Duality and Hardy Spaces on Levi-Flat Domains with Corners

by

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DEDICATION

To my parents, who are primarily responsible for the innumerable opportunies I have been fortunate enough to have. I would also like to thank my grandparents, for giving my parents the same. And of course my siblings, Emma and Noah for their constant friendship and support.

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TABLE OF CONTENTS

LIST OF FIGURES

Figure No.

NOTATION

$\mathbf{z}, [\mathbf{z}]$	Homogeneous coordinates on $\mathbb{P}^n$.
$\mathbf{w}, [\mathbf{w}]$	Homogeneous coordinates on the dual $(\mathbb{P}^n)^*$, identified with the hyperplane $\mathbf{w} \cdot \mathbf{z} = \mathbf{0}$.
$\vec{\boldsymbol{z}}, \vec{\boldsymbol{w}}$	Affine coordinates $\left(\frac{z_i}{z_0}\right)_{i=1}^n, \left(\frac{w_i}{-w_0}\right)_{i=1}^n$ respectively.
$\ell_{\mathbf{w}}, \ell_{\vec{\boldsymbol{w}}}$	Complex hyperplanes $\mathbf{w} \cdot \mathbf{z} = 0,\ \vec{\boldsymbol{z}} \cdot \vec{\boldsymbol{w}} = 1$ respectively.
$\mathbf{e}_j$	Homogeneous coordinates corresponding to the j^{th} standard basis vector in $\mathbb{C}^{n+1}$.
$\vec{\boldsymbol{e}}_j$	The j^{th} standard basis vector in $\mathbb{C}^n$.
I	The incidence locus, corresponding to $\{\mathbf{z} \cdot \mathbf{w} = 0\} \subset \mathbb{P}^n \times (\mathbb{P}^n)^*$.
ω	The Universal Cauchy-Fantappié-Leray form.
Ω	A pseudoconvex domain in $\mathbb{P}^n$.
S	The boundary of Ω.
S_{sm}	The smooth locus of S.
S_{sing}	The singular locus of$S$.
S_n	The skeleton of Ω.
Ω^*	The dual of Ω, usually assumed to have non-empty interior.
S^*	The boundary of Ω^*.
I_S	The incidence manifold corresponding to $S, I \cap (S \times S^*)$.
$\mathcal{O}(\Omega)$	For Ω open, the space of functions holomorphic on Ω. For Ω closed, the space of functions holomorphic in an open neighborhood of Ω.
$d\vec{\boldsymbol{z}}_j, d\vec{\boldsymbol{w}}_j$	$\prod_{k=1,k\neq j}^{n} dz_j, \quad \prod_{k=1,k\neq j}^{n} dw_j.$
$\det_j(\mathbf{W})$	For an $n \times (n+1)$ matrix $\mathbf{W}$, the determinant of $\mathbf{W}$ with the j^{th} column

	removed.
T^{-t}, M^{-t}	For a projective transformation T, with associated lift M, the map and the lift, respectively, on the dual space.
$T_{\mathbf{z}}(H)$	The tangent space to H at $\mathbf{z}$.
J	The map induced on tangent vectors by multiplication by i.
$T_{\mathbf{z}}(H) \cap JT_{\mathbf{z}}(H)$	The maximal complex subspace of the tangent space at $\mathbf{z}$.
$TC_{\mathbf{z}}(E)$	The tangent cone of a set E at $\mathbf{z}$.
Δ^m	The m-simplex $\{\vec{\boldsymbol{t}} \in \mathbb{R}^m \mid \sum t_i = 1, t_i \geq 0\}$.
$\mathbb{D}^n$	The polydisk $\{\vec{\boldsymbol{z}} \in \mathbb{C}^n \mid \lvert z_i\rvert < 1\}$.
$b\mathbb{D}^n, S^n$	The distinguished boundary of the bidisk$\{\vec{\boldsymbol{z}} \in \mathbb{C}^n \mid \lvert z_i\rvert = 1\}$, written as the latter when emphasizing the group structure.
$\lvert\lvert \cdot \rvert\rvert_\infty$	The norm $\lvert\lvert\vec{\boldsymbol{z}}\rvert\rvert_\infty := \max_i \lvert z_i\rvert$.
d	The total derivative operator.
∂	The differential operator $\partial\rho := \sum_j \frac{\partial\rho}{\partial z_j} dz_j$when applied to function. The topological boundary when applied to a set.
β	The multivariate β-function.

ABSTRACT

In this thesis we develop a Hardy space theory for piece-wise smooth Levi-flat domains. We use the Cauchy-Fantappié pairing to produce projectively invariant reproducing kernels for such domains, along with families of measures to define Hardy spaces on the original domain and its dual. We study the qualitative properties of the pairing, and show that many of the properties from the smooth, strongly $\mathbb{C}$-convex case continue to hold. However, the maps between Hardy spaces are in general not isomorphisms, but are injective with dense image. Lastly, we demonstrate how to construct projectively invariant Hardy spaces in two variables.

CHAPTER I

Introduction

We begin with a discussion of Hardy spaces and duality in one variable as a primer for the theory in several variables. The reader unfamiliar with the notions of Hardy spaces, $\mathbb{C}$-convexity or our treatment of line bundles may want to consult Chapter II first. After summarizing some of the known results, we give an exposition of the new results contained within this thesis.

Suppose we have bounded domain $\Omega \subset \hat{\mathbb{C}}$ containing the origin whose boundary is a simple closed curve γ, oriented as the boundary of Ω. Then we have a Cauchy kernel which reproduces holomorphic functions on the interior of γ, namely

$$\mathscr{C}_+(f)(\tau) := \frac{1}{2\pi i}\int_\gamma f(z)\frac{1}{z-\tau}dz$$
$$\mathscr{C}_+(f) = f \text{ for } f \in \mathcal{O}(\Omega) \cap L^2(\gamma, |dz|).$$

See section 2.2 for full definitions. Let Ω^* denote the interior of the complement $\hat{\mathbb{C}}\backslash\Omega$, and set $w = \frac{1}{z}$, so that w is a holomorphic coordinate for Ω^*. Note that $\Omega^{**} = \Omega$, partially justifying calling Ω^* the *dual* of Ω. We re-write the Cauchy kernel slightly to highlight all the necessary ingredients for the generalization to several variables.

$$\mathscr{C}_+(f)(\tau) = \frac{1}{2\pi i}\int_\gamma f(z)\frac{1}{1-w\tau}\frac{dz}{z}$$

We have a linearly invariant 1-form $\frac{dz}{z}$ and a family of rational functions, holomorphic on Ω^*, namely $g_\tau(w) := \frac{1}{1-w\tau}, \tau \in \Omega$ which can be integrated over $\partial\Omega$ to reproduce the values of holomorphic f with L^2-boundary values. Written this way, we obtain a corresponding Cauchy kernel with a reproducing property for holomorphic functions $g(w)$ on the dual Ω^* (the minus sign coming from the orientation of γ)

$$\mathscr{C}_- g(\tau) = \frac{-1}{2\pi i}\int_\gamma g(w)\frac{1}{1-z\tau}\frac{dw}{w}$$
$$\mathscr{C}_- g = g \text{ for } g \in \mathcal{O}(\Omega) \cap L^2(\gamma, |dw|).$$

Put another way, given a bounded domain Ω containing the origin, we have two Hardy spaces

$$H^2(\Omega) := \mathcal{O}(\Omega) \cap L^2(\gamma, |dz|), H^2(\Omega^*) := \mathcal{O}(\Omega^*) \cap L^2(\gamma, |dw|),$$

a linearly invariant kernel

$$\frac{1}{2\pi i}\frac{dz}{z},$$

a pairing,

$$\langle\langle , \rangle\rangle : H^2(\Omega, |dz|) \times H^2(\Omega^*, |dw|) \to \mathbb{C}$$
$$\langle\langle f, g\rangle\rangle := \int_\gamma f \cdot g \frac{1}{2\pi i}\frac{dz}{z}$$

and two simple families of rational functions, parameterized by each domain, holomorphic on the dual, which reproduce holomorphic functions on each domain. Each family is given by

$$g_\tau(\cdot) = \frac{1}{1 - \langle \tau, \cdot \rangle}$$

where $\tau \in \Omega$ or $\tau \in \Omega^*$.

Now let us relax the somewhat restrictive conditions that Ω be bounded containing the origin. For any domain $\Omega \subset \hat{\mathbb{C}}$ with smooth boundary, a reproducing pairing should still exist, albeit slightly modified. Let $\mathbf{z} = [z_0 : z_1]$ be homogeneous coordinates on $\hat{\mathbb{C}}$. For each $\mathbf{z}$, there is a unique $\mathbf{w} = [w_0 : w_1]$ such that $\mathbf{w} \cdot \mathbf{z} = w_0 z_0 + z_1 w_1 = 0$, so that $\mathbf{w}$ is a function of $\mathbf{z}$ and vice versa. Written in the affine coordinates $z = \frac{z_1}{z_0}, w = \frac{w_1}{-w_0}$, we get $w \cdot z = 1$, i.e. $w = \frac{1}{z}$. Define $\Omega^* := \{\mathbf{w} | \mathbf{z}(\mathbf{w}) \notin \Omega\}$, and note that $\Omega^* \subset (\mathbb{P}^1)^*$ and is closed, and $\Omega^{**} = \Omega$. Define the $\mathcal{O}_{\mathbb{P}^1}(1,0) \otimes \mathcal{O}_{(\mathbb{P}^1)^*}(1,0)$-valued 1-form $\omega = \frac{z_0 w_0}{(2\pi i)} w\, dz$. One can verify directly that ω is projectively invariant, and thus given sections $f \in O_{\partial\Omega}(-1,0), g \in O_{\partial\Omega^*}(-1,0)$, we have a well-defined projectively invariant pairing

$$\langle\langle f, g\rangle\rangle = \int_{\partial\Omega} f \cdot g\, \omega.$$

Furthermore, if we set $g_{\boldsymbol{\tau}}(\cdot) = \frac{1}{\langle \boldsymbol{\tau}, \cdot \rangle}$, we have the reproducing properties

$$f(\boldsymbol{\tau}) = \int_{\partial\Omega} f \cdot g_{\boldsymbol{\tau}}\, \omega \quad \boldsymbol{\tau} \in \Omega,\ f \in \mathcal{O}_{\overline{\Omega}}(-1,0)$$
$$g(\boldsymbol{\tau}) = \int_{\partial\Omega} g_{\boldsymbol{\tau}} \cdot g\, \omega \quad \boldsymbol{\tau} \in \Omega^*,\ g \in \mathcal{O}_{\Omega^*}(-1,0)$$

These identities are simply the Cauchy formula when written in the affine coordinates $z_0 = 1, w_0 = -1$. The expressions $|z_0^2\, dz|$ and $|w_0^2\, dw|$ define invariant $O(1,1)$-valued 1-forms, giving rise to the norms

$$||f||^2 := \int_{\partial\Omega} |f|^2 |z_0^2 \, dz| \quad f \in O_{\partial\Omega}(-1,0)$$
$$||g||^2 := \int_{\partial\Omega^*} |g|^2 |w_0^2 \, dw| \quad g \in O_{\partial\Omega^*}(-1,0)$$

Using these norms to define the corresponding L^2 and Hardy spaces, we get a *duality pairing* between $H^2(\Omega, |z_0^2 \, dz|)$ and $H^2(\Omega^*, |w_0^2 \, dw|)$, that is

$$\inf_{g \in H^2(\Omega^*), ||g||=1} \sup_{f \in H^2(\Omega^*), ||f|| \leq 1} |\langle\langle f, g \rangle\rangle| = \inf_{f \in H^2(\Omega), ||f||=1} \sup_{g \in H^2(\Omega^*), ||g|| \leq 1} |\langle\langle f, g \rangle\rangle| > 0$$

In particular, the open mapping theorem implies that the pairing induces an isomorphisms $H^2(\Omega, |z_0^2 dz|)^* \cong H^2(\Omega^*, |w_0^2 dw|), H^2(\Omega^*, |w_0^2 dw|)^* \cong H^2(\Omega, |z_0^2 dz|)$ (see page 4 of [Zim90]). The proof requires two elementary identities

$$\langle\langle \mathscr{C}_+ f, g \rangle\rangle = \langle\langle f, \mathscr{C}_- g \rangle\rangle, f \in L^2(\gamma, |z_0^2 dz|), g \in L^2(\gamma, |w_0^2 dw|)$$
$$|\frac{dz}{z^2}| = |dw| \Rightarrow |\omega| = \sqrt{|z_0^2 dz|}\sqrt{|w_0^2 dw|}.$$

Applying Cauchy-Schwartz to the following point-wise identity,

$$|fg\omega| = |f|\sqrt{|z_0^2 dz|}|g|\sqrt{|w_0^2 dw|}$$
$$\Rightarrow |\langle\langle f, g \rangle\rangle| \leq ||f|| \cdot ||g||$$

with equality when $g = \overline{f}$ on γ. Finally, we need the boundedness of the Cauchy operator on $L^2(\gamma, |dz|)$. For γ Lipschitz, this is a classical result established by Coifman, McIntosh, and Meyer [CMM82] building on the foundational work of Calderón [Cal77]. For an accessible treatment in the case of analytic curves, see Theorem 4.1 of [Bel16]. Following [Bar15], we have for

$f \in H^2(\Omega, |z) -^2 dz|), ||f|| = 1$

$$
\begin{aligned}
1 = ||f|| &= \max_{g\in L^2(\gamma,|dw|),||g||\leq 1} |\langle\langle f, g\rangle\rangle| \\
&= \max_{g\in L^2(\gamma,|dw|),||g||\leq 1} |\langle\langle \mathscr{C}_+ f, g\rangle\rangle| \\
&= \max_{g\in L^2(\gamma,|dw|),||g||\leq 1} |\langle\langle f, \mathscr{C}_- g\rangle\rangle| \\
&\leq \sup_{g\in H^2(\Omega^c),||g||\leq ||\mathscr{C}_|} |\langle\langle f, g\rangle\rangle| \\
&\leq ||\mathscr{C}_-|| \sup_{g\in H^2(\Omega^c),||g||\leq 1} |\langle\langle f, g\rangle\rangle|.
\end{aligned}
$$

We have re-framed aspects of the one variable theory in such a way that we can now analogously describe the theory in several variables. Let $\mathbb{P}^n$ have homogeneous coordinates $\mathbf{z} = [z_0 : \ldots : z_n]$, and let $(\mathbb{P}^n)^*$ denote the set of hyperplanes in $\mathbb{P}^n$, with coordinates given by $\mathbf{w} \cdot \mathbf{z} = 0$. Define the incidence locus $I := \{\mathbf{w} \cdot \mathbf{z} = 0\} \subset \mathbb{P}^n \times (\mathbb{P}^n)^*$. On I, there is a unique (up to constants) projectively invariant $\mathcal{O}_{\mathbb{P}^n}(n, 0) \otimes \mathcal{O}_{(\mathbb{P}^n)^*}(n, 0)$-valued $(2n-1)$ form ω, called the Universal Cauchy-Fantappié-Leray Form, or Universal CFL form, which will play the role of $\frac{dz}{z}$. Given a strongly $\mathbb{C}$-convex domain $\Omega \subset \mathbb{P}^n$ with C^2 boundary S, let Ω^* denote the set of complex hyperplanes external to Ω, called the *dual* of Ω. The boundary $S^* := \partial\Omega^*$ is C^2 and strongly $\mathbb{C}$-convex, and we have duality maps $S \to S^*$ and $S \to S^*$ given by mapping $\mathbf{z} \in S$ to the maximal complex subspace of S at $\mathbf{z}$, namely $T_{\mathbf{z}}(S) \cap JT_{\mathbf{z}}(S)$ and vice versa. From this description and the symmetry of the relation $\mathbf{z} \cdot \mathbf{w} = 0$, it follows that the duality maps are inverses of each other. Let $I_S := I \cap (S \times S^*)$ be the graph of the duality map and set $g_{\boldsymbol{\tau}}(\cdot) := \frac{1}{\langle \boldsymbol{\tau}, \cdot\rangle^n}$. There exists a unique separately continuous projectively invariant pairing defined by

$$
\begin{aligned}
&\langle\langle , \rangle\rangle : \mathcal{O}_\Omega(-n, 0) \times \mathcal{O}_{\Omega^*}(-n, 0) \to \mathbb{C} \\
&\langle\langle f, g\rangle\rangle := \int_{I_S} f \cdot g \, \omega
\end{aligned}
$$

with the reproducing properties.

$$
\begin{aligned}
\langle\langle f, g_{\boldsymbol{\tau}}\rangle\rangle &= f(\boldsymbol{\tau}) \quad f \in \mathcal{O}_\Omega(-n, 0),\ \boldsymbol{\tau} \in \Omega \\
\langle\langle g_{\boldsymbol{\tau}}, g\rangle\rangle &= g(\boldsymbol{\tau}) \quad g \in \mathcal{O}_{\Omega^*}(-n, 0),\ \boldsymbol{\tau} \in \Omega^*
\end{aligned}
$$

We shall refer to the functionals $\langle\langle \cdot, g_{\boldsymbol{\tau}}\rangle\rangle$ and $\langle\langle g_{\boldsymbol{\tau}}, \cdot\rangle\rangle$ as the *Leray Kernel*. The pairing exists for general open or closed $\mathbb{C}$-convex sets Ω, but we shall additionally assume that Ω^* is non-empty. By taking $\mathbf{w} \in int(\Omega^*)$ and moving the corresponding hyperplane $\ell_{\mathbf{w}}$ to infinity, this is equivalent to Ω being bounded in some affine coordinates.

By finding a positive $O_S(n,n)$-valued $2n-1$-form μ_S on S and the same on S^*, one can define projectively invariant L^2 and Hardy spaces using the norms

$$||f||_2^2 = \int_S |f|^2 \mu_S \quad f \in O_S(-n,0)$$
$$||g||_2^2 = \int_{S^*} |g|^2 \mu_{S^*} \quad g \in O_{S^*}(-n,0)$$

and then investigate how the norms of the corresponding Leray kernels relate to the projective geometry of the domain Ω. When Ω is C^2 and strongly $\mathbb{C}$-convex, a preferred choice of measure is set forth in [Bar15], as $\varphi_S^{\frac{-2n}{n+1}} \mu_{S,Fef}$, where φ_S is an explicitly defined projectively invariant function on S, and $\mu_{S,Fef}$ is the Fefferman form (see [KN99]) on S. With respect to this measure, the pairing $\langle\langle , \rangle\rangle$ is a duality pairing, meaning

$$\inf_{||g||=1} \sup_{||f|| \le 1} |\langle\langle f, g\rangle\rangle| = \inf_{||f||=1} \sup_{||g|| \le 1} |\langle\langle f, g\rangle\rangle| > 0.$$

This result relies on the boundedness of the Leray kernel (see [KS78] or Theorem 1 of [LS13]) and then following the same proof as in the one variable case.

In this thesis, we consider a class of $\mathbb{C}$-convex domains with Levi-flat, singular boundaries that we dub $\mathbb{C}$-polytopes. Proposition 3.2.7 gives an explicit description of the boundary of the dual domain S^*, which when combined with Theorem 2.4.9, permits the calculation of a reproducing kernel concentrated on the skeleton of Ω in Theorem 4.1.1. We propose a family of measures to define substitute Hardy spaces on Ω and Ω^*, and summarize the main properties of the corresponding pairing in Theorem 4.2.2. The pairing between these Hardy spaces has similar invariance and reproducing properties as in the smooth, strongly $\mathbb{C}$-convex case, but we show the pairing is in general not a duality pairing by demonstrate for a large class of examples that

$$\inf_{||g||=1} \sup_{||f|| \le 1} |\langle\langle f, g\rangle\rangle| = 0.$$

However, the induced maps between the Hardy spaces and the corresponding duals are not far from isomorphisms, namely they are injective with dense image. Lastly, we demonstrate how to build projectively invariant Hardy spaces on $\mathbb{C}$-polytopes, or more generally, piece-wise smooth pseudoconvex domains, when $n = 2$.

CHAPTER II

Background Material

First some notational conventions. Let $\mathbf{z}$ denote homogeneous coordinates on $\mathbb{P}^n$. The hyperplane at infinity will correspond to $z_0 = 0$, so standard affine coordinates are $\vec{\boldsymbol{z}} = (\frac{z_j}{z_0})$. Give the space of hyperplanes $(\mathbb{P}^n)^*$ homogeneous coordinates $\mathbf{w}$, identifying $\mathbf{w}$ with the hyperplane $\ell_{\mathbf{w}} := \{\mathbf{z}|\mathbf{w} \cdot \mathbf{z} = 0\}$. In the affine coordinates $\vec{\boldsymbol{w}} = (\frac{w_i}{-w_0})$, this amounts to identifying $\vec{\boldsymbol{w}}$ with $\ell_{\vec{\boldsymbol{w}}} := \{\vec{\boldsymbol{z}}|\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{z}} = 1\}$. In $(\mathbb{P}^n)^*$, the hyperplane at infinity $w_0 = 0$ corresponds to hyperplanes in $\mathbb{P}^n$ passing through the origin, all other hyperplanes are parameterized in these affine coordinates. We will interchangeably use $\langle , \rangle$ and $\cdot$ for the standard dot product, depending on which is easier to read in context.

We use the shorthand $d\vec{\boldsymbol{z}} := dz_1 \wedge \ldots \wedge dz_n$ and $d\vec{\boldsymbol{z}}_{[j]} := dz_1 \wedge \ldots \hat{dz_j} \ldots \wedge dz_n$, where the $\hat{}$ indicates omission.

Throughout it will be useful to have a notion of transversality that applies to the intersection of multiple hypersurfaces which we obtain from Shifrin [Shi]. Suppose we have k hypersurfaces $H_1, \ldots, H_k$ intersecting at a point $\vec{\boldsymbol{z}} \in \mathbb{C}^n$, and let $N_{\vec{\boldsymbol{z}}}(H_j)$ denote the stalk of the normal bundle at $\vec{\boldsymbol{z}}$ and let $\mathbb{C}N_{\vec{\boldsymbol{z}}}(H_j)$ denote the stalk of the complex normal bundle at $\vec{\boldsymbol{z}}$. The intersection is *transverse* if $\bigoplus_j N_{\vec{\boldsymbol{z}}}(H_j)$ has rank k. In this case, the intersection is a manifold of dimension $2n-k$. The intersection is *complex transverse* if $\bigoplus_j \mathbb{C}N_{\vec{\boldsymbol{z}}}(H_j)$ has complex rank k. We again conclude that the intersection is a manifold of dimension $2n - k$ whose maximal complex subspace at $\vec{\boldsymbol{z}}$ has dimension $2n - 2k$.

2.1 Projective Transformations and Line Bundles

Let $T : \mathbb{P}^n \to \mathbb{P}^n$ be a projective transformation. Choose an invertible matrix $(M_{i,j})_{i,j=0}^n$ with $\det(M) = 1$ which descends to T. Any two choices of M differ by a $n + 1$ root of unity, and for the formulas we care about the ambiguity will wash out. In the standard affine coordinates,

$$T(z_1, \ldots, z_n) = \frac{1}{M_{0,0} + \sum_{j=1}^n M_{0,j} z_j}(M_{1,0} + \sum_{j=1}^n M_{1,j} z_j, \ldots, M_{n,0} + \sum_{j=1}^n M_{n,j} z_j).$$

Given a set $E \subset \mathbb{P}^n$ such that $T(E) \subset \mathbb{C}^n$ and integers j, k, let $O_E(j,k)$ denote the space of continuous functions on the cone over E satisfying

$$F(\lambda z_0, \ldots, \lambda z_n) = \lambda^j \overline{\lambda^k} F(z_0, \ldots z_n).$$

Working in standard affine coordinates, we can identify F with the continuous function f on E, $f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n)$ and obtain the following transformation law

$$(T^* f)(\vec{z}) = (M_{0,0} + \sum_{i=1}^{n} M_{0,i} z_i)^j \overline{(M_{0,0} + \sum_{i=1}^{n} M_{0,i} z_i)^k} f(T(\vec{z})).$$

If E is open, $\mathcal{O}_E(j,0)$ will denote the space of sections over holomorphic on E, and if E is closed $\mathcal{O}_E(j,0)$ will denote the space of sections holomorphic in some neighborhood of E. With this convention, expressions like z_j can be thought of as sections of $\mathcal{O}_{\mathbb{P}^n}(1,0)$. Likewise, an expression $f d\vec{z}$ defined on E in the affine plane can be identified with the section F of $O_E(-n-1, 0)$, $F(z_0, \ldots, z_n) := z_0^{-n-1} f(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0})$. Given $F \in O_E(j,k)$, we have well-defined expressions $\overline{F} \in O_E(k,j), |F|^2 \in O_E(j+k, j+k)$, etc. More generally, if $G \in O_E(j', k')$ then $FG \in O_E(j+j', k+k')$. Lastly we remark that a section of $O_E(0,0)$ is nothing but a continuous function on E, and so an $O_E(0,0)$-valued k-form is just an ordinary k-form.

For T and M as above, there is a unique transformation $T^{-t} : (\mathbb{P}^n)^* \to (\mathbb{P}^n)^*$ preserving the incidence locus I, or so that $T\mathbf{z} \cdot T^{-t}\mathbf{w} = \mathbf{z} \cdot \mathbf{w}$. The matrix $M^{-t} := {}^t M^{-1}$ descends to T^{-t}, and the two operations commute, explaining the notation. We will need the notion of a projectively invariant $\mathcal{O}_{\mathbb{P}^n}(n,0) \otimes \mathcal{O}_{(\mathbb{P}^n)^*}(n,0)$-valued $(2n-1)$-form. This means we have an expression

$$f\ g\ \eta, \quad f \in \mathcal{O}_{\mathbb{P}^n}(n,0), g \in \mathcal{O}_{(\mathbb{P}^n)^*}(n,0), \eta \in \bigwedge^{2n-1} T^*(\mathbb{P}^n \times (\mathbb{P}^n)^*).$$

Pulling back such an expression means pulling back f via T, g via T^{-t} and η via (T, T^{-t}).

2.2 Hardy Spaces

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 strongly $\mathbb{C}$-convex boundary $S := \partial\Omega$. Given a non-negative measure μ equivalent to surface measure on S, define the L^2-space $L^2(S, \mu)$ using the norm

$$||f||_\mu^2 := \int_S |f|^2 d\mu.$$

We define the *Hardy space* $H^2(\Omega, \mu)$ to be the closure of $\mathcal{O}_{\overline{\Omega}}(0,0)$ with respect to this norm. Intuitively the Hardy space is the space of holomorphic functions on Ω with L^2-boundary values, and in one variable this is made precise by the fact that the restriction map $\mathcal{O}_{\overline{\Omega}}(0,0) \to L^2(S, |dz|)$ is injective, and given an element $f \in H^2(\Omega, |dz|)$, there is a holomorphic function on Ω which has radial limits agreeing a.e. with f. For more on equivalent definitions and properties of Hardy spaces, see Chapters 1-3 of [Dur70].

If instead of a measure μ we are given a positive $O_S(j,j)$-valued $2n-1$-form σ, we can define a norm on $O_S(-j,0)$ via

$$||f||_\mu^2 := \int_S |f|^2 \sigma.$$

The Hardy space is then the closure of $\mathcal{O}_{\overline{\Omega}}(-j,0)$ with respect to this norm. Note that the integrand is a $O_S(0,0)$-valued $(2n-1)$-form, i.e. this is simply an ordinary integral and there is no ambiguity when using a lift M of a projective transformation T to pull back the integrand.

2.3 Notions of Convexity and Duality

We now define notions of convexity and duality that are crucial to the theory in several variables. Let $\Omega \subset \mathbb{P}^n$ be an open domain. Ω is *pseudoconvex* if there exists a plurisubharmonic exhaustion, or equivalently if Ω is a *domain of holomorphy*. Without getting in too deep of a discussion here, it suffices to say that such domains have a long history in the subject. They are the natural domains for studying holomorphic function theory in several variables and the $\overline{\partial}$-equation. For a thorough treatment of the subject, see [Hö90].

Ω is $\mathbb{C}$-*convex* if the intersection of Ω with every projective line is either connected and simply connected or empty. Ω is $\mathbb{C}$-*linearly convex* if Ω^c is a union of complex hyperplanes. The *dual complement* or *dual* of Ω, denoted Ω^* is the set $\{\mathbf{w} | \ell_{\mathbf{w}} \cap \Omega = \emptyset\}$. Set $S := \partial\Omega$, then the *dual* of S is $S^* := \partial\Omega^*$. For $\mathbb{C}$-linearly convex Ω, $S^* = \{\mathbf{w} | \ell_{\mathbf{w}} \cap S \neq \emptyset,\ \ell_{\mathbf{w}} \cap \Omega = \emptyset\}$. It's not difficult to see that $\mathbb{C}$-convexity is a stronger notion than pseudoconvexity. $\mathbb{C}$-convexity is an important notion, as it is a necessary and sufficient condition to be able to solve linear, constant coefficient, holomorphic partial differential equations. See chapter 4 of [APS04].

This notation will be fixed throughout, that is Ω will always be a pseudoconvex domain in $\mathbb{P}^n$ with boundary S, and the dual will always be Ω^* with boundary S^*. Additionally, we will also always assume that the interior of Ω^* is non-empty. The geometric properties of Ω, S will vary based on the context.

That these are the right notions of complex projective convexity is partially justified by the following theorem.

Theorem 2.3.1. *Duality of $\mathbb{C}$-linearly Convex Domains ([APS04] Theorem 2.3.9)*
Let Ω be a $\mathbb{C}$-convex domain. Then Ω^* is $\mathbb{C}$-linearly convex, that is $\Omega^{**} = \Omega$.

The first statement of Theorem 2.3.1 can be thought of as a complex version of Hahn-Banach.

The second is a complex version of duality for the Legendre transform. Assuming certain boundary regularity, there are theorems in the other direction as well. See [APS04] Chapter II for more details.

2.4 Invariant Forms, Reproducing Kernels

The reproducing property of the pairing comes from the notion of a generating form which we now describe.

Definition 2.4.1. *Generating Form*
Let $\vec{\boldsymbol{\tau}} \in \Omega$ with S piece-wise smooth, and $\vec{\boldsymbol{w}}$ be a vector valued function in $C^1(S)$ satisfying $\langle \vec{\boldsymbol{w}}, \vec{\boldsymbol{z}} - \vec{\boldsymbol{\tau}} \rangle \neq 0$. The Generating Form *for $\vec{\boldsymbol{\tau}}$, $\omega_{\vec{\boldsymbol{\tau}}}$ is defined by*

$$\omega_{\vec{\boldsymbol{\tau}}}(\vec{\boldsymbol{z}} - \vec{\boldsymbol{\tau}}, \vec{\boldsymbol{w}}) = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{k=1}^{n} (-1)^k w_k d\vec{\boldsymbol{w}}_{[k]} \wedge d\vec{\boldsymbol{z}}}{\langle \vec{\boldsymbol{w}}, \vec{\boldsymbol{z}} - \vec{\boldsymbol{\tau}} \rangle^n}.$$

Theorem 2.4.2. *([AY83] Lemma 3.3 and Corollary 3.6)*
In the above definition, $\omega_{\vec{\boldsymbol{\tau}}}$ does not depend on choice of $\vec{\boldsymbol{w}}$ and we have the reproducing property

$$f(\vec{\boldsymbol{\tau}}) = \int_S f\omega_{\vec{\boldsymbol{\tau}}},\ f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega}).$$

A generating form is a recipe for constructing any number of reproducing kernels on S depending on choice of $\vec{\boldsymbol{w}}$. The choice of interest to us is set forth by Leray, where one uses the hyperplane tangent at $\vec{\boldsymbol{z}}$ to S to produce $\vec{\boldsymbol{w}}$. However, there are other interesting choices, and in fact one can replace S by any smooth cycle h homologous to $\vec{\boldsymbol{w}}(S)$ ([AY83] Corollary 3.7). We demonstrate how this technique is used to construct the Bergman-Weil kernel, as the same idea will be used later.

Given a domain of holomorphy $D \subset \mathbb{C}^n$, and $N > n$ holomorphic functions W_j on D, an *analytic polyhedron* Ω is a relatively compact finite union of connected components of $\{|W_j| < 1\}$ compactly contained in D. Given a strictly increasing tuple $\vec{\boldsymbol{j}} = (j_1, \dots, j_k)$, the *edge* $\sigma_{\vec{\boldsymbol{j}}} := \{|W_{j_l}| = 1, l = 1, \dots, k\} \cap \overline{\Omega}$. A *face* is an edge corresponding to a tuple of length 1. An analytic polyhedron is a *Weil domain* if all faces meets complex transversely. In particular, $\dim(\sigma_{\vec{\boldsymbol{j}}}) = 2n - k$ for $k \leq n$. The *skeleton* S_n is the set of n-dimensional edges.

Definition 2.4.3. *Bergman-Weil Kernel*
Let Ω be a Weil domain defined by $\{|W_j| < 1\}$, $f \in \mathcal{O}(D) \cap C(\overline{D})$. Define $P_{i,j}$ by Hefer's lemma ([AY83] Thm 25.2)

$$W_j(\vec{\boldsymbol{z}}) - W_j(\vec{\boldsymbol{\tau}}) = \sum_{i,j} (z_i - \tau_i) P_{i,j}(\vec{\boldsymbol{z}}, \vec{\boldsymbol{\tau}}).$$

The Bergman-Weil kernel is defined for $z \in \sigma_{\vec{\boldsymbol{j}}} \cap S_n$ as

$$\mathscr{B}_\Omega(\vec{\boldsymbol{z}}, \vec{\boldsymbol{\tau}}) = \frac{1}{(2\pi i)^n} \frac{\det\left(P_{i,\vec{\boldsymbol{j}}}\right)}{\prod_k (W_{j_k}(\vec{\boldsymbol{z}}) - W_{j_k}(\vec{\boldsymbol{\tau}}))} d\vec{\boldsymbol{z}}.$$

Theorem 2.4.4. *Bergman-Weil Representation ([AY83] Theorem 9.1)*
Let Ω be a Weil polyhedron, $f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$. Then we have the following reproducing formula.

$$f(\vec{\boldsymbol{\tau}}) = \int_{S_n} f(\vec{\boldsymbol{z}}) \mathscr{B}_\Omega(\vec{\boldsymbol{z}}, \vec{\boldsymbol{\tau}})$$

We give the proof, as similar proofs are given in Chapter 3. The idea is to define an appropriate cycle to integrate over in $\vec{\boldsymbol{w}}$-space using Theorem 2.4.2.

Lemma 2.4.5. *Let $\Delta^n(\vec{\boldsymbol{q}}_1, \ldots, \vec{\boldsymbol{q}}_n)$ denote the convex hull of $\vec{\boldsymbol{q}}_j \in \mathbb{C}^n$. Then*

$$\int_{\Delta^n(\vec{\boldsymbol{q}}_1,\ldots,\vec{\boldsymbol{q}}_n)} \sum_k (-1)^{k-1} w_k dw_{[j]} = \frac{1}{(n-1)!} \det(\vec{\boldsymbol{q}}_1, \ldots, \vec{\boldsymbol{q}}_n).$$

Proof. For a linear map A, we have the identity

$$A^*(\sum_k (-1)^{k-1} w_k \vec{\boldsymbol{w}}_{[k]}) = \det(A)(\sum_k (-1)^{k-1} w_k d\vec{\boldsymbol{w}}_{[k]}).$$

The is easily seen by checking it holds for generators of the linear group. For transpositions and scaling coordinates this is clear. It remains to check basic row and column operations. By the previous remarks, it suffices to check when A is the map $w_1' = w_1 + aw_2$, $w_j' = w_j, j \neq 1$, but a direct computation verifies the claim.
We have

$$\begin{aligned}\int_{\Delta^n(\vec{\boldsymbol{q}}_1,\ldots,\vec{\boldsymbol{q}}_n)} \sum_k (-1)^{k-1} w_k d\vec{\boldsymbol{w}}_{[k]} &= \det(\vec{\boldsymbol{q}}_1, \ldots, \vec{\boldsymbol{q}}_n) \int_{\Delta^n(\vec{e}_1,\ldots,\vec{e}_n)} \sum_{k=1}^n (-1)^{k-1} w_k d\vec{\boldsymbol{w}}_{[k]} \\ &= \frac{1}{(n-1)!} \det(\vec{\boldsymbol{q}}_1, \ldots, \vec{\boldsymbol{q}}_n).\end{aligned}$$

$\square$

Proof. Proof of Theorem 2.4.4
Fix $\vec{\boldsymbol{\tau}} \in \Omega$. On the face $S_j := \{|W_j| = 1\}$ of Ω, define the map

$$\vec{\boldsymbol{w}}^j = \left(\frac{P_{1,j}}{W_j(\vec{\boldsymbol{z}}) - W_j(\vec{\boldsymbol{\tau}})}, \ldots, \frac{P_{n,j}}{W_j(\vec{\boldsymbol{z}}) - W_j(\vec{\boldsymbol{\tau}})} \right).$$

Each $\vec{\boldsymbol{w}}^j$ satisfies $\langle \vec{\boldsymbol{w}}^j, \vec{\boldsymbol{z}} - \vec{\boldsymbol{\tau}} \rangle = 1$, whose image is a manifold in $\vec{\boldsymbol{w}}$-space lying over the face S_j. The idea is to stitch these manifolds together over each edge to get a cycle in $\vec{\boldsymbol{w}}$ space. Over a point $\vec{\boldsymbol{z}}$ in the edge $S_{\vec{\boldsymbol{j}}}$ let $\vec{\boldsymbol{w}}$ take values in $\Delta^n(\vec{\boldsymbol{w}}^{j_1}(\vec{\boldsymbol{z}}), \ldots, \vec{\boldsymbol{w}}^{j_n}(\vec{\boldsymbol{z}}))$. Taking the union of all these simplices gives an appropriate cycle in $\vec{\boldsymbol{w}}$-space, although note that $\vec{\boldsymbol{w}}$ is no longer a function of $\vec{\boldsymbol{z}} \in S$. The integral in Theorem 2.4.2 can now be evaluated using Lemma 2.4.5. The integral vanishes on edges that are of dimension $< n$, since the expression involves a wedge of n differentials in the $\vec{\boldsymbol{z}}$-variables, but these take values in a manifold of dimension $< n$. On edges that are dimension $2n - k > n$, note that the $\vec{\boldsymbol{w}}^j$ are holomorphic functions of $\vec{\boldsymbol{z}}$ and functions of $k - 1$ other independent parameters. So in all, $d\vec{\boldsymbol{w}}_{[k]} \wedge d\vec{\boldsymbol{z}}$ consists of at most $n + k - 1 < 2n - 1$ independent differentials, and thus must vanish. Applying theorem 2.4.2 and lemma 2.4.5,

$$\begin{aligned} f(\vec{\boldsymbol{\tau}}) &= \frac{(n-1)!}{(2\pi i)^n} \sum_{j_1 < j_2 < \ldots < j_m} \int_{S_{j_1,\ldots,j_m}} f(\vec{\boldsymbol{z}}) \left(\int_{\Delta^n(\vec{\boldsymbol{w}}^{j_1},\ldots,\vec{\boldsymbol{w}}^{j_m})} \sum_k (-1)^{k-1} w_k d\vec{\boldsymbol{w}}_{[k]} \right) d\vec{\boldsymbol{z}} \\ &= \frac{1}{(2\pi i)^n} \sum_{j_1 < j_2 < \ldots < j_n} \int_{S_{j_1,\ldots,j_n}} f(\vec{\boldsymbol{z}}) \frac{\det\left(P_{i,\vec{\boldsymbol{j}}}\right)}{\prod_k (W_{j_k}(\vec{\boldsymbol{z}}) - W_{j_k}(\vec{\boldsymbol{\tau}}))} d\vec{\boldsymbol{z}}. \end{aligned}$$

$\square$

Remark 2.4.6. *If $\Omega = \mathring{\gamma}_1 \times \ldots \times \mathring{\gamma}_n$ where each γ_i is a simple analytic closed curve in $\mathbb{C}$, then we may take $W_j(\vec{\boldsymbol{z}}) = W_j(z_j)$. For the decomposition from Hefer's lemma, we have*

$$W_j(z_j) - W_j(\tau_j) = (z_j - \tau_j) P_{j,j}$$

implying

$$\frac{P_{j,j}}{W_j(z_j) - W_j(\tau_j)} = \frac{1}{z_j - \tau_j}.$$

It follows that for product domains the Bergman-Weil kernel is just the multivariate Cauchy kernel.

We now come the Fantappié-Leray part of the story. Let

$$\pi : \mathbb{C}^{n+1} \backslash \{0\} \times \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n \times (\mathbb{P}^n)^*$$

and set

$$\begin{aligned} \tilde{I} &:= \pi^{-1}(I) \\ \tilde{\omega}(\mathbf{z}, \mathbf{w}) &:= \frac{1}{(2\pi i)^n} \left(\sum_{j=0}^n z_j dw_j \right) \wedge \left(\sum_{j=0}^n dz_j \wedge dw_j \right)^{n-1} \Bigg|_{\tilde{I}}. \end{aligned}$$

First we observe that $\tilde{\omega}(\mathbf{z}, \mathbf{w}) = (-1)^n \tilde{\omega}(\mathbf{w}, \mathbf{z})$. This follows from differentiating the relation $\sum_{j=0}^{n} z_j w_j$ defining $\tilde{I}$, so that $\sum_{j=0}^{n} z_j dw_j = -\sum_{j=0}^{n} w_j dz_j$. We also claim that if $\lambda \in C^1(\tilde{I})$ then $\tilde{\omega}(\lambda \mathbf{z}, \mathbf{w}) = \lambda^n \tilde{\omega}(\mathbf{z}, \mathbf{w})$. We have

$$\sum_{j=0}^{n} d(\lambda z_j) \wedge dw_j = \sum_{j=0}^{n} \lambda dz_j \wedge dw_j + d\lambda \wedge \sum_{j=0}^{n} z_j dw_j.$$

The second term vanishes when computing $\tilde{\omega}$, so the claim is proved. By the symmetry of $\tilde{\omega}$, the same holds in the $\mathbf{w}$ variable.

Definition 2.4.7. *Universal Cauchy-Fantappié-Leray Form ([APS04] Ch. 3, [Bar15] Sec. 7)*
The Universal CFL *form is the restriction to I of the $\mathcal{O}_{\mathbb{P}^n}(n,0) \otimes \mathcal{O}_{(\mathbb{P}^n)^*}(n,0)$-valued $2n-1$ form ω defined by*

$$\pi^* \omega = \frac{1}{(2\pi i)^n} \left(\sum_{j=0}^{n} z_j dw_j \right) \wedge \left(\sum_{j=0}^{n} dz_j \wedge dw_j \right)^{n-1}.$$

On the coordinate patch $U_{j,k} := \{z_j \neq 0, w_k \neq 0\}$, it is given by the expression

$$\omega_{j,k} = \frac{z_j^n w_k^n}{(2\pi i)^n} \left(\sum_{l=0}^{n} \frac{z_l}{z_j} d\left(\frac{w_l}{w_k}\right) \right) \wedge \left(\sum_{l=0}^{n} d\left(\frac{z_l}{z_j}\right) \wedge d\left(\frac{w_l}{w_k}\right) \right)^{n-1}.$$

Proposition 2.4.8. *ω is the unique (up to a constant) projectively invariant $\mathcal{O}_{\mathbb{P}^n}(n,0) \otimes \mathcal{O}_{(\mathbb{P}^n)^*}(n,0)$-valued $(2n-1)$ form on I.*

Proof. It suffices to check that $\tilde{\omega}$ is invariant under $SL_{n+1}(\mathbb{C})$, which acts on $\tilde{I}$ via $M(\mathbf{z}, \mathbf{w}) \mapsto (M\mathbf{z}, M^{-t}\mathbf{w})$ The first factor is clearly invariant, and one can check that the second factor is invariant under permutations, diagonal matrices and basic row/column operations. Uniqueness follows from homogeneity of the incidence manifold. □

Theorem 2.4.9. *([APS04] Theorem 3.1.7)*
Let $\Omega \subset \mathbb{P}^n$ be an open $\mathbb{C}$-convex set, and set $g_{\boldsymbol{\tau}}(\cdot) := \frac{1}{\langle \boldsymbol{\tau}, \cdot \rangle^n}$. Then there exists a unique, separately continuous $\mathbb{C}$-bilinear pairing $\langle\langle , \rangle\rangle : \mathcal{O}_{\Omega}(-n,0) \times \mathcal{O}_{\Omega^*}(-n,0) \to \mathbb{C}$ so that

$$f(\boldsymbol{\tau}) = \langle\langle f, g_{\boldsymbol{\tau}} \rangle\rangle \text{ for } \boldsymbol{\tau} \in \Omega,\ f \in \mathcal{O}_{\Omega}(-n,0)$$
$$g(\boldsymbol{\tau}) = \langle\langle g_{\boldsymbol{\tau}}, g \rangle\rangle \text{ for } \boldsymbol{\tau} \in \Omega^*,\ g \in \mathcal{O}_{\Omega^*}(-n,0).$$

The pairing is realized by the following procedure. Choosing $\mathbf{p} \in \Omega$ and $\mathbf{q} \in \Omega^*$, show that

$$I_{\mathbf{p},\mathbf{q}} := \{(\mathbf{z}, \mathbf{w}) \in I | \mathbf{z} \neq \mathbf{p}, \mathbf{w} \neq \mathbf{q}\}$$

is homotopic to the $2n-1$ sphere, thus the top dimensional homology group $H_{2n-1}(I_{\mathbf{p},\mathbf{q}}) \cong \mathbb{Z}$. Choose a cycle D whose homology class corresponds to the generator of $H_{2n-1}(I_{\mathbf{p},\mathbf{q}})$, and such that f, g are defined on D. The pairing is given by

$$\langle\langle f, g\rangle\rangle = \int_D f \cdot g \, \omega$$

with the reproducing property coming from Theorem 2.4.2.

In the case when Ω is C^2 and has strongly $\mathbb{C}$-convex boundary, there is an obvious choice for D, namely $I_S := I \cap (S \times S^*)$. To be more precise, we have diffeomorphisms

$$\mathscr{D} : S \to S^*$$
$$\mathscr{D}^* : S^* \to S$$

given by mapping $\mathbf{z} \in S$ to the maximal complex subspace of the tangent space, $T_{\mathbf{z}}(S) \cap JT_{\mathbf{z}}(S)$ and the same for $\mathscr{D}^*$. It follows from the uniqueness of this subspace and the symmetry of the relation $\mathbf{z} \cdot \mathbf{w} = 0$ that these maps are inverses of each other. The orientation on I_S is induced by orienting S as the boundary of Ω, and using the diffeomorphism $\mathbf{z} \mapsto (\mathbf{z}, \mathscr{D}(\mathbf{z}))$ to orient I_S. Given a defining function ρ and writing these maps in affine coordinates, we have

$$\mathscr{D}(\vec{\boldsymbol{z}}) = \frac{\partial\rho}{\langle \partial\rho, \vec{\boldsymbol{z}}\rangle}$$

Using this formula to pull back the integral produces the *Leray kernel*.

$$\mathscr{L}(\vec{\boldsymbol{z}}, \vec{\boldsymbol{\tau}}) := \frac{1}{(2\pi i)^n} \frac{\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}}{\langle \partial\rho, \vec{\boldsymbol{z}} - \vec{\boldsymbol{\tau}}\rangle^n}$$

with the reproducing property for $f \in \mathcal{O}(\overline{\Omega})$, $\vec{\boldsymbol{\tau}} \in \Omega$

$$f(\vec{\boldsymbol{\tau}}) = \int_S f(\vec{\boldsymbol{z}}) \, \mathscr{L}(\vec{\boldsymbol{z}}, \vec{\boldsymbol{\tau}})$$

In section 4.1, we will produce a parameterization of I_S when Ω is piece-wise smooth, and use this parameterization to calculate a reproducing kernel for such domains.

As in the one variable case, we can factor the universal CFL form as a product of projective invariants on S and S^*.

Theorem 2.4.10. *([Bar15])*
Let Ω be a domain in $\mathbb{P}^n$ with C^2 strongly $\mathbb{C}$-linearly convex boundary. There exists an explicit

projectively invariant function φ_S so that

$$|\omega| = \varphi_S^{\frac{-n}{n+1}} \mu_{S,Fef}^{\frac{1}{2}} \varphi_{S^*}^{\frac{-n}{n+1}} \mu_{S^*,Fef}^{\frac{1}{2}}.$$

Here $\mu_{S,Fef}$ is the Fefferman form, a (somewhat) well-known invariant $O_S(n,n)$-valued $2n-1$ form. See [KN99] for construction of the $2n-1$ form, or [Bar15] for treatment as an $O_S(n,n)$-valued $2n-1$ form. The positive bundle-valued form $\varphi_S^{\frac{-2n}{n+1}} \mu_{S,Fef}$ connects optimally with the pairing in the sense that if we use it to define the corresponding L^2 and Hardy spaces, then the identity

$$|fg\,\omega| = |f||g| \varphi_S^{\frac{-n}{n+1}} \mu_{S,Fef}^{\frac{1}{2}} \varphi_{S^*}^{\frac{-n}{n+1}} \mu_{S^*,Fef}^{\frac{1}{2}}$$

implies by Cauchy-Schwartz that

$$|\langle\langle f, g\rangle\rangle| \leq ||f||\;||g||.$$

CHAPTER III

Geometry of $\mathbb{C}$-Polytopes

Suppose we want to set up a projective Hardy space theory for a $\mathbb{C}$-convex domain Ω which has Levi-flat boundary. Let's make a few observations about what examples the theory should encompass. First, by taking a hyperplane outside Ω and moving it to infinity, we may assume that Ω is bounded. It then follows from the maximum principle that the boundary S must have singularities. Here we define a class of $\mathbb{C}$-convex domains with extremely nice singularities which will allow us to describe the geometry of the boundary well enough such that in Chapter 4, we can produce an explicit pairing with functions on the dual which realizes the pairing of Theorem 2.4.9, compute a version of the Leray Kernel for these domains, and define analogous Hardy spaces.

3.1 Definitions

An analytic hypersurface H is *Levi-flat* at $\vec{\boldsymbol{p}}$ if there are a holomorphic coordinates $\vec{\boldsymbol{z}}$ at $\vec{\boldsymbol{p}}$ so that $H = \{Re(z_n) = 0\}$. H is *Levi-flat* if every point is Levi-flat. In this case there is a corresponding *Levi foliation* by complex manifolds locally of the form $z_n = it,\ t \in \mathbb{R}$.

Ω is *piece-wise smooth* if there are finitely many connected smooth hypersurfaces H_j so that $S \subset \cup_j H_j$ and each intersection $\cap_{l=1}^{k} H_{j_k}$ is complex transverse for $k \leq n$ and real transverse for $k > n$, and for each $\vec{\boldsymbol{z}} \in S$, there is a sufficiently small neighborhood U so that $\Omega \cap U$ is connected. Ω is *piece-wise Levi-flat* if in addition each H_j is Levi-flat.

The *smooth locus* of S is $S_{sm} := \cup_j (H_j \backslash \cup_{k\neq j} H_k)$. The *singular locus* of S is $S_{sing} := S \backslash S_{sm}$. It follows that an *edge* $S_{\vec{\boldsymbol{j}}} := H_{j_1} \cap \ldots \cap H_{j_k}$ has dimension $2n - k$, and the edges may be ordered by inclusion. The *skeleton* S_n is the union of the n-dimensional edges. At a complex transverse intersection, an edge of dimension $2n-k$ has maximal complex subspace of real dimension $2n-2k$. In particular, the skeleton is *totally real*, that is $T_{\vec{\boldsymbol{z}}}(S_n) \cap JT_{\vec{\boldsymbol{z}}}(S_n) = \{\vec{\boldsymbol{z}}\}$.

Definition 3.1.1. *Strong Tangents*
For a piece-wise smooth domain Ω, $\vec{\boldsymbol{w}}$ is a strong tangent *at $\vec{\boldsymbol{z}} \in S$ if $\ell_{\vec{\boldsymbol{w}}}$ is the maximal complex subspace of one of the H_j containing $\vec{\boldsymbol{z}}$.*

Definition 3.1.2. *Weak Tangents*
Let Ω be a piece-wise smooth domain, $\vec{z} \in S$, $S_{\vec{j}}$ the minimal edge containing $\vec{z}$. A hyperplane $\vec{w}$ is an interior weak tangent *at $\vec{z}$ if the tangent cone $TC_{\vec{z}}(\ell_{\vec{w}} \cap \Omega)$ is contained in $T_{\vec{z}}(S_{\vec{j}})$. A hyperplane $\vec{w}$ is a* weak tangent *if it is in the closure of the interior weak tangents and it is not a strong tangent. Denote by $W_{\vec{z}}(\Omega)$ the set of weak tangents at $\vec{z}$.*

See Section 3.9 of [Mor16] for more on tangent cones. In particular, if a hyperplane $\vec{w}$ through $\vec{z}$ avoids Ω, then $\vec{w} \in W_{\vec{z}}(\Omega)$.

Definition 3.1.3. *$\mathbb{C}$-Polytope*
Ω is a $\mathbb{C}$-polytope *if the following properties hold*

a) Ω is $\mathbb{C}$-convex

b) S is piece-wise Levi-flat

c) For every $\vec{z} \in S$, denote by $S_{\vec{j}}$ the minimal edge containing $\vec{z}$. For every weak interior tangent $\vec{w} \in W_{\vec{z}}(\Omega)$, $\ell_{\vec{w}} \cap \overline{\Omega} \subset T_{\vec{z}}(S_{\vec{j}}) \cap JT_{\vec{z}}(S_{\vec{j}})$.

A few observations and comments are in order. First, observe that definition 3.1.3 is projectively invariant since the right hand side $T_{\vec{z}}(S_{\vec{j}}) \cap JT_{\vec{z}}(S_{\vec{j}})$ is the intersection of all strong tangents at $\vec{z}$. Second, for $\vec{z}$ in the skeleton S_n, $T_{\vec{z}}(S_{\vec{j}}) \cap JT_{\vec{z}}(S_{\vec{j}}) = \{\vec{z}\}$, and thus all interior weak tangents intersect $\overline{\Omega}$ precisely at $\vec{z}$. This property is necessary to calculate the reproducing kernels of Chapter 4. One undesirable property of the stated definition is it does not necessarily hold under intersection, however in Prop. 3.2.10 a sufficient criteria to be a $\mathbb{C}$-polytope is given, and we show that this criteria holds under appropriate intersections. The main examples we have in mind are convex product domains $\Omega = \mathring{\gamma}_1 \times \ldots \times \mathring{\gamma}_n$, where each γ_j is a convex analytic curve, appropriate intersections of such domains, and their projective images (see Prop. 3.2.11 and the final statement of 3.2.10).

3.2 Main Results

Lemma 3.2.1. *Barrett's Lemma*
If Ω is piece-wise Levi-flat and a strong tangent avoids Ω at smooth point $\vec{z}$ then the leaf of the Levi-foliation through $\vec{z}$ consists of subsets of a hyperplane. In particular, if Ω is a $\mathbb{C}$-polytope then all leaves of the Levi-foliation are subsets of complex hyperplanes.

Proof. We prove the first statement, and the second will follow since $\mathbb{C}$-polytopes are assumed to be $\mathbb{C}$-convex, and therefore $\mathbb{C}$-linearly convex by Theorem 2.3.1. Choose local analytic coordinates centered at $\vec{z}$ so that S is given by $Re(z_1) = 0$. In these analytic coordinates, the hyperplane tangent to S at $\vec{z}$ is given by $z_1 = f(z_2, \ldots z_n)$ where f holomorphic and $\partial f(0) = 0$. By the open mapping theorem, if f is non-constant we will have solutions to $f(z_2, \ldots, z_n) = t$ for $0 < |t| << 1$, so the only way the complex hyperplane can avoid Ω is if $f \equiv 0$. This in turn implies that the hyperplane $z_1 = 0$ lies in the boundary of the surface, i.e. it is one of the leaves of the Levi foliation. $\square$

Proposition 3.2.2. *Suppose Ω is pseudoconvex with piece-wise smooth boundary, $\vec{z} \in S$, with $\{H_l, l = 1, \ldots, k\}$ a minimal set of boundary hypersurfaces at $\vec{z}$. Suppose U is sufficiently small that $U \backslash \cup_l H_l$ has 2^k components, and $\Omega \cap U$ is connected. Then $\Omega \cap U$ coincides with one of the components of $U \backslash \cup_l H_l$.*

We need some definitions and lemmas before the proof. To see why the pseudoconvexity assumption is crucial, consider the domain $\Omega \subset \mathbb{C}^2$ where the complement of Ω is $\{Im(z_1) \geq 0, Im(z_2) \geq 0\}$. This is the prototypical example of a complex boundary wedge, defined below and depicted in Figure 3.1.

Definition 3.2.3. *Complex Boundary Wedge*
A domain Ω has a complex boundary wedge *at $\vec{z} \in S$ if there exists a neighborhood $U \ni p$ and two real hypersurfaces H_1, H_2 satisfying*

- *$S \cap U \subset H_1 \cup H_2$.*
- *H_1 and H_2 intersect complex transversely at $\vec{z}$.*
- *Given defining functions ρ_j for H_j on U such that $\Omega \cap U \supset \{\rho_1 < 0, \rho_2 < 0\}$, then $\Omega^c \cap U = \{\rho_1 \geq 0, \rho_2 \geq 0\}$.*

Lemma 3.2.4. *Pseudoconvex Domains Cannot Have a Complex Boundary Wedge*

The main idea is to use the Kontinuitätssatz, see Figure 3.1.

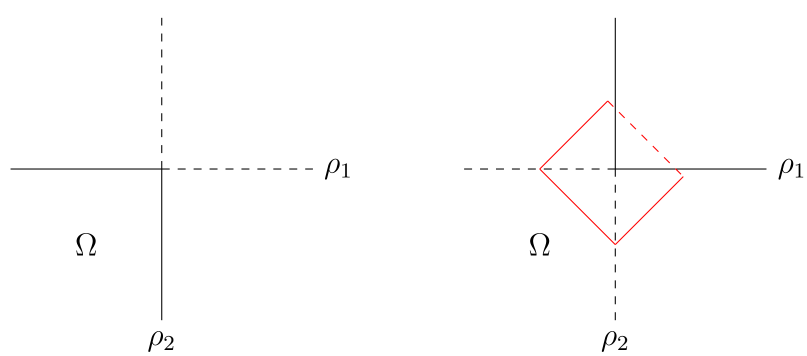

Figure 3.1: Ω piece-wise smooth and pseudoconvex on the left, Ω with a complex boundary wedge on the right. Boundary of family of holomorphic disks in solid red, dashed line represents final disk leaving the domain Ω.

Proof. We first prove the statement when $n = 2$, and the general statement will follow.

The idea is if Ω contains a complex boundary wedge, produce a family of holomorphic disks $F_t : \overline{\mathbb{D}} \to \mathbb{C}^2, 0 \leq t \leq 1$ such that $F_0(\mathbb{D}) \subset\subset \Omega$ and $F_t(\partial\mathbb{D}) \subset\subset \Omega$ for all t while $F_1(\mathbb{D}) \cap int(\Omega^c) \neq \emptyset$, violating the so-called Kontinuitätssatz (see Theorem 3.3.5 of [Kra08]).

By changing coordinates, assume $\vec{\boldsymbol{z}} = 0$ and $T_0(H_j) = \{Im(z_j) = 0\}$, the neighborhood U is a ball of radius 1, $\max_{\vec{\boldsymbol{z}} \in S} dist(\vec{\boldsymbol{z}}, T_0(H_1) \cup T_0(H_2))$ is arbitrarily small. We construct a family F_t as above for when Ω is the complement of $\{Im(z_j) \geq 0\}$, and it follows from our choice of coordinates that F_t will have the desired properties for Ω as well.

Consider four points $p_1, \ldots, p_4$ on $\partial\mathbb{D}$ in clockwise order. Take two line segments, C_1 from p_1 to p_4 and C_2 from p_2 to p_3. Let $C_1(t), C_2(t), 0 \leq t \leq 1$ be homotopies of C_1, C_2 respectively, relative to each endpoint such that $C_j(0) = C_j$, $C_j(t)$ is a segment of a circle for each $t > 0$ and the $C_j(1)$ cross transversely in the interior of the $\mathbb{D}$ bounding a set E compactly contained in $\mathbb{D}$. Let $f_1(z,t)$ be the Möbius transformation so that $Im(f_1(C_1(t), t)) = 0$ and $Im(f_1(z,t)) < 0$ below $C_1(t)$. Let $f_2(z,t)$ be the Möbius transformation so that $Im(f_2(C_2(t), t)) = 0$ and $Im(f_2(z,t)) < 0$ above $C_2(t)$. It follows that $F_t(z) := (f_1(z,t), f_2(z,t))$ has the desired properties. See Figure 3.2 for construction of the family of analytic disks.

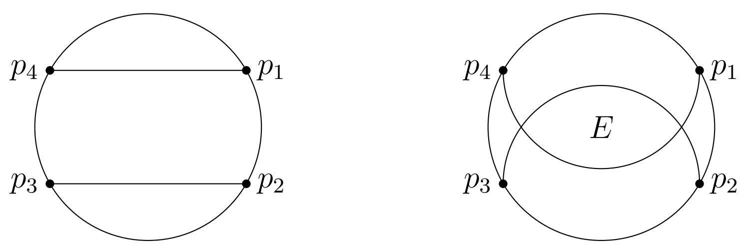

Figure 3.2: Construction of the family of holomorphic disks, F_0 on the left, F_1 on the right.

The general case follows by considering the intersection of Ω with a complex plane of dimension 2 which is transverse to H_1 and H_2. $\square$

Proof. Proof of Prop. 3.2.2

The idea is to show that if the conclusion is false, then Ω must contain a complex wedge.

Assume $\vec{\boldsymbol{z}} = 0$, and each H_l has defining function ρ_l. Since all the H_l are real-transverse, $\cup_{l=1}^k H_l$ divides U into 2^k components $\{\vec{\boldsymbol{z}} | sgn(\rho_l(\vec{\boldsymbol{z}})) = \epsilon_l\}, \vec{\boldsymbol{\epsilon}} \in \{-1,1\}^k$. Let C_k denote the hypercube graph on $\{-1,1\}^k$, and let $G(\Omega \cap U)$ be the induced graph on the vertices $\vec{\boldsymbol{\epsilon}}$ such that the component of $U \setminus \cup_{l=1}^k \{\rho_l = 0\}$ corresponding to $\vec{\boldsymbol{\epsilon}}$ is contained in Ω. Let C denote the smallest sub-hypercube containing $G(\Omega \cap U)$. Our goal is to show that C consists of precisely one vertex, so assume this is not the case.

Since $\Omega \cap U$ is connected, $G(\Omega \cap U)$ is connected. Since $G(\Omega \cap U)$ cannot be all of C, (otherwise there are extraneous H_l that we could discard) there must be two adjacent vertices in C_k such that one vertex is in $G(\Omega \cap U)$ and one is not. By applying an automorphism of the cube we may assume that the all 1's vector $\vec{\mathbf{1}} \in G(\Omega \cap U)$ and $(-1, 1, \ldots, 1) \notin G(\Omega \cap U)$. Since $G(\Omega \cap U)$ is contained in no smaller hypercube, $G' := G(\Omega \cap U) \cap \{\vec{\boldsymbol{\epsilon}} | \epsilon_1 = -1\} \neq \emptyset$. Let γ be a shortest path from $\vec{\mathbf{1}}$ to G'. γ must contain at least 3 vertices, and consider the end of the path. Up to permuting coordinates, the last three vertices must be

$$(1, -x_2, x_3, \ldots, x_k) \to (1, x_2, x_3, \ldots, x_k) \to (-1, x_2, x_3, x_4, \ldots, x_k).$$

The vertex $\vec{\boldsymbol{v}}^4 := (-1, -x_2, x_3, \ldots, x_k)$ cannot be in $G(\Omega \cap U)$ since then there would be a shorter path to G'. Let $\vec{\boldsymbol{v}}^1, \vec{\boldsymbol{v}}^2, \vec{\boldsymbol{v}}^3$ be the three last vertices of γ. By definition then Ω contains the components $\{sgn(\vec{\boldsymbol{z}}) = \vec{\boldsymbol{v}}^j\}, j = 1, \ldots, 3$ and not $\{sgn(\vec{\boldsymbol{z}}) = \vec{\boldsymbol{v}}^4\}$, and therefore Ω contains a complex wedge. □

We are now in a position to describe the weak tangents of a pseudoconvex domain with piecewise smooth boundary. We start with a simple example that will be useful later.

Proposition 3.2.5. *Weak Tangents of a Cone*
Suppose Ω is the cone $\{Im(\vec{\boldsymbol{\alpha}}_{\boldsymbol{j}} \cdot \vec{\boldsymbol{z}}) < 0, j = 1, \ldots, m\}$. The weak tangents through 0 coincide with the positive cone over the α_j,

$$C := \{\vec{\boldsymbol{w}} | \vec{\boldsymbol{w}} = \sum t_j \vec{\boldsymbol{\alpha}}_{\boldsymbol{j}}, t_j \geq 0\}$$

temporarily identifying $\vec{\boldsymbol{w}}$ with the line $\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{z}} = 0$.

Proof. Suppose we take a line of the form $\sum t_j \vec{\boldsymbol{\alpha}}_j \cdot \vec{\boldsymbol{z}} = 0$, $t_j \geq 0$, then we have that $\sum t_j Im(\vec{\boldsymbol{\alpha}}_j \cdot \vec{\boldsymbol{z}}) = 0$, so for at least one j, $Im(\vec{\boldsymbol{\alpha}}_j \cdot \vec{\boldsymbol{z}}) \leq 0$ and the line avoids Ω.
Now suppose that we take a line $\vec{\boldsymbol{\alpha}}$ which is not in C. Since C is convex, by Hahn-Banach there is a real hyperplane $\{\vec{\boldsymbol{w}} | Im(\vec{\boldsymbol{z}} \cdot \vec{\boldsymbol{w}}) = 0\}$ so that $Im(\vec{\boldsymbol{z}} \cdot C) < 0$ and $Im(\vec{\boldsymbol{z}} \cdot \vec{\boldsymbol{\alpha}}) > 0$. The first condition implies that $\vec{\boldsymbol{z}} \in \Omega$, and the second condition implies that $\vec{\boldsymbol{\alpha}}$ cannot be written as a positive linear combination of the $\vec{\boldsymbol{\alpha}}_j$. □

Remark 3.2.6. *Letting $\Delta^m := \{\vec{\boldsymbol{t}} \in \mathbb{R}^n | \sum t_j = 1, t_j \geq 0\}$ be the standard m-simplex, prop 3.2.5 gives a convenient way of parameterizing the weak tangents of the cone via*

$$\begin{aligned} \Delta^m &\to W_0(\Omega) \\ \vec{\boldsymbol{t}} &\mapsto \sum t_j \vec{\boldsymbol{\alpha}}_j. \end{aligned}$$

The vertices of Δ^m map to the strong tangents of Ω, and similarly the boundary of Δ^m maps to the weak tangents lying over the corresponding edge of the cone. The cone C is unaffected by replacing

$\vec{\boldsymbol{\alpha}}_j$ with a positive multiple of $\vec{\boldsymbol{\alpha}}_j$, so for computations we may assume, for example, that $|\vec{\boldsymbol{\alpha}}_j| = 1$.

For the shifted cone $\Omega' = \Omega + \vec{\boldsymbol{z}}$, we can parameterize the weak tangents in homogeneous coordinates. In the standard affine and homogeneous coordinates, the weak tangents are parameterized via

$$\begin{aligned}\Delta^m &\to W_{\vec{\boldsymbol{z}}}(\Omega')\\ \vec{\boldsymbol{t}} &\mapsto [-\sum t_j\vec{\boldsymbol{\alpha}}_j \cdot \vec{\boldsymbol{z}} : \sum t_j\vec{\boldsymbol{\alpha}}_j] = \mathbf{w}\\ &= (\frac{\sum t_j\vec{\boldsymbol{\alpha}}_j}{\sum t_j\vec{\boldsymbol{\alpha}}_j \cdot \vec{\boldsymbol{z}}}) = \vec{\boldsymbol{w}}.\end{aligned}$$

Proposition 3.2.7. *Description of Weak Tangents at Transverse Intersection*
Let Ω be a pseudoconvex domain with piece-wise smooth boundary. Suppose $H_{j_k}, k = 1, \ldots, l$ are the boundary Levi-flat hypersurfaces containing $\vec{\boldsymbol{z}}$, each with defining function ρ_{j_k}. Then we can parameterize the weak tangents at $\vec{\boldsymbol{z}}$ via the map

$$\begin{aligned}\Delta^m &\to W_{\vec{\boldsymbol{z}}}(\Omega)\\ \vec{\boldsymbol{t}} &\mapsto [-\sum_k t_k\partial\rho_{j_k} \cdot \vec{\boldsymbol{z}} : \sum_k t_k\partial\rho_{j_k}]\\ &= \left(\frac{\sum_k t_k\partial\rho_{j_k}}{\sum_k t_k\partial\rho_{j_k} \cdot \vec{\boldsymbol{z}}}\right).\end{aligned}$$

In particular, the vertices of Δ^m map to the strong tangents at $\vec{\boldsymbol{z}}$, and the union of the strong and weak tangents give a smooth cycle in $(\mathbb{P}^n)^*$.

Proof. First shift $\vec{\boldsymbol{z}}$ to the origin, and just for the purposes of this proof, encode hyperplanes through the origin via $\ell_{\vec{\boldsymbol{w}}} := \{\vec{\boldsymbol{z}} | \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{z}} = 0\}$. This encodes hyperplanes uniquely up to multiplying $\vec{\boldsymbol{w}}$ by a scalar. For each k, write

$$\rho_{j_k} = -Im(\vec{\boldsymbol{\alpha}}_k \cdot \vec{\boldsymbol{z}}) + h.o.t.$$

For each k, let $P_{\vec{\boldsymbol{\alpha}}_k}$ denote the orthogonal projection onto the real hyperplane $T_{\vec{\boldsymbol{0}}}(H_{j_k})$. We claim that there are constants $C > 0, 0 < c << 1$ such that if $||\vec{\boldsymbol{z}}|| < c$ and $Im(\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{z}}) > C(P_{\vec{\boldsymbol{\alpha}}_k}\vec{\boldsymbol{z}})^2$ then $\vec{\boldsymbol{z}} \notin \Omega$.

We prove this for a fixed k, then the constants C and c can be chosen to be the max and min, respectively, of the all the chosen constants. Fixing k, choose an orthogonal change of coordinates so that $T_{\vec{\boldsymbol{0}}}(H_{j_k}) = \mathbb{C}^{n-1} \times \mathbb{R}$. Thinking of H_{j_k} as a graph over $z_1, \ldots, z_{n-1}, Re(z_n)$, we have

$$Im(z_n) = \sum_{i,j=1}^{n-1} L_{i,j}z_i\overline{z_j} + \sum_{i,j=1}^{n-1} Re(Q_{i,j}z_iz_j) + \sum_{i=1}^{n-1} Im(R_iz_i) \cdot Re(z_n) + \tilde{R} \cdot Re(z_n)^2 + h.o.t.$$

The right hand side is a quadratic form in $z_1, \ldots, z_{n-1}, Re(z_n)$, so taking C larger than the largest eigenvalue of this quadratic form suffices.

Now suppose that we take $\vec{\boldsymbol{w}}$ to be in the interior of the cone over the $\vec{\boldsymbol{\alpha}}_k$, i.e. of the form $\vec{\boldsymbol{w}} = \sum_k t_k \vec{\boldsymbol{\alpha}}_k$, $t_k > 0$. We want to show that $TC_{\vec{\boldsymbol{z}}}(\Omega \cap \ell_{\vec{\boldsymbol{w}}}) \subset T_{\vec{\boldsymbol{z}}}(S_{\vec{\boldsymbol{j}}}) = \{Im(\vec{\boldsymbol{\alpha}}_k \cdot \vec{\boldsymbol{z}}) = 0, k = 1, \ldots, m\}$. Suppose there were a sequence $\vec{\boldsymbol{z}}_s \to 0$ with $\vec{\boldsymbol{z}}_s \cdot \vec{\boldsymbol{w}} = 0$, $\vec{\boldsymbol{z}}_s \in \Omega$ and $\frac{\vec{\boldsymbol{z}}_s}{|\vec{\boldsymbol{z}}_s|} \to \vec{\boldsymbol{p}} \notin \{Im(\vec{\boldsymbol{\alpha}}_k \cdot \vec{\boldsymbol{z}}) = 0, k = 1, \ldots, m\}$. It follows from $\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{p}} = 0$ that there is at least one k with $Im(\vec{\boldsymbol{\alpha}}_k \cdot \vec{\boldsymbol{p}}) > 0$. We then have for $0 < t < 1$ and a constant K that $Im(\vec{\boldsymbol{\alpha}}_k \cdot t \cdot \vec{\boldsymbol{p}}) > t \cdot K \cdot ||\vec{\boldsymbol{p}}||$. It then follows for $s >> 1$ that

$$Im(\vec{\boldsymbol{\alpha}}_l \cdot \vec{\boldsymbol{z}}_s) > \tilde{K} \cdot ||\vec{\boldsymbol{z}}_s|| > C||P_{\vec{\boldsymbol{\alpha}}_k}(\vec{\boldsymbol{z}}_s)||^2$$

contradicting the fact that $\vec{\boldsymbol{z}}_s \in \Omega$.

Now take $\vec{\boldsymbol{w}}$ to not be in the closure of the positive cone over the α_j. From proposition 3.2.5, we know that $\ell_{\vec{\boldsymbol{w}}}$ will intersect the interior of the cone $\{Im(\vec{\boldsymbol{\alpha}}_l \cdot \vec{\boldsymbol{z}}) \leq 0\}$, and from here it's easy to see that $TC_0(\Omega \cap \vec{\boldsymbol{w}})$ is not contained in $T_0(S_{\vec{\boldsymbol{j}}})$. (For example, take a complex line $L \subset \ell_{\vec{\boldsymbol{w}}}$) intersecting the interior of the cone. The intersection of L with the cone is a cone in $\mathbb{C}$ agreeing up to first order with $L \cap \Omega$).

Completing the proof just requires shifting the origin back to the original point $\vec{\boldsymbol{z}}$ as in remark 3.2.6. $\square$

Remark 3.2.8. *For S Levi-flat, take $\rho = f + \overline{f}$ for f holomorphic. Applying Prop. 3.2.7 at a smooth point (in the case $l = 1$), $\vec{\boldsymbol{w}} = \vec{\boldsymbol{w}}(\vec{\boldsymbol{z}}) = \frac{\partial f}{\partial f \cdot \vec{\boldsymbol{z}}}$. In particular, for $\vec{\boldsymbol{z}}$ in a leaf of the Levi foliation, $\vec{\boldsymbol{w}}$ is a holomorphic function of $\vec{\boldsymbol{z}}$. For S piece-wise Levi-flat, taking $\rho_{j_k} = f_{j_k} + \overline{f_{j_k}}$, Prop. 3.2.7 implies that holomorphic differentials dw_l can be written as a sum of holomorphic differentials in $\vec{\boldsymbol{z}}$ and l other differentials in $\vec{\boldsymbol{t}}$. This fact will be used in the proof of Theorem 4.1.1 to deduce that the Universal CFL form vanishes on high-dimensional edges.*

Proposition 3.2.9. *If Ω is a $\mathbb{C}$-polytope, then every strong and weak tangent avoids Ω. If Ω is a $\mathbb{C}$-polytope then every image of Ω by a projective transformation is a $\mathbb{C}$-polytope.*

Proof. If $\vec{\boldsymbol{z}} \in S_{sm}$ the first statement follows from the $\mathbb{C}$-linear convexity of Ω, so assume $\vec{\boldsymbol{z}} \in S_{sing}$. Since Ω is open, the condition that $\ell_{\vec{\boldsymbol{w}}} \cap \Omega$ is non-empty is an open condition on $\vec{\boldsymbol{w}}$. Thus if there were a weak tangent such that the intersection is non-empty, by density we could find an interior weak tangent so that the intersection is also non-empty, so assume $\vec{\boldsymbol{w}}$ is an interior weak tangent. Then $\ell_{\vec{\boldsymbol{w}}} \cap \Omega$ is an open subset of $\ell_{\vec{\boldsymbol{w}}}$ contained in a proper subspace $T_{\vec{\boldsymbol{z}}}(S_{\vec{\boldsymbol{j}}}) \cap JT_{\vec{\boldsymbol{z}}}(S_{\vec{\boldsymbol{j}}})$ which is absurd.

The notions of weak and strong tangents are projectively invariant. $T_{\vec{\boldsymbol{z}}}(S_{\vec{\boldsymbol{j}}}) \cap JT_{\vec{\boldsymbol{z}}}(S_{\vec{\boldsymbol{j}}})$ is the intersection of all strong tangents at $\vec{\boldsymbol{z}}$, and thus also projectively invariant. $\square$

Proposition 3.2.10. *Sufficient Condition to be a $\mathbb{C}$-polytope*
Ω is a $\mathbb{C}$-polytope if there is a choice of projective coordinates such that

a) Ω is piece-wise Levi-flat.

b) Ω is convex.

c) If $\vec{z} \in S$ is contained in the minimal edge $S_{\vec{j}}$, then $T_{\vec{z}}(S_{\vec{j}}) \cap \overline{\Omega} \subset T_{\vec{z}}(S_{\vec{j}}) \cap JT_{\vec{z}}(S_{\vec{j}})$.

Furthermore, if there is a choice of projective coordinates so that items $b)$ and $c)$ hold simultaneously for Ω_1, Ω_2 and the intersection $\Omega_1 \cap \Omega_2$ is piece-wise Levi-flat then item $c)$ holds for $\Omega_1 \cap \Omega_2$. In particular, $\Omega_1 \cap \Omega_2$ is a $\mathbb{C}$-polytope.

Proof. Let $\vec{z} \in S$, make a linear shift so that $\vec{z} = 0$ and identify $\vec{w}$ with complex lines through the origin via $\vec{w} \cdot \vec{z} = 0$ exactly as in Prop. 3.2.5. By convexity Ω will be contained in the real half-spaces $\{Im(\vec{w}_{j_k} \cdot \vec{z}) < 0\}$, and we may write our interior weak tangent as $\vec{w} = \sum_k t_k \vec{w}_{j_k}, t_k > 0$. Suppose we have a point $\vec{p} \in \ell_{\vec{w}} \cap \overline{\Omega}$. From the equation $\vec{w} \cdot \vec{p} = 0$, it follows then that $Im(\vec{w}_{j_k} \cdot \vec{p}) = 0$ (if there were a k such that $Im(\vec{w}_{j_k} \cdot \vec{p}) < 0$, there would have to be some k' with $Im(\vec{w}_{j_k} \cdot \vec{p}) > 0$, implying $p \notin \overline{\Omega}$). Thus $\vec{p} \in T_0(S_{\vec{j}})$, and by item $c)$, $\vec{p} \in T_0(S_{\vec{j}}) \cap JT_0(S_{\vec{j}})$.

For the last statement, suppose $\vec{z} \in \partial(\Omega_1 \cap \Omega_2) \subset S_1 \cup S_2$. If $\vec{z} \notin S_1$ or S_2, there is nothing to prove, so assume $\vec{z} \in S_1 \cap S_2$ and is contained in the minimal edges σ_1, σ_2. Since the intersection is transverse, the corresponding minimal edge in $\partial(\Omega_1 \cap \Omega_2)$ is $\sigma_1 \cap \sigma_2$, and item $c)$ holding under intersection follows easily from the equality $T_{\vec{z}}(\sigma_1 \cap \sigma_2) = T_{\vec{z}}(\sigma_1) \cap T_{\vec{z}}(\sigma_2)$. □

Combined with the following proposition, we get a somewhat large class of examples of $\mathbb{C}$-polytopes.

Proposition 3.2.11. *Let $n > 1$. If $\Omega = \mathring{\gamma}_1 \times \ldots \times \mathring{\gamma}_n$ where each γ_j is an analytic curve in $\mathbb{C}$, then Ω is a $\mathbb{C}$-polytope iff each γ_j is convex.*

The proof will demonstrate why being a $\mathbb{C}$-polytope does not necessarily hold under intersection. In particular, the property of weak tangents avoiding Ω does not necessarily hold under intersection. Take $\Omega_1 = \mathring{\gamma}_1 \times \mathbb{C}$ where γ_1 is not convex and $\Omega_2 = \mathbb{C} \times \mathring{\gamma}_2$. Ω_1 has smooth boundary, with the strong tangent at $\vec{p} = (p_1, p_2) \in \gamma_1 \times \mathbb{C}$ defined by $z_1 = p$ obviously avoiding Ω, and since the boundary is smooth there are no interior weak tangents to check. The same argument shows that the strong tangents of $\Omega_1 \cap \Omega_2 = \mathring{\gamma}_1 \times \mathring{\gamma}_2$ avoid $\Omega_1 \cap \Omega_2$, but the proof below will produce a weak tangent that intersects $\Omega_1 \cap \Omega_2$.

Proof. Proof of Prop. 3.2.11

Begin with the case when γ_1 is not convex. Obviously conditions $a), b), c)$ do not hold in the standard affine coordinates, but it's somewhat less obvious they cannot hold in any projective

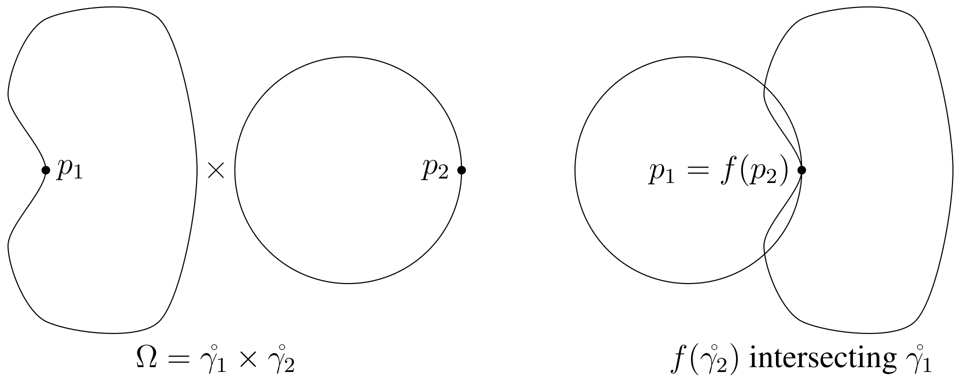

Figure 3.3: Affine map corresponding to a weak tangent intersecting the interior of $\Omega = \mathring{\gamma}_1 \times \mathring{\gamma}_2$ when γ_1 is not convex. Note that the tangent cone of the intersection near $p_1 = f(p_2)$ corresponds to the totally real 2-plane $T_{p_1}(\gamma_1) \times T_{p_2}(\gamma_2)$, so the corresponding complex line is indeed a weak tangent.

coordinates. To show Ω cannot be a $\mathbb{C}$-polytope, we produce a weak tangent which intersects Ω in violation of proposition 3.2.9. Assume that $0 \in \Omega$, so that every line outside Ω can be uniquely represented as $\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{z}} = 1$. The problem can be reduced to the case $n = 2$ by only considering lines where $w_j = 0, j \geq 3$. We will produce a weak tangent $\vec{\boldsymbol{w}} \in W_{\vec{\boldsymbol{z}}}(\Omega)$ of this form which intersects the interior of Ω.

The defining equation reads $w_1 z_1 + w_2 z_2 = 1$. Solving for z_1 produces $z_1 = \frac{1 - w_2 z_2}{w_1}$. In this way, identify the line $\vec{\boldsymbol{w}}$ with the affine function of one variable $f(z) = \frac{1 - w_2 z}{w_1}$. Taking a point $\vec{\boldsymbol{p}} = (p_1, p_2) \in \gamma_1 \times \gamma_2$, a line passing through $\vec{\boldsymbol{p}}$ avoiding $\mathring{\gamma}_1 \times \mathring{\gamma}_2$ is the same as the graph of an affine $f : \mathbb{C} \to \mathbb{C}$ so that $f(p_2) = p_1$ and $f(\mathring{\gamma}_2) \cap \mathring{\gamma}_1 = \emptyset$. A weak tangent is the graph of an affine $f : \mathbb{C} \to \mathbb{C}$ so that $f(p_2) = p_1$ and $f'(p_2) \cdot \gamma_2'(p_2) = -r \cdot \gamma_1'(p_1)$, where $r > 0$ and each γ_j is oriented clockwise, see Figure 3.3.

Assuming that $\mathring{\gamma}_1$ is not convex, set p_1, p_1' to be points in the boundary γ_1 connected by a line segment contained in the exterior of $\mathring{\gamma}_1$ and so that this line segment is real-transverse to $\gamma_1'(p_1)$. Take a point $p_2 \in \gamma_2$ and a linear map in one variable $f(z) = az + b$ so that

- $f(p_2) = p_1$
- $f'(p_2) \cdot \gamma_2'(p_2) = -r \cdot \gamma_1'(p_1), r > 0$
- $|f'| >> 1$.

In particular, choose $|f'|$ so large so that $f(\mathring{\gamma}_2)$ contains p_1'. If we write $f(z) = az + b$, then the line $z_2 = az_1 + b$ intersects $\mathring{\gamma}_1 \times \mathring{\gamma}_2$ at $(p_1', f^{-1}(p_1'))$ it follows from proposition 3.2.7 that this line is in $W_{\vec{\boldsymbol{p}}}(\Omega)$, yielding the promised contradiction to Prop. 3.2.9.

In the case when each γ_j is convex, items $a)$ and $b)$ of Prop. 3.2.10 are immediate. Analyticity implies that each γ_j is strictly convex, that is it contains no line segments, and item $c)$ follows. $\square$

CHAPTER IV

Duality for $\mathbb{C}$-Polytopes

It is a general fact that on the Levi-flat portion of S, $\mu_{S,Fef} = 0$. Examining Theorem 2.4.10 suggests that good measures to use for a Hardy space theory should be supported on the singular locus of S, and likewise Theorem 2.4.4 indicates that the accompanying reproducing kernel should be concentrated on the skeleton. In this chapter we will indeed produce such measures and kernels, and examine the qualitative properties of the pairing.

4.1 The Leray Kernel

Theorem 4.1.1. *Let Ω be a $\mathbb{C}$-polytope, and let $\mathbf{w}_j = \mathbf{w}_j(\mathbf{z}), j = 1, \ldots, n$ denote the strong tangents at points $\mathbf{z} \in S_n$ not contained in any smaller edge of S. We have a projectively invariant $O_{S_n}(n,0)$-valued n-form*

$$\frac{(-1)^n}{(2\pi i)^n} \frac{\det_0(\mathbf{w}_j)}{\prod_j \langle \mathbf{w}_j, \boldsymbol{\tau} \rangle} d\vec{\boldsymbol{z}}$$

with the reproducing property for $f \in \mathcal{O}_{\overline{\Omega}}(-n, 0)$

$$f(\boldsymbol{\tau}) = \frac{(-1)^n}{(2\pi i)^n} \int_{S_n} f(\mathbf{z}) \frac{\det_0(\mathbf{w}_j)}{\prod_j \langle \mathbf{w}_j, \boldsymbol{\tau} \rangle} d\vec{\boldsymbol{z}}$$

.

The idea is simple; once projective invariance is established, it suffices to check the equality of the integrands in Theorems 4.1.1 and 2.4.9 in a preferred choice of coordinates. First we point out that in the above formula, the $\mathbf{w}_j$ are "row vectors", with $\det_0(\mathbf{w})$ meaning the determinant of the last n columns. We emphasize also that pulling back by a transformation T requires also pulling back $\boldsymbol{\tau}$ via T. Care must be taken when numbering the strong tangents, as choosing a different ordering may result in a different sign. The correct ordering will be induced by parameterizing I_S appropriately, see the discussion under Theorem 2.4.9. Notice also $\Omega = \mathring{\gamma}_1 \times \ldots \times \mathring{\gamma}_n$ where each γ_i is a convex curve then the strong tangents are $\mathbf{w}_j = \mathbf{e}_0 - \frac{1}{z_j}\mathbf{e}_j$. Substituting into the integrand of Theorem 4.1.1 produces the multivariate Cauchy formula. The proof requires a few lemmas.

Lemma 4.1.2. *Projective Cramer's Rule*
Let $\mathbf{z} \in \mathbb{P}^n, \mathbf{w}_j \in (\mathbb{P}^n)^*$ *be a set of independent hyperplanes such that* $\mathbf{w}_j \cdot \mathbf{z} = 0$*. Let* $\mathbf{W}$ *be the* $n \times (n+1)$ *matrix with the* $\mathbf{w}_j$ *as rows. Then*

$$z_j \mathrm{det}_0(\mathbf{W}) = (-1)^j z_0 \mathrm{det}_j(\mathbf{W})$$

.

Proof. We check the equality on the dense open set $I \cap U_{0,0}$. In the standard affine coordinates, we have the matrix equation.

$$\begin{bmatrix} \frac{w_{1,1}}{-w_{1,0}} & \frac{w_{1,2}}{-w_{1,0}} & \cdots & \frac{w_{1,n}}{-w_{1,0}} \\ \frac{w_{2,1}}{-w_{2,0}} & \frac{w_{2,2}}{-w_{2,0}} & \cdots & \frac{w_{2,n}}{-w_{2,0}} \\ \vdots & & & \vdots \\ \frac{w_{n,1}}{-w_{n,0}} & \frac{w_{n,2}}{-w_{n,0}} & \cdots & \frac{w_{n,n}}{-w_{n,0}} \end{bmatrix} \begin{bmatrix} \frac{z_1}{z_0} \\ \frac{z_2}{z_0} \\ \vdots \\ \frac{z_n}{z_0} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Cramer's rule gives

$$\frac{z_1}{z_0} = \frac{\begin{vmatrix} 1 & \frac{w_{1,2}}{-w_{1,0}} & \cdots & \frac{w_{1,n}}{-w_{1,0}} \\ 1 & \frac{w_{2,2}}{-w_{2,0}} & \cdots & \frac{w_{2,n}}{-w_{2,0}} \\ \vdots & & & \vdots \\ 1 & \frac{w_{n,2}}{-w_{n,0}} & \cdots & \frac{w_{n,n}}{-w_{n,0}} \end{vmatrix}}{\begin{vmatrix} \frac{w_{1,1}}{-w_{1,0}} & \frac{w_{1,2}}{-w_{1,0}} & \cdots & \frac{w_{1,n}}{-w_{1,0}} \\ \frac{w_{2,1}}{-w_{2,0}} & \frac{w_{2,2}}{-w_{2,0}} & \cdots & \frac{w_{2,n}}{-w_{2,0}} \\ \vdots & & & \vdots \\ \frac{w_{n,1}}{-w_{n,0}} & \frac{w_{n,2}}{-w_{n,0}} & \cdots & \frac{w_{n,n}}{-w_{n,0}} \end{vmatrix}}$$

Rearranging this yields $z_1 \det_0(\mathbf{W}) = -z_0 \det_1(\mathbf{W})$. The same argument works for arbitrary j. ∎

Lemma 4.1.3. *Simplex Calculation*
Fix $\vec{\boldsymbol{z}}$ *and suppose* $\vec{\boldsymbol{w}}_i = \vec{\boldsymbol{w}}_i(\vec{\boldsymbol{t}}), \vec{\boldsymbol{t}} \in \Delta^n$ *are functions satisfying* $\vec{\boldsymbol{z}} \cdot \vec{\boldsymbol{w}} = 1$ *and that* $w_i(\vec{\boldsymbol{e}}_j) = \begin{cases} \frac{1}{z_i} & i = j \\ 0 & i \neq j \end{cases}$*. Then*

$$I(w_1, \ldots, w_n) := \int_{\Delta^n} \frac{1}{(1 - \langle \vec{\boldsymbol{\tau}}, \vec{\boldsymbol{w}} \rangle)^n} \left(\prod_{j=1}^{n-1} z_j dw_j \right) = \frac{(-1)^n}{(n-1)!} \prod_{j=1}^{n} \frac{1}{1 - \frac{\tau_j}{z_j}}.$$

Proof. The proof is induction on n. For $n = 2$, this reduces to calculating

$$\int_{t_1=0}^{t_1=1} \frac{1}{(1-(w_1\tau_1+w_2\tau_2))^2} z_1 dw_1.$$

Keeping in mind that $w_2 = \frac{1-w_1z_1}{z_2}$, we get

$$\left[\frac{-1}{(1-(w_1\tau_1+w_2\tau_2))}\frac{-1}{\tau_1-\frac{z_1}{z_2}\tau_2}\cdot z_1\right]_{t_1=0}^{t_1=1} = \frac{1}{(1-\frac{\tau_1}{z_1})(1-\frac{\tau_2}{z_2})}.$$

For the induction step, we will use Stoke's theorem. Set $\partial\Delta_i^n := \Delta^n \cap \{t_i = 0\}$ oriented as the boundary of Δ^n, and let φ denote the integrand in the lemma. Thinking of w_n as a function of the other w_j, we have $\frac{\partial w_n}{\partial w_{n-1}} = -\frac{z_{n-1}}{z_n}$. Define the primitive for φ

$$\eta := \frac{(-1)^{n-1}}{(n-1)(\tau_{n-1}+\tau_n(\frac{-z_{n-1}}{z_n}))}\frac{1}{(1-\langle\vec{\tau},\vec{w}\rangle)^{n-1}} z_1\dots z_{n-1} dw_1\wedge\dots\wedge dw_{n-2}.$$

Note that η is only non-zero along $\partial\Delta_i^n$ for $i = n-1, n$. Keeping in mind the orientation, we have

$$\int_{\partial\Delta_{n-1}^n}\eta = \frac{(-1)^{n-2}}{(n-1)}\frac{z_{n-1}}{\tau_{n-1}+\tau_n(\frac{-z_{n-1}}{z_n})}(-1)^{n-2} I(w_1,\dots,w_{n-2},w_n)$$
$$\int_{\partial\Delta_{n}^n}\eta = \frac{(-1)^{n-2}}{(n-1)}\frac{z_{n-1}}{\tau_{n-1}+\tau_n(\frac{-z_{n-1}}{z_n})}(-1)^{n-1} I(w_1,\dots,w_{n-2},w_{n-1}).$$

By Stoke's

$$\int_{\Delta^n}\varphi = \frac{1}{(n-1)(\frac{\tau_{n-1}}{z_{n-1}}-\frac{\tau_n}{z_n})}\left(I(w_1,\dots,w_{n-2},w_n) - I(w_1,\dots,w_{n-2},w_{n-1})\right).$$

From induction, along with the identity

$$\prod_{j\neq n}\frac{1}{1-\frac{\tau_j}{z_j}} - \prod_{j\neq n-1}\frac{1}{1-\frac{\tau_j}{z_j}} = \frac{\frac{\tau_{n-1}}{z_{n-1}}-\frac{\tau_n}{z_n}}{\prod_{j=1}^n 1-\frac{\tau_j}{z_j}}$$

we are done. □

Proof. Proof of Thm. 4.1.1

First we verify the projective invariance. Let T be a projective transformation with associated matrix M. It suffices to check invariance for T in a generating set.

Case 1: T is a linear transformation in $\vec{z}$, that is M is of the form

$$\left(\begin{array}{c|c} \det\left(\tilde{M}\right)^{\frac{-1}{n+1}} & \mathbf{0} \\ \hline \mathbf{0} & \det\left(\tilde{M}\right)^{\frac{-1}{n+1}}\tilde{M} \end{array}\right), \quad \tilde{M} \in GL_n(\mathbb{C})$$

$$\begin{aligned} T^*(\frac{\det_0(\mathbf{w}_j)}{\prod_j \langle \mathbf{w}_j, \boldsymbol{\tau}\rangle} d\vec{z}) &= \frac{\det_0(\mathbf{w}_j \circ M^{-1})}{\prod_j \langle M^{-t}\mathbf{w}_j, M\boldsymbol{\tau}\rangle} \det\left(\tilde{M}\right) d\vec{z} \\ &= \det\left(\tilde{M}\right)^{\frac{n}{n+1}} \frac{\det_0(\mathbf{w}_j)}{\prod_j \langle \mathbf{w}_j, \boldsymbol{\tau}\rangle} d\vec{z}. \end{aligned}$$

Case 2: T is a linear shift, that is $T(\mathbf{z}) = \mathbf{z} + \lambda z_0 \mathbf{e}_j$.

Case 1 allows us to assume that $j = 1$. The dual transformation is given by $T^{-t}(\mathbf{w}) = \mathbf{w} - \lambda w_j \mathbf{e}_0$, so one column operation shows that $T^* \det_0(\mathbf{w}_j) = \det_0(\mathbf{w}_j \circ T^{-1}) = \det_0(\mathbf{w}_j)$.
Case 3: T is an inversion, that is $T(\mathbf{z}) = \mathbf{z} + \lambda z_j \mathbf{e}_0$.

Again, assume that $j = 1$. The dual transformation is given by $T^{-t}(\mathbf{w}) = \mathbf{w} + \lambda w_0 \mathbf{e}_1$. Applying Lemma 4.1.2,

$$\begin{aligned} T^*(\det_0(\mathbf{w}_j)) &= \det_0(\mathbf{w}_j) - \lambda \det_1(\mathbf{w}_j) \\ &= (1 + \lambda\frac{z_1}{z_0})\det_0(\mathbf{w}_j) \\ T^*(\frac{\det_0(\mathbf{w}_j)}{\prod_j \langle \mathbf{w}_j, \boldsymbol{\tau}\rangle} d\vec{z}) &= (1 + \lambda\frac{z_1}{z_0})^n \frac{\det_0(\mathbf{w}_j)}{\prod_j \langle \mathbf{w}_j, \boldsymbol{\tau}\rangle} d\vec{z}. \end{aligned}$$

We have the desired invariance, so now we establish that ω vanishes on points $(\vec{z}, \vec{w}) \in I_S$ where $\vec{z} \in S \backslash S_n$ and at points $\vec{z}$ in edges of real dimension $< n$. At a point where $\vec{z}$ is in an edge of dimension $< n$, then clearly $d\vec{z} = 0$. Conversely, at a point where $\vec{z}$ is in an edge of dimension $2n - k > n$, it follows from Prop 3.2.7, recalling the fact that S_i is Levi-flat, that the strong tangents $\vec{w}_i^S$ are holomorphic functions of $\vec{z}$, and thus $\vec{w}$ depends holomorphically on $\vec{z}$ and the $k - 1$ independent parameters $\vec{t}$ (see remark 3.2.8). Thus there are at most $n + k - 1 < 2n - 1$ independent differentials, so $d\vec{w}_{[j]} \wedge d\vec{z} = 0$.

By appropriate projective transformations, we may assume that Ω is bounded containing the origin, that is $(\Omega \times \Omega^*) \subset\subset U_{0,0}$. In these affine coordinates, we claim ω is equal to the expression

$$\omega = \frac{(-1)^{\frac{n^2-n-2}{2}}(n-1)!}{(2\pi i)^n} \prod_{i=1}^{n-1} z_i d\vec{w}_{[n]} \wedge \frac{d\vec{z}}{\prod_{i=1}^n z_n}.$$

To see this, first make the substitution $\frac{1}{z_n} = w_n + \sum_{j=1}^{n-1} \frac{w_j z_j}{z_n}$ into the right hand side. Then observe by differentiating the relation $\vec{\boldsymbol{z}} \cdot \vec{\boldsymbol{w}} = 1$, that $z_j dw_j = -z_n dw_n +$ terms that involve $dw_j, j \neq n$ and dz_k. These extra terms cancel upon substitution. The calculations described are below.

$$\begin{aligned}
\frac{(-1)^{\frac{n^2-n-2}{2}}(n-1)!}{(2\pi i)^n}\frac{1}{z_n}d\vec{\boldsymbol{w}}_{[n]} \wedge d\vec{\boldsymbol{z}} &= (w_n + \sum_{j=1}^{n-1}\frac{w_j z_j}{z_n})d\vec{\boldsymbol{w}}_{[n]} \wedge d\vec{\boldsymbol{z}} \\
&= \frac{(-1)^{\frac{n^2-n-2}{2}}(n-1)!}{(2\pi i)^n}\left(w_n d\vec{\boldsymbol{w}}_{[n]} \wedge d\vec{\boldsymbol{z}} + \sum_{j=1}^{n-1} w_j(-1)^{n-j} d\vec{\boldsymbol{w}}_{[j]} \wedge d\vec{\boldsymbol{z}}\right) \\
&= \frac{(-1)^{\frac{n^2-3n-2}{2}}(n-1)!}{(2\pi i)^n}\sum_{j=1}^{n}(-1)^j w_j d\vec{\boldsymbol{w}}_{[j]} \wedge d\vec{\boldsymbol{z}}
\end{aligned}$$

To see that this is indeed the Universal CFL form, take the expression in Def. 2.4.7 in the coordinate chart $U_{0,0}$ (remembering that $w_0 = -1$), apply the symmetry to the $\vec{\boldsymbol{z}}$ and $\vec{\boldsymbol{w}}$ variables and expand.

To conclude, the vanishing of ω away from the skeleton implies

$$\int_{(S\times S^*)\cap I} f(\vec{\boldsymbol{z}})g_{\vec{\boldsymbol{\tau}}}(\vec{\boldsymbol{w}})\omega_{0,0} = \int_{(S_n\times S^*)\cap I} f(\vec{\boldsymbol{z}})g_{\vec{\boldsymbol{\tau}}}(\vec{\boldsymbol{w}})\omega_{0,0}.$$

Writing this in local coordinates, applying Prop. 3.2.7, we obtain

$$\begin{aligned}
f(\vec{\boldsymbol{\tau}}) &= \int_{(S_n\times S^*)\cap I} f(\vec{\boldsymbol{z}})g_{\vec{\boldsymbol{\tau}}}(\vec{\boldsymbol{w}})\omega_{0,0} \\
&= \frac{(-1)^{\frac{n^2-n-2}{2}}(n-1)!}{(2\pi i)^n}\int_{S_n} f(\vec{\boldsymbol{z}})\left(\int_{\Delta^n}\frac{1}{(1-\langle\vec{\boldsymbol{\tau}},\vec{\boldsymbol{w}}\rangle)^n}\prod_{j=1}^{n-1}(z_j dw_j)\right)\frac{d\vec{\boldsymbol{z}}}{\prod_{j=1}^n z_j}.
\end{aligned}$$

By a projective change of coordinates, assume we are looking near a point $\vec{\boldsymbol{p}} = (1,\ldots,1) \in S_n$ and $\vec{\boldsymbol{w}}_j(\vec{\boldsymbol{p}}) = \vec{\boldsymbol{e}}_j$, then use Lemma 4.1.3 to evaluate the inner integral at $\vec{\boldsymbol{p}}$ and verify that it does indeed agree with the n-form in Theorem 4.1.1. $\square$

4.2 Substitute Hardy Spaces

To set up projectively invariant Hardy space on Ω and Ω^*, ideally we would like a positive, projectively invariant $O_{S_n}(n,n)$-valued n-form μ_1 and a positive, projectively invariant $O_{S^*}(n,n)$-

valued $2n-1$ form μ_2 defining the norms

$$||f||_2^2 := \int_{S_n} |f|^2 \mu_1 \quad f \in O_{S_n}(-n,0)$$
$$||g||_2^2 := \int_{S^*} |g|^2 \mu_2 \quad g \in O_{S^*}(-n,0)$$

giving corresponding L^2 and Hardy spaces. On S^*_{sm} there is such a form, namely the Fefferman form. One might also want to use Barrett's preferred measure (page 19 of [Bar15]), but the invariant φ_{S^*} vanishes in this case. On S_n, the natural guess is $|z_0^{n+1} d\vec{z}|$, but this is an invariant $O(\frac{n+1}{2}, \frac{n+1}{2})$-valued n-form. There are two approaches to fix this, either construct a $O_{S_n}(n,n)$-valued n-forms, or construct substitute L^2 spaces. In the next section we will accomplish the former when $n=2$, but here we explore the latter option, and construct somewhat satisfactory substitute L^2-spaces whose topologies are projectively invariant. Recall that $I_S := (S \times S^*) \cap I$, and ω is the Universal CFL-form (see Def. 2.4.7).

Definition 4.2.1. *Let Ω be a $\mathbb{C}$-polytope, $\pi : \mathbb{C}^{n+1}\backslash\{0\} \times \mathbb{C}^{n+1}\backslash\{0\}$. For $(\mathbf{p}, \mathbf{q}) \in \pi^{-1}(\Omega \times int(\Omega^*))$ define the norms*

$$||f||^2_{\mathbf{p},\mathbf{q}} := \int_{I_S} |f|^2 \left(\frac{|\mathbf{q}\cdot\mathbf{z}|}{|\mathbf{p}\cdot\mathbf{w}|}\right)^{\frac{n}{2}} |\omega|,\ f \in O_{S_n}(-n,0)$$
$$||g||^2_{\mathbf{p},\mathbf{q}} := \int_{I_S} |g|^2 \left(\frac{|\mathbf{p}\cdot\mathbf{w}|}{|\mathbf{q}\cdot\mathbf{z}|}\right)^{\frac{n}{2}} |\omega|,\ g \in O_{S^*}(-n,0).$$

Define $L^2(S_n, \mathbf{p}, \mathbf{q}), L^2(S^*, \mathbf{p}, \mathbf{q})$ as the closure with respect to each norm, and define the Hardy spaces $H^2(\Omega, \mathbf{p}, \mathbf{q}), H^2(\Omega^*, \mathbf{p}, \mathbf{q})$ as the L^2-closure of $\mathcal{O}_{\overline{\Omega}}(-n,0)$ and $\mathcal{O}_{\Omega^*}(-n,0)$ respectively.

With these definitions, we get a $\mathbb{C}$-bilinear pairing on the Hardy spaces with similar properties as in the C^2 strongly $\mathbb{C}$-convex case.

Theorem 4.2.2. *Qualitative Properties of the Pairing*
Define the $\mathbb{C}$-bilinear pairing

$$\langle\langle , \rangle\rangle : L^2(S_n, \mathbf{p}, \mathbf{q}) \times L^2(S^*, \mathbf{p}, \mathbf{q}) \to \mathbb{C}$$
$$\langle\langle f, g\rangle\rangle := \int_{I_S} f \cdot g\, \omega.$$

a) $\langle\langle , \rangle\rangle$ is separately continuous.

b) We have the reproducing properties

$$f(\boldsymbol{\tau}) = \langle\langle f, g_{\boldsymbol{\tau}}\rangle\rangle \textit{ for } \boldsymbol{\tau} \in \Omega,\ f \in H^2(\Omega, \mathbf{p}, \mathbf{q})$$
$$g(\boldsymbol{\tau}) = \langle\langle g_{\boldsymbol{\tau}}, g\rangle\rangle \textit{ for } \boldsymbol{\tau} \in \Omega^*,\ g \in H^2(\Omega^*, \mathbf{p}, \mathbf{q}).$$

c) The pairing induces injective maps

$$H^2(\Omega, \mathbf{p}, \mathbf{q}) \hookrightarrow H^2(\Omega^*, \mathbf{p}, \mathbf{q})^*$$
$$H^2(\Omega^*, \mathbf{p}, \mathbf{q}) \hookrightarrow H^2(\Omega, \mathbf{p}, \mathbf{q})^*$$

with dense image.

d) If $f \in \mathcal{O}_{\overline{\Omega}}(-n, 0)$, $g \in \mathcal{O}_{\overline{\Omega^*}}(-n, 0))$ and $S_2 := \partial\Omega_2$ is homologous to S in the domain of f, and S_2^* is homologous to S^* in the domain of g then $\langle\langle f, g\rangle\rangle_{I_S} = \langle\langle f, g\rangle\rangle_{I_{S_2}}$.

e) This is not *in general a duality pairing. When Ω is a product of bounded convex curves containing the origin, taking $\mathbf{p} = \mathbf{q} = (1, 0, \dots, 0)$, we have*

$$\inf_{g \in H^2(\Omega^*, \mathbf{p}, \mathbf{q}), ||g||=1} \ \sup_{f \in H^2(\Omega, \mathbf{p}, \mathbf{q}), ||f|| \le 1} |\langle\langle f, g\rangle\rangle| = 0.$$

In the C^2 strongly $\mathbb{C}$-convex case, one gets bounded operators $L^2(\Omega, \mu) \to H^2(\Omega, \mu)$ by pairing with $g_{\boldsymbol{\tau}}$ and letting $\boldsymbol{\tau}$ tend to the boundary [LS13]. This is far from obvious, as the integrals will develop singularities as $\boldsymbol{\tau}$ tends to the boundary. We emphasize here that we are not claiming that we get even unbounded operators $L^2 \to H^2$. It follows from item e) and following Barrett's proof in the introduction that if in fact this operator gives a map $L^2(S^*, \mathbf{p}, \mathbf{q}) \to H^2(\Omega^*, \mathbf{p}, \mathbf{q})$, it is necessarily unbounded.

Before proving the theorem, we collect some basic propositions and lemmas.

Proposition 4.2.3. *Let $(\mathbf{p}, \mathbf{q}), (\mathbf{p}', \mathbf{q}') \in \pi^{-1}(\Omega \times int(\Omega^*))$. There exists a constants C_1, C_2, depending on $(\mathbf{p}, \mathbf{q}), (\mathbf{p}', \mathbf{q}')$, so that for all $f \in L^2(S, \mathbf{p}, \mathbf{q}), g \in L^2(S^*, \mathbf{p}, \mathbf{q})$*

$$||f||_{\mathbf{p},\mathbf{q}} \le C_1 ||f||_{\mathbf{p}',\mathbf{q}'}$$
$$||g||_{\mathbf{p},\mathbf{q}} \le C_2 ||f||_{\mathbf{p}',\mathbf{q}'}.$$

Thus the L^2 and H^2 spaces above coincide as topological vector spaces for different choices of $(\mathbf{p}, \mathbf{q})$. Given a projective transformation $T : \Omega \to T(\Omega)$, with lift M, we have $||T^*(f)||_{\mathbf{p},\mathbf{q}} = ||f||_{M\mathbf{p}, M^{-t}\mathbf{q}}$.

Proof. Since $\mathbf{q}, \mathbf{q}' \in \Omega^*, \mathbf{q}' \cdot \mathbf{z} \neq 0$ and $\mathbf{q} \cdot \mathbf{z} \neq 0$ on $\overline{\Omega}$, and by compactness there are k, K so that $0 < k \le |\frac{\mathbf{q} \cdot \mathbf{z}}{\mathbf{q}' \cdot \mathbf{z}}| \le K$. The same argument applies to the forms $\mathbf{p} \cdot \mathbf{w}, \mathbf{p}' \cdot \mathbf{w}$, and the first claim follows. The second claim is applying the transformation laws for the various bundles, remembering that $|\omega|$ is projectively invariant. □

The previous proposition implies that the qualitative properties of the pairing are unchanged if we restrict ourselves to the case $\mathbf{p} = \mathbf{q} = (1, 0, \dots, 0)$, i.e. Ω is bounded containing the origin.

Lemma 4.2.4. *Pairing Calculation*
Let $\Omega = \mathbb{D}^n$ so that Ω^* is the ℓ^1-ball, and define the constants $c_n = \frac{(-1)^{\frac{n^2-n-2}{2}}}{(2\pi i)^n}$. We have

$$\langle\langle z_0^{-n}\vec{\boldsymbol{z}}^{\vec{\boldsymbol{a}}}, w_0^{-n}\vec{\boldsymbol{w}}^{\vec{\boldsymbol{b}}}\rangle\rangle = c_n \delta_{\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}} \cdot \beta(\vec{\boldsymbol{a}} + \vec{\mathbf{1}})$$

where $\beta(a_1, \dots, a_n) = \frac{\prod \Gamma(a_i)}{\Gamma(\sum a_i)}$

Proof. We can parameterize the ℓ^1-sphere by $\Delta^n \times b\mathbb{D}^n$ via the map $(\vec{\boldsymbol{t}}, \vec{\boldsymbol{z}}) \mapsto (t_i \overline{z_i})$, where $\Delta^n \times \vec{\boldsymbol{z}}$ parameterizes $W_{b\mathbb{D}^n}(\vec{\boldsymbol{z}})$. In these coordinates, we have $dw_i = t_i d\overline{z_i} + \overline{z_i} dt_i$. Substituting these expressions into ω and expanding, every term with $d\overline{z_i} \wedge d\vec{\boldsymbol{z}}$ vanishes since the $\vec{\boldsymbol{z}}$ takes value in a real n-dimensional manifold. Keeping in mind that $\overline{z_i} = \frac{1}{z_i}$, we are left with

$$\omega = (z_0 w_0)^n c_n d\vec{\boldsymbol{t}}_{[n]} \wedge \frac{d\vec{\boldsymbol{z}}}{z_1 \dots z_n} = (z_0 w_0)^n c_n d\vec{\boldsymbol{t}}_{[n]} \wedge d\vec{\boldsymbol{\theta}}$$

where θ_i is the angle along the circle in the i^{th} component. We calculate

$$\langle\langle z_0^{-n}\vec{\boldsymbol{z}}^{\vec{\boldsymbol{a}}}, w_0^{-n}\vec{\boldsymbol{w}}^{\vec{\boldsymbol{b}}}\rangle\rangle = c_n \int_{b\mathbb{D}^n} \vec{\boldsymbol{z}}^{\vec{\boldsymbol{a}}}\overline{\vec{\boldsymbol{z}}}^{\vec{\boldsymbol{b}}} \left(\int_{\Delta^n} \vec{\boldsymbol{t}}^{\vec{\boldsymbol{b}}} d\vec{\boldsymbol{t}}_{[n]} \right) d\vec{\boldsymbol{\theta}}.$$

From the S^n-invariance of the $\vec{\boldsymbol{z}}$-integral, we see that the integral is zero unless $\vec{\boldsymbol{a}} = \vec{\boldsymbol{b}}$. It just remains to show

$$\int_{\Delta^n} \vec{\boldsymbol{t}}^{\vec{\boldsymbol{a}}} d\vec{\boldsymbol{t}}_{[n]} = \beta(a_1 + 1, \dots, a_n + 1)$$

which is a standard fact about multivariate Dirichlet distributions, see section 2.1 of [TT11]. □

For the calculations that follow $\vec{\mathbf{1}} := (1, 1, \dots, 1)$ is the all-ones vector of length n.

Lemma 4.2.5. *Asymptotic Estimate*
There is a constant C_n depending on n so that

$$\lim_{m\to\infty} \frac{\beta((m+1) \cdot \vec{\mathbf{1}})}{m^{-\frac{n-1}{2}} n^{-nm}} = C_n$$

Proof. The main ingredient is Stirling's approximation.

$$\begin{aligned}\beta((m+1)\cdot\vec{\mathbf{1}}) = \frac{(m!)^n}{(mn+n-1)!} &\sim \frac{m^{n/2}(m/e)^{mn}}{\sqrt{mn+n-1}(\frac{mn+n-1}{e})^{mn+n-1}} \\ &\sim C_n m^{-\frac{n-1}{2}} n^{-nm}\left(\frac{m}{m+1-\frac{1}{n}}\right)^{mn} \\ &\sim C_n m^{-\frac{n-1}{2}} n^{-nm}\end{aligned}$$

□

Lemma 4.2.6. *Function Construction*

Let S be a smooth manifold, f_k a finite collection of smooth real-valued functions each with a minimum at $\vec{\boldsymbol{z}}_0$, $M = f_k(\vec{\boldsymbol{z}}_0)$ for all k. Then there exists a smooth function g so that $\max_k f_k \leq g$, $g(\vec{\boldsymbol{z}}_0) = M$ and $\vec{\boldsymbol{z}}_0$ is non-degenerate for g, that is Hessian of g at $\vec{\boldsymbol{z}}_0$ is strictly positive definite.

Proof. Take (real) coordinates $\vec{\boldsymbol{x}}$ centered at $\vec{\boldsymbol{z}}_{\mathbf{0}}$, then we have the Taylor expansion $f_k(\vec{\boldsymbol{x}}) = M + \vec{\boldsymbol{x}}^t H_k \vec{\boldsymbol{x}} + o(||\vec{\boldsymbol{x}}||^3)$, where H is positive semi-definite. Letting λ_k denote the largest eigenvalue of H_k, we have $\vec{\boldsymbol{x}}^t H_k \vec{\boldsymbol{x}} < \lambda_k ||\vec{\boldsymbol{x}}||^2$. So pick λ with $\lambda > \max_k \lambda_k$ and pick g to be smooth with local expansion $M + \lambda \vec{\boldsymbol{x}}^t \vec{\boldsymbol{x}} + o(||\vec{\boldsymbol{x}}||^3)$. □

Lemma 4.2.7. *Laplace Bounds*

Let f be a non-negative smooth function on the n-dimensional compact manifold S, μ a measure mutually absolutely continuous with respect to Lebesgue. Suppose $N := \{\vec{\boldsymbol{z}} | f(\vec{\boldsymbol{z}}) = \sup_S f\}$ is a non-degenerate critical manifold of dimension k, meaning the Hessian of f has rank $n-k$ at every point of N. Then there exists a constants $C_1, C_2 > 0$ so that for $m >> 1$, we have

$$C_1 \sup(e^{m\cdot f}) m^{-\frac{n-k}{2}} \geq \int_S e^{mf} d\mu \geq C_2 \sup(e^{m\cdot f}) m^{-\frac{n-k}{2}}.$$

Proof. Cover M with finitely many coordinate patches U_j so that either $U_j \cap N = \emptyset$ or N is given by the last $n-k$ coordinates being 0, and take a partition of unity subordinate to the covering φ_j. On each U_j, with $U_j \cap N = \emptyset$ we have

$$|\int_{U_j} \varphi_j e^{m\cdot f} d\mu| \leq C \sup_S e^{m\cdot(f-\epsilon)}$$

and so in the limit, none of these integrals will affect our estimates.

For U_j intersecting N, write $d\mu = h d\vec{\boldsymbol{x}}$ and first applying Fubini

$$\int_{U_j} \varphi_j e^{m\cdot f} d\mu = \int_{x_1,\ldots,x_k} \left(\int_{x_{k+1},\ldots,x_n} e^{m\cdot f} \varphi_j \cdot h \, dx_{k+1} \ldots dx_n \right) dx_1 \ldots dx_k$$

For the inner integrand, we are in exactly the situation where the Laplace method applies, and we obtain constants $C_{1,j}, C_{2,j} > 0$ so that

$$C_{2,j} \sup(e^{m\cdot f}) m^{-\frac{n-k}{2}} \leq \int_{x_{k+1},\ldots,x_n} e^{m\cdot f} \varphi_j \cdot h \, dx_{k+1} \ldots dx_n \leq C_{1,j} \sup(e^{m\cdot f}) m^{-\frac{n-k}{2}}$$

Finishing the integration and then summing up the estimates from each coordinate patch produces the result. □

Proof. Proof of Theorem 4.2.2

a) Continuity follows from applying Cauchy-Schwartz to the point-wise identity

$$|f||g||\omega| = |f| \left(\frac{|\mathbf{q} \cdot \mathbf{z}|}{|\mathbf{p} \cdot \mathbf{w}|} \right)^{\frac{n}{4}} |g| \left(\frac{|\mathbf{p} \cdot \mathbf{w}|}{|\mathbf{q} \cdot \mathbf{z}|} \right)^{\frac{n}{4}} |\omega|$$

implying

$$|\langle\langle f, g \rangle\rangle| \leq ||f||_{\mathbf{p},\mathbf{q}} ||g||_{\mathbf{p},\mathbf{q}}.$$

b) This is the reproducing property of Theorem 2.4.9.

c) Injectivity follows from the following general fact; given two Hilbert spaces H_1, H_2 and a non-degenerate pairing $\langle\langle , \rangle\rangle : H_1 \times H_2 \to \mathbb{C}$, then the map $H_1 \to H_2^*$ has dense image. Suppose for contradiction that the image of H_1 in H_2^* were contained in a proper closed subspace K. Then there exists $h_2 \in H_2$ so that $K(h_2) = 0$, violating the non-degeneracy assumption.

d) The integrand is holomorphic of maximal degree, thus closed.

e) Let $\Omega = \mathring{\gamma}_1 \times \ldots \mathring{\gamma}_n$ be a bounded convex product domain containing the origin. We will take $\mathbf{p} = \mathbf{q} = (1, 0, \ldots, 0)$, dropping subscripts for convenience. Let k denote the number of γ_i which are circles centered at the origin. Note when applying lemma 4.2.7 to expressions of the form $\int_S |\vec{\boldsymbol{z}}|^m$, the critical manifold has dimension k, and further assume that this manifold is non-degenerate (equivalently, $|\cdot|$ has non-degenerate minima on each γ_i). Set $r_i = \inf_{\vec{\boldsymbol{z}} \in S} |z_i|$ so that $\sup_S |\vec{\boldsymbol{z}}^{-m\cdot\vec{\mathbf{1}}}| = \vec{\boldsymbol{r}}^{-m\cdot\vec{\mathbf{1}}}$, $\sup_S(||z||_\infty^{-1})^{-1} = \max_i r_i$. By scaling each γ_i and applying prop 4.2.3, we may assume that all r_i are equal to r, and that the first k γ_i are circles.
Set $d_m = ||(w_0)^{-n} \vec{\boldsymbol{w}}^{m\cdot\vec{\mathbf{1}}}||^{-1}$, and let $f_m \in H^2(S_n)$ be such that $||f_m|| = 1$ and

$$\sup_{f \in H^2(S), ||f|| \leq 1} |\langle\langle f, d_m (w_0)^{-n} \vec{\boldsymbol{w}}^{m\cdot\vec{\mathbf{1}}} \rangle\rangle| \leq |\langle\langle f_m, d_m (w_0)^{-n} \vec{\boldsymbol{w}}^{m\cdot\vec{\mathbf{1}}} \rangle\rangle| + \frac{1}{m}.$$

We may further assume that f_m is entire by approximating $f_m((1+\epsilon)\vec{\boldsymbol{z}})$ uniformly on $\overline{\Omega}$ by an entire function, see Theorem 3.1.3 in [APS04]. Writing $f_m = (z_0)^{-n} \sum a_{\vec{\alpha}} \vec{\boldsymbol{z}}^{\vec{\alpha}}$, it follows from lemma 4.2.4 and property d) that

$$\langle\langle f_m, d_m \vec{\boldsymbol{w}}^{m\cdot\vec{\mathbf{1}}}\rangle\rangle = \langle\langle (z_0)^{-n} a_{m\cdot\vec{\mathbf{1}}} \vec{\boldsymbol{z}}^{m\cdot\vec{\mathbf{1}}}, d_m (w_0)^{-n} \vec{\boldsymbol{w}}^{m\cdot\vec{\mathbf{1}}}\rangle\rangle = a_{m\cdot\vec{\mathbf{1}}} \cdot d_m \cdot c_n \beta((m+1)\cdot\vec{\mathbf{1}}).$$

We begin by estimating the $|a_m|$. By differentiating the usual Cauchy formula and appealing to lemma 4.2.7, we have

$$\begin{aligned} a_m &= |c_n| \int_{S_n} (z_0)^n f_m(\vec{\boldsymbol{z}}) \prod_{i=1}^{n} \vec{\boldsymbol{z}}^{-m\cdot\vec{\mathbf{1}}} d\vec{\boldsymbol{\theta}} \\ |a_m| &\le |c_n| ||f_m|| (\int_S |\vec{\boldsymbol{z}}^{-2m\cdot\vec{\mathbf{1}}}| d\vec{\boldsymbol{\theta}})^{\frac{1}{2}} \\ &\le C \sup_S |\vec{\boldsymbol{z}}^{-m}| m^{-\frac{n-k}{4}} = C r^{mn} m^{-\frac{n-k}{4}}. \end{aligned}$$

Now we need estimates on the asymptotics of d_m. Set $\vec{\boldsymbol{w}}_i(\vec{\boldsymbol{z}}) = \frac{\vec{\boldsymbol{e}}_i}{z_i}$, so $\vec{\boldsymbol{w}}_i$ is the i^{th} strong tangent at $\vec{\boldsymbol{z}}$. Using Remark 3.2.6 with $\alpha_i = \frac{|z_i|}{z_i}$,

$$\begin{aligned} d_m^{-2} &= \int_{S^*} |\vec{\boldsymbol{w}}|^{2m} |\frac{w_0}{z_0}|^{\frac{n}{2}} |\omega| \\ &= \int_S \int_{\Delta^n} \frac{\vec{\boldsymbol{t}}^{2m\cdot\vec{\mathbf{1}}}}{|\sum_i t_i \alpha_i z_i|^{2mn}} |\frac{w_0}{z_0}|^{\frac{n}{2}} |\omega| \\ &\ge \beta((2m+1)\cdot\vec{\mathbf{1}}) \int_S ||\vec{\boldsymbol{z}}||_\infty^{-2mn}. \end{aligned}$$

Note that since the first k factors are circles which have constant radius r, and all other curves are distance $\ge r$ away from 0, we have $||\vec{\boldsymbol{z}}||_\infty = \max_i |z_i| = \max_{i>k} |z_i|$. Each $|z_i|$ has a minimum r, so by lemma 4.2.6, there exists a smooth function $g = g(z_{k+1}, \ldots z_n)$ so that $||\vec{\boldsymbol{z}}||_\infty \le g$, $\min(g) = \min(||\vec{\boldsymbol{z}}||_\infty)$ implying $\sup(\frac{1}{g}) = \sup(||\vec{\boldsymbol{z}}||_\infty^{-1}) = r^{-1}$. Lastly observe that the level set of $g^{-1} = r^{-1}$ has dimension k and applying lemma 4.2.7, we have

$$\begin{aligned} \beta((2m+1)\cdot\vec{\mathbf{1}}) \int_S ||\vec{\boldsymbol{z}}||_\infty^{-2mn} &\ge \beta((2m+1)\cdot\vec{\mathbf{1}}) \int_S g^{-2mn} \\ &\ge \beta((2m+1)\cdot\vec{\mathbf{1}}) r^{-2mn} m^{-\frac{n-k}{2}} \end{aligned}$$

implying

$$d_m \le r^{mn} m^{\frac{n-k}{4}} \beta((2m+1)\vec{\mathbf{1}})^{-\frac{1}{2}}.$$

Putting it all together and applying lemma 4.2.5,

$$\begin{aligned}a_{m\cdot\vec{\mathbf{1}}}\cdot d_m\cdot\beta((m+1)\cdot\vec{\mathbf{1}}) &\leq C_4\cdot\beta((m+1)\cdot\vec{\mathbf{1}})\beta((2m+1)\cdot\vec{\mathbf{1}})^{-\frac{1}{2}}\\ &\leq C_4 m^{\frac{-1}{4}}.\end{aligned}$$

□

Example 4.2.8. *If we had a duality pairing, it would follow from the open mapping theorem that the image of $H^2(S^*)$ is all of $H^2(S_n)^*$. We demonstrate by example that this is not the case. Consider $\Omega = \mathbb{D}^2$ so that $\Omega^* = \ell^1 := |\{(w_1, w_2)||w_1| + |w_2| < 1\}|$, and consider the functional*

$$f \mapsto \int_{b\mathbb{D}^2} f\overline{\sum_{m=1}^{\infty}\frac{1}{m}z_1^m}d\theta_1 d\theta_2.$$

By Lemma 4.2.4, the only candidate power series in $H^2(l^1)$ is $\sum_{m\in\mathbb{Z}_{\geq 0}}\frac{m+1}{m}w_1^m$. All the w_1^m are orthogonal by symmetry, so we have

$$\begin{aligned}||\sum_m \frac{m+1}{m}w_1^m||^2 &= \sum_m \frac{(m+1)^2}{m^2}||w_1^m||^2\\ &= \sum_m \frac{(m+1)^2}{m^2}\beta(2m+1,1)\\ &= \sum_m \frac{(m+1)^2}{m^2(2m+1)}\end{aligned}$$

which diverges. Taking this example further, we see that if $g = \sum_{m_1,m_2\in\mathbb{Z}_{\geq 0}} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}$, the only candidate in $H^2(\partial\ell^1)$ for the functional

$$f \mapsto \int_{b\mathbb{D}^2} f\overline{g}d\theta_1 d\theta_2$$

is given by the formal series

$$\tilde{g} = \sum_{m_1,m_2\in\mathbb{Z}_{\geq 0}} \frac{\overline{a_{m_1,m_2}}}{\beta(m_1,m_2)} w_1^{m_1} w_2^{m_2}.$$

The set of monomials $\{w_1^{m_1}w_2^{m_2}\}$ are mutually orthogonal by the S^2 symmetry of ℓ^1, and

$$||w_1^{m_1}w_2^{m_2}||^2 = \beta(2m_1, 2m_2).$$

We see that the image of $H^2(\partial\ell^1)^*$ in $H^2(b\mathbb{D}^2)$ consists of power series

$$g = \sum_{m_1,m_2\in\mathbb{Z}_{\geq 0}} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}$$

where

$$\sum_{m_1,m_2\in\mathbb{Z}_{\geq 0}} \frac{\beta(2m_1,2m_2)}{\beta(m_1,m_2)^2} |a_{m_1,m_2}|^2 < \infty.$$

4.3 Invariant Sections and Hardy Spaces when $n = 2$

In this section, we describe how to construct a completely projectively invariant Hardy space when $n = 2$. We would like a positive, projectively invariant $O_{S_2}(2,2)$-valued 2-form, and as previously described $|z_0^2 d\vec{z}|$ takes values in the wrong bundle. However, if we construct a projectively invariant section of $O_{S_2}(j,j)$ for $j \neq 0$, then the form $|\sigma|^{\frac{3}{2j}}|z_0^2 d\vec{z}|$ will have values in the correct bundle.

Proposition 4.3.1. *Let $\Omega \subset \mathbb{P}^2$ be a piece-wise smooth pseudoconvex domain. Then there exists a projectively invariant section $\sigma \in O_{S_2}(\frac{3}{2},\frac{3}{2})$.*

Proof. Let $\mathbf{z} \in S_2$ with boundary surfaces H_1, H_2, and by Lemma 3.2.2, we can make a projective change of coordinates so that $\mathbf{z} = 0$ and $TC_0(\overline{\Omega}) = \{Im(z_1) \leq 0, Im(z_2) \leq 0\}$. This choice uniquely determines the projective coordinates up the the action of the group generated by the following matrices.

$$I_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \lambda \in \mathbb{C},\ S_r := \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix},\ r \in \mathbb{R}_{>0},\ W := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Write $z_j = x_j + iy_j$ so that on $H_1 \cap H_2$, we have

$$\begin{aligned} y_1 &= a_1 x_1^2 + b_1 x_1 x_2 + c_1 x_2^2 + h.o.t. \\ y_2 &= a_2 x_2^2 + b_2 x_1 x_2 + c_2 x_1^2 + h.o.t. \end{aligned}$$

Using Mathematica, we can compute how the normal form changes by pulling back by each of the above matrices. For convenience, identify the normal form with the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

We write down the result of the computations, which can be found in the Appendix.

$$W^* \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{pmatrix}$$
$$I_\lambda^* \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 + Im(\lambda) & b_1 & c_1 \\ a_2 & b_2 + Im(\lambda) & c_2 \end{pmatrix}$$
$$S_r^* \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 r & b_1 & \frac{c_1}{r} \\ a_2 & b_2 r & c_2 r^2 \end{pmatrix}$$

It follows that the quantity $\sigma := (b_1 - a_2)(b_2 - a_1)$ is a projectively invariant section of $O_{S_2}(\frac{3}{2}, \frac{3}{2})$. □

We remark that when $\Omega = \mathring{\gamma}_1 \times \mathring{\gamma}_2$, it follows that $\sigma = 4\kappa_1\kappa_2$, where κ_j is the *signed* curvature of each γ_j, and thus is everywhere positive exactly when Ω is a $\mathbb{C}$-polytope. An interesting, related question is given a totally real submanifold S of $\mathbb{P}^2$, are there any projectively invariant sections of $O_S(j, j)$? This can be approached in the same method as above, with the additional matrices

$$S_r, r \in \mathbb{R}_{<0},\ H_r := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix},\ r \in \mathbb{R}\backslash\{0\}.$$

In the Appendix, it is calculated that

$$H_r^* \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} =$$
$$\begin{pmatrix} a_1 + rb_1 + r^2 c_1 & b_1 + 2rc_1 & c_1 \\ a_2 - rc_1 & b_2 + 2ra_2 - rb_1 + 2r^2 c_1 & c_2 + rb_2 + r^2 a_2 - r(a_1 + rb_1 + r^2 c_1) \end{pmatrix}.$$

APPENDIX

Projective Normal Form Calculations

```
PullBack[expr1_, expr2_, vars_, M_]:=Module[{z1, z2, w1, w2, x1, y1, x2, y2, t, newexpr1,
    newexpr2},
z1 = x1 + I y1;
z2 = x2 + I y2;
w1 = (M[[1, 2]] + M[[2, 2]] * z1 + M[[3, 2]] * z2)/(M[[1, 1]] + M[[2, 1]] * z1 + M[[3, 1]] * z2);
w2 = (M[[1, 3]] + M[[2, 3]] * z1 + M[[3, 3]] * z2)/(M[[1, 1]] + M[[2, 1]] * z1 + M[[3, 1]] * z2);

(*Substituting in Real, Imaginary parts of transformed variables into the normal form*)
newexpr1 = expr1/.Thread[vars->{ComplexExpand[Re[w1]], ComplexExpand[Re[w2]],
    ComplexExpand[Im[w1]], ComplexExpand[Im[w2]]}];
newexpr2 = expr2/.Thread[vars->{ComplexExpand[Re[w1]], ComplexExpand[Re[w2]],
    ComplexExpand[Im[w1]], ComplexExpand[Im[w2]]}];

(*Discarding all terms of degree > 2, remembering that y1, y2 have degree2*)
newexpr1 = Normal[Series[newexpr1/.Thread[{y1, y2, x1, x2}->{t^2 * y1, t^2 * y2, t * x1,
    t * x2}], {t, 0, 2}]]/.t->1;
newexpr2 = Normal[Series[newexpr2/.Thread[{y1, y2, x1, x2}->{t^2 * y1, t^2 * y2, t * x1,
    t * x2}], {t, 0, 2}]]/.t->1;

(*Substituting in original variables*)
```

```
{Collect[newexpr1/.Thread[{x1, x2, y1, y2}->vars], vars], Collect[newexpr2/.Thread[{x1, x2, y1,
    y2}->vars], vars]}
]
```

```
PullBack[a1 * s1^2 − t1, a2 * s2^2 − t2, {s1, s2, t1, t2}, {{1, 0, 0}, {−a1 * I, 1, 0},
    {−a2 * I, 0, 1}}]
```

$\{-\mathrm{a2s1s2} - \mathrm{t1}, -\mathrm{a1s1s2} - \mathrm{t2}\}$

Pullback Rules Under Different Generators

```
PullBack[a1 * s1^2 + b1 * s1 * s2 + c1 * s2^2 − t1, a2 * s2^2 + b2 * s1 * s2 + c2 * s1^2 − t2,
    {s1, s2, t1, t2}, {{1, 0, 0}, {λ, 1, 0}, {0, 0, 1}}]
```

$\left\{\mathrm{a1s1}^2 + \mathrm{b1s1s2} + \mathrm{c1s2}^2 - \mathrm{t1}, \mathrm{c2s1}^2 + \mathrm{b2s1s2} + \mathrm{a2s2}^2 - \mathrm{t2}\right\}$

```
PullBack[a1 * s1^2 + b1 * s1 * s2 + c1 * s2^2 − t1, a2 * s2^2 + b2 * s1 * s2 + c2 * s1^2 − t2,
    {s1, s2, t1, t2}, {{1, 0, 0}, {I * λ, 1, 0}, {0, 0, 1}}]
```

$\left\{\mathrm{b1s1s2} + \mathrm{c1s2}^2 - \mathrm{t1} + \mathrm{s1}^2(\mathrm{a1} + \lambda), \mathrm{c2s1}^2 + \mathrm{a2s2}^2 - \mathrm{t2} + \mathrm{s1s2}(\mathrm{b2} + \lambda)\right\}$

```
PullBack[a1 * s1^2 + b1 * s1 * s2 + c1 * s2^2 − t1, a2 * s2^2 + b2 * s1 * s2 + c2 * s1^2 − t2,
    {s1, s2, t1, t2}, {{1, 0, 0}, {0, r, 0}, {0, 0, 1}}]
```

$\left\{\mathrm{a1}r^2\mathrm{s1}^2 + \mathrm{b1}r\mathrm{s1s2} + \mathrm{c1s2}^2 - r\mathrm{t1}, \mathrm{c2}r^2\mathrm{s1}^2 + \mathrm{b2}r\mathrm{s1s2} + \mathrm{a2s2}^2 - \mathrm{t2}\right\}$

```
res = PullBack[a1 * s1^2 + b1 * s1 * s2 + c1 * s2^2 − t1, a2 * s2^2 + b2 * s1 * s2 + c2 * s1^2
    −t2, {s1, s2, t1, t2}, {{1, 0, 0}, {0, 1, r}, {0, 0, 1}}]
```

$\left(\mathrm{a1} + \mathrm{b1}r + \mathrm{c1}r^2\right)\mathrm{s1}^2 + (\mathrm{b1} + 2\mathrm{c1}r)\mathrm{s1s2} + \mathrm{c1s2}^2 - \mathrm{t1}, \left(\mathrm{c2} + \mathrm{b2}r + \mathrm{a2}r^2\right)\mathrm{s1}^2 +$

$(\mathrm{b2} + 2\mathrm{a2}r)\mathrm{s1s2} + \mathrm{a2s2}^2 - r\mathrm{t1} - \mathrm{t2}$

```
Collect[res[[2]] − r * res[[1]], {s1, s2, t1, t2}]
```

$$\left(\mathrm{c2}+\mathrm{b2}r+\mathrm{a2}r^2-r\left(\mathrm{a1}+\mathrm{b1}r+\mathrm{c1}r^2\right)\right)\mathrm{s1}^2+(\mathrm{b2}+2\mathrm{a2}r-r(\mathrm{b1}+2\mathrm{c1}r))\mathrm{s1}\mathrm{s2}+(\mathrm{a2}-\mathrm{c1}r)\mathrm{s2}^2-\mathrm{t2}$$

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