Presented at Dynamic Walking on July 15th 2021

Constant speed gaits should work across all speeds

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I. Introduction

It is well-known that swimming in high-viscosity fluids and slithering on granular media are "*Stokesian*". In Stokesian systems, the body frame $g \in \mathcal{G}$ (where $\mathcal{G}$ is some matrix Lie group) and body motion is governed by an equation of the form $g^{-1}\dot{g} = A(r)\dot{r}$ where $r \in \mathcal{Q}$ is the instantaneous shape of the swimmer. An immediate and important corollary of this fact is that the motion is "*geometric*": it depends only on the shape of the motion but not its pace. It is in this sense that we use the word "geometric" here. Under geometric motion, going through a shape change twice as fast (e.g. walking with twice the stride frequency) moves the body through the world along the same trajectory twice as fast.

In high-viscosity swimming and slithering the linear velocity dependence of Stokes drag exactly yields geometric motion. Here we demonstrate that velocity *independent* static and dynamic Coulomb friction also yield geometric motion under certain conditions. This formulation provides a common mathematical framework to examine swimming, slithering, and walking locomotion.

II. Model

Assume legs whose point of contact moves at $\dot{q}_k$ w.r.t. to the body frame, and whose point of contact is at q_k w.r.t. to the body frame. The points of contact in the world frame are therefore $p_k = gq_k$. They are moving with speed $\dot{p}_k = \dot{g}q_k + g\dot{q}_k$. Let us consider various types of friction interaction between the legs and the environment; the only assumption we make is that the same kind of friction applies to all the legs. Under that assumption, consider a speed change of the form $q'_k(t) = q_k(\tau(t))$. Define $\alpha(t) := \frac{d}{dt}\tau(t)$, giving $\dot{q}'_k = \alpha\dot{q}_k$.

A. Viscous friction

For viscous friction, each leg generates a force $F_k = -C_{D,k}(T,r)\dot{p}_k$ for $C_{D,k}(T,r)$ the drag matrix of leg k when the body is at location T and has shape r (note: the q_k are assumed to be determined by r). The choice of $\dot{g}' = \alpha\dot{g}$ would produce $\dot{p}'_k = \alpha\dot{p}_k$, leading to $F'_k = \alpha F_k$, showing that a geometric solution will satisfy the equations.

Constant speed motion requires force balance and moment balance. Both those equations are homogeneous in F_k, and so hold for F'_k since they held for F_k. We conclude that scaling up the body speed maintains force balance.

For a constant α the forces increase by a factor α just as the duration of a cycle decreases by factor α, giving the same net impulse at all values of α.

B. Dynamic Coulomb Friction

Assume instead that all legs generate dynamic Coulomb friction forces $F_k = -\mu_k N_k \dot{p}_k \|\dot{p}_k\|^{-1}$ where μ_k is the dynamic friction coefficient, and N_k is the normal force on that leg. Consider a speed change that maintains $N'_k(t) := N_k(\tau(t))$. The choice of $\dot{g}' = \alpha\dot{g}$ would produce $\dot{p}'_k = \alpha\dot{p}_k$ and would therefore create the same friction forces $F'_k = F_k$. Thus, if previously the system satisfied some equations w.r.t T, r, F_k, it still satisfies the same equations. If it was in force balance before, it still is now too, so any constant velocity motion achievable before is still achievable in a geometrically identical way. If a given net force and torque on the CoM was produced before, the same one is produced now too, except the duration it acts will be reduced by a factor of α. For dynamic Coulomb friction the invariance is very strong – **any** equation that depends only on T, r, F_k would still hold.

C. Static Coulomb Friction

Last, but not least, we have the case of purely static friction. In the mechanics literature this is sometimes referred to as a kinematic system performing piece-wise holonomic motion. In that case, $\dot{p}_k = 0$ for all k. The scaling $\dot{g}' = \alpha\dot{g}$ will maintain that equality. Thus if a constant velocity motion is possible, it is possible in a geometrically similar form at all speeds that can actually be produced.

III. Mixed Friction Types

The Stokesian property just described can break down if multiple kinds of friction participate in producing the motion, however that can easily be avoided by choosing an appropriate policy. Consider mixtures of static and dynamic friction or static and viscous friction. If the legs that are under static friction are not used for propulsion at all – only for weight bearing – then they do not contribute to horizontal plane force balance and moment balance. In that case, the horizontal motion would still be Stokesian: some legs would be static with respect to the environment but produce no propulsion, whereas the legs that slip with respect to the world frame will produce propulsion through a single kind of friction and therefore do so in a Stokesian way.

IV. Conclusion

A consequence of this analysis is that regardless of the type of friction involved, it should be possible to take any constant speed periodic gait discovered while moving slowly, and then utilise that gait at any speed. In essence, constant speed gaits always work the same way at all speeds.