Managing students’ non-canonical approaches to solving equations: Linguistic resources teachers use to respond to students’ mathematical work

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Abstract: A historical review of mathematics curriculum suggests a long-standing canonical method to solving equations that teachers often see as “the” way to solve equations. In this paper, we examine data collected from a nationally-distributed sample of 524 secondary mathematics teachers who responded to a set of scenario-based survey items that represent the instructional situation of solving equations. The items featured scenarios in which students presented non-canonical solution methods and asked participants to share how they would respond to that situation. Using a framework that draws on systemic functional linguistics, we describe patterns in the various linguistic resources teachers use to manage the situation. While closed moves are frequently used to avoid discussion of non-canonical methods, our results suggest that teachers find ways to make regular use of: (1) closed moves when accommodating students’ non-canonical solution methods and (2) open moves when steering the conversation back to the canonical solution method.

Background and Framework

While policy documents have been crafted to provide numerous visions for mathematics instruction in the U.S. (NCTM, 1991, 2014)—such visions have yet to become a regular state of affairs in actual classrooms. This is nowhere less true than teachers’ instructional practices of responding to students’ mathematical contributions (Author, 2016) where they tend to be overly evaluative and propagate standard teaching routines—praising only those contributions that correctly carry out previously-demonstrated procedures while dismissing contributions that use alternative, but correct, methods (Ball, 1997; Crespo, 2002). Teachers’ stalwart commitment to a single procedure encourages students to learn rotely—undermining the development of conceptual understanding and flexible thinking (Hiebert & Carpenter, 1992).

In the case of solving equations in Algebra 1, a historical review of the mathematics curriculum suggests a long-standing canonical method (Author, 2015a) to solving equations that
teachers often see as “the” way to solve equations (Authors, 2019a). This method has been described by scholars as containing the following steps: (1) use the distributive property to clear out grouping symbols (when applicable), (2) simplify expressions on each side of the equation, (3) use the addition and subtraction properties of equality to isolate the variable from the constants, and (4) use the multiplication and division properties of equality to solve for the unknown variable (Authors, 2015a; Star & Seifert, 2006).

While research has suggested many teachers prefer to spend class time on the canonical method (Author, 2019a), our knowledge of classrooms helps us to understand that, regardless of their preferences, teachers sometimes have to make on-the-spot decisions about how to handle non-canonical solution methods offered by students. This study investigates the linguistic resources teachers use when responding to non-canonical solution methods in the instructional situation of solving equations: including those responses that manage to make use of students’ alternative contributions as well as those that do not. In this paper, we examine data collected from a nationally-distributed sample of 524 secondary mathematics teachers who responded to a set of scenario-based survey items that each featured an embedded, rich-media representation of the instructional situation of solving equations (Herbst, 2006). Within these items, teachers were asked to share how they would respond to scenarios in which a student presents a non-canonical method for solving an equation on the board.

**Theoretical Framework:** While teachers’ instructional decisions are commonly modelled based on individual characteristics, such as teachers’ knowledge, beliefs, and dispositions (Schoenfeld, 2010), other factors need to be taken into consideration. Phenomena such as *cultural scripts* (Hiebert & Stigler, 2000) and *lesson signatures* (Givvin et al., 2005) provides...
evidence that the norms of teaching can be distinguished across cultural lines, which suggests that teaching is as much a socially-defined activity as it is individual.

The theory of practical rationality (Herbst & Chazan, 2012) accounts for teachers’ decision making using both individual and social resources. It does this using the two primary building blocks of (1) Brousseau’s (1997) notion of didactical contract, and (2) Herbst’s (2006) notion of instructional situation. Brousseau’s concept of didactical contract identifies relationships between the teacher, her students, and the content in ways that tacitly regulate the ways that the teacher and students are expected to act within instructional exchanges (Herbst, 2003). Herbst’s notion of instructional situation takes note of the way the didactical contract is shaped within the set of recurring situations within a course of study. For example, the theory posits the set of norms for solving equations in algebra differs from the set of norms for doing proof in geometry and these differences impact both the teachers’ and students’ understanding of what kind of work is necessary for the teacher to claim the student has learnt what is expected of them (Herbst, 2006; Herbst & Chazan, 2012). In this way, the normative and routine nature of these instructional situations create a stable social resource that can be used by teachers and students to know how to act within a given situation.

In the case of the instructional situation of solving equations, the canonical method plays a crucial role in defining the norms of the situation. As such, it is normative for the teacher to devote class time to supporting students in learning the canonical method; and non-normative for teachers to spend time exploring alternative solutions. To be clear, the norms of the situation are not deterministic, even for teachers with strong preferences for the canonical method. For example, when faced with the situation of having a shy student at the board presenting a non-canonical method, a teacher who might normally feel quite strongly about adhering to the situational norms may respond in ways that accommodate the student’s work to avoid
embarrassing the student. Teachers have resources they can use to navigate such situations. For example, at least some portion of the reform literature has aimed to delineate specific linguistic resources teachers can use to shift their practices of responding to supporting less normative practices (e.g., O’Connor & Michaels, 2019).

**Research Questions:** In prior work, we have shown that when confronted with non-canonical student solutions in the instructional situation of solving equations, teachers’ responses can be parsed into one of three categories—those responses where the teacher: a) complies with the norm by finding a way to move quickly back to the canonical method, b) repairs the task by finding a way to make slight accommodations for a non-canonical solution such as ensuring each step of the students’ process was justified before moving on, and, c) repairs the situation by making large accommodations for a non-canonical solution such as switching the focus of the lesson towards that solution. When making slight or large accommodations we mean, for example, the teachers might accept for discussion solution methods involving mathematical steps that create equivalent equations even if they don’t follow the canonical method, or they might go further by shifting to a discussion of a method that helps determine the answer that makes a true mathematical statement while not using typical equation solving techniques (e.g. graphical, tabular, guess and check, or other symbolic manipulation not fully endorsed as “algebraic”). In this paper, we ask: what are the various linguistic resources teachers use to: (a) comply with the norms of that situation?, (b) repair the task?, and (c) repair the situation?

**Methodology and Data Sources**

**Participants:** Data used in this paper come from a nationally-distributed sample of 524 secondary mathematics teachers from 47 states who were invited by email and received an honorarium for participation. The sample teachers included 59.6% female, 40.1% male, and 0.36% other or no answer; 83.58% White, 7.3% Black, 2% Hispanic, 2.8% Asian, 0.89% Other.
They had an average of 14.32 years of experience ($SD=8.68$) ranging from 1 to 40 years.

Participants were invited to partake in a total of 27 open-ended scenario-based instruments—one of which, the Algebra-Equations Decision Instrument, we focus on here.

**Instrument:** As part of their participation in the Algebra-Equations Decision Instrument, each participant was provided with four rich-media, scenario-based items; each containing a classroom scenario that played out across several storyboard frames. Such multimedia representations have been found effective at gauging participant teachers’ decision-making (Authors, 2015b). Each scenario begins with a teacher posing a solving equations task and includes a moment in which a student is called to the board to share and the student subsequently describes a method. In all cases, the students’ solution was both mathematically correct and non-canonical (Authors, 2019b). For example, in one set of slides, the teacher poses $27x-36= 9(x+5)$, and a student volunteer approaches the board to share their solution where they first divide the entire equation by 9 to get $3x-4 = x + 5$. After viewing each scenario, study participants were asked to respond to the following open-ended prompt: “Please describe the action you would do next and your reasons for doing this action”. Participants’ open-ended responses are the focus of our analysis for this paper.

**Data Corpus & Analytical Method:** In total, the corpus contains 2,087 participant responses: some included a single “next action” ($n=1,530$), while others included no action ($n=94$) or a more detailed sequence of moves ($n=463$). Among the single actions responses, some avoided addressing the students’ method ($n=251$). The present analysis focused on those responses that managed to address the students’ method with a single “next action” ($n=1,279$) and proceeded in two parallel phases. In phase one, we coded responses according to the degree the participant indicated they would direct the class towards the canonical method or towards the offered non-canonical method provided by the student in each scenario.
In phase two, we used a previously-established coding scheme that augments a framework developed by teachers (Authors, 2020) with functional classifications drawn from a linguistic framework (Eggins & Slade, 2005). The framework comprises two sets of nested functional categories which organize responding moves according to how they shape the discourse. The first functional categorization (open/close) distinguishes between moves that prolong or curtail the discussion of the prior contribution; the second distinguishes between moves that demonstrate a willingness to accept the contribution (support, confront) or defers responsibility for responding by asking other students’ to react to the contribution (invite). Altogether, these functional categories produce the following six functional categories: curtail the interaction by supporting the contribution (close-support), curtail the interaction by confronting the contribution (close-confront), defer responsibility for responding by suggesting other students curtail the interaction (close-invite), extend the interaction by supporting the contribution (open-support) extend the interaction by confronting the contribution (open-confront) and defer responsibility for responding by suggesting other students prolong the interaction (open-invite). Details about the first and second phases of the coding can be found in our earlier work (Authors, 2019b & Authors, 2020), but will also be illustrated with examples in the results section. After both phases of coding were complete, we examined patterns in the frequency of overlap of codes to help answer the research questions.

Analysis & Results

From the 1,279 responses we coded, 599 (47%) contained descriptions of actions that comply with the norms of the situation—finding ways to move quickly back to the canonical

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1 In this paper, we elect to use open and close rather than aligning with Eggins and Slade’s (2004) words rejoinder and response, respectively. We do this to avoid confusion that could result from the use of the word “response”—which already has such a prevalent use in the mathematics education literature that is not synonymous with Eggins and Slade’s use.
method. Of these 599 responses, the majority (n=404, 67%) represent actions that fit into the linguistic category of close-confront. Some of these close-confront responses took on the form of a tell (e.g., *I would work through it using another method that is more routine*) while others took the form of a negative evaluation (e.g., *Since the bell rang I would make a note to bring up the same problem next class period and start off by solving it the right way -- meaning the way the students were used to*). Still others took a softer form, soliciting the class for a different solution (e.g., *I would ask if anyone in the class solved the problem a different way so that we could discuss the more traditional method*). Of course, close-confront moves are not the only way that teachers can manage to comply (see Figure 2).

<table>
<thead>
<tr>
<th>Linguistic Function</th>
<th>Participant Response</th>
<th>% of responses</th>
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</thead>
<tbody>
<tr>
<td>Close-Support</td>
<td><em>I would explain while that works there’s a much simpler way to solve the equation.</em></td>
<td>17%</td>
</tr>
<tr>
<td>Close-Invite</td>
<td><em>Have someone else share their method and show how it shows the same thing as what orange just did</em></td>
<td>5%</td>
</tr>
<tr>
<td>Open-Support</td>
<td><em>I would ask Blue: ‘why did you not divide (x+5) by 9 also on the right side?’</em></td>
<td>5%</td>
</tr>
</tbody>
</table>

*Figure 2. Examples of responses distinct from closed-confront that teachers used to comply with the situation*

From the 1,279 total responses, 430 (34%) contained descriptions of actions that represent mild breaches of the norms of the instructional situation (repair the task)—providing some slight accommodations for the student’s non-canonical solution. Nearly a third of those 430 responses (n=156, 36%) fit into the linguistic category of open-support. Some represented the teacher asking the student to clarify or justify aspects of the student’s non-canonical solution (e.g., *Have the student explaining reiterate the step and make sure the class understands*) while others represented the teacher resolving the uncertainty in the room by some reassurance about
the mathematical appropriateness of the method (e.g., *I would explain that as long as the same action is performed to each side of the equation that method is valid*). That said, teachers sometimes found other ways, beyond *open-support* moves, to *repair the task* (see Figure 3).

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<tr>
<td>Open-Invite</td>
<td>It's not clear what 'dividing everything by 9' means so prompt students to ask questions of the student.</td>
<td>27%</td>
</tr>
<tr>
<td>Close-Support</td>
<td>Go over how each term changes when you divide it by 9.</td>
<td>22%</td>
</tr>
<tr>
<td>Close-Invite</td>
<td>I would ask the students for homework to write down whether or not they thought the solution on the board was correct and if they could get the same solution algebraically.</td>
<td>7%</td>
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</tbody>
</table>

*Figure 3.* Examples of responses distinct from *open-support* that teachers used to *repair the task*

From the 1,279 responses, the remaining 250 (20%) responses contained descriptions of actions containing larger breaches of the instructional situation (*Repair the Situation*)—making large accommodations for the student’s non-canonical solution. Nearly half (n=135, 54%) of those 250 responses fit into the linguistic category of *close-invite*. Some of these responses represented the teacher asking other students to evaluate the contribution (e.g., *I would ask the students to discuss at their tables what was on the board and see if they agree or disagree with what is on the board*) while others represented the teacher requesting other students or the class take up the strategy on another problem (e.g., *I would give them another problem similar to the one Orange did and see if they can duplicate the process*). Again, not all of the responses describing actions that breach the situation were categorized as *close-invite* (see Figure 4).

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I would have students discuss in pairs what they think Blue did. 20%

I would answer the students questions about why certain procedures were done in the problem. 14%

I would ask the student (with help from the class) to justify using mathematical properties or concepts each step. 7%

Figure 4. Examples of responses distinct from close-invite that teachers used to repair the situation

In this section, we have shown that the modal teacher response to students’ non-canonical methods comply with norms of the situation (47%) and the preponderance of those responses take up the form of moves that belong to the close-confront functional category (67%). We have also shown that teachers sometimes elect to make small accommodations for students’ non-canonical methods (repair the task, 34%), and when they manage to do so they tend to use moves that belong to the open-support functional category (36%). That said, nearly half of the repair the task responses were accomplished with moves that fall into the two other functional categories of open-invite (27%) and close-support (22%). Finally, in 20% of the responses, we see teachers make sweeping accommodations for students’ non-canonical methods with a repair the situation; and in the majority of those responses, teachers elect to use moves that belong to the close-invite functional category (54%).

Discussion, Conclusion and Significance

Despite the calls of reformers for teachers to embrace the open discussion of students’ solutions, irrespective of whether they are canonical or not, research has routinely reported that teachers elect to dismiss non-canonical methods in favor of the canonical one. The theory of instructional situations and practical rationality has suggested teachers are often operating in contexts in which they feel responsible for maintaining the norms of the situation, which favors the canonical method. That said, we see in this data some promise in that a small majority of
teachers’ responses (54%) deviate from the norms of the situation by making some kind of accommodations for students’ non-canonical methods. Yet, teachers’ willingness to use open and/or supportive moves is mostly restricted to those instances when they are making only slight accommodations of students’ non-canonical methods (repair the task). In contrast, when a teacher takes the risk of making a significant accommodation for a students’ non-canonical method (repair the situation), they tend to use closed moves—albeit they often elect to use closed invitations. Yet even the invitational nature of these more accommodating moves allow the teacher to maintain some semblance of control of the situation by sanctioning a narrow platform from which students can react to the non-canonical solution presented (e.g., requesting students evaluate, add on to, or replicate the method). These results support our prior hypotheses (Author, 2009) that even when teachers are willing to engage with students’ non-canonical methods, there are important tensions in doing that.

In closing, one of the primary ways that reformers have sought to combat teachers’ problematic patterns of responding is with the prescription of alternative discursive moves that encourage teachers to use more open or invitational moves when responding to students’ work. The results from the second and third research question casts at least some suspicion on the efficacy of such prescriptions. These results suggest that teachers can and do find ways to: (1) make regular use of closed moves to repair the situation—in which they in some serious way abandon the canonical solution method and (2) make regular use of open moves to repair the task—often steering the conversation back to the canonical solution method. These findings are reminiscent of earlier work in the field that looked critically at reform recommendations (Cohen, 1990; Chazan & Ball, 1999). In closing, we suggest teachers need more than prescriptions for action and propose that more work is needed to understand the rationality of practice.

References


