightarrow m$ may be lost (unless $\omega(1-s) \equiv \dfrac{1}{\omega(s)}$). That said, the formula for the slope-based limit values is independent of the measure (provided that it's in the family), while the formulas for the horizontal and vertical cases only depend on the order q rather than the function ω. Moreover, these formulas work for all domains in $\mathcal{R}$ as opposed to just l_p balls. This suggests a fairly general result as far as the essential norm is concerned.

Theorem II.6. *Let $D \in \mathcal{R}$ have a boundary endowed with a measure of order $q \leq -1$ or $q = 0$ such that the boundedness condition (1.8) holds. Then the essential norm $\|\mathbb{L}\|_{e,q}$ is attained by one of the following three options:*

- $n \to \infty, m = 0$*, in which case*

$$\|\mathbb{L}\|_{e,q}^2 = \frac{1}{4}\Gamma(1+q(\frac{1}{p_2}-\frac{1}{p_2^*}))\Gamma(1+q(\frac{1}{p_2^*}-\frac{1}{p_2}))p_2^{1+q(\frac{1}{p_2}-\frac{1}{p_2^*})}{p_2^*}^{1+q(\frac{1}{p_2^*}-\frac{1}{p_2})}.$$

- $m \to \infty, n = 0$*, in which case*

$$\|\mathbb{L}\|_{e,q}^2 = \frac{1}{4}\Gamma(1+q(\frac{1}{p_1}-\frac{1}{p_1^*}))\Gamma(1+q(\frac{1}{p_1^*}-\frac{1}{p_1}))p_1^{1+q(\frac{1}{p_1}-\frac{1}{p_1^*})}{p_1^*}^{1+q(\frac{1}{p_1^*}-\frac{1}{p_1})}.$$

- $\frac{n}{m} \to x \in (0,\infty)$*, in which case*

$$\|\mathbb{L}\|_{e,q}^2 = \phi_D(x) = \frac{1}{2}\sqrt{p(\frac{x}{1+x})p^*(\frac{x}{1+x})} = \|\mathbb{L}\|_{\min}^2.$$

Remark II.7. In other words, if the essential norm is attained on either the line $m = \infty$ or the line $n = \infty$, then it is necessarily corresponds to one of the two vertices $(0, \infty), (\infty, 0)$ (both if $p(0) = p(1)$ or $p(0) = p^*(1)$). See Figure 2.1.

Proof. It is enough to prove that for all $1 < p < \infty, q \leq -1$ ($q = 0$ has been worked out) the function

$$G_{p,q}(x) = \frac{1}{\Gamma(x+1)^2}\Gamma(\frac{2x}{p} + 1 + q(\frac{1}{p} - \frac{1}{p^*}))\Gamma(\frac{2x}{p^*} + 1 + q(\frac{1}{p^*} - \frac{1}{p}))(\frac{p}{2})^{\frac{2x}{p}1+q(\frac{1}{p}-\frac{1}{p^*})} \times (\frac{p^*}{2})^{\frac{2x}{p^*}1+q(\frac{1}{p^*}-\frac{1}{p})}$$

is decreasing for $x \geq 0$, since this function describes the limit values along $n = \infty$ for $p = p_0$ and as well as along $m = \infty$ for $p = p_1$ (using integer $x \geq 0$) according to (1.7). Taking the logarithmic derivative, we want to prove that it is negative, i.e.

$$-2\psi(x+1)+\frac{2}{p}\psi(\frac{2x}{p}+1+q(\frac{1}{p}-\frac{1}{p^*}))+\frac{2}{p^*}\psi(\frac{2x}{p^*}+1+q(\frac{1}{p^*}-\frac{1}{p}))+\frac{2}{p}\log\frac{p}{2}+\frac{2}{p^*}\log\frac{p^*}{2} < 0.$$

Rearranging and dividing by 2, we get

$$\frac{1}{p}[\psi(\frac{2x}{p}+1+q(\frac{2}{p}-1))+\log p]+\frac{1}{p^*}[\psi(\frac{2x}{p^*}+1+q(\frac{2}{p^*}-1)+\log p^*] < \psi(x+1)+\log 2.$$

Once again, this will follow if we prove that the function

$$j_{x,q}(y) := \psi(\frac{2x}{y} + 1 + q(\frac{2}{y} - 1)) + \log y$$

is concave, since the RHS of the previous inequality is exactly $j_{x,q}(2)$ and $p\frac{1}{p} + p^*\frac{1}{p^*} = 2$. Taking the second derivative, we get

$$j''_{x,q}(y) = \frac{1}{y^2}[\frac{4(x+q)}{y}\psi'(\frac{2n}{y} + q(\frac{2}{y} - 1) + 1) + (\frac{2(x+q)}{y})^2\psi''(\frac{2n}{y} + q(\frac{2}{y} - 1) + 1) - 1],$$

and so if we write $v = \frac{4(x+q)}{y}$, it is enough to prove that

$$2v\psi'(v + 1 - q) + v^2\psi''(v + 1 - q) < 1$$

for all $v > 0, q \leq -1$. Note that the case $q = 0$ follows from (2.1) and the recursion formula for polygamma functions (which follows from the identity $\Gamma(z+1) = z\Gamma(z)$), i.e.

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}},$$

since then we have

$$2v\psi'(v+1) + v^2\psi''(v+1) = 2v(\psi'(v) - \frac{1}{v^2}) + v^2(\psi''(v) + \frac{2}{v^3}) = 2v\psi'(v) + v^2\psi''(v) < 1.$$

We first show that the case $q = -1$ follows from the above, again using the recursion formula (applied twice)

$$2v\psi'(v+2) + v^2\psi''(v+2) = 2v(\psi'(v) - \frac{1}{v^2} - \frac{1}{(v+1)^2}) + v^2(\psi''(v) + \frac{2}{v^3} + \frac{2}{(v+1)^3})$$
$$= 2v\psi'(v) + v^2\psi''(v) + -\frac{2v}{(v+1)^2} + \frac{2v^2}{(v+1)^3} < 1 + \frac{2v}{(v+1)^2}(\frac{v}{v+1} - 1) < 1,$$

since $\frac{v}{v+1} < 1$ for $v > 0$. Now write $a = 1 - q$. If we show that

$$\frac{\partial}{\partial a}(2v\psi'(v+a) + v^2\psi''(v+a)) < 0$$

for all $v > 0, a \geq 2$, this will imply the desired inequality for all $v > 0, q \leq -1$ and we'll be done. First, we have

$$\frac{\partial}{\partial a}(2v\psi'(v+a) + v^2\psi''(v+a)) = 2v\psi''(v+a) + v^2\psi'''(v+a).$$

We will use the inequalities (see [5])

$$\frac{(k-1)!}{v^k} + \frac{k!}{2v^{k+1}} \leq (-1)^{k+1}\psi^{(k)}(v) \leq \frac{(k-1)!}{v^k} + \frac{k!}{v^{k+1}}$$

for $v > 0, k = 2, 3$, giving us

$$2v\psi''(v+a) + v^2\psi'''(v+a) \leq -2v(\frac{1}{(v+a)^2} + \frac{1}{(v+a)^3})$$
$$+v^2(\frac{2}{(v+a)^3} + \frac{6}{(v+a)^4}) = -\frac{2v}{(v+a)^4}(a^2 + a + (a-2)v) < 0$$

for $v > 0, a \geq 2$. $\square$

Remark II.8. Computer graphing suggests that $2v\psi'(v+1-q)+v^2\psi''(v+1-q)<1$ for all $v>0$, $q<\epsilon$, for some $0.25<\epsilon<0.3$. It also appears that the inequality is reversed for $\epsilon<q<\delta$, where $0.8<\delta<0.85$, giving us increasing behavior for $G_{p,q}$. Finally, the inequality holds for $q>\delta$ for sufficiently large v (which may be enough for decreasing behavior of $G_{p,q}$). Proving this is a topic for further exploration.

Examples

- For l_p balls the essential norm is attained at both vertices for $q\leq -1$. Indeed, as in the case $q=0$, the limit values on the arc are constant and the decreasing behavior along both lines $m=\infty, n=\infty$ shows that the essential norm is at attained at the vertices. We also have

$$\|\mathbb{L}\|^2_{\min}=\|\mathbb{L}\|_{e,q}=\frac{\sqrt{pp^*}}{2}$$

  for $q=0$, and in particular the ratio $\frac{\|\mathbb{L}\|_{\min}}{\|\mathbb{L}\|_{e,q}}$ has no positive lower bound independent of the osculation function $p(s)$, where s was defined in (1.3).

- It is easy to manipulate the function $p(s)$ without changing the prescribed boundary values so that the essential norm is attained at any desired slope $u>0$. Simply pick a piecewise C^1 function $p(s)>1$ on [0,1] (Hölder is enough for $\mathbb{L}$ to be bounded) such that $p(s)p^*(s)$ has a maximum at $\frac{u}{1+u}$ and

$$p(\frac{u}{1+u})p^*(\frac{u}{1+u})>\max_{j=1,2}4G^2_{p_j,q}(0).$$

  In particular, it is possible to have $\|\mathbb{L}\|_{\min}=\|\mathbb{L}\|_{e,q}$ regardless of q. A piecewise linear choice for $p(s)$ might be the simplest for further study.

- For balls with mixed exponents given by

$$B_{p_1,p_2}=\{(z_1,z_2)\in\mathbb{C}^2:|z_1|^{p_1}+|z_2|^{p_2}<1\},$$

with $p_1, p_2 > 1$, the essential norm is attained at one of the vertices. This is because the function $\phi_{B_{p_1,p_2}}(u)$ is monotone in this case, and we still have for $q \leq -1$

$$\phi_D(0) = \lim_{x\to\infty} \sqrt{G_{p_1,q}(x)} \leq G_{p_1,q}(0),$$
$$\phi_D(\infty) = \lim_{x\to\infty} \sqrt{G_{p_2,q}(x)} \leq G_{p_2,q}(0).$$

In the case $q = 0$ the above still holds, and moreover

$$\|\mathbb{L}\|^2_{\min} = \|\mathbb{L}\|_{e,q} = \max_{j=1,2} \frac{\sqrt{p_j p_j^*}}{2}.$$

We call this the square root phenomenon for $q = 0$. A weakened version can be carried over to all domains in $\mathcal{R}$.

Theorem II.9. *For any $D \in \mathcal{R}$, the square root of the essential Leray norm (with respect to $L^2(S, \mu_0)$) lies in the essential spectrum.*

Proof. Working with $\mathbb{L}^*_{\mu_0}\mathbb{L}$ again, the essential norm either corresponds to a point on $n = \infty$, $m = \infty$ or the slope-based part. In the first two cases, that would mean it corresponds to either $m = 0$, $n = \infty$ or the reverse, since the formula for the vertical/horizontal parts is exactly the same as for l_p balls (except p_1 and p_2 may differ), and we've already proven the decreasing behavior along those lines. It follows that the square root is attained via slope ∞ (for $m = 0$) or 0 (for $n = 0$), since

$$\phi_D(0) = \sqrt{\frac{p_1 p_1^*}{4}}, \quad \phi_D(\infty) = \sqrt{\frac{p_2 p_2^*}{4}}.$$

Otherwise, the essential norm corresponds to some slope $u_e \in (0, \infty)$, which is the global maximum of ϕ_D. If

$$\sqrt{\phi_D(u_e)} < \phi_D(0),$$

then

$$\phi_D(u_e) < \frac{p_1 p_1^*}{4},$$

which is a another limit value and hence we have a contradiction. By the intermediate value theorem applied to ϕ_D (a continuous function), it follows that the square root of the essential norm corresponds to some slope in $(0, u_e)$. In fact, by applying this argument to (u_e, ∞), it follows that the square root also occurs in that interval.

□

Chapter III

Can You Hear the Shape of a Sufficiently Smooth and Convex Reinhardt Domain?

For the Laplacian operator with Dirichlet boundary conditions, the spectrum is not known to generally classify the domain unless we restrict the problem to a rather small class (for example, see S. Zelditch's work in [6]). The known counterexamples are special, which raises the possibility that even for other operators the inverse spectral problem has a positive solution in some sense.

In our case, the group on $\tilde{\mathcal{R}}$ for which the Leray kernel transforms nicely, is generated by all coordinate dilations and the coordinate reflection $R(z_1, z_2) = (z_2, z_1)$ (and obviously rotations in either variable, which preserve any domain in $\tilde{\mathcal{R}}$). Also, passing to the polar (dual) domain preserves the spectrum when the measure is μ_0, which is easy to check as the integrals in (1.5) are swapped under duality. For this reason and its simpler form, we restrict or attention to the spectrum of $\mathbb{L}^*_{\mu_0}\mathbb{L}$, which we call the *Reinhardt-Leray spectrum* (note that that the actual spectrum of $\mathbb{L}$ is trivially $\{0, 1\}$, since it is a skew projection). In particular, we work with $q = 0$.

Unlike the Laplacian, the eigenvalues are not ordered, but rather given as function values on a lattice. Sticking to this marking means remembering the toroidal frequencies associated with the eigenvalues, and gives us a better chance of recovering the domain from its marked spectrum.

The slope approach

As mentioned before, by considering sequences of lattice points with a convergent slope $u \in [0, \infty]$, we arrive at the function

$$\phi_D : [0, \infty] \to [1, \infty), \quad \phi_D(x) = \frac{\sqrt{p(\frac{x}{1+x})p^*(\frac{x}{1+x})}}{2}.$$

Note that $p \mapsto pp^* = \dfrac{p^2}{p-1}$ is a covering map of degree 2 with $p \mapsto p^*$ as the deck transformation. Thus, it is clear that using ϕ_D, the marked spectrum recovers $p(s)$ up to $p \mapsto p^*$ for all $s \in [0, 1]$. Does this mean that this rather simple recovery method works up to coordinate dilations and the duality map (reflection entails reflecting the marking, giving $p(1 - s)$), as one would hope based on (1.4)?

No, because in general the equation $p(s) = 2$ has $k \geq 0$ solutions in $(0, 1)$, and these solutions partition $[0, 1]$. We can apply $p \mapsto p^*$ on any subinterval independently of the other ones, giving us 2^{k+1} options for p given the same ϕ_D. They all satisfy continuity and the boundary conditions, although if we impose a differentiability condition, that could help (but even if we don't mind the loss of generality, there are still pesky exceptions involving critical points). If $k = 0$, this ambiguity doesn't arise at all. For finite $k > 0$ (and the infinite case under a natural constraint), it turns out that if you basically add another term to the asymptotic expansion, it will reveal the difference between the two domains up to coordinate dilations and duality. There's something special about the case $k = 1$ in that a single eigenvalue (of a certain kind) detects the difference.

Some geometry

By the way, whether or not $p(s) > 2$ or $p(s) < 2$ can be described geometrically as follows: Consider the osculating (to first order) dilated l_2 ball in $\mathbb{C}^2$ of the form $a|z_1|^2 + b|z_2|^2 = 1$ $(a, b > 0)$ at a point (z_1, z_2) away from the axes with $(|z_1|, |z_2|) =$

$(r_1(s), r_2(s))$ for $s \in (0,1)$. If $p(s) > 2$, this dilated l_2 ball is locally inside the Reinhardt domain (globally if the inequality holds for all $s \in (0,1)$). If the inequality is reversed, then the dilated l_2 ball contains the Reinhardt domain locally (globally). See Appendix B for the proof. We call the former case Reinhardt convexity and the latter Reinhardt concavity.

This can be compared with a vertex of a plane curve, where by definition the first derivative of curvature is 0, and equivalently the curve has a 4-point contact with the osculating circle at that point (as opposed to just 3-point contact in the generic case).

Definition III.1. If we have $p(s_0) = 2$, we call s_0 a *Reinhardt vertex*. If $p \equiv 2$ on a subinterval, we only count the endpoints as Reinhardt vertices. For a domain $D \in \tilde{\mathcal{R}}$, we denote its set of Reinhardt vertices by V_D.

Note that V_D is an invariant of essentially isospectral domains in $\tilde{\mathcal{R}}$, i.e. domains with the same pp^* function, since the vertices correspond to solutions of $pp^* = 4$.

Theorem III.2. *Let $\mathcal{R}_1$ denote the collection of all domains $D \in \tilde{\mathcal{R}}$ such that $card(V_D \setminus \{0,1\}) = 1$. Denote by DS the space of double sequences $\{a_{n,m}\}_{n,m \in \mathbb{Z}_{\geq 0}}$ with non-negative elements. Then the marked spectrum map $\chi : \mathcal{R}_1 \to DS$ given by $\chi(D) = \{\lambda_{n,m}\}_{\mathbb{Z}^2_{\geq 0}}$ is injective modulo coordinate dilations and duality.*

Remark III.3. For essentially isospectral domains, we can distinguish between them (modulo coordinate dilations and duality) using just a single eigenvalue of the form $\lambda_{n,0}$ or $\lambda_{0,m}$, where $n, m \in \mathbb{N}$. If the unique Reinhardt vertex in $(0,1)$ happens to be rational, then $\lambda_{n,m}$ also suffices for any $n, m \in \mathbb{N}$ such that $\dfrac{n}{n+m} = a$.

Proof. We work by steps.

Step 1

Let $D, \tilde{D} \in \mathcal{R}_1$ have the same marked spectrum, and let $p(s)$, $\tilde{p}(s)$ be the respective osculating parameter functions. We know that both domains share the same Reinhardt vertex $a \in (0,1)$. Thanks to duality, we may assume $\tilde{p}(s) = p(s) \geq 2$ on $[0,a]$ (we can arrange for $p(s) \geq 2$ and $\tilde{p}(s) \geq 2$ on $[0,a]$ separately). Then either $\tilde{p}(s) = p(s)$ on $(a,1]$ and we are done, or $\tilde{p}(s) = p^*(s)$ on $(a,1]$. We assume the latter.

Step 2

We observe that for $s \in [0,a]$ we have (using (1.4) for $r_1(s), r_2(s)$)

$$\tilde{r_1}(s) = \alpha r_1(s), \quad \tilde{r_2}(s) = r_2(s),$$

for the constant $\alpha = \exp(\int_a^1 (\frac{1}{tp(t)} - \frac{1}{tp^*(t)})dt)$. On (a,1] we have

$$\tilde{r_1}(s) = \frac{s}{r_1(s)}, \quad \tilde{r_2}(s) = \beta \frac{1-s}{r_2(s)},$$

for the constant $\beta = \exp(\int_0^a (\frac{1}{(1-t)p^*(t)} - \frac{1}{(1-t)p(t)})dt)$.

Step 3

Now we want to show that $\forall n \in \mathbb{N} \quad \tilde{\lambda}_{n,0} - \lambda_{n,0} \neq 0$. The calculation for $\lambda_{0,m} - \lambda_{0,m}$ is similar, but we can simply consider the reflected domains $R(D), R(\tilde{D})$ which swap the indices. Using (1.5), we get that $\frac{1}{(n+1)^2}(\tilde{\lambda}_{n,0} - \lambda_{n,0})$ is given by

$$\begin{aligned}
&\left(\alpha^{2n}\int_0^a r_1^{2n}(s)ds + \int_a^1 (\frac{s}{r_1(s)})^{2n}ds\right) \times \left(\alpha^{-2n}\int_0^a (\frac{s}{r_1(s)})^{2n}ds + \int_a^1 r_1^{2n}(s)ds\right) \\
&- \left(\int_0^a r_1^{2n}(s)ds + \int_a^1 r_1^{2n}(s)ds\right) \times \left(\int_0^a (\frac{s}{r_1(s)})^{2n}ds + \int_a^1 (\frac{s}{r_1(s)})^{2n}ds\right) \\
&= \left(\alpha^{2n}\int_0^a r_1^{2n}(s)ds - \int_0^a (\frac{s}{r_1(s)})^{2n}ds\right) \times \left(\int_a^1 r_1^{2n}(s)ds - \alpha^{-2n}\int_a^1 (\frac{s}{r_1(s)})^{2n}ds\right).
\end{aligned} \tag{3.1}$$

Step 4

If we can show that

$$\forall s \neq a \quad \alpha r_1(s) \neq \frac{s}{r_1(s)},$$

then it will follow that both factors in (3.1) are non-zero and thus

$$\forall n \in \mathbb{N} \quad \tilde{\lambda}_{n,0} \neq \lambda_{n,0}.$$

Taking the logarithmic derivative of the quotient of both sides, we get

$$\frac{d}{ds} \log \frac{\alpha r_1(s)}{\frac{s}{r_1(s)}} = \frac{2r_1'(s)}{r_1(s)} - \frac{1}{s} = \frac{1}{s}(\frac{2}{p(s)} - 1).$$

This means that the quotient is decreasing on $[0, a]$ and is either monotone or constant on $[a, 1]$ (since $p(s) \neq 2$ unless $p(s) \equiv 2$ on $(a, 1)$ by assumption, and the latter implies $\tilde{D} = D$). Note that $\alpha r_1(a) = \dfrac{a}{r_1(a)}$ by continuity of $\tilde{r_1}(s)$ at $s = a$. Thus

$$\forall s \neq a \quad \alpha r_1(s) \neq \frac{s}{r_1(s)},$$

except in the trivial case $\tilde{D} = D$. □

Theorem III.4. *Let $\mathcal{R}_m$ denote the collection of all domains $D \in \tilde{\mathcal{R}}$ such that V_D can be represented as a monotone (possibly finite) one-sided sequence $\{v_n\}$. Then the marked spectrum map $\chi : \mathcal{R}_m \to DS$ given by $\chi(D) = \{\lambda_{n,m}\}_{\mathbb{Z}^2_{\geq 0}}$ is injective modulo coordinate dilations and duality.*

Remark III.5. Note that if we compare a domain from $\mathcal{R}_m$ with a domain from $\tilde{\mathcal{R}} \backslash \mathcal{R}_m$, they certainly don't have the same marked spectrum since they have different sets of Reinhardt vertices. We need to restrict to $\mathcal{R}_m$ for this particular proof, but that is not to say that the condition is necessary for injectivity (an open question).

Proof. Again, we work by steps.

<u>Step 1</u>

Let $D, \tilde{D} \in \mathcal{R}_m$ have the same marked spectrum, and let $p(s)$, $\tilde{p}(s)$ be the respective osculating parameter functions. We know that $pp^* = \tilde{p}\tilde{p}^*$ and in particular, both domains share the same Reinhardt vertices. Without loss of generality, there are at least two Reinhardt vertices in $(0,1)$ (due to the previous theorem), the corresponding sequence is increasing (otherwise, consider the reflected domains) and for small s we have $\tilde{p}(s) = p(s)$ (otherwise, we can replace D by D^*). Now let $[a,b]$ be the closest interval to 0 such that a, b are Reinhardt vertices (for both domains) and $\tilde{p}(s) = p^*(s) \neq 2$ on (a,b) (if it doesn't exist, then $\tilde{D} = D$ and we are done).

<u>Step 2</u>

We recall from the proof of Theorem 45 in [2] that we have

$$\int_0^1 r_1^{2n}(s) r_2^{2m}(s) ds \sim \sqrt{\frac{\pi nm}{(n+m)^3} p(s_x)} r_1^{2n}(s_x) r_2^{2m}(s_x),$$

$$\int_0^1 (\frac{s}{r_1(s)})^{2n} (\frac{1-s}{r_2(s)})^{2m} ds \sim \sqrt{\frac{\pi nm}{(n+m)^3} p^*(s_x)} (\frac{s_x}{r_1(s_x)})^{2n} (\frac{1-s_x}{r_2(s_x)})^{2m},$$

as $n, m \to \infty$ with $\frac{n}{m} \to x \in [0, \infty]$, where $s_x = \frac{x}{1+x}$ is the limit point of the (unique) maxima for these integrands. The above estimates hold even if we replace the integral bounds $0, 1$ by any c, d such that $\frac{x}{1+x} \in (c,d)$, due to the Laplace-like method involved.

<u>Step 3</u>

We observe that for $s \in [0,a]$ we have by assumption $\tilde{p}(s) = p(s)$, yielding (using formulas (1.4) for $r_1(s), r_2(s)$)

$$\tilde{r_1}(s) = \alpha r_1(s), \quad \tilde{r_2}(s) = r_2(s),$$

for the constant $\alpha = \exp(\int_a^1 (\frac{1}{tp(t)} - \frac{1}{t\tilde{p}(t)}) dt)$. On [a,b] we have

$$\tilde{r_1}(s) = \beta \frac{s}{r_1(s)}, \quad \tilde{r_2}(s) = \gamma \frac{1-s}{r_2(s)},$$

for similar constants $\beta, \gamma > 0$. We write

$$f_{n,m}(s) = r_1^{2n}(s) r_2^{2m}(s), \quad g_{n,m}(s) = (\frac{s}{r_1(s)})^{2n}(\frac{1-s}{r_2(s)})^{2m}.$$

We define $\tilde{f}_{n,m}, \tilde{g}_{n,m}$ similarly.

Step 4

Now we want to improve the estimate for $\tilde{\lambda}_{n,m}$. Splitting each integral in the product from (1.5) into three pieces and using the previous step, we get

$$\begin{aligned}
&\left(\frac{n!m!}{(n+m+1)!}\right)^2 \tilde{\lambda}_{n,m} = \left(\int_0^1 \tilde{f}_{n,m} ds\right)\left(\int_0^1 \tilde{g}_{n,m} ds\right) \\
&= \left(\int_0^a f_{n,m} ds\right)\left(\int_0^a g_{n,m} ds\right) + (\frac{\alpha}{\beta})^{2n}\gamma^{-2m}\left(\int_0^a f_{n,m} ds\right)\left(\int_a^b f_{n,m} ds\right) \\
&+ (\frac{\alpha}{\beta})^{-2n}\gamma^{2m}\left(\int_0^a g_{n,m} ds\right)\left(\int_a^b g_{n,m} ds\right) + \text{error},
\end{aligned} \tag{3.2}$$

where the error terms are products of integrals that are negligible (in the sense of little o notation) compared to those listed above. This is due to Watson's lemma (see Theorem 15.2.7 in [7] for a stronger version), as follows.

Theorem III.6. *Let f be continuous on $[c,d]$ such that it attains a unique global minimum at c and $f'(0) > 0$ exists. If h is bounded, Lebesgue-measurable on $[c,d]$ and continuous at 0, then*

$$\int_c^d e^{-mf(x)} h(x) dx \sim \frac{h(c)e^{-mf(c)}}{mf'(c)} \quad as \quad m \to \infty.$$

Step 5 (Application)

Let $u > 0$ be a rational number (to simplify the argument) such that $\dfrac{x}{1+x} \in (0,a)$. We apply the lemma to

$$f_u(s) = -2u \log r_1(s) - 2 \log r_2(s), \quad h \equiv 1, \quad c = a, \quad d = b.$$

Then letting $n = mu, m \to \infty$, we get

$$\int_a^b f_{n,m}(s)ds \sim \frac{r_1(a)^{2n} r_2(a)^{2m} a(1-a)p(a)}{2m(a-(1-a)u)}.$$

In fact, we're considering a subsequence $\{m_k\}_{k\in\mathbb{N}}$ for which $m_k u \in \mathbb{N}$, but we omit the subscript. For the record, $f_u(s)$ is increasing on $[a, 1]$, since we have for all $u \in [0, a]$

$$f_u'(s) = -2\frac{r_1'(s)}{r_1(s)}(u - \frac{s}{1-s}) > 0.$$

Similarly, using $g_u(s) = -2u\log(\frac{s}{r_1(s)}) - 2\log(\frac{1-s}{r_2(s)})$ we get

$$\int_a^b g_{n,m}(s)ds \sim \frac{(\frac{a}{r_1(a)})^{2n}(\frac{1-a}{r_2(a)})^{2m} a(1-a)p^*(a)}{2m(a-(1-a)u)}.$$

If you replace the bounds a, b by $b, 1$, the new estimates are relatively negligible since $-f_u(s), -g_u(s)$ are decreasing on $[a, 1]$.

Step 6

Going back to (3.2), writing a similar computation for $\lambda_{n,m}$ and then taking the difference, we get

$$\left(\frac{n!m!}{(n+m+1)!}\right)^2\left(\tilde{\lambda}_{n,m} - \lambda_{n,m}\right) = \Bigg((\frac{\alpha}{\beta})^{2n}\gamma^{-2m}\int_0^a f_{n,m}(s)ds \\ -\int_0^a g_{n,m}(s)ds\Bigg) \times \Bigg(\int_a^b f_{n,m}(s)ds - (\frac{\beta}{\alpha})^{2n}\gamma^{2m}\int_a^b g_{n,m}(s)ds\Bigg) + error. \quad (3.3)$$

We want to show that the product is asymptotically larger than the error, which is

$$O(\frac{1}{m^{1.5}}\max\{(\frac{\beta}{\alpha})^{2n}\gamma^{2m}g_{n,m}(s_x)g_{n,m}(b), f_{n,m}(s_x)g_{n,m}(b), (\frac{\alpha}{\beta})^{2n}\gamma^{-2m}f_{n,m}(s_x)f_{n,m}(b), \\ g_{n,m}(s_x)f_{n,m}(b)\}). \quad (3.4)$$

Note that by Step 2 (which works with different bounds as $s_x \in (0, a)$ by design), we have

$$
\begin{aligned}
&(\frac{\alpha}{\beta})^{2n}\gamma^{-2m}\int_0^a f_{n,m}(s)ds - \int_0^a g_{n,m}(s)ds \\
&\sim \sqrt{\frac{\pi nm}{(n+m)^3}}\left(\sqrt{p(s_x)}(\frac{\alpha}{\beta}r_1(s_x))^{2n}(\gamma^{-1}r_2(s_x))^{2m} - \sqrt{p^*(s_x)}(\frac{s_x}{r_1(s_x)})^{2n}(\frac{1-s_x}{r_2(s_x)})^{2m}\right).
\end{aligned}
\tag{3.5}
$$

Since $p(s) \neq 2$ on $(0, a)$, one of these two exponential terms dominates. If $p(s) > 2$, then the latter term dominates (see below). Even without that, there is no cancellation since $p(s_x) \neq 2$.

As seen in Step 5, if we apply Watson's lemma to each integral separately and take the difference, the estimates cancel out since $p(a) = 2 = p^*(a)$. If we replace the lower bound a by any $t \in (a, b)$, we get an estimate we can work with, but for that we need to know that on (a, b) we have

$$(\frac{\alpha}{\beta})^{2n}\gamma^{-2m}f_{n,m}(s) \neq g_{n,m}(s),$$

to avoid any cancellation. Note that we get equality for $s = a$ and if we take the logarithmic derivative of the quotient, we get

$$2m(\frac{2}{p(s)} - 1)(\frac{x}{s} - \frac{1}{1-s}) \neq 0.$$

This is because $s_x = \dfrac{x}{1+x} \notin (a, b)$ by assumption.

<u>Wrapping up</u>

Plugging (3.5) into (3.3) and applying Watson's lemma to the second factor (for $[t, b]$), we get for $n = mu, \quad m \to \infty$

$$
\begin{aligned}
\left(\frac{n!m!}{(n+m+1)!}\right)^2 |\tilde{\lambda}_{n,m} - \lambda_{n,m}| \gtrsim \frac{1}{m^{1.5}} \max\{&(\frac{\beta}{\alpha})^{2n}\gamma^{2m}g_{n,m}(s_x)g_{n,m}(t), \\
f_{n,m}(s_x)g_{n,m}(t), (\frac{\alpha}{\beta})^{2n}\gamma^{-2m}&f_{n,m}(s_x)f_{n,m}(t), g_{n,m}(s_x)f_{n,m}(t)\}.
\end{aligned}
$$

The error resulting from (3.4) is negligible as the functions

$$-f_u = \frac{1}{m}\log(f_{n,m}), \quad -g_u = \frac{1}{m}\log(g_{n,m})$$

are both decreasing on $[s_x, 1]$. It follows that the eigenvalues differ for sufficiently large m (possibly depending on u). $\square$

Corollary III.7. *If two domains $D, \tilde{D} \in \mathcal{R}_m$ have the same marked spectrum outside a finite set, i.e. $\exists N \in \mathbb{N} \quad \forall n, m > N \quad \tilde{\lambda}_{n,m} = \lambda_{n,m}$, then they are the same up to coordinate dilations and duality. This implies that the image of $\chi : \mathcal{R}_m \to DS$ is not invariant under any finite permutation $P \neq Id$ (i.e. $P = Id$ outside a non-empty finite set), and in fact for any such P and any $D \in \mathcal{R}_m$ we have $P(\chi(D)) \notin Im(\chi)$.*

Proof. The previous proof still applies as we only need asymptotics for it. If a finite permutation of some spectrum (for a domain in $\mathcal{R}_m$) corresponds to another domain in $\mathcal{R}_m$, we immediately see that this permutation is trivial. $\square$

Chapter IV

A Spectral Analysis of Rigid Hartogs Domains in $\mathbf{C}^2$

4.0.1 Leray spectrum for rigid Hartogs domains

This chapter is part of an ongoing project with L. Edholm (Theorem IV.5 is based on his contribution). First, let $\gamma \geq 1$ and consider the real hypersurface

$$M_\gamma = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_2) = |z_1|^\gamma\}.$$

Reparametrize it with $\zeta = (r_\zeta e^{i\theta_\zeta}, s_\zeta + i r_\zeta^\gamma)$. The corresponding Leray-Levi measure is then (up to a constant)

$$d\sigma(\zeta) = r_\zeta^{\gamma-1} ds_\zeta \wedge dr_\zeta \wedge d\theta_\zeta.$$

We also call it the pairing measure for reasons that will become clear. The following is due to D. Barrett and L. Edholm (the proof will appear in [8]):

Theorem IV.1. *The eigenvalues of $\mathbb{L}_\sigma^* \mathbb{L}$ for M_γ are given by*

$$C_\gamma(k) = \frac{1}{k!^2} \Gamma(\frac{2k}{\gamma} + 1) \Gamma(\frac{2k}{\gamma^*} + 1) (\frac{\gamma}{2})^{\frac{2k}{\gamma}+1} (\frac{\gamma^*}{2})^{\frac{2k}{\gamma^*}+1},$$

for integer $k \geq 0$. For a fixed $\gamma > 1$, $C_\gamma(k)$ is a non-increasing function of k and this implies

$$\|\mathbb{L}\|_{L^2(M_\gamma, \sigma)} = \|\mathbb{L}\|_{L^2(M_\gamma, \sigma), e} = \sqrt{C_\gamma(0)} = \frac{\sqrt{\gamma \gamma^*}}{2}.$$

It is also worth noting that the square root phenomenon from the previous chapter also occurs in this setting. That is, we have

$$\lim_{k\to\infty} C_\gamma(k) = \sqrt{C_\gamma(0)}.$$

As in the setting of Reinhardt domains, we can identify the LHS as a variant of the essential norm. We will do so later in a more general context, which we introduce now.

Definition IV.2. We define a class of domains in $\mathbb{C}^2$ and a useful subclass.

- A *rigid Hartogs domain* in $\mathbb{C}^2$ is given by

  $$D = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_2) > f(|z_1|)\},$$

  where $f : [0,\infty) \to [0,\infty)$. Such a domain is rotationally invariant in z_1 and translation invariant (by real scalars) in z_2.

- We denote by $\tilde{\mathcal{H}}$ the subclass of rigid Hartogs domains where $f \in C^1[0,\infty) \cap C^2(0,\infty)$ satisfies

  $$f'(0) = 0, \quad \forall x > 0 \quad f''(x) > 0.$$

Remark IV.3. The condition $f'(0) = 0$ is needed for C^1 smoothness of the boundary hypersurface S, and also to organize our treatment of duality. We will see that a stronger condition is needed for $\mathbb{L}$ to be L^2 bounded with respect to a natural family of measures.

Definition IV.4. Let D be a rigid Hartogs domain D corresponding to $f(r)$. We consider a family of measures that are invariant under the natural automorphisms (i.e. rotations in z_1 and real translations in z_2), given by

$$d\sigma_{d,g}(\zeta) = ds_\zeta \wedge d\tilde{\sigma}_{d,g}(r_\zeta) \wedge d\theta_\zeta,$$

where $d \in \mathbb{R}$, $g : [0, \infty] \to (0, \infty)$ is continuous and

$$d\tilde{\sigma}_{d,g}(r_\zeta) = ((r_\zeta f'(r_\zeta)')^d g(r_\zeta) dr_\zeta.$$

The special case $\sigma \coloneqq \sigma_{1,1}$ is in fact a Leray-Levi measure (we will check this), which we will call the *pairing measure* as before.

The following theorem extends L. Edholm's work for σ to the above family (the adjustments are subtle).

Theorem IV.5. *For a domain $D \in \tilde{\mathcal{H}}$ with boundary S endowed with $\sigma_{d,g}$ as above, we define the adjoint operator $\mathbb{L}^*_{\sigma_{d,g}}$ relative to the subspace $L^2(S, \sigma_{d,g})$. Then there exists a decomposition $\mathbb{L} = \bigoplus_{k=-\infty}^{\infty} \mathbb{L}_k$ such that the $\mathbb{L}^*_{k,\sigma_{d,g}} \mathbb{L}_k$ are unitarily equivalent to rank 1 projections $P_{k,d,g}$ on $L^2((-\infty, 0) \times (0, \infty))$ corresponding to some functions $v_k(\xi, r)$. More precisely*

$$(P_{k,d,g} w)(\xi, r) = B_{d,g}(\xi, k) \langle w(\xi, \cdot), \kappa_k(\xi, \cdot) \rangle_\sigma v_k(\xi, r),$$

where

$$\begin{aligned} B_{d,g}(\xi, k) &= \eta_k^2(\xi) \left\| \tau_k(\xi, \cdot) \right\|^2_{\tilde{\sigma}_{d,g}}, \\ \eta_k(\xi) &= \tfrac{(-2\pi\xi)^{k+1}}{k!} \mathbb{1}_{(-\infty,0)}, \\ \tau_k(\xi, r) &= r^k e^{2\pi\xi f(r)}, \\ \kappa_k(\xi, r) &= (f'(r))^k e^{2\pi\xi(rf'(r) - f(r))} \end{aligned}$$

for $\xi < 0$ and integer $k \geq 0$.

Proof. First, let $\rho(z) = f(|z_1|) - \mathrm{Im}(z_2) = f(\sqrt{z_1 \overline{z_1}}) - \frac{z_2 - \overline{z_2}}{2i}$. In order to compute the Leray kernel, we need $\partial\rho, \overline{\partial}\partial\rho$. We have

$$\partial\rho(\zeta) = \frac{f'(|\zeta_1|)}{2} \sqrt{\frac{\overline{\zeta_1}}{\zeta_1}} d\zeta_1 + \frac{i}{2} d\zeta_2, \quad \overline{\partial}\partial\rho(\zeta) = \frac{1}{4}(f''(|\zeta_1|) + \frac{f'(|\zeta_1|)}{|\zeta_1|}) d\overline{\zeta_1} \wedge d\zeta_1.$$

Now using the reparametrization, we have

$$d\zeta_1 = e^{i\theta_\zeta} dr_\zeta + i r e^{i\theta_\zeta} d\theta_\zeta, \quad d\zeta_2 = ds_\zeta + i f'(r_\zeta) dr_\zeta,$$

which gives

$$j^*(\partial\rho \wedge \overline{\partial}\partial\rho)(\zeta) = -\frac{1}{4}(r_\zeta f''(r_\zeta) + f'(r_\zeta)) ds_\zeta \wedge dr_\zeta \wedge d\theta_\zeta = -\frac{1}{4} d\sigma(\zeta).$$

This justifies the claim that the pairing measure σ is a Leray-Levi measure. We also need to calculate the denominator of the Leray kernel $L(\zeta, z)$ for $z \in S \setminus \{\zeta\}$, i.e.

$$\begin{aligned} \langle \partial\rho(\zeta), \zeta - z \rangle &= \frac{1}{2} f'(r_\zeta) e^{-i\theta_\zeta}(r_\zeta e^{i\theta_\zeta} - r_z e^{i\theta_z}) + \frac{i}{2}(s_\zeta - s_z + i(f(r_\zeta) - f(r_z))) \\ &= \frac{1}{2}(f(r_z) - f(r_\zeta) + f'(r_\zeta)(r_\zeta - r_z e^{i(\theta_z - \theta_\zeta)}) + \frac{i}{2}(s_\zeta - s_z). \end{aligned}$$

Taking the real part, we get for all $r_z > r_\zeta$

$$\begin{aligned} &\forall \theta_z, \theta_\zeta \in \mathbb{R} \quad \mathrm{Re}\langle \partial\rho(\zeta), \zeta - z \rangle = \frac{1}{2}(f(r_z) - f(r_\zeta) + f'(r_\zeta)(r_\zeta - r_z \cos(\theta_z - \theta_\zeta)) > 0 \\ &\iff f(r_z) - f(r_\zeta) > f'(r_\zeta)(r_z - r_\zeta) \iff \frac{f(r_z) - f(r_\zeta)}{r_z - r_\zeta} > f'(r_\zeta). \end{aligned}$$

By Lagrange's mean value theorem, the LHS of the final inequality is equal to $f'(r_m)$ for some $r_m \in (r_\zeta, r_z)$. Since f' is an increasing function ($f'' > 0$), we are done. Now, if $r_z < r_\zeta$ then final inequality direction is reversed and we get $f'(r_m) < f'(r_\zeta)$, which works again since $r_m < r_\zeta$. Finally, if $r_z = r_\zeta$, either we get $\cos(\theta_z - \theta_\zeta) < 1$ making it so that the inequality is unchanged (doesn't become an equality), or

$$\mathrm{Im}\langle \partial\rho(\zeta), \zeta - z \rangle = s_\zeta - s_z \neq 0.$$

Either way, we have $\langle \partial\rho(\zeta), \zeta - z \rangle \neq 0$ for $z \in S \setminus \{\zeta\}$. Now, write

$$A = f(r_z) - f(r_\zeta) + f'(r_\zeta) r_\zeta + i(s_\zeta - s_z), \quad B = r_z f'(r_\zeta).$$

Then

$$\langle \partial\rho(\zeta), \zeta - z \rangle = \frac{1}{2}(A - B e^{i(\theta_z - \theta_\zeta)}).$$

Plugging into (1.2) we get

$$\begin{aligned}L(\zeta,z) &= \frac{d\sigma(\zeta)}{4\pi^2(A-Be^{i(\theta_z-\theta_\zeta)})^2} = \frac{d\sigma(\zeta)}{4\pi^2A^2(1-\frac{B}{A}e^{i(\theta_z-\theta_\zeta)})^2} \\ &= \frac{1}{4\pi^2A^2}\sum_{k=0}^{\infty}(k+1)(\frac{B}{A})^k e^{ik(\theta_z-\theta_\zeta)}d\sigma(\zeta),\end{aligned} \tag{4.1}$$

where we have used the fact that $|\frac{B}{A}|<1$ (unless $(r_z,s_z)=(r_\zeta,s_\zeta)$) as well as the Taylor series (with radius of convergence 1)

$$\frac{1}{(1-z)^2}=\sum_{k=0}^{\infty}(k+1)z^k.$$

Now, let $h(\zeta)\in L^2(S,\sigma_{d,g})$. Considering the periodic dependence on θ_ζ, we may represent h as a Fourier series of the form

$$h(\zeta)=\sum_{j=-\infty}^{\infty}h_j(s_\zeta,r_\zeta)e^{ij\theta_\zeta}.$$

Plugging (4.1) and the Fourier series into (1.1), we carry out the term-by-term integration. This can be justified by removing from S the image of $(r_z-\epsilon,r_z+\epsilon)\times\mathbb{R}\times(0,2\pi)$, guaranteeing $|\frac{B}{A}|\le t_\epsilon<1$, and then letting $\epsilon\to 0$. Using the fact $\int_0^{2\pi}e^{ik(\theta-\alpha)}d\theta=0$ for all $k\in\mathbb{Z}\setminus\{0\},\alpha\in\mathbb{R}$, we get

$$(\mathbb{L}h)(z)=\int\limits_S h(\zeta)L(\zeta,z)=\sum_{k=0}^{\infty}(I_kh)(s_z,r_z)e^{ik\theta_z}, \tag{4.2}$$

where

$$\begin{aligned}(I_kh)(s_z,r_z) &= \frac{k+1}{2\pi}\int\limits_{r_\zeta=0}^{\infty}\int\limits_{s_\zeta=-\infty}^{\infty}\frac{B^k}{A^{k+2}}h_k(s_\zeta,r_\zeta)(r_\zeta f'(r_\zeta)'ds_\zeta dr_\zeta \\ &= \frac{k+1}{2\pi}r_z^k i^{k+2}\int\limits_{r_\zeta=0}^{\infty}(f'(r_\zeta)^k(r_\zeta f'(r_\zeta)'\int\limits_{s_\zeta=-\infty}^{\infty}\frac{h_k(s_\zeta,r_\zeta)}{(s_z-s_\zeta+iC)^{k+2}}ds_\zeta dr_\zeta, \\ C &= f(r_z)-f(r_\zeta)+f'(r_\zeta)r_\zeta.\end{aligned}$$

We identify $\int\limits_{-\infty}^{\infty}\frac{h_k(s_\zeta,r_\zeta)}{(s_z-s_\zeta+iC)^{k+2}}ds_\zeta$ as a convolution h_k*G_k, where $G_k(s)=\frac{1}{(s+iC)^{k+2}}$. To undo the convolution, we need to conjugate $\mathbb{L}$ by a Fourier transform for s_ζ. First, recall the unitary Fourier transform on $\mathbb{R}$ and its inverse:

$$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(\xi)e^{-2\pi i\xi s}ds,$$

$$\mathcal{F}^{-1}f(\xi) = \int_{-\infty}^{\infty} f(\xi)e^{2\pi i\xi s}ds.$$

$\mathcal{F}$ satisfies $\mathcal{F}^{-1}(f_1 * f_2) = \mathcal{F}^{-1}(f_1)\mathcal{F}^{-1}(f_2)$. Also, we may compute $\mathcal{F}^{-1}(G_k)(\xi)$ by contour integration. For $\xi \geq 0$ we may integrate on a half-circle (centered at 0) in the upper half-plane, giving us 0 since the pole $s = -iC$ is in the lower half-plane (the contribution of the half-circle tends to 0 as the radius tends to ∞). If $\xi < 0$, we need to use the half-circle in the lower half-plane and apply the residue theorem (remembering to account for the orientation). In total, we get

$$\mathcal{F}^{-1}G_k(\xi) = \begin{cases} 0 & \xi \geq 0 \\ -\frac{(2\pi i)^{k+2}\xi^{k+1}}{(k+1)!}e^{2\pi\xi C} & \xi < 0 \end{cases}$$

Applying $\mathcal{F}^{-1}$ on (4.2) (it commutes with the r_ζ-integration) and using the above, we get after some simplification

$$\mathcal{F}^{-1}(I_k h)(\xi, r_z) = \begin{cases} 0 & \xi \geq 0 \\ \frac{(-2\pi\xi)^{k+1}}{k!}r_z^k e^{2\pi\xi f(r_z)}\int_0^\infty (f'(r_\zeta)^k(r_\zeta f'(r_\zeta)'e^{2\pi C}\mathcal{F}^{-1}h_k(\xi, r_\zeta)dr_\zeta & \xi < 0 \end{cases}$$

For every $k \in \mathbb{Z}$, let $\mathbb{L}_k$ be the restriction of $\mathbb{L}$ to the subspace V_k of functions in $L^2(S, \sigma_{d,g})$ of the form $h_k(s, r)e^{ik\theta}$. We have $\mathbb{L} = \bigoplus_{k\in\mathbb{Z}} \mathbb{L}_k$. Using the definitions of $C, \eta_k, \tau_k, \kappa_k$, we can interpret the above as

$$\mathcal{F}^{-1}\mathbb{L}_k\mathcal{F}(h_k(\xi, r_z)e^{ik\theta_z}) = \eta_k(\xi)\tau_k(r_z)\langle h_k(\xi, \cdot), \kappa_k(\xi, \cdot)\rangle_{\tilde{\sigma}}e^{ik\theta_z},$$

where $\langle \cdot, \cdot \rangle_{\tilde{\sigma}}$ denotes integration of the product (taking the complex conjugate of the second argument) on $(0, \infty)$ against $d\tilde{\sigma}(r_\zeta) = (r_\zeta f'(r_\zeta)'dr_\zeta$. To compute the adjoint relative to $L^2(S,\sigma_{d,g})$ we need to manipulate $\langle\cdot,\cdot\rangle_{\sigma_{d,g}}$. For every $k\in\mathbb{Z}$ and $h_k e^{ik\theta_z}, u_k e^{ik\theta_z}\in L^2(S,\sigma_{d,g})$, we get (using Fubini's theorem and writing ξ instead of s_z, as well as $V:=(-\infty,0)\times(0,\infty)^2$)

$$
\begin{aligned}
&\frac{1}{2\pi}\langle\mathcal{F}^{-1}\mathbb{L}_k\mathcal{F}h_k e^{ikz}, u_k e^{ikz}\rangle_{\sigma_{d,g}} = \frac{1}{2\pi}\int\limits_{z\in S}\eta_k(\xi)\tau_k(\xi,r_z)\langle h_k,\kappa_k\rangle_{\tilde\sigma(r_\zeta)}u_k(\xi,r_z)d\sigma_{d,g}(z)\\
&=\iiint\limits_V h_k(\xi,r_\zeta)\eta_k(\xi)\tau_k(\xi,r_z)\kappa_k(\xi,r_\zeta)u_k(\xi,r_z)(r_\zeta f'(r_\zeta)'((r_z f'(r_z))')^d g(r_z)dr_\zeta dr_z d\xi\\
&=\int\limits_{-\infty}^{0}\int\limits_{0}^{\infty}h_k(\xi,r_\zeta)\eta_k(\xi)\kappa_k(\xi,r_\zeta)\left(\int\limits_0^\infty u_k(\xi,r_z)\tau_k(\xi,r_z)((r_z f'(r_z))')^d g(r_z)dr_z\right)d\tilde\sigma(r_\zeta)d\xi\\
&=\frac{1}{2\pi}\langle h_k e^{ik\theta_z},\mathcal{F}^{-1}\mathbb{L}^*_{k,\sigma_{d,g}}\mathcal{F}(u_k e^{ik\theta_z})\rangle_{\sigma_{d,g}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{F}^{-1}\mathbb{L}^*_{k,\sigma_{d,g}}\mathcal{F}(u_k e^{ik\theta_z}) &= \frac{\eta_k(\xi)\kappa_k(\xi,r_z)}{((r_z f'(r_z))')^{d-1}g(r_z)} < u_k(\xi,\cdot),\tau_k(\xi,\cdot) >_{\tilde\sigma_{d,g}} e^{ik\theta_z}\\
\mathcal{F}^{-1}\mathbb{L}^*_{k,\sigma_{d,g}}\mathbb{L}_k\mathcal{F}(h_k e^{ik\theta_z}) &= \frac{\eta_k^2(\xi)\kappa_k(\xi,r_z)}{((r_z f'(r_z))')^{d-1}g(r_z)}\left\|\tau_k(\xi,\cdot)\right\|^2_{\tilde\sigma_{d,g}}\langle h_k(\xi,\cdot),\kappa_k(\xi,\cdot)\rangle_{\tilde\sigma}e^{ik\theta_z}.
\end{aligned}
$$

Finally, conjugate the operator by multiplication by $e^{i\theta_z}$ and set

$$v_k(\xi,r):=\frac{\kappa_k(\xi,r)}{(rf'(r)')^{d-1}g(r)}.$$

$\square$

Corollary IV.6. *The image of the function*

$$C_{d,g}(\xi,k):=\eta_k^2(\xi)\left\|\tau_k(\xi,\cdot)\right\|^2_{\tilde\sigma_{d,g}}\left\|\kappa_k(\xi,\cdot)\right\|^2_{\tilde\sigma_{2-d,1/g}} \tag{4.3}$$

is dense in the spectrum of $\mathbb{L}^*_{\sigma_{d,g}}\mathbb{L}$. *The multiplier* $C_{d,g}(\xi,k)$ *is called the symbol function associated with* D *and* $\sigma_{d,g}$.

Proof. Clearly, we can restrict our attention to the subspace spanned by v_k. Now note that for all $k\in\mathbb{Z}_{\geq 0}$

$$\eta_k^2(\xi)\left\|\tau_k(\xi,\cdot)\right\|^2_{\tilde\sigma_{d,g}}\langle v_k(\xi,\cdot),\kappa_k(\xi,\cdot)\rangle_{\tilde\sigma}=\eta_k^2(\xi)\left\|\tau_k(\xi,\cdot)\right\|^2_{\tilde\sigma_{d,g}}\left\|\kappa_k(\xi,\cdot)\right\|^2_{\tilde\sigma_{2-d,1/g}}.$$

Any number that isn't a cluster point of the image is a regular point for $\mathbb{L} *_{\sigma_{d,g}} \mathbb{L}$.

Remark IV.7. Why do we call σ the pairing measure? Without getting into much detail, it suffices to point out that for all $d \in \mathbb{R}$ and continuous $g : (0, \infty) \to (0, \infty)$, we have

$$\sigma = \sqrt{\sigma_{d,g}\sigma_{2-d,\frac{1}{g}}}.$$

The corresponding measures $\tilde{\sigma}_{d,g}, \tilde{\sigma}_{2-d,\frac{1}{g}}$ are paired in the spectrum formula (4.3).

Remark IV.8. For M_γ the symbol function is independent of ξ and we get the eigenvalues $C_\gamma(k)$. This is relatively easy to verify by a change of variable in each integral. We will see in the next chapter that ξ-independence characterizes M_γ. This doesn't necessarily mean that no other rigid Hartogs domain has at least one eigenvalue, but the spectrum isn't pure point.

4.0.2 An osculation function

We'd like to have an analogue of the function $p(s)$, which recovers convex Reinhardt domains up to dilations. The most intuitive approach is to use translated and dilated versions of M_γ as model domains, osculating to the second order. This will give rise to a function $\gamma(r)$, which we call an osculation function for a domain $D \in \tilde{\mathcal{H}}$. Now set up $j(r) = a + br^\gamma$ as the defining function for the translated and dilated osculating M_γ domain, where $a \in \mathbb{R}, b > 0$. We can compute γ (a, b are of little interest) by writing the second order osculation conditions at point $(r, f(r))$ on the graph $S \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})$ (this is enough due to the rotation invariance and translation invariance of both D and M_γ), namely

$$f(r) = j(r) = a + br^\gamma,$$

$$f'(r) = j'(r) = b\gamma r^{\gamma-1},$$

$$f''(r) = j''(r) = b\gamma(\gamma-1)r^{\gamma-2}.$$

We can divide the third equation by the second one, giving us after rearranging

$$\gamma(r) = 1 + \frac{rf''(r)}{f'(r)}. \tag{4.4}$$

Note that $\gamma : (0,\infty) \to (1,\infty)$ is continuous. Also, we have

$$d\sigma_{d,g}(r) = (f'(r)\gamma(r))^d g(r)dr.$$

As promised, (4.4) can be inverted (up to constants) by subtracting 1, dividing by r, integrating, applying $x \mapsto e^x$ to both sides, and then integrating again. This yields

$$f(r) = C\int\limits_0^r \exp\left(\int_1^s \frac{\gamma(t)-1}{t}dt\right)ds + D, \tag{4.5}$$

where $C = f'(1)$ (1 is an arbitrary choice) and $D = f(0)$ are independent of $\gamma(r)$. This shows that $\gamma(r)$ recovers the domain up to dilations in z_2 and translations in z_2 (imaginary translations by the above, while real ones are obvious). As for dilations in z_1 (rotations are obvious), for all $c > 0$ we have that $f(r) \mapsto f(cr)$ yields $\gamma(r) \mapsto \gamma(cr)$. Indeed, by (4.4), if we let $\tilde{\gamma}(r)$ correspond to $f(cr)$, then we have

$$\tilde{\gamma}(r) = 1 + \frac{r(c^2 f''(cr))}{cf'(cr)} = \gamma(cr).$$

4.0.3 Duality

We want to understand the projective dual domain of a domain $D \in \tilde{\mathcal{H}}$. Is it also a rigid Hartogs domain for some affinization? First, recall that we have

$$\rho(z) = \text{Im}|z_2| - f(|z_1|),$$

$$\nabla\rho(z) = (-\frac{1}{2}f'(r)e^{-i\theta}, \frac{1}{2i}).$$

Writing down the hyperplane equation for $w = (w_1, w_2)$ at the point $(re^{i\theta}, a + if(r))$ where $r \geq 0$ and $a, \theta \in \mathbb{R}$, we get

$$\begin{aligned} -\frac{1}{2}f'(r)e^{-i\theta}(w_1 - re^{i\theta}) + \frac{1}{2i}(w_2 - a - if(r)) = 0 \\ -f'(r)e^{-i\theta}w_1 - iw_2 = f(r) - rf'(r) - ia \end{aligned}$$

We get $[f'(r)e^{-i\theta} : i : rf'(r) - f(r) + ia]$ in projective coordinates, and we may take an affinization by the dropping the middle coordinate and multiplying the other two by i, giving us $[f'(r)e^{-i(\theta - \frac{\pi}{2})} : -a + i(rf'(r) - f(r))]$ as a parametrization of this affinization of the dual domain. What's special about this choice is that it corresponds to a rigid Hartogs domain with a defining function $\rho^*(z) = \mathrm{Im}(z_2) - f^*(|z_1|)$, where we have for all $r \geq 0$

$$f^*(f'(r)) = rf'(r) - f(r). \tag{4.6}$$

We will simply call this the dual domain and denote it by D^*. Note that f^* is the Legendre transform of f (not to be confused with the Laplace transform). However, there are many other duals in $\mathbf{CP}^n$ and even in $\mathbb{C}^n$, which are obtained by applying projective isomorphisms to D^*. Only coordinates dilations and translations in z_2 yield other rigid Hartogs domains, but we single out D^*.

Lemma IV.9. *Let $D \in \tilde{\mathcal{H}}$ correspond to the defining function f and osculation function γ, such that its dual D^* corresponds to functions f^*, γ^*. Then $D^* \in \tilde{\mathcal{H}}$.*

Proof. We only need to check the differentiability conditions at this point. Using the chain rule on (4.6), we get for all $r > 0$

$$(f^*)'(f'(r))f''(r) = rf''(r) \Rightarrow (f^*)'(f'(r)) = r, \tag{4.7}$$

which shows that f^* is $C^1(0, \infty)$ and (f^*) is increasing. Since f^* is certainly continuous at 0, it follows by a limit argument (using the mean value theorem, also named after Legendre) that $(f^*)'(0) = 0$. Finally, applying the chain rule again shows that for $r > 0$ we have

$$(f^*)''(f'(r))f''(r) = 1 \Rightarrow (f^*)''(f'(r)) = \frac{1}{f''(r)} > 0.$$

This implies that $(f^*)''$ and is positive and continuous on $(0, \infty)$. □

Auxiliary Parameter

It is sometimes useful to work with the parameter

$$u = rf'(r). \tag{4.8}$$

Since this defines u as an increasing function of r, the inverse $r = r(u)$ exists and we can differentiate (4.8) with respect to u, giving us

$$\begin{aligned} 1 &= (f'(r(u) + r(u)f''(r(u)))r'(u) \\ \Rightarrow r'(u) &= \frac{1}{f'(r(u)) + r(u)f''(r(u))} = \frac{1}{f'(r(u))\gamma(r(u))}. \end{aligned}$$

Now write $y(u) = f(r(u))$. Then clearly

$$y'(u) = f'(r(u))r'(u) = \frac{1}{\gamma(r(u))}. \tag{4.9}$$

Lemma IV.10. *Let $D \in \tilde{\mathcal{H}}$ correspond to f with osculation function γ and u-parametrization $(r(u), y(u))$. Then we have*

1. *The u-parametrization for the dual domain D^* is given by $(r^*(u), y^*(u))$, where*

$$r^*(u) = f'(r(u)) = \frac{y'(u)}{r'(u)}, \quad y^*(u) = u - y(u).$$

2. *If $\hat{\gamma}$ is the osculation function for D^*, then*

$$\forall u > 0 \quad \hat{\gamma}(r^*(u)) = \gamma^*(r(u)) := \frac{\gamma(r(u))}{\gamma(r(u)) - 1}.$$

Proof. Note that by (4.8) and (4.7), we have

$$u = r(u)f'(r(u)) = f'(r(u))(f^*)'(f'(r(u))).$$

By (4.8) applied to D^*, we have

$$r^*(u)(f^*)'(r^*(u)) = u = f'(r(u))(f^*)'(f'(r(u))).$$

Since $r \mapsto r(f^*)'(r)$ is increasing, the above and the chain rule imply

$$r^*(u) = f'(r(u)) = \frac{y'(u)}{r'(u)}.$$

It follows by the definitions of $y^*(u), y(u)$ and (4.8) that

$$y^*(u) = (f^*)(r^*(u)) = (f^*)(f'(r(u))) = r(u)f'(r(u)) - f(r(u)) = u - y(u).$$

Finally, if we apply (4.9) to D^*, we get

$$\frac{1}{\hat{\gamma}(r^*(u))} = (y^*)'(u) = \frac{d}{du}(u - y(u)) = 1 - \frac{1}{\gamma(r(u))},$$

which implies $\quad \hat{\gamma}(r^*(u)) = \gamma^*(r(u))$ for all $u > 0$. □

Remark IV.11. For M_γ we get

$$u = \gamma r^\gamma \Rightarrow r(u) = (\frac{u}{\gamma})^{\frac{1}{\gamma}}, \quad y(u) = \frac{u}{\gamma}.$$

Hence

$$y'(u) = \frac{1}{\gamma} \Rightarrow \gamma(r(u)) \equiv \gamma$$

as expected. Moreover, the dual domain is parametrized by

$$r^*(u) = \frac{\frac{1}{\gamma}}{(\frac{u}{\gamma})^{\frac{1}{\gamma}-1}\frac{1}{\gamma^2}} = \gamma^{\frac{1}{\gamma}} u^{\frac{1}{\gamma^*}}, \quad y^*(u) = u - \frac{u}{\gamma} = \frac{u}{\gamma^*}.$$

Comparing the two parametrizations, we see that $(M_\gamma)^*$ is a dilation of of M_{γ^*} (in z_1 only).

4.0.4 Asymptotics and Boundedness

We want to compute all limit values of the symbol function $C_{d,g}(\xi, k)$ given by (4.3). Consider a seqence $((\xi_j, k_j))_{j=0}^{\infty}$, where $\xi_j < 0, k_j \in \mathbb{Z}_{\geq 0}$ for all $j \in \mathbb{Z}_{\geq 0}$. Much like the Reinhardt setting, by potentially passing to a subseqence (without changing the subscript) we have four options other than (uninteresting) limits in $(-\infty, 0) \times \mathbb{Z}_{\geq 0}$:

1. $\xi_j \to -\infty, \quad k_j \equiv k_0.$
2. $\xi_j \to 0, \quad k_j \equiv k_0.$
3. $\lim\limits_{j\to\infty} \xi_j \in (-\infty, 0], \quad k_j \to \infty.$
4. $\lim\limits_{j\to\infty} \frac{k_j}{\xi_j} \in [-\infty, 0], \quad k_j, |\xi_j| \to \infty.$

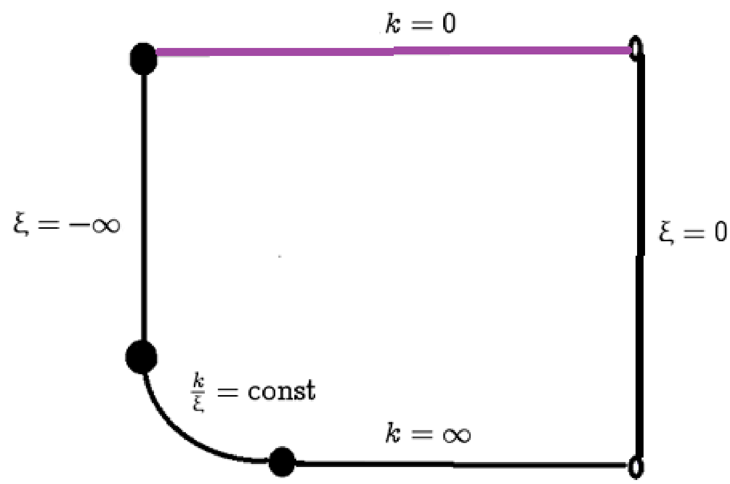

Figure 4.1: Here we have discrete points lying on three relevant lines ($k = 0$ is a regular case), with those at infinity being joined by a continuous arc for the slope-based limit values.

Note that option 4 follows from the compactness of $[-\infty, 0]$, since we can arrange for the sequence of slopes $\frac{k_j}{\xi_j}$ to be convergent (possibly to $-\infty$) by passing to a subseqence (a generic sequence with $k_j, |\xi_j| \to \infty$ has multiple partial limits for the slope). In any other case, either ξ_j or k_j is bounded (both in option 2) and we can pass to a subsequence satisfying one of the three options.

Let's write down (4.3) more explicitly:

$$C_{d,g}(\xi,k) = \frac{(2\pi\xi)^{2k+2}}{k!^2} I_{D,d,g}(\xi,k) I^*_{D,2-d,\frac{1}{g}}(\xi,k), \tag{4.10}$$

where

$$I_{D,d,g}(\xi,k) = \int_0^\infty r^{2k} e^{4\pi\xi f(r)} ((rf'(r))')^d g(r) dr,$$
$$I^*_{D,2-d,\frac{1}{g}}(\xi,k) = \int_0^\infty (f'(r))^{2k} e^{4\pi\xi(rf'(r)-f(r)}((rf'(r))')^{2-d} \frac{1}{g(r)} dr$$

As the notation implies, the two integrals are related via duality. Specifically, by using $u = rf'(r)$, we see that for all $\xi < 0, k \in \mathbb{Z}_{\geq 0}, d \in \mathbb{R}$ and continuous $g : (0,\infty) \to (0,\infty)$, we have

$$I_{D,d,g}(\xi,k) = \int_0^\infty (r(u))^{2k} e^{4\pi\xi y(u)} (f'(r(u))\gamma(r(u))^{d-1} g(r(u)) du, \tag{4.11}$$

$$I^*_{D,2-d,\frac{1}{g}}(\xi,k) = \int_0^\infty (r^*(u))^{2k} e^{4\pi\xi y^*(u)} (f'(r(u))\gamma(r(u))^{1-d} \frac{1}{g(r(u))} du. \tag{4.12}$$

Note that we have used the identity $f'(r)\gamma(r) = f'(r) + rf''(r) = (rf'(r))'$. It follows that for $d = 1$ we have

$$I^*_{D,1,\frac{1}{g}}(\xi,k) = I_{D^*,1,\frac{1}{g}}(\xi,k). \tag{4.13}$$

For $d \neq 1$ we need to be more careful. The factors γ^{d-1}, g will end up being canceled out by $\gamma^{1-d}, \frac{1}{g}$ in the asymptotics (some conditions on g are required), but $(f')^{d-1}$ and $(f')^{1-d}$ act differently if $f'(0) = 0$. We need a definition and a lemma to pinpoint the role of f':

Definition IV.12. We will denote by $\mathcal{H}$ the subclass of all $D \in \tilde{\mathcal{H}}$ whose corresponding $\gamma : (0,\infty) \to (1,\infty)$ has a continuous (hence bounded) extension to $[0,\infty]$, and

- $\gamma_0 \coloneqq \gamma(0) > 1, \quad \gamma_\infty \coloneqq \gamma(\infty) > 1.$

- $\int_0^1 \frac{\gamma(r)-\gamma_0}{r} dr, \quad \int_1^\infty \frac{\gamma(r)-\gamma_\infty}{r} dr$ converge.

Remark IV.13. The above definition is completely analogous to that of $\mathcal{R}$ from the Reinhardt setting. These conditions are natural and useful for the asymptotic computations, but they might not be necessary for the limit values to be bounded. Also note that the first integral condition is weaker than the condition from Dini's test for pointwise convergence of the Fourier series of $\gamma(r)$ at $r = 0$, namely

$$\int_0^1 \frac{\omega_\gamma(\delta; 0)}{\delta} d\delta < \infty,$$

where $\omega_\gamma(\delta; 0) = \max_{\epsilon \leq \delta} |\gamma(\epsilon) - \gamma_0|$ is the modulus of continuity at 0. The second condition is a loose variant for ∞.

Lemma IV.14. *For a domain $D \in \mathcal{H}$, we have*

1.

$$\exists \gamma_1, \gamma_2 > 1 \quad \exists C_1, C_2, D > 0 \quad \forall r \geq 0 \quad C_1 r^{\gamma_1} + D \leq f(r) \leq C_2 r^{\gamma_2} + D.$$

2. $\lim\limits_{r \to 0} \frac{f(r)}{r^{\gamma_0}}$ *and* $\lim\limits_{r \to 0} \frac{f'(r)}{r^{\gamma_0 - 1}}$ *are positive and finite.*

3. $\lim\limits_{r \to \infty} \frac{f(r)}{r^{\gamma_\infty}}$ *and* $\lim\limits_{r \to \infty} \frac{f'(r)}{r^{\gamma_\infty - 1}}$ *are positive and finite.*

Proof. 1. Let $\gamma_1 = \min\limits_{r \in [0,\infty]} \gamma(r) > 1, \quad \gamma_2 = \max\limits_{r \in [0,\infty]} \gamma(r) > 1$. By the definition of γ, we have

$$\gamma_1 \leq 1 + \frac{r f''(r)}{f'(r)} \leq \gamma_2,$$

and if we follow the derivation of (4.5), we get

$$C_1 r^{\gamma_1} + D \leq f(r) \leq C_2 r^{\gamma_2} + D$$

for $C_j = \frac{f'(1)}{\gamma_j}, \quad D = f(0).$

2. Note that by l'Hôpital's rule, it is enough to show this for the second limit. By (4.5) we have
$$\lim_{r\to 0}\frac{f'(r)}{r^{\gamma_0-1}} = C\exp\left(\int_0^1 \frac{\gamma_0-\gamma(t)}{t}dt\right)ds.$$
By assumption, the limit is finite and positive since $\int_0^1 \frac{\gamma_0-\gamma(t)}{t}dt$ converges.

3. Similar to the above. Just swap 0 with ∞.

□

Remark IV.15. If $g(r)$ is bounded away from 0 and ∞, then the first part implies the convergence of $I_j(\xi,k)$ for all $j\in\{1,2\},\xi<0$ and large enough $k\in\mathbb{Z}_{\geq 0}$. Indeed:
$$I_j(\xi,k)\leq \|g\|_\infty\int_0^\infty r^{2k}e^{4\pi\xi(C_1r^{\gamma_1}+D)}(C_2\gamma_2^2r^{\gamma_2-1})^d dr$$
converges if $2k+d(\gamma_2-1)>-1$ (otherwise it is not locally integrable at 0). The second integral can be treated similarly, or we can just use duality and swap $d, g(r)$ with $2-d, \frac{1}{g(r)}$. We can make this condition more precise by considering local behavior at 0 only (replacing γ_2 with γ_0), and plug in $k=0$ to obtain a convergence condition for all $k\in\mathbb{Z}_{\geq 0}$. Doing this for both integrals yields
$$d(\gamma_0-1)>-1,\quad (2-d)(\gamma_0^*-1)>-1,$$
or equivalently
$$1-\gamma_0^*<d<1+\gamma_0. \tag{4.14}$$

We still need one more tool. This is a variant of the Laplace method that allows for more flexibility than usual. For the proof, we modify the proof of Theorem 15.2.2 in [7].

Theorem IV.16. *Let $(a_k(x))_{k\in\mathbb{N}}$ and $\tilde{h}(x)$ be measurable real-valued functions on $\mathbb{R}$ such that for all $k\in\mathbb{N}$ the function $a_k(x)$ has a unique maximum point $x_k\in\mathbb{R}$, and moreover the following conditions hold:*

1. $\lim_{k\to\infty} x_k = \tilde{x}$ *is finite and* $\tilde{h}(\tilde{x}) \neq 0$.

2. *There exists some open interval* $I = (\tilde{x} - 2R, \tilde{x} + 2R)$*, such that* $\tilde{h}(x)$ *is continuous on* I *and for all sufficiently large* k, $a_k \in C^2(I)$ *and*

$$\exists D > 0 \quad a_k''(x) < -D$$

for all $x \in I$.

3. *There exist* $k_0 \in \mathbb{N}$ *and a measurable function* $a(x)$ *such that* $a_k(x) \leq a(x)$ *for all sufficiently large* k*, and* $\int_{\mathbb{R}} e^{k_0 a(x)} \tilde{h}(x) dx < \infty$.

4. $A \coloneqq \lim_{k\to\infty} a_k''(\tilde{x})$ *is finite and negative, while* $a_k(x_k)$ *is bounded from below.*

5. $a_k''(x)$ *are equicontinuous at* $\tilde{x}$*, i.e.* $\omega_{a_k''}(\delta, \tilde{x})$ *is uniformly bounded in* k *for some* $\delta > 0$.

Then

$$\lim_{k\to\infty} \sqrt{k} e^{-k a_k(\tilde{x})} \int_{\mathbb{R}} e^{k a_k(x)} \tilde{h}(x) dx = \sqrt{\frac{2\pi}{|A|}} \tilde{h}(\tilde{x}).$$

Proof. Assume $\tilde{x} = 0$ without loss of generality (otherwise shift all the functions and I). For large $k \in \mathbb{N}$ we have $|x_k| < \frac{R}{2}$, making the contribution outside $I_k \coloneqq (x_k - R, x_k + R)$ relatively small, as we will see. By the third assumption, there exist $k_0 \in \mathbb{N}$ and a majorant $a(x)$ such that for all large k we have $\int\limits_{\mathbb{R}} e^{k a_k(x)} \tilde{h}(x) dx < \int\limits_{\mathbb{R}} e^{k a(x)} \tilde{h}(x) dx < \infty$. Then, for large k, the second assumption implies the existence of some $c > 0$ such that $a_k(x) < a_k(x_k) - c$ for $x \in \mathbb{R}$ with $|x - \tilde{x}| > \frac{R}{2}$. This is due to a first order Taylor approximation at 0 with a Lagrange reminder, which yields for all $x < -\frac{R}{2}$

$$a_k(x) < a_k(\frac{R}{2}) \leq a_k(x_k) - \frac{D}{2}(\frac{R}{2} - x_k)^2 \leq a_k(x_k) - \frac{DR^2}{16}.$$

This is because $x_k > -\frac{R}{2}$ and $(\frac{R}{2} - x_k)^2 \geq \frac{R^2}{8}$ for large k. The case $x > \frac{R}{2}$ is similar.

$$\int\limits_{|x-x_k|>R} e^{ka_k(x)}\tilde{h}(x)dx \leq e^{(k-k_0)(a_k(x_k)-c)} \int\limits_{|x|>\frac{R}{2}} e^{k_0 a_k(x)}\tilde{h}(x)dx = const \times e^{(k-k_0)(a_k(x_k)-c)}$$

$$= o(\frac{1}{\sqrt{k}}e^{ka_k(x_k)}),$$

with the last step relying on $a_k(x_k)$ being bounded from below as per the fourth assumption. Now we can focus on I_k, which is contained in I for large k. Using the change of variable $y = \sqrt{k}(x - x_k)$, we get

$$\sqrt{k} \int\limits_{|x-x_k|<R} e^{k(a_k(x)-a_k(x_k))}\tilde{h}(x)dx = \int\limits_{|y|<\sqrt{k}R} e^{k(a_k(x_k+\frac{y}{\sqrt{k}})-a_k(\tilde{x}))}\tilde{h}(x_k + \frac{y}{\sqrt{k}})dy.$$

Note that the uniform bound on a_k'' (second assumption) implies, again using a first order Taylor approximation at x_k with a Lagrange remainder, that

$$\exists D > 0 \quad a_k(x_k + \frac{y}{\sqrt{k}}) \leq a_k(x_k) - \frac{D}{2}(\frac{y}{\sqrt{k}})^2,$$

which in turns means that integrand is dominated by the integrable Gaussian function $e^{-\frac{Dy^2}{2}}$ times the constant $\max\limits_{x\in I} |\tilde{h}(x)|$. Thus, by the dominated convergence theorem we can take a limit under the integral sign, giving us

$$\int\limits_{|y|<\sqrt{k}R} e^{k(a_k(\frac{y}{\sqrt{k}})-a_k(x_k))}\tilde{h}(x_k + \frac{y}{\sqrt{k}})dy \xrightarrow[k\to\infty]{} \int_{\mathbb{R}} e^{\frac{Ay^2}{2}}\tilde{h}(0)dy = \sqrt{\frac{2\pi}{|A|}}\tilde{h}(0).$$

To be precise about the pointwise convergence, we use a first order Taylor approximation at x_k with a Lagrange remainder. For a fixed $y \in \mathbb{R}$ and for all $k \in \mathbb{N}$, there exists $\tilde{y_k}$ such that $|\tilde{y_k} - x_k| < |y - x_k|$ and

$$k(a_k(x_k + \frac{y}{\sqrt{k}}) - a_k(x_k)) = \frac{a_k''(x_k + \frac{\tilde{y}_k}{\sqrt{k}})}{2}y^2 = \frac{A + o(1)}{2}y^2 = \frac{Ay^2}{2} + o(1)$$

as $k \to \infty$. Here we used the fourth assumption for A, as well as the fifth assumption about the equicontinuity of a_k'' at 0. $\square$

We can apply this theorem to the asymptotic computation of type 4 (where the slope is convergent), but some restriction is needed (the slopes $0, -\infty$ are special cases).

Theorem IV.17. *Assume $d \in \mathbb{R}$ satisfies (4.14), $g : (0,\infty) \to (0,\infty)$ is continuous and $\lim_{j\to\infty} \frac{k_j}{\xi_j} = -2\pi\tilde{u}$, where $\tilde{u} \in (0,\infty)$, $\quad |\xi_j|, k_j \xrightarrow[j\to\infty]{} \infty$. Then for any $D \in \tilde{\mathcal{H}}$, we have*

$$\lim_{j\to\infty} C_{d,g}(\xi_j, k_j) = \frac{\sqrt{\gamma(r(\tilde{u}))\gamma^*(r(\tilde{u}))}}{2} = \frac{\gamma(r(\tilde{u}))}{2\sqrt{\gamma(r(\tilde{u}))-1}}.$$

Remark IV.18. This formula is analogous to the one from the Reinhardt case, with γ replacing p and $r(\tilde{u})$ replacing $\frac{u}{1+u}$. This shows that the function γ being bounded away from 1 and ∞ (which is encoded in the definition of $\mathcal{H}$) is a necessary condition for L^2 boundedness of $\mathbb{L}$ with respect to any $\sigma_{d,g}$. This condition in turn implies $f'(0) = 0$ due to (4.5), since $\int_0^1 \frac{\gamma(t)-1}{t} dt = \infty$ by a simple integral comparison.

Proof. We may assume $k_j = j$ without loss of generality, since otherwise any subsequence of $(k_j)_{j\in\mathbb{N}}$ has an increasing subsequence k_{j_l} (since $\lim_{j\to\infty} k_j = \infty$). The previous theorem applies in that case. Indeed, the proof doesn't change if we swap k with k_{j_l} and let $l \to \infty$, but another way to see this is by expanding (ξ_{j_l}, k_{j_l}) to $(\tilde{\xi}_k, k)$, simply by letting

$$\tilde{\xi}_k := -\frac{k}{2\pi\tilde{u}}$$

for any $k \notin \{k_{j_l}\}_{l\in\mathbb{N}}$, and $\tilde{\xi}_k := \xi_k$ otherwise. We obviously have $\lim_{k\to\infty} \frac{k}{\tilde{\xi}_k} = -2\pi\tilde{u}$ by construction, and if $\{C_{d,g}(\tilde{\xi}_k, k)\}_{k\in\mathbb{N}}$ converges, so does $\{C_{d,g}(\xi_{k_l}, k_l)\}_{l\in\mathbb{N}}$ as a subsequence.

Thus, we may as well identify k with j, giving us $\lim_{k\to\infty} \frac{k}{\xi_k} = -2\pi\tilde{u}$. Now define the sequence of functions $(a_k(r))_{k\in\mathbb{Z}}$ by

$$a_k(r) := 2\log(r) + \frac{4\pi\xi_k}{k} f(r).$$

Note that for $\tilde{h}(r) := ((f'(r)\gamma(r))^d g(r)$ we have

$$I_{d,g}(\xi_k, k) = \int_0^\infty e^{ka_k(r)}\tilde{h}(r)dr,$$
$$a_k'(r) = \frac{2}{r} + \frac{4\pi\xi_k}{k}f'(r),$$
$$a_k''(r) = -\frac{2}{r^2} + \frac{4\pi\xi_k}{k}f''(r).$$

We see that each a_k has a unique maximum r_k satisfying $r_k f'(r_k) = -\frac{k}{2\pi\xi_k}$. It is also clear that $a_k'' < 0$ since $\xi_k < 0$, $f'' > 0$. Moreover, by assumption

$$r_k f'(r_k) = -\frac{k}{2\pi\xi_k} \xrightarrow[k\to\infty]{} \tilde{u}.$$

Recalling that $r \mapsto rf'(r)$ has a continuous inverse $u \mapsto r(u)$, we conclude that $r_k \xrightarrow[k\to\infty]{} r(\tilde{u})$. All five assumptions in Theorem IV.16 are are relatively easy to check. Any bounded interval $I \subset (0,\infty)$ containing $r(\tilde{u})$ works. Certainly, the functions a_k'' are uniformly bounded and equicontinuous on I, as are a_k, leading to assumptions 2, 4 and 5. Regarding the third one, a majorant is easy to find since $\frac{\xi_k}{k} < -M$ for some $M > 0$, so we set

$$a(x) := 2\log(r) - Mf(r)$$

and k_0 is arbitrary due to (4.14). So then applying the theorem, we get

$$I_{d,g}(\xi_k, k) = \sqrt{\frac{2\pi}{|A|k}}(f'(r(\tilde{u}))\gamma(r(\tilde{u})))^d g(r(\tilde{u}))e^{a_k(r_k)} + o(\frac{1}{\sqrt{k}}e^{a_k(r_k)}), \tag{4.15}$$

where

$$A = -\frac{2}{r(\tilde{u})^2} - \frac{2}{u}f''(r(\tilde{u})) = -\frac{2}{r(\tilde{u})^2}\gamma(r(\tilde{u})).$$

For the second integral, a direct approach is problematic since $f'(r)$ is not necessarily twice differentiable. Using u as a parameter doesn't help since the functions

$r^*(u), y^*(u)$ aren't necessarily in C^2, either. Instead, we will use the change of variable $t = f'(r)$ (denoting the inverse by $r(t)$), yielding

$$I^*_{d,g}(\xi_k, k) = \int_0^\infty e^{kb_k(t)} \frac{(f'(\tilde{r}(t))\gamma(\tilde{r}(t)))^{2-d}}{g(\tilde{r}(t))f''(\tilde{r}(t))} dt,$$

where

$$b_k(t) := 2\log(t) + \frac{4\pi\xi_k}{k} f^*(t),$$
$$b_k'(t) = \frac{2}{t} + + \frac{4\pi\xi_k}{k}(f^*)'(t).$$

Note that by duality $r = (f^*)'(f'(r))$, which implies a correspondence between the maximum r_k of a_k and the maximum t_k of b_k, for all $k \in \mathbb{Z}_{\geq 0}$. Specifically, we have $t_k = f'(r_k)$. It follows that the we can repeat the previous computation, replacing by $\gamma(r(u))$ by $\gamma^*(r(u))$, $r(\tilde{u})$ by $r^*(\tilde{u})$, d by $2-d$, and g by $\frac{1}{gf''}$. This gives us

$$I^*_{d,g}(\xi_k, k) = \sqrt{\frac{2\pi}{|\tilde{A}|k}} \frac{(f'(r(\tilde{u}))\gamma(r(\tilde{u})))^{2-d}}{g(r(\tilde{u}))f''(r(\tilde{u}))} e^{b_k(t_k)} + o(\frac{1}{\sqrt{k}} e^{b_k(t_k)}), \tag{4.16}$$

where

$$\tilde{A} = -\frac{2}{r^*(\tilde{u})^2}\gamma^*(r(\tilde{u})).$$

Finally, plugging (4.15) and (4.16) into (4.10) and using Stirling's formula $k! \sim \sqrt{2\pi k}(\frac{k}{e})^k$ as well as the fact

$$a_k(r_k) + b_k(t_k) = 2\log(r_k f'(r_k)) + \frac{4\pi\xi_k}{k} r_k f'(r_k) = 2\log(\frac{k}{2\pi|\xi_k|}) - 2,$$

we get

$$\lim_{k\to\infty} C_{d,g}(\xi_k,k) = \lim_{k\to\infty} \frac{(2\pi\xi_k)^{2k+2}}{k!^2}\frac{2\pi}{k\sqrt{A\tilde{A}}}(\frac{k}{2\pi|\xi_k|})^{2k}e^{-2k}\frac{(f'(r(\tilde{u})\gamma(r(\tilde{u})))^2}{f''(r(\tilde{u}))}$$
$$= \lim_{k\to\infty} \frac{(2\pi\xi_k)^2k^{2k}e^{-2k}}{2\pi k^2(\frac{k}{e})^{2k}}\frac{\pi\tilde{u}}{\sqrt{\gamma(r(u))\gamma^*(r(u))}}\frac{(f'(r(\tilde{u})\gamma(r(\tilde{u})))^2}{f''(r(\tilde{u}))}$$
$$= \lim_{k\to\infty}\left(\frac{2\pi\xi_k}{k}\right)^2\frac{\tilde{u}(f'(r(\tilde{u}))\gamma(r(\tilde{u})))^2}{2\sqrt{\gamma(r(\tilde{u}))\gamma^*(r(\tilde{u}))}f''(r(\tilde{u})}$$
$$= \frac{(f'(r(\tilde{u}))\gamma(r(\tilde{u})))^2}{2\tilde{u}\sqrt{\gamma(r(\tilde{u}))\gamma^*(r(\tilde{u}))}f''(r(\tilde{u}))} = \frac{\sqrt{\gamma(r(\tilde{u}))\gamma^*(r(\tilde{u}))}}{2},$$

since $\frac{k}{2\pi|\xi_k|}\xrightarrow[k\to\infty]{}\tilde{u} = r(\tilde{u})f'(r(\tilde{u}))$, and by (4.4)

$$\gamma^*(r(\tilde{u})) = \frac{\gamma(r(\tilde{u}))}{\gamma(r(\tilde{u}))-1} = \frac{f'(r(\tilde{u}))\gamma(r(\tilde{u}))}{r(\tilde{u})f''(r(\tilde{u}))} = \frac{(f'(r(\tilde{u})))^2\gamma(r(\tilde{u}))}{\tilde{u}f''(r(\tilde{u}))}.$$

□

We want to address two more cases.

Theorem IV.19. *Define the function*

$$J_\gamma(k) = \frac{\gamma^{\frac{2k}{\gamma}+\frac{d-1}{\gamma^*}+1}(\gamma^*)^{\frac{2k+1-d}{\gamma^*}+1}}{(k!)^2 2^{2k+2}}\Gamma(\frac{2k}{\gamma}+\frac{d-1}{\gamma^*}+1)\Gamma(\frac{2k+1-d}{\gamma^*}+1).$$

Then for $D\in\mathcal{H}$ and the measure $\sigma_{d,g}$, where $d\in\mathbb{R}$ and $g:(0,\infty)\to(0,\infty)$ has a positive continuous extension to $[0,\infty]$, we get the following asymptotic results:

1. $C_{d,g}(\xi_j,k_j)\to J_{\gamma_0}(k_0)$ *as* $\xi_j\to-\infty$, $k_j\equiv k_0$.
2. $C_{d,g}(\xi_j,k_j)\to J_{\gamma_\infty}(k_0)$ *as* $\xi_j\to 0$, $k_j\equiv k_0$.

We need a technical lemma first:

Lemma IV.20. *If $f(r) = cr^\gamma + o(r^\gamma)$ for $c>0$ as $r\to 0$ (or $r\to\infty$) and $f(r)$ is invertible near* 0 *(∞), then $f^{-1}(r) = (\frac{r}{c})^{\frac{1}{\gamma}} + o(r^{\frac{1}{\gamma}})$ as $r\to 0$ ($r\to\infty$).*

Proof. Let $0<c_1<c<c_2$. For $r>0$ small enough (large enough) we have

$$c_1r^\gamma < f(r) < c_2r^\gamma.$$

Taking the inverse functions of all sides reverses the inequalities, yielding

$$(\frac{1}{c_2})^{\frac{1}{\gamma}} r^{\frac{1}{\gamma}} < f^{-1}(r) < (\frac{1}{c_2})^{\frac{1}{\gamma}} r^{\frac{1}{\gamma}}.$$

Dividing by $r^{\frac{1}{\gamma}}$ reveals the desired result, since $\frac{1}{c_1}, \frac{1}{c_2}$ are arbitrarily close to $\frac{1}{c}$.

Moving onto the proof of the theorem:

Proof. We treat each remaining case separately in terms of the asymptotic strategy for $I_{d,g}(\xi_j, k_j)$.

The case $\xi_j \to -\infty, \quad k_j \equiv k$

By Lemma IV.14 we have $f(r) = c_0 r^{\gamma_0} + o(r^{\gamma_0})$ as $\to 0$ for some $\gamma_0 > 1$. This will allow us to replace $f(r), f'(r)$ with $c_0 r^{\gamma_0}, c_0\gamma_0 r^{\gamma_0 - 1}$ to obtain the principal part of the asymptotic expansion of $I_{d,g}(\xi_j, k_j)$ in (4.10), which we will then justify by estimating the error. We may as well assume $c_0 = 1$ since we will see that the only dependence on D is via γ_0, which is invariant under scaling (alternatively, the symbol function is dilated horizontally, so the limit is unaffected). But as a first step, we use a change of variable $t = 4\pi f(r)$ to get

$$I_{d,g} = \int_0^\infty r^{2k_0} e^{4\pi\xi_j f(r)} ((rf'(r))')^d g(r) dr = \frac{1}{4\pi} \int_0^\infty \varphi_{k_0}(\frac{t}{4\pi}) e^{\xi_j t} dt \tag{4.17}$$

where $\varphi_{k_0}(t) := (f^{-1}(t))^{2k_0} (f'(f^{-1}(t)))^{d-1} (\gamma(f^{-1}(t)))^d g(f^{-1}(t))$.

While it is possible to use a particular version of Watson's lemma here (see Theorem 15.2.7 in [7]), we will use a somewhat different approach that can also be applied to the next case ($\xi_j \to 0$). By a combination of Lemma IV.14, Lemma IV.20 and the continuity assumptions on γ, g at 0, we have for small $t > 0$

$$f^{-1}(t) \sim t^{\frac{1}{\gamma_0}}, \quad f'(f^{-1}(t)) \sim \gamma_0 (t^{\frac{1}{\gamma_0}})^{\gamma_0 - 1} = t^{\frac{1}{\gamma_0^*}}, \quad g(f^{-1}(t)) \sim g(0)$$

and thus

$$\varphi_{k_0}(t) \sim \tilde{\varphi}_{k_0}(t) := t^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}}\gamma_0^{2d-1}g(0),$$

where $\sim$ denotes that the limit of the quotient is 1 as $t \to 0$. Let $\tilde{I}_{d,g}$ denote the respective integral for M_{γ_0}. Then by (4.17) we get

$$\begin{aligned}\tilde{I}_{d,g} &= \frac{1}{4\pi}\int_0^\infty \tilde{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi t}dt = \frac{1}{4\pi}\int_0^\infty (\frac{t}{4\pi})^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}}\gamma_0^{2d-1}g(0)e^{\xi_j t}dt \\ &= \frac{\gamma_0^{2d-1}g(0)\Gamma(\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}+1)}{(4\pi|\xi_j|)^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}+1}}.\end{aligned}$$

It is tempting to simply swap $\gamma_0 \mapsto \gamma_0^*, d \mapsto 2-d, g \mapsto \frac{1}{g}$ (or wherever both γ_0, d are involved, only swap d) to obtain $\tilde{I^*}_{d,g}$, but that would correspond to $M_{\gamma_0^*}$ rather than the Legendre transform, which is off by a dilation factor. By Remark IV.11 $(M_\gamma)^*$ is obtained from M_{γ^*} by a horizontal dilation of a factor $\gamma_0^{\frac{1}{\gamma_0}}\gamma_0^{*\frac{1}{\gamma_0^*}}$, so it is given by $\mathrm{Im} z_2 = \left(\frac{|z_1|}{\gamma_0^{\frac{1}{\gamma_0}}(\gamma_0^*)^{\frac{1}{\gamma_0^*}}}\right)^{\gamma_0^*}$. Then by (4.17) and the dilation behavior of $I_{d,g}$ in (4.10), we have

$$\tilde{I}^*_{2-d,\frac{1}{g}} = \left(\gamma_0^{-\frac{\gamma_0^*}{\gamma_0}}(\gamma_0^*)^{-1}\right)^{-\frac{2k_0}{\gamma_0^*}+\frac{1-d}{\gamma_0}} \frac{(\gamma_0^*)^{2(2-d)-1}\frac{1}{g(0)}\Gamma(\frac{2k_0}{\gamma_0^*}+\frac{(2-d)-1}{\gamma_0^*}+1)}{(4\pi|\xi_j|)^{\frac{2k_0}{\gamma_0^*}+\frac{(2-d)-1}{\gamma_0^*}+1}}.$$

The above two approximations yield the desired approximation for $C_{d,g}(\xi_j, k_0)$ once we plug them into (4.10), but this relies on showing that the error $I_{d,g} - \tilde{I}_{d,g}$ is small relative to $\tilde{I}_{d,g}$, or equivalently $|\xi_j|^{-\frac{2k_0}{\gamma_0}-\frac{d-1}{\gamma_0^*}-1}$, for large $|\xi_j|$. This will also imply the corresponding approximation result for $I^*_{2-d,\frac{1}{g}}$. Now write $\hat{\varphi}_{k_0}(t) := |\varphi_{k_0}(t) - \tilde{\varphi}_{k_0}(t)|$. We know that for small $t > 0$ we have $\hat{\varphi}_{k_0}(t) = o(t^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}})$. Then for small $\epsilon > 0$ we have

$$\begin{aligned}&\int_0^\infty \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j t}dt \leq \int_0^\epsilon \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j t}dt + \int_\epsilon^\infty \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j t}dt \\ &= o\left(\int_0^\infty (\frac{t}{4\pi})^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}} e^{\xi_j t}dt\right) + e^{\frac{\epsilon\xi_j}{2}}\int_\epsilon^\infty \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j(t-\frac{\epsilon}{2})}dt = o\left(|\xi_j|^{-\frac{2k_0}{\gamma_0}-\frac{d-1}{\gamma_0^*}-1}\right).\end{aligned}$$

The last step can be justified as follows: Note that for any fixed $\epsilon > 0$ we have that $\int_\epsilon^\infty \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j(t-\frac{\epsilon}{2})}dt$ is $O(1)$ as it is decreasing in $|\xi_j|$, and clearly $e^{\frac{\epsilon\xi_j}{2}} = o(|\xi_j|^{-a})$ for any $a > 0$. Thus, we can pick $\epsilon > 0$ according to the estimate $\hat{\varphi}_{k_0}(t) = o(t^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}})$.

<u>The case $\xi_j \to 0, \quad k_j = k$</u>

This case is analogous to the above. We approximate $f(r)$ near ∞ rather than 0 (which is why γ_∞ replaces γ_0 in the formula), and without loss of generality we may assume $f(r) \sim r^{\gamma_\infty}$ for large $r > 0$. The only non-trivial change lies in how we estimate $\int_0^\infty \hat{\varphi}_{k_0}(\frac{t}{4\pi})dt$, where this time we have $\hat{\varphi}_{k_0}(t) = o(t^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}})$ for large $t > 0$. Now fix a large $M > 0$.

$$\int_0^\infty \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j t}dt \le \int_0^M \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j t}dt + \int_M^\infty \hat{\varphi}_{k_0}(\frac{t}{4\pi})e^{\xi_j t}dt$$
$$\le \int_0^M \hat{\varphi}_{k_0}(\frac{t}{4\pi})dt + o\left(\int_0^\infty (\frac{t}{4\pi})^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}} e^{\xi_j t}dt\right) = o\left(|\xi_j|^{-\frac{2k_0}{\gamma_0}-\frac{d-1}{\gamma_0^*}-1}\right).$$

Note that $\int_0^M \hat{\varphi}_{k_0}(\frac{t}{4\pi})dt$ doesn't depend on ξ_j, making it trivially $o\left(|\xi_j|^{-\frac{2k_0}{\gamma_0}-\frac{d-1}{\gamma_0^*}-1}\right)$ as $\xi_j \to 0$. We can pick $M > 0$ according to the estimate $\hat{\varphi}_{k_0}(t) = o(t^{\frac{2k_0}{\gamma_0}+\frac{d-1}{\gamma_0^*}})$. □

Remark IV.21. We have not addressed Case 3 ($\lim_{j\to\infty} \xi_j \in (-\infty, 0], k \to \infty$) or Case 4 with 0 or ∞ as the slope. Based on computations for the pairing measure σ, we conjecture that the formula for Case 4 extends to $0, \infty$, while Case 3 can be obtained by taking the limit of the formula in Case 2 as $k_0 \to \infty$ (so there is no dependence on $\lim_{j\to\infty} \xi_j$), yielding

$$\lim_{j\to\infty} C_{d,g}(\xi_j, k_j) = \frac{\sqrt{\gamma_\infty\gamma_\infty^*}}{2}(\gamma_\infty - 1)^{\frac{d-1}{\gamma_\infty^*}}.$$

Should this be true, then based on (4.14) and a similar analysis for the formulas in Theorem IV.19, for any continuous $g : [0, \infty] \to (0, \infty)$, $\mathbb{L}$ is bounded with respect to $L^2(S, \sigma_{d,g})$ if and only if

$$1 - w(\gamma_0, \gamma_\infty) < d < 1 + w(\gamma_0, \gamma_\infty),$$

where $w(\gamma_0, \gamma_\infty) \coloneqq \min_{j \in \{0,\infty\}} \gamma_j^*$. Note that this condition is vacuous for $d \in [0, 2]$.

Chapter V

Can You Hear the Shape of a Sufficiently Smooth and Convex Rigid Hartogs Domain?

In Chapter 3 we discussed our ability to recover a convex Reinhardt domain (in $\tilde{\mathcal{R}}$) from its marked spectrum. Theorem III.4 shows that the marked spectrum map is injective on a fairly large subclass $\mathcal{R}_m$, where the set of Reinhardt vertices for each domain is relatively tame (can be represented as a monotone sequence). The point of this chapter is to essentially convert Chapter 3 into the setting of rigid Hartogs domains, or more specifically the subclass $\tilde{\mathcal{H}}$. Recall that $D \in \tilde{\mathcal{H}}$ is C^1 (except potentially at ∞), C^2 away from $z_1 = 0$ and strictly convex.

The group on $\tilde{\mathcal{H}}$ for which the Leray kernel transforms nicely, is generated by all coordinate dilations and imaginary translations in z_2 (under which $\tilde{\mathcal{H}}$ is invariant), and uninterestingly, rotations in z_1 and real translations in z_1 (which preserve each rigid Hartogs domain). Also, passing to the dual domain preserves the spectrum when the measure is σ, which is easy to check as the integrals in (4.3) are swapped under duality according to (4.13). As in Chapter 3, we focus on this one special measure due to its duality properties and simpler form, and call the spectrum of $\mathbb{L}_\sigma^* \mathbb{L}$ the *Hartogs-Leray spectrum.* In particular, we work with $d = 1$.

Denote the symbol function for a domain D whose boundary is endowed with σ by $C(\xi, k)$. This information is richer than the image of the symbol function counting multiplicities, or any variant thereof.

Setup

The key fact that makes the conversion between the chapters natural is Theorem IV.17. First, it is very useful to work with the composition $\gamma_c(u) \coloneqq \gamma(r(u))$. Then

$$\Phi_D(u) = \frac{\sqrt{\gamma_c(u)\gamma_c^*(u)}}{2},$$

is given directly by $C(\xi, k)$ via limits. As in the Reinhardt case, $\gamma \mapsto \gamma\gamma^*$ is a covering map of degree 2 with $\gamma \mapsto \gamma^*$ as the deck transformation. Thus, it is clear that using Φ_D, the semi-marked spectrum map given by $D \mapsto C(\xi, k)$ recovers $\gamma_c(u)$ up to $\gamma_c(u) \mapsto \gamma_c^*(u)$ for any subset of $u \in (0, \infty)$ such that the resulting function is continuous. The choice of $(0, \infty)$ yields the dual domain D^* by Lemma IV.10. In general, if we can recover $\gamma_c(u)$, we can generate the domain as follows. Integrating (4.9), we have

$$y(u) = \int_0^u \frac{1}{\gamma_c(t)} dt + y(0). \tag{5.1}$$

This is compatible with what we already know: Imaginary shifts in z_2 preserve not only the spectrum but also the symbol function, while dilations in z_2 preserve the spectrum but dilate the symbol function. On the other hand:

$$\frac{r'(u)}{r(u)} = \frac{f'(r(u))r'(u)}{r(u)f'(r(u))} = \frac{y'(u)}{u} = \frac{1}{u\gamma_c(u)}.$$

Integrating from some $u_0 \in (0, 1)$ to u and then applying $x \mapsto e^x$ to both sides, we get

$$r(u) = r(u_0) \exp\left(\int_{u_0}^u \frac{1}{t\gamma_c(t)} dt\right). \tag{5.2}$$

Thus, the problem boils down to the recovery of $\gamma_c(u)$. The sets of solutions to $\gamma_c(u) = 2$ plays a pivotal role.

Some geometry

We have a geometric description in this setting, too. Consider the osculating (to first order) vertically dilated and shifted M_2 domain in $\mathbb{C}^2$ at a point $(z_1, z_2) \neq (0,0)$ with $(|z_1|, |z_2|) = (r(u), y(u))$ for $u > 0$. If $\gamma_c(u) > 2$, this dilated M_2 domain locally contains the given rigid Hartogs domain (globally if the inequality holds for all $u \in (0,\infty)$). If the inequality is reversed, then the dilated M_2 ball is locally inside the rigid Hartogs domain (globally). See Appendix B for proof. We call the former case Hartogs convexity and the latter Hartogs concavity.

Definition V.1. If we have $p(u_0) = 2$, we call u_0 a *Hartogs vertex*. If $r(u) \equiv 2$ on a subinterval, we only count the endpoints as Hartogs vertices. For a domain $D \in \mathcal{H}$, we denote its set of Hartogs vertices by V_D (not to be confused with the Reinhardt notation, based on the context).

Note that V_D is an invariant of essentially isospectral domains in $\tilde{\mathcal{H}}$, i.e. domains with the same Φ_D function, since the vertices correspond to solutions of $\gamma\gamma^* = 4$.

Theorem V.2. *Let $\mathcal{H}_1$ denote the collection of all domains $D \in \tilde{\mathcal{H}}$ such that card$(V_D \setminus \{0,\infty\}) = 1$. Let $\mathcal{S}$ denote the space of sequences (indexed by $k \in \mathbb{Z}_{\geq 0}$) of continuous functions on $(-\infty, 0)$. Then the semi-marked spectrum map $\Omega : \mathcal{H}_1 \to \mathcal{S}$ given by $\Omega(D) = C(\xi, k)$ is injective modulo dilations in z_1, translations in z_2 and duality.*

Proof. We convert the proof of Theorem III.2.

Step 1

Let $D, \tilde{D} \in \mathcal{H}_1$ have the same semi-marked spectrum, and let $\gamma_c(u)$, $\tilde{\gamma}_c(u)$ be the respective osculation functions. We know that both domains share the same Hartogs vertex $a \in (0,\infty)$. Thanks to duality, we may assume $\tilde{\gamma}_c(u) = \gamma_c(u) > 2$ on $(0,a)$ (we can arrange for $\gamma_c(u) > 2$ and $\tilde{\gamma}_c(u) > 2$ on $(0,a)$ separately). Then either

$\tilde{\gamma}_c(u) = \gamma_c(u)$ on $(a, 1)$ and we are done, or $\tilde{\gamma}_c(u) = \gamma_c^*(u)$ on (a, ∞). We assume the latter.

Step 2

We observe that for $u \in (0, a]$ we can use (5.1) and (5.2) with $u_0 = a$, to obtain

$$\tilde{y}(u) = y(u), \quad \tilde{r}(u) = r(u),$$

On (a, ∞) we have

$$\tilde{y}(u) = u - y(u) + \alpha, \quad \tilde{r}(u) = \beta \frac{u}{r(u)}$$

for the constants $\alpha \coloneqq 2y(a) - a, \beta \coloneqq \frac{r(a)^2}{a}$ (this can be verified using the integral formula, or just by continuity of $y(u), r(u)$ at $u = a$).

Step 3

Now we want to show that for all $\xi < 0 \quad \tilde{C}(\xi, 0) \neq C(\xi, 0)$. Using (4.10), we get that $\dfrac{1}{(2\pi\xi)^2}(\tilde{C}(\xi, 0) - C(\xi, 0))$ is given by

$$\begin{aligned}
&\left(\int_0^a e^{4\pi\xi y(u)} du + e^{4\pi\alpha\xi} \int_a^\infty e^{4\pi(u-y(u))} du \right) \times \left(\int_0^a e^{4\pi\xi(u-y(u))} du \right. \\
&\left. + e^{-4\pi\alpha\xi} \int_a^\infty e^{4\pi\xi y(u)} du \right) - \left(\int_0^a e^{4\pi\xi y(u)} du + \int_a^\infty e^{4\pi\xi y(u)} du \right) \\
&\times \left(\int_0^a e^{4\pi\xi(u-y(u))} du + \int_a^\infty e^{4\pi\xi(u-y(u))} du \right) \qquad (5.3) \\
&= \left(\int_0^a e^{4\pi\xi y(u)} du - e^{4\pi\alpha\xi} \int_0^a e^{4\pi\xi(u-y(u))} du \right) \times \left(e^{-4\pi\alpha\xi} \int_a^\infty e^{4\pi\xi y(u)} du \right. \\
&\left. - \int_a^\infty e^{4\pi\xi(u-y(u))} du \right).
\end{aligned}$$

Step 4

If we can show that

$$\forall u \neq a \quad y(u) \neq u - y(u) + \alpha,$$

then it will follow that both factors in (5.3) are non-zero and thus

$$\forall \xi < 0 \quad \tilde{C}(\xi, 0) \neq C(\xi, 0).$$

Note that since for all $u < a$ we have $\gamma_c(u) > 2$ by assumption, then

$$y'(u) = \frac{1}{\gamma_c(u)} < 1 - \frac{1}{\gamma_c(u)} = \frac{d}{du}(u - y(u) + \alpha).$$

This shows that for $u < a$ we have $y(u) > u - y(u) + \alpha$, and similarly, the inequality is reversed for $u > a$. $\square$

Theorem V.3. *Let $\mathcal{H}_m$ denote the collection of all domains $D \in \tilde{\mathcal{H}}$ such that V_D can be represented as a monotone (possibly finite) one-sided sequence (v_n). As before, let $\mathcal{S}$ denote the space of sequences (indexed by $k \in \mathbb{Z}_{\geq 0}$) of continuous functions on $(-\infty, 0)$. Then the semi-marked spectrum map $\Omega : \mathcal{H}_m \to \mathcal{S}$ given by $\Omega(D) = C(\xi, k)$ is injective modulo dilations in z_1, translations in z_2 and duality.*

Proof. Convert the proof of Theorem III.4 much like the previous proof. Note that if the sequence of Hartogs vertices is decreasing, we can't bypass this (i.e. make it increasing) by a reflection argument in this setting. Luckily, the proof works in reverse order of the subintervals with subtle changes.

$\square$

Corollary V.4. *If two domains $D, \tilde{D} \in \mathcal{H}_m$ have symbol functions that coincide outside a compact set, i.e. $\exists K \in \mathbb{N} \quad \forall k > K, \xi < -K \quad C(\xi, k) = \tilde{C}(\xi, k)$, then they are the same up to dilations in z_1, translations in z_2 and duality.*

Proof. The proof of the above theorem only relies on asymptotics as $\xi, k \to \infty$. $\square$

Corollary V.5. *Let $D \in H_m$. If the symbol function $C(\xi, k)$ is independent of ξ, then D is an M_γ domain up to coordinate dilations and translations in z_2. Otherwise, the continuous spectrum of $\mathbb{L}_\sigma^* \mathbb{L}$ is non-empty.*

Proof. For such a domain, the symbol function must coincide with the symbol function of M_{γ_0} (or equivalently M_{γ_∞}), since the boundary values of a symbol function are determined by γ_0 (at ∞) and γ_∞ (at 0) by Theorem IV.19. Then by Theorem V.3, the conclusion is immediate. $\square$

Chapter VI

Future Directions

6.0.1 Open questions

The following is a list of open questions related to the preceding chapters:

1. For domains in $\mathcal{R}$, what can we say about the Leray norm? Is it ever attained (only) by an eigenvalue? How is it affected by the boundary measure?

2. How do we prove that Theorem II.6 holds for some $\epsilon > 0$ and all $q < \epsilon$? Is the case $q \geq \epsilon$ essentially the same because there is (seemingly) no local maximum that is also global?

3. In $\mathcal{R}$, can we find a positive lower bound for the ratio $\frac{\|\mathbb{L}\|_{e,q}}{\|\mathbb{L}\|}$? At least for $q = 0$?

4. Can we classify domains in $\mathcal{R}$ that have (preferably isolated) eigenvalues of infinite multiplicity for some measure? Is the l_2 ball the only domain (up to coordinate dilations) with this property for μ_0?

5. Is there a domain in $\mathcal{H}$ that has at least one eigenvalue (with the eigenfunction being in $L^2(S, \sigma)$), other than $\{M_\gamma\}_{\gamma>1}$ and their dilations?

6. What happens to the spectral recovery problem outside of $\mathcal{R}_m$ and similarly outside $\mathcal{H}_m$? In particular, what happens when both endpoints are accumulation points of V_D?

7. In $\mathcal{R}_m$, what happens when we remove the marking? Are there, at least for some

domains, transformations that permute the Reinhardt-Leray spectrum other than reflection? If so, what transformations do they induce on $p(s)$?

8. In $\mathcal{H}_m$, are dilations in z_2 the only way to modify the symbol function while preserving the Hartogs-Leray spectrum?

9. Can we describe the image of the marked spectrum map or symbol map? This might be easier to answer when restricting to $\mathcal{R}$ and $\mathcal{H}$, as opposed to $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{H}}$.

10. Does $\mathbb{L}^*\mathbb{L}$ always have a basis of eigenfunctions for a bounded, strongly $\mathbb{C}$-convex domains with a C^2 or smoother boundary? Note that this is the case for domain in $\tilde{\mathcal{R}}$ despite the potential failure of the aforementioned conditions on the axes.

11. What happens in higher dimensions? In terms of the spectrum, boundedness, norms, spectral recovery problem and related concepts.

6.0.2 Convex Reinhardt domains in higher dimensions

A brief survey of an ongoing project: Given a convex Reinhardt domain $D \in \mathbb{C}^3$, the Leray eigenvalues (for a measure analoguous to μ_0) can can be written down using a parametrization of $bD \cap \mathbb{R}^3$, like s defined more explicitly in (1.3). This parametrization is obtained by the map $(r_1, r_2, r_3) \mapsto (\log r_1, \log r_2, \log r_3)$, followed by the Gauss map normalized relative to the l_1 norm. In this way, the 2-simplex

$$\Delta = \{(s, t, u) : s + t + u = 1, s, t, u > 0\}$$

parametrizes $S \cap \mathbb{R}^3$, where $S = bD$. Then the eigenvalues can be shown to be given by

$$\lambda_{n,m,k} = \left(\frac{(n+m+k+2)!}{n!m!k!} \right)^2 \int_0^1 \int_0^{1-t} r_1^{2n}(s,t)) r_2^{2m}(s,t) r_3^{2k}(s,t) ds dt$$
$$\times \int_0^1 \int_0^{1-t} \left(\frac{s}{r_1(s,t)} \right)^{2n} \left(\frac{t}{r_2(s,t)} \right)^{2m} \left(\frac{1-s-t}{r_3(s,t)} \right)^{2k} ds dt.$$

(The generalization to $\mathbb{C}^n$ is natural, where the domain of integration is an $(n-1)$-simplex.)

Formulas for the essential spectrum can be obtained, but they are somewhat inelegant as they use the auxiliary functions $p_j(s,t)$ given by

$$\frac{1}{sp_1(s,t)} = \frac{\partial \log r_1(s,t)}{\partial s}, \quad \frac{1}{tp_2(s,t)} = \frac{\partial \log r_2(s,t)}{\partial t}.$$

This representation singles out s,t and r_1, r_2 (so there are 8 similar representations). It may be possible to use two alternative functions of a more canonical (geometric) nature, which may simplify the formulas. It's also worth noting that in $\mathbb{C}^n$ that for what *appears* to be the essential norm (at least based on the intuition of the $n=2$ case), as many as $n-1$ of its fractional powers of occur in the essential spectrum.

6.0.3 Semi-marked range problem

Recall the 8th open question: Are coordinate dilations in z_2 the only way to modify the symbol function while preserving the spectrum? Less loosely, could there be two domains that are not essentially isospectral, and yet for each $k \in \mathbb{Z}_{\geq 0}$ (the marking) the functions $C(\cdot,k), \tilde{C}(\cdot,k)$ have the same range counting multiplicities? We call this the semi-marked range problem for $\tilde{\mathcal{H}}$. It seems that there are families of such domains for which $C(\cdot,k)$ is monotone for any fixed $k \in \mathbb{Z}_{\geq 0}$, and its image is an interval that depends only on k (i.e. is the same for all domains in said family). To be specific, for a fixed $\gamma > 2$ ($\gamma < 2$ is similar), consider the 1-parameter family of domains corresponding to the function family

$$f_t(r) = r^2 + tr^\gamma = t(r^\gamma + \frac{1}{t}r^2)$$

for $t > 0$. If t is small we get perturbations of M_2, in the sense that $f_t(r) \xrightarrow[t\to 0]{} r^2$ locally uniformly. For large t, we can think of these domains as dilated perturbations of M_γ. Note that (even by inspection due to asymptotics) we have that

$$\gamma_t(0) = 2, \quad \gamma_t(\infty) = \gamma$$

are independent of t (they are swapped if $\gamma < 2$), which implies that the boundary values of the symbol functions (that are determined by the above boundary values) are also independent of t. If we can show that at least for two different $t > 0$ values $C_t(\cdot, k)$ are monotone for all $k \in \mathbb{Z}_{\geq 0}$, then we are done as then the domains have the same marked range, but their symbol functions are different even modulo dilations. The monotoncity of a given symbol function is determined by the partial logarithmic derivative

$$\frac{\partial \log(C_t)}{\partial \xi}(\xi, k) = \frac{\partial \log(I_t)}{\partial \xi}(\xi, k) + \frac{\partial \log(I_t^*)}{\partial \xi}(\xi, k) - \frac{2k+2}{|\xi|}.$$

Fixing $k > 0, \xi < 0$, a careful asymptotic expansion (using a variant of Watson's method) yields for some function $c(\gamma) > 0$ (for $\gamma \neq 2$)

$$\frac{\partial \log(C_t)}{\partial \xi}(\xi, k) = -c(\gamma)\left(\frac{k}{t|\xi|}\right)^{\frac{2}{\gamma}} - c(\gamma^*)\left(\frac{k}{t|\xi|}\right)^{\frac{2}{\gamma^*}} + o\left(\left(\frac{k}{t|\xi|}\right)^{\frac{2}{\gamma}} + \left(\frac{k}{t|\xi|}\right)^{\frac{2}{\gamma^*}}\right)$$

as $t \to \infty$. For $k = 0$ it turns out that we get the same type of estimate as $k = 1$ (with a modified $c(\gamma)$). The problem is that the error terms depend on the slope $\frac{k}{|\xi|}$ (or $\frac{1}{|\xi|}$ for $k = 0$), so we need a different approach for large slopes (and small $|\xi|$ for $k = 0$) in order to show monotonicity for large t and all $k \geq 0, \xi < 0$.

It is possible that perturbations of this kind could turn out to be a general way to create many such families (perhaps "dense" in some sense) for the semi-marked range problem.

Appendices

Appendix A

Proof of Lemma II.2

We want to prove that

$$h_{a,b}(x) = \psi(\frac{a}{x}) - \psi(\frac{b}{x})$$

is convex for $a > b > 0$ and $x > 0$. The proof is due to I. Pinelis.

Proof. It is enough to prove that the function

$$\delta(a,x) := \frac{\partial}{\partial a}\frac{\partial^2}{\partial x^2}\psi(\frac{a}{x})$$

is increasing in $a > 0$ for each $x > 0$. Writing $v := \frac{a}{x}$, we get

$$\delta(a,x) = \frac{\epsilon(v)}{x^2},$$

where $\epsilon(v) := \psi''(v)v^2 + 2\psi'(v)v$. We need to show that $\epsilon'(v) > 0$ for all $v > 0$. The integral representation (2.2) yields

$$\epsilon'(v) = \int_0^\infty U(z)(2 - 4vz + v^2z^2)e^{-vz}dz,$$

where $U(z) := \frac{z}{1-e^{-z}}$. Integrating by parts twice, we have

$$\epsilon'(v) = \int_0^\infty U'(z)z(-2+vz)e^{-vz}dz = \int_0^\infty U''(z)z^2e^{-vz}dz, \qquad \text{(A.1)}$$

where

$$U''(z) = \frac{e^{-2z}(2 + e^z(z-2) + z)}{(1-e^{-z})^3} = \frac{e^{-2z}}{(1-e^{-z})^3}\int_0^z (z-t)te^t dt > 0$$

for $z > 0$. Thus, the integrand in the RHS of (A.1) is positive on $(0, \infty)$, which completes the proof .

$\square$

Appendix B

Vertex Geometry

Recall the discussion in Chapter 3 about the geometry of Reinhardt vertices.

Lemma B.1. *Let $D \in \tilde{R}$. Consider the osculating (to first order) dilated l_2 ball in $\mathbb{C}^2$ of the form $a|z_1|^2 + b|z_2|^2 = 1$ $(a, b > 0)$ at a point (z_1, z_2) away from the axes with $(|z_1|, |z_2|) = (r_1(s), r_2(s))$ for $s \in (0,1)$. If $p(s) > 2$, this dilated l_2 ball is locally inside the Reinhardt domain (globally if the inequality holds for all $s \in (0,1)$). If the inequality is reversed, then the dilated l_2 ball contains the Reinhardt domain locally (globally).*

Proof. Due to rotation invariance, we can focus on $D \cap \mathbb{R}^2_{>0}$. We need a formula from ([2]). First, let $\breve{p}(r_1) := p(s(r_1))$, where $s(r_1)$ is the inverse of the function $s(r_1)$ defined in (1.3) (note that in the paper the notation for $p, \breve{p}$ is reversed). Then, by the paper, we have

$$\breve{p}(r_1) = 1 + \frac{r_1\phi(r_1)\phi''(r_1)}{\phi'(r_1)(\phi(r_1) - r_1\phi'(r_1))},$$

where $r_2 = \phi(r_1)$ defines D and ϕ is C^2 away from $\{0,1\}$ with $\phi'' < 0$, and is C^1 away from 1. Assume that $p(s) > 2$ for $s \in I_s \subset (0,1)$. Then on some corresponding interval $r_1(I_s)$, we get

$$\frac{r_1\phi(r_1)\phi''(r_1)}{\phi'(r_1)(\phi(r_1) - r_1\phi'(r_1))} > 1.$$

Multiplying by $\frac{\phi(r_1)-r_1\phi'(r_1)}{r_1\phi(r_1)} > 0$ yields

$$\frac{\phi''(r_1)}{\phi'(r_1)} > \frac{1}{r_1} - \frac{\phi'(r_1)}{\phi(r_1)}.$$

Integrating and applying $x \mapsto e^x$ to both sides gives for some constants $C_s, c_s > 0$ (depending only on s) and $r_1 \in r_1(I_s)$

$$\phi'(r_1) \geq \frac{C_s r_1}{\phi(r_1)} \implies \frac{\phi^2(r_1)}{2} \geq \frac{C_s r_1^2}{2} + c_s \implies \phi(r_1) > \sqrt{C_s r_1^2 + 2c_s}.$$

Imposing the first order initial conditions, we get for all $s \in (0,1)$

$$C_s = \frac{\phi(r_1(s))\phi'(r_1(s))}{r_1(s)} < 0, \quad 2c_s = \phi(r_1(s))(\phi(r_1(s)) - r_1(s)\phi'(r_1(s)) > 0.$$

This shows that $\phi_s(r_1) := \sqrt{C_s r_1^2 + 2c_s}$ defines the unique dilated l_2 ball that osculates D to first order at $(r_1(s), r_2(s))$. We have $\phi(r_1) > \phi_s(r_1)$, so D contains this dilated l_2 ball near $(r_1(s), r_2(s))$. If $p(s) > 2$ for all $s \in (0,1)$, then the containment is global. Finally, if $p(s) < 2$ all inequalities are reversed, so the conclusion is that D is contained in the dilated l_2 ball (locally or globally).

$\square$

Remark B.2. For points on the axes, the osculating dilated l_2 ball fails to be unique (there's a 1-parameter family of such balls). The calculation above actually gives the degenerate case of a line, for which it is already obvious that the tangent hyperplane lies above the domain (if considering $s = 0$) or to the right (if $s = 1$) regardless of the p value.

It remains to prove the analogous lemma for rigid Hartogs domains in $\tilde{\mathcal{H}}$.

Lemma B.3. *Let $D \in \tilde{\mathcal{H}}$. Consider the osculating (to first order) vertically dilated and shifted M_2 domain in $\mathbb{C}^2$ at a point (z_1, z_2) away from the z_2 axis with $(|z_1|, |z_2|) = (r(u), y(u))$ for $u > 0$. If $\gamma_c(u) > 2$, this dilated M_2 domain locally contains the* *given rigid Hartogs domain (globally if the inequality holds for all $u \in (0,\infty)$). If the inequality is reversed, then the dilated M_2 ball is locally inside the rigid Hartogs domain (globally).*

Proof. Once again, we can focus on $D \cap \mathbb{R}^2_{>0}$. Assume $\gamma_c(u) > 2$. Using the computation behind (4.5) with a slight tweak, integrating on the interval around $r(u)$ where $\gamma(r) > 2$. Then

$$1 + \frac{rf''(r)}{f'(r)} > 2 \iff \frac{f''(r)}{f'(r)} > \frac{1}{r},$$

which yields after integration and applying $x \mapsto e^x$

$$f'(r) > C_u r$$

for some constant $C_u = f'(r(u)) > 0$ depending only on u. Integrating again gives us

$$f(r) > \frac{C_u r^2}{2} + D_u.$$

Clearly, the RHS defines a vertical shift of a dilated M_2 domain, which osculates D to first order at the starting point. Then D is contained in this domain near this point. If $\gamma(r) > 2$ on $(0,1)$ we get global containment, and if $\gamma(r) < 2$ on some interval, the inequalities are reversed.

$\square$

Remark B.4. Here the case $z_1 = 0$ is exceptional since there is a 1-parameter family of vertically dilated and shifted M_2 domains that osculate a given $D \in \tilde{\mathcal{H}}$ to first order at a point $(0, z_2)$. The above computation gives a hyerplane lying under D regardless of $\gamma(0)$.

Bibliography

[1] Loredana Lanzani and Elias M. Stein. *The Cauchy Integral in $\mathbb{C}^n$ for domains with minimal smoothness.* Advances in Mathematics 264 (2004), 776–830.

[2] David E. Barrett and Loredana Lanzani. *The Leray transform on weighted boundary spaces for convex Reinhardt domains.* Journal of Functional Analysis 257 (9) (2009), 2780-2819.

[3] R. Michael Range. *Holomorphic Functions and Integral Representations in Several Complex Variables.* Springer Science+Business Media, LLC (1986).

[4] Mats Andersson, Mikael Passare and Ragnar Sigurdsson. *Complex Convexity and Analytic Functionals.* Progress in Mathematics 225, Springer Basel AG (2000).

[5] Milton Abramowitz and Irene Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* United States Department of Commerce, National Bureau of Standards (1964).

[6] Steve Zelditch. *Inverse spectral problem for analytic domains, II: $\mathbb{Z}_2$-symmetric domains.* Annals of Mathematics 170 (2009), 205–269.

[7] Barry Simon. *Advanced Complex Analysis: A Comprehensive Course in Analysis, Part 2B.* American Mathematical Society (2015).

[8] David E. Barrett and Luke D. Edholm. *High-frequency behavior of the Leray transform: Model hypersurfaces and projective duality*. Preprint, to appear.

[9] Charles Fefferman. *Parabolic invariant theory in complex analysis*. Advances in Mathematics 31 (1979), 131-262.