On Information Policy Design and Strategic Interactions

by

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ABSTRACT

My dissertation investigates the design of information policy in three different types of strategic interactions as specified below:

Chapter II: "*Performance Evaluation Design in Dynamic Incentive Contracts*", examines how to motivate employees in an organization by strategically disclosing evaluations. Specifically, consider a continuous-time principal-agent setting where the agent exerts effort to generate output and the principal subjectively evaluates and pays for the agent's performance. If the principal can commit, what type of evaluation system should she implement? I show that the optimal evaluation system is not based directly on the output, but rather on an adjusted evaluation of output. In particular, it assigns inflated evaluations when the agent's continuation value is low and deflated evaluations when the continuation value is high. Adding such adjustment into evaluations allows the principal to recalibrate the agent's continuation value, which improves the contract by capturing gains from concavification that are not feasible in contracts directly based on output. As a result, adjusting evaluations also induces weakly higher volatility in the agent's continuation value even though the agent is risk-averse. Moreover, I show that additional contractual possibilities such as leaving the firm for an outside option, promotion, and reciprocity in output, could result in strengthening different evaluation biases at the optimum. My results help explain evaluation biases that have been empirically observed in appraisal systems.

Chapter III: "*Persuasion of Interacting Receivers*", investigates how the consideration of realistic features could complicate the structure of information policy in situations of strategic interaction. Its main contribution is to propose a general framework that allows a

formal characterization for several relevant features and provides a tractable way to design optimal information policy in such settings. Specifically, I consider a setting where a sender could communicate privately with multiple receivers before they engage in a one-shot strategic interaction. To understand optimal information policy, standard approaches would focus on signals that recommend actions. However, if there are realistic features outside the scope of the standard model, such a focus could be suboptimal. I consider the following four features: (i) the equilibrium selection rule may be different from the sender-preferred one, (ii) receivers may have private information, (iii) receivers may have non von Neumann–Morgenstern utilities, and (iv) receivers may have heterogeneous priors. I establish a generalized obedience principle. In this version, the sender recommends actions and *conjectures*. I further provide a sufficient condition under which it is without loss of generality to reduce the messages involved in any signal from continuum to countably many. The construction of the proof provides a tractable way to compute optimal signals. I also apply my result to study information policy design in two applications in which the equilibrium selections differ from the sender-preferred selection and receivers may be privately informed.

Chapter IV:"*Derandomized Persuasion Mechanisms*", investigates when focusing only on information policy that either fully reveals or (partially) pools the underlying states without adding extra noise is without loss of generality. I consider a setting where one sender can communicate with several privately informed receivers through a persuasion mechanism before the receivers play a game. I show that for any potentially randomized persuasion mechanism, under certain conditions, there is an *effectively equivalent* derandomized persuasion mechanism, and these two mechanisms have the same set of equilibria. I exhibit the usefulness of my result in a specific disclosure problem, where I apply my result to derandomize optimal disclosure mechanisms. Overall, this paper provides a rationale for the fact that persuasion mechanisms are often deterministic in practice.

CHAPTER I

Introduction

Information asymmetry is present in almost all kinds of economic activities. When some parties have private information crucial to the outcome, they could exploit such an advantage by strategically providing information. The uninformed understand the conflict of interests and would try to unravel such manipulation. My dissertation investigates information policy design from economic and technical aspects in three different strategic situations, and I will describe them in detail below.

A lot of institutions nowadays conduct evaluations on their employees. Given that those evaluations are usually tied to employees' compensation, the disclosure policy of the evaluation outcome would greatly affect the employees' incentives. In Chapter II, "*Performance Evaluation Design in Dynamic Incentive Contracts*", I study the design of evaluation schemes in a dynamic contracting setting. My analysis is based on a continuous-time Principal-Agent setting with two-sided private information, where the agent privately knows his effort level and the principal privately knows the noisy output that the agent generates. A good example for such a setting would be the following: Consider a customer service agent whose duty is to answer customers' calls. He knows how much effort he puts in to help customers with their problems, but the manager (or the principal) does not know. To evaluate the quality of the agent's service, the manager asks customers to fill out a short survey about their satisfaction right after being served. The result of the survey is only observable by the manager, not the agent.

My result shows that the principal can benefit from evaluating the agent not directly based on the output he generated. In particular, the optimal evaluation scheme assigns inflated evaluations when the agent's continuation value is low, and deflated evaluations when the continuation value is high. In doing so, the principal could extend the duration in which the agent exerts high effort, therefore strictly improving productivity. Despite the agent's risk aversion, adding evaluation bias induces weakly higher volatility in the agent's continuation value than the benchmark when the principal bases her evaluation directly on the output. If the principal has the flexibility to choose report strategies, I show that it is optimal for her to file frequent instantaneous reports about the agent's evaluation. In an extension, I examine the effect of additional contractual possibilities, such as promotion or the agent quitting the firm, and show that the above result is robust under these considerations. My result could explain the long-existing evaluation bias in practical appraisal systems, and provides a rationale behind the fact that many companies now switch from an annual report to the frequent performance evaluation.

In Chapter III, "*Persuasion of Interacting Receivers*", I study information design in non-standard settings. In standard settings we can use either the concavification approach, or in a somewhat broader range of settings we can use what is called the "revelation principle for information design", or the "obedience principle". This principle says that it is without loss of generality to assume that the messages the sender sends to the receivers are action recommendations in the equilibrium she picks and the receivers are willing to follow such recommendations. However, it is known in the literature that the obedience principle does not hold more generally. I study how the optimal information structure may change when the standard conditions for the obedience principle are violated. Specifically, I consider an information design setting where there are multiple receivers, who may then interact in a game after receiving their messages from the sender. Moreover, the game I consider has the following four features: (1) the equilibrium selection rule may differ from the sender-preferred one; (2) receivers may have their private information; (3) receivers may have non-expected utility and (4) receivers may have heterogeneous prior beliefs.

The first result of my paper is a generalized obedience principle. In this version, the information structure recommends both action and conjectures to each receiver. The term "conjecture" is defined as a receiver's belief over the following three things: the underlying state, the other receivers' belief hierarchies, and their actions. I show when we can simplify these recommendations further by reducing its cardinality from potentially a continuum to countable. The proof provides an algorithm for computing an optimal information structure.

I use the above result to compute the optimal information structure in two applications. The first application considers the case of a politician who would like to persuade privately-informed voters to support her proposal. Moreover, I assume that the voters will abstain when they perceive their votes are not pivotal. Under this assumption, the sender cannot always select the equilibrium she favors.[1] Given these nonstandard features, the design problem in this setting cannot be solved by standard approaches in the literature. My result shows that the optimal information structure strategically coordinates different voters and pools unfavorable states with favorable states in such a way that each voter, whenever pivotal, finds it weakly better to vote for the politician's proposal. Thus the information the sender disseminates not only influences voters' belief about the underlying state but also what other voters will do. The second application considers a bank regulator disclosing a stress test result to privately informed investors. In particular, the regulator wants to minimize the possibility of a bank run under a sender-worse equilibrium selection. In this case, belief hierarchies matter for characterizing investors' behavior under the sender-worse equilibria. By applying the result, I show that the optimal information structure in such a setting disseminates information more than merely action recommendations because it needs to influence each investor's belief about others' lower order beliefs.

In Bayesian Persuasion models, we often find persuasion mechanisms that involve randomization. But the literature has not established in general whether this randomization is necessary or can be purified away. Say a signal is derandomized if it either fully reveals the state or pools over several states without adding extra noise. Such signals are easy to design and implement in reality. In Chapter IV, "*Derandomized Persuasion Mechanisms*", I investigate when it is without loss of generality to use persuasion mechanisms that only implement derandomized signals. The setting I examine is the following: There is a sender and multiple privately-informed receivers. The sender could commit to a persuasion mechanism that allows two-way communications between her and each receiver. Specifically, the mechanism first solicits private reports from each receiver and then picks a signal that reveals some private information, potentially different, to each receiver about the underlying state. After communicating with the mechanism, receivers then engage in a one-shot game.

I provide a set of conditions under which for any persuasion mechanism, we can find

[1] For example, even if the politician recommends everyone to vote for him, since no one's vote is pivotal it is possible will vote and the proposal will not pass.

a derandomized persuasion mechanism such that the set of equilibria under the original mechanism is preserved under the new one. Moreover, the set of conditions is tight in the sense that whenever one condition is violated, there may exist a counterexample. This result provides a useful simplification technique to derive optimal information policy. Overall, this chapter provides a rationale for the fact that persuasion mechanisms are often derandomized in practice.

CHAPTER II

Performance Evaluation Design in Dynamic Incentive Contracts

2.1 Introduction

Most workers are evaluated on subjective criteria, and firms design contracts that specify how to evaluate employees and how to pay based on those evaluations (Prendergast 1999). Commitment often plays a key role in the design; many companies, such as Intel (Lawler 2003) and Amazon (Del Rey 2021), assign subjective evaluations based on a predetermined distribution.[1] Evaluating employees is thus a committed procedure that transforms the performance-relevant data into the final evaluation. There is widely documented evidence that firms make adjustments in such evaluation procedures. Given that the initial feedbacks usually have summarised all the information about employees' performance, it seems a mystery how firms could benefit from adding adjustments into the evaluation. This is the main research question this paper would like to study.

Consider a continuous-time moral hazard model with a risk-neutral Principal and a risk-averse Agent who privately chooses instantaneous effort of binary value to generate a noisy and unobservable output process X. The principal holds a belief P about how the output evolves and commits to an evaluation system and a compensation system. The evaluation system generates a noisy evaluation process X^α, specifies a belief Q on how the future evaluation would unfold, and reports some information Y about the evaluation. The compensation system pays the agent based on the report Y. This is analogous to that the

[1] In fact, by 2012, an estimated 60% of Fortune 500 firms adopt some form of forced rankings (Kwoh 2012).

employer sets employees' expectations about the evolution of the evaluation by announcing the evaluation plan at the beginning. Let the evaluation process X^α evolve as follows: $\mathrm{d}X_t^\alpha = \mathrm{d}X_t + \alpha_t \,\mathrm{d}t$, where α is the adjustment added into the evaluation. To avoid that the principal could obtain better information about the agent's effort via evaluation systems, I restrict the distribution of the evaluation X^α under the chosen belief Q to be the same as that of the output X under Principal's belief P. Moreover, I assume that Agent has limited liability, adopts the belief Q specified by the system and acts in his own best interest.

To put this setting into some context, consider the following example of a customer service agent whose job is to answer customers' calls. The agent knows how much effort he puts in to help customers with their problems, but the manager does not know. To evaluate the service quality, the principal asks customers to fill out a short survey about their satisfaction right after being served. The result of the survey is only observable by the manager but not the agent. It is natural for the manager to relate the compensation to the results of the surveys in the hope of motiving the agent to work hard. This conduct, however, brings extra costs, since the agent is risk-averse and customers' response is only a noisy signal about his effort. The question is whether evaluating the agent not directly based on the survey outcome could reduce such costs in some way.

My main result shows that introducing adjustments into the evaluation system is beneficial for the principal. In particular, the optimal incentive contract has the following three features: First, the optimal evaluation system does not evaluate Agent directly based on the output: the system will add the maximum positive adjustment into the evaluation when Agent's expected future payoff is low and will add the maximum negative adjustment when his expected future payoff is high. Second, despite the agent's risk aversion, the volatility of his continuation payoff may be higher at the optimum than the benchmark case when the principal bases the evaluation directly on the output. Third, the principal truthfully reports the agent's instantaneous evaluation.

Why is adding adjustments to the evaluation beneficial in the above situation? Intuitively, the principal could delay terminating the employment relationship by adding adjustment into evaluations and therefore improve her payoff. Such termination arises naturally in this relationship because the agent has limited liability and is risk-averse with diminishing marginal utility. Given his limited liability, the agent's utility cannot drop below 0 since otherwise, he could shirk forever to get at least 0. Thus the principal necessarily fires the agent when his continuation value drops to 0. On the other hand, given the agent's risk aversion with diminishing marginal utility, when his continuation value gets too high, he needs to be retired since it is very costly to induce him to exert high effort. The first feature of the optimal contract is to add leniency bias for low continuation values and severity bias for high continuation values, to lean against firing and retiring. However, extending the expected employment duration in this way creates nonconcavities in the principal's value function. There is room for improvement by concavification. So the second role of the evaluation process is to insert what amounts to conditional randomization to concavify the principal's value function. This generates the second feature of the optimal contract, in which the agent's continuation value is weakly noisier at the optimum.

My result justifies the existence of the evaluation biases that have been documented in the empirical literature (cf. Landy and Farr 1980, Jawahar and Williams 1997, Prendergast 1999, Moers 2005, Frederiksen et al. 2017). In addition, my main result implies that not directly basing the evaluation on the output could provide partial insurance for Agent's firing risk, which aligns with the empirical findings in Gibbs et al. (2004) and Bol and Smith (2011). Also, the following phenomenon coincides with my result on the optimality of frequent reporting: more than $1/3$ of U.S. companies have abandoned the traditional annual review approach and adopt a new evaluation policy that provides frequent and exhaustive reports to the employees' performance. These companies include General Electric, Adobe, and Microsoft; see Cappelli and Tavis (2016) and Baldassarre and Finken (2015) for more discussion.

In an extension, I study performance evaluation design under three realistic contractual possibilities: (i) the reciprocal effect of evaluation biases on performance; (ii) Agent can be promoted; and (iii) Agent can quit the firm and receive a positive outside option. My result shows that, depending on the situation, leniency or severity bias may be strengthened in the optimal incentive contract. This result suggests that employers should treat different evaluation biases differently in a context-based manner.

LITERATURE REVIEW

There are five strands in the literature relevant to my paper. First, this paper relates the literature of contracting with subjective evaluations, where the principal privately observes the output, with or without monitoring costs, and the compensation is based on the report the principal made ("pay for performance"). The most relevant papers are those with commitments on performance evaluation schemes such as Georgiadis and Szentes (2020) and Zábojník (2014). Similar to my focus, Georgiadis and Szentes (2020) study a continuous-time moral hazard model with the principal committing to a subjective evaluation scheme; but their consideration is on the tradeoff between the monitoring and information acquisition cost in the optimal contract, which differs from mine.[2] Zábojník (2014) studies a two-period model where production is affected by both Agent's effort and his ability. The principal privately observes the agent's ability after period 1 and then provides feedback through the subjective evaluation, which affects the agent's future effort provision. This feedback role of evaluation is absent in my model since my agent does not have a private type.

The majority in this literature assume that the principal cannot commit to messaging strategy, thus reneging becomes an issue. See, for instance, MacLeod (2003), Levin (2003), Fuchs (2007) and Zhu (2020). My result is starkly different from the results in this literature: for instance, the optimal subjective rating without commitment may be less informative (cf. MacLeod 2003, Fuchs 2007 and Zhu 2020). Also the optimal contract without commitment on messaging strategy has a money-burning feature (cf. MacLeod 2003 and Fuchs 2007), which is absent from my setting. There is another strand of literature showing that introducing subjective evaluation could improve the contract by (i) mitigating the incentive distortions based on the distortionary objective measure (Baker et al. 1994) or (ii) allowing Principal to utilize non-contractible information (Baiman and Rajan 1995 and Hayes and Schaefer 2000); or (iii) reducing risks (Gibbs et al. 2004). My result also provides a new explanation for firms not basing evaluation directly on the outputs.

The second strand is the literature of Bayesian persuasion and information design pioneered by Kamenica and Gentzkow (2011) and Rayo and Segal (2010). In particular, my work is most relevant to the strand that studies persuasion in dynamic moral hazard problems, including Ely (2017) and Renault et al. (2017) studying a myopic agent, Ely and Szydlowski (2020) (the principal has no new information after time 0), Orlov (2020) (the principal chooses to publicly monitor a fraction of projects and only learn about the quality of the monitored projects) and Smolin (2020) (the principal does not directly observe output but commits to a dynamic policy of public tests to inform both players). In all these works,

[2]Their setting also differs from mine in that the agent's initial effort level determines the mean of the output for the whole course.

the evaluation is objective: there is an underlying state and Agent has his belief about its distribution. Different from these works, my setting allows Principal to influence Agent's belief about the underlying state through the evaluation system.

As evaluation biases are widely documented, there is a large strand of literature that studies subjective performance evaluation bias in the pay-for-performance context in both theory and practice. The literature also provides justifications for the existence of evaluation biases and many adopt behavioral and cognitive procedures. Several studies point out that an evaluator who is not the owner of the firm may assign lenient evaluations because of collusion or favoritism (Tirole 1986; Prendergast and Topel 1993; De Chiara and Livio 2017). Some argue that social influence could cause leniency bias (Judge and Ferris 1993; Giebe and Gürtler 2012). Another justification is that subjective evaluations are prone to information inaccuracy and information is costly to acquire, so it is convenient to assign average or lenient evaluation to avoid confrontations (Bol 2011). Relatedly, Golman and Bhatia (2012) argue that the rater feels worse about unfavorable errors than about favorable errors, so she will assign ratings biased upwards. Leniency bias may also be a result of luck: Bol and Smith (2011) show that when evaluations are related to an objective measure in which an uncontrollable stochastic factor (luck) plays a role, supervisors evaluations are higher when the agent has bad luck (this factor decreases the agent's objective measure), but are not lower if he has good luck. Other justifications also come from behavioral considerations, such as loss-aversion, from the agent's side (Marchegiani et al. 2016). All these potential explanations are excluded in my setting. My result could justify evaluation biases without a behavioral agent or a behavioral principal, information acquisition cost, or information inaccuracy, which differs from the existing justifications.

The fourth strand of the literature is the continuous-time standard moral hazard with publicly monitoring outputs, such as Holmstrom and Milgrom (1987), Schattler and Sung (1993), Sannikov (2008) and Cvitanić et al. (2018). In particular, my work builds on the work of Sannikov (2008). The main difference from this strand in that the principal in my setting could have the power in determining the agent's initial belief and how to evaluate the agent.

Lastly, my work relates to the literature that studies a privately informed principal in mechanism design, including Myerson (1983), Maskin and Tirole (1990) and Maskin and Tirole (1992). In that literature, Principal is informed at the beginning about the state and then picks a mechanism. Thus the choice reflects the principal's private knowledge. But the principal in my work has no private information at the time when she commits.

This paper is organized as follows: Section 3.2 introduces the model. Section 2.3 characterizes Agent's incentive compatibility and studies canonical evaluation systems. Section 2.4 formalizes Principal's problem and derives the benchmark case. Section 2.5 studies the optimal incentive contract. Section 2.6 extends the model to study the reaction of optimal incentive contracts under three contractual possibilities. Section 2.7 is a conclusion. The proofs are all collected in the appendix.

2.2 Model

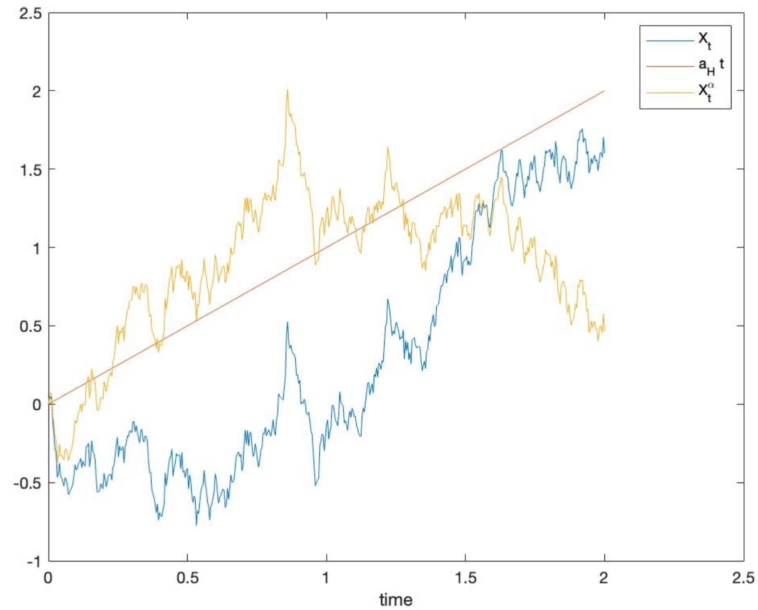
There is a principal and an agent, both von Neumann–Morgenstern utility maximizers. Principal has a project and hires Agent to work in an infinite time horizon $[0, \infty)$. Agent exerts costly effort a_t at each time t that Principal cannot observe, where a_t takes binary value $\{0, a_H\}$ with $a_H > 0$ and the cost of efforts is $c(0) = 0$ for zero effort and $c(a_H) > 0$ for the high effort.[3] Principal commits to a contract that governs Agent's information and determines his compensation as I will describe below.

At each instant t, Agent produces a cumulative output process X_t unobservable to both parties, which represents the true production of the agent at that instant.[4] The cumulative output X is a stochastic process in a fixed filtered probability space $(\Omega, \mathcal{F}_\infty, \mathbb{F}, P)$ with Ω the space of all possible continuous functions on $[0, \infty)$. Moreover, the evolution of X is determined by Agent's effort process a_t and an exogenous noise W such that $\mathrm{d}X_t = a_t \, \mathrm{d}t + \sigma \, \mathrm{d}W_t$, where the process W is a 1-dimensional Brownian motion under the Wiener measure P that satisfies $W_t(\omega) = \omega_t$ for any t with $W_0 = 0$. For brevity, I will henceforth call X "the output". Let $\mathbb{F}$ be the filtration generated by W. The constant $\sigma > 0$ is commonly known.

For contracting purposes, Principal can choose and commit to a costless noisy evaluation system that generates contractible reports about Agent's performance evaluation. In reality,

[3] Although it is conceptually straightforward to extend the model to the case where the feasible effort set is continuous in $\mathbb{R}$, there will be extra computationally complications to understand the equilibrium effort provision in the optimal incentive contract. Hence we will focus on binary-effort choices to keep our intuition simple and tractable.

[4] Given that Principal commits to the chosen evaluation and compensation systems without relying on the realization of X at time 0, Principal doesn't need to be able to observe X.



This figure captures the idea of how the evaluation is conducted. Let the x-axis indicate the time and let the y-axis indicate the value of the output or evaluation processes. For illustration purpose, I set $\sigma = 1$, $a_H = 1$, $\underline{\alpha} = -3$, $\overline{\alpha} = 3$ and $\alpha_t(\omega) = 3 * cos(2t)$ for any t and ω. Let the blue path be an output realization path, and let the yellow one be the realization of the associated evaluation process. The gap between these two paths is due to the adjustment process.

Figure 2.1: How to conduct the evaluation process

this system could be the HR department in a firm. A chosen noisy evaluation system generates an evaluation process X^α such that $\mathrm{d}X_t^\alpha = a_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^\alpha$, where W^α is an *evaluation noise* that satisfies $\mathrm{d}W_t^\alpha = \alpha_t\,\mathrm{d}t + \mathrm{d}W_t$ with α an $\mathbb{F}$-progressively measurable *adjustment process* taking value from a fixed interval $[\underline{\alpha}, \overline{\alpha}]$. Here $\underline{\alpha}$ and $\overline{\alpha}$ are fixed constants that satisfy $\underline{\alpha} \leq 0 \leq \overline{\alpha}$. By such a definition, the adjustment α could be informative about the agent's effort. As for the evaluation noise W^α, it is unobservable to both parties with $\mathbb{F}^\alpha$ its generated filtration. I assume that the system specifies a probability measure Q equivalent to P (i.e., Q is absolutely continuous w.r.t P and P is absolutely continuous w.r.t Q) such that W^α is a Brownian motion in $(\Omega, \mathcal{F}_\infty^\alpha, \mathbb{F}^\alpha, Q)$. This assumption implies that the evaluation cannot be more informative about the agent's effort than the output process. The above are common knowledge except that Agent does not observe the realized evaluation X^α.

Then the evaluation system reports to Agent her evaluation using a *report process* Y that discloses some information about the evaluation X^α without adding extra noise. Hence, Y is a stochastic process in $(\Omega, \mathcal{F}^\alpha_\infty, \mathbb{F}^\alpha, Q)$ adapted to the filtration generated by X^α. Let $\mathbb{F}^Y$ be the filtration generated by Y. Agent calculates his expected payoff given the specifications in the system and acts at the best of his interests.

Denote an evaluation system as (α, Q, Y). Say an evaluation system (α, Q, Y) has reports *independent of history* if for any $s < t$, the report information $\mathcal{F}^Y_t \backslash \mathcal{F}^Y_s$ is independent of $\mathcal{F}^\alpha_s$ under measure Q. Intuitively, it requires the reports to base on new information. Throughout the paper, I focus on evaluation systems that satisfy this condition. This focus is actually without loss of generality since I allow the principal to compensate the agent based on the entire report history.

Recall that Principal commits to a contract at the outset. A contract consists of an evaluation system (α, Q, Y) and a compensation system that includes a reward process $B := \{B_t\}_{t\geq 0}$ and an effort recommendation process $A := \{A_t\}_{t\geq 0}$, both $\mathbb{F}^Y$-adapted. Based on the entire report history $\{Y_s\}_{s\geq 0}$, the contract then recommends an effort level A_t to Agent and pays a reward $B_t \in [0, \bar{b}]$ at each time t with $\bar{b}$ a fixed constant. Note that the effort recommendation process $\{A_t\}_{t\geq 0}$ can differ from the agent's actual effort process $\{a_t\}_{t\geq 0}$, and only the agent's actual effort $\{a_t\}_{t\geq 0}$ can determine the output X and the evaluation processes X^α.

Principal and Agent discount their payoffs exponentially with the same discounting rate r with $0 < r$. Principal is risk-neutral and maximizes the expected present value of innovations in X_t less the reward payments

$$E^P \left[\int_0^\infty e^{-rs} \, \mathrm{d}X_s - \int_0^\infty e^{-rs} B_s \, \mathrm{d}s \right]. \tag{2.1}$$

I assume that Agent is risk-averse, consumes the compensation immediately, and does not have hidden savings. Specifically, Agent's Bernoulli utility takes a separable form $u(\cdot) - c(\cdot)$ where $c(\cdot)$ is the cost of Agent's effort, and $u(\cdot)$ is the utility from reward which is a twice continuously differentiable strictly increasing concave function on $[0, \infty)$ with the following assumptions: (i) $u(0) = 0$; (ii) $\lim_{b\to\infty} u(b) \to \infty$, and (iii) $\lim_{b\to\infty} u'(b) \to 0$. Given the assumptions on utility, there always exists a bound $\bar{b}$ such that $u'(\bar{b}) = \frac{c(a_H)}{a_H}$ holds and I will fix the choice of $\bar{b}$ such that this condition is satisfied. We will see below that this condition will ensure that the upper bound is never reached.

Agent will calculate his payoff in the alternative probability space $(\Omega, \mathcal{F}^\alpha, \mathbb{F}^\alpha, Q)$ specified by the evaluation system. Given the compensation system (B, A), suppose that the agent follows the effort recommendation process, then his continuation payoff at each time t, conditioning on the current information is as follows:

$$E^Q \left[\int_t^\infty e^{-r(s-t)} \left(u\left(B_s\right) - c(A_s) \right) \mathrm{d}s \mid \mathcal{F}_t^Y \right]. \tag{2.2}$$

Given the contract, a recommended effort process $A := \{A_s\}_{s \geq 0}$ is *incentive compatible* ("IC") for Agent if and only if at each time t, $\{A_s\}_{s \geq t}$ maximizes expression (2.2). An *incentive contract* is a contract that recommends incentive compatible effort process to Agent. Moreover, I assume that both the principal and the agent obtain 0 if the employment relationship does not form at time 0.

Principal's problem is to determine an optimal incentive contract that maximizes her payoff. I assume that Agent will take Principal's recommended effort process as long as it is incentive compatible to do so; and that the choice of parameters is such that it is strictly optimal for Principal to hire Agent with some contracts rather than walk away.[5]

2.3 Preparation

I will first introduce a preparation lemma that shows Principal's feasible choice of evaluation systems for any adjustment process α is non-empty.

Lemma II.1. *For any adjustment process α, there exists a unique probability measure Q on Ω that satisfies the following: for any time $t \in [0, \infty)$ and $S \in \mathcal{F}_t$,*

$$Q(S) = E^P \left[\mathbb{1}_S \exp\left(-\int_0^t \alpha_s \, \mathrm{d}W_s - \frac{1}{2} \int_0^t |\alpha_s|^2 \, \mathrm{d}s \right) \right] \tag{2.3}$$

such that the evaluation noise W_s^α satisfying $\mathrm{d}\, W_s^\alpha := \alpha_s \, \mathrm{d}s + \mathrm{d}W_s$, $s \in [0, \infty)$ is a Brownian motion under Q. Moreover, Q is equivalent to P.

The above lemma further illustrates how to induce a different belief Q given any adjustment.

[5]Similar to Demarzo and Sannikov (2006), Sannikov (2008) and Zhu (2013), I do not assume the presence of an independent public randomization device to keep the model simple. This is actually without loss of generality in my setting since one will see later that Principal's value function is concave.

Let the density function of Q with respect to P conditional on the information flow be $\{Z_s\}_{s\geq 0}$. Suppose both the principal and the agent hold the same belief P at time 0. To induce belief Q, the principal can tell the agent that she will evaluate the agent's compensation by re-weighting the future events with the stochastic weights Z_t. Accepting this piece of information will move the agent's belief from P to Q. Moreover, the proof of Lemma II.1 shows that Z_t is an $\mathbb{F}$-martingale under P, meaning that the future weight is a mean-preserving spread conditional on any current weight, i.e., $E^P[Z_{t+\Delta} \mid \mathcal{F}_t] = Z_t$ for any Δ. Hence, introducing stochastic weights amounts to adding extra noise into Agent's belief.

Now we will consider the characterization of Agent's promised continuation value from Principal's point of view. Not knowing the underlying noise, Principal has to infer the noise from the realized evaluation X^α and the effort process A that she believes Agent is taking. Thus I will add a subscript A on the expectation whenever necessary (i.e., E_A) to indicate the dependence of Principal's perception of Agent's effort A. In the equilibrium, Principal could correctly anticipate Agent's choice of effort.

Given an arbitrary contract (α, Q, Y, B, A), I define Agent's promised continuation payoff based on Principal's information at time t as follows:

$$V_t(B, A, \alpha) := E_A^Q \left[\int_t^\infty e^{-r(s-t)} \big(u(B_s) - c(A_s) \big) \, \mathrm{d}s \mid \mathcal{F}_t^\alpha \right]. \tag{2.4}$$

Principal could set up the contract to retire or fire Agent at any desirable time. When the right time comes, the contract recommends Agent shirk forever and rewards him a constant perpetual amount, which is zero for firing and strictly positive for retiring.

Why would the principal ever want to fire or retire the agent? Recall that the agent is protected by limited liability. By shirking forever, the agent can guarantee his continuation value to be at least 0. Hence whenever Agent's continuation value hits 0, it is necessary for Principal to fire Agent so that his continuation value is never negative. We define the firing time τ^f as follows:

$$\tau^f := \inf \big\{ s \in (0, \infty) \mid V_s(B, A, \alpha) = 0 \big\}. \tag{2.5}$$

On the other hand, the principal would like to retire the agent when his continuation value

V_t is too high. This is because the agent's instantaneous consumption B_t at a high V_t will also be large at most of the future moments, and given that Agent's utility function is gradually flattened out, it will be very costly to incentivize the agent to work hard at such a high V_t. So it is better to retire the agent when V_t rises to a high level that the cost of inducing the agent's high effort exceeds its benefit. Let τ^r be the stopping time that Principal optimally retires Agent.

When retiring a risk-averse Agent, it is optimal to pay him a perpetual constant reward flow. Let $p(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ be the *retirement value function* such that, given Agent's continuation value v, $p(v)$ is the present value of the total payment Agent receives after retirement at continuation value v. The fact that the horizon is infinite implies that p is stationary with respect to time. Based on the property of Agent's utility, p is a nonnegative-valued twice differentiable convex function that satisfies $p(0) = 0$.

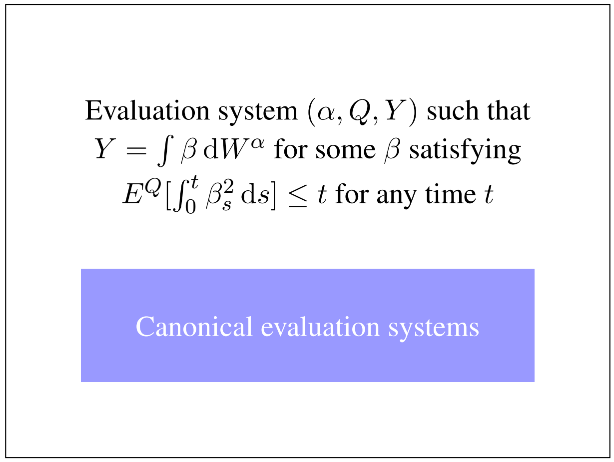
2.3.1 Canonical evaluation scheme

The agent knows his own actions so, under any contract, he will continuously update his belief about evaluation noise from the report. Say an evaluation system (α, Q, Y) is *canonical*, if the report process Y coincides with Agent's correct belief of the evaluation noise conditional on his available information, i.e., $Y_t = E^Q[W_t^\alpha | \mathcal{F}_t^Y]$ for any time t.

For any evaluation system, Agent can always transform the report into the corresponding expected evaluation noise under which the evaluation system becomes canonical. Call such a translation *a canonical transformation*. The following proposition provides a martingale characterization for canonical transformations of any evaluation system.

Proposition II.1. *For any evaluation system, let (α, Q, Y) be its canonical transformation. Then there exists a unique $\mathbb{F}^Y$-adapted β_t such that $Y_t = \int_0^t \beta_s \, \mathrm{d}W_s^\alpha$ at any time t. Moreover, $E^Q[Y_t^2] = E^Q[\int_0^t \beta_s^2 \, \mathrm{d}s] \leq t$ at any time t.*

Proposition II.1 identifies an inclusion relation for canonical evaluation systems (see Figure 2.2). Say β in the above proposition the *variational coefficient* for the given evaluation system. Such coefficient reflects how the report would behave path-wisely due to the evolution of the realized evaluation in this system. Given any evaluation system, the canonical transformation is unique, thus its corresponding variational coefficient is also unique. Thus the problem of choosing a report Y boils down to choosing an appropriate



The rectangle represents the set of evaluation schemes admitting a martingale representation with some $\mathbb{F}^Y$-adapted variational coefficient β that satisfies $E^Q[\int_0^t \beta_s^2 \, \mathrm{d}s] \leq t$ for any time t. The purple rectangle represents the set of canonical evaluation systems.

Figure 2.2: The inclusion relation presented in Proposition II.1

variational coefficient β (subject to a canonical transformation).

I further focus on variational coefficients β taking value from the interval $[-1, 1]$, under which, any such choice of β satisfies $E^Q[\int_0^t \beta_s^2 \, \mathrm{d}s] \leq t$ for any time t. We will see later that it is actually without loss of generality.[6]

2.3.2 Agent's incentive compatibility

The following lemma characterises the evolution of the above Agent's continuation value based on Principal's information, which follows from Sannikov (2008).

Lemma II.2. *Fix an arbitrary contract (α, Q, Y, B, A). There exists a $\mathbb{F}^{\alpha}$-progressively measurable process K such that*

$$\mathrm{d}\, V_t(B, A, \alpha) = (rV_t(B, A, \alpha) - u(B_t) + c(A_t))\, \mathrm{d}t + K_t(\mathrm{d}\, X_t^{\alpha} - A_t \, \mathrm{d}t) \tag{2.6}$$

[6]The restriction that the variational coefficient β_t takes value in $[-1, 1]$ is without loss of generality for the following reason: suppose that we allow a general domain $[-\overline{\beta}, \overline{\beta}]$ with some constant $\overline{\beta} > 1$ and ignore the canonical constraint for now. Following the same analysis, we can derive that an optimal variational coefficient always takes the upper bound value $\overline{\beta}$. However, to satisfy the canonical requirement of the evaluation systems, the constraint $E^P[\int_0^t \beta_s^2 \, \mathrm{d}s] \leq t$ must satisfy for every t. This means that the optimal variational coefficient will not be feasible unless the upper bound $\overline{\beta} \leq 1$.

holds Q-almost everywhere. Moreover, the above K is unique $\mathrm{d}Q \times \mathrm{d}t$-a.s., and it satisfies $E_A^Q[\int_0^t K_s^2 \,\mathrm{d}s] < \infty$ for all $t \in [0, \infty)$.

Based on Equation (2.6), the change of the continuation value can be decomposed into the following three parts:

$$\underbrace{\mathrm{d}\, V_t(B, A, \alpha)}_{\text{the change of continuation value}} = \Big(\underbrace{rV_t(B, A, \alpha)}_{\text{the interest rate of deferring}} - \underbrace{(u(B_t) - c(A_t))}_{\text{the net surplus agent gets}} \Big) \,\mathrm{d}t + \underbrace{K_t(\mathrm{d}\, X_t^\alpha - A_t \,\mathrm{d}t)}_{\text{the changes due to noise}}, \text{ for } Q - a.s,$$

where the expression in blue is essentially the evaluation noise. Say the above K in Equation (2.6) the *sensitivity coefficient*, since it stipulates how much Agent's continuation value will change given the fluctuation of evaluations. We will see later that K plays a critical role in characterizing the agent's incentive under the given contract.

Building on the idea of the above lemma, the following lemma further shows that if the principal cannot manipulate the agent's prior belief, then the adjustment α_t has no effect in changing the evolution of the agent's continuation value. Without loss of generality, we will fix the agent's prior belief to be P.

Lemma II.3. *Suppose that Agent's belief is fixed at P and the adjustment process can be any $\mathbb{F}$-adapted process α satisfying $\mathbb{F}^\alpha = \mathbb{F}$. Then for any contract (α, P, Y, B, A) with the above fixed (α, P), there exists a $\mathbb{F}$-progressively measurable process K such that*

$$\mathrm{d}\, V_t(B, A, \alpha) = (rV_t(B, A, \alpha) - u(B_t) + c(A_t)) \,\mathrm{d}t + K_t(\mathrm{d}\, X_t - A_t \,\mathrm{d}t) \tag{2.7}$$

holds P-almost everywhere.

The above lemma imposes that $\mathbb{F}^\alpha = \mathbb{F}$, which guarantees that the principal does not evaluate the agent based on better information about his effort than that in the output.

Let Agent's promised continuation value conditional *on his own information* from the report be

$$V_t^Y(B, A, \alpha) := E^Q\left[\int_t^\infty e^{-r(s-t)}\big(u(B_s) - c(A_s)\big)\,\mathrm{d}s \mid \mathcal{F}_t^Y\right]. \tag{2.8}$$

The following proposition shows that, assuming the sensitivity coefficient is adapted to

Agent's information, the process $V_t^Y(B, A, \alpha)$ admits a similar representation as identified in Lemma IV.1. More importantly, it characterizes Agent's incentive compatibility with the variational and sensitivity coefficients we defined above.

Proposition II.2. *Fix an arbitrary contract* (α, Q, Y, B, A). *Let* β *be the variational coefficient and let* K *be the sensitivity coefficient. Suppose that* K *is* $\mathbb{F}^Y$*-adapted. Then the evolution of* $V^Y(B, A, \alpha)$ *satisfies*

$$\mathrm{d}\, V_t^Y(B, A, \alpha) = (rV_t^Y(B, A, \alpha) - u(B_t) + c(A_t))\, \mathrm{d}t + \sigma K_t \beta_t \, \mathrm{d}W_t^\alpha. \tag{2.9}$$

Moreover, for any time t*, the effort process* A_t *is incentive compatible if*

$$A_t \cdot K_t\beta_t - c(A_t) \geq \max_{a \in \{0, a_H\}} (a \cdot K_t\beta_t - c(a)), \text{ for } Q\text{-almost everywhere } \omega. \tag{2.10}$$

Remark II.1. The above incentive compatibility characterization in (2.10) does not restrict to the binary effort. Similar to Sannikov (2008), this characterization applies to any compact set of feasible effort levels in $\mathbb{R}$ with the smallest element 0.

The next lemma connects the characterizations under the measure Q to that under Principal's belief P, which prepares us for Principal's problem in the next section.

Lemma II.4. *Suppose that* P *is equivalent to* Q*. For any* $\mathbb{F}^\alpha$*-adapted stochastic processes* $V_t(B, A, \alpha)$ *and* K*, the following hold:*

(i) (2.6) *is satisfied* Q*-a.s if and only if it is satisfied* P*-a.s;*

(ii) (2.9) *is satisfied* Q*-a.s if and only if it is satisfied* P*-a.s;*

(iii) (2.10) *is satisfied* Q*-a.s if and only if it is satisfied* P*-a.s.*

2.3.3 Different types of evaluation biases

Say an evaluation scheme (α, Q, Y) is *truthful* if $\alpha \equiv 0$, P-almost everywhere and $Q = P$. Moreover, a contract is *output-based* if its evaluation scheme is truthful. In this case, Agent's continuation value is measured under the true production noise W and Principal's belief P. In general, an evaluation scheme need not be truthful.

The following two types of performance evaluation biases are widely documented in practice: leniency bias and severity bias. The literature has proposed several operational definitions of these evaluation biases; see Saal et al. (1980) for a detailed summary and discussion. Conceptually, one could consider leniency (or severity) bias as the tendency to assign a higher (or lower) evaluation than is warranted by employees' performance.

Given any evaluation scheme, conditional on a realized path ω, we say the evaluation scheme is *lenient* at a specific time t for this path ω if the adjustment of the evaluation process at time t is strictly greater than 0, i.e., $\alpha_t(\omega) > 0$; say the evaluation scheme is *severe* at time t for that path if the adjustment of the evaluation process at time t is strictly less than 0, i.e., $\alpha_t(\omega) < 0$. This definition of leniency or severity biases is aligned with that in Sharon and Bartlett (1969) and Bernardin et al. (1976), which says "Leniency (or severity) error is defined as a shift in mean ratings from the midpoint of the rating scale in the favorable (or unfavorable) direction".

2.4 Principal's problem and the benchmark contracts

Building on Lemma II.1, the principal essentially controls processes $\{A_s, B_s, \alpha_s, K_s, \beta_s\}_{s\geq 0}$ that take values from the following compact sets respectively:

$$\alpha_s \in [\underline{\alpha}, \overline{\alpha}],\ K_s \in [\epsilon, \overline{K}],\ \beta_s \in [-1, 1],\ B_s \in [0, \bar{b}],\ A_s \in \{0, a_H\} \text{ for any } s \geq 0.$$

To prevent the nonexistence of optimal incentive contracts, it is necessary to impose a strictly positive lower bound ϵ on the domain of the sensitivity coefficient K. Specifically, we pick an ϵ small enough such that $\frac{c(a_H)}{a_H} > \epsilon > 0$ and an upper bound $\overline{K}$ large enough such that $\overline{K} > \frac{c(a_H)}{a_H}$.

From now on, I will focus on contracts in which the sensitivity coefficient is adapted to the report information. We will see later that such a focus is in fact without loss of generality, since the optimal incentive contract always truthfully reports the realized evaluation so the information flow generated by the report and evaluation will coincide, i.e., $\mathbb{F}^Y = \mathbb{F}^\alpha$.

Recall that τ^r is the stopping time for the principal to retire the agent. Say a control $\Phi := \{A_s, B_s, \alpha_s, K_s, \beta_s, \tau^r\}_{s\geq 0}$ is *admissible* if (i) $\{\alpha_s\}_{s\geq 0}$ is adapted to the filtration $\mathbb{F}$; (ii) $\{\beta_s\}_{s\geq 0}$ and τ^r are adapted to filtration $\mathbb{F}^\alpha$; and (iii) $\{A_s, B_s, K_s\}_{s\geq 0}$ is adapted to the filtration $\mathbb{F}^Y$ such that the following equation has a unique strong solution:

$$\mathrm{d}\, V_s(B, A, \alpha) = (rV_s(B, A, \alpha) - u(B_s) + c(A_s))\,\mathrm{d}s + \sigma K_s(\mathrm{d}\, W_s + \alpha_s \,\mathrm{d}\, s).$$

Given an admissible control $\Phi = \{A_s, B_s, \alpha_s, K_s, \beta_s, \tau^r\}_{s\geq 0}$, recall that τ^f is the initial time that Agent's continuation value hits 0. Then the principal's expected payoff is

$$\mathscr{E}(v, \Phi) := E_A^P \left[\int_0^{\tau^r \wedge \tau^f} e^{-rs}\,\mathrm{d}X_s - e^{-rs} B_s \,\mathrm{d}s - e^{-r(\tau^r\wedge\tau^f)} p(V_{\tau^r\wedge\tau^f}(B, A, \alpha)) \right], \tag{2.11}$$

where $V_t(B, A, \alpha)$ is the process starting from the initial value v. Denote by $G(\cdot)$ the principal's value function such that given any initial value v, $G(v)$ is the principal's maximal expected payoff among all admissible controls Φ subject to the incentive compatibility constraint:

$$A_s \cdot K_s\beta_s - c(A_s) \geq \max_{a\in\{0,a_H\}} (aK_s\beta_s - c(a)).$$

An admissible control Φ^* is said to be optimal if it satisfies the incentive compatibility constraint and $\mathscr{E}(v, \Phi^*) = G(v)$. Let Γ be the feasible domain of incentive compatible effort-sensitivity pairs such that

$$\Gamma := \left\{ (a_H, k\beta) \middle| \overline{K} \geq k\beta \geq \frac{c(a_H)}{a_H} \right\} \cup \left\{ (0, k\beta) \middle| \epsilon \leq k\beta \leq \frac{c(a_H)}{a_H} \right\}. \tag{2.12}$$

My first objective is to prove that the value function G is twice continuously differentiable in $(0, v^*)$ for some *retirement boundary* v^* which further solves the following Hamilton–Jacobi–Bellman (HJB) equation:

$$rG(v) = \sup_{\substack{(a,k\beta)\in\Gamma, b\in[0,\bar{b}] \\ \beta\in[-1,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} G'(v)(rv - u(b) + c(a) + \sigma k\alpha) + \frac{G''(v)}{2}(\sigma k)^2 + a - b \tag{2.13}$$

with the retirement boundary v^* characterized by the following condition:

$$G(0) = 0, G(v^*) = -p(v^*) \text{ and } G'(v^*) = -p'(v^*). \tag{2.14}$$

The following results provide a characterization of the principal's value function and the above HJB equation: Theorem II.1 shows that such HJB equation with the boundary condition has a unique, well-defined solution; and Theorem II.2 shows that such solution characterizes the principal's value function.

Theorem II.1. *There exists a unique twice continuously differentiable and strictly concave function G that solves HJB equation* (2.13) *with $G \geq -p$ and the boundary condition* (2.14) *holds for v^* where $v^* = \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* = 0$ otherwise.*

Theorem II.2. *Suppose that G is the solution to the HJB equation* (2.13) *with the boundary condition* (2.14) *satisfied by v^* where $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* := 0$ otherwise. Then for any initial agent's continuation value v, the principal's maximum expected payoff under admissible IC controls is $G(v)$ if $v \leq \frac{u(\bar{b})}{r}$, and $-p(v)$ if $v > \frac{u(\bar{b})}{r}$. In particular, the value function is twice continuously differentiable and it is the unique solution of the HJB equation* (2.13) *with the boundary condition* (2.14) *satisfied by the above v^*.*

Recall that the agent's outside option is 0, so the initial value the principal promises the agent will always start from the position maximizing her ex ante payoff, that is, $\max\{0, G'^{-1}(0)\}$. As in the condition that $G(v^*) = -p(v^*)$, the principal's value function must be strictly negative when she retires the agent. Theorem II.2 thus implies that it is optimal for the principal to retire the agent whenever his continuation value process hits v^*. Moreover, this further implies that whenever $v^* = 0$, the principal is better off not hiring the agent. We will henceforth focus on describing the structure of the optimal contracts before the agent gets retired, i.e., when the agent's continuation value is less than v^*.

2.4.1 Benchmark: optimal output-based incentive contract

Say a contract is *output-based* if the system evaluates the agent directly based on the output, i.e., $\underline{\alpha} = \overline{\alpha} = 0$ and the system specifies Principal's belief P as Agent's belief. As a benchmark, I study the structure of optimal output-based incentive contracts below.

Based on the characterization in Theorem II.1, the following proposition characterizes the explicit form of the optimal output-based incentive contract.

Proposition II.3. *Suppose that $\underline{\alpha} = \overline{\alpha} \equiv 0$ and $Q = P$. Let $F(\cdot)$ be the unique solution to equation* (2.13) *with $F \geq -p$ and the boundary condition* (2.14) *satisfied by v^* where*

$v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid F(v) = -p(v)\}$ *if* $\{v \in (0, \frac{u(\bar{b})}{r}] \mid F(v) = -p(v)\} \neq \varnothing$ *and* $v^* := 0$ *otherwise. Consider the following policy for any* v *such that* $0 \leq v \leq v^*$*:* $\beta^*(v) = 1$*,*

$$\begin{cases} a^*(v) = a_H, k^*(v) = \frac{c(a_H)}{a_H} & \frac{F''(v)}{2}(\epsilon\sigma)^2 \leq F'(v)c(a_H) + \frac{F''(v)}{2}(\frac{c(a_H)}{a_H}\sigma)^2 + a_H \\ a^*(v) = 0, k^*(v) = \epsilon & \textit{otherwise} \end{cases},$$

$$b^*(v) = \begin{cases} 0 & u'_+(0) \leq \frac{1}{-F'(v)} \textit{ or } F'(v) \geq 0 \\ u'^{-1}(\frac{1}{-F'(v)}) & \frac{1}{-F'(v)} \in (u'_-(\bar{b}), u'_+(0)) \\ \bar{b} & \textit{otherwise} \end{cases}. \tag{2.15}$$

Given the initial value $v_0 := \max\{0, F'^{-1}(0)\}$*, then there exists a unique strong solution to* $\mathrm{d}\, V_s = (rV_s - u(b^*(V_s)) + c(a^*(V_s)))\,\mathrm{d}s + \sigma k^*(V_s)\,\mathrm{d}\, W_s$*, and the optimal output-based incentive contract is characterized by the control* $(\beta^*(V_t), a^*(V_t), b^*(V_t), k^*(V_t))_{t\geq 0}$.

By the above theorem, the principal retires the agent at the continuation value $v^* = \frac{u(b^*(v^*))}{r}$. Recall that we show in Theorem II.1 that the retirement bound $v^* < \frac{u(\bar{b})}{r}$, which further implies that the agent is optimally retired before receiving the upper bound $\bar{b}$. Moreover, the immediate reward b^* in the optimal output-based incentive contract is an increasing function with respect to Agent's continuation value. Hence we can further divide the range of continuation values in which Agent remains employed into two distinct intervals, based on whether the immediate compensation is 0 or not. Let the division point be b_0 such that The principal pays Agent zero immediate reward if and only if his continuation value is in $[0, b_0]$. Note that the above theorem shows that the principal pays the agent zero immediate compensation when her value function is increasing, implying that it is cheaper to motivate the agent with future payoffs than immediate compensation in this region.

To incentivize Agent to exert high effort, Principal must expose Agent to compensation risk, which is costly given that Agent is risk-averse. In the above benchmark contract, the principal minimizes the exposure of the agent's risk conditional on maintaining his incentive. To achieve her goal, there are two options at the principal's disposal: she could either reduce the compensation sensitivity or the report variation. My result shows that the principal strictly prefers the former option: The optimal output-based incentive contract fully reveals the evaluation to Agent, which provides the maximum amount of noise to allow the compensation sensitivity to decrease to a minimum level. Therefore, the principal does not benefit from reporting less information. The optimal output-based incentive contract achieves the same payoff as that under the optimal incentive contract when the output is publicly observable (Sannikov (2008)). The canonical reporting requirement of the evaluation system in this contract is also satisfied.

In a different setting with a risk-neutral Agent, Zhu (2013) shows that it is optimal for Agent to shirk frequently. It is interesting to know whether switching Agent's effort between working and shirking is optimal in my setting. To investigate this problem, we first rewrite the HJB equation (2.13) by plugging the policy $(a^*(\cdot), b^*(\cdot), k^*(\cdot))$ in Proposition II.3:

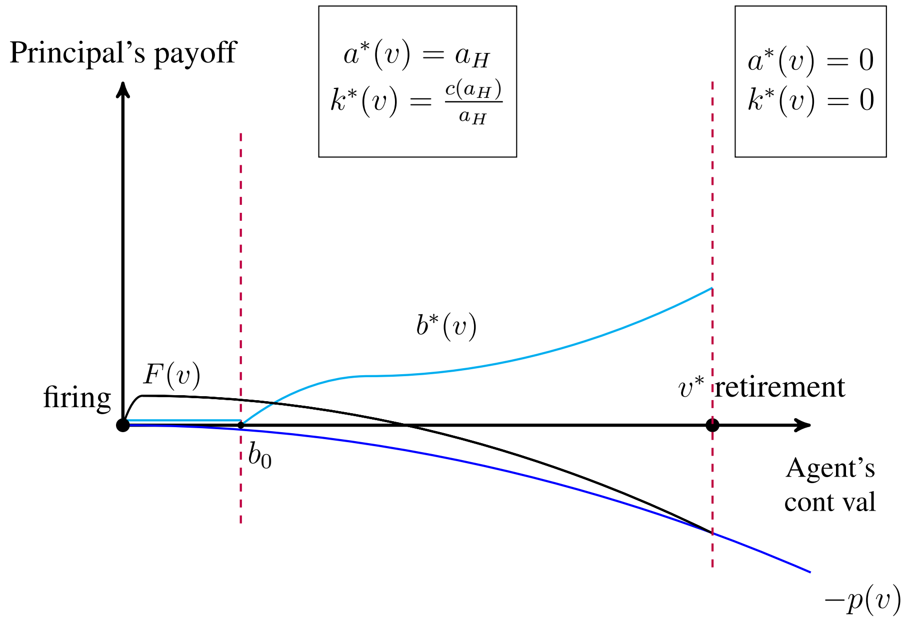
$$rF(v) = \max\Big\{F'(v)(rv - u(b^*(v))) + \frac{F''(v)}{2}(\epsilon\sigma)^2 - b^*(v), \\ F'(v)(rv - u(b^*(v)) + c(a_H)) + \frac{F''(v)}{2}(\frac{c(a_H)}{a_H}\sigma)^2 + a_H - b^*(v)\Big\}. \tag{2.16}$$

The following proposition answers this question and shows that it is always optimal for Principal to recommend high effort a_H before either firing or retiring occurs. Hence, different from Zhu (2013), the shirking in my setting is an absorbing state. I further represent the optimal output-based incentive contract in Figure 2.3.

Proposition II.4. *The optimal output-based incentive contract always implements effort a_H before firing or retiring.*

The following result builds on the above results and shows that if we take away the principal's control on the agent's prior belief, then the benefit of adding adjustment into the evaluation also vanishes. Again we will fix the agent's belief to be P and assume that $\mathbb{F}^\alpha = \mathbb{F}$, under which the (adjusted) evaluation does not provide more information about the agent's effort than the output.

Theorem II.3. *Suppose that Agent's belief is fixed to be P and the adjustment process can be any $\mathbb{F}$-adapted process α satisfying $\mathbb{F}^\alpha = \mathbb{F}$. Then the optimal incentive contract is the optimal output-based contract.*



This figure presents the optimal incentive output-based contract. The black parabolic curve is Principal's value function; the blue curve is the retirement value function $-p(\cdot)$ with v^* the optimal retirement boundary. The cyan curve is the optimal reward function $b^*(\cdot)$. Before retiring or firing, Agent always exerts high effort. The optimal sensitivity coefficient is the minimum conditional on maintaining Agent's incentives and the optimal report is to fully reveal the evaluation.

Figure 2.3: Optimal incentive output-based contract

2.5 Optimal incentive contract

We will now turn to the general case when the principal has the flexibility to add adjustments and change the agent's prior belief subject to the constraints we describe. Let us first introduce a condition that could simplify the exposition.

Condition II.1. Under the given primitives, for any C^2-function $G(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and any function $\alpha^*(\cdot) : \mathbb{R}_+ \to [\underline{\alpha}, \overline{\alpha}]$, say $G(\cdot)$ and $\alpha^*(\cdot)$ satisfy Condition II.1 if one of the following is satisfied:

(i) $G'(v)(c(a_H) + \sigma(\frac{c(a_H)}{a_H} - \epsilon)\alpha^*(v)) + a_H + \frac{G''(v)}{2}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) \geq 0$ and $\epsilon > \frac{-G'(v)\alpha^*(v)}{G''(v)\sigma}$;

(ii) $G'(v)c(a_H) + a_H + \frac{G''(v)}{2}(\sigma\frac{c(a_H)}{a_H} + \frac{G'(v)\alpha^*(v)}{G''(v)})^2 \geq 0$ and $\epsilon \leq \frac{-G'(v)\alpha^*(v)}{G''(v)\sigma} < \frac{c(a_H)}{a_H}$;

(iii) $\frac{c(a_H)}{a_H} \leq \frac{-G'(v)\alpha^*(v)}{G''(v)\sigma}$.

The following theorem characterizes the structure of the optimal incentive contract.

Theorem II.4. *Let $G(\cdot)$ be the unique solution to equation* (2.13) *with $G \geq -p$ and the boundary condition* (2.14) *satisfied by v^* where $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* := 0$ otherwise. Suppose that G is third continuously differentiable. Consider the following policies for any v such that $0 \leq v \leq v^*$: $\beta^*(v) = 1$,*

$$b^*(v) = \begin{cases} 0 & u'_+(0) \leq \frac{1}{-G'(v)} \text{ or } G'(v) \geq 0 \\ u'^{-1}(\frac{1}{-G'(v)}) & \frac{1}{-G'(v)} \in (u'_-(\bar{b}), u'_+(0)) \\ \bar{b} & \text{otherwise} \end{cases} \qquad \alpha^*(v) = \begin{cases} \overline{\alpha} & G'(v) > 0 \\ \underline{\alpha} & G'(v) < 0 \\ [\underline{\alpha}, \overline{\alpha}] & G'(v) = 0 \end{cases};$$

$$\begin{cases} a^*(v) = a_H, k^*(v) = \min\left\{\overline{K}, \max\left\{\frac{G'(v)\alpha^*(v)}{-G''(v)\sigma}, \frac{c(a_H)}{a_H}\right\}\right\} & G \text{ and } \alpha^* \text{ satisfy Condition } II.1 \\ a^*(v) = 0, k^*(v) = \max\{\frac{G'(v)\alpha^*(v)}{-G''(v)\sigma}, \epsilon\} & \text{otherwise} \end{cases},$$

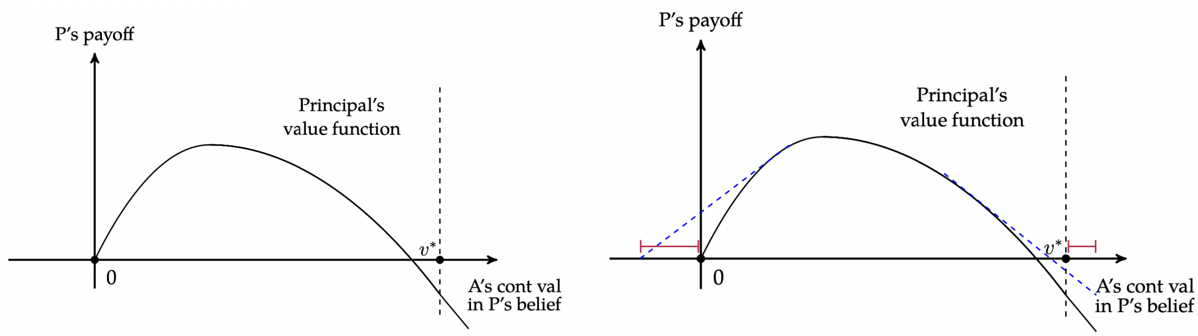
Given the initial value $v_0 := \max\{0, G'^{-1}(0)\}$, then there exists a unique strong solution to $\mathrm{d}\, V_s = (rV_s - u(b^*(V_s)) + c(a^*(V_s)))\, \mathrm{d}s + \sigma k^*(V_s)\, \mathrm{d}(W_s + \alpha^*(V_s)\, \mathrm{d}\, s)$, and the optimal incentive contract is characterized by the control $(\beta^*(V_t), \alpha^*(V_t), a^*(V_t), b^*(V_t), k^*(V_t))_{t \geq 0}$.

Why introducing evaluation biases can benefit the principal? The main intuition is as follows: the employment relationship between the principal and the agent must end at some point since the agent either gets fired due to a sequence of bad luck or gets retired due to a sequence of good luck. The principal has a zero outside option, who is thus better off by adding leniency bias for low continuation value and severity bias for high continuation value to reduce the probability of labor turnover. Hence, adding biases could improve productivity in my setting.

To graphically illustrate the idea, let us start with the benchmark case and consider the principal's value function under the optimal output-based incentive contract; see the left subfigure (a) in Figure 4.1. In this case, the relationship terminates at two endpoints. If we allow the principal to adjust evaluations around these termination points, she could extend the agent's employment duration.[7] For a visual description of such change, see Figure 4.1 (b). Such extension further induces nonconcavities in the principal's value function, which

[7] Those extensions are excluded in the employment range in optimal output-based contracts due to the violation of constraints such as limited liability.

creates room for improvement by concavification. The adjustment process could serve as a randomization device in concavifying the value function. That randomization is desirable also explains the related observation that, even though Agent is risk-averse, the volatility with respect to Agent's continuation value is weakly higher when there are nonzero adjustments.[8]



(a) Benchmark: optimal output-based contract (b) Improvement

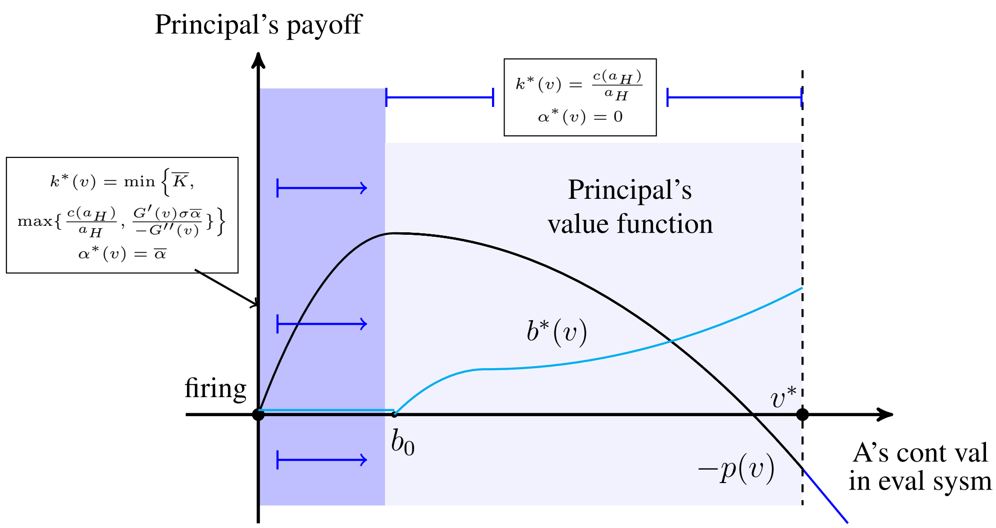
This figure presents how an evaluation system could improve from the benchmark contract. By allowing biased evaluation around the firing and retiring points, the principal could extend the agent's employing range by going beyond the original termination points. Let the red segment on the x-axis be such an extension. The blue dashed line is the gain from concavifying the value function.

Figure 2.4: How the evaluation could improve from the benchmark

One interpretation of adding adjustments is to add conditional purified noise (the evaluation biases) into the realized outputs. Such interpretation places this paper in the dynamic persuasion literature, where Ely (2017) studies the case of discontinuous production technology and the output directly affects the agent's payoff. The discontinuity of the production creates a jump in the principal's preliminary continuation payoff in Ely (2017) where the agent is about to quit the relationship (see also Figure 2 in Ely 2017). Such a jump leads to nonconcavity and thus creates room for persuasion. Different from Ely (2017), the technology (production and evaluation) are continuous in my setting and the agent's payoff does not directly affect by the output. Adding in the insight of Theorem II.3, the principal in such a setting could benefit from adding adjustments into evaluation only if she could also change the agent's prior belief.

[8] Ishida (2012) shows that when the agent's self-esteem plays a role, adding noise into the compensation can benefit the principal since it could reduce the agent's self-handicapping and thus improve his average effort. My result reveals a different reason for adding noise into compensation without relying on such behavioral assumptions.

From the above theorem, the evaluation bias at each point takes the value of either the upper or the lower bound $\overline{\alpha}$ and $\underline{\alpha}$, i.e., a bang-bang solution. In particular, Principal exhibits the maximum leniency in evaluation when Agent's continuation value is lower than v_0; and exhibits the maximum severity bias when it is higher than v_0, where v_0 is the maximizer of the principal's value function. While the agent's continuation value is less than v_0, the associated immediate reward equals 0. This means that lenient evaluation only increases Agent's continuation value but not his immediate payment, which allows Principal to partially insure Agent from downside risks and defer firing. At the other end when the agent's continuation value is high, severe evaluations serve to delay Agent's retirement and extend his employment duration. The following Figure 2.5 to 2.7 further give visual representations about these evaluation biases in the optimal incentive contract when the agent exerts high effort during his employment according to three situations: (i) $\overline{\alpha} > \underline{\alpha} = 0$; (ii) $\underline{\alpha} < 0 < \overline{\alpha}$ and (iii) $0 = \overline{\alpha} > \underline{\alpha}$, respectively.

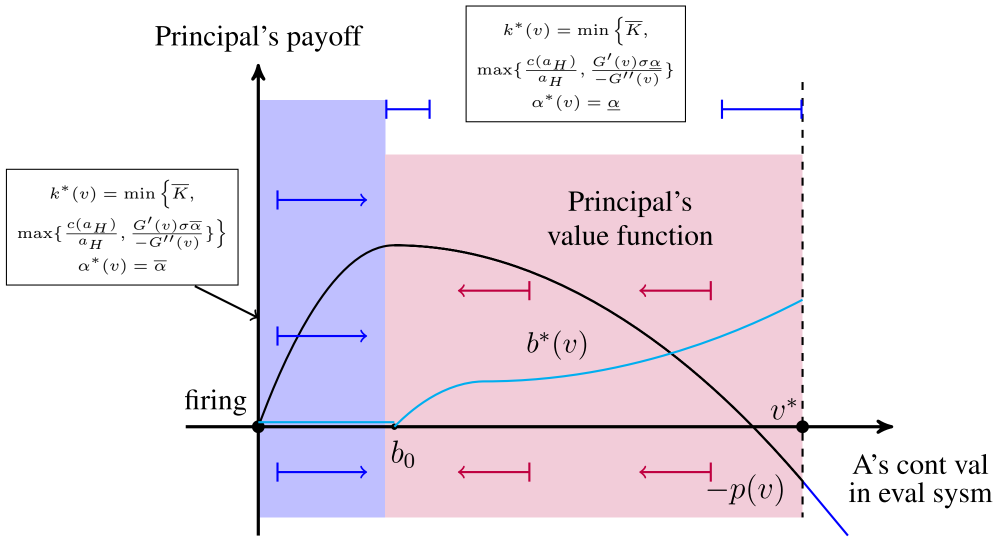


The evaluation scheme adds the maximum leniency bias in the deep blue region and is truthful in the light blue region. The volatility of Agent's continuation value process is weakly greater than that in the optimal output-based contract only in the deep blue region. The light blue curve indicates the reward policy.

Figure 2.5: When $\overline{\alpha} > \underline{\alpha} = 0$

We will see below that the optimal incentive contract always implements high effort provided that some technical conditions hold.

Theorem II.5. *Let G be the unique strictly concave function that solves the HJB equa-*



The optimal evaluation system adds the maximum leniency bias in the blue region and the maximum severity bias in the red region. The volatility of Agent's continuation value process is weakly greater than that in the optimal output-based contract almost everywhere. The light blue curve is the reward policy.

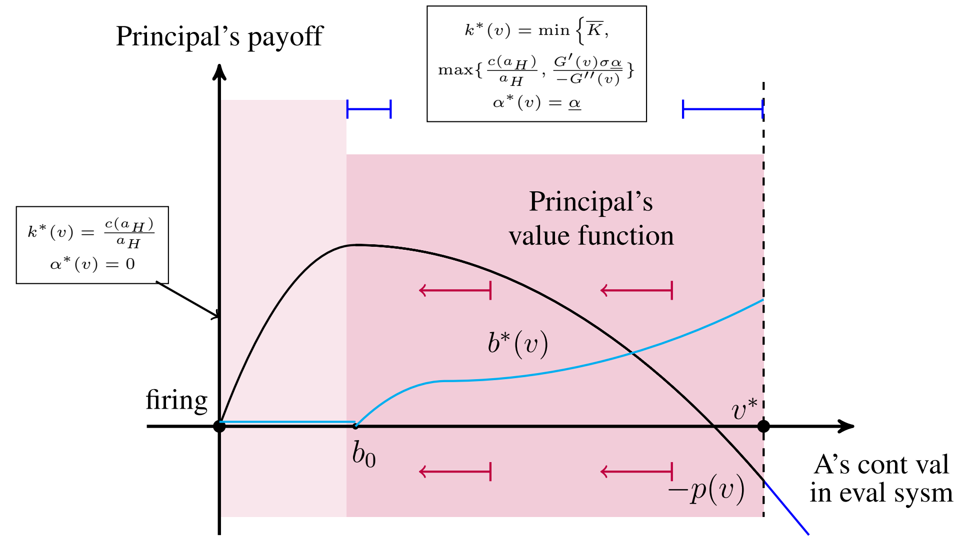
Figure 2.6: When $\overline{\alpha} > 0 > \underline{\alpha}$

tion (2.13) *with* $G \geq -p$ *and the boundary condition* (2.14) *satisfied by* v^* *where* $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid F(v) = -p(v)\}$ *if* $\{v \in (0, \frac{u(\bar{b})}{r}] \mid F(v) = -p(v)\} \neq \varnothing$ *and* $v^* := 0$ *otherwise. Assume that* $G \in C^3([0,\infty))$, $G'(0)\sigma\overline{\alpha} + \frac{G''(0)}{2}\sigma^2\epsilon \neq 0$ *and* $G'(v)c(a_H) + a_H + \frac{G''(v)}{2}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) \neq 0$ *for any* $v \in (G'^{-1}(0), v^*)$. *Then the optimal incentive contract always implements the high effort* a_h *before firing or retiring the agent.*

While evaluation bias has been discussed in the agency literature on subjective performance evaluation, it arises from Principal's incentive to underreport regardless of Agent's continuation values.[9] Different from a cheap-talk principal, a principal with commitment would like to under-report only when Agent's continuation value is high enough. Moreover, my result shows that, depending on the context, the principal will exhibit different types of evaluation biases. This helps explain the existence of evaluation bias documented in the empirical literature (Landy and Farr 1980, Rynes et al. 2005, Prendergast 1999).

Lastly, the above theorem also shows that the optimal information policy is to fully reveal the

[9]Several works show that without commitment, principals who are the residual claimants of the firm have incentives to under-report agents' performance to keep compensation costs down (cf. Bull 1983; MacLeod and Malcomson 1989; Baker et al. 1994). This reflects the major concern of reneging on bonuses in the relational contracting literature (cf. Levin 2003).



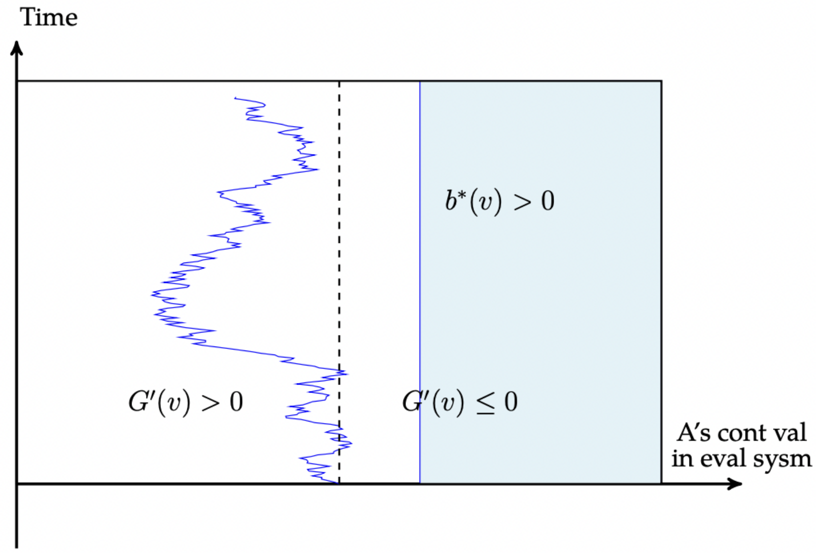
The evaluation scheme adds the maximum severity bias in the deep red region and is truthful in the light red region. The volatility of Agent's continuation value process is weakly greater than that in the optimal output-based contract only in the deep red region. The light blue curve is again the reward policy.

Figure 2.7: When $0 = \overline{\alpha} > \underline{\alpha}$

instantaneous evaluation to Agent. This result starkly contrasts with the non-commitment case, where it can be optimal that no information until termination is revealed by the principal, so as to reuse punishments across different periods (Fuchs 2007). Why is fully revealing information optimal with commitment? Intuitively, we may describe the reasons as follows: essentially, Agent only cares about his compensation. So information revelation without payment consequence will not affect Agent's effort provision. Nevertheless, revealing more information is weakly better for the principal, since it allows more possibilities to design compensation systems. Hence, a principal who could fully commit to information policy could benefit from revealing more evaluation information to the agent.

2.5.1 Towards the extreme

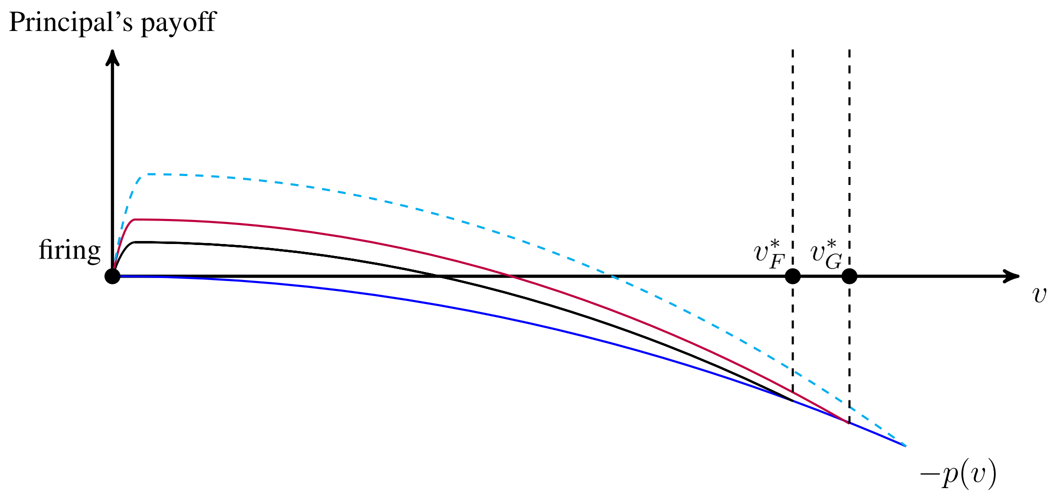
The evaluation bias in the optimal incentive contract is either the lower bound or the upper bound. How would the agent's continuation value process under the optimal contract behave differently when we take both bounds to infinity. As the upper bound $\overline{\alpha}$ increases (resp. lower bound $\underline{\alpha}$ decreases), an immediate observation is that the duration of V_t staying within the region $\{v \mid G'(v) > 0\}$ (resp. $\{v \mid G'(v) < 0\}$) decreases. When $\overline{\alpha}$ goes to ∞,



Let the cyan region be the region where Agent gets a positive amount of payment. The dashed line indicates the value v_0 at which $G'(v_0) = 0$. As $\underline{\alpha} \to -\infty$ and $\overline{\alpha} \to \infty$, the left side of the cyan region will converge to the middle dashed line so Agent is more likely to get paid. And the continuation value process reflects quickly after entering either the left region $\{v \mid G'(v) > 0\}$ or the right region $\{v \mid G'(v) \leq 0\}$.

Figure 2.8: As $\underline{\alpha} \to -\infty$ and $\overline{\alpha} \to \infty$

the continuation value process can only stay in the region $\{v \mid G'(v) > 0\}$ very briefly. Because of the increasingly large leniency bias, the probability of firing is also decreasing to zero. Similarly, as $\underline{\alpha} \searrow -\infty$, the continuation process will exit soon after it enters the region $\{v \mid G'(v) < 0\}$ and the probability of retiring is also decreasing to zero, as the severity bias gets infinitely large. As we take both $\underline{\alpha} \to -\infty$ and $\overline{\alpha} \to \infty$, the continuation value process V_t will rapidly fluctuate between $\{v \mid G'(v) > 0\}$ and $\{v \mid G'(v) < 0\}$ (recall that the volatility of Agent's continuation value is the maximum between the benchmark value and $\frac{G'(v)\alpha}{-G''(v)\sigma}$ when nonzero evaluation bias α is present). Moreover, Principal's value function becomes very steep so that whenever V_t enters the region in $\{v \mid G'(v) < 0\}$ where Agent gets positive payment, Agent's continuation value then immediately gets pushed back to the middle area. In this way, Agent's continuation value fluctuates rapidly between the above two regions during which he still gets nonnegative payment.



The red curve is Principal's value function G under optimal incentive contract, and the black curve is her value function F under the optimal output-based contract. The two points, v_G^* and v_F^*, are retirement boundary points under G and F respectively, which satisfy $v_G^* > v_F^*$. As the range $[\underline{\alpha}, \overline{\alpha}]$ with $\underline{\alpha} \leq 0 < \overline{\alpha}$ expands, Principal's value function increases pointwisely, as shown by the dot cyan curve.

Figure 2.9: The change of the principal's value function as we reduce $\underline{\alpha}$ and increase $\overline{\alpha}$

2.5.2 Comparison

This section compares the optimal incentive contract and the benchmark output-based contract. Let $G(\cdot)$ and $F(\cdot)$ be Principal's value functions under these two contracts respectively. From the definition of the principal's value function, the principal's value under the optimal incentive contract is always better than that under the benchmark contract, i.e., $G(v) \geq F(v)$ must satisfy at each Agent's continuation value v. Recall that the retirement boundary point must satisfy the smooth-pasting condition and the retirement profit function $-p(\cdot)$ remains the same for both cases. With G pointwisely dominating F, we can further conclude that the optimal incentive contract has a greater retirement bound as compared to that under the benchmark contract. Figure 2.9 illustrates such dominance. Moreover, as the domain of the adjustment process expands, the principal's value function is pointwisely increasing and dominates those with a smaller domain of adjustments. To graphically capture such increments, we include an additional dot cyan curve in Figure 2.9.

2.5.3 Mean exit time

Given any retirement bound v^*, it is also possible to characterize the average of the first time of Agent's continuation value process exits the employment domain $[0, v^*]$ due to either firing and retiring. I define the *first passage time* or *first exit time* to be the stopping time that records the first time that Agent's continuation value process V_t exits the employment domain:

$$\tau_v^*(\omega) := \inf\{t \geq 0 \mid V_t(\omega) = 0 \text{ or } V_t(\omega) = v^*, V_0(\omega) = v\}.$$

The mean exit time is defined as $\tau(v) := E^P[\tau_v^*] = E^P[\inf\{t \geq 0 : V_t \notin [0, v^*]\} | V_0 = v]$. Recall that in the optimal incentive contract, Agent always exerts high effort before being either retired or fired. Thus the mean exit time directly reflects the average production generated by this relationship.

Proposition II.5. *Given the optimal Markovian policy* $(b^*(\cdot), a^*(\cdot), \alpha^*(\cdot), k^*(\cdot))$ *in Theorem II.4 with the retirement bound* v^**, the mean exit time is the solution of the following boundary value problem: for any* $v \in (0, v^*)$*,*

$$-(rv - u(b^*(v)) + c(a^*(v)) + \sigma k^*(v)\alpha^*(v))\frac{\mathrm{d}\tau(v)}{\mathrm{d}v} - (\sigma k^*(v))^2 \frac{\mathrm{d}^2 \tau(v)}{(\mathrm{d}v)^2} = 1,$$
$$\tau(v) = 0, v \in \{0, v^*\}. \tag{2.17}$$

2.6 Evaluation biases under extra contractual possibilities

Numerous studies observe and document evaluation biases in the appraisal systems (cf. Landy and Farr 1980, Jawahar and Williams 1997, Prendergast 1999, Moers 2005, Frederiksen et al. 2017). We can see from the previous section that the optimal incentive contract exhibits leniency bias and severity bias, which justifies the existence of these evaluation biases. Do evaluation biases remain optimal as we introduce additional contractual possibilities? If not, how will they change?

In this section, I consider three types of realistic contractual possibilities, and how their presence may affect evaluation biases at the optimum, including (i) there is the reciprocity effect of evaluation biases on employment relationship that directly impacts the output; (ii)

Agent can get promoted before retirement, and (iii) Agent has a strictly positive outside option who can quit the firm at any time. The results show that evaluation biases remain a robust feature of the optimal incentive contract. Moreover, the above possibilities can strengthen specific evaluation biases depending on the context.

2.6.1 When evaluation biases have reciprocity effect on outputs

We will first consider the case when evaluation biases can have positive effects on employee performance if they could enhance the employees' perceptions of fairness. Empirical evidence in the literature shows that leniency bias could strengthen the employment relationship, which could have a positive behavioral effect on employees' performance. On the contrary, severity bias, even applying to only a fraction of employees, undermines the perception of fairness and negatively affects employees' performance.[10]

I explicitly model about the influence of such reciprocity on the output in the following parsimonious model such that the output evolves as $\mathrm{d}X_t^\gamma = \gamma(\alpha_t)A_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$, where $\gamma(\cdot)$ is the reciprocal effect of evaluation bias on output. I assume that $\gamma(\cdot)$ is nonnegative, increasing, concave differentiable function on $[\underline{\alpha}, \overline{\alpha}]$ with $\gamma(0) = 1$. The rest of the whole model remains the same, so the reciprocity effect does not directly affect Agent's evaluation or compensation.

For convenience, given any function $k(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and any continuously differentiable function $G(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, we define the function $\alpha_\gamma^{k,G}(\cdot)$ such that

$$\alpha_\gamma^{k,G}(v) := \begin{cases} \overline{\alpha} & \frac{-G'(v)\sigma k(v)}{a_H} < \gamma'(\overline{\alpha}) \text{ or } G'(v) \geq 0 \\ \gamma'^{-1}\left(\frac{-G'(v)\sigma k(v)}{a_H}\right) & \frac{-G'(v)\sigma k(v)}{a_H} \in (\gamma'(\overline{\alpha}), \gamma'(\underline{\alpha})) \\ \underline{\alpha} & \frac{-G'(v)\sigma k(v)}{a_H} > \gamma'(\underline{\alpha}) \end{cases}.$$

With Γ defined in (2.12), the following proposition characterizes the optimal incentive contract in such a situation.

[10] Bol (2011) shows that it is positively associated with employees' performance. From a comparative aspect, Marchegiani et al. (2016) show that failing to reward a deserving agent under a severe contract is significantly more detrimental to effort provision than rewarding an undeserving agent under a lenient contract. The effect of evaluation bias involves severe bias that is negatively associated with the performance (cf. Ahn et al. 2010; Bol 2011).

Proposition II.6. *Suppose that $G(\cdot)$ is the unique solution to the following equation:*

$$rG(v) = \sup_{\substack{(a,k\beta)\in\Gamma, b\in[0,\overline{b}]\\ \beta\in[-1,1],\alpha\in[\underline{\alpha},\overline{\alpha}]}} G'(v)(rv - u(b) + c(a) + \sigma k\alpha) + \frac{G''(v)}{2}(\sigma k)^2 + \gamma(\alpha)a - b, \tag{2.18}$$

and the boundary conditions $G(0) = 0, G(v^*) = -p(v^*)$ and $G'(v^*) = -p'(v^*)$ are satisfied by some v^* where $v^* := \arg\min\{v \in (0, \frac{u(\overline{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\overline{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* := 0$ otherwise. Assume that G is third continuously differentiable. Consider the following policies for any $v \leq v^*$: $\beta^*(v) = 1$,

$$b^*(v) = \begin{cases} 0 & u'_+(0) \leq \frac{1}{-G'(v)} \text{ or } G'(v) \geq 0 \\ u'^{-1}(\frac{1}{-G'(v)}) & \frac{1}{-G'(v)} \in (u'_-(\overline{b}), u'_+(0)) \\ \overline{b} & \text{otherwise} \end{cases};$$

if G and $\alpha_\gamma^{k^*,G}(v)$ satisfy Cond II.1,

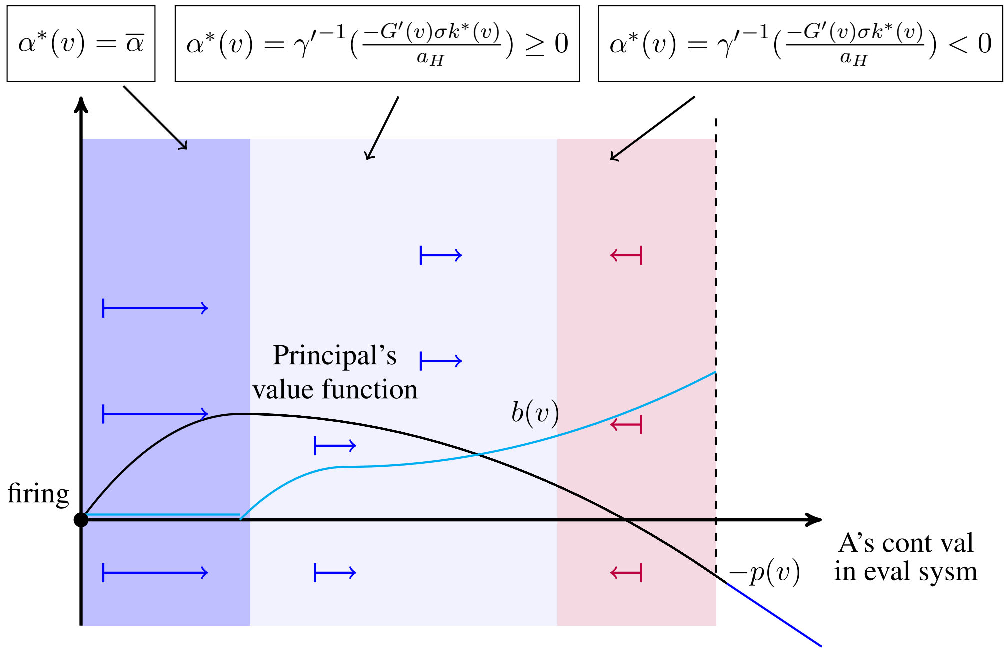
$$a^*(v) = a_H, k^*(v) = \min\left\{\overline{K}, \max\left\{\frac{G'(v)\alpha^*(v)}{-G''(v)\sigma}, \frac{c(a_H)}{a_H}\right\}\right\}, \alpha^*(v) = \alpha_\gamma^{k^*,G}(v),$$

otherwise,

$$a^*(v) = 0, k^*(v) = \max\left\{\frac{G'(v)\alpha^*(v)}{-G''(v)\sigma}, \epsilon\right\}, \alpha^*(v) = \overline{\alpha} + \mathbb{1}_{(0,\infty)}(G'(v))(\underline{\alpha} - \overline{\alpha}).$$

Given the initial value $v_0 := \max\{0, G'^{-1}(0)\}$, then there exists a unique strong solution to $\mathrm{d}\, V_s = (rV_s - u(b^*(V_s)) + c(a^*(V_s)))\,\mathrm{d}s + \sigma k^*(V_s)\,\mathrm{d}(W_s + \alpha^*(V_s)\,\mathrm{d}\, s)$, and the optimal incentive contract is characterized by the control $(\beta^*(V_t), \alpha^*(V_t), a^*(V_t), b^*(V_t), k^*(V_t))_{t\geq 0}$.

The above proposition shows that the optimal incentive contract in this variant again fully reveals the real-time evaluation to Agent. Compared to the case without reciprocity effects on productions, a major difference is that there is an overall pointwise shift of evaluation bias toward leniency in the optimal incentive contract. Whenever $G'(v) \geq 0$, the optimal evaluation bias is still the upper bound $\overline{\alpha}$. But there exists an additional region $\{v \mid G'(v) < 0, \alpha^*(v) \geq 0\}$ in which Principal is willing to evaluate Agent with moderate leniency. Accordingly, the reciprocal effect attenuates the severity bias in both shrinking



This example presents the case when the equilibrium effort is always a_H.

Figure 2.10: An example of optimal incentive contract with the reciprocity effect

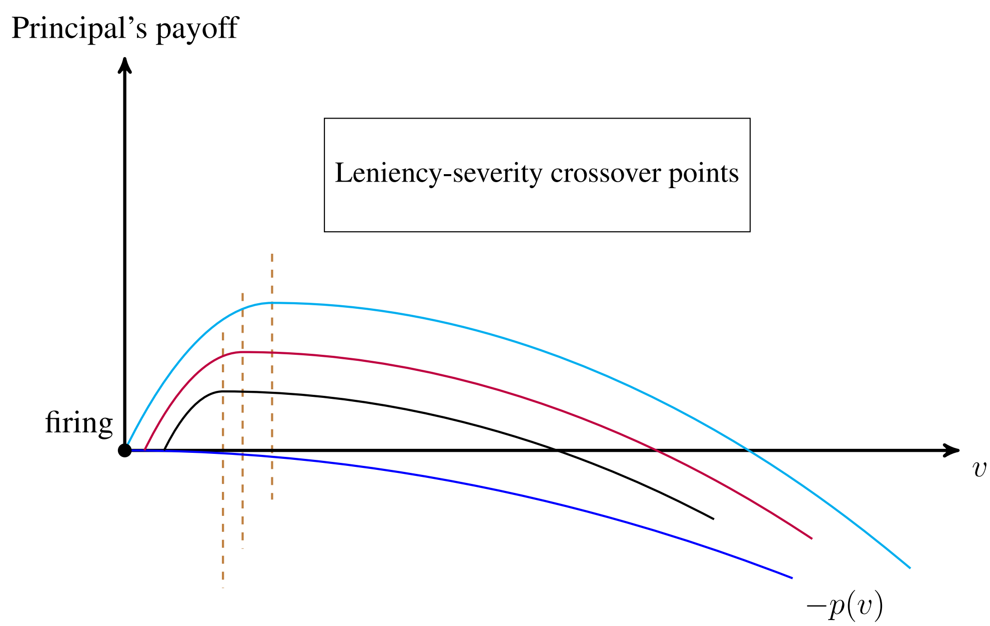
the region that exhibits severity bias and reducing its intensity. Figure 2.10 presents a possible situation when the reciprocity effect of evaluation biases could directly affect the production and the agent always exerts high effort before being fired or retired.

2.6.2 Agent can quit at any time

We now consider the situation when Agent can quit at any time and receive his outside option. In this case, the domain of Principal's value function does not include any region below Agent's outside option, since otherwise, the agent will quit immediately when his continuation value drops to the outside option.

In this situation, the derivation of the optimal incentive contract remains the same as our main result. The following proposition further gives a comparison result on the principal's value functions as the agent's outside option varies.

Proposition II.7. *Given any two outside options w_1 and w_2 such that $0 \leq w_1 < w_2$, let the principal's value function be G_1 and G_2, respectively. Then $G_1(v) \geq G_2(v)$ for any $v \in (w_2, \infty)$. Moreover, if $G_2'(w_2) > 0$, then $G_1(v) > G_2(v)$ and $G_1'(v) > G_2'(v)$ for any*



In the above figure, the cyan, red and black parabolic curves are Principal's value functions with the agent's outside option W_1, W_2 and W_3, respectively. I label the crossover point from leniency to severity for these three value functions in the brown dashed line, which is moving to the left.

Figure 2.11: When the agent has different outside options

$v \in (w_2, \infty)$.

Figure 2.11 presents Principal's value functions according to three different Agent's outside options: $\{W_j\}_{j=1}^3$ such that $0 = W_1 < W_2 < W_3$. The figure illustrates that as Agent's outside option goes up, Principal becomes less lenient in evaluating Agent. In particular, the leniency bias weakens in the pointwise sense that whenever the principal is lenient to an agent with a high outside option, she must be lenient to an agent with a low outside option. Similarly, severity bias strengthens: whenever Principal is severe to an agent with a low outside option, she must be severe to an agent with a high outside option who has the same continuation value. This implies that the principal tends to be more lenient to the agent if the employment relationship is easier to sustain.

The figure further implies that with a low outside option, an agent has a greater initial continuation value as compared to when he has a high outside option. Thus lowering the agent's outside option provides a Pareto improvement in this relationship since it generates a higher ex-ante expected payoff for both the principal and the agent.

2.6.3 Principal has the opportunity to promote Agent

Suppose that the principal has an opportunity to promote the agent to a new position by training. We model this case in a similar way as the extension in Sannikov (2008). Assume that the training cost is a perpetual constant cost flow with $r \cdot C > 0$ at each instant starting from the moment when the agent gets a promotion, where r is the discounting rate served as a normalization. The promotion permanently increases the agent's productivity of effort from a_H to $\hat{a}_H$ with $\hat{a}_H > a_H$ and $c(\hat{a}_H) = c(a_H)$. In this case, we again fix the agent's outside option to be 0. What does the optimal incentive contract, especially the biases in its evaluation system, look like in this setting?

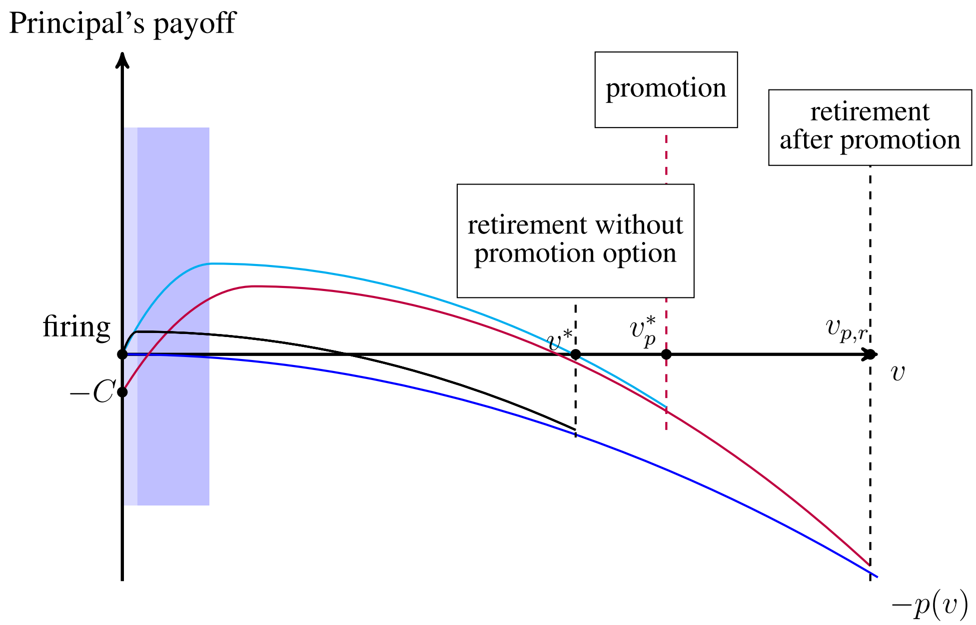
We will use the same notation $G(\cdot)$ as the principal's value function in the case without any promotion opportunity. Under the promotion opportunity, let $G_p(\cdot)$ be the principal's value function after promoting the agent and let $\widehat{G}(\cdot)$ be the principal's value function before such promotion. The following proposition provides a pointwise comparison of the principal's value function with the promotion to that without such promotion.

Proposition II.8. *(i)* $G_p(v) + C > G(v)$ *for any* $v \in (0, \infty)$*;*

(ii) $\widehat{G}(v) \geq G(v)$ *for any* $v \in (0, \infty)$*. Moreover, if there exists a point* $\hat{v} \in (0, \infty)$ *such that* $G_p(\hat{v}) > -p(\hat{v})$*, then* $\widehat{G}(v) > G(v)$ *and* $\widehat{G}'(v) > G'(v)$ *for any* $v \in (0, \infty)$*.*

The condition in the above proposition (ii) says that there exists some continuation value at which to promote the agent is strictly better for the principal than to retire the agent. If this condition is violated, then the principal always prefers to retire the agent and the promotion opportunity will not be exercised in all cases.

To illustrate the result, I further depict the above value functions in Figure 2.12. We can see that, with the presence of the promotion option, Principal effectively delays the retirement as compared to the case without. The crossover point from leniency bias to severity bias is pushed further to the right, implying that leniency bias strengthens and severity bias weakens pointwisely. In conclusion, Principal is more lenient in evaluating Agent with promotion opportunities than that under the case without.



This figure illustrates the optimal incentive contract with the promotion opportunity. The red parabolic curve is Principal's value function after promotion G_p; the blue curve is the principal's retirement profit function $-p(\cdot)$. The cyan curve is Principal' value function $\widehat{G}$ and the point indicated by the red dashed line is the optimal promotion point v_p^* and the optimal retirement point after promotion is $v_{p,r}$. The deep purple rectangle area indicates the expansion of leniency bias from the case without promotion to the one with promotion.

Figure 2.12: Optimal incentive contract with the promotion opportunity

2.6.4 Summary

From the above three possible model variants, we can see that either leniency or severity bias can be a feature in the optimal contract. This prediction contrasts with the conventional view that evaluation biases are detrimental. Moreover, the result implies that in certain environments, leniency bias, if in a correct dose, could be positively associated with production, which is supported by empirical evidence.[11]

My result thus suggests that, in assessing the effectiveness of the subjective performance evaluation within an organization, employers should be alert about the type of evaluation biases and whether it is in the optimal amount.

[11]For instance, Bol (2011) shows that leniency bias has a positive effect on motivating employees within a large bank in the Netherlands. The empirical evidence also documents the negative correlation between evaluation biases involving severity bias and the effort provision (see, Ahn et al. 2010; Bol 2011; Trapp and Trapp 2019; Marchegiani et al. 2016).

2.7 Conclusion

This paper suggests that the principal could benefit from not directly basing her evaluation on the output. The optimal incentive contract will assign lenient evaluation to Agent with low continuation value and severe evaluation to Agent with high continuation value. Adding biases into evaluation help Principal to extend the duration of the employment relationship as compared to the situation without. Also, there is room for concavification so the optimal incentive contract induces weakly higher volatility in Agent's continuation value as compared to the benchmark.

The result is useful in understanding the effect of the evaluation biases observed in the practical appraisal system. In particular, my result supports the view that evaluation not directly based on the output could provide partial insurance for the agent from the downside risk. The results in the extension also imply that evaluation biases could be a robust feature at the optimum, which further sheds light on how to improve the design of evaluation schemes. For the designer of a practical appraisal system, it is important to separate different evaluation biases. Sometimes, it may be worth considering to introduce evaluation biases into the system.

CHAPTER III

Persuasion of Interacting Receivers

3.1 Introduction

Many practical operations involve a sender providing information to multiple receivers before they engage in strategic interaction. Persuasion schemes in these situations often provide receivers with abundant information not only about the underlying state but also how others may act or believe. For instance, many online shopping websites disclose their existing consumers' reviews and the number of past purchases. Such disclosure allows the sellers to influence potential buyers by existing buyers' beliefs and actions. Similarly, political campaigns would invite celebrities to publicly endorse the candidate, in the hope of influencing voters' choice by the celebrities' beliefs. However, many in the literature have focused on situations where the sender provides only action recommendations. Departing from those focuses, our paper investigates the situations that go beyond the standard model with realistic features playing a critical role in determining the outcome of the receivers' game. In particular, we provide a new view that could unify several classes of Bayesian persuasion problems that have been studied separately, which contributes to a more coherent understanding of the persuasion literature.

In this paper, we study information design settings where multiple receivers are engaged in a strategic interaction with the following features: (i) the equilibrium selection may differ from the sender-preferred equilibrium selection, (ii) receivers may have private information about their preferences and the underlying states, (iii) receivers may have non von Neumann–Morgenstern utilities, and (iv) receivers may have heterogeneous prior beliefs. Given that the literature has no consensus on how to select equilibria (cf. Samuelson 1998), we propose a notion that describes equilibrium selections as selection criteria that eliminate unqualified equilibria. Our notion is quite flexible in describing different equilibrium selections, including the sender-preferred, sincere voting, and the sender-adverse selection in the literature.

Given the above features we introduce, the analogous revelation principle in the information design literature may not apply, in other words, the sender in our setting may benefit from sending messages beyond action recommendations. To better describe information and endow the messages with more structures, we further propose a more general message space that could explicitly capture both the basic uncertainty (i.e., underlying state and receivers' private information) and strategic uncertainty. In particular, we use a *conjecture*, a concept modified from epistemic game theory literature (see Dekel et al. 2007 and Chen et al. 2017) to characterize strategic uncertainty that comes from strategic reasoning. A conjecture consists of a receiver's belief about the underlying state, others' belief hierarchies, and actions. The message space we propose thus contains recommendations of conjectures and actions to each receiver type. Say a signal is *canonical* if it adopts the above message space and disseminates consistent information.

Our first result establishes a generalized obedience principle, which states that it is without loss of generality to restrict attention to canonical signals in our setting. Building on this result, we consider when one can simplify the signal structures further by restricting attention to canonical signals with countably many messages. This question is nontrivial in general since the literature has demonstrated that when the equilibrium selection differs from the sender-preferred one, an optimal information structure could employ uncountably many messages even though there are finitely many underlying states (see, for instance, Ali et al. 2021). We provide a sufficient condition for the equilibrium selection under which to employ countably many messages is without loss of generality for the sender. The key is to require certain stability properties to hold in the equilibrium selection. The construction of the proof further provides a tractable method to compute optimal signals in the general settings we describe above.

We illustrate the usefulness of our result in two applications where the equilibrium selection differs from the sender-preferred selection and receivers may have private information. Application 1 studies the optimal information policy in soliciting privately informed voters' support who will express their opinion only when they are pivotal; Application 2 is about disclosing a stress test result that minimizes the possibility of a bank run with privately informed investors under a sender-worse selection. Given that our applications have nonstandard features such as that the selection criteria differ from the sender-preferred one, they cannot be solved by standard approaches in the literature. We also provide a discussion about how our method connects to the existing approaches in the literature.

3.1.1 Literature Review

This paper belongs to the literature of information design and Bayesian persuasion. The literature, pioneering by Kamenica and Gentzkow (2011) and Rayo and Segal (2010), models the communication game as a sender committing to a probabilistic device that reveals partial information regarding the underlying state that receivers care about. Specifically, our paper connects this literature in two different levels:

At the first level, it belongs to the stream interested in a general methodological investigation of the settings with multiple receivers engaged in strategic interaction. Under the sender-preferred selection, Bergemann and Morris (2016, 2019) and Taneva (2018) establish the analogous revelation principle ("direct approach") in the standard static setting in which it is without loss of generality for a sender to recommend actions. Similarly, in the dynamic environment, Ely (2017) discusses what he calls the obfuscation principle in the setting of multiple agents. Doval and Ely (2020) studies the solution concept of coordinated equilibrium in an extensive-form game in which the canonical implementation of the equilibrium is that the sender recommends a sequence of actions. However, it is also known in the literature that the revelation principle does not generally apply. Among the counterexamples, the revelation principle does not apply in situations where the selection rule is sender-worst as the example in Mathevet et al. (2020) (the intuition is similar to the full implementation literature). And it does not apply to the situation where receivers' preference is psychological (see Lipnowski and Mathevet (2018)). Our paper is one step to generalize the idea of the canonical implementation to these natural situations where the standard revelation principle does not apply.

When going out of the standard settings, progress on studying interacting receivers is made mostly in specific environments (see the discussions in Kamenica (2019)). The specificities include binary action, specific payoff structure for receivers (e.g. supermodular), or signal with specific structures (e.g. public signal, iid signal). Many papers that study interacting receivers focus on the specific voting environment and consider how to persuade voters.

The works on persuading voters include Arieli and Babichenko (2019)(a receiver's payoff depends on their own action), Alonso and Camara (2016b) (public signals), Bardhi and Guo (2018) (a receiver's payoff depends on the outcome of the social choice rule) and Wang (2015) (iid signals). Relatedly, Application 1 in our paper considers how to persuade voters with the equilibrium selection that voters will vote only when they are pivotal with some positive probability and receivers may have private information about their preference.

A different well-known method in solving Bayesian persuasion problems is the concavification method. For a single receiver setting, see Kamenica and Gentzkow (2011) (a common prior,) and Alonso and Camara (2016a) (heterogeneous priors). This method is generalized by Mathevet et al. (2020) to the setting with multiple receivers under a common prior. In particular, their method can be used to study the situation with equilibrium selection differing from the sender-preferred one. For other methods in persuasion problem with a single receiver, there are works that investigate the methodology from a different dual angle under the sender-preferred selection rule, including Dworczak and Martini (2019), Dworczak and Kolotilin (2019), Galperti and Perego (2018).[1]

In the second level, our paper connects with the literature that studies any of the three features we mentioned above. Specifically:

For non-sender-preferred equilibrium selection, a well-studied alternative selection is the sender-adverse selection. The study of this selection on supermodular games can be found in Inostroza and Pavan (2020), Morris et al. (2020) and the example in Mathevet et al. (2020). Relatedly, Ziegler (2020) studies adverse selection with a different solution concept that relies only on first-order beliefs. In our Application 2, we substantially extend the example in Mathevet et al. (2020), and apply our result to compute the optimal signal. There are other selections such as sincere voting (see, for instance, Alonso and Camara (2016b) and Titova (2021)). Application 1 in our paper also has the equilibrium selection that relates to this voting rule.

For privately informed receivers, there are many works studying economics implications on specific contexts with focuses substantially different from this paper. To list a few: Rayo and Segal (2010), Guo and Shmaya (2019), Kolotilin et al. (2017) and Kolotilin (2018), etc.

[1] Different from our discrete setting, these investigations often allow continuum state spaces but may have other restrictions on player's utility functions or actions.

In the application section, we study two applications with some receivers having private information: in Application 1, the private information is about receivers' preference, and in Application 2, the private information is about the underlying state.

For strategic information disclosure to psychological receivers, there are two papers that we are aware of using Bayesian persuasion to model information disclosure to a psychological audience: Ely et al. (2015) and Lipnowski and Mathevet (2018). In particular, Ely et al. (2015) analyze the optimal information revelation for an audience who has a preference for noninstrumental information in a dynamic setting, and Lipnowski and Mathevet (2018) analyze a setting of a single psychological receiver whose utility is aligned with the sender. Both of them adopt some extended versions of the concavification method. We study an example (Example III.5) modified from Lipnowski and Mathevet (2018) and determine the optimal signal in such a context.

This paper also borrows important conceptual insight from the epistemic game theory literature, especially for the concepts of conjectures. Several works, including Ely and Peski (2006), Dekel et al. (2007), Liu (2009) and Chen et al. (2017), bring valuable insight in considering a canonical representation in capturing explicitly basic underlying uncertainties and the strategic uncertainty that comes from strategic reasoning.

This paper is organized as follows: we introduce the model and the definition of equilibrium selection criterion in Section 3.2. We present our main results in Section 3.3. Section 3.4 includes two applications. We discuss the relationship between our method and the currently existing approaches and possible extensions in Section 3.5. All proofs are collected in the appendix.

3.2 The Model

Let Ω be a set of finite underlying states. There is a sender ("Player 0") and a finite set of receivers $\mathcal{I}$, each endowed with a set of finite primitive types $\mathbb{T}_i, i \in \mathcal{I}$, where $\mathbb{T} := \prod_{i\in\mathcal{I}} \mathbb{T}_i$ is the product space of primitive types. Initially, players have no information about their primitive types. It is common knowledge that Nature will select a primitive type profile $\tau := (\tau_i)_{i\in\mathcal{I}}$ according to a rule and privately informs each receiver i about his type τ_i. Nature's rule π^N is also commonly known, and conditioning on each realized state $\omega \in \Omega$, $\pi^N(\cdot|\omega)$ is a distribution over $\mathbb{T}$. Before Nature draws receivers' primitive type profile, each player i has a prior belief $\mu_i^0 \in \Delta(\Omega \times \mathbb{T})$ consistent with the known Nature's rule but potentially different across players. For any realized primitive type τ_i, let $\mu^0_{i|\tau_i} \in \Delta(\Omega \times \mathbb{T})$ be the intermediate belief of Receiver i condition on τ_i updated by Bayes' rule. As in Kamenica (2019), private information that receivers might have could be one of the following cases: receivers' preferences (e.g., Rayo and Segal (2010)) or the state of the world (e.g., Guo and Shmaya (2019)). The receivers' private information in our model could be in any or both of the above two situations, depending on whether the primitive types are directly payoff relevant or not. For convenience, our treatment will mainly base on the first case, which also applies to the second case with some adaptations.

Each receiver i has a finite set of actions, denoted as A_i. Let $A := \prod_{i\in\mathcal{I}} A_i$ be the space of receivers' action profiles. In the main setting, we will restrict attention to Players are von Neumann-Morgenstern utility maximizers, and each player i has a Bernoulli utility function $u_i : \Omega \times A \times \mathbb{T} \to \mathbb{R}$. Allowing for non von Neumann-Morgenstern players whose higher-order beliefs directly affect their utilities raises no conceptual difficulties for our results but requires heavier notations. We will provide a discussion on extending our results to the case where players have non von Neumann-Morgenstern utilities in Section 3.5.

The sender can design a signal that generates some information about the state. A generic signal is a pair $(M := \prod_{i\in\mathcal{I}} M_i, \pi)$ which consists of a message space M and a rule $\pi : \Omega \to \Delta(M)$ that specifies the distribution of messages conditional on the realized state. Let $\pi_i : \Omega \to \Delta(M_i)$ be the marginal conditional distribution of π on M_i. Moreover, I assume that the sender's rule π is independent of Nature's rule π^N.

This game proceeds in five stages: (i) The sender chooses a message space M and commits to a rule π; (ii) Nature picks a state ω and, given ω, a primitive type profile $(\tau_i)_{i\in\mathcal{I}}$ according to π^N and a message profile $m = (m_i)_{i\in\mathcal{I}}$ according to π; (iii) Nature privately informs each receiver of his own realized type τ_i; (vi) each receiver i privately observes her individual message m_i and updates his belief according to Bayes' rule; (v) receivers take simultaneous actions and payoffs are realized.

It is common knowledge that all players are rational and understand the model (including all prior beliefs, utility functions and the equilibrium selection rule). We use the Bayesian Nash equilibrium as the solution concept for the receivers' game: given any signal $(\prod_{i\in\mathcal{I}} M_i, \pi)$, denote by $\sigma := (\sigma_i)_{i\in\mathcal{I}}$ a strategy profile where $\sigma_i : M_i \times \mathbb{T}_i \to \Delta(A_i)$ for receiver i. Given strategy profile σ played in the receivers' game, let $E^\pi[u_i(\sigma_{-i}, \tau_{-i}, \sigma_i)|m_i, \tau_i]$ be receiver

i's expected utility conditional on the realized message and the primitive type.[2]

Definition III.1. A strategy profile σ^* is a Bayesian Nash equilibrium ("BNE") if for any receiver $i \in \mathcal{I}$,

$$E^{\pi}[u_i(\sigma^*_{-i}, \tau_{-i}, \sigma^*_i)|m_i, \tau_i] = \max_{a_i \in A_i} E^{\pi}[u_i(\sigma^*_{-i}, \tau_{-i}, a_i)|m_i, \tau_i],$$

for every $\tau_i \in \mathbb{T}_i$ and $m_i \in M_i$.

3.2.1 The notion of equilibrium selection

It is often the case that the receivers' game has more than one equilibrium. Many works in the Bayesian persuasion literature consider that Sender could select the equilibrium she prefers. This selection is termed "the sender-preferred selection" (see Kamenica 2019 and Bergemann and Morris 2019), which implicitly assumes that receivers obey the action recommendations as long as it satisfies the Bayesian incentive compatibility constraints. Nevertheless, the increasing awareness of the relevance of equilibrium selection rules such as the sender-adverse selection has led to the examination of other selection rules in the literature (see Mathevet et al. 2020, Inostroza and Pavan 2020, Ziegler 2020 and Morris et al. 2020).

The first challenge of considering equilibrium selection is that there is a lack of consensus among game theorists on an explicit definition of such a notion (cf. Samuelson 1998).[3] In this paper, we propose a notion that can provide a formal description of equilibrium selection rules. The idea behind our notion is aligned with equilibrium refinements literature, which specifies selection criteria to rule out unqualified equilibrium. In particular, a selection criterion in our setting is an axiomatic element on receivers' behavior. Moreover, which selection should be present in the receivers' game is determined by factors outside the game such as personal characteristics, culture, life experience, etc.

In games of incomplete information, when one deviates from the sender-preferred selection, belief hierarchies matter for the equilibrium selection since each receiver's action could

[2] As a convention, for message m'_i realized with 0 probability (i.e., $\sum_{\omega \in \Omega} \pi_i(m'_i|\omega) \operatorname{marg}_{\Omega} \mu^0_{i|\tau_i}(\omega) = 0$), we set Receiver i's conditional expected payoff under this message to be the lower bound of their Bernoulli utility regardless of their primitive type.

[3] In fact, the problem of equilibrium selection has been heavily examined in the game theory literature. The refinements literature and the evolutionary games' literature both have contributed to this topic; see Samuelson (1998) and references therein.

depend on his or her belief about how others believe about what the others will believe and act. Our notion of equilibrium selection thus builds on the concepts from epistemic game theory literature. To prepare us for this notion, we first provide a very brief review of the basics that serve as our building blocks.

3.2.1.1 Preliminaries

The basic uncertainty that each receiver i is facing comes from two sources: the underlying state Ω and his opponents' primitive types $\mathbb{T}_{-i}$, both assumed to be finite. We thus take $\Omega \times \mathbb{T}_{-i}$ to be the complete underlying set that summarizes all relevant basic uncertainty receiver i cares about.

Remark III.1 (when "a primitive type" is private information about the state)**.** In this case, each receiver i's utility is $u_i : \Omega \times A \to \mathbb{R}$, the basic uncertainty is the underlying state in Ω. Here, the primitive types τ_i can be interpreted as private information for updating reciever i's prior over Ω and thus will be reflected automatically on his first-order belief.

For any receiver $i \in \mathcal{I}$, a first order belief is an element of $T_i^1 := \Delta(\Omega \times \mathbb{T}_{-i})$, and for $k \geq 2$, a k-order belief is an element of $T_i^k := \Delta(\Omega \times \mathbb{T}_{-i} \times \prod_{l=1}^{k-1} T_{-i}^l)$. Following Brandenburger and Dekel (1993), an infinite hierarchy of beliefs $t_i = (t_i^1, t_i^2, ...)$ is *coherent* if for every $k \geq 1$, $\mathrm{marg}_{T_i^{k-1}} t_i^k = t_i^{k-1}$, where $\mathrm{marg}_{T_i^{k-1}}$ denotes the marginal on the space T_i^{k-1}. A tedious calculation shows the common knowledge of coherency condition in Brandenburger and Dekel (1993) is satisfied in our setting.

Denote by T_i the set of receiver i's infinite belief hierarchies with $T_i \subseteq \prod_{k=1}^{\infty} T_i^k$ that satisfies the common knowledge of coherency condition, i.e., each receiver's higher-order beliefs are commonly known to be compatible with the lower-order beliefs. Note that T_i^k and T_i, as spaces of probability measures on compact sets, are compact in weak* topology.[4]

By Proposition 2 in Brandenburger and Dekel (1993) (or Theorem 2.9 in Mertens and Zamir, 1985), there exists a homeomorphism $\psi_i : T_i \to \Delta(\Omega \times \mathbb{T}_{-i} \times T_{-i})$. Therefore, the explicit and implicit approaches are equivalent and we can treat infinite belief hierarchies as "universal types" (see Harsanyi, 1967). As such, we can consider the uncertainty over belief hierarchies by universal types, and denote as $\psi_i(t_i)$ the associated belief of type t_i about the state and the universal types of the other players.

[4]See Mertens and Zamir (1985) or Brandenburger and Dekel (1993) for further details.

We define the conjecture of each receiver i to be ν_i, which is an element in $V_i := \Delta(\Omega \times \mathbb{T}_{-i} \times T_{-i} \times A_{-i})$, modified from the literature to describe the perception of receiver i with type τ_i about the joint distribution regarding the basic uncertainty (the underlying state and others' primitive types), others' universal types and actions.

Definition III.2. For each receiver i , given any of his primitive type τ_i and universal type $t^i_{\tau_i} \in T_i$, his conjecture $\nu^i_{\tau_i} \in V_i$ is consistent with his belief $\psi_i(t^i_{\tau_i})$ if, for each $\omega \in \Omega$, $\tau_{-i} \in \mathbb{T}_{-i}$ and measurable subset $E_{-i} \subseteq T_{-i}$,

$$\nu^i_{\tau_i}(\{\omega\} \times \{\tau_{-i}\} \times E_{-i} \times A_{-i}) = \psi_i(t^i_{\tau_i})\Big(\{\omega\} \times \{\tau_{-i}\} \times E_{-i}\Big).$$

Since a receiver with different primitive types may form different intermediary beliefs over the basic uncertainty $\Omega \times \mathbb{T}_{-i}$, which could result in different conjectures even given a same message by the sender. Thus we introduce the space of conjecture menus for each receiver i, denoted as $\mathbb{T}_i \times V_i$, where ν^i is a generic element and $\nu^i := \prod_{\tau_i \in \mathbb{T}_i} \nu^i_{\tau_i}$ with each $\nu^i_{\tau_i}$ the conjecture of receiver i with primitive type τ_i.

For each receiver i with primitive type τ_i, let $B_{i,\tau_i}(\nu^i_{\tau_i})$ be the corresponding set of receiver i's mixed strategy best responses to conjecture $\nu^i_{\tau_i}$ and type τ_i. Given that receivers' actions are finite, then we can view $B_{i,\tau_i}(\nu^i_{\tau_i})$ is a subset in $\mathbb{R}^{|A_i|}$. We are now ready to introduce the notion of equilibrium selection in our model.

3.2.1.2 Equilibrium selection criteria

In our setting, how to select equilibrium is determined by how each receive will react based on their information, which is considered an axiomatized element outside the receivers' game. To formally describe such a selection, we borrow the insights from Jehiel (2005) and Jehiel and Koessler (2008) and propose the notion of selection criteria. Specifically, a selection criterion specifies (i) how to categorize each receivers' conjectures into analogy classes, and (ii) assign each class with a subset of selected actions.

The implicit assumption behind our notion is that each receiver with a similar interpretation of information (summarized by conjectures) will behave similarly such that given their conjectures, they will consider only a subset of all possible actions in playing the receivers' game. A selection criterion thus gives explicit instructions on how to think about the "similarities" of conjectures by bundling them into analogy classes and also which actions each class may take. This notion provides a simplified representation for the idea of equilibrium selection which could bring tractability to describe the idea of equilibrium selection in complex strategic situations.

Specifically, for each receiver i, a *behavioral analogy class* is a product set $p_i \times S_i$ with $p_i \subseteq V_i$ and $S_i \in 2^{A_i}$. Let $\mathscr{P}_i$ be the set of all possible countable partitions on the conjecture space V_i. Then we define a selection criterion as follows:

Definition III.3. A *selection criterion* $\zeta := (\zeta_i)_{i\in\mathcal{I}}$ specifies a partition $P_i \in \mathscr{P}_i$ for each receiver i such that it is a collection of behavioral analogy classes induced by P_i for each receiver i, i.e., $\zeta_i := \{p_i \times S_i(p_i) \text{ for some } S_i(p_i) \in 2^{A_i}\backslash\{\varnothing\} \mid p_i \in P_i\}_{i\in\mathcal{I}}$. Moreover, if the above $\{P_i\}_{i\in\mathcal{I}}$ satisfies $\#|P_i| < \infty$ for all $i \in \mathcal{I}$, then the above ζ is a *finite selection criterion.*

For any behavioral analogy class $p_i \times S_i$, say an element (ν_i, α_i) is in this behavioral analogy class if $\nu_i \in p_i$ and $\alpha_i \in \Delta(S_i)$. A selection criterion selects a subset of equilibria from any given set of equilibria by eliminating those unqualified. Specifically, given an arbitrary signal, say a selection criterion ζ selects an equilibrium under this signal if for each primitive type of receiver i and any equilibrium conjecture ν_i, its equilibrium action α_i is in the action set specified in ζ_i. Formally,

Definition III.4. Fix a selection criterion $\zeta = (\zeta_i)_{i\in\mathcal{I}}$. Given any signal $(\prod_{j\in\mathcal{I}} M_j, \pi)$ and an associated BNE $\sigma^* := (\sigma_i^*)_{i\in\mathcal{I}}$, let $\nu_i^{\pi,\sigma^*}(m_i)$ be receiver i's conjecture given message $m_i \in M_i$ under equilibrium σ^*. Then we say σ^* *survives* the selection criterion ζ if for any $m_i \in M_i$, the equilibrium conjecture-action pair $(\nu_i^{\pi,\sigma^*}(m_i), \sigma_i^*(m_i, \tau_i))$ for each primitive type τ_i is in some behavioral analogy class in ζ_i.

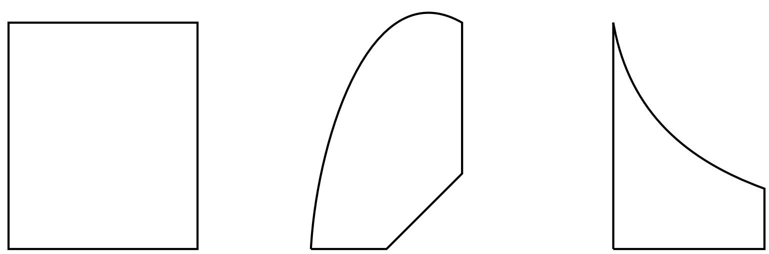
Our selection criterion helps to exclude unqualified equilibria, which, in the same vein as the idea of equilibrium refinements literature, does not guarantee uniqueness. If multiple equilibria survive this selection criterion, then we assume the tie-breaking rule is that the sender could choose her preferred equilibrium among the survived ones.

Remark III.2. For exposition convenience, throughout the text, we will focus on the selection criteria that apply to all receivers' primitive types. The results can be modified in a straightforward way to extend to the case where selection criteria could tailor the selection to each each receiver primitive type.

We now introduce an important class of selection criteria that we work with throughout the paper. First let us introduce the concept of regular sets.

Definition III.5 (regular boundary)**.** A set $\tilde{S}$ in a topological space X has a *regular boundary* if there exist finitely many continuous functions $f_j : X \to \mathbb{R}$ for $j = 1, 2, ..., N$ such that the closure of $\tilde{X}$ is $\cap_{j=1}^{N} f_j^{-1}([0, \infty))$.

Figure 3.1 provides three explicit examples to illustrate the above concept.



The boundaries of the above sets are defined by finitely many continuous functions.

Figure 3.1: Sets with regular boundaries

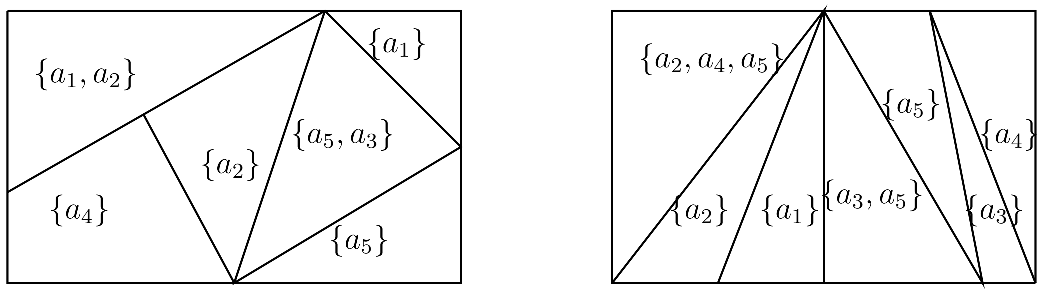
Definition III.6 (regular selection criterion)**.** A selection criterion $\zeta := \{p_i \times S_i(p_i)$ with $S_i(p_i) \in 2^{A_i} \mid p_i \in P_i\}_{i \in \mathcal{I}}$ is *regular* provided that each $p_i \in P_i$ is a convex set with a regular boundary in V_i for any $i \in \mathcal{I}$.

For visualization, we present two examples of regular selection criteria in Figure 3.2: we use a rectangle to represent the entire conjecture space for an individual receiver, and the partition of the conjectures as to the partition of the rectangle. We also associate each selected action set specified by the criterion with its partition component.

For non-regular selection criteria, it is possible to transform them into regular selection criteria via further partitioning on their partition components. Our results are thus immediately applicable to these types of non-regular selection criteria.

3.2.1.3 Some examples of equilibrium selections in the literature

In the following, we provide four examples to illustrate how the notion of selection criteria could describe some common equilibrium selection we have seen in the literature.



Let action set be $A := \{a_1, ..., a_5\}$. In the left rectangle of this figure, we can see that the selection criterion associates the actions $\{a_1, a_2\}$ to the partition component in the upward left corner. Similarly, this criterion associates action a_1 with the partition component in the upward right corner of the same rectangle. The right rectangle presents another regular selection criterion.

Figure 3.2: Regular selection criterion

Example III.1 (sender-preferred)**.** The most well-known selection rule in the Bayesian persuasion literature is the sender-preferred selection, as introduced in Kamenica and Gentzkow (2011). Under this selection, the sender could pick any equilibrium she prefers. The trivial selection criterion $\{V_i \times A_i\}$ that imposes no restriction on the equilibrium will always describe the idea that the sender could always select her preferred outcome.

Example III.2 (sincere voting)**.** Building on the empirical evidence that voters derive utility from expressing support for one of the candidates in large elections, it is common to adopt the sincere voting rule to predict the election outcomes (see, for instance, Alonso and Camara 2016b and Titova 2021). Our notion can capture such a selection directly. Suppose the underlying state is binary $\theta \in \{L, R\}$, and a sender would like to persuade a set $\mathcal{I}$ of voters, each with a binary action $\{Y, N\}$, to pass a law. Let u_i be the utility function for each voter i. The sincere voting rule requires each voter to select "Y" if and only if given the sender's message, to pass the law is better than not to pass under its posterior belief.

Suppose the law will pass if all voters vote Y and will not pass if all voters vote N. Let ρ_i^V be a subset of voter i's conjecture space such that

$$\rho_i^V := \{v_i \in V_i \mid E_{\mathrm{marg}_{\Delta(\Omega)} v_i}[u_i(Y^{\mathcal{I}}, \theta)] \geq E_{\mathrm{marg}_{\Delta(\Omega)} v_i}[u_i(N^{\mathcal{I}}, \theta)]\},$$

where $\mathrm{marg}_{\Delta(\Omega)} v_i$ is the projection of v_i on $\Delta(\Omega)$. The following selection criterion can capture the sincere voting rule: $\{\rho_i^V \times \{Y\}, (V_i \backslash \rho_i^V) \times \{N\}\}_{i \in \mathcal{I}}$.

Example III.3 (sender-worst)**.** One way to think about sender-worst selection, borrowing the idea from the global game literature, is to select the action profile such that each action is uniquely rationalizable given the corresponding agent's belief hierarchies. For illustration purposes, we consider the application in Mathevet et al. (2020) and describe it using the notion of selection criteria.

The underlying state θ takes a binary value $\{-1, 2\}$. There is a sender who could like to persuade two receivers with binary action $\{I, N\}$ to take action I as much as possible. Receivers would take action I if action I is uniquely rationalizable given his belief hierarchies. Mathevet et al. (2020) characterize the set of belief hierarchies for each receiver i, denoted as ρ_i, under which action I is uniquely rationalizable. Let ρ_i^V be a subset of V_i for receiver i such that

$$\rho_i^V := \{v_i \in V_i \mid \psi_i^{-1}\left(\text{marg}_{\Delta(\Omega\times T_{-i})} v_i\right) \in \rho_i\},$$

where $\psi_i : T_i \to \Delta(\Omega \times T_{-i})$ is the homeomorphism. Then the selection criterion $\{\rho_i^V \times \{I\}, (V_i \backslash \rho_i^V) \times \{N\}\}_{i=1,2}$ captures the sender-worse selection in this example.

Example III.4 (skeptical posture)**.** In a seller-buyer context, Milgrom and Roberts (1986) capture the sophisticated buyer's behavior in the equilibrium selection with the notion "skeptical posture". In particular, a buyer exhibits a skeptical posture if he always chooses a belief that minimizes the purchased quantity among all possible beliefs given the seller's information. Our notion of selection criteria is also useful to formally describe such a skeptical buyer's purchase behavior.

Consider the setting of one seller and one buyer where the seller owns products with unknown quality x that takes finitely many values $X := \{x_1, x_2, ..., x_N\}$ with $x_1 < x_2 < ... < x_N$. Let $A := \{q(x_n)\}_{n=1,...,N}$ be all possible purchase quantities such that if the buyer knows the quality is x_n, then the quantity $q(x_n)$ is the best choice. Moreover, the quantities satisfy that $q(x_1) < q(x_2) < ... < q(x_N)$. In this case, the conjecture of the buyer degenerates to his belief over X. Let $\Delta(X)$ be the space of the buyer's all possible beliefs. Define $\tilde{\phi} : \Delta(X) \to \{q(x_n)\}_{n=1,...,N}$ such that $\tilde{\phi}(\mu) = \min_{x\in \text{supp}\,\mu} q(x)$ for any $\mu \in \Delta(X)$. Then the selection criterion

$$\left\{\tilde{\phi}^{-1}(q(x_1)) \times \{q(x_1)\}, \tilde{\phi}^{-1}(q(x_2)) \times \{q(x_2)\}, ..., \tilde{\phi}^{-1}(q(x_N)) \times \{q(x_N)\}\right\}$$

captures the behavior of a skeptical buyer in such a setting.

3.3 Main Results

3.3.1 Canonical Signals

To tackle the information design problem with private signals, we introduce a convenient representation for arbitrary signals, which we call the canonical form. Let $M_i^c := \mathbb{T}_i \times V_i \times \Delta(A_i)$ be the product space of each receiver i's conjectures and actions across its possible primitive types. Intuitively, each element $m_i \in M_i^c$ is a menu of recommended conjectures and types for each possible primitive types ("menu") for this receiver. Let $M^c := \prod_{i \in \mathcal{I}} M_i^c$ be the product of such recommendations across all possible receivers.

Definition III.7 (canonical signal)**.** A signal is *canonical* if it is a pair (M^c, π^c) such that for each receiver i who gets message $\prod_{\tau_i \in \mathbb{T}_i}(\tau_i, \nu^i_{\tau_i}, \alpha^i_{\tau_i})$, given all other receiver complying with their recommendations, each primitive type τ_i finds its recommended conjecture $\nu^i_{\tau_i}$ in the menu consistent with that derived from the Bayes' rule under π, and $\alpha^i_{\tau_i}$ is a best response given $\nu^i_{\tau_i}$.

Intuitively, a canonical signal is a direct private recommendation to each receiver primitive type about their posterior conjecture and action such that, believing others' compliance, each receiver primitive type will comply.

Recall that the standard definition of any equilibrium outcome is its induced joint distribution of state and action profiles. With this definition, however, the heterogeneity of players' priors may lead to different understanding among players given any equilibrium. The following definition we introduce is stronger than the standard version, which avoids such inconsistency. For any signal (M, π) and any associated Bayesian Nash equilibrium σ, we define $O^{M,\pi}$ as its *equilibrium outcome* where $O^{M,\pi} : \Omega \times \mathbb{T} \to \Delta(A)$ such that $O^{M,\pi}(\omega, \tau) = \sum_{m \in M} \sigma(m, \tau)\pi(m \mid \omega)$ for each $\omega \in \Omega$ and $\tau \in \mathbb{T}$. Moreover, we say two equilibrium outcomes O_1 and O_2 are the same if and only if $O_1(\omega, \tau) = O_2(\omega, \tau)$ for any ω and τ. We are now ready to introduce our first proposition.

Proposition III.1. *Fix an arbitrary regular selection criterion ζ. For any signal (M, π) and any associated Bayesian Nash equilibrium σ that survives ζ, there exists a canonical signal in which the direct recommendations constitute a BNE that survives ζ and achieves the same outcome as that under σ given the signal (M, π). Therefore the sender's expected payoff remains the same under both signals.*

The intuition of the proof is as follows: Consider an arbitrary signal $s := (M, \pi)$ that induces a Bayesian Nash equilibrium σ surviving rule ζ. We could construct its canonical counterpart $s^c := (M^c, \pi^c)$ such that for each message the signal s sends to receiver i at any state, its canonical counterpart s^c sends the corresponding menu of posterior conjectures and action recommendations with the same conditional probability at that state. This canonical implementation pools each set of messages that induce an identical menu of conjecture and action recommendations into a single message. We then verify such a pooling does not change the resulted menu of conjecture and action. Thus the resulted signal (M^c, π^c) is canonical. Moreover, given that the original equilibrium survives the rule ζ and the two signals have identical conjectures and actions in the equilibria, the equilibrium recommended by (M^c, π^c) would survive ζ as well. The preservation of the equilibrium outcomes also follows directly.

3.3.2 Countable canonical signals

The literature has established that, if the equilibrium selection differs from the sender-preferred one, the optimal signal may have uncountably many messages even with finitely many underlying states.[5] Our first result shows that, with a regular selection criterion, we can focus on canonical signals without loss of generality. We will consider pure strategy BNEs to avoid exposition complications. To simplify the structure of canonical signals further, we investigate when it is without loss of generality to consider canonical signals with at most countably many messages. Formally, a canonical signal is a *countable canonical signal* if the support of its conditional distribution on each realized state is a countable set. Our second main result provides a sufficient condition for the equilibrium selection that allows restricting one's attention to countable canonical signals.

3.3.2.1 From uncountable to countable messages: the result

To formally present such a condition, let us first introduce the following preliminary notions. Let $M_i^{p,c} := \mathbb{T}_i \times V_i \times A_i$ be the space that consists of all possible recommended menus for each receiver i. Denote $M^{p,c}$ to be the product across receivers $M^{p,c} := \prod_{j\in\mathcal{I}} M_j^{p,c}$. Let

$$\mathscr{M} := ((\Delta(\Omega \times \mathbb{T}))^{\mathcal{I}} \times (\Delta(\mathbb{T}))^{\Omega} \times (\Delta(M^{p,c}))^{\Omega}$$

[5] For example, Ali et al. (2021) show that, under the adverse equilibrium selection, there is a continuum of messages in the optimal information structure even though the underlying state is of binary values.

be the set that collects all possible combinations of receivers' prior belief profile, the Nature's rule and the sender's signal with the message space $M^{p,c}$. Denote $\mathscr{M}^c \subseteq \mathscr{M}$ as the largest subset such that for any $((\mu_j)^{\mathcal{I}}, \pi^N, \pi) \in \mathscr{M}^c$, π is a canonical signal given the prior $(\mu_j)^{\mathcal{I}} := (\mu_j)_{j\in\mathcal{I}}$ and the Nature's rule π^N assuming receivers' obedience.

Let $\mathcal{B}(V_j)$ be the set of all Borel measurable subsets of V_j for each $j \in \mathcal{I}$. To simplify the exposition, we define the following correspondence

$$\Lambda : (\Delta(M^{p,c}))^{\Omega} \times \cup_{j\in\mathcal{I}}(\mathcal{B}(V_j) \times A_j)^{\mathbb{T}_j} \twoheadrightarrow (\Delta(M^{p,c}))^{\Omega} \tag{3.1}$$

such that for any $\left(\pi, (S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j}\right)$ with $(S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j} := (S_{\tau_j} \times \{a_{\tau_j}\})_{\tau_j\in\mathbb{T}_j}$ for some $j \in \mathcal{I}$, define $\Lambda(\pi, (S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j})$ to be a subset of $(\Delta(M^{p,c}))^{\Omega}$ in which each signal is the same as π except that it replaces all the messages within $(S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j}$ in π with a uniquely distinguishable message for receiver j. Building on this correspondence, define the following mapping

$$\chi_i : (V_i \times A_i)^{\mathbb{T}_i} \times \mathscr{M} \times \cup_{j\in\mathcal{I}}((\mathcal{B}(V_j))^{\mathbb{T}_j}) \times \cup_{j\in\mathcal{I}}((A_j)^{\mathbb{T}_j}) \to (V_i)^{\mathbb{T}_i}$$

such that for each $((v_{\tau_i}, a_{\tau_i})^{\mathbb{T}_i}, (\mu_{j'})^{\mathcal{I}}, \pi^N, \pi, (S_{\tau_{\hat{j}}})^{\mathbb{T}_{\hat{j}}}, (a_{\tau_j})^{\mathbb{T}_j})$:

if $((\mu_{j'})^{\mathcal{I}}, \pi^N, \pi) \in \mathscr{M}^c$ and $\hat{j} = j$, i.e., $(S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j} \in (\mathcal{B}(V_j) \times A_j)^{\mathbb{T}_j}$, then define

$$\chi_i((v_{\tau_i}, a_{\tau_i})^{\mathbb{T}_i}, (\mu_{j'})^{\mathcal{I}}, \pi^N, \pi, (S_{\tau_{\hat{j}}})^{\mathbb{T}_{\hat{j}}}, (a_{\tau_j})^{\mathbb{T}_j})$$

to be receiver i's correct conjecture menu of getting message $(v_{\tau_i}, a_{\tau_i})^{\mathbb{T}_i}$ under any signal in $\Lambda(\pi, (S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j})$ given all receivers' obedience; for any other case, define

$$\chi_i((v_{\tau_i}, a_{\tau_i})^{\mathbb{T}_i}, (\mu_{j'})^{\mathcal{I}}, \pi^N, \pi, (S_{\tau_{\hat{j}}})^{\mathbb{T}_{\hat{j}}}, (a_{\tau_j})^{\mathbb{T}_j}) := (v_{\tau_i})^{\mathbb{T}_i}.$$

Building on the above χ_i, define the following correspondence

$$\Upsilon_i : (V_i)^{\mathbb{T}_i} \times \cup_{j\in\mathcal{I}}((\mathcal{B}(V_j))^{\mathbb{T}_j}) \twoheadrightarrow (V_i)^{\mathbb{T}_i}$$

such that for each pair $((v_{\tau_i})^{\mathbb{T}_i}, (S_{\tau_j})^{\mathbb{T}_j})$ with $(S_{\tau_j})^{\mathbb{T}_j} \in (\mathcal{B}(V_j))^{\mathbb{T}_j}$ for some $j \in \mathcal{I}$:

$$\Upsilon_i((v_{\tau_i})^{\mathbb{T}_j}, (S_{\tau_j})^{\mathbb{T}_j}) :=$$
$$\left\{ \chi_i((v_{\tau_i}, a_{\tau_i})^{\mathbb{T}_i}, (\mu_{j'})^{\mathcal{I}}, \pi^N, \pi, (S_{\tau_j})^{\mathbb{T}_j}, (a_{\tau_{\hat{j}}})^{\mathbb{T}_{\hat{j}}}) \;\middle|\; \begin{array}{l} (a_{\tau_i})^{\mathbb{T}_i} \in (A_i)^{\mathbb{T}_i} \\ ((\mu_{j'})^{\mathcal{I}}, \pi^N, \pi) \in \mathscr{M}^c \\ (a_{\hat{j}})^{\mathbb{T}_{\hat{j}}} \in (A_{\hat{j}})^{\mathbb{T}_{\hat{j}}}, \forall \hat{j} \end{array} \right\}.$$

We are now ready to introduce the sufficient condition, which requires stability in the equilibrium selection.

Definition III.8 (stability)**.** Given a regular selection criterion $\zeta := (\zeta_i)_{i\in\mathcal{I}}$, let $P := (P_i)_{i\in\mathcal{I}}$ such that each P_i is the partition of the conjecture space specified by ζ_i for each receiver i. Let $\tilde{\mathcal{B}}(P) := \cup_{j\in\mathcal{I}}\{(S_{\tau_j})_{\tau_j\in\mathbb{T}_j} \mid S_{\tau_j} \subseteq \mathcal{B}(p_j), p_j \in P_j\}$, where $\mathcal{B}(p_j)$ is the set of all Borel measurable subsets of p_j for each $j \in \mathcal{I}$. Say ζ is *stable* provided that for any receiver i and any menu of partition components $(p_{\tau_i})^{\mathbb{T}_i} := (p_{\tau_i})_{\tau_i\in\mathbb{T}_i} \in (P_i)^{\mathbb{T}_i}$, if a conjecture menu $(v_{\tau_i})^{\mathbb{T}_i} \in (p_{\tau_i})^{\mathbb{T}_i}$, then $\Upsilon_i((v_{\tau_i})^{\mathbb{T}_i}, S) \subseteq (p_{\tau_i})^{\mathbb{T}_i}$ for any $S \in \tilde{\mathcal{B}}(P)$.

Intuitively, if a regular selection criterion ζ is stable, then for each receiver, coarsening any receiver's information within the partitions in ζ will not change his conjecture's partition location in any possible cases. Such stability brings tractabilities, and we can often construct stable selection criteria to describe equilibrium selections commonly seen in the literature. In particular, all selection criteria in Section 3.2.1.3 satisfy this condition. We will revisit some of these examples in our application section.

Proposition III.2. *Fix an arbitrary stable regular selection criterion ζ. For any canonical signal (M^c, π^c) which recommends a pure strategy BNE σ that survives ζ, there exists a countable canonical signal π^s which recommends a pure strategy BNE that survives ζ and achieves the same outcome as that under σ and (M^c, π^c). Therefore the sender's expected payoff remains the same under both signals.*

In the following, we will illustrate how to construct a countable canonical signal based on a given canonical signal which preserves the sender's expected payoff under a stable regular selection criterion. Such a construction will further provide a tractable way to compute the optimal signal in our setting.

3.3.2.2 From uncountable to countable messages: the construction

First let us introduce the concept of strategic partitions. Recall that $B_{i,\tau_i}(\nu^i_{\tau_i})$ is the set of mixed strategy best responses to conjecture $\nu^i_{\tau_i}$ of receiver i with primitive type τ_i.

Definition III.9 (strategic partition)**.** Given a realized primitive type τ_i of receiver i, *a strategic partition* $\xi_i(\tau_i)$ *for* τ_i is the following special behavioral analogy classes such that

$$\xi_i(\tau_i) := \{p_i \times S_i \subseteq V_i \times (2^{A_i} \backslash \{\varnothing\}) \mid \nu_i \in p_i \text{ iff } B_{i,\tau_i}(\nu_i) = \Delta(S_i)\}.$$

The strategic partition is a special family of behavioral analogy classes that captures Bayesian incentive compatibility for receiver i with the given primitive type. Specifically, such partition is based on different forms of the complete set of best responses. We will further define a partition finer than the strategic partition by taking both the selection criterion and the Bayesian incentive compatibility into consideration at the same time.

Definition III.10 (ζ-strategic partition)**.** Given a regular selection criterion ζ, let $\zeta_i \cap \xi_i(\tau_i)$ be the ζ-*strategic partition* for receiver i with τ_i, where

$$\zeta_i \cap \xi_i(\tau_i) := \left\{ p_i \times S_i \subseteq V_i \times 2^{A_i} \mid \begin{array}{l} \forall\, p_i \times S_i, \exists \text{ two behavioral analogy classes} \\ p_i^1 \times S_i^1 \in \xi_i(\tau_i) \text{ and } p_i^2 \times S_i^2 \in \zeta_i \\ \text{s.t. } S_i = S_i^1 \cap S_i^2 \neq \varnothing,\ p_i = p_i^1 \cap p_i^2 \end{array} \right\}.$$

In the Appendix, Section B.1.1, we will provide an example to illustrate both strategic partitions and ζ-strategic partitions in the context of disclosing stress test results.

Given any regular selection criterion ζ, for any receiver i, say any product across primitive types for their ζ-strategic partition components is a ζ-*strategic partition component menu* for this receiver ("partition menu" for short). Let Γ_i be the collection of all such partition menus, i.e.,

$$\Gamma_i := \{(\tau_i, p_{\tau_i} \times S_{\tau_i})_{\tau_i \in \mathbb{T}_i} \mid p_{\tau_i} \times S_{\tau_i} \in \zeta_i \cap \xi_i(\tau_i)\}.$$

Before we introduce our key lemma in this section, let us first introduce two correspondences to simplify notations. Let $\mathcal{S}((\Delta(M^{p,c}))^{\Omega})$ be all possible subset of $(\Delta(M^{p,c}))^{\Omega}$. Given the correspondence Λ in the previous section (see Equation (3.1)), we define the following correspondence

$$\overline{\Lambda} : \mathcal{S}((\Delta(M^{p,c}))^{\Omega}) \times \cup_{j\in\mathcal{I}}(\mathcal{B}(V_j) \times A_j)^{\mathbb{T}_j} \twoheadrightarrow (\Delta(M^{p,c}))^{\Omega} \tag{3.2}$$

such that for any $\left(S^s, (S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j}\right)$,

$$\overline{\Lambda}\left(S^s, (S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j}\right) := \cup_{\pi\in S^s}\left(\Lambda\left(\pi, (S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j}\right)\right).$$

Based on the above $\overline{\Lambda}$, we can define the following correspondence

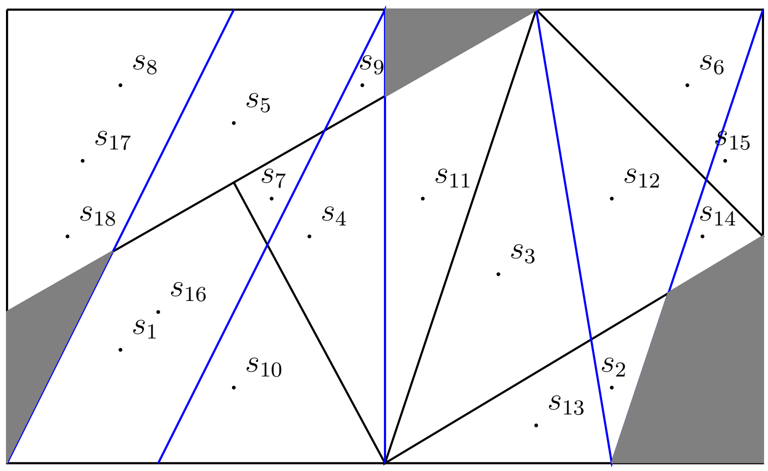
$$\hat{\Lambda} : (\Delta(M^{p,c}))^{\Omega} \times \cup_{j\in\mathcal{I}}(\mathcal{B}(V_j) \times 2^{A_j})^{\mathbb{T}_j} \twoheadrightarrow (\Delta(M^{p,c}))^{\Omega} \tag{3.3}$$

such that for any $\left(\pi, (S_{\tau_j}, S^A_{\tau_j})^{\mathbb{T}_j}\right)$, define $\hat{\Lambda}(\pi, (S_{\tau_j}, S^A_{\tau_j})^{\mathbb{T}_j})$ to be the resulted set after iteratively applying $\overline{\Lambda}$ to the result of its previous round with $(S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j}$ for an $a_{\tau_j} \in (S^A_{\tau_j})^{\mathbb{T}_j}$ that has not been used by all the previous rounds until we exhaust such possibility of getting a new a_{τ_j}; moreover, such iteration will start from an initial set $\overline{\Lambda}\left(\pi, (S_{\tau_j}, a^0_{\tau_j})^{\mathbb{T}_j}\right)$ with some $a^0_{\tau_j} \in (S^A_{\tau_j})^{\mathbb{T}_j}$. Intuitively, each $\hat{\Lambda}(\pi, (S_{\tau_j}, S^A_{\tau_j})^{\mathbb{T}_j})$ is a subset of signals in which each member is the same as π except that, for each $(a_{\tau_j})_{\tau_j\in\mathbb{T}_j} \in (S^A_{\tau_j})_{\tau_j\in\mathbb{T}_j}$, it replaces all the messages within $(S_{\tau_j}, a_{\tau_j})^{\mathbb{T}_j}$ in π with a uniquely distinguishable message for receiver j.

The basic idea of reducing the number of messages from uncountably many to countably many is via pooling them in a countable way. The following proposition, based on the above notions, provides further guidance on how to pool messages without affecting receivers' behaviors under the given equilibrium selection.

Proposition III.3. *Fix an arbitrary stable regular selection criterion ζ and any canonical signal (M^c, π^c) which recommends a pure strategy BNE σ that survives ζ. For any receiver i and an arbitrary partition menu $(\tau_i, p_{\tau_i} \times S^A_{\tau_i})_{\tau_i\in\mathbb{T}_i}$, consider any $\pi^{m,i} \in \hat{\Lambda}(\pi, (p_{\tau_i} \times S^A_{\tau_i})_{\tau_i\in\mathbb{T}_i})$. Then there exists a pure strategy BNE under the signal $(M^c, \pi^{m,i})$ that survives ζ and achieves the same outcome as that under σ and (M^c, π^c).*

Intuitively, the above proposition says that, the sender could simplify any canonical signal by arbitrarily choosing a receiver i and a partition menu $(\tau_i, p_{\tau_i} \times S_{\tau_i})_{\tau_i\in\mathbb{T}_i}$ such that: if the realized message induces a conjecture in the chosen partition component for each receiver i's primitive type τ_i, the sender will inform this receiver only the message's partition location (for each τ_i) and the action recommendation menu; otherwise, the sender will fully inform him the exact message (conjecture-action recommendation menu). Say $(\tau_i, \nu_{\tau_i}, a_{\tau_i})_{\tau_i \in \mathbb{T}_i}$ is a *sample* of the above partition menu if $(\tau_i, \nu_{\tau_i}, a_{\tau_i})_{\tau_i \in \mathbb{T}_i} \in (\tau_i, p_{\tau_i} \times S_{\tau_i})_{\tau_i \in \mathbb{T}_i}$. Moreover, say $M_i^{c,s}$ is a *sample collection* of receiver i if, given each action recommendation menu $(a_{\tau_i})_{\tau_i \in \mathbb{T}_i}$, it collects a unique sample for each element $(\tau_i, p_{\tau_i} \times S_{\tau_i})_{\tau_i \in \mathbb{T}_i}$ that satisfies $(a_{\tau_i})_{\tau_i \in \mathbb{T}_i} \in (S_{\tau_i})_{\tau_i \in \mathbb{T}_i}$ in collection of partition menus Γ_i.



To make the illustration clean, we present the sample for a single receiver primitive type only (and omit all the associated action sets). The ζ-strategic partition for a given receiver primitive type divides its conjecture space into finitely many regular sets. The above partition on a rectangle is an analogy of such a partition. In particular, black lines represent the partition induced by the selection criterion; blue lines represent the partition induced by the strategic partition, and the grey areas represent those components of which the associated action set is empty. Then the collection of points $\{s_1, ..., s_{18}\}$ is a sample of the given partition (of the conjectures), with each s_i corresponding to a unique action in its associated set. As shown in the top left component, a behavioral analogy class may have multiple representative elements since it is associated with more than one action. A sample collection thus collects samples defined above across receiver primitive types.

Figure 3.3: A sample of a partition menu

We can further extend the argument of Proposition III.3 to any receiver and any set of menus of ζ-strategic partitions by pooling messages based on all possible combinations of this receiver's partition menus and action recommendation menus, and we will have at most countably many such combinations. This extended argument, subject to a canonical transformation, concludes Proposition III.2 that it is without loss of generality to focus on canonical signals with countably many messages. Moreover, we have a shaper result that it is without loss of generality to focus on canonical signals with their message space a sample collection for each receiver. This result could provide a way to compute the optimal signal in our setting. Figure 3.3 further provides a graphic demonstration of what it looks like for a canonical signal to use a unique sample for each menu in Γ_i for each receiver i.

3.4 Application

This section is to demonstrate how one could use the insight of the proof of Proposition III.2 to solve complicated persuasion problems. In particular, we present the following two applications in which the equilibrium selection is different from the sender-preferred one, and receivers are either privately informed about their preference (application 1) or the underlying state (application 2). Given these nonstandard features, existing standard approaches are not applicable for these applications.

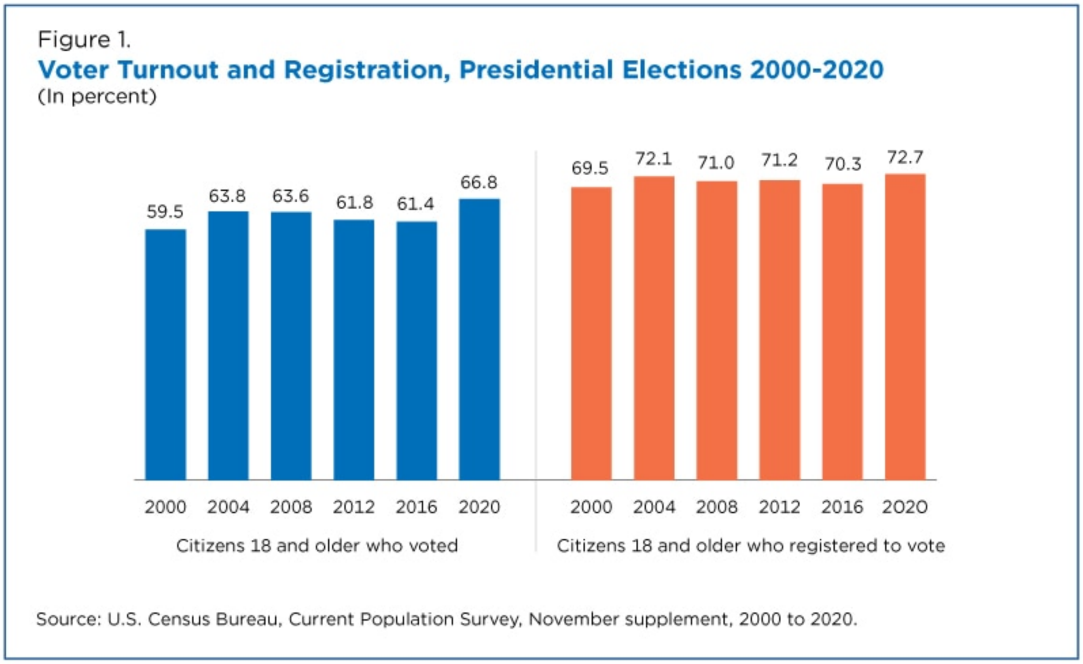
3.4.1 Application 1: persuading voters to vote

The presidential election in the United States always attracts enormous attention from all over the world. The literature points out that political parties may sway the election outcome by employing targeted advertising, which allows private communication to voters. In particular, political targeted advertising not only tries to solicit support from voters but also persuade them to register and vote. As observed, the turnout rates of the past elections are not very high (see Figure 3.4). Thus a political candidate who could persuade a significant amount of the no-turnout voters to vote for them may secure a winning position.

In this application, we consider a political voting game with target advertising where a politician ("the sender") wants to improve the voter turnout rate as well as the votes supporting her proposal. The focus is to understand the optimal advertising scheme in this setting.

There are four possible states of the world: $\Omega := \{1, 2, 3, 4\}$. A politician proposes a legislative bill to 3 voters.[6] These voters will jointly determine whether to adopt the bill. All voters have three actions $A := \{\text{Yes ("Y"), No ("N"), Absentee ("E")}\}$. The social choice $X(a)$ is the majority rule such that the bill is passed as long as there are at least 2 voters voting for its adoption: $X(a) = 1$ if at least 2 voters vote "Y"; otherwise $X(a) = 0$.

[6] To consider only three voters is for exposition elegance. One may interpret them as representative voters, who in fact represent many voters with similar or even identical preference in the game.



Note that even at the highest point, the actual turnout rate is a mere 66.8 percent. See more in https://www.census.gov/library/stories/2021/04/record-high-turnout-in-2020-general-election.html.

Figure 3.4: The record of the turnout rate for general elections from 2000 to 2020

Let action profile be $a := (a_i)_{i=1}^3$. The payoffs of each voter are specified as follows: if the law is not adopted, i.e., $X(a) = 0$, then $u_i(a, \omega) \equiv 0$ for all voters. If the law is adopted (i.e., $X(a) = 1$), then

$$u_1(a,\omega) = \begin{cases} 1 & \omega = 1 \\ 0 & \omega = 3 \\ -1 & \text{otherwise} \end{cases} \qquad u_2(a,\omega) = \begin{cases} 1 & \omega = 2 \\ 0 & \omega = 4 \\ -1 & \text{otherwise} \end{cases}$$

Receiver 1's payoff Receiver 2's payoff

Receiver 3 is a swing voter who has a type $\tau \in \mathbb{T} := \{1, 2\}$, each realized with $\frac{1}{2}$ probability, such that $u_3(a, \tau, \omega) = u_\tau(a, \omega)$ for any $\tau \in \mathbb{T}$. The politician earns a payoff of 1 if $X = 1$ regardless of the true state. It is common knowledge that everyone is Bayesian rational with a common prior $\mu^0 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

Equilibrium selection: each individual voter will cast its vote if and only if it believes that of at least 50 percent it is a pivotal voter. In such a case, an individual voter will vote Y as long as Y is its weakly best response.[7]

3.4.1.1 Solution

We will first present two benchmark cases: no information and full information.

No information: Given the common prior is $\mu^0 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, whoever pivotal will only vote N. Hence regardless of each receiver's belief about whether they are pivotal or not, the bill will not pass will probability 1.

Full information: Suppose the sender must fully reveal the state. Note that, when $\omega \in \{1, 3\}$, receiver 1 and 3 of type 1 will vote Y and they are the only supporters; and $\omega \in \{2, 4\}$, receiver 2 and 3 of type 2 will vote Y and they are the only supporters. Note that receiver 3 is equally likely to be either type 1 or type 2. Thus, by providing appropriate coordination in the message, the maximum payoff sender could achieve by fully revealing the state is $\frac{1}{2}$.

Now we are ready to present the optimal advertising scheme derived from our approach, and place all the details in the Appendix, Section B.3.1.

Optimal advertising scheme: By applying our approach, one can find a deterministic optimal signal that gives the sender a payoff of 1. This optimal signal is present in the following table:

	voter 1	voter 2	voter 3
state 1	$c_{1,Y}$	$c_{2,E}$	$c_{3,YY}$
state 2	$c_{1,E}$	$c_{2,Y}$	$c_{3,YY}$
state 3	$c_{1,Y}$	$c_{2,Y}$	$c_{3,EE}$
state 4	$c_{1,Y}$	$c_{2,Y}$	$c_{3,EE}$

We can interpret voters' beliefs and actions under the above signal as follows: for voter 1, whenever he gets a private message $c_{1,E}$, he is sure that the state is 2 and both voter 2 and 3 (both types) will vote Y. Hence voter 1 is not pivotal and he will be absent from voting. For

[7] Given that each individual will only express its view if it believes its opinion determines the outcome of at least 0.5 probability, the equilibrium selection in this setting implies that each voter, if he ever votes, is sincere (the sincerely voting rule).

the rest situation voter 1 will get private message $c_{1,Y}$, in which situation he believes that with probability $\frac{1}{3}$, the realized state is 1 and voter 2 will be absent but voter 3 (both types) will vote Y and with probability $\frac{2}{3}$, the realized state is either 3 or 4 with equal chance, and voter 3 (both types) will be absent but voter 2 will vote Y. So in both cases, voter 1 is pivotal and he will vote Y as well. Note that under state 4, voter 1's utility of passing the bill is actually -1. Nevertheless, the sender strategically pools this unfavorable state with favorable states 1 and 3 in a way that voter 1 finds it weakly better to vote Y. The beliefs and actions of voter 2 and 3 under the above signal can be interpreted similarly.

3.4.2 Application 2: Stress Test Minimizes Bank Run

Let ω be the bank's fundamental value, which is -1 if the state of the bank is low and is 2 if it is high, i.e.,

$$\omega \in \Omega := \{-1\ (\text{“}L\text{”}), 2\ (\text{“}H\text{”})\}.$$

A policymaker ("player 0") is conducting a stress test for the bank to get information regarding ω, the result of which will be privately informed to two major investors in the financial market. Among them, investors 1 has his independent private information resources regarding the state of the bank, which is commonly known to be a signal π^N committed by Nature that sends a private message of either $\{h, l\}$ with precision $\frac{2}{3}$ to investor 1, that is, $\pi_i^N(h|H) = \pi_i^N(l|L) = \frac{2}{3}$. Based on their information, the two receivers will then decide whether to run on the bank or wait, i.e., $a_1, a_2 \in A := \{R, W\}$. Receivers' payoff matrix is as follows.

(u_1, u_2)	W	R
W	(ω, ω)	$(\omega - 1, 0)$
R	$(0, \omega - 1)$	$(0, 0)$

Regardless of the state, the policymaker strictly prefers receivers to wait. We specify her utility as $v(W, W) = 2, v(W, R) = v(R, W) = 1$ and $v(R, R) = 0$. All players share a common prior μ_i^0 :=Prob($\{\omega = H\}) = 0.3$, $i \in \{0, 1, 2\}$ at the beginning of the game.

The timeline of this game is as follows: (i) The policymaker designs and commits to a stress test π; (ii) Investor 1 receives his private information from Nature under π^N and updates his prior μ_1^0 to an intermediate belief; (iii) The test result is realized and private messages are sent to receivers; (iv) Investors observe their private message(s), update their belief accordingly, and take actions that maximize their expected payoff.

Moreover, if there are multiple equilibria, then one of the worst equilibria for the policymaker is selected. This example is a substantial extension of the application in Mathevet et al. (2020). Indeed, by allowing receivers to have extra information from Nature, the sender's ex ante expected payoff in this case lies strictly within the corresponding convex hull and thus the concavification method does not apply.

3.4.2.1 Solution

Given that only investor 1 has private information, denote investor 1's intermediary belief as $\mu^0_{1|\tau_1} := \Pr(\{\omega = H \mid \tau_1\})$ for any $\tau_1 \in \{h, l\}$. We first present two benchmarks: no information and full information.

No information: Without persuasion the policymaker gets 0: Investor 1's intermediate belief will be either $\mu^0_{1|h} = \frac{6}{13}$ (if he receives message h) or $\mu^0_{1|l} = \frac{3}{17}$ (if he receives message l).[8] Investor 2 has no private information with prior $\mu^0_2 = 0.3$. Thus the worst equilibrium is both receivers run on the bank.

Full information: Note that the dominant strategy for each investor is R under state L and W under state H. Thus the policymaker can guarantee both players to wait when the underlying state is high. With the truth-telling stress test, the policymaker gets an expected payoff of 0.6.

We present the optimal stress test scheme derived from our approach below with all the details placed in the Appendix, Section B.3.2.

[8]The explicit calculation of the intermediate belief of receiver 1 is as follows: upon receiving message h, receiver 1 will update his belief to $\mu^0_{1|h}$ with

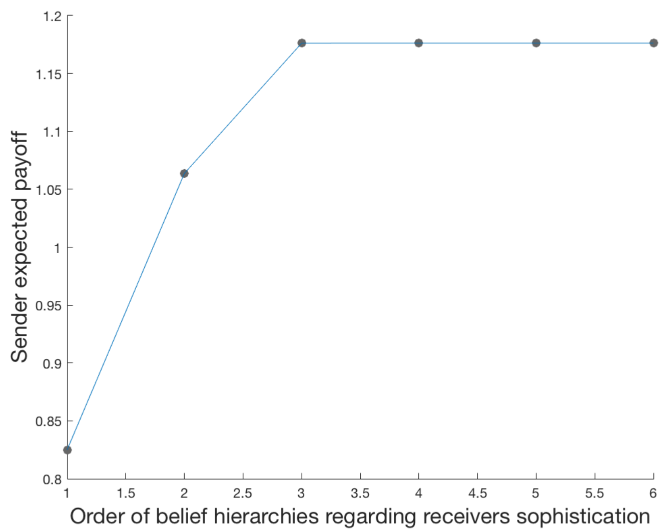
$$\mu^0_{1|h} = \frac{\mu^0_1 \cdot \pi^N(h|H)}{\mu^0_1 \cdot \pi^N(h|H) + (1-\mu^0_1) \cdot \pi^N(h|L)} = \frac{6}{13};$$

upon receiving message l, receiver 1 will update his belief to $\mu^0_{1|l}$ with

$$\mu^0_{1|l} = \frac{\mu^0_1 \cdot \pi^N(l|H)}{\mu^0_1 \cdot \pi^N(l|H) + (1-\mu^0_1) \cdot \pi^N(l|L)} = \frac{3}{17}.$$

Optimal stress test scheme:

To design an algorithm, we will restrict to finitely many orders of belief hierarchies and consider the maximum belief orders up to K. As we increase such maximum belief order K, the policy maker's payoff also increases, since the optimal signal can take more and more belief hierarchies into account. The following Figure 3.5 shows that the policymaker's maximum expected payoff is converging to ≈ 1.1762 as K increases. We can see that the persuasion gain is exhausted at the third-order belief hierarchies.



This figure shows the policymaker's maximum expected payoff from persuasion increases as the maximum belief order increases. Especially, the persuasion gain is exhausted at the third-order belief hierarchies.

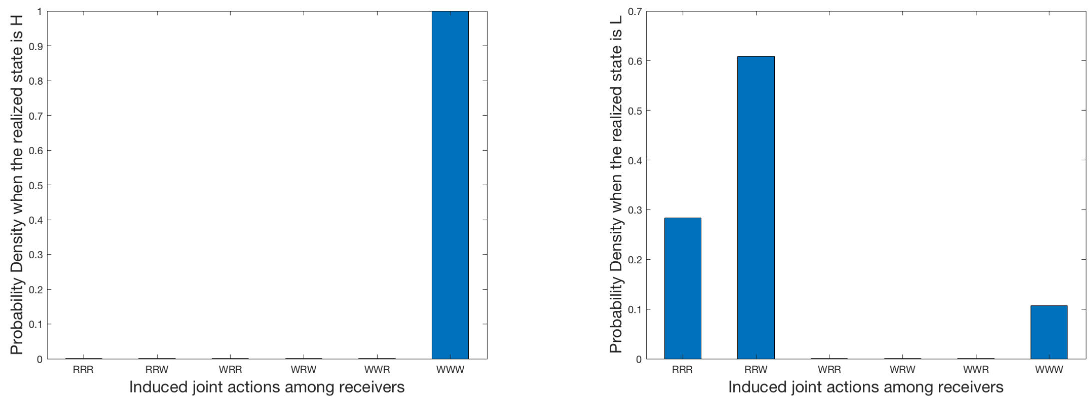
Figure 3.5: Sender's gain increases as the maximum belief order K increases

Figure 3.6 further presents the detailed distribution of actions induced by the optimal stress test scheme when $K = 3$.

3.5 Discussions

3.5.1 Psychological receivers

Our result can be extended naturally to the setting where receivers have psychological preferences. Borrowing the modeling choice in Geanakoplos et al. (1989), a psychological



The figures show the distributions of joint actions induced by the optimal signal when the realized state is H (left) and L (right). On the x-axis, each action profile is labeled in the following way: (a_h, a_l, a_2), where we aggregate the probability over messages according to their recommended action profile.

Figure 3.6: Distribution of joint actions induced by the optimal signal

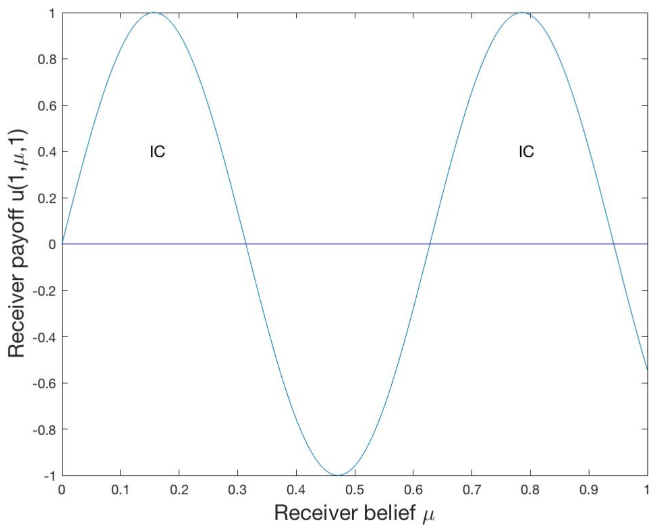
receiver i's payoff not only depends on everyone's action but also on what everyone thinks (reflected in his belief hierarchies): $u_i : T_i \times A \to \mathbb{R}$, $i \in \mathcal{I}$. Unlike the case of expected utilities, we may need extra regularity assumption for receivers' preference since, the psychological preference may not guarantee the regularity of the induced strategic partition. Under additional regularity assumption, we can then modify our proof and our main insight remains the same in this setting.

To illustrate how to apply our insight to psychological games, we will consider an example with a single psychological receiver. In such a setting, Lipnowski and Mathevet (2018) provide a counterexample that the analogous revelation principle fails and the straightforward signal may be suboptimal. The following example is inspired by their work, where Lipnowski and Mathevet (2018) consider the case when the sender's utility is reduced from the receiver's utility function while our example allows the sender's preference to be misaligned with the receiver's. This example provides a simple demonstration of how to determine an optimal signal with a psychological receiver using our approach.

Example III.5. A sender and a psychological receiver are present in an uncertain environment, where the underlying state Ω and the action space A are specified as $\Omega = A = \{0, 1\}$. Players have a common prior $\mu_s^0(\omega = 1) = \mu_r^0(\omega = 1) = 0.5$. Let μ indicate the probability that the receiver believes the underlying state $\omega = 1$. The receiver's utility is described as follows

$$u(a,\mu,\omega) = \begin{cases} \sin(10\mu) & a=\omega=1 \\ 0 & \text{otherwise} \end{cases}.$$

The sender earns a payoff of 1 if the receiver takes action $a=1$ regardless of the state, and 0 otherwise. This example adopts the sender-preferred selection, thus it is regular and stable. Given that there is only one receiver, we can focus on first-order belief only rather than the entire belief hierarchies. Whenever his posterior belief lands on the region $(0, \frac{\pi}{10}) \cup (\frac{\pi}{5}, \frac{3\pi}{10})$, the receiver will take action 0; otherwise he will take action 1. In particular, the receiver is indifferent between action 0 and 1 whenever his belief is in $\{0, \frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}\}$.



This figure shows the receiver's utility is positive when his (posterior) belief $\mu \in (0, \frac{\pi}{10}) \cup (\frac{\pi}{5}, \frac{3\pi}{10})$. These two regions are thus incentive compatible with respect to action $a=1$.

Figure 3.7: Incentive Compatible Regions Supporting Receiver's Action $a=1$

To apply our insight to this example, we need to define a regular strategic partition. One such version will be as follows:

$$\Big\{(0, \frac{\pi}{10}) \times \{1\},\ (\frac{\pi}{5}, \frac{3\pi}{10}) \times \{1\},$$
$$\{0\} \times \{0,1\}, \{\frac{\pi}{10}\} \times \{0,1\},\ \{\frac{\pi}{5}\} \times \{0,1\},\ \{\frac{3\pi}{10}\} \times \{0,1\},$$
$$(\frac{\pi}{10}, \frac{\pi}{5}) \times \{0\},\ (\frac{3\pi}{10}, 1] \times \{0\}\Big\}.$$

Given the sender-preferred selection, the ζ-strategic partition (and its associated collection of partition menus) is also the strategic partition itself. By slightly extending the proof of Proposition III.2 to this setting, we can show it is sufficient to focus on canonical signals with their message spaces a sample collection of the above ζ-strategic partition. A straightforward calculation shows that the optimal canonical signal uses only two messages $\{0.3\} \times \{1\}$ and $\{\frac{2}{3}\} \times \{1\}$, with the following structure:

$$\pi(\{0.3\} \times \{1\}|\omega = 1) = \frac{3}{11};\ \pi(\{0.3\} \times \{1\}|\omega = 0) = \frac{7}{11}.$$

Under this signal, with probability $\frac{4}{9}$, the receiver updates to the posterior belief $\mu_r^1(\omega = 1) = 0.3$ and with probability $\frac{5}{9}$ he updates his posterior belief to $\mu_r^2(\omega = 1) = 0.66$. Thus it guarantees the sender a payoff of 1.

Note that the analogous revelation principle result still fails in our example: by replacing the messages in the above signal π with its recommended action, then π reduces to an uninformative signal, and the receiver will take action 0 under his prior 0.5, leaving the sender a payoff of 0.

3.5.2 Extending to public persuasion

The insight of our main result can be extended to the setting where the sender's signal is observed publicly by adding the extra constraint that the messages that everyone receives must be the same. Moreover, higher-order beliefs more than second-order do not play an important role here since the realized posterior belief for each receiver is common knowledge in the public persuasion setting. Thus one could reduce the message space in such a setting to the space of $\Delta(\Omega \times T^1_{-i} \times A_{-i})$ for each receiver i and apply the same techniques we derive previously.

A different form of public persuasion problem may be that the realized outcome is determined by some nonlinear social choice rule, such as the scoring rules, single transferable vote, plurality with run-off, etc. However, if one could incorporate the social choice into receivers' utility with which receivers' transformed utility directly depends on the action profile, then we can transform these problems into the standard public persuasion setting and solve it with our approach.

3.5.3 Why not define selection criteria on belief hierarchies?

Our notion of selection criteria is defined on conjectures. An alternative candidate of such a notion could be to define the selection criterion on the space of coherent belief hierarchies or universal types. Compared to our current version, however, this candidate may have the limitation of failing to capture the benefit of using redundant belief hierarchies as a correlating device for the sender's benefit.[9]

To see this point more clearly, we provide the following example which shows that the selection criterion in our setting captures the strategic relevance brought by redundant types but the above alternative definition does not.

Example III.6. There is no uncertainty, i.e., underlying state space is $\Omega = \{\omega\}$. A sender (the social planner) is present in a society consisting of two receivers, each endowed with an action set $A_i = \{L, R\}$, $i \in \{1, 2\}$. The sender's payoff is the sum of all receivers' payoffs, i.e., $u_s(\omega, a_1, a_2) = u_1(\omega, a_1, a_2) + u_2(\omega, a_1, a_2)$, and the sender could commit to a private signal. Receivers' payoff matrix is described below.

$\omega, (u_1, u_2)$	L	R
L	(6,6)	(1,8)
R	(8,1)	(0,0)

It is known that there exist three Nash equilibria (L, R), (R, L) and $(\frac{1}{3}L + \frac{2}{3}R, \frac{1}{3}L + \frac{2}{3}R)$ under any signal. There exists a signal which recommends a correlated equilibrium (a BNE under this signal) as follows:

$$\pi((L, L)|\omega) = \frac{1}{5}, \pi((L, R)|\omega) = \frac{2}{5}, \pi((R, L)|\omega) = \frac{2}{5}.$$

Suppose that we define the selection criterion on universal type space (i.e, associate the selected action set to partition components on the universal type space). Under this signal with the degenerated universal type, then we can only select one of the Nash equilibria in this example. By defining the selection criterion on conjectures, we can find a selection

[9]This point has been raised in Dekel et al. (2007) with a remark that "redundant types can serve as a correlating device, and so these types are not truly "redundant" unless the addition of correlating devices has no effect".

criterion ζ that selects the above correlated equilibrium, which is specified below:

$$\Big\{\{\delta_R\} \times \{R\}, \{\delta_L\} \times \{R\}, \{\frac{2}{3}\delta_R + \frac{1}{3}\delta_L\} \times \{L\},$$
$$\{b\delta_R + (1-b)\delta_L \mid \frac{2}{3} < b < 1\} \times \{L\}, \{b\delta_R + (1-b)\delta_L \mid 0 < b < \frac{2}{3}\} \times \{R\}\Big\}.$$

3.5.4 On the relation of the two well-known approaches

There are two major approaches in the information design literature: the concavification method and the direct approach with signal recommending actions. We provide a brief discussion about how our method relates to these well-known approaches.

The concavification method, proposed by Kamenica and Gentzkow (2011), and generalized by Mathevet et al. (2020), is based on a nice geometric transformation that relaxes the incentive compatible constraint to a necessary condition (the Bayes plausible condition) from Bayes rule. As this condition is relaxed from the original constraints, the concavification approach in fact solves a relaxed problem.[10] It is known that the solution identified by the concavification method may not be feasible for the original problem, and a well-known counterexample would be the "agreeing to disagree" in Aumann (1976) (see also the discussion in Mathevet et al. (2020)). The usage of conjectures as messages in our approach is inspired by the fact that the concavification method transforms the problem into designing the distribution of posterior beliefs.

Another well-known method, the direct approach, allows one to restrict attention to signals that recommend actions only. Such a restriction is without loss of generality in situations where the classical obedience principle holds (that is, Bayesian games with sender-preferred equilibrium selection, see, for instance, Kamenica and Gentzkow (2011), Bergemann and Morris (2016) and Taneva (2018)). However this approach breaks down in more general settings, such as the case with psychological receivers (Lipnowski and Mathevet (2018)), or when the equilibrium selection differs from the sender-preferred (Mathevet et al. (2020)). Our approach is a generalized direct approach, which identifies the extra factors along with the action recommendation, that a message should include.

[10]Except for the single receiver case, where Kamenica and Gentzkow (2011) show that to focus on the Bayes plausibility condition is without loss of generality.

3.6 Conclusion

Practical persuasion schemes are often complicated which may involve using some receivers' actions or beliefs to persuade other receivers. This paper proposes a new view that could unify the setting with multiple interacting receivers with any or all of the following features: (i) receivers have non von Neumann–Morgenstern utilities, (ii) an equilibrium selection rule other than the designer-preferred equilibrium selection may apply, (iii) receivers have private information about their preferences and the underlying states, and (iv) with heterogeneous priors. We establish a generalized obedience principle and provide a sufficient condition under which it is without loss of generality to restrict attention to canonical signals with countably many messages. This further provides a tractable way to determine optimal information revelation policy in such a setting.

We use our method to analyze two applications. Application 1 considers a politician who wants to maximize the probability of passing a bill by persuading voters to cast the ballot for her using private targeted advertising. Voters may be privately informed about their preference, who will vote only when they believe their votes are pivotal. Application 2 considers a pessimistic policy-maker, who designs an optimal stress test to privately inform each of two investors about the financial health of a bank. The policy-maker wants to minimize the probability of a bank run. Our method could provide explicit numerical and analytical solutions for the optimal information structure for the sender in both these situations. These examples cannot be solved by pre-existing standard approaches in the literature. We hope that our paper could help to bring the theory one step closer to understanding persuasion in the real world.

CHAPTER IV

Derandomized Persuasion Mechanisms

4.1 Introduction

In practice, persuasion mechanisms often have a simple and deterministic structure. Sometimes, types considered similar are assigned the same categorical identifier. For example, schools use letter grades to evaluate the performance of students, and bond rating agencies adopt coarse ratings to measure the creditworthiness of bonds. Sometimes, some types are fully revealed and others are pooled without introducing extra noise. For example, with bank stress tests, banks that pass the test are pooled, while those that fail may have their types revealed. A variety of recent papers have observed that the optimal persuasion mechanism is deterministic in several specific economic environments.[1] This paper uncovers a general underlying principle behind this phenomenon and provides tight conditions under which an optimal persuasion mechanism is deterministic, or *derandomized*. In fact, under these conditions, the sets of outcomes induced by derandomized and randomized persuasion mechanisms are equivalent.

To make our statement concrete, consider a setting where a sender, who has the ability to generate information about the underlying state Ω, engages in two-way communication with multiple interacting receivers who have private types. For simplicity, we fix the message spaces for both parties: for receivers, there is an exogenously given report system which specifies the set of feasible reports each type of each receiver can make; for the sender, there is a fixed finite message set $\hat{A}$ which specifies all feasible messages she can send. A

[1] For example, Kolotilin (2018), Guo and Shmaya (2019), Wei and Green (2019), and Dworczak and Martini (2019) point out that the optimal persuasion mechanism takes a special deterministic structure under certain conditions. See the literature review for more discussion.

persuasion mechanism π is a menu of signals that collects receivers' reports from the report system, and then selects a signal (i.e., a conditional distribution $\Omega \to \Delta(\hat{A})$). A persuasion mechanism π is *derandomized* if for any realized report profile $\hat{t} \in \widehat{T}$, the selected signal conditional on every state realization is a degenerate distribution over message space $\hat{A}$ (i.e., $\pi(\cdot|\hat{t}, \cdot) : \Omega \to \{\delta_{\hat{a}} \mid \hat{a} \in \hat{A}\}$). We use Bayesian Nash equilibrium as the solution concept within the receivers' game.

The main result provides conditions under which it is possible to derandomize all signals in a persuasion mechanism in an *effectively equivalent* sense. Under those conditions, for any persuasion mechanism π, we can find a derandomized persuasion mechanism $\bar{\pi}$ such that for each receiver, when holding others' strategy fixed (subject to a message isometric transformation), in any decision problem determined by the type profile realization, the highest payoff he is able to achieve is the same under $\bar{\pi}$ and π. Moreover, $\bar{\pi}$ preserves the set of Bayesian Nash equilibria under π, as well as the equilibrium payoff of each type of each receiver and that of the sender in any specific equilibrium. The key conditions for our main results include: (i) the underlying state space is atomless; (ii) receivers' actions and possible signal realizations are finite; (iii) the players' utilities are *pseudo-separable* (that is, separable in the underlying states and receivers' type profiles). These conditions are tight, in the sense that if any of them is violated, one could find a counterexample. We provide such counterexamples in Section 4.4.1. An important assumption is that the underlying state and receivers' private information are independent, which could be relaxed to allow some interdependence.

This problem contains two technical aspects: a signal-by-signal derandomization and a measurable selection among derandomized signals. For signal derandomization, a standard tool is the Lyapunov theorem, well-known in the game theory literature. The result is immediate for the simplest private Bayesian persuasion model with one receiver who has no private information (i.e., no types), where the players hold an atomless common prior over underlying states. However, when the type space is infinite as in our setting, it is known that the Lyapunov theorem fails in general. To overcome this difficulty, we propose an approach that identifies payoff-relevant characteristics of a given persuasion mechanism; based on such identification, we could construct another derandomized persuasion mechanism with the same payoff characteristics. A key step in our construction is to solve a related measurable selection problem, for which we borrow the insight of the measurable choice theorem of Mertens (2003). To our knowledge, such an approach of derandomizing persuasion mechanisms is new in the literature.

From a constructive angle, we demonstrate the usefulness of our result on an information disclosure problem. Our result allows us to focus on derandomized persuasion mechanisms, which makes the sender's problem tractable. A derandomized persuasion mechanism in this setting collects only signals with finite partition structures on states (i.e., such a signal sends messages according to the partition in which the realized state is located). We also show that under certain conditions, our result is applicable to derandomize *experiments*, i.e., the disclosure mechanism that discloses information without soliciting the receiver's reports. For a parametrized example inspired by Kolotilin et al. (2017), we numerically derive an optimal information disclosure mechanism using a different and more straightforward method.

4.1.1 Literature Review

This paper provides a systematic study of persuasion mechanism derandomization, allowing multiple interacting privately informed receivers in the Bayesian persuasion and information design context (Kamenica and Gentzkow (2011) and Rayo and Segal (2010)). Especially, our set-up relates to two specific strands of this literature: the first one is the strand that studies receivers with private information about their preferences, including Rayo and Segal (2010), Kolotilin et al. (2017), Kolotilin (2018), and Guo and Shmaya (2019); second, it relates to the literature that studies Bayesian persuasion problem with a large underlying state space, such as Gentzkow and Kamenica (2016),Kolotilin et al. (2017), Dworczak and Martini (2019), and Dworczak and Kolotilin (2019). In particular, Kolotilin (2018), Guo and Shmaya (2019), Wei and Green (2019) and Dworczak and Martini (2019) also have relevant results in a single receiver setting that the optimal mechanism takes a derandomized structure under certain conditions.[2] By allowing multiple privately informed receivers, our setting goes outside their frameworks. We can still draw interesting implications of our result in some of their settings. Specifically, the settings in Kolotilin et al. (2017) and Wei and Green (2019) have the following common features: the distribution of underlying

[2]To be exact, Kolotilin (2018) and Dworczak and Martini (2019) have pointed out that, under certain curvature of the sender's payoff function, the optimal mechanism may be deterministic within specific regions in the posterior mean (state) space. Guo and Shmaya (2019) show that under the increasing monotone likelihood ratio condition the optimal mechanism takes the form of nested intervals. Wei and Green (2019) show that under a monotone hazard rate condition the optimal persuasion mechanism has a deterministic cutoff structure.

states is atomless, the receiver has a binary action and a linear utility, the sender's utility separates the receiver's type and the underlying states, and the receiver's private type is independent of the state. In these settings, our result implies that it is without loss of generality to work directly with derandomized mechanisms. Guo and Shmaya (2019) allow nonseparable utility functions, but their result requires an increasing monotone likelihood ratio condition. Dworczak and Martini (2019) consider the setting where the receiver's best response depends on the posterior mean of the state who has no private information. They show that if utility function u is *regular* and affine-closed, then for any continuous and full-support prior there exists an optimal signal that is a monotone partitional signal.[3] Our result adds new insights that derandomized persuasion mechanisms may still be optimal when the receiver's utility violates the regularity properties required therein. Nevertheless, the aforementioned works do not intend to provide a general study of persuasion mechanism derandomization, so many focus on specific environments such as binary action and certain monotonicity conditions. Our main result allows arbitrarily finite actions and does not rely on any monotonicity condition.

The mechanism design literature that studies mechanism equivalence is also related, especially the strand that debates whether randomization creates extra benefit for the designer such as McAfee and McMillan (1988), Strausz (2006), Manelli and Vincent (2006), Manelli and Vincent (2007), Hart and Reny (2015) and Chen et al. (2019). In particular, Chen et al. (2019) establish the equivalence of stochastic and deterministic mechanisms in a general environment using a mutual purification technique. However, their technique is not applicable here since we allow nonseparability among receivers' type profiles.[4] Moreover, there is a fundamental difference between persuasion mechanism derandomization and similar topics in mechanism design literature, since a standard mechanism can assign the allocation but a persuasion mechanism cannot.[5]

[3] In Dworczak and Martini (2019), a utility function u is *regular* provided that (i) it is upper semicontinuous with at most finitely many one-sided jump discontinuities at interior points $y_1, ..., y_k \in (0, 1)$ and has bounded slope (i.e., Lipschitz continuous) in each (y_i, y_{i+1}), with $y_0 = 0$ and $y_{k+1} = 1$; (ii) there exists a finite partition of $[0, 1]$ into intervals such that u is either strictly convex, strictly concave, or affine on each interval in that partition.

[4] The separability condition in Chen et al. (2019) requires each receiver's payoff to be separable in his own type t_i and the types of the rest receivers t_{-i} (see Definition 4 therein).

[5] The coordination mechanism in Myerson (1982) also recommends actions. On mechanism comparison, however, the literature mainly focuses on the standard mechanisms such as auctions that directly assign the allocation. The notion of coordination mechanism is still different from that of persuasion mechanism, since the sender in a persuasion setting has extra information not available to the players. This is the new feature of the information design problem added to the problem of communication in games (see Bergemann and

From a methodological viewpoint, our result is technically related to the purification literature, especially the Lyapunov theorem and the related result in Dvoretzky et al. (1950) and Dvoretzky et al. (1951). However, the Lyapunov theorem fails in infinite-dimensional settings, so the measurable selection insight in Mertens (2003) helps us to overcome this difficulty. Also, as a natural comparison between persuasion mechanisms in our setting should build on signal/experiment comparisons, the comparison we proposed is inspired by the well-known works including Blackwell (1952), Blackwell (1953) and Lehmann (1988).

Organization: The rest of the paper is organized as follows: In the next section, we introduce our model. Section 4.3 presents our main result and in Section 4.3.2, we apply our result to a specific disclosure model and derive an explicit solution of a parametrized case. Section 4.4 provides all the counterexamples and two extensions. All proofs are collected in the appendix.

4.2 Model

There is a sender (Player 0) and a finite set of receivers, denoted as $\mathcal{I}$. The underlying state space is a measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a countably generated σ-algebra.[6] Let μ^{Ω} be a common prior on Ω. Receivers have private information: each receiver $i \in I$ has a private type t_i from a space T_i. Let T_i be a polish space with $\mathcal{T}_i$ the associated Borel σ-algebra. Denote by $T := \prod_{i\in\mathcal{I}} T_i$ the receivers' joint type space. Let $\mathcal{T} := \otimes_{i\in\mathcal{I}}\mathcal{T}_i$ be the associated product σ-algebra and let μ^T be the common prior type distribution over T. Each receiver i chooses an action from a finite set A_i; denote by $A := \prod_{i\in\mathcal{I}} A_i$ the set of action profiles. Each player is a von Neumann-Morgenstern utility maximizer with a bounded measurable utility function $u_i : T \times \Omega \times A \to \mathbb{R}$ for $i \in \mathcal{I} \cup \{0\}$. For expositional purposes, throughout the text, we assume that the players have a common prior $\mu = \mu^{\Omega} \times \mu^T$ such that type profiles and underlying states are distributed independently (although receivers' types may be correlated with each other). As noted in Remark IV.1 below, the proof of Theorem IV.1 is provided for a more general case in which some interdependence between ω and t is allowed.

Receivers may have restrictions in mimicking others' types. This restriction could come

Morris 2019).

[6]For example, the Borel σ-algebra of the n-dimensional Euclidean space $\mathbb{R}^n$ is countably generated.

from the partial verifiability of information: for instance, the IRS could audit unusual income report claims; it could also come from the feasibility of reports: for instance, many surveys allow consumers to report their satisfaction level in integers only. We introduce the notion of a report system to describe such restrictions. Let each receiver i's report space be $(\widehat{T}_i, \widehat{\mathcal{T}}_i)$, where $\widehat{T}_i$ is a Polish space and $\widehat{\mathcal{T}}_i$ is the corresponding Borel σ-algebra. Denote by $\widehat{T} := \prod_{i \in \mathcal{I}} \widehat{T}_i$ the set of all possible report profiles from the system. Let $\mathfrak{T} := \prod_{i \in \mathcal{I}} \mathfrak{T}_i$ be a report system such that for each type t_i of each receiver i, each $\mathfrak{T}_i : T_i \twoheadrightarrow \widehat{T}_i$ is a closed-valued measurable correspondence specifying the set of reports he is able to file. Moreover, we assume that $\mathfrak{T}$ is exogenously given.

Players communicate through a *persuasion mechanism*, denoted as π, under a fixed report system $\mathfrak{T}$ and a fixed finite message space for the sender (denoted as $\hat{A}$). Following Kamenica and Gentzkow (2011), a *signal* is a conditional distribution $\Omega \to \Delta(\hat{A})$. The persuasion mechanism, once committed by the sender, asks for each receiver i's report and then selects a signal according to receivers' reports, i.e., $\pi : \widehat{T} \times \Omega \to \Delta(\hat{A})$, and for each report profile $\hat{t}$, $\pi(\cdot|\hat{t}, \cdot)$ is a signal. We further assume that $\hat{A}$ satisfies $|\hat{A}| \geq |A|$, so signals that recommend actions are always feasible for the sender.

The game proceeds as follows: (i) the sender commits to a persuasion mechanism; (ii) each receiver, based on his type realization and subject to report system $\mathfrak{T}$, submits a feasible report to the sender; (iii) given the realized underlying state and the joint report, the persuasion mechanism selects a signal; (v) nature picks a realization given the signal; (vi) each receiver privately observes his individual signal realization, and takes an action that maximizes his expected utility.

Definition IV.1. We say a signal is *derandomized* if conditional on each ω, the signal sends out one single message with probability 1, i.e., $\Omega \to \{\delta_{\hat{a}} | \hat{a} \in \hat{A}\} \subsetneq \Delta(\hat{A})$, where $\delta_{\hat{a}}$ is a Dirac measure on message $\hat{a} \in \hat{A}$. A persuasion mechanism π is *derandomized* if $\pi(\cdot|\hat{t}, \cdot)$ is a derandomized signal for every realized report $\hat{t} \in \widehat{T}$.

Each receiver's strategy includes two components: a reporting strategy which files a report based on his realized private type, and an action strategy that responses to the observed signal realization based on the updated belief and his knowledge. We formally define these two strategies below:

Definition IV.2. Given the report system $\mathfrak{T}$, a reporting strategy of receiver i is a mapping

r_i such that $r_i : T_i \times \widehat{\mathcal{T}_i} \to [0,1]$ and (i) for any measurable set $E_i \in \widehat{\mathcal{T}_i}$, $r_i(\cdot, E_i)$ is $\mathcal{T}_i$-measurable; (ii) for any t_i, $r_i(t_i, \cdot)$ is a probability measure on the set $\mathfrak{T}_i(t_i)$.

Let $\mathrm{Gr}\,\mathfrak{T}_i$ be the graph of $\mathfrak{T}_i$, i.e., $\mathrm{Gr}\,\mathfrak{T}_i := \cup_{t_i \in T_i}(\{t_i\} \times \mathfrak{T}_i(t_i))$. An action strategy of receiver i is a measurable mapping such that $\sigma_i : \mathrm{Gr}\,\mathfrak{T}_i \times \hat{A}_i \to \Delta(A_i)$. Let R_i be the set of feasible reporting strategies for receiver $i \in \mathcal{I}$, and let Σ_i be the set of feasible action strategies for receiver $i \in \mathcal{I}$.

Consider a strategy profile (r, σ) under a persuasion mechanism π: we say a strategy (r_i, σ_i) for receiver i is *interim incentive compatible* given others' strategy profile (r_{-i}, σ_{-i}) if

$$E^{\pi}[u_i(\sigma_i, \sigma_{-i})|r_i, r_{-i}, t_i] = \sup_{\hat{\sigma}_i \in \Sigma_i, \hat{r}_i \in R_i} E^{\pi}[u_i(\hat{\sigma}_i, \sigma_{-i})|\hat{r}_i, r_{-i}, t_i] \text{ for all } t_i \in T_i. \quad (4.1)$$

For each receiver i to have the incentive to choose an action strategy σ_i, it must be the case that, conditional on each receiver type, his reporting strategy, and the signal realization, receiver i always prefer σ_i to any other action strategy. This statement is captured by inequality (4.1) that each receiver i chooses an action strategy that maximizes its interim utility, as discussed in Bergemann and Morris (2016). Now we are ready to introduce our solution concept for the receivers' game.

Definition IV.3 (equilibrium)**.** A strategy profile (r, σ) is a Bayesian Nash equilibrium (BNE) under a persuasion mechanism π if each receiver's strategy is interim incentive compatible given others' strategies.

The definition of BNE employed here imposes sequential rationality and consistency of beliefs on the equilibrium path, as well as on the off-equilibrium path after a receiver deviates from his equilibrium reporting strategy.[7] However, it imposes no restrictions on receivers' behavior if some message arises that is supposed to be absent unless other receivers deviate. Upon the appearance of such a message, a receiver could play sequentially irrationally.

Due to the lack of certain regularity conditions (e.g., a compact report space or continuous utility functions), a BNE may not exist even though we allow receivers to play behavioral

[7]Our definition of BNE is slightly stronger than a traditional BNE, since we impose consistency of any receiver's belief off the equilibrium path after his own deviation.

strategies. The existence of BNE, though of great importance, is not the focus of this paper.[8] We derive our results under the implicit assumption that at least one BNE exists.

4.2.1 Comparing persuasion mechanisms

Similar to the comparison of mechanisms, it is natural to try comparing two persuasion mechanisms by comparing the corresponding designer's payoffs under their induced outcome distributions (i.e., the joint distribution of action and state). However, this is not well-defined since persuasion mechanisms cannot assign outcomes of the receivers' game. When there are multiple possible outcomes, it is unclear which one should be selected for calculating the designer's payoff. Given the fact that persuasion mechanisms are menus of signals, another natural comparison of persuasion mechanisms is to compare signals listed in their menus. We define our comparison in this way.

Our definition borrows the insights of signal comparison. Recall that the literature has proposed various kinds of partial orders to compare signals, among which the most well-known is perhaps Blackwell's ordering. Lehmann (1988) proposes a localized comparison notion based on signal effectiveness. His definition, adapted to this setting, says that a signal Y is more effective than a signal X with respect to a specific class of decision problems (*a problem* is specified as a set of feasible actions A and a utility function $u(\omega, a)$) if for any problem in this class, given any action strategy σ based on X, there exists another action strategy σ based on Y such that, for each realization of the state ω, receiver's payoff condition on ω under strategy σ' and signal Y is always weakly better than that under σ and X.

In contrast to the pointwise comparisons in Lehmann (1988), our comparison is based on a given prior to make the comparison feasible for a large set of signals. We say a signal Y is *more effective* than a signal X w.r.t. the class of decision problems *under a prior* μ if for any problem in this class, for any decision procedure σ based on X, there exists a (possibly randomized) procedure σ' based on Y such that given the prior μ, the expected payoff of the receiver with strategy σ' under signal Y is weakly better than that under σ based on signal X. Note that such comparison is robust under isometric transformations of signals (i.e., there is a bijective mapping between the message spaces of two signals under which their signal structure are essentially the same).

[8] For related literature, one may refer to Balder (1988) for a general result on the existence of the Bayesian Nash equilibrium in behavioral strategies in games with incomplete information.

Based on the comparison of signals in the decision framework, we define the comparison of persuasion mechanisms in a game environment as follows: Given other receivers' type profile realization t_{-i} and strategy profile (r_{-i}, σ_{-i}), let $E^{\pi}[u_i(\sigma_i, \sigma_{-i})|r_i, r_{-i}, t_i, t_{-i}]$ be the expected utility of type t_i of receiver i under π who plays strategy (r_i, σ_i).

Definition IV.4. For two persuasion mechanisms π_1 and π_2, we say π_1 is *more effective* than π_2 for receiver $i \in \mathcal{I}$ (denoted as $\pi_1 \succcurlyeq_i \pi_2$) under prior μ if:

there exists a bijection mapping $\psi : \prod_{i\in\mathcal{I}} \hat{A}_i \to \prod_{i\in\mathcal{I}} \hat{A}_i$ (isometric mapping between messages) such that for any strategy $(r_i, \sigma_i) \in R_i \times \Sigma_i$ under π_2, there exists a strategy $(r_i', \sigma_i') \in R_i \times \Sigma_i$ under the persuasion mechanism $\pi_1 \circ \psi$ (with the message isometric transformation ψ) such that

$$E^{\pi_2}[u_i(\sigma_i, \sigma_{-i})|r_i, r_{-i}, t_i, t_{-i}] \leq E^{\pi_1\circ\psi}[u_i(\sigma_i', \sigma_{-i})|r_i', r_{-i}, t_i, t_{-i}],$$

for any realized type t_i, any others' realized type profile $t_{-i} \in T_{-i}$ and strategy profile $r_{-i} \in R_{-i}, \sigma_{-i} \in \Sigma_{-i}$.

Two persuasion mechanisms π_1 and π_2 are *effectively equivalent* if $\pi_1 \succcurlyeq_i \pi_2$ and $\pi_1 \preccurlyeq_i \pi_2$ for each receiver $i \in \mathcal{I}$.

Intuitively, a persuasion mechanism π_1 is more effective than another persuasion mechanism π_2 for a receiver i, if for any reporting strategy r_i in π_2, he is able to find a strategy r_i' in π_1 (subject to a isometric transformation) such that the signal generated by r_i' under π_1 is more effective than that generated by r_i under π_2 for all possible decision problems he may face, where the class of decision problems is parametrized by his own type realization t_i, others' type realizations t_{-i} and the strategy profile (r_{-i}, σ_{-i}).

4.3 Effectively equivalent signal derandomization

Section 4.3 is organized as follows: Section 4.3.1 shows the main result and its implications; Section 4.3.2 provides an application for our main result.

4.3.1 Main result

From now on, whenever we say "all players" or "each player", we mean the statement also applies to the sender. We first introduce an important condition of our main result

Definition IV.5 (pseudo-separable utility)**.** A player i's utility function is *pseudo-separable* if his utility function can be written as $u_i(t, \omega, a) = \sum_{n=1}^{N} f_{i,n}(\omega, a) \cdot g_{i,n}(t, a)$, where N is a positive integer.

Pseudo-separable utility functions are common in the literature; in fact, several well-known works, including Rayo and Segal (2010), Kolotilin et al. (2017), and Kolotilin (2018), consider a privately informed receiver whose utility function is either $a \cdot (\omega - t)$ or $a \cdot (\omega - t)^2$: the former represents a receiver who prefers the state above his private reserved value and the latter represents a receiver who wants to match the state. These are special forms of pseudo-separable utility functions.[9] The following theorem shows that a persuasion mechanism can be derandomized in an effectively equivalent way if all players' utility functions are pseudo-separable.

Theorem IV.1. *Suppose that μ^{Ω} is atomless, the message set $\hat{A}$ is finite, and all players have pseudo-separable utility functions. For any persuasion mechanism π, there exists an effectively equivalent derandomized persuasion mechanism $\overline{\pi}$ such that*

(i) any BNE under π is still a BNE under $\overline{\pi}$, and vice versa;

(ii) for any such π and $\overline{\pi}$ under which (r^*, σ^*) is a BNE, the expected equilibrium payoff of each type of each receiver in (r^*, σ^*) is the same under π and under $\overline{\pi}$, and the expected equilibrium payoff of the sender in (r^*, σ^*) is the same under π and under $\overline{\pi}$.

To understand our result, consider a simple private Bayesian persuasion model (with a singleton type) where there is a receiver who has finitely many actions and the underlying state space is an interval in $\mathbb{R}$, such as the setting in Gentzkow and Kamenica (2016). If the common prior is atomless, then any signal with finite realizations, such as a straightforward signal (a signal that recommends actions), can be derandomized by the result in Dvoretzky et al. (1950) based on the Lyapunov theorem.[10] The resulted derandomized signal is effectively equivalent to the original one. However, recall that in our setting, there are infinitely many types. In this case, the Lyapunov theorem breaks down and we cannot generalize the above intuitive argument to our main theorem. We propose the following

[9]For the second example $a \cdot (\omega - t)^2$, one way of writing it into the pseudo-separable form is by setting $N = 3$, $f_1(\omega, a) = a\omega^2$, $f_2(\omega, a) = a\omega$, $f_3(\omega, a) = 1$; $g_1(t) = 1$, $g_2(t) = 2t$, and $g_3(t) = at^2$.

[10]One needs to take into account the combination between the realized message and the actual action the receiver are taking in the signal.

approach to overcome this difficulty and prove our main theorem. The intuition of the proof is described as follows:

Let $\mathbf{e}_k \in \mathbb{R}^{|\hat{A}|}$ be the standard unit vector with all coordinates zero except that the k-th coordinate is 1. A persuasion mechanism $\tilde{\pi}$ is derandomized if and only if for every state and report profile, $\tilde{\pi}$ recommends a specific action, i.e., $\tilde{\pi}(\cdot \mid \hat{t}, \omega) \in \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{|\hat{A}|}\}$ for every ω and $\hat{t}$. Consider a fixed persuasion mechanism π. Let $\tilde{\mathbf{N}}$ be a $\mathbb{R}^{|\hat{A}|}$-valued constant correspondence on $\widehat{T} \times \Omega$ such that $\tilde{\mathbf{N}}(\cdot, \cdot) \equiv \left\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{|\hat{A}|}\right\}$. To find an effectively equivalent derandomized persuasion mechanism, our approach includes identifying receivers' payoff relevant characteristics under π, and determining a selection of the correspondence $\tilde{\mathbf{N}}$ that preserves all these characteristics simultaneously.

To be more precise, consider the expression of each receiver's "posterior" payoff: given an arbitrarily fixed receivers' strategy profile $(r := \prod_{i \in \mathcal{I}} r_i, \sigma := \prod_{i \in \mathcal{I}} \sigma_i)$, condition on type profile $t := (t_i)_{i \in \mathcal{I}}$, message $\hat{a}_i$ and report $\hat{t}_i$, each receiver i's expected payoff under π condition on the above can be written in the following form:

$$\sum_{\hat{a}_{-i}} \frac{\int_{\widehat{T}_{-i}} \sum_{a \in A} \overbrace{\int_{\Omega} u_i(\omega, t, a) \pi(\hat{a} \mid \hat{t}, \omega) \,\mathrm{d}\mu^{\Omega}(\omega)}^{\text{Term } A} \sigma_i(a_i, \hat{a}_i, t_i, \hat{t}_i) \cdot \sigma_{-i}(a_{-i}, \hat{a}_{-i}, t_{-i}, \hat{t}_{-i}) r_{-i}(t_{-i}, \mathrm{d}\hat{t}_{-i})}{\int_{\widehat{T}_{-i}} \underbrace{\int_{\Omega} \pi(\hat{a}_{-i}, \hat{a}_i \mid \hat{t}_{-i}, \hat{t}_i, \omega) \,\mathrm{d}\mu^{\Omega}(\omega)}_{\text{Term } B} r_{-i}(t_{-i}, \mathrm{d}\hat{t}_{-i})}.$$

The payoff relevant characteristics we refer to are Term A and B. In particular, if we could find a derandomized persuasion mechanism $\overline{\pi}$ such that for any possible combination of message profile $\hat{a}$, type profile t and report profile $\hat{t}$, the following holds for each receiver i:

$$\begin{aligned} &\text{Term } A: \int_{\Omega} u_i(\omega, t, a) \pi(\hat{a} \mid \hat{t}, \omega) \,\mathrm{d}\mu^{\Omega}(\omega) = \int_{\Omega} u_i(\omega, t, a) \overline{\pi}(\hat{a} \mid \hat{t}, \omega) \,\mathrm{d}\mu^{\Omega}(\omega) \\ &\text{Term } B: \int_{\Omega} \pi(\hat{a} \mid \hat{t}, \omega) \,\mathrm{d}\mu^{\Omega}(\omega) = \int_{\Omega} \overline{\pi}(\hat{a} \mid \hat{t}, \omega) \,\mathrm{d}\mu^{\Omega}(\omega), \end{aligned} \tag{4.2}$$

then we could show that $\overline{\pi}$ is effectively equivalent to π, and also the other statements in Theorem IV.1. Based on the pseudo-separable structure of receivers' payoff functions, we identify a selection of the above $\tilde{\mathbf{N}}$ that establishes the above equation (4.2), borrowing the measurable selection insight of Mertens (2003). Such a construction is demonstrated in the proof of Theorem IV.1′ in the appendix.

Remark IV.1. We actually prove the theorem for the more general situation where players have subjective priors absolutely continuous with respect to some underlying measure $\mu^\Omega \times \mu^T$, and some interdependence between the underlying state ω and the type profile t is allowed.[11] The set-up in the main text is therefore a special case. In the general case, the proof requires some additional technical conditions (including separability of the corresponding Radon–Nikodym densities) that are satisfied given the assumptions in the main text. We place the more general proof of Theorem IV.1 in the Appendix, Section C.1.

Based on the above insight, we may provide a simplification for the definition of signals with rich structures proposed by Gentzkow and Kamenica (2017). Under their definition, a signal is a finite partition of $\Omega \times [0, 1]$, where every partition element is a signal realization. In the original definition of Gentzkow and Kamenica (2017), Ω is a finite state space. Here we extend it to a general underlying state space $(\Omega, \mathcal{F}, \mu^\Omega)$. Suppose that the common prior μ^Ω is atomless, by Theorem IV.1, we could simplify the above definition into a finite partition on Ω instead of the product set $\Omega \times [0, 1]$. For example, if the underlying state space is $\Omega = [0, 1]$ with the common prior the Lebesgue measure, then one could define a signal as a finite partition on $\Omega = [0, 1]$ instead of the product set $\Omega \times [0, 1]$.

4.3.2 Application

To explain our result from a constructive angle, in this section we consider a persuasion situation as that in Kolotilin et al. (2017). With a different approach, Kolotilin et al. (2017) characterize the structure of optimal persuasion mechanism, which may be stochastic. Our main theorem shows that it is without loss of generality to focus on derandomized persuasion mechanisms in such a setting. We also apply our result to study the derandomization of a different disclosure mechanism that discloses the same information to all receiver types.

The following setting is from Kolotilin et al. (2017): There is a sender and a privately-informed receiver. The underlying state space is an interval $\Omega := [\underline{\omega}, \overline{\omega}]$. The set of the receiver's possible private type is the interval $T := [\underline{t}, \overline{t}]$. It is common knowledge that random variables ω and t are independent. Let the cumulative distribution function (CDF) of ω and t be G^Ω and G^T, respectively. Assume G^Ω is atomless. The receiver has two

[11] Nevertheless, we cannot allow full generality in the interdependence between ω and t. Later we will provide two counterexamples (Example IV.4 and IV.5) showing that such a derandomization result may fail if ω and t are correlated.

actions $\{0, 1\}$. The sender gets utility 1 if the receiver takes action 1, and gets 0 utility otherwise. The receiver's utility is $u(\omega, t, a) = a \cdot (\omega - t)$. We fix the sender's message space to be action recommendations, i.e., $\{0, 1\}$, and restrict attention to signals that recommend actions. The tie-breaking rule is that a receiver type takes action 1 whenever indifferent.

Say a persuasion mechanism is *incentive compatible* if telling one's true type and obeying the recommendation is incentive compatible for each receiver type. This setting satisfies the prerequisites of Theorem IV.1. It is therefore without loss of generality for the sender to focus on derandomized incentive compatible persuasion mechanisms.

Corollary IV.1. *For any incentive compatible persuasion mechanism, there exists an incentive compatible derandomized persuasion mechanism under which the sender achieves the same payoff as that under the given mechanism.*

Derandomized persuasion mechanisms consist of signals that have a partitional structure on states, i.e., for each signal in the mechanism, there is a binary partition $\{S, \Omega \backslash S\}$ of the state for some subset $S \subseteq \Omega$ such that the signal sends message 1 if and only if the realized state is $\omega \in S$. In other words, derandomized persuasion mechanisms collect signals that divide Ω into two partition components and send binary messages to inform the receiver which partition component the realized state is in.

We now turn to a different disclosure mechanism that discloses the same information to all receiver types, which we call the experiments. An *experiment* communicates a one-way message to the receiver and do not require the receiver to report his type. Kolotilin et al. (2017) show that a sender achieves her maximum payoff in this setting with experiments. We will consider how to apply our result to derandomize experiments at the optimum.

Without loss of generality, we can focus on canonical experiments. A *canonical experiment* takes the following form: $\rho : \Omega \to \Delta(\Omega)$ such that, conditional on the realized ω, $\rho(\omega)$ is a distribution of posterior means. The realized posterior mean is disclosed to all receiver types. Given any canonical experiment, it is incentive compatible for every receiver type below the realized posterior mean to take action 1 and for every receiver type above the realized posterior mean to take action 0. A canonical experiment ρ is derandomized if for each ω, $\rho(\omega)$ is a Dirac measure of Ω. Say an experiment is *optimal* if it achieves the same payoff as that under optimal persuasion mechanisms.

We provide two possible ways to apply our result to derandomize (canonical) experiments at the optimum. Recall that it is without loss of generality to focus on derandomized persuasion mechanisms. The first way is to consider, whether an optimal derandomized persuasion mechanism could be transformed into a (canonical) experiment under some circumstances. The following corollary, adding in the result in Guo and Shmaya (2019), gives an affirmative answer under the situation when derandomized persuasion mechanism is in a cutoff form.

Corollary IV.2. *Suppose that π is an optimal derandomized persuasion mechanism that satisfies the following cutoff property: for any ω, if $\pi(\omega, t) = 1$, then for any $t' \leq t$, $\pi(\omega, t') = 1$. Then there exists an optimal derandomized canonical experiment.*

Alternatively, if the primitives are sufficiently regular so that the optimal experiment has a special structure, our result may be applicable. This provides a second way to derandomize experiments. We introduce a regularity condition that allows us to derandomize experiments in this way.

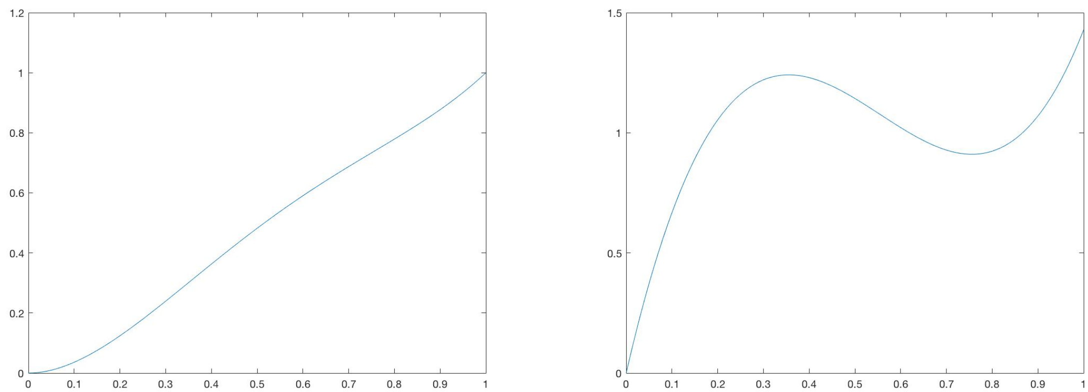
Definition IV.6. The CDF of receiver type variable G^T is *partitional regular* on $[\underline{t}, \overline{t}]$ if there exists a finite partition of $[\underline{t}, \overline{t}]$ that divides $[\underline{t}, \overline{t}]$ into finitely many intervals such that G^T is either strictly concave or strictly convex on each interval in that partition.

In the following, we will show that under the above condition (plus a few minor assumptions), the optimal experiment can be derandomized. To see how our result is applicable: borrowing the insight from Dworczak and Martini (2019), we show that the optimal experiment in this setting will truthfully reveal some states; for states not truthfully revealed, it only sends messages within a fixed finite set. Thus we could apply our result to the region consisting of states that are not truthfully revealed.

Proposition IV.1. *Suppose that G^T is continuously differentiable and partitional regular on $[\underline{t}, \overline{t}]$ and that G^Ω has full support. Then there exists an optimal derandomized canonical experiment.*

The above results also provide a derandomized approach to explore the optimal persuasion mechanism in this setting. Kolotilin et al. (2017) show that if the density function for receiver's type is single-peaked, the optimal mechanism is derandomized. In fact, the optimal mechanism fully reveals states below a certain threshold and pools those above (see their Theorem 2 and Example 1). Guo and Shmaya (2019) show that under certain assumptions, the optimal disclosure mechanism takes a derandomized nested interval form. We consider the following parametrized example, in which the density function has multiple peaks and the assumptions in Guo and Shmaya (2019) is violated.[12] We apply our result and explicitly compute an optimal persuasion mechanism for this example.

Example IV.1. Let the underlying state space be $\Omega = [-1, 1]$ with the uniform CDF $\mathrm{d}G^{\Omega}(\omega) = 0.5\,\mathrm{d}\omega$. Let the type space be $T = [0, 1]$ with the CDF of type variable G^T such that $G^T(x) = \frac{29x^2}{7} - \frac{40x^3}{7} + \frac{18x^4}{7}$. The density function $\frac{\mathrm{d}G^T(t)}{\mathrm{d}t} = \frac{216}{21}(t-\frac{1}{6})(t-\frac{2}{3})(t-\frac{5}{6}) + \frac{20}{21}$. We plot both G^T and its density function in the following Figure 4.1.



(a) G^T (b) the density function

Note that G^T is partitional regular on $[0, 1]$: it is strictly convex on the interval $[0, \frac{1}{18}(10 - \sqrt{13})]$; strictly concave on the interval $(\frac{1}{18}(10 - \sqrt{13}), \frac{1}{18}(10 + \sqrt{13}))$; and strictly convex on the interval $[\frac{1}{18}(10 + \sqrt{13}), 1]$.

Figure 4.1: The plots of G^T and its density function

The sender's payoff under two benchmark cases—full revelation and no information—will be 0.233333 and 0, respectively, implying that the sender optimally discloses at least some information. The prerequisite conditions in Proposition IV.1 are satisfied in this example. By Proposition IV.1, there exists an optimal derandomized canonical experiment. Denote this optimal derandomized canonical experiment as $\rho^* : \Omega \to \Omega$, and we solve its explicit

[12] In particular, Assumption 3 in Guo and Shmaya (2019) requires the existence of a reference state ω_0 such that receiver's payoff is negative for all types whenever the realized state $\omega \leq \omega_0$ and positive whenever $\omega \geq \omega_0$. This assumption is violated in the parametrized example we provided. In the discussion of their assumptions, Guo and Shmaya (2019) also mention that the optimal IC mechanism in Kolotilin et al. (2017) does not always recommend accepting on intervals (see Section 5.1).

form as follows:

$$\rho^*(\omega) = \begin{cases} 0.5403, & \text{for } \omega \in [0.0806, 1] \\ \omega, & \text{for } \omega \in [-1, 0.0806) \end{cases}.$$

The above optimal derandomized canonical experiment ρ^* implies the following persuasion mechanism π^* is optimal: if the realized state $\omega \in [0.0806, 1]$, then π^* recommends all receiver type under the cutoff 0.5403 to take action 1 and the rest to take action 0; if $\omega \in [0, 0.0806)$, then π^* recommends the receiver type that below the state ω to take action 1 and those above the state to take action 0; for the rest state, π^* recommends all receiver type to take action 0. The sender's maximal payoff under π^* is 0.2427. The details of the derivation can be found in Appendix, Section C.2.1.

4.4 Counterexamples and extensions

4.4.1 Counterexamples

There are three conditions crucial for our results: (i) the measure μ^Ω on the underlying state space is atomless; (ii) the receivers' actions and the possible signal realizations are finite; (iii) the receivers' utility is pseudo-separable. This section provides three counterexamples to illustrate that our results may not hold if these conditions are violated. The proofs of this section are collected in Appendix, Section C.3.

There is a straightforward counterexample when the measure on state space is atomic. The judge-prosecutor example in Kamenica and Gentzkow (2011) satisfies the conditions of pseudo-separability and finiteness except for the condition that the prior is atomless. In that famous example, Kamenica and Gentzkow (2011) show that an optimal signal cannot be a derandomized signal.[13]

4.4.1.1 Infinitely many signal realizations

The finiteness condition on receivers' signal realizations is also crucial. The following counterexample, motivated by Shen et al. (2019), shows that if a persuasion mechanism could select signals with infinitely many realizations, then our result may not hold.

[13] It is known that any probability measure μ can be decomposed into a purely atomic part and a nonatomic part (see, for instance, Johnson 1970). Our result will be applicable to the nonatomic part of a probability measure (with some rescaling).

Example IV.2 (infinitely many signal realizations)**.** Let the underlying state Ω be $[0, 1]$ with the Lebesgue measure λ. There are two receivers and a sender.[14] Both receiver 1 and 2 have two actions $\{0, 1\}$. Each receiver gets utility 1 if his opponent takes action 1 and gets 0 otherwise. It is common knowledge that receiver 1's prior belief about the underlying state $\lambda_1 := \lambda$, and receiver 2's belief about the underlying state is $\mathrm{d}\lambda_2 := 2\omega\,\mathrm{d}\lambda(\omega)$. The sender has prior λ, who gets 1 if both receivers take action 1, and gets 0 otherwise. Let message space be $[0, 1]$. Then we claim the following hold:

(i) there exists a signal $\pi : \Omega \to \Delta([0, 1])$ public observable to both receivers that induces the message distribution $\mu := (\mu_1, \mu_2)$ for receiver 1 and 2 where $\mu_1 = \lambda$, and $\mathrm{d}\mu_2 = |4\omega - 2|\,\mathrm{d}\lambda$.

(ii) there does not exist a derandomized persuasion mechanism with the above message space $\hat{A}$ that is effectively equivalent to π.

4.4.1.2 Non-pseudo-separable utility

The following counterexample violates the third condition, by having a receiver whose utility is not pseudo-separable.

Example IV.3. There is one sender and one receiver, who has a private type. The underlying state space and the receiver's private type space are unit intervals, i.e., $\Omega = T = [0, 1]$. It is commonly known that the random variables ω and t are independent and both are uniformly distributed. We allow the sender to have any bounded measurable utility function. The receiver has two actions $\{0, 1\}$, and utility function $u(\omega, t, a)$ where

$$u(\omega, t, a) := \begin{cases} \mathbb{1}_{\{(\omega', t') | \omega' \geq t'\}}(\omega, t) & \text{if he takes action } a = 1; \\ \mathbb{1}_{\{(\omega', t') | \omega' < t'\}}(\omega, t) & \text{if he takes action } a = 0. \end{cases}$$

For any type t, if he always takes action 0 ignoring any information he may receive, his expected payoff is t, and if he always takes action 1 then his payoff is $1 - t$. Consider a signal π with the sender's message set $\{0, 1\}$ with the following structure: $\pi(0|\omega) = 0.8$ and $\pi(1|\omega) := 0.2$ if $\omega < 0.5$, and $\pi(1|\omega) = 0.8$ and $\pi(0|\omega) = 0.2$ if $\omega \geq 0.5$. We claim that there does not exist a derandomized signal that is effectively equivalent to π.

[14] We do not know if there is a counterexample in a single receiver setting. But the breakdown of the result in this multi-receiver environment is due to the incompatibility of beliefs rather than the role of randomization in coordination.

4.4.1.3 When the underlying state correlates with receivers' private information

The independence between the underlying state and receivers' type profile can be weakened, as shown in Theorem IV.1$'$ in the Appendix. However, we cannot relax this assumption completely. We provide two different reasons why such independence is required.

The first reason is aligned with the idea in Aumann (1987) that randomization could provide coordination among players in the multi-player environment. As in Example IV.4, when all players' information is perfectly correlated, then derandomized persuasion mechanisms cannot be equivalent to random mechanisms. The second reason is more fundamental, which says that independence allows the joint density function to separate the types and states. If such a separation fails, as in Example IV.5, then the derandomization result could fail even in a single receiver setting.

Example IV.4. There is a sender (the social planner), and two receivers, each with an action set $A_i = \{L, R\}$, $i \in \{1, 2\}$. The underlying state space is $\Omega = [0, 1]$. All three players have perfect information, and thus the underlying state and receivers' information are perfectly correlated. The sender's payoff is the sum of all receivers' payoffs, i.e., $u_s(\omega, a_1, a_2) = u_1(\omega, a_1, a_2) + u_2(\omega, a_1, a_2)$. The receivers' payoff matrix is described below. Whenever receivers are indifferent, they take the sender-preferred actions.

$\omega, (u_1, u_2)$	L	R
L	(6ω,6ω)	(1ω,8ω)
R	(8ω,1ω)	(0,0)

In this example, the optimal signal is the following: conditional on any realized ω, with probability 0.2, the sender privately recommends both receivers to play L; with probability 0.4, the sender privately recommends the first receiver to play L and the second receiver to play R and with probability 0.4, the sender privately recommends the first receiver to play R and the second receiver to play L. Such a signal would induce receivers to play a correlated equilibrium at each realized state. But in this case, any incentive compatible derandomized signal would be a recommendation of playing pure strategy Nash equilibrium at each realized state. Thus the optimal signal achieves a strictly higher payoff than any derandomized signal.

Example IV.5. There is one sender, and one receiver who has a private type. The underlying state space and the receiver's private type space are unit intervals, i.e., $\Omega = T = [0, 1]$. It is commonly known that the joint distribution of random variables ω and t is $f(\omega, t)\, \mathrm{d}\omega\, \mathrm{d}t$ where $f(\omega, t) = \frac{2}{3} \cdot (2\mathbb{1}_{\{(\omega',t')|\omega' \geq t'\}}(\omega, t) + \mathbb{1}_{\{(\omega',t')|\omega' < t'\}}(\omega, t))$. We allow the sender to have any bounded measurable utility function. The receiver has two actions $\{0, 1\}$, and utility function $u(\omega, t, a)$ where $u(\omega, t, a) := a \cdot (\omega - t)$.

For any type t, if he always takes action 0 ignoring any information he may receive then his expected payoff is 0, and if he always takes action 1 then his payoff is $\frac{2+t^2-4t}{4-2t}$. Consider a signal π with the sender's message set $\{0, 1\}$ with the following structure: $\pi(0|\omega) = 0.8$ and $\pi(1|\omega) := 0.2$ if $\omega < 0.5$, and $\pi(1|\omega) = 0.8$ and $\pi(0|\omega) = 0.2$ if $\omega \geq 0.5$. We claim that there does not exist a derandomized signal that is effectively equivalent to π.

4.4.2 Extensions

4.4.2.1 Persuasion mechanism with transfers

Our results can be extended to the setting where there is a transfer scheme and receivers have to pay report-dependent fees for the signals they get (see, for instance, Wei and Green 2019 and Li and Shi 2017). Under the transfer scheme, receivers' reports enter into players' payoffs in an additive and separable way. The conclusions of our results still hold as long as the conditions in Theorem IV.1 are satisfied.

4.4.2.2 The sender has a general utility function

It is natural to conjecture that our main results would still hold if one relaxed the restrictions on the sender's utility. However, this is not true: without those regularity conditions, a crucial step that involves an exchange of the order of integrating over Ω and T may break down.

For this more general case, one way to bypass this obstacle is to introduce some regularity to reporting strategies. Say a reporting strategy profile is *decomposable* if there exists a σ-finite measure λ on $\widehat{T}$ such that for almost every t, $r(t, \mathrm{d}\hat{t})$ can be decomposed into a purely atomic or discrete measure and a finite measure that is absolutely continuous with respect to λ. This condition permits analogs of our main results where we relax the sender's utility to be merely bounded and measurable. Specifically, with the other prerequisites remaining the same, the previous conclusions still hold in the relaxed version, except that the preservation of the sender's payoff is weakened to only arbitrarily finitely many BNEs with *decomposable* reporting strategies. This restriction may not be stringent since there is a large class of qualified strategies, such as pure reporting strategy and those absolutely continuous with respect to a fixed probability measure given any type profile. Moreover, this restriction does not constrain a sender who relies on the revelation principle to focus only on the truthful, obedient equilibrium.

4.5 Conclusion

It is quite natural for a practical designer to restrict attention to derandomized persuasion mechanisms, since they are relatively simple to design and implement in practice. However, such restriction may lose generality, incurring suboptimality.

This paper proposes a way to compare different persuasion mechanisms based on their information effectiveness from the receivers' viewpoint. More importantly, building on such comparison criteria, we provide tight conditions under which a potentially stochastic persuasion mechanism is effectively equivalent to some derandomized persuasion mechanism. Our results enhance the understanding of when it is without loss of generality to restrict attention to derandomized persuasion mechanisms, which may provide a simplified solution to the sender's optimization problem in practice.

CHAPTER V

Conclusion

This dissertation focuses on the design of information policy in strategic settings.

Chapter II shows that a principal could benefit from not evaluating the agent directly based on the output. In particular, the optimal incentive contract assigns lenient evaluation when the agent's continuation value is low and severe evaluation when it is high. Adding biases into evaluation helps the principal to extend the duration of the employment relationship, which also weakly increases the resulted agent's continuation value process as compared to the situation without. The above result voices against conducting the evaluation based on noisy objective indexes, since shielding the agent partially from the downside risk could improve productivity. This view aligns with the existing empirical evidence. My result provides further insights on how to design evaluation schemes for organizations to achieve higher efficiency under extra contractual possibilities.

Chapter III examines information design settings with any or all of the following features: (i) receivers have non von Neumann–Morgenstern utilities, (ii) an equilibrium selection rule other than the designer-preferred equilibrium selection may apply, and (iii) receivers have private information. I propose a general message space to describe the multi-level basic and strategic uncertainties, which captures the richness of information. In such a framework, I establish a generalized obedience principle and show when it is without loss of generality to restrict attention to simple canonical signals. This further provides a tractable way to determine optimal information policy. I apply my approach to analyze two applications that cannot be solved by pre-existing approaches in the literature. My result helps to design algorithms that explicitly compute optimal information policies in both applications.

Chapter IV investigates when restricting attention to persuasion mechanisms that either fully reveal or pool underlying states is without loss of generality. Given that such mechanisms are relatively simple to design and implement in practice, it is quite useful for a designer to know when such a restriction will not incur suboptimality. I provide tight conditions under which a potentially stochastic persuasion mechanism is equivalent to some derandomized persuasion mechanism. My results enhance the understanding of the role randomization plays in persuasion mechanisms, which could be utilized to further simplify the designer's problem. This result also justifies the wide adoption of derandomized persuasion mechanisms in practice.

APPENDICES

APPENDIX A

Appendix for Chapter II

A.1 Proofs of Section 2.3

Proof of Lemma II.1. Consider the probability space $(C([0,\infty)), \mathcal{B}(C([0,\infty))), P)$ with P the Wiener measure and $W_t(\omega) = \omega_t$ for any $t \in (0,\infty)$. Let $\mathbb{F}$ be the filtration generated by W, and hence $\mathcal{F}_\infty = \mathcal{B}(C([0,\infty)))$. By definition, any adjustment process α satisfies the Novikov condition. Define a new process $Z_t(\alpha)$ as follows: for any time $t \in [0,\infty)$,

$$Z_t(\alpha) := \exp\left(-\int_0^t \alpha_s \, \mathrm{d}W_s - \frac{1}{2}\int_0^t |\alpha_s|^2 \, \mathrm{d}s\right).$$

Thus by Corollary 3.5.13 in Karatzas and Shreve (1991), $Z_t(\alpha)$ is a $\mathbb{F}$-martingale.

By Corollary 3.5.2 in Karatzas and Shreve (1991), there exists a unique probability measure Q satisfying $Q(S) = E^P[1_S Z_t(\alpha)], \forall\, S \in \mathcal{F}_t$ for any fixed time t with $0 \le t < \infty$ such that the process $\{W^\alpha_s, \mathcal{F}_s \mid 0 \le s < \infty\}$ satisfying $\mathrm{d}\, W^\alpha_s := \alpha_s \, \mathrm{d}s + \mathrm{d}W_s$ is a Brownian motion on $(\Omega, \mathbb{F}, \mathcal{F}_\infty, Q)$.

Lastly, we will show Q is also equivalent to the measure P. Note that for any t and W, $Z_t(\alpha)$ is defined by an exponential function, which is always of strictly positive value. By such definition, for any time $t \in [0,\infty)$ and $S \in \mathcal{F}_t$, $Q(S) = 0$ if and only if $P(S) = 0$. Hence we conclude the proof. $\square$

Proof of Proposition II.1. Recall that I assume the report is independent of history, i.e., for any $s < t$, $\mathcal{F}_t^Y \backslash \mathcal{F}_s^Y$ is independent of $\mathcal{F}_s^\alpha$ under Q. Then for any $s < t$,

$$\begin{aligned}
&E^Q\left[Y_t \mid \mathcal{F}_s^\alpha\right] = E^Q\left[E^Q[W_t^\alpha|\mathcal{F}_t^Y] \mid \mathcal{F}_s^\alpha\right] && \text{by the definition that } Y \text{ is canonical}\\
=&E^Q\left[E^Q[W_t^\alpha - W_s^\alpha + W_s^\alpha|\mathcal{F}_s^Y \vee (\mathcal{F}_t^Y \backslash \mathcal{F}_s^Y)] \mid \mathcal{F}_s^\alpha\right]\\
=&E^Q\left[E^Q[W_t^\alpha - W_s^\alpha + W_s^\alpha|\mathcal{F}_s^\alpha] \mid \mathcal{F}_s^Y \vee (\mathcal{F}_t^Y \backslash \mathcal{F}_s^Y)\right]\\
=&E^Q[E^Q[W_s^\alpha|\mathcal{F}_s^\alpha] \mid \mathcal{F}_s^Y \vee (\mathcal{F}_t^Y \backslash \mathcal{F}_s^Y)] && W_t^\alpha \backslash W_s^\alpha \text{ is independent of } \mathcal{F}_s^\alpha \text{ under } Q\\
=&E^Q[E^Q[W_s^\alpha|\mathcal{F}_s^\alpha] \mid \mathcal{F}_s^Y] && \mathcal{F}_t^Y \backslash \mathcal{F}_s^Y \text{ is independent of } \mathcal{F}_s^\alpha \text{ under } Q\\
=&E^Q[W_s^\alpha \mid \mathcal{F}_s^Y] = Y_s; && \text{by the tower property that } \mathcal{F}_s^Y \subseteq \mathcal{F}_s^\alpha.
\end{aligned}$$

Hence, $Y_t = E^Q[W_t^\alpha|\mathcal{F}_t^Y]$ is a $\mathbb{F}^\alpha$-martingale under measure Q. By the martingale representation theorem, there exists a $\mathbb{F}^\alpha$-adapted process β_t such that $Y_t = \int_0^t \beta_s \, \mathrm{d}W_s^\alpha$. By Theorem 27, PP 71 in Protter (2004), $Y^2 - [Y]$ is a $\mathbb{F}^\alpha$-martingale under measure Q. Thus $E^Q[Y_t^2] = E^Q[[Y_t]]$ for any time t. The square-integrability is implied by Jensen's inequality and Itô isometry:

$$E^Q[[Y_t]] = E^Q[(E^Q[W_t^\alpha|\mathcal{F}_t^Y])^2] \leq E^Q[(W_t^\alpha)^2] = t < \infty.$$

We will next show that β_t must be $\mathbb{F}^Y$-adapted. Given that Y is $\mathbb{F}^Y$-adapted process, its quadratic variation is also $\mathbb{F}^Y$-adapted. Since $Y_t = \int_0^t \beta_s \, \mathrm{d}W_s^\alpha$ is a continuous square-integrable martingale, by (2.19), PP138, Karatzas and Shreve (1991), its quadratic variation is $\int_0^t \beta_s^2 \, \mathrm{d}s$. This also implies that β_t must be $\mathbb{F}^Y$-adapted (by taking derivative with respect to t and then a square root). Hence we conclude the proof. □

Proof of Lemma II.2. Recall that the evaluation system generates an independent Brownian noise W^α in the filtered probability space $(\Omega, \mathcal{F}^\alpha, \mathbb{F}^\alpha, Q)$, and the report Y is an $\mathbb{F}^\alpha$-adapted process. Thus the compensation-effort pair (B, A) adapted to $\mathbb{F}^Y$ is also $\mathbb{F}^\alpha$-adapted. Proposition 1 in Sannikov (2008) in the appendix is directly applicable, by which, $V_t(B, A, \alpha)$ admits the following representation:

$$\mathrm{d}\, V_t(B, A, \alpha) = (rV_t(B, A, \alpha) - u(B_t) + c(A_t))\, \mathrm{d}t + \sigma K_t \, \mathrm{d}W_t^\alpha. \tag{A.1}$$

The $\mathrm{d}Q \times \mathrm{d}t$ uniqueness of K follows from the second part of Theorem 4.15, Karatzas and Shreve (1991). Therefore, we conclude the proof. □

Proof of Lemma II.3. The proof is a modification of Proposition 1, Sannikov (2008). For any arbitrary contract (α, Y, B, A), the agent's total payoff conditional on the information at time t, denoted as TV_t, is the following:

$$TV_t(B, A, \alpha) = \int_0^t e^{-rs}(u(B_s) - c(A_s))\,\mathrm{d}s + e^{-rt}V_t(B, A, \alpha),$$

which implies that

$$\begin{aligned}\mathrm{d}TV_t(B, A, \alpha) &= e^{-rt}(u(B_t) - c(A_t))\,\mathrm{d}t + \mathrm{d}\big(e^{-rt}V_t(B, A, \alpha)\big)\\ &= e^{-rt}(u(B_t) - c(A_t))\,\mathrm{d}t - re^{-rt}V_t(B, A, \alpha) + e^{-rt}\,\mathrm{d}V_t(B, A, \alpha).\end{aligned} \tag{A.2}$$

By the assumption that $\mathbb{F}^\alpha = \mathbb{F}$ and the definition that

$$V_t(B, A, \alpha) := E_A^P\left[\int_t^\infty e^{-r(s-t)}\big(u(B_s) - c(A_s)\big)\,\mathrm{d}s\Big|\mathcal{F}_t\right],$$

the stochastic process TV is an $\mathbb{F}$-martingale. Then by the martingale representation theorem, we have

$$\mathrm{d}TV_t(B, A, \alpha) = \sigma K_t e^{-rt}\,\mathrm{d}W_t. \tag{A.3}$$

By substituting that $\mathrm{d}W_t = \frac{1}{\sigma}(\mathrm{d}X_t - A_t\,\mathrm{d}t)$ into (A.3) and combine with (A.2), then we arrive at Equation (2.7). □

Proof of Proposition II.2. By definition, $V_t^Y(B, A, \alpha) = E^Q[V_t(B, A, \alpha)|\mathcal{F}_t^Y]$. For convenience, let (α, Q, Y) be the canonical evaluation scheme with the variational coefficient β. Hence, $Y_t = E^Q[W_t^\alpha|\mathcal{F}_t^Y]$. Based on Lemma II.2, $\mathrm{d}Y_t = \beta_t\,\mathrm{d}W_t^\alpha = \frac{\beta_t}{\sigma}(\mathrm{d}X_t^\alpha - A_t\,\mathrm{d}t)$.

By Lemma II.2, equation (2.6) holds Q almost everywhere, which is equivalent to the following: for $Q - a.s$,

$$\begin{aligned}&V_t(B, A, \alpha) - V_0(B, A, \alpha)\\ &= \int_0^t \big(rV_s(B, A, \alpha) - u(B_s) + c(A_s)\big)\,\mathrm{d}s + \int_0^t \sigma K_s\,\mathrm{d}\,W_s^\alpha,\ \forall\, t \in [0, \infty).\end{aligned} \tag{A.4}$$

Recall that the pair (B, A) is $\mathbb{F}^Y$-adapted. Thus by projecting the above equation onto $\mathcal{F}_t^Y$

at any fixed time t, we have the following:

$$V_t^Y(B,A,\alpha) - V_0(B,A,\alpha) = E^Q\left[V_t(B,A,\alpha) \mid \mathcal{F}_t^Y\right] - V_0(B,A,\alpha)$$
$$= \int_0^t \left(rE^Q\left[V_s(B,A,\alpha) \mid \mathcal{F}_t^Y\right] - u(B_s) + c(A_s)\right) \mathrm{d}s + \sigma E^Q\left[\int_0^t K_s \,\mathrm{d}\, W_s^\alpha \middle| \mathcal{F}_t^Y\right]. \tag{A.5}$$

Consider the first term $E^Q\left[V_s(B,A,\alpha) \mid \mathcal{F}_t^Y\right]$ for any $0 \leq s < t$. By the assumption that $\mathcal{F}_t^Y \backslash \mathcal{F}_s^Y$ is independent of $\mathcal{F}_s^Y$, the following holds:

$$\begin{aligned} &E^Q\left[V_s(B,A,\alpha) \mid \mathcal{F}_t^Y\right] \\ =&E^Q\left[V_s(B,A,\alpha) | \mathcal{F}_s^Y \vee (\mathcal{F}_t^Y \backslash \mathcal{F}_s^Y)\right] \\ =&E^Q\left[V_s(B,A,\alpha) | \mathcal{F}_s^Y\right] = V_s^Y(B,A,\alpha). \end{aligned} \tag{A.6}$$

Next we will show that the second term $E^Q\left[\int_0^t K_s \,\mathrm{d}\, W_s^\alpha \middle| \mathcal{F}_t^Y\right] = \int_0^t K_s \,\mathrm{d}\, Y_s$. We first need to introduce two definitions of metrics for stochastic processes in Karatzas and Shreve (1991) for the fixed probability filtered space $(\Omega, \mathcal{F}_\infty^\alpha, \mathbb{F}^\alpha, Q)$: For any square-integrable martingale M and the fixed time t, let $\|M\|_t := \sqrt{E^Q[M_t^2]}$. Also let $\|M\| := \sum_{n=1}^\infty \frac{\|M\|_n \wedge 1}{2^n}$. The second metric is defined based on a continuous square integrable martingale M as follows: for any measurable, $\mathbb{F}^\alpha$-adapted process Z, define

$$[Z]^M := \sum_{n=1}^\infty \frac{1 \wedge [Z]_n^M}{2^n} \text{ with } [Z]_t^M := \sqrt{E^Q\left[\int_0^t Z_s^2 \,\mathrm{d} < M >_s\right]} \text{ for any } t \in [0, \infty). \tag{A.7}$$

A process is called *simple*, denoted as $\widehat{K}$, if there exists a strictly increasing sequence of real numbers $\{t_m\}_{m=0}^\infty$ with $t_0 = 0$ and $\lim_{m\to\infty} t_m = \infty$, a sequence of random variables $\{\xi_m\}$, and a nonrandom constant $C < \infty$ with $\sup_{\omega\in\Omega, m\geq 0} |\xi_m(\omega)| \leq C$, such that ξ_m is $\mathcal{F}_{t_m}^\alpha$-measurable for every $m \geq 0$ and

$$\widehat{K}_t(\omega) = \xi_0(\omega)\mathbb{1}_0(t) + \sum_{i=0}^\infty \xi_i(\omega)\mathbb{1}_{(t_i,t_{i+1}]}(t); 0 \leq t < \infty, \omega \in \Omega. \tag{A.8}$$

See Definition 3.2.3 in Karatzas and Shreve (1991), PP 132.

For the square integrable K, by Proposition 3.2.6 in Karatzas and Shreve (1991), PP 134, there exists a sequence of simple processes $\{K^{(n)}\}_{n=1}^{\infty}$, each in the form of (A.8), such that $\lim_{n\to\infty}[K^{(n)} - K]^{W^\alpha} = 0$ with the metric defined in (A.7).

Also by the definition of the stochastic integral (see Definition 3.2.9, PP 139, Karatzas and Shreve 1991), the stochastic integral of K with respect to the Brownian motion W^α is the unique, square-integrable martingale which satisfies $\lim_{n\to\infty} \| \int K^{(n)} \, \mathrm{d}W^\alpha - \int K \, \mathrm{d}W^\alpha \| = 0$ for every sequence of simple processes $\{K^{(n)}\}_{n=1}^{\infty}$ with $\lim_{n\to\infty}[K^{(n)} - K]^{W^\alpha} = 0$. Based on such convergence, for the above sequence of simple processes $\{K^{(n)}\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}[K^{(n)} - K]^{W^\alpha} = 0$, the following convergence also holds: for any arbitrarily fixed t,

$$\begin{aligned}
&\left\| E^Q\left[\int_0^t K_s^{(n)} \, \mathrm{d}\, W_s^\alpha \middle| \mathcal{F}_t^Y\right] - E^Q\left[\int_0^t K_s \, \mathrm{d}\, W_s^\alpha \middle| \mathcal{F}_t^Y\right] \right\|_t \\
=& \sqrt{E^Q\left[\left(E^Q\left[\int_0^t K_s^{(n)} \, \mathrm{d}\, W_s^\alpha \middle| \mathcal{F}_t^Y\right] - E_A^Q\left[\int_0^t K_s \, \mathrm{d}\, W_s^\alpha \middle| \mathcal{F}_t^Y\right]\right)^2\right]} \\
\leq& \sqrt{E^Q\left[E^Q\left[\left(\int_0^t K_s^{(n)} \, \mathrm{d}\, W_s^\alpha - \int_0^t K_s \, \mathrm{d}\, W_s^\alpha\right)^2 \middle| \mathcal{F}_t^Y\right]\right]} \\
=& \| \int K^{(n)} \, \mathrm{d}\, W^\alpha - \int K \, \mathrm{d}\, W^\alpha \|_t \to 0 \text{ as } n \to \infty,
\end{aligned} \tag{A.9}$$

where the convergence in the last equation is due to the definition of $\int_0^t K_s \, \mathrm{d}W_s^\alpha$ that $\lim_{n\to\infty} \| \int K^{(n)} \, \mathrm{d}W^\alpha - \int K \, \mathrm{d}W^\alpha \| = 0$ whenever $\lim_{n\to\infty}[K^{(n)} - K]^{W^\alpha} = 0$.

Recall that K is $\mathbb{F}^Y$-adapted. Hence, for each n, each ξ_m^n is $\mathcal{F}_{t_m}^Y$-measurable for every $m \geq 0$. For any n, by projecting $\int K^{(n)} \, \mathrm{d}W^\alpha$ onto $\mathcal{F}_t^Y$ at any fixed time t, we have the following equation:

$$
\begin{aligned}
&E^Q\left[\int_0^t K_s^{(n)}\,\mathrm{d}\,W_s^\alpha\Big|\mathcal{F}_t^Y\right]\\
=&E^Q\left[\xi_0^{(n)}(\omega)\mathbb{1}_0(t)+\sum_{i=0}^\infty \xi_i^{(n)}(\omega)(W_{t_{i+1}}^\alpha-W_{t_i}^\alpha)\Big|\mathcal{F}_t^Y\right] \qquad \text{(A.10)}\\
=&\xi_0^{(n)}(\omega)\mathbb{1}_0(t)+\sum_{i=0}^\infty \xi_i^{(n)}(\omega)\left(E^Q\left[W_{t_{i+1}}^\alpha\Big|\mathcal{F}_t^Y\right]-E^Q\left[W_{t_i}^\alpha\Big|\mathcal{F}_t^Y\right]\right).
\end{aligned}
$$

For any $t_i < t$, $E_A^Q\left[W_{t_i}^\alpha\Big|\mathcal{F}_t^Y\right] = E_A^Q\left[W_{t_i}^\alpha\Big|\mathcal{F}_{t_i}^Y \vee (\mathcal{F}_t^Y\backslash\mathcal{F}_{t_i}^Y)\right]$. Recall that Y is a "canonical" report and that $\mathcal{F}_t^Y\backslash\mathcal{F}_s^Y$ is independent of $\mathcal{F}_s^\alpha$ under Q. Hence, $E^Q\left[W_{t_i}^\alpha\Big|\mathcal{F}_t^Y\right] = E^Q\left[W_{t_i}^\alpha\Big|\mathcal{F}_{t_i}^Y\right] = Y_{t_i}$ for any $t_i < t$. Combining this equality with equation (A.10), we have

$$
E^Q\left[\int_0^t K_s^{(n)}\,\mathrm{d}\,W_s^\alpha\Big|\mathcal{F}_t^Y\right] = \xi_0^{(n)}(\omega)\mathbb{1}_0(t)+\sum_{i=0}^\infty \xi_i^{(n)}(\omega)\left(Y_{t_{i+1}}-Y_{t_i}\right) = \int_0^t K_s^{(n)}\,\mathrm{d}\,Y_s. \tag{A.11}
$$

Recall that we assume that $\sup_{t\in[0,\infty),\omega\in\Omega}|\beta_t(\omega)| \leq 1$ and that the sequence $\{K^{(n)}\}_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty}[K^{(n)}-K]^{W^\alpha} = 0$. Thus for any $t \in [0,\infty)$,

$$
\begin{aligned}
[K^{(n)}-K]_t^Y :&= \sqrt{E^Q[\int_0^n (K_s^{(n)}-K_s)^2\,\mathrm{d}< Y >_s]}\\
&= \sqrt{E^Q[\int_0^n (K_s^{(n)}-K_s)^2\beta_s^2\,\mathrm{d}< W^\alpha >_s]} \qquad \text{(A.12)}\\
&\leq [K^{(n)}-K]^{W^\alpha} \to 0
\end{aligned}
$$

Therefore, by the definition of $\int_0^t K_s\,\mathrm{d}\,Y_s$, we have $\|\int_0^t K_s^{(n)}\,\mathrm{d}\,Y_s - \int_0^t K_s\,\mathrm{d}\,Y_s\| \to 0$ as $n \to \infty$. Combine the above equality (A.12), (A.11) and (A.9), and by the Q-almost everywhere uniqueness of the limit defined by $\|\cdot\|$, we have

$$E_A^Q\left[\int_0^t K_s \,\mathrm{d}\, W_s^\alpha \Big| \mathcal{F}_t^Y\right] = \int_0^t K_s \,\mathrm{d}\, Y_s = \int_0^t K_s\beta_s \,\mathrm{d}\, W_s^\alpha \tag{A.13}$$

for Q-almost everywhere. By plugging (A.6) and (A.13) in (A.5), we have

$$\begin{aligned} &V_t^Y(B,A,\alpha) - V_0^Y(B,A,\alpha) \\ =&\int_0^t \Big(rV_s^Y(B,A,\alpha) - u(B_s) + c(A_s)\Big) \,\mathrm{d}s + \sigma\int_0^t K_s\beta_s \,\mathrm{d}\, W_s^\alpha. \end{aligned} \tag{A.14}$$

Therefore, (2.9) follows directly from (A.14).

In the rest of the proof, we will characterize Agent's incentive compatibility constraint. The proof borrows the same insight of appendix, Proposition 2 in Sannikov (2008). Given the above A and at any initial time $t_0 \in [0,\infty)$, consider an alternative $\mathbb{F}^Y$-adapted effort strategy $\hat{A}$. Define by $\widehat{TV}_t$ Agent's total expected payoff starting from time t_0, who adopts $\hat{A}$ before time $t \geq t_0$ then follow the strategy A after time t, i.e.,

$$\widehat{TV}_t := \int_{t_0}^t e^{-r(s-t_0)}\Big(u(B_s) - c(\hat{A}_s)\Big)\,\mathrm{d}s + e^{-r(t-t_0)}V_t^Y(B,A,\alpha).$$

Thus $\mathrm{d}\,\widehat{TV}_t = e^{-r(t-t_0)}\Big(u(B_t) - c(\hat{A}_t)\Big)\,\mathrm{d}t + \mathrm{d}(e^{-r(t-t_0)}V_t^Y(B,A,\alpha))$. By (2.9), we have

$$\mathrm{d}\widehat{TV}_t = e^{-r(t-t_0)}(c(A_t) - c(\hat{A}_t) + K_t\beta_t(\hat{A}_t - A_t))\,\mathrm{d}t + e^{-r(t-t_0)}K_t\beta_t\sigma\,\mathrm{d}W_t^\alpha, \tag{A.15}$$

where the drift of $\mathrm{d}\widehat{TV}_t$ the impact of such a secret deviation.

We now verify that Agent's incentive compatibility is satisfied if and only if condition (2.10) is satisfied.

For the necessary condition: At any starting time t_0 as specified above, if condition (2.10) is violated for the recommended effort A in a set of positive measure during time $[t_0,\infty)$, then we choose the alternative effort strategy $A^* \in \mathcal{A}^{\mathbb{F}_\tau}$ that maximizes $K_t\beta_t\tilde{A}_t - h(\tilde{A}_t)$ for all t. By equation (A.15), the drift of $\mathrm{d}\widehat{TV}_t$ is always nonnegative and is positive on a set of strictly positive measure. Thus there exists a time t^* such that $E^Q[\widehat{TV}_{t^*}] > E^Q[\widehat{TV}_{t_0}] = V_{t_0}^Y(B,A,\alpha)$. Therefore, starting at time t_0, the recommended effort process A is not optimal.

For sufficient condition: if A satisfies condition (2.10), then similar to the above, we can show that for any alternative effort process $\hat{A}$, the resulted total value $\widehat{TV}$ has a nonpositive drift. Thus for any time t, $E^Q[\widehat{TV}_t] \le E^Q[\widehat{TV}_{t_0}] = V^Y_{t_0}(B, A, \alpha)$. Hence, standing at any initial time t_0, A is at least as good as any alternative $\mathbb{F}^Y$-adapted effort process $\hat{A}$. Hence, we conclude the proof. □

Proof of Lemma II.4. (i) Suppose that $V_t(B, A, \alpha)$ and K satisfy equation (2.6) Q-almost everywhere, so the set of paths that violate that above equation (2.6) is of Q-zero-measure. Given that P is equivalent to Q, P and Q have the same set of zero-measure events. The set of paths that violate equation (2.6) is also of P-zero-measure. The same argument applies to the other direction. (ii) and (iii) can also be shown similarly. □

A.2 Proofs of Section 2.4

A.2.1 The proofs of Theorem II.1 and II.2

We first define the following operator: Given any tuple $(a, k, b, \alpha) \in \{0, a_H\} \times [\epsilon, \overline{K}] \times [0, \bar{b}] \times [\underline{\alpha}, \overline{\alpha}]$, let $\mathcal{H}_{a,k,b,\alpha}$ be a function such that

$$\mathcal{H}_{a,k,b,\alpha}(v, G(v), G'(v)) := \frac{-2G'(v)(rv - u(b) + c(a) + \sigma k \alpha) - 2(a - b) + 2rG(v)}{(\sigma k)^2}. \tag{A.16}$$

With the above notation, we further abbreviate the infimum of $\mathcal{H}_{a,k,b,\alpha}$ of (a, k, b, α) over the feasible range to be the following:

$$\inf \mathcal{H}(v, G(v), G'(v)) := \inf_{\substack{(a,k\beta)\in\Gamma, b\in[0,\bar{b}] \\ \beta\in[-1,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} \mathcal{H}_{a,k,b,\alpha}(v, G(v), G'(v)).$$

The HJB equation can be rewritten equivalently as the following equation $G''(v) = \inf \mathcal{H}(v, G(v), G'(v))$. We first have the following lemma:

Lemma A.1. *For any initial condition $y^0 \in \mathbb{R}^2$, $G'' = \inf \mathcal{H}(v, G, G')$ has a unique twice continuously differentiable solution $G(\cdot, y^0)$ on $[0, \infty)$ satisfying $y^0 = (G(0, y^0), G'(0, y^0))$. Moreover, $G(\cdot, y^0)$ is uniformly continuous in y^0.*

Proof. For any $y := (y_1, y_2) \in \mathbb{R}^2$, we define $\tilde{H}(v, y) := \big(y_2, \inf \mathcal{H}(v, y)\big)$. Let $\|\cdot\|$ be the norm in $\mathbb{R}^2$. It is easy to check that on any compact set $S_0 \subseteq [0, \infty)$, there exists some positive constants M_1 and M_2 (depending on S_0) such that for every $v \in S_0$ the following holds: (i) $\|\tilde{H}(v, y)\| \leq M_1(1 + \|y\|)$; (ii)$\|\tilde{H}(v, y) - \tilde{H}(v, \tilde{y})\| \leq M_2\|y - \tilde{y}\|$; (iii) $\tilde{H}(v, y)$ is continuous in v for each y. See also the proof of Proposition 2 in Strulovici and Szydlowski (2015) for more details for such verifications (Note that Assumption 1-3 therein are satisfied in my setting). Hence the statement of this lemma follows directly from Lemma 4 and 5, Appendix B, in Strulovici and Szydlowski (2015). □

Lemma A.2. *For any initial condition $y^0 \in \mathbb{R}^2$, there exists a twice continuously differentiable strictly concave solution $G(\cdot)$ to the HJB equation* (2.13) *satisfying* $(G(0), G'(0)) = y^0$.

Proof. Given any initial condition y^0, Lemma A.1 shows that there exists a unique twice continuously differentiable solution to the HJB equation, denoted as $G(\cdot)$. We will show that the solution G is strictly concave, which is divided into the following two steps.

First step: We will first show that there does not exist a continuation value $v \in (0, \infty)$ such that $G''(v) = 0$. Suppose to the contrary that there exists one value $v_0 \in [0, \infty)$ such that $G''(v_0) = 0$. This implies that $G''(v_0) = 0 = \inf \mathcal{H}(v_0, G(v_0), G'(v_0))$. Then consider the affine function $f(v) = G(v_0) + G'(v_0)(v - v_0)$. A straightforward calculation generates the following equation: for any (a, k, b, α), $\mathcal{H}_{a,k,b,\alpha}(v, f(v), f'(v)) = \mathcal{H}_{a,k,b,\alpha}(v_0, G(v_0), G'(v_0))$. Hence, for any $v \in [0, \infty)$,

$$f''(v) = 0 = G''(v_0) = \inf \mathcal{H}(v_0, G(v_0), G'(v_0)) = \inf \mathcal{H}(v, f(v), f'(v)).$$

Hence, f is a solution to the HJB equation that satisfies the initial value $f(v_0) = G(v_0)$ and $f'(v_0) = G'(v_0)$. By the uniqueness of the solution for a given value $(v_0, G(v_0), G'(v_0))$, this implies that the solution to the HJB equation on $(0, \infty)$ is a straight line.[1] However, given that the maximum production is bounded by $\frac{a_H}{r}$, Principal's value function cannot go to infinity. Hence we must have $f'(v_0) \leq 0$, which implies that the maximum is achieved at $f(0) = 0$. But this contradicts the prerequisite assumption that it is strictly optimal for Principal to hire Agent than no trade. So we have a contradiction.

[1]For the uniqueness, any given value will work; it needs not be the initial value only.

Second step: To show this, first note that $G''(0) < 0$. The proof is the following: by taking the special value $(b = 0, a = 0, \beta = 1, k \in (0, \frac{c(a_H)}{a_H}), \alpha = 0)$, $\mathcal{H}_{a,k,b,\alpha}(0, G, G') = 0$. Thus $G''(0) = \inf \mathcal{H}(0, G, G') \leq 0$. By the first step, it cannot be the case $G''(0) = 0$. Thus we have $G''(0) < 0$.

Now, suppose that if there exists v_1 such that $G''(v_1) \geq 0$, then by the continuity of function $G''(\cdot)$, there must exist a point v_1 such that $G''(v_1) = 0$. This again contradict with the first step that it cannot be the case $G''(v_0) = 0$ for some $v_0 \in [0, \infty)$. Therefore, it must be the case that $G''(v) < 0$ for any v. ☐

The next lemma follows directly from Lemma 11, Appendix F in Strulovici and Szydlowski (2015).

Lemma A.3. *For any given initial conditions y_0, let $G(\cdot, y_0)$ be the unique twice differentiable solution on $[0, \infty)$ to the equation $G''(v) = \inf \mathcal{H}(v, G(v), G'(v))$. For any two initial values $y_0 = (y_0^1, y_0^2)$ and $\hat{y}_0 := (\hat{y}_0^1, \hat{y}_0^2)$ such that $y_0^1 = \hat{y}_0^1$ and $y_0^2 > \hat{y}_0^2$, the corresponding solutions to the given equation then satisfy $G(v, y_0) > G(v, \hat{y}_0)$ for any $v \in [0, \infty)$.*

Next we will prove that there exists a unique pair of a twice differentiable strictly concave function G and a retirement boundary v^* that satisfies the given HJB equation and the boundary condition.

First, we introduce the following lemma. Recall that $-p(\cdot)$ is the payment for the agent's retirement. Given $\bar{b}$ is the maximum amount that the principal could pay at each instant, we will set the upper bound of the agent's continuation value to be $\frac{u(\bar{b})}{r}$ for now. Later, Lemma A.6 will provide a formal argument that the principal optimally retires the agent whenever his continuation value exceeds $\frac{u(\bar{b})}{r}$ so setting such an upper bound is without loss of generality.

Lemma A.4. *For the initial value problem $G''(v) = \inf \mathcal{H}(v, G(v), G'(v))$ with the initial value $y^0 := (0, s)$, the solution G satisfies the following: there exists a function $\tilde{u}$ such that $G \geq \tilde{u}$ on $[0, \frac{u(\bar{b})}{r}]$ and $\tilde{u}' \geq 0$ on $[0, M_1 \log\left(\frac{s+M_2}{M_2}\right)]$ for some strictly positive constant M_1 and M_2.*

Proof. Note that Assumption 1-3 in Strulovici and Szydlowski (2015) are satisfied in my setting. Following the same procedure in the proof of Proposition 2 in Strulovici and

Szydlowski (2015), we can verify that $-\inf \mathcal{H}$ satisfies Condition 1-3 therein, thus the above statement follows directly from the proof of Lemma 7, Appendix B, Strulovici and Szydlowski (2015). □

Lemma A.5. *There exists a unique twice continuously differentiable strictly concave solution G to $G''(v) = \inf \mathcal{H}(v, G(v), G'(v))$ such that $G(v) \geq -p(v)$ and $G'(v^*) = -p'(v^*)$ for the boundary point v^*, where $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* = 0$ otherwise.*

Proof. Given any initial value $(0, 0, G'(0))$, the unique solution G to the given initial value problem is a strictly concave C^2-function. It is straightforward to verify that a necessary condition of $G \geq -p$ on is that $G'(0) \geq -p'(0)$, which we will impose for the rest of the proof. Consider the special initial value $(0, 0, -p'(0))$, let the unique solution be G_0. We divide the possible situations into two different cases.

First case: If the unique solution satisfies $G_0(v) > -p(v)$ for all $v \in (0, \frac{u(\bar{b})}{r})$. Then the above statement holds for G_0 with $v^* = 0$. The uniqueness of such a pair follows directly from Lemma A.3.

Second case: Otherwise, there exists some $v' \in (0, \frac{u(\bar{b})}{r})$ such that $G_0(v') = -p(v')$. In this case, if we keep increasing the initial slope $G'(0)$, then by Lemma A.3, the corresponding unique solution to (2.13) is increasing in $G(v)$ for each v. If we raise the initial slope $G'(0)$ large enough, by Lemma A.4, the lower boundary $\tilde{u}$ in Lemma A.4 can be increasing on $(0, \frac{u(\bar{b})}{r})$ and thus the solution to the initial value problem G satisfies $G(v) > -p(v)$ for all $v \in (0, \frac{u(\bar{b})}{r})$. Suppose that s^* is the largest initial value of $G'(0)$ such that if we raise $G'(0) > s^*$, then the resulted solution to (2.13) has no intersection with $-p(\cdot)$ on $(0, \frac{u(\bar{b})}{r})$. Then given initial value $(G(0), G'(0)) = (0, s^*)$, the solution must be tangential to $-p(\cdot)$ at $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$, otherwise we can find a contradiction that there exists a strictly lager initial value of $G'(0)$ under which the resulted solution and $-p(\cdot)$ has nonempty intersections. Thus $G(v^*) = -p(v^*)$ and $G'(v^*) = -p'(v^*)$ must hold. The uniqueness of the pair G and v^* in this case is as follows: Recall that $G'(0) \geq -p'(0)$ must hold (in order for $G \geq -p$). For any initial slope $-p'(0) \leq G'(0) < s^*$, by definition the smallest intersection of G and $-p$ is not tangent so the requirement in the statement does not satisfy; for any initial slope $G'(0) > s^*$, then $G > -p$ on $(0, \frac{u(\bar{b})}{r})$ but $G'(0) > -p'(0)$ so the requirement does not satisfy, either. Hence, the pair (G, v^*) that satisfies the statement of the lemma is unique.

Hence we conclude our proof. $\square$

Proof of Theorem II.1. The result follows from Lemma A.5. $\square$

We now turn to the proof of Theorem II.2.

Lemma A.6. *For any agent's initial continuation value v if $v > \frac{u(\bar{b})}{r}$, then Principal achieves a payoff of at most $-p(v)$.*

Proof. We will show that under any contract (α, Q, Y, B, A) with the agent's initial value $v \geq \frac{u(\bar{b})}{r}$, regardless of whether this contract satisfies the incentive compatibility constraint. Principal gets at most $-p(v)$. Recall that the reward upper bound $\bar{b}$ must satisfy the condition $u'(\bar{b}) = \frac{c(a_H)}{a_H}$.

The proof is as follows: Let $b_0 := r \cdot p(v)$. By such definition, $v = \frac{u(b_0)}{r}$. Given that $v \geq \frac{u(\bar{b})}{r}$, we have $b_0 \geq \bar{b}$ and thus $u'(b_0) \leq u'(\bar{b})$. Hence

$$\begin{aligned}
v &= E^Q\left[\int_0^\infty e^{-rs}(u(B_s) - c(A_s))\,\mathrm{d}s\right] \\
&\leq E^Q\left[\int_0^\infty e^{-rs}(u(b_0) + u'(b_0)(B_s - b_0) - u'(\bar{b}) \cdot A_s)\,\mathrm{d}s\right] \\
&\leq E^Q\left[\int_0^\infty e^{-rs}(u(b_0) + u'(b_0)(B_s - b_0) - u'(b_0) \cdot A_s)\,\mathrm{d}s\right] \\
&= v - u'(b_0)(E^Q\left[\int_0^\infty e^{-rs}(A_s - B_s)\,\mathrm{d}s\right] + \frac{b_0}{r}),
\end{aligned}$$

where the first inequality is by the concavity of u and $u'(\bar{b}) = \frac{c(a_H)}{a_H}$ and the second inequality is by that $u'(b_0) \leq u'(\bar{b})$). The above inequality further implies that

$$E^Q\left[\int_0^\infty e^{-rs}(A_s - B_s)\,\mathrm{d}s\right] \leq -\frac{b_0}{r} = -p(v).$$

Hence Principal can get at most $-p(v)$ for any initial value $v > \bar{v}$.

The above result implies that whenever Agent's continuation value satisfies $v > \bar{v}$, he needs to be retired immediately, in other words, the optimal retirement bound v^* must satisfy

$v^* \leq \bar{v}$. So we conclude our result. $\square$

It is also important to verify that the solution on $[0, \frac{u(\bar{b})}{r}]$ to the HJB equation (2.13) with the boundary condition (2.14) is indeed Principal's value function. The following lemma presents this verification argument.

Lemma A.7. *Suppose that G is the solution to the HJB equation* (2.13) *with the boundary condition* (2.14) *satisfied by v^* where $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* := 0$ otherwise. Then for any initial agent's continuation value $v \in [0, \frac{u(\bar{b})}{r}]$, the principal's maximum expected payoff under admissible controls is $G(v)$.*

Proof. Given that the domain $[0, \frac{u(\bar{b})}{r}]$ is compact, so the solution G to equation (2.13) naturally satisfies the linear growth condition: $|G(v)| \leq M_0(1 + |v|)$ for some positive constant M_0. Thus by Lemma 2 and 3, Strulovici and Szydlowski (2015), the principal's value function equals to the above solution G pointwisely on $[0, \frac{u(\bar{b})}{r}]$. In particular, Lemma 3 therein provides a way to construct an optimal admissible control. Under such optimal admissible control and consider the optimal stopping problem. Note that Assumption 2-4, Strulovici and Szydlowski (2015), are satisfied in my setting, by Proposition 7, G is the value function of the optimal stopping problem. Hence we conclude our proof. $\square$

Proof of Theorem II.2. The proof follows from Lemma A.6 and A.7. $\square$

A.2.2 The proofs of Proposition II.3, Proposition II.4 and Theorem II.3

Proof of Proposition II.3. It is straightforward to verify that for each $v \leq v^*$, the given tuple $(\beta^*(v), k^*(v), b^*(v), a^*(v))$ solves the maximization within the HJB equation (2.13). It is also straightforward to verify that the assumptions in Theorem 3 in Strulovici and Szydlowski (2015) are satisfied, by which, there exists a unique strong solution V_t to equation , the Markov control $(\beta^*(V_t), a^*(V_t), b^*(V_t), k^*(V_t))$ is admissible and optimal. Moreover, by Proposition II.2, the incentive compatibility is satisfied so the contract characterized by the above control is also incentive compatible. $\square$

Proof of Proposition II.4. Let v^* be the optimal retirement bound. Recall that the HJB equation (2.13) can be rewritten into the following form:

$$rF(v) = \max\Big\{F'(v)(rv - u(b^*(v))) + \frac{F''(v)}{2}(\epsilon\sigma)^2 - b^*(v),$$
$$F'(v)(rv - u(b^*(v)) + c(a_H)) + \frac{F''(v)}{2}(\frac{c(a_H)}{a_H}\sigma)^2 + a_H - b^*(v)\Big\}, \tag{A.17}$$

where we call the first equation in the right-side of equation (A.17) the shirking ODE and the second equation the working ODE.

In the following I will show that $rF(v) \geq \sup_{b\in[0,\bar{b}]}\big(F'(v)(rv - u(b)) - b\big)$ holds for all $v \in [0, v^*]$ where "=" holds only at $v = 0$ or $v = v^*$. Note that the right hand-side is strictly greater than the shirking ODE . If the above claim is true, then by implementing shirking Principal will never achieve the optimum value on $(0, v^*)$. Thus the effort 0 will not be optimal for Principal before stopping occurs.

Recall that the optimal reward $b^*(v)$ solves $\sup_{b\in[0,\bar{b}]}\big(F'(v)(rv - u(b)) - b\big)$ for any v. Depending on whether the optimal reward is strictly positive or not, we can divide the range of continuation value $[0, v^*]$ into two intervals. Let b_0 be the threshold such that $b^*(v) = 0$ for any $v \in [0, b_0]$ and $b^*(v) > 0$ for $v \in (b_0, v^*]$.

For any $v \in [0, b_0]$, the optimal reward $b^*(v) = 0$. Thus on this interval $rF(v) - \sup_{b\in[0,\bar{b}]}\big(F'(v)(rv - u(b)) - b\big) = rF(v) - F'(v)rv$. Note that the derivative function $-F''(v)rv \geq 0$. Hence it is an increasing function. With the initial condition $F(0) = 0$, the minimum in this region $[0, b_0]$ is achieved at 0. Hence for any $v \in [0, b_0]$,

$$rF(v) - \sup_{b\in[0,\bar{b}]}\big(F'(v)(rv - u(b)) - b\big) \geq 0,$$

with "=" achieved at $v = 0$.

The rest case is when the continuation value v is in $(b_0, v^*]$, then $b^*(v) > 0$. Furthermore, I claim the following:

Claim A.1. *For an arbitrarily fixed* $b \in [0, \bar{b}]$*,* $rF(v) > \big(F'(v)(rv - u(b)) - b\big)$ *holds for all* $v \in (b_0, v^*)$*.*

If the above claim is true, then it is not possible for any $v' \in (b_0, v^*)$ to satisfy that

$rF(v') = \sup_{b\in[0,\bar{b}]} \big(F'(v')(rv' - u(b)) - b\big)$: suppose to the contrary that $rF(v') = \sup_{b\in[0,\bar{b}]} \big(F'(v')(rv' - u(b)) - b\big)$ for some value $v' \in (b_0, v^*)$. Given the compact interval $[0, \bar{b}]$ and that the function $\big(F'(v)(rv - u(b)) - b\big)$ is continuous with respect to b, there must exist b' such that $rF(v') = \big(F'(v')(rv' - u(b')) - b'\big)$. Thus we arrive at a contradiction to the given claim. Hence we can conclude the proof by proving this claim, and combining it with the above result for the case of $[0, b_0]$.

The rest is to prove the above claim. For the given arbitrarily fixed b, consider the function $rF(v) - F'(v)(rv - u(b)) + b$ for all $v \in (b_0, v^*)$. By taking the derivative of function $rF(v) - F'(v)(rv - u(b)) + b$ with respect to v, we have $-F''(v)(rv - u(b))$. By Theorem II.1, $F(\cdot)$ is strictly concave, $F''(\cdot) < 0$. There are two possible cases: $\frac{u(b)}{r} < v^*$ or $\frac{u(b)}{r} \geq v^*$. In the first case, $-F''(v)(rv - u(b))$ is strictly negative on $\Big(b_0, \frac{u(b)}{r}\Big)$, zero at $\frac{u(b)}{r}$ and strictly positive on $\Big(\frac{u(b)}{r}, v^*\Big)$. Then the following holds:

$$\begin{aligned}
&\inf_{v\in(b_0,v^*)} \big(rF(v) - F'(v)(rv - u(b)) + b\big) \\
&\geq \left(rF(\frac{u(b)}{r}) - F'(\frac{u(b)}{r})(r\frac{u(b)}{r} - u(b)) + b\right) \quad \text{(by evaluating at the minimizer } \frac{u(b)}{r}) \\
&= rF(\frac{u(b)}{r}) + b > -rp(\frac{u(b)}{r}) + b \qquad \text{(by that } F(\frac{u(b)}{r}) > -p(\frac{u(b)}{r})) \\
&= -b + b = 0 \qquad\qquad \text{(by the definition of } p(\cdot)).
\end{aligned}$$

In the second case, $-F''(v)(rv-u(b))$ is strictly negative on $[b_0, v^*)$. By a similar derivation, for any $v \in (b_0, v^*)$,

$$\begin{aligned}
&\big(rF(v) - F'(v)(rv - u(b)) + b\big) \\
&> \big(rF(v^*) - F'(v^*)(rv^* - u(b)) + b\big) \geq \left(rF(\frac{u(b)}{r}) - F'(\frac{u(b)}{r})(r\frac{u(b)}{r} - u(b)) + b\right) \\
&= rF(\frac{u(b)}{r}) + b \geq -rp(\frac{u(b)}{r}) + b = -b + b = 0.
\end{aligned}$$

Hence we conclude the claim. □

Proof of Theorem II.3. From Lemma II.3, the stochastic equation that governs the evolution of the agent's continuation value depends only on the external noise W (without the drift term $\sigma\alpha\,\mathrm{d}t$). In this case, the canonical report to the agent is $Y_t = E^P[W_t|\mathcal{F}_t^Y], \forall t$. The agent's incentive compatibility constraint is characterized by Proposition II.2. Based on that the agent's prior belief is P, Principal's problem in this situation will be equivalent to setting $\overline{\alpha} = \underline{\alpha} = 0$ in (2.11). Hence the optimal incentive contract is the optimal output-based contract derived in Proposition II.3. $\square$

A.3 Proofs in Section 2.5

Proof of Theorem II.4. It is straightforward to verify that the given policy $(b^*(\cdot), \alpha^*(\cdot), k^*(\cdot))$ maximizes the HJB equation (2.13). Note that the choice of β could give the principal some flexibility to switch the choice of effort from a_H to 0 in situations when the optimal $k^*(v)$ supports the high effort, without violating the IC constraint (note that $k^*(v)$ is independent of the choice of $a^*(v)$). However, I claim that the inequality $G'(v)c(a_H) \geq -a_H$ holds for any $v \in [0, v^*]$ (its proof will be present at the end), which implies that it is not profitable for the principal to switch a_H to 0 when $k^*(v)$ supports high effort. Hence, the given tuple $(a^*(\cdot), k^*(\cdot), b^*(\cdot), \alpha^*(\cdot))$ with $\beta^* \equiv 1$ maximizes the optimization within the HJB equation (2.13) for every v.

Given that G is third continuously differentiable, we can verify that the assumptions in Theorem 3 in Strulovici and Szydlowski (2015) are satisfied. By this theorem, we can conclude that there exists a unique strong solution V_t to the equation

$$\mathrm{d}\, V_s = (rV_s - u(b^*(V_s)) + c(a^*(V_s)))\,\mathrm{d}s + \sigma k^*(V_s)\,\mathrm{d}(W_s + \alpha^*(V_s)\,\mathrm{d}\, s),$$

and the Markov control $(\beta^*(V_t), \alpha^*(V_t), a^*(V_t), b^*(V_t), k^*(V_t))_{t\geq 0}$ is admissible and optimal. Moreover, by Proposition II.2, the incentive compatibility is satisfied so the contract characterized by the above control is also incentive compatible.

Lastly, we present the proof of the inequality $G'(v)c(a_H) \geq -a_H$ as follows: Recall that $\overline{v} := \frac{u(\overline{b})}{r}$ with $u'(\overline{b}) = \frac{c(a_H)}{a_H}$. By Theorem II.1, G is a strictly concave function and by Theorem II.2, $v^* \leq \overline{v}$ holds. Thus for any $v \in [0, v^*]$, the following inequality holds:

$$G'(v)c(a_H) + a_H \geq G'(v^*)c(a_H) + a_H = -p'(v^*)c(a_H) + a_H \geq -p'(\overline{v})c(a_H) + a_H = 0.$$

Hence we complete the proof. $\square$

Let v_0 be the particular continuation value such that $G'(v_0) = 0$. By the strict concavity of G, this v_0 is unique. Note that when $\overline{\alpha} = 0$, then the proof on $[0, v_0]$ follows directly from that in Proposition II.4; similar for the case when $\underline{\alpha} = 0$ on $[v_0, v^*]$. In the following, we will solely focus on the case when $\overline{\alpha} > 0$ and $\underline{\alpha} < 0$. The proof of Theorem II.5 is divided into the following lemmas.

Lemma A.8. *Suppose that* $G'(0)\sigma\epsilon\overline{\alpha} + \frac{G''(0)}{2}(\sigma\epsilon)^2 \neq 0$. *Then the optimal incentive contract recommends the effort level* a_H *at the agent's continuation value* 0.

Proof. Note that the HJB equation (2.13), when evaluated at $v = 0$, generates the following inequality with $G(0) = 0$:

$$\sup_{\substack{(0,k\beta)\in\Gamma, b\in[0,\bar{b}]\\ \beta\in[-1,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} - rG(0) + G'(0)(\sigma k\overline{\alpha}) + \frac{G''(0)}{2}(\sigma k)^2 \leq 0. \tag{A.18}$$

Since $\sup_{k\in[0,\overline{K}]}\left(G'(0)(\sigma k\alpha) + \frac{G''(0)}{2}(\sigma k)^2\right) \geq 0$, thus for the above inequality (A.18) to hold, the true maximizer in $[0, \overline{K}]$ that solves $\sup_{k\in[0,\overline{K}]}\left(G'(0)(\sigma k\alpha) + \frac{G''(0)}{2}(\sigma k)^2\right)$ must be strictly less than ϵ, thus

$$\arg\max\left\{k \in [\epsilon, \overline{K}] \mid G'(0)(\sigma k\alpha) + \frac{G''(0)}{2}(\sigma k)^2\right\} = \epsilon.$$

Moreover, the assumption that $G'(0)\sigma\epsilon\overline{\alpha} + \frac{G''(0)}{2}(\sigma\epsilon)^2 \neq 0$ implies that strict inequality must hold in the above inequality (A.18). Thus the optimal effort level cannot be 0 at the continuation value 0. We conclude the proof. □

Lemma A.9. *Suppose that* $G'(0)\sigma\epsilon\overline{\alpha} + \frac{G''(0)}{2}(\sigma\epsilon)^2 \neq 0$. *Then the optimal incentive contract recommends the effort level* a_H *for any continuation value* $v \in [0, v_0]$.

Proof. The proof is by contradiction. By Lemma A.8, the optimal effort is a_H at continuation value 0. Suppose to the contrary that the optimal effort would switch from a_H to 0 when v increases from 0 to v_0, with the first switching point continuation value v_1. Let the interval $[v_1, v_2] \subseteq [0, v_0]$ to be an interval in which the agent optimally shirks before switching back to a_H. Recall that on $[0, v_0]$, the optimal compensation is $b^*(v) = 0$ and the evaluation bias is $\alpha^*(v) = \overline{\alpha}$. Then the HJB equation for any $v \in [v_1, v_2]$ takes the following form:

$$-rG(v)+G'(v)(rv+\sigma k^*(v)\overline{\alpha})+\frac{G''(v)}{2}(\sigma k^*(v))^2=0, \tag{A.19}$$

where $k^*(v)=\max\{\frac{-G'(v)\overline{\alpha}}{G''(v)\sigma},\epsilon\}$.

Let us first consider the case when $k^*(v)=\frac{-G'(v)\overline{\alpha}}{G''(v)\sigma}$ on $[v_1,v_1+\hat{\epsilon}]$ for some $\hat{\epsilon}>0$. Now consider the switching point v_1, at which either condition (ii) or (iii) must hold. Note that $rG(v)-G'(v)rv$ is increasing, thus the above equation (A.19) implies that $G'(v)\sigma k^*(v)\overline{\alpha}+\frac{G''(v)}{2}(\sigma k^*(v))^2=\frac{-(G'(v)\overline{\alpha})^2}{2G''(v)}$ is also increasing. Moreover, given that $G'(v)$ is a decreasing function, this further implies that $\frac{-G'(v)}{G''(v)}$ must be increasing.

Suppose that working condition (iii) $\frac{c(a_H)}{a_H}\leq\frac{-G'(v)\alpha^*(v)}{G''(v)\sigma}$ holds right at the switching point v_1, then given that $\frac{-G'(v)}{G''(v)}$ is increasing on $[v_1,v_1+\hat{\epsilon}]$, condition (iii) still holds on $[v_1,v_1+\hat{\epsilon}]$. Hence it cannot be that $a^*(v)=0$ on $[v_1,v_1+\hat{\epsilon}]$, a contradiction. Suppose that the working condition (ii) $G'(v)c(a_H)+a_H+\frac{G''(v)}{2}(\sigma\frac{c(a_H)}{a_H}+\frac{G'(v)\alpha^*(v)}{G''(v)})^2\geq 0$ and $\epsilon\leq\frac{-G'(v)\alpha^*(v)}{G''(v)\sigma}<\frac{c(a_H)}{a_H}$ are satisfied at the switching point v_1. By applying the envelope theorem and taking the derivative of the equation (A.19) on both sides, we have the following:

$$\underbrace{G''(v)(rv+\sigma k^*(v)\overline{\alpha})}_{<0}+\frac{G'''(v)}{2}(\sigma k^*(v))^2=0,$$

which implies that $G'''(v)>0$. The first sub-condition in condition (ii) can be transformed into the following:

$$\frac{G'(v)c(a_H)}{-G''(v)}+\frac{a_H}{-G''(v)}\geq\frac{1}{2}(\sigma\frac{c(a_H)}{a_H}+\frac{G'(v)\alpha^*(v)}{G''(v)})^2. \tag{A.20}$$

Given that the above inequality (A.20) holds at v_1, with that $\frac{G'(v)}{-G''(v)}$ is increasing and $-G''(v)$ is decreasing on $[v_1,v_1+\hat{\epsilon}]$, the left handside of the above inequality is increasing but the right handside is decreasing. Hence the above inequality still holds for $[v_1,v_1+\hat{\epsilon}]$. Therefore, the optimal effort on $[v_1,v_1+\hat{\epsilon}]$ is a_H, which is again a contradiction.

Lastly we consider the case when $k^*(v)=\epsilon$ on $[v_1,v_1+\hat{\epsilon}']$ for some $\hat{\epsilon}'>0$. Now consider the switching point v_1, at which condition (i) holds. Note that $rG(v)-G'(v)rv$ is increasing, thus the above equation (A.19) implies that

$$G'(v)\sigma\epsilon\overline{\alpha} + \frac{G''(v)}{2}(\sigma\epsilon)^2 = G'(v)\left(\sigma\epsilon\overline{\alpha} + \frac{G''(v)}{2G'(v)}(\sigma\epsilon)^2\right)$$

is also increasing. Moreover, given that $G'(v)$ is a decreasing function, this further implies that $\frac{G''(v)}{G'(v)}$ must be increasing. Suppose that working condition (i)

$$(c(a_H) + \sigma(\frac{c(a_H)}{a_H} - \epsilon)\alpha^*(v)) + \frac{a_H}{G'(v)} + \frac{G''(v)}{2G'(v)}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) \geq 0$$

and $\epsilon > \frac{-G'(v)\alpha^*(v)}{G''(v)\sigma}$ hold right at the switching point v_1. Given that $\frac{G''(v)}{G'(v)}$ is increasing on $[v_1, v_1 + \hat{\epsilon}']$ and $G'(v)$ is decreasing and nonnegagive on $[0, v_0]$, condition (i) still holds on $[v_1, v_1 + \hat{\epsilon}]$. Hence it cannot be that $a^*(v) = 0$ on $[v_1, v_1 + \hat{\epsilon}]$, which is again a contradiction.

In conclusion, the optimal effort is a_H at any continuation value $v \in [0, v_0]$. □

Lemma A.10. *Suppose that* $G'(v)c(a_H) + a_H + \frac{G''(v)}{2}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) \neq 0$ *for any* $v \in (v_0, v^*)$*. For any continuation value* v *with* $v \in [v_0, v^*)$*, the optimal incentive contract recommends the effort level* a_H*.*

Proof. By Lemma A.9, the optimal effort at the continuation value v_0 must be a_H. Moreover, by the assumption that $G'(v_0)c(a_H) + a_H + \frac{G''(v_0)}{2}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) \neq 0$ and the HJB equation, we have $G'(v_0)c(a_H) + a_H + \frac{G''(v_0)}{2}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) > 0$ holds at v_0. By that G is C^3 and that for any $v \in (v_0, v^*)$,

$$G'(v)c(a_H) + a_H + \frac{G''(v)}{2}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) \neq 0,$$

hence we have that

$$G'(v)c(a_H) + a_H + \frac{G''(v)}{2}\sigma^2((\frac{c(a_H)}{a_H})^2 - \epsilon^2) > 0 \text{ on } (v_0, v^*). \tag{A.21}$$

For any continuation value $v \in [v_0, v^*)$, we will now divide the possible situation into the following two cases: If $\frac{-G'(v)\underline{\alpha}}{G''(v)\sigma} \geq \frac{c(a_H)}{a_H}$, then by Theorem II.4, the optimal effort is a_H. Otherwise, $\frac{-G'(v)\underline{\alpha}}{G''(v)\sigma} < \frac{c(a_H)}{a_H}$, then let $k^*(v) := \max\{\frac{-G'(v)\underline{\alpha}}{G''(v)\sigma}, \epsilon\}$. Given that G is strictly concave, that $G' \leq 0$ on (v_0, v^*), and by Inequality (A.21), the following inequality holds:

$$\begin{aligned} & G'(v)(c(a_H) + \sigma \frac{c(a_H)}{a_H}\underline{\alpha}) + a_H + \frac{G''(v)}{2}\sigma^2(\frac{c(a_H)}{a_H})^2 \\ & > \frac{G''(v)}{2}\sigma^2(k^*(v))^2 + G'(v)\sigma k^*(v)\underline{\alpha}. \end{aligned}$$

Thus, the above inequality further implies the following

$$\begin{aligned} 0 = & \sup_{\substack{(a,k\beta)\in\Gamma, b\in[0,\bar{b}] \\ \beta\in[-1,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} -rG(v) + G'(v)(rv - u(b) + c(a) + \sigma k\alpha) + \frac{G''(v)}{2}(\sigma k)^2 + a - b \\ \geq & \sup_{b\in[0,\bar{b}]} -rG(v) + G'(v)(rv - u(b) + c(a) + \sigma\frac{c(a_H)}{a_H}\alpha) + \frac{G''(v)}{2}(\sigma\frac{c(a_H)}{a_H})^2 + a - b \\ > & \sup_{b\in[0,\bar{b}]} -rG(v) + G'(v)(rv - u(b) + \sigma k^*(v)\alpha) + \frac{G''(v)}{2}(\sigma k^*(v))^2 - b. \end{aligned}$$

By the above inequality, 0 cannot be the optimal effort. Thus the optimal effort must be a_H. Hence we conclude the proof. □

Proof of Theorem II.5. The proof follows from Lemma A.9 and A.10. □

Proof of Proposition II.5. The result for mean exit time for diffusion process is shown in Result 7.1, PP 239, Pavliotis (2014). □

A.4 Proofs in Section 2.6

A.4.1 Proofs in Section 2.6.1

Note that we can characterize Agent's continuation value in the same way as that in the main derivation. Hence Principal's problem can be written in a similar way as (2.11), and by a similar derivation as that in Section 2.4, we can define the corresponding HJB equation as follows: Fix a small ϵ such that $\overline{K} > \frac{c(a_H)}{a_H} > \epsilon > 0$. Let $\Gamma := \{(a_H, k\beta) | \overline{K} \geq k\beta \geq$

$\frac{c(a_H)}{a_H}\} \cup \{(0, k\beta) | \epsilon \leq k\beta \leq \frac{c(a_H)}{a_H}\}$, then

$$rG(v) = \sup_{\substack{(a,k\beta)\in\Gamma, b\in[0,\bar{b}] \\ \beta\in[-1,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} G'(v)(rv - u(b) + c(a) + \sigma k\alpha) + \frac{G''(v)}{2}(\sigma k)^2 + \gamma(\alpha)a - b, \tag{A.22}$$

$$G(0) = 0, G(v^*) = -p(v^*) \text{ and } G'(v^*) = -p'(v^*). \tag{A.23}$$

We can similarly show that the above HJB equation characterizes the principal's value function in this case.

Lemma A.11. *(i) There exists a unique twice continuously differentiable and strictly concave function G that solves HJB equation* (A.22) *with $G \geq -p$ and the boundary condition* (A.23) *holds for v^* where $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* := 0$ otherwise.*

(ii) Suppose that G is the solution to the HJB equation (A.22) *with the boundary condition* (A.23) *satisfied by v^* where $v^* := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G(v) = -p(v)\} \neq \varnothing$ and $v^* := 0$ otherwise. Then for any initial agent's continuation value v, the principal's maximum expected payoff under admissible controls is $G(v)$ if $v \leq \frac{u(\bar{b})}{r}$, and $-p(v)$ if $v > \frac{u(\bar{b})}{r}$. In particular, the value function is twice continuously differentiable and it is the unique solution of the HJB equation* (A.22) *with the boundary condition* (A.23) *satisfied by the above v^*.*

Proof. The proof is similar to that in Theorem II.1 and II.2 so we omit it here. □

Proof of Proposition II.6. It is straightforward to verify that the given policy solves the optimization within the HJB equation (A.22). Given Lemma A.11 , the rest of the proof follows similar to that in Theorem II.4. □

A.4.2 Proofs in Section 2.6.2

Proof of Proposition II.7. By the definition, it is straightforward to verify that given any agent's continuation value v, the set of admissible controls under the situation when the agent's outside option is w_1 is weakly larger than that under the outside option w_2. Hence $G_1(v) \geq G_2(v)$ for any $v \in [w_2, \infty)$. So we showed the first statement.

For the second statement, recall from Theorem II.1 and II.2 that the value function G_1 and G_2 are twice continuously differentiable strictly concave unique classical solution to the equation $G'' = \inf \mathcal{H}(v_0, G, G')$ given the initial value $(G_1(w_2), G_1'(w_2))$ and $(G_2(w_2), G_2'(w_2))$ respectively, where

$$\inf \mathcal{H}(v, G(v), G'(v)) := \inf_{\substack{(a,k\beta)\in\Gamma, b\in[0,\bar{b}] \\ \beta\in[-1,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} \mathcal{H}_{a,k,b,\alpha}(v, G(v), G'(v)),$$

and for any tuple $(a, k, b, \alpha) \in \{0, a_H\} \times [\epsilon, \overline{K}] \times [0, \bar{b}] \times [\underline{\alpha}, \overline{\alpha}]$, $\mathcal{H}_{a,k,b,\alpha}$ is a function such that

$$\mathcal{H}_{a,k,b,\alpha}(v, G(v), G'(v)) := \frac{-2G'(v)(rv - u(b) + c(a) + \sigma k\alpha) - 2(a - b) + 2rG(v)}{(\sigma k)^2}.$$

Then we claim that $G_1(w_2) > G_2(w_2)$ holds. To see this, suppose that $G_1(w_2) = G_2(w_2) = 0$. By the strictly concavity, $G_1(w_1) = 0$ and $G_1(w_2) = 0$ imply that $G_1 \geq 0$ on $[w_1, w_2)$. However, $G_2'(w_2) \geq 0$ and that $G_1 \geq G_2$ on $[w_2, \infty)$ imply that $G_1'(w_2) \geq G_2'(w_2) > 0$. That $G_1 \geq 0$ on $[w_1, w_2)$ combining with that $G_1'(w_2) > 0$ with $G_1(w_2) = G_2(w_2) = 0$ implies the non-concavity of G_1 at w_2. This is a contradiction with that G_1 is a twice continuously differentiable strictly concave function on $[w_1, \infty)$. So we have $G_1(w_2) > G_2(w_2)$.

The rest of the proof will show that $G_1(v) > G_2(v)$ and $G_1'(v) > G_2'(v)$ for any $v \in (w_2, \infty)$. Suppose that to the contrary, $G_1(v) > G_2(v)$ or $G_1'(v) > G_2'(v)$ is violated for some $v \in (w_2, \infty)$. Then there must exist a point $\hat{v}' \in [w_2, \infty)$ at which $G_1(\hat{v}') > G_2(\hat{v}')$ is violated. Let $v' := \inf\{v \in [w_2, \infty) \mid G_1'(v) > G_2'(v)\}$. It is straightforward to verify that the following inequality holds:

$$\begin{aligned} G_1''(v') &= \inf \mathcal{H}(v', G_1(v'), G_1'(v')) \\ &> \inf \mathcal{H}(v', G_2(v'), G_2'(v')) = G_2''(v'). \end{aligned}$$

Since $G_1'(\cdot)$ hits $G_2'(\cdot)$ at v' the first time before which time $G_1'(\cdot) > G_2'(\cdot)$ and $G_1(\cdot) > G_2(\cdot)$, we have a contradiction with the above inequality. Hence we conclude that $G_1(v) > G_2(v)$ and $G_1'(v) > G_2'(v)$ for any $v \in (w_2, \infty)$. □

A.4.3 Proofs in Section 2.6.3

We can solve Principal's problem in two steps: first we can characterize the principal's problem after promoting the agent with the corresponding HJB equation as follows:

$$rG_p(v) = \sup_{\substack{(a,k\beta)\in\Gamma', b\in[0,\bar{b}] \\ \beta\in[0,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} G_p'(v)(rv - u(b) + c(a) + \sigma k\alpha) + \frac{G_p''(v)}{2}(\sigma k)^2 + a - b - rC \tag{A.24}$$

$$G_p(0) = -C, G_p(v_{p,r}) = -p(v_{p,r}) \text{ and } G_p'(v_{p,r}) = -p'(v_{p,r}), \tag{A.25}$$

where $\Gamma' = \{(\hat{a}_H, k\beta)|\overline{K} \geq k\beta \geq \frac{c(\hat{a}_H)}{\hat{a}_H}\} \cup \{(0, k\beta)|\epsilon \leq k\beta \leq \frac{c(\hat{a}_H)}{\hat{a}_H}\}$ for some ϵ and $\overline{K}$ such that $0 < \epsilon < \frac{c(\hat{a}_H)}{\hat{a}_H} < \overline{K}$.

Equivalently, we can write the above HJB equation into the following form: let $\widehat{G}_p(v) := G_p(v) + C$ and $\hat{p}(v) := p(v) - C$ for any $v \in [0, \infty)$, then

$$r\widehat{G}_p(v) = \sup_{\substack{(a,k\beta)\in\Gamma', b\in[0,\bar{b}] \\ \beta\in[0,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} \widehat{G}_p'(v)(rv - u(b) + c(a) + \sigma k\alpha) + \frac{\widehat{G}_p''(v)}{2}(\sigma k)^2 + a - b \tag{A.26}$$

$$\widehat{G}_p(0) = 0, \widehat{G}_p(v_{p,r}) = -\hat{p}(v_{p,r}) \text{ and } G_p'(v_{p,r}) = -\hat{p}'(v_{p,r}), \tag{A.27}$$

where $\Gamma' = \{(\hat{a}_H, k\beta)|\overline{K} \geq k\beta \geq \frac{c(\hat{a}_H)}{\hat{a}_H}\} \cup \{(0, k\beta)|\epsilon \leq k\beta \leq \frac{c(\hat{a}_H)}{\hat{a}_H}\}$ for some ϵ and $\overline{K}$ such that $0 < \epsilon < \frac{c(\hat{a}_H)}{\hat{a}_H} < \overline{K}$.

By following the same procedure of the proof to Theorem II.1 and II.2 on the equivalent equation (A.26) and the boundary condition (A.27), we can show that the value function G_p is the unique classical solution to the above HJB equation (A.24) with the boundary condition (A.25) satisfied by the retirement bound $v_{p,r}$ where $v_{p,r} := \arg\min\{v \in (0, \frac{u(\bar{b})}{r}] \mid G_p(v) = -p(v)\}$ if $\{v \in (0, \frac{u(\bar{b})}{r}] \mid G_p(v) = -p(v)\} \neq \varnothing$ and $v_{p,r} := 0$ otherwise. Based on this conclusion, we will first prove Proposition II.8 (i).

Proof of Proposition II.8 (i). We start with the observation that $G'(0) > 0$ must hold: suppose to the contrary that $G'(0) \leq 0$ holds. Then by Theorem II.1 and II.2 that G is strictly concave, $G'(v) < 0$ for all $v \in (0, \infty)$. This contradicts our basic assumption that the principal prefers to hire the agent at the beginning.

Note that the incentive compatibility constraint for high effort is relaxed in the case of after promoting the agent. Hence for any admissible control in the benchmark case (without promotion), we can find a corresponding control by replacing the effort level from either a_H or 0 to $\hat{a}_H$ in the given effort process based on incentive compatibility constraint. Given that $c(\hat{a}_H) = c(a_H) > 0$, the evolution of the agent's continuation value remains the same under such replacement and thus the separation time due to either firing or retirement remains the same. Moreover, $G'(0) > 0$ implies that the agent exerts high effort during the employment relationship in the benchmark case (without promotion opportunities). Therefore, given any control in the benchmark case, the corresponding control defined above will generate a strictly higher Principal's payoff under the case after promotion if we don't calculate the training cost. Hence by the definition of the value function, $G_p + C > G$ holds on $(0, \infty)$. $\square$

Based on the value function for a trained agent, Principal should decide on the timing of the promotion or retirement for an untrained agent in the following way. Given the promotion opportunity, Principal can choose a stopping payoff by optimizing among either retiring or promoting Agent. Let $G_0(v) := \max\{-p(v), G_p(v)\}$ be the optimal stopping payoff that summarizes the options between retirement and promotion. We can characterize the principal's problem before promoting the agent with the corresponding HJB equation as follows:

$$r\widehat{G}(v) = \sup_{\substack{(a,k\beta)\in\Gamma, b\in[0,\bar{b}] \\ \beta\in[0,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} \widehat{G}'(v)(rv - u(b) + c(a) + \sigma k\alpha) + \frac{\widehat{G}''(v)}{2}k^2 + a - b, \quad \text{(A.28)}$$

$$\widehat{G}(0) = 0, \widehat{G}(v_p^*) = G_0(v_p^*) \text{ and } \widehat{G}'(v_p^*) = G_0'(v_p^*), \quad \text{(A.29)}$$

where $\Gamma := \{(a_H, k\beta) | \overline{K} \geq k\beta \geq \frac{c(a_H)}{a_H}\} \cup \{(0, k\beta) | \epsilon \leq k\beta \leq \frac{c(a_H)}{a_H}\}$ for some ϵ small enough and $\overline{K}$ large enough such that $0 < \epsilon < \frac{c(a_H)}{a_H} < \overline{K}$.

Proof of Proposition II.8 (ii). Note that (A.28) is the same as (2.13) except that we have a different termination payment G_0. By the proof similar to Theorem II.1 and II.2, the value function $\widehat{G}$ in this case is still the unique classical solution to the equation $\widehat{G}'' = \inf \mathcal{H}(v_0, \widehat{G}, \widehat{G}')$, where

$$\inf \mathcal{H}(v, G(v), G'(v)) := \inf_{\substack{(a,k\beta)\in\Gamma, b\in[0,\bar{b}] \\ \beta\in[-1,1], \alpha\in[\underline{\alpha},\overline{\alpha}]}} \mathcal{H}_{a,k,b,\alpha}(v, G(v), G'(v)),$$

and for any tuple $(a, k, b, \alpha) \in \{0, a_H\} \times [\epsilon, \overline{K}] \times [0, \bar{b}] \times [\underline{\alpha}, \overline{\alpha}]$, $\mathcal{H}_{a,k,b,\alpha}$ is a function such that

$$\mathcal{H}_{a,k,b,\alpha}(v, G(v), G'(v)) := \frac{-2G'(v)(rv - u(b) + c(a) + \sigma k\alpha) - 2(a - b) + 2rG(v)}{(\sigma k)^2}.$$

Recall that for the value function G in the benchmark case, Lemma A.5 shows that G is obtained by increasing the initial slope $G'(0)$ to the maximum value while satisfying the condition that $\{v \in [0, \infty) \mid G(v) = -p(v)\} \neq \varnothing$ for the resulted solution G. Similar to Lemma A.5, the value function $\widehat{G}$ before the promotion is to increase the initial slope $\widehat{G}'(0)$ to the maximum value while satisfying the condition that $\{v \in [0, \infty) \mid \widehat{G}(v) = G_0(v)\} \neq \varnothing$ for the resulted solution $\widehat{G}$.

Given that the terminal payoff $G_0 \geq -p$, the maximum value of initial slope such that $\{v \in [0, \infty) \mid \widehat{G}(v) = G_0(v)\} \neq \varnothing$ is higher than that such that $\{v \in [0, \infty) \mid G(v) = -p(v)\} \neq \varnothing$. By the same argument as that in Proposition II.7 with $\widehat{G}'(0) \geq G'(0)$, we can show that the solution $\widehat{G} \geq G$ on $[0, \infty)$. Moreover, if there exists a point $\hat{v}$ such that $G_p(\hat{v}) - C > -p(\hat{v})$, then $\widehat{G}'(0) > G'(0)$. By the same argument as that in Proposition II.7, we have $\widehat{G}(v) > G(v)$ and $\widehat{G}'(v) > G'(v)$ for any $v \in [0, \infty)$. □

APPENDIX B

Appendix for Chapter III

B.1 Proofs and Extra Examples of Section 3.3.2

B.1.1 An example illustration of strategic partitions

We illustrate the strategic partition in a setting with one receiver without any primitive types.

Example B.1. Let ω be the bank's fundamental value with two possible realizations $\Omega = \{-1, 2\}$ with equal probability. A policymaker is conducting a stress test for the bank to get information regarding ω, the result of which will be privately informed to two receivers. Receivers will then decide whether to run on the bank or wait, i.e., $A_1 = A_2 := \{R, W\}$ with the payoff matrix:

(u_1, u_2)	W	R
W	(ω, ω)	$(\omega - 1, 0)$
R	$(0, \omega - 1)$	$(0, 0)$

Let $S_1 := \{\eta \in \Delta(\Omega \times \{R, W\}) \mid \eta(-1, W) + \eta(2, W) > 0.5\}$. Whenever a receiver's beliefs is in S_1, action W is uniquely optimal. Let $S_2 = \{\eta \in \Delta(\Omega \times \{R, W\}) \mid \eta(-1, W) + \eta(2, W) < 0.5\}$, under which R is uniquely optimal. Let $S_3 = \Delta(\Omega \times \{R, W\}) \backslash (S_1 \cup S_2)$ under which both actions are optimal. Note that $\{S_1, S_2, S_3\}$ is a partition of $\Delta(\Omega \times \{R, W\})$, and the strategic partition in this example is presented as follows: for any $i \in \{1,2\}$,

$$\xi_i := \left\{ \begin{array}{l} \{\nu_i \in V_i \mid \operatorname{marg}_{\Delta(\Omega \times A_{-i})} \nu_i \in S_1\} \times \{W\}, \\ \{\nu_i \in V_i \mid \operatorname{marg}_{\Delta(\Omega \times A_{-i})} \nu_i \in S_2\} \times \{R\}, \\ \{\nu_i \in V_i \mid \operatorname{marg}_{\Delta(\Omega \times A_{-i})} \nu_i \in S_3\} \times \{R, W\} \end{array} \right\}.$$

B.2 Proofs of Section 3.3.2

Proof of Proposition III.3. Fix any stable regular selection criterion ζ and any canonical signal $(\pi^c, M^c := \prod_{i \in \mathcal{I}} M_i^c)$ with the recommendations σ constituting a BNE that survives ζ.

Let us arbitrarily pick a receiver i and an arbitrary partition menu $\Psi^i := (\tau_i, p_{\tau_i} \times S_{\tau_i})_{\tau_i \in \mathbb{T}_i} \in \Gamma_i$. Based on our choice, consider the modified signal $\pi^{m,i} \in \hat{\Lambda}(\pi, (p_{\tau_i} \times S_{\tau_i}^A)_{\tau_i \in \mathbb{T}_i})$ such that $\pi^{m,i}$ has the same structure as π except that given each $a^i := (a_{\tau_i})_{\tau_i \in \mathbb{T}_i}$ it pools all messages in $\mathcal{M}_i(\Psi^i, a^i) := \{(\tau_i, v_{\tau_i}, a_{\tau_i})_{\tau_i \in \mathbb{T}_i} \mid v_{\tau_i} \in p_{\tau_i}\}$ (if $\mathcal{M}_i(\Psi^i, a^i) \neq \varnothing$) into a unique message, denoted as $m_i^p \in M_i^c$, for receiver i. That $\mathcal{M}_i(\Psi^i, a^i) \neq \varnothing$ and that the above canonical signal recommendations a BNE surviving ζ imply that $a^i \in (S_{\tau_i})_{\tau_i \in \mathbb{T}_i}$.

Suppose that there exists at least one underlying state ω, such that $\pi_i(\mathcal{M}_i(\Psi^i, a^i)|\omega) > 0$ and $\#|\mathcal{M}_i(\Psi^i, a^i)| > 1$. Consider the above modified signal $\pi^{m,i}$. For any message m_j in $\pi^{m,i}$, define a mapping $\mathcal{C} : \cup_{i \in \mathcal{I}} M_i^c \to \cup_{i \in \mathcal{I}} M^c$ which indicates its *corresponding message* in π as follows: if $m_j \notin \mathcal{M}_i(\Psi^i, a^i)$, then its corresponding message in π is itself, i.e., $\mathcal{C}(m_j) = m_j$; otherwise, the corresponding message $\mathcal{C}(m_j) = m_i^p$.

By the stability condition, if every receiver primitive type obeys its action recommendation under the modified signal $\pi^{m,i}$ in all cases, then for each message m_j, the correct conjecture of receiver i induced by m_j under $\pi^{m,i}$ will still be in the same behavioral analogy class in ζ as that under its corresponding message $\mathcal{C}(m_j)$ in π. Thus if we can show that the recommendation action profile in the modified signal $\pi^{m,i}$ is indeed incentive compatible so that the recommendations constitute a BNE. Then this equilibrium survives ζ.

We now show that the recommended action under $\pi^{m,i}$ is indeed incentive compatible given others' obedience. First we consider each primitive type τ_j and any message $m_j \notin \mathcal{M}_i(\Psi^i, a^i)$. Note that, the conditional distribution of this message under signal $\pi^{m,i}$ is the same as that of its corresponding message $\mathcal{C}(m_j)$ under π. Assuming that the rest receiver primitive types obey their action recommendations, then the primitive type τ_j's belief over $\Omega \times A_{-j}$ under the modified signal $\pi^{m,i}$ remains the same as that under $\mathcal{C}(m_j)$ in signal π. Given that the recommended action is incentive compatible under the original signal π, taking the recommended action in the modified signal $\pi^{m,i}$ for such receiver primitive type is, therefore, incentive compatible given others' obedience.

Then we only need to consider the incentive compatibility for any receiver i's primitive type τ_i and message m_i^p under signal $\pi^{m,i}$. Assume that all other receiver primitive types comply with their action recommendations. Given that $\pi^{m,i}$ is equivalent to pooling all messages in $\mathcal{M}_i(\Psi^i, a^i)$ in π into a single message m_i^p and keep the rest the same, receiver i's belief over $\Omega \times A_{-i}$ under the message m_i^p will be a convex combination of those induced by the messages in $\mathcal{M}_i(\Psi^i, a^i)$ under π. Note that the strategic partition component is convex, and such pooling is among messages within a single partition component. Hence the belief induced by message m_i^p will not change the incentive compatibility of the recommended action for the primitive type τ_i under $\pi^{m,i}$. Thus taking the recommended action in the modified signal $\pi^{m,i}$ for such receiver primitive type is still incentive compatible given others' obedience.

By applying the above argument to each receiver j's primitive type, we can conclude that the action recommendation menu in every message under the modified signal $\pi^{m,i}$ is incentive compatible for each receiver. Hence, there exists a pure strategy BNE under $(\pi^{m,i}, M^c)$ that survives ζ under which each receiver obeys the action recommendation. Hence, it achieves the same outcome as that under σ and (M^c, π^c). □

Proof of Proposition III.2. Fix any stable regular selection criterion ζ and any canonical signal $(\prod_{i\in\mathcal{I}} M_i, \pi)$ with the recommendations σ constituting a BNE that survives the selection criterion ζ. For any receiver i and any partition menu, by Proposition III.3, we can construct another signal that achieves the same outcome as that under σ and (M^c, π^c). Now consider the following signal $(\hat{\pi}, M^c)$: given each action recommendation menu $(a_{\tau_i})_{\tau_i\in\mathbb{T}_i}$ for each receiver i, $\hat{\pi}$ pools all messages in $\mathcal{M}_i := \{(\tau_i, v_{\tau_i}, a_{\tau_i})_{\tau_i\in\mathbb{T}_i} \mid v_{\tau_i} \in p_{\tau_i}\}$ (if $\mathcal{M}_i \neq \varnothing$) into a unique message for all possible ζ-strategy partition component menu for receiver i. Hence, by applying Proposition III.3 repeatedly to different possible combination of receivers and theri partition menus, we can eventually show that there exists a a pure strategy BNE under the signal $(\hat{\pi}, M^c)$ that survives ζ and achieves the same outcome as σ under $(\prod_{i\in\mathcal{I}} M_i, \pi)$. Then there exists a pure strategy BNE under $(\hat{\pi}, M^c)$ that survives

ζ and achieves the same outcome as that under σ and (M^c, π^c). So the conclusion of Proposition III.2 follows after a canonical transformation of the above signal $(\hat{\pi}, M^c)$. $\square$

B.3 Solutions in Section 3.4

B.3.1 Detailed derivations in Section 3.4.1

Given that receiver will vote if and only if with at least 0.5 probability it is a pivotal voter, an effective targeted advertisement must convince the targeted individual voter about two things: (i) with at least 0.5 probability, there is exactly one of the rest receivers who would vote Y; (ii) given the advertisement, voting Y is indeed a best response under the voter's updated belief about the underlying states.

Let voter i's belief be $\mu^i := (\mu^i_1, \mu^i_2, \mu^i_3)$ for any $i \in \{1, 2, 3\}$. We can calculate that, conditional on being pivotal, for voter 1 to vote Y, its posterior belief must be in the belief region $\Phi_1 := \{\mu^1 \mid 2\mu^1_1 + \mu^1_3 \geq 1\}$ and let $\partial\Phi_1 := \{\mu^1 \mid 2\mu^1_1 + \mu^1_3 = 1\}$ be its boundary. Similarly, conditional on being pivotal, for voter 2 to vote Y, its posterior belief must be in the belief region $\Phi_2 := \{\mu^2 \mid 2\mu^2_2 + \mu^2_4 \geq 1\}$ with $\partial\Phi_2$ its boundary.

We will first characterize the selection criterion. Define the following set of conjectures of a voter i under which he believes that he is pivotal of at least 0.5 probability: Let $A^{PV} := \{YE\} \cup \{YN\} \cup \{EY\} \cup \{NY\}$, then define

$$PV_i := \left\{ \nu_i \in V_i \mid \operatorname{marg}_{\Delta(A_{-i})} \nu_i \left(A^{PV} \right) \geq 0.5 \right\}.$$

For any receiver i's conjecture ν_i, let $\operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot | A^{PV}) \in \Delta(\Omega)$ be the first-order belief induced by ν_i conditional on this receiver being pivotal (whenever well-defined). We can specify the equilibrium selection criterion as follows: for each receiver $i \in \{1, 2, 3\}$,

$$\zeta_i := \Bigg\{ \Big\{ \nu_i \mid \nu_i \in PV_i, \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_i \backslash \partial\Phi_i \Big\} \times \{Y, N\}, \\ \Big\{ \nu_i \mid \nu_i \in PV_i, \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_i^c \Big\} \times \{Y, N\}, \\ \Big\{ \nu_i \mid \nu_i \in PV_i, \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \partial\Phi_i \Big\} \times \{Y\}, \\ \Big\{ \nu_i \mid \nu_i \notin PV_i \Big\} \times \{E\} \Bigg\}.$$

It is immediate that this selection criterion is finite and regular. Moreover, it is stable. We will use the following intuitive pooling argument to illustrate this point. Consider any canonical signal π. For any voter $i \in \{1, 2\}$, suppose m_i and m_i' are two messages with the same action recommendation in the behavioral analogy class $\Big\{ \nu_i \mid \nu_i \in PV_i, \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_i \backslash \partial\Phi_i \Big\} \times \{Y, N\}$. Note that the selection criterion ζ_i only imposes restrictions on the first-order beliefs over A_{-i} and the conditional first-order belief over Ω. Since both the projection of PV_i on $\Delta(A_{-i})$ and the set $\Phi_i \backslash \partial\Phi_i$ are convex, if we modify the signal π by pooling these two messages into one unique distinguishable message, say m_i, then under the modified signal and the imposed obedience, receiver i's correct conjecture under message m_i (with the recommended action Y) is still in the same behavioral analogy class. Similarly, for any other receiver j ($j \neq i$), such modification will not change its first-order belief on A_{-j} for any message m_j; so his correct conjecture under the modified signal also remains in the same behavioral analogy class in all possible cases. Thus the stability condition follows from applying a similar argument to all possible subsets of messages with the same action recommendation within any behavioral analogy class for any voter.

We now define the strategic partitions based on set Φ_i: for each receiver $i \in \{1, 2\}$,

$$\Bigg\{ \Big\{ \nu_i \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_i \backslash \partial\Phi_i \Big\} \times \{Y\}, \Big\{ \nu_i \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \notin \Phi_i \Big\} \times \{N\}, \\ \Big\{ \nu_i \mid \operatorname{marg}_{\Delta(A_{-i})} \nu_i(\Omega \times \mathbb{T}_{-i} \times \{YY\}) = 1 \text{ or } \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \partial\Phi_i \Big\} \times \{Y, E, N\} \Bigg\};$$

For receiver 3, the strategic partition for his primitive type τ_i, $i \in \{1, 2\}$, is the same as that for receiver i. Based on the above and note that we also assume that whenever the voter is indifferent, he will choose, we can define their collection of partition menus Γ_i as follows: for each receiver $i \in \{1, 2\}$,

$$\Gamma_i := \Bigg\{ \Big\{ \nu_i \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_i \Big\} \times \{Y\}, \\ \Big\{ \nu_i \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \notin \Phi_i \Big\} \times \{N\}, \\ \Big\{ \nu_i \mid \operatorname{marg}_{\Delta(A_{-i})} \nu_i(\Omega \times \mathbb{T}_{-i} \times \{YY\}) = 1 \Big\} \times \{E\} \Bigg\}.$$

For convenience, denote the above behavioral analogy classes in the presented order as $C_{i,j}$, $i \in \{1, 2\}$, $j \in \{Y, N, E\}$.

For voter 3, note that for any signal, the conjecture given any message is identical for both primitive types except that his best response differs due to the difference of his utility. We then write down a simplified version of collection of partition menus (as compared to its original definition) for voter 3 as follows:[1]

$$\Gamma_3 := \Bigg\{ \left\{ \begin{bmatrix} 1 & \nu_i \times \{Y\} \\ 2 & \nu_i \times \{Y\} \end{bmatrix} \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_1 \cap \Phi_2, \nu_i \in PV_i \right\}, \\ \left\{ \begin{bmatrix} 1 & \nu_i \times \{Y\} \\ 2 & \nu_i \times \{N\} \end{bmatrix} \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_1 \cap \Phi_2^c, \nu_i \in PV_i \right\}, \\ \left\{ \begin{bmatrix} 1 & \nu_i \times \{N\} \\ 2 & \nu_i \times \{Y\} \end{bmatrix} \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_1^c \cap \Phi_2, \nu_i \in PV_i \right\}, \\ \left\{ \begin{bmatrix} 1 & \nu_i \times \{N\} \\ 2 & \nu_i \times \{N\} \end{bmatrix} \mid \operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot|A^{PV}) \in \Phi_1^c \cap \Phi_2^c, \nu_i \in PV_i \right\}, \\ \left\{ \begin{bmatrix} 1 & \nu_i \times \{E\} \\ 2 & \nu_i \times \{E\} \end{bmatrix} \mid \operatorname{marg}_{\Delta(A_{-i})} \nu_i(\Omega \times \mathbb{T}_{-i} \times \{YY\}) = 1 \right\} \Bigg\}.$$

Denote the above behavioral analogy classes as $C_{3,j}$, $j \in \{YY, YN, NY, NN, EE\}$.

[1] Theoretically, a selection criterion for voter 3 should take all the possibilities, including the heterogeneous conjectures as part of its components. We omit such behavioral analogy classes in the exposition since in this application, there is no equilibrium in which the conjecture of voter 3 differs for different types. Thus these omitted classes are redundant in all possible cases.

By Proposition III.3 and the proof of Proposition III.2, it is without loss of generality to focus on canonical signals in which the canonical message space for each voter is a sample collection of ζ-strategic partition component menus. Hence we can specify the messages space as follows:

$$\mathcal{M} := \left\{ \begin{bmatrix} c_{1,Y} \\ c_{1,N} \\ c_{1,E} \end{bmatrix} \times \begin{bmatrix} c_{2,Y} \\ c_{2,N} \\ c_{2,E} \end{bmatrix} \times \begin{bmatrix} c_{3,YY} \\ c_{3,YN} \\ c_{3,NY} \\ c_{3,NN} \\ c_{3,EE} \end{bmatrix} \right\},$$

where $c_{i,j}$ is a sample of the partition menu $C_{i,j}$ for any $i \in \{1,2,3\}$ and any j in the corresponding set. Note that if receiver 3 is of type 1, then the bill will pass if any of the following subset of message profiles is realized:

$$\mathcal{M}_1 := \left\{ \begin{array}{c} \begin{bmatrix} c_{1,N} \\ c_{1,E} \end{bmatrix} \times \begin{bmatrix} c_{2,Y} \end{bmatrix} \times \begin{bmatrix} c_{3,YY} \\ c_{3,YN} \end{bmatrix}, \quad \begin{bmatrix} c_{1,Y} \end{bmatrix} \times \begin{bmatrix} c_{2,N} \\ c_{2,E} \end{bmatrix} \times \begin{bmatrix} c_{3,YY} \\ c_{3,YN} \end{bmatrix}, \\ \begin{bmatrix} c_{1,Y} \end{bmatrix} \times \begin{bmatrix} c_{2,Y} \end{bmatrix} \times \begin{bmatrix} c_{3,NY} \\ c_{3,NN} \\ c_{3,EE} \end{bmatrix} \end{array} \right\}.$$

Similarly, if receiver 3 is of type 2, then the bill will pass if any of the following subset of message profiles is realized:

$$\mathcal{M}_2 := \left\{ \begin{array}{c} \begin{bmatrix} c_{1,N} \\ c_{1,E} \end{bmatrix} \times \begin{bmatrix} c_{2,Y} \end{bmatrix} \times \begin{bmatrix} c_{3,YY} \\ c_{3,NY} \end{bmatrix}, \quad \begin{bmatrix} c_{1,1} \end{bmatrix} \times \begin{bmatrix} c_{2,2} \\ c_{2,3} \end{bmatrix} \times \begin{bmatrix} c_{3,1} \\ c_{3,3} \end{bmatrix}, \\ \begin{bmatrix} c_{1,Y} \end{bmatrix} \times \begin{bmatrix} c_{2,Y} \end{bmatrix} \times \begin{bmatrix} c_{3,YN} \\ c_{3,NN} \\ c_{3,EE} \end{bmatrix} \end{array} \right\}.$$

For exposition simplicity, define the following two notions: for any i, j,

$$\operatorname{marg}_{\Delta(A_{-i})} C_{i,j} := \{\operatorname{marg}_{\Delta(A_{-i})} \nu_i \mid \nu_i \in C_{i,j}\}$$

and

$$\operatorname{marg}_{\Delta(\Omega)} C_{i,j} | A^{PV_i} := \{\operatorname{marg}_{\Delta(\Omega)} \nu_i(\cdot | A^{PV_i}) \mid \nu_i \in C_{i,j}\}.$$

For any given signal π with the above message space $\mathcal{M}$, given any message $c_{i,j}$, denote by $\beta_i^A(\cdot|c_{i,j},\pi)$ the posterior belief of receiver i over $\Delta(A_{-i})$; similarly, denote by $\beta_i^\Omega(\cdot|A^{PV_i},c_{i,j},\pi)$ the posterior belief of receiver i over $\Delta(\Omega)$ when getting message $c_{i,j}$ conditional on he being pivotal. By focusing on canonical signals using a sample collection to message each receiver, the politician then solves the following linear programming problem:

$$\max_{\pi}\sum_{\omega\in\Omega}\Big(\sum_{(c_{i,j_i})_{i=1}^3\in\mathcal{M}_1}\pi((c_{i,j_i})_{i=1}^3|\omega)+\sum_{(c'_{i,j_i})_{i=1}^3\in\mathcal{M}_2}\pi((c'_{i,j_i})_{i=1}^3|\omega)\Big)\mu^0(\omega)0.5 \tag{B.1}$$

$$0\le\pi((c_{i,j_i})_{i=1}^3|\omega)\le 1,(c'_{i,j_i})_{i=1}^3\in\mathcal{M}, \tag{B.2}$$

$$\sum_{j=1}^{3}\pi_i(c_{i,j}|\omega)=1,\forall\,\omega\in\Omega,i\in\{1,2\},\sum_{j=1}^{5}\pi_3(c_{3,j}|\omega)=1,\ \forall\,\omega\in\Omega, \tag{B.3}$$

$$\beta_i^A(\cdot|c_{i,j},\pi)\in\mathrm{marg}_{\Delta(A_{-i})}\,C_{i,j}\text{ for any }i,j, \tag{B.4}$$

$$\beta_i^\Omega(\cdot|A^{PV_i},c_{i,j},\pi)\in\mathrm{marg}_{\Delta(\Omega)}\,C_{i,j}|A^{PV_i}\text{ for any }i,j. \tag{B.5}$$

Note that Constraint (B.4) to (B.5) are linear due to the Bayes' rule.

B.3.2 Detailed derivations in Section 3.4.2

The worst equilibrium selection in this game will coordinate both players on the running equilibria whenever this is a possible option. Thus we need to identify the set of belief hierarchies under which the unique rationalizable action is to wait.

We define explicitly receivers' primitive type spaces for exposition convenience. Let $\mathbb{T}_1:=\{h,l\}$ and let $\mathbb{T}_2$ be a singleton set. For each receiver i, the underlying uncertainty space is $\Omega\times\mathbb{T}_{-i}$ and denote T_i^k as receiver i's k-order of belief for any $k\ge 1$.

Note that if receiver i's first-order belief is in $\Phi_i^1=\{t_i^1\in T_i^1|t_i^1(H)\ge\frac{2}{3}\}$, $i\in\{1,2\}$, then W is the unique optimal strategy regardless of the other receiver's primitive type or action. Hence W is the uniquely rationalizable action if receiver i's first-order belief is in the above set.[2]

Given any integer $K\ge 1$ and the k-order belief set Φ_i^k defined above for any $1\le k\le K$,

[2]We will always assign the boundary of the (k-order) belief sets to the action W to have a well-defined solution; otherwise, one may be able to implement the signal as close to the boundary as possible but cannot hit the boundary exactly.

the worst equilibrium is that each receiver will believe that the other will only play W if and only if his lower-order beliefs are in $\cup_{k=1}^{K}\Phi_i^k$. Thus W is uniquely optimal if and only if the following inequality holds:

$$\begin{aligned}2\cdot t_i^{K+1}(\{H\}\times\mathbb{T}_{-i}\times\overbrace{\cup_{k=1}^{K}\Phi_{-i}^k}^{\text{playing } W})+1\cdot t_i^{K+1}(\{H\}\times\mathbb{T}_{-i}\times\{R\})\\ -1\cdot t_i^{K+1}(\{L\}\times\mathbb{T}_{-i}\times\underbrace{\cup_{k=1}^{K}\Phi_{-i}^k}_{\text{playing } W})-2\cdot t_i^{K+1}(\{L\}\times\mathbb{T}_{-i}\times\{R\})\geq 0.\end{aligned}$$

By simplifying the above inequality, we have $t_i^{K+1}(\{\omega = H\}) + \frac{1}{3}t_i^{K+1}(\Omega\times\mathbb{T}_{-i}\times\cup_{k=1}^{K}\Phi_{-i}^k)\geq\frac{2}{3}$. Thus we can iteratively determine the $(K+1)$-order belief set under which the action W is uniquely rationalizable (denoted as Φ_i^{K+1}) as follows:

$$\Phi_i^{K+1} := \left\{ t_i^{K+1}\in T_i^{K+1} \mid \begin{array}{l} t_i^{K+1}(\{\omega=H\})+\frac{1}{3}t_i^{K+1}(\Omega\times\mathbb{T}_{-i}\times\cup_{k=1}^{K}\Phi_{-i}^k)\geq\frac{2}{3} \\ \operatorname{marg}_{T_i^j} t_i^{K+1}\notin\Phi_i^j, 1\leq j\leq K \end{array} \right\}.$$

W is receiver i's unique optimal strategy if his $(K+1)$-order belief is in Φ_i^{K+1}.

We now define the selection criterion ζ that describes the sender-worse selection in this game. For exposition convenience, let $\widehat{\Phi}_i^k$ be the conjecture set associated to each Φ_i^k for any $k\geq 1$ such that

$$\widehat{\Phi}_i^k := \left\{ v_i\in V_i \mid \left(\operatorname{marg}_{T_i^k}\psi_i^{-1}\left(\operatorname{marg}_{\Delta(\Omega\times\mathbb{T}_{-i}\times T_{-i})} v_i\right)\right)\in\Phi_i^k\right\}.$$

Intuitively, $\widehat{\Phi}_i^k$ collects all the conjectures with which the k-order belief is in Φ_i^k. Denote by $\widehat{\Phi}_i^0 := \cap_{k=1}^{\infty}(\widehat{\Phi}_i^k)^c$. Thus the selection criterion is the following:[3] for any $i\in\{1,2\}$,

$$\zeta_i := \left\{\widehat{\Phi}_i^0\times\{R\}, \widehat{\Phi}_i^k\times\{W\}, k\geq 1\right\}. \tag{B.6}$$

Lemma B.1. *The selection criterion ζ defined in* (B.6) *is regular and stable.*

Proof. It is straightforward to see that ζ is regular. We will show it is stable below. Essentially, the stability condition says that for any canonical signal, if we modify this

[3] Again, we assign the boundary to the sender-preferred action so as to have a well-defined solution.

signal by pooling any subset of messages (according to action recommendations) for an individual receiver within any partition component, the resulted conjecture under such modification will remain in the same partition component for everyone in all possible cases. We will prove this statement by induction.

First consider when such pooling happens for the subset of messages in $(S^1_{\tau_1} \times \{W\})_{\tau_1 \in \mathbb{T}_1}$ with each subset $S^1_{\tau_1} \subseteq \widehat{\Phi}^1_1$, $\tau_1 \in \mathbb{T}_1$. It is straightforward to check that pooling messages within $(S^1_{\tau_1} \times \{W\})_{\tau_1 \in \mathbb{T}_1}$ for receiver 1 will result in the first-order belief still within $\widehat{\Phi}^1_1$ for both primitive types of receiver 1. For receiver 2, his first-order belief will not be affected by the above modification so if his conjecture was in $\widehat{\Phi}^1_2$, it remains in there and if it stays out previously, then it stays out in the modified signal as well. Consider the effect of such modification to the second-order beliefs for both receivers. Given that receiver 2's first-order belief does not change, the second-order belief of each primitive type of receiver 1 regarding whether it is within or outside Φ^2_1 does not change. For receiver 2, recall that $\widehat{\Phi}^2_2$ is defined as the conjectures induced by the set of belief hierarchies Φ^2_2 defined as follows:

$$\Phi^2_2 := \left\{ t^2_2 \in T^2_2 \mid \begin{array}{l} t^2_2(\{\omega = H\}) + \frac{1}{3} t^2_2(\Omega \times \mathbb{T}_1 \times \Phi^1_1) \geq \frac{2}{3} \\ \operatorname{marg}_{T^1_2} t^2_2 \notin \Phi^1_2 \end{array} \right\}.$$

Given the fact that the above pooling does not change receiver 1's first-order beliefs regarding whether it is within or outside Φ^1_1, hence for any message $m_2 \in \Phi^2_2$, the receiver 2's correct second-order belief under such pooling still remains in Φ^2_2. Similarly, for any $m_2 \notin \Phi^2_2$ (that is, under message m_2, either $\operatorname{marg}_{T^1_2} t^2_2 \in \Phi^1_2$ or $t^2_2(\{\omega = H\}) + \frac{1}{3} t^2_2(\Omega \times \mathbb{T}_1 \times \Phi^1_1) < \frac{2}{3}$ is satisfied), the receiver 2's correct second-order belief under such pooling still satisfy either of the above expressions. So his correct second-order belief under such pooling is still not in Φ^2_2 and the resulted conjecture is still not in $\widehat{\Phi}^2_2$. A similar argument can be applied to that receiver 1's correct third-order belief regarding whether it locates within Φ^3_1. Similarly for the other receiver and the other higher-order beliefs as well. We can further apply similar arguments to any other subsets of $(S_{\tau_i} \times \{a_{\tau_i}\})_{\tau_i \in \mathbb{T}_i}$, $i \in \{1, 2\}$ to conclude all possible cases of pooling within the partition component Φ^1_1 and Φ^1_2.

For any belief order $K \geq 1$, suppose that for any $k \leq K$ and $i \in \{1, 2\}$, if we modify the signal by pooling all messages within a subset of messages in $(S^k_{\tau_i} \times \{W\})_{\tau_i \in \mathbb{T}_i}$ with each subset $S^1_{\tau_i} \subseteq \widehat{\Phi}^k_i$, $\tau_i \in \mathbb{T}_i$, then all receivers' correct conjectures do not change their location in their partition components in the selection criterion.

Now consider the order $K+1$ and a subset of messages in $(S_{\tau_1}^{K+1} \times \{W\})_{\tau_1 \in \mathbb{T}_1}$ with each subset $S_{\tau_1}^{K+1} \subseteq \widehat{\Phi}_1^{K+1}$, $\tau_1 \in \mathbb{T}_1$. Note that $S_{\tau_1}^{K+1} \cap \widehat{\Phi}_1^k = \varnothing$ for any $1 \leq k \leq K$. So by the induction hypothesis, under such pooling, the resulted conjecture still remains outside $\widehat{\Phi}_1^k$ for any $k \geq 1$. Again by the induction hypothesis (that the above pooling does not change receiver i's k-order beliefs regarding whether it is within or outside Φ_1^k for any $k \leq K$), we can further conclude that pooling over messages that previously satisfying $t_1^{K+1}(\{\omega = H\}) + \frac{1}{3} t_1^{K+1}(\Omega \times \mathbb{T}_1 \times \cup_{k=1}^K \Phi_k^1) \geq \frac{2}{3}$ will result in the pooled message still satisfies the same inequality. Thus such pooling does not change the partition component that receiver 1's $(K+1)$-order beliefs previously locates as well. By the induction hypothesis, under such pooling, the receiver 2's correct $(K+1)$-order belief still remains in the same partition component as they previously locate in all possible cases. Similar argument applies to all both receivers' high-order beliefs and the rest possible cases of pooling as well. So we conclude the proof. □

Let us now define the strategic partitions for any receiver primitive type as follows: Recall that we denote the best response given a conjecture ν_i to be $B_i(\nu_i)$ (note that here receivers' primitive types do not affect their utility function and lead to the same best responses for different primitive types).

$$\Big\{ \{\nu_i \in V_i \mid B_i(\nu_i) = \{R, W\}\} \times \{R, W\},$$
$$\{\nu_i \in V_i \mid B_i(\nu_i) = \{W\}\} \times \{W\},$$
$$\{\nu_i \in V_i \mid B_i(\nu_i) = \{R\}\} \times \{R\}\}.$$

By intersecting the above two partitions, we can obtain the ζ-strategic partitions and derive their associated collection of partition menus. The following observation could help us to simplify the messages further by pointing out that if a more pessimistic type of receiver 1 (who receives message l) agrees to wait, the optimistic type (who receives message h) would be willing to wait as well.

Lemma B.2. *Given the selection criterion ζ in* (B.6)*, for any canonical signal π with countably many messages, if it is optimal for receiver 1 with primitive type l to wait conditional on receiving a message m_1, then it is optimal for receiver 1 with primitive type h to wait conditional on receiving the same message as well.*

Proof of Lemma B.2. We will prove this lemma by showing that, for any signal π, if the message m_1 leads to receiver 1's primitive type l to have j-order belief $t_{1,l}^j \in \Phi_1^j$ for any $j \geq 1$, then the corresponding j-order belief of primitive type h also satisfies $t_{1,h}^j \in \Phi_1^j$ under the same message.

We first consider the case when $j = 1$: Recall that both types see the same message m_1. Suppose that $t_1^{l,1} \in \Phi_1^1$, i.e, $\pi(m_1|H)\mu_{1|l}^0 \geq 2 \cdot \pi(m_1|L)(1 - \mu_{1|l}^0)$. Note that $\mu_{1|h}^0 = \frac{6}{13}$ and $\mu_{1|l}^0 = \frac{3}{17}$, we have $\pi(m_1|H)\mu_{1|h}^0 \geq 2 \cdot \pi(m_1|L)(1 - \mu_{1|h}^0)$ and thus $t_1^{h,1} \in \Phi_1^1$ as well.

We now consider the case when $K > 1$: Let M_2 be the whole message space for receiver 2. For any integer $K > 1$, let $M_2^{K-1}(W)$ be the messages for receiver 2 in π such that for each $m_2 \in M_2^{K-1}(W)$, there exists k with $1 \leq k \leq K - 1$ where the k-order belief of receiver 2 given m_2 satisfies $t_2^k \in \Phi_2^k$.

For the given message m_1, suppose that $t_1^{l,K} \in \Phi_1^K$; then we have

$$\left(\sum_{m_2 \in M_2} \pi\left((m_1, m_2)|H\right) + \sum_{m_2 \in M_2^{j-1}(W)} \pi\left((m_1, m_2)|H\right) \right) \mu_{1|l}^0$$
$$\geq \left(\sum_{m_2 \in M_2} \pi\left((m_1, m_2)|L\right) + \sum_{m_2 \in M_2^{j-1}(W)^c} \pi\left((m_1, m_2)|L\right) \right) (1 - \mu_{1|l}^0).$$

By the above and that $\mu_{1|h}^0 \geq \mu_{1|l}^0$, the following holds:

$$\left(\sum_{m_2 \in M_2} \pi\left((m_1, m_2)|H\right) + \sum_{m_2 \in M_2^{j-1}(W)} \pi\left((m_1, m_2)|H\right) \right) \mu_{1|h}^0$$
$$\geq \left(\sum_{m_2 \in M_2} \pi\left((m_1, m_2)|L\right) + \sum_{m_2 \in M_2^{j-1}(W)^c} \pi\left((m_1, m_2)|L\right) \right) (1 - \mu_{1|h}^0).$$

The above inequality implies that for the given message m_1 in π, $t_1^{h,j} \in \Phi_1^j$ holds for receiver 1 with primitive type h. Therefore it is optimal for him to wait as well. $\square$

Based on the above lemma, we can write down the simplified version of the collection of partition menus associated to ζ-strategic partition as follows:

$$\Gamma_1 = \left\{ \begin{bmatrix} h & v & W \\ l & \hat{v} & W \end{bmatrix} \mid \begin{array}{l} v \in \widehat{\Phi}_1^{k_h}, \hat{v} \in \widehat{\Phi}_1^{k_l} \\ k_h, k_l \geq 1 \end{array} \right\} \bigcup \left\{ \begin{bmatrix} h & v & W \\ l & \hat{v} & R \end{bmatrix} \mid \begin{array}{l} v \in \widehat{\Phi}_1^{k_h}, \hat{v} \in \widehat{\Phi}_1^{0} \\ k_h \geq 1 \end{array} \right\}$$
$$\bigcup \left\{ \begin{bmatrix} h & v & R \\ l & \hat{v} & R \end{bmatrix} \mid v \in \widehat{\Phi}_1^{0}, \hat{v} \in \widehat{\Phi}_1^{0} \right\};$$
$$\Gamma_2 = \left\{ \widehat{\Phi}_2^0 \times \{R\}, \widehat{\Phi}_2^{k'} \times \{W\}, k' \geq 1 \right\}.$$

For simplicity, denote by $C^1_{k_l,k_h}$ the receiver 1's partition menu with $v_{k_h} \in \widehat{\Phi}_1^{k_h}$ and $v_{k_l} \in \widehat{\Phi}_1^{k_l}$ and denote by C^2_k the receiver 2's partition menu with $v_k \in \widehat{\Phi}_2^k$. Given that there are countablely many elements in ζ-strategic partition, to make our problem computer solvable, we will fix a finite integer K and only consider the order of beliefs up to K.

By Lemma B.1 that the selection criterion is regular and stable, we can apply Proposition III.3 to conclude that it is without loss of generality to focus on canonical signals with the message space a sample collection. By Lemma B.2, we can further focus on the following subset of sample collection: Let $M_1 := \{m_{k,\hat{k}}, k, \hat{k} \geq 1, m_{0,k'}, k' \geq 0\}$ such that $m_{k,\hat{k}} \in C^1_{k,\hat{k}}$ for $k, \hat{k} \geq 0$ and let $M_2 := \{m_{k'}, k' \geq 0\}$ with $m_{k'} \in C^2_{k'}$ for any $k' \geq 0$. From now on we will focus on canonical signals with the message space $M_1 \times M_2$.

Given any signal π with message space $M_1 \times M_2$, for any receiver i and any message $m_i \in M_i$, denote by $v_i^{\tau_i}(\cdot|m_i, \pi)$ the associated conjecture of receiver 1 with type $\tau_i \in \mathbb{T}_i$ by receiving m_i. For convenience, define $E_{\overline{\pi}}[V|(m^1_{k,\hat{k}}, m^2_{k'}), \omega]$ as the policymaker's expected value conditional on the underlying state ω and realized message $(m^1_{k,\hat{k}}, m^2_{k'})$ where

$$E_{\overline{\pi}}[V|(m^1_{k,\hat{k}}, m^2_{k'}), \omega] = \begin{cases} 2, & 1 \leq k, \hat{k}, k' \leq K \\ \overline{\pi}(h|\omega)2 + \overline{\pi}(l|\omega) & k = 0, 1 \leq \hat{k} \leq K, 1 \leq k' \\ \overline{\pi}(h|\omega) & k = 0, 1 \leq \hat{k} \leq K, k' = 0 \\ 0 & k = \hat{k} = k' = 0 \end{cases}.$$

The following linear programming with the constraint of consistency and regularity yields an optimal K-order signal.

$$\max_{\pi} \sum_{\omega \in \Omega} \mu_0^0(\omega) \left(\sum_{(m^1_{k,\hat{k}}, m^2_{k'}) \in M_1 \times M_2} \pi(m^1_{k,\hat{k}}, m^2_{k'} | \omega) \cdot E_{\overline{\pi}}[V | (m^1_{k,\hat{k}}, m^2_{k'}), \omega] \right)$$

subject to

$$\begin{cases} 0 \leq \pi(m^1_{k,\hat{k}}, m^2_{k'} | \omega) \leq 1; m^1_{k,\hat{k}} \in M_1, m^2_{k'} \in M_2, \omega \in \Omega; \\ \sum_{m^1_{k,\hat{k}} \in M_1, m^2_{k'} \in M_2} \pi((m^1_{k,\hat{k}}, m^2_{k'}) | \omega) = 1, \omega \in \Omega; \\ \begin{bmatrix} h & v_1^h(\cdot | m^1_{k,\hat{k}}, \pi) & W \\ l & v_1^l(\cdot | m^1_{k,\hat{k}}, \pi) & W \end{bmatrix} \in C^1_{k,\hat{k}}, \begin{bmatrix} h & v_1^h(\cdot | m^1_{0,k'}, \pi) & W \\ l & v_1^l(\cdot | m^1_{0,k'}, \pi) & R \end{bmatrix} \in C^1_{0,k'} \\ \begin{bmatrix} h & v_1^h(\cdot | m^1_{0,0}, \pi) & R \\ l & v_1^l(\cdot | m^1_{0,0}, \pi) & R \end{bmatrix} \in C^1_{0,0}, \ \forall 1 \leq k', \hat{k}, k \leq K \\ (v_2(\cdot | m^2_k, \pi), W) \in C^2_k, \ \forall 1 \leq k \leq K \\ (v_2(\cdot | m^2_0, \pi), R) \in C^2_0 \end{cases}.$$

APPENDIX C

Appendix for Chapter IV

C.1 Proof of Theorem IV.1

Before the main proof, we first introduce a technical tool from Mertens (2003). A measurable correspondence $\mathbf{N}$ from $E \times \Omega$ to $\mathbb{R}^l$ is $\mathbf{P}$-integrable if, for any measurable selection $\mathbf{f}$ of $\mathbf{N}$ (denoted as $f \in N$), $\mathbf{f}(\omega, e)$ is $\mathbf{P}(\mathrm{d}\omega|e)$-integrable for any $e \in E$.

Lemma C.1. *Let $(\Omega, \mathcal{A})$ and $(E, \mathscr{E})$ be measurable spaces with $\mathcal{A}$ separable. Suppose that $\mathbf{P}(\mathrm{d}\omega|e)$ is a $\mathbb{R}^k$-valued bounded kernel.*[1] *Let $\mathbf{N}$ be a $\mathbf{P}$-integrable measurable correspondence from $(E \times \Omega, \mathscr{E} \otimes \mathcal{A})$ to $K^*(\mathbb{R}^l)$. Define $\int \mathbf{N}\, \mathrm{d}\mathbf{P}$ as the map from E to the subsets of $\mathbb{R}^{l \cdot k}$ where $\int \mathbf{N}\, \mathrm{d}\mathbf{P}(e) := \left\{ \int \mathbf{f}(\omega, e)\mathbf{P}(\mathrm{d}\omega|e) \middle| \mathbf{f} \in \mathbf{N} \right\}$.*

(1) Fix an arbitrary element $e \in E$. Let $\mathbf{f}$ be a measurable selection of the convex hull of $\mathbf{N}(e, \cdot)$ and $\mathbf{f} \in \mathbf{N}(e, \cdot)$ on each atom of $\mathbf{P}(\cdot|e)$. For any $\mathbb{R}^l$-valued bounded measurable function $\mathbf{u}$, then $\int \mathbf{u} \cdot \mathbf{f}\ \mathbf{P}(\mathrm{d}\omega|e) \in \left(\int \mathbf{u} \cdot \mathbf{N}\, \mathrm{d}\mathbf{P} \right)(e)$.

(2) Fix an arbitrary $\mathbb{R}^{l \cdot M}$-valued function $\mathbf{g}^M := (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_M)$ such that each $\mathbf{g}_m$, $1 \le m \le M$, is a $\mathbb{R}^l$-valued $\mathscr{E} \otimes \mathcal{A}$-measurable bounded function. Let $\int \mathbf{g}^M \odot \mathbf{N}\, \mathrm{d}\mathbf{P}$ be a mapping from E to the subsets of $\mathbb{R}^{M \cdot k}$ such that

$$\int \mathbf{g}^M \odot \mathbf{N}\, \mathrm{d}\mathbf{P} : e \to \left\{ \int (\mathbf{g}_1 \cdot \mathbf{f}'(\omega, e), ..., \mathbf{g}_M \cdot \mathbf{f}'(\omega, e))\mathbf{P}(\mathrm{d}\omega|e) \middle| \mathbf{f}' \in \mathbf{N} \right\}.$$

[1] A $\mathbb{R}^k$-valued *kernel* $\mathbf{P}(\cdot|\cdot)$ is a map from $E \times \mathcal{A} \to \mathbb{R}^k$ such that (i) for any $e \in E$, $\mathbf{P}(\cdot|e)$ is a $\mathbb{R}^k$-valued measure on $(\Omega, \mathcal{A})$; (ii) $\mathbf{P}(A|\cdot)$ is $\mathscr{E}$-measurable, $\forall A \in \mathcal{A}$. A kernel $\mathbf{P}(\cdot|\cdot)$ is *bounded* if there exists a constant C such that $\|\mathbf{P}(A|e)\| \le C$ for all $A \in \mathcal{A}$ and $e \in E$.

Denote by $\mathbf{F}^v \subseteq E \times \mathbb{R}^{M\cdot k}$ its graph with $\mathcal{F}^v$ the corresponding σ-algebra. Then

(i) $(\int \mathbf{g}^M \odot \mathbf{N}\,\mathrm{d}\mathbf{P})$ *is an $\mathscr{E}$-measurable mapping to $K^*(\mathbb{R}^{M\cdot k})$, and $\mathbf{F}^v$ is a measurable set in $E \times \mathbb{R}^{M\cdot k}$;*

(ii) there exists a measurable, $\mathbb{R}^l$-valued function $\mathbf{h}$ on $(\mathbf{F}^v \times \Omega, \mathcal{F}^v \otimes \mathcal{A})$ such that for any $\mathbf{x} \in \mathbf{F}^v \subseteq \mathbb{R}^{M\cdot k}$, the following assertions hold: (i) $\mathbf{h}(e,\mathbf{x},\omega) \in \mathbf{N}(e,\omega)$; and (ii) $\mathbf{x} = \int \mathbf{g}^M(e,\omega) \odot \mathbf{h}(e,\mathbf{x},\omega)\mathbf{P}(\mathrm{d}\omega|e)$, where $\mathbf{g}^M(e,\omega) \odot \mathbf{h}(e,\mathbf{x},\omega) := (\mathbf{g}_1(e,\omega)\cdot\mathbf{h}(e,\mathbf{x},\omega), ..., \mathbf{g}_M(e,\omega)\cdot\mathbf{h}(e,\mathbf{x},\omega))$.

Proof of Lemma C.1. (1) Let $\tilde{\mathbf{P}}(\cdot|\cdot)$ be a $\mathbb{R}^{l\cdot k}$-valued map where

$$\tilde{\mathbf{P}}(\mathrm{d}\omega|e) := (u_n \mathbf{P}(\mathrm{d}\omega|e))_{1\le n\le l}.$$

Given the boundedness of $\mathbf{P}$ and $\mathbf{u}$, $\tilde{\mathbf{P}}$ is also a bounded kernel. Moreover, for the fixed element $e \in E$, an atom of $\tilde{\mathbf{P}}(\cdot|e)$ must be an atom of $\mathbf{P}(\cdot|e)$ as well. For any measurable selection $\mathbf{f} := (f_1, ..., f_l)$ from the convex hull of $\mathbf{N}(e,\cdot)$ such that $\mathbf{f} \in \mathbf{N}(e,\cdot)$ on the atoms of $\mathbf{P}(\mathrm{d}\omega|e)$, then $\mathbf{f}$ is also a measurable selection of the convex hull of $\mathbf{N}(e,\cdot)$ such that $\mathbf{f} \in \mathbf{N}(e,\cdot)$ on the atoms of $\tilde{\mathbf{P}}(\mathrm{d}\omega|e)$. Let $\int \mathbf{N}\,\mathrm{d}\tilde{\mathbf{P}}(e) := \left\{\int \mathbf{f}'(\omega,e)\tilde{\mathbf{P}}(\mathrm{d}\omega|e)\middle|\mathbf{f}' \in \mathbf{N}\right\}$. Hence by Theorem (2), Mertens (2003), $\int \mathbf{f}\,\mathrm{d}\tilde{\mathbf{P}}(e) \in \int \mathbf{N}\,\mathrm{d}\tilde{\mathbf{P}}(e)$. Given that such inclusion holds for each term in Kronecker product, which implies $\int \mathbf{u}\cdot\mathbf{f}(\omega)\mathbf{P}(\mathrm{d}\omega|e) \in (\int \mathbf{u}\cdot\mathbf{N}\,\mathrm{d}\mathbf{P})(e)$.

(2) Fix an arbitrary M-component vector function $\mathbf{g}^M := (\mathbf{g}_1, ..., \mathbf{g}_M)$ which satisfies that each $\mathbf{g}_m$ is a $\mathbb{R}^l$-valued $\mathscr{E} \otimes \mathcal{A}$-measurable bounded function. Define a $\mathbb{R}^M$-valued correspondence $\hat{\mathbf{N}}$ from $(E \times \Omega, \mathscr{E} \otimes \mathcal{A})$ to $K^*(\mathbb{R}^M)$ such that for each $(e,\omega) \in E \times \Omega$, $\hat{\mathbf{N}}(e,\omega) := \{(\mathbf{g}_1(e,\omega)\cdot\mathbf{a}, ..., \mathbf{g}_M(e,\omega)\cdot\mathbf{a})|\mathbf{a} \in \mathbf{N}(e,\omega)\}$. By given conditions, the above correspondence $\hat{\mathbf{N}}$ is $\mathbf{P}$-integrable and $\hat{\mathbf{N}}$ is compact-valued.

(i) By Theorem (1) (a) and (b), Mertens (2003), $(\int \hat{\mathbf{N}}\,\mathrm{d}\mathbf{P})$ is an $\mathscr{E}$-measurable mapping to $K^*(\mathbb{R}^{M\cdot k})$, and its graph is a measurable set in $E \times \mathbb{R}^{M\cdot k}$.

(ii) Denote by $\hat{\mathbf{F}}$ the graph of $\int \hat{\mathbf{N}}\,\mathrm{d}\mathbf{P}$ with its sub σ-algebra $\hat{\mathcal{F}} \subseteq \mathscr{E} \otimes \mathcal{B}(\mathbb{R}^{M\cdot k})$. By Theorem (3), Mertens (2003), there exists a measurable, $\mathbb{R}^M$-valued function $\hat{\mathbf{f}}$ on $(\hat{\mathbf{F}} \times$

$\Omega, \hat{\mathcal{F}} \otimes \mathcal{A})$ such that $\hat{\mathbf{f}}(e, \hat{\mathbf{x}}, \omega) \in \hat{\mathbf{N}}(e, \omega)$ and

$$\hat{\mathbf{x}} = \int \hat{\mathbf{f}}(e, \hat{\mathbf{x}}, \omega)\mathbf{P}(\mathrm{d}\omega|e). \tag{C.1}$$

For each element $(e, \hat{\mathbf{x}}, \omega)$, by the definition of $\hat{\mathbf{N}}(e, \omega)$ and that $\hat{\mathbf{f}}(e, \hat{\mathbf{x}}, \omega) \in \hat{\mathbf{N}}(e, \omega)$, there exists an element $\hat{\mathbf{a}}'_{e,\hat{\mathbf{x}},\omega} \in \hat{\mathbf{N}}(e, \omega)$ such that $\hat{\mathbf{f}}(e, \hat{\mathbf{x}}, \omega) = (\mathbf{g}_1 \cdot \hat{\mathbf{a}}'_{e,\hat{\mathbf{x}},\omega}, ..., \mathbf{g}_M \cdot \hat{\mathbf{a}}'_{e,\hat{\mathbf{x}},\omega})$. Based on the above $\hat{\mathbf{f}}$, define a correspondence $\mathbf{H} : E \times \hat{\mathbf{F}} \times \Omega$ to $\mathbb{R}^l$ such that for any $(e, \hat{\mathbf{x}}, \omega)$,

$$\mathbf{H}(e, \hat{\mathbf{x}}, \omega) := \{\hat{\mathbf{a}}''_{e,\hat{\mathbf{x}},\omega} \in \mathbf{N}(e, \omega) | (\mathbf{g}_n \cdot \hat{\mathbf{a}}''_{e,\hat{\mathbf{x}},\omega})_{1 \leq k \leq M} = \hat{\mathbf{f}}(e, \hat{\mathbf{x}}, \omega)\}.$$

By such a definition, $\mathbf{H}(e, \hat{\mathbf{x}}, \omega)$ is measurable, nonempty-valued and compact-valued (for measurability see Corollary 18.8 in Aliprantis and Border 2006). Therefore, Kuratowski–Ryll-Nardzewski Selection Theorem implies that we could find a pointwise measurable selector $\mathbf{h}$ of $\mathbf{H}$. Therefore, for each $(e, \hat{\mathbf{x}}, \omega)$, $\mathbf{h}(e, \hat{\mathbf{x}}, \omega) \in \mathbf{N}(e, \omega)$, and by (C.1) and that $\mathbf{h} \in \mathbf{H}$ pointwisely, we have $\hat{\mathbf{x}} = \int \mathbf{g}^M(e, \omega) \odot \mathbf{h}(e, \hat{\mathbf{x}}, \omega)\mathbf{P}(\mathrm{d}\omega|e)$. □

We introduce a more general environment under which our main results still hold, of which the setting in the main text (with independent information) is a special case.

Definition C.1. The setting is of *interdependent information with players holding subjective priors* ("IWSP"), if the following hold: let $(T \times \Omega, \mathcal{T} \otimes \mathcal{F}, \mu^T \times \mu^\Omega)$ be the product probability space of state and types. Assume that μ^Ω is atomless. Denote by $\tilde{\mu}_i \in \Delta(T \times \Omega)$ the subjective belief of each player i, $i \in \mathcal{I} \cup \{0\}$. Assume that $\tilde{\mu}_i$ is absolutely continuous with respect to the product probability measure $\mu^T \times \mu^\Omega$ with the density function $\ell_i(t, \omega)$, i.e., $\mathrm{d}\tilde{\mu}_i = \ell_i(t, \omega)\,\mathrm{d}(\mu^T \times \mu^\Omega)$. For each $i \in \mathcal{I} \cup \{0\}$, the density function $\ell_i(t, \omega)$ is assumed to be bounded and measurable.

We will prove the conclusion of Theorem IV.1 in the IWSP setting, which we numbered as Theorem IV.1′. We say the density function of a player i's prior is *separable* if $\ell_i(t, \omega)$ can be written into the following form: $\ell_i(t, \omega) = \sum_{n=1}^{N} \ell_i^n(\omega) \cdot \tilde{\ell}_i^n(t)$, where N is a positive integer; ℓ_i^n and $\tilde{\ell}_i^n$, $1 \leq n \leq N$, are bounded measurable functions. In such a general setting, this separability condition of density functions is required.

Theorem IV.1′. *Suppose that the setting is of IWSP and the density function of each player's prior is separable, then the statement of Theorem IV.1 holds.*

Proof of Theorem IV.1'. For convenience, let N be the largest index such that for each player $i \in \mathcal{I} \cup \{0\}$, $\ell_i(t,\omega) := \sum_{\tilde{n}=1}^{N} \ell_i^{\tilde{n}}(\omega) \cdot \tilde{\ell}_i^{\tilde{n}}(t)$, and $u_i(t,\omega,a) := \sum_{n=1}^{N} f_{i,n}(\omega,a) \cdot g_{i,n}(t,a)$ with bounded function $f_{i,n}$, $g_{i,n}$, $\ell_i^{\tilde{n}}$ and $\tilde{\ell}_i^{\tilde{n}}$ for $1 \leq n \leq N$.

Let $P(\cdot|\cdot)$ be a stochastic kernel from $\widehat{T}$ to $\Delta(\Omega)$ such that $P(\cdot|\cdot) \equiv \mu^{\Omega}(\cdot)$. For convenience, we label the elements in $\hat{A}$ and A as $\hat{A} := \{\hat{a}^1, \hat{a}^2, ..., \hat{a}^{|\hat{A}|}\}$ and $A := \{a^1, ..., a^{|A|}\}$, respectively. For any m with $1 \leq m \leq |\hat{A}|$, let $\mathbf{e}_m \in \mathbb{R}^{|\hat{A}|}$ be a coordinate vector such that all components in $\mathbf{e}_m$ are zero except that its m-th component is 1. Define a $\mathbb{R}^{|\hat{A}|}$-valued $\widehat{\mathcal{T}} \otimes \mathcal{F}$-measurable constant correspondence $\tilde{\mathbf{N}}$ on $\widehat{T} \times \Omega$ such that $\tilde{\mathbf{N}}(\cdot,\cdot) \equiv \left\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{|\hat{A}|}\right\}$.

We will now construct a vector-valued mapping $\mathbf{F}^{\mathcal{N}} := (\mathbf{F}_1, \mathbf{F}_2, ..., \mathbf{F}_{\mathcal{N}})$ with each $\mathbf{F}_k$ a $\mathbb{R}^{|\hat{A}|}$-valued mapping and the largest index $\mathcal{N} := (|A|N+1) \cdot |\hat{A}|N(|\mathcal{I}|+1)$. Before the explicit construction, we define the following two sets for indexing purposes: let S_1 and S_2 be finite sets of integer-valued vectors, where

$$S_1 := \left\{(\hat{j}, j, i, n, \tilde{n}) \in \mathbb{N}^5 \middle| 1 \leq \hat{j} \leq |\hat{A}|, 1 \leq j \leq |A|, 0 \leq i \leq |\mathcal{I}|, 1 \leq n, \tilde{n} \leq N\right\};$$
$$S_2 := \left\{(\hat{j}, i, \tilde{n}) \in \mathbb{N}^3 \middle| 1 \leq \hat{j} \leq |\hat{A}|, 0 \leq i \leq |\mathcal{I}|, 1 \leq \tilde{n} \leq N\right\}.$$

We label the elements in $S_{j'}$ as $S_{j'} := \{s^{j',1}, s^{j',2}, ..., s^{j',|S_{j'}|}\}$ for $j' = 1, 2$, with cardinalities $|S_1| = |A||\hat{A}|N^2(|\mathcal{I}|+1)$ and $|S_2| = |\hat{A}|N(|\mathcal{I}|+1)$. We are now ready to define each $\mathbf{F}_k : \widehat{T} \times \Omega \to \mathbb{R}^{|\hat{A}|}$ for any $1 \leq k \leq \mathcal{N}$. For any integer k,

(1) if the integer k satisfies that $1 \leq k \leq |S_1|$, then we define $\mathbf{F}_k$ based on the vector $s^{1,k} := (\hat{j}_k, j_k, i_k, n_k, \tilde{n}_k)$ as follows:

$$\mathbf{F}_k(\hat{t},\omega) := f_{i_k,n_k}(\omega, a_{j_k}) \ell_{i_k}^{\tilde{n}_k}(\omega) \cdot \mathbf{e}_{\hat{j}_k} \text{ for any } (\hat{t},\omega). \tag{C.2}$$

(2) if the integer k satisfies that $|S_1|+1 \leq k \leq |S_1|+|S_2|$, then we define $\mathbf{F}_k$ based on the vector $s^{2,\hat{k}} := (\hat{j}_{\hat{k}}, i_{\hat{k}}, \tilde{n}_{\hat{k}})$ with $\hat{k} := k - |S_1|$ as follows:

$$\mathbf{F}_k(\hat{t},\omega) := \ell_{i_{\hat{k}}}^{\tilde{n}_{\hat{k}}}(\omega) \cdot \mathbf{e}_{\hat{j}_{\hat{k}}} \text{ for any } (\hat{t},\omega). \tag{C.3}$$

Based on the above $\mathbf{F}^{\mathcal{N}}$, we further define the following vector-valued integral $\int \mathbf{F}^{\mathcal{N}} \odot \tilde{\mathbf{N}} \,\mathrm{d}P := (\int \mathbf{F}_k \cdot \tilde{\mathbf{N}} \,\mathrm{d}P)_{1 \leq k \leq \mathcal{N}}$ such that each $\int \mathbf{F}_k \cdot \tilde{\mathbf{N}} \,\mathrm{d}P : \widehat{T} \twoheadrightarrow \mathbb{R}$ is a measurable correspondence such that

$$\int \mathbf{F}_k \cdot \tilde{\mathbf{N}} \,\mathrm{d}P : \widehat{T} \twoheadrightarrow \Big\{ \int_\Omega \mathbf{F}_k(\hat{t},\omega) \cdot \mathbf{h}(\hat{t},\omega) P(\mathrm{d}\omega|\hat{t}) \Big| \\ \mathbf{h} \text{ is an } \widehat{\mathcal{T}} \otimes \mathcal{F}\text{-measurable selection from } \tilde{\mathbf{N}} \Big\}.$$

Denote its graph with the corresponding σ-algebra as $(\mathbf{G}^u, \mathcal{G}^u)$. Since any persuasion mechanism π is a $\widehat{\mathcal{T}} \otimes \mathcal{F}$-measurable function from $\widehat{T} \times \Omega \to [0,1]^{|\hat{A}|}$ such that $\sum_{\hat{j}=1}^{|\hat{A}|} \pi(\hat{a}^{\hat{j}}|\hat{t},\omega) \equiv 1$ for any $(\hat{t},\omega)$, by definition $\pi(\cdot|\hat{t},\omega)$ is in the convex hull of $\tilde{\mathbf{N}}(\hat{t},\omega)$ for any $(\hat{t},\omega)$. By (1) of Lemma C.1 and given that $P(\cdot|\cdot) \equiv \mu^\Omega(\cdot)$ is atomless, we can conclude that, fix any $\hat{t}$,

$$\int_\Omega \mathbf{F}_k(\hat{t},\omega) \cdot \pi(\cdot|\hat{t},\omega) P(\mathrm{d}\omega|\hat{t}) \in \int \mathbf{F}_k \cdot \tilde{\mathbf{N}} \,\mathrm{d}P(\hat{t}) \text{ for each } 1 \le k \le \mathcal{N}.$$

By (2i) of Lemma C.1, we could further conclude that $\int_\Omega \mathbf{F}_k(\hat{t},\omega) \cdot \pi(\cdot|\hat{t},\omega) P(\mathrm{d}\omega|\hat{t})$ is $\widehat{\mathcal{T}}$-measurable. Moreover, by (2ii) of Lemma C.1, there exists a measurable, $\mathbb{R}^{|\hat{A}|}$-valued function $\mathbf{h}$ on $(\mathbf{G}^u \times \Omega, \mathcal{G}^u \otimes \mathcal{F})$ such that $\mathbf{x} \in \mathbf{G}^u$, $\mathbf{h}(\hat{t},\mathbf{x},\omega) \in \tilde{\mathbf{N}}(\hat{t},\omega)$ and

$$\mathbf{x} = \int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega) \odot \mathbf{h}(\hat{t},\mathbf{x},\omega) P(\mathrm{d}\omega|\hat{t}) \text{ for any } \hat{t}. \tag{C.4}$$

By substituting $\mathbf{x}$ with the expression $\int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega) \odot \pi(\cdot|\hat{t},\omega) P(\mathrm{d}\omega|\hat{t})$ in the above equation, for any $\hat{t}$, we have

$$\begin{aligned} &\int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega) \odot \pi(\cdot|\hat{t},\omega) P(\mathrm{d}\omega|\hat{t}) \\ =&\int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega) \odot \mathbf{h}\Big(\hat{t}, \int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega) \odot \pi(\cdot|\hat{t},\omega) P(\mathrm{d}\omega|\hat{t}), \omega\Big) P(\mathrm{d}\omega|\hat{t}). \end{aligned} \tag{C.5}$$

Let $\overline{\pi}$ be a function from $\widehat{T} \times \Omega \to \mathbb{R}^{|\hat{A}|}$ such that

$$\overline{\pi}(\cdot|\hat{t},\omega) := \mathbf{h}\left(\hat{t}, \int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega) \odot \pi(\cdot|\hat{t},\omega) P(\mathrm{d}\omega|\hat{t}), \omega\right) \text{ for any } (\hat{t},\omega). \tag{C.6}$$

That $\mathbf{h}(\hat{t},\mathbf{x},\omega) \in \tilde{\mathbf{N}}(\hat{t},\omega)$ pointwisely implies that $\overline{\pi}(\cdot|\hat{t},\omega) \in \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{|\hat{A}|}\}$ for all $(\hat{t},\omega)$, and therefore $\overline{\pi}$ is a derandomized persuasion mechanism. By the measurability of $\mathbf{h}$ and $\int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega) \odot \pi(\cdot|\hat{t},\omega) P(\mathrm{d}\omega|\hat{t})$, the measurability of $\overline{\pi}$ is immediate. Moreover, by

Equation (C.5) and (C.6), we could conclude that for any $\hat{t}$,

$$\int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega)\odot\pi(\cdot|\hat{t},\omega)P(\mathrm{d}\omega|\hat{t}) = \int_\Omega \mathbf{F}^{\mathcal{N}}(\hat{t},\omega)\odot\overline{\pi}(\cdot|\hat{t},\omega)P(\mathrm{d}\omega|\hat{t}). \tag{C.7}$$

By replacing each $\mathbf{F}_k$ in Equation (C.7) with its explicit form for $1 \le k \le \mathcal{N}$, we have the following equations: for any $i \in \mathcal{I}\cup\{0\}$, any $\hat{t}\in\widehat{T}$, $1 \le j \le |A|, 1 \le \hat{j} \le |\hat{A}|$, and $1 \le n, \tilde{n} \le N$, we have

$$\int_\Omega f_{i,n}(\omega,a^j)\,\ell_i^{\tilde{n}}(\omega)\,\pi(\hat{a}^{\hat{j}}|\hat{t},\omega)\mu^\Omega(\mathrm{d}\omega) = \int_\Omega f_{i,n}(\omega,a^j)\,\ell_i^{\tilde{n}}(\omega)\,\overline{\pi}(\hat{a}^{\hat{j}}|\hat{t},\omega)\mu^\Omega(\mathrm{d}\omega) \tag{C.8}$$

$$\int_\Omega \ell_i^{\tilde{n}}(\omega)\,\pi(\hat{a}^{\hat{j}}|\hat{t},\omega)\mu^\Omega(\mathrm{d}\omega) = \int_\Omega \ell_i^{\tilde{n}}(\omega)\,\overline{\pi}(\hat{a}^{\hat{j}}|\hat{t},\omega)\mu^\Omega(\mathrm{d}\omega). \tag{C.9}$$

By Equation (C.8), for $i \in \mathcal{I}\cup\{0\}$, any $\hat{t}\in\widehat{T}$, $1 \le j \le |A|, 1 \le \hat{j} \le |\hat{A}|$, we have

$$\begin{aligned}
&\int_\Omega u_i(t,\omega,a^j)\pi(\hat{a}^{\hat{j}}|\hat{t},\omega)\,\ell_i(t,\omega)\mu^\Omega(\mathrm{d}\omega)r(t,\mathrm{d}\hat{t}) \\
=&\sum_{\tilde{n}=1}^N\sum_{n=1}^N \tilde{\ell}_i^{\tilde{n}}(t)\,g_{i,n}(t,a^j)\int_\Omega f_{i,n}(\omega,a^j)\,\pi(\hat{a}^{\hat{j}}|\hat{t},\omega)\,\ell_i^{\tilde{n}}(\omega)\mu^\Omega(\mathrm{d}\omega) \\
=&\sum_{\tilde{n}=1}^N\sum_{n=1}^N \tilde{\ell}_i^{\tilde{n}}(t)\,g_{i,n}(t,a^j)\int_\Omega f_{i,n}(\omega,a^j)\,\overline{\pi}(\hat{a}^{\hat{j}}|\hat{t},\omega)\,\ell_i^{\tilde{n}}(\omega)\mu^\Omega(\mathrm{d}\omega) \\
=&\int_\Omega u_i(t,\omega,a^j)\pi(\hat{a}^{\hat{j}}|\hat{t},\omega)\,\ell_i(t,\omega)\mu^\Omega(\mathrm{d}\omega)r(t,\mathrm{d}\hat{t}).
\end{aligned} \tag{C.10}$$

Similarly, by Equation (C.9), for $i \in \mathcal{I}\cup\{0\}$, any $\hat{t}\in\widehat{T}$ and $1 \le \hat{j} \le |\hat{A}|$, we have

$$\int_\Omega \ell_i(t,\omega)\,\pi(\hat{a}^{\hat{j}}|\hat{t},\omega)\mu^\Omega(\mathrm{d}\omega) = \int_\Omega \ell_i(t,\omega)\,\overline{\pi}(\hat{a}^{\hat{j}}|\hat{t},\omega)\mu^\Omega(\mathrm{d}\omega).$$

Our conclusion follows directly from Lemma C.2 presented below. □

Lemma C.2. *In the setting of IWSP and players have separable densities, suppose that for two persuasion mechanisms π and $\overline{\pi}$, the following hold for any $i \in \mathcal{I}\cup\{0\}$, any $t \in T$,*

$\hat{t} \in \widehat{T}$, $\hat{a} \in \hat{A}$, and $a \in A$:

$$\int_{\Omega} u_i(t,\omega,a)\,\ell_i(t,\omega)\,\pi(\hat{a}|\hat{t},\omega)\mu^{\Omega}(\mathrm{d}\omega) = \int_{\Omega} u_i(t,\omega,a)\,\ell_i(t,\omega)\,\overline{\pi}(\hat{a}|\hat{t},\omega)\mu^{\Omega}(\mathrm{d}\omega) \quad \text{(C.11)}$$

$$\int_{\Omega} \ell_i(t,\omega)\,\pi(\hat{a}|\hat{t},\omega)\mu^{\Omega}(\mathrm{d}\omega) = \int_{\Omega} \ell_i(t,\omega)\,\overline{\pi}(\hat{a}|\hat{t},\omega)\mu^{\Omega}(\mathrm{d}\omega). \quad \text{(C.12)}$$

Then (i) π is effectively equivalent to $\overline{\pi}$; (ii) any BNE under π is still a BNE under $\overline{\pi}$, and vice versa; (iii) moreover, for any BNE (r^*, σ^*) under π, then the expected equilibrium payoff of each type of each receiver in (r^*, σ^*) is the same under π and under $\overline{\pi}$, and the expected equilibrium payoff of the sender in (r^*, σ^*) is the same under π and under $\overline{\pi}$.

Proof. Fix an arbitrary strategy profile $(\tilde{\sigma}, \tilde{r})$. For any $i \in \mathcal{I}$ and the type profile $\tilde{t}$, any signal realization profile $\hat{a}'$ and the action profile a', let $\tilde{\sigma}(\tilde{t}, \tilde{r}, \hat{a}')(a')$ be the probability of playing a' given the above $(\tilde{\sigma}, \tilde{r}, \tilde{t})$, i.e. $\tilde{\sigma}(\tilde{t}, \tilde{r}, \hat{a}')(a') := \prod_{j=1}^{|\mathcal{I}|} \int_{\widehat{T}_j} \tilde{\sigma}_j(\tilde{t}_j, \hat{t}_j, \hat{a}'_j)(a'_j)\tilde{r}_j(\tilde{t}_j, \mathrm{d}\hat{t}_j)$. By (C.11), the following holds

$$\begin{aligned}
&\int_{\widehat{T}}\int_{\Omega} \left(\tilde{\sigma}(\tilde{t}, \tilde{r}, \hat{a}')(a')\right) u_i(\tilde{t},\omega,a')\pi(\hat{a}'|\hat{t},\omega)\,\ell_i(\tilde{t},\omega)\mu^{\Omega}(\mathrm{d}\omega)\tilde{r}(\tilde{t},\mathrm{d}\hat{t}) \\
=&\int_{\widehat{T}}\int_{\Omega} \left(\tilde{\sigma}(\tilde{t}, \tilde{r}, \hat{a}')(a')\right)\; u_i(\tilde{t},\omega,a')\,\overline{\pi}(\hat{a}'|\hat{t},\omega)\,\ell_i(\tilde{t},\omega)\mu^{\Omega}(\mathrm{d}\omega)\tilde{r}(\tilde{t},\mathrm{d}\hat{t}),
\end{aligned} \quad \text{(C.13)}$$

Given the above $(\tilde{r}, \tilde{t})$, $\hat{a}' \in \hat{A}$ and $i \in \mathcal{I}$, by taking the expectations over $\widehat{T}$ in (C.12) according to $\tilde{r}(\tilde{t}, \mathrm{d}\hat{t})$, we have

$$\int_{\Omega}\int_{\widehat{T}} \pi(\hat{a}'|\hat{t},\omega)\,\ell_i(\tilde{t},\omega)\tilde{r}(\tilde{t},\mathrm{d}\hat{t})\mu^{\Omega}(\mathrm{d}\omega) = \int_{\Omega}\int_{\widehat{T}} \overline{\pi}(\hat{a}'|\hat{t},\omega)\,\ell_i(\tilde{t},\omega)\tilde{r}(\tilde{t},\mathrm{d}\hat{t})\mu^{\Omega}(\mathrm{d}\omega). \quad \text{(C.14)}$$

Recall that $E^{\pi}[u_i(\tilde{\sigma}_i, \tilde{\sigma}_{-i})|\tilde{r}_i, \tilde{r}_{-i}, \tilde{t}]$ (resp. $E^{\overline{\pi}}[u_i(\tilde{\sigma}_i, \tilde{\sigma}_{-i})|\tilde{r}_i, \tilde{r}_{-i}, \tilde{t}]$) is receiver i's conditional expected utility on type profile $\tilde{t}$ in persuasion mechanism π (resp. $\overline{\pi}$). Label the elements in $\hat{A}$ and A as $\hat{A} := \{\hat{a}^1, ..., \hat{a}^{|\hat{A}|}\}$ and $A := \{a^1, ..., a^{|A|}\}$. By Equation (C.14) and (C.13), given the above $(\tilde{\sigma}, \tilde{r})$, for any $i \in \mathcal{I}$ and any type profile $\tilde{t}$, we have

$$
\begin{aligned}
&E^{\pi}[u_i(\tilde{\sigma}_i, \tilde{\sigma}_{-i})|\tilde{r}_i, \tilde{r}_{-i}, \tilde{t}] \\
=&\sum_{\hat{j}=1}^{|\hat{A}|}\sum_{j=1}^{|A|} \frac{\int_{\widehat{T}}\int_{\Omega} \left(\tilde{\sigma}(\tilde{t}, \tilde{r}, \hat{a}^{\hat{j}})(a^j)\right) u_i(\tilde{t}, \omega, a^j)\pi(\hat{a}^{\hat{j}}|\hat{t}, \omega)\, \ell_i(\tilde{t}, \omega)\mu^{\Omega}(\mathrm{d}\omega)\tilde{r}(\tilde{t}, \mathrm{d}\hat{t})}{\sum_{\hat{j}=1}^{|\hat{A}|}\int_{\Omega}\int_{\widehat{T}} \pi(\hat{a}^{\hat{j}}|\hat{t}, \omega)\, \ell_i(\tilde{t}, \omega)\tilde{r}(\tilde{t}, \mathrm{d}\hat{t})\mu^{\Omega}(\mathrm{d}\omega)} \\
=&\sum_{\hat{j}=1}^{|\hat{A}|}\sum_{j=1}^{|A|} \frac{\int_{\widehat{T}}\int_{\Omega} \left(\tilde{\sigma}(\tilde{t}, \tilde{r}, \hat{a}^{\hat{j}})(a^j)\right) u_i(\tilde{t}, \omega, a^j)\overline{\pi}(\hat{a}^{\hat{j}}|\hat{t}, \omega)\, \ell_i(\tilde{t}, \omega)\mu^{\Omega}(\mathrm{d}\omega)\tilde{r}(\tilde{t}, \mathrm{d}\hat{t})}{\sum_{\hat{j}=1}^{|\hat{A}|}\int_{\Omega}\int_{\widehat{T}} \overline{\pi}(\hat{a}^{\hat{j}}|\hat{t}, \omega)\, \ell_i(\tilde{t}, \omega)\tilde{r}(\tilde{t}, \mathrm{d}\hat{t})\mu^{\Omega}(\mathrm{d}\omega)} \\
=&E^{\overline{\pi}}[u_i(\tilde{\sigma}_i, \tilde{\sigma}_{-i})|\tilde{r}_i, \tilde{r}_{-i}, \tilde{t}].
\end{aligned}
\tag{C.15}
$$

Thus by definition, $\pi \succcurlyeq_i \overline{\pi}$ and $\pi \preccurlyeq_i \overline{\pi}$ for each $i \in \mathcal{I}$, and we could conclude that π is effectively equivalent to $\overline{\pi}$.

(ii) Building on the above proof, for any strategy profile (r, σ), for any $t_i \in T_i$ and $i \in \mathcal{I}$, by taking the expectation of t_{-i} over T_{-i} in equation (C.15),

$$E^{\pi}[u_i(\sigma_i, \sigma_{-i})|r, t_i] = E^{\overline{\pi}}[u_i(\sigma_i, \sigma_{-i})|r, t_i]. \tag{C.16}$$

By equation (C.16), if receivers follow the strategy (r, σ), then the interim expected payoff of type t_i of receiver i is the same under π and under $\overline{\pi}$. Now consider any BNE $(r^* = (r_i^*)_{i\in\mathcal{I}}, \sigma^* = (\sigma_i^*)_{i\in\mathcal{I}})$ under π, then the following incentive compatibility constraint holds for any type t_i of any receiver i:

$$E^{\pi}[u_i(\sigma_i^*, \sigma_{-i}^*)|r^*, t_i] \geq \sup_{\hat{\sigma}_i\in\Sigma_i, \hat{r}_i\in R_i} E^{\pi}[u_i(\hat{\sigma}_i, \sigma_{-i}^*)|\hat{r}_i, r_{-i}^*, t_i].$$

By (C.16), the same strategy profile (r^*, σ^*) is also incentive compatible under persuasion mechanism $\overline{\pi}$, and thus (r^*, σ^*) is also a BNE under $\overline{\pi}$. Similarly, if (r^*, σ^*) is a BNE under $\overline{\pi}$, then it is also a BNE under π.

(iii) For any BNE (r^*, σ^*), (C.16) implies that the equilibrium payoff of each type of each receiver is the same under π and $\overline{\pi}$. The preservation of the sender's expected utility in (r^*, σ^*) under π and $\overline{\pi}$ is follows by taking expectation of Equation (C.11) over A, $\hat{A}$ and $\widehat{T}$ for $i = 0$. That is, $E^{\pi}[u_0(\sigma^*)|r^*] = E^{\overline{\pi}}[u_0(\sigma)|r]$. Thus we conclude our proof. □

C.2 Proofs of Section 4.3.2

Proof of Corollary IV.1. It follows directly from Theorem IV.1. □

Proof of Corollary IV.2. Given the cutoff-form of the optimal derandomized persuasion mechanism π, define a derandomized experiment $\overline{\pi}$ as follows: $\overline{\pi}(\omega) := \sup\{t \in [\underline{t}, \overline{t}] \mid \pi(\omega, t) = 1\}$. The construction of $\overline{\pi}$ implies that, if all receiver types follow the recommendation, then $\overline{\pi}$ achieves the same payoff for the sender as that under π. We will show $\overline{\pi}$ is incentive compatible below.

Note that the ranking assumption in Proposition 5.1, Guo and Shmaya (2019) is satisfied in this setting.[2] By Proposition 5.1 in Guo and Shmaya (2019), given that the persuasion mechanism π is incentive compatible and in a cutoff-form, the resulted experiment $\overline{\pi}$ is incentive compatible (i.e., by $\overline{\pi}$ fully disclosing the entire realized recommendation menu to all receive types, it is incentive compatible for each receiver type to follow its corresponding recommendation on the menu). The optimality of π implies that the derandomized experiment $\overline{\pi}$ is also optimal. □

Proof of Proposition IV.1. By Theorem 1 and 2 in Kolotilin et al. (2017), it is without loss of generality to focus on experiments. We will focus on canonical experiments that disclose posterior means to all receiver types. Each canonical experiment induces a distribution of posterior means such that it is a mean-preserving contraction of G^{Ω}. Let $v(\cdot)$ be the sender's payoff function defined on posterior means. For each realized posterior mean x, the sender's payoff is $v(x) = G^T(x)$. Then this problem can be transformed into the sender choosing an optimal CDF of posterior means F such that

$$\max_F \int_{\underline{\omega}}^{\overline{\omega}} v(x)\,\mathrm{d}F(x) \text{ subject to that } G^{\Omega} \text{ is a mean-preserving spread of } F. \tag{C.17}$$

By our assumption, $v(\cdot)$ on $[\underline{\omega}, \overline{\omega}]$ satisfies the regular condition in Dworczak and Martini (2019). By Corollary 1 in Dworczak and Martini (2019), there exists a convex and continuous function ("price function") $p(\cdot) : [\underline{\omega}, \overline{\omega}] \to \mathbb{R}$ with $p(x) \geq v(x)$ for any $x \in \Omega$ such that

[2] Guo and Shmaya (2019) also point out this relationship with the setting of Kolotilin et al. (2017) right after their Proposition 5.1.

F is optimal if and only if it satisfies the following conditions:

$$\operatorname{supp} F \subseteq \left\{x \in [\underline{\omega}, \overline{\omega}] \mid v(x) = p(x)\right\}, \tag{C.18}$$

$$\int_{\underline{\omega}}^{\overline{\omega}} p(x)\,\mathrm{d}F(x) = \int_{\underline{\omega}}^{\overline{\omega}} p(x)\,\mathrm{d}G^{\Omega}(x), \tag{C.19}$$

$$G^{\Omega} \text{ is a mean-preserving spread of } F. \tag{C.20}$$

Given the price function p that satisfies (C.18)-(C.20) for some optimal CDF of posterior mean F, by Proposition 2 in Dworczak and Martini (2019), there exists a coarsest partition $\underline{\omega} = x_0 < x_1, .. < x_n = \overline{\omega}$ such that on each $[x_i, x_{i+1}]$, either (i) or (ii) of the following holds:

(i) p is strictly convex on $[x_i, x_{i+1}]$, $p(x) = v(x)$ and $F(x) = G^{\Omega}(x)$ for all $x \in [x_i, x_{i+1}]$;

(ii) p is affine on $[x_i, x_{i+1}]$ and $[x_i, x_{i+1}]$ is a maximal interval on which p is affine. Then $F(x_i) = G^{\Omega}(x_i)$, $F(x_{i+1}) = G^{\Omega}(x_{i+1})$, $\int_{x_i}^{x_{i+1}} x\,\mathrm{d}G^{\Omega}(x) = \int_{x_i}^{x_{i+1}} x\,\mathrm{d}F(x)$, and $p(x') = v(x')$ for at least one $x' \in [x_i, x_{i+1}]$.

Given that the price function p is convex, p is either strictly convex or affine in each partitional interval $[x_i, x_{i+1}]$. That the above (i) or (ii) must be satisfies implies that F must be a mean-preserving contraction of G^{Ω} conditional on each partitional interval $[x_i, x_{i+1}]$. Then:

For the interval $[x_i, x_{i+1}]$ on which p is strictly convex, by (i), we have $F(x) = G^{\Omega}(x)$ for any $x \in [x_i, x_{i+1}]$. Thus to truthfully reveal the underlying state on $[x_i, x_{i+1}]$ is one way to achieve the optimal posterior distribution F on $[x_i, x_{i+1}]$ given G^{Ω}. Note that the above properties further implies that the mass of $\{x \in [\underline{\omega}, \overline{\omega}] \mid p \text{ strictly convex}\}$ is the same under F and under G^{Ω}. Similarly, the mass of $\{x \in [\underline{\omega}, \overline{\omega}] \mid p \text{ affine}\}$ is the same under F and under G^{Ω}.

Consider the interval $[x_i, x_{i+1}]$ on which p is affine. If $p(x) = v(x) = 0$ on $[x_i, x_{i+1}]$, then we set $F(x) = G^{\Omega}(x)$ on $x \in [x_i, x_{i+1}]$. By such a setting and combining with the fact that (i) or (ii) is satisfied in each of the rest partitional intervals, we can verify F satisfies condition (C.18)-(C.20). Hence, F is optimal by Corollary 1 in Dworczak and Martini (2019). Thus to truthfully reveal the underlying state on $[x_i, x_{i+1}]$ is one way to achieve the optimal posterior distribution.

Consider the interval $[x_i, x_{i+1}]$ on which p is affine such that $p(x) > 0$ for some $x \in [x_i, x_{i+1}]$. Then by the partitional regularity of G^T, we can divide $[\underline{t}, \overline{t}]$ into finitely many intervals such that G^T is either strictly concave or strictly convex on each interval. Based on such a partition, we can further divide the interval $[x_i, x_{i+1}]$ into several subintervals such that each subinterval is either (i) entirely outside $[\underline{t}, \overline{t}]$ or (ii) when inside $[\underline{t}, \overline{t}]$, then $v(\cdot)$ is either strictly convex or strictly concave on this subinterval. Recall that $p(x) \geq v(x), \forall\, x \in [x_i, x_{i+1}]$ and $p(x') > 0$ for some $x' \in [x_i, x_{i+1}]$. With p affine on $[x_i, x_{i+1}]$, for each subinterval within $[x_i, x_{i+1}]$, there will be at most one point x' in the entire subinterval such that $p(x') = v(x')$. Given that there are finitely many such intervals, the set $\{x \in \Omega \mid p(x) = v(x)\}$ will be finite. By (C.18), the support of the optimal F on $[x_i, x_{i+1}]$ will be at most finitely many points. This implies that the optimal experiment sends out finitely many messages on this region. Hence, by Theorem IV.1, there exists a derandomized experiment on the region when $p(v)$ is affine that preserves the sender's payoff.

Combining with the above three situations, we conclude that in this setting, there exists an optimal canonical experiment that is derandomized. □

C.2.1 Derivation of the optimal derandomized canonical experiment

Given a realized posterior mean x, let the sender's payoff function be $v(x)$, where $v(x) := G^T(x) = \frac{29x^2}{7} - \frac{40x^3}{7} + \frac{18x^4}{7}$ for any $x \in [0,1]$ and $v(x) = 0$ for any $x \in [-1, 0)$. By Proposition IV.1, there exists an optimal derandomized canonical experiment.

We further claim that we can find an optimal derandomized canonical experiment, denoted as ρ^*, that takes the following form: there exist two points x_1 and x_2 with $-1 \leq x_1 < x_2 \leq 1$ such that ρ^* fully reveal the state if $x \in [-1, x_1]$ or $x \in [x_2, 1]$ and ρ^* pools all the states in $[x_1, x_2]$.

To prove this claim, let G^{ρ^*} be the CDF of ρ^*. Given that $v(\cdot)$ satisfies the regularity conditions in Dworczak and Martini (2019), by their Corollary 1, there exists a convex and continuous function ("price function") $p(\cdot) : [-1, 1] \to \mathbb{R}$ with $p(x) \geq v(x)$ for any $x \in \Omega$

such that G^{ρ^*} satisfies

$$\operatorname{supp} G^{\rho^*} \subseteq \left\{x \in [-1,1] \mid v(x) = p(x)\right\}, \tag{C.21}$$

$$\int_0^1 p(x)\,\mathrm{d}G^{\rho^*}(x) = \int_0^1 p(x)\,\mathrm{d}G^{\Omega}(x), \tag{C.22}$$

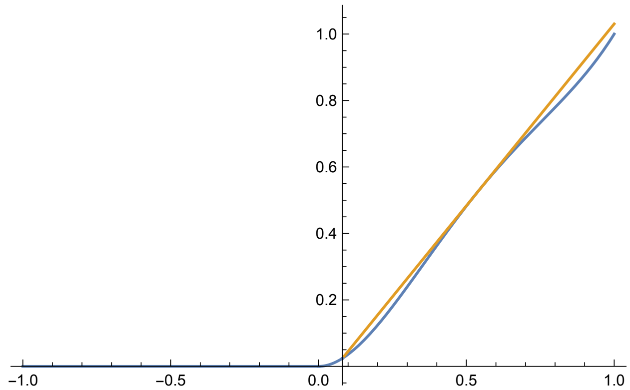
$$G^{\Omega} \text{ is a mean-preserving spread of } G^{\rho^*}. \tag{C.23}$$

Similar to that in the proof of Proposition IV.1, we can divide $[-1,1]$ into finitely many intervals such that on each interval, either p is strictly convex or that p is affine. Given that $v(\cdot)$ is of convex-concave-convex form and that $p = v$ if p is strictly convex, there exists an interval such that p is affine on this interval and it includes the entire concave region of $v(\cdot)$. By Proposition 1 in Dworczak and Martini (2019), the price function $p(\cdot)$ satisfies the following problem

$$\min_{p} \int_{\underline{\omega}}^{\overline{\omega}} p(x)\,\mathrm{d}G^{\Omega}(x) \text{ subject to } p \text{ being convex and } p \geq v. \tag{C.24}$$

The above minimization problem further implies that the price function must take the following form: there exist two points x_1 and x_2 with $0 \leq x_1 < x_2 \leq 1$ such that $p(x) = v(x) = 0$ on $[-1,0]$; $p(x)$ is strictly convex on $[0,x_1]$ and on $[x_2,1]$; $p(x)$ is affine on $[x_1,x_2]$. By Proposition 2 in Dworczak and Martini (2019), when p is strictly convex on a partitional interval $[x_i,x_{i+1}]$ such that it is a maximal interval on which p is strictly convex, then the optimal G^{ρ^*} satisfies that $G^{\rho^*}(x) = G^{\Omega}(x)$ for any $x \in [x_i,x_{i+1}]$. As shown in the proof of Proposition IV.1, when $p = v = 0$ on a partitional interval $[x_i,x_{i+1}]$ such that it is a maximal interval on which $p = v = 0$ holds, then there exists an optimal distribution of posterior means G^{ρ^*} such that $G^{\rho^*}(x) = G^{\Omega}(x)$ for any $x \in [x_i,x_{i+1}]$. Thus one way to achieve the optimum is to fully reveal the state for $x \in [-1,x_1] \cup [x_2,1]$. Moreover, that $v(\cdot)$ is in the convex-concave-convex form implies there will be at most one point $x \in (x_1,x_2)$ such that $p(x) = v(x)$ on the interval (x_1,x_2). Thus to pool the state within (x_1,x_2) is the only way to satisfy the necessary condition $\operatorname{supp} G^{\rho^*} \subseteq \left\{x \in [\underline{\omega},\overline{\omega}] \mid v(x) = p(x)\right\}$. Hence we prove our claim.

We then search for optimal derandomized canonical experiments in the above described structure with the aid of numerical methods. The solution shows that the optimal ρ^* pools state from $x_1 = 0.806$ to $x_2 = 1$, and fully reveal the states below 0.0806. We further plot the price function p that solves the above problem (C.24) in the following figure.



On interval $(0.0806, 1]$, $p(\cdot)$ is affine, represented as the orange straight line in the above figure. For any $x \leq 0.0806$, $p(x) = v(x)$, represented in the blue curve.

Figure C.1: The shape of price function $p(\cdot)$

C.3 Proofs of Section 4.4.1

Our proofs use the following two definitions (Definition 3.1 and Definition 3.4) in Shen et al. (2019). For an arbitrary probability space $(\Omega, \mathcal{F}, P)$, denote by $\mathcal{L}^1(\Omega; \mathbb{R}^n)$ the set of all integrable n-dimensional random vectors defined on $(\Omega, \mathcal{F}, P)$.

Definition C.2 (convex order)**.** Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ be two probability spaces. For random vectors $X \in \mathcal{L}^1(\Omega_1; \mathbb{R}^n)$ and $Y \in \mathcal{L}^1(\Omega_2; \mathbb{R}^n)$, we say $X|_{P1} \preccurlyeq_{cx} Y|_{P_2}$ if $E^{P_1}[f(X)] \leq E^{P_2}[f(Y)]$ for all convex functions $f : \mathbb{R}^n \to \mathbb{R}$.

Let $\mathcal{M}(\Omega_1)$ and $\mathcal{M}(\Omega_2)$ be the sets of probability measures on two arbitrary measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$, respectively.

Definition C.3 (dominated in heterogeneity)**.** We say $(P_1, ..., P_n) \in \mathcal{M}^n(\Omega_1)$ is dominated by $(Q_1, ..., Q_n) \in \mathcal{M}^n(\Omega_2)$ in heterogeneity, denoted by $(P_1, ..., P_n) \preccurlyeq_h (Q_1, ..., Q_n)$, if

$$\left(\frac{\mathrm{d}P_1}{\mathrm{d}P}, ..., \frac{\mathrm{d}P_n}{\mathrm{d}P}\right)\Big|_P \preccurlyeq_{cx} \left(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, ..., \frac{\mathrm{d}Q_n}{\mathrm{d}Q}\right)\Big|_Q,$$

for some $P \in \mathcal{M}(\Omega_1)$ dominating $(P_1, ..., P_n)$ and $Q \in \mathcal{M}(\Omega_2)$ dominating $(Q_1, ..., Q_n)$.

Proof for the claim in Example IV.2. (i) Recall that $\lambda_1 := \lambda$ and $\mathrm{d}\lambda_2 := 2\omega \, \mathrm{d}\lambda(\omega)$. Let ι

be an independent random variable ("randomization device") on the unit interval $[0,1]$ with the Lebesgue measure λ^ι. Note that the pair $(\lambda_2 \times \lambda^\iota, \lambda_1 \times \lambda^\iota)$ of atomless probability measures on $\Omega \times [0,1]$ satisfies the definition of conditionally atomless.[3] By Example 3.8 in Shen et al. 2019, the given pair (μ_2, μ_1) with $\mu_1 = \lambda$, and $\mathrm{d}\mu_2 = |4\omega - 2|\,\mathrm{d}\lambda$ satisfies that $(\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}, 1)|_{\mu_1} \preccurlyeq_{cx} (\frac{\mathrm{d}\lambda_2}{\mathrm{d}\lambda_1}, 1)|_{\lambda_1}$. Note that the densities satisfy $\frac{\mathrm{d}\lambda_2 \times \lambda^\iota}{\mathrm{d}\lambda_1 \times \lambda^\iota}|_{\lambda_1 \times \lambda^\iota} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}\lambda_1}|_{\lambda_1}$, which by Definition C.3 implies that $(\mu_2, \mu_1) \preccurlyeq_h (\lambda_2 \times \lambda^\iota, \lambda_1 \times \lambda^\iota)$. Then by Theorem 3.17 in Shen et al. (2019), (μ_2, μ_1) and $(\lambda_2 \times \lambda^\iota, \lambda_1 \times \lambda^\iota)$ are compatible.[4] Hence there exists a signal $\pi : \Omega \times [0,1] \to [0,1]$ which discloses identical information to both receivers such that $\lambda_2 \times \lambda^\iota \circ \pi^{-1} = \mu_2$ and $\lambda_1 \times \lambda^\iota \circ \pi^{-1} = \mu_1$. Thus, (μ_1, μ_2) are the message distributions of both receiver 1 and 2 induced by π under the pair of their initial beliefs.

(ii) We will show by contradiction that there does not exist a derandomized signal that is effectively equivalent to the signal π in (i). Suppose that there exists such a derandomized signal $g : \Omega \to [0,1]$ (for simplicity, we embed the signal isometric transformation into g) that is effectively equivalent to π. For any measurable set $\tilde{B} \in \sigma([0,1])$, consider the pure strategy equilibrium that both receiver 1 and 2 take the action 1 if and only if the signal realization $\hat{a} \in \tilde{B}$. Then under π, receiver 1's payoff is $\mu_1(\tilde{B})$ and receiver 2's payoff is $\mu_2(\tilde{B})$. Similarly, under the derandomized signal g and the above strategy, receiver 1's expected payoff is $\lambda_1 \circ g^{-1}(\tilde{B})$ and receiver 2's expected payoff is $\lambda_2 \circ g^{-1}(\tilde{B})$. That π and g are effectively equivalent implies that, the same strategy $1_{\tilde{B}}(\hat{a})$ of receiver i will give receiver $-i$ the same payoff under π and under g. Thus, for any measurable set $\tilde{B}$, $\mu_2(\tilde{B}) = \lambda_2 \circ g^{-1}(\tilde{B})$ and $\mu_1(\tilde{B}) = \lambda_1 \circ g^{-1}(\tilde{B})$. However, Example 3.8 in Shen et al. (2019) shows that there does not exist a function $g : \Omega \to [0,1]$ such that $\lambda_1 \circ g^{-1} = \mu_1$ and $\lambda_2 \circ g^{-1} = \mu_2$. We have a contradiction. □

Proof of Example IV.3. The proof is by contradiction. Suppose to the contrary that there exists a derandomized signal $\overline{\pi}$ (for simplicity, we embed the signal isometric transformation into $\overline{\pi}$) that is effectively equivalent to π, where $\overline{\pi}(\cdot) := \mathbb{1}_S(\cdot)$ for some fixed measurable set S. There must exist a pure strategy that is optimal under the given mechanism π, which is one of the following: (i) $\sigma(\cdot) \equiv \delta_0$; (ii) $\sigma(\cdot) \equiv \delta_1$; (iii) $\sigma(1) \equiv \delta_1$ and $\sigma(0) \equiv \delta_0$; (iv)

[3]Definition 3.12 in Shen et al. 2019: $(Q_1, ..., Q_n) \in \mathcal{M}^n(\Omega_1)$ is conditionally atomless if there exist $Q \in \mathcal{M}(\Omega_1)$ dominating $(Q_1, ..., Q_n)$ and $X \in L^0(\Omega_1; R)$ such that under Q, X is continuously distributed and independent of $(\frac{\mathrm{d}Q_1}{\mathrm{d}Q}, ..., \frac{\mathrm{d}Q_n}{\mathrm{d}Q})$.

[4]Definition 2.1 in Shen et al. (2019): $(Q_i)_{i \in J} \subseteq \mathcal{M}(\Omega_1)$ and $(F_i)_{i \in J} \subseteq \mathcal{M}(\Omega_2)$ are compatible if there exists a random variable X on $(\Omega_1, \mathcal{A}_1)$ such that F_i is the distribution of X under Q_i for each $i \in J$.

$\sigma(1) \equiv \delta_0$ and $\sigma(0) \equiv \delta_1$. We will consider only the types in $(0.5, \frac{11}{16})$. For any $t \in (0.5, \frac{11}{16})$, one could check that the optimal pure strategy under π is to follow the recommendation that gives him the following payoff

$$\int_{\Omega} \left(\mathbb{1}_{[t,1]}(\omega) \cdot 0.8 + \mathbb{1}_{[0.5,t]}(\omega) \cdot 0.2 + 0.8 \cdot \mathbb{1}_{[0,0.5)}(\omega)\right) \mathrm{d}\omega = 1.1 - 0.6t.$$

Similarly, there must exist a pure strategy that is optimal under $\overline{\pi}$. Given that π and $\overline{\pi}$ are effectively equivalent, an optimal pure strategy under $\overline{\pi}$ must give this type t the same payoff as well. For any $t \in (0.5, \frac{11}{16})$, the strategies $\sigma \equiv \delta_0$ and $\sigma \equiv \delta_1$ cannot be optimal under $\overline{\pi}$, since the respective payoffs of these two strategies are t and $1 - t$. So we consider the two only possible cases for the optimal strategy under $\overline{\pi}$: one is always following the recommendation (obedient strategy), i.e., $\sigma(1) \equiv \delta_1$ and $\sigma(0) \equiv \delta_0$; the other is always defying the recommendation (defiant strategy), i.e., $\sigma(1) \equiv \delta_0$ and $\sigma(0) \equiv \delta_1$. By playing the obedient strategy, each type $t \in (0.5, \frac{11}{16})$ gets the following expected payoff:

$$\begin{aligned} u^o(t) &:= \int_{\Omega} \Big(\overline{\pi}(\omega) \cdot 1_{[t,1]}(\omega) + \big(1 - \overline{\pi}(\omega)\big) \cdot 1_{[0,t)}(\omega)\Big) \,\mathrm{d}\omega \\ &= \int_{\Omega} (-2 \cdot \overline{\pi}(\omega) + 1) \cdot 1_{[0,t)}(\omega) \,\mathrm{d}\omega + \int_{\Omega} \overline{\pi}(\omega) \,\mathrm{d}\omega. \end{aligned} \tag{C.25}$$

Similarly, by playing the defiant strategy, type $t \in (0.5, \frac{11}{16})$ gets the following payoff:

$$\begin{aligned} u^d(t) &:= \int_{\Omega} \big((1 - \overline{\pi}(\omega)) \cdot 1_{[t,1]}(\omega) + \overline{\pi}(\omega) \cdot 1_{[0,t)}(\omega)\big) \,\mathrm{d}\omega \\ &= \int_0^t \big(-2 \cdot (1 - \overline{\pi}(\omega)) + 1\big) \,\mathrm{d}\omega + \int_{\Omega} \big(1 - \overline{\pi}(\omega)\big) \,\mathrm{d}\omega. \end{aligned} \tag{C.26}$$

Note that the function $u^o(t)$ and $u^d(t)$ are almost everywhere differentiable with respect to t. Let T_O be the set of types for whom the obedient strategy is strictly better than the defiant strategy, i.e., $T_O := \{t \in (0.5, \frac{11}{16}) \mid u^o(t) > u^d(t)\}$. Let T_D be the set of types for whom the defiant strategy is strictly better than the obedient strategy, i.e., $T_D := \{t \in (0.5, \frac{11}{16}) \mid u^o(t) < u^d(t)\}$. Let T_B be the set of types for whom both strategies are equally optimal, i.e., $T_B := \{t \in (0.5, \frac{11}{16}) \mid u^o(t) = u^d(t)\}$. Based on the above argument that $\sigma \equiv 1$ and $\sigma \equiv 0$ are both suboptimal for any type in $(0.5, \frac{11}{16})$, thus $(0.5, \frac{11}{16}) = T_O \cup T_D \cup T_B$, and at least one of them must be of strictly positive measure, which contradicts with the following Claim C.1 and thus to assume the existence of an effectively equivalent derandomized signal in the first place will lead to a contradiction. The rest is to prove the claim.

Claim C.1. *The set T_O, T_D and T_B are of zero measure.*

The proof of Claim C.1 is by contradiction. Consider the case that T_O is of strictly positive measure and note that by definition T_O is an open set. Given that the maximum payoff under $\overline{\pi}$ must equal to that under π, i.e., $1.1 - 0.6t = u^o(t),$ for any $t \in T_O$. Thus $u^{o\prime}(t) = -0.6$ for any $t \in T_O$, i.e., $-0.6 = -2 \cdot \overline{\pi}(t) + 1$ for every $t \in T_O$, and thus $\overline{\pi}(t) = 0.8$ for every $t \in T_O$. This contradicts with the definition that $\overline{\pi}$ is derandomized, i.e., $\overline{\pi} \in \{0, 1\}$ for almost everywhere. For the second case that T_D is of strictly positive measure, we can use similar argument to derive $\overline{\pi}(t) = 0.2$ for every $t \in T_D$, which again leads to a contradiction. Since both T_O and T_D are of zero measure, then we have T_B is only different from $(0.5, \frac{11}{16})$ by a zero measure set. However, the same argument as above implies $\overline{\pi} \notin \{0, 1\}$ for almost everywhere $\omega \in T_B$, which is again a contradiction. So we have shown that all of these sets cannot be of positive measure. ☐

Proof of Example IV.5. The logic behind the proof is quite similar to Example IV.3 and is also done by contradiction. Suppose to the contrary that there exists a derandomized signal $\overline{\pi}$ (for simplicity, we embed the signal isometric transformation into $\overline{\pi}$) that is effectively equivalent to π, where $\overline{\pi}(\cdot) := \mathbb{1}_S(\cdot)$ for some fixed measurable set S.

For each type t, its belief about the underlying state condition on type t will be the following distribution: $\frac{f(t,\omega)}{\frac{2}{3}(2-t)}\,\mathrm{d}\omega$. There must exist a pure strategy that is optimal under π. We will consider only the types in $(0.5, 0.6)$. For any $t \in (0.5, 0.6)$, one could check that the optimal pure strategy under π is to follow the recommendation that gives him the payoff:

$$\frac{\int_0^{0.5}(\omega - t)0.2\,\mathrm{d}\omega + \int_{0.5}^{t} 0.8(\omega - t)\,\mathrm{d}\omega + \int_t^1 1.6(\omega - t)\,\mathrm{d}\omega}{2-t} = \frac{0.725 + 0.4t^2 - 1.3t}{2-t}.$$

Similarly, there must exist a pure strategy that is optimal under $\overline{\pi}$. Given that π and $\overline{\pi}$ are effectively equivalent, an optimal pure strategy under $\overline{\pi}$ must give this type t the same payoff as well. For any $t \in (0.5, 0.6)$, the strategies $\sigma \equiv \delta_0$ and $\sigma \equiv \delta_1$ cannot be optimal under $\overline{\pi}$, since both $\frac{0.725+0.4t^2-1.3t}{2-t} > 0$ and $\frac{0.725+0.4t^2-1.3t}{2-t} > \frac{2+t^2-4t}{4-2t}$ hold for any $t \in (0.5, 0.6)$. So we consider the two only possible cases for the optimal strategy under $\overline{\pi}$: the obedient strategy and the defiant strategy. Under the obedient strategy, each type $t \in (0.5, 0.6)$ gets a payoff:

$$u^o(t) := \frac{\int_t^1 2\overline{\pi}(\omega)(\omega - t)\,\mathrm{d}\omega + \int_0^t \overline{\pi}(\omega)(\omega - t)\,\mathrm{d}\omega}{2 - t} \tag{C.27}$$

Similarly, by playing the defiant strategy, type $t \in (0.5, 0.6)$ gets the following payoff:

$$u^d(t) := \frac{\int_t^1 2(1 - \overline{\pi}(\omega))(\omega - t)\,\mathrm{d}\omega + \int_0^t (1 - \overline{\pi}(\omega))(\omega - t)\,\mathrm{d}\omega}{2 - t} \tag{C.28}$$

Note that the function $u^o(t)$ and $u^d(t)$ are almost everywhere differentiable with respect to t. Let T_O be the set of types for whom the obedient strategy is strictly better than the defiant strategy, i.e., $T_O := \{t \in (0.5, 0.6) \mid u^o(t) > u^d(t)\}$. Let T_D be the set of types for whom the defiant strategy is strictly better than the obedient strategy, i.e., $T_D := \{t \in (0.5, 0.6) \mid u^o(t) < u^d(t)\}$. Let T_B be the set of types for whom both strategies are equally optimal, i.e., $T_B := \{t \in (0.5, 0.6) \mid u^o(t) = u^d(t)\}$. Based on the above argument that $\sigma \equiv 1$ and $\sigma \equiv 0$ are both suboptimal for any type in $(0.5, 0.6)$, thus $(0.5, 0.6) = T_O \cup T_D \cup T_B$, and at least one of them must be of strictly positive measure, which contradicts with the following Claim C.2 and thus to assume the existence of an effectively equivalent derandomized signal in the first place will lead to a contradiction. The rest is to prove the claim.

Claim C.2. *The set T_O, T_D and T_B are of zero measure.*

The proof of Claim C.2 is again by contradiction. Consider the case that T_O is of strictly positive measure and note that by definition T_O is an open set. Given that the maximum payoff under $\overline{\pi}$ must equal to that under π, i.e., $\frac{0.725+0.4t^2-1.3t}{2-t} = u^o(t)$, for any $t \in T_O$, which implies that $\int_t^1 2\overline{\pi}(\omega)(\omega - t)\,\mathrm{d}\omega + \int_0^t \overline{\pi}(\omega)(\omega - t)\,\mathrm{d}\omega = 0.725 + 0.4t^2 - 1.3t$, for any $t \in T_O$. By taking derivative w.r.t t twice, the following holds: $\overline{\pi}(t) = 0.8$, for any $t \in T_O$. This is a contradiction. For the second case that T_D is of strictly positive measure, we can use similar argument to derive $\overline{\pi}(t) = 0.2$ for every $t \in T_D$, which again is a contradiction. Since both T_O and T_D are of zero measure, then we have T_B only different from $(0.5, 0.6)$ by a zero measure set. However, the same argument as above implies that $\overline{\pi} \notin \{0, 1\}$ for almost everywhere $\omega \in T_B$, which is again a contradiction. So we have shown that all of these sets cannot be of positive measure. □

BIBLIOGRAPHY

BIBLIOGRAPHY

Ahn, T. S., Hwang, I., and Kim, M.-I. (2010). The impact of performance measure discriminability on ratee incentives. *THE ACCOUNTING REVIEW*, 85(2):389–417.

Ali, S. N., Haghpanah, N., Lin, X., and Siegel, R. (2021). How to Sell Hard Information*. *The Quarterly Journal of Economics*, 137(1):619–678.

Aliprantis, C. D. and Border, K. C. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag, Berlin, Heidelberg, 3 edition. https://doi.org/10.1007/3-540-29587-9.

Alonso, R. and Camara, O. (2016a). Bayesian Persuasion with Heterogeneous Priors. *Journal of Economic Theory*, 165:672–706.

Alonso, R. and Camara, O. (2016b). Persuading Voters. *American Economic Review*, 165(11):3590–3605.

Arieli, I. and Babichenko, Y. (2019). Private Bayesian persuasion. *Journal of Economic Theory*, 182:185–217.

Aumann, R. J. (1976). Agreeing to disagree. *The Annals of Statistics*, 4(6):1236–1239.

Aumann, R. J. (1987). Correlated equilibrium as an expression of Bayesian rationality. *Econometrica*, 55(1):1–18. https://doi.org/10.2307/1911154.

Baiman, S. and Rajan, M. V. (1995). The informational advantages of discretionary bonus schemes. *The Accounting Review*, 70(4):557–579.

Baker, G., Gibbons, R., and Murphy, K. J. (1994). Subjective performance measures in optimal incentive contracts. *The Quarterly Journal of Economics*, 109(4):1125–1156.

Baldassarre, L. and Finken, B. (2015). Ge's real-time performance development.

Balder, E. J. (1988). Generalized equilibrium results for games with incomplete information. *Math. Oper. Res.*, 13(2):265–276. https://doi.org/10.1287/moor.13.2.265.

Bardhi, A. and Guo, Y. (2018). Modes of persuasion toward unanimous consent. *Theoretical Economics*, 13(3).

Bergemann, D. and Morris, S. (2016). Bayes correlated equilibrium and the comparison of information structures in games. *Theoretical Economics*, 11:487–522. https://doi.org/10.3982/TE1808.

Bergemann, D. and Morris, S. (2019). Information design: A unified perspective. *Journal of Economic Literature*, 1:1–57. https://doi.org/10.1257/jel.20181489.

Bernardin, H. J., LaShells, M. B., Smith, P. C., and Alvares, K. M. (1976). Behavioral expectation scales: Effects of developmental procedures and formats. *Journal of Applied Psychology*, 61(1):75–79.

Blackwell, D. (1952). Comparison of experiments. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, pages 93–102. University of California Press, Berkeley and Los Angeles. https://doi.org/10.2307/2334042.

Blackwell, D. (1953). Equivalent comparisons of experiments. *Ann. Math. Statistics*, 24:265–272. https://doi.org/10.1214/aoms/1177729032.

Bol, J. C. (2011). The determinants and performance effects of managers? performance evaluation biases. *THE ACCOUNTING REVIEW*, 86(5):1549–1575.

Bol, J. C. and Smith, S. D. (2011). Spillover effects in subjective performance evaluation: Bias and the asymmetric influence of controllabilit. *THE ACCOUNTING REVIEW*, 86(4):1213–1230.

Brandenburger, A. and Dekel, E. (1993). Hierarchies of beliefs and common knowledge. *Journal of Economic Theory*, 59:189–198.

Bull, C. (1983). Implicit contracts in the absence of enforcement and risk aversion. *The American Economic Review*, 73(4):658–671.

Cappelli, P. and Tavis, A. (2016). The performance management revolution.

Chen, Y.-C., He, W., Li, J., and Sun, Y. (2019). Equivalence of Stochastic and Deterministic Mechanisms. *Econometrica*, 87(4):1367–1390. https://doi.org/10.3982/ECTA14698.

Chen, Y.-C., Tillio, A., Faingold, E., and Xiong, S. (2017). Characterizing the strategic impact of misspecified beliefs. *Review of Economic Studies*, 84:1424–1471.

Cvitanić, J., Possamaï, D., and Touzi, N. (2018). Dynamic programming approach to principal-agent problems. *Finance Stoch.*, 22(1):1–37.

De Chiara, A. and Livio, L. (2017). The threat of corruption and the optimal supervisory task. *Journal of Economic Behavior & Organization*, 133(C):172–186.

Dekel, E., Fudenberg, D., and Morris, S. (2007). Interim correlated rationalizability. *Theoretical Economics*, 2:15–40.

Del Rey, J. (2021). Bias, disrespect, and demotions: Black employees say amazon has a race problem. *Vox*.

Demarzo, P. M. and Sannikov, Y. (2006). Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61(6):2681–2724.

Doval, L. and Ely, J. C. (2020). Sequential Information Design. *Econometrica*, 88(6):2575–2608.

Dvoretzky, A., Wald, A., and Wolfowitz, J. (1950). Elimination of randomization in certain problems of statistics and of the theory of games. *Proceedings of the National Academy of Sciences of the United States of America*, 36:256–260. https://doi.org/10.1073/pnas.36.4.256.

Dvoretzky, A., Wald, A., and Wolfowitz, J. (1951). Relations among certain ranges of vector measures. *Pacific J. Math.*, 1:59–74. https://doi.org/10.2140/pjm.1951.1.59.

Dworczak, P. and Kolotilin, A. (2019). The persuasion duality. *working paper*. https://arxiv.org/abs/1910.11392.

Dworczak, P. and Martini, G. (2019). The simple economics of optimal persuasion. *Journal of political economy*, 127(5). https://doi.org/10.1086/701813.

Ely, J. (2017). Beeps. *American Economic Review*, 107(11):31–53.

Ely, J., Frankel, A., and Kamenica, E. (2015). Suspense and surprise. Journal of Political Economy, 123(1).

Ely, J. and Peski, M. (2006). Hierarchies of belief and interim rationalizability. *Theoretical Economics*, 1(1):19–65.

Ely, J. C. and Szydlowski, M. (2020). Moving the Goalposts. *Journal of Political Economy*, 128(2):468–506.

Frederiksen, A., Lange, F., and Kriechel, B. (2017). Subjective performance evaluations and employee careers. *Journal of Economic Behavior & Organization*, 134:408–429.

Fuchs, W. (2007). Contracting with repeated moral hazard and private evaluations. *American Economic Review*, 97(4):1432–1448.

Galperti, S. and Perego, J. (2018). A dual perspective on information design. Working Paper (September 2018). Retrieved from http://www.jperego.com/files/papers/GP_lp/draft.pdf.

Geanakoplos, J., Pearce, D., and Stacchetti, E. (1989). Psychological games and sequential rationality. *Games and Economic Behavior*, 1:60–79.

Gentzkow, M. and Kamenica, E. (2016). A Rothschild-Stiglitz Approach to Bayesian Persuasion. *American Economic Review, Papers & Proceedings*, 106(5):597–601. https://doi.org/10.1257/aer.p20161049.

Gentzkow, M. and Kamenica, E. (2017). Bayesian persuasion with multiple senders and rich signal spaces. *Games and Economic Behavior*, 104(5):411–429. https://doi.org/10.1016/j.geb.2017.05.004.

Georgiadis, G. and Szentes, B. (2020). Optimal monitoring design. Econometrica, 88(5).

Gibbs, M., Merchant, K. A., der Stede, W. A. V., and Vargus, M. E. (2004). Determinants and effects of subjectivity in incentives. *The Accounting Review*, 79(2):409–436.

Giebe, T. and Gürtler, O. (2012). Optimal contracts for lenient supervisors. *Journal of Economic Behavior & Organization*, 81(2):403–420.

Golman, R. and Bhatia, S. (2012). Performance evaluation inflation and compression. *Accounting, Organizations and Society*, 37(1):534–543.

Guo, Y. and Shmaya, E. (2019). The interval structure of optimal disclosure. *Econometrica*, 87(2):653–675. https://doi.org/10.3982/ECTA15668.

Harsanyi, J. (1967). Games with Incomplete Information Played by “Bayesian” Players, I-III. *Management Science*, 14:159–182, 320–334, 486–502.

Hart, S. and Reny, P. J. (2015). Maximal revenue with multiple goods: nonmonotonicity and other observations. *Theoretical Economics*, 10(3). https://doi.org/ 10.3982/TE1517.

Hayes, R. M. and Schaefer, S. (2000). Implicit contracts and the explanatory power of top executive compensation for future performance. *The RAND Journal of Economics*, 31(2):273–293.

Holmstrom, B. and Milgrom, P. (1987). Aggregation and Linearity in the Provision of Intertemporal Incentives. *Econometrica*, 55(2):303–328.

Inostroza, N. and Pavan, A. (2020). Persuasion in global games with application to stress testing. *working paper*.

Ishida, J. (2012). Contracting with self-esteem concerns. *Journal of Economic Behavior & Organization*, 81(2):329–340.

Jawahar, I. M. and Williams, C. R. (1997). Where all the children are above average: The performance appraisal purpose effect. *Personnel Psychology*, 50(4):905–926.

Jehiel, P. (2005). Analogy-based expectation equilibrium. *Journal of Economic Theory*, 123(2):81–104.

Jehiel, P. and Koessler, F. (2008). Revisiting games of incomplete information with analogy-based expectations. *Games and Economic Behavior*, 62(2):533–557.

Johnson, R. A. (1970). Atomic and nonatomic measures. *Proc. Amer. Math. Soc.*, 25:650–655. https://doi.org/10.2307/2036664.

Judge, T. A. and Ferris, G. R. (1993). Social context of performance evaluation decisions. *The Academy of Management Journal*, 36(1):80–105.

Kamenica, E. (2019). Bayesian Persuasion and Information Design. *Annual Review of Economics*, 11:249–272.

Kamenica, E. and Gentzkow, M. (2011). Bayesian Persuasion. *American Economic Review*, 101(6):2590–2615. https://doi.org/10.1257/aer.101.6.2590.

Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer-Verlag Berlin Heidelberg New York, 2 edition.

Kolotilin, A. (2018). Optimal information disclosure: A linear programming approach. *Theoretical Economics*, 13(2):607–635. https://doi.org/10.3982/TE1805.

Kolotilin, A., Mylovanov, T., Zapechelnyuk, A., and Li, M. (2017). Persuasion of a privately informed receiver. *Econometrica*, 85(6):1949–1964. https://doi.org/10.3982/ECTA13251.

Kwoh, L. (2012). 'rank and yank' retains vocal fans.

Landy, F. J. and Farr, J. L. (1980). Performance rating. *Psychological Bulletin*, 87(1):72–107.

Lawler, I. E. E. (2003). Reward Practices and Performance Management System Effectiveness. *Organizational Dynamics*, 34(2):396?404.

Lehmann, E. L. (1988). Comparing location experiments. *Ann. Statist.*, 16(2):521–533. https://doi.org/10.1214/aos/1176350818.

Levin, J. (2003). Relational incentive contracts. *American Economic Review*, 93(3):835–857.

Li, H. and Shi, X. (2017). Discriminatory information disclosure. *American Economic Review*, 107(11):3363–3385. https://doi.org/10.1257/aer.20151743.

Lipnowski, E. and Mathevet, L. (2018). Disclosure to a psychological audience. AEJ: Microeconomics, 10.

Liu, Q. (2009). On redundant types and bayesian formulation of incomplete information. *Journal of Economic Theory*, 144:2115–2145.

MacLeod, W. B. (2003). Optimal contracting with subjective evaluation. *American Economic Review*, 93(1):216–240.

MacLeod, W. B. and Malcomson, J. M. (1989). Implicit contracts, incentive compatibility, and involuntary unemployment. *Econometrica*, 57(2):447–480.

Manelli, A. M. and Vincent, D. R. (2006). Bundling as an optimal selling mechanism for a multiple-good monopolist. *Journal of Economic Theory*, 127(1):1–35. https://doi.org/10.1016/j.jet.2005.08.007.

Manelli, A. M. and Vincent, D. R. (2007). Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. *Journal of Economic Theory*, 137(1):153–185. https://doi.org/10.1016/j.jet.2006.12.007.

Marchegiani, L., Reggiani, T., and Rizzolli, M. (2016). Loss averse agents and lenient supervisors in performance appraisal. *Journal of Economic Behavior & Organization*, 131(PA):183–197.

Maskin, E. and Tirole, J. (1990). The Principal-Agent Relationship with an Informed Principal: The Case of Private Values. *Econometrica*, 58(2):379–409.

Maskin, E. and Tirole, J. (1992). The Principal-Agent Relationship with an Informed Principal, II: Common Values. *Econometrica*, 60(1):1–42.

Mathevet, L., Perego, J., and Taneva, I. (2020). On Information Design in Games. Journal of Political Economy, 128(4).

McAfee, R. P. and McMillan, J. (1988). Multidimensional incentive compatibility and mechanism design. *Journal of Economic Theory*, 46(2):335–354. https://doi.org/10.1016/0022-0531(88)90135-4.

Mertens, J.-F. (2003). A measurable "measurable choice" theorem. *In: Neyman A., Sorin S. (eds) Stochastic Games and Applications. NATO Science Series, (Series C: Mathematical and Physical Sciences)*, 507:107–130. https://doi.org/10.1007/978-94-010-0189-2_9.

Mertens, J.-F. and Zamir, S. (1985). Formulation of Bayesian analysis for games with incomplete information. *International Journal of Game Theory*, 14(1):475–488.

Milgrom, P. and Roberts, J. (1986). Relying on the information of interested parties. *The RAND Journal of Economics*, 17(1):18–32.

Moers, F. (2005). Discretion and bias in performance evaluation: The impact of diversity and subjectivity. *Accounting, Organizations and Society*, 30(1):67–80.

Morris, S., Oyama, D., and Takahashi, S. (2020). Implementation via information design in binary-action supermodular games. *working paper*.

Myerson, R. (1982). Optimal coordination mechanisms in generalized principal-agent problems. *Journal of Mathematical Economics*, 10:67–81. https://doi.org/10.1016/0304-4068(82)90006-4.

Myerson, R. B. (1983). Mechanism Design by an Informed Principal. *Econometrica*, 51(6):1767–1797.

Orlov, D. (2020). Frequent monitoring in dynamic contracts. *working paper*.

Pavliotis, G. A. (2014). *Stochastic processes and applications*, volume 60 of *Texts in Applied Mathematics*. Springer, New York. Diffusion processes, the Fokker-Planck and Langevin equations.

Prendergast, C. (1999). The provision of incentives in firms. *Journal of Economic Literature*, 37(1):7–63.

Prendergast, C. and Topel, R. (1993). Discretion and bias in performance evaluation. *European Economic Review*, 37(2-3):355–365.

Protter, P. E. (2004). *Stochastic Integration and Differential Equations*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg New York, 2.1 edition.

Rayo, L. and Segal, I. (2010). Optimal information disclosure. Journal of Political Economy, 118(5). https://doi.org/10.1086/657922.

Renault, J., Solan, E., and Vieille, N. (2017). Optimal dynamic information provision. *Games and Economic Behavior*, 104(C):329–349.

Rynes, S. L., Gerhart, B., and Parks, L. (2005). Personnel psychology: Performance evaluation and pay for performance. *Annual Review of Psychology*, 56:571–600.

Saal, F. E., Downey, R. G., and Lahey, M. A. (1980). Rating the ratings: Assessing the psychometric quality of rating data. *Psychological Bulletin*, 88(2):413–428.

Samuelson, L. (1998). *Evolutionary Games and Equilibrium Selection*. The MIT Press, second edition. Economic Learning and Social Evolution.

Sannikov, Y. (2008). A continuous-time version of the principal-agent problem. *The Review of Economic Studies*, 75:957–984.

Schattler, H. and Sung, J. (1993). The First-Order Approach to the Continuous-Time Principal-Agent Problem with Exponential Utility. *Journal of Economic Theory*, 61(2):331–371.

Sharon, A. T. and Bartlett, C. J. (1969). Effect of instructional conditions in producing leniency on two types of rating scales. *Personnel Psychology*, 22(3):251–263.

Shen, J., Shen, Y., Wang, B., and Wang, R. (2019). Distributional compatibility for change of measures. *Finance Stoch.*, 23(3):761–794. https://doi.org/10.1007/s00780-019-00393-4.

Smolin, A. (2020). Dynamic evaluation design. *American Economic Journal: Microeconomics, Conditionally Accepted*.

Strausz, R. (2006). Deterministic versus stochastic mechanisms in principal-agent models. *Journal of Economic Theory*, 128(1):306–314. https://doi.org/10.1016/j.jet.2004.11.008.

Strulovici, B. and Szydlowski, M. (2015). On the smoothness of value functions and the existence of optimal strategies in diffusion models. *Journal of Economic Theory*, 159:1016–1055. Symposium Issue on Dynamic Contracts and Mechanism Design.

Taneva, I. (2018). Information design. Forthcoming at AEJ: Microeconomics. Retrieved from https://docs.google.com/viewer?a=v&pid=sites&srcid=ZGVmYXVsdGRvbWFpbnxpdGFuZXZhMTN8Z3g6N2Q4YTEwYjExODNiMmJlMw.

Tirole, J. (1986). Hierarchies and Bureaucracies: On the Role of Collusion in Organizations. *Journal of Law, Economics, and Organization*, 2(2):181–214.

Titova, M. (2021). Persuasion with verifiable information. *working paper*.

Trapp, I. and Trapp, R. (2019). The psychological effects of centrality bias: an experimental analysis. *J Bus Econ*, 89(2):155–189.

Wang, Y. (2015). Bayesian persuasion with multiple receivers. Working Paper (February 2015). Retrieved from http://research.soe.xmu.edu.cn/repec/upload/201503241124351199.pdf.

Wei, D. and Green, B. (2019). (Reverse) price discrimination with information design. Working Paper (December 2019).

Zábojník, J. (2014). Subjective evaluations with performance feedback. *The RAND Journal of Economics*, 45(2):341–369.

Zhu, J. Y. (2013). Optimal contracts with shirking. *The Review of Economic Studies*, 80(2 (283)):812–839.

Zhu, J. Y. (2020). A better monitoring . . .worse productivity? *manuscript*.

Ziegler, G. (2020). Adversarial bilateral information design. *working paper*.